

Multivariable Calc 2

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1. By saying $\nabla f = \lambda \nabla g$, we are saying that ∇g is a constant multiple of ∇f . Since they are both vectors, they can have a cross product. This cross product then must become 0 because they are constant multiples.

$$f(x, y) = xy \quad \nabla f = \langle y, x \rangle$$

$$g(x, y) = x^2 + y^2 - 9 \quad \nabla g = \langle 2x, 2y \rangle$$

$$y = \lambda 2x \quad x = \lambda 2y \rightarrow \frac{y}{2x} = \frac{x}{2y}$$

$$x = y \quad 2x^2 = 9 \rightarrow x = y = \pm \frac{3}{\sqrt{2}}$$

2. (a)

$$T(x, y, z) = x^2 + 2y^2 - 3z + 1$$

$$\Delta T = \langle 2x, 4y, -3 \rangle$$

$$\left\langle \frac{2}{\sqrt{13}}, 0, -\frac{3}{\sqrt{13}} \right\rangle$$

- (b)

$$T(3, 2, 1) = 15 \quad 17 = x^2 + y^2$$

$$T(\sqrt{15}, 1, 1) = 15$$

$$\left\langle \frac{3 - \sqrt{15}}{\sqrt{1 + (3 - \sqrt{15})^2}}, \frac{1}{\sqrt{1 + (3 - \sqrt{15})^2}}, 0 \right\rangle$$

3.

$$f(x, y) = e^{2x-y-2} + y + \sin(x-1) \quad x(t) = \cos(5t), y(t) = \sin(5t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = 2e^{2x-y-2} + \cos(x-1) \quad \frac{dx}{dt} = -5\sin(t)$$

$$\frac{\partial f}{\partial y} = -e^{2x-y-2} + 1 \quad \frac{dy}{dt} = 5\cos(t)$$

$$(2e^{2x-y-2} + \cos(x-1)) \cdot -5\sin(t) + (-e^{2x-y-2} + 1) \cdot 5\cos(t) = \frac{df}{dt}$$

4.

$$f(x, y) = xy + x + 2y \quad g(x, y) = xy - 4$$

$$\Delta f = \langle y+1, x+2 \rangle \quad \Delta g = \langle y, x \rangle$$

$$\frac{y+1}{y} = \frac{x+2}{x} \rightarrow x = 2y$$

$$2y^2 = 4 \rightarrow y = \sqrt{2}, x = 2\sqrt{2}$$

5. s

6.

$$f : x^2 + y^2 + z^2 - 9 = 0 \quad g : z - x^2 - y^2 + 3 = 0$$

$$\nabla f = \langle 2x, 2y, 2z \rangle \quad \nabla g = \langle -2x, -2y, 1 \rangle$$

Tangent plane of f and g respectively: $4(x-2) - 2(y+1) + 4(z-2) = 0$

$$-4(x-2) + 2(y+1) + z - 2 = 0$$

$$\vec{n}_1 = \langle 4, -2, 4 \rangle \quad \vec{n}_2 = \langle -4, 2, 1 \rangle$$

$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|}$$

$$\cos(\theta) = \frac{-16}{6\sqrt{21}}$$

7. (a)

$$\begin{aligned}
 f(x, y, z) &= x^3 + y^3 - z^3 & f(9, 10, 12) &= 1 \\
 f_x &= 3x^2 & f_y &= 3y^2 & f_z &= -3z^2 \\
 L(x, y, z) &= f(9, 10, 12) + f_x(x - 9) + f_y(y - 10) + f_z(z - 12) \\
 L(x, y, z) &= 1 + 243(x - 9) + 300(y - 10) - 432(z - 12) \\
 L(9.001, 10.02, 12.001) &= 1 + 243 \cdot .001 + 300 \cdot .02 - 432 \cdot .001 = 6.811 \\
 f(9.001, 10.02, 12.001) &= 6.822999
 \end{aligned}$$

(b)

$$\begin{aligned}
 f(x, y) &= x\sqrt{y} & f(3, 169) &= 3\sqrt{169} \\
 f_x &= \sqrt{y} = \sqrt{169} & f_y &= \frac{x}{2\sqrt{y}} = \frac{3}{2\sqrt{169}} \\
 L(x, y) &= 3\sqrt{169} + \sqrt{169}(x - 3) + \frac{3}{2\sqrt{169}}(y - 169) \\
 L(3.141, 163) &= 40.1407 & f(3, 169) &= 40.1016
 \end{aligned}$$

8.

$$\begin{aligned}
 \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \cdot \nabla f &= 2 & \left\langle \frac{-4}{5}, \frac{3}{5} \right\rangle \cdot \nabla f &= 2 \\
 \frac{3}{5}f_x + \frac{4}{5}f_y &= 2 & \frac{-4}{5}f_x + \frac{3}{5}f_y &= -1 \\
 f_x(0, 0) &= 2 & f_y(0, 0) &= 1 \\
 2x + y + 1 &= L(x, y) & L(0.06, .08) &= 1.2
 \end{aligned}$$

9. The interpretation of f_I is that as p is kept constant, the rate of change of consumption is dependent on the change in I which is essentially the change in consumption of beef at 3\$/lb as income per year increases. For instance, if C has units of $\frac{lb}{y}$ - which is derived from household income per year over price of beef per pound - then the partial derivative in respect to I where p is kept constant would result in units of $\frac{lb}{\$}$ because of the definition of the partial derivative: $\lim_{h \rightarrow 0} \left(\frac{f(I+h, p) - f(I, p)}{h} \right)$. Plugging in the units shows that it must demonstrate the change in consumption of beef at p per increase of household income per year.

$$f_p = \frac{4.97 - 5.00}{.5} = -.06$$

10.

$$f_x = \frac{1}{2} \quad f_y = 1 \quad f_{xx} = -\frac{1}{4} \quad f_{yy} = -1$$

$$\begin{aligned} Q_f &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \\ &\quad \frac{1}{2}f_{xx}(x_0, y_0)(x - x_0)^2 + \\ &\quad f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \\ &\quad \frac{1}{2}f_{yy}(x_0, y_0)(y - y_0)^2 \end{aligned}$$

$$\begin{aligned} Q_f &= 1 + \frac{1}{2}(x - 1) + y - \frac{1}{8}(x - 1)^2 - \frac{1}{2}(x - 1)y - \frac{1}{2}(y)^2 \\ Q_f(1, 0) &= 1 \checkmark \end{aligned}$$

11. (a)

$$\begin{aligned} f(x, y) &= 2x^2 + y^4 - 4xy \\ f_x &= 4x - 4y = 0 \quad f_y = 4y^3 - 4x = 0 \\ &(-1, -1), (0, 0), (1, 1) \end{aligned}$$

(b)

$$\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

(c)

$$\begin{aligned} \mathbf{D} &= f_{xx}f_{yy} - f_{xy}^2 = 48y^2 - 16 \\ D(-1, -1) &> 0, f_{xx} > 4 \rightarrow \text{local minimum} \\ D(0, 0) &< 0 \rightarrow \text{saddle point} \\ D(1, 1) &> 0, f_{xx} > 4 \rightarrow \text{local minimum} \end{aligned}$$

(d)

$$-\nabla f = -\langle 4x - 4y, 4y^3 - 4x \rangle \quad -\nabla f(3, 2) = -\langle 4, 20 \rangle$$

12.

$$f(x, y) = e^{\frac{x}{y}} \quad f(tx, yx) = e^{\frac{tx}{yx}} = e^{\frac{x}{y}} = t^0 f(x, y) \quad n = 0$$

$$f_x(x, y) = \frac{e^{\frac{x}{y}}}{y} \quad f_y(x, y) = \frac{-xe^{\frac{x}{y}}}{y^2}$$

$$\frac{xe^{\frac{x}{y}}}{y} - \frac{xe^{\frac{x}{y}}}{y^2} = (0)f(x, y)$$