Multivariable Calc 2

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December 19, 2018

1. By saying $\nabla f = \lambda \nabla g$, we are saying that ∇g is a constant multiple of ∇f. Since they are both vectors, they can have a cross product. This cross product then must become 0 because they are constant multiples.

$$f(x,y) = xy \quad \nabla f =< y, x >$$

$$g(x,y) = x^2 + y^2 - 9 \quad \nabla g =< 2x, 2y >$$

$$y = \lambda 2x \quad x = \lambda 2y \rightarrow \frac{y}{2x} = \frac{x}{2y}$$

$$x = y \quad 2x^2 = 9 \rightarrow x = y = \pm\frac{3}{\sqrt{2}}$$

2. (a)

$$T(x,y,z) = x^2 + 2y^2 - 3z + 1$$

$$\Delta T =< 2x, 4y, -3 >$$

$$< \frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}} >$$

(b)

$$T(3,2,1) = 15 \quad 17 = x^2 + y^2$$

$$T(\sqrt{15}, 1, 1) = 15$$

$$< \frac{3-\sqrt{15}}{\sqrt{1+(3-\sqrt{15})^2}}, \frac{1}{\sqrt{1+(3-\sqrt{15})^2}}, 0 >$$

3.

$$f(x,y) = e^{2x-y-2} + y + \sin(x-1) \quad x(t) = \cos(5t), y(t) = \sin(5t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = 2e^{2x-y-2} + \cos(x-1) \quad \frac{dx}{dt} = -5\sin(t)$$

$$\frac{\partial f}{\partial y} = -e^{2x-y-2} + 1 \quad \frac{dy}{dt} = 5\cos(t)$$

$$\left(2e^{2x-y-2} + \cos(x-1)\right) \cdot -5\sin(t) + \left(-e^{2x-y-2} + 1\right) \cdot 5\cos(t) = \frac{df}{dt}$$

4.

$$f(x,y) = xy + x + 2y \quad g(x,y) = xy - 4$$

$$\Delta f =< y+1, x+2 > \quad \Delta g =< y, x >$$

$$\frac{y+1}{y} = \frac{x+2}{x} \rightarrow x = 2y$$

$$2y^2 = 4 \rightarrow y = \sqrt{2}, x = 2\sqrt{2}$$

5. (a) S
   (b) V
   (c) W
   (d) Y
   (e) Q
   (f) P
   (g) X

6.

$$f : x^2 + y^2 + z^2 - 9 = 0 \quad g : z - x^2 - y^2 + 3 = 0$$

$$\nabla f =< 2x, 2y, 2z > \quad \nabla g =< -2x, -2y, 1 >$$

Tangent plane of f and g respectively: $4(x-2) - 2(y+1) + 4(z-2) = 0$

$$-4(x-2) + 2(y+1) + z - 2 = 0$$

$$\vec{n_1} =< 4, -2, 4 > \quad \vec{n_2} =< -4, 2, 1 >$$

$$\cos(\theta) = \frac{\vec{n_1} \cdot \vec{n_2}}{\parallel \vec{n_1} \parallel \cdot \parallel \vec{n_2} \parallel}$$

$$\cos(\theta) = \frac{-16}{6\sqrt{21}}$$

7. (a)

$$f\,(x, y, z) = x^3 + y^3 - z^3 \quad f\,(9, 10, 12) = 1$$

$$f_x = 3x^2 \quad f_y = 3y^2 \quad f_z = -3z^2$$

$$L(x, y, z) = f(9, 10, 12) + f_x(x-9) + f_y(y-10) + f_z(z-12)$$

$$L(x, y, z) = 1 + 243(x-9) + 300(y-10) - 432(z-12)$$

$$L(9.001, 10.02, 12.001) = 1{+}243{*}.001{+}300{*}.02{-}432{*}.001 = 6.811$$

$$f(9.001, 10.02, 12.001) = 6.822999$$

(b)

$$f(x, y) = x\sqrt{y} \quad f(3, 169) = 3\sqrt{169}$$

$$f_x = \sqrt{y} = \sqrt{169} \quad f_y = \frac{x}{2\sqrt{y}} = \frac{3}{2\sqrt{169}}$$

$$L(x, y) = 3\sqrt{169} + \sqrt{169}(x-3) + \frac{3}{2\sqrt{169}}(y-169)$$

$$L(3.141, 163) = 40.1407 \quad f(3, 169) = 40.1016$$

8.

$$< \frac{3}{5}, \frac{4}{5} > \cdot \nabla f = 2 \qquad < \frac{-4}{5}, \frac{3}{5} > \cdot \nabla f = 2$$

$$\frac{3}{5} f_x + \frac{4}{5} f_y = 2 \quad \frac{-4}{5} f_x + \frac{3}{5} f_y = -1$$

$$f_x(0, 0) = 2 \quad f_y(0, 0) = 1$$

$$2x + y + 1 = L(x, y) \quad L(0.06, .08) = 1.2$$

9. The interpretation of f_I is that as p is kept constant, the rate of change of consumption is dependent on the change in I which is essentially the change in consumption of beef at 3$/lb as income per year increases. For instance, if C has units of $\frac{lb}{y}$ - which is derived from household income per year over price of beef per pound - then the partial derivative

in respect to I where p is kept constant would result in units of $\frac{lb}{\$}$ because of the definition of the partial derivative: $\lim_{h\to 0}(\frac{f(I+h,p)-f(I,p)}{h})$. Plugging in the units shows that it must demonstrate the change in consumption of beef at p per increase of household income per year.

$$f_p = \frac{4.97 - 5.00}{.5} = -.06$$

10.

$$f_x = \frac{1}{2} \quad f_y = 1 \quad f_{xx} = -\frac{1}{4} \quad f_{yy} = -1$$

$$\begin{aligned} Q_f = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) +& \\ \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 +& \\ f_{xy}(x_0, y_0)(x - x_0)(y - y_0) +& \\ \frac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2& \end{aligned}$$

$$Q_f = 1 + \frac{1}{2}(x-1) + y - \frac{1}{8}(x-1)^2 - \frac{1}{2}(x-1)y - \frac{1}{2}(y)^2$$

$$Q_f(1, 0) = 1 \checkmark$$

11. (a)

$$f(x, y) = 2x^2 + y^4 - 4xy$$

$$f_x = 4x - 4y = 0 \quad f_y = 4y^3 - 4x = 0$$

$$(-1, -1), (0, 0), (1, 1)$$

(b)

$$\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

(c)

$$\mathbf{D} = f_{xx} f_{yy} - f_{xy}^2 = 48y^2 - 16$$

$$D(-1, -1) > 0, f_{xx} > 4 \to \text{local minimum}$$

$$D(0, 0) < 0 \to \text{saddle point}$$

$$D(1, 1) > 0, f_{xx} > 4 \to \text{local minimum}$$

(d)

$$-\nabla f = - < 4x - 4y, 4y^3 - 4x > \quad -\nabla f(3,2) = - < 4, 20 >$$

12.

$$f(x,y) = e^{\frac{x}{y}} \quad f(tx, yx) = e^{\frac{tx}{ty}} = e^{\frac{x}{y}} = t^0 f(x,y) \quad n = 0$$

$$f_x(x,y) = \frac{e^{\frac{x}{y}}}{y} \quad f_y(x,y) = \frac{-xe^{\frac{x}{y}}}{y^2}$$

$$\frac{xe^{\frac{x}{y}}}{y} - \frac{xe^{\frac{x}{y}}}{y^2} = (0)f(x,y)$$