Machine Learning Group
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Week 1: Euclidean Vector Spaces

Homework

Solutions must be submitted on ISIS until Tuesday May 28th, at 10am.

Exercise 1 (10 Points)

1. Let $V = (\mathcal{V}, +, \cdot)$ be a real vector space. Which of the following statements is false?
   - ☐ For all $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ it holds that $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
   - ☐ For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ it holds that $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
   - ☐ For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ it holds that $(\boldsymbol{u}\boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v}\boldsymbol{w})$
2. Which of the following sets together with standard addition and scalar multiplication does not form a real vector space?
   - ☐ The set of imaginary numbers $\{iy \in \mathbb{C} \mid y \in \mathbb{R}\}$
   - ☐ The set of real-valued, positive functions $\{f : \mathbb{R}^n \to [0, +\infty]\}$
   - ☐ The set of all polynomials of degree $\leq n$ $\{f : \mathbb{R} \to \mathbb{R} \mid f(x) = \sum_{p=0}^{n} a_p x^p, a_p \in \mathbb{R}\}$
3. Let V be a real vector space as in question 1. Which of the following statements is true?
   - ☐ $\{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}\} = \mathcal{V}$
   - ☐ $\{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}\} = \mathcal{V} \times \mathcal{V}$
   - ☐ $\{\lambda \boldsymbol{v} \mid \boldsymbol{v} \in \mathcal{V}, \lambda \in \mathbb{R}\} = \mathbb{R} \times \mathcal{V}$
4. Which of the following subsets of $\mathbb{R}^n$ is a vector subspace?
   - ☐ $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = \ldots = x_n\}$
   - ☐ $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1^2 = x_2^2\}$
   - ☐ $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = 1\}$
5. Which of the following means that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, all $\boldsymbol{v}_i \neq \boldsymbol{0}$ are linearly independent?
   - ☐ $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n = 0$ if and only if $\lambda_1 = \ldots = \lambda_n = 0$
   - ☐ If $\lambda_1 = \ldots = \lambda_n = 0$, then $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n = 0$
   - ☐ $\lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n = 0$ for all $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$
6. Which of the following vectors form a basis of the vector space of polynomials of degree 2?
   - ☐ $f_1(x) = x^2 + x - 2,\ f_2(x) = 2x^2 + 2x - 4,\ f_3(x) = 3x + 2$
   - ☐ $f_1(x) = 5x^2 - 2,\ f_2(x) = 3x,\ f_3(x) = 1$
   - ☐ $f_1(x) = x^2 + 3x + 1,\ f_2(x) = x^2 - 1,\ f_3(x) = x + 1,\ f_4(x) = x - 1$
7. What is the dimension of the vector space $\{\boldsymbol{x} \in \mathbb{R}^3 \mid x_1 + 2x_2 = 0, x_1 + x_3 = 0\}$?
   - ☐ 1 ☐ 2 ☐ 3
8. Which of the following statements is true?
   - ☐ If $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defines a scalar product on $\mathbb{R}^n$, then it holds that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \ldots + x_n y_n$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
   - ☐ $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := x_1 y_1 + \ldots + x_n y_n$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ defines a scalar product on $\mathbb{R}^n$.
   - ☐ In order for a mapping $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ to define a scalar product on $\mathbb{R}^n$, it is a necessary condition that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. (The Kronecker symbol δ_{ij} is defined as $\delta_{ii} := 1$ and $\delta_{ij} := 0$ for $i \neq j$.)
9. Which of the following mappings does not define a scalar product on the vector space $L_2(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$?
   - ☐ $\langle f, g \rangle := \int_{-\infty}^{+\infty} f(x)g(x)dx$ for all $f, g \in L_2(\mathbb{R})$
   - ☐ $\langle f, g \rangle := \int_{-\infty}^{+\infty} \exp(-x^2) f(x)g(x)dx$ for all $f, g \in L_2(\mathbb{R})$
   - ☐ $\langle f, g \rangle := \int_{-\infty}^{+\infty} (f(x) + g(x))dx$ for all $f, g \in L_2(\mathbb{R})$

10. Let $\mathbb{R}^2$ be a Euclidean vector space with the standard scalar product. Which of the following vectors form an orthonormal basis?

☐ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Exercise 2 (5 Points)

Let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathcal{V}$ be pairwise orthogonal vectors. *Prove* the following generalized version of the Pythagorean theorem:

$$\left\| \sum_{i=1}^{n} \boldsymbol{v}_i \right\|^2 = \sum_{i=1}^{n} \|\boldsymbol{v}_i\|^2$$

Exercise 3 (5 Points)

Let V be a vector space and $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset V$ a basis. This implies that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent and any $\boldsymbol{v} \in V$ can be written as a linear combination of the $\boldsymbol{v}_i$, i.e., there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\boldsymbol{v} = \lambda_1 \boldsymbol{v}_1 + \ldots + \lambda_n \boldsymbol{v}_n$.
Show: For all $\boldsymbol{v} \in V$ the $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are uniquely determined.