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# Week 1: Euclidean Vector Spaces

## Exercise 1

1. The Cartesian product $A_1 \times \cdots \times A_n$ of sets $A_1, \ldots, A_n$ is defined as

   ☐ $A_1 \times \cdots \times A_n \coloneqq A_1 \setminus (A_2 \cup \ldots \cup A_n)$
   ☐ $A_1 \times \cdots \times A_n \coloneqq \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$
   ☐ $A_1 \times \cdots \times A_n \coloneqq \{a_1 \cdot a_2 \cdot \ldots \cdot a_n \mid a_1 \in A_1, \ldots, a_n \in A_n\}$

2. Let $n \in \mathbb{N} \setminus \{0\}$. Then $\mathbb{R}^n$ contains

   ☐ $n$ real numbers ☐ $n$-tuples of real numbers ☐ $n$-tuples of vectors

3. Which of the following sets together with standard addition and scalar multiplication does not form a real vector space?

   ☐ The set of integers $\mathbb{Z}$
   ☐ The set of complex numbers $\mathbb{C}$
   ☐ The set of real-valued, continuous functions $\{f \colon \mathbb{R}^n \to \mathbb{R} \mid f \text{ continous }\}$

4. Scalar multiplication in a real vector space $V = (\mathcal{V}, +, \cdot)$ is given by a mapping

   ☐ $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$ ☐ $\mathbb{R} \times \mathcal{V} \to \mathcal{V}$ ☐ $\mathbb{R} \times \mathcal{V} \to \mathbb{R}$

5. How many vector subspaces does $\mathbb{R}^2$ have?

   ☐ two: $\{\mathbf{0}\}$ and $\mathbb{R}^2$
   ☐ four: $\{\mathbf{0}\}$, $\mathbb{R} \times \{0\}$, $\{0\} \times \mathbb{R}$ (the axes), $\mathbb{R}^2$
   ☐ infinitely many

6. Which of the following subsets of $\mathbb{R}^2$ is not a vector subspace?

   ☐ $\{\mathbf{0}\}$ ☐ $\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = 2x_2\}$ ☐ $\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = x_2 + 1\}$

7. For which of the following objects does it make sense to say that they are "linearly independent"?

   ☐ Elements of a vector space $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{V}$
   ☐ Real numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$
   ☐ The linear combination $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n$

8. Which of the following vectors form a basis of $\mathbb{R}^2$ ?

   ☐ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

9. What is the scalar product of the vectors $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ ?

   ☐ 3 ☐ 5 ☐ 7

10. Which of the following statements is false?

    ☐ For any Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ the function $\|\cdot\|$ defined by $\|\boldsymbol{v}\| \coloneqq \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$ $(\boldsymbol{v} \in \mathcal{V})$ satisfies the properties of a norm.
    ☐ For any Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ it holds that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle \geq 0$ for all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.
    ☐ Let $M$ be a subspace of an Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ . Then

    $$\{\boldsymbol{v} \in \mathcal{V} \mid \forall \boldsymbol{u} \in M \colon \boldsymbol{v} \perp \boldsymbol{u}\}$$

    is also subspace of $V$.

## Exercise 2

Consider $(\mathbb{R} \setminus \{-1\}, \star)$, where $a \star b := ab + a + b$ with $a, b \in \mathbb{R} \setminus \{-1\}$.

1. *Show* that $(\mathbb{R} \setminus \{-1\}, \star)$ is an Abelian group.
2. *Solve* $3 \star x \star x = 15$ in the Abelian group $(\mathbb{R} \setminus \{-1\}, \star)$.
3. Is $(\mathbb{R} \setminus \{-1\}, +, \star)$ a field? Justify!

## Exercise 3

*Show:* If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$, the following holds for all $\boldsymbol{x} \in \mathcal{V}$:

$$\boldsymbol{x} = \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{v}_i \rangle \, \boldsymbol{v}_i$$

*Hint:* Establish first that $\boldsymbol{x}$ can be expressed as $\boldsymbol{x} = \lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n$. Then show that $\langle \boldsymbol{x}, \boldsymbol{v}_i \rangle = \lambda_i$ for all $1 \leq i \leq n$ .

## Exercise 4

Let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space and $U$ an $r$-dimensional vector subspace $\mathcal{U} \subseteq \mathcal{V}$ with orthonormal basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$. The orthogonal projection of a vector $\boldsymbol{v} \in \mathcal{V}$ onto $\mathcal{U}$ is given by

$$p(\boldsymbol{v}) := \sum_{i=1}^{r} \langle \boldsymbol{v}, \boldsymbol{u}_i \rangle \, \boldsymbol{u}_i$$

1. *Compute* the orthogonal projection of the vector $(25, 0)^\top$ onto the subspace spanned by the vector $(3, 4)^\top$. Visualize the subspace and the projection in a drawing with both vectors.
2. Let $\boldsymbol{x} \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. *Show:*
$$p(\lambda \boldsymbol{x}) = \lambda p(\boldsymbol{x}).$$
3. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. *Show:*
$$p(\boldsymbol{x} + \boldsymbol{y}) = p(\boldsymbol{x}) + p(\boldsymbol{y}).$$

## Exercise 5

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^n$.

1. *Show* that the mapping
$$\begin{aligned} k \colon \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\ (\boldsymbol{x}, \boldsymbol{y}) &\mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \end{aligned}$$
does <u>not</u> define a scalar product on $\mathbb{R}^2$.
2. Consider the mapping $\Phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ with
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2} \cdot x_1 x_2 \end{bmatrix}.$$
*Show:*
$$\langle \Phi(\boldsymbol{x}), \Phi(\boldsymbol{y}) \rangle = k(\boldsymbol{x}, \boldsymbol{y})$$

## Exercise 6

The 1-norm is often used for finding sparse solutions to an optimization problem (vectors or matrices with many entries equal to zero). This will be demonstrated in the following exercise:
We are looking for a vector $\boldsymbol{w} = (x, y)^\top \in \mathbb{R}^2$, which solves the optimization problem

$$\max_{\boldsymbol{w}} f(\boldsymbol{w}) \qquad \text{s.t.} \quad \|\boldsymbol{w}\| = 1$$

Consider $f(x, y) = \frac{1}{2}x + y$ and compare the solutions to this optimization problem for the 1- and the 2-norm.

1. Draw the set of all points on the $x$-$y$-plane that have 2-norm equal to 1 (i.e. the $\ell_2$ unit circle).
$$C_2 := \left\{ \boldsymbol{w} \in \mathbb{R}^2 \mid \|\boldsymbol{w}\|_2 = \sqrt{x^2 + y^2} = 1 \right\}.$$

2. Draw the set of all points on the $x$-$y$-plane that have 1-norm equal to 1 (i.e. the $\ell_1$ unit circle).

$$C_1 := \left\{ \boldsymbol{w} \in \mathbb{R}^2 \mid \|\boldsymbol{w}\|_1 = |x| + |y| = 1 \right\}.$$

3. Draw contour lines $c = f(x, y)$ for $c = 0.5$, $c = 1$ and $c = 1.1$.

4. Where in your drawing does the solution to the optimization problem lie for the 1-norm? Where does it lie for the 2-norm?

## Exercise 7

Given training data (red and blue) a new data point (green) should be classified using the $k$-NN algorithm. Which label does the new data point receive for $k \in 1, 2, 5$?