Machine Learning Group
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Week 2: Matrices

Homework

Solutions must be submitted on ISIS until Tuesday, June 4th at 10am.

Exercise 1 (6 Points)

1. The rank of $\begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}$ is

   ☐ 1 ☐ 3 ☐ 4

2. For any symmetric, invertible matrix $A \in \mathbb{R}^{n\times n}$ and vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ it holds that:

   ☐ $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, A\boldsymbol{w}\rangle$
   ☐ $\langle A\boldsymbol{v}, A\boldsymbol{w}\rangle = \langle \boldsymbol{v}, \boldsymbol{w}\rangle$
   ☐ $\langle A\boldsymbol{v}, \boldsymbol{w}\rangle = \left\langle \boldsymbol{v}, A^{-1}\boldsymbol{w}\right\rangle$

3. Which of the following statements is <u>true</u>? For any square $n \times n$ matrix A it holds that:

   ☐ $\det A = 0 \Rightarrow \operatorname{rank} A = 0$
   ☐ $\det A = 0 \Leftrightarrow \operatorname{rank} A < n$
   ☐ $\operatorname{rank} A = n \Rightarrow \det A = n$

4. Which of the following matrices is orthogonal?

   ☐ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

5. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$ be an arbitrary square matrix with full rank. Which of the following mappings from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ defines a scalar product on $\mathbb{R}^n$?

   ☐ $f(\boldsymbol{x}, \boldsymbol{y}) := \langle A\boldsymbol{x}, \boldsymbol{y}\rangle$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
   ☐ $g(\boldsymbol{x}, \boldsymbol{y}) := \langle A\boldsymbol{x}, A\boldsymbol{y}\rangle$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
   ☐ $h(\boldsymbol{x}, \boldsymbol{y}) := \left\langle A\boldsymbol{x}, A^\top\boldsymbol{y}\right\rangle$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

6. Which of the following sets of $n \times n$ together with matrix addition and scalar multiplication do <u>not</u> form a real vector space?

   ☐ The set of symmetric matrices $\{A \in \mathbb{R}^{n\times n} \mid A^\top = A\}$
   ☐ The set of orthogonal matrices $\{A \in \mathbb{R}^{n\times n} \mid AA^\top = A^\top A = I_n\}$
   ☐ The set of upper triangle matrices $\{A \in \mathbb{R}^{n\times n} \mid A_{ij} = 0 \text{ for } i > j\}$

Exercise 2 (6 Points)

1. What transformation is described by the following matrix? What is the determinant?

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

2. *Show:* Orthogonal matrices have determinant 1 or -1.

   Hint: Show that

$$\det R = (\det R)^{-1}$$

   for any orthogonal matrix $R \in \mathbb{R}^{n\times n}$.

Exercise 3 (8 Points)

Let $\mathcal{U}$ be an r-dimensional vector subspace of $\mathbb{R}^n$ with basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\} \subset \mathbb{R}^n$. Let $U := (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \in \mathbb{R}^{n \times r}$ be the matrix whose columns are the $\boldsymbol{u}_i$. The matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{U}$ is given by $P := U(U^\top U)^{-1} U^\top$.

1. *Compute* the product $P \cdot P$. What does the result mean intuitively?

2. *Show:* If it holds that $r = n$, then P is the identity matrix, i.e., $P = I$.

3. What is the 3×3 matrix that describes the orthogonal projection onto the vector subspace spanned by the vectors $\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$?