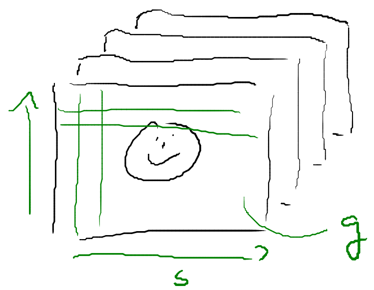
Orthogonale regression



$g \in \mathbb{R}^n$ $\quad n = s \cdot h$

Daten $g_i \in \mathbb{R}^n$ $\quad m \ll n$

$i \in [0, \dots, m-1]$

„Gesicht g zuordnen"

$$\operatorname*{argmin}_i \; \| g - g_i \| \qquad O(nm)$$

$f: \mathbb{R}^o \to \mathbb{R}^n$ „Deskriptoren" Dimension $o \le m \ll n$

$$f(x) = \left.\begin{matrix} c_{00} x_0 + c_{01} x_1 + \dots + s_0 \\ c_{10} x_0 + c_{11} x_1 + \dots + s_1 \\ \vdots \\ c_{n-1,0} x_0 + c_{n-1,1} x_1 + \dots + s_{n-1} \end{matrix}\right\} \underbrace{\boxed{C} x + s}_{\text{Linearer Unterraum}}$$

$x \in \mathbb{R}^o$

$$\| f(x) - f(x_i) \| = \| Cx + s - (Cx_i + s) \| = \| C(x - x_i) \| = \| x - x_i \|$$

Annahme: $C^T C = I$

$$C^T C x + C^T s = C^T g \qquad x = \underbrace{(C^T C)^{-1}}_{I} C^T (g - s)$$

$$\| C(x - x_i) \|^2 = (x - x_i)^T \underbrace{C^T C}_{I} (x - x_i) = (x - x_i)^T (x - x_i) = \| x - x_i \|^2$$

Residuum

$$r_i = f(x_i) - g_i = C x_i + s - g_i = C C^T (g_i - s) + s - g_i$$

$$= (CC^T - I)(g_i - s)$$

$$\operatorname*{argmin}_{s,\, C^T C = I} \sum_i r_i^T r_i = \sum_i (g_i - s)^T (CC^T - I)(CC^T - I)(g_i - s)$$

$$= \quad " \quad (C\underbrace{C^T C}_{I} C^T - CC^T - CC^T + I) \quad "$$

$$\boxed{= \sum_i (g_i - s)^T \boxed{(I - CC^T)} (g_i - s)}$$

?

Eigenzerlegung von Matrizen A^TA, AA^T

$M^T = M \qquad M = Q \Lambda Q^T, \quad Q^TQ = I = QQ^T$

$$Q^TMQ = \Lambda = \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix} \qquad |\lambda_0| \geq |\lambda_1| \geq \dots$$

$$0 \leq x^TA^TAx = x^TQ\Lambda \underbrace{Q^Tx}_{y} = y^T \Lambda y \Rightarrow \lambda_i \geq 0$$

$$A^TAx = \lambda x \qquad AA^T\underbrace{Ax}_{y} = \lambda \underbrace{Ax}_{y} \Rightarrow AA^Ty = \lambda y$$

Eigenwert λ von A^TA
ist Eigenwert von AA^T „Kernel-Trick"

$\boxed{I - CC^T} \Rightarrow$ PSD!

$$C^TC = I \Rightarrow \lambda_i = 1 \Rightarrow CC^T \quad \lambda_i = 1, 0$$

$$I - CC^T = I - Q\Lambda Q^T = QIQ^T - Q\Lambda Q^T = Q(I - \Lambda)Q^T$$

Mittelwertbefreiung

$$\underset{s}{\arg\min} \sum_i (s_i - s)^T \underbrace{(I - cc^T)}_{M} (s_i - s)$$

Annahme: $\sum_i s_i = 0$ „mittelwertbefreit"

$$\sum_i \left(s_i^T M s_i - s_i^T M s - s^T M s_i + s^T M s \right)$$

$$\sum_i \left(\underbrace{s_i^T M s_i}_{\text{konstant}} + s^T M s \right) - \cancel{\left(\sum s_i\right)} M s - s^T M \cancel{\left(\sum s_i\right)}$$

$$\hookrightarrow \underset{s}{\arg\min}\ s^T M s \Rightarrow s = 0$$

$$\sum (s_i - s) = 0 \Rightarrow s = \frac{1}{m} \sum_i s_i$$

Bestimmung von C

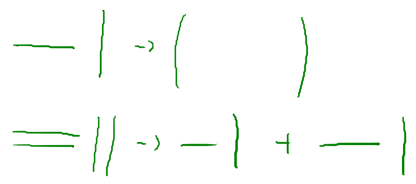
$y_i = g_i - S$, S ist Mittelwert von g_i

$$\underset{C^TC = I}{\arg\min} \sum_i y_i^T (I - CC^T) y_i$$

$$\underset{C^TC = I}{\arg\min} \sum_i \underbrace{y_i^T y_i}_{\text{konstant}} - \sum_i y_i^T CC^T y_i$$

$$\underset{C^TC = I}{\arg\max} \sum_i y_i^T CC^T y_i$$

Einfacher: suche $c \in \mathbb{R}^n$



$$\underset{c^Tc = 1}{\arg\max} \sum_i y_i^T c c^T y_i$$

$$\sum_i c^T y_i y_i^T c$$

$$c^T \left(\sum_i y_i y_i^T\right) c$$

$$c^T Y Y^T c$$

$$c^T Q \Lambda \underbrace{Q^T c}_{d}$$

$d = Q^T c$

$c = Qd$

$$\underset{d^Td = 1}{\arg\max} \; d^T \Lambda d$$

$c^Tc = d^T Q^T Q d$
$= d^T d$

$$d^T \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix} d \qquad \lambda_0 \geq \lambda_1 \geq \dots$$

$d_0^2 \lambda_0 + d_1^2 \lambda_1 + \dots$

$$\Rightarrow d = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow c = Qd = q_0$ (die erste Spalte von Q)

$c_0 = c$ $\quad c_1$? $\quad c_1^T c_1 = 1 \quad c_1^T c_0 = 0$

$$\underset{\substack{d_1^T d_1 = 1 \\ d_1^T d_0 = 0}}{\arg\max} \; d^T \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix} d$$

$$\Rightarrow d_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$\Rightarrow c_1 = q_1$ (die zweite Spalte von Q)

Was gilt für $C \in \mathbb{R}^{n \times 2}$?

$\vdots$

$$\underset{D^TD = I}{\arg\max} \; D^T \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix} D$$

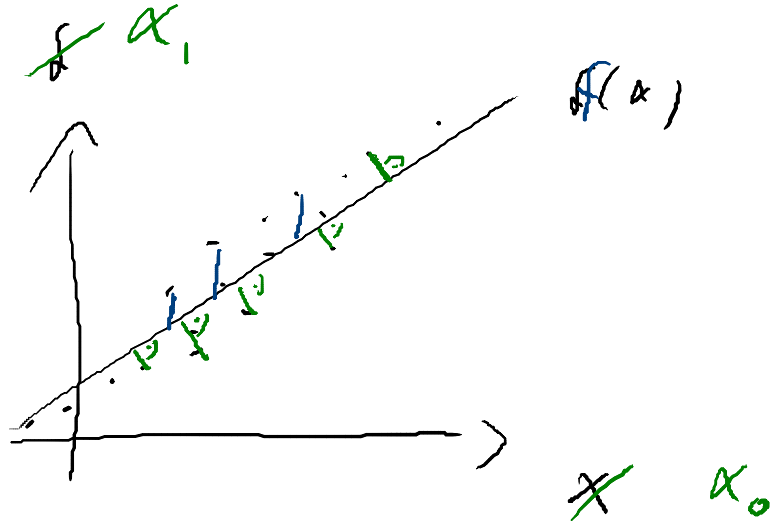
$$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Zusammenfassung:

Daten $g_i \in \mathbb{R}^n$, n groß, $i \in (0, \dots, m-1)$

1. Bestimme Mittelpunkt $s = \frac{1}{m} \sum_i g_i$
2. $y_i = g_i - s$ (mittelwertbefreit $\sum y_i = 0$)
3. $Y = (y_0, y_1, \dots) \to Y Y^T$
4. $Y Y^T = \boxed{Q \Lambda Q^T} \Rightarrow C = (q_0, q_1, q_2, \dots)$

Orthogonale Regression



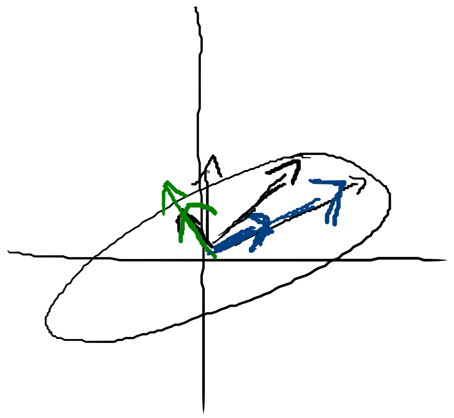
Power-Methode

("von Mises-Verfahren")

$$y^{(k+1)} = \frac{A y^{(k)}}{\| A y^{(k)} \|} = \frac{x^{(k)}}{\| x^{(k)} \|} \qquad x^{(k)} = A y^{(k)}$$

$$A = V \begin{pmatrix} \lambda_0 & & \\ & \lambda_1 & \\ & & \ddots \end{pmatrix} (v_0, v_1, \dots)$$

$y^{(0)} = \sum_i \alpha_i v_i$ Annahme $\alpha_0 \neq 0$



$$x^{(k)} = A^k y^{(0)} = \sum_i \alpha_i A^k v_i = \sum_{i=0} \alpha_i \lambda_i^k v_i$$

$$= \lambda_0^k \left(\alpha_0 v_0 + \sum_{i=1} \alpha_i \underbrace{\boxed{\left(\frac{\lambda_i}{\lambda_0} \right)^k}}_{<1} v_i \right)$$

$|\lambda_0| > |\lambda_i|$

$$\lim_{k \to \infty} x^{(k)} = \lambda_0^k \left(\alpha_0 v_0 + 0 \right) = \lambda_0^k \alpha_0 v_0$$