

Problem 3

Total: 15 points

Use Pumping Lemma for CFLs to prove the following language is not context-free.

$$C = \{0^n 10^{2n} 10^{3n} \mid n \geq 0\}$$

Solution: C is not context-free. We will prove it via contradiction.

Suppose C is context-free. Then \exists pumping length p so that C satisfies the pumping lemma for CFL for all $s \in C$ where $|s| \geq p$.

Let $s = 0^p 10^{2p} 10^{3p}$, which is clearly in C and $|s| = p + 1 + 2p + 1 + 3p = 6p + 2 > p$. By the pumping lemma, $s = uvxyz$ where 1) $uv^i xy^i z \in C$ for all $i \geq 0$, 2) $|vy| > 0$, and 3) $|vxy| \leq p$. But because of our choice on s , any split of s into 5 parts satisfying the three conditions falls under two cases.

1. v, y contains only 0s: Since each part of the string separated by 1's contains at least p 0s, we must have vxy contained in at most two of those three parts. So, either pumping down or up will cause the number of 0s to change in at least one of the parts but will also leave the number of 0s unchanged in at least one of the parts, breaking the proportion of 0s in the string.
2. v or y contains 1: Since any string in C has exactly two 1s, pumping down causes the resulting string to have only one 1, which cannot be in C .

Since each of the two cases contradicts the fact that s can be pumped, C must be non-context-free. \square

Problem 4

Total: 15 points

Let $\Sigma = \{1, 2, 3, 4\}$ and $E = \{w \in \Sigma^* \mid \text{in } w, \text{ the number of 1s equals the number of 2s, and the number of 3s equal the number of 4s}\}$. Show that E is not context free.

Solution: E is not context-free. We will prove it via contradiction.

Suppose E is context-free. Then \exists pumping length p so that E satisfies the pumping lemma for CFL for all $s \in E$ where $|s| \geq p$.

Let $s = 1^p 3^p 2^p 4^p$, which is clearly in E and $|s| = 4p > p$. By the pumping lemma, $s = uvxyz$ where 1) $uv^i xy^i z \in E$ for all $i \geq 0$, 2) $|vy| > 0$, and 3) $|vxy| \leq p$. But because of our choice on s , any split of s into 5 parts satisfying the three conditions falls under two cases.

1. vxy is contained over a substring containing a single type of symbol: Pumping down will only cause the number exactly one symbol to change and leave the other three unchanged. Therefore, we cannot enforce the language's requirement.
2. vxy is contained over a substring containing a pair of unrelated symbols like 1&3, 3&2, or 2&4: Although v and y may contain up to two symbols in this case, since these are unrelated, pumping down will cause the number of symbols of the related pairs like 1&2 and 3&4 to be unequal.

Since each of the two cases contradicts the fact that s can be pumped, E must be non-context-free. \square