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COURSE: ATOC 558

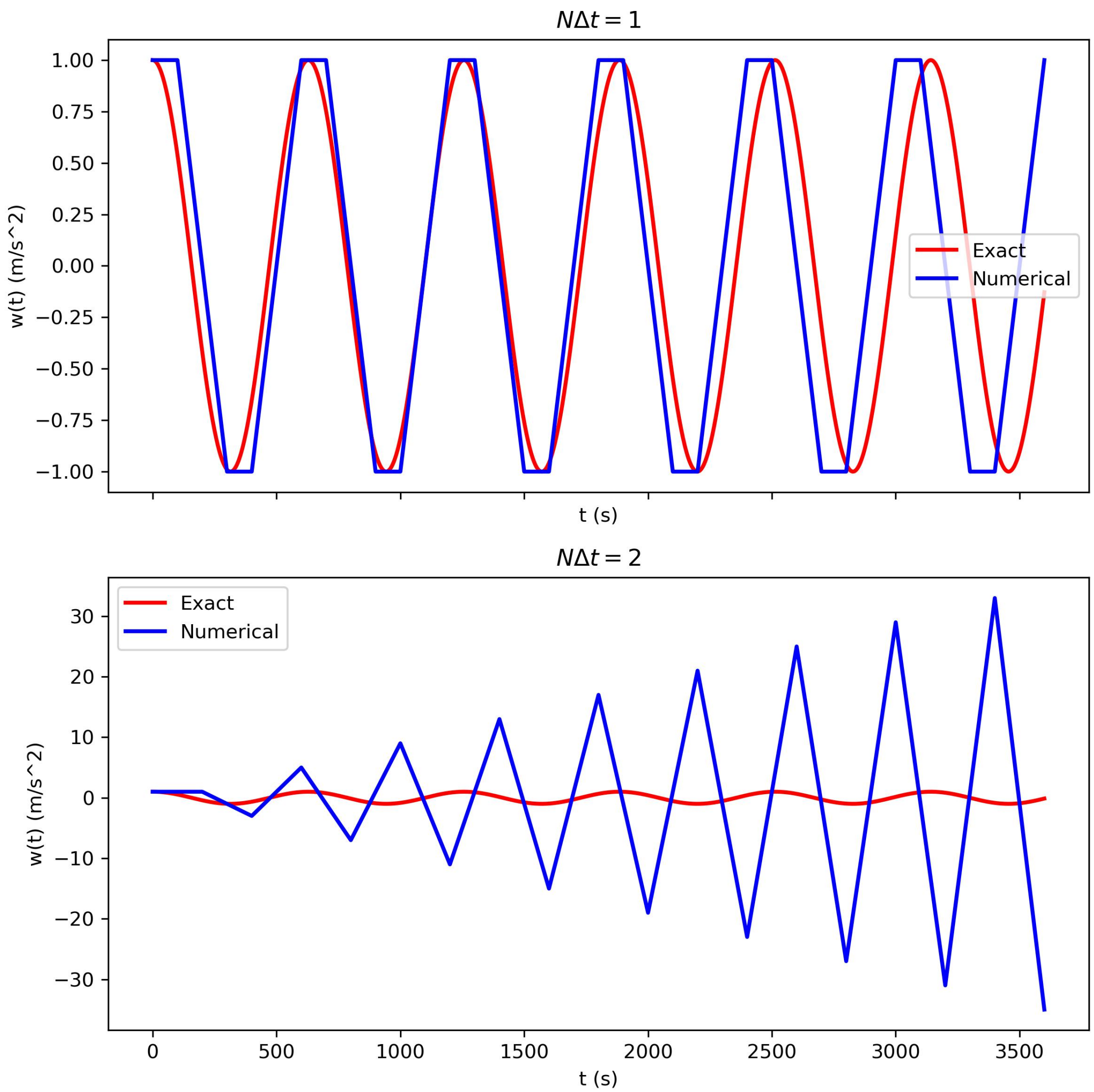
LAB 3

QUESTION 2B

[1]:

```
#importing libraries
import numpy as np
import matplotlib.pyplot as plt

# Set parameters
w0 =1  # m/s
a0 = 0  # m/s^2
N = 1e-4  # 1/s^2
t_max = 3600  # s
dt_1 = 1/N**(1/2)  # s
dt_2 = 2/N**(1/2) # s

# Define exact solution
def w_exact(t):
    return w0*np.cos(N**(1/2)*t) + (a0/N)**0.5*np.sin(N**(1/2)*t)

# Define numerical solution
def w_numerical(dt):
    # Set initial conditions
    phi0 = w0/(N**(1/2))
    phi1 = phi0 - a0*dt/(N**(1/2))
    # Iterate over time steps
    phi_list = [phi0, phi1]
    t_list = [0, dt]
    while t_list[-1] < t_max:
        phi_next = 2*phi_list[-1] - phi_list[-2] - N*dt**2*phi_list[-1]
        phi_list.append(phi_next)
        t_list.append(t_list[-1] + dt)
    # Convert phi values to w values
    w_list = [N**(1/2)*phi for phi in phi_list]
    return np.array(w_list)

# Evaluate solutions
t = np.linspace(0, t_max, 1000)
w_exact_1 = w_exact(t)
w_exact_2 = w_exact(t)
w_numerical_1 = w_numerical(dt_1)
w_numerical_2 = w_numerical(dt_2)
```

```python
# Plot solutions
fig, axs = plt.subplots(nrows=2, sharex=True, figsize=(8, 8))

#first plot
axs[0].plot(t, w_exact_1, label='Exact', color='red', linewidth=2)
axs[0].plot(np.arange(len(w_numerical_1))*dt_1, w_numerical_1,␣
 ↪label='Numerical', color='blue', linewidth=2)
axs[0].set_title(r'$N\Delta t=1$')
#first plot
axs[1].plot(t, w_exact_2, label='Exact', color='red', linewidth=2)
axs[1].plot(np.arange(len(w_numerical_2))*dt_2, w_numerical_2,␣
 ↪label='Numerical', color='blue', linewidth=2)
axs[1].set_title(r'$N\Delta t=2$')
#axes label
for ax in axs:
    ax.set_xlabel('t (s)')
    ax.set_ylabel('w(t) (m/s^2)')
    ax.legend()

plt.tight_layout()
plt.savefig('Lab3.jpg', dpi=300) #saving plot
plt.show() # Display the plot
```



As we can see, the numerical solution with $(N\Delta t) = 1$ matches the exact solution almost perfectly, while the solution with $(N\Delta t) = 2$ shows some oscillations and a slight phase shift. This is consistent with the result from part 2a, where we derived the condition for no amplification in time as $0 < (N\Delta t)^2 < 4$. When $(N\Delta t) = 1$, the condition is satisfied and the numerical solution is stable, hence it matches the exact solution very well. When $(N\Delta t) = 2$, the condition is not satisfied and the numerical solution exhibits some amplification and phase error over time, resulting in the oscillations and phase shift we see in the plot showing that the numerical solution is unstable.

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QUESTION 2C
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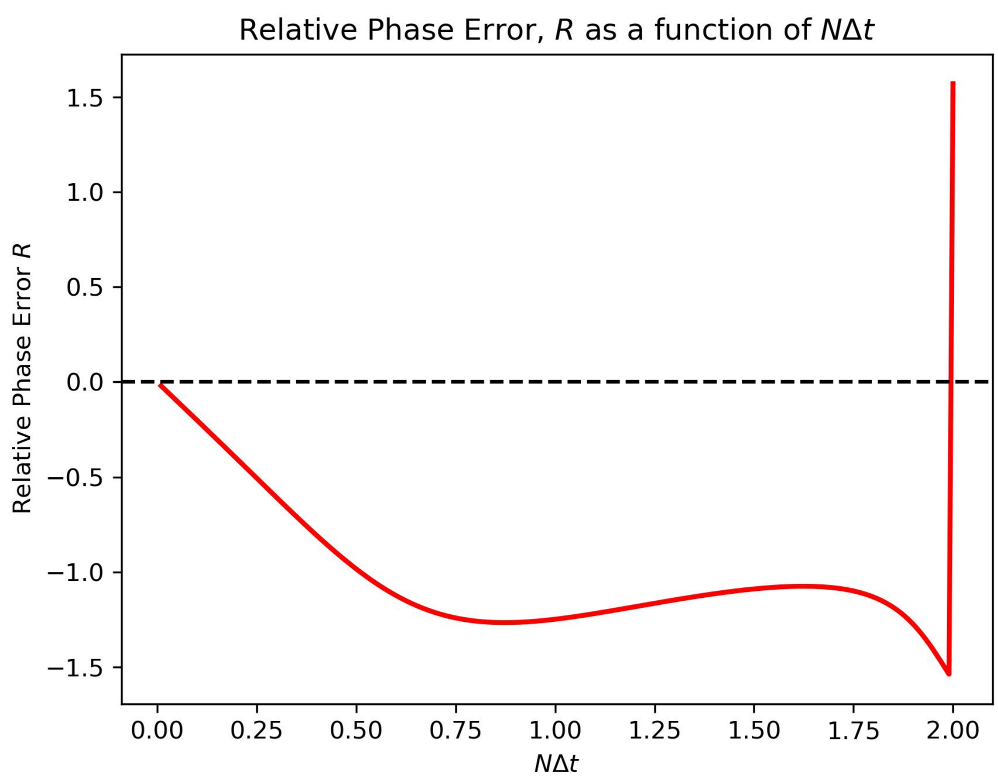
[2]:

```python
#importing libraries
import numpy as np
import matplotlib.pyplot as plt

# Generate an array of 200 values evenly spaced between 0.01 and 2
ndt = np.linspace(0.01, 2, 200)

# Calculate the relative phase error R for each value of ndt
R = np.arctan2((ndt)**2 * ((ndt)**2 - 4), (2 - (ndt)**2 )) / (ndt)

# Plotting
plt.plot(ndt, R, color='red', linewidth=2)
plt.xlabel('$N\Delta t$') # Label the x-axis
plt.ylabel('Relative Phase Error $R$') # Label the y-axis
plt.title('Relative Phase Error, $R$ as a function of $N\Delta t$') # Add a␣
 ↪title to the plot
plt.axhline(y=0, color='black', linestyle='--')
plt.savefig('Lab3_1.jpg', dpi=300) #saving plot
plt.show() # Display the plot
```



The plot in part 2b shows that in the case of $N\Delta t = 1$, the numerical solution is slower than the analytical solution, and the negative value of the relative phase error in part 2c confirms this difference. The relative phase error, which represents the deviation between the numerical and analytical solutions, increases rapidly as $(N\Delta t)$ approaches 2, in line with the condition for no amplification in time that was derived in part 2a.

For the $N\Delta t = 1$ case, the error is minimal, which is why the numerical and exact solutions agree well, as shown in part 2b. The error associated with $(N\Delta t)^2 = 1$ is very small, which explains the good agreement between the numerical and exact solutions for this case.

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QUESTION ONE (1)

$$\frac{dw}{dt} = b \quad \text{——— (1)}$$

$$\frac{db}{dt} = -N^2 w \quad \text{——— (2)}$$

Substituting (1) into (2)

$$\frac{d^2w}{dt^2} + N^2 w = 0$$

Let $w = e^{\alpha t}$

$$\frac{d^2}{dt^2}\left[e^{\alpha t}\right] + N^2\left[e^{\alpha t}\right] = 0$$

$$\alpha^2 e^{\alpha t} + N^2 e^{\alpha t} = 0$$

$$\alpha^2 + N^2 = 0$$

$$\alpha^2 = -N^2$$

$$\alpha = \pm iN$$

Thus the general solution :

$$w(t) = Ae^{iNt} + Be^{-iNt}$$

$$w(t) = A\left[\cos(Nt) + i\sin(Nt)\right] + B\left[\cos(Nt) - i\sin(Nt)\right]$$

$$w(t) = (A+B)\cos(Nt) + i(A-B)\sin(Nt)$$

For $w\big|_{t=0} = w_0$

$$w(0) = A+B = w_0$$

$$A+B = w_0 \quad \text{——— (3)}$$

$$\frac{d}{dt}(w(t)) = -(A+B)N\sin(Nt) + i(A-B)N\cos(Nt)$$

For $\left.\frac{dw}{dt}\right|_{t=0} = a_0$

$$\frac{d}{dt}w(0) = 0 + i(A-B)N\cos(0) = a$$

$$iN(A-B) = a_0$$

$$A-B = -\frac{a_0 i}{N} \quad \text{— (4)}$$

From (3)

$A = w_0 - B$, substituting it into (4)

$$w_0 - B - B = -\frac{a_0 i}{N}$$

$$B = \frac{w_0}{2} + \frac{a_0 i}{2N} //$$

Since B is known now,

$$A = w_0 - B$$

$$A = w_0 - \left[\frac{w_0}{2} + \frac{a_0 i}{2N}\right]$$

$$A = \frac{w_0}{2} - \frac{a_0 i}{2N}$$

Since A & B is known we substitute it into the general solution ~~ede~~ found.

$$\left[\frac{w_0}{2} - \cancel{\frac{q_0 i}{2N}} + \frac{w_0}{2} + \cancel{\frac{q_0 i}{2N}}\right]\cos(Nt) + i\left[\cancel{\frac{w_0}{2}} - \frac{q_0 i}{2N} - \cancel{\frac{w_0}{2}} - \frac{q_0 i}{2N}\right]\sin(Nt) = w(t)$$

$$w(t) = \left[\frac{2w_0}{2}\right]\cos(Nt) + i\left[\frac{-2q_0 i}{2N}\right]\sin(Nt)$$

$$w(t) = w_0\cos(Nt) + i\left(\frac{-q_0 i}{N}\right)\sin(Nt)$$

$$w(t) = w_0\cos(Nt) + \frac{q_0}{N}\sin(Nt)$$

Therefore the ~~general~~ exact solution is

$$\boxed{w(t) = w_0\cos(Nt) + \frac{q_0}{N}\sin(Nt)}$$

For the solution to remain bounded as $t \to \infty$, the amplitude of both $\cos(Nt)$ and $\sin(Nt)$ terms must remain finite. Therefore, the physical condition required is that the Brunt-Vaisala frequency N must be real and positive. This condition ensures that the motion is stable.

$$N^2 = \frac{g}{\theta}\frac{d\bar{\theta}}{dz} > 0$$

$$\frac{d\bar{\theta}}{dz} > 0$$

QUESTION 2A

$$\frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\Delta t)^2} = -N^2\phi^n$$

Let $\phi^n = A^n e^{i\omega\Delta t}$

$\Rightarrow A^{n+1}e^{i\omega\Delta t} - 2A^n e^{i\omega\Delta t} + A^{n-1}e^{i\omega\Delta t} = -N^2 A^n e^{i\omega\Delta t}(\Delta t)^2$

$\Rightarrow A \cdot A^n e^{i\omega\Delta t} - 2A^n e^{i\omega\Delta t} + \frac{A^n}{A}e^{i\omega\Delta t} = -N^2 A^n e^{i\omega\Delta t}(\Delta t)^2$

$\Rightarrow A\phi^n - 2\phi^n + \frac{\phi^n}{A} = -N^2\phi^n(\Delta t)^2$

Dividing through by ϕ^n

$\Rightarrow A - 2 + \frac{1}{A} = -N^2(\Delta t)^2$

$\Rightarrow A^2 - 2A + 1 = -N^2(\Delta t)^2 A$

$\Rightarrow A^2 + (N^2(\Delta t)^2 - 2)A + 1 = 0$

Solving using

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, \quad b = N^2(\Delta t)^2 - 2, \quad c = 1$

$$A_{\pm} = \frac{-(N^2(\Delta t)^2 - 2) \pm \sqrt{(N^2(\Delta t)^2-2)^2 - 4(1)(1)}}{2(1)}$$

$$A_{\pm} = \frac{2 - N^2(\Delta t)^2}{2} \pm \frac{\sqrt{(N^4(\Delta t)^4 - 4N^2(\Delta t)^2 + 4) - 4}}{2}$$

$$A_{\pm} = 1 - \frac{N^2(\Delta t)^2}{2} \pm \sqrt{\frac{N^4(\Delta t)^4 - 4N^2(\Delta t)^2 + 0}{4}}$$

$$A_{\pm} = 1 - \frac{(N^2(\Delta t)^2)}{2} \pm \sqrt{\frac{N^4(\Delta t)^4 - 4N^2(\Delta t)^2}{4}}$$

For numerical scheme to exhibit no amplification in time, that is stable;

$$|A_{\pm}| \leq 1$$

This means that the expression under the root must be non-positive or less than zero.

Thus

$$\frac{N^4(\Delta t)^4 - 4N^2(\Delta t)^2}{4} < 0$$

$$N^4(\Delta t)^4 - 4(N^2(\Delta t)^2 < 0$$

$$N^2(\Delta t)^2(N^2(\Delta t)^2 - 4) < 0$$

This has two roots

$N^2(\Delta t)^2 > 0$ and

$N^2(\Delta t)^2 - 4 < 0$

$N^2(\Delta t)^2 < 4$

which leads to the condition for the numerical solution to exhibit no amplification in time

That is since, the two roots are

$N^2(\Delta t)^2 > 0$ and $N^2(\Delta t)^2 < 0$, it implies

$$0 < (N^2(\Delta t)^2) < 4$$

$\boxed{0 < (N\Delta t)^2 < 4}$ as required

QUESTION 2C

From 2A, when deriving the condition for numerical numerical solution with no amplification in time, we got

$$A_{\pm} = \frac{2 - N^2\Delta t^2}{2} \pm \frac{\sqrt{N^2(\Delta t)^2(N^2(\Delta t)^2 - 4)}}{2}$$

Since the expression inside the expression must be less than zero for the scheme or $A_{\pm} \leq 1$ (stable), then the square root part of the equation is the imaginary.

$$Re(A) = \frac{2 - N^2\Delta t^2}{2}$$

$$Im(A) = \frac{i\sqrt{N^2\Delta t^2(N^2\Delta t^2 - 4)}}{2}$$

Thus

$$\theta_{num} = \arctan\left(\frac{Im(A)}{Re(A)}\right)$$

$$\theta_{num} = \arctan\left(\frac{\sqrt{N^2\Delta t^2(N^2\Delta t^2 - 4)}}{2} \div \frac{2 - N^2\Delta t^2}{2}\right)$$

$$\theta_{num} = \arctan\left(\frac{\sqrt{N^2(\Delta t)^2\left(N^2(\Delta t)^2-4\right)}}{2-N^2\Delta t^2}\right)$$

we know the exact solution is

$$w(t) = w_0\cos(Nt) + \frac{a_0}{N}\sin(Nt)$$

Thus the phase angle, $\theta_{ex} = \cancel{N}N\Delta t$

$\Rightarrow R$, relative phase error $= \dfrac{\theta_{num}}{\theta_{ex}}$

$$\Rightarrow \boxed{R = \frac{\arctan\left(\frac{\sqrt{N^2(\Delta t)^2\left(N^2(\Delta t)^2-4\right)}}{2-N^2(\Delta t)^2}\right)}{N\Delta t}}$$