INTERNAL WAVES AT BUTE INLET, BRITISH COLUMBIA, CANADA

WAVES AND STABILITY (ATOC 513)
SEMESTER PROJECT

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1 Introduction

Bute Inlet is a steep fjord-like inlet located on the west coast of Canada and is known for its strong tidal currents and steep topography, making it an ideal location for the study of internal waves. It may be classified as an estuary in the sense that the fresh water in it is measurably diluted by seawater [1]. Bute Inlet is one of the largest fiords on the mainland coast, of British Columbia: it discharges into a complex system of channels at the north end of the Gulf of Georgia at a latitude of about $50°30'N$ [2]. Like the other inlets of British Columbia and southern Alaska, it represents a preglacial valley that has been subsequently modified by glacial erosion.

Freshwater enters Bute Inlet from many rivers and streams, largely from the head of the inlet, whereas seawater enters the inlet from the neighbouring coastal region, primarily from the inlet mouth (Figures 1 and 2). The Homathko River, which flows into the Bute Inlet near its head, is the largest river to do so. Other notable rivers that flow into the inlet's head include the Orford River and the Southgate River. The seawater that enters Bute Inlet comes from the close-by Johnstone Strait, a confined passageway that links the Queen Charlotte Strait to the northwest with the Strait of Georgia to the northeast. The tides in the Johnstone Strait push seawater into Bute Inlet through its mouth, which is located near the coastal town of Powell River. In addition to the main rivers, Bute Inlet also receives numerous smaller streams(Figure 1 and 2) that may be dry or have very low flow during the late summer months, but become more significant during winter and early summer when snow at higher elevations begins to melt. It is the overall amount of freshwater flowing into the inlet that gives it the characteristics of an estuary.

Internal waves are gravity waves that develop within the body of a fluid rather than on its surface. They are a result of oscillations in fluid density brought on by changes in salinity or temperature with depth. Internal waves can exist thanks to the Bute Inlet's ideal conditions. With two distinct layers—a deep layer of salty water from the mouth and an upper layer of water with reduced salinity (freshwater) from the head—its head and mouth serve as an interchange point in the flow between bodies of water. Bute Inlet stratification is unique, with a surface layer of fresh water from nearby mountains and a deeper layer of saltwater from the ocean. The density difference between these two layers creates a strong stratification (Figure 3), which affects the generation and propagation of internal waves. The mean flow is composed of the difference in flow between these two bodies. The inflow of freshwater and seawater into Bute Inlet creates a dynamic environment that is rich in nutrients and supports a diverse array of marine life.

When a ship moves through the water, it generates a wake that can interact with the underlying stratified water column. This interaction produce disturbances that propagate as internal waves, which then modify surface currents and convergence patterns. Due to these circumstances, the internal density interface exhibits a very potent dead water effect linked to large-amplitude wave activity [3]. Surface slicks in the wake of a ship in Bute Inlet can provide indirect evidence of the presence of internal waves. These slicks may appear as elongated or patchy features on the water surface, with distinct boundaries between the smooth, glassy water inside the slicks and the rougher water outside. The orientation, spacing, and shape of the slicks

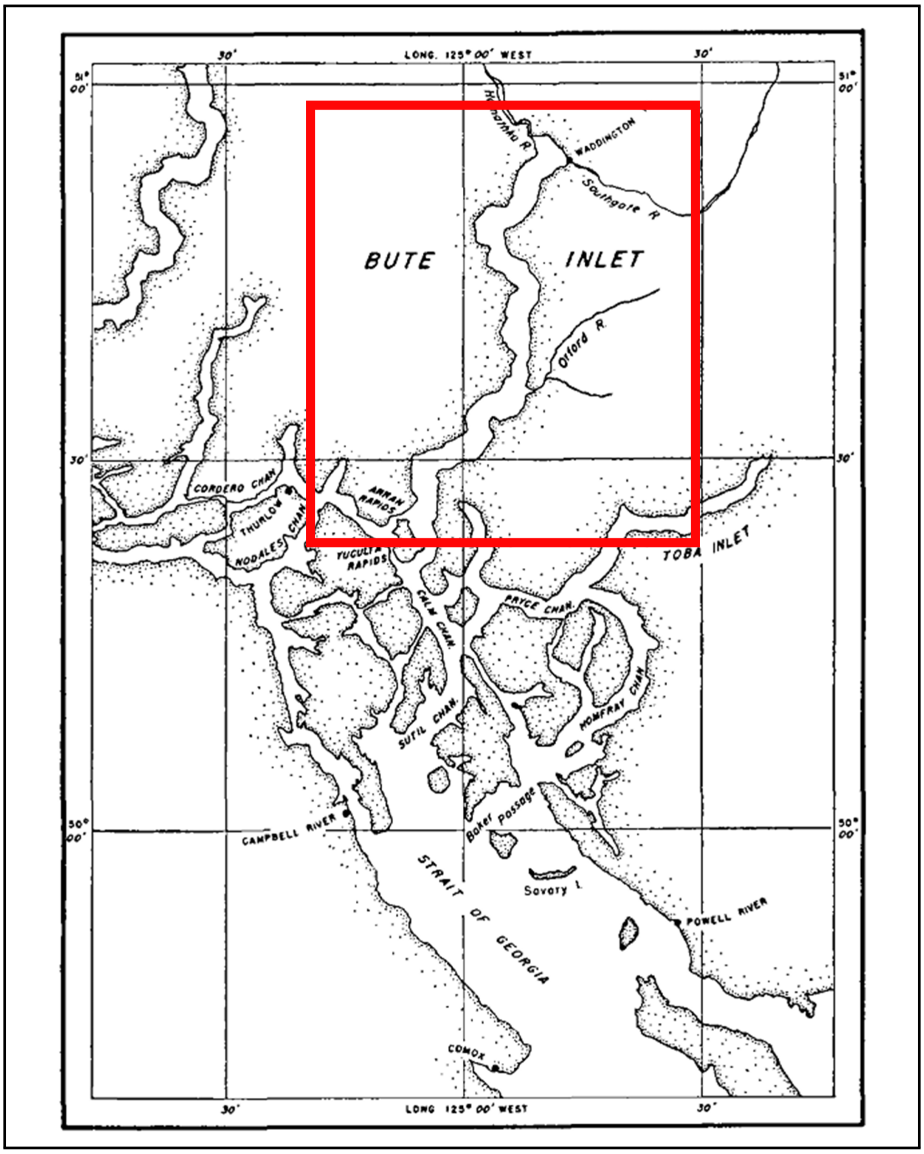


Figure 1: Chart of Bute Inlet showing adjacent channels, rapids, and rlvers (Susumu Tabata and George L Pickard (1957))

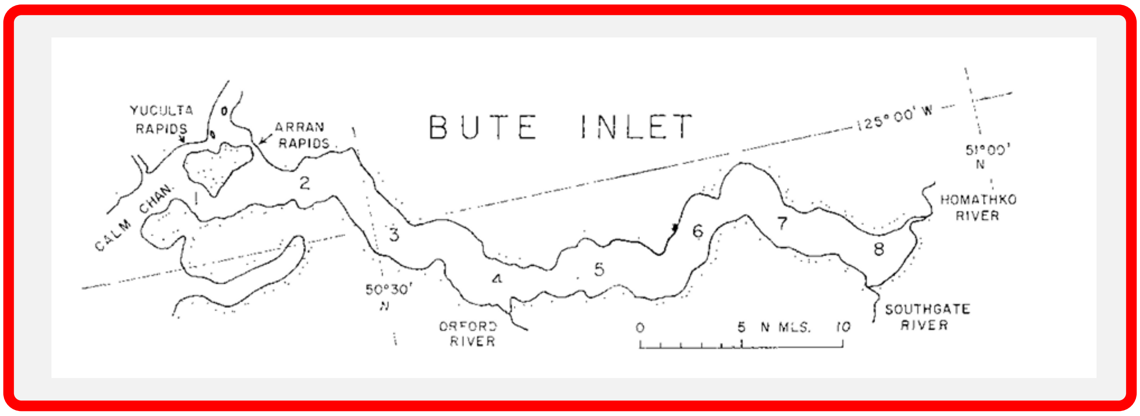


Figure 2: Bute Inlet

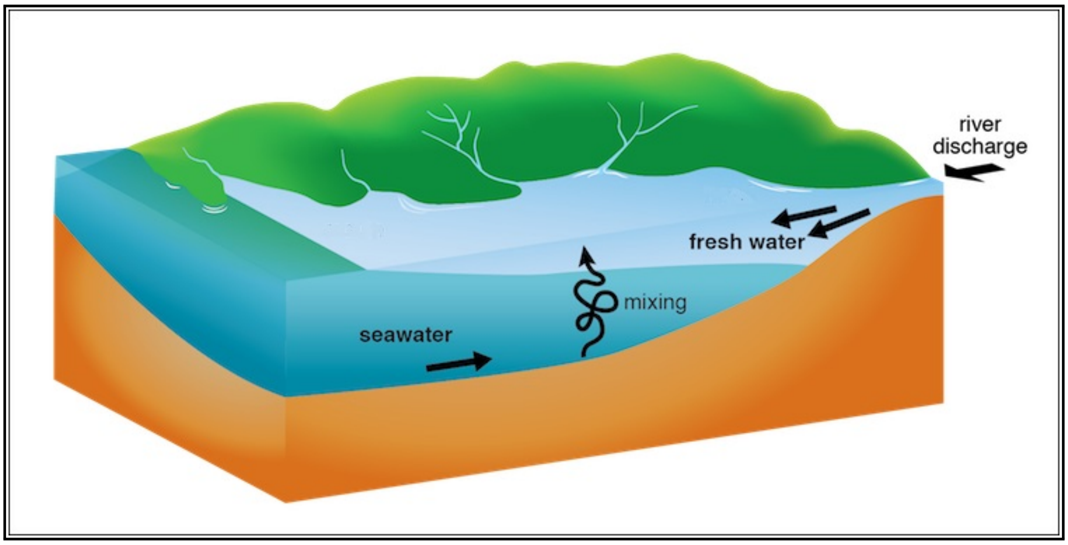


Figure 3: Layer of fresh and seawater (`https://mississippiriverdelta.org`)

can provide clues about the wavelength, direction, and amplitude of the underlying internal waves. This research will examine the Bute Inlet's internal wave creation using the dead-water phenomena.

2 Mathematical Formulation

2.0.1 Two Layer Fluid System

When there are two layers with differing densities, the internal wave's orbital motion is above and below the interface in the opposite direction. As the internal wave moves along, its upper layer alternately gets thinner and thicker. This causes water at the surface to diverge and converge (see figure 4).

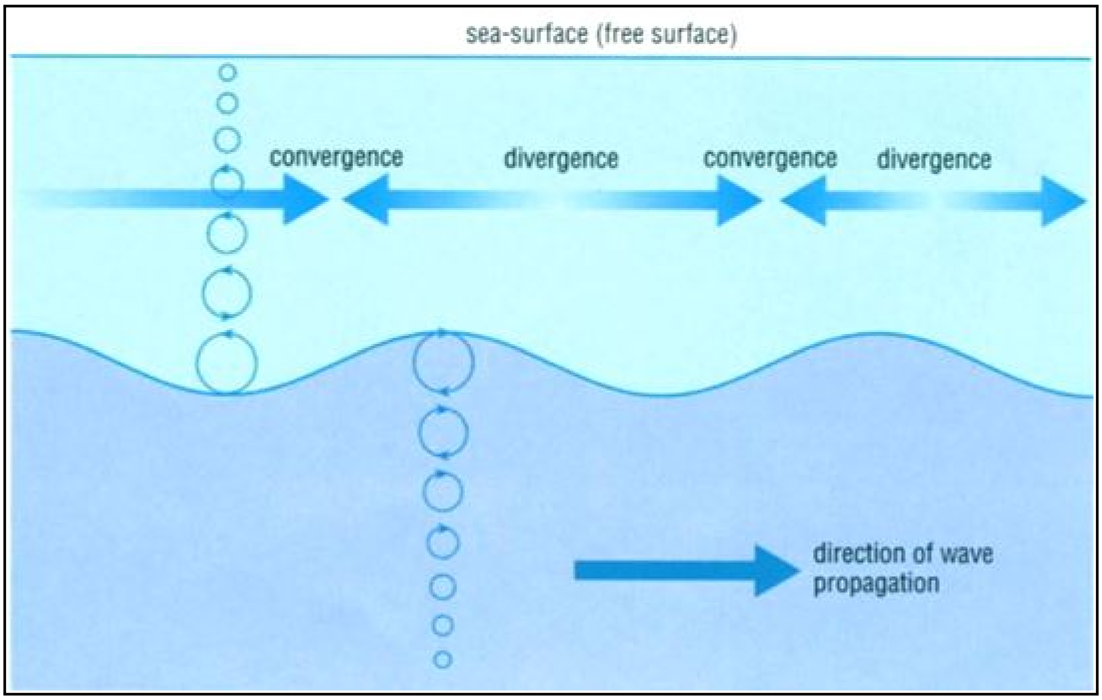


Figure 4: Two Layer Fluid (`https://oceanwiki.ethz.ch`)

To mathematically simulate internal waves, we first take into account a two-layer fluid, or a fluid having two layers, each with a distinct density, with the denser layer at the bottom. Since this enables us to ignore the effects of mixing, we will proceed as though the layers are immiscible. We shall use a subscript 1 to indicate the top fluid layer and a subscript 2 to indicate the bottom fluid layer. The factors in the issue are depicted in the diagram below. (Figure 5).

If we assume that the fluid is incompressible and irrotational then the potential function for each layer solves Laplace's equation.

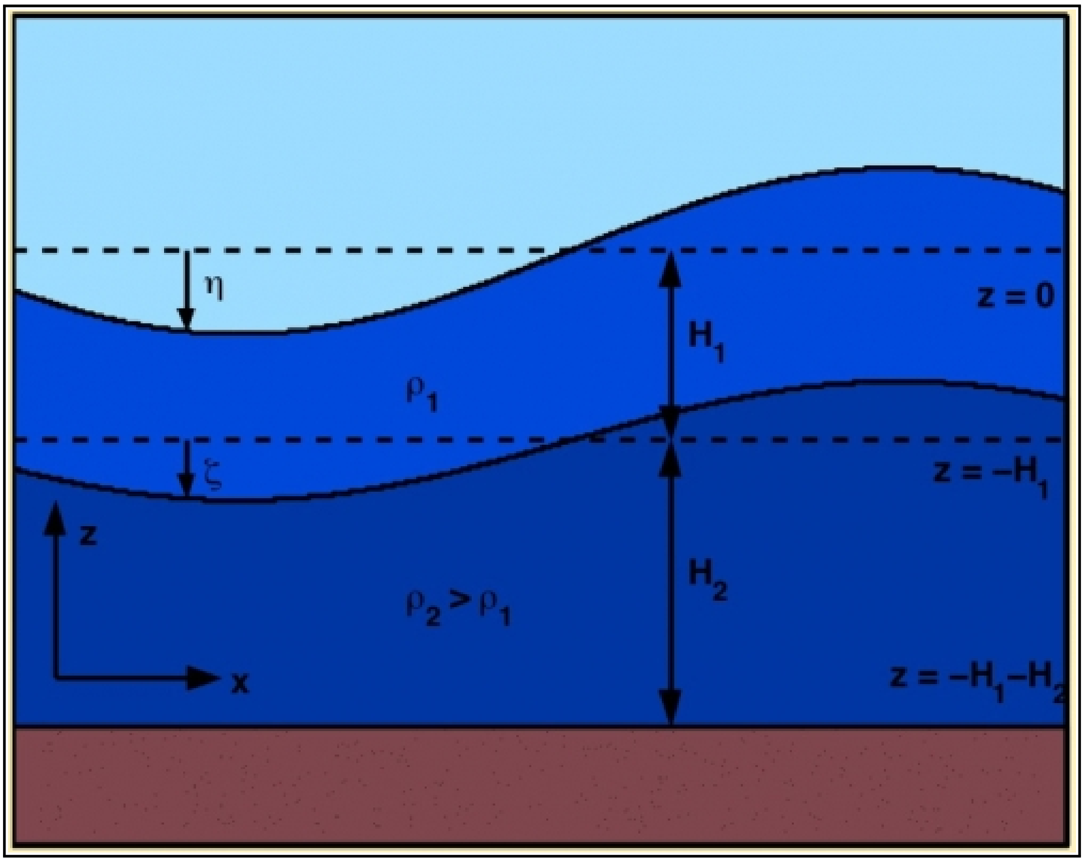


Figure 5: Two Layer Fluid Set-Up (`https://uwaterloo.ca`)

$$\nabla^2\phi_1 = 0, \qquad \nabla^2\phi_2 = 0 \tag{1}$$

We must now examine the boundary conditions for the problem at hand. We demand that there be no gaps between the two fluids at the interface, which entails that the vertical velocities in both layers must be equal and that the pressure at the contact be continuous. As a result, the contact has the following two boundary conditions, which we refer to as the kinematic and dynamic boundary conditions, respectively:

$$\frac{\partial\phi_1}{\partial z} = \frac{\partial\phi_2}{\partial z} = \frac{\partial\zeta}{\partial t} \qquad\qquad z = -H_1 \tag{2}$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \zeta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g \zeta \qquad z = -H_1 \tag{3}$$

For the boundary conditions at the surface and on the bottom we get the following conditions, in linearized form.

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} \qquad z = 0 \tag{4}$$

$$\frac{\partial \phi_1}{\partial t} + g\eta = 0 \qquad z = 0 \tag{5}$$

$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \eta}{\partial t} \qquad z = -H_2 \tag{6}$$

If we assume that the solution has the form:

$$\eta = ae^{i(kx-wt)} \text{ and } \zeta = be^{i(kx-wt)} \tag{7}$$

then from the pressure condition at the interface we can get the required dispersion relation after some work:

$$\left(\frac{\omega^2}{gk}\right)\left(\frac{\omega^2}{gk}[\rho_1 \sinh kH + \rho_2 \cosh kH - (\rho_2 - \rho_1)\sinh kH]\right) = 0 \tag{8}$$

Note that $H = H_1 + H_2$. From this dispersion relation we get two different relationships between the wavenumber and the frequency which means there are two different waves that can exist at the interface. In the **barotropic mode**,

$$\omega^2 = gk \tag{9}$$

surfaces of constant density and surfaces of constant pressure coincide, hence this mode behaves similar to the deep water gravity wave. The other mode is referred to as the **baroclinic mode**.

$$\omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh kH}{\rho_1 \sinh kH + \rho_2 \cosh kH} \tag{10}$$

In this mode, the surface wave and the interface wave are out of phase, and the disturbance amplitude is generally larger at the interface than at the surface. Now based on (equation 10), the phase velocity reads as

$$c(k) = \frac{\omega}{k} = \sqrt{\frac{g}{k}\frac{(\rho_2 - \rho_1)\sinh kH}{\rho_1 \sinh kH + \rho_2 \cosh kH}} \tag{11}$$

In the long-wave ($k \to 0$) limit, the relationship takes the form of equation (1); therefore,

$$c(k) \equiv c_0 = \sqrt{g' H_r} \tag{12}$$

with

$$g\frac{\rho_2 - \rho_1}{\rho_1} \tag{13}$$

and

$$H_r = \frac{H_1 H_2}{H_1 + H_2} = \frac{H_1 H_2}{H} \tag{14}$$

The formula for the wave speed is similar to the one of shallow water waves with g' replacing g and H_r instead of H. For surface gravity waves the density of the air is so much smaller than the density of water so that g' is basically equivalent to g. But for internal waves, the difference is much smaller and therefore g' has to be used. Because g' is much smaller than g, the wave speed of internal waves is much smaller than the one of surface waves.

2.0.2 Continuously Stratified Fluid

The Bute inlet is characterized by strong stratification during Summer, thus it is very unlikely that the water will consist precisely of two or more distinct layers. It is more likely that the water is continuously stratified. Internal waves in a stratified fluid have peculiar properties compared to surface gravity waves. The structure and dispersion relation of internal waves in a uniformly stratified fluid is found through the solution of the linearized conservation of mass, momentum, and internal energy equations assuming the fluid is incompressible and the background density varies by a small amount (the Boussinesq approximation). we get the following governing equations:

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{15}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \tag{16}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g \tag{17}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \bar{\rho}}{\partial z} w = 0 \tag{18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{19}$$

in which ρ is the peturbatiob density, p is the pressure, and (u, v, w) is the velocity. The ambient density changes linearly with height as given by $\bar{\rho}$ and ρ_0.

Through quite a bit of manipulation of the governing equations we can determine the following expression for the vertical velocity:

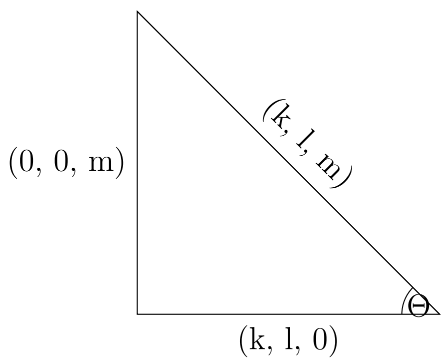
$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2 \nabla_H^2 w = 0 \tag{20}$$

From this expression assuming the vertical velocity field has the form:

$$w = w_0 e^{i(kx+ly+mz-wt)} \tag{21}$$

We can get the dispersion relation as:

$$\omega^2 = N^2 \frac{k^2+l^2}{k^2+l^2+m^2} = N^2 \cos^2 \Theta \tag{22}$$



3 Conclusion

This study looked into the occurrence and movement of internal waves in the Bute Inlet, a fjord-like inlet on Canada's west coast. Internal waves, which are essential for mixing and spreading nutrients throughout the water column, are best studied at Bute Inlet due to its distinctive terrain and powerful tidal currents. Internal wave behaviour in two-layer fluid systems and continuously stratified fluids can be understood using the mathematical concepts offered in this study.

Dispersion relations for both barotropic and baroclinic modes, as well as the phase velocity of the internal waves, were derived as a result of our investigation of a two-layer fluid system. Additionally, we expanded our investigation to the more plausible scenario of a continuously stratified fluid, and as a result, we were able to determine a governing equation for the vertical velocity and a dispersion relation that takes the stratification into account.

These mathematical formulations help us comprehend the complex physics of internal waves in Bute Inlet and offer a structure for further study on how internal waves affect the local marine ecosystem. Further research may help us better understand the mechanisms underlying the generation and propagation of internal waves. The dead-water phenomenon seen in the inlet may be connected to the large-amplitude wave activity. In the end, this information can aid in the development of policies to lessen the possible negative effects of human activities on the nearby marine environment as well as more efficient management of the inlet's precious resources.

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