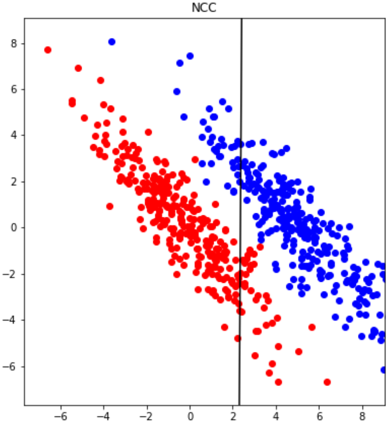
Cognitive Algorithms: Tutorial 2

Linear Discriminant Analysis

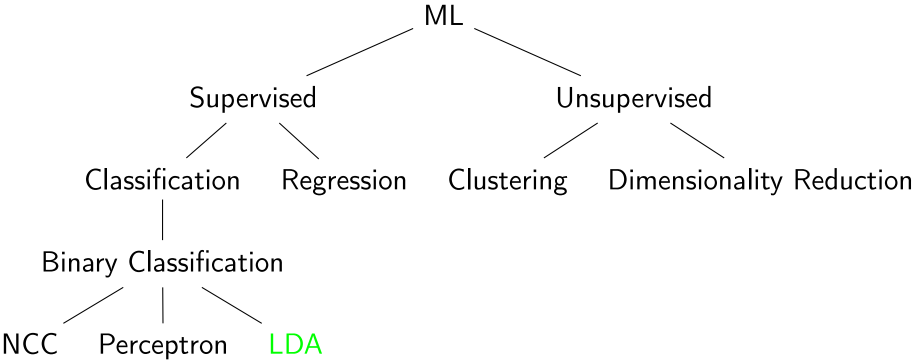
Joanina, Ken, Augustin

Problems with NCC



Content

The Tree of CA



Random variables

- Formalization of objects or quantities that depend on randomness

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$$X : \Omega \to \mathbb{R}$$

Common Statistics

- Expected Value

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Covariance Matrix

- A set of d random variables has a covariance matrix:

$$\begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \dots & \mathrm{Cov}(X_1, X_d) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_d, X_1) & \dots & \mathrm{Cov}(X_d, X_d) \end{bmatrix}$$

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$$=$$

$$\begin{bmatrix} \mathrm{Var}(X_1) & \dots & \mathrm{Cov}(X_1, X_d) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(X_d, X_1) & \dots & \mathrm{Var}(X_d) \end{bmatrix}$$

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$$\overset{\text{Task 1.2}}{=}$$

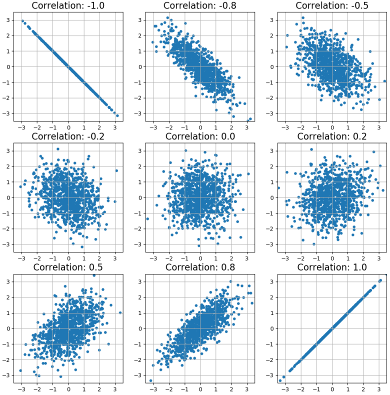
$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Common Statistics

- Expected Value $\hat{=}$ mean
  $\hat{=}$ "What value can we on average expect?"
- Variance
- Covariance

$\Longrightarrow$ How does this relate to the real world?

Correlation visualized (for Gaussians)



Positive semi-definite matrices

- A Matrix $\mathsf{A} \in \mathbb{R}^{d\times d}$ is positive semi-definite iff

$$\mathsf{x}^\mathsf{T}\mathsf{A}\mathsf{x} \geq 0 \quad \text{for all } \mathsf{x} \in \mathbb{R}^d$$

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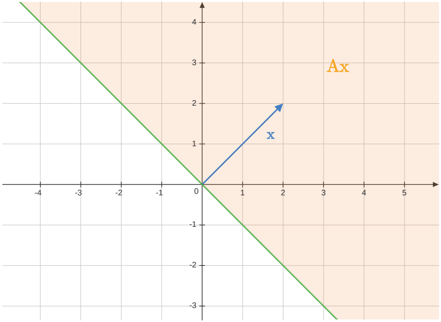
$$x^T A x \geq 0 \quad \text{for all } x \in \mathbb{R}^d$$

- What does this mean?

$$x^T A x = \langle x, Ax \rangle = \underbrace{\|x\|}_{\geq 0} \underbrace{\|Ax\|}_{\geq 0} \cos(\alpha)$$

Positive semi-definite matrices II

- Ax is in the half space of x or orthogonal to x!



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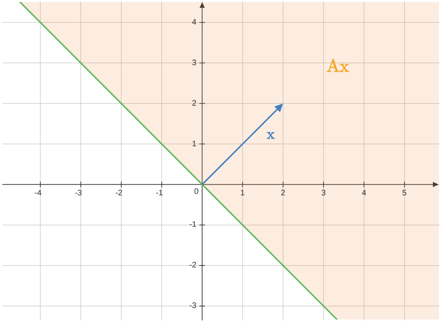

- A **symmetric** matrix is PSD iff all eigenvalues are non-negative

Task 3 - PSD: Pretty Sweet, Dude!

We care in particular about PSD matrices because all covariance matrices are PSD. One interesting property about PSD matrices is that all of their eigenvalues are non-negative. In particular it holds that all real, symmetric matrices A with non-negative eigenvalues are PSD, and vice versa. Prove this statement in both directions!

Task: Possible or not

- Given three 1-dimensional random variables X, Y and Z with the following properties:

$$\mathrm{Var}(X) = \mathrm{Var}(Y) = \mathrm{Var}(Z) = 10, \ \mathrm{Cov}(X, Y) = 8,$$

$$\mathrm{Cov}(X, Z) = 1, \ \mathrm{Cov}(Y, Z) = 8$$

- Is this possible?

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$$C = \begin{bmatrix} 10 & 8 & 1 \\ 8 & 10 & 8 \\ 1 & 8 & 10 \end{bmatrix}$$

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- Is this possible?

$$\mathrm{C} = \begin{bmatrix} 10 & 8 & 1 \\ 8 & 10 & 8 \\ 1 & 8 & 10 \end{bmatrix}$$

- Show that C is not positive semi-definite!

Method 1: Quadratic form

- Find a vector $x \in \mathbb{R}^3$ such that $x^T C x < 0$

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- For example:

$$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 8 & 1 \\ 8 & 10 & 8 \\ 1 & 8 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -4$$

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- Only works well for disproving

Method 2: Eigenvalues

- Eigenvalues of C:

$$\lambda(C) \approx \{-0.82,\ 9,\ 21.82\}$$

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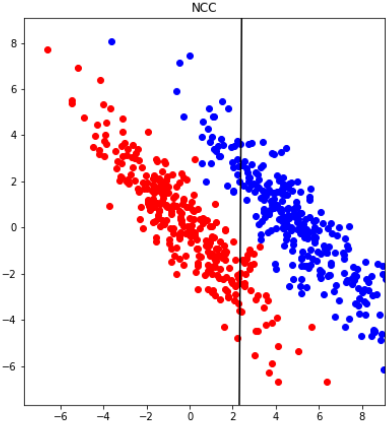

The Fisher criterion

$$\mathsf{w}^* = \arg\max_{\mathsf{w}} \frac{\mathsf{w}^\mathsf{T} \mathsf{S}_\mathrm{b} \mathsf{w}}{\mathsf{w}^\mathsf{T} \mathsf{S}_\mathrm{w} \mathsf{w}}$$

Orthogonal Matrices

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- $U^T = U^{-1}$

Eigendecomposition

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$\implies$ Always possible if matrix is real and symmetric!