A Framework for the Price of Anarchy of Welfare and Revenue of Auctions

George Noarov and Daniel Braga

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1 Introduction

In these lecture notes that supplement our COS 597F presentation, we provide a condensed discussion of the paper titled *Price of Anarchy for Auction Revenue* by Hartline, Hoy, and Taggart (2014). This paper presents a framework for finding the Price of Anarchy of welfare and revenue of Bayes-Nash equilibria in First-Price Auctions and, more generally, in winner-pays-bid auctions and a variety of related mechanisms.

We restrict our attention to the derivations of the Price of Anarchy of Welfare and Revenue of single-item First-Price Auctions, since they have all the important features that any more general application of the discussed framework has. We note that our task is mainly to show what exact assumptions each part of the mentioned framework depends on, as well as to give an idea of how each of the parts is extended to bound the Price of Anarchy of revenue rather than welfare.

Our roadmap is as follows: first, we introduce the framework in extreme brevity, by only mentioning the key ideas. Then in the subsequent section we flesh out how its main two parts - Value Covering and Revenue Covering - are derived in the context of welfare. After that, we show all the important steps needed to extend the results to Price of Anarchy of revenue. Following that, we discuss important aspects of the framework such as its generalizability and their implications for possible future research, based on our own attempts.

2 Some Notation and Definitions

At first, let us set the stage for the following discussion in this paper. We will focus, as mentioned before, on single-item First-Price Auctions in the Bayesian context. Our notation associated with the FPA mechanism is standard: The mechanism itself is (X, P), and there are n bidders. Each bidder has a distribution F_i over its set of values. The F_i's are assumed independent but are not necessarily identical. The value of bidder i we will denote v_i, $x_i(v_i)$ being the corresponding probability of allocation of the item to bidder i, in expectation over the randomness of all other bidders' types. Further, $u_i(v_i)$ denotes the utility of bidder i and $p_i(v_i)$ the price bidder i has to pay, again both in expectation over the randomness of all other bidders' types. Note that in these definitions, we are treating the actual bidding strategies of the bidders as implicit. We define, additionally, for each bidder i, $b_i(v_i)$ to be its bid. *Note:* we will sometimes also abuse the above notation by writing $x_i(b_i)$, which is the allocation probability to bidder i given bid b_i.

We define the Price of Anarchy of Welfare of Bayes-Nash equilibria as follows for First-Price Auctions:

Definition 1. *The* Price of Anarchy of Welfare *of a FPA is*

$$\max_{s\text{: Bayes-Nash equil. for } M \text{ with dist. } F} \frac{\textit{WELFARE}(OPT, F)}{\textit{WELFARE}(M, s, F)}.$$

Here, $F = \times_i F_i$ is the distribution over all bidders' types, M is the FPA mechanism, OPT is the welfare-optimal auction (in this case, the optimal auction will be a variant of VCG).

Similarly, we define the Price of Anarchy of Revenue of Bayes-Nash equilibria as follows for First-Price Auctions:

Definition 2. *The* Price of Anarchy of Revenue *of a FPA is*

$$\max_{s\text{: Bayes-Nash equil. for } M \text{ with dist. } F} \frac{\textit{REVENUE}(OPT, F)}{\textit{REVENUE}(M, s, F)}.$$

Here, $F = \times_i F_i$ is the distribution over all bidders' types, M is the FPA mechanism, OPT is the welfare-optimal auction (in this case, the optimal auction will be dependent on the probabilities vector F). In addition, F is assumed regular*, in the usual Myersonian sense.*

The notion of Price of Anarchy quantifies inefficiency of equilibria in auctions. In our case we study this inefficiency in terms of both the welfare and revenue in BNE.

3 Brief Summary of the Framework for Welfare

We consider the framework presented in the Hartline et al. paper. We begin by super-briefly covering the key parts of it.

3.1 Threshold bids

The key new notion that we will need to define for this framework is that of threshold bids, which simply mean in this context that *a bidder can only win if its bid exceeds the bids made by all other bidders.* Thus, the maximum of the other bids can be viewed as a threshold that any bidder needs to overcome in order to get allocated the item.

Definition 3. *For any bidder i, its* threshold bid *is defined as*

$$\tau_i(v_{-i}) = \max_{j \neq i} b_j(v_j).$$

Further, the expected threshold bid *is defined as*

$$T_i = \mathbb{E}_{v_{-i}}[\tau_i(v_{-i})].$$

3.2 Value Covering

This is the first part of the framework. It captures the phenomenon that the utility and the threshold bid of a bidder i approximate that bidder's value. Algebraically, we have:

Lemma 4 (Value Covering)**.** *For any bidder i in any Bayes-Nash equilibrium of the FPA, it holds that*

$$u_i(v_i) + T_i \geq \frac{e-1}{e} v_i.$$

The validity of this inequality only depends on the fact that the bidders are in a BNE. The only property of BNEs used to derive Value Covering is that by definition of BNE, any bidder's utility in equilibrium weakly dominates its expected utility from bidding any other bid b: $u_i(v_i) \geq (v_i - b)x_i(b)$.

3.3 Revenue Covering

This is the second part of the framework. This part says that for any bidder, its threshold bid T_i is at most the revenue of the entire auction. Formally,

Lemma 5 (Revenue Covering)**.** *For any bid distribution B, the revenue of the FPA satisfies*

$$\textit{REVENUE}(FPA) \geq \sum_i T_i y_i,$$

where y is any feasible allocation.

We recall that a *feasible* allocation y in the context of single-item auctions satisfies that the y_i's sum to at most 1.

We remark, importantly, that the derivation of the Revenue Covering inequality *only depends on the format of the auction.* For FPAs it holds quite trivially, but in other common settings it may not.

3.4 Merging Value and Revenue Covering

This corresponds to the step where we apply both the inequalities above and get a bound on the welfare Price of Anarchy. Let us show how it is done:

Lemma 6 (Combined Inequality)**.** *When both value and revenue covering inequalities hold, one has*

$$\textit{UTIL}(FPA) + \textit{REVENUE}(FPA) \geq \mathbb{E}\left[\sum_i \big(u_i(v_i) + T_i x_i^*(v_i)\big)\right] \geq \frac{e-1}{e}\textit{WELFARE}(OPT),$$

where $x_i^*(\cdot)$ denotes the allocation probabilities of the welfare-optimal auction, and the expectation in the middle is taken over the randomness in the types of all bidders.

Note that UTIL(FPA) + REVENUE(FPA) = WELFARE(FPA), from which we obtain

Corollary 7. *The Price of Anarchy of Welfare of FPA is bounded according to*

$$\textit{PoA}_{Welfare}(FPA) = \frac{\textit{WELFARE}(OPT)}{\textit{WELFARE}(FPA)} \leq \frac{e}{e-1}.$$

Essentially, Lemma 6 hinges on

- Revenue covering for its left inequality, where we simply bound the sum of expected threshold bids T_i weighted by the optimal allocation probability x_i^* from above by the revenue of the FPA.

- Value covering for its right inequality, where bringing in the optimal allocation probability $x_i^*(v_i)$ for bidder i still has the value covering hold in the form of:

  $$u_i(v_i) + T_i x_i^*(v_i) \geq \frac{e-1}{e} v_i x_i^*(v_i).$$

  After summing over all bidders and taking the expectation, the RHS of the resulting value covering inequality will be $\frac{e-1}{e}\mathbb{E}\left[\sum_i v_i x_i^*(v_i)\right]$, which is exactly the welfare of the optimal auction WELFARE(OPT).

4 Filling Details into the Framework Summary

We now delve somewhat deeper into the proofs of the steps above. Indeed, we need to do so in order to understand what can be modified in the statements and proofs of Value and Revenue Covering in order to bound the Price of Anarchy of *revenue* rather than welfare.

4.1 Computing Threshold Bids

Let us see how we can calculate T_i, the expected threshold bid of any bidder i.

Lemma 8.

$$T_i = \int_0^1 t_i(x)dx,$$

where we define $t_i(x) = \min\{a|x_i(a) \geq x\}$*, the minimum taken over all possible bids* a *that bidder* i *can make.*

Proof. Very simple probabilistic argument. Note that the expected threshold is equal to

$$T_i = \int_0^\infty \text{Pr}(T_i > y)dy = \int_0^\infty \text{Pr}(i \text{ bids } y \text{ but doesn't get the item}) = \int_0^\infty (1 - x_i(y))dy.$$

Now, consider the inverse function of function $x_i(y)$ (y ranges over all possible bids for bidder i). This function is easy to see to be precisely $t_i(x)$ (x ranges over all possible allocation probabilities, i.e. over $[0, 1]$). Hence, $T_i = \int_0^\infty (1 - x_i(y))dy = \int_0^1 t_i dx$, Qed. $\square$

Note that the integrand, the function t_i, has the following intuitive meaning: $t_i(x)$ is how much bidder i needs to bid in order to have probability x of winning the item.

4.2 Proving Value Covering: where $\frac{e}{e-1}$ Comes From

From this integral characterization, we can lower-bound T_i and obtain the Value-Covering lemma. Namely, using the BNE condition for bidder i, we have for any $x \in [0, 1]$ that $u_i(v_i) \geq (v_i - t_i(x))x$. This inequality is true because it just says that in the BNE, the utility of bidder i is at least as much as i would get by bidding $t_i(x)$, the bid that ensures allocation probability x.

Rearranging this BNE condition, we have that $t_i(x) \geq \max\{v_i - \frac{u_i(v_i)}{x}, 0\}$, and now we can integrate this inequality over all $x \in [0, 1]$, getting

$$T_i = \int_0^1 t_i(x)dx \geq \int_{\frac{u_i(v_i)}{v_i}}^1 \left(v_i - \frac{u_i(v_i)}{x}\right) = v_i - u_i(v_i)\left(1 - \ln \frac{u_1(v_i)}{v_i}\right).$$

Thus, $u_i(v_i) + T_i \geq v_i + u_i \ln \frac{u_i}{v_i}$. Optimizing over u_i for fixed v_i, we have that $u_i(v_i) + T_i \geq \frac{e-1}{e} v_i$.

4.3 Proving Revenue Covering

Proving Revenue Covering turns out to be even easier than the last derivation. Indeed, for any bidder i, T_i is the expected maximum over all bids but one, so it is weakly dominated by the revenue of the FPA, since REVENUE(FPA) is the maximum expected bid.

5 Now Finally, Extending to Revenue

In Section 3.4, we learned how to combine Value Covering and Revenue Covering into an inequality giving the bound on the ratio of the optimal welfare and the welfare of a BNE. The easiest idea how to replace welfare with revenue everywhere in these formulas, comes, as always in the single-item case, from Myerson's Theorem. Specifically, recall that

Theorem 9 (Myerson, 81)**.** *Let $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ define the* virtual value *of bidder i. Then the expected virtual welfare $\mathbb{E}_v[\varphi_i(v_i)x_i(v_i)]$ equals the revenue of the auction.*

- So why don't we just replace v_i everywhere with $\varphi_i(v_i)$ in Value and Revenue covering inequalities? Let's try to do it like this: Since $u_i(v_i) \leq v_i x_i(v_i)$, we can rewrite Value Covering as

  $$v_i x_i(v_i) + T_i \geq \frac{e-1}{e} v_i.$$

  Now because $\varphi_i(v_i) \leq v_i$, substituting virtual values in place of values would only make the inequality weaker, and we would have obtained:

  Lemma 10 ("Virtual Value" Covering)**.** *For all i,*

  $$\phi(v_i)x_i(v_i) + T_i \geq \frac{e-1}{e}\phi(v_i).$$

  After taking expectations like in Section 3.4 we could have obtained revenue instead of welfare on the left and right sides of the combined inequality...

- No! The "Virtual Value" Covering inequality only approximates revenue among bidders with $\varphi_i(v_i) \geq 0$. Otherwise, the RHS is negative and we cannot apply the reasoning from Section 3.4.

- Thus, to salvage the possible transition to virtual values, we now assume that our FPA has *Myerson reserves*: for each bidder i, there is a *reserve price* r_i so that if bidder i bids lower than that, it cannot get the item. We call our new auction FPA_r

- However, we cannot introduce reserve prices without further modifications. Now, Revenue Covering falls apart.
  - What is the intuition behind Revenue Covering? *When all bidders have high thresholds (implying it is difficult for them to get the item), then the auction has high revenue*. But now, since we have reserve prices, then no matter how high the threshold bids T_i are, the Revenue Covering inequality may be false: maybe no bidder gets the item since they all haven't met their reserves, then REVENUE$(FPA_r) = 0$.
  - The real problem that we need to fix is therefore: how to redefine expected threshold bid T_i, so that revenue covering would hold? Note that right now, Lemma 8 is wrong: if the bid of bidder i is below the reserve, then the real threshold that prevents i from winning the item is not the maximum of the other bids, but rather the reserve r_i. Thus, below $x = x_i(r_i)$, t_i should be redefined as $t_i(x) = r_i$.
  - Now, we have that $T_i = r_i x_i(r_i) + \int_{x_i(r_i)}^1 t_i(x)dx$.
  - Denote the truncation of the integral for T_i that appears above as:

    $$T_i^{r_i} = \int_{x_i(r_i)}^1 t_i(x)dx.$$

It turns out that having implemented these two fixes:

1. Replace FPA by FPA_r, the First-Price auction with reserve prices r.
2. Define the updated notion of expected threshold bid as $T_i^{r_i} = \int_{x_i(r_i)}^1 t_i(x)dx$.

... We can actually adapt the welfare framework from above!

5.1 Value and Revenue Covering, Updated

First, let us reprove Revenue Covering. It requires no modification besides changing T_i to $T_i^{r_i}$.

Lemma 11 (Revenue Covering - 2). *For any bid distribution B, the revenue of the FPA_r auction with reserves r satisfies*

$$\textit{REVENUE}(FPA_r) \geq \sum_i T_i^{r_i} y_i,$$

where y is any feasible allocation.

Proof. For every bidder separately, REVENUE$(FPA_r) \geq T_i^{r_i}$. Indeed, whenever bidder i cannot get the item due to other higher bids (and *not* due to the reserve r_i), then the auction derives at least as much revenue as the contribution $T_i^{r_i}$ of other bids higher than r_i to bidder i's threshold. This proves the Revenue covering property. □

The Value Covering inequality changes to exactly the same format as we had anticipated, except one more condition is added: $v_i \geq r_i$.

Lemma 12 (Value Covering - 2). *For any bidder i in any Bayes-Nash equilibrium of the FPA_r, such that $v_i \geq r_i$ and $\varphi(v_i) \geq 0$, it holds that*

$$\varphi_i(v_i)x_i(v_i) + T_i^{r_i} \geq \frac{e-1}{e}\phi_i(v_i).$$

Proof. We want to use the previous Value Covering if possible. For that, we would need to connect T_i and $T_i^{r_i}$. Note that if bidder i's bids are such that it always loses due to competition and never loses due to the reserve r_i, then $T_i = T_i^{r_i}$. Otherwise, we note that

$$u_i(v_i) + T_i = u_i(v_i) + r_i x_i(r_i) + T_i^{r_i} \leq u_i(v_i) + p_i(v_i) + T_i^{r_i} = v_i x_i(v_i) + T_i^{r_i}.$$

Indeed, $r_i x_i(r_i) \leq p_i(v_i)$ holds because $v_i \geq r_i$ so it is a best response in BNE for bidder i to bid above r_i. And $p_i(v_i) + u_i(v_i) = v_i x_i(v_i)$.

Thus we can use the previous Value Covering, to obtain

$$v_i x_i(v_i) + T_i^{r_i} \geq u_i(v_i) + T_i \geq \frac{e-1}{e} v_i.$$

Now we just need to replace both v_i's with $\varphi(v_i)$ so that

$$v_i x_i(v_i) + T_i^{r_i} \geq \frac{e-1}{e} v_i \quad \longrightarrow \quad \varphi_i(v_i)x_i(v_i) + T_i^{r_i} \geq \frac{e-1}{e}\phi_i(v_i).$$

But this can surely be done since $v_i \geq \varphi_i(v_i)$. This concludes the proof. □

Note: Since we assume regular value distribution F_i for every bidder, then in fact using the *monopoly reserves* $r_i^* = \varphi_i^{-1}(0)$ will exclude exactly the agents who have negative virtual values. Thus, Value Covering as stated right above actually holds for all bidders with $v_i \geq r_i^*$.

5.2 Obtaining the Revenue Bound

Finally, we need as before to combine the Value and Revenue covering. We need to exercise care since we only have Value Covering for bidders with $v_i \geq r_i^*$.

Theorem 13. *In any BNE of the FPA_{r^*} with monopoly reserves and regularly distributed bidders' values, the revenue of that BNE $\leq \frac{2e}{e-1}$-approximates the revenue of the revenue-optimal auction.*

Proof. Taking the sum of the Value Covering inequality over all bidders with $v_i \geq r_i^*$, we have

$$\sum_{i:v_i \geq r_i^*} \varphi_i(v_i)x_i(v_i) + \sum_{i:v_i \geq r_i^*} T_i^{r^*} \geq \frac{e-1}{e} \sum_{i:v_i \geq r_i^*} \varphi_i(v_i).$$

We let as in the analogous proof in Section 3.4, $x^*(\cdot)$ be the optimal allocation. Since $\varphi_i(v_i)x_i(v_i) \geq 0$ and since the allocation is feasible, we can directly get after taking expectations over all values v

$$\mathbb{E} \sum_{i:v_i \geq r_i^*} \varphi_i(v_i)x_i(v_i) + \mathbb{E} \sum_{i:v_i \geq r_i^*} T_i^{r^*} x_i^*(v_i) \geq \frac{e-1}{e} \mathbb{E} \sum_{i:v_i \geq r_i^*} \varphi_i(v_i)x_i^*(v_i).$$

Here, the first term on the left is exactly REVENUE(FPA_{r^*}) and the term on the right is the virtual welfare of the revenue-optimal auction, which is equal to the revenue REVENUE(OPT) of the revenue-optimal auction. So,

$$\text{REVENUE}(FPA_{r^*}) + \mathbb{E} \sum_{i:v_i \geq r_i^*} T_i^{r^*} x_i^*(v_i) \geq \frac{e-1}{e} \text{REVENUE}(OPT).$$

The term that still is in expectation can be bounded from above by REVENUE(FPA_{r^*}) by directly applying the Revenue Covering lemma. Hence,

$$2 \cdot \text{REVENUE}(FPA_{r^*}) \geq \frac{e-1}{e} \text{REVENUE}(OPT).$$

□

5.3 How Did We Do It?

Now is the time to reflect on what tools we actually needed for shifting the framework to bound Revenue Price of Anarchy.

- The good definition of the new threshold bids T_i^r, together with switching to an FPA with reserves, surprisingly did most of the job!
- To prove Value Covering, we did not nee any algebraic properties of virtual values $\varphi_i(v_i)$, except that $v_i \geq \varphi_i(v_i)$. When proving Value Covering, we still essentially only depended on the BNE condition holding for each player individually, and not on the auction format.
- Revenue Covering was again proved using nothing but the format of the auction. Again, the hard job to make this possible was done by defining threshold bids T_i^r compatibly with the format of an FPA with reserves.

6 Directions in the Multiple-Item Setting

Hartline et. al also give the first steps into analyzing the usefulness of the ideas of value covering and revenue covering for scenarios with more than one item. The auction analyzed then awards each item to the person who bid the most for it and this person is simply charged his bid. In this section, we will explore Hartline et. al's results for multiple items. Before proceeding we must add the concept of μ-Revenue covering.

Definition 14. *A mechanism M is said to be μ-Revenue covered if for any bid distribution G and feasible allocation y,*

$$\mu Rev(M, G) \geq \sum_i T_i y_i.$$

Truth is, we were quite lucky that FPA is revenue covered. The intuition behind this concept is that if a bidder is having a really hard time buying an item (has to bid very high and therefore have a high T_i), that also means that the Revenue is very high. However, if we look to auctions like the Second-Price auction, it is easy to see that that is not the case. If a bidder A bids in a very high interval and his competitor B bids in a low interval, B will have a hard time buying the item, but that doesn't translate in revenue, since A pays the bid of B and not its own bid. This is why we are considering μ-revenue covered auctions. Even if we cannot guarantee $\mu = 1$, many of the results from past sections will extend to these auctions. In particular, we have equivalent theorems for the welfare and revenue of μ-covered winner-pays-bid auctions.

Theorem 15. *The welfare of any μ-revenue covered winner-pays-bid mechanism is a $\mu\frac{e}{e-1}$ approximation to the welfare of any other mechanism.*

This comes from an increment of a coefficient μ in lemma 6, which will only change the sum $\sum_i T_i y_i$ and the right side of the inequality, adding the μ to the fraction $\frac{e}{e-1}$. We can also derive a second theorem regarding revenue, which is our main goal. Once again, their idea mostly depends on virtual values and Myerson's characterization. Therefore, an important part here is to consider winner-pays-bid auctions with reserve prices, just as we did before, which translates into the following similar looking equation

Theorem 16. *The revenue of any μ-covered winner-pays-bid mechanism with regular bidders and monopoly reserves is a $(\mu + 1)\frac{e}{e-1}$-approximation to the revenue of the optimal mechanism.*

Finally, we still need a concept of greedy algorithms, which will help us generate μ-revenue covered auctions. In general, greedy algorithms will order the bidders, then go through such order and allocate elements to each bidder, as long is it doesn't break any feasibility constraint of the auction (e.g. give an indivisible item to two people). The formal definition is:

Definition 17. *The greed by priority algorithm is given by a profile $\phi = (\phi_1, ..., \phi_n)$ of nondecreasing priority functions mapping bids for each bidder i to real numbers. It proceeds in the following way:*

1. *Sort bidders in nonincreasing order of priority $\phi_i(a_i)$*
2. *Initialize the set of winners $S = 0$*
3. *For each bidder i in sorted order: if $S \cup \{i\}$ is feasible, $S = S \cup \{i\}$*
4. *Return S*

This concept of greedy algorithms is especially useful when dealing with matroid settings and single-minded combinatorial auctions (items are indivisible and allocated to at most one bidder). One example that we saw in our first lectures is that is a greedy algorithm from Lehmann and Shohan (2002), which gave a priority functions of $\phi_i = \frac{a_i}{\sqrt{|S_i|}}$ resulting in a $\sqrt{m}$-approximation.

The last useful connection that I will draw in this section in why greedy algorithms are relevant for μ-revenue covered auctions.

Theorem 18. *For any feasibility environment $\mathcal{F}$ and greedy α-approximation algorithm $\mathcal{A}$ for $\mathcal{F}$, the winner-pays-bid mechanism which allocates according to $\mathcal{A}$ is α-revenue covered.*

Therefore, one useful idea to proceed from here is how to develop good greedy algorithms for distinct auction restrains.

7 Where To Go From Here

So far we have heavily used virtual values in our transitions from welfare approximations to revenue approximations. On one hand, one of the most important limitations of virtual values is that they are uni-dimensional. On the other hand, the only properties we used from virtual values is that $\varphi(v) \leq v$ and that we can use Myerson's characterization to translate virtual surplus as revenue in BNE settings. One possible direction from here is to find a more general version of virtual values that translates well those two properties. In class, for example, we saw the following extention when discussing the Lagrangian Duality framework for multiple-item auctions:

Definition 19. *(Flow) A profile of Lagrangian multipliers λ is a flow if for all i, v_i,*

$$f_i(v_i) + \sum_{v_i'} \lambda_i(v_i', v_i) = \sum_{v_i'} \lambda_i(v_i, v_i')$$

Definition 20. *(Virtual Value) For any flow λ, define the associated Virtual Valuation function as*

$$\Phi^{\lambda}_{i,v_i}(S) := v_i(S) - \frac{\sum_{v_i'} (v_i'(S) - v_i(S)) \cdot \lambda_i(v_i', v_i)}{f_i(v_i)}.$$

This is an interesting virtual value, because it satisfies the first constraint that value v upper-bounds it. However, the Myerson translation of the virtual value doesn't quite land and leads simply to

$$\text{REVENUE} \leq \text{VIRTUAL WELFARE}$$

The not so great part of this is that, since most of our lemmas and theorems required transitioning from value to virtual value and therefore from welfare to revenue, it is more important for us to find a treatment of these new virtual values that provide a lower bound for the revenue.

7.1 Additive Buyers

It is useful to analyze specific cases instead of just jumping into general asymmetrical bidders. Since buyers are additive, we know that bidder i's value for a certain item doesn't affect his valuation for another item. Therefore one possible auction in the multiple-case setting is to run a FPA auction with reserve prices for each item separately. This will yield the $\frac{e}{e-1}$ and $\frac{2e}{e-1}$ approximations that we had for welfare and revenue for each one of the items. This means that when we sum up revenue and welfare across all the items we get that the total welfare and total revenue of this auction also have the same $\frac{e}{e-1}$ and $\frac{2e}{e-1}$ approximations (respectively) that the single-item auction we did before had. An interesting follow-up question that remains is whether we can organize the auction in different ways to get an even better approximation for the case of additive bidders.

8 Conclusion

Analyzing Price of Anarchy for First-Price auctions is a good way to develop intuition for how to transition from welfare approximations to revenue approximations using the concepts of Value covering and Revenue covered auctions. As we move from FPA into other auctions, the value covering property remains because it almost exclusively depends on the Bayes-Nash Equilibrium conditions for individual bidders, but Revenue covering becomes more difficult. One main idea to be explored is whether we can give new meaning to virtual values in multiple-item settings.

9 Section

THANK YOU FOR A GREAT COURSE, MATT!
Georgy and Daniel