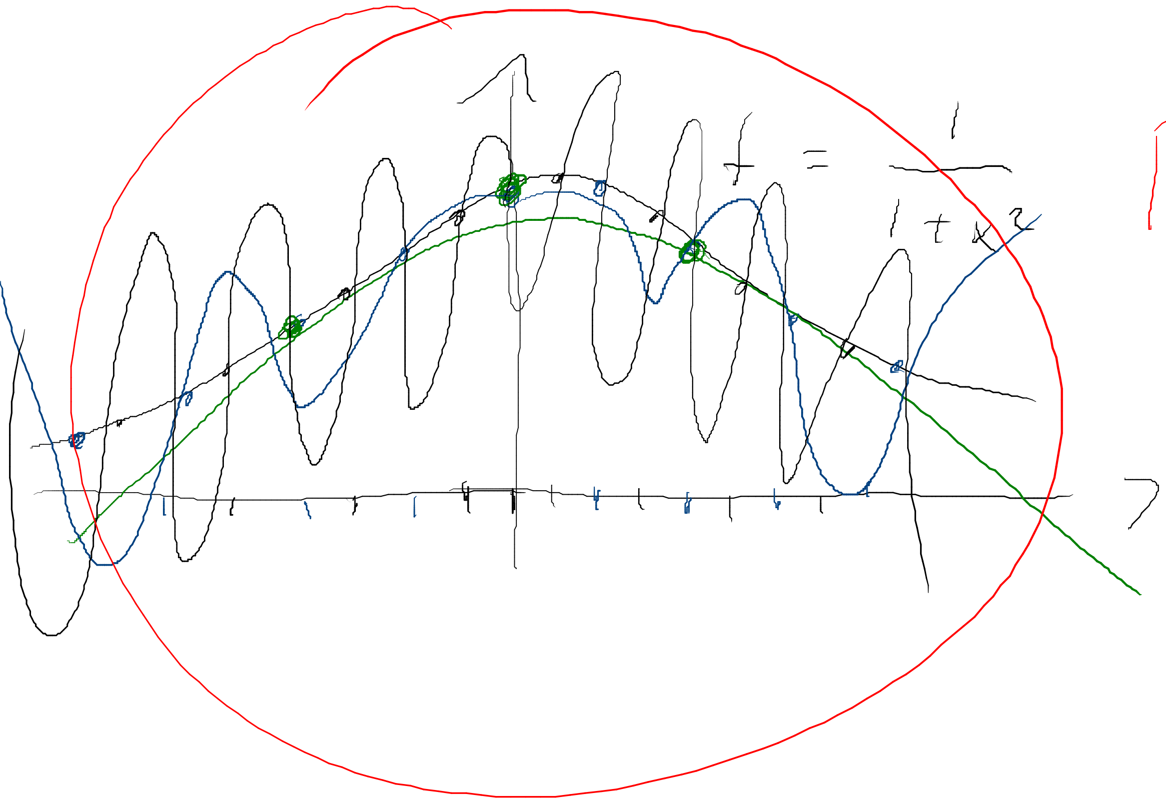
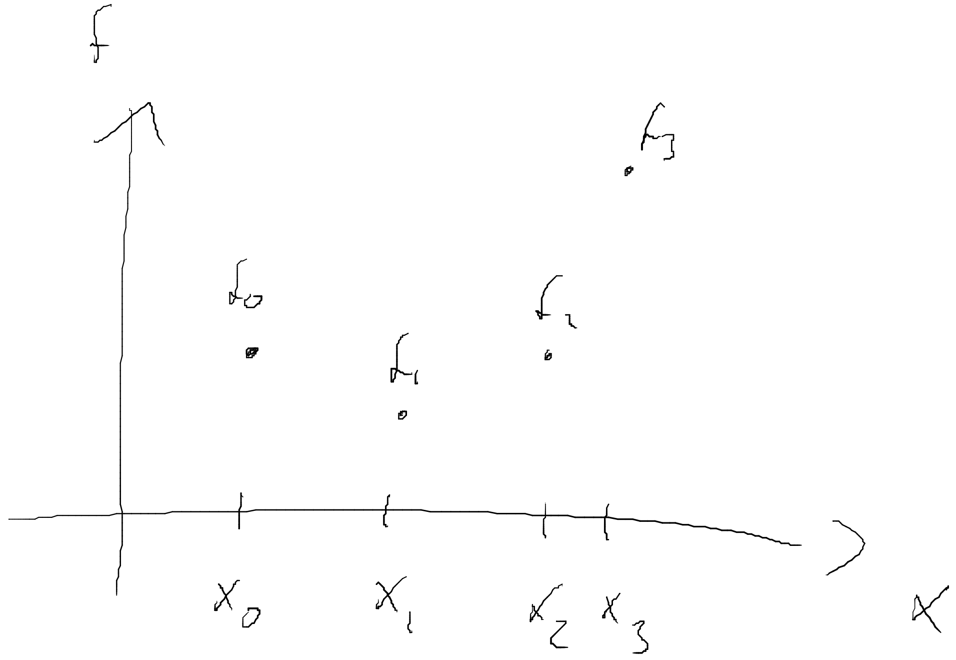
Interpolation



Runge Phänomen



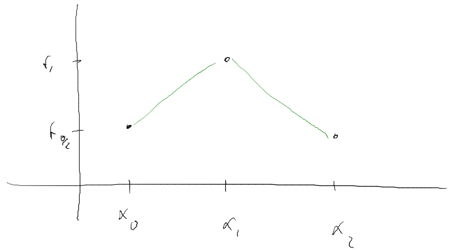
Gesucht: $f : \mathbb{R} \to \mathbb{R}$

$$f(x_i) = f_i$$

Stückweise Polynominterpolation

Stückweise Lineare: $x_i \le x < x_{i+1}$

$$f(x) = f_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + f_{i+1} \frac{x - x_i}{x_{i+1} - x_i}$$

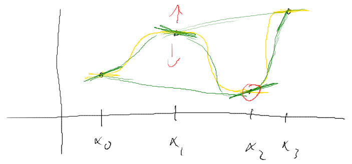


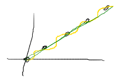
Splines

Polynomstücke $\quad f_i(x) = (1, x, x^2, x^3)\, c_i$

$$f_i'(x) = (0, 1, 2x, 3x^2)\, c_i$$

$$f_i''(x) = (0, 0, 2, 6x)\, c_i$$





$x_0 \; \dots \; x_1 \quad x_2 \quad \dots \quad x_{n-1}$

n Stützstellen

$n-1$ Segmente (Polynomstücke)

$4n-4$ Variablen

$f_i(x_i) = f_i \qquad (1, x_i, x_i^2, x_i^3)\, c_i \qquad \boxed{2n-2}$

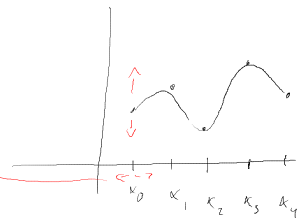
$f_i(x_{i+1}) = f_{i+1} \qquad (1, x_{i+1}, x_{i+1}^2, x_{i+1}^3)\, c_{i+1}$

$f_i'(x_{i+1}) = f_{i+1}'(x_{i+1}) \qquad (0, 1, 2x_{i+1}, 3x_{i+1}^2)\, c_i = (0, 1, 2x_{i+1}, 3x_{i+1}^2)\, c_{i+1} \qquad \boxed{n-2}$

$f_i''(x_{i+1}) = f_{i+1}''(x_{i+1}) \qquad (0, 0, 2, 6x_{i+1})\, c_i = (0, 0, 2, 6x_{i+1})\, c_{i+1} \qquad \boxed{n-2}$

$4n-6$

$$\begin{pmatrix}
1 & x_0 & x_0^2 & x_0^3 & & & & & \\
1 & x_1 & x_1^2 & x_1^3 & & & & & \\
0 & 1 & 2x_1 & 3x_1^2 & 0 & -1 & -2x_1 & -3x_1^2 & \\
0 & 0 & 2 & 6x_1 & 0 & 0 & -2 & -6x_1 & \\
 & & & & 1 & x_1 & x_1^2 & x_1^3 & \\
 & & & & 1 & x_2 & x_2^2 & x_2^3 & \\
 & & & & 0 & 1 & 2x_2 & 3x_2^2 & 0 \; -1 \; -2 \dots \\
 & & & & & \vdots & & &
\end{pmatrix}
\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{pmatrix}
=
\begin{pmatrix} f_0 \\ f_1 \\ 0 \\ 0 \\ f_1 \\ f_2 \\ 0 \\ \vdots \end{pmatrix}$$



Randbedingungen

1. Natürliche Rand Bdg. $\quad f_0''(x_0) = 0 = f_{n-2}''(x_{n-1})$

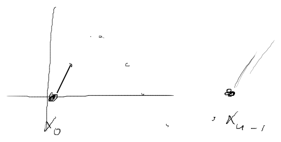
$$\begin{pmatrix} 0 & 0 & 2 & 6x_0 & \dots & & & & \\ & & & & \dots & 0 & 0 & 2 & 6x_{n-1} \end{pmatrix} \Bigg| \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. Vorgabe der Steigungen in x_0, x_{n-1}

3. Periodische Rand Bdg. $\quad f_0 = f_{n-1}$

$$f_0'(x_0) = f_{n-2}'(x_{n-1})$$

$$f_0''(x) = f_{n-2}''(x_{n-1})$$

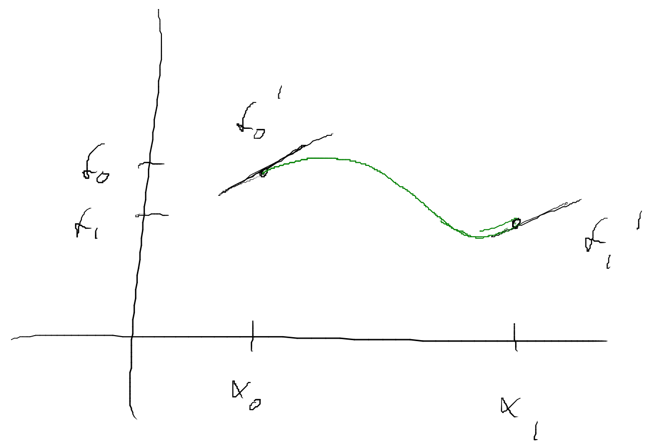


$$\begin{pmatrix} 0 & 1 & 2x_0 & 3x_0^2 & \dots & 0 & -1 & -2x_{n-1} & -3x_{n-1}^2 \\ 0 & 0 & 2 & 6x_0 & \dots & 0 & 0 & -2 & -6x_{n-1} \end{pmatrix} \Bigg| \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hermite-Interpolation

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2$$



$$f_0 = c_0 + c_1 x_0 + c_2 x_0^2 + c_3 x_0^3$$

$$f_1 = c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3$$

$$f_0' = c_1 + 2c_2 x_0 + 3c_3 x_0^2$$

$$f_1' = c_1 + 2c_2 x_1 + 3c_3 x_1^2$$

$$\rightarrow \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 0 & 1 & 2x_1 & 3x_1^2 \end{pmatrix} C = \begin{pmatrix} f_0 \\ f_1 \\ f_0' \\ f_1' \end{pmatrix}$$

$$x_0 = 0 \quad , \quad x_1 = 1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}^{-1}$$

$$f(x) = (1, x, x^2, x^3)\, C = (1, x, x^2, x^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_0' \\ f_1' \end{pmatrix}$$