Nullstellen

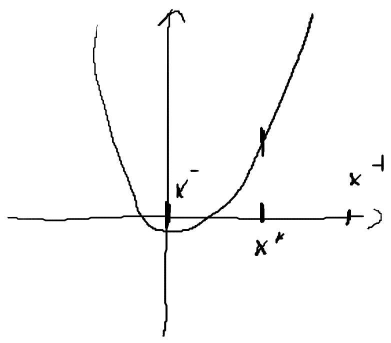
$$f(x) = g(x) \qquad x \in \mathbb{R}^n$$

$$\underbrace{f(x) - g(x)}\, = 0 \qquad f, g : \mathbb{R}^n \to \mathbb{R}^n$$

Annahme $f: \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = 0$

Suche: $\quad x^+ : f(x^+) > 0$

$\qquad\qquad x^- : f(x^-) < 0$



$f(x) = x^2 - \varphi, \quad \varphi > 0$

NST f : $x = \pm\sqrt{\varphi}$

$f(x) < 0 \quad]-\varphi, \varphi[$

Bisektion

$$x^* = \frac{x^+ + x^-}{2}$$

$f(x^*) > 0 \qquad x^+ \leftarrow x^*$

$f(x^*) < 0 \qquad x^- \leftarrow x^*$

$|f(x^*)| < \varepsilon$ $\quad$ Abbruchbed. I $\qquad f(x) = f(x^*) + f'(x^*)(x - x^*) + \dots$

$|x^+ - x^-| < \varepsilon$ $\quad$ Abbruchbed. II $\qquad$ worst case optimal

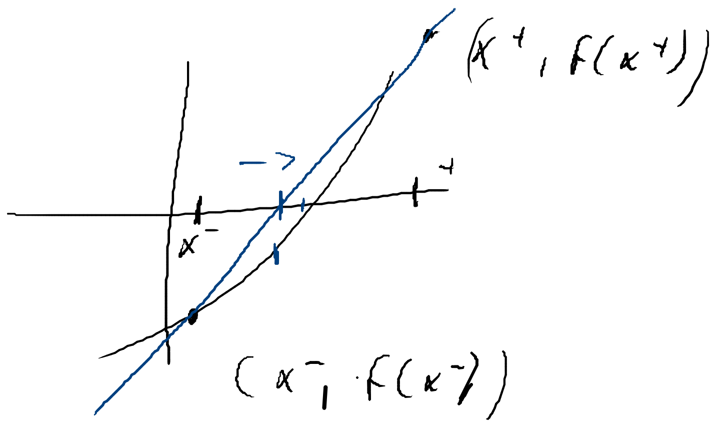
Regula Falsi

$$(1-\lambda)\, f(x^-) + \lambda\, f(x^+) = 0$$

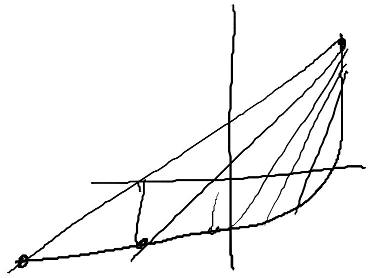
$$\lambda = \frac{f(x^-)}{f(x^-) - f(x^+)}$$

$$x^* = \left(1 - \frac{f(x^-)}{f(x^+) - f(x^-)}\right) x^- + \frac{f(x^-)}{f(x^+) - f(x^-)}\, x^+$$

$$= \frac{x^+ f(x^-) - x^- f(x^+)}{f(x^+) - f(x^-)}$$



$$m = \frac{f(x^-) - f(x^+)}{x^- - x^+}$$

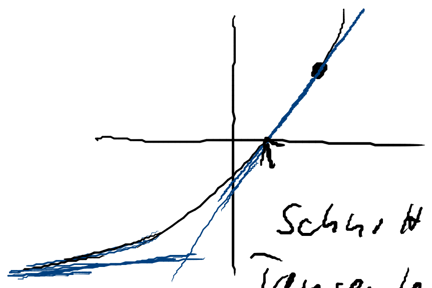


$$|f(x)| < \varepsilon \cdot \max\left(1, \left|\frac{f(x^-) - f(x^+)}{x^- - x^+}\right|\right)$$

Newton-Verfahren

$$0 = f(x) = f(x^*) + f'(x^*)(\underline{x} - x^*) + \ldots$$

$$x = x^* - \frac{f(x^*)}{f'(x^*)}$$



Schnitt von Tangente und $f=0$

Startwert x_0

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad |x_i - x_{i+1}|$$

NST

$$0 = f(x^*) = f(x_i) + f'(x_i)(x^* - x_i) + \frac{1}{2} f''(r_i)(x^* - x_i)^2, \quad r_i \in [x_i, x^*] \cup [x^*, x_i]$$

$$0 = \cancel{f(x_i)} + \cancel{f'(x_i)}\left(x^* - x_{i+1} - \frac{\cancel{f(x_i)}}{\cancel{f'(x_i)}}\right) + \frac{1}{2} f''(r_i)(x^* - x_i)^2$$

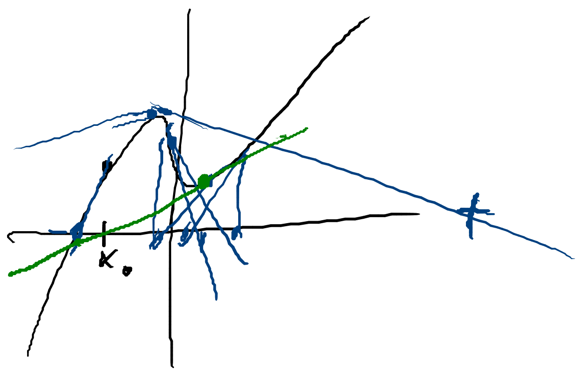
$$\underline{|x^* - x_{i+1}|} = \frac{1}{2} \frac{f''(r_i)}{f'(x_i)} \underline{(x^* - x_i)}^2$$

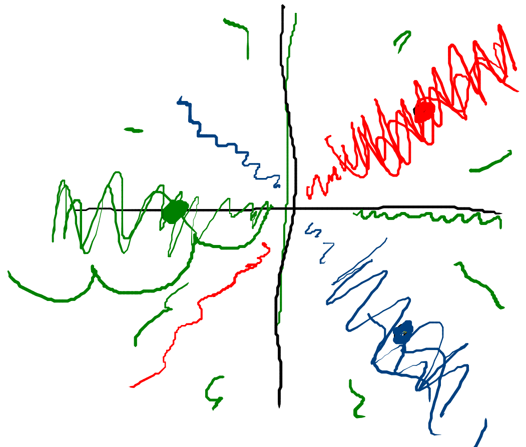
Bis. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

Newton $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \frac{1}{65536}$

Bisektion konvergiert linear
Newton-Verf. " quadratisch

Problem: Newton-Verf. konvergiert nicht





Newton-Fraktal

Newton-Verfahren in nD

$f: \mathbb{R}^n \to \mathbb{R}^m$

$f_0(x)$
$f_1(x)$
$f_2(x)$

$f_c : \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial f_c}{\partial x_i}$ partielle Ableitung

Bsp $f(x_0, x_1) = x_0^2 + x_0 x_1 + x_1^2$

$\frac{\partial f}{\partial x_0} = 2x_0 + x_1$

$\frac{\partial f}{\partial x_1} = x_0 + 2x_1{}^2$

$$x_d^{(i+1)} = x_d^i - \left(\frac{f_0(x^i)}{\frac{\partial f_0(x^i)}{\partial x_d}} + \frac{f_1(x^i)}{\frac{\partial f_1(x^i)}{\partial x_d}} + \dots \right)$$

$$J_f = \begin{pmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \frac{\partial f_0}{\partial x_2} & \dots \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots \\ \frac{\partial f_2}{\partial x_0} & & & \\ \vdots & & & \end{pmatrix}$$

Jacobi-Matrix

Newton-Schritt $x^{i+1} = x^i - J_f^{-1}(x^i) f(x^i)$

$\Delta x = x^{i+1} - x^i$

$\Delta x = x^{i+1} - x^i = -J_f^{-1} f$

$J_f(x^i)\,\Delta x = -f(x^i)$

$f: \mathbb{C} \to \mathbb{C}$

$$x_{i+1} = x_i - \frac{f(x_i)\,\overline{f'(x_i)}}{|f'(x_i)|^2}$$

$f: \mathbb{R}^n \to \mathbb{R}$

$J_f = g \in \mathbb{R}^n$ $\qquad g^+ g = 1$ $\qquad g^+ = \frac{g^T}{\|g\|^2}$

$$x^{i+1} = x^i - \frac{g^T f(x^i)}{\|g\|^2} = x^i - \frac{\nabla f(x^i)^T f(x^i)}{\|\nabla f(x^i)\|^2}$$