COMP 6651: Worksheet 1

Growth of functions: Some useful relationships are listed below.

1. Any positive exponential function grows faster than any polynomial. That is, for any real constants a and b such that $a > 1$, $n^b = o(a^n)$
2. Any positive polynomial function grows faster than any polylogarithmic function. That is, for any real constants $a, b > 0$, $\log^b n = o(n^a)$.
3. $2^n = o(n!)$
4. $n! = o(n^n)$
5. $a^{\log_b n} = n^{\log_b a}$
6. $\log(n!) = \Theta(n \log n)$

Problems:

1. Consider the following functions of n.

   (a) $f_1(n) = 100n^2$
   (b) $f_2(n) = n^2 + 1000n$
   (c) $f_3(n) = \begin{cases} 1000n & \text{if } n \text{ is odd} \\ n^3 & \text{if } n \text{ is even} \end{cases}$
   (d) $f_4(n) = \begin{cases} n & \text{if } n \leq 100 \\ n^3 & \text{if } n > 100 \end{cases}$

   Indicate for each pair i and j if $f_i(n)$ is $O(f_j(n))$.

2. Indicate for each pair of functions (A, B) if A is $O, \Omega, \Theta, o, \omega$ of B. Assume that $k \geq 1; \epsilon > 0; c > 1$ are constants.

A	B	O	o	Ω	ω	Θ
$\log^k n$	n^ϵ					
n^k	c^n					
$\sqrt{n}$	$n^{\sin n}$					
2^n	$2^{n/2}$					
$n^{\log m}$	$m^{\log n}$					
$\log(n!)$	$\log(n^n)$					

3. Consider the following statements:

   (A) $f(n) = o(g(n))$

   (B) $f(n) = O(g(n))$

   (a) Prove: (A) implies (B).

   (b) Prove: (B) does not imply (A).

4. Let $f(n), g(n) > 1$ and consider the following statements:

   (A) $f(n) = o(g(n))$

   (B) $\ln f(n) = o(\ln g(n))$

   (a) Prove: (A) does not imply (B).

   (b) Prove: (B) does not imply (A).

5. Recall the notion of *asymptotic equality*: for $f, g : \mathbb{N} \to \mathbb{N}$ we say that f is asymptotically equal to g (notation: $f \sim g$) if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

   Prove:

   (a) $n^3 - 10n \sin n + 284 \sim n^3$

   (b) $\sqrt{n+1} - \sqrt{n} \sim 1/(2\sqrt{n})$

   (c) $\ln(1 + 1/n) \sim 1/n$

6. Consider the following two statements:

   (A) $f(n) = \Theta(g(n))$

   (B) $\ln f(n) \sim \ln g(n)$

   Prove:

   (a) (B) does not follow from (A). (Give a counterexample.)

   (b) (B) does follow from (A) if we assume that $f(n) \to \infty$.

7. Recall Stirling's formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

   Use Stirling's formula to show that $\log(n!) \sim n \log n$.

8. Prove that there exist constants c, d such that

$$\frac{\binom{2n}{n}}{4^n} \sim cn^d.$$

   Determine the values of c and d. (Hint: use Stirling's formula.)

9. The Prime Number Theorem (PNT) states that

$$\pi(x) \sim \frac{x}{\ln x},$$

   where $\pi(x)$ denotes the number of primes $\leq x$. (So, for instance, $\pi(10) = 4, \pi(100) = 25, \pi(\pi) = 2$). Write down a random $(0, 1)$-string x of length 1000. Interpret x as a number in binary. Estimate the probability that x is prime using the PNT assuming that 2^{1000} is "sufficiently large".

10. Prove by mathematical induction each of the following:

    (a) $\Sigma_{i=1}^{n} i = n(n+1)/2$
    (b) $\Sigma_{i=1}^{n} i^2 = n(2n+1)(n+1)/6$
    (c) $\Sigma_{i=0}^{n-1} 2^i = 2^n - 1$
    (d) $\Sigma_{i=1}^{n} (2i-1) = n^2$
    (e) $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$
    (f) for every $x \neq 1$ we have $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$
    (g) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
    (h) for $n \geq 7$ we have $3^n < n!$
    (i) for $n \geq 5$ we have $2^n > n^2$
    (j) $\lfloor \frac{n}{2} \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$
    (k) $n^3 + 2n$ is divisible by 3
    (l) a decimal number is divisible by 3 if and only if the sum of its digits is divisible by 3

11. Recall the notion of *divisibility*: we say that the integer d divides the integer n (notation: $d|n$), if there exists an integer x such that $dx = n$. We say that d is a *divisor* of n and n is a *multiple* of d.

    (a) True or false: $0|0$?
    (b) What are multiples of $d = 0$?
    (c) What are multiples of $d = 1$?
    (d) What are multiples of $d = -1$?
    (e) What are divisors of $n = 0$?
    (f) What are divisors of $n = 1$?
    (g) What are divisors of $n = -12$?
    (h) Prove: if $a|b$ and $b|a$ then $a = \pm b$.
    (i) Prove: if $d|a$ and $d|b$ then $d|a \pm b$.
    (j) Prove: the number of positive divisors of the positive integer n is less than $2\sqrt{n}$.

12. Let a, b, m be integers. We say that a is congruent to b modulo m (notation: $a \equiv b \pmod{m}$) if $m|a-b$.

    (a) Prove: day k and day ℓ of a given month fall on the same day of the week if and only if $k \equiv \ell \pmod{7}$.
    (b) Prove: if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a \pm c \equiv b \pm d \pmod{m}$ and $a \cdot c \equiv b \cdot d \pmod{m}$.
    (c) Prove: if $a \equiv b \pmod{m}$ then $\gcd(a, m) = \gcd(g, m)$.

13. Suppose we draw n straight lines on a plane so that every pair of lines interest but no three lines intersect in a common point. Into how many regions do these n lines divide the plane? Hint: write a recurrence and solve it.

14. Consider a staircase with n stairs.

(a) Write down the recurrence for the number of ways to ascend the staircase if we can climb by 1 stair or by 2 stairs in each step. Don't forget the base case(s).

(b) Solve the above recurrence using the method of characteristic polynomial.

(c) Do you recognize this recurrence? It has a special name.

15. Using the method of characteristic polynomials solve the following recurrences:

    (a) $T(0) = 2, T(1) = 3$ and $T(n+2) = 3T(n) - 2T(n+1)$

    (b) $T(0) = 0, T(1) = 1$ and $T(n+2) = 4T(n+1) - 4T(n)$

    Master theorem: *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function and let $T(n)$ be defined on the non-negative integers by the recurrence:*

    $$T(n) = aT(n/b) + f(n)$$

    where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$, Then $T(n)$ can be bounded asymptotically as follows:

    (a) If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$ then $T(n) = \Theta(n^{\log_b a})$.

    (b) If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log n)$.

    (c) If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n then $T(n) = \Theta(f(n))$.

16. Use the master theorem above to give tight asymptotic bounds for the following recurrences:

    (a) $T(n) = 4T(n/2) + n$

    (b) $T(n) = 4T(n/2) + n^2$

    (c) $T(n) = 4T(n/2) + n^3$

17. Write down a recurrence for the number of different ways of making change for n\$ using (unlimited number of) 1\$, 5\$ and 20\$ bills. Do not forget the base case(s).

18. Let $M_1, \ldots, M_n$ be a sequence of n matrices so that the following multiplication makes sense (dimensions need to be compatible):

    $$M_1 \times M_2 \times \cdots \times M_n.$$

    One can parenthesize this expression in many different ways, e.g., for $n = 3$ we have $(M_1 \times M_2) \times M_3$ or $M_1 \times (M_2 \times M_3)$. Let $T(n)$ denote the number of different ways to place parenthesis. Write down the recurrence for $T(n)$. Do not forget the base case(s).