Gleitkommazahlen

$m \cdot b^e \qquad 1 \le m < b$

Mantisse „Zähler"   „Nenner"

$$G(b,n_m) = \left\{ x \in \mathbb{Q} : x = \sum_{i=0}^{n_m-1} x_i \cdot b^{i+e-n_m+1}, \; x_i \in [0,\dots,b-1], \; e \in \mathbb{Z} \right\}$$

$$G: \mathbb{Q} \to \mathbb{G} \qquad G(x) = \hat{x} \qquad \operatorname*{argmax}_{x \in \mathbb{G}} \hat{x} \le x$$

Abstand von GKZ

$$\begin{array}{rccccl} & x_{n_m-1}, & x_{n_m-2} & \dots & (x_0+1) & \cdot\, b^e \\ - & x_{n_m-1}, & x_{n_m-2} & \dots & x_0 & \cdot\, b^e \\ \hline & 0, & 0 & \dots 0 & 1 & \cdot\, b^e \end{array}$$

$$1 \cdot \boxed{b^{e-(n_m-1)}}$$

$G(10,1)$

$$\underbrace{0{,}1 \quad 0{,}2}_{0{,}1} \quad 0{,}3 \quad \dots \quad \underbrace{0{,}8 \quad 0{,}9}_{0{,}1} \quad \underbrace{1 \quad 2}_{1} \underbrace{\quad 3}_{1} \quad \dots \quad \underbrace{8 \quad 9}_{1} \quad \underbrace{10 \quad 20}_{10} \quad 30$$

$G(3-\varepsilon) = 2 \qquad |3-\varepsilon-2| = 1-\varepsilon$

$G(30-\varepsilon) = 20 \qquad |30-\varepsilon-20| = 10-\varepsilon$

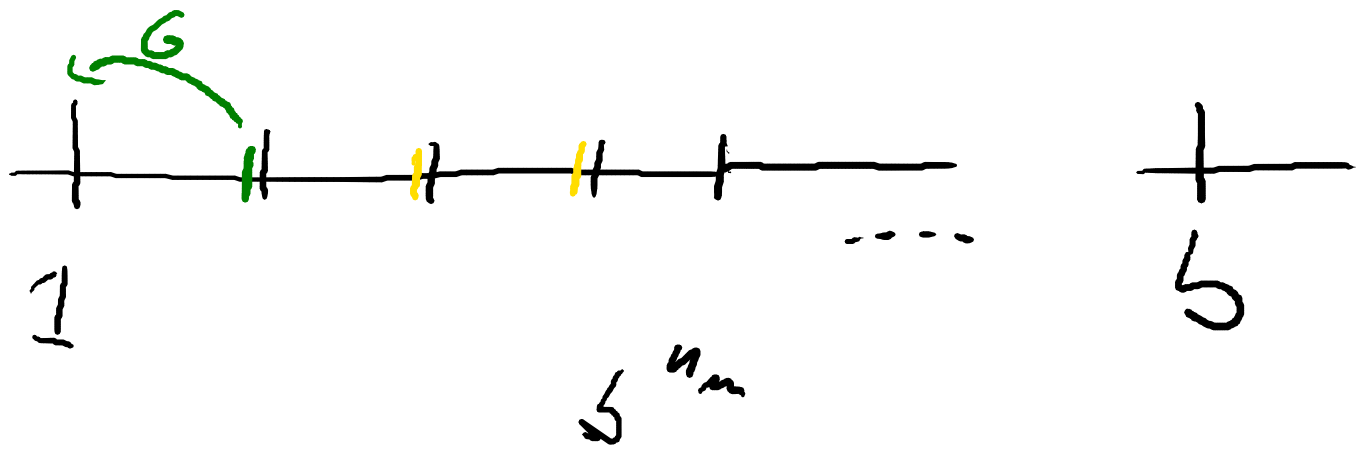
Relativer Fehler

$$\delta = \frac{|x - \hat{x}|}{|x|}$$

$$\varepsilon = \max_{x \in \mathbb{Q}^+} \frac{|x - G(x)|}{|x|} = \max_{x \in \mathbb{Q}^+} \frac{x - G(x)}{x}$$

Maschinengenauigkeit (von $G(b, n_m)$)

$$\varepsilon \le \frac{b^{e-(n_m-1)}}{b^e} = b^{-n_m+1}$$



$$\varepsilon = \operatorname*{argmin}_{x \in \mathbb{Q}} G(1+x) \neq 1$$

Rechenoperationen – Fehler

$x = g_x + r_x \qquad g_x = G(x)$

$y = g_y + r_y \qquad g_y = G(y)$

$0 \le r_x, r_y < 1 \cdot b^e \cdot b^{-n_m+1}$

$x * y \ne G(x) * G(y) \ne$
$\ne G(x * y)$
$\ne G(G(x) * G(y))$

~~Assoziativgesetz~~ ~~$(x \cdot y)z = x \cdot (y \cdot z)$~~
~~$(x+y)+z = x+(y+z)$~~

~~Distributiv~~ ~~$x(y+z)$~~ $= xy + xz$

Produkt

$x \cdot y = (g_x + r_x)(g_y + r_y)$

$G(x) \cdot G(y) = g_x g_y$

$$\frac{g_x r_y + g_y r_x + r_x r_y}{g_x g_y + \boxed{\cancel{g_x r_y + g_y r_x + r_x r_y}}}$$

$$\frac{g_x b^e b^{-n_m+1} + g_y b^e b^{-n_m+1} + b^{2e} b^{2(n_m+1)}}{g_x g_y}$$

$$\frac{\cancel{b b^e} b^{-n_m+1} + \cancel{b b^e} b^{-n_m+1} + \ldots}{\cancel{b^e} \quad \cancel{b^e}}$$

$< (2 b^{-n_m+1} + b^{-n_m+1} \cdot b^{-n_m+1})$

$\quad \varepsilon \qquad \varepsilon \qquad \varepsilon$

$b(2\varepsilon + \varepsilon^2)$

Division

Addition

$x+y = g_x + r_x + g_y + r_y$

$G(x) + G(y) = g_x + g_y$

$$\frac{r_x + r_y}{g_x + r_x + g_y + r_y}$$

$$\frac{2 \cancel{b^e} b^{-n_m+1}}{\cancel{2 b^e}} \quad \varepsilon$$

Subtraktion

$$\frac{|r_x - r_y|}{g_x - g_y + r_x - r_y} = \frac{|r_x - r_y|}{x - y}$$

$$\frac{b^e b^{-n_m+1}}{\mu}, \quad \mu > 0$$

$\quad x_{n_m-1}, \; x_{n_m-2} \; \cdots \; x_1 \; x_0$

$- \; y_{n_m-1}, \; y_{n_m-2} \; \cdots \; y_1 \; y_0$

$\quad 0, \; 0 \; \cdots \; 0 \; 1 \; 1$

Auslöschung

Reele Zahlen $\mathbb{R}$

$$ax^2+bx+c=0$$

(a, b, c)

algebraische Zahlen

$$x \in \mathbb{R}$$

$$|x-q| < \varepsilon$$

$q \in \mathbb{Q}$ $\quad$ $\varepsilon, x \in \mathbb{R}$

irrational

$$x^2 = 2$$

$$\left(\frac{p}{q}\right)^2 = 2$$

$$p^2 = 2q^2 \Rightarrow p^2 \text{ gerade}$$

$$\cancel{p = 2n+1}$$

$$\cancel{p^2 = (2n+1)^2 = 4n^2+4n+1}$$

$$(2p')^2 = 2q^2$$

$$4p'^2 = 2q^2$$

$$2p'^2 = q^2$$

$$\boxed{x^2 = -1}$$

$$\mathbb{C} = \{(a,b)\}$$

Komplexe Zahlen

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\mathbb{I} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$a\mathbb{I} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left(a\mathbb{I} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\left(c\mathbb{I} + d\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = ac\,\mathbb{I} +$$

$$ad(\quad) +$$

$$bc(\quad) +$$

$$bd \cdot -\mathbb{I}$$

$$= (ac - bd)\,\mathbb{I}$$

$$+ (ad + bc)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$