The direction functor for Schreier extensions of monoids

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It is a classical result in Homological Algebra (due to N. Yoneda) that the Ext^n groups, which are usually defined as right derived functors in an abelian setting, admit a description in terms of *n-extensions*, i.e. (equivalence classes of) exact sequences. This approach has also been fruitfully applied to the study of cohomological theories for some highly frequented non-abelian algebraic structures, such as groups, Lie algebras and rings.

It was shown in [B99, BR07] that a unified treatment of these and many other non-abelian cohomological theories is possible, by resorting to the so-called *direction functors*.

Indeed, if $\mathcal{C}$ is a Barr-exact and Mal'tsev category, the 1-dimensional direction functor d_1 associates with every object $X \in \mathcal{C}$ endowed with a (necessarily unique) Mal'tsev operation $p : X^3 \to X$ and such that the terminal morphism $X \to 1$ is a regular epimorphism, an internal abelian group $d_1(X) \in \mathbf{Ab}(\mathcal{C})$, in such a way that the fibres $d_1^{-1}(A)$ of d_1 inherit a canonical symmetric and (bi)closed monoidal structure: the class $\pi_0\big(d_1^{-1}(A)\big)$ of the connected components of $d_1^{-1}(A)$ is thus canonically an abelian group, which, for example, in the cases $\mathcal{C} = \mathbf{Gp}/G$ or $\mathcal{C} = \mathbf{Lie}_K/\mathfrak{h}$ is isomorphic to the standard second cohomology groups $H^2(G, A)$ and $H^2(\mathfrak{h}, A)$, respectively.

Similarly, if $n \geq 2$, the higher dimensional direction functors d_n's allow for a conceptual description of the higher cohomology groups $H^{n+1}(G, A)$, $H^{n+1}(\mathfrak{h}, A)$, as well as for the construction of the classical long cohomology sequence.

Now, in the context of Algebraic Topology, cohomology monoids have been introduced as a generalisation of the usual singular cohomology groups of a topological space, proving to have a higher classification power. Following the same spirit, cohomology monoids $H^n(M, A)$ (with coefficients in semimodules) were introduced in [P77] as a generalisation of the classical cohomology groups of a group. These cohomology monoids also find arithmetic applications in the theory of the Brauer group and in Galois cohomology.

In [P18], the second cohomology monoid $H^2(M, A)$ was shown to admit a description in terms of Schreier extensions of the monoid M by the M-semimodule A, which in the case of groups are the same as the usual group extensions.

The aim of this talk is to present a generalisation of the direction functor [B99] in the case of Schreier extensions of monoids, showing that we can consider a new functor d which coincides with the classical direction functor d_1 on the slice category $\mathbf{Gp}/G$ when it is applied to group extensions, and whose fibres still inherit a canonical symmetric monoidal structure. The commutative monoid which results by considering the class of connected components of the fibre $d^{-1}(A)$ is then precisely the cohomology monoid $H^2(M, A)$ of [P18]. This is a joint work with Andrea Montoli and Diana Rodelo.

References

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