PACM Proposal: Scoring Rules that Incentivize Precision

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October 2019

This project is supervised by Professor Matt Weinberg (COS), and the reader is Professor Mark Braverman. The model investigated in the project was formulated in collaboration with Eric Neyman and Matt Weinberg.

1 Introduction to the Model

The model arises in the following context: suppose a weather agency desires to solicit an expert's prediction of the probability of rain tomorrow. The prediction comes in the form of a single number $\hat{p}$, representing the expert's belief about the probability of rain, and we model the occurrence of rain tomorrow as a Bernoulli random variable $\mathcal{X}$ with unknown parameter p. In order to reward accurate predictions and penalize inaccurate ones, the weather agency utilizes a *scoring rule*, which formally is a function $f : [0,1] \to \mathbb{R}$, such that if the expert predicts $\hat{p}$ and it rains the next day, then the expert receives $f(\hat{p})$ dollars, while if it does not rain the next day, then the expert receives $f(1-\hat{p})$ dollars.

Traditionally, decision theory has focused on scoring rules that are *proper*. That is, if the expert's actual belief is $\tilde{p}$, then a scoring rule f is proper if it incentivizes the expert to report truthfully: for any $\tilde{p} \in [0,1]$, it is the unique maximizer of $R(p) = \tilde{p}f(p) + (1-\tilde{p})f(1-p)$, the expected reward from the expert's ex-ante point of view. Three classic proper scoring rules are the Brier, logarithmic, and spherical scoring rules.

However, what has been left out of this picture is the simple fact that the expert's belief, even if reported truthfully, will be of little value unless the expert is reasonably familiar with the true distribution of $\mathcal{X}$. For example, if the expert has not done any research into tomorrow's weather conditions, then a proper scoring rule would nudge them towards truthfully reporting $\hat{p} = \frac{1}{2}$, which corresponds to the uniform prior on the probability of rain tomorrow. Few would disagree that this is not a very useful, even if truthful, prediction.

In order to be able to measure the extent to which the expert's prediction is well-informed, we introduce the following model. The expert starts off having a uniform Bayesian prior on the true parameter of $\mathcal{X}$. Then, he or she performs statistical sampling from the true distribution $\mathcal{X}$, obtaining independently distributed sample points $X_1, \ldots, X_n$. The true belief $\hat{p}$ of the expert is then defined, according to Laplace's rule of succession, as $\frac{\sum_{i=1}^{n} X_i + 1}{n+2}$ (observe that this is also very close to the Maximum Likelihood Estimator of the parameter of $\mathcal{X}$). Furthermore, to account for the effort involved in performing the sampling, we assume that the expert incurs a constant cost of $c > 0$ dollars per sample.

Further, we measure how close the prediction is to the true value of the parameter according to the L1 norm of the difference $\hat{p} - p$, that is, as $\mathbb{E}[|\hat{p} - p|]$, where the expectation is with respect to the randomness of the samples. This measure decreases as the number of samples, n, increases. Therefore, assuming the expert reports truthfully, it is in the weather agency's interest to encourage the expert to draw more.

Now, we pose the natural question: how to quantify the degree to which a given *proper* scoring rule incentivizes the expert to draw as many samples as possible? To answer this question, we must define how the expert's sampling is influenced by a particular scoring rule

f. We introduce the notion of a *locally adaptive expert*, who draws samples one at a time, and after drawing the jth sample ($i \geq 1$) proceeds to draw the next sample if and only if the resulting expected increase in reward, with respect to the current Laplace belief (as defined above), exceeds the cost of an additional flip c. Expressed as a formula, the condition is that $\mathbb{E}[R(\frac{\sum_{i=1}^{j+1} X_i+1}{j+3})] - R(\frac{\sum_{i=1}^{j} X_i+1}{j+2}) \geq c$. The expectation is with respect to the random variable X_{j+1}, which is Bernoulli-distributed with parameter $\frac{\sum_{i=1}^{j} X_i+1}{j+2}$.

With the above definition, it is clear that choosing between any two proper scoring rules f, g, the weather agency would like to use the one that would, for fixed cost c of a sample, result in a smaller value of $\mathbb{E}[|\hat{p} - p|]$, where in this case $\hat{p}$ is the prediction of the locally adaptive expert, and the expectation is over all trajectories of the locally adaptive process corresponding to the given scoring rule. However, it is unclear whether, for two fixed proper scoring rules f and g, it must be the case that one of these two rules is better, in the above sense, for *all* values of $c > 0$. Therefore, it is not obvious whether any pair of scoring rules would be comparable in the above sense.

2 Results

We establish the following theorem.

Theorem 1. *Suppose the actual probability p of rain tomorrow is itself distributed uniformly on $[0, 1]$. Denote the cost-per-sample by $c > 0$. For any proper scoring rule f such that its associated reward function $R = pf(p) + (1-p)f(1-p)$ satisfies certain technical conditions, we have*

$$\lim_{c \to 0} c^{-1/4} \mathbb{E}\Big[\, |\hat{p} - p| \,\Big] = \int_{q=0}^{1} \sqrt[4]{\frac{2q(1-q)}{R''(q)}} dq,$$

where the expectation is with respect to p and the randomness of the locally adaptive process.

In other words, this theorem says that asymptotically as c tends to 0, the measure of the average closeness of the expert's prediction to the true value of the parameter p is inverse proportional to the fourth root of the cost. The coefficient of proportionality only depends on the scoring rule used. The importance of this result lies in its direct consequence.

Definition 1. *For any scoring rule f, define its* incentivization index *to be*

$$I(f) := \int_{q=0}^{1} \sqrt[4]{\frac{2q(1-q)}{R''(q)}} dq.$$

Corollary 1. *For any two proper scoring rules f_1 and f_2 with $I(f_1) < I(f_2) < \infty$ and whose reward functions R_f and R_g satisfy certain technical conditions, there exists a constant $c_{f_1,f_2} > 0$ such that for any cost-per-sample $c \in (0, c_{f_1,f_2})$,*

$$\mathbb{E}\Big[\, |\hat{p}_{f_1} - p| \,\Big] < \mathbb{E}\Big[\, |\hat{p}_{f_2} - p| \,\Big],$$

where $\hat{p}_{f_i}$ denotes the prediction of the locally adaptive expert under rule f_i, and each expectation is with respect to the uniform distribution of the true probability of rain p and the randomness in the respective locally adaptive process.

In other words, under some technical assumptions, for any two proper scoring rules with finite incentivization indices, the one that has a smaller incentivization index is more precise *for all small enough costs-per-sample.* Thus, there is a natural sense in which, subject to technical conditions, the weather agency can compare any two proper scoring rules in terms of how well they encourage the locally adaptive expert to be precise.