Set - 12 : Exploring Capital-Labor Dynamics in the Solow Growth Model

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CS302, Modelling and Simulation

This assignment investigates the mathematical foundations of Robert Solow's economic growth theory, with a particular emphasis on the use of nonlinear dynamics in economic modeling. It centers on the behavior of the capital-to-labor ratio r(t)r(t) within the Solow growth framework, integrating Harrod's natural growth model for labor and a Cobb-Douglas production function. Through both analytical and numerical methods, the study explores key growth patterns and the transitions between different growth phases.

I. EQUATIONS

The Solow growth model explains long-term economic growth through capital accumulation, labor growth, and productivity. This lab models the capital-to-labor ratio $r(t)$, focusing on interactions between capital and labor without technological progress.

- Output (Y) depends on capital (C) and labor (L) via a Cobb-Douglas function.
- Labor grows exponentially as per Harrod's natural growth model.
- A fraction (s) of output is reinvested to grow capital.

Production Function

The production output is given by the Cobb-Douglas production function:

$$Y(t) = F(C, L) = C^\alpha L^{1-\alpha}, \quad 0 < \alpha < 1$$

Labor Growth (Harrod's Natural Growth Model)

Labor grows exponentially:

$$\dot{L} = nL, \quad L(t) = L_0 e^{nt}, \quad 0 < n < 1$$

Capital Growth

The rate of capital accumulation is proportional to the savings from production:

$$\dot{C} = sY, \quad 0 < s < 1$$

Capital-to-Labor Ratio Dynamics

By defining $r = \frac{C}{L}$, the dynamics of r are described by:

$$\dot{r} = sr^\alpha - nr$$

Integral Solution for $r(t)$

The analytical solution to the above differential equation is:

$$r(t) = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} \left[1 + Ae^{-n(1-\alpha)t}\right]^{\frac{1}{1-\alpha}}$$

where A is an integration constant given by:

$$A = \left(\frac{n}{s}\right) r_0^{1-\alpha} - 1$$

Transition Time

The transition from early power-law growth to stabilization occurs at:

$$t_{\text{trans}} = \frac{1}{n(1-\alpha)} \ln\left(\frac{-A}{1-\alpha}\right)$$

II. GRAPHS

The following parameter values were used throughout the assignment:

- $s = 0.15$ (Fraction of production for capital)
- $n = 0.03$ (Natural growth rate of labor)
- $\alpha = 0.5$ (Cobb-Douglas parameter)

A. Question 1

We plot $\dot{r}$ versus r to analyze the phase diagram, identify fixed points, and verify intersections by graphing $y_1 = sr^\alpha$ and $y_2 = nr$.

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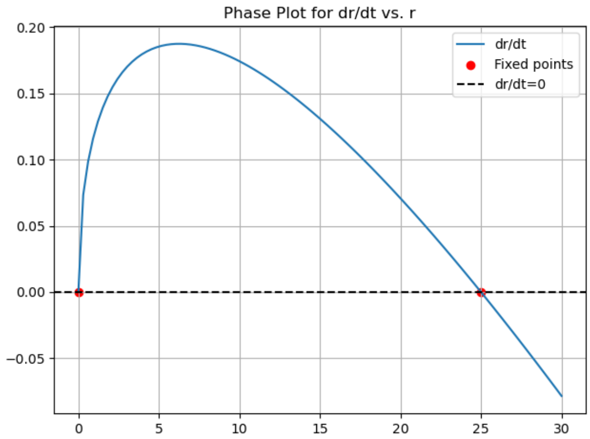

FIG. 1: Phase Diagram of $\dot{r}$ vs r

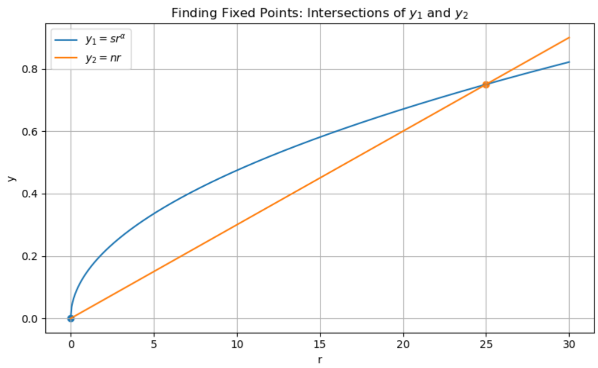

FIG. 2: Graphs of y_1 and y_2 vs r

The phase diagram reveals two fixed points, which correspond to the intersections of y_1 and y_2, aligning with theoretical predictions.

B. Question 2

In this step, we numerically solve for $r(t)$ using Euler's method with initial condition $r(0) = r_0 = 0$, and compare it with the analytical solution. We choose a time step of $\Delta t = 0.0333s$. The results are shown below:

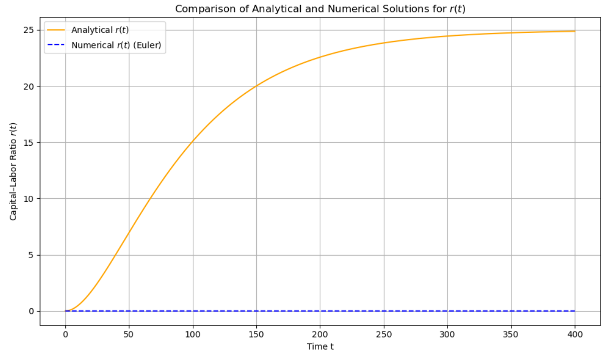

FIG. 3: Numerical and Analytical Solutions with $r_0 = 0$ and $\Delta t = 0.0333s$. Transition time $t_{\text{trans}} \approx 46.210s$ (normal scale)

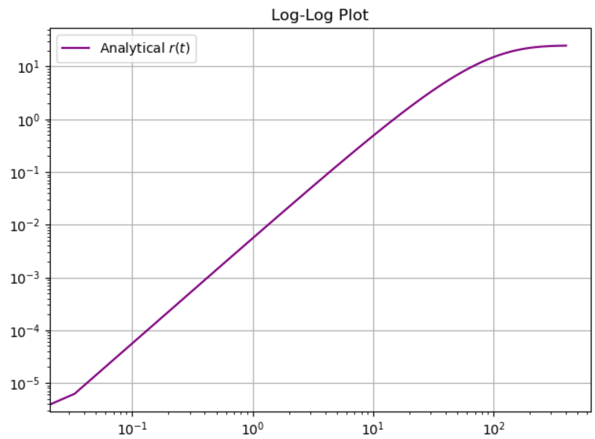

FIG. 4: Log-log Plot of Numerical and Analytical Solutions for $r(t)$ with $r_0 = 0$ and $\Delta t = 0.0333s$. Transition time $t_{\text{trans}} \approx 46.210s$

C. Question 3

In this step, we repeat the procedure with the initial condition $r(0) = r_0 = 0.001$, and explore the behavior for other small values of r_0, much smaller than the transition value $r_{\text{trans}} = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}}$. We also note that the admissible range for r_0 is $0 < r_0 < \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}}$, as this ensures a valid solution within the model constraints.

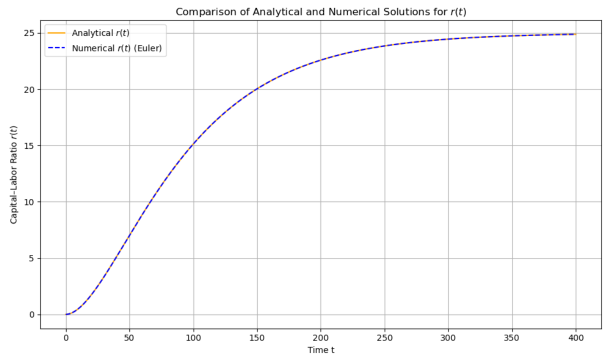

FIG. 5: Comparison of Numerical and Analytical Solutions for $r(t)$ with $r_0 = 0.001$ and $\Delta t = 0.0333s$. Transition time $t_{\text{trans}} \approx 45.787s$ (normal scale)

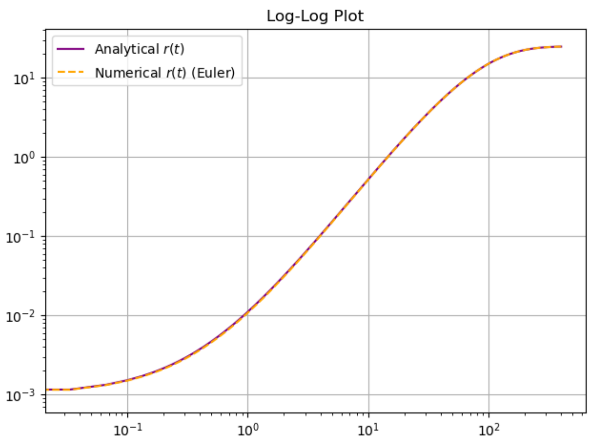

FIG. 6: Comparison of Numerical and Analytical Solutions for $r(t)$ with $r_0 = 0.001$ and $\Delta t = 0.0333s$. Transition time $t_{\text{trans}} \approx 45.787s$ (log-log scale)

III. CONCLUSION

- When $r(0) = 0$, the numerical method remains static, failing to represent the initial growth phase depicted by the analytical solution.
- For $r(0) > 0$, the numerical approximation captures the expected growth and closely tracks the analytical behavior.
- Both numerical and analytical solutions converge to the same steady-state, confirming the model's long-term stability.