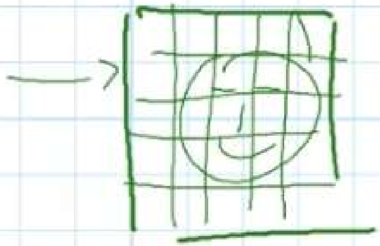
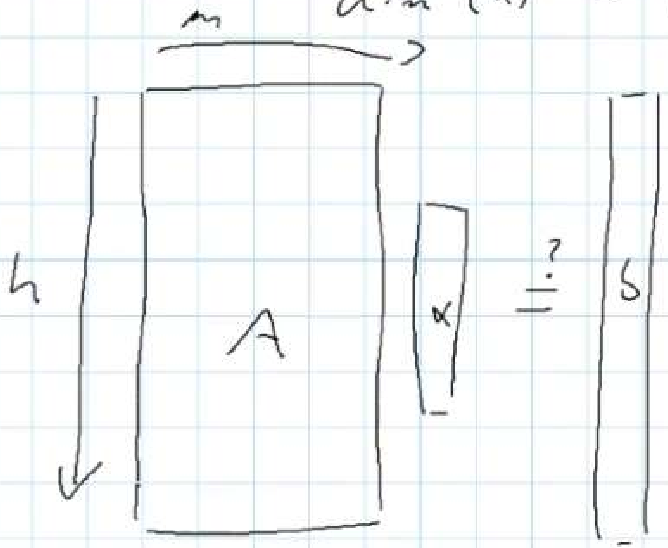
Lineare Ausgleichsrechnung



$$Ax = b$$

Zeile $\hat{=}$ Strahl

$\dim(x)$ $\hat{=}$ Anzahl der Kästchen



$A \in \mathbb{R}^{n \times m}$

$x \in \mathbb{R}^{m}$

$b \in \mathbb{R}^{n}$

$$r = Ax - b +$$

$$\|r\|_2 = \sqrt{r_0^2 + r_1^2 + \dots + r_{n-1}^2}$$

$$\arg\min_{x \in \mathbb{R}^m} \|r\|_2$$

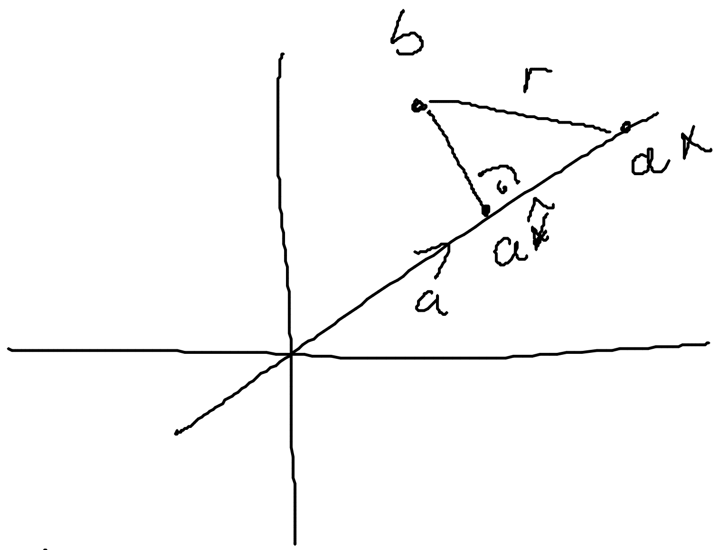
$$\iff$$

$$\arg\min_{x \in \mathbb{R}} \|r\|_2^2$$

$$Ax \overset{?}{=} b \qquad A \in \mathbb{R}^{n \times m},\ b \in \mathbb{R}^n,\ x \in \mathbb{R}^m$$

$$r = Ax - b$$

$$\underset{x \in \mathbb{R}^m}{\arg\min} \ \|Ax - b\|_2^2 = \|r\|_2^2$$



Gerade $\quad ax, \quad a \in \mathbb{R}^2, x \in \mathbb{R}, b \in \mathbb{R}^2$

$$\|r\|_2^2 = \|a\hat{x} - b\|^2 + \|a\hat{x} - ax\|^2$$
$$= \|a\hat{x} - b\|^2 + \|a(\hat{x} - x)\|^2 \quad \Rightarrow \quad x = \hat{x}$$

$$0 = a^T r = a^T(ax - b) = a^T a\, x - a^T b \quad \Leftrightarrow$$

$$a^T a\, x = a^T b \quad \rightarrow \quad x = \frac{a^T b}{a^T a}$$

Gilt für alle a!

$$A^T A x = A^T b \qquad \text{Normalengleichung}$$

$$x = (A^T A)^{-1} A^T b$$

$$A^T (Ax - b) = 0$$

$$\begin{pmatrix} a_0^T \\ a_1^T \\ \vdots \end{pmatrix} (Ax - b) = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$Ax = \underbrace{A (A^T A)^{-1} A^T}_{\text{Projektor}} b$$

$$P = A (A^T A)^{-1} A^T$$

$$P^2 = P$$

Eigenschaften von A^TA

Symmetrische $\quad (A^TA)^T = A^T A^{T^T} = A^TA$

Positiv semidefinit (PSD) $\quad x^T A^T A x = (Ax)^T Ax = y^T y = \|y\|^2 \geq 0$

A hat vollen Spaltenrang $\rightarrow$ A^TA regulär

$\hat{=}$ Spalten linear unabhängig $\qquad \hat{=}$ $\det(A^TA) \neq 0$

$A^TA x \neq 0 \iff x \neq 0$

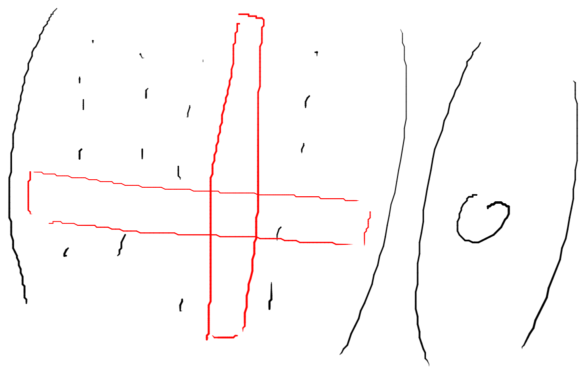
$\Rightarrow x^T A^T A x > 0 \qquad$ Positiv definit (PD)

Eigenschaften von PD-Matrizen $\qquad B = A^T A$

$$(0,\dots,0,1,0\dots0)\, B \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} > 0 \quad \Rightarrow \quad s_{ii} > 0$$

$$(x_0 \cdots x_i, 0, x_{i+2} \cdots)\, B \begin{pmatrix} x_0 \\ \vdots \\ x_i \\ 0 \\ x_{i+2} \\ \vdots \end{pmatrix} > 0 \quad \Rightarrow$$

Löschen von Spalte/Zeile i erhält PD



B ist PD dann auch $s \circ B$ für $s > 0$

A, B sind PD $\qquad$ " $\qquad (A+B) \qquad x^T(A+B)x = \underbrace{x^T A x}_{>0} + \underbrace{x^T B x}_{>0}$

$\Rightarrow \quad sA + tB$ PD für $s, t > 0$

$\det(B) > 0$

$$\det(\underbrace{I(1-t) + B \cdot t}_{PD}) \neq 0 \quad \text{für } t \in [0,1]$$

keine Nullstellen in $[0,1]$

Cholesky

$$B = A^T A \qquad B = L L^T$$

$$\begin{pmatrix} b_{00} & b_{10} & b_{20} & \cdots \\ b_{10} & b_{11} & b_{21} & \\ b_{20} & b_{21} & b_{22} & \\ \vdots & & & \ddots \end{pmatrix} = \begin{pmatrix} l_{00} & 0 & 0 & \cdots \\ l_{10} & l_{11} & 0 & \\ l_{20} & l_{21} & l_{22} & \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} l_{00} & l_{10} & l_{20} & \cdots \\ 0 & l_{11} & l_{21} & \cdots \\ 0 & 0 & l_{22} & \cdots \\ \vdots & & & \end{pmatrix} = \begin{matrix} l_{00}^2 & l_{00} \cdot l_{10} & l_{00} l_{20} & l_{00} \cdot l_{30} \\ (l_{10} \cdot l_{00}) & l_{10}^2 + l_{11}^2 & l_{10} l_{20} + l_{11} l_{21} & \cdots \end{matrix}$$

$$b_{00} = l_{00}^2 \quad \to \quad l_{00} = \sqrt{b_{00}}$$

$$b_{10} = l_{00} \cdot l_{10} \quad \to \quad l_{10} = \frac{b_{10}}{l_{00}}$$

$$b_{11} = l_{10}^2 + l_{11}^2 \quad \to \quad l_{11} = \sqrt{b_{11} - l_{10}^2} \qquad 0 < b_{11} - l_{10}^2 = b_{11} - \frac{b_{10}^2}{l_{00}^2} = b_{11} - \frac{b_{10}^2}{b_{00}}$$

$$b_{i1} = l_{10} l_{i0} + l_{11} l_{i1} \quad \to \qquad \iff \quad b_{00} b_{11} - b_{10}^2 > 0$$

$$l_{i1} = \frac{b_{i1} - l_{10} l_{i0}}{l_{11}}$$

$$l_{ii} = \sqrt{b_{ii} - \sum_{k=0}^{i-1} l_{ik}^2}$$

$$l_{ij} = \frac{1}{l_{ii}} \left(b_{ij} - \sum_{k=0}^{i-1} l_{ik} l_{jk} \right)$$

$$Ax \neq b \quad \to \quad A^T A x = A^T b \quad \xrightarrow{\text{Cholesky}} \quad L L^T x = c \quad (= A^T b)$$

$$y = L^T x \quad \to \quad L y = c \quad \text{Vorwärtseins.} \to y$$

$$y \to \quad L^T x = y \quad \text{Rückwärtseins.} \to x$$

$$(a_0 x_0 - 5)^2 \qquad 5$$

$$+ (a_1 x_1 - 5)$$

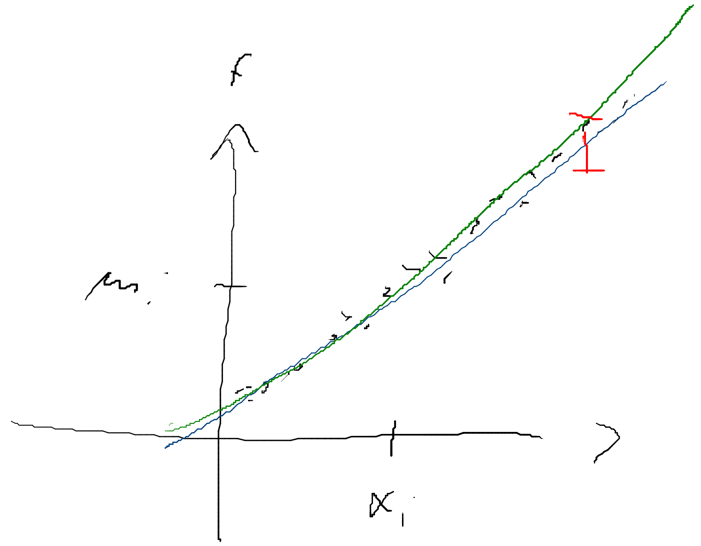
$$(a_0 x_0 + c_0 - 5)^2 + (a_1 x_1 + c_1 - 5)^2$$

$$B = U^T D U \qquad U^T U = I$$

$$B^{-1} = U^T D^{-1} U$$

$$B B^{-1} = \underbrace{U^T \underbrace{D \underbrace{U U^T}_{} D^{-1}}_{I} U}_{I}$$

Lineare Regression



Gesucht: $f(x_i) = m_i$

$$\underset{f \in \Theta}{\arg\min} \sum_i \left(f(x_i) - m_i \right)^2$$

$$f(x) = c_0 + c_1 \cdot x + c_2 x^2$$

$$\left.\begin{aligned} m_0 &= f(x_0) = c_0 + c_1 x_0 + c_2 x_0^2 \\ m_1 &= f(x_1) = c_0 + c_1 x_1 + c_2 x_1^2 \\ &\vdots \end{aligned}\right\} \begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix}$$

$$X^T X c = X^T m$$

$$f(x) = \sum_i c_i B_i(x)$$

$$B_i(x) = 1,\ x,\ x^2,\ \sin(x),\ e^{-x}$$