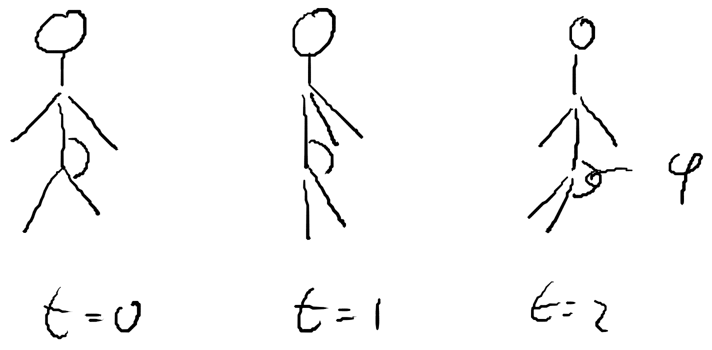
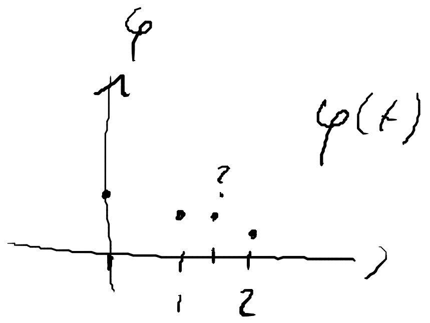
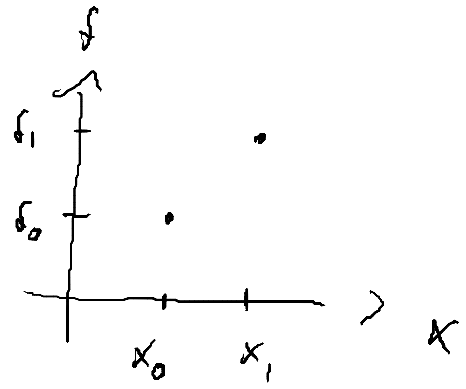
Interpolation





Lineare Interpolation



geg. $(x_0, f_0), (x_1, f_1)$

ges. $f(x) = c_0 + c_1 x = (1, x)\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

$$= (1, x)\, C$$

$$\left.\begin{aligned} f_0 &= c_0 + c_1 x_0 \\ f_1 &= c_0 + c_1 x_1 \end{aligned}\right\} \quad \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$C = \frac{1}{x_1 - x_0}\begin{pmatrix} x_1 & -x_0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$f(x) = \frac{1}{x_1 - x_0}(1, x)\begin{pmatrix} x_1 & -x_0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$= \left(\frac{x_1 - x}{x_1 - x_0},\ \frac{x - x_0}{x_1 - x_0} \right)\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

$$= f_0 \cdot \frac{x_1 - x}{x_1 - x_0} + f_1 \cdot \frac{x - x_0}{x_1 - x_0}$$

Polynom - Interpolation

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots = (1, x, x^2, x^3, \dots) \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

$$= (1, x, x^2, x^3, \dots)\, c$$

$$\left.\begin{aligned} f_0 &= c_0 + c_1 x_0 + c_2 x_0^2 + \dots \\ f_1 &= c_0 + c_1 x_1 + c_2 x_1^2 + \dots \\ f_2 &= c_0 + c_1 x_2 + c_2 x_2^2 + \dots \end{aligned}\right\} \begin{pmatrix} 1 & x_0 & x_0^2 & \dots \\ 1 & x_1 & x_1^2 & \dots \\ 1 & x_2 & x_2^2 & \dots \\ \vdots & & & \end{pmatrix} c = f$$

n Koeffizienten
n Bedingungen

$$V(x_0, x_1, x_2, \dots, x_{n-1}) = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & & & & \\ 1 & x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

Vandermonde-Matrix

Polynome Grad $n-1$

$$\det V(x_0, x_1) = x_1 - x_0$$

Induktion $\quad \det V(x_0, \dots, x_{n-1}) = \prod_{0 \le i < j < n} (x_i - x_j)$

Lagrange - Interpolation

2 Punkte $\rightarrow$ $f_0 \cdot \underline{\frac{x_1 - x}{x_1 - x_0}} + f_1 \cdot \underline{\frac{x - x_0}{x_1 - x_0}}$

als.
$$f(x) = \underline{f_0 \cdot l_0(x) + f_1 \cdot l_1(x)} + f_2 \cdot l_2(x) + \dots$$
$$= \boxed{\sum_{i=0}^{n-1} f_i \cdot l_i(x)}$$

$$l_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$l_i(x) = \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdot \dots \cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \dots$$

$$\boxed{l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n-1} \frac{x - x_j}{x_i - x_j}}$$

Polynom

Grad $n-1$ ✓

Lagrange Basisfunktionen

$$l_i(x_i) = \prod \frac{x_i - x_j}{x_i - x_j} = \underline{1}$$

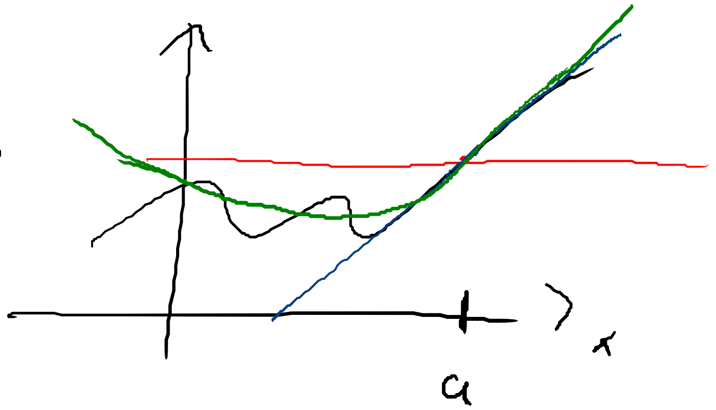
$$\underset{j \neq i}{l_i(x_j)} = \frac{x_j - x_j}{x_i - x_j} \prod \dots = 0$$

Satz von Taylor

Gegeben f

Ges. Polynom p

$p \approx f$



$$f(x) \approx f(a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(a) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + R_n(x)$$

Lagrange Restglied

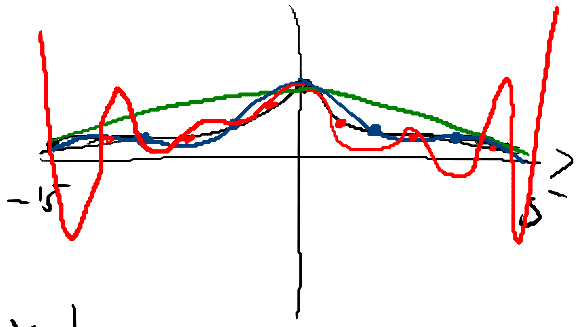
$$R_n(x) = \frac{|f^{(n+1)}(z)|}{(n+1)!}(x-a)^{n+1}, \quad \exists\, z \in [x,a] \cup [a,x]$$

Annahme $\quad |f^{(i)}(x)| \leq L \qquad x \in [a,b]$

$\rightarrow \quad R_n \leq L \cdot \frac{(x-a)^{n+1}}{(n+1)!} \leq L \cdot \frac{(b-a)^{n+1}}{(n+1)!} \leq \varepsilon$

Runge - Phänomen

$$f(x) = \frac{1}{1+x^2} \qquad x \in [-5,5]$$



$$f(x) = \underbrace{\sum_i f(x_i)\, l_i(x)}_{p_n(x)} + \frac{f^{(n+1)}(z)}{(n+1)!} \prod_{j=0}^{n} (x-x_j)$$

$$\exists\, z \in [a,b]$$

$$\sup |f(x) - p_n(x)| \le \underbrace{\left(\max \frac{f^{(n+1)}(z)}{(n+1)!} \right)}_{\text{gegeben durch } f} \cdot \underbrace{\left(\max \prod_{j=0}^{n} (x-x_j) \right)}_{\text{bestimmt durch } \{x_j\}}$$

$$G_n(x) = \prod_{j=0}^{n} (x-x_j)$$

wähle x_j so,

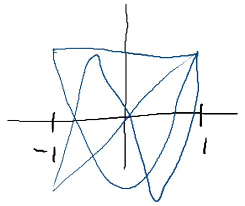
dass $|G_n|$ für $x \in [a,b]$ möglichst klein wird!

Tschebyschev Polynom

$T_0(x) = 1$
$T_1(x) = x$
$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$ $\quad n \geq 1$

T_n hat Grad n

$T_2(x) = 2x\, T_1(x) - T_0(x)$
$= 2x^2 - 1$
$T_3(x) = 2x\, T_2(x) - T_1(x)$
$= 2x(2x^2-1) + x$
$= 4x^3 - 2x - x$



für $x \in [-1,1]$ $\quad T_n(x) = \cos(n \operatorname{acos} x)$

Nullstellen $\quad n \cdot \operatorname{acos} x = (j + \frac{1}{2})\pi$

$x_j = \cos\left(\frac{(j+\frac{1}{2})\pi}{n}\right)$

n Nullstellen in $[-1, 1]$

Extremstellen $\quad T_n'(x) = -\sin(n \operatorname{acos} x)\,\frac{-n}{\sqrt{1-x^2}} \quad \Rightarrow \quad x \in]-1, 1[$

$n \cdot \operatorname{acos} x = k\pi$

$x_k = \cos\left(\frac{k\pi}{n}\right)$ $\quad n+1$ verschiedene NSt von T'

ohne ± 1 $\quad \Rightarrow \quad n-1$ " Extremstellen von T

Einsetze $\quad T_n(x_k) = \cos(\cancel{n}\ \cancel{\operatorname{acos}}\ \cancel{\cos}\ \frac{k\pi}{\cancel{n}}) = \cos(k\pi) = (-1)^k$

Betrachte nur Polynome $\quad p(x) = 1x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0$

$\min_{x \in [-1,1]} |p(x)|$? $\quad T_n \to c_n = 2^{n-1}$

$2^{1-n} T_n$ haben $c_n = 1$

$|2^{1-n} T_n(x)| \leq 2^{1-n}$ für $x \in [-1, 1]$

Annahme $\quad |p(x)| < 2^{1-n}$

$q(x) = 2^{1-n} T_n(x) - p(x)$ $\quad$ Grad $n-1$

x_k $\quad q(x_k) > 0$ wenn $T_n(x_k)$ maximal $\quad n+1$
$\quad q(x_k) < 0$ $\quad T_n(x_k)$ minimal

$\rightarrow$ n Nullstellen in $q(x)$ in $[-1, 1]$ $\quad$ ↯

Gute Abstastpunkte für Lagrange-Interpolation im Intervall $[-1, 1]$ sind Nst. von T_n

für $[a, b]$

$$\boxed{x_j = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{(j+\frac{1}{2})\pi}{n}\right)} \;!$$