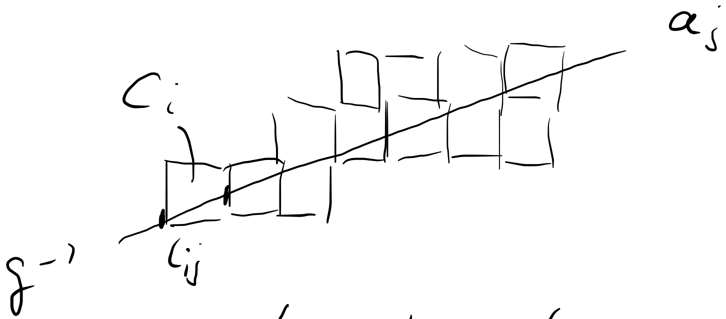
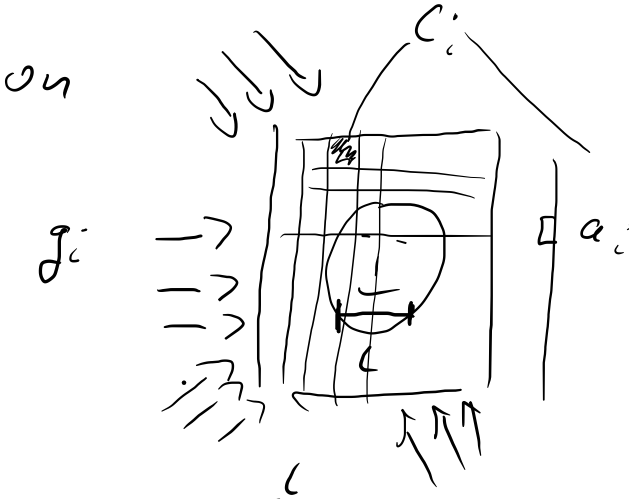
LGS Motivation





$2g \to \boxed{}_{c} \to 2a$

$a = c\,g$

$g \to \boxed{}\boxed{} \quad c^2 g$

$\boxed{} \quad c^{1/2} g$

$a = c^l g$

$\boxed{\text{Ziel: } c \in \mathbb{R}^n}$

$$a_0 = g \cdot c_0^{l_{00}} \cdot c_1^{l_{01}} \cdot c_2^{l_{02}} \cdots$$

$$\log \frac{a_0}{g} = l_{00} \log c_0 + l_{01} \cdot \log c_1 + l_{02} \cdot \log c_2 + \ldots$$

$$a_0' = l_{00}\, c_0' + l_{01}\, c_1' + l_{02}\, c_2' + \ldots$$

0,0 0,1 0,2 0,3
1,0 1,1

0 1 2 3
4 5 6 7
8

$$\begin{pmatrix} l_{00} & l_{01} & l_{02} & \cdots \\ l_{10} & l_{11} & l_{12} & \cdots \\ & & & \end{pmatrix} \begin{pmatrix} c_0' \\ c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} a_0' \\ a_1' \\ a_2 \end{pmatrix}$$

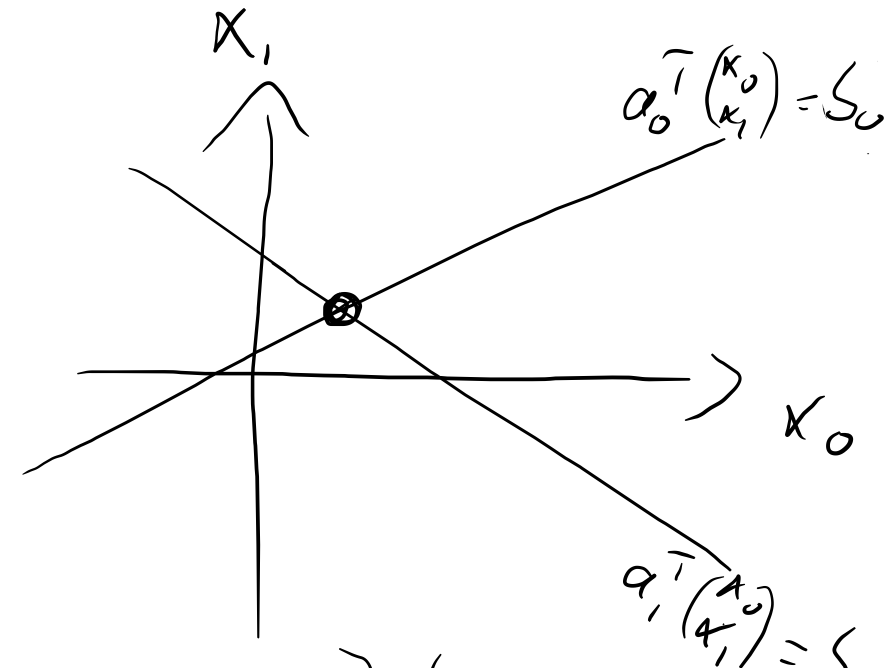
$$L\,c' = a'$$

LGS Interpretation $Ax = b$

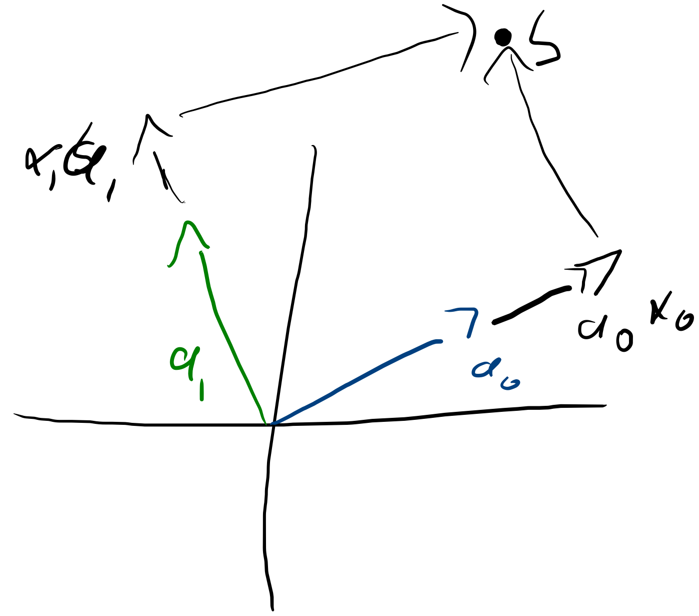
Zeilen $\begin{pmatrix} a_0^T \\ a_1^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$

$a_{00} x_0 + a_{01} x_1 = b_0 \iff x_1 = \frac{b_0 - a_{00} x_0}{a_{01}}$

$= -\frac{a_{00}}{a_{01}} x_0 + \frac{b_0}{a_{01}}$

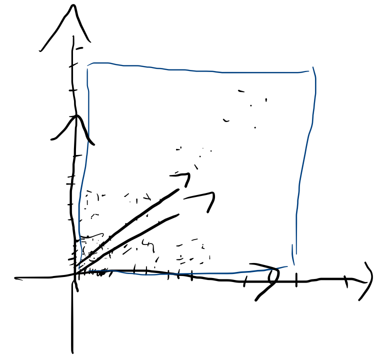


Spalten $\left(\underline{a_0}, \underline{a_1}\right) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = b$



LGS in GKZ

$A \in \mathbb{G}^{n \times n}$ $\quad n = 2 \quad$ $x \in \mathbb{G}^n$



$\mathbb{G}^2$

$Ax = y \in \mathbb{G}^n$

$Ax' = y' \in \mathbb{G}^n \qquad \exists\, x \neq x' \to y = y'$

$Ax \overset{?}{=} s \qquad A \in \mathbb{G}^{n \times n},\ x, s \in \mathbb{G}^n,\ \|s\| > 0$

$Ax^* = s \qquad x^* \in \mathbb{Q}^n$

$r = Ax - s$

$$\frac{\|r\|}{\|s\|} = \frac{\|Ax - s\|}{\|s\|} = \frac{\|Ax - Ax^*\|}{\|Ax^*\|} = \frac{\|A(x - x^*)\|}{\|Ax^*\|} = \frac{c_{x-x^*} \cdot \|x - x^*\|}{c_{x^*} \|x^*\|}$$

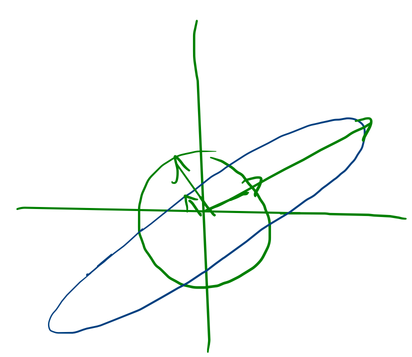
$\|Ay\| \qquad \|A(sy)\| = s \cdot \|Ay\| \qquad c_y = \|Ay\|,\ \|y\| = 1$

$\Rightarrow \|Ay\| = c_y \|y\|$

$$\Rightarrow \frac{\|r\|}{\|s\|} \leq \frac{\max_{\|y\|=1} \|Ay\| \cdot \|x - x^*\|}{\min_{\|y\|=1} \|Ay\| \cdot \|x^*\|} \approx \boxed{\frac{\max_{\|y\|=1} \|Ay\|}{\min_{\|y\|=1} \|Ay\|}} \cdot \varepsilon$$

Kondition von A

$$\kappa(A) = \frac{\max_{\|y\|=1} \|Ay\|}{\min_{\|y\|=1} \|Ay\|} \geq 1$$



$$\kappa(A) = \kappa(A^{-1})$$

Lösen von LGS

Diagonalform

$$\begin{pmatrix} a_{00} & 0 & 0 & \cdots \\ 0 & a_{11} & 0 & \cdots \\ 0 & 0 & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{pmatrix} \longrightarrow \begin{array}{l} x_0 = \frac{s_0}{a_{00}} \\ x_1 = \frac{s_1}{a_{11}} \\ \vdots \end{array}$$

$a_{ii} = 0,\ s_i \neq 0 \rightarrow$ keine Lösung

$a_{ii} = 0,\ s_i = 0 \rightarrow$ unendlich viele

Dreiecksform

untere Dreiecksm.

$$\begin{pmatrix} a_{00} & 0 & 0 & \cdots \\ a_{10} & a_{11} & 0 & \cdots \\ a_{20} & a_{21} & a_{22} & 0 \cdots \\ & & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{pmatrix} \longrightarrow \begin{array}{l} x_0 = \frac{s_0}{a_{00}} \\ x_1 = (s_1 - a_{10} \cdot x_0)/a_{11} \\ x_2 = (s_2 - a_{20}x_0 - a_{21}x_1)/a_{22} \\ \vdots \end{array}$$

„Vorwärtseinsetzen"

obere Dreiecksm.

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\ 0 & a_{11} & a_{12} & a_{13} & \cdots \\ 0 & 0 & a_{22} & a_{23} & \cdots \\ & & \vdots & & \\ 0 & \cdots & & 0 & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

$$x_{n-1} = \frac{s_{n-1}}{a_{n-1,n-1}}$$

$\vdots$

„Rückwärtseinsetzen"

Gaußsches Eliminationsverfahren

$$A^{(k)} = \begin{pmatrix} a_{00}^0 & a_{01}^0 & a_{02}^0 & \cdots \\ 0 & a_{11}^1 & a_{12}^1 & \cdots \\ 0 & 0 & \ddots & \\ & & \boxed{\begin{matrix} a_{kk}^k \\ a_{k+1,k}^k \\ a_{k+2,k}^k \\ \vdots \end{matrix}} & \cdots \\ & 0 \to & & \\ & 0 \to & & \end{pmatrix} x = \begin{pmatrix} s_0^k \\ s_1^k \\ s_2^k \\ \vdots \end{pmatrix}$$

$$m_{ik} = -\frac{a_{k+i,k}}{a_{kk}^k}$$

$$A^{(k+1)} = L^{(k)} A^{(k)}$$

$$L^k = \begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & 0 & & \\ 0 & 0 & 1 & & & & \\ & & & \ddots & & & \\ & 0 & & & 1 & 0 & \\ & & & & m_{0k} & 1 & \\ & & & & m_{2k} & 0 & 1 \\ & & & & m_{3k} & 0 & 0 & 1 \end{pmatrix}$$

$$L^{(k)} e_j = e_j \quad j \neq k$$

$$e_j = (0, \ldots, 0, \underset{j}{1}, 0, \ldots 0)$$

$$\min_{\|y\|=1} L^{(k)} y \leq 1$$

$$L^{(k)} e_k = (0, \ldots 0, 1, m_{1k}, m_{2k} \ldots)^T$$

$$\max_{\|y\|} L^{(k)} y \geq \sqrt{1 + \sum_{i=1} \frac{a_{k+i,k}^2}{a_{kk}^2}} = \sqrt{\sum_{i=0} \frac{a_{k+i,k}^2}{a_{kk}^2}} = \frac{\sqrt{\sum a_{k+i,k}^2}}{|a_{kk}|}$$

$$\kappa(L^{(k)}) \geq \frac{\sqrt{}}{|a_{kk}|}$$

→ wähle $|a_{kk}|$ möglichst groß