Stable Matchings in Balanced and Unbalanced Two-sided Markets with Random Preferences

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1 Introduction

Consider two finite sets M and W. We call vertices on the M-side *men* and vertices on the W-side *women*[1]. We will assume that $|M|\leq|W|$, i.e. there are at most as many men as there are women. Furthermore, let each man $i \in [|M|]$ have a strict preference ordering $\succ_i$ over the women, and each woman $j \in [|W|]$ have a strict preference ordering $\succ'_j$ over the men. What we have constructed is called a *two-sided matching market*. The market is *balanced* if $|M|=|W|$, otherwise it is *unbalanced*.

A *matching*[2] $\mathcal{M}$ in this market is an injection from the set of men to the set of women. For any man m, $\mathcal{M}(m)$ is called his *wife*, and for any woman w, $\mathcal{M}^{-1}(w)$ is called her *husband*[3]. A matching is *stable* if there is no *blocking pair*, i.e. a man m and a woman w such that both m prefers w to his wife in $\mathcal{M}$ and w prefers m to her husband in $\mathcal{M}$.

The aim of this paper is to introduce the reader to a class of highly interesting and technically demanding problems naturally arising in this setting. Namely, instead of treating men and women's preferences $\{\succ_i\}_{i\in[|M|]}$, $\{\succ'_j\}_{j\in[|W|]}$ as deterministic, let us assume each man and woman's preference list is generated according to a certain probability distribution independently of the other preference lists.

As a consequence of the randomness in preference orderings, we can now ask the following questions about stable matchings in the market:

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[1] We will refer to a generic man sometimes as $i \in [|M|]$, and sometimes as $m \in M$ or $m_i \in M$; this carries over to their preference lists, which could be denoted as $\succ_i$, or $\succ_{m_i}$, or $\succ_m$. The same convention holds for the women.

[2] Note that our definition of a matching requires that it covers every vertex on the smaller side of G. However, in other contexts, matchings may not be required to have this property: e.g., a matching could be empty.

[3] We will also call a husband or a wife a *spouse*.

- Fix a matching $\mathcal{M}$ (e.g., such that $\mathcal{M}(m_i) = w_i$ for $i \in [|M|]$). Now randomly generate all the preference lists. What is the probability that $\mathcal{M}$ is stable?

- Randomly generate all the preference lists. Let X be the random variable denoting the resulting number of stable matchings. What is $E[X]$? Does X concentrate around its expected value?

- For any matching $\mathcal{M}$, define the *rank* of a man $m \in M$ as 1 + the number of women higher than $\mathcal{M}(m)$ on his preference list $\succ_m$ (thus, if he gets his most preferred woman as his wife then $\text{rank}_m(\mathcal{M}) = 1$). Define the rank of each woman in the same way. Now, let the average rank of the men be $\text{rank}_{M,ave}(\mathcal{M}) = \frac{1}{|M|} \sum\limits_{m \in M} \text{rank}_m(\mathcal{M})$. Denote by $\bar{W}$ the set of unmatched women under $\mathcal{M}$, and let the average rank of matched women be $\text{rank}_{W,ave}(\mathcal{M}) = \frac{1}{|W/\bar{W}|} \sum\limits_{w \in W/\bar{W}} \text{rank}_w(\mathcal{M})$. Intuitively, $\text{rank}_{M,ave}(\mathcal{M})$ and $\text{rank}_{W,ave}(\mathcal{M})$ describe how well matching $\mathcal{M}$ treats men (resp. women) on average: the lower the average rank of the men (resp. women), the more favorable $\mathcal{M}$ is to the men (resp. women).

  What can we say about the range of the values of the average rank of the men (resp. of the women) over all possible stable matchings, given independently and randomly chosen preference lists?

Each of these questions should be quantitatively understood as asking whether the quantity of interest (i.e. $\Pr[M$ is stable$]$ for the first question, $E[X]$ for the second question, the range of $\text{rank}_{M,ave}$ for the third question) can be expressed, with high probability (w.h.p.), as a function of the sizes $|M|, |W|$ of the sets of men and women, as $\min(|M|, |W|) \to \infty$.

Specifically, in this paper we will scrutinize these questions with respect to two different models for generating preference orderings. The first one is classic and considers preference orderings that are uniformly random. This model has received attention since the 1970s and until now. It is correspondingly well explored.

In fact, for this model of uniformly random preferences, the three questions above are answered very differently when $|M| = |W|$ (the market is balanced) and when $|M| \neq |W|$ (the market is unbalanced). It turns out, for the unbalanced case, stable matchings are almost unique w.h.p. and, correspondingly, the best and worst possible average rank of men (women) over all stable matchings are essentially the same [AKL17]. This is in sharp contrast with earlier results [Pit92], that showed that there exist relatively many stable matchings in the balanced case. We will state these results precisely in Section 2, and we will discuss techniques that worked in the unbalanced case in Section 3.

Therefore, the three questions asked above can be sliced along this new dimension, to produce a very interesting, and currently open, question:

- Consider a random generation of preference lists different from the uniformly random generation just described. How different are the answers to the three above questions between the balanced and the unbalanced case? In particular, is it still true that in the unbalanced case, there will be much fewer stable matchings than in the balanced case?

In the alternative model we consider, each man and each woman is viewed as a Poisson clock with a certain rate. Each man then independently starts all women's Poisson clocks and forms his preference list according to the order in which the women's clocks went off. Each women goes through the same procedure but with respect to the men's Poisson clocks. This second model is quite natural but not well explored. We will discuss our attempts to proceed in this direction in Section 4. We conclude our exploration of all these fascinating topics in Section 5.

2 The Probabilistic Models and Overview of Known Results

We proceed to define formally two natural models for the probabilistic generation of preference lists. We then describe some classical and some state-of-the-art results that address the questions from above for the Uniform Preferences Model.

2.1 Two Probabilistic Models

We consider two models for randomly generating preference lists.

Uniform Preferences Model In this model, each $\succ_i$ and each $\succ'_j$ is generated independently as follows. For each $i \in [|M|]$, $\succ_i$ is chosen uniformly at random from the possible $|W|!$ preference orderings over the set of all women. For each $j \in [|W|]$, $\succ'_j$ is chosen uniformly at random from the possible $|M|!$ preference orderings over the set of all men.

Poisson Clocks Model In this model, we assign each man and each woman a personal *rate*. For man i, this rate is denoted $\lambda_i \geq 0$, and for woman j it is denoted $\lambda'_j \geq 0$. For an illustration, see Figure 1. Now, here is how man i's preference list is generated: man i considers an independent homogeneous Poisson process $\mathcal{P}'_j$ with rate λ'_j associated with each woman j. He takes the Poisson process $\mathcal{P}$ that is the superposition of these $|W|$ Poisson processes. He initializes his preference list to be empty. Now, he goes through all arrivals of $\mathcal{P}$ starting from the earliest. At each arrival, he identifies the woman j whose Poisson process generated this arrival. If woman j is already included in man i's preference list, then he goes to the next arrival of $\mathcal{P}$, otherwise he appends woman j to the end of his preference list - that is, as his least favorite woman so far. Once all women have been included in his preference list, he terminates.

For each woman j, construct her preference list in the same way as described above for the men, namely by creating and superimposing independent homogeneous Poisson processes $\mathcal{P}_i$, of respective rates λ_i, for all men, and then considering their earliest arrivals. We will refer to each Poisson process $\mathcal{P}_i$ or $\mathcal{P}'_j$ as a "Poisson clock", and their arrivals as "ringings". In this terminology, for each man, to construct his preference ordering, we start an independent Poisson clock $\mathcal{P}'_j$ for each woman. His preference ordering is then the following: given women j_1, j_2, he prefers j_1 to j_2 if and only if j_1's clock $\mathcal{P}'_{j_1}$ rings before j_2's clock $\mathcal{P}'_{j_2}$. The women decide their preference lists in the same manner.

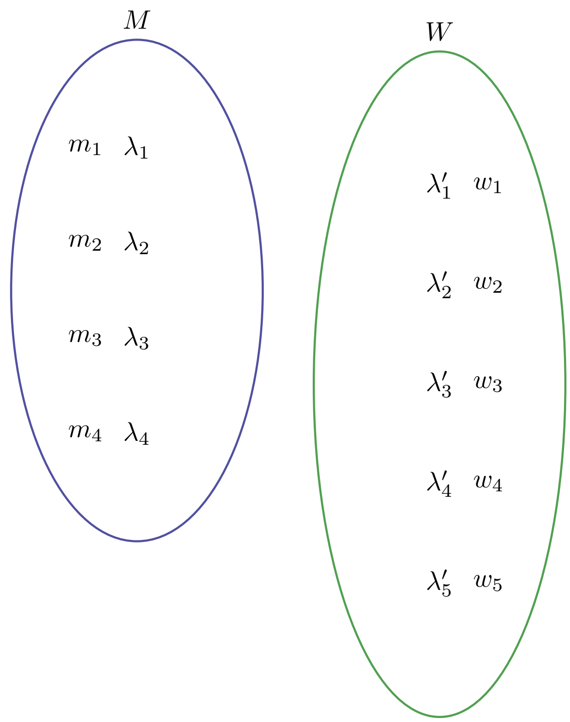

Figure 1: The Poisson Clocks Model: Men's and Women's Rates

Remark 1. *We would like to note that the Poisson Clocks model has* $|M|+|W|$ *degrees of freedom, expressed as parameters* λ_i*'s and* λ'_j*'s. This is very nice, because this offers us enough flexibility to model unbalanced matching markets: e.g., take* $n+k$ *men and* $n+k$ *women, and then, for the women* j *from* $n+1$ *through* $n+k$*, let their parameters* $\lambda'_j \to 0$*. This means that when the men will be building their preference lists, all these "extra" women will most likely always come after the women* 1 *through* n *in the preference ordering of any man. Therefore, they will always be unmatched, and none of them will be likely to create a blocking pair together with any of the men. Thus, the considered market is essentially equivalent to an unbalanced Poisson Clocks matching market on* $n+k$ *men and* n *women.*

Also, note that of course the Poisson Clocks model subsumes the Uniform

Preferences model, by simply setting all the parameters (all the λ_i's and all the λ_j's) equal.

2.2 Classical and New Results for the Uniform Preferences Model

We will now introduce the following definitions. Fix the preference orderings. Then for any man m, a woman w is his *stable partner* if the matchup (m, w) belongs to at least one stable matching. This definition of course also applies to define stable partners of women. The *Men-Optimal Stable Matching*, or MOSM for short, is the (unique, and existing, as shown in [GS62]) stable matching such that every man is matched with his best stable partner (i.e. his most preferred woman among all his stable partners). Consequently, in particular, the MOSM has the lowest average rank of all men. The Woman-Optimal Stable Matching (WOSM) is defined in the same way. Here is a fact proved in [GS62]: the MOSM is at the same time the *woman-pessimal* stable matching, that is, the stable matching where each woman gets her least preferred stable match. Analogously, the WOSM is the men-pessimal stable matching. To reiterate, we have

$$\mathrm{rank}_{M,ave}(MOSM) \leq \mathrm{rank}_{M,ave}(\mathcal{M}) \leq \mathrm{rank}_{M,ave}(WOSM),$$

$$\mathrm{rank}_{W,ave}(WOSM) \leq \mathrm{rank}_{W,ave}(\mathcal{M}) \leq \mathrm{rank}_{M,ave}(MOSM),$$

where $\mathcal{M}$ is any stable matching.

The thorough study of the questions listed in the Introduction was begun by Boris Pittel in the 1980's, inspired by Donald Knuth's book *On Stable Marriages*, which appeared in print for the first time in 1976 and was originally in French. Here is the summary of Pittel's results from [Pit92] and [Pit89]:

Theorem 2.1 (Stable Matchings in the Balanced Case, [Pit92])**.** *Say there are n men and n women, and their preference lists are all uniformly random and independent from each other. Then*

- *The expected number of stable matchings is asymptotic to $\frac{n \log n}{e}$ as $n \to \infty$.*
- *The worst and best average men's ranks a.s. satisfy that:* $\mathrm{rank}_{M,ave}(WOSM)$ *is asymptotic to $\frac{n}{\log n}$ as $n \to \infty$, and* $\mathrm{rank}_{M,ave}(MOSM)$ *is asymptotic to* $\log n$ *as $n \to \infty$. Of course, the situation for the women is symmetric since this is the balanced case.*
- *As $n \to \infty$, the total number of stable partners for any fixed man (woman) is asymptotically normal with mean and variance* $\log n$*.*
- *In the MOSM, the smallest rank of a man is w.h.p. on the order of* $(\log n)^2$.

Intuitively, this theorem claims that there are many stable matchings; that the stable matchings are diverse, in that each man and each woman have multiple stable partners; that in the MOSM, the average rank of men is quite favorable, while the average rank of women is not much better than the maximum possible value (n); and that not only is the average man well-off in terms of rank in the MOSM ($\log n$ on average), but also the worst-case scenario is very favorable, in that the worst rank a man could get is on the order of $(\log n)^2$.

In proving this result, Pittel relied on the following fundamental and long-known result proven by Knuth in the 1970s:

Theorem 2.2 (Knuth Integral Formula, [Knu97])**.**

$$\underbrace{\int_0^1 \cdots \int_0^1}_{2n\ variables} \prod_{1\le i\ne j\le n} (1 - x_i y_j) dx_1 \dots dx_n dy_1 \dots dy_n$$

is the probability that any of the $n!$ matchings on n men and n women is stable under the Uniform Preferences model.

Remark 2. *In fact, Pittel used, along with the Knuth Integral formula in its original form, also its following variant. Fixing not only $|M| = |W| = n$, but also two numbers $n \le k, l \le n^2$, it is possible to extend Knuth's formula to compute the probability that a given matching $\mathcal{M}$ is stable* and *the average rank of the men's side is $\frac{k}{n}$,* and *the average rank of the women's side is $\frac{l}{n}$. The extension is straightforward, so in the subsequent sections of this paper, including the Our Results section which discusses the Knuth Integral Formula for the Poisson Clocks model, we only talk about the original rather than the extended formula.*

Now, we note that the above results only extend to balanced matching markets. However, what if the number of (without loss of generality) women is greater than the number of men by 1 or more? It turns out, surprisingly, that in this case almost the exact opposites of the claims made in Theorem 2.1 hold. This was proved recently by Ashlagi et al. We give the exact statement of the result.

Theorem 2.3 (Stable Matchings in the Unbalanced Case, [AKL17])**.** *Let $\epsilon > 0$, and consider a sequence of matching markets with n men and $n + k_n$ women, indexed by $n \in \mathbb{N}$, where $\{k_n\}_{n=1}^{\infty}$ is an arbitrary sequence of positive integers. There exists some $N \in \mathbb{N}$ such that for all $n > N$, it holds with very high probability[4] that:*

- *In every stable matching $\mathcal{M}$, the average rank of men is*

$$\mathrm{rank}_{M,ave}(\mathcal{M}) \le (1+\epsilon)\frac{1}{n}(n+k)\left(\log \frac{n+k}{k}\right)$$

while the average rank of the matched women is

$$\mathrm{rank}_{W,ave}(\mathcal{M}) \ge (1-\epsilon)\frac{n}{1 + (1+\epsilon)\frac{1}{n}(n+k)\log\frac{n+k}{n}}.$$

[4]I.e. with probability at least $1 - \exp(-(\log n)^{0.4})$.

- *The average rank of men has a very narrow range:*

$$\frac{\text{rank}_{M,ave}(WOSM)}{\text{rank}_{M,ave}(MOSM)} \leq 1 + (\log n)^{-0.4},$$

and so does the average rank of matched women:

$$\frac{\text{rank}_{W,ave}(WOSM)}{\text{rank}_{W,ave}(MOSM)} \geq 1 - (\log n)^{-0.4}.$$

- *Having more than one stable partner is only possible for less than $\frac{n}{\sqrt{\log n}}$ men and less than $\frac{n}{\sqrt{\log n}}$ women. In fact, the total number of stable partners over all men (resp. women) is at most $n + \frac{n}{\sqrt{\log n}}$. Hence, it cannot be the case that there are at least a few men or women with a lot of stable partners.*

This theorem essentially states that in an unbalanced market, for uniformly random preferences, stable matchings are almost unique. The intuition for this comes from the Rural Hospital Theorem, which is often misattributed to Alvin E. Roth, but in fact was proven by McVitie and Wilson much earlier:

Theorem 2.4. *In the unbalanced case above, where there are n men and $n+k$ women, fix any preference lists, and consider any stable matching. Denote (as above) by $\bar{W}$ the set of unmatched women under this matching. Then, it turns out that in any other stable matching as well, the set of unmatched women will be exactly the same, $\bar{W}$.*

How is this related to the unbalanced case theorem above? It helps us establish the following. Take any unbalanced market T, and let its corresponding balanced market T' be obtained from T by removing the never-stably-matched-up women, $\bar{W}$. Intuitively, the stable matchings in T are in bijective correspondence with those matchings in the balanced market T' where the men's average rank is *so low that no man prefers his current wife to any woman in $\bar{W}$*. The latter matchings are clearly rare due to the low-rank restriction, and thus so are stable matchings in unbalanced markets.

Indeed, on the one hand, the Rural Hospital theorem implies that any stable matching in the unbalanced market T with n men and $n+k$ women induces a stable matching in the balanced market T' obtained from T by removing the women in $\bar{W}$. Conversely, take the balanced market T' on n men and n women. *If* no man prefers any women from $\bar{W}$ to his current wife, then if we add back the women from $\bar{W}$, this will not create any blocking pairs and so the same matching that was stable in T' will be stable in T.

In the following section, we will take a deeper dive into these results and show the intuition behind and some details of their proofs.

3 Techniques and Methods for Unbalanced Matching Markets with Uniform Preferences

3.1 Pittel's Results

Pittel obtained his results in [Pit92] and [Pit89] by using analytic tools. As mentioned above, his starting point was Theorem 2.2, the Knuth Integral Formula, along with its variations. He analysed the integral in these formulas analytically to establish sharp asymptotic bounds, and expressed various quantities that characterize average men's and women's ranks in stable matchings through these formulas. Furthermore, Pittel's works share two important tools with Ashlagi et. al's paper. These are

- *rotations*, which provide a way to modify a given stable matching that is not the WOSM so as to tranisition to another stable matching, where the average rank of the women has improved.
- The so-called Fundamental Algorithm that allows to compute the MOSM.

We discuss both these tools below, in the context of the Ashlagi et al. paper, that makes a more interesting (from an algorithmic point of view) use of these two tools.

3.2 Ashlagi et al.'s Results

Below in this section, we will briefly discuss [AKL17] in order to demonstrate tools that allow to resolve the questions stated in Section 1 for uniformly random preferences. Recall that Theorem 2.3 implies w.v.h.p. (with very high probability) that in unbalanced matching markets, there are very few stable matchings, and that the best and the worst possible men's (resp. women's) average rank is almost the same. At a high level, in order to prove this, it suffices to

- consider the MOSM and compute the average men's and women's ranks there;
- find a sequence of stable matchings that will, in a certain sense, "connect" the MOSM to the WOSM through the sequence of *all* other stable matchings;
- start at the MOSM and follow this sequence of stable matchings through to the WOSM; along the way, take note of the change in the average rank of the men and of the women;
- note that the changes observed in the previous step were small, and conclude that the WOSM has approximately the same average men's and women's ranks, as well as that the path between MOSM and WOSM is short, and hence that there are very few stable matchings.

The first of these steps is the actual computation of the best possible average men's rank, the worst possible average women's rank, etc., and uses a well-known algorithm for reaching the MOSM along with tools for its analysis such as the coupon collector's problem. The succeeding steps are all, in essence, achieved via concentration bounds.

3.2.1 Algorithms for generating the MOSM

The men-optimal stable matching can be efficiently computed using a number of efficient and simple algorithms. The first one historically and in terms of popularity is the Gale-Shapley algorithm. It is provided here.

Algorithm 1 Gale-Shapley Deferred Acceptance Algorithm.

- Start with all men and all women unmatched.
- Repeat for rounds $t = 1, 2, \ldots$ until all men are matched:
  - Take all currently unmatched men $m_1, \ldots, m_k$ and make m_i propose to the most favorite woman w_i in his list out of those he has not proposed to yet.
  - Each woman $w \in \{w_i\}_{i=1}^k$ considers all proposals she has received, and picks her most favorite man m_w out of those. Now if she is unmatched, she marries m_w. If she is matched to man m'_w then she unmatches m'_w and marries m_w if and only if she prefers the applicant to her current husband.

The second algorithm is called the McVitie-Wilson algorithm, or the Fundamental algorithm. It was invented about 10 years later and presented in [MW71]. Its main difference from Gale-Shapley algorithm is that it only asks one unmatched man to propose at a time. This makes it a more appropriate algorithm to study probabilistically. Indeed, in contrast, Gale-Shapley asks all currently unmatched men to propose simultaneously, and thus in each round each woman may get more than one proposal.

Algorithm 2 McVitie-Wilson Fundamental Algorithm.

- Start with all men and all women unmatched.
- Initialize m to be an arbitrary man (who is thus currently unmatched).
- Repeat for rounds $t = 1, 2, \ldots$ until all men are matched:
  - Make m propose to the most favorite woman w_m in his list out of those he has not proposed to yet.
    * If woman w_m is unmatched, she marries m. Now, set $m \leftarrow$ an arbitrary remaining unmatched man.
    * Suppose w_m is matched to man m'_w. If she prefers the applicant to her current husband, she unmatches m'_w and marries m_w. In that case, set $m \leftarrow m'_w$. Otherwise, she rejects the applicant; m is unchanged and we continue into round $t + 1$, where m will be proposing to his next favorite woman.

We now define a *rotation* of a stable matching as a cyclic sequence $(m_1, w_1), \ldots, (m_r, w_r)$ (so that $(m_{r+1}, w_{r+1}) = (m_1, w_1)$) of matched-up pairs, such that for each i, woman $i + 1$ is the most preferred woman for man i among those woman who he prefers his wife to and who prefer m_i over their current husband. Rotations have the following crucial property: if we now break all the marriages in the rotation and, for each $i \in [r]$, marry m_i to w_{i+1}, then the resulting matching will be stable, and the average rank of the women will have decreased.

Now, we present Ashlagi's algorithm:

Algorithm 3 Ashlagi et al. Algorithm for Generating the MOSM and then traversing all stable matchings leading down to the WOSM.

- Apply the Fundamental Algorithm to reach the MOSM.
- Initialize the set of "finalized" women for whom there exists no stable matching where they would receive a better partner, to be $S = \bar{W}$ (by the Rural Hospital Theorem, these woman are never matched up stably, hence cannot get better stable partners in the future). When S is full, we have reached the WOSM.
- Do the following, until the WOSM is reached:
  - Take the woman outside S who has received the most proposals so far, and unmatch (divorce) her from her husband m. Let the current set of divorced women be $V \leftarrow (w)$.
  - Let m propose to his next most preferred woman. If that woman is in V and prefers m to her current husband, we have found a rotation involving part of the women in V. Rotate, which makes us transition to the next stable matching, and backtrack all other divorces that we have done in this phase. Start a new phase.
  - If that new woman is not in V and prefers m, then she rejects her current husband, and we replace m with that husband and keep going.
  - If the new woman is in S, then take all the women in V, add them to S, backtrack all changes in this phase, and restart a new phase.
  - If the new woman rejects m, he keeps proposing.

This algorithm relies on the following fact from [IM05]: that if the woman w whose marriage we broke induces a chain of proposals that does not contain a rotation and ends in a man proposing to some woman in S, then w currently is married to her favorite stable partner, and hence should be included in S.

4 Our Results

For our own research, we have focused on the Poisson Clocks Model. Specifically, we determined that we will try to rederive Pittel's results from [Pit92] for this model instead of the Uniform Preferences Model that he considered. Pittel's derivations are based on the Knuth Integral Formula. We recall that this formula computes the probability that the generic matching $M = \{(m_1, w_1), \ldots, (m_n, w_n)\}$ is stable.

We prove that

Theorem 4.1. *In the Poisson Clocks Model, the probability that the generic* *matching* $M = \{(m_1, w_1), \dots, (m_n, w_n)\}$ *is stable is*

$$\Pr[M \textit{ is stable}] = \prod_{i=1}^{n}(\lambda_i \lambda_i') \int_{(x_1,\dots,x_n)\in[0,\infty)^n} \int_{(y_1,\dots,y_n)\in[0,\infty)^n} dx_1 \dots dx_n dy_1 \dots dy_n \times$$
$$\times\, e^{-\sum_{i=1}^n \lambda_i' x_i - \sum_{i=1}^n \lambda_i y_i} \prod_{i\neq j}(e^{-\lambda_j' x_i} + e^{-\lambda_i y_j} - e^{-\lambda_j' x_i - \lambda_i y_j}).$$

Proof. The proof is relatively simple compared to the original derivation by Knuth in [Knu97] which used the inclusion-exclusion formula. It is based on the new proof of the Knuth Integral Formula given in [Pit89]. Denote by $X_{i,j}$ the Poisson Clock time of woman j wrt. man i, and define $Y_{i,j}$ as the Poisson clock time of man i wrt. woman j. First, let us, for each man m_i, sample the Poisson Clock time $X_{i,i} = x_i$ of his match, woman w_i, and for each woman w_i, sample the Poisson Clock time $Y_{i,i} = y_i$ of her match, man m_i. Then, let us condition on these values, which allows us to treat these times as fixed. Now,

$$\Pr[M \text{ is stable}] =$$
$$\int_{(x_1,\dots,x_n)\in[0,\infty)^n} \int_{(y_1,\dots,y_n)\in[0,\infty)^n} dF_{\lambda_1'}(x_1) \dots dF_{\lambda_n'}(x_n) dF_{\lambda_1}(y_1) \dots dF_{\lambda_n}(y_n) \times$$
$$\times \Pr[M \text{ is stable}|X_{1,1} = x_1, \dots, X_{n,n} = x_n, Y_{1,1} = y_1, \dots, Y_{n,n} = y_n]. \tag{1}$$

Here, our notation is that $F_\lambda(x) = \Pr[Expon(\lambda) \leq x] = 1 - e^{-\lambda x}$ is the CDF of an exponential variable with mean $\frac{1}{\lambda}$.

Let us thus compute the conditional probability in this expression. This can be done easily if we notice that M will be stable if and only if none of the pairs (m_i, w_j) is blocking, for $i \neq j, (i,j) \in [n]^2$. Denoting the following set $A = \{X_{1,1} = x_1, \dots, X_{n,n} = x_n, Y_{1,1} = y_1, \dots, Y_{n,n} = y_n\}$, we have

$$\begin{aligned}\Pr[(m_i, w_j) \text{ is blocking}|A] &= \Pr[m_i \text{ prefers } w_j \text{ to } w_i \bigwedge w_j \text{ prefers } m_i \text{ to } m_j|A] \\ &= \Pr[m_i \text{ prefers } w_j \text{ to } w_i|A] \Pr[w_j \text{ prefers } m_i \text{ to } m_j|A]\end{aligned}$$

due to independence of the two events $\{m_i \text{ prefers } w_j \text{ to } w_i\}$ and $\{w_j \text{ prefers } m_i \text{ to } m_j\}$, conditioned on A. Note that

$$\Pr[m_i \text{ prefers } w_j \text{ to } w_i|A] = \Pr[X_{i,j} < X_{i,i}|A] = \Pr[X_{i,j} < x_i] = 1 - e^{\lambda_j' x_i}.$$

In the same way,

$$\Pr[w_j \text{ prefers } m_i \text{ to } m_j|A] = \Pr[Y_{i,j} < Y_{j,j}|A] = \Pr[Y_{i,j} < y_i] = 1 - e^{\lambda_i y_j}.$$

From the last three displays, we deduce that

$$\Pr[(m_i, w_j) \text{ is blocking}|A] = (1 - e^{\lambda_j' x_i})(1 - e^{\lambda_i y_j}),$$

and consequently

$$\begin{aligned}\Pr[(m_i, w_j) \text{ is not blocking}|A] &= 1 - \Pr[(m_i, w_j) \text{ is blocking}|A] \\ &= e^{-\lambda'_j x_i} + e^{-\lambda_i y_j} - e^{-\lambda'_j x_i - \lambda_i y_j}.\end{aligned}$$

Thus,

$$\begin{aligned}\Pr[M \text{ is stable}|A] &= \prod_{(m_i,w_j):i\neq j} \Pr[(m_i, w_j) \text{ is not blocking}|A] \\ &= \prod_{i\neq j}(e^{-\lambda'_j x_i} + e^{-\lambda_i y_j} - e^{-\lambda'_j x_i - \lambda_i y_j}).\end{aligned}$$

Now integrating according to Equation 1, we obtain precisely the declared formula for the probability of M being stable, Qed. $\square$

Now we would like to simplify the integral expression that we have obtained. As it turns out, this simplification, which we present below, does not seem to simplify the matters in terms of estimating the asymptotics of the integral formula. However, it is, perhaps appealingly, expressed algorithmically rather than algebraically.

Theorem 4.2. *The probability that the matching* $M = \{(m_1, w_1), \ldots, (m_n, w_n)\}$ *is stable under the Poisson Clocks Model can be computed as follows.*

Algorithm 4 Computing the probability that the matching $M = \{(m_1, w_1), \ldots, (m_n, w_n)\}$ is stable under the Poisson Clocks Model.

Let L_i, $i \in [n]$, be the "box" of each man where he will "store" a subset of the λ'_j's, for $j \in [n]$. Also, let L'_j, $j \in [n]$, be the "box" of each woman where she will "store" a subset of the λ_i's, for $i \in [n]$.

Enumerate all possible ways to flip $2\binom{n}{2} = n^2 - n$ coins P_l, where each $P_l = 0, 1,$ or 2 with probabilities $1/3, 1/3, 1/3$, by $t \in [3^{n^2-n}]$. For each t:

- Initialize the sign variable to be $s = +1$.
- Initialize the boxes so that for each $i \in [n]$, $L_i = \{\lambda'_i\}$, and $L'_i = \{\lambda_i\}$.
- Enumerate all $n^2 - n$ unordered pairs (i, j), $i \neq j$, by $l = 1, \ldots, n^2 - n$. For each l:
  - Let (i, j) be the pair of indices enumerated by l.
  - If $P_l = 0$, then add λ'_j to box L_i.
  - If $P_l = 1$, then add λ_i to box L'_j.
  - If $P_l = 2$, then add λ'_j to box L_i, add λ_i to box L'_j, and flip the sign by setting $s \leftarrow -s$.
- For each $i \in [n]$, compute $\mathrm{Sum}(L_i) = \sum_{i \in [n]: \lambda'_i \in L_i} \lambda'_i$. Also, for each $i \in [n]$, compute $\mathrm{Sum}(L'_i) = \sum_{i \in [n]: \lambda_i \in L'_i} \lambda_i$.
- Form the fraction $F_t \leftarrow \frac{s}{\prod_{i \in [n]} \mathrm{Sum}(L_i) \prod_{i \in [n]} \mathrm{Sum}(L'_i))}$

Output the resulting formula $(\prod_{i=1}^{n} (\lambda_i \lambda'_i)) \cdot (\sum_{t \in [3^{n^2-n}]} F_t)$.

Let us note the overall pattern of the algorithm: we claim that

$$\Pr[M \text{ is stable}] / (\prod_{i=1}^{n} (\lambda_i \lambda'_i))$$

is a sum of 3^{n^2-n} fractions, each of which has numerator ± 1 and denominator a positive integer. Zooming into the denominator of each of the fractions, it will be the product of $2n$ terms. One for each woman and one for each man. For each fraction, Algorithm 4 "assembles" it step by step: it determines the overall sign of the fraction, as well as each of the $2n$ product terms in the denominator. We further can notice that each of these product terms will be a sum of either each of the λ_i's with coefficients 0 or 1, or a sum of each of the λ'_i's with coefficients 0 or 1. Metaphorically, the algorithm body above refers to the process of determining the coefficients of the λ_i's (or the λ'_i's) for each product term as creating a "box" corresponding to this term and then gradually filling this box with the λ_i's (or the λ'_i's) that will have coefficient 1 in the term's sum.

Let us now look at an example of the calculation performed in Theorems 4.1 and 4.2. Note that, due to the quick growth of the number of fractions in the resulting sum, this example is actually the only one that can be conveniently worked out on the limited space that we have.

Example 4.1. Consider a 2 men, 2 women matching, where the men have parameters λ_1, λ_2 and the women have parameters λ'_1, λ'_2. Using Theorem 4.1, we can calculate the probability that the matching $(m_1, w_1), (m_2, w_2)$ is stable to be

$$\begin{aligned}
&\lambda_1\lambda_2\lambda'_1\lambda'_2 \int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\int_{y_1=0}^{\infty}\int_{y_2=0}^{\infty} (e^{-\lambda'_1 x_2} + e^{-\lambda_2 y_1} - e^{-\lambda'_1 x_2 - \lambda_2 y_1}) \times \\
&\quad \times (e^{-\lambda'_2 x_1} + e^{-\lambda_1 y_2} - e^{-\lambda'_2 x_1 - \lambda_1 y_2}) e^{-\lambda'_1 x_1 - \lambda'_2 x_2} e^{-\lambda_1 y_1 - \lambda_2 y_2} dy_2 dy_1 dx_2 dx_1 \\
&\quad = \lambda_1\lambda_2\lambda'_1\lambda'_2 \Bigg[\frac{1}{(\lambda'_1+\lambda'_2)^2\lambda_1\lambda_2} + \frac{1}{(\lambda'_1+\lambda'_2)\lambda'_2(\lambda_1+\lambda_2)\lambda_2} \\
&\qquad + \frac{1}{(\lambda'_1+\lambda'_2)\lambda'_1(\lambda_1+\lambda_2)\lambda_1} + \frac{1}{\lambda'_1\lambda'_2(\lambda_1+\lambda_2)^2} + \frac{1}{(\lambda'_1+\lambda'_2)^2(\lambda_1+\lambda_2)^2} \\
&\qquad - \frac{1}{(\lambda'_1+\lambda'_2)^2(\lambda_1+\lambda_2)\lambda_2} - \frac{1}{(\lambda'_1+\lambda'_2)\lambda'_1(\lambda_1+\lambda_2)^2} \\
&\qquad - \frac{1}{(\lambda'_1+\lambda'_2)^2(\lambda_1+\lambda_2)\lambda_1} - \frac{1}{(\lambda'_1+\lambda'_2)\lambda'_2(\lambda_1+\lambda_2)^2} \Bigg].
\end{aligned}$$

The integration was done by first expanding the product in the integrand into the sum of nine terms each of the form $\pm e^{-ax_1-bx_2-cy_1-dy_2}$, then applying linearity of integration so as to integrate each term separately, and finally using separation of variables on each term. For example, the first term in the sum is equal to

$$\begin{aligned}
&\lambda_1\lambda_2\lambda'_1\lambda'_2 \left(\int_0^{\infty} e^{-(\lambda'_1+\lambda'_2)x_1} dx_1\right)\left(\int_0^{\infty} e^{-(\lambda'_1+\lambda'_2)x_2} dx_2\right)\left(\int_0^{\infty} e^{-\lambda_1 y_1} dy_1\right)\left(\int_0^{\infty} e^{-\lambda_2 y_2} dy_2\right) \\
&= \lambda_1\lambda_2\lambda'_1\lambda'_2 \left[\frac{1}{(\lambda'_1+\lambda'_2)^2\lambda_1\lambda_2}\right].
\end{aligned}$$

This calculation thus led us precisely to the output of the algorithm in Theorem 4.2.

Proof of Theorem 4.2. Proving that the Algorithm in Theorem 4.2 generates an expression equal to the formula given in Theorem 4.1 is very similar to what we did in the Example above. Namely, considering the integrand in Theorem 4.1's statement, we see that after expanding the $2\binom{n}{2}$ parentheses, we will obtain 3^{n^2-n} different summands. The integral of each of these summands, H_t, can be computed by noting its separability along all of its variables $x_1, \ldots, x_n, y_1, \ldots, y_n$, and then using, for each of these variables, that $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}$ for $a > 0$ a constant. Indeed, as a result of multiplying the $2n$ evaluated integrals together, we will get a fraction whose denominator is a product of $2n$ terms, one for each variable, such that each of these terms is the sum of a subset of the λ_i's or the λ'_j's, correspondingly. Due to the exponential factors in the integrand outside the parentheses, we have for every i, that λ'_i

is in man i's term's sum, just as for every j, we have that λ_j is in woman j's term's sum. Other than that, which parameters will be summed up in each of the terms is determined based on which of the three exponentials inside each of the 3^{n^2-n} parentheses are included into the current summand H_t we are considering. This is precisely what the Algorithm in Theorem 4.2 computes using the language of boxes and ternary coin flips: the boxes store the λ's for each of the terms in the denominator of the summand H_t, while coin flips determine which of the three exponentials from each parenthesis H_t gets. Finally, the sign rule simply says that the sign of the summand H_t is the parity of the number of parentheses from which the term with the 'minus' sign (of the form $e^{-\lambda'_j x_i - \lambda_i y_j}$) is selected. $\square$

5 Conclusions and Future Research Directions

In this paper, we have exposed the reader to a spectrum of results and approaches to stable matchings in two-sided matching markets that are induced by randomly generated preference orderings. As mentioned before, the Uniform Preferences model is very simple and well-studied, but it is the only model that has yielded to researchers' efforts. Therefore, the Poisson Clocks model, perhaps the second most natural in this setting, remains to be explored, in all its flexibility stemming from the fact that each man or woman has an individual rate. Questions of interest mentioned in the Introduction are, however, worth answering even for scenarios when the number of effective parameters of the Poisson Clocks model is reduced - say, half the women share one rate and the other half has another rate.

The approach to the Knuth Integral Formula that we have developed may be of use to devise analytic estimation tools to answer these questions, that do not require using Ashlagi's algorithmic approach. On the other hand, as mentioned above, our 4.2 provides a simple, closed-form formula for the probability of a matching to be stable, and yet its asymptotic behaviour is highly mysterious. Indeed, each of the fractions F_t, that the formula is made of, shows up several times in the resulting summation (albeit with the same sign, as trivially witnessed by the algorithm in Theorem 4.2). There are exponentially many such fractions, and it is clear that there is a lot of cancellation happening, but we have not been able to find a good way to bijectively match up the various fractions so as to expose the asymptotic behavior of the probability.

Thus, for future research, we recommend that both the analytic approach through the new variant of the Knuth Integral Formula given here, and the algorithmic approach due to Ashlagi, which traces the path between the Men-Optimal and the Women-Optimal Stable matchings, be considered.

6 Pledge

I pledge my honor that this paper represents my own work, in accordance with University regulations. Acknowledgements: Thank you to Daniel Lima Braga and Seth Glas Lovelace for reading this manuscript and giving helpful comments.

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