QUESTION ONE (1)

$$\frac{dw}{dt} = b \quad \text{——— } (1)$$

$$\frac{db}{dt} = -N^2 w \quad \text{——— } (2)$$

Substituting (1) into (2)

$$\frac{d^2 w}{dt^2} + N^2 w = 0$$

Let $w = e^{\alpha t}$

$$\frac{d^2}{dt^2}\left[e^{\alpha t}\right] + N^2\left[e^{\alpha t}\right] = 0$$

$$\alpha^2 e^{\alpha t} + N^2 e^{\alpha t} = 0$$

$$\alpha^2 + N^2 = 0$$

$$\alpha^2 = -N^2$$

$$\alpha = \pm iN$$

Thus the general solution :

$$w(t) = Ae^{iNt} + Be^{-iNt}$$

$$w(t) = A\left[\cos(Nt) + i\sin(Nt)\right] + B\left[\cos(Nt) - i\sin(Nt)\right]$$

$$w(t) = (A+B)\cos(Nt) + i(A-B)\sin(Nt)$$

For $w\big|_{t=0} = W_0$

$$w(0) = A+B = W_0$$

$$A+B = W_0 \quad \text{——— } (3)$$

$$\frac{d}{dt}(w(t)) = -(A+B)N\sin(Nt) + i(A-B)N\cos(Nt)$$

For $\left.\frac{dw}{dt}\right|_{t=0} = a_0$

$$\frac{d}{dt}w(0) = 0 + i(A-B)N\cos(0) = a$$

$$iN(A-B) = a_0$$

$$A-B = -\frac{a_0 i}{N} \quad \text{— (4)}$$

From (3)

$A = w_0 - B$, substituting it into (4)

$$w_0 - B - B = -\frac{a_0 i}{N}$$

$$B = \frac{w_0}{2} + \frac{a_0 i}{2N}$$

Since B is known now,

$$A = w_0 - B$$

$$A = w_0 - \left[\frac{w_0}{2} + \frac{a_0 i}{2N}\right]$$

$$A = \frac{w_0}{2} - \frac{a_0 i}{2N}$$

Since A & B is known we substitute it into the general solution ~~ee~~ found.

$$\left[\frac{w_0}{\cancel{2}} - \cancel{\frac{q_0 i}{2N}} + \frac{w_0}{2} + \cancel{\frac{q_0 i}{2N}}\right]\cos(Nt) + i\left[\cancel{\frac{w_0}{2}} - \frac{q_0 i}{2N} - \cancel{\frac{w_0}{2}} - \frac{q_0 i}{2N}\right]\sin(Nt) = w(t)$$

$$w(t) = \left[\frac{2w_0}{2}\right]\cos(Nt) + i\left[\frac{-2q_0 i}{2N}\right]\sin(Nt)$$

$$w(t) = w_0\cos(Nt) + i\left(\frac{-q_0 i}{N}\right)\sin(Nt)$$

$$w(t) = w_0\cos(Nt) + \frac{q_0}{N}\sin(Nt)$$

Therefore the ~~general~~ exact solution is

$$\boxed{w(t) = w_0\cos(Nt) + \frac{q_0}{N}\sin(Nt)}$$

For the solution to remain bounded as $t \to \infty$, the amplitude of both $\cos(Nt)$ and $\sin(Nt)$ terms must remain finite. Therefore, the physical condition required is ~~that~~ the Brunt-Vaisala frequency N must be real and positive. This condition ensures that the motion is stable.

$$N^2 = \frac{g}{\theta}\frac{d\bar{\theta}}{dz} > 0$$

$$\frac{d\bar{\theta}}{dz} > 0$$

QUESTION 2A

$$\frac{\phi^{n+1} - 2\phi^{n} + \phi^{n-1}}{(\Delta t)^2} = -N^2\phi^n$$

Let $\phi^n = A^n e^{i\omega\Delta t}$

$\Rightarrow A^{n+1}e^{i\omega\Delta t} - 2A^n e^{i\omega\Delta t} + A^{n-1}e^{i\omega\Delta t} = -N^2 A^n e^{i\omega\Delta t}(\Delta t)^2$

$\Rightarrow A.A^n e^{i\omega\Delta t} - 2A^n e^{i\omega\Delta t} + \frac{A^n}{A} e^{i\omega\Delta t} = -N^2 A^n e^{i\omega\Delta t}(\Delta t)^2$

$\Rightarrow A\phi^n - 2\phi^n + \frac{\phi^n}{A} = -N^2\phi^n(\Delta t)^2$

Dividing through by ϕ^n

$\Rightarrow A - 2 + \frac{1}{A} = -N^2(\Delta t)^2$

$\Rightarrow A^2 - 2A + 1 = -N^2(\Delta t)^2 A$

$\Rightarrow A^2 + (N^2(\Delta t)^2 - 2)A + 1 = 0$

Solving using

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, \quad b = N^2(\Delta t)^2 - 2, \quad c = 1$

$$A_{\pm} = \frac{-(N^2(\Delta t)^2-2) \pm \sqrt{(N^2(\Delta t)^2-2)^2-4(1)(1)}}{2(1)}$$

$$A_{\pm} = \frac{2-N^2(\Delta t)^2}{2} \pm \frac{\sqrt{(N^4(\Delta t)^4-4N^2(\Delta t)^2+4)-4}}{2}$$

$$A_{\pm} = 1-\frac{N^2(\Delta t)^2}{2} \pm \sqrt{\frac{N^4(\Delta t)^4-4N^2(\Delta t)^2+0}{4}}$$

$$A_{\pm} = 1-\frac{(N^2(\Delta t)^2)}{2} \pm \sqrt{\frac{N^4(\Delta t)^4-4N^2(\Delta t)^2}{4}}$$

For numerical scheme to exhibit no amplification in time, ~~that~~ is stable;

$$|A_{\pm}| \leq 1$$

This means that the expression under the root must be non-positive or less than zero.

Thus

$$\frac{N^4(\Delta t)^4-4N^2(\Delta t)^2}{4} < 0$$

$$N^4(\Delta t)^4-4(N^2(\Delta t)^2 < 0$$

$$N^2(\Delta t)^2(N^2(\Delta t)^2-4) < 0$$

This has two roots

$N^2(\Delta t)^2 > 0$ and

$N^2(\Delta t)^2 - 4 < 0$

$N^2(\Delta t)^2 < 4$

which leads to the condition for the numerical solution to exhibit no amplification in time

That is since, the two roots are

$N^2(\Delta t)^2 > 0$ and $N^2(\Delta t)^2 < 0$, it implies

$0 < (N^2(\Delta t)^2) < 4$

$\boxed{0 < (N\Delta t)^2 < 4}$ as required

QUESTION 2C

From 2A, when deriving the condition for numerical numerical solution with no amplification in time, we got

$$A_{\pm} = \frac{2 - N^2\Delta t^2}{2} \pm \frac{\sqrt{N^2(\Delta t)^2(N^2(\Delta t)^2 - 4)}}{2}$$

Since the expression inside the expression must be less than zero for the scheme or $A_{\pm} \leq 1$ (stable), then the square root part of the equation is the imaginary.

$$Re(A) = \frac{2 - N^2\Delta t^2}{2}$$

$$Im(A) = \frac{i\sqrt{N^2\Delta t^2(N^2\Delta t^2 - 4)}}{2}$$

Thus

$$\theta_{num} = \arctan\left(\frac{Im(A)}{Re(A)}\right)$$

$$\theta_{num} = \arctan\left(\frac{\sqrt{N^2\Delta t^2(N^2\Delta t^2 - 4)}}{2} \div \frac{2 - N^2\Delta t^2}{2}\right)$$

$$\theta_{num} = \arctan\left(\frac{\sqrt{N^2(\Delta t)^2\left(N^2(\Delta t)^2-4\right)}}{2-N^2\Delta t^2}\right)$$

we know the exact solution is

$$w(t) = w_0 \cos(Nt) + \frac{a_0}{N}\sin(Nt)$$

Thus the phase angle, $\theta_{ex} = N\Delta t$

$\Rightarrow$ R, relative phase error $= \dfrac{\theta_{num}}{\theta_{ex}}$

$$\Rightarrow \boxed{R = \frac{\arctan\left(\dfrac{\sqrt{N^2(\Delta t)^2\left(N^2(\Delta t)^2-4\right)}}{2-N^2(\Delta t)^2}\right)}{N\Delta t}} \;//$$