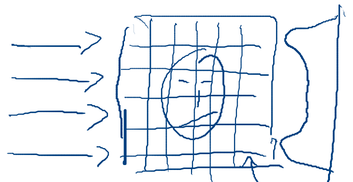
LGS : Motivation



Dichte konstant

$\downarrow$

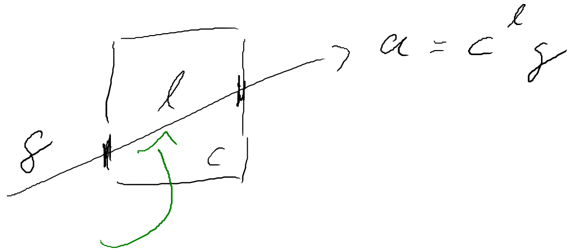
$x \in \mathbb{R}^n$

$2g \to \square \quad 2a \qquad 2a = \cancel{2g} - c$

$\boxed{a = c \cdot g}$

$g \to \square\square \quad 1+1=2 \quad c^2 g$

$g \to \square \quad 0{,}5 \quad \sqrt{c}\, g$



$$a_i = c_0^{l_{i,0}} \cdot c_1^{l_{i,1}} \cdot c_2^{l_{i,3}} \cdots \cdot g$$

$$\frac{a_i}{g} = c_0^{l_{i0}}\, c_1^{l_{i1}} \cdot c_2^{l_{i2}}$$

$$\underbrace{\log \frac{a_i}{g}}_{a_i'} = l_{i0} \cdot \underbrace{\log c_0}_{c_0'} + l_{i1} \cdot \underbrace{\log c_1}_{c_1'} + l_{i2} \cdot \underbrace{\log c_2}_{c_2'} + \dots$$

$$a_i' = l_{i0}\, c_0' + l_{i1}\, c_1' + l_{i2} \cdot c_2^2 + \dots$$

$$\begin{pmatrix} l_{00} & l_{10} & l_{20} & \cdots \\ l_{01} & l_{11} & l_{21} & \cdots \\ & & & \end{pmatrix} \begin{pmatrix} c_0' \\ c_1' \\ c_2' \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0' \\ a_1' \\ a_2' \end{pmatrix}$$

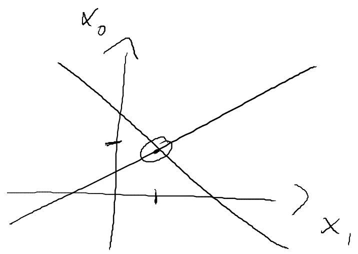
$$Lc = a$$

Interpretation von LGS $\quad Ax = b$

Zeilen

$$\begin{pmatrix} a_0^T \\ a_1^T \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

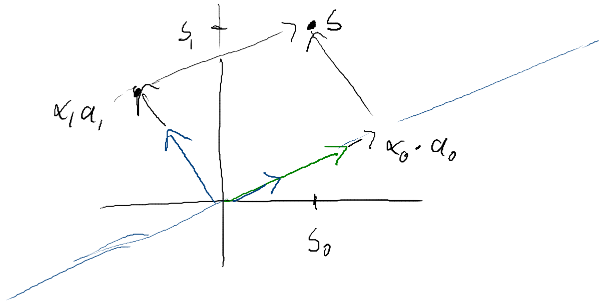
$$a_{00} x_0 + a_{01} x_1 - b_0 = 0$$
$$a_{10} x_0 + a_{11} x_1 - b_1 = 0$$



$$\boxed{x_0} = \frac{b_0 - a_{01} x_1}{a_{00}} = \frac{b_0}{a_{00}} - \frac{a_{01}}{a_{00}} \boxed{x_1}$$
$$\boxed{x_0} = \frac{b_1 - a_{11} x_1}{a_{10}} = \frac{b_1}{a_{10}} - \frac{a_{11}}{a_{10}} \boxed{x_1}$$

Spalten

$$\begin{pmatrix} \vec{a}_0 , \vec{a}_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \vec{b} \qquad x_0 \cdot \vec{a}_0 + x_1 \vec{a}_1 = \vec{b}$$



schwarz	rot
rot	blau
blau	grün
grün	gelb

LGS in GKZ

$A \cdot x \,, \quad A \in G^{n \times n} \,, \quad x \in G^n \quad \to$

$Ax = Ax' \qquad x \neq x'$

$|G^2| = 10000$

$x \in G^2$

G^2

$Ax = b \qquad x, b \in G^n$

$r = Ax - b \qquad Ax^* = b \qquad x^* \in \mathbb{Q}^n$

$$\frac{\|r\|}{\|b\|} = \frac{\|Ax - b\|}{\|b\|} = \frac{\|Ax - Ax^*\|}{\|Ax^*\|} = \frac{\|A(x - x^*)\|}{\|Ax^*\|}$$

$G(Ax) \in G^2$

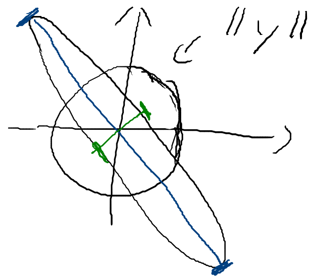
Ax

$\|Ay\| = c_y \|y\| \qquad c_y = \|Ay\| \,, \quad \|y\| = 1$

$$\|A(x - x^*)\| \leq \max_{\|y\|=1} \|Ay\| \cdot \|x - x^*\|$$

$$\|Ax^*\| \geq \min_{\|y\|=1} \|Ay\| \cdot \|x^*\|$$

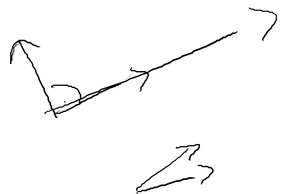
$$\frac{\|A(x - x^*)\|}{\|Ax^*\|} \leq \frac{\quad}{\quad} = \varepsilon \cdot \frac{\max_{\|y\|=1} \|Ay\|}{\min_{\|y\|=1} \|Ay\|}$$



$\|y\| = 1$

$$\kappa(A) = \frac{\max_{\|y\|=1} \|Ay\|}{\min_{\|y\|=1} \|Ay\|} \geq 1$$

Kondition



$$\kappa(A) = \kappa(A^{-1})$$

Lösen von LGS

$Dx = s$

$$\begin{pmatrix} d_0 & & & \\ & d_1 & & \\ & & d_2 & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \end{pmatrix} \qquad x_i = \frac{s_i}{d_i}$$

$$\begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & 0 \cdots & 0 \\ a_{20} & a_{21} & a_{22} & 0 \cdots 0 \\ \vdots & & & \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \end{pmatrix}$$

$x_0 = s_0 / a_{00}$

$x_1 = \dfrac{s_1 - a_{10} x_0}{a_{11}}$

$x_2 = \dfrac{s_2 - a_{20} x_0 - a_{21} x_1}{a_{22}}$

untere Dreiecksmatrix — Vorwärtseinsetzen

obere " $\rightarrow$ Rückwärtseinsetzen

Gauß-Elimination

$$A^{k} = \begin{pmatrix} a_{00}^{(0)} & a_{01}^{(0)} & \cdots & \\ 0 & a_{11}^{(1)} & \cdots & \\ & & a_{kk}^{(k)} & \cdots \\ & & a_{k+1,k}^{(k)} & \cdots \\ & & \vdots & \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & -1 & 5 \\ \underbrace{-1}_{} & -1 & 2 & -4 \\ -1 & 2 & -3 & 1 \end{matrix}$$

$=0$

$$\begin{matrix} 1 & 0 & -1 & 5 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & -4 & 6 \end{matrix}$$

$$\boxed{m_{ik} = -\frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}} \overset{?}{=} 0$$

$$L^{(k)} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & 0 & m_{k+1,k} & 1 & 0 \\ & & 0 & m_{k+2,k} & 0 & 1 \\ & & 0 & \vdots & & \end{pmatrix}$$

$$A^{(k+1)} = L^{(k)} A^{(k)}$$

$$b^{(k+1)} = L^{(k)} b^{(k)}$$

$$\kappa(L) = \frac{\max}{\min}$$

$$\min_{\|y\|=1} \|L^{(k)} y\| \leq 1$$

$$\max_{\|y\|=1} \|L^{(k)} y\| \geq \sqrt{1 + m_{k+1,k}^2 + m_{k+2,k}^2} = \sqrt{\sum \frac{a_{ik}^2}{a_{kk}^2}} = \frac{\sqrt{\sum a_{ik}^2}}{k \cdot |a_{kk}|}$$

$\rightarrow$ Wähle $|a_{kk}|$ möglichst groß