Machine Learning Group
Fakulty IV, Technische Universität Berlin
Prof. Dr. Klaus-Robert Müller
Email: klaus-robert.mueller@tu-berlin.de

Week 2: Matrices

Exercise 1

1. A mapping $f\colon V \to W$ between two real vector spaces V and W is linear if
   - ☐ $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda f(\boldsymbol{x}) + \mu f(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V, \lambda, \mu \in \mathbb{R}$.
   - ☐ f is a matrix.
   - ☐ the image of f is a vector subspace of W.
2. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

   ☐ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$
3. For which of the following 3×3 matrices A does it hold that $AB = BA = B$ for all $B \in \mathbb{R}^{3\times 3}$:

   ☐ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ☐ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ☐ $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
4. Let $A, B \in \mathbb{R}^{2\times 3}$ be 2×3 matrices. Then

   ☐ $A + B \in \mathbb{R}^{2\times 3}$ ☐ $A + B \in \mathbb{R}^{4\times 6}$ ☐ $A + B \in \mathbb{R}^{4\times 9}$
5. For $A \in \mathbb{R}^{m\times n}$ it holds that
   - ☐ A has m rows and n columns
   - ☐ A has n rows and m columns
   - ☐ The rows of A have length m and the columns of A have length n.
6. Which of the following is <u>not</u> a property of matrix multiplication?

   ☐ Associative property ☐ Commutative property ☐ Distributive property
7. For every square $n \times n$ matrix A it holds that
   - ☐ $\operatorname{rank} A = n \Rightarrow A$ is invertible, but there also exist invertible A with $\operatorname{rank} A \neq n$.
   - ☐ A is invertible $\Rightarrow \operatorname{rank} A = n$, but there also exist A with $\operatorname{rank} A = n$ which are not invertible.
   - ☐ $\operatorname{rank} A = n \Leftrightarrow A$ is invertible
8. Which of the following statements is <u>true</u> for all $A, B, C \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$?
   - ☐ $\det(A + B) = \det A + \det B$
   - ☐ $\det \lambda A = \lambda \det A$
   - ☐ $\det(ABC) = \det A \det B \det C$
9. Which of the following statements is true for all invertible $A, B \in \mathbb{R}^{n\times n}$?

   ☐ $\det(A^{-1}BA) = \det A \det B$ ☐ $\det(A^{-1}BA) = \det A$ ☐ $\det(A^{-1}BA) = \det B$

10. $\det \begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix} =$

☐ 0 ☐ λ ☐ λ^3

Exercise 2

Describe in words or draw the mappings represented by the following matrices. Compute the determinants of the matrices.

1. $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$
2. $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Exercise 3

Let $R \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, i.e., $RR^\top = R^\top R = I$. Show: Multiplication with R is invariant to the scalar product of two vectors, i.e., for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ it holds that

$$\langle R\boldsymbol{x}, R\boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle\,.$$

Exercise 4

Show that any square matrix M can be written as the sum of a symmetric matrix M_s and an anti-symmetric matrix M_a, i.e., $M = M_s + M_a$ with $M_s^\top = M_s$ and $M_a^\top = -M_a$ *Hint:* Construct a symmetric and an anti-symmetric matrix based on M. First express $M^\top$ with respect to M_s and M_a. Then express M_s with respect to M and $M^\top$.

Exercise 5

We have seen last week that the orthogonal projection onto a vector subspace is a linear transformation. The goal of this exercise is to find the matrix representation of an orthogonal projection. For this we consider $\mathbb{R}^n$ with standard scalar product. Let $\mathcal{U}$ be an r-dimensional vector subspace of $\mathbb{R}^n$ with basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\} \subseteq \mathbb{R}^n$. Let $U := (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \in \mathbb{R}^{n\times r}$ be the matrix whose columns are the $\boldsymbol{u}_i$. Because $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r \in \mathbb{R}^n$ form a basis, they must be linearly independent, but they do not need to be orthonormal (as in exercise 4 from last week). Let $\boldsymbol{v} \in \mathbb{R}^n$ be an arbitrary vector and $P\boldsymbol{v}$ its orthogonal projection onto $\mathcal{U}$. The goal of this exercise is to compute P from U. For this we first establish that $P\boldsymbol{v}$ has the following two properties:

- Because $P\boldsymbol{v}$ is an *orthogonal* projection, $\boldsymbol{v} - P\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$, i.e.,
$$U^\top(P\boldsymbol{v} - \boldsymbol{v}) = \boldsymbol{0} \tag{1}$$
- Because $P\boldsymbol{v}$ is a projection onto $\mathcal{U}$, $P\boldsymbol{v}$ is contained in $\mathcal{U}$, i.e., there exist $c_1, \ldots, c_r \in \mathbb{R}$ such that $P\boldsymbol{v} = c_1\boldsymbol{u}_1 + \ldots + c_r\boldsymbol{u}_r$. If we define $\boldsymbol{c} := (c_1, \ldots, c_r)^\top$, we can express this as follows:
$$P\boldsymbol{v} = U\boldsymbol{c} \tag{2}$$

Using this knowledge complete the following exercises:

1. Consider your drawing from last week which shows the vector $\boldsymbol{v} = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$ and its orthogonal projection $P\boldsymbol{v} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ onto the subspace spanned by the vector $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. *Visualize* equations (1) and (2) in your drawing.
2. Based on equations (1) and (2) *show* that
$$\mathbf{c} = (U^\top U)^{-1} U^\top \boldsymbol{v}.$$
3. Then, *show* that this implies
$$P = U(U^\top U)^{-1} U^\top.$$
4. What is the dimensionality of $U^\top U$ and P?
5. *Compute* the determinant of P (for $r < n$).
6. Does the assumption that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r \in \mathbb{R}^n$ form an orthonormal basis of $\mathcal{U}$ simplify the computation of P?
7. *Construct* the 2×2 matrix which describes the orthogonal projection onto the vector subspace spanned by $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.