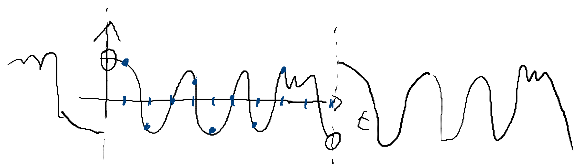
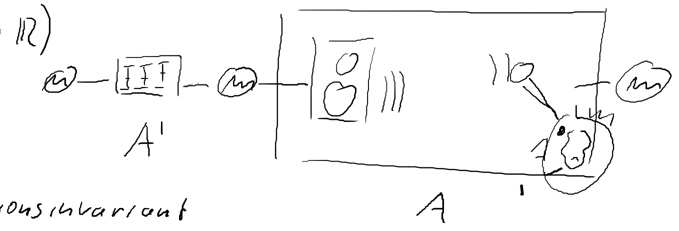
Signale

$s: \mathbb{R} \to \mathbb{R}$



$A: (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})$

↑ Filter



Zeitinvariant / ortsinvariant / translationsinvariant

stationär

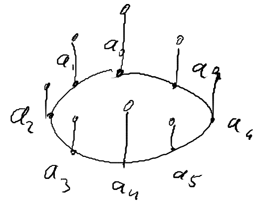
$$A \circ s(t-\varepsilon) = (A \circ s)(t-\varepsilon)$$

Linear

$$A \circ (\alpha \cdot s(t)) = \alpha \cdot (A \circ s(t)) \quad , \quad A \circ (s(t) + s'(t)) = A \circ s(t) + A \circ s'(t)$$

Diskret: $s \in \mathbb{R}^n$ $\qquad s = (s_0, \dots, s_{n-1})$

↑ Zeit/Ort



Filtern $\boxed{A}s$, $A \in \mathbb{R}^{n \times n}$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & & & & \\ 0 & 0 & \dots & & & 0 & 1 \\ 1 & 0 & 0 & \dots & & 0 & 0 \end{pmatrix}}_{= Z} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_0 \end{pmatrix}$$

$Z^k s \to$ Verschiebung um k

$$\boxed{A Z^k s = Z^k A s}$$

$$a' = \begin{pmatrix} a_0 \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad Z A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = Z \begin{pmatrix} a_0 \\ a_{n-1} \\ \vdots \\ a_1 \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} = A \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = A Z \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \\ a_{n-1} & a_0 & a_1 & \dots \\ a_{n-2} & a_{n-1} & a_0 & \\ \vdots & \vdots & \vdots & \\ a_2 & & & \\ a_1 & a_2 & a_3 & \end{pmatrix} = a_0 \cdot \underline{I} + a_1 \cdot Z + a_2 \cdot Z^2 + \dots + a_{n-1} \cdot Z^{n-1}$$

$$= \sum_{i=0}^{n-1} a_i Z^i$$

Eigenschaften von $A = \sum_i a_i Z^i$

Eigenvektoren v von Z

$Zv = \lambda v$

$$Z^k v = Z^{k-1} \cdot Zv = Z^{k-1} \cdot \lambda v = Z^{k-2} \cdot Z \lambda v = Z^{k-2} \lambda^2 v = \dots = \lambda^k v$$

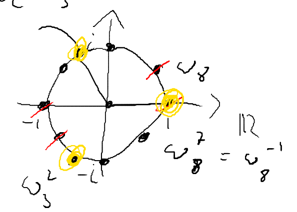
$$A \cdot v = \left(\sum_i a_i Z^i\right) v = \sum_i a_i Z^i v = \sum_i a_i \lambda^i \cdot v = (\underbrace{\sum_i a_i \lambda^i}_{\text{Skalar}}) v$$

Eigenwerte von $Z \in \mathbb{R}^{n \times n}$ $\quad Z^n = I \quad \to \quad Z^n v = \lambda^n \cdot v = I v$

$\Rightarrow \lambda^n = 1$ Einheitswurzeln

$\boxed{\omega_n = e^{\frac{2\pi i}{n}}}$, $\omega_n^2 = e^{\frac{2\pi i \cdot 2}{n}}, \dots, \omega_n^{n-1}, \omega_n^n$ $\quad i \in \mathbb{C}, i^2 = -1$

$\boxed{\omega_n^n = 1}$, $(\omega_n^2)^n = \omega_n^{2n} = \omega_n^n \cdot \omega_n^n = 1$



$\lambda^n = 1 \qquad \lambda = \omega_n^k, \; k = 0, 1, \dots, n-1$

$\overline{a + bi} = a - bi$

Eigenvektoren von Z

$Zv = \omega_n^k v \quad \to \quad \underbrace{v_{i-1}}_{Z \text{ anwenden}} = \omega_n^k v_i$

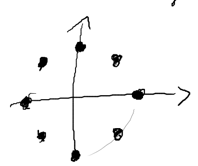
$$\boxed{v_k = \begin{pmatrix} 1 \\ \omega_n^k \\ \omega_n^{2k} \\ \omega_n^{3k} \\ \vdots \\ \omega_n^{(n-1)k} \end{pmatrix}}$$

Eigenvektoren von A

$$v_j^* v_k = 1 + \overline{\omega^j}\omega^k + \overline{\omega^{2j}}\omega^{2k} + \dots + \overline{\omega^{(n-1)j}}\omega^{(n-1)k}$$
$$= 1 + \omega^{-j}\omega^k + \omega^{-2j}\omega^{2k} + \dots + \omega^{-(n-1)j}\omega^{(n-1)k}$$
$$= 1 + \omega^{k-j} + \omega^{2(k-j)} + \dots + \omega^{(n-1)(k-j)}$$

$k = j \quad \Rightarrow v_k^* v_k = 1 + 1 + \dots + 1 = n$

$k \neq j \quad \Rightarrow v_j^* v_k = 0$



$(v_0, v_1, v_2, \dots, v_{\frac{n}{2}}, \dots, v_{n-1})$

Konstante, „Frequenz" 1, hohen Frequenzen, „Frequenz" -1