Problem 1

Total: 20 points (5 points each)
Answer the following exercises/problems in the book:

1. Exercise **0.1f** (page 25)
2. Exercise **0.1e** (page 25)
3. Exercise **0.6d** (page 26)
4. Exercise **0.6e** (page 26)

Solution:

1. The empty set.
2. The set of palindromes over the binary alphabet.
3. The range is {6, 7, 8, 9, 10}
   The domain is {(1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 6)(2, 7), (2, 8), (2, 9), (2, 10), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (5, 6), (5, 7), (5, 8), (5, 9), (5, 10)}
4. $g(4, f(4)) = g(4, 7) = 8$

Problem 2

Total: 40 points (20 points each)
Prove the following by mathematical induction. For each solution, please specify your (1) base case; (2) induction hypothesis; and (3) inductive step.

1. For all $n \in \mathbb{N}$:
$$5^n + 5 < 5^{n+1}$$
2. For all $n \in \mathbb{N}$:
$$\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$$

Solution:

- We will prove by induction on n that, for all $n \in \mathbb{N}$
$$5^n + 5 < 5^{n+1} \tag{1}$$
Basis step: We first prove that the statement holds when $n = 1$.
$$5^1 + 5 = 10 < 25 = 5^{1+1}$$

When $n = 1$, the left side of the inequality is $5^1 + 5 = 10$, and the right side is $5^{1+1} = 25$. The inequality holds for $n = 1$.
***Induction Hypothesis*:** Assume that the inequality holds when $n = k$ for some $k \in \mathbb{N}$.

$$5^k + 5 < 5^{k+1}$$

***Inductive step*:** We need to prove that the inequality holds for $k + 1$ from our induction hypothesis. That is, we need to show

$$5^{k+1} + 5 < 5^{k+2} \tag{2}$$

Let us start from the LHS of (2).

$$\begin{aligned} 5^{k+1} + 5 &= 5(5^k + 5) - 20 && \text{(by algebra)} \\ &< 5(5^{k+1}) - 20 && \text{(by induction hypothesis)} \\ &= 5^{k+2} - 20 \\ &< 5^{k+2} \end{aligned}$$

We have derived the RHS of (2) from its LHS using the induction hypothesis. Thus, (1) holds for $k + 1$ if it holds for k. Therefore, by the principle of mathematical induction, we have shown (1) is true for all $n \in \mathbb{N}$. $\square$

- We will prove by induction on n that, for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2} \tag{3}$$

***Basis step*:** We first prove that the statement holds when $n = 1$.

$$(-1)^1 1^2 = -1 = (-1)^1 \frac{1(1+1)}{2}$$

***Induction Hypothesis*:** Assume that the statement holds when $n = k$ for some $k \in \mathbb{N}$.

$$\sum_{i=1}^{k} (-1)^i i^2 = (-1)^k \frac{k(k+1)}{2}$$

***Inductive step*:** We now need to prove that (3) also works when $n = k+1$ from the induction hypothesis. That is,

$$\sum_{i=1}^{k+1} (-1)^i i^2 = (-1)^k \frac{(k+1)(k+2)}{2} \tag{4}$$

Let us start from the LHS of (4).

$$
\begin{aligned}
\sum_{i=1}^{k+1}(-1)^i i^2 &= \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2 \\
&= (-1)^k \frac{k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \\
&\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
&= (-1)^{k+1}(k+1)\Big(-\frac{k}{2} + k + 1\Big) \qquad \text{(by algebra)} \\
&= (-1)^{k+1}(k+1)(\frac{-k+2k+2}{2}) \\
&= (-1)^{k+1}\frac{(k+1)(k+2)}{2}
\end{aligned}
$$

We have derived the RHS of (4) from its LHS using the induction hypothesis. Thus, (3) holds for $k+1$ if it holds for k. Therefore, by the principle of mathematical induction, we have shown (3) is true for all $n \in \mathbb{N}$. $\square$