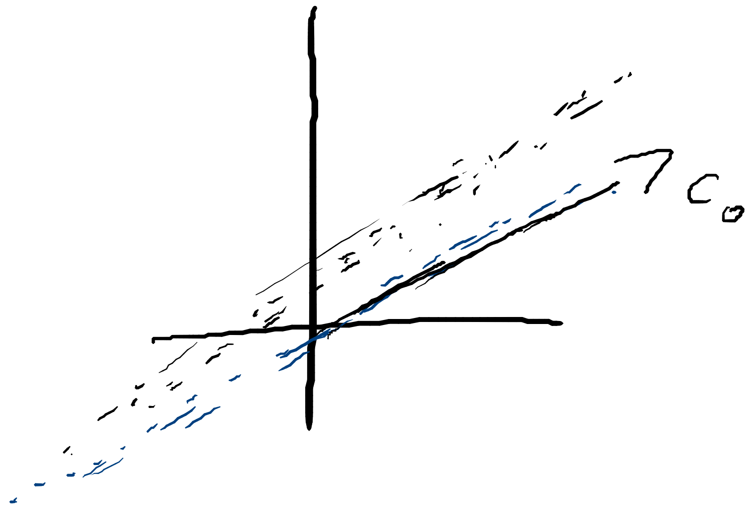
O-Basis für Vektoren $y_i \in \mathbb{R}^n$

$Y = (y_0, y_1, y_2, \dots)$

$YY^T = Q \Lambda Q^T$

c_i Spalten von Q



$y^T = X$

$$v_i^T y y^T v_i = v_i^T X^T X v_i = (Xv_i)^T (Xv_i) = \| X v_i \|^2$$

$$\boxed{\underset{\|v_i\|=1}{\arg\max} \; \| X v_i \|} \qquad \sigma_i u_i = X v_i \;, \quad \|u_i\| = 1$$

$$\sigma_i u_i^T \sigma_j u_j = (Xv_i)^T (Xv_j) = v_i^T X^T X v_j$$

$$= v_i^T Q \Lambda Q^T v_j = e_i^T \Lambda e_j = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}$$

$$\|\sigma_i u_i\|^2 = \sigma_i^2 u_i^T u_i = \sigma_i^2 = \lambda_i \qquad \sigma_i = \sqrt{\lambda_i}$$

$$X X^T (X v_i) = \lambda_i (X v_i) \qquad X X^T u_i = \lambda_i u_i \qquad \boxed{\underset{\|u_i\|=1}{\arg\max} \; \| u_i^T X \|}$$

$\mathrm{rang}(X) = r$

$$U \begin{bmatrix} \Sigma & 0 \end{bmatrix} = X V$$

$$X = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^T$$

$$X = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T$$

$$X = U \Sigma V^T$$

$$X = U \Sigma V^T$$

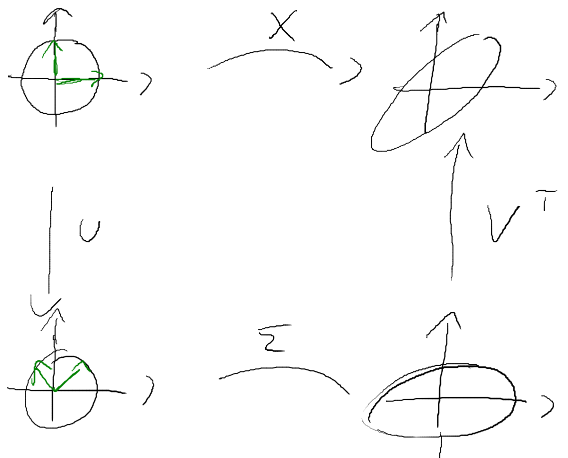
SVD

Singulärwertzerlegung

rechten Singulärvektoren

linken "

Singulärwerte $\quad \sigma_i \geq 0$



Anwendungen SVD

$$X = U \Sigma V^T$$

$$\sigma_0 \geq \sigma_1 \geq \dots \sigma_{r-1} > \sigma_r, \sigma_{r+1}, \dots = 0$$

$$\Sigma^+ = \begin{pmatrix} \sigma_0^{-1} & & & & & \\ & \sigma_1^{-1} & & & & \\ & & \ddots & & & \\ & & & \sigma_{r-1}^{-1} & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & \ddots \end{pmatrix}$$

$$X^+ = V \Sigma^+ U^T \qquad \text{Pseudoinverse}$$

$X \in \mathbb{R}^{n \times n}$

rang $(X) = n$

$\Rightarrow X^+ = X^{-1}$

$$X X^+ = U \Sigma \underbrace{V^T V}_{I} \Sigma^+ U^T = U \Sigma \Sigma^+ U^T = U \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} U^T$$

$A x \approx b \qquad x = A^+ b \leftarrow$ immer sinnvoll

1. $Ax = b$ hat keine Lösung $\rightarrow$ minimiere $\|Ax - b\|$
2. $Ax = b$ hat 1 Lösung $\rightarrow$ $x = A^{-1} b$
3. $Ax = b$ hat viele Lösungen $\rightarrow$ $\underset{Ax=b}{\arg\min} \|x\|$

$$\|Ax - b\| = \|U \Sigma V^T x - b\|$$

$$= \|U^T(U \Sigma V^T x - b)\| = \|\Sigma V^T x - U^T b\| \qquad x' = V^T x \quad b' = U^T b$$

$$= \|\Sigma x' - b'\|$$

$$\|\Sigma x' - b'\|^2 = \sum_{i=0}^{r-1} (\sigma_i x_i' - b_i')^2 + \boxed{\sum_{i=r}^{n-1} (-b_i')^2}$$

$x_i' = b_i' / \sigma_i$

$x_i', i \geq r \rightarrow x_i' = 0 \rightarrow$ min $\|x'\|$

$$V^T x = x' = \Sigma^+ b' = \Sigma^+ U^T b$$

$$x = \underbrace{V \Sigma^+ U^T}_{A^+} b$$

Homogene LGS

$$Ax = 0 \qquad x \neq 0 \quad \rightarrow \quad Ax = 0 \quad \rightarrow \quad A(\lambda x) = 0$$

$$\underset{\|x\|=1}{\operatorname{argmin}} \|Ax\|$$

$$x' = V^T x$$

$$\underset{\|x\|=1}{\operatorname{argmin}} \|U \Sigma V^T x\| = \operatorname{argmin} \|\Sigma V^T x\| = \underset{\|x'\|}{\operatorname{argmin}} \|\Sigma x'\| = \sigma_0 x_0' + \sigma_1 x_1' + \dots$$

$$\sigma_0 \geq \sigma_1 \geq \dots$$

$$\Rightarrow x_0 = 0, x_1 = 0, \dots, x_{n-1} = 1$$

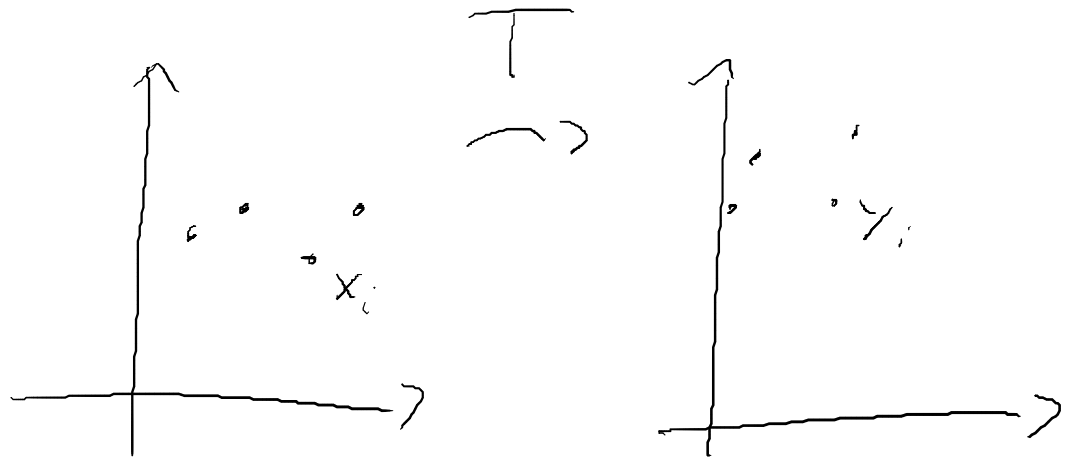
$$x = V_{n-1} \longleftarrow x' = e_{n-1}$$

Singulärvektor zum kleinsten S-Wert

$x_i, y_i \in \mathbb{R}^n$

$$\underset{R^T R = I}{\operatorname{argmin}} \sum \| y_i - R x_i \|^2$$

$(\det R = +1)$



$Y = (y_0, y_1, \dots) \qquad X = (x_0, x_1, \dots)$

$T \doteq R$

$XY^T \in \mathbb{R}^{n \times n}$

$XY^T = U \Sigma V^T$

Orthogonals Procustes Problem

$\det XY^T = 1 \rightarrow R = UV^T$

$= -1 \rightarrow R = U \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} V^T$