The Higher Structure of Martingales

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The problem. One of the main feature of stochastic processes is that they generally exhibit *memory*: the transition probabilities at each step do not only depend on the current state, but also, possibly, on the previous ones. This behavior, which is completely invisible in the deterministic case, is an obstruction to describing stochastic processes in terms of categorical composition, i.e. as 'sequences of arrows'.

The idea. We can model systems with memory, instead of using categories, with more general *simplicial sets* [3]. More in detail, let $\mathbb{T}$, which we think of as 'time', denote $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$ with their usual order, and let $\Delta_{\mathbb{T}}$ be the simplicial set which maps n to the set of monotone maps $(n+1) \to \mathbb{T}$. We can view a $\mathbb{T}$-indexed stochastic process as a map of simplicial sets from $\Delta_{\mathbb{T}}$ to the simplicial set of n-ary joint distributions on a space X:

$$\begin{tikzcd} \ldots \arrow[r] & P(X \times X \times X) \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r] & P(X \times X) \arrow[l, shift left=2] \arrow[l, shift left=4] \arrow[r, shift left=2] \arrow[r] & PX \arrow[l, shift left=2] \end{tikzcd}$$

The memory-less systems (Markov processes) are the ones that factor through the nerve of a category, i.e. which have no non-degenerate higher cells.

The bar construction and martingales. Given a monad (T, μ, ν) and a T-algebra (A, a), the *bar construction* is the simplicial object constructed as follows:

$$\begin{tikzcd} \ldots \arrow[r] & TTTA \arrow[r, shift left=6, "\mu"] \arrow[r, shift left=3, "T\mu"] \arrow[r, "TTa"] & TTA \arrow[l, shift left=3, "T\eta"] \arrow[l, shift left=6, "TT\eta"] \arrow[r, shift left=3, "\mu"] \arrow[r, "Ta"] & TA \arrow[l, shift left=3, "T\eta"] \end{tikzcd}$$

For probability monads, it is known that the 1-cells of this simplicial object are closely related to conditional expectation of random variables [4]. Here we extend this correspondence to the entirety of the stochastic process, and show that $\Delta_{\mathbb{T}}$-shapes in the bar construction are exactly discrete and continuous-time *martingales*, which are some of the most important and well studied processes in probability theory.

It is known that, even for well behaved probability monads, the bar constructions are very far from being nerves of categories or even quasi-categories [2]. This has to do with the nontrivial role of memory, and its precise homotopy-theoretic properties still remain to be studied in detail.

References

[1] Paolo Perrone, *The Higher Structure of Martingales.* In preparation.

[2] Carmen Constantin, Paolo Perrone, Tobias Fritz and Brandon T. Shapiro, *Partial Evaluations and the Compositional Structure of the Bar Construction.* TAC 39(11), 2023, available at arXiv:2009.07302.

[3] Aziz Kharoof and Cihan Okay, *Simplicial distributions, convex categories and contextuality.* Submitted (2022), available at arXiv:2211.00571.

[4] Tobias Fritz and Paolo Perrone, *Monads, partial evaluations, and rewriting.* Proceedings of MFPS (2020), available at arXiv:1810.06037.

[5] Paolo Perrone, *Categorical probability and stochastic dominance in metric spaces.* PhD thesis, University of Leipzig (2018), available at paoloperrone.org/phdthesis.pdf.