# An Improved Algorithm for Computing Approximate Equilibria in Weighted Congestion Games

Yiannis Giannakopoulos[1], Georgy Noarov[2*], Andreas S. Schulz[1],
[2]Princeton University [1]Technical University of Munich
*gnoarov@princeton.edu

## 1 Introduction

*Congestion games* model situations where a set of selfish agents compete over a set of shared resources. *Network routing games* are an important special case: in an undirected network $G$ with source $s$ and sink $t$, each player chooses a simple path from $s$ to $t$. The more congested an edge in $G$ becomes, the bigger delay (i.e. cost) is experienced by those players whose paths go through that edge. Each player's delay is the sum of the delays on all edges along its path.

Generally, a polynomial weighted congestion game of degree $d$ is defined as the tuple $(N, E, \{w_u\}_{u\in N}, \{S_u\}_{u\in N}, \{c_e\}_{e\in E})$. Here, $N$ is the set of players, $E$ is the set of resources, $w_u > 0$ is the weight of player $u$, $S_u \subseteq E$ is the strategy space of player $u$, and $c_e$ is the polynomial, of degree at most $d$, cost function of resource $e$. In a state $s$ of the game, each player $u$ pays $C_u(s) = w_u \sum_{e\in s_u} c_e(\sum_{u'\in N: e\in s_{u'}} w_{u'})$, where $s_k \in S_k$ denotes the strategy of player $k \in N$ in the state $s$.

In this project, we study *approximate equilibria* of these games.

**Definition 1.** *A state $s$ of a (polynomial weighted) congestion game $\mathcal{G}$ is called a $\rho$-*approximate (pure Nash) equilibrium *of $\mathcal{G}$ if in this state, no player $u \in N$ can unilaterally deviate and improve its cost by more than a factor of $\rho$. (A 1-approximate equilibrium of $\mathcal{G}$ is simply a pure Nash equilibrium.)*

#### Price of Anarchy of Equilibria

The notion of *Price of Anarchy* of congestion games was introduced by Koutsoupias and Papadimitriou (STACS'99) as the ratio between the worst social cost in an equilibrium and the optimal social cost. Thus, the Price of Anarchy quantifies the inefficiency of equilibria in congestion games. A generalization that is useful for the study of approximate equilibria is:

**Definition 2.** *The Price of Anarchy of $\rho$-approximate Nash equilibria in weighted congestion games of degree $d$ is*

$$PoA_d(\rho) := \sup_{\mathcal{G}} \max_{s} \frac{\sum_{u\in N} C_u(s)}{\sum_{u\in N} C_u(s_{opt})},$$

*where game $\mathcal{G}$ ranges over all polynomial weighted congestion games of degree $d$, state $s$ ranges over all $\rho$-approximate equilibria in $\mathcal{G}$, and $s_{opt}$ is a socially optimal state of $\mathcal{G}$ (i.e. minimizes the sum of players' costs).*

#### Potential Functions of Congestion Games

In the case of unweighted congestion games (i.e. those where $w_u = 1$ for every player $u \in N$), Rosenthal (Int'l J. of GT, 1973) found that for every such game $\mathcal{G}$, there exists a function $\Phi_{\mathcal{G}}$ over the states of $\mathcal{G}$, called the *potential function* of $\mathcal{G}$, which has the following property: for any states $s, s'$ that only differ by a unilateral deviation of a player $u \in N$,

$$\Phi(s) - \Phi(s') = C_u(s) - C_u(s').$$

Unfortunately, a potential function does not in general exist when $d > 1$, as implied by the work of Harks and Klimm (Mathematics of OR, 2012).

A recent result by Christodoulou et al. [2], however, is that for $d > 1$, there exists a $(d+1)$-approximate potential $\Phi'_{\mathcal{G}}$, in the sense that the property

$$\sum_{u\in N} C_u(s) \le \Phi'_{\mathcal{G}}(s) \le (d+1) \sum_{u\in N} C_u(s)$$

holds for all states $s$ of the game.

## 2 Computation of Equilibria in Congestion Games

In general, computing equilibria in congestion games or determining whether they exist is a computationally hard problem. Fabrikant et al. (STOC'04) showed that computing exact equilibria is PLS-complete. Worse still, Skopalik and Vöcking (STOC'08) showed that for *any* polynomially computable $\rho$, it is PLS-complete to compute a $\rho$-approximate equilibrium. Dunkel and Schulz (Mathematics of OR, 2008) demonstrated strong NP-completeness of determining the existence of exact equilibria.

### 2.1 Caragiannis et al. [1]: Efficient Computation of $d^{2d+o(d)}$-Approximate Equilibria in Degree-$d$ Polynomial Weighted Congestion Games

This was one of the first positive algorithmic results for equilibrium computation in congestion games. The key ideas are:

1. Given the input game $\mathcal{G}$, one can construct a special game $\Psi$ on the same state space as $\mathcal{G}$, such that: 1) $\Psi$ has a potential function, and 2) each $\rho$-approximate equilibrium in $\Psi$ is a $d!\rho$-approximate equilibrium in $\mathcal{G}$.
2. Now, it suffices to compute a $d^{d+o(d)}$-approximate equilibrium in $\Psi$. This is done by splitting the set of players into batches $B_1, \ldots, B_m$ (for some $m$), in decreasing order of their payments in a given initial state of $\Psi$. Now, process the batches one-by-one, so that at step $i$: 1) players from batches $B_i, B_{i+1}$ are allowed to sequentially perform individual best responses, provided they would gain "a lot" from best-responding; and 2) players who moved in batches $B_1, \ldots, B_{i-1}$ do not move anymore.
3. The approximation guarantee of $d^{d+o(d)}$ for game $\Psi$ is obtained by making use of the exact potential of $\Psi$ to analyze how players' payments change throughout the algorithm.

## 3 Our Results

We improve the algorithm by Caragiannis et al. by obviating the need for the game $\Psi$, which was mainly necessary for the analysis of the algorithm due to its exact potential property. Instead, we apply a modified version of that algorithm (as described in item 2. above) to the input game $\mathcal{G}$ itself. To analyze it, we reinterpret the Christodoulou et al. [2] approximate potential. The improved approximation guarantee $d^{d+o(d)}$ (in contrast with the previous guarantee of $d^{2d+o(d)}$) hinges on our results (of independent interest) regarding the approximate Price of Anarchy, given in Theorem 2 below.

**Theorem 1.** *Given any state $s$ of a polynomial weighted congestion game of degree $d$, one can efficiently compute a polynomially long chain of players' individual best responses leading from $s$ to a $d^{d+o(d)}$-approximate equilibrium of $\mathcal{G}$.*

**Theorem 2.**

$PoA_d(\rho) = \Phi_{d,\rho}^{d+1}$, *where $\Phi_{d,\rho}$ is the unique positive root of $\rho(x+1)^d = x^{d+1}$.*

*Additionally,*

$$PoA_d(\rho) \le \left(\frac{d}{\mathcal{W}(d/\rho)}\right)^{d+1}, \text{ where } \mathcal{W}(\cdot) \text{ is the Lambert-W function.}$$

### References

[1] Caragiannis et al. Approximate pure Nash equilibria in weighted congestion games: Existence, efficient computation, and structure. *EC'15*

[2] Christodoulou et al. The price of stability of weighted congestion games. *ICALP'18*