Gleitkommazahlen

$56 = 5{,}6 \cdot 10^1$

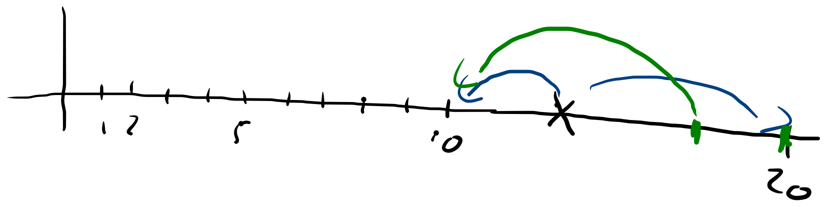
$m \cdot b^e \qquad 1 \le m < b$

Mantisse   Exponent

$$\mathbb{G}(b, n_m) = \left\{ x \in \mathbb{Q}^+ : \; x = \sum_{i=0}^{n_m - 1} x_i \, b^{\,i+e-n_m+1}, \; e \in \mathbb{Z}, \; x_i \in [0, \dots, b-1] \right\}$$

$G: \mathbb{Q} \to \mathbb{G} \qquad G(x) = \hat{x} \qquad \underset{\hat{x} \in \mathbb{G}}{\operatorname{argmax}} \; \hat{x} \le x$

$\mathbb{G}(10, 1)$



Absolute Fehler von G: $|x - G(x)|$

Worst-case (für Abrunden) $G(\hat{x} - \varepsilon) \qquad \hat{x} \in \mathbb{G}, \; \varepsilon > 0 \in \mathbb{Q}$

$$\begin{array}{rcccccl} & x_{n_m-1}, & x_{n_m-2} & \dots & x_1 & x_0+1 & \cdot\, b^e \\ - & x_{n_m-1}, & x_{n_m-2} & \dots & x_1 & x_0 & \cdot\, b^e \\ \hline & 0, & 0 & \dots & 0 & 1 & \cdot\, b^e \end{array} \quad \to \quad 1 \cdot \boxed{b^{\,e-(n_m-1)}}$$

-> Abstand, absoluter Fehler von G ist abhängig von e in $m \cdot b^e$

Gleitkommazahlen

$56 = 5{,}6 \cdot 10^1$

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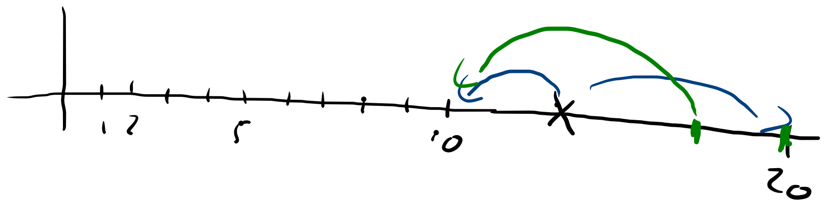

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Relativer Fehler

$$\delta = \frac{|x - \hat{x}|}{|x|}$$

$$\frac{|x - G(x)|}{|x|}$$

$x = x' \cdot b^e \quad 1 \leq x' < b$

$G(x) = G(x' \cdot b^e) = G(x')\, b^e$

$$\frac{|x' \cdot b^e - G(x') \cdot b^e|}{|x' \cdot b^e|} = \frac{|x' - G(x')|}{|x'|}$$

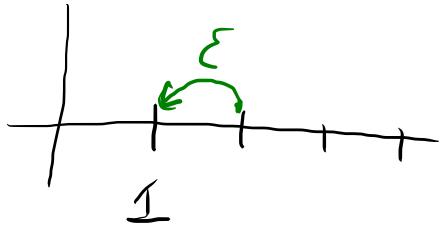
$\Rightarrow$ relativer Fehler von G ist unabhängig von e

$$\varepsilon \geq \frac{|x - G(x)|}{|x|} = \frac{|x' - G(x')|}{|x'|}$$

$|x' - G(x')|$ -> groß ->

$|x'|$ -> klein -> $x' = 1$

$$\varepsilon = \frac{b^{-n_m+1}}{1}$$ ← $\boxed{\text{Maschinengenauigkeit}}$



$$\varepsilon' = \underset{x \in \mathbb{Q}}{\arg\min}\ G(1+x) > 1$$

Fehler Sei Elementaroperationen

$$x * y \neq G(x * y) \neq G(x) * G(y) \neq G(G(x) * G(y)) \qquad G(10,1)$$

Assoziativgesetz $\cancel{(x * y) * z = x * (y * z)}$

Distributivg. $\cancel{(x+y)z = xz + yz}$

$$\frac{|x * y - G(G(x) * G(y))|}{|x * y|} \geq \frac{|x * y - G(x) * G(y)|}{|x * y|}$$

$x = g_x + r_x \qquad g_x = m_x b^{e_x} \in G, \qquad 0 \le r_x < b^{e_x - n_m + 1}$

$y = g_y + r_y \qquad g_y = m_y \cdot b^{e_y} \qquad r_y < b^{e_y - n_m + 1}$

Produkt (Division)

$$x \cdot y = (g_x + r_x)(g_y + r_y) = g_x g_y + r_x g_y + g_x r_y + r_x r_y$$

$$G(x) \cdot G(y) = g_x g_y$$

$$\delta = \frac{r_x g_y + r_y g_x + r_x r_y}{g_x g_y + r_x g_y + r_y g_x + r_x r_y} \to \begin{matrix} r_x, r_y = b^{e - n_m + 1} \\ g_x, g_y = 1 \cdot b^e \end{matrix}$$

$$\to \begin{matrix} r_x, r_y = 0 \\ g_x, g_y = 1 \cdot b^e \end{matrix}$$

$$\to \frac{b^{2e - n_m + 2} + b^{2e - n_m + 2} + b^{2(e - n_m + 1)}}{b^{2e}} = 2 \underbrace{b^{-n_m + 1}}_{\varepsilon} \cdot b + \underbrace{b^{2(-n_m + 1)}}_{\varepsilon^2}$$

Addition

$$x + y = g_x + r_x + g_y + r_y$$

$$G(x) + G(y) = g_x + g_y$$

$$\delta = \frac{r_x + r_y}{g_x + g_y + r_x + r_y} \le \frac{2 b^{e - n_m + 1}}{2 b^e} = b^{-n_m + 1} = \varepsilon$$

Differenz

$(x \geq y) \qquad x - y = g_x + r_x - g_y - r_y$

$$G(x) - G(y) = g_x - g_y$$

$$\delta = \frac{|r_x - r_y|}{g_x + r_x - g_y - r_y} = \frac{|r_x - r_y|}{x - y} \le \frac{b^{e - n_m + 1}}{\mu}$$

$x = g \to r_x = 0$

$y = g - \mu \to r_y = b^{e - n_m + 1}$

↯ Auslöschung

$$ax^2 + bx + c = 0$$

$x \in \mathbb{R}$, $q \in \mathbb{Q}$

$$|x - q| < \varepsilon$$

Für jedes $\varepsilon \in \mathbb{R}$ und $x \in \mathbb{R}$ gibt es ein $q \in \mathbb{Q}$

Problem 1 $\quad x^2 - 2 = 0$

$$\Leftrightarrow x^2 = 2$$

$$\left(\frac{p}{q}\right)^2 = 2$$

kürzen $\rightarrow$ p und q sind nicht beide gerade

$$p^2 = 2q^2$$

$\rightarrow$ p gerade

$p = 2p'$

$$(2p')^2 = 4p'^2 = 2q^2$$

$$\Leftrightarrow 2p'^2 = q^2$$

$\rightarrow$ q gerade

↯

$$\Rightarrow x \in \mathbb{R}$$

$$ax^2 + bx + c = 0$$

Problem $x^2 = -1$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{I}$$

$$\mathbb{C} = \{(a, b)\}$$

$$a\underline{I} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(a,b) + (c,d) = (a+c, b+d)$$

$$(a,b) \cdot (c,d) = (ac - bd, ad + bc)$$

$$\left(a\underline{I} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\left(c\underline{I} + d\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = ac\,\underline{I} + ad\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + bc\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + bd \cdot (-\underline{I})$$

$$= (ac - bd)\,\underline{I} + (ad + bc)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$