Techniques in Artificial Intelligence

Propositional Logic Reasoning

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This Lecture

- Propositional Logic Reasoning Using Resolution
  - Resolution Rule
  - Conjunctive Normal Form
  - Resolution Algorithm

Recall: Propositional Logic Inference

To answer queries such as:
"Is ψ entailed by KB?"
"Is ψ valid?"

We studied two sound and complete inference procedures for propositional logic

1. *Natural deduction:* To automate it, we need an algorithm that "searches" through all possible sequences of natural deduction rules applied to KB until we find ψ or keep going indefinitely (generating more and more entailed sentences)
   - Can be improved by using special heuristics (rules that we know, from experience, will get us to the solution quickly)

Recall: Propositional Logic Inference

2. *Truth table enumeration:* To automate it, we need an algorithm that generates and evaluates all possible interpretations of propositions in our language
   - Can be improved by ignoring *irrelevant* propositions (that do not appear in the sentence)

There are other ways of automating propositional logic inference

We will look at an inference procedure called *resolution*

Unit Resolution Rule

A **literal** l is a proposition or the negation of a proposition

Literals l and m are called **complementary literals** if one is the negation of the other (e.g. P and $\neg P$)

Unit Resolution inference rule:

Let $l_1, \ldots l_k$ and m be literals
And let l_i and m be complementary literals.

$$\frac{l_1 \vee \ldots \vee l_k, \qquad m}{l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k}$$

Example:

$$\frac{P \vee Q, \qquad \neg Q}{P}$$

General Resolution Rule

Resolution inference rule:

Let $l_1, \ldots l_k$ and $m_1, \ldots m_n$ be literals
And let l_i and m_j be complementary literals.

$$\frac{l_1 \vee \ldots \vee l_k, \qquad m_1 \vee \ldots \vee m_n}{l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n}$$

The new clause is called the **resolvent**

Example:

$$\frac{P \vee Q \vee S, \qquad \neg Q \vee R}{P \vee S \vee R}$$

Conjunctive Normal Form

A **conjunction** of literals is a sentence of the form: $l_1 \wedge \ldots \wedge l_k$

A **disjunction** of literals (or a **clause**) is a sentence of the form: $l_1 \vee \ldots \vee l_k$

Note that the resolution rule applies to two *disjunctions* of propositions only

Conjunctive Normal Form

A formula is in **conjunctive normal form (CNF)** if it is a conjunctions of disjunction of literals, i.e. if it is of the form:

$$A_1 \wedge \ldots \wedge A_n$$

where each A_i, $i = 1, \ldots, n$ is of the form

$$l_1 \vee \ldots \vee l_k$$

where each l_j, $j = 1, \ldots, k$, is a literal

Example:

$$(P \vee Q) \wedge (R \vee S \vee T) \wedge (\neg S \vee P)$$

Every sentence of propositional logic can be transformed into a logically equivalent sentence in CNF

Converting to CNF

Procedure for converting to CNF

1. Use equivalence $(A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$ to eliminate $\leftrightarrow$
2. User equivalence $A \rightarrow B \equiv \neg A \vee B$ to eliminate $\rightarrow$
3. Use De Morgan's laws
   $\neg(A \wedge B) \equiv \neg A \vee \neg B$ and
   $\neg(A \vee B) \equiv \neg A \wedge \neg B$
   to push $\neg$ immediately before propositions
4. Use equivalence $\neg\neg A \equiv A$ to eliminate double negation
5. Use the distributive law $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ to finalise the conversion

Note: distributive law $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ is not actually needed

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

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$$(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg P_2 \rightarrow P_3))) \quad \textit{S 1}$$

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

$(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg P_2 \rightarrow P_3)))$ *S 1*

$(\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3)))$ *S 2*

Converting to CNF (Example)

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$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 4*

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$(\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3)))$ *S 2*

$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 4*

$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 2*

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

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$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 4*
$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 2*
$((\neg\neg P_1 \vee \neg(P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 3*

Converting to CNF (Example)

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$(\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3)))$	*S 2*
$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 4*
$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 2*
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$((P_1 \vee (\neg P_2 \wedge \neg P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$	*S 3*

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

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$((\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3)))$ *S 2*

$((\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 4*

$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 2*

$((\neg\neg P_1 \vee \neg(P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 3*

$((P_1 \vee (\neg P_2 \wedge \neg P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3)))$ *S 3*

$(((P_1 \vee \neg P_2) \wedge (P_1 \vee \neg P_3)) \vee P_4) \wedge ((\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee (P_2 \vee P_3)))$ *S 5*

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

$$(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg P_2 \rightarrow P_3))) \quad S\,1$$

$$(\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3))) \quad S\,2$$

$$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3))) \quad S\,4$$

$$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \quad S\,2$$

$$((\neg\neg P_1 \vee \neg(P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \quad S\,3$$

$$((P_1 \vee (\neg P_2 \wedge \neg P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \quad S\,3$$

$$(((P_1 \vee \neg P_2) \wedge (P_1 \vee \neg P_3)) \vee P_4) \wedge ((\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee (P_2 \vee P_3))) \quad S\,5$$

$$(((P_1 \vee \neg P_2) \vee P_4) \wedge ((P_1 \vee \neg P_3) \vee P_4))$$
$$\wedge((\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee (P_2 \vee P_3))) \quad S\,5$$

Converting to CNF (Example)

Convert this sentence to CNF: $(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \leftrightarrow P_4$

$$(\neg P_1 \wedge (\neg P_2 \rightarrow P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg P_2 \rightarrow P_3))) \qquad S\ 1$$

$$(\neg P_1 \wedge (\neg\neg P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (\neg\neg P_2 \vee P_3))) \qquad S\ 2$$

$$(\neg P_1 \wedge (P_2 \vee P_3)) \rightarrow P_4) \wedge (P_4 \rightarrow (\neg P_1 \wedge (P_2 \vee P_3))) \qquad S\ 4$$

$$(\neg(\neg P_1 \wedge (P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \qquad S\ 2$$

$$((\neg\neg P_1 \vee \neg(P_2 \vee P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \qquad S\ 3$$

$$((P_1 \vee (\neg P_2 \wedge \neg P_3)) \vee P_4) \wedge (\neg P_4 \vee (\neg P_1 \wedge (P_2 \vee P_3))) \qquad S\ 3$$

$$(((P_1 \vee \neg P_2) \wedge (P_1 \vee \neg P_3)) \vee P_4) \wedge ((\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee (P_2 \vee P_3))) \qquad S\ 5$$

$$(((P_1 \vee \neg P_2) \vee P_4) \wedge ((P_1 \vee \neg P_3) \vee P_4))$$
$$\wedge((\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee (P_2 \vee P_3))) \qquad S\ 5$$

$$(P_1 \vee \neg P_2 \vee P_4) \wedge (P_1 \vee \neg P_3 \vee P_4) \wedge (\neg P_4 \vee \neg P_1) \wedge (\neg P_4 \vee P_2 \vee P_3)$$

A Closer Look at Resolution

Take the proof $P, P \rightarrow Q \vdash Q$

Let's try to do it using the resolution rule

Recall that $(P \rightarrow Q) \equiv (\neg P \vee Q)$

So have two clauses: P and $\neg P \vee Q$ that can be resolved:

$$\frac{P, \qquad \neg P \vee Q}{Q}$$

So, resolution is a disguised form of *modus ponens*

Inference Using Resolution

Question: $KB \models \psi$?

Method: The method is based on *proof-by contradiction*

1. Take $KB \wedge \neg\psi$
2. Convert $KB \wedge \neg\psi$ to CNF
3. Apply the resolution inference rule iteratively to the resulting clauses (whenever there are complementary literals) until either:
   - two clauses are resolved to the empty clause (i.e. contradiction), therefore we conclude ψ; or
   - cannot apply resolution rule any more, therefore conclude $\neg\psi$

Resolution (cont.)

Why is resolving to the empty clause a contradiction?

A disjunction is true only if at least one of its literals are true.
So if we get the empty clause, none of the literals is true

Resolution is sound and complete, (and someone proved it)

Resolution (Example)

Let $KB = (P \leftrightarrow (Q \vee R)) \wedge \neg P$

Want to prove that $KB \models \alpha$ where $\alpha = \neg Q$

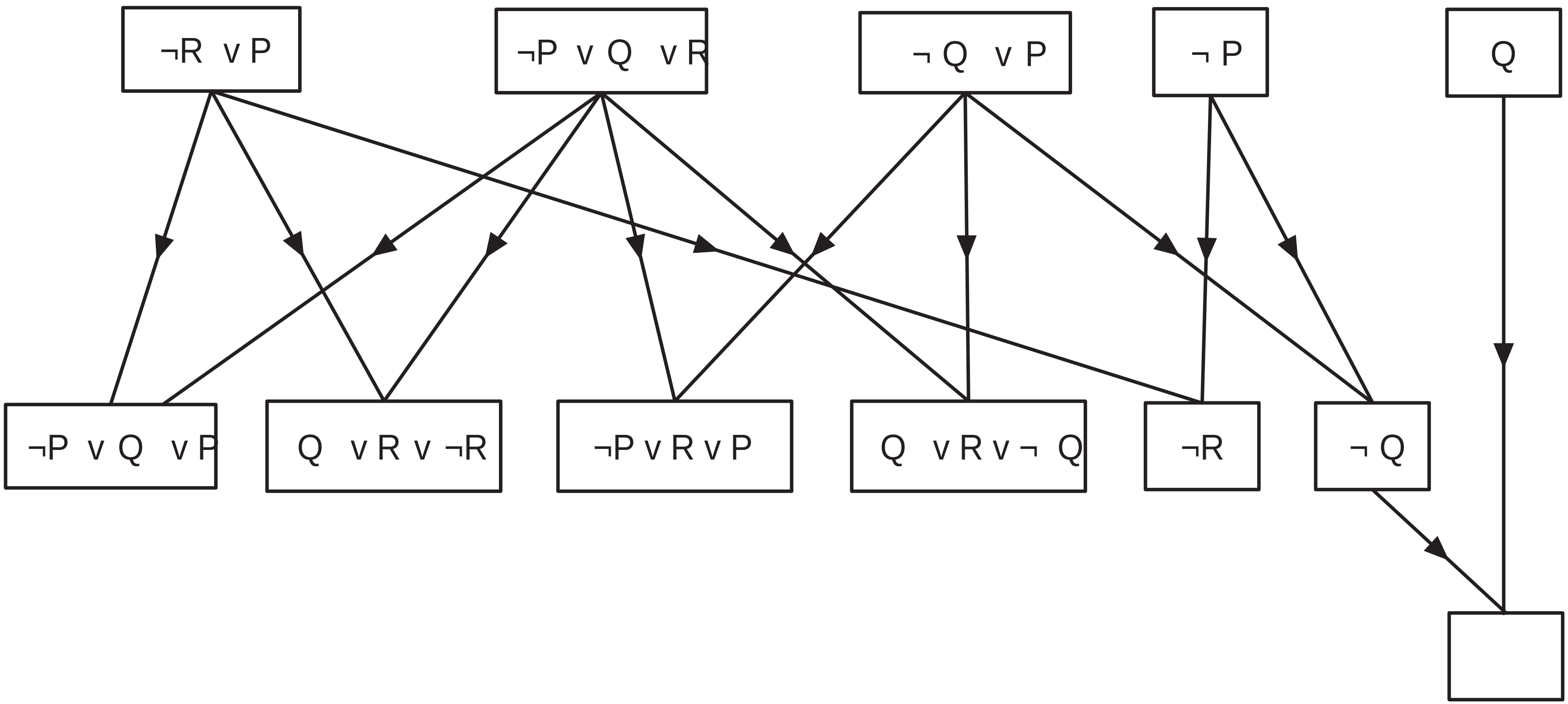
I.e. we should prove $KB \wedge \neg\alpha$:

I.e. we should prove $(P \leftrightarrow (Q \vee R)) \wedge \neg P \wedge \neg\neg Q$

Converting the above sentence to CNF (exercise), we have:

$(\neg R \vee P) \wedge (\neg P \vee Q \vee R) \wedge (\neg Q \vee P) \wedge \neg P \wedge Q$

Resolution (Example, continued)



Note that many resolutions were pointless. E.g. clause $\neg P \vee Q \vee P$ is equivalent to **True** $\vee\, Q$, which is equivalent to **True**. This is not very helpful, so any clause with two complementary literals can be discarded

Resolution Always Terminates

Resolution Closure of a set of clauses S, written $RC(S)$, is the set of all clauses derivable by repeated application of the resolution rule to clauses in S.

$RC(S)$ is finite, because we can only generate a finite set of distinct clauses from the propositions in S. Hence, resolution always terminates

Summary

- Propositional Logic Reasoning Using Resolution
  - Resolution Rule
  - Conjunctive Normal Form
  - Resolution Algorithm
- Required reading:
  - Stuart Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall, 3rd Edition, 2010
    - Sections 7.5.1 and 7.5.2