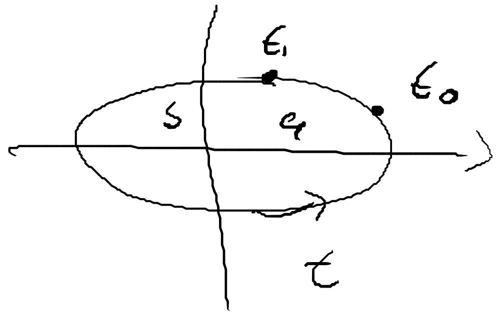
Numerische Integration



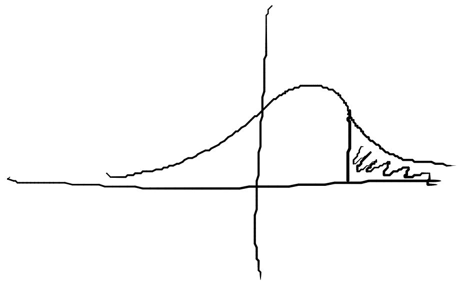
elliptisches Integral

$$l = \int_{t_0}^{t_1} \| -a \sin(t), b \cdot \cos(t) \| \, dt = \int_{t_0}^{t_1} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt$$

$$x(t), y(t) = a \cdot \cos(t), b \cdot \sin(t)$$

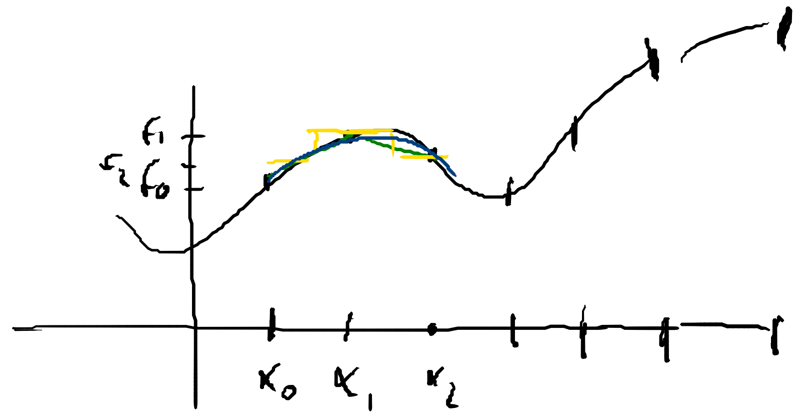
Fehlerfunkt.

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$



Integration $\Leftarrow$ Interpolation

$$\int_a^b f(x)\,dx$$



1. Intervallteilung
2. Polynominterpolation in $[a,b]$ mit Lagrange

$$f(x) \approx \sum_i w_i f_i = \sum_i f(x_i)\cdot l_i(x_i)$$

$\nwarrow$ Lagrange-B.-Fkt.

$$\int_a^b f(x)\,dx \approx \int_a^b \sum_i f(x_i)\, l_i(x)\,dx =$$

$$\sum_i f(x_i) \int_a^b l_i(x)\,dx =$$

$$(b-a)\sum_i f(x_i)\,\frac{1}{(b-a)}\int_a^b l_i(x)\,dx =$$

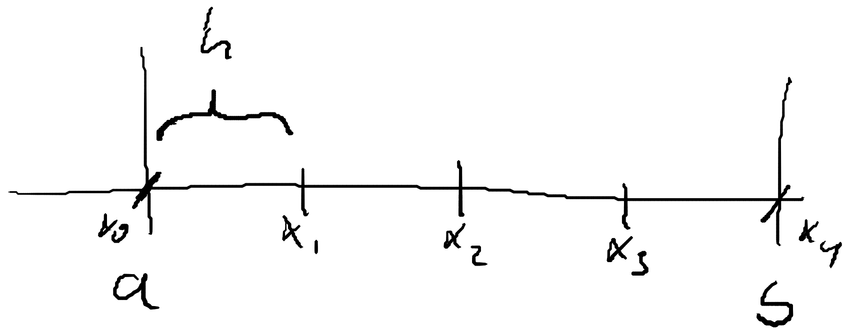
$$(b-a)\sum_i w_i f(x_i)$$

$$w_i = \frac{1}{b-a}\int_a^b l_i(x)\,dx$$

$\Leftarrow$ Einfach!

l_i ist ein Polynom

Newton – Cotes

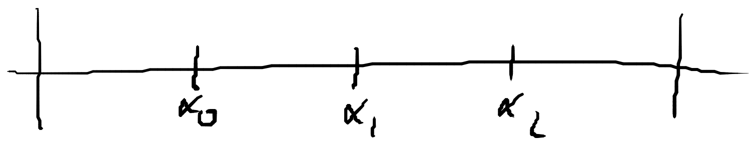


abgeschlossene Newton – Cotes

1. $x_0 = a$, $h = \frac{b-a}{n}$

$\to n+1$ Stützstellen $x_i = a + ih$

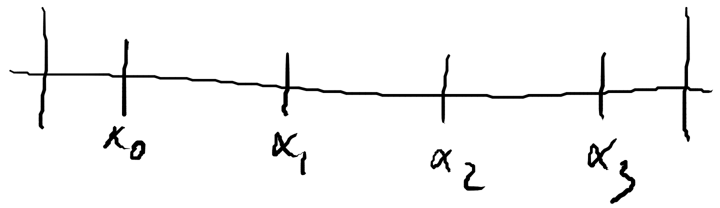
2. $h = \frac{b-a}{n+2}$, $x_i = a + (i+1)h$



offenen Newton-Cotes

3. $h = \frac{b-a}{n+1}$, $x_0 = a + \frac{h}{2}$

$x_i = a + \frac{h}{2} + ih$



Mac Laurin

n	abg. NC	off. NC	MC
0	–	1	1
1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
2	$\frac{1}{6}, \frac{4}{6}, \frac{1}{6}$	$\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
3	$\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$	$\frac{11}{24}, \frac{1}{24}, \frac{1}{24}, \frac{11}{24}$	

Mittelpunktsregel

Gauß - Integration

Mit $n+1$ Stützstellen kann man Polynome vom Grad $2n+1$ exakt integrieren

$f(x) = p_{n+1}(x)\,q(x) + r(x)$ Polynome von Grad n

bestimmte Wahl für $[a,b]$ Polynom vom Grad $n+1$

$$\int_a^b f(x)\,dx = \int p_{n+1}(x)\,q(x) + r(x)\,dx = \underbrace{\int p_{n+1}(x)\,q(x)\,dx}_{0} + \int r(x)\,dx$$

$$= \sum_i w_i\, f(x_i) = \underbrace{\sum w_i\, p_{n+1}(x_i)\,q(x_i)}_{0} + \sum w_i\, r(x_i)$$

Idee

1. wähle p_{n+1} so, dass $\int p_{n+1}(x)\,q(x) = 0$ ✓ -> wähle p_{n+1} als Basiselement einer orthogonalen Poly-B.
2. wähle x_i so, dass $\sum w_i\, p_{n+1}(x_i)\,q(x_i) = 0$

-> $p_{n+1}(x_i) = 0$

x_i sind Nullstellen von $p_{n+1}(x)$

Achtung NST von $p_{n+1}(x)$ müssen alle reel sein und in $[a,b]$ liegen

$$\int_a^b f(x)\,dx = \int_a^b \cancel{p_{n+1}(x)\,q(x)} + r(x)\,dx = \int_a^b r(x)\,dx$$

$$r(x) = \sum r(x_i)\, l_i(x)$$

$$\int_a^b \sum r(x_i)\, l_i(x) = \sum r(x_i) \underbrace{\int_a^b l_i(x)\,dx}_{w_i}$$

$$\sum r(x_i)\, w_i = \sum \left(r(x_i) + \underbrace{p_{n+1}(x_i)\,q(x_i)}_{0}\right) w_i = \sum f(x_i)\, w_i$$

Gewichte Gauß Integration

$$f(x) \approx w(x)\left(p_{n+1}(x)\, q(x) + r(x)\right) \qquad w(x) \geq 0 \text{ in } [a,b]$$

$$\int_a^b f(x)\,dx = \int w(x)\, p_{n+1}(x)\, q(x) + \int w(x)\, r(x) =$$

$$\sum w_i f(x_i) = \sum w_i \cancel{w(x_i)\, p_{n+1}(x_i)}\, q(x_i) + \sum w_i\, w(x_i)\, r(x_i)$$

w-orthogonale Polynombasen

$$\langle f, g \rangle_w = \int_a^b w(x)\, f(x)\, g(x)\, dx$$

Beispiel: $w = \frac{1}{\sqrt{1-x^2}}$ $\quad [a,b] = [-1, 1]$

→ Tschebyschev-Polynome

$f, g \in \mathbb{R}^n$

$f^T g$

$g^T D g$

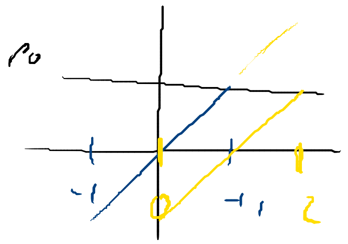
$D = \mathrm{diag}(d_0, d_1, \ldots)$

$d_i \geq 0$

Orthogonale Polynombasen

$$\langle f, g \rangle = \int_a^b f(x) \cdot g(x) \, dx$$

$$\langle f, g \rangle = 0 \quad \text{orthogonal}$$



$p_0 = 1$

$$p_1 = x - \frac{\langle p_0, x \rangle}{\langle p_0, p_0 \rangle} p_0$$

$$p_2 = x^2 - \frac{\langle p_0, x^2 \rangle}{\langle p_0, p_0 \rangle} p_0 - \frac{\langle p_1, x^2 \rangle}{\langle p_1, p_1 \rangle} p_1 \qquad \text{Gram-Schmidt}$$

$\vdots$

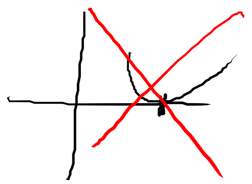
$$p_i = x^i - \sum_{j=0}^{i-1} \frac{\langle p_j, x^i \rangle}{\langle p_j, p_j \rangle} p_j$$

-> ist orthogonal zu $p_{i-1}, p_{i-2}, p_{i-3}, \ldots, p_1, p_0$

-> ist orthogonal zu jedem Polynom bis zum Grad $(i-1)$

Nullstellen?

m Stellen $v_j, 0 \le j < m$ in $[a,b]$ Vorzeichenwechsel von p_i



$$p^*(x) = (x - v_0)(x - v_1) \cdots (x - v_{m-1})$$

Polynom Grad m

$$\langle p^*, p_i \rangle = \int_a^b p^*(x) \, p_i(x) \, dx$$

In jedem Intervall $[v_j, v_{j+1}]$ haben p^* und p_i keinen Vorzeichenwechsel