Tutorial Session 1: Linear Classification

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Organizational Remarks

- **Lecture**:
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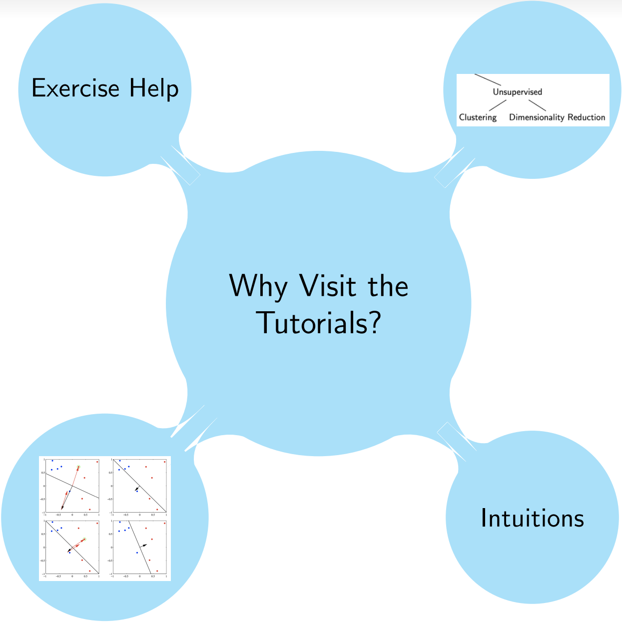
  Augustin (augustin.krause@campus.tu-berlin.de)
- We see each other every two weeks!

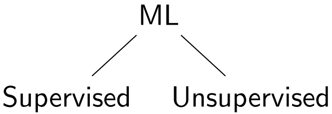

Exercise Help
Unsupervised
Clustering
Dimensionality Reduction
Why Visit the Tutorials?
Intuitions

What is an 'ML-Model'?

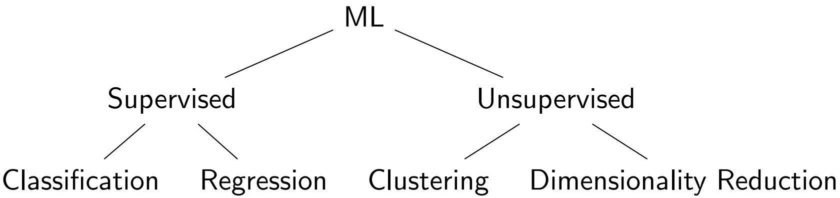
The Tree of CA

ML

The Tree of CA



The Tree of CA



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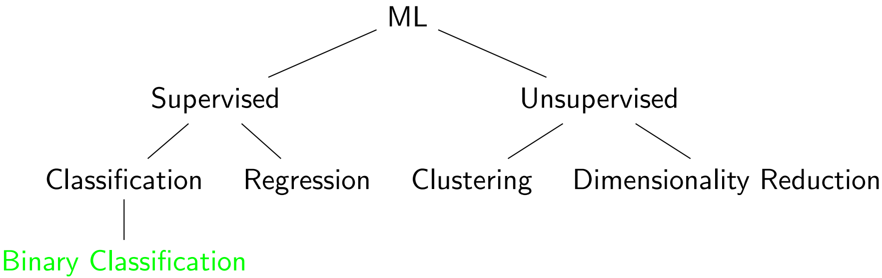
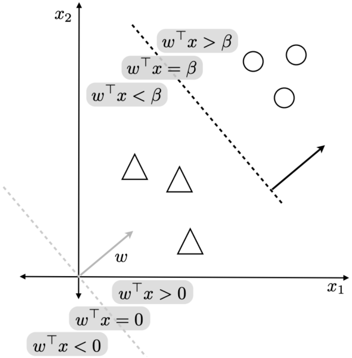


Interpretation: The Decision Boundary



$\mathbf{x}, \mathbf{w} \in \mathbb{R}^D, \beta \in \mathbb{R}$

$$\mathbf{w}^\top \mathbf{x} - \beta = \begin{cases} > 0 & \text{if } \mathbf{x} \text{ belongs to o} \\ < 0 & \text{if } \mathbf{x} \text{ belongs to } \Delta \end{cases}$$

Points on the decision boundary satisfy $y(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} - \beta = 0$

The Tree of CA

- ML
  - Supervised
    - Classification
      - Binary Classification
        - NCC
    - Regression
  - Unsupervised
    - Clustering
    - Dimensionality Reduction

Task 1 - Example Prototype Classifier

Consider the following data points:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\mathbf{x}_1$ and $\mathbf{x}_2$ belong to class -1, while $\mathbf{x}_3$ and $\mathbf{x}_4$ belong to class $+1$.

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1. Compute the class means $\mathbf{w}_{-1}$ and $\mathbf{w}_{+1}$.
2. Compute the classification boundary $\mathbf{w}^\top \mathbf{x} - \beta = 0$ of the prototype classifier.

Task 1 - Example Prototype Classifier

3. For each point, compute the assigned class label $\mathrm{sign}(\mathbf{w}^\top \mathbf{x} - \beta)$. Are all points classified correctly?

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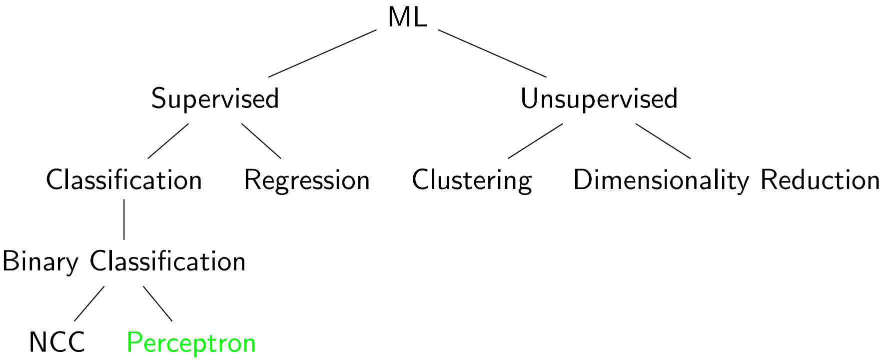
3. For each point, compute the assigned class label $\text{sign}(\mathbf{w}^\top \mathbf{x} - \beta)$. Are all points classified correctly?
4. Sketch the data points, their class means $\mathbf{w}_{-1}$ and $\mathbf{w}_{+1}$, the normal vector $\mathbf{w}$, and the classification boundary.

The Tree of CA



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   Pick a random misclassified data point $\mathbf{x}_m$

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$$
\begin{aligned}
\mathcal{E}_m(\mathbf{w}) &= -\mathbf{w}^\top \mathbf{x}_m y_m \\
\nabla\mathcal{E}_m(\mathbf{w}) &= -\mathbf{x}_m y_m \\
\mathbf{w}^{\text{new}} &\leftarrow \mathbf{w}^{\text{old}} - \eta\nabla\mathcal{E}_m(\mathbf{w}^{\text{old}}) = \mathbf{w}^{\text{old}} + \eta\mathbf{x}_m y_m
\end{aligned}
$$

Task 3 - Convergence of the Perceptron

1. We denote a hyperplane by $\mathbf{w}^\top \mathbf{x} = 0$. Show that there exists a $\mathbf{w}_{\text{sep}}$ such that:

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2. Given a current $\mathbf{w}_{\text{old}} \in \mathbb{R}$, the perceptron algorithm identifies a point $\mathbf{x}_m$ that is misclassified, and produces the update rule $\mathbf{w}_{\text{new}} = \mathbf{w}_{\text{old}} + \eta \mathbf{x}_m y_m$. Using this update rule, show that

$$\|\mathbf{w}_{\text{new}} - \mathbf{w}_{\text{sep}}\|^2 \leq \|\mathbf{w}_{\text{old}} - \mathbf{w}_{\text{sep}}\|^2 - \|\mathbf{x}_m\|^2 \tag{1}$$

This implies that the perceptron algorithm converges to a separating hyperplane in a finite number of steps.

Task 2 - The linear classification boundary

Consider a linear classification boundary $\mathbf{w}^\top \mathbf{x} - \beta = 0$. Draw a sketch in 2D to visualize the classification boundary and answer the following questions:

1. Suppose $\beta = 0$ and $\|\mathbf{w}\| = 1$. How large is the distance of a point $\mathbf{z}$ to the classification boundary?
2. How large is the distance of a point $\mathbf{z}$ to the classification boundary if $\|\mathbf{w}\| = 1$ but $\beta \neq 0$?
3. How large is the distance of a point $\mathbf{z}$ to the classification boundary for arbitrary β and $\mathbf{w}$?
4. How large is the distance between a classification boundary $\mathbf{w}$ and the origin for arbitrary β and $\mathbf{w}$?
5. Is β in the general case the intercept of the classification boundary $\mathbf{w}^\top \mathbf{x} - \beta = 0$ with the x_2-axis? If yes, explain why. If not, give a counter-example.

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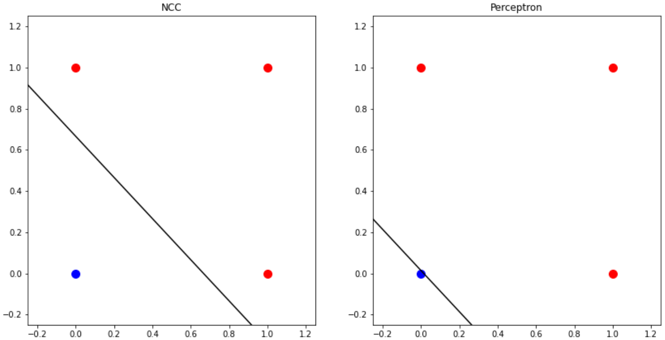
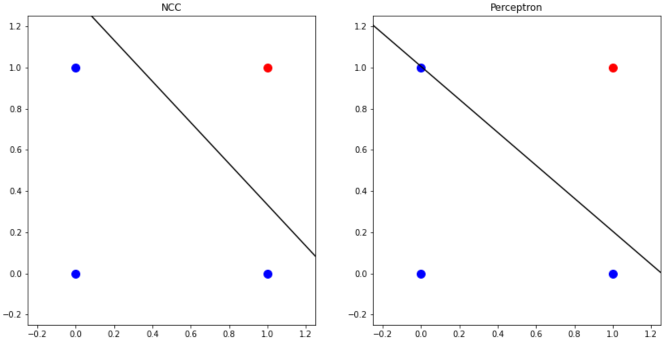
3. arbitrary β and $\mathbf{w}$
4. distance between boundary and the origin for arbitrary β and $\mathbf{w}$
5. Is β intercept of classification boundary with the x_2-axis?

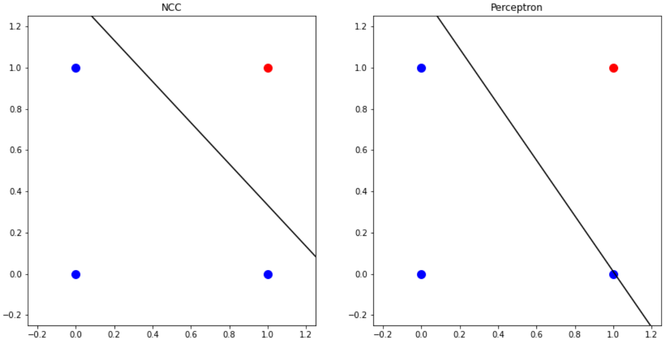
NCC and Perceptron: OR



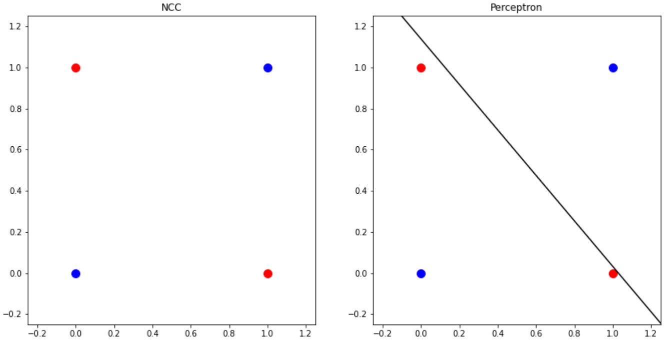
NCC and Perceptron: AND (1)



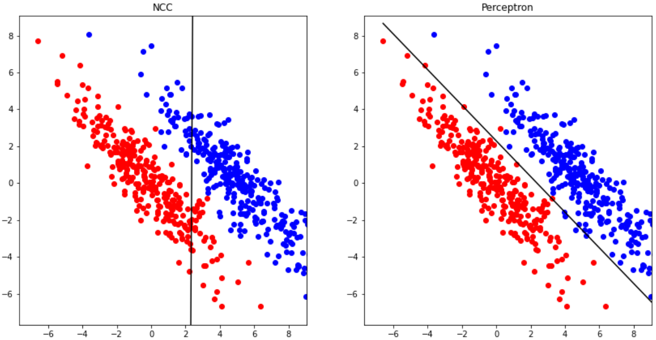
NCC and Perceptron: AND (2)



NCC and Perceptron: XOR



NCC and Perceptron



NCC and Perceptron

	NCC	**Perceptron**
Problem	Classification	Classification(, Regression)
Model	$y = \text{sign}(\mathbf{w}^T\mathbf{x})$	$y = f(\mathbf{w}^T\mathbf{x})$
Error	distance to $\mathbf{w}_{+1}, \mathbf{w}_{-1}$	$-\sum_{m\in M}\mathbf{w}^T\mathbf{x}_m y_m$
Optimization	closed form	SGD
Result	always the same	can differ
Application	Cancer Prediction[1]	NLP[2]

[1] (Tibshirani et al., 2002)
[2] (Collins, 2002)

References

Michael Collins. Discriminative training methods for hidden markov models: Theory and experiments with perceptron algorithms. In *Proceedings of the ACL-02 conference on Empirical methods in natural language processing-Volume 10*, pages 1–8. Association for Computational Linguistics, 2002.

Robert Tibshirani, Trevor Hastie, Balasubramanian Narasimhan, and Gilbert Chu. Diagnosis of multiple cancer types by shrunken centroids of gene expression. *Proceedings of the National Academy of Sciences*, 99(10): 6567–6572, 2002.