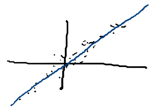
Singulärwertzerlegung



$Y = (y_0, y_1, y_2, \dots)$ ← mittelwertbefr. Daten

$YY^T = Q \Lambda Q^T$ $\qquad Q = (q_0, q_1, q_2, \dots)$ ← ges. Basis

$X = Y^T$

gesucht: Basis $V = (v_0, v_1, \dots)$ $\quad V$ ist orthogonal, $\|v_i\| = 1$, $v_i^T v_j = 0 \; i \neq j$

$v_i^T YY^T v_i = v_i^T X^T X v_i = (Xv_i)^T X v_i = \|Xv_i\|^2$

$\underset{\|v_i\|=1}{\arg\max} \|Xv_i\|$ $\qquad \boxed{\sigma_i u_i = X v_i}$ $\qquad \|u_i\| = 1 \quad \sigma_i \geq 0$

$\sigma_i u_i^T \sigma_j u_j = (Xv_i)^T (Xv_j) = v_i^T X^T X v_j$

$= v_i^T YY^T v_j = \underbrace{v_i^T Q}_{} \Lambda Q^T v_j$

$= d_i^T \Lambda d_j \quad \Rightarrow d_{i,j} = e_{i,j}$

$\Rightarrow e_i^T \Lambda e_j = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}$

$\|\sigma_i u_i\|^2 = \sigma_i u_i^T u_i \sigma_i = \lambda_i$

$\Rightarrow \sigma_i^2 = \lambda_i \quad \Rightarrow \sigma_i = \sqrt{\lambda_i}$

$\underset{\|u_i\|=1}{\arg\max} \|u^T X\|$

rang(X) = r < n

$U \, [\Sigma \; 0] = X V$

$V \in \mathbb{R}^{n \times n}$

$[\Sigma \; 0] = \begin{pmatrix} \sigma_0 & & & \\ & \sigma_1 & & \\ & & \sigma_2 & \\ & & & \ddots \end{pmatrix} \; 0$

$\sigma_0 \geq \sigma_1 \geq \dots \geq \sigma_{r-1} = 0 = \sigma_r = \dots$

$U \, [\Sigma \; 0] \, V^T = X$

$U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T = X$

linke Singulärvekt.

rechte Singulärvekt.

$U \Sigma V^T = X$

$U \Sigma V^T = X$

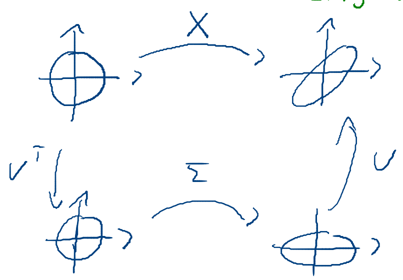
reduzierte SVD

Singulärwertzerlegung

U, V orthogonal

Σ diagonal $\quad \sigma_i \geq 0$

Singulärwerte



Anwendungen der SVD $\qquad X = U \Sigma V^T$

$$\Sigma = \begin{pmatrix} \sigma_0 & & & & \\ & \sigma_1 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \qquad \sigma_0 \geq \sigma_1 \geq \dots \sigma_{r-1} > \sigma_r = \sigma_{r+1} = \dots = 0$$

$$\Sigma^+ = \begin{pmatrix} 1/\sigma_0 & & & & & \\ & 1/\sigma_1 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_{r-1} & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix}$$

$X^+ = V \Sigma^+ U^T$ Pseudoinverse

$$XX^+ = U \Sigma \underbrace{V^T V}_{I} \Sigma^+ U^T = U \Sigma \Sigma^+ U^T = U \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix} U^T$$

$\Rightarrow X \in \mathbb{R}^{n \times n}$, $\operatorname{rang}(X) = n \rightarrow X^+ = X^{-1}$

LGS

$Ax \approx b \qquad x = A^+ b$ ist immer ok!

1. $Ax = b$ hat genau eine Lösung $\rightarrow$ $x = A^+ b$ ist die ges. Lsg.
2. $Ax = b$ hat keine Lösung $\rightarrow$ minimiere $\|Ax - b\|$
3. $Ax = b$ hat viele Lösungen $\rightarrow$ $\underset{Ax=b}{\operatorname{argmin}} \|x\|$

$$\begin{aligned} \|Ax - b\| &= \| U \Sigma V^T x - b \| \\ &= \| U^T (U \Sigma V^T x - b) \| = \| \Sigma V^T x - U^T b \| \\ &= \| \Sigma x' - b' \| \end{aligned} \qquad \begin{aligned} x' &= V^T x \\ b' &= U^T b \end{aligned}$$

$$\|\Sigma x' - b'\|^2 = \underbrace{\sum_{i=0}^{r-1} (\sigma_i x_i' - b_i')^2}_{x_i' = \frac{b_i'}{\sigma_i}} + \underbrace{\sum_{i=r}^{n-1} (-b_i')^2}_{x_i' \text{ egal} \rightarrow x_i' = 0}$$

$\sigma_i x_i' = b_i'$
$x_i' = b_i' / \sigma_i$

$$V^T x = x' = \Sigma^+ b' = \Sigma^+ U^T b$$

$$\rightarrow x = V \Sigma^+ U^T b = A^+ b$$

Homogene LGS

$$Ax = 0 \quad \text{für} \quad x \neq 0$$

$Ax = 0 \to A(\lambda x) = 0$

$\underset{\|x\|=1}{\operatorname{argmin}} \|Ax\|$

$x' = V^T x$

$$\underset{\|x\|=1}{\operatorname{argmin}} \|U \Sigma V^T x\| = \underset{\|x\|=1}{\operatorname{argmin}} \|\Sigma V^T x\| = \underset{\|x\|=1}{\operatorname{argmin}} \|\Sigma x'\|$$

$$= \left((\sigma_0 x_0')^2 + (\sigma_1 x_1')^2 + \dots \right)^{1/2}$$

$\sigma_0 \geq \sigma_1 \geq \dots \geq \sigma_{n-1} \Rightarrow x_0' = \dots = x_{n-2}' = 0, \; x_{n-1}' = 1$

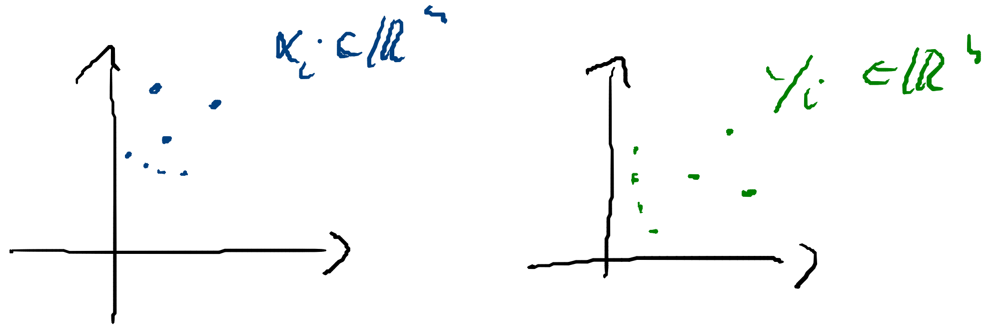
$$x' = e_{n-1}$$

$$x = Vx' = Ve_{n-1}$$

$\Rightarrow$ die letzte Spalte von V

$\to$ der rechte Singulärvekt. zum kleinsten Singulärwert.

Orthogonale Procustes Problem



$X = (x_0, x_1, \dots)$

$Y = (y_0, y_1, \dots)$

$T x_i \overset{!}{\approx} y_i \qquad \sum_i \|T x_i - y_i\|^2 \to \min_T$

$$XY^T = U \Sigma V^T$$

$\det(XY^T) \geq 0$ ges. Rotation ist UV^T

ansonsten $U \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} V^T$