Theory of Computation

Lecture 1: Introduction

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Content of Lecture 1

- Course Outline
- Overview of this course
- Introduction to Mathematical Notations and Terminology

Course Outline

- Aim of this course
  - Equip students with the knowledge of fundamental properties of computer hardware and software
  - To answer:
    - **What are the capabilities and limitations of computers?**
- Three major subjects in this course
  - Automata and Language
  - Computability
  - Complexity

Why we need Mathematics?

- Without Mathematics, difficult to **define behavior of an object precisely**
- For software engineering, one still needs mathematical or equivalent model to describe the behavior or functionalities of a software
- **In this course, we define the model of behavior of a computer**

Relevance to Graduate Students

- When one wants to publish a good journal paper, he/she needs to **define the problem and the solution precisely**
- For comparing two software tools or algorithms, one can not just state that *A* is faster than *B* by 2 seconds in my machine. Instead, you need to state and **compare their computational complexity**
- For a difficult problem, you need to indicate that this problem is very hard, by mathematical analysis, and then propose a simplified algorithm to deal with it
- For problem that is not computable or is highly complex, one should propose an **computable algorithm** with a **reasonable complexity** by trading-off to have sub-optimal solution instead of perfect one

Course Outline

- Text Book:
  - Michael Sipser, "Introduction to the Theory of Computation", Second Edition, MIT Press.
- Website:
  - http://miles.hitsz.edu.cn/course/tc0607
- Contact us:
  - C308, 15:00 to 16:00 Wed

Course Outline

- Grading

Attendance	10%
3 Assignments	30%
Mid-term Test	25%
Final Exam	35%

Overview

- Automata and Language
- Computability
- Complexity

Computability

- Determine whether a problem is computable (solvable) or not by a computer algorithm

- e.g. Is it computable?
  - Determining whether a mathematical statement is true or false

Complexity

- Determine whether a problem is easy to solve or hard to solve

- e.g. Is it easy to solve?
  - Scheduling all the classes of a university without any two classes take place in the same room and at same time
  - Sorting all students in a university according to their height

Complexity

- Usually, easy computational problem is preferred

- In Cryptography, hard computational problem is preferred
  - Prevent secret to be accessed without password or secret code

Automata and Language

- As the theories of computability and complexity require **a precise definition of a computer**, so this is where we begin:
  - What is a computer?
    - We can treat it as a mathematical theory which is complex. Here we call it computational model.

- There are several different computational models, depending on the features we want to focus on.

- Automata theory deals with the definition and properties of computational models.
  - Finite Automaton and Regular Language
  - Context-Free Grammar

Finite Automaton and Regular Language

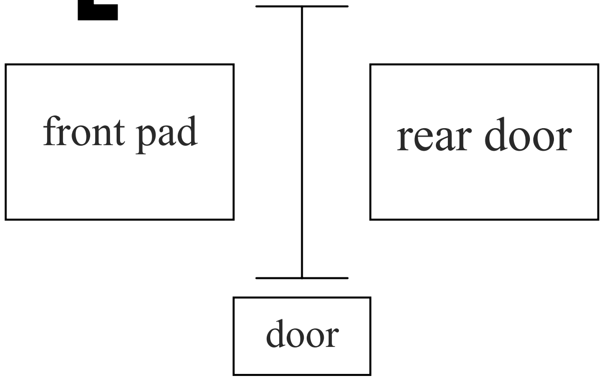
- **Finite automata are good models for computers with an extremely limited amount of memory.**

- Example: the controller for an automatic door.
  - Door swing open when sensing a person is in front
  - Controller can hold the door open long enough for the person to pass.

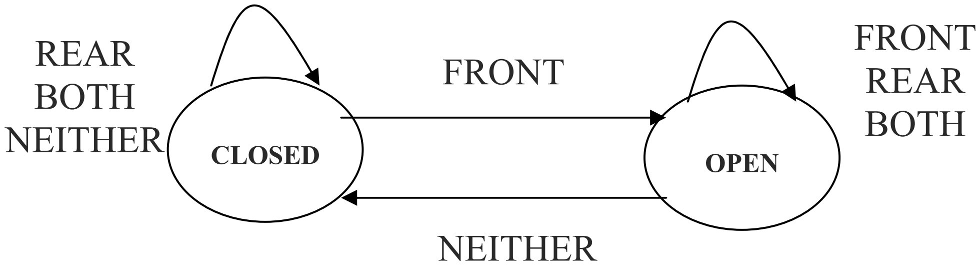
- Applications:
  - Text processing
  - Compiler
  - Hardware Design

- More Detail in Chapter 1.

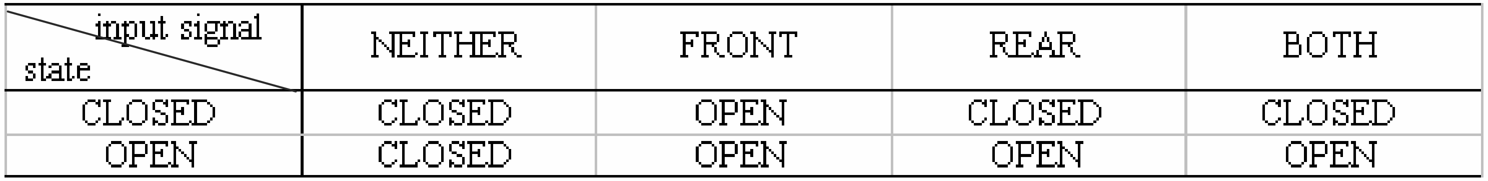
Finite Automaton of Automatic door



Top view



State diagram



input signal / state	NEITHER	FRONT	REAR	BOTH
CLOSED	CLOSED	OPEN	CLOSED	CLOSED
OPEN	CLOSED	OPEN	OPEN	OPEN

Transition table of the door

It only has 1-bit memory

Context-Free Grammar and Context-Free Language

- Context-Free Grammar is a more powerful **method of describing languages**. It can describe certain features that have a recursive structure.

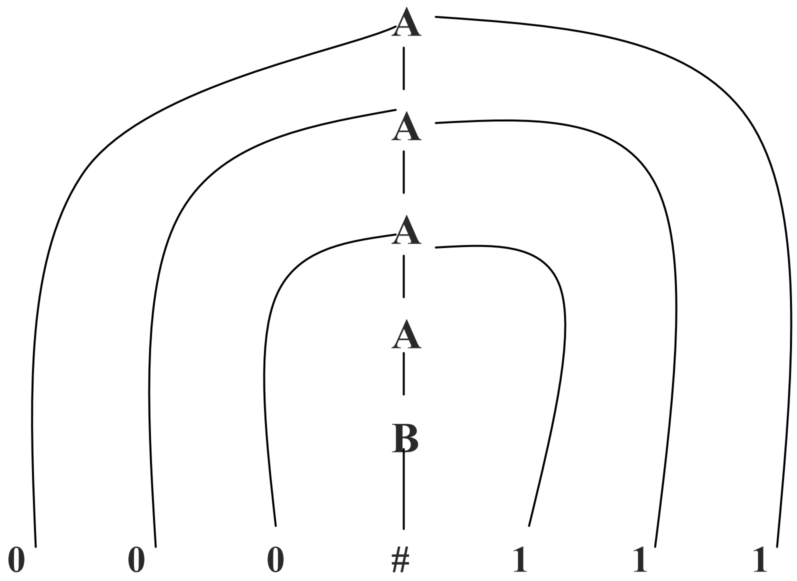
- Applications:
  - Programming Language
  - Artificial Intelligence

- Example:

$$G1:\ A \rightarrow 0A1,\ A \rightarrow B,\ B \rightarrow \#$$

- More Detail in Chapter 1.

Context-Free Grammar and Context-Free Language



Mathematical Notations and Terminology

- As in any mathematical subjects, we begin with the definition on notation and terminology
  - Set and Sequence
  - Function and Relation
  - Graph
  - String and Language
  - Boolean Logics
  - Definition, Theorem and Proof

Sets

- Set is a group of objects represented as a unit
- The set A has three elements $\qquad A = \{1,2,3\}$
- 1 is member (or element) of A $\qquad 1 \in \{1,2,3\}$
- 4 is not member of A $\qquad 4 \notin \{1,2,3\}$

Sets

- The set $\{1\}$ is a proper subset of A $\quad \{1\} \subset \{1,2,3\}$
- The set A is a subset of A $\quad \{1,2,3\} \subseteq \{1,2,3\}$
- Intersection of two sets $\quad \{1,2\} \cap \{2,3\} = \{2\}$
- Union of two sets $\quad \{1,2\} \cup \{2,3\} = \{1,2,3\}$

Sets

- Complement of A denotes all elements that are not belonging to set A

$$4 \in \overline{A}$$

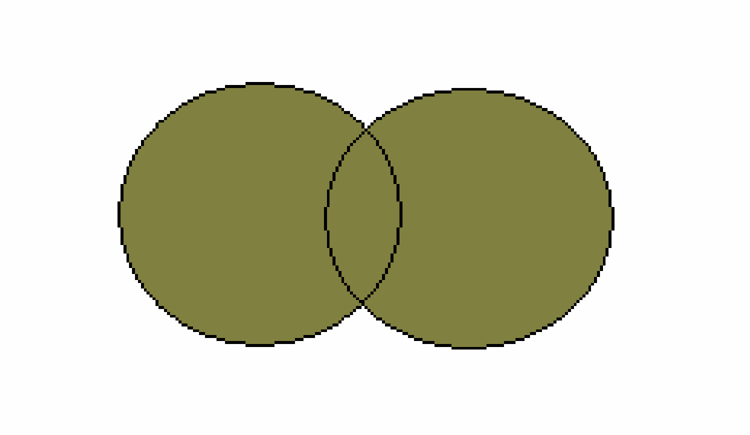
- Power Set of A denotes set of all subset of A

power set of A $\quad P(A)$ or $2^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

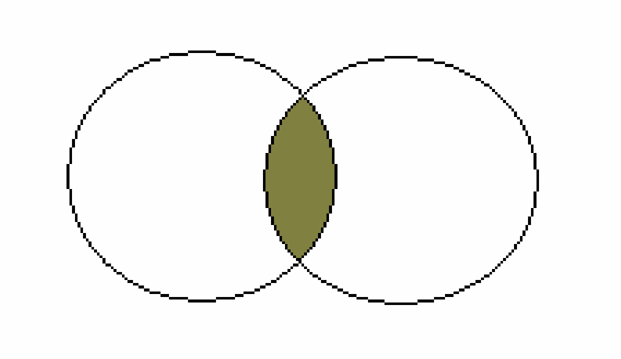
- Empty Set is a set with zero elements $\quad \phi = \{\ \}$

Sets

- Venn diagram



$A \cup B$



$A \cap B$

Sets

- Cartesian Product of *A* and *B* (*A*x*B*)

$$A = \{1,2,3\} \qquad B = \{a,b,c,d\}$$

$$A \times B = \begin{Bmatrix} (1,a),(1,b),(1,c),(1,d),(2,a),(2,b), \\ (2,c),(2,d),(3,a),(3,b),(3,c),(3,d) \end{Bmatrix}$$

$$A \times A \times A = A^3$$

Sequences

- A Sequence of objects is a list of these objects in some order

$$S = (3,1,2)$$

- Sequence S is a 3-tuple
  - 2-tuple is also known as pair
- **Set** does not care the order and repeat of elements, but **Sequence** does

Functions and Relations

- A Function (f) is an object that sets up an input-output relationship
    - x is an input value and y is an output value

$$f(x) = y \qquad\qquad f : D \to R$$

- The set (D)of all possible inputs of f is called its Domain
- The set (R) of all possible outputs of f is called its Range
    - A function may not necessarily use all the elements of its range
        - R denotes the set of all real numbers

$$abs : \Re \to \Re \qquad\qquad add : \Re \times \Re \to \Re$$

- Unary function, Binary function, n-ary function

Functions and Relations

- Predicate or Property is a function whose range is {True, False}

  $even(4) = TRUE$ $even(5) = FALSE$

- Infix notation $x + z = y$

- Prefix notation $add(x, z) = y$

- Binary function, n-ary function

Functions and Relations

- Equivalent Relation denotes two objects being equal in some feature

- A binary relation R is an equivalent relation if R satisfies:
  - R is reflexive if for every x, xRx
  - R is symmetric if for every x and y, xRy implies yRx
  - R is transitive if for every x, y and z, xRy and yRz implies xRz

Functions and Relations

On class exercise

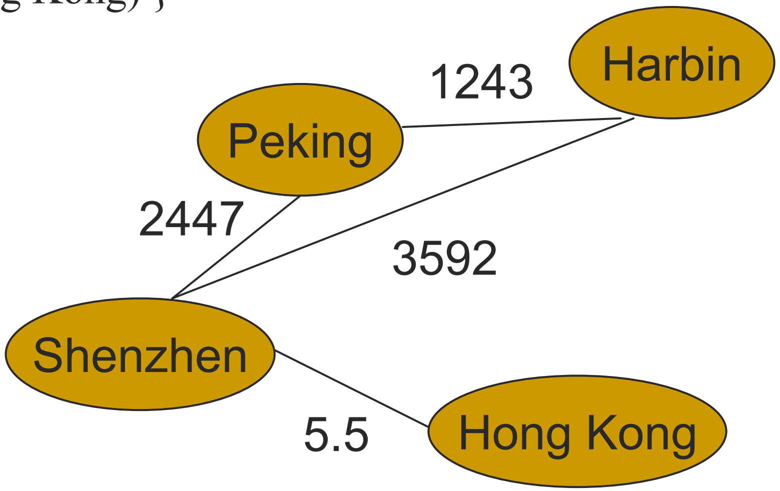
- Let *R* be a binary relationship on two integers. For integers *a* and *b, aRb* if 3 divides (*a* - *b*). Is *R* an equivalence relation?
  - E.g. 3 divides (4 - 1) results an integer 1 and thus 4*R*1 is true.

Graph

- **Undirected graph** or **graph** is a set of points with lines connecting some of the points.
- The points are called nodes or vertices, the lines are called edges.
- Degree of the node is the number of edges at this node

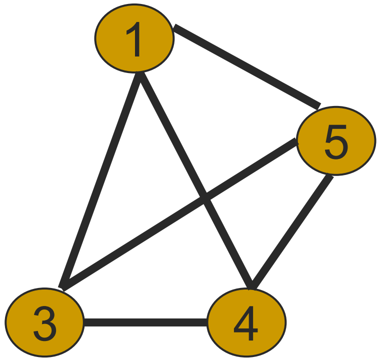
Graph

- Represents of graph
  - Sets: $G=(V,E)$ V is the set of nodes, E is the set of edges
    - ({ Harbin, Peking, Shenzhen, Hong Kong }, { (Harbin, Peking), (Harbin, Shenzhen), (Peking, Shenzhen), (Shenzhen, Hong Kong) }
  - Labeled graph:

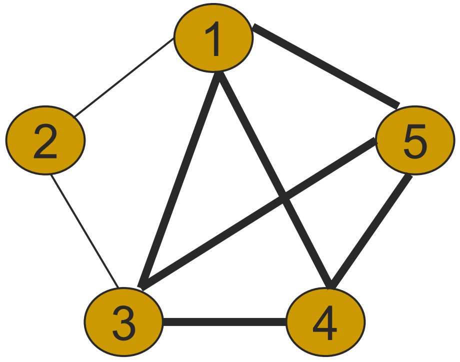


Graph

- *G* is a subgraph of *H* if the nodes of *G* are a subset of the nodes of *H*, and the edges of *G* are the edges of *H* on the corresponding nodes



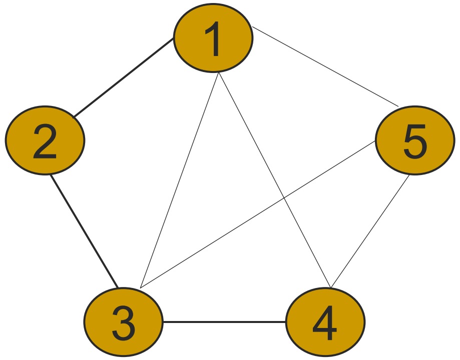
G



H

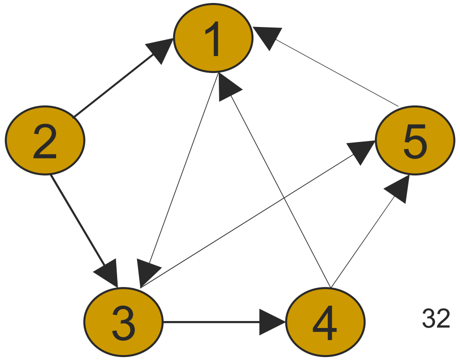
Graph

- A path is a sequence of nodes connected by edges.
  - 1-3-1-5
- A simple path is a path that doesn't repeat any nodes.
  - 1-2-3-4-5
- A path is a cycle if it starts and ends in the same node.
  - 1-5-3-4-5-1
- A simple cycle is a cycle that contains at least three nodes and repeats only the first and last nodes.
  - 1-2-3-4-5-1



Graph

- If every two nodes have a path between them, the graph is connected.
- A graph is a tree if it is connected and has no simple cycles.
  - Root, leaves.
- If the graph has arrows instead of lines, the graph is a directed graph.
  - Outdegree, indegree.
  - Directed path
  - Strongly connected



String and languages

- Alphabet is a finite set that is nonempty.
  - $\Sigma_1 = \{0,1\}$
  - $\Gamma_1 = \{a,b,c,...,x,y,z\}$
- The members of the alphabet are the symbols.
- String is a finite sequence of symbols from that alphabet, usually written next to one another and not separated by commas.
  - $w = w_1 w_2 ... w_n \quad |w| = n$
  - Reverse of w: $w^{\Re} = w_n w_{n-1} ... w_1$
  - Empty string : ε
  - String z is a substring of w if z apears consecutively within w.

String and languages

- $x = x_1 x_2 \ldots x_m, y = y_1 y_2 \ldots y_n$
  - Concatenation of x and y *is* $xy = x_1 x_2 \ldots x_m y_1 y_2 \ldots y_n$
  - $x^k = \overbrace{xx\ldots x}^{k}$
- The lexicographic ordering of strings is the same as the dictionary ordering, except that shorter strings precede longer strings.
  - lexicographic ordering over $\Sigma_1 = \{0,1\}$ is

    $$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$$
- A language is a set of strings.

Boolean logic

- Boolean logic is built around {TRUE, FALSE}.
- Boolean operation:
  - Negation (NOT)
    - $\neg 0=1,\ \neg 1=0$
  - Conjunction (AND)
    - $0\wedge 0=0,\ 0\wedge 1=0,\ 1\wedge 0=0,\ 1\wedge 1=1$
  - Disjunction (OR)
    - $0\vee 0=0,\ 0\vee 1=1,\ 1\vee 0=1,\ 1\vee 1=1$
  - Exclusive (XOR)
    - $0\oplus 0=0, 0\oplus 1=1, 1\oplus 0=1, 1\oplus 1=0,$
  - Equality($\leftrightarrow$)
    - $0\leftrightarrow 0=1, 0\leftrightarrow 1=0, 1\leftrightarrow 0=0, 1\leftrightarrow 1=1,$
  - Implication($\rightarrow$)
    - $0\rightarrow 0=1, 0\rightarrow 1=1, 1\rightarrow 0=0, 1\rightarrow 1=1,$

Boolean logic

- Equal relations:
  - $\mathrm{P} \vee \mathrm{Q} \qquad \neg(\neg P \wedge \neg Q)$
  - $\mathrm{P} \rightarrow \mathrm{Q} \qquad \neg P \vee Q$
  - $\mathrm{P} \leftrightarrow \mathrm{Q} \qquad (\mathrm{P} \rightarrow \mathrm{Q}) \wedge (\mathrm{Q} \rightarrow \mathrm{P})$
  - $\mathrm{P} \oplus \mathrm{Q} \qquad \neg(P \leftrightarrow Q)$
  - $\mathrm{P} \wedge (\mathrm{Q} \vee \mathrm{R}) \qquad (\mathrm{P} \wedge \mathrm{Q}) \vee (\mathrm{P} \wedge \mathrm{R})$
  - $\mathrm{P} \vee (\mathrm{Q} \wedge \mathrm{R}) \qquad (\mathrm{P} \vee \mathrm{Q}) \wedge (\mathrm{P} \vee \mathrm{R})$

Definitions, theorems and proofs

- Definition
- Theorem, lemma, corollary
- Proofs:
  - Proof by construction
  - Proof by contradiction
  - Proof by induction

Definitions, theorems and proofs

- **Definitions** describe the objects and notions that we used
  - **Precision** is essential to any mathematical definition
  - **Mathematical Statement** state the property of a defined object
- **Proof** is a convincing logical argument that a statement is true
  - In mathematics, argument must be airtight and **convincing in absolute sense**
  - Evidence plays no role and we demand proof beyond any doubt

Definitions, theorems and proofs

- **Theorem** is a mathematical statement proved true
- **Lemma** is a statement that we proved to assist in the proof of another more significant statement
- A theorem or its proof may allow us to conclude easily that another related statement is true and it is called **Corollary**

- The only way to determine the truth or falsity of a mathematical statement is with a mathematical proof.

Proof by construction

- **proof by construction**
  - Many theorem state that a particular type of object exists.
  - One way to prove the theorems that state a particular type of object exists is by demonstrating how to construct the object

Proof by construction

- Example 1:

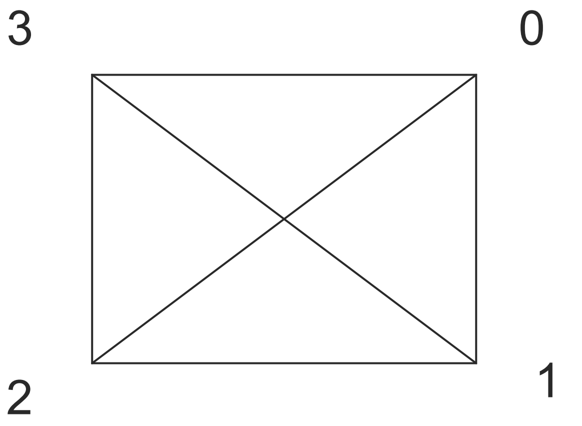
  Prove for each even number *n* greater than 2, there exists a 3-regular graph (a graph with degree 3 for all of its nodes) with *n* nodes.

- Proof:

  Let *n* be an even number and *n* > 2. Construct graph *G* =(*V*,*E*) with n nodes as follows. The set of nodes of *G* is *V*={0,1,…,*n*-1}, and the set of edges of *G* is the set

$$E = \{\{i, i+1\} \mid for\ \ 0 \le i \le n-2\} \cup \{\{n-1, 0\}\} \cup \{\{i, i+n/2\} \mid for\ \ 0 \le i \le n/2-1\}$$

Proof by construction

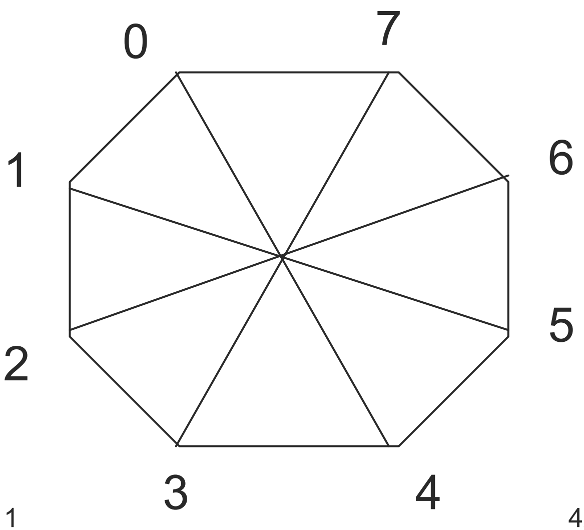


$\{\{i,i+1\} \mid for\ 0 \le i \le n-2\} \cup \{\{n-1,0\}\}$

These edges are the edges between adjacent pairs around the circle

$\{\{i,i+n/2\} \mid for\ 0 \le i \le n/2-1\}$

These edges are the edges between nodes on opposite sides of the circle.

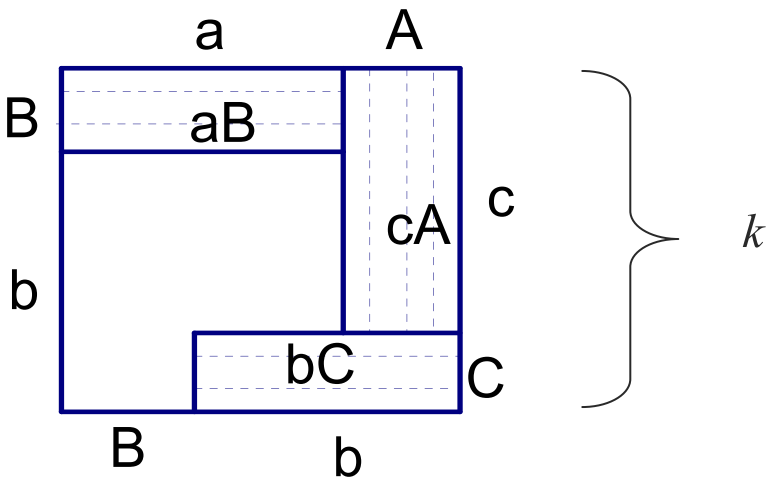


So this construction satisfies that every node in G has degree 3.

Proof by construction

- Example 2:

  Positive numbers a、b、c、A、B、C satisfy $a+A=b+B=c+C=k$, now prove $aB+bC+cA<k^2$



Proof by contradiction

- Proof by contradiction:
  - we assume that the theorem is false and then show that this assumption leads to an obviously false consequence, called a contradiction.

Notice the remark

Proof by contradiction

- Example:
  Prove $\sqrt{2}$ is irrational.

- Proof:
  Assume is a rational number and such that $\sqrt{2} = m/n$ where both m and n are integers.

  If both m and n are divisible by the same integer greater than 1, doing it does not change the value of the fraction. Now, at least one of m and n must be an odd number.

  Then, multiply both sides of the equation by n , $n\sqrt{2} = m$

  and square both sides to obtain $2n^2 = m^2$

Proof by contradiction

Because m^2 is 2 times the integer n^2, we know that m^2 is even. Therefore m is even. So we can write $m = 2k$ for some integer k. Thus,

$$2n^2=(2k)^2$$
$$n^2=2k^2$$

So n is even too. This is contradict to the predict that at least one of m and n must be an odd number. So $\sqrt{2}$ can't be written as a fraction. So $\sqrt{2}$ is irrational.

Proof by induction

- Proof by induction is an method used to show that all elements of an infinite set have a specified property.
- Take the infinite set to be the natural numbers, $N = \{1,2,3,\dots\}$, and say that the property is called P. Our goal is to prove that $P(k)$ is true for each natural number k.

Proof by induction

- Steps:
  - Basis

    Proves that $P(1)$ is true.
  - Induction step

    Proves that for each $i \geqslant 1$, if $P(i)$ is true, then so is $P(i+1)$.

Proof by induction

- Example:

  A monthly payments of home mortgages problem.

  Let P be the *principal*, the amount of the original loan, I be the yearly *interest rate*. Y be the *monthly payment*. M be the *monthly multiplier* ($M = 1 + I / 12$)

  P_t be the amount of loan outstanding after *the t*-th month.

  so we have $P_0 = P$ and $P_t = P_{t-1}M - Y$

- To prove: for each $t \geq 0$,

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

Proof by induction

- Proof:

  Basis:

  prove that the formula is true for $t = 0$, that is

  (remark: $M^0=1$)

$$P_0 = PM^0 - Y\left(\frac{M^0 - 1}{M - 1}\right) = P$$

This holds true because we have defined $P_0 = P$. So, we have proved the basis of the induction is true.

Proof by induction

- **Induction step:**

  For each $k \geq 0$ assume that the formula is true for $t = k$ *and* show that it is true for $t = k + 1$.

  we have
$$P_{k+1} = P_k M - Y = \left[PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)\right] M - Y$$
$$= PM^{k+1} - Y\left(\frac{M^{k+1} - M}{M - 1}\right) - Y\left(\frac{M - 1}{M - 1}\right)$$
$$= PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

  Thus the formula is correct for $t = k + 1$, which proves the theorem.

Theory of Computation

Lecture 2: Regular Language
(Part I)

Lecturer: Dr. Ng, Wing Yin

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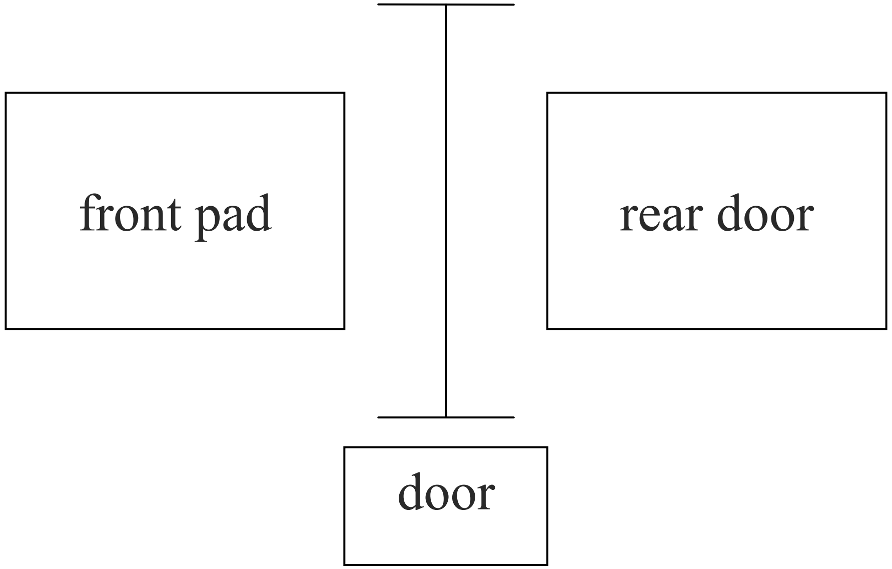
Content of Lecture 2

- Finite Automata
- Nondeterminism
- Regular Expressions
- Nonregular Languages

Review the example

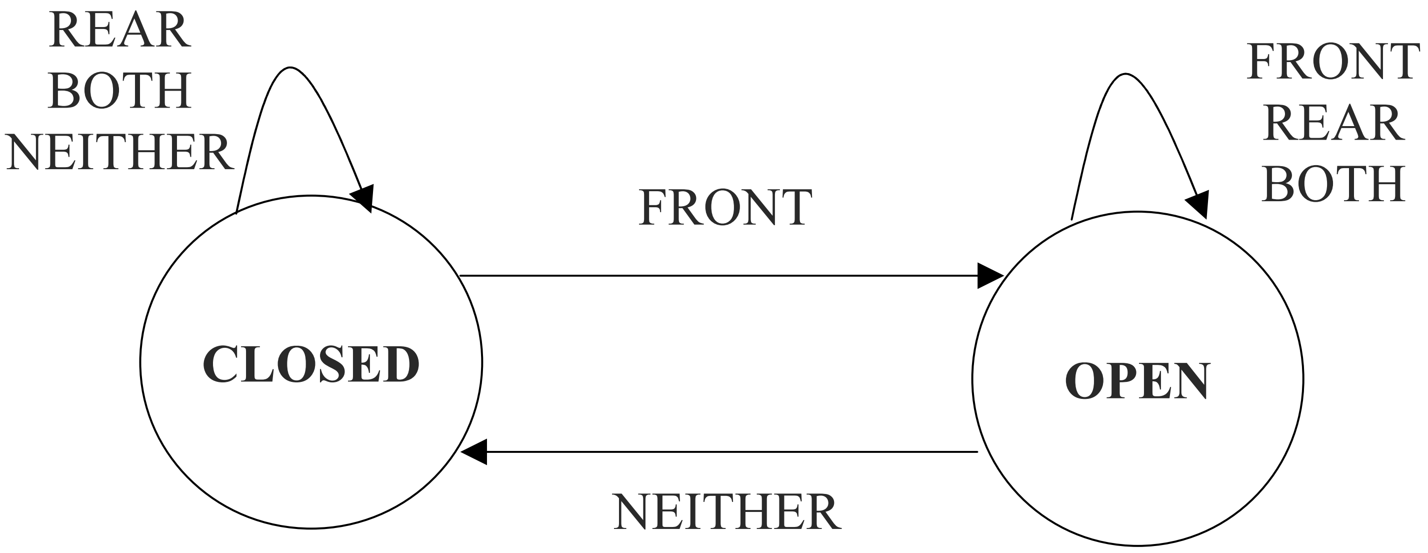
- The controller for an automatic door.
  - Door swing open when sensing a person is standing in front of it
  - Controller holds the door open long enough for the person to pass.
- Controller is in either of two states
  - Open
  - Closed
- Possible input conditions
  - Front
  - Rear
  - Both
  - Neither
- The controller moves from state to state, depending on the input it receives.

The controller for an automatic door



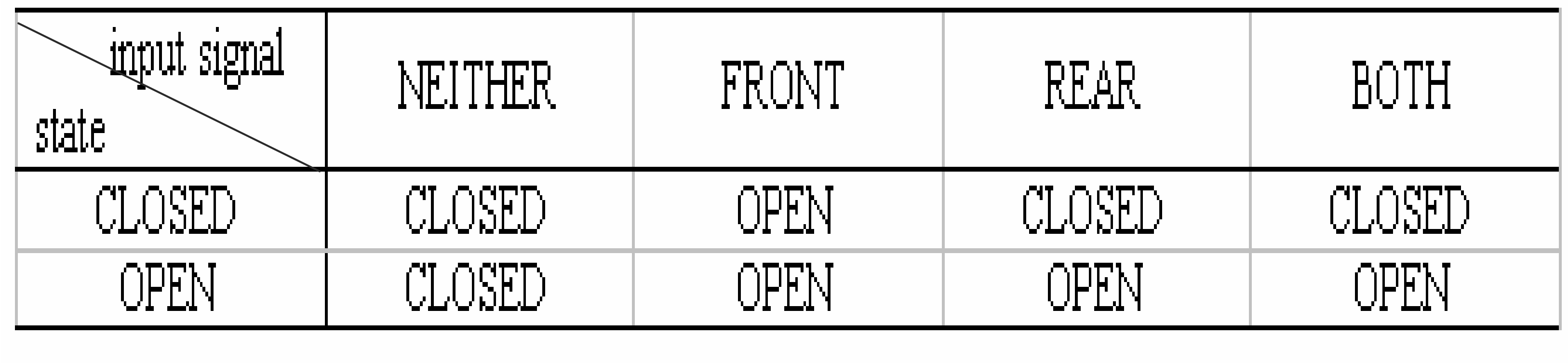
Top view of the system

The controller for an automatic door



State diagram

The controller for an automatic door



input signal state	NEITHER	FRONT	REAR	BOTH
CLOSED	CLOSED	OPEN	CLOSED	CLOSED
OPEN	CLOSED	OPEN	OPEN	OPEN

Transition table of the door

The controller for an automatic door

- Example:
  - Start in state **closed**
  - Receive the series of input signals FRONT, REAR, NEITHER, FRONT, BOTH, NEITHER, REAR and NEITHER,
  - Controller will go through the series of states CLOSED (starting), OPEN, OPEN, CLOSED, OPEN, OPEN, CLOSED, CLOSED, AND CLOSED

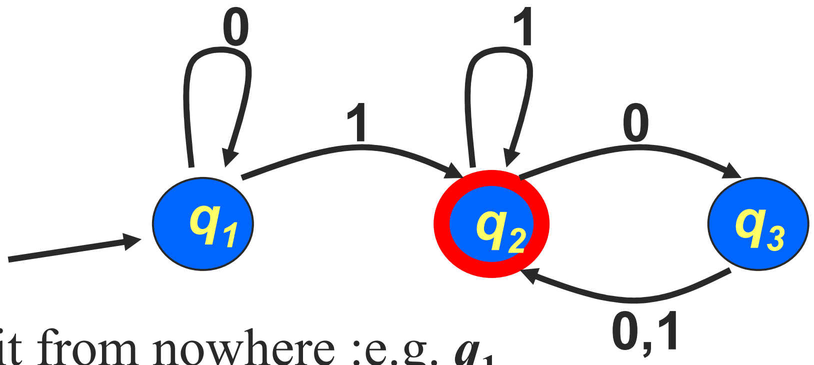
Finite Automata (FA)

- The daily devices that can be demonstrated by FA
  - Accutron (electronic watch):
    - e.g: there four keys on the Accutron, then there are four possible input, and there are some states.
  - Recorder: the transform keys:
    - ►. ►►. ◄◄. O. ■ are the possible input.
  - TV telecontrol
    - the telecontrol of multilayer menu.
  - Elevator

Finite Automata

- Briefly, FA:
  - Current State + Input → Next State
- As a computer, FA:
  - can read
  - can not write
  - no memory (only register to store states and input)
  - can not perform reasoning
  - no imagination
  - That is, computation here is to judge the last state.

Finite Automata



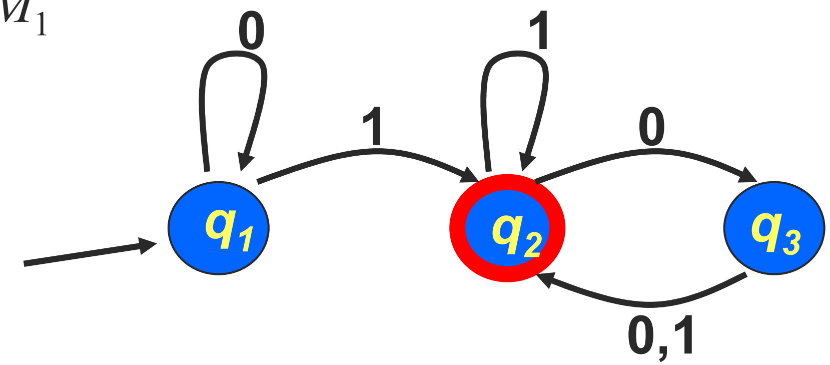
- **State diagram:**
  - We use M_1 to denote a FA (NFA: uses N_1)
- **States**:
  - 3 states: q_1, q_2, q_3
- **Start state**:
  - indicated by the arrow pointing at it from nowhere :e.g. q_1
- **Accept state**:
  - the one with a double circle. e.g.: q_2
- **Inputs**:
  - Strings: a finite list of symbols from alphabet $\sum$
- **Transitions**
  - The arrows going from one state to another
- **Output:**
  - **Accept** or **reject**

Finite Automata

- The process procedure:
  - Begins in M_1's start state
  - Repeat until the last symbol:
    - Read one symbol from the input string from left to right in turns.
    - M_1 moves from one state to another along the transition that has that symbol as its label.
  - M_1 produces its output, the output is accept if M_1 is now in an accept state and reject if it is not.

Finite Automata

- E.g.: We input a string 1101 to the FA M_1



- The processing proceeds:
  - Start in state $\boldsymbol{q_1}$.
  - Read 1, follow transition from $\boldsymbol{q_1}$. **to** $\boldsymbol{q_2}$;
  - Read 1, follow transition from $\boldsymbol{q_2}$. **to** $\boldsymbol{q_2}$;
  - Read 0, follow transition from $\boldsymbol{q_2}$. **to** $\boldsymbol{q_3}$;
  - Read 1, follow transition from $\boldsymbol{q_3}$. **to** $\boldsymbol{q_2}$.
  - **Accept** because M_1 is in an accept state q_2 at the end of the input.

- In fact, M_1 accepts any string that ends with a 1, and also any string that ends with an even number of 0s following the last 1.

Formal Definition of FA

- A finite automaton is a 5-tuple $(Q, \sum, \delta, q_0, F)$, where
  1. Q is a finite set called the states,
  2. $\sum$ is a finite set called the alphabet,
  3. $\delta : Q \times \sum \rightarrow Q$ is the transition function
  4. $q_0 \in Q$ is the start state, and
  5. $F \subseteq Q$ is the set of accept states.

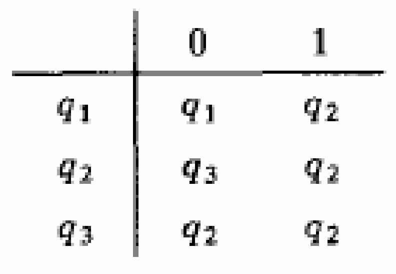
- Transition function is denoted by δ
  - $\delta(x, 1) = y$ means there is a arrow from a state x to a state y labeled with the input symbol 1.

Formal Definition of FA

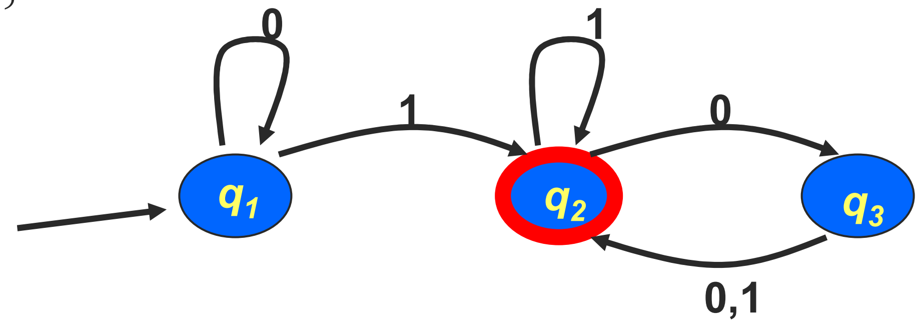
- E.g. we formally describe M_1 as following:

$M_1 = (Q, \sum, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3\}$
2. $\sum = \{ 0, 1 \}$
3. δ is described as
4. q_1 is the start state, and
5. $F = \{q_2\}$



	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2



Formal Definition of FA

- If A is the set of all strings that machine M accepts, we say that A is the language of machine M , *or* M recognizes A.

$$L(M)=A$$

- If the machine accepts no strings, it still recognizes one language:

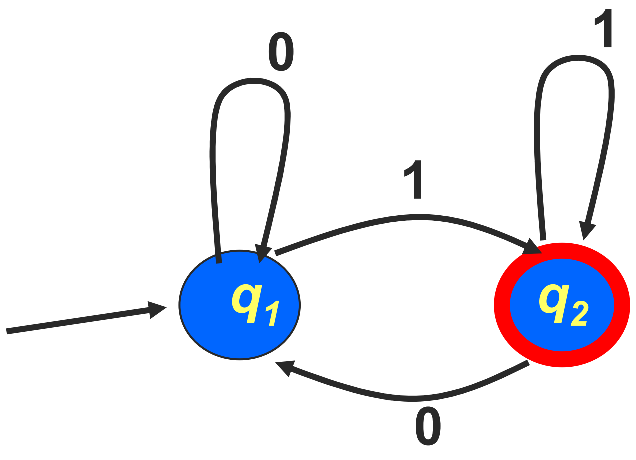
  empty language Φ

- E.g.:
  - $A = \{w|\ w$ contains at least one 1 and also an even number of 0s follow the last 1 $\}$, Then

    $L(M)=A$, or M recognizes A.

Finite Automata

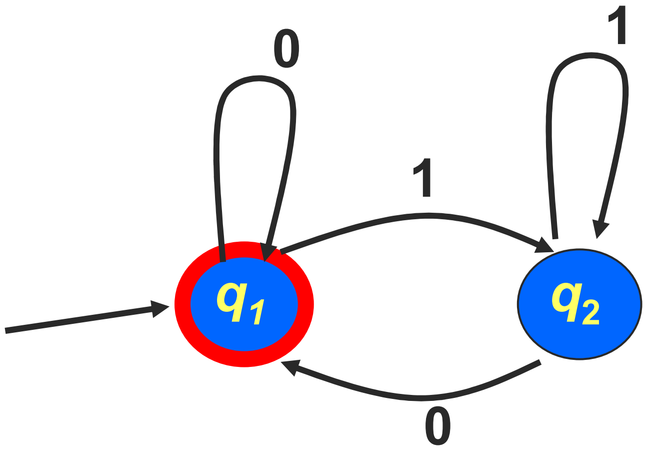
- On class exercises:
  - Find the transition tables and languages that FAs accept



Exercise 1

Finite Automata

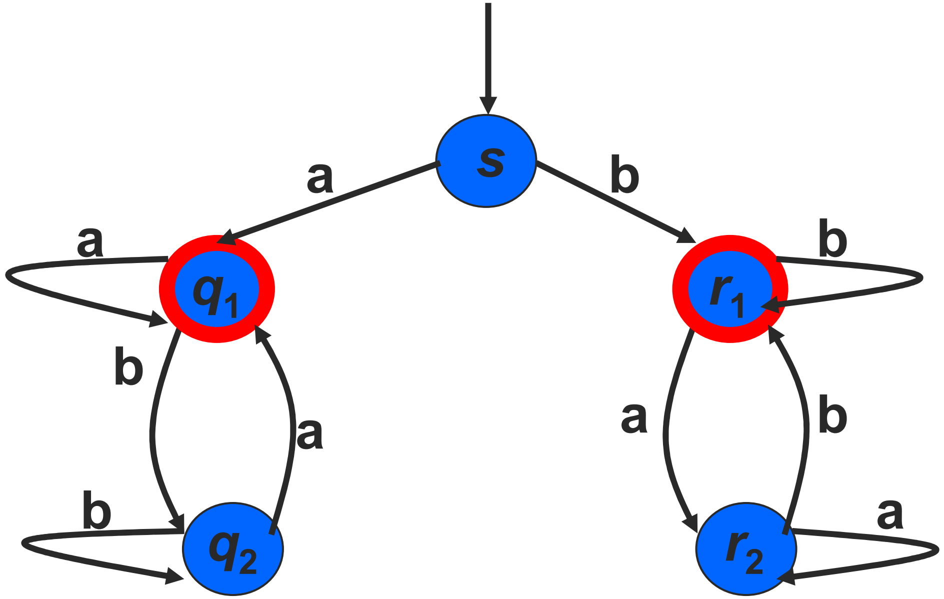
- On class exercise:



Exercise 2

Finite Automata

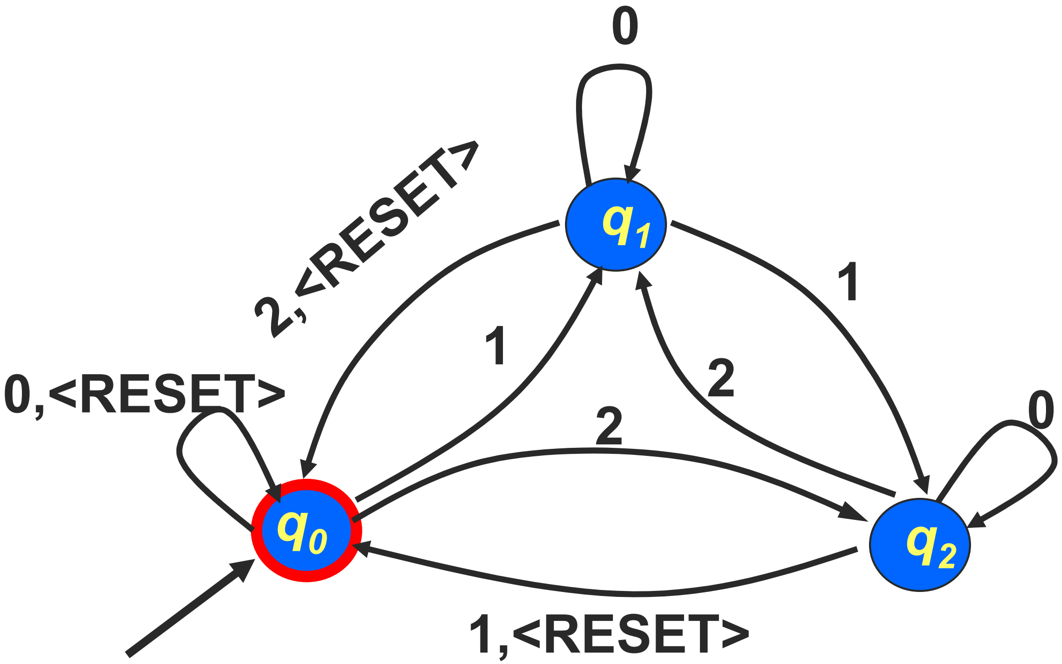
- On class exercise:



Exercise 3

Finite Automata

- On class exercise:



Exercise 4

Formal Definition of Computation

- Let $M = (Q, \sum, \delta , q_0, F)$ be a FA and let $w=w_1w_2 \ldots w_n$ be a string where each w_i is a member of the alphabet $\sum$. Then M accepts w if a sequence of states $r_0,r_1, \ldots r_n$ in Q exists with three conditions:,
  - ○ $r_0 = q_0$
  - ○ $\delta\ (r_i, w_{i+1}) = r_{i+1}$, for $i = 0,\ldots,n$-1, and
  - ○ $r_n \in F$
- So we can say that M recognizes language A if

$$A=\{w \mid M \text{ accepts } w\}$$

- A language is called a regular language if some finite automaton recognizes it.

Designing Finite Automata

- To design a FA, put yourself in the place of the machine you are trying to design, see how you would do to perform the machine's task.
- So pretending yourself to be the automaton, receive an input string, see the symbols one by one, and then must decide whether the string seen so far is in the language. You (FA) don't know when the end of the string is coming, so you must always be ready with the answer.

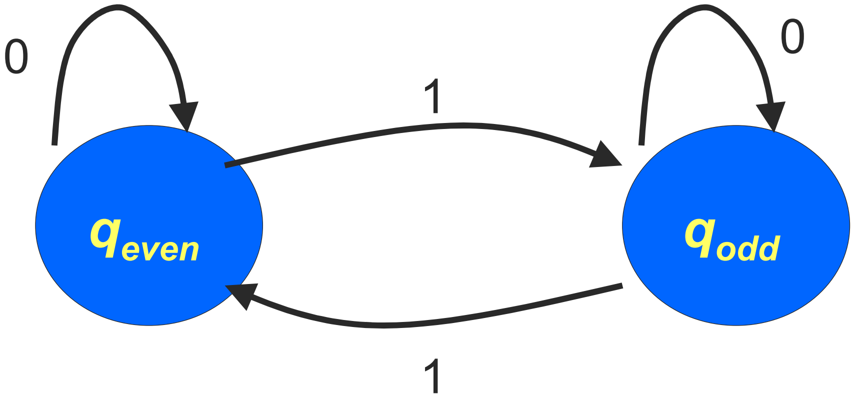
Designing Finite Automata

How to design a FA:

- $A = \{ w \mid w$ composes only 1 or 0, and has odd number of 1s$\}$, design a FA M to recognizes A.
  - Pretending to be the automaton, get an input string of 0s and 1s symbol by symbol.
  - Remember whether the number of 1s seen so far is even or odd (two possibilities), if you read a 1, flip the answer, a 0, leave the answer as is.
  - Assign a state to each of the possibilities.

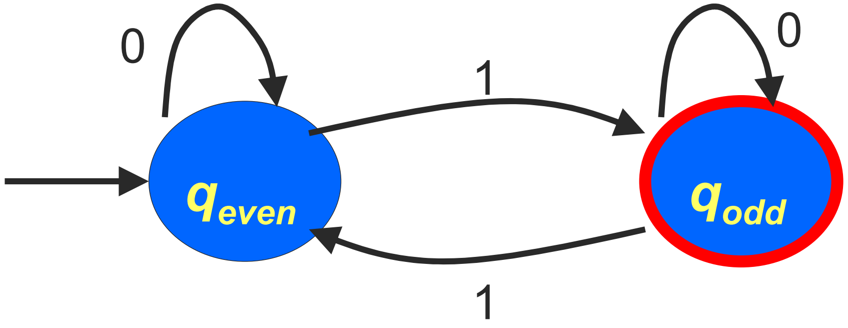
Designing Finite Automata

- Assign the transitions by seeing how to go from one possibility to another upon reading a symbol.



Designing Finite Automata

- Determine the start state that is get no symbols (ε), so that is zero 1, so the start state is q_{even}
- Determine the accept states to be those corresponding to possibilities where you want to accept the input string, so the accept states is q_{ood}



Designing Finite Automata

- On class exercise:

Design a finite automaton *E* to recognize the regular language of all strings that contain the string 001 as a substring.

Regular operations

	Basic objects	Tools(Operations for manipulating)
Arithmetic	Numbers	e.g. +, /
Computation theory	Languages	e.g. U, ◦ ,*

- Let A and B be languages. We define the regular operations union, concatenation, and star as follows.
  - **Union:** $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$
  - **Concatenation**: $A \circ B = \{ xy \mid x \in A \text{ and } y \in B \}$
  - **Star:** $A^* = \{x_1 x_2 \ldots x_n \mid k \geqslant 0 \text{ and each } x_i \in A \}$

Note:When $k = 0$, for * operation, empty string $\varepsilon \in A^*$

Regular operations

- Example:

  ∑={ a, b, … , z }, A = { red, blue },B = { dog, bird, cat }

  - $A \cup B$ = { red, blue, dog, bird, cat }
  - $A \circ B$ = { reddog, redbird, redcat, bluedog, bluebird, bluecat }
  - A^* = { ε, red, blue, redred, redblue, bluered, blueblue, redredred, redredblue, redbluered, redblueblue, … }

Regular operations

- A collection of objects is closed under some operation if applying that operation to members of the collection returns an object still in the collection.
- The collection of regular languages is closed under all three of the regular operations: union, concatenation and star.

Regular operations

- Theorem: the class of regular languages is closed under the union operation.
- Proof:
  - We prove that if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$
  - Suppose M_1 recognizes A_1 and M_2 recognizes A_2, we demonstrate a FA M to recognize $A_1 \cup A_2$. This is a proof by construction.

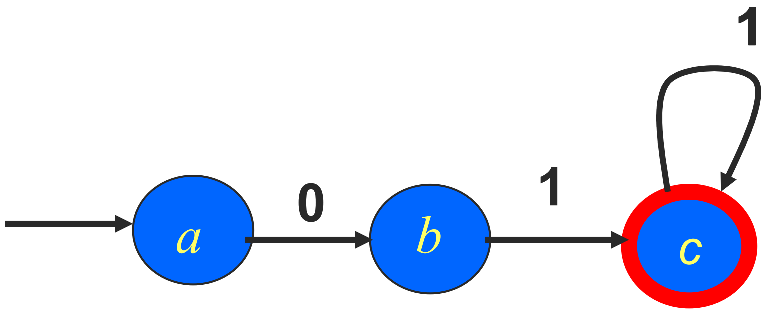
Regular operations

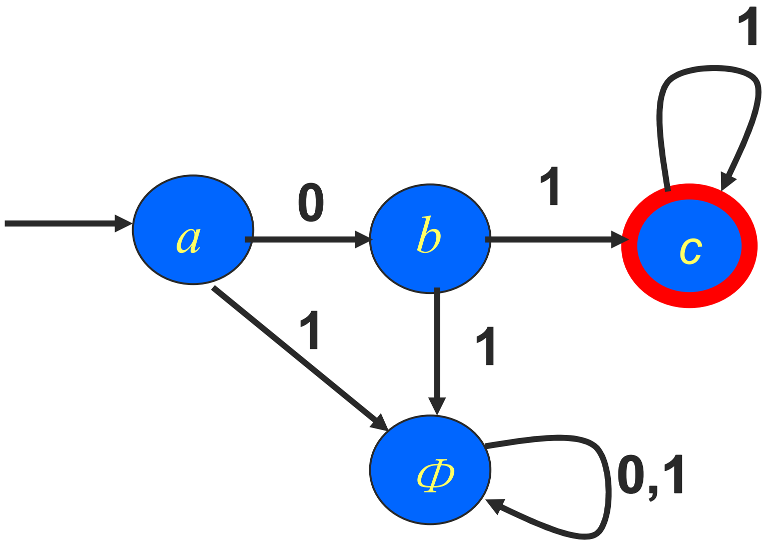
- $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and $M = (Q, \Sigma, \delta, q_0, F)$, thus

1. $Q = Q_1 \times Q_2 = \{ (r_1, r_2) \mid r_1 \in Q_1$ and $r_2 \in Q_2\}$
2. We assume M_1 and M_2 have the same input alphabet Σ , so for M, the alphabet is still Σ
3. For each $(r_1, r_2) \in Q$ and each a $\in \Sigma$, let $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta(r_2, a))$
4. $q_0 = (q_1, q_2)$
5. $F = (F_1 \times Q_2) \cup (Q_1 \times F_2) = \{ (r_1, r_2) \mid r_1 \in F_1$ or $r_2 \in F_2\}$

Regular operations

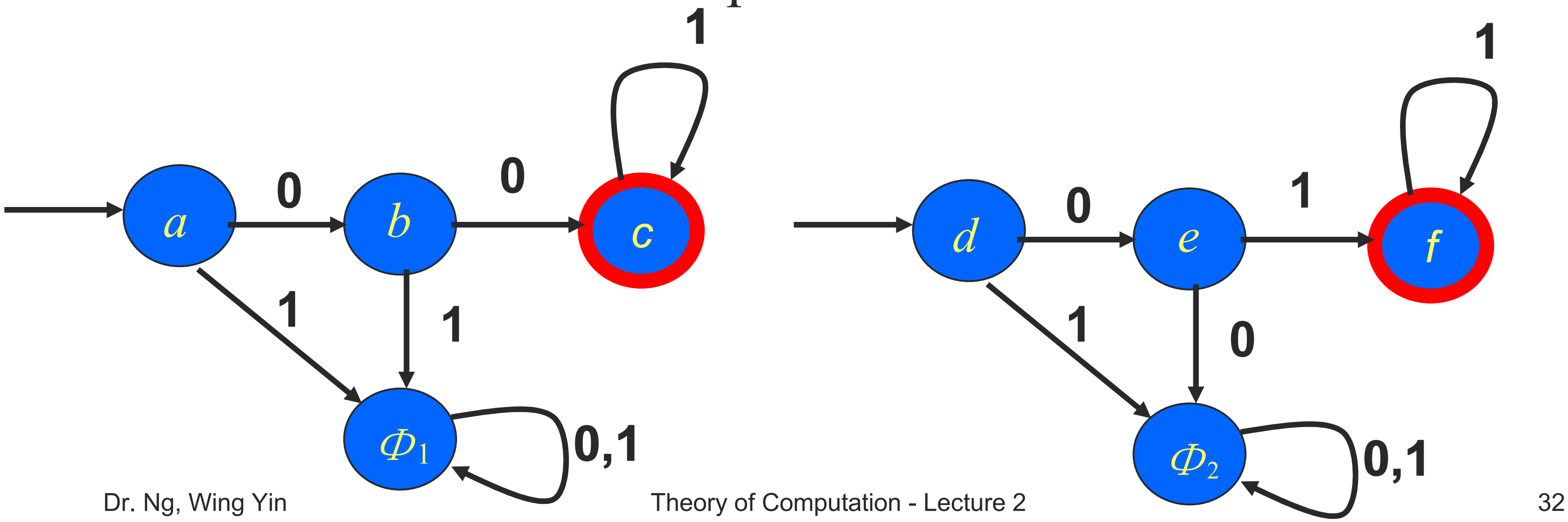
- To construct the union operation, we introduce the Φ (a null state) where we omit in the transition.



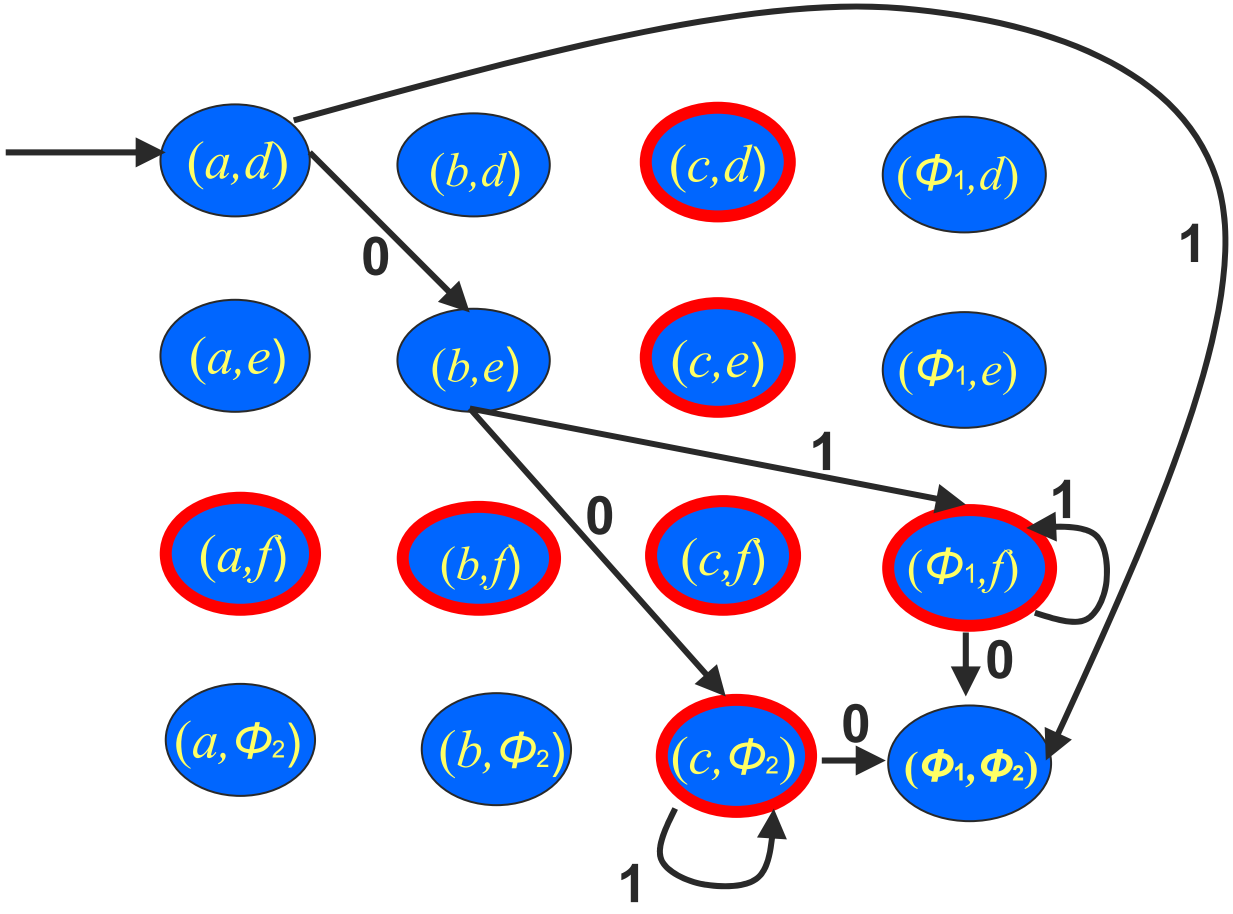


Regular operations

- A_1={ language that the strings are 00 followed with any number of 1s}, construct M_1 accepts A_1
- A_2={ language that the strings are 01 followed with any number of 1s}, construct M_2 accepts A_2
- Now construct M accepts $A_1 \cup A_2$



Regular operations

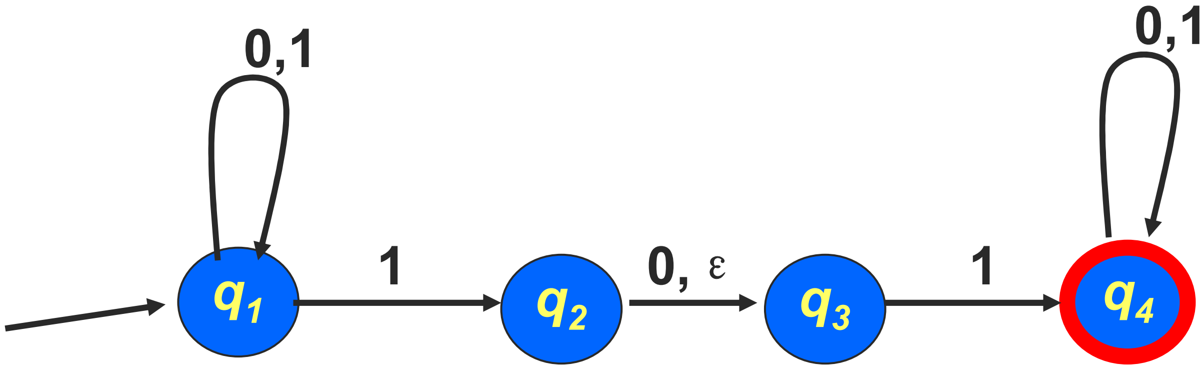


Regular operations

- Theorem: the class of regular languages is closed under the concatenation operation.
- Proof:
  - Now, instead of constructing automaton M to accept its input if either M_1 or M_2 accept, it must accept if its input can be broken into two pieces, where M_1 accepts the first piece an M_2 accepts the second piece. But M does not need to know where to break its input.
  - To solve this problem , we introduce a new technique called nondeterminism.

Nondeterminism

- Deterministic and nondeterministic
  - we call it deterministic computation, if we know what the next state will be. While we have several choices for the next state at any point in a nondterministic computation.
  - So we call the FA that performs deterministic computation a DFA, and the FA that performs nondeterministic computation a NFA.
  - Nondeterminism is a generalization of determinism, so do the NFA and DFA
- An example of nondeterministic finite automaton (NFA)



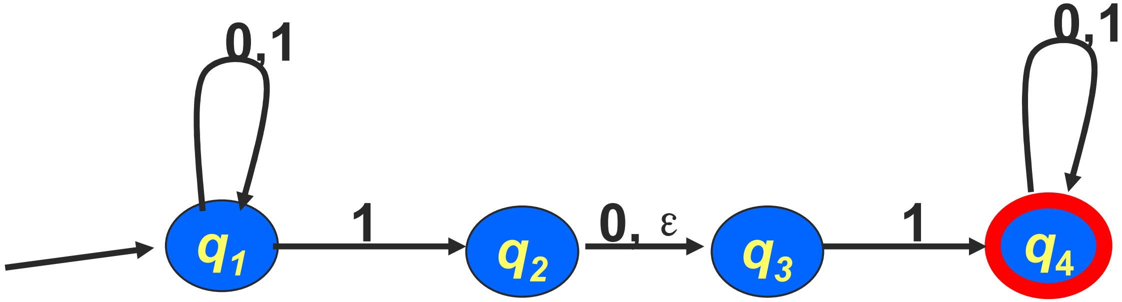
Nondeterminism

- The difference between DFA and NFA
  1. Every state of a DFA always has exactly one exiting transition arrow for each symbol in the alphabet.

     A state of a NFA may have zero, one or many exiting arrows for each alphabet symbol.
  2. In a DFA, labels on the transition arrows are symbols from the alphabet, while ε is added in a NFA.

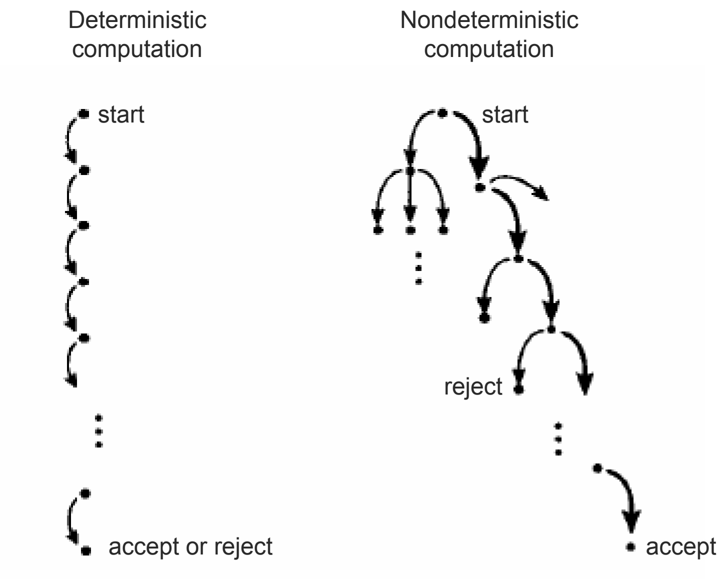
Nondeterminism

- NFA running mechanism:
  - When NFA reads a symbol and is at a state with multiple out-going arrows with this symbol, NFA **splits into multiple copies of itself**. **Remember these copies and follows all the possibilities in parallel.**
    - If there are subsequent choices, the machine splits again.
    - If the next input symbol doesn't appear on any of the arrows exiting the state, that copy of the machine dies, along with the branch of the computation associated with it.
    - If **any one** of these copies of the machine is in an accept state at the end of the input, the NFA accepts the inputs string.
    - If a state with an ε **arrow** is encountered, the machine **splits into multiple copies without reading any input**, one following each of the existing ε arrow and one staying at the current state.



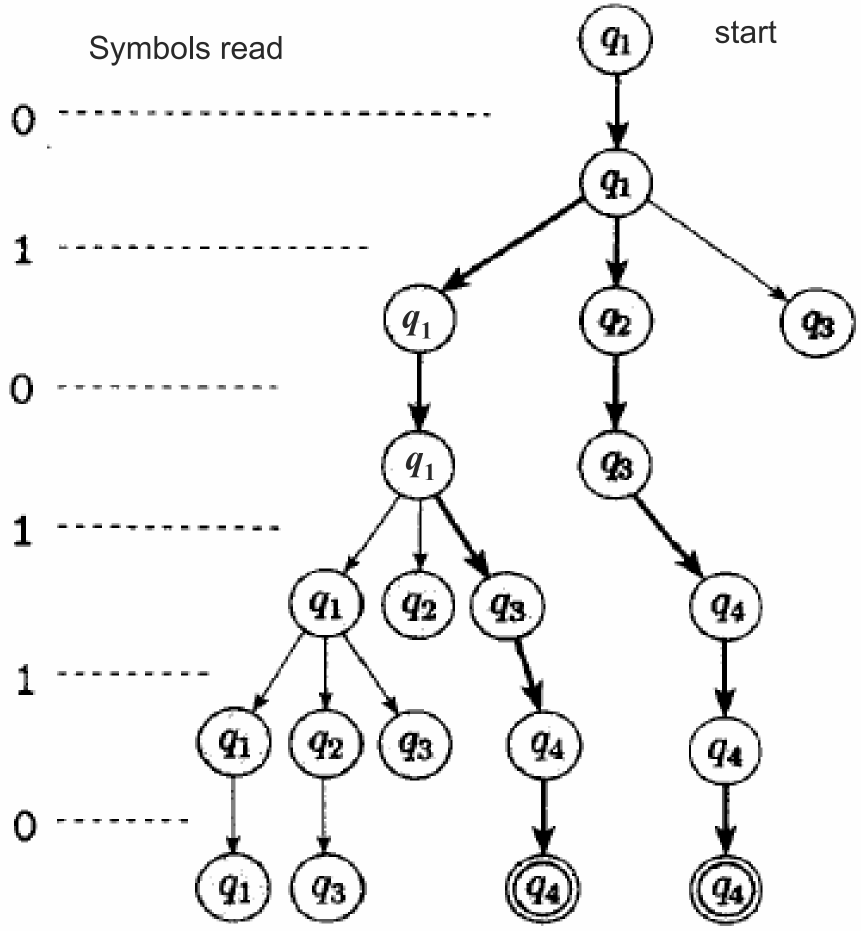
Nondeterminism

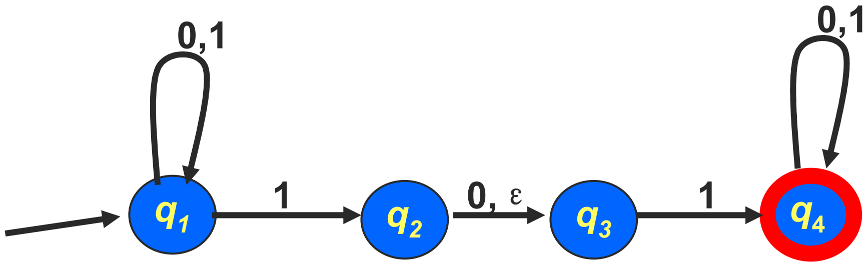
- NFA running mechanism:
  - Tree of possibilities.
    - The root of the tree corresponds to the start of the computation.
    - Every branching point in the tree corresponds to a point in the computation at which the machine has multiple choices.
    - The machine accepts if at least one of the computation branches ends in an accept state.



Nondeterminism

- For input 010110



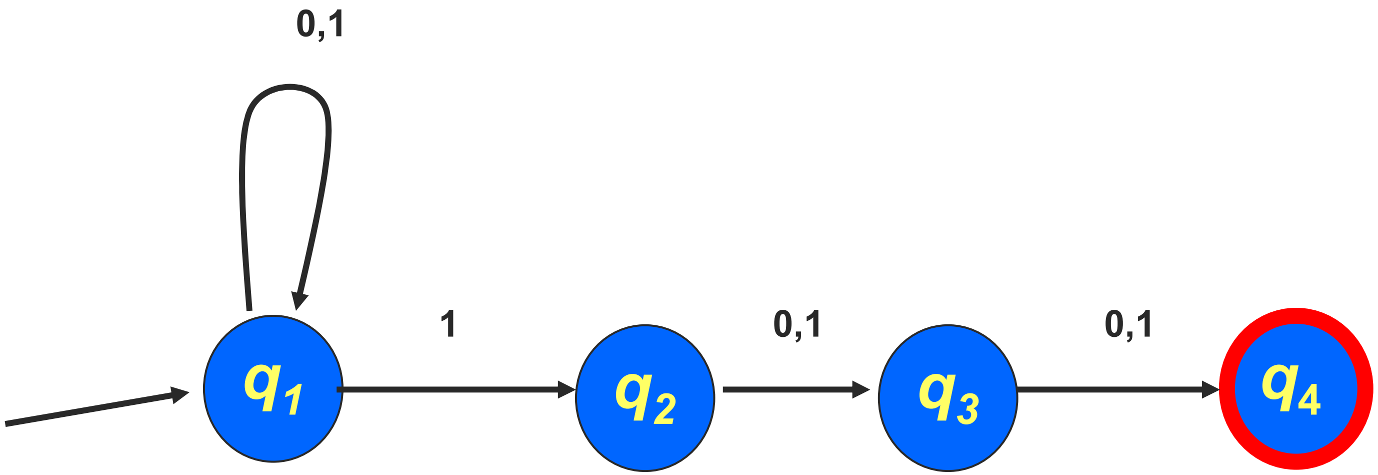


Nondeterminism

- Every NFA can be converted into an equivalent DFA.
  - Constructing NFAs is sometimes easier than directly constructing DFAs.
  - An NFA may be much smaller than its deterministic counterpart, or its functioning may be easier to understand.

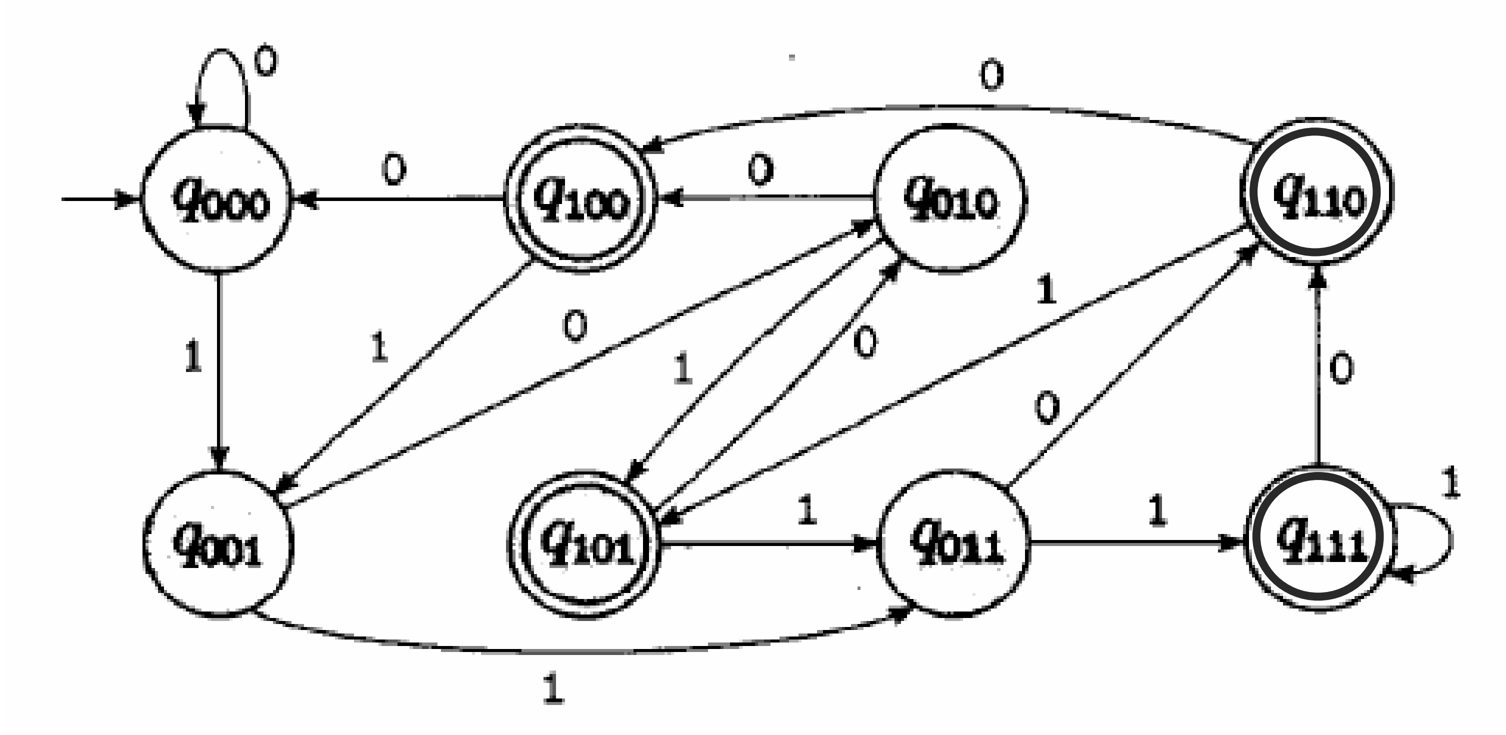
Nondeterminism

- Examples:
  - *A* is the language consisting of all strings over {0,1} containing a 1 in the third position from the end. The following four-state NFA N_2 recognizes *A*.



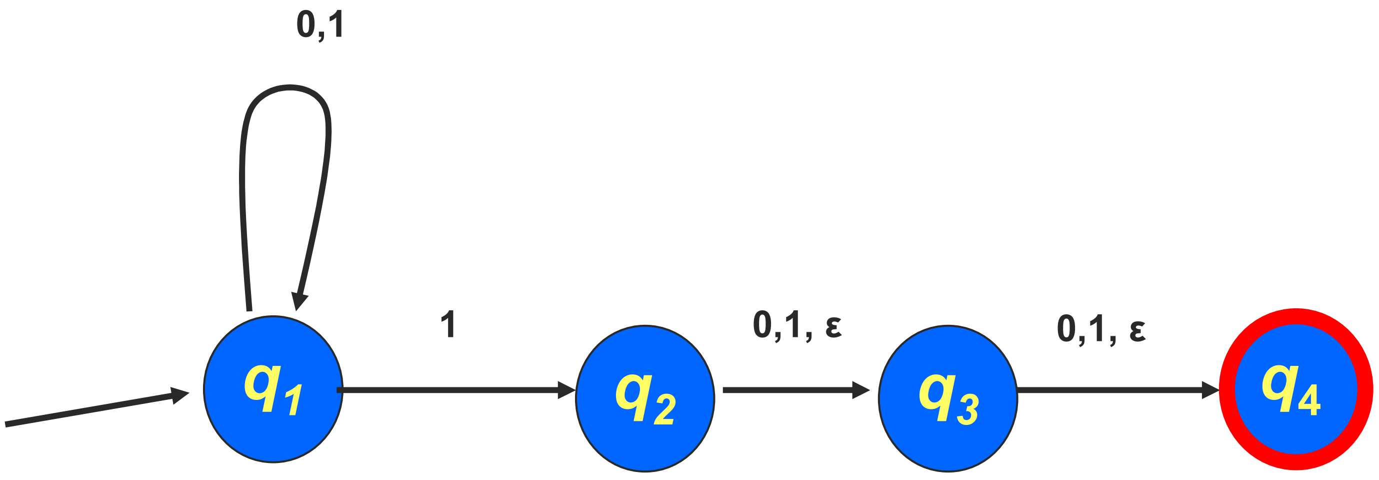
Nondeterminism

- The following eight states DFA recognizes A too.



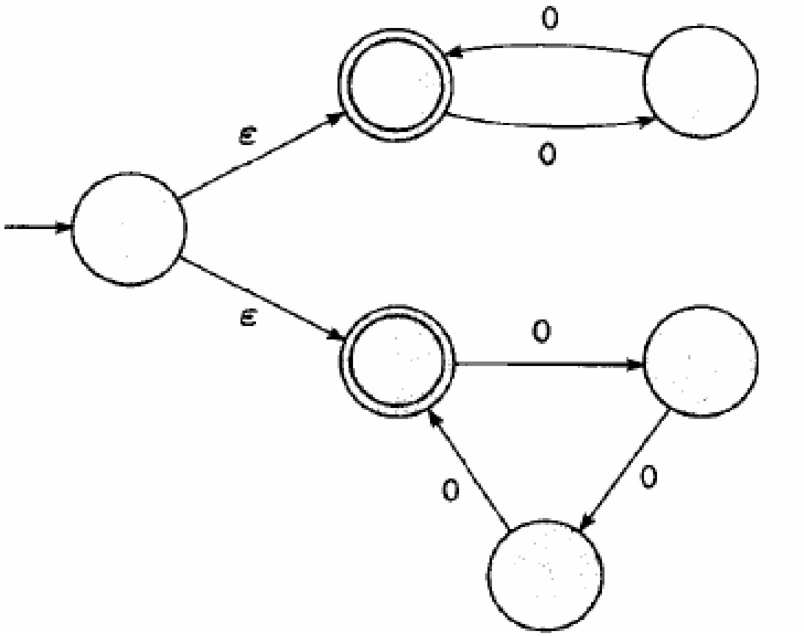
Nondeterminism

- By comparing diagrams from previous pages, we find that understanding the function of a NFA is much easier.
- What language could the following NFA recognize and what is the corresponding DFA?



Nondeterminism

- A_3 is the language that has an input alphabet {0} and accepts all strings of the form 0^k where k is a multiple of 2 or 3. The following NFA N_3 recognizes A_3.



  - We can see the convenience of using ε arrows.

Formal definition of NFA

- Let **P**(Q) be the power set of Q which is the collection of all subsets of Q, $\sum_{\varepsilon} = \sum \cup \{ \varepsilon \}$
- A nondeterministic finite automaton is a 5-tuple (Q, $\sum$, δ , q_0, F) , where
  1. Q is a finite set of states,
  2. $\sum$ is a finite alphabet,
  3. $\delta : Q \times \sum_{\varepsilon} \rightarrow$ **P**(Q) is the transition function
  4. $q_0 \in Q$ is the start state, and
  5. $F \subseteq Q$ is the set of accept states.

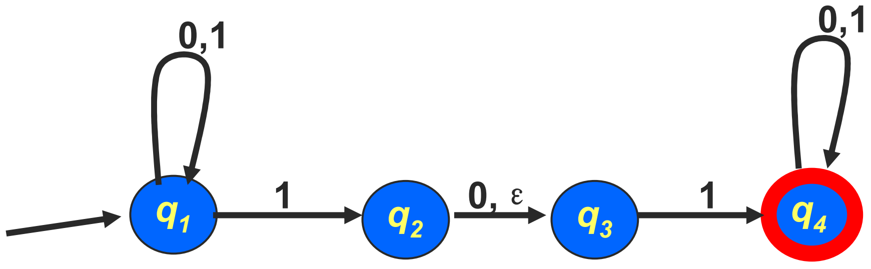
Formal definition of NFA

- Example:

$N_1 = (Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$
2. $\Sigma = \{0, 1\}$
3. δ is described as
4. q_1 is the start state, and
5. $F = \{q_4\}$

δ	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	Φ
q_2	$\{q_3\}$	Φ	$\{q_3\}$
q_3	Φ	$\{q_4\}$	Φ
q_4	$\{q_4\}$	$\{q_4\}$	Φ



Formal definition of computation of NFA

- Let $N = (Q, \sum, \delta, q_0, F)$ be a NFA and let $w=y_1y_2 \ldots y_n$ be a string where each y_i is a member of the alphabet $\sum_\varepsilon$. Then N accepts w if a sequence of states $r_0, r_1, \ldots r_m$ in Q exists with three conditions:,
  - $r_0 = q_0$
  - $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0,\ldots,m-1$, and
  - $r_m \in F$
- So we can say that N recognizes language A if

$$A=\{w \mid N \text{ accepts } w\}$$

Equivalence of NFAs and DFAs

- Two machines are equivalent if they recognize the same language.

- Theorem:
  - Every nondeterministic finite automaton has an equivalent deterministic finite automaton.
- Proof:
  - Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA recognizing some language A. We construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$ recognizing A.

Equivalence of NFAs and DFAs

- Consider the case where N has no ε arrow.
  - $Q' = \mathbf{P}(Q)$
  - $\delta'(R, a) = \{ q \in Q \mid q \in \delta(r, a)$ for some $r \in R\}$

    $= \quad \delta'(R,a) = \bigcup_{r \in R} \delta(r,a)$
    - R is a state of M, it is also a set of states of N.
    - When M reads a symbol a in state R, it shows where a takes each state in R.
    - Because each state may go to a set of states, we take the union of all these sets.
  - $q_0' = \{q_0\}$
  - $F' = \{ R \in Q' \mid R$ contains an accept state of $N \}$

Equivalence of NFAs and DFAs

- Consider the ε arrows.
  - For $R \subseteq Q$ (or $R \in \mathbf{P}(Q)$), let
  - $E(R)$={ q | q can be reached from R by traveling along 0 or more ε arrows}
  - Change
    - $\delta'(R, a) = \{ q \in Q \mid q \in \mathrm{E}(\delta(r, a)) \text{ for some } r \in R\}$
    - $q_0' = \mathrm{E}(\{q_0\})$

- We have now completed the construction of the DFA *M* that simulates the NFA *N*.
- The construction of *M* obviously works correctly. At every step in the computation of *M* on an input, it clearly enters a state that corresponds to the subset of states that *N* could be in at that point. Thus, our proof is complete.

Equivalence of NFAs and DFAs

- Corollary

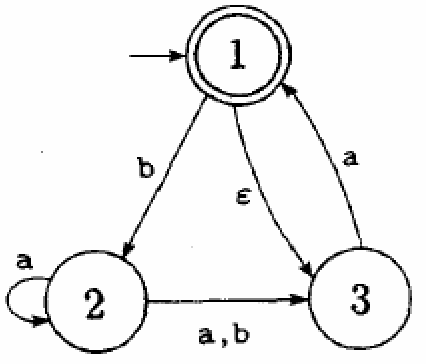
  A language is regular if and only if some nondeterministic finite automaton recognizes it.

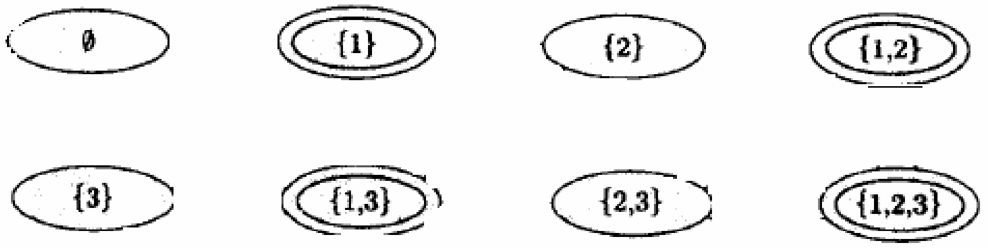
Equivalence of NFAs and DFAs

- Example
- Given the NFA N_4 as following, now construct the equivalent DFA D.
- Solution:

  $N_4 = (Q, \{a,b\}, \delta, 1, \{1\})$, where $Q=\{1,2,3\}$

  - The states of D is $\mathbf{P}(Q)$:
    - $\{ \Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
  - The start state is $E(\{1\}) = \{1,3\}$
  - The accept state is $\{ \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\} \}$



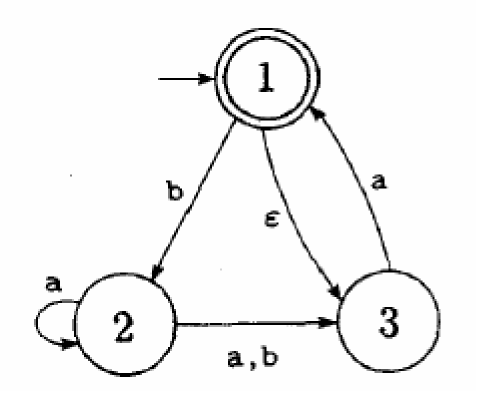


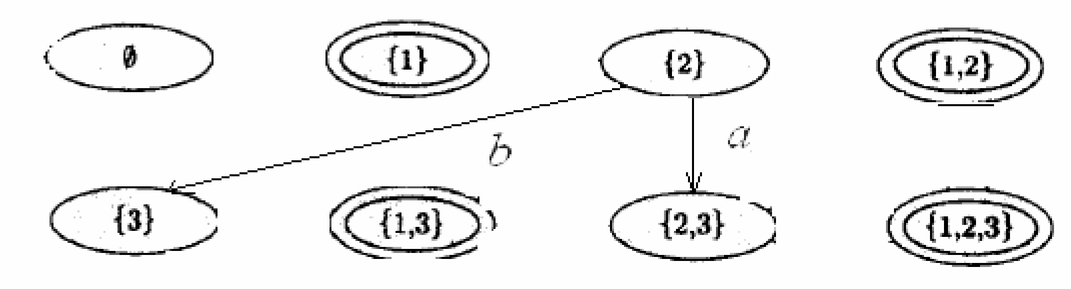
Equivalence of NFAs and DFAs

- Determine the transition function
  - {2} with a input a,
    - by $\delta'(R, a) = \{ q \in Q \mid q \in E(\delta(r, a)) \text{ for some } r \in R\}$

      $\delta'(\{2\}, a) = \{2\} \cup \{3\} = \{2, 3\}$
  - {2} with a input b,

    $\delta'(\{2\}, b) = \{3\}$



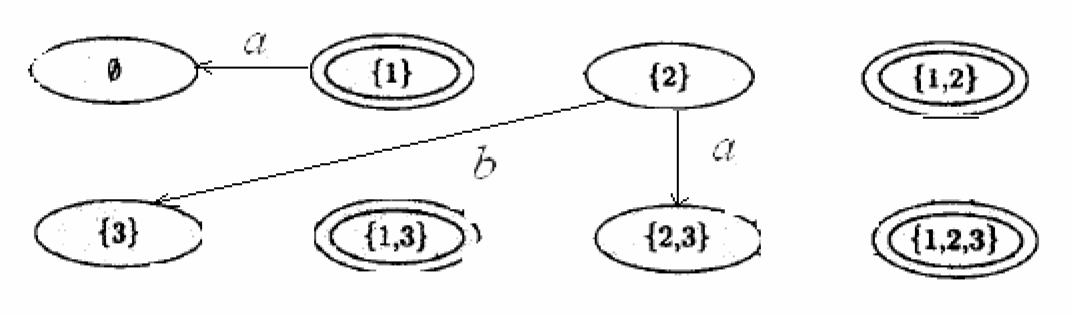


Equivalence of NFAs and DFAs

-

with a input *a*

$$\delta'(\{1\}, a) = \Phi$$



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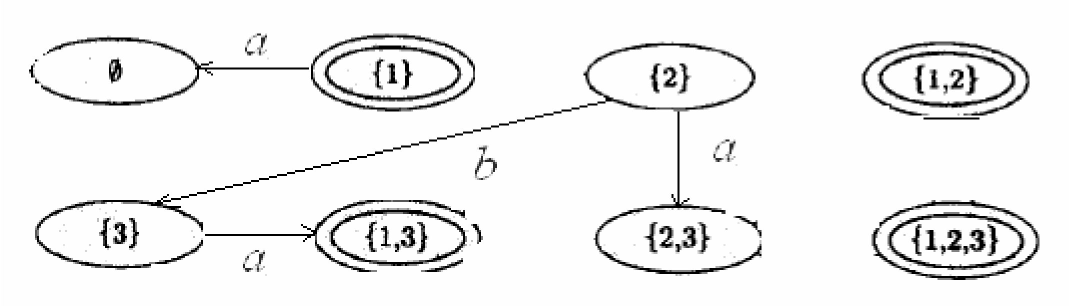
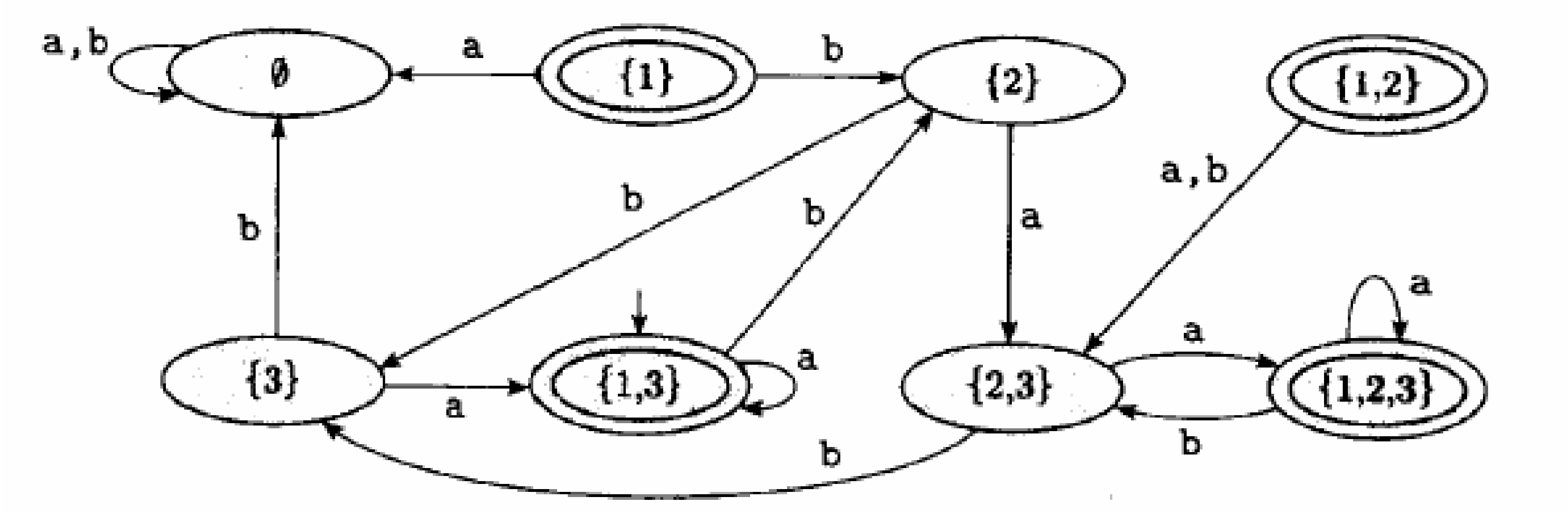
with a input *a*

$$\delta'(\{3\}, a) = E(\{1\}) = \{1, 3\}$$



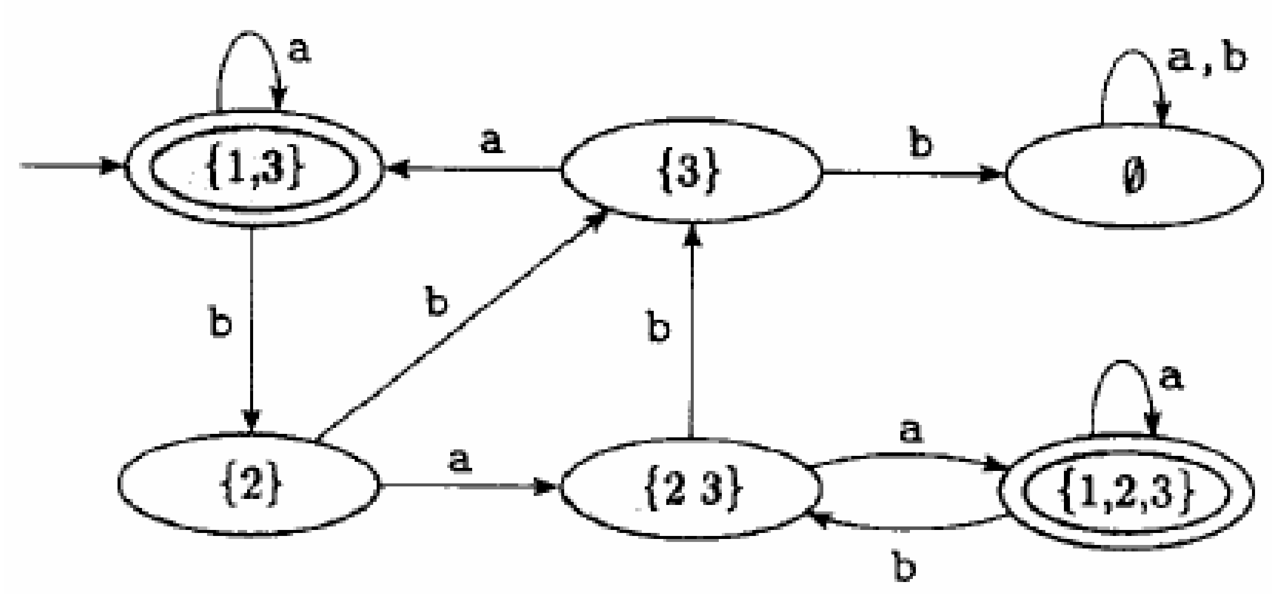
Equivalence of NFAs and DFAs

- Finally



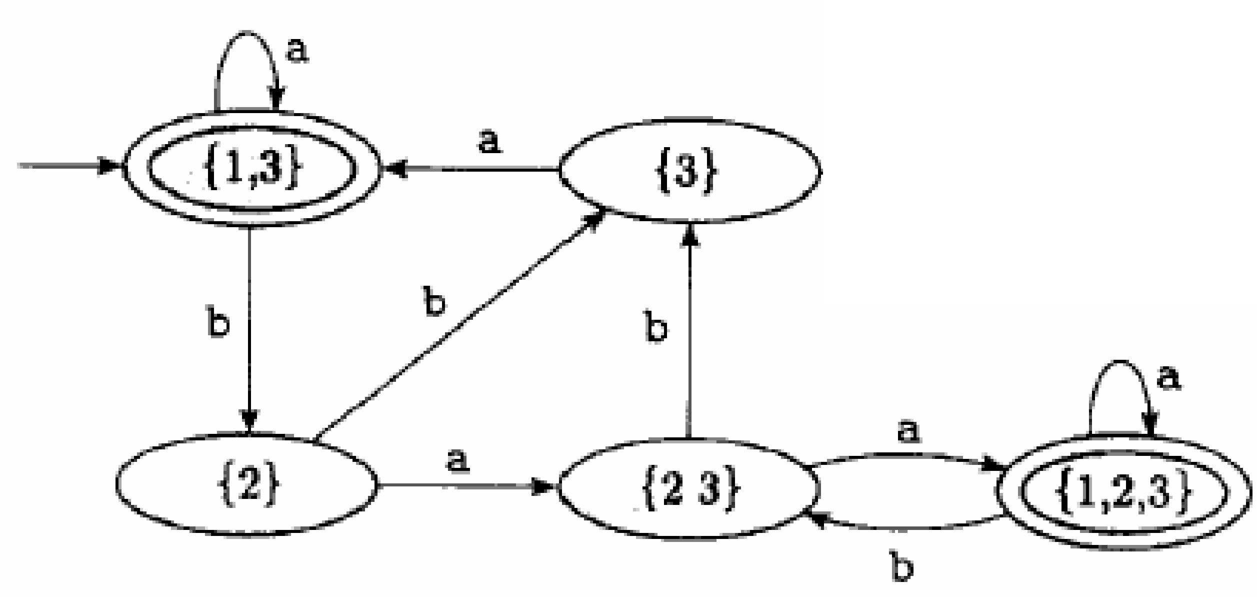
Equivalence of NFAs and DFAs

- Simplify the machine by removing those nodes with no in-arrow, i.e. states {1}, and {1,2}.



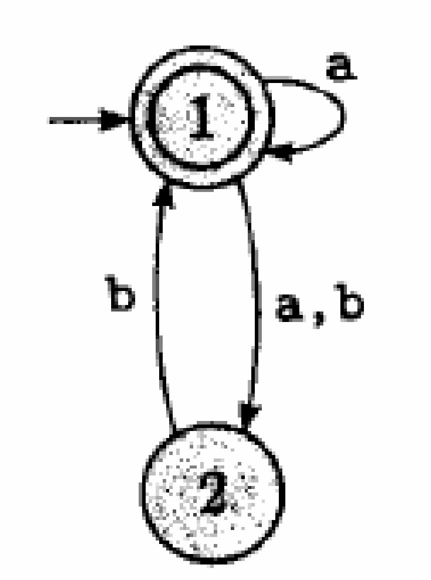
Equivalence of NFAs and DFAs

- Further simplify the machine by removing some “dead looping node”.



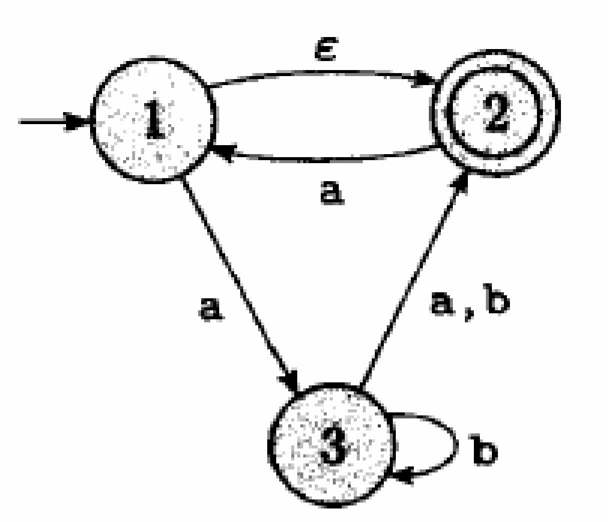
Equivalence of NFAs and DFAs

- On class exercise
  1. Construct DFA equivalent to the following NFA



Equivalence of NFAs and DFAs

2. Construct DFA equivalent to the following NFA



Theory of Computation

Lecture 3: Regular Language (Part II)

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Content of Lecture 3

- Nondeterminism (continued)
- Regular Expressions
- Nonregular Languages

Closure under the regular operations

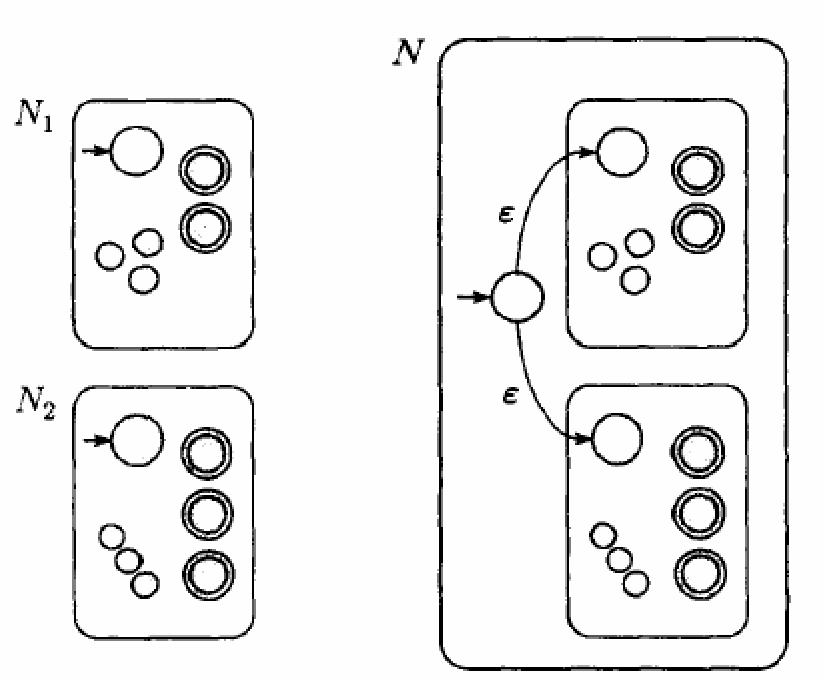
- After learnt the powerful NFA, we will proof again the closure under Union operation in a much easier way and complete the proofs for the other two operations

Closure under the regular operations

- Theorem: The class of regular languages is closed under the union operation.
- Proof:
  - We prove that if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$
    - Suppose N_1 recognizes A_1 and N_2 recognizes A_2, we demonstrate a NFA N to recognize $A_1 \cup A_2$. This is a proof by construction, too.
    - The new machine has a new start state that branches to the start states of the old machines with ε arrows.

Closure under the regular operations

- In this way the new machine nondeterministically guesses which of the two machines accepts the input. If one of them accepts the input, *N* will accept it, too.



Closure under the regular operations

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ and $N = (Q, \Sigma, \delta, q_0, F)$, thus

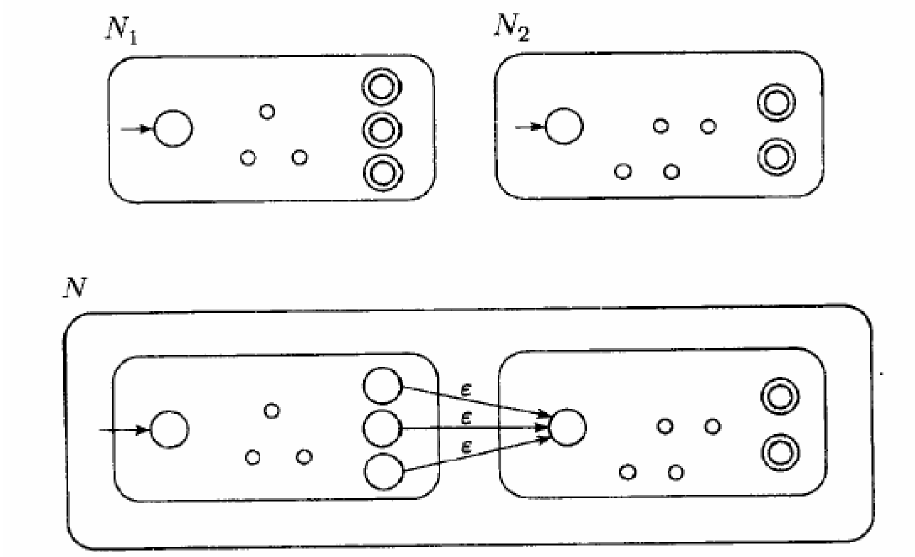
1. $Q = \{q_0\} \cup Q_1 \cup Q_2$
2. Σ
3. Transition function:

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \phi & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

4. q_0
5. $F = F_1 \cup F_2$

Closure under the regular operations

- Theorem: The class of regular languages is closed under the concatenation operation.
- Proof:
  - We prove that if A_1 and A_2 are regular languages, so is $A_1 \circ A_2$



Closure under the regular operations

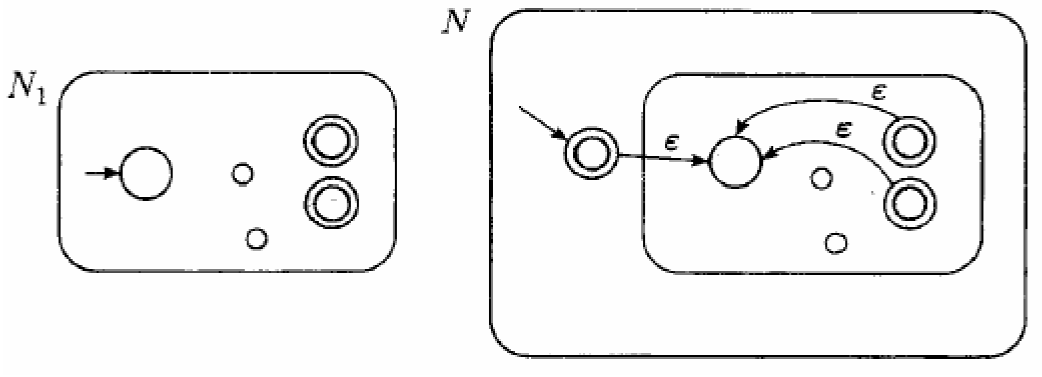
- $N_1 = (Q_1, \sum, \delta_1, q_1, F_1)$, $N_2 = (Q_2, \sum, \delta_2, q_2, F_2)$ and $N = (Q, \sum, \delta, q_0, F)$ to recognize $A_1 \circ A_2$, thus
  1. $Q = Q_1 \cup Q_2$
  2. $\sum$
  3. Transition function:

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

  4. $q_0 = q_1$
  5. $F = F_2$

Closure under the regular operations

- Theorem: The class of regular languages is closed under the star operation.
- Proof:
  - We prove that if A_1 is regular language, so is A_1^*



Closure under the regular operations

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1{}^*$, thus

1. $Q = \{ q_0 \} \cup Q_1$
2. Σ
3. Transition function:

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \phi & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

4. q_0
5. $F = \{ q_0 \} \cup F_1$

Regular expressions

	operations	expression	value
Arithmetic	e.g. +, /	(6 + 9) / 3	5
Computation theory	e.g., ∘ ,*	(0 U 1) *	Language consisting of all possible strings of 0s and 1s.

- Regular expressions provide a powerful method for describing patterns in which the users may want the strings be.
  - Applications
    - modern programming languages, such as PERL
    - text editors.

Formal definition of a regular expressions

- Say that R is a regular expression if R is
  - a for some a in the alphabet $\sum$
  - ε
  - Φ
  - $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
  - $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
  - $R_1{}^*$, where R_1 is a regular expression.

Regular expressions

- Notes:
  - a and ε represent the languages $\{a\}$ and $\{\varepsilon\}$.
  - $\mathit{\Phi}$ represent empty language.
  - ε is an empty string, and $\mathit{\Phi}$ contains no string.
  - Parentheses in an expression may be omitted. If they are, evaluation is done in the precedence order: star, then concatenation, then union.
  - Let R^+ be shorthand for RR^*, so it is the language that has all strings that are 1 or more concatenations of strings from R. $R^+ \cup \varepsilon = R^*$
  - R^k is shorthand for the concatenation of k R's with each other.
  - a regular expression----R

    the language a regular expression describes---- $L(R)$

Regular expressions

- Exercises:

0^*10^*	$\{w \mid w$ contains a single $1\}$
$\Sigma^*1\Sigma^*$	$\{w \mid w$ has at least one $1\}$
$\Sigma^*001\Sigma^*$	$\{w \mid w$ contains the string 001 as a substring$\}$
$(01^+)^*$	$\{w \mid$ every 0 in w is followed by at least one $1\}$
$(\Sigma\Sigma)^*$	$\{w \mid w$ is a string of even length$\}$
$(\Sigma\Sigma\Sigma)^*$	$\{w \mid w$ the length of w is a multiple of three.$\}$

(remarks: $\Sigma^* = (0 \cup 1)^*$)

Regular expressions

- Exercises:

$01 \cup 10$	$\{w \mid 01, 10\}$
$0\Sigma^* 0 \cup 1\Sigma^* 1 \cup 0 \cup 1$	$\{w \mid w \text{ starts and ends with the same symbol}\}$
$(0 \cup \varepsilon)1^*$	$01^* \cup 1^*$
$(0 \cup \varepsilon)(1 \cup \varepsilon)$	$\{\varepsilon, 0, 1, 01\}$
$1^* \phi$	ϕ
ϕ^*	$\{\varepsilon\}$

Regular expressions

- Note:

$$\begin{cases} R \cup \phi = R \\ R \circ \varepsilon = R \end{cases}$$

$$\text{but} \begin{cases} R \cup \varepsilon \neq R & \text{e.g. } R = 0, L(R) = \{0\},\ L(R \cup \varepsilon) = \{0, \varepsilon\} \\ R \circ \phi \neq R & \text{e.g. } R = 0, L(R) = \{0\},\ L(R \circ \phi) = \phi \end{cases}$$

Regular expressions

- Elemental objects in a programming language is tokens.
- Variable names and constants in programming language are tokens, they can be described with regular expressions.
- A numerical constant can be described as a member of the language
  - $(+\cup - \cup \varepsilon)(D^{+} \cup D^{+}.D^{*} \cup D^{*}.D^{+})$
    where $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Once the syntax of the tokens of the programming language have been described with regular expressions, automatic systems can generate the lexical analyzer, which is part of a compiler processing the input program.

Regular expressions vs FA

- Theorem: A language is regular if and only if some regular expression describes it.
  - language that a regular expression describe → regular language
  - regular language → a regular expression can describe it

Regular expressions

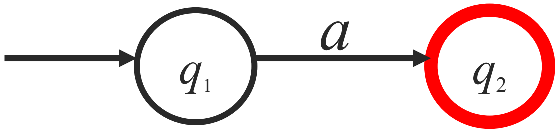
- If a language is described by a regular expression, then it is regular.

  Proof: (R → NFA → A)

  1) R → NFA

  We consider the six cases in the formal definition of regular expressions.

  - $R = a$, Then $L(R) = \{a\}$. The following NFA recognizes $L(R)$

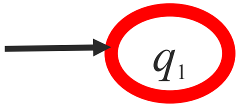


- $N=(\{q_1,q_2\},\sum, \varepsilon ,q_1,\{q_2\})$, where

$$\begin{cases} \delta(q_1,a)=\{q_2\} & \\ \delta(r,b)=\phi & \text{for } r \neq q_1 \text{ or } b \neq a \end{cases}$$

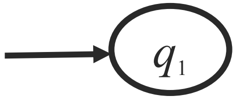
Regular expressions

- $R = \varepsilon$ .Then $L(R) = \{ \varepsilon \}$,and the following NFA recognizes $L(R)$



  - $N = (\{q_1\},\Sigma, \varepsilon ,q_1,\{q_1\})$, where $\delta(r,b) = \Phi$ for any r and b.

- $R = \Phi$. Then $L(R) = \Phi$, and the following NFA recognizes $L(R)$.



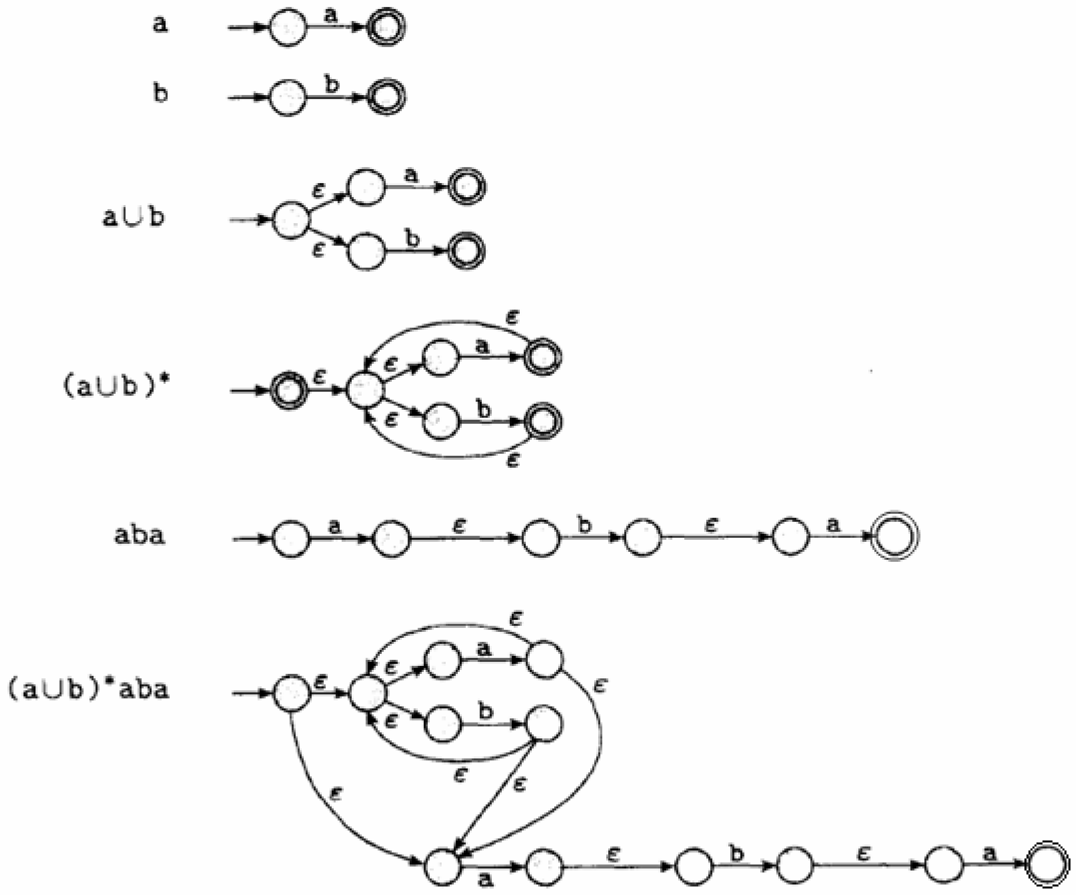
  - $N=(\{q_1\},\Sigma, \varepsilon ,q_1 , \Phi)$, where $\delta(r,b)= \Phi$ for any r and b.

- $R = (R_1 \cup R_2)$,
- $R =(R_1 \circ R_2)$,
- $R =R_1*$,
  - We can prove by construction as we have done in the proofs of closure.

- By the definition of regular language, we can get NFA → A .
- We finish the proof.

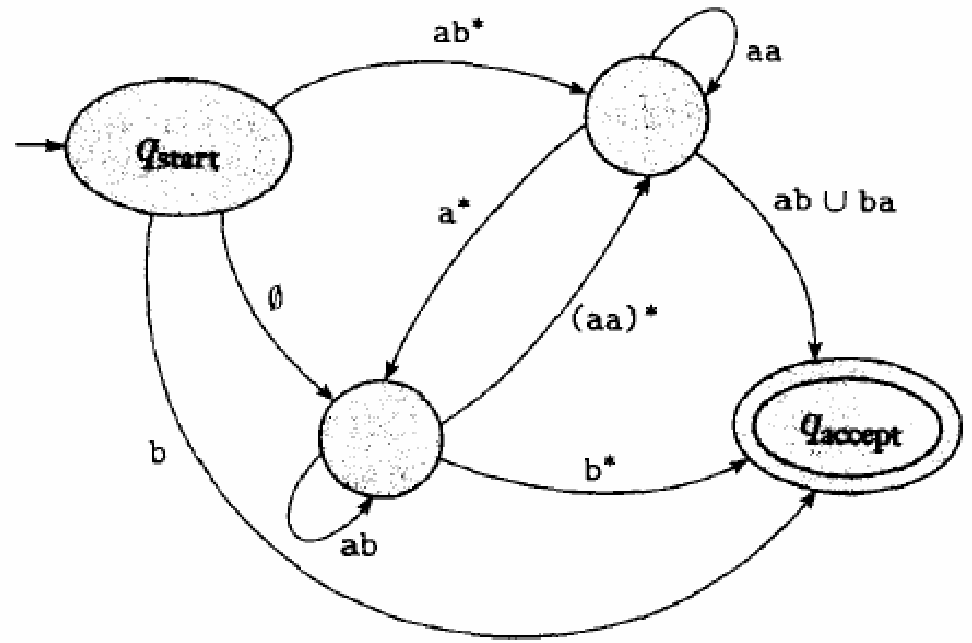
Regular expressions

- Example to convert $(a \cup b)^*aba$ into an NFA N.



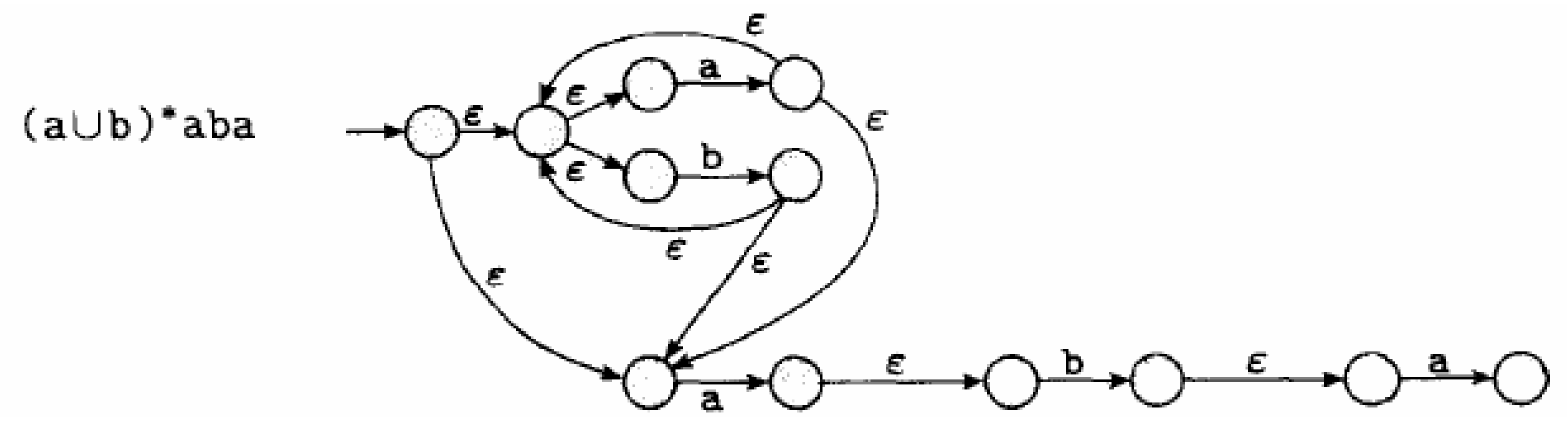
Regular expressions

- Generalized Nondeterministic Finite Automaton (GNFA)
  - GNFA is simply nondeterministic finite automata wherein the transition arrows may have any regular expressions as labels, instead of only member of the alphabet or ε .
  - A GNFA may have several different ways to process the same input string.

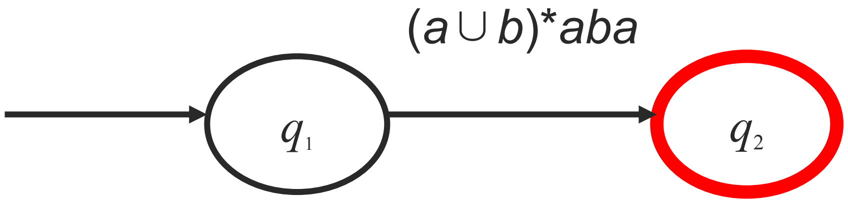


Regular expressions

DFA



GNFA



Regular expressions

- For the purpose of using GNFA to prove the next theory, we require that GNFA always have the following special form:
  - The start state has transition arrows going to every other state but no arrows coming in from any other state.
  - There is only a single accept state, and it has arrows coming in from every other state but no arrows going to any other state. Further more, the accept state is not the same as the start state.
  - Except for the start and accept states, one arrow goes from every state to every other state and also from each state to itself.

Regular expressions

- The formal definition of GNFA
  - A generalized nondeterministic finite automaton G is a 5-tuple $(Q, \sum, \delta, q_{start}, q_{accept})$, where

    1. Q is a finite set states,
    2. $\sum$ is the input alphabet,
    3. $\delta : (Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \rightarrow R$ is the transition function
    4. q_{start} is the start state, and
    5. q_{accept} is the accept state.

    Note: R is the collection of all regular expressions over the alphabet $\sum$

Regular expressions

- The formal definition of computation of GNFA
  - A GNFA accepts a string w *in* $\sum^*$ if $w=w_1w_2 \ldots w_n$,where each w_i is *in* $\sum^*$ a sequence of states $q_0,q_1, \ldots q_k$ exists such that

    1. $q_0 = q_{\text{start}}$ is the start state.
    2. $q_k = q_{accept}$ is the accept state
    3. For each i, we have wi $\in L(R_i)$, where $R_i = \delta(q_{i-1},q_i)$; in other words, R_i is the expression on the arrow from q_{i-1} to q_i.

Regular expressions

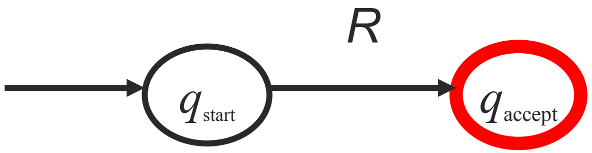
- If a language is regular, then it is described by a regular expression.
- Proof: (A → DFA → GNFA → R)
  - By the definition of regular language, A →DFA
  - DFA → GNFA
    - Add a new start state with an ε arrow to the old start state
    - Add a new accept state with an ε arrows to the old accept states
    - If any arrows have multiple labels (or if there are multiple arrows going between the same two states in the same direction), we replace each with a single arrow whose label is the union of the previous labels.
    - Add arrows labeled Φ between states that had no arrows.

Regular expressions

- GNFA → R

$$\text{GNFA} \xrightarrow{\text{CONVERT}(G)} R$$

- We denote the conversion as CONVERT(G) which takes a GNFA as the input and returns an equivalent regular expression.
  - Let k be the number of states of G.
  - $k = 2$,



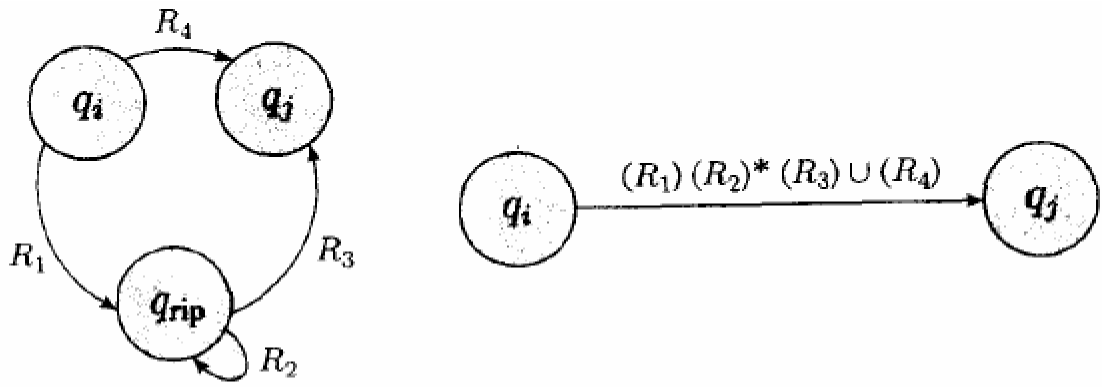
  - $k > 2$, we select any state $q_{\text{rip}} \in Q$ different from q_{start} and q_{accept} and let $G' = (Q', \Sigma, \delta', q_{start}, q_{accept})$, where $Q' = Q' - \{q_{\text{rip}}\}$, for any $q_{\text{i}} \in Q - \{q_{\text{accept}}\}$ and any $q_{\text{j}} \in Q - \{q_{\text{start}}\}$ let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$$

Where $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$, $R_4 = \delta(q_i, q_j)$,

  - Compute CONVERT(G') and return this.

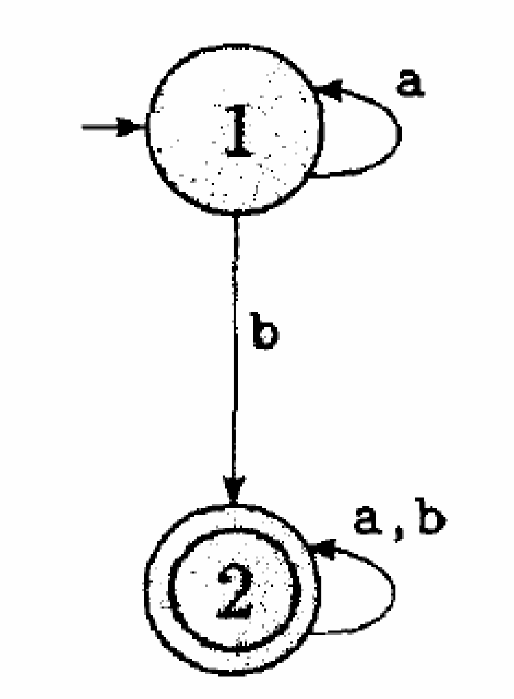
Regular expressions



- Note: we can prove by induction that for any GNFA *G*, CONVERT(*G*) is equivalent to *R*. (p.74 in English version, p. 44 in Chinese version)
- Finish the whole proof.

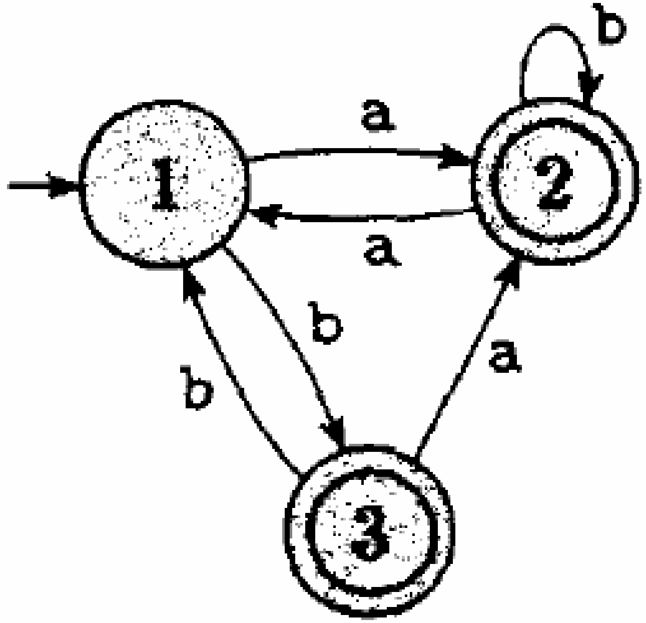
Regular expressions

- On class exercise:
  - Convert the DFA into a regular expression.



Regular expressions

- On class exercise:
  - Convert the DFAs into a regular expressions.



Regular expressions

- On class exercise:
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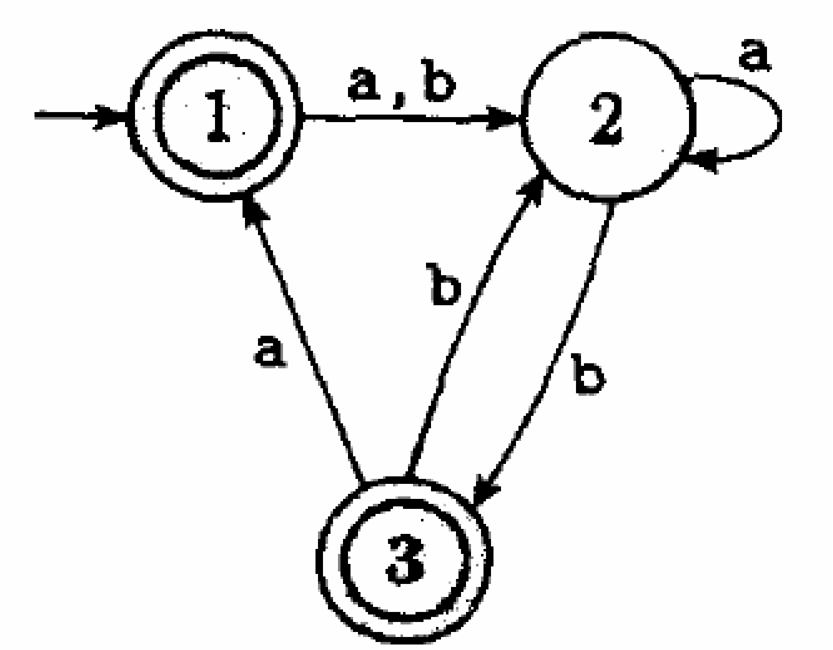


Nonregular languages

- Finite automata has finite states, so there are some languages that FA can't recognize.
  - E.g. $B = \{0^n1^n \mid n \geqslant 0\}$
- To prove a language is not a nonregular, we state a property that a regular language possesses.
- **The pumping lemma for regular languages:**
  - **All strings in a regular language can be "pumped" if they are at least as long as a certain special value, called the pumping length.**
- **This lemma means that such string contains a section that can be repeated any number of times with the resulting string remaining in the language.**

Nonregular languages

Pumping lemma:

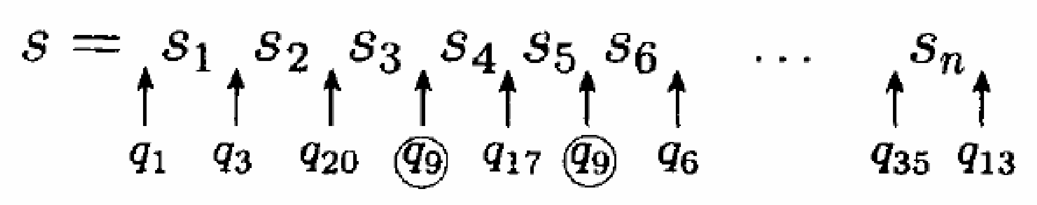
- If A is a regular language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into three pieces, $s=xyz$, satisfying the following conditions:
  - For each $i \geqslant 0$, $xy^iz \in A$
  - $|y|>0$, and
  - $|xy| \leqslant p$

Nonregular languages

- Note: The complete proof is provided on p.79-80 in English version, p. 48 in Chinese version, and we only provide the idea of proof in this lecture notes.

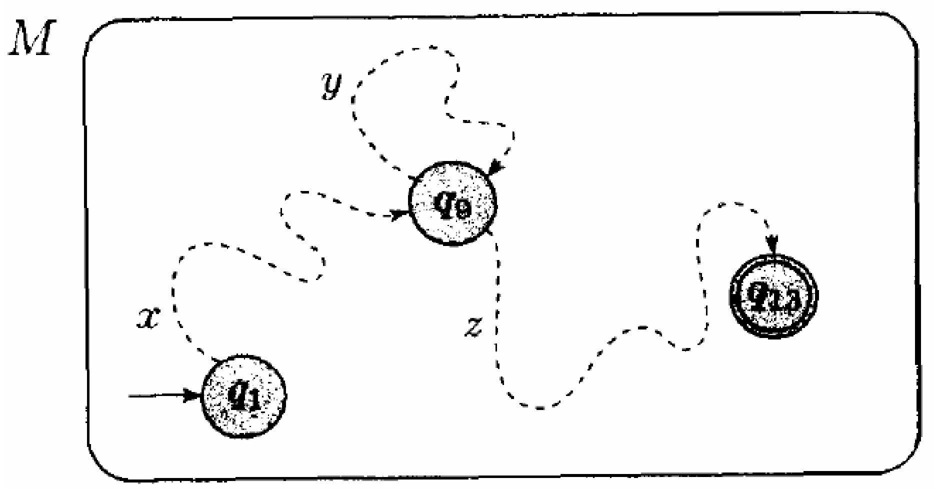
Nonregular languages

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that recognizes A.
- Assign the pumping length p to be the number of states of M.
- We now prove a string s ($|s| = n$, and $|s| \geqslant p$) belonging to A, can be divided into 3 pieces xyz satisfying the three conditions.
- Suppose through states sequence $q_1, q_{k1}, \ldots, q_{kn}$, where $q_{kn} \in F$, so there are n+1 states, and as n $\geqslant p$, so $n+1 > p$. By the pigeonhole principle, the sequence must contain a repeated state.
- Suppose state q_9 is the state that repeats first..


$s = s_1 \; s_2 \; s_3 \; s_4 \; s_5 \; s_6 \; \ldots \; s_n$
q_1
q_3
q_{20}
q_9
q_{17}
q_9
q_6
q_{35}
q_{13}

Nonregular languages

- Let $x = [q_1, q_9]$ $y = [q_{10}, q_9]$ $z = [q_9, q_{13}]$



- By condition 1, we can see M accept $xyyz$, so does xy^iz.
- Obviously $|y| > 0$
- As state q_9 is the state that repeats first, by pigeonhole principle, the first $p+1$ states in the sequence must contain a repetition. So $|xy| \leqslant p$.

Nonregular languages

- Using pumping lemma to prove a language *B* is not regular.
  - Assume *B* is regular to prove by contradiction.
  - Use pumping lemma to guarantee the existence of a pumping length *p* such that all strings of length *p* or greater in *B* can be pumped.
  - Find a string *s* in *B* that has length *p* or greater but that cannot be pumped.
  - Demonstrate that *s* cannot be pumped by considering all ways of dividing s into *x*, *y*, and *z* and, for each such division, finding a value *i* where $xy^iz \notin B$. Therefore, there is a contradiction.

Nonregular languages

Example:

- $B = \{0^n1^n \mid n \geqslant 0\}$ is not regular.
- Proof:
- Assume that B is regular.
  - Let p be the pumping length given by the pumping lemma.
  - Choose s to be the string 0^p1^p . Because s is a member of B and s has length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geqslant 0$ the string x is in B. so
    - y consists only of **0**s. In this case the string $xyyz$ has more **0**s than **1**s and so is not a member of B, violating condition 1.
    - y consists only of 1s.more **1**s than **0**s.
    - y consists both **0**s and **1**s. $xyyz$ may have the same number of **0**s and **1**s, but there must be a **1** before **0**, and so is not a member of B.
  - Thus a contradiction is unavoidable, B is not regular.

Nonregular languages

On class exercise:

- Prove that $B_1 = \{0^n 1^n 2^n | \, n \geqslant 0\}$ is not regular.

Nonregular languages

On class exercise:

- Prove that $B_2 = \{ww \mid w \in \{0,1\}^*\}$ is not regular.

Theory of Computation

Lecture 4: Context-free languages

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Content of Lecture 4

- Context-free Grammars
- Pushdown Automata
- Non-context-free Languages

Context-free Grammars

- Context-free grammars are a more powerful method of describing languages.
- Context-free grammars can describe certain features that have a recursive structure.
- E.g. human languages: the relationship of terms such as *noun*, *verb*, and *preposition* and their respective *phrases* leads to a natural recursion because *noun* phrases may appear inside *verb* phrases and vice versa. Context-free grammars can capture important aspects of these relationships

Context-free Grammars

- E.g. Another important application of context-free grammar is the specification and compilation of programming languages. A grammar for a programming language often appears as a reference for learning the language syntax. So most compilers and interpreters contain *parser* which extracts the meaning of a program prior to generating the compiled code or performing the interpreted execution.

Formal Definition of a context-free grammar

- A context-free grammar is a 4-tuple (V, ∑, R, S),where
  - ○ V is a finite set called the variables,
  - ○ ∑ is a finite set, disjoint from V, called the terminals,
  - ○ R is a finite set of rules (production), with each rule being a variable and a string of variables and terminals, and
  - ○ $S \in V$ is the start variable.

Context-free Grammars

- If u, v and w are strings of variables and terminals, and $A \rightarrow w$ is a rule of the grammar, we say that uAv yields uwv, written

$$uAv \Rightarrow uwv$$

- If $u = v$ or if a sequence u_1, u_2,…, u_k exists for $k \geqslant 0$ and $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_k \Rightarrow v$ we say that u derives v, written

$$u \overset{*}{\Rightarrow} v$$

Context-free Grammars

- The collection of languages associated with context-free grammars are context-free languages. So the language of the grammar is $\{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}$

Context-free Grammars

- The process to generate the strings of the context-free language:
  1. Write down the start variable.
  2. Find a variable that is written down and a rule that starts with that variable. Replace the written variable with the right-hand side of that rule.
  3. Repeat step 2 until no variables remain.

Context-free Grammars

- Example1 : a context-free grammar G_1

$$A \to 0A1$$

$$A \to B$$

$$B \to \#$$

- We can present it formally as following:
  - $G_1 = (V, \sum, R, S)$, $V=\{A,B\}$, $\sum=\{0, 1, \#\}$, the production is

$$S \to 0S1 \mid B$$

$$B \to \#$$

- Note:
  - **We abbreviate several rules with the same left-hand variable, into a single line using the symbol "|" as an "or".**
  - **We usually write the start variable on left of the first rule**

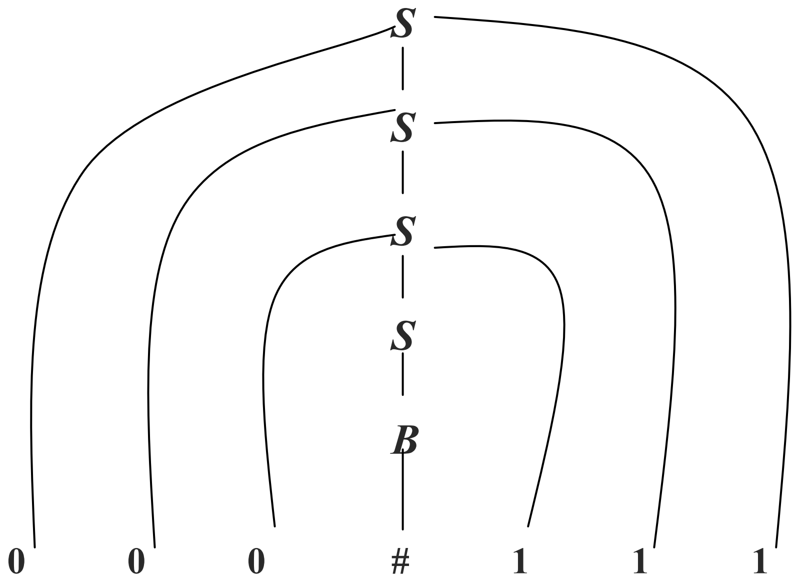
Context-free Grammars

- We can represent using a parse tree.

  E.g. the by grammar G1 generate string 000#111.

$$S \rightarrow 0S1 \mid B$$

$$B \rightarrow \#$$



Context-free Grammars

- Example 2: a context-free grammar G_2

$$
\begin{aligned}
\langle\text{SENTENCE}\rangle &\rightarrow \langle\text{NOUN-PHRASE}\rangle\langle\text{VERB-PHRASE}\rangle \\
\langle\text{NOUN-PHRASE}\rangle &\rightarrow \langle\text{CMPLX-NOUN}\rangle | \langle\text{CMPLX-NOUN}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{VERB-PHRASE}\rangle &\rightarrow \langle\text{CMPLX-VERB}\rangle | \langle\text{CMPLX-VERB}\rangle\langle\text{PREP-PHRASE}\rangle \\
\langle\text{PREP-PHRASE}\rangle &\rightarrow \langle\text{PREP}\rangle\langle\text{CMPLX-NOUN}\rangle \\
\langle\text{CMPLX-NOUN}\rangle &\rightarrow \langle\text{ARTICLE}\rangle\langle\text{NOUN}\rangle \\
\langle\text{CMPLX-VERB}\rangle &\rightarrow \langle\text{VERB}\rangle | \langle\text{VERB}\rangle\langle\text{NOUN-PHRASE}\rangle \\
\langle\text{ARTICLE}\rangle &\rightarrow \text{a} \mid \text{the} \\
\langle\text{NOUN}\rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{flower} \\
\langle\text{VERB}\rangle &\rightarrow \text{touches} \mid \text{likes} \mid \text{sees} \\
\langle\text{PREP}\rangle &\rightarrow \text{with}
\end{aligned}
$$

Context-free Grammars

- Strings in $L(G_2)$ include
  - a boy sees
  - the boy sees a flower
  - a girl with a flower likes the boy

Context-free Grammars

- Example 3: a context-free grammar G_3= ($\{S\}$, $\{a,b\}$, R, S), with the rule set: $S \rightarrow \mathrm{a}S\mathrm{b} \mid SS \mid \varepsilon$
  - This grammar generates strings such as abab, aaabbb, and aababb.
  - You can see more easily what this language is if you think of **a** as a left parenthesis "("" and **b** as a right parenthesis")". $(\;)(\;),(((\;))),((\;)(\;))$
  - So $L(G_3)$ is the language of all strings of properly nested parentheses.

Context-free Grammars

- Example 4: a context-free grammar $G_4 = (V, \Sigma, R, <\text{EXPR}>)$ where

  $V =$ {<EXPR>, <TERM>, <FACTOR>},

  $\Sigma = \{A, +, \times, (,)\}$,

  The rules are

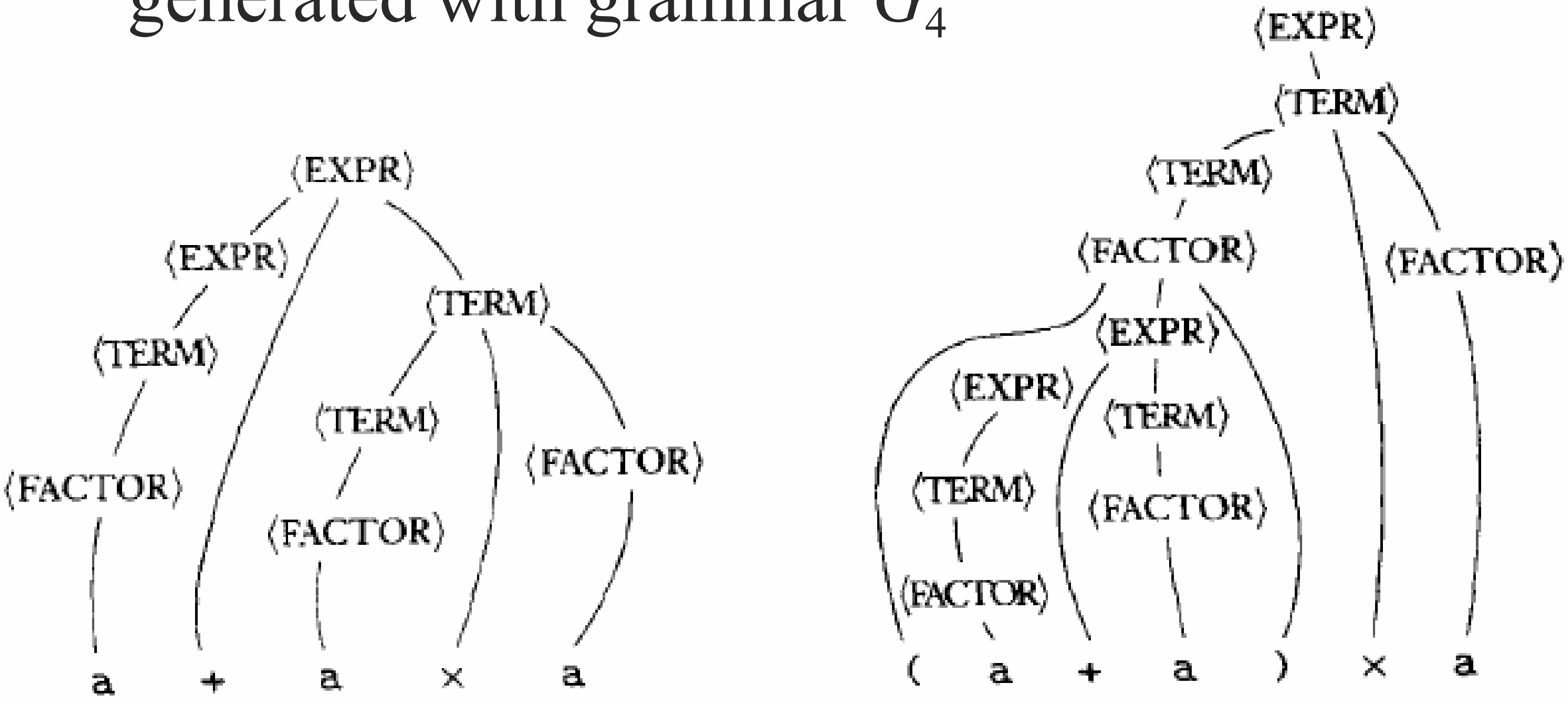
$$<\text{EXPR}> \to <\text{EXPR}> + <\text{TERM}> \mid <\text{TERM}>$$

$$<\text{TERM}> \to <\text{TERM}> \times <\text{FACTOR}> \mid <\text{FACTOR}>$$

$$<\text{FACTOR}> \to (<\text{EXPR}>) \mid \text{a}$$

Context-free Grammars

- a + a × a and (a + a) × a can be generated with grammar G_4



Designing context-free grammars

- For we are more accustomed to programming a machine for specific tasks than we are to describing languages with grammars. So designing a CFG is even harder than a FA.

Designing context-free grammars

- **Technique one:**
  - To remember the substring:
  - Certain CFLs contain strings with two substrings that require the machine to remember an unbounded amount of information about one of the substring *s*, in order to verify the corresponding properly to the other substring.
  - Then you can construct a CFG by using a rule of the form

$$R \rightarrow uRv$$

  - E.g. $\{0^n 1^n \mid n \geq 0\}$ $\qquad S_1 \rightarrow 0S_1 1 \mid \varepsilon$

    here CFG remembers the number of 0s in order to verify that it equals the number of 1s.

Designing context-free grammars

- **Technique two:**
  - Break into simpler pieces:
  - Many CFLs are the union of simple CFLs. So break into simpler pieces, then construct individual grammars for each piece. Merge them into a grammar for the original language by combining their rule and then adding the new rule

$$S \to S_1 \mid S_2 \mid \ldots \mid S_k$$

  - where S_i are the start variables for the individual grammars.

Designing context-free grammars

- Example:
  - The language $\{0^n 1^n \mid n \geq 0\} \cup \{1^n 0^n \mid n \geq 0\}$
  - For $\left\{0^n 1^n \mid n \geq 0\right\}$

$$S_1 \rightarrow 0S_1 1 \mid \varepsilon$$

  - For $\left\{1^n 0^n \mid n \geq 0\right\}$

$$S_2 \rightarrow 1S_2 0 \mid \varepsilon$$

  - So adding the new rule, we get

$$S \rightarrow S_1 \mid S_2$$

$$S_1 \rightarrow 0S_1 1 \mid \varepsilon$$

$$S_2 \rightarrow 1S_2 0 \mid \varepsilon$$

Designing context-free grammars

- **Technique three:**
  - For regular language, construct DFA first
  - Then, we convert the DFA to a CFG
  - Make a variable R_i for each state q_i of the DFA.
  - Add the rule $R_i \to aR_j$ to the CFG if $\delta(q_i, a) = q_j$
  - Add the rule $R_i \to \varepsilon$ if q_i is an accept state.
  - Make R_0 the start variable, where q_0 is the start state of the DFA

Designing context-free grammars

- Technique four:
  - Recursive structure.
  - E.g. Example 4, any time the symbol ***a*** appear, an entire parenthesized expression might appear recursively instead.
  - To achieve this affect, place the variable symbol generating the structure in the location of the rules corresponding to where that structure may recursively appear.

Designing context-free grammars

- On class exercise:
- Given that ∑=(0,1)
- {*w*| *w* starts and ends with the same symbol }

Designing context-free grammars

- On class exercise:
- $\{w|\ w$ contains at least three 1s $\}$

Ambiguity

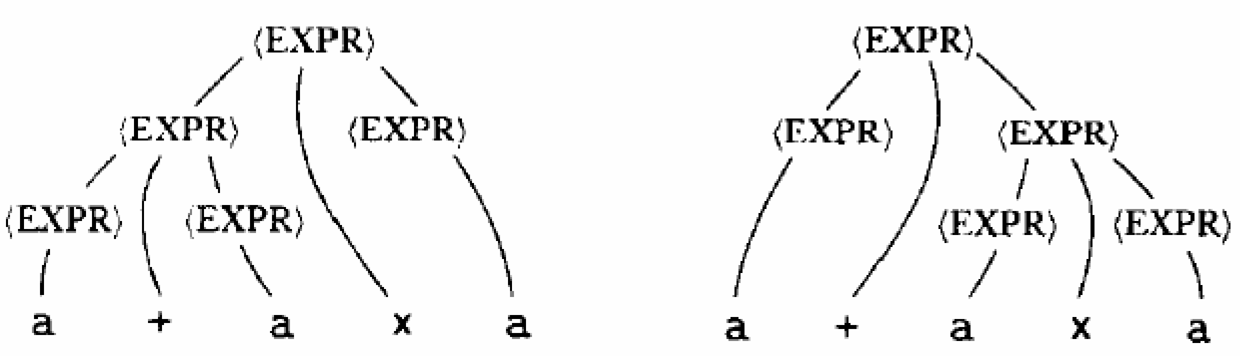
- In some grammar, the same string can be generated in several different way (have several different parse trees), so have several different meanings.
  - ○ **Note: two parse trees, not two ways to derive.**
- If a grammar generates the same string in several different way, we say that the string is derived **ambiguously** in that grammar.

Ambiguity

- E.g.

$$< \text{EXPR} > \to < \text{EXPR} > + < \text{EXPR} > | < \text{EXPR} > \times < \text{EXPR} > | (< \text{EXPR} >) | a$$

- to generate a + a × a



- This grammar doesn’t capture the usual precedence relations, but Example 4 does.

Ambiguity

- E.g
  - Example 2 is an ambiguous grammar too.
  - the girl touches the boy with the flower is ambiguous.

Ambiguity

- A derivation of a string w in a grammar G is a leftmost derivation if at every step the leftmost remaining variable is the one being replaced.
- Definition:
  - A string w is derived ambiguously in context-free grammar G if it has two or more different leftmost derivations.
  - Grammar G is ambiguous if it generates some string ambiguously.

Ambiguity

- If a CFL can be generated only by ambiguous grammars, then it is called inherently ambiguous.
- E.g. $\{a^i b^j c^k \mid i = j \text{ or } i = k\}$
- Illustrate it later.

Chomsky normal form

- If we add more constrains to the CFG, then we can design and use it more easily.
- **Chomsky normal form:**
  - A context-free grammar is in Chomsky normal form if every rule is of the form

$$A \to BC$$

$$A \to a$$

  - where
    - ***a*** is any terminal
    - *A*, *B*, and *C* are any variables---except that *B* and *C* may not be the start variable
    - Permit rule $S \to \varepsilon$, only when *S* is the start variable.

Chomsky normal form

- Theorem: Any context-free language is generated by a context-free grammar in Chomsky normal form.
- Idea:
  - Add a new start variable
  - Eliminate all ε rules of the form $A \rightarrow \varepsilon$
  - Eliminate all unit rules of the form $A \rightarrow B$

Chomsky normal form

- Proof

  to explain with an example :

- 1) add a new start variable S_0 and the rule $S_0 \to S$, where S was the original start variable.

$$S \to ASA \mid aB$$
$$A \to B \mid S$$
$$B \to b \mid \varepsilon$$

$$S_0 \to S$$
$$S \to ASA \mid aB$$
$$A \to B \mid S$$
$$B \to b \mid \varepsilon$$

Chomsky normal form

- 2) remove $A \to \varepsilon$, where A is not a start variable. For each occurrence of an A on the right, we add a new rule with the deleted occurrence.

$S_0 \to S$

$S \to ASA \mid aB$

$A \to B \mid S$

$B \to b \mid \varepsilon$

Remove rule $B \to \varepsilon$

$S_0 \to S$

$S \to ASA \mid aB \mid a$

$A \to B \mid S \mid \varepsilon$

$B \to b \boxed{\mid \varepsilon}$

Chomsky normal form

Remove rule $A \rightarrow \varepsilon$

$S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS \mid S$

$A \rightarrow B \mid S \mid \varepsilon$

$B \rightarrow b$

Chomsky normal form

- 3) remove $A \rightarrow B$, when a rule $B \rightarrow u$ appears, we add the rule $A \rightarrow u$ unless this was a unit rule previously removed.

Remove rule $S \rightarrow S$

$S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS \boxed{\mid S}$

$A \rightarrow B \mid S$

$B \rightarrow b$

Chomsky normal form

Remove rule $S_0 \rightarrow S$

$$S_0 \rightarrow \boxed{S} \mid ASA \mid aB \mid a \mid SA \mid AS$$
$$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$$
$$A \rightarrow B \mid S$$
$$B \rightarrow b$$

Remove rule $A \rightarrow B$

$$S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$$
$$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$$
$$A \rightarrow \boxed{B} \mid S \mid b$$
$$B \rightarrow b$$

Remove rule $A \rightarrow S$

$$S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$$
$$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$$
$$A \rightarrow \boxed{S} \mid b \mid ASA \mid aB \mid a \mid SA \mid AS$$
$$B \rightarrow b$$

Chomsky normal form

- 4) convert all remaining rules into the proper form. Replace rule $A \rightarrow u_1u_2\ldots u_k$ with the rules $A \rightarrow u_1A_1, A_1 \rightarrow u_2A_2, \ldots, A_{k-2} \rightarrow u_{k-1}u_k$ together with $U_i \rightarrow u_i$

$$S_0 \rightarrow AA_1 \mid UB \mid a \mid SA \mid AS$$
$$S \rightarrow AA_1 \mid UB \mid a \mid SA \mid AS$$
$$A \rightarrow b \mid AA_1 \mid UB \mid a \mid SA \mid AS$$
$$B \rightarrow b$$
$$A_1 \rightarrow SA$$
$$U \rightarrow a$$

Chomsky normal form

- Now, try to convert it by yourself

$$S \to ASA \mid aB$$

$$A \to B \mid S$$

$$B \to b \mid \varepsilon$$

Chomsky normal form

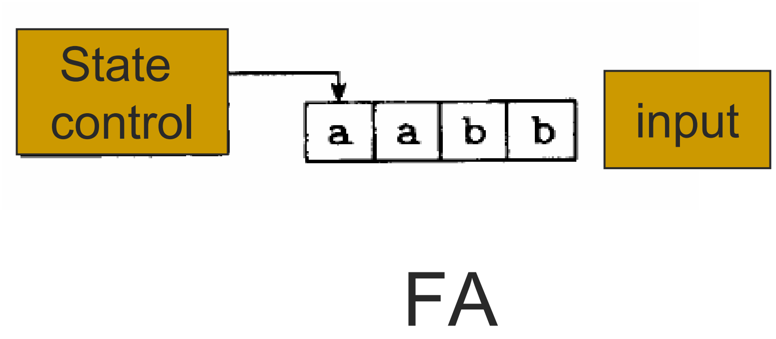
- Convert the following CFG into equivalent CFG in Chomsky normal form.

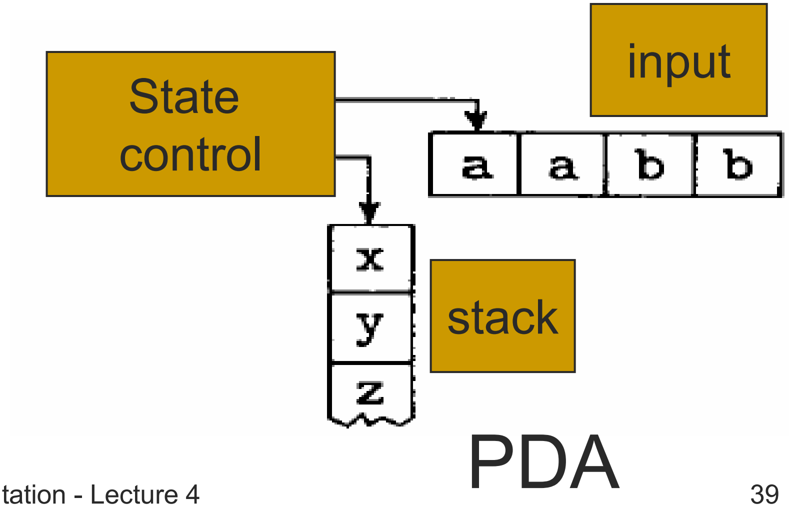
$$A \to BAB \mid B \mid \varepsilon$$

$$B \to 00 \mid \varepsilon$$

Pushdown automata (PDA)

- Pushdown automata are like NFA but have an extra component called a **stack**.
- Stack:
  - Last in, first out (LIFO)
  - Operation: push, pop





Pushdown automata

- DFA and NFA are equivalent
- For PDA, the deterministic PDA and the nondeterministic are not equivalent.
- Nondeterministic PDA recognizes certain languages that no deterministic PDA can recognize (we can’t prove this fact now).
- As we will find that the nondeterministic PDA are equivalent in power to CFG, so we focus on nondeterministic PDA in the following section.

Formal definition of pushdown automata

- We define a alphabet for stack Γ
- Now for every transition, we will consider the state, the input and the stack state. So the domain is $Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon$ to determine the next state and the change of stack state $Q \times \Gamma_\varepsilon$
- As the nondeterminism in model, finally the transition function has the form:

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathrm{P}(Q \times \Gamma_\varepsilon)$$

Formal definition of pushdown automata

- Definition
- A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ and F are all finite sets, and
  - Q is the set of states,
  - Σ is the input alphabet,
  - Γ is the stack alphabet,
  - $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \mathrm{P}(Q \times \Gamma_\varepsilon)$ is the transition funtion,
  - q_0 is the start state, and
  - $F \subseteq Q$ is the set of accept states.

Formal definition of computation

- It accepts input w if w can be written as $w = w_1 w_2 \ldots w_m$, where each $w_i \in \Sigma_\varepsilon$ and sequences of states $r_0, r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ exist that satisfy the following three conditions.
- $r_0 = q_0$ and $s_0 = \varepsilon$ (M starts out in the start state and with an empty stack)
- For $i = 0, \ldots, m-1, r$ we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$ where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ (moves)
- $r_m \in F$ (accept occurs)

Formal definition of pushdown automata

- Example 1: PDA for $L = \{0^n 1^n \mid n \geq 0\}$

  $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, F)$

  where $Q = \{q_1, q_2, q_3, q_4\}$

  $\Sigma = \{0, 1\}$

  $\Gamma = \{0, \$\}$

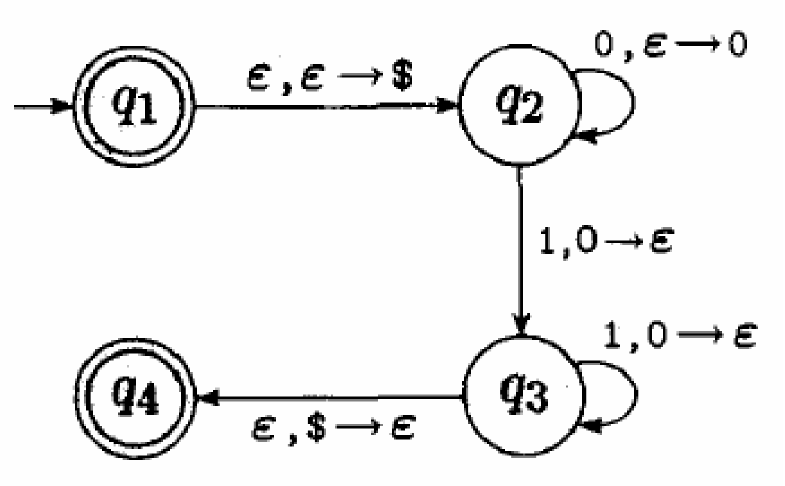
  $F = \{q_1, q_4\}$

- Transition function is described in the following (blank stands for Φ)

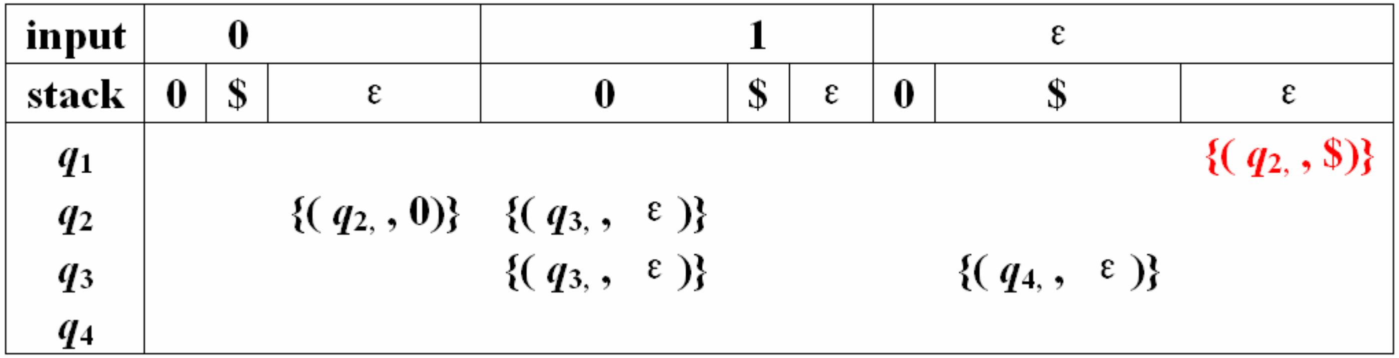
input	**0**			**1**			ε		
stack	**0**	**$**	ε	**0**	**$**	ε	**0**	**$**	ε
q_1									{(q_2, , $)}
q_2			{(q_2, , 0)}	{(q_3, , ε)}					
q_3				{(q_3, , ε)}				{(q_4, , ε)}	
q_4									

Example of pushdown automata

- The state graph is as following:
- On the transition arrow, $a, b \rightarrow c$ denotes that when the machine is **reading an input *a*** from the input, and **pop the symbol *b*** from the top of the stack and **push *c*** to the stack.


q_1
$\varepsilon, \varepsilon \rightarrow \$$
q_2
$0, \varepsilon \rightarrow 0$
$1, 0 \rightarrow \varepsilon$
q_3
$1, 0 \rightarrow \varepsilon$
q_4
$\varepsilon, \$ \rightarrow \varepsilon$

Example of pushdown automata



input	0			1			ε		
stack	0	\$	ε	0	\$	ε	0	\$	ε
q_1									{($q_{2,}$, \$)}
q_2			{($q_{2,}$, 0)}	{($q_{3,}$, ε)}					
q_3				{($q_{3,}$, ε)}				{($q_{4,}$, ε)}	
q_4									

- We can see the PDA M_1 initially push a special symbol \$ on the stack to allow the PDA to test for an empty stack.

Formal definition of pushdown automata

- Example 2: PDA for

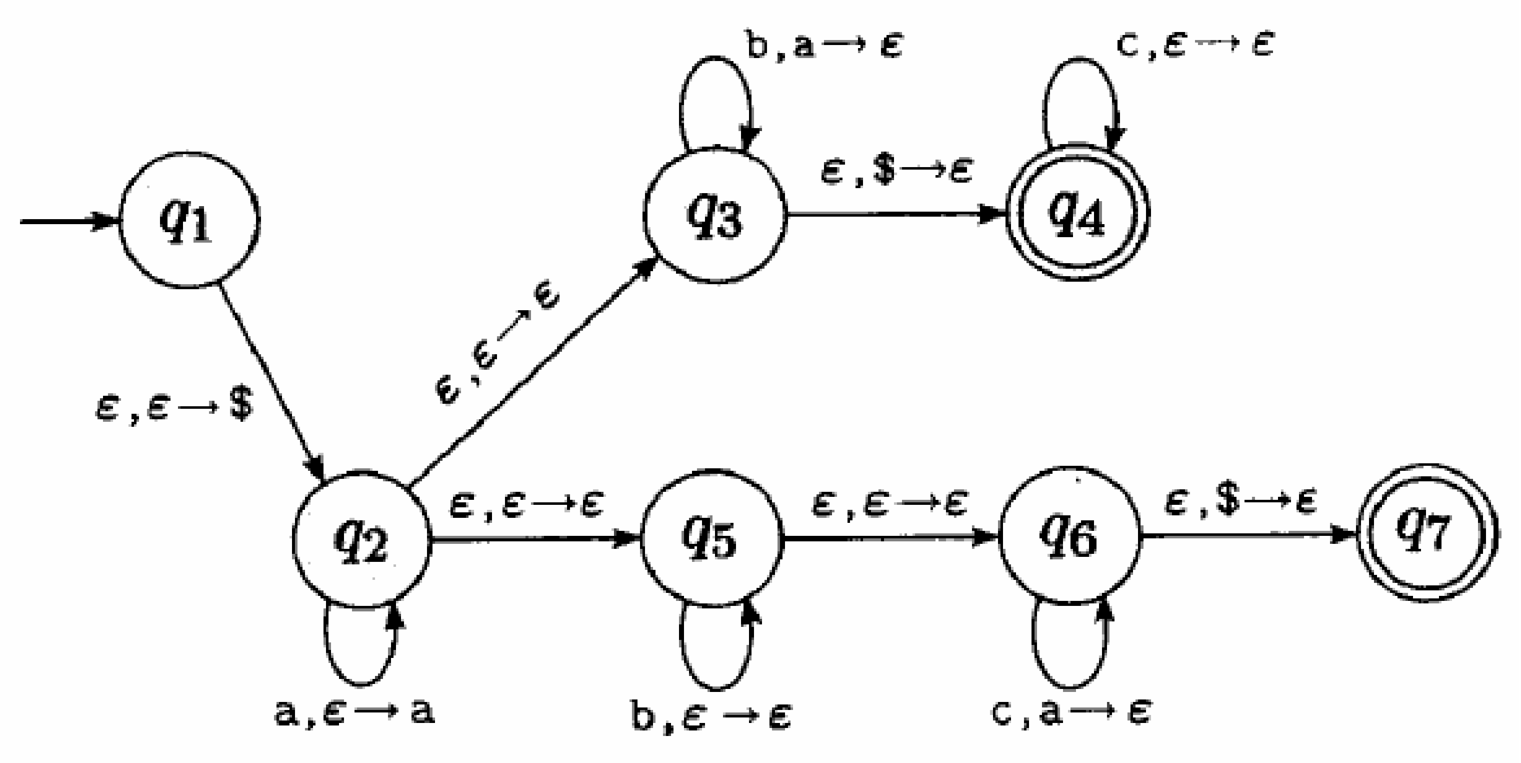
$$L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$

- This is a language that the number of ***b*** or ***c*** equivalent to the number of ***a***. so PDA have to guess whether to match the ***a***'s with the ***b***'s or with the ***c***'s.

Example of pushdown automata

- PDA for

$L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$

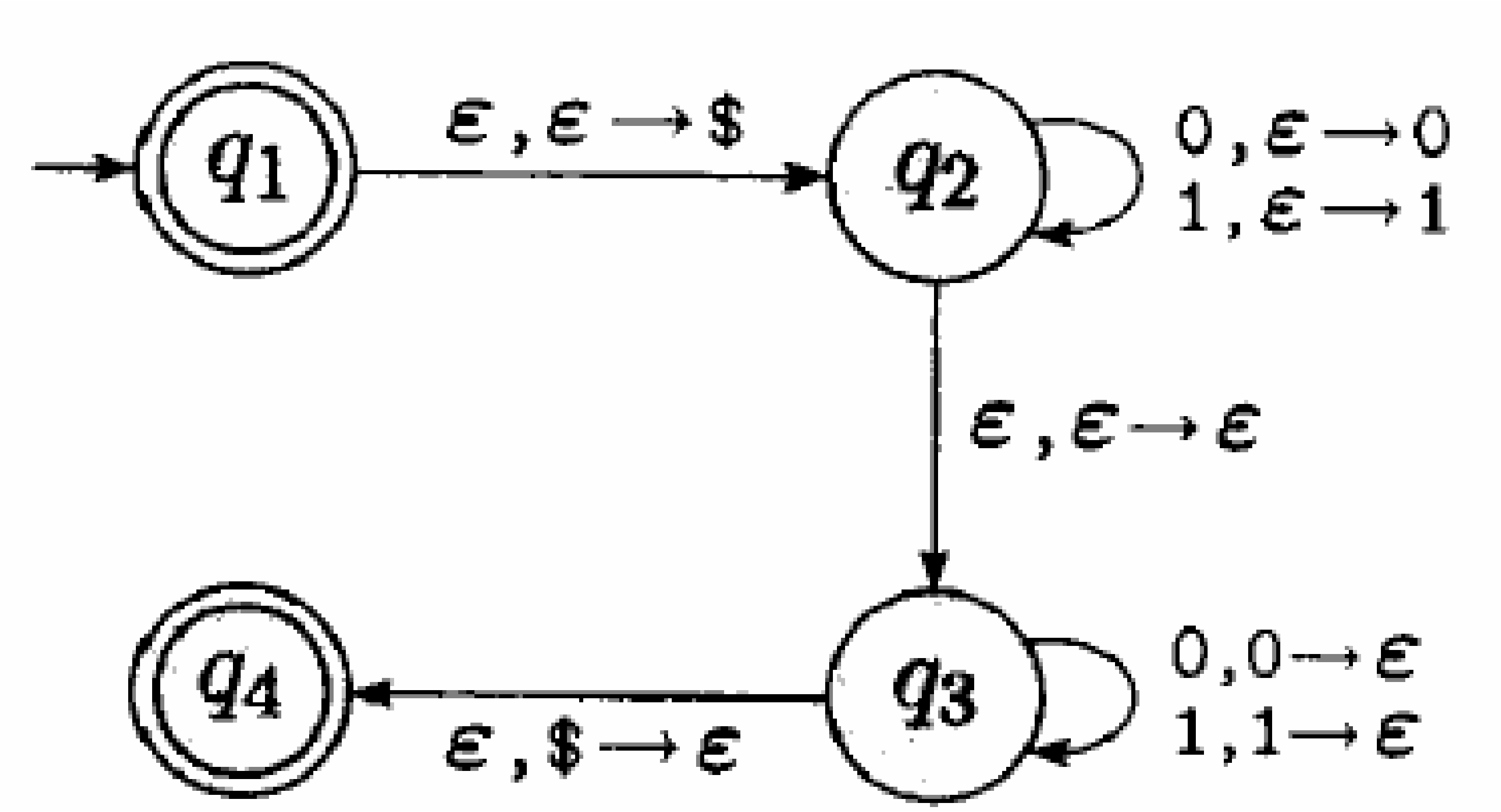


Formal definition of pushdown automata

- Example 3: PDA for $L = \{ww^{\Re} \mid w \in \{0,1\}^*\}$
- PDA have to guess whether it is reaching the middle of the string.

Formal definition of pushdown automata

- Example 3: PDA for $L = \{ww^{\Re} \mid w \in \{0,1\}^*\}$


q_1
$\varepsilon, \varepsilon \rightarrow \$$
q_2
$0, \varepsilon \rightarrow 0$
$1, \varepsilon \rightarrow 1$
$\varepsilon, \varepsilon \rightarrow \varepsilon$
q_3
$0, 0 \rightarrow \varepsilon$
$1, 1 \rightarrow \varepsilon$
q_4
$\varepsilon, \$ \rightarrow \varepsilon$

PDA is equivalence with CFG

- Theorem: A language is context free if and only if some pushdown automaton recognizes it.
- The proof is partitioned into two parts:
  - If a language is context free, then some pushdown automaton recognizes it.
  - If a pushdown automaton recognizes some language, then it is context free.

Non-context-free languages

- Pumping lemma for context-free language: every context-free language has a special value called the ***pumping length*** such that all longer strings in the language can be "pumped."
- Here the definition means that the string can be divided into five parts so that the second part and the fourth parts may be repeated together any number of times and the resulting string still remains in the language.

Non-context-free languages

- Pumping lemma for context-free language:

  If *A* is a context-free language, then there is a number *p* (the pumping length) where, if s is any string in *A* of length at least *p*, then s may be divided into five pieces *s=uvxyz* satisfying the conditions

$$\text{for each } i \geq 0, uv^i xy^i z \in A$$

$$|xy| > 0, \text{and}$$

$$|vxy| \leq p$$

Theory of Computation

Lecture 5: The Church-Turing Thesis

Lecturer: Dr. Ng, Wing Yin
TA: Ms. Sun, Binbin

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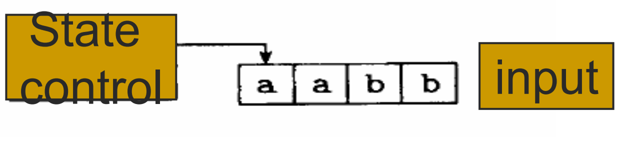
Harbin Institute of Technology Shenzhen Graduate School.

Content of Lecture 5

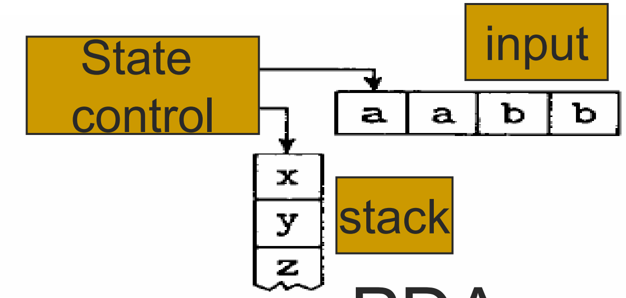
- Turing Machines
- Variants of Turing Machines
- The definition of algorithm

Turing Machines (TM)

- So far have learned **FA** and **PDA**.
- In section 2.3, the pumping lemma for CFL, we conclude that there are some language that is not context-free.
- E.g. $\{a^n b^n c^n \mid n \geq 0\}$ is not a context-free language.
- Both of FA and PDA are too restricted to serve as models of general purpose computers.



FA



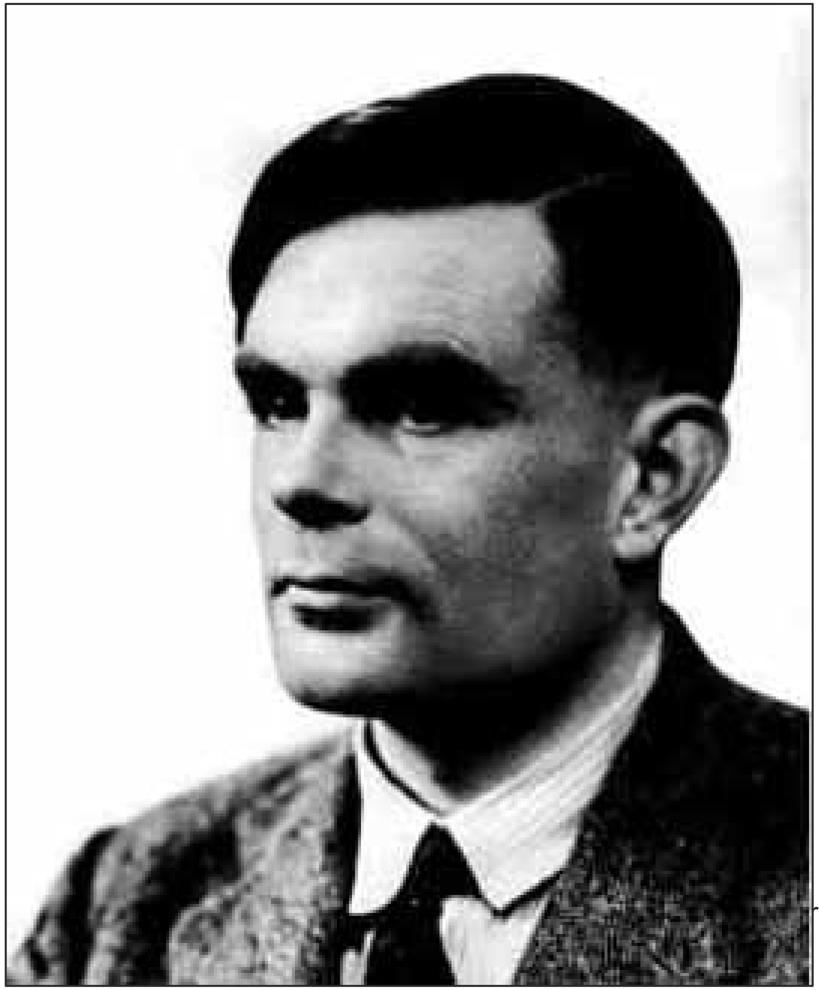
PDA

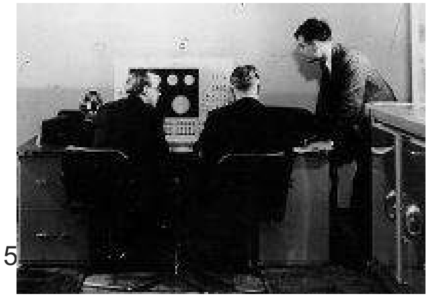
$B = \{abc\}$

Turing Machines

- We know that the Turing model has become the standard model in theoretical computer science.
- In 1936, Alan Turing proposed his abstract model for computation in his article "*On Computable Numbers,*".
- A Turing machine is a much more accurate model of a general purpose computer. It can do everything that a real computer can do.

Alan Turing 1912~1954



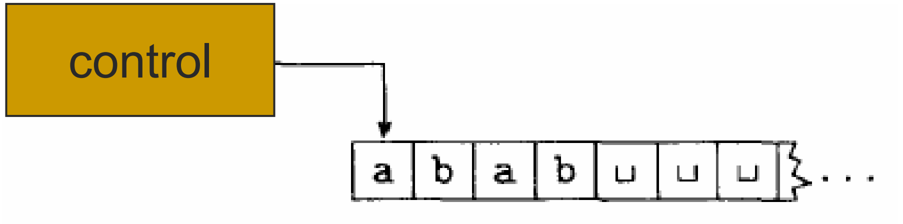


Turing Machines

- **A.M. Turing Award** is ACM's most prestigious technical award which is accompanied by a prize of $100,000. It is given to an individual selected for contributions of a technical nature made to the computing community. The contributions should be of lasting and major technical importance to the computer field. Financial support of the Turing Award is provided by the Intel Corporation.

Turing Machines

- The Turing machine model uses an infinite tape as its unlimited memory.



- There is a tape head that can
  - Read symbols,
  - write symbols, and
  - move around on the tape.

Turing Machines

- Initially the tape contains only the input string and is blank (␣)every where else.
- Then the machine continues computing until it decides to produce an output.
- The outputs ***accepts*** and ***reject*** are obtained by entering designated accepting and rejecting states.
- Therefore, if the machine doesn't enter accepting or a rejecting state, it will go on forever, never halting.

Turing Machines

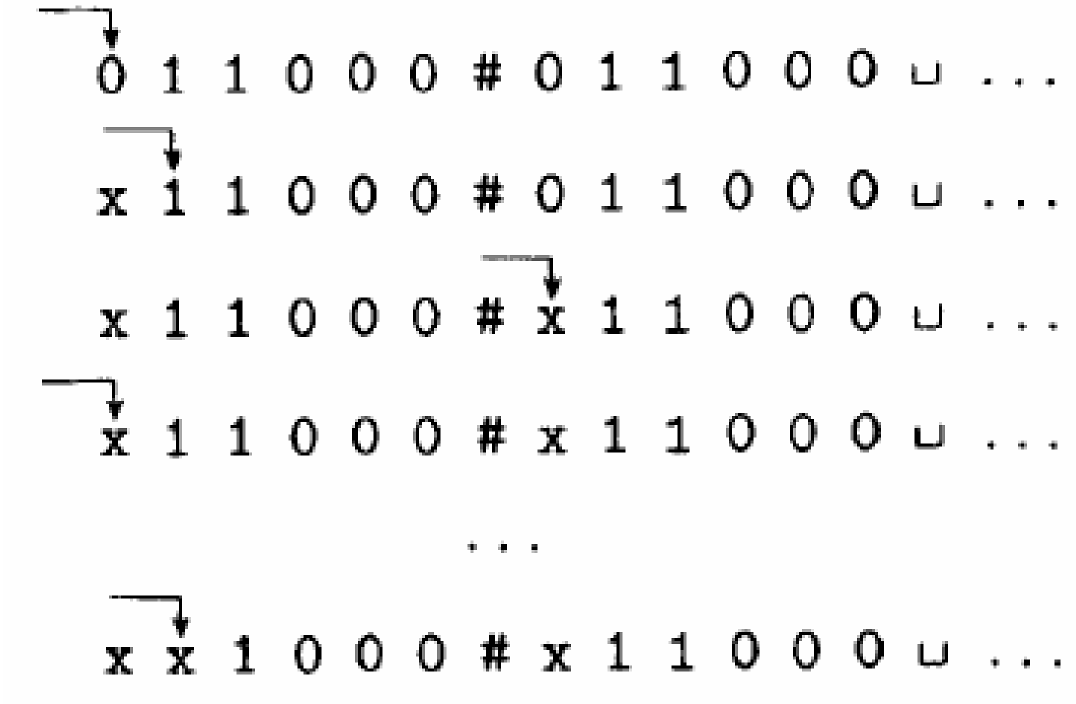
- The differences between FA and TM:
  - A Turing machine can both write on the tape and read from it.
  - The read-write head can move both to the left and to the right.
  - The tape is infinite.
  - The special states for rejecting and accepting take effect immediately.

Turing Machines

- Example: $B = \{w\#w \mid w \in \{0,1\}^*\}$
- So when you want to design a TM M_1 to recognize *B*, what should you do?
- We can do it as following:
  - Zigzags to the corresponding places on the two sides of the #
  - Determines whether they match.
  - Crosses off ($\times$) each symbol as it is examined.
  - If it crosses off all symbols, then accept, while if it discovers a mismatch, reject.

Turing Machines

- E.g. a string **011000#011000**



0 1 1 0 0 0 # 0 1 1 0 0 0 ␣ . . .

x 1 1 0 0 0 # 0 1 1 0 0 0 ␣ . . .

x 1 1 0 0 0 # x 1 1 0 0 0 ␣ . . .

x 1 1 0 0 0 # x 1 1 0 0 0 ␣ . . .

. . .

x x 1 0 0 0 # x 1 1 0 0 0 ␣ . . .

Turing Machines

- In summary, M_1's algorithm is as follows.
- M_1 =“On input string w:
  - Zigzag across the tape to corresponding positions on either positions on either side of the # symbol:
    - check whether these positions contain the same symbol.
    - If they not, or if no # is found, reject.
    - Cross off ($\times$) symbols as they are checked to keep track of which symbols correspond.
  - When all symbols to the left of the # have been crossed off, check for any remaining symbols to the right of the #.
    - If any symbols remain, reject
    - otherwise, accept.”

Formal definition of a Turing Machines

Note before the definition:

- Though we give the formal definition to a TM, in actuality we almost never give formal descriptions of Turing machines because they tend to be very big.
- For the transition function δ in a TM definition, it takes the form: $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$
  - where ***L*** or ***R*** indicates a move to the left or right.
  - E.g. $\delta(q, a) = \delta(r, b, L)$ indicates the machine writes the symbol *b* replacing the *a*, and goes to state *r*.

Formal definition of a Turing Machines

- A Turing machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where Q, Σ and Γ are all finite sets, and
  - Q is the set of states,
  - Σ is the input alphabet not containing the blank symbol $\sqcup$,
  - Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$
  - $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
  - q_0 is the start state, and
  - $q_{accept} \in Q$ is the set of accept states, and
  - $q_{reject} \in Q$ is the set of reject states, where $q_{accept} \neq q_{accept}$

Formal definition of a Turing Machines

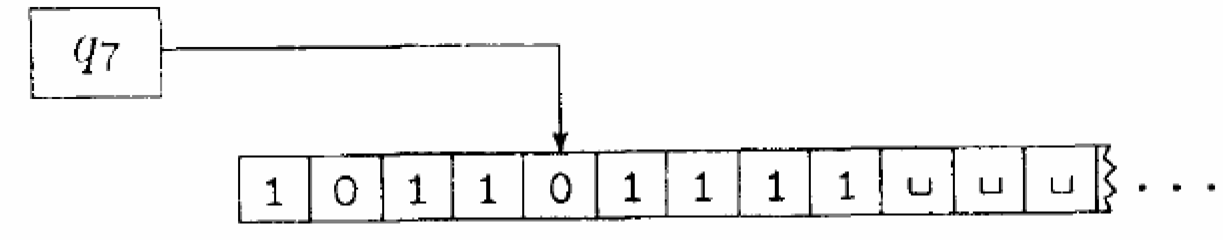
Note behind the definition

- For an input $w = w_1 w_2 \ldots w_n \in \Sigma^*$, takes the left n places of the tape.
- as $\sqcup \notin \Sigma$, the first blank appearing on the tape marks the end of the input.
- If the read-write head is on the left-hand end of the tape, even the transition function indicates ***L***, the head stays in the same place for that move.
- The computation continues until it enters either the accept or reject states at which point it halts. If neither occurs, *M* goes on forever.

Formal definition of a Turing Machines

Configuration

- As a Turing machine computes, a setting of the following three items is called a configuration
  - the current state, the current tape contents, and the current head location
- E.g. $1011q_7 01111$ Represents the configuration when the tape is 101101111, the current state is q_7 , and the head is currently on the second 0.



Formal definition of a Turing Machines

Yields

- Configuration C_1 yields configuration C_2 if the Turing machine can legally go from C_1 to C_2 in a single step.
- Formally:
- Suppose that we have *a*, *b*, and *c* in Γ , as well as *u* and *v* in Γ^* and states q_1 and q_2 . In that case uaq_ibv and uq_iacv are two configurations. Say that

$$uaq_ibv \text{ yields } uq_jacv \quad \text{if } \delta(q_i, b) = (q_j, c, L)$$

$$uaq_ibv \text{ yields } uacq_jv \quad \text{if } \delta(q_i, b) = (q_j, c, R)$$

Formal definition of a Turing Machines

- So for the two ends of the tape

$$q_i bv \ \text{ yields } q_j cv \quad \text{if } \delta(q_i, b) = (q_j, c, L)$$

$$uaq_i \ \text{ yields } ua\sqcup q_j \quad \text{if } \delta(q_i, b) = (q_j, c, R)$$

note $uaq_i\sqcup$ is equivalent to uaq_i

Formal definition of computation for a TM

- The start configuration of $\boldsymbol{M}$ on input w is the configuration q_0w, which indicates that the machine is in the start state q_0 with its head at the leftmost position on the tape.
- In an accepting configuration the state of the configuration is q_{accept} .
- In a rejecting configuration the state of the configuration is q_{reject} .
  - Accepting and rejecting configurations are halting configurations and do not yield further configurations.

Formal definition of a Turing Machines

- Let Q' be Q without q_{accept} and q_{reject}.
- The transition function in definition is changed to

  $Q' \times \Gamma \rightarrow \mathrm{Q} \times \Gamma \times \{L,R\}$

- A Turing machine M accepts input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists, where
  - C_1 is the start configuration of M on input w.
  - Each C_i yields C_{i+1}, and
  - C_k is an accepting configuration.

Formal definition of a Turing Machines

- The collection of strings that *M* accepts is the language of *M*, or the language recognized by *M*, denoted *L*(*M*)
- Call a language Turing-recognizable if some Turing machine recognizes it.

Formal definition of a Turing Machines

Decide

- Outcomes of a Turing machines may be: accept, reject, loop (here loop means TM doesn’t halt).
- The Turing machine that halt on all inputs, and never loop is called deciders.
- A decider that recognizes some language also is said to decide that language.
- Call a language Turing-decidable or simply decidable if some Turing machine decides it.

Example of Turing machine

Describe certain Turing machine

- Formal definition
  - Describe all the seven parts of the TM.
  - Big even for a tiny TM.
- Higher level descriptions
  - Precise enough
  - Easier to understand.

Example of Turing machine

- Example 1: $A = \{0^{2^n} \mid n \geq 0\}$

  Design a TM M_2 to recognize A.

High level description:

M_2= “On input string w:

1. Sweep left to right across the tape, crossing off every other 0.
2. If in stage 1 the tape contained a single 0, ***accept***.
3. If in stage 1 the tape contained more than a single 0 and the number of 0s was odd, ***reject***.
4. Return the head to the left-hand end of the tape.
5. Go to stage 1.”

Example of Turing machine

- Formal definition of $M_2 = (Q, \Sigma, \Gamma, \delta, q_1, q_{accept}, q_{reject})$

  $Q = \{q_1, q_2, q_3, q_4, q_5, q_{accept}, q_{reject}\},$

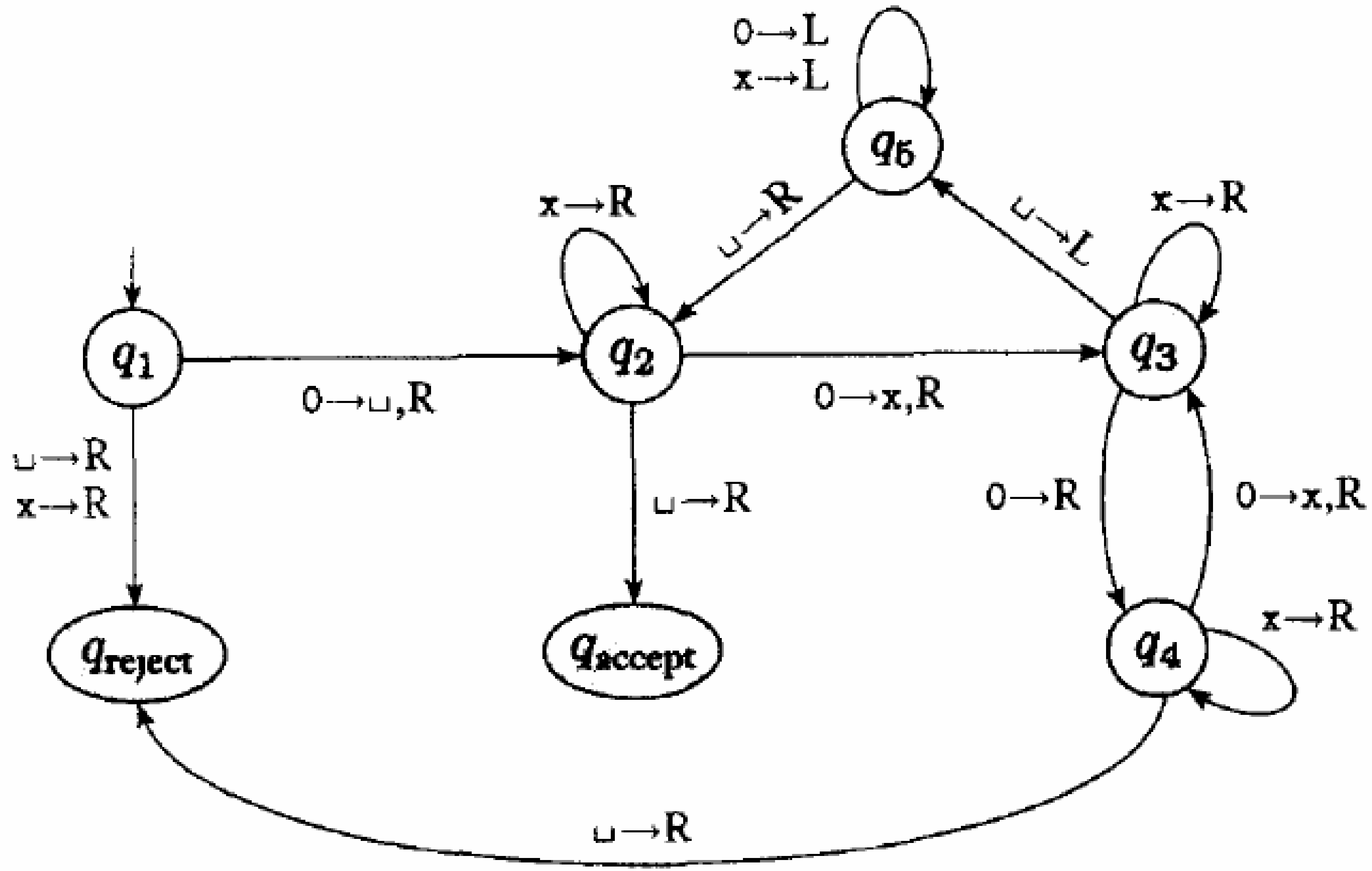
  $\Sigma = \{0\},$

  $\Gamma = \{0, \mathbf{x}, \sqcup\}$

  We describe δ with the state diagram.

  $q_1, q_{accept}, q_{reject}$ are the start, accept and reject states.

Example of Turing machine



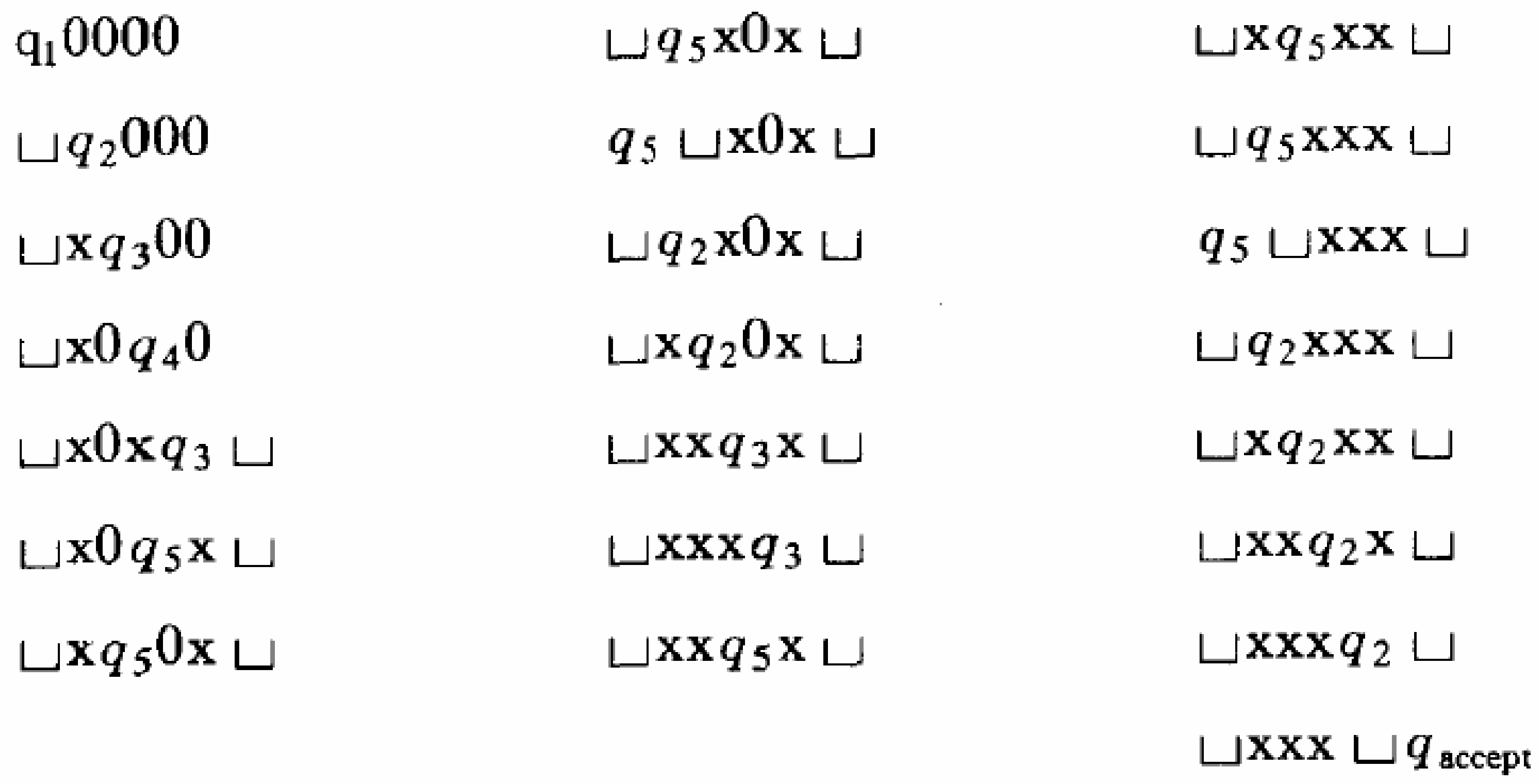
Example of Turing machine

The beginning transition:

- Here write a blank symbol over the leftmost **0** on the tape so that it can find the left-hand end of the tape later.
- We would normally use a more suggestive symbol such as # for the left-hand end delimiter, here use the blank here to keep the tape alphabet, and hence the state diagram, small.

Example of Turing machine

- Example string: 0000



$q_1 0000$

$\sqcup q_2 000$

$\sqcup \mathrm{x} q_3 00$

$\sqcup \mathrm{x} 0 q_4 0$

$\sqcup \mathrm{x} 0 \mathrm{x} q_3 \sqcup$

$\sqcup \mathrm{x} 0 q_5 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_5 0 \mathrm{x} \sqcup$

$\sqcup q_5 \mathrm{x} 0 \mathrm{x} \sqcup$

$q_5 \sqcup \mathrm{x} 0 \mathrm{x} \sqcup$

$\sqcup q_2 \mathrm{x} 0 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_2 0 \mathrm{x} \sqcup$

$\sqcup \mathrm{xx} q_3 \mathrm{x} \sqcup$

$\sqcup \mathrm{xxx} q_3 \sqcup$

$\sqcup \mathrm{xx} q_5 \mathrm{x} \sqcup$

$\sqcup \mathrm{x} q_5 \mathrm{xx} \sqcup$

$\sqcup q_5 \mathrm{xxx} \sqcup$

$q_5 \sqcup \mathrm{xxx} \sqcup$

$\sqcup q_2 \mathrm{xxx} \sqcup$

$\sqcup \mathrm{x} q_2 \mathrm{xx} \sqcup$

$\sqcup \mathrm{xx} q_2 \mathrm{x} \sqcup$

$\sqcup \mathrm{xxx} q_2 \sqcup$

$\sqcup \mathrm{xxx} \sqcup q_{\mathrm{accept}}$

Example of Turing machine

- Example 2: $C = \{a^i b^j c^k \mid i \times j = k, \text{and } i, j, k \geq 1\}$
  Design a TM M_3 to recognize B.

High level description:

M_3 = "On input string w:

1. Scan the input from left to right to determine whether it is a member of $a^+b^+c^+$ and ***reject*** if it isn't.
2. Return the head to the left-hand end of the tape.

Example of Turing machine

3. Cross off an *a* and scan to the right until *ab* occurs. Shuttle between the *b*'s and the *c*'s, crossing off one of each until all *b*'s are gone. If all *c*'s have been crossed off and some *b*'s remain, ***reject***.
4. Restore the crossed off *b*'s and repeat stage 3 if there is another *a* to across off. If all *a*'s have been crossed off, determine whether all *c*'s also have been crossed off. If yes, ***accept***; otherwise, reject."

Example of Turing machine

- Example 3: element distinctness problem

$$E = \left\{ \# x_1 \# x_2 \# x_3 ... \# x_l \mid \text{each } x_i \in \{0,1\} \text{ and } x_i \neq x_j \ \ \text{for each } i \neq j \right\}$$

Design a TM M_4 to recognize E.

High level description:

M_4= "On input string w:

1. Place a mark on top of the leftmost tape symbol. If that symbol was a blank, ***accept***. If that symbol was a #, continue with the next stage. Otherwise, ***reject***.
2. Scan right to the next # and place a second mark on top of it. If no # is encountered before a blank symbol, only x_1 was present, so ***accept***."

Example of Turing machine

3. By zigzagging, compare the two strings to the right of the marked #s. If they are equal, ***reject***.
4. Move the rightmost of the two marks to the next # symbol to the right. If no # symbol is encountered before a blank symbol, move the leftmost mark to the next # to its right and the rightmost mark to the # after that. This time, if no # is available for the rightmost mark, all the strings have been compared, so ***accept***.
5. Go to stage 3."

Example of Turing machine

- On class exercise:

$$\left\{0^n 1^n \mid n \geq 0\right\}$$

Variants of Turing machines

- Variants of Turing machines:
  - Multitape Turing machines,
  - Nondeterminism Turing machines.
  - Enumerators
- Robustness:
  - The original model and its reasonable variants **recognize the same class of languages**.
  - FA and PDA are somewhat robust models .
  - TM have an astonishing degree of robustness.

Variants of Turing machines

- Think about that the head may **stay put** without moving to the left or right after each step. This model is equivalent to the original one.
- We can convert any TM with the “stay put” feature to one that doesn’t have it. We do so by replacing each stay put transition with two transitions, one that moves to the right and the second back to the left.

Multitape Turing machines (MTM)

- **MTM have several tapes**.
- Each tape has its own head for reading and writing.
- Initially the input appears on tape 1, and the others start out blank.
- The transition function is changed to allow for reading, writing, and moving the heads on some or all of the tapes simultaneously.

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$$

Multitape Turing machines (MTM)

$$\delta(q_i, a_1, a_2, \ldots a_k) = \delta(q_i, b_1, b_2, \ldots b_k, \underbrace{L, R, \ldots, L}_{k})$$

means that, if the machine is in state q_i and heads 1 trough k are reading symbols a_1 though a_k, the machine goes to state q_j ,writes symbols b_1 through b_k ,and directs each head to move left or right, or to stay put, as specified.

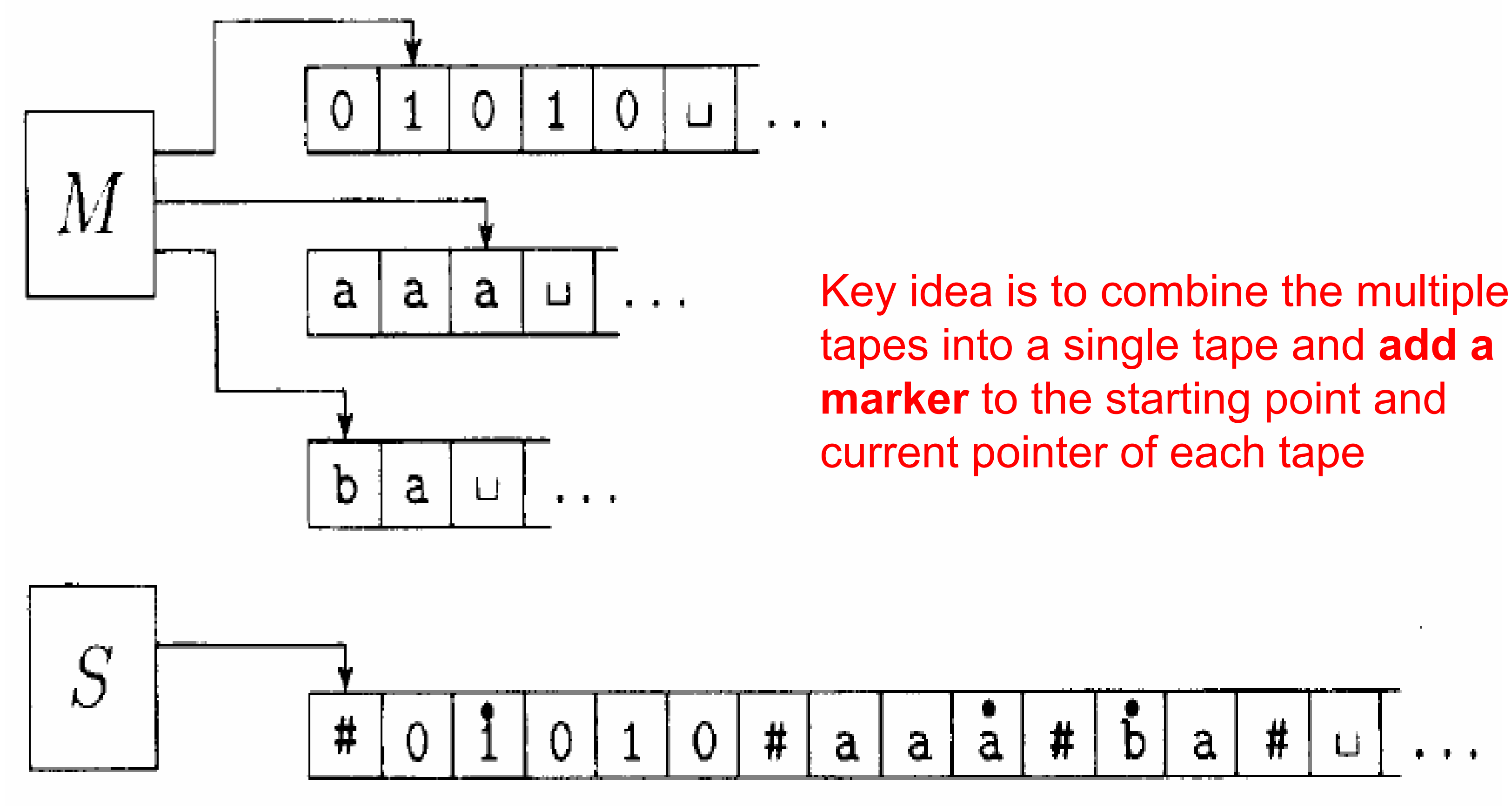
Multitape Turing machines (MTM)

- MTM is more powerful than ordinary Turing machines?
- **No!!**
- Theorem:
- Every Multitape Turing machine has an equivalent single-tape Turing machine.

Multitape Turing machines (MTM)

- The proof is to show how to convert a multitape TM *M* to an equivalent single-tape TM *S*.
- The key idea is to show how to simulate *M* with *S*.
- Read Ep.151, and Cp.93-94

Multitape Turing machines (MTM)



Key idea is to combine the multiple tapes into a single tape and **add a marker** to the starting point and current pointer of each tape

Nondeterministic Turing machines (NTM)

- A **nondeterministic Turing machine** is defined as: at any point in a computation the machine may proceed according to several possibilities.
- The transition function has the form:

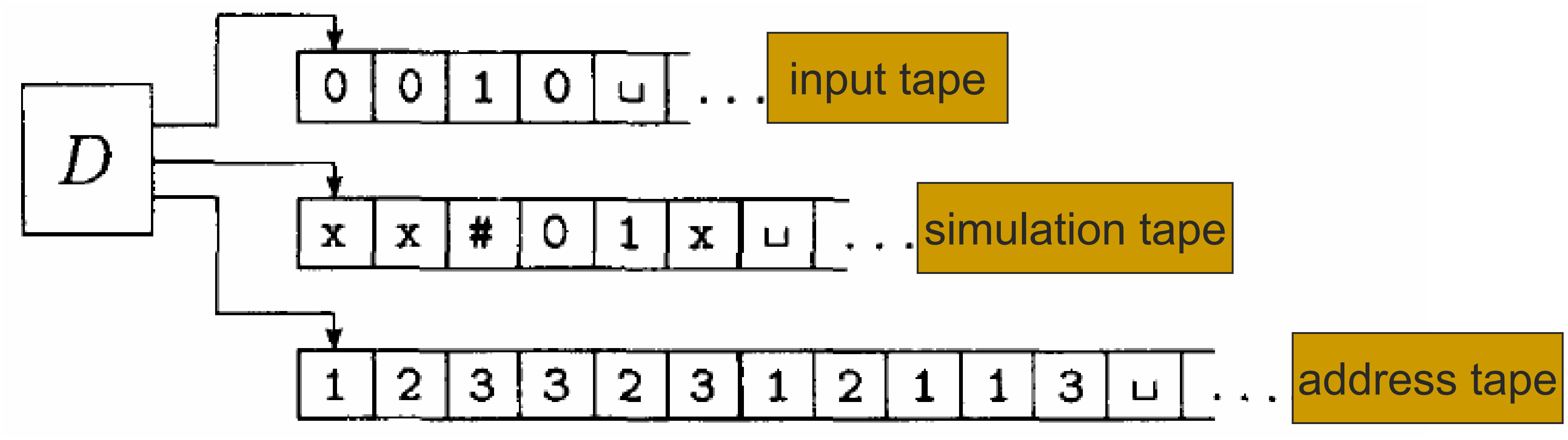
$$\delta : Q \times \Gamma \rightarrow \mathrm{P}\left(Q \times \Gamma \times \{L, R\}\right)$$

Nondeterministic Turing machines (NTM)

- Theorem: Every nondeterministic Turing machine has an equivalent deterministic Turing machine.
- The key idea is to simulate any NTM *N* with a deterministic TM *D*.
  - We take *D* to try all possible branches of *N*'s nondeterministic computation.
  - If *D* ever finds the accept state on one of these branches, *D* accepts.
  - Otherwise, *D*'s simulation will not terminate.

Nondeterministic Turing machines (NTM)

- A simulation tape is added to record the nondeterministic branches of the NTM



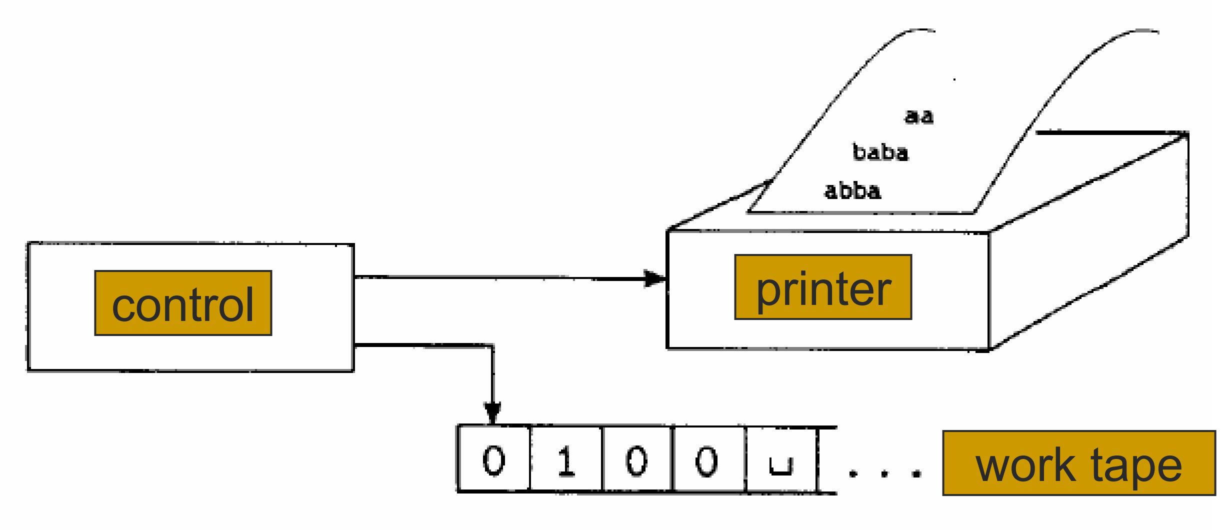
Enumerators

- Another Turing machine variant which have an attached printer.
- **Enumerator** can use the printer as an output device to print string.
- The language enumerated by the machine is the collection of all the strings that it eventually prints out.
- It does not intake any input and print can print the strings in the language in any order

Enumerators

On class exercise:

- Give the formal definition of enumerator!!



Enumerators

- Theorem: A language is Turing-recognizable if and only if some enumerator enumerates it.

Other equivalent variant

- There are many other models of general purpose computation. Some of these models are very much like TM, but others are quite different.
- All these machines share the essential feature of TM, namely unrestricted access to unlimited memory.
- This essential feature makes these models to be equivalent in power.

Other equivalent variant

- Example:
  - When we want to write an **algorithm**, ordinarily we can use the programming language that we are familiar with, such as C, java, Pascal.
  - This means that these programming languages describe exactly the same class of algorithm.
- Any two computational models that satisfy certain reasonable requirements can simulate one another and hence are equivalent in power.

Definition of Algorithm

- Informally, an algorithm is a collection of simple instructions for carrying out some task.
- E.g.
  - to find the prime numbers,
  - to find the greatest common divisors.
  - To calculate the value of π (Zu Chongzhi)
  - To calculate square roots. (Abū ‘Abd Allāh Muhammed)

Definition of Algorithm

- Hilbert's problems
- In 1900, mathematician David Hilbert (1862-1943) delivered a now-famous address at the International Congress of Mathematicians in Paris. In the twenty-three mathematical problems he proposed to be challenge for the coming century, the 10th problem is about algorithms.
- This problem is to devise an algorithm to test whether a *polynomial* has an *integral root*.

Definition of Algorithm

- polynomial

$$6x^3yz^2 + 7xy - x^4$$

- coefficient
- root $6x^3yz^2 + 7xy - x^4 = 0$

  $x = 0$ is a integral root.

Definition of Algorithm

- We now know it is impossible to find an algorithm that achieves this.
- To prove that an algorithm does not exist requiring a precise definition of algorithm.
- This is Church-Turing thesis.

Intuitive notion of algorithms equals Turing machine algorithms

Definition of Algorithm

- So the Hilbert's 10th problem can be defined as: whether the set D is decidable.
- $D=\{p \mid p$ is a polynomial with an integral root$\}$

Definition of Algorithm

- Consider a simpler problem: constrain the polynomials to have only a single variable x.
- $D_1 = \{p \mid p$ is a polynomial over x with an integral root$\}$
- The TM M_1 recognizes D_1

M_1 =" The input is polynomial p over the variable x.

1. Evaluate p with x set successively to the values 0,1,-1,2,-2,…. If at any point the polynomial evaluates to 0, accept."

Definition of Algorithm

- We can design TM M that modify the TM M_1 to test all possible setting of its variables to integral values.
- By the Matijasevic theorem, we can convert M_1 to be a decider, but can not convert M.

Definition of Algorithm

- By the Church-Turing thesis, now the Turing machine serves as a precise model for the definition of algorithm.
- Now we concentrate on algorithm.
- The first thing to do is to standardize the way we describe Turing machine algorithms.

Definition of Algorithm

- **Formal description**
  - $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$
  - Lowest level, highly detailed and precise
- **Implementation description**
  - Use English prose to describe the way that the TM moves its head and the way that it stores data on its tape.
- **High-level description**
  - Use English prose to describe an algorithm, ignoring the implementation details. We don't have to mention how the machine manages its tape or head.

Definition of Algorithm

- Now the input is changed to be object. So the first step is to convert object to strings.
- $\langle O \rangle$ stands to convert an object O into a string. If several object$O_1, O_2, ...O_k$ into a single string is $\langle O_1, O_2, ...O_k \rangle$

Definition of Algorithm

- Second describe TM algorithms with an indented segment of text within quotes.
  - Here we can break the algorithm into stages, each involving many individual steps of the TM's computation.
  - The first line of the algorithm describes the input to the machine.

Definition of Algorithm

- Example: Let A be the language consisting of all strings representing undirected graphs that are connected. Give the high-level description of the TM M to recognize A.

$$A = \left\{ \langle G \rangle \mid G \text{ is a connected undirected graph} \right\}$$

Definition of Algorithm

- Solution:

M="On input <*G*>:

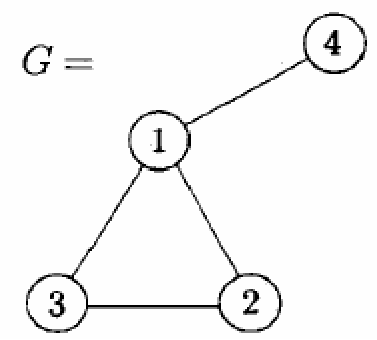
1. Select the first node of *G* and mark it.
2. Repeat the following stage until no new nodes are marked:

   For each node in *G*, mark it if it is attached by an edge to a node that is already marked.

3. Scan all the nodes of *G* to determine whether they all are marked. If they are, ***accept***; other wise, ***reject***."

Definition of Algorithm

- A instance of *G*.
- <G> denotes by the list of nodes <1,2,3,4> followed by the edge pair.



$\langle G \rangle =$ `(1,2,3,4)((1,2),(2,3),(3,1),(1,4))`

Theory of Computation

Lecture 6: Decidability

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Content of Lecture 5

- Decidable languages
- Halting problem
  - Diagonalization method
  - The halting problem is undecidable
  - A Turing-unrecognizable language

Decidable language

- You should have heard that some famous scientists are not famous for they solve some problem, but for proving that some problem are unsolvable.
- Though much of computer science is devoted to solving problems, one should still aware that some problems are unsolvable.
- Knowing when a problem is algorithmically unsolvable is useful
  - You realize that the problem must be simplified or altered before you can find an algorithmic solution.
  - Cultural: a glimpse of the unsolvable problem can stimulate your imagination and help you gain an important perspective on computation.

Decidable language

- Recall your memory:
  - Decidable (here means TM-decidable): No loop, always make a decision to accept or reject.
- We focus on decidability on the familiar language
  - Regular language
  - Context-free language
- Notice:
  - Though automata and grammar can be recognized by limited storage machine, but some of them are not decidable by algorithms.
  - These two languages have close relation to some applications such as recognizing and compiling programs in a programming language.

Decidable problems concerning regular languages

Acceptance problem for automata

- To test whether a particular automaton accepts a given string. A_{****} contains the encodings of all automata(****) together with strings that the automata accept. e.g.
- Acceptance problem for DFAs

$$A_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

- Acceptance problem for NFAs

$$A_{NFA} = \{\langle B, w\rangle \mid B \text{ is a NFA that accepts input string } w\}$$

- Acceptance problem for REXs (regular expressions)

$$A_{REX} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$$

Decidable problems concerning regular languages

$$A_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

- Theorem: A_{DFA} is a decidable language
- The problem of testing whether a DFA B accepts an input w is the same as the problem of testing whether $\langle B, w\rangle$ is a member of the language A_{DFA} .
- Hence this theorem shows that the problem of testing whether a given finite automaton accepts a given string is decidable.

Decidable problems concerning regular languages

A_{DFA} is a decidable language.

- Proof idea:

We simply need to present a TM M that decides A_{DFA}

M="On input<B, w>,where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, accept. If it ends in a nonaccepting state, reject."

Decidable problems concerning regular languages

Proof:

- Examine the input <B, w>. It is a representation of a DFA B together with a string w, where B= (Q, ∑, δ , q_0, F) and $w \in \Sigma^*$
- We present a TM M that decides A_{DFA} . M performs as following:

Decidable problems concerning regular languages

- M=“On input<B, w>,where B is a DFA and w is a string:

1. Simulate B on w with the help of two pointers: $P_q \in Q$ for the internal state of the DFA, and $P_w \in \{0,1,\ldots,|w|\}$ for the position on the string. While we increase P_w from 0 to $|w|$, we change P_q according to the input letter w_{Pw} and the transition function value $\delta(P_q, w_{Pw})$.
2. When M finishes processing the last symbol of w, M ***accepts*** the input if B is in an accepting state; M ***rejects*** the input if B is in a nonaccepting state.”

Decidable problems concerning regular languages

$$A_{NFA} = \{\langle B, w \rangle \mid B \text{ is a NFA that accepts input string } w\}$$

Theorem: A_{NFA} is a decidable language

- We can also prove it in the way of simulating.
- Here we use the simulating TM *M* in the last proof to construct the TM *N* (if we demonstrate it in programming language, *M* is a subroutine of *N*.)

Decidable problems concerning regular languages

Proof:

We present a TM *N* that decides A_{NFA} :

N="On input <*B*, *w*>,where *B* is a NFA and *w* is a string:

1. Convert NFA *B* to an equivalent DFA *C*.
2. Run TM *M* from the last proof on input <*C*, *w*>.
3. If *M* accepts , ***accept***; otherwise, ***reject***."

Decidable problems concerning regular languages

$$A_{REX} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$$

- Theorem: A_{REX} is a decidable language

Proof:

We present a TM *P* that decides A_{REX} :

N="On input<*R*, *w*>,where *R* is a regular expression and *w* is a string:

1. Convert regular expression *R* to an equivalent NFA *A*.
2. Run TM *N* on input <*A*, *w*>.
3. If *N* accepts , ***accept***; otherwise, ***reject***."

Decidable problems concerning regular languages

Emptiness testing for automaton:

- To test whether a particular automaton accepts none string.
- E_{****} contains the encodings of all automata(****) that recognize none strings.
- E.g. $E_{DFA} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \phi\}$

Decidable problems concerning regular languages

$$E_{DFA} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \phi\}$$

Theorem: E_{DFA} is a decidable language.

- Proof:
- T="On input <A> where A is a DFA:

1. Mark the start state of A,
2. Repeat until no new states get marked:
   - Mark any state that has a transition coming into it from any state that is already marked.
3. If no accept state is marked, ***accept***; otherwise, ***reject***.

Decidable problems concerning regular languages

Equivalent testing for automaton:

- To test whether two particular automaton accepts the same string.
- E.g.

$$EQ_{DFA} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

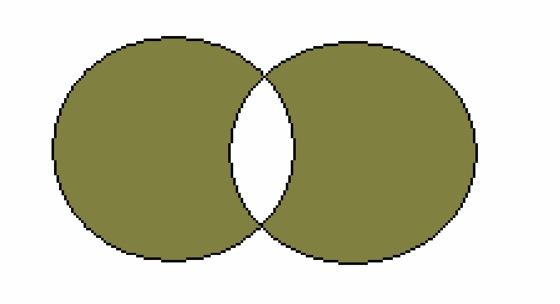
Decidable problems concerning regular languages

$$EQ_{DFA} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

Theorem: EQ_{DFA} is a decidable language.

Proof:

- We proof the symmetric difference of $L(A)$ and $L(B)$ are empty instead of proving $L(A)=L(B)$.



Decidable problems concerning regular languages

- F = "On input $<A, B>$ where A and B are DFAs:

1. Construct DFA C to recognize the language $\left(L(A)\cap\overline{L(B)}\right)\cup\left(L(B)\cap\overline{L(A)}\right)$.
2. Run TM T from the last empty testing theorem on input $<C>$.
3. If T accepts, ***accept***; otherwise, ***reject*.**"

Decidable problems concerning context-free languages

- Acceptance problem for context-free grammar.

$$A_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$$

- Theorem:

A_{CFG} is a decidable language.

Decidable problems concerning context-free languages

Proof idea:

- If we use *G* to go through all derivations to determine whether the result is a derivation of *w*.
- This idea doesn’t work because there may be infinitely many derivations to be tried.
- This idea gives a Turing machine that is a recognizer, but not a decider. So we can’t use it to prove “decidable”.

Decidable problems concerning context-free languages

- We convert the CFG *G* in the Chomsky normal form.
- As we can easily prove that *G* can be converted into the Chomsky normal form, by which any derivation of *w* has 2|*w*|-1 steps.

Chomsky normal form

- **Chomsky normal form:**
  - A context-free grammar is in Chomsky normal form if every rule is of the form

    $$A \rightarrow BC$$

    $$A \rightarrow a$$

  - where
    - ***a*** is any terminal
    - ***A***, ***B***, and ***C*** are any variables---except that ***B*** and ***C*** may not be the start variable
    - Permit rule $S \rightarrow \varepsilon$, only when ***S*** is the start variable.

Decidable problems concerning context-free languages

Proof:

- The TM S for A_{CFG} is as following:
- S="On input $<G,w>$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2|w|-1$ steps, except if $|w|=0$, then instead list all derivations with 1 step.
3. If any of these derivations generate w, ***accept***; if not, ***reject***."

Decidable problems concerning context-free languages

- This theorem is applicable for the compiler of a programming language.
- This theorem is correct for PDA too, for the equivalence of CFG and PDA.

Decidable problems concerning context-free languages

- Empty problem for context-free grammar.

$$E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \phi\}$$

- Theorem:

$$E_{CFG} \text{ is a decidable language.}$$

Decidable problems concerning context-free languages

- For we have infinite number of *w*, we can't do as we do in the proof of equivalent of DFA.
- When we present the TM *R*, firstly, we have to mark all the terminal symbols in the grammar. Then, it scan all the rules of the grammar.
- If it ever finds a rule that permits some variable to be replaced by some string of symbols all of which are already marked, the algorithm knows that this variable should be marked, too.

Decidable problems concerning context-free languages

Proof:

- R="On input <*G*>, where *G* is a CFG:

1. Mark all terminal symbols in *G*.
2. Repeat until no new variables get marked:
   - Mark any variable *A* where *G* has a rule $A \rightarrow U_1U_2...U_k$ and each symbol $U_1,U_2...,U_k$ has already been marked.
3. If the start variable is not marked, ***accept***; otherwise, ***reject***."

Decidable problems concerning context-free languages

- Notice:

$$EQ_{CFG} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

EQ_{CFG} is not decidable.

- For we can prove the class of context-free language is not closed under complementation or intersection.
- So we can't prove it as we do for the finite automaton machines.

Decidable problems concerning context-free languages

Theorem:

- Every context-free language is decidable.

Proof:

- We use the TM S that we designed to decide A_{CFG}.
- Let G be a CFG for A and design a TM M_G that decides A. We build a copy of G into M_G. It works as follows:

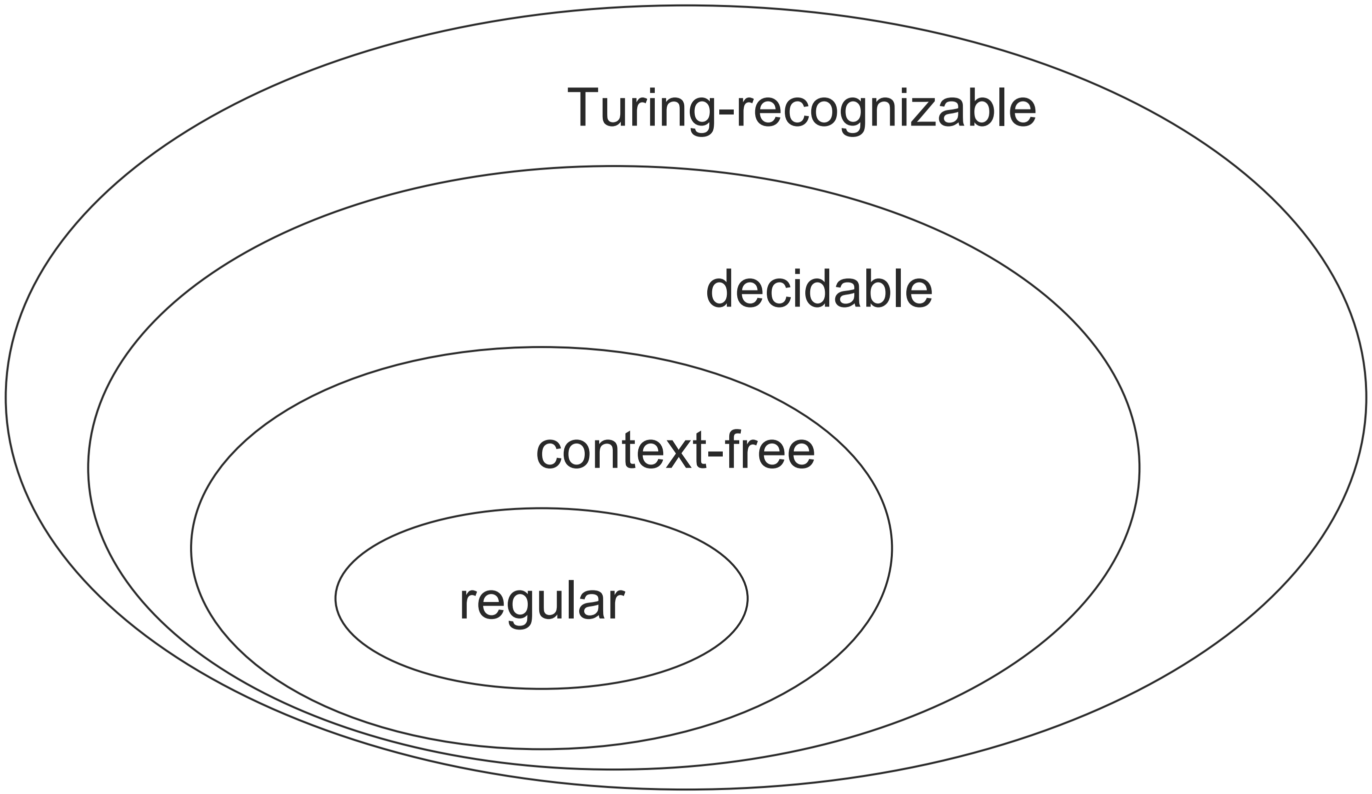
Decidable problems concerning context-free languages

- M_G =“On input string w:

1. Run TM S on input<G, w>
2. If this machine accepts, ***accept***; if it rejects, ***reject***.”

Notice: from these theorem we can recognize the relations among these languages better

Decidable problems concerning context-free languages



The halting problem

- The first problem that we will prove undecidability:
- Determining whether a Turing machine accepts a given input string.

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts input string } w\}$$

The halting problem

- Theorem: A_{TM} is not decidable.
- We have to prepare some knowledge before proving this theorem (Halting problem):
  - countable.

The halting problem

We can easily prove that A_{TM} is Turing-recognizable.

- *U*="On input <*M*, *w*>, where *M* is a TM and *w* is a string:

1. Simulate *M* on input *w*.
2. If *M* accepts, ***accept***; if it rejects, ***reject***"

The halting problem

- We can see that U is not a decider, for it may get into a loop on an input string w.
- If the algorithm has some way to determine that M was not halting on w, it could reject. Then it is decidable, but it can't.
- So A_{TM} is called **halting problem**.

The Diagonalization method

- Diagonalization method is discovered by Georg Cantor in order to solve the problem of measuring the sizes of infinite sets.
- E.g. the set of even integers $E=\{2,4,6,\ldots\}$ and the set of natural numbers $N=\{1,2,3,\ldots\}$, give the relation of $|E|$ and $|N|$.
- $|E| < |N|$????
- Cantor uses "correspondence" to define this problem.

The Diagonalization method

- Cantor

The Diagonalization method

Definition:

- Assume that we have sets A and B and a function f from A to B.
- Say that f is ***one-to-one*** if it never maps two different elements to the same place.

  if $f(a) \neq f(b)$ whenever $a \neq b$

- Say that f is ***onto*** if it hits every elements of B

  if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$

The Diagonalization method

- Say that A and B are the ***same size*** if there is a one-to-one, onto function (correspondence)

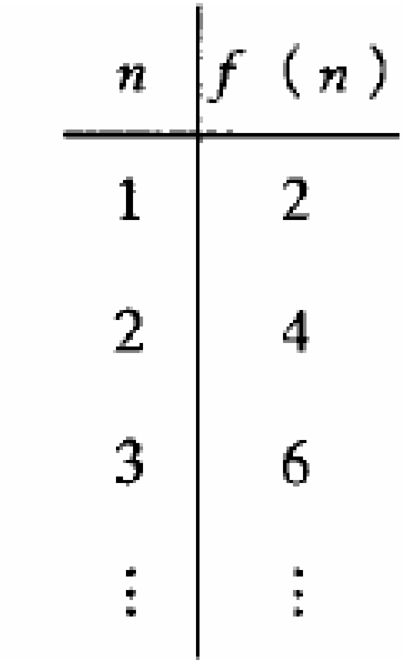
$$f : A \rightarrow B$$

- A function that is both one-to-one and onto is called a ***correspondence***. (So in a correspondence every element of A maps to a unique element of B and each element of B has a unique element of A mapping to it.)
- A correspondence is simply a way of ***pairing the elements of A with the elements of B***.

The Diagonalization method

- E.g. the set of even integers E={2,4,6,…} and the set of natural numbers N={1,2,3,…}, give the relation of |E| and |N|.
- Using Cantor's definition of size:
  - *Correspondence:*

    $e = f(n) = 2n$ *where* $n \in N$, $e \in E$
- So N and E have the same size.



n	$f(n)$
1	2
2	4
3	6
⋮	⋮

The Diagonalization method

Definition:

- A set A is ***countable*** if either it is finite or it has the same size as $\boldsymbol{N}$.

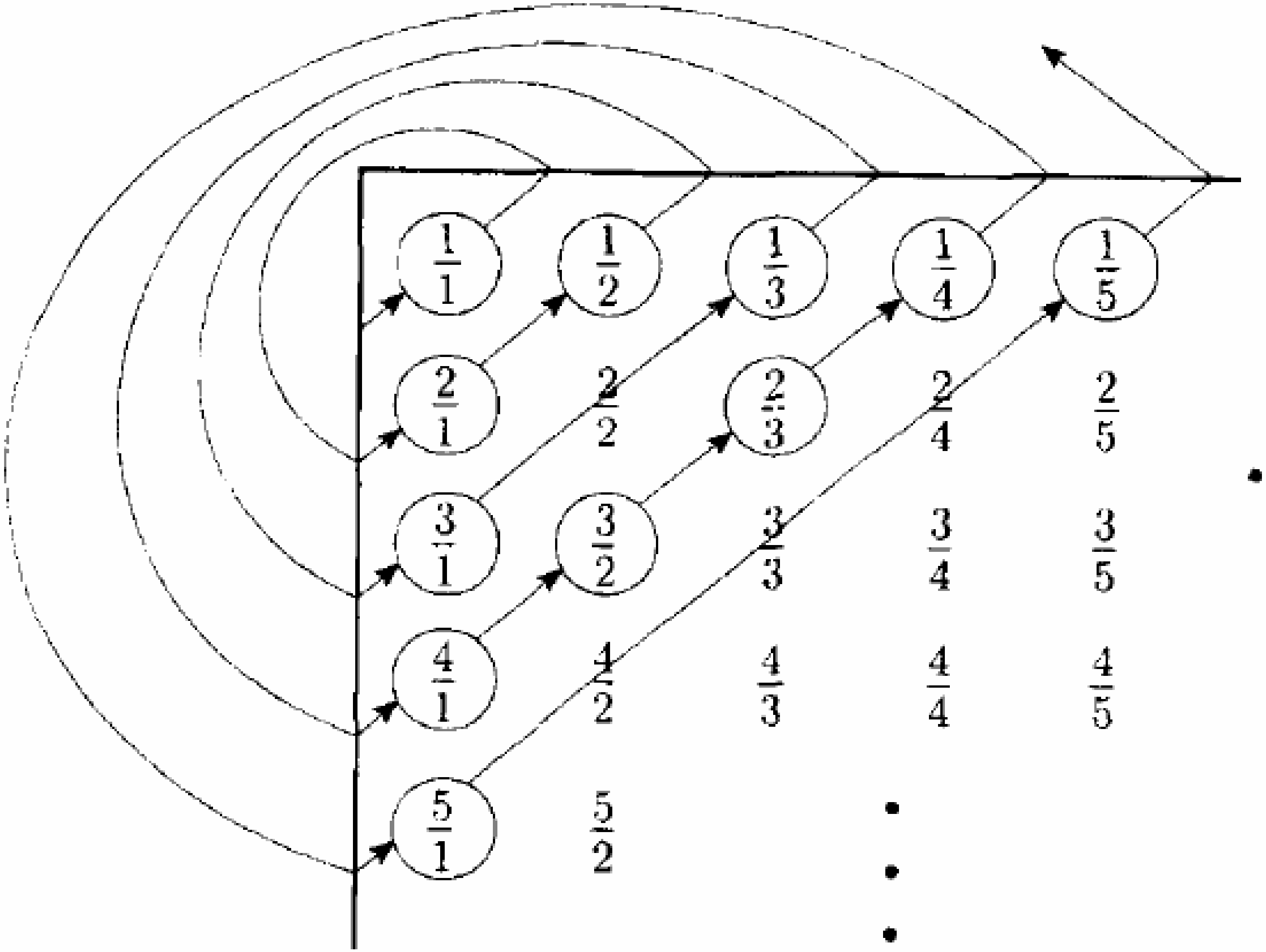
The Diagonalization method

Example 1:

The set of positive rational numbers $\boldsymbol{Q}$ is countable

- We represent $\boldsymbol{Q}$ in the fraction form $\boldsymbol{Q}=\{\ m/n \mid m,n \in \boldsymbol{N}\}$
- List all elements of $\boldsymbol{Q}$, then pair the elements of $\boldsymbol{Q}$ with the number of $\boldsymbol{N}$.
- Construct an infinite matrix containing all the positive rational numbers, the *i*th row contains all numbers with numerator i and the *j*th column has all numbers with denominator j.

The Diagonalization method



The Diagonalization method

Example 2:

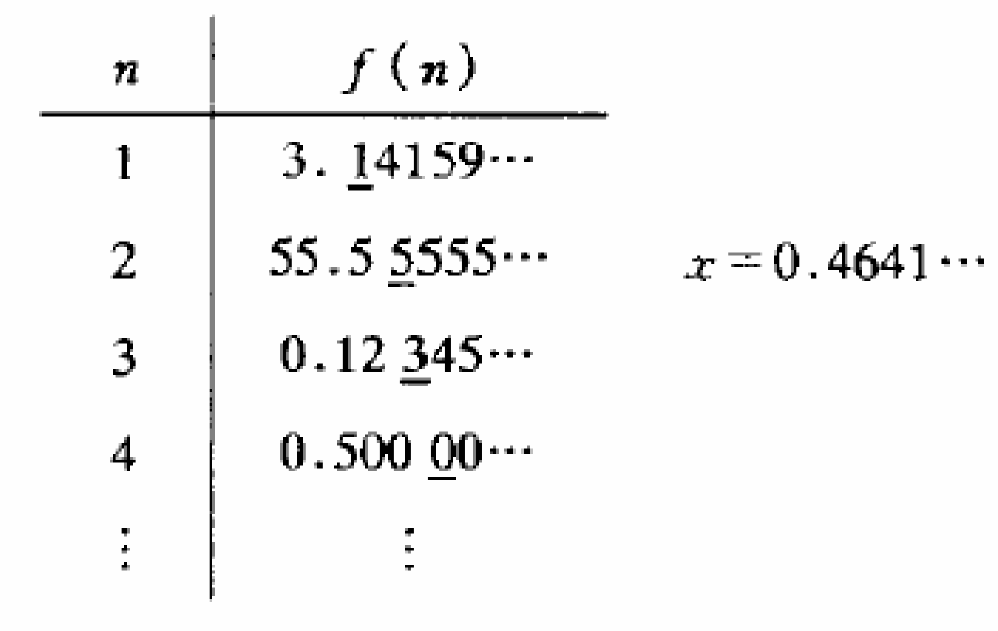
Real number set ***R*** is uncountable.

Proof: (prove by contradiction)

- Suppose ***R*** is countable, and there is a correspondence *f* existed between ***N*** and ***R***.
- We try to find an element *x* in ***R*** that is not paired with any element in ***N***.

The Diagonalization method

- Construct x by every digit.
- Choose each digit of x to make x different from one of the real numbers that is paired with an element of N.



n	$f(n)$
1	$3.\underline{1}4159\cdots$
2	$55.5\underline{5}555\cdots$
3	$0.12\underline{3}45\cdots$
4	$0.500\underline{0}0\cdots$
$\vdots$	$\vdots$

$x = 0.4641\cdots$

The Diagonalization method

- As x is not $f(n)$ for any n because it differs from $f(n)$ in the nth fractional digit. So x has no corresponding element n in N. There is a contradiction.
- So Real number set is uncountable.

The Diagonalization method

On class exercise:

- Prove that the set of all infinite binary sequences is uncountable.
  - (note: infinite binary sequence is an unending sequence of 0s and 1s.)
- E.g.

$$\underbrace{0000...}_{\text{infinite}},\ \underbrace{1000...}_{\text{infinite}},\underbrace{010...}_{\text{infinite}},...$$

Turing-unrecognizable language

Corollary:

Some languages are not Turing-recognizable.

Proof:

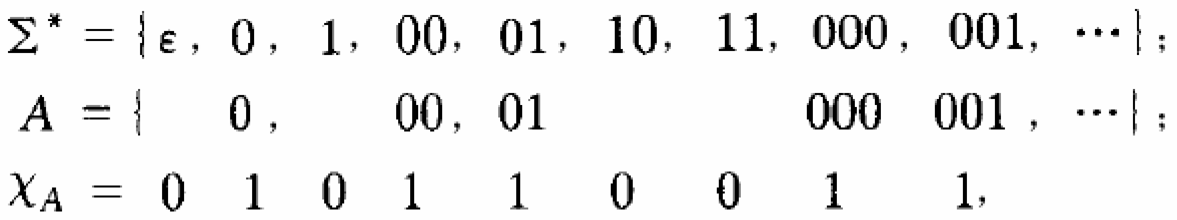
- The set of all strings Σ^* is countable, for any alphabet Σ.
  - We may form a list of Σ^* by writing down all strings of length 0, length 1, and so on.

Turing-unrecognizable language

- The set of all Turing machines is countable.
  - Each Turing machine M has an encoding into a string <*M*>.
  - Omit those strings that are not legal encodings of Turing machines, we can list all Turing machines.
- The set of all languages (***L***) is uncountable.
  - The set of all infinite binary sequences (***B***) is uncountable.
  - The correspondence of the set of all languages over alphabet Σ (***L***) with ***B***.

Turing-unrecognizable language

- Now prove the correspondence.
- Let $\Sigma^* = \{s_1, s_2, \ldots\}$, then each language $A \in \boldsymbol{L}$ has a unique sequence in $\boldsymbol{B}$ by the following method.
- The ith bit of that sequence is a 1 if $s_i \in A$ and is a 0 if $s_i \notin A$. Denoted as X_A, and we it the characteristic sequence of A.



$\Sigma^* =$	$\{\varepsilon,$	$0,$	$1,$	$00,$	$01,$	$10,$	$11,$	$000,$	$001,$	$\cdots\};$
$A =$	$\{$	$0,$		$00,$	01			000	$001,$	$\cdots\};$
$\chi_A =$	0	1	0	1	1	0	0	1	$1,$	

Turing-unrecognizable language

- We can’t put the uncoutable ***L*** (the set of all languages) into a correspondence with the countable ***M*** (set of all Turing machines).
- So there are some languages that are not recognized by any Turing machine.

The Halting problem is undecidable

Theorem

A_{TM} is not decidable.

Proof:

- Assume A_{TM} is decidable and prove by contradiction.
- Suppose *H* is a decider for A_{TM} .

The Halting problem is undecidable

- *H*="On input <*M*, *w*>, where *M* is a TM and *w* is a string:

1. Simulate *M* on input *w*.
2. If *M* accepts, ***accept***; if it rejects, ***reject***."

The Halting problem is undecidable

- Construct a new TM *D* with *H* as a subroutine.
- *D*="On input <*M*>, where *M* is a TM

1. Run *H* on input <*M*,< *M*>>.
2. Output the opposite of what *H* outputs; that is, if *H* accepts, ***reject*** and if *H* rejects, ***accept***."

In summary:

$$D(\langle M\rangle)=\begin{cases} accept & \text{if } M \text{ does not accept } \langle M\rangle \\ reject & \text{if } M \text{ accept } \langle M\rangle \end{cases}$$

The Halting problem is undecidable

- Take its own description <*D*> as input:

$$D(\langle D\rangle)=\begin{cases} accept & \text{if } D \text{ does not accept } \langle D\rangle \\ reject & \text{if } D \text{ accept } \langle D\rangle \end{cases}$$

- No matter what *D* does, it is forced to do the opposite, which is obviously a contradiction.
- So neither TM *D* nor TM *H* can exist.

The Halting problem is undecidable

‘Acceptance behavior’ of M_i on $<M_j>$

TM

strings

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	accept		accept		
M_2	accept	accept	accept	accept	
M_3					$\cdots$
M_4	accept	accept			
$\vdots$			$\vdots$		$\ddots$

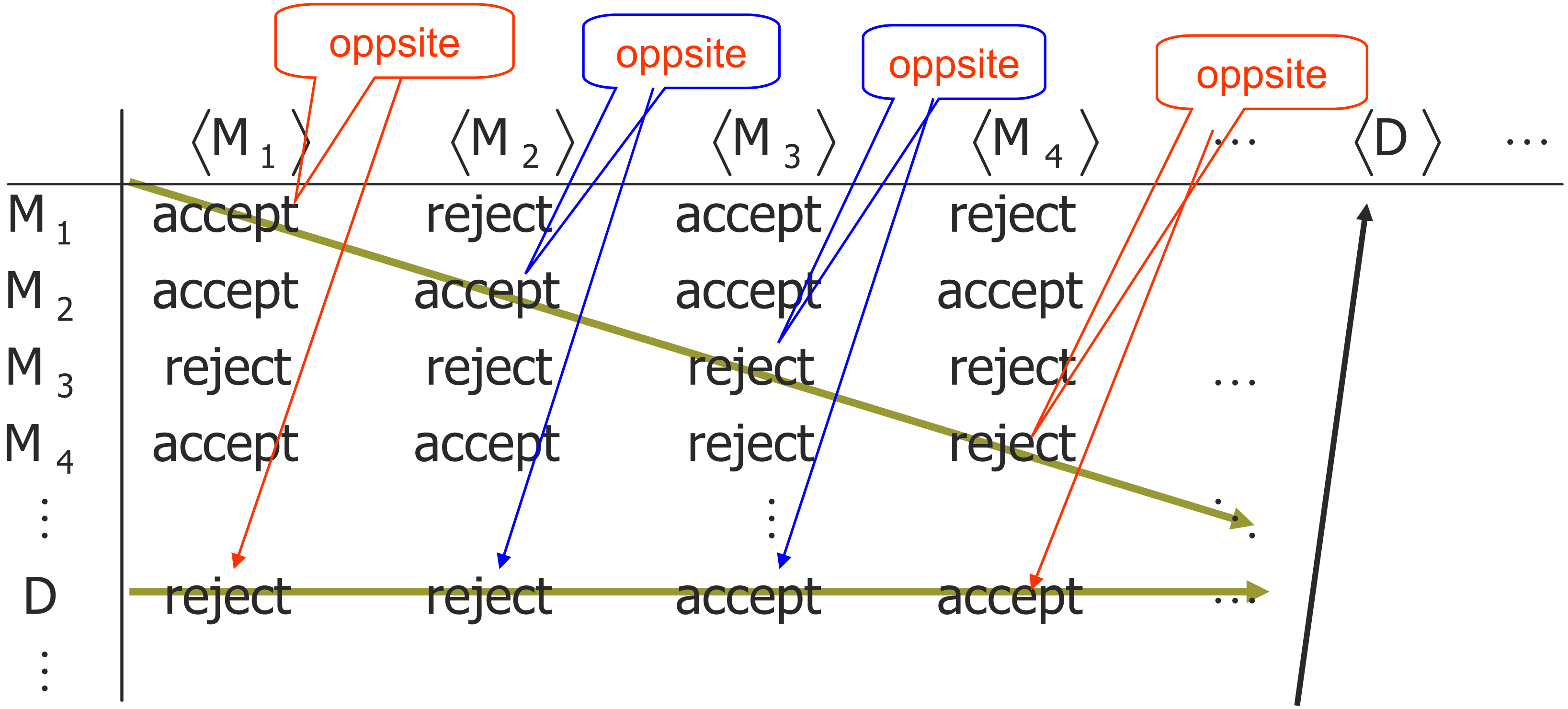
The Halting problem is undecidable

- Output of *H*

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	accept	reject	accept	reject	
M_2	accept	accept	accept	accept	
M_3	reject	reject	reject	reject	$\cdots$
M_4	accept	accept	reject	reject	
$\vdots$			$\vdots$		$\ddots$

The Halting problem is undecidable

- A contradict occurs if we take <*D*> as input.



	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	accept	reject	accept	reject			
M_2	accept	accept	accept	accept			
M_3	reject	reject	reject	reject	$\cdots$		
M_4	accept	accept	reject	reject			
$\vdots$			$\vdots$		$\ddots$		
D	reject	reject	accept	accept	$\cdots$		
$\vdots$							

Disagreeing *D* has to occur in list as well…

Co-Turing-recognizable language

Definition:

- A language is co-Turing-recognizable if it is the complement of a Turing-recognizable language.

Theorem:

- A language is decidable **iff** it is Turing-recognizable and co-Turing-recognizable.

Co-Turing-recognizable language

- → is easy to prove.
- Now show how to prove ←
- Suppose A and $\overline{A}$ are Turing-recognizable. Let M_1 be the recognizer of A and M_2 be the recognizer of $\overline{A}$.
- Let M be a two-tape Turing machine.

Co-Turing-recognizable language

- M="On input string w:

1. Run M_1 on the first tape, and Run M_2 on the second tape. (run in parallel)
2. If M_1 accepts, ***accept***; if M_2 accepts, ***reject***."

- As the string of A will be accepted by M_1. And the string not in A will be accepted by M_2. So M always halts and is a decider of A.

Co-Turing-recognizable language

- $\overline{A_{TM}}$ is not Turing-recognizable.
- Proof:
- A_{TM} is Turing-recognizable, if $\overline{A_{TM}}$ also were Turing-recognizable, A_{TM} would be decidable.
- As A_{TM} is not decidable, so $\overline{A_{TM}}$ must not be Turing-recognizable.

Theory of Computation

Lecture 7: Reducibility

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Content of Lecture 7

- Undecidable problems from language theory
  - Reduction
  - Computation history
- Post correspondence problem
- Mapping reducibility

Reducibility

- We have used the prove by construction method proving that some problems are decidable:

  $A_{DFA}, A_{NFA}, A_{REX}, E_{DFA}, EQ_{DFA}$

  A_{CFG}, E_{CFG}, every CFL

- The problem that we have proved to be undecidable is the halting problem A_{TM}

Reducibility

- In this chapter we first introduce a method called reduction, to prove that some problems are undecidable.
- A reduction is a way of converting one problem ***A*** to another problem ***B*** in such a way that a solution to problem ***B*** can be used to solve the problem ***A***.
- When ***A*** is reducible to ***B***, solving ***A*** cannot be harder than solving ***B*** because a solution to ***B*** gives a solution to ***A***.

Reducibility

- Example:
  - Problem ***A***: measuring the area of a rectangle
  - Problem ***B***: measuring length and width of a rectangle.
  - We can reduce ***A*** to ***B***.
- Example 2:
  - Problem ***A***: solving a system of linear equations.
  - Problem ***B***: inverting a matrix.
  - We can reduce ***A*** to ***B***.

Reducibility

- On the decidability theory and the complexity theory, reducibility is important.
- If $\boldsymbol{A}$ is undecidable and reducible to $\boldsymbol{B}$, $\boldsymbol{B}$ is undecidable too. This is key to proving that various problems are undecidable while we know that the halting problem A_{TM} is undecidable.

Undecidable problems from language theory

- $HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

So $HALT_{\text{TM}}$ determine whether a Turing machine halts on a given input.

- Prove $HALT_{\text{TM}}$ is undecidable.

Proof:

- This proof is by contradiction. Assume that $HALT_{\text{TM}}$ is decidable. After showing that A_{TM} is reducible to $HALT_{\text{TM}}$, we can conclude that A_{TM} is decidable, so there is a contradiction.

Undecidable problems from language theory

- Assume that TM *R* decides *HALT*$_{TM}$, Now construct TM *S* to decide *A*$_{TM}$, with *S* operating as follows:
- *S*="On input <*M*,*w*>, an encoding of a TM *M* and a string *w*:
  1. Run TM *R* on input <*M*,*w*>.
  2. If *R* rejects, ***reject***.
  3. If *R* accepts, simulate *M* on *w* until it halts.
  4. If *M* has accepted, ***accept***; if *M* has rejected, ***reject***."

Undecidable problems from language theory

- So from now on, we have a common strategy to prove a problem is undecidable, except for the undecidability of A_{TM} itself, which is proved directly via the diagonalization.

Undecidable problems from language theory

- $E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \phi\}$
- So E_{TM} determine whether a Turing-recognizable language is empty.
- Prove E_{TM} is undecidable.
- Proof:
- Ordinarily, A_{TM} means that for the encoding <*M*,*w*>, we can't decide (no loop) whether *M* recognize *w*.

Undecidable problems from language theory

- Assume that TM *R* decides E_{TM}, Now use *R* to construct TM *S* to decide A_{TM} .
- Instead of running *R* directly on <*M*>, we run *R* on a modification of <*M*>. For we want to judge the empty corresponding to recognize *w*. so we modify <*M*> to guarantee that *M* rejects all strings except *w*, and work as usual on input *w*.

Undecidable problems from language theory

- Modified machine M_1 works as following:
- M_1 ="On input x:

1. If $x \neq w$, ***reject***.
2. If $x = w$, run M on w and ***accept*** if M does."

Undecidable problems from language theory

- S=“On input <M,w>, an encoding of a TM M and a string w:
  1. Run TM R on input $< M_1 >$.
  2. If R accepts, ***reject***; if R rejects, ***accept***.”

Rice’s theorem

- The problem of testing whether the language of a Turing machine is a simpler language (such as regular language, context-free language, decidable language, finite language) is undecidable.
- Rice’s theorem: the problem of testing any property of the languages recognized by Turing machines is undecidable.

Rice's theorem

$$REGULAR_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$$

- So $REGULAR_{\mathrm{TM}}$ determine whether a given Turing machine has a equivalent finite automaton. This kind of problem is to determine whether a given Turing machine recognizes a language that also can be recognized by a simpler computational model.
- Prove $REGULAR_{\mathrm{TM}}$ is undecidable.

Rice's theorem

Proof:

- Assume that TM R decides $REGULAR_{\text{TM}}$, Now use R to construct TM S to decide A_{TM} .
- Instead of running R directly on <M>, we run R on a modification of <M> so that the resulting TM recognizes a regular language if and only if M accepts w. We design M_2 to recognize the nonregular language $\{0^n1^n \mid n \geq 0\}$ if M does not accept w, and to recognize the regular language Σ^* if M accepts w.

Rice's theorem

- Modified machine M_2 works as following:
- M_2 ="On input x:
  1. If x has the form 0^n1^n , ***accept***.
  2. If x does not have this form, run M on w and ***accept*** if M does."

Rice’s theorem

- *S*=“On input <*M*,*w*>, an encoding of a TM *M* and a string *w*:
  1. Run TM *R* on input $< M_2 >$.
  2. If *R* accepts , ***accept***; if *R* rejects, ***reject***.”

Undecidable problems from language theory

- On class exercise:
- Prove that

$CF_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a context-free language}\}$

is undicidable

Undecidable problems from language theory

- So far, we reduce to the halting problem A_{TM} . And use the method of proving by contradiction to A_{TM}. Of course we can also reducing from some other undecidable language such as E_{TM}, which may be more convenient.

A_{TM}

Undecidable problems from language theory

- $EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$
- EQ_{TM} determines the equivalence of two Turing machines.
- Prove EQ_{TM} is undecidable.

Undecidable problems from language theory

Proof:

- Assume that TM R decides EQ_{TM}, now use R to construct TM S to decide E_{TM} .
- S="On input <M>, where M is a TM:
  1. Run TM R on input <M, M_1 >, where M_1 is a TM that rejects all inputs.
  2. If R accepts , ***accept***; if R rejects, ***reject***."

Reductions via computation histories

- The computation history for a Turing machine on an input is simply the sequence of configurations that the machine goes through as it processes the input. It is a complete record of the computation of this machine.
- The computation history method is an important technique for proving that A_{TM} is reducible to certain languages. It is often useful when the problem to be shown undecidable involves testing for the existence of something.

Reductions via computation histories

- Let M be a Turing machine and w an input string. An accepting computation history for M on w is a sequence of configurations, $C_1, C_2, \ldots, C_l$,
  - C_1 is the start configuration of M on w,
  - C_l is an accepting configuration of M,
  - each C_{i+1} legally follows from C_i according to the rules of M.
- A rejection computation history for M on w is defined similarly, except that C_l is a rejecting configuration.
- So accepting computation histories are finite sequences.
- If M doesn't halt on w, no accepting or rejecting computation history exists.

Reductions via computation histories

- A **linear bounded automaton (LBA)** is a restricted tape of Turing machine wherein the tape head isn't permitted to move off the portion of the tape containing the input. If the machine tries to move its head off either end of the input, the head stays where it is, in the same way that the head will do as on the left-hand end of an ordinary Turing machine's tape.

Reductions via computation histories

- Note:
- LBA is not equivalent to ordinary TM. LBAs, with limited memory, are still quite powerful.
- $A_{DFA}, A_{CFG}, E_{DFA}, E_{CFG}$ and every CFL can be decided by an LBA.

A_{TM}

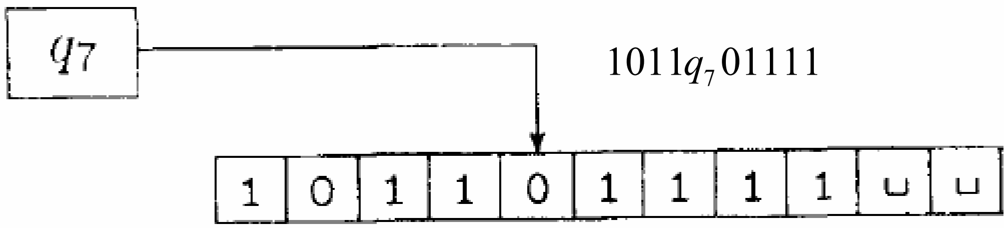
Reductions via computation histories

Lemma:

- Let M be an LBA with q states and g symbols in the tape alphabet. There are exactly qng^n distinct configurations of M for a tape of length n.

Proof:

- As the configuration consists of the state of the control, position of the head and contents of the tape, so it is easy to conclude this lemma.

- 

$1011q_7 01111$

Reductions via computation histories

$A_{LBA} = \{\langle M, w\rangle \mid M \text{ is an LBA that accepts string } w\}$

- Prove A_{LBA} is decidable.
- Proof:
- On LBA M which has finite configuration, when M computes on w, it goes from configuration to configuration. If M ever repeats a configuration it would go on to repeat this configuration over and over again and thus be in a loop. So only a limited amount of time is available to M before it will enter some configuration that it has previously entered. If we test by simulating M for qng^n steps, we can determine if it is looping.

Reductions via computation histories

- L determines A_{LBA}
- L = "On input $<M,w>$, an encoding of a LBA M and a string w:
  1. Simulate M on w for qng^n steps or until it halts.
  2. If M has halted, ***accept*** if it has accepted and ***reject*** if it has rejected. If it has not halted, ***reject***."

Reductions via computation histories

- $E_{LBA} = \{\langle M \rangle \mid M \text{ is an LBA and } L(M) = \phi\}$
- Prove E_{LBA} is undecidable.
- Proof:
- Assume that TM R decides E_{LBA}, Now use R to construct TM S to decide A_{TM} .

Reductions via computation histories

- First, construct a LBA B to test whether $L(B)$ is empty.
- B recognizes all accepting computation histories for M on w.
- If M accepts w, this language contains on string (the sequence of computation configuration) and so is nonempty.
- If M does not accept w, this language is empty.

Reductions via computation histories

- Now construct *B*:
- Suppose *x* is an accepting computation history for *M* on *w*.

$$x\text{:}\quad \#\underbrace{\qquad\quad}_{C_1}\#\underbrace{\qquad\quad}_{C_2}\#\underbrace{\qquad\quad}_{C_3}\#\ \cdots\ \#\underbrace{\qquad\quad}_{C_l}\#$$

- So *B* will accept *x* too.

Reductions via computation histories

- First, B breaks up x according to the delimiters into strings C_1, $C_2,\ldots,C_l$. Then B determines whether the C_i satisfy the three conditions of an accepting computation history.
  - ○ C_1 is the start configuration of M on w,
  - ○ each C_{i+1} legally follows from C_i according to the rules of M,
  - ○ C_l is an accepting configuration of M.

Reductions via computation histories

- B has the string $q_0w_1w_2\ldots w_n$ which is the start configuration C_1 built in, so the first condition can be checked.
- B can check the third condition by scanning C_l for q_{accept}.
- According to the transition function of M, B check C_i and C_{i+1} for $i = 1,\ldots n\text{-}1$.

Reductions via computation histories

- Therefore *S* computes as following:
- *S*="On input <*M*,*w*>, an encoding of a TM *M* and a string *w*:
  1. Construct LBA *B* from *M* and *w*
  2. Run TM *R* on input < *B* >.
  3. If *R* accepts, ***reject***; if *R* rejects, ***accept***."

Reductions via computation histories

- $ALL_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$
- ALL_{LBA} determines whether a context-free grammar generates all possible strings.
- Prove ALL_{LBA} is undecidable.
- Proof:
- Assume that TM R decides ALL_{LBA}, Now use R to construct TM S to decide A_{TM} .
- Note: the reduction from A_{TM} via computation histories.

Reductions via computation histories

- Construct a CFG G for a TM M and an input w. G generates all strings if and only if M does not accept w. This means that G generate all strings that are not accepting computation histories for M on w.
- An accepting computation history for M on w appears as $\#C_1\#C_2\#\ldots\#C_l$.

Reductions via computation histories

- The string that G recognized should be testing in the following step:
  - Test if starts with C_1. If does not, ***accept***, else
  - Test if end with an accepting configuration C_l, If does not, ***accept***, else
  - Test if every C_i properly yields C_{i+1} under the rules of M. If does not, ***accept***, else ***reject***.

Reductions via computation histories

- Instead of actually constructing G, we construct a PDA D, then we certainly can convert D to G.
- As there are three conditions to test, so we have to design D having three copies.
    - One checks on whether the beginning of the input string is C_1.
    - One checks on whether the input string ends with a configuration containing the accept state, q_{accept}.
    - One checks on whether every C_i properly yields C_{i+1} under the rules of M

Reductions via computation histories

- The third copy works as following:
  - Scanning the input until it nondeterministically decides that it has come to C_i.
  - Push C_i onto the stack until it comes to the end as marked by the # symbol.
  - Pops the stack to compare with C_{i+1}. (in the comparing process, the poping symbol of stack is inverse to the order of C_i. so modify by the following method:

Reductions via computation histories

- ○ Method 1: in the beginning

$$\#\underbrace{\longrightarrow}_{C_1}\#\underbrace{\longleftarrow}_{C_2^{\mathcal{R}}}\#\underbrace{\longrightarrow}_{C_3}\#\underbrace{\longleftarrow}_{C_4^{\mathcal{R}}}\#\ \cdots\ \#\underbrace{\qquad\quad}_{C_l}\#$$

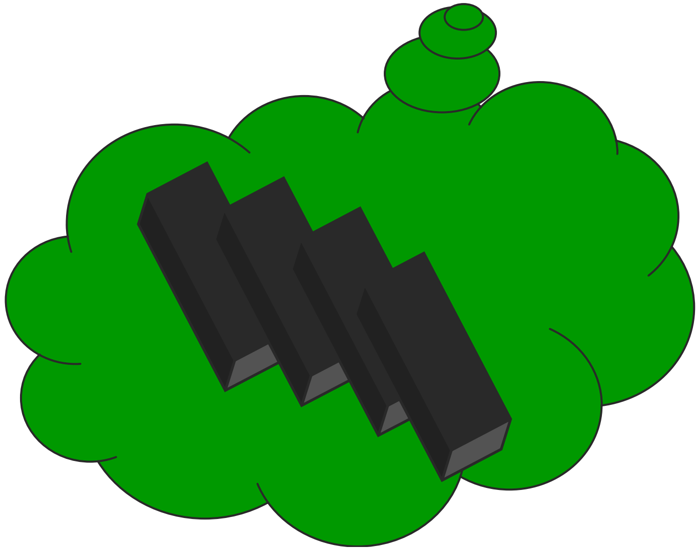
- ○ Method 2: in every step

  copy C_i to another stack, then C_i is in its order again.

- D accepts if it is a mismatch or an improper update.

Post correspondence problem

- Post correspondence problem (PCP) is in fact an undecidable problem concerning simple manipulations of strings. Here we demonstrate it as a puzzle.
- There is a collection of dominos.



Not this dominos

Post correspondence problem

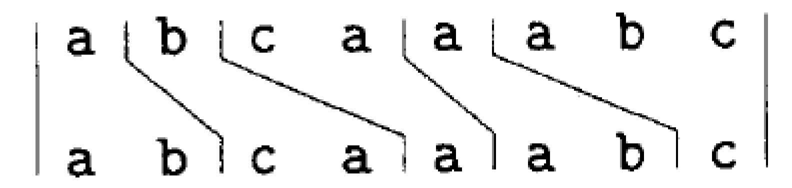
- Each dominos containing two strings, one on each side.
- E.g. a collection of dominos looks like
- $$\left\{ \left[\frac{\mathbf{b}}{\mathbf{ca}}\right], \left[\frac{\mathbf{a}}{\mathbf{ab}}\right], \left[\frac{\mathbf{ca}}{\mathbf{a}}\right], \left[\frac{\mathbf{abc}}{\mathbf{c}}\right] \right\}$$
- The task is to list the match which is a list of these dominos (repetitions permitted) so that the string we get by reading off the symbols on the top is the same as the string of symbols on the bottom.

Post correspondence problem

- A match for this puzzle:

$$\left[\frac{\mathbf{a}}{\mathbf{ab}}\right], \left[\frac{\mathbf{b}}{\mathbf{ca}}\right], \left[\frac{\mathbf{ca}}{\mathbf{a}}\right], \left[\frac{\mathbf{a}}{\mathbf{ab}}\right], \left[\frac{\mathbf{abc}}{\mathbf{c}}\right]$$

- Or written as



Post correspondence problem

- For some collections of dominos finding a match may not be possible for example, the collection (every top string is longer than the corresponding bottom string)

$$\left\{ \left[\frac{\mathbf{ca}}{\mathbf{a}} \right], \left[\frac{\mathbf{acc}}{\mathbf{ba}} \right], \left[\frac{\mathbf{abc}}{\mathbf{ab}} \right] \right\}$$

Post correspondence problem

Formally, PCP is a collection P of dominos:

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

and a match is a sequence i_1, i_2,…, i_l, where $t_{i_1} t_{i_2} \cdots t_{i_l} = b_{i_1} b_{i_2} \cdots b_{i_l}$ The problem is to determine whether P has a match.

PCP = { <P> | P is an instance of the Post correspondence problem with a match}

Post correspondence problem

- Theorem: *PCP* is undecidable.

Proof:

- Assume that TM *R* decides *PCP*, Now use *R* to construct TM *S* to decide A_{TM} .
- Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$
- Construct an instance of PCP *P* that has a match iff *M* accepts *w*.

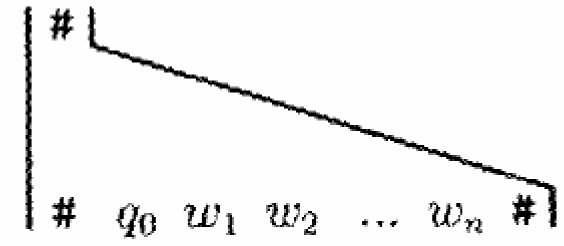
Post correspondence problem

- Construct an instance P' of MPCP where MPCP = { <*P*> | *P* is an instance of the Post correspondence problem with a match that starts with the first domino}
- The following seven parts accomplishes the simulation. (explain in example)

Post correspondence problem

Part 1:

- Put $\left[\frac{\#}{\# q_0 w_1 w_2 \ldots w_n \#}\right]$ into as the first domino $\left[\frac{t_1}{b_1}\right]$.



- The dominos of parts 2,3, and 4 extend the match by adding the next configuration after the current one.

Post correspondence problem

Part 2: (handles head motions to the right)

- For every $a, b \in \Gamma$ and every $q, r \in Q$ where $q \neq q_{reject}$

$$\text{if } \delta(q,a) = (r,b,R), \text{put} \left[\frac{qa}{br}\right] \text{ into } P'$$

Post correspondence problem

Part 3: (handles head motions to the left)

For every $a, b, c \in \Gamma$ and every $q, r \in Q$ where

$q \neq q_{reject}$

$$\text{if } \delta(q, a) = (r, b, L), \text{put} \left[\frac{cqa}{rcb} \right] \text{ into } P'$$

Post correspondence problem

Part 4: (handles the tape cells not adjacent to the head)

- For every $a \in \Gamma$

$$\text{put} \left[\frac{a}{a} \right] \text{ into } P'$$

Post correspondence problem

- Part 5: (adding the symbol #)

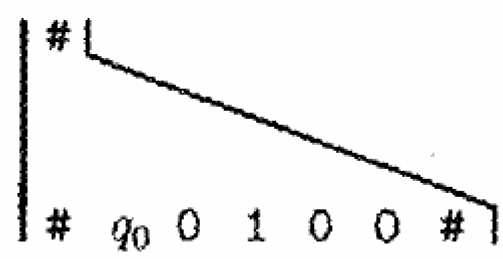
$$\text{put}\left[\frac{\#}{\#}\right]\text{ and }\left[\frac{\#}{\sqcup\#}\right]\text{into } P'$$

Post correspondence problem

- Example:

$$\Gamma = \{0, 1, 2, \Box\}, w = 0100, \delta(q_0, 0) = (q_7, 2, R)$$

- Part 1:

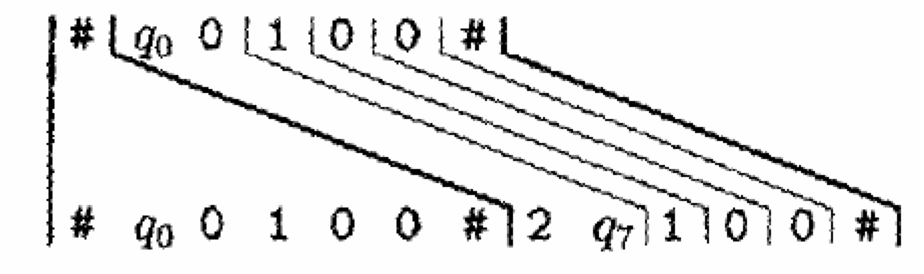


Post correspondence problem

- Part 2: put $\left[\frac{q_0 0}{2q_7}\right]$

- Part 4: put $\left[\frac{1}{1}\right],\left[\frac{0}{0}\right],\left[\frac{0}{0}\right]$

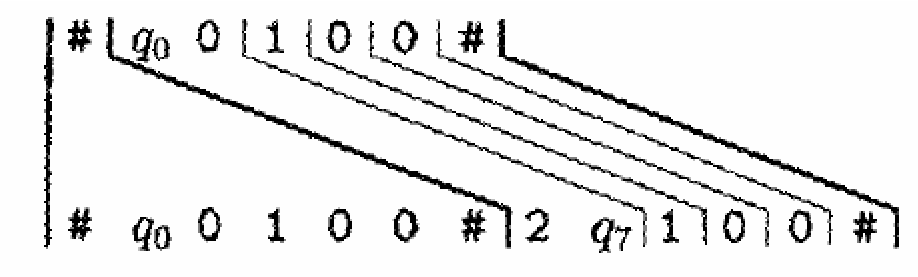
- Part 5: put $\left[\frac{\#}{\#}\right]$



Post correspondence problem

- On class exercise:

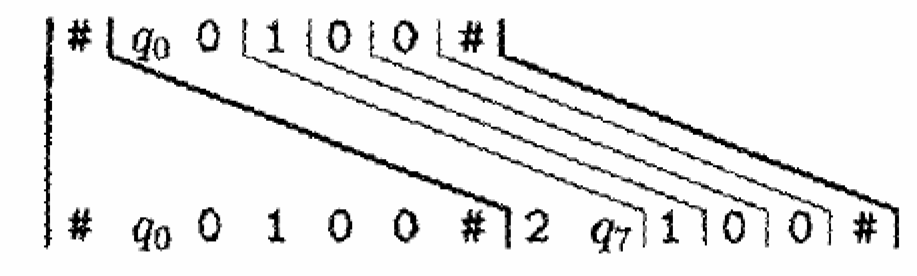
$$\delta(q_7, 1) = (q_5, 0, R)$$

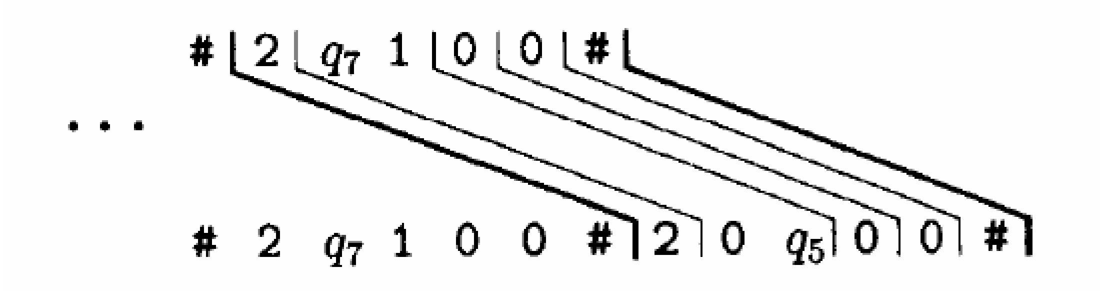


Post correspondence problem

- On class exercise:

$$\delta(q_7, 1) = (q_5, 0, R)$$





Post correspondence problem

Part 6:

- For every $a \in \Gamma$

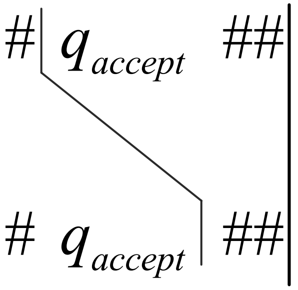
$$\text{put} \left[\frac{aq_{accept}}{q_{accept}} \right] \text{ and } \left[\frac{q_{accept}a}{q_{accept}} \right] \text{ into } P'$$

Post correspondence problem

Part 7:

- Finally adding domino

$\left[\dfrac{q_{accept}\#\#}{\#}\right]$ to complete the match: $\cdots$

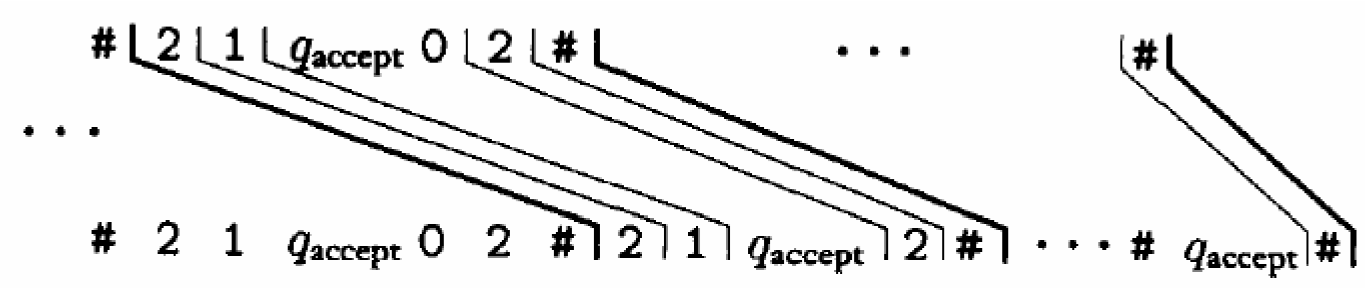


$\# \; q_{accept} \; \#\#$

$\# \; q_{accept} \; \#\#$

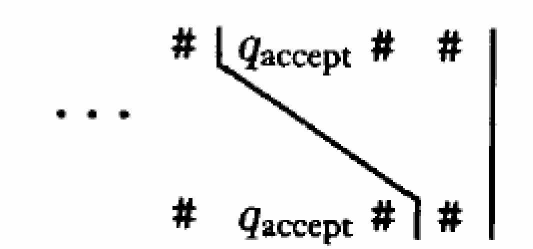
Post correspondence problem

- Example:
- Part 6:



# 2 1 q_{accept} 0 2 # . . . #
. . .
# 2 1 q_{accept} 0 2 # 2 1 q_{accept} 2 # . . . # q_{accept} #

- Part 7:



Post correspondence problem

- Now to convert P' to P.
- Key idea is to build the requirement of string with the first domino directly into the problem.
- define $\bigstar u = *u_1 * u_2 * u_3 * \ldots * u_n$

$$u\bigstar = u_1 * u_2 * u_3 * \ldots * u_n *$$

$$\bigstar u \bigstar = *u_1 * u_2 * u_3 * \ldots * u_n *$$

Post correspondence problem

- So if P' is the collection

$$P' = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

- then P is the collection

$$P = \left\{ \left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{*\square}{\square}\right] \right\}$$

Mapping reducibility

- We will define reduction here.
- A simple type of reducibility is mapping reducibility.
- Roughly speaking, being able to reduce problem A to B by using a mapping reducibility means that a computable function exists that converts instances of problem A to instances of problem B.

Mapping reducibility

- A function $f:\Sigma^* \rightarrow \Sigma^*$ is a computable function if some Turing machine *M*, on every input *w*, halts with just *f*(*w*) on its tape.
- All usual arithmetic operations on integers are computable functions.
- Then we can design a machine to take input <*m*,*n*> and give the result *m*+*n*.

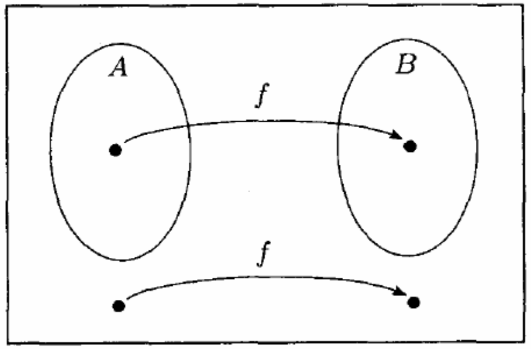
Mapping reducibility

- Computable functions may be transformations of machine descritions.
- For example, one computable function *f* takes input *w* and returns the description of a Turing machine <*M*′ > if *w* = <*M*> is an encoding of a Turing machine *M*. The machine *M*′ is a machine that recognizes the same language as M, but never move its head off the left-hand end of its tape. The function *f* accomplishes this task by adding several states to the description of *M*. The function returns ε if *w* is not a legal encoding of a Turing machine.

Formal definition of mapping reducibility

- Language A is mapping reducible to language B, writtern $A \le_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$ where for every w,

$$w \in A \Leftrightarrow f(w) \in B$$



- The function f is called the reduction of A to B.

Formal definition of mapping reducibility

- A mapping reduction of A to B provides a way to convert questions about membership testing in A to membership testing in B: to test whether $w \in A$, we use the reduction f to map w to $f(w)$ and test whether $f(w) \in B$.

Formal definition of mapping reducibility

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable.

Proof:

- Let M be the decider for B and f be the reduction from A to B.
- We describe a decider N for A as follows:
- N="On input w:
  - ○ Compute $f(w)$.
  - ○ Run M on input $f(w)$ and output whatever M outputs."

Formal definition of mapping reducibility

- Example:
- Show the mapping reducibility from A_{TM} to $HALT_{\mathrm{TM}}$.
- Design a computable function f that takes input $<M,w>$ and returns output $<M',w'>$, where $<M,w> \in A_{\mathrm{TM}}$ if and only if $<M',w'> \in HALT_{\mathrm{TM}}$.

Formal definition of mapping reducibility

- The following machine F computes *f*:
- *F*="On input <*M*,*w*>:
  - Construct the following machine M' ,
    - M' ="On input *x*:
      1. Run *M* on *x*.
      2. If *M* accepts, ***accept***.
      3. If *M* rejects, enter a loop.
  - Outputs < M' ,*w*>"

Mapping reducibility

- Mapping reducibility is sensitive to complementation operation.
- We can use mapping reducibility to prove non-recognizability of certain language.
  - Note: A language is decidable **iff** it is Turing-recognizable and co-Turing-recognizable.

Mapping reducibility

- Theorem: If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.
- Corollary: If $A \leq_m B$ and A is Turing-recognizable, then B is Turing-recognizable.

Mapping reducibility

- Theorem: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

Proof: EQ_{TM} is not Turing-recognizable.

- The reducing function *f* reduce A_{TM} to $\overline{EQ_{TM}}$
- *F*="On input <*M*,*w*> where *M* is a TM,*w* is a string:
  - Construct the following two machines M_1 and M_2 ,
    - M_1="On any input:
      1. ***Reject***."
    - M_2="On any input:
      1. Run *M* on *w*. If it accepts, ***accept***."
  - Outputs $< M_1 , M_2 >$."

Mapping reducibility

- The reducing function *g* reduce A_{TM} to EQ_{TM}
- *G*="On input <*M*,*w*> where *M* is a TM,*w* is a string:
  - Construct the following two machines M_1 and M_2 ,
    - M_1="On any input:
      1. ***Accept***."
    - M_2="On any input:
      1. Run *M* on *w*. If it accepts, ***accept***."
  - Outputs < M_1 ,M_2 >."

Mapping reducibility

- Since $A_{TM} \leq_m \overline{EQ_{TM}}$

  So, $\overline{A_{TM}} \leq_m EQ_{TM}$

  Therefore, EQ_{TM} is not Turing Recognizable

- Since $A_{TM} \leq_m EQ_{TM}$

  So, $\overline{A_{TM}} \leq_m \overline{EQ_{TM}}$

  Therefore, $\overline{EQ_{TM}}$ is not Turing Recognizable

Theory of Computation

Lecture 8:
Advanced topics in computability theory

Lecturer: Dr. Ng, Wing Yin
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Harbin Institute of Technology Shenzhen Graduate School.

Content of Lecture 8

- Recursion Theorem
- Decidability of Logical Theories
- Turing Reducibility
- Definition of Information
  - Minimal length description
  - Optimality of the definition
  - Incompressible strings and randomness

Recursion theorem

- In mathematics, the recursion is defined as:

  $$f(n) = g\left(f(n-1)\right)$$

  where $g(n)$ is a function that not related to $f(n)$

- E.g.:

  Fibonacci sequence:

  $$f(n) = f(n-1) + f(n-2)$$

  where $f(0) = 1, f(1) = 1$

Recursion theorem

- In the theory of computability, recursion theorem is important.
- Recursion theorem can be applied to:
  - Theory of mathematical logic
  - Theory of self-producing systems
  - Computer viruses

Recursion theorem

- A famous paradox:
  1. Living things are machines.
  2. Living things can self-reproduce.
  3. Machines cannot self-reproduce.
- I is correct as organisms is believed to operate in a mechanistic way.
- II is correct.
- It seemed III is incorrect according to our normal knowledge. But it is true according to the recursion theorem.

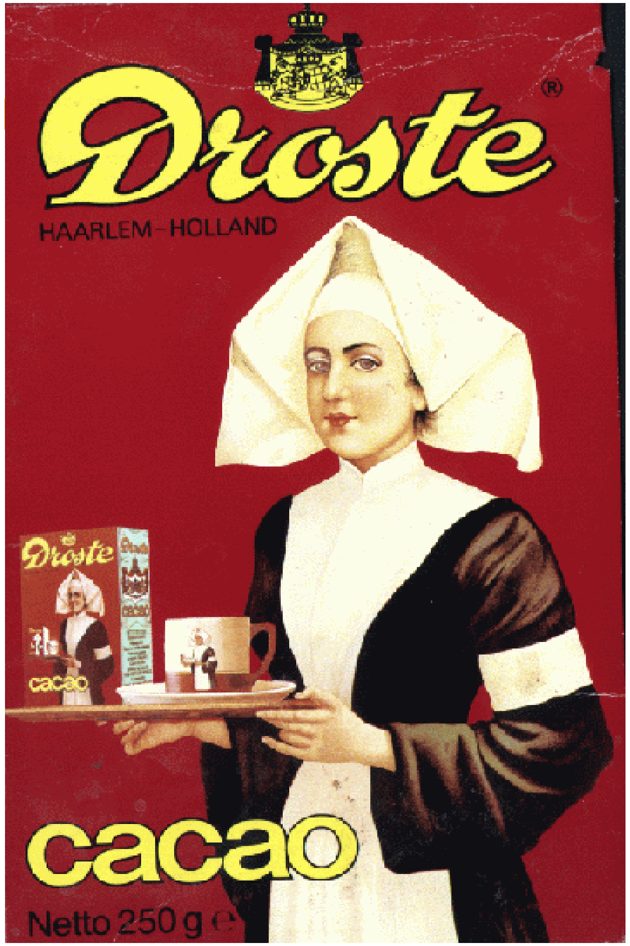
Self-reference

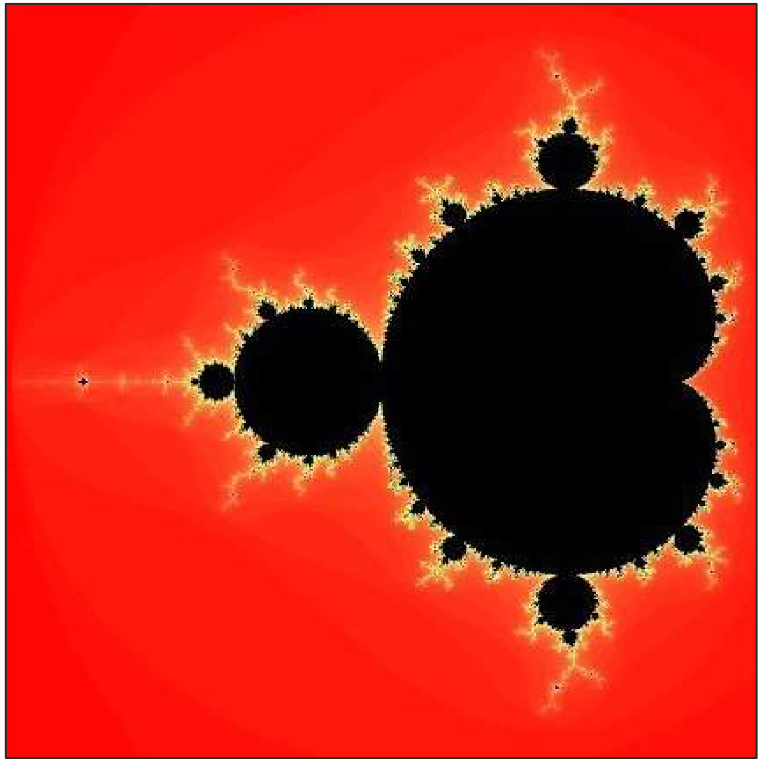
- We now design a Turing machine called *SELF*:
  - Ignore its input
  - Print out a copy of its own description.
- In short, *Q* accepts <*Q*>.

Self-reference

- Recall that we have used the reduction method to prove the undecidability of some problems based on the halting problem A_{TM}.
- In fact, we prove the undecidability of A_{TM} using the self-reference method.
- $D = \{ <M> \mid \text{TM } M \text{ does not accept } <M> \}$

Self-reference





Self-reference

- To describe *SELF*, introduce the following lemma:

  There is a computable function $q:\Sigma^* \rightarrow \Sigma^*$, where if w is any string, $q(w)$ is the description of a Turing machine P_w that prints out w and then halts.
- The proof is easy.

Self-reference

- Proof:
- We can take any string w and construct Turing machine P_w that build w into a table, and simply output w when started (on any input).
- Q=“On input string w:
  1. Construct the following Turing machine P_{w}.

     P_{w} =“On any input:
     - ○ Erase input.
     - ○ Write w on the tape.
     - ○ Halt.
  2. Output < P_w >.”

Self-reference

- The Turing machine SELF is in two parts, *A* and *B*.
  - The job of *A* is to print out a description of *B*.
  - The job of *B* is to print out a description of *A*.
- So job of *SELF* is to print <*SELF*>=<*AB*>.

Self-reference

- Construct Turing machine $P_{<B>}$ to define A:

  Describe $P_{<B>}$ by a computational function $q(<B>)$.

- It is easy to see that A can finish its job: prints out $<B>$.

Self-reference

- Circular definition: to define an object in terms of itself.
- We can't define *B* (use a computational function *q*(<*A*>))as we have done to *A* to avoid the circular definition.
- Define *B* as: *B* computes *A* from the output that *A* produces.
- As <*B*> is on the tape of machine *A* when *A* halts. And we can define *q*(<*B*>) = <*A*> to get the description of *A*.

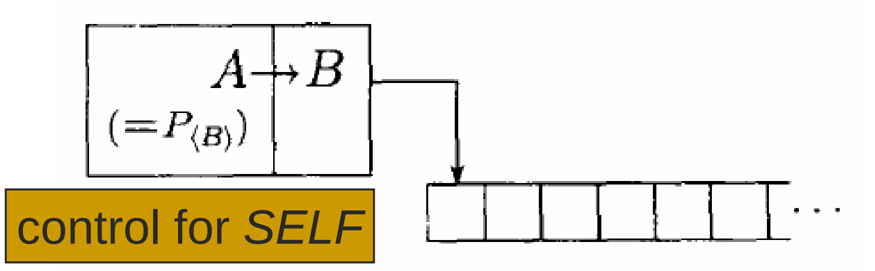
Self-reference

For all:

- $A = P_{<B>}$,
- B = "On input $<M>$, where M is a portion of a TM:
  1. Compute $q(<M>)$.
  2. Combine the result with $<M>$ to make a complete TM.
  3. Print the description of this TM and halt."

Self-reference

The schematic of *SELF*:



control for *SELF*

1. First *A* runs. It prints <*B*> on the tape.
2. *B* starts. It looks at the tape and finds its input, <*B*>.
3. B calculates q(<*B*>)=(<*A*>) and combines that with <*B*> into a TM description, <*SELF*>.
4. *B* prints this description and halts.

Self-reference

Example:

- Print out this sentence.
  - As "this" has no counterpart, so this sentence can not be easily translated into programming language.
- Print out two copies of the following, the second one in quotes:" Print out two copies of the following, the second one in quotes:"
  - In this sentence, the self-reference is replaced with the same construction used to make the TM *SELF*.
  - *A*: "Print out two copies of the following, the second one in quotes:"
  - *B*: Print out two copies of the following, the second one in quotes:

Recursion theorem

- Recursion theorem extends the *SELF* technique to go on computation instead of merely printing.
- Recursion theorem: Let *T* be a Turing machine that computes a function $t:\Sigma^* \times \Sigma^* \to \Sigma^*$. There is a Turing machine *R* that computes a function $r:\Sigma^* \to \Sigma^*$, where for every *w*,

$$r(w) = t(\langle R\rangle, w)$$

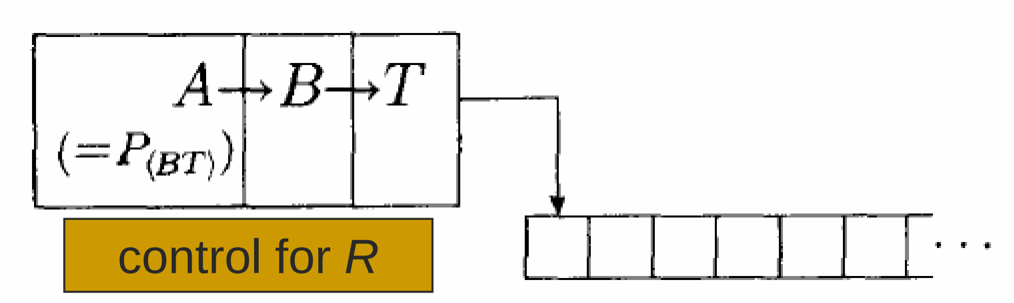
Recursion theorem

- The theorem states that construct a machine *T* which receives the description of the machine as an extra input, then construct a new machine *R*, which operates exactly as *T* does but with *R*’s description filled in automatically.

Recursion theorem

Proof:

- Construct *R* in three parts: *A*, *B* and *T*.



Recursion theorem

- $A = P_{<BT>}$ (defined by $q(<BT>)$), and after the running of A, $w<BT>$ is left on the tape.
- B combines A, B and T into a single machine and obtains its description $<ABT>=<R>$.
- Finally, it encodes $<R>$ with w, places the resulting string $< R , w >$ on the tape, and T runs.

Recursion theorem

- Recursion theorem is a handy tool for solving certain problems concerning the theory of algorithms.
- Using recursion theorem to design a TM *M*:
  - Print out its own description <*M*> as TM SELF done.
  - Count the number of states in <*M*>.
  - Simulate <*M*>.

Recursion theorem

For *SELF*:

- *SELF* = “ on any input:
  1. Obtain, via the recursion theorem, own description <*SELF*>.
  2. Print <*SELF*>.

Recursion theorem

On class exercise:

- Design TM *COUNT* to count the states number in <*COUNT*>.

Recursion theorem

- Computer virus:
  - Inactive when standing alone as a piece of code
  - When placed in a host computer and it will infect the host computer.
  - And then being activated and transmit copies of themselves to other accessible machines.

Recursion theorem

Virus: (Autoexec.BAT)   copy from disk A to disk B

```
Echo This is a Virus program
    IF exist b:\autoexec.bat  goto  Virus
    Goto No_Virus
    : Virus
    B:
    Rename autoexec.bat  auto.bat        //replace
    Copy a:\autoexec.bat  b:             //copy itself
    Echo I am Virus
    Del *.exe                //destroyment
    : No_virus
A:
\Auto          //run “autoexec.bat”，display as if nothing
has happened
```

Recursion theorem

- Using Recursion theorem to prove the undecidability of A_{TM}.

Proof:

- Assume that Turing machine H decides A_{TM}, for the purposes of obtaining a contradiction. We construct the following machine B.

Recursion theorem

- *B* = “On input *w*:

1. Obtain, via the recursion theorem, own description <*B*>.
2. Run *H* on input <*B*,*w*>.
3. Do the opposite of what *H* says: ***accept*** if *H* rejects and ***reject*** if *H* accepts.”

- On *w* we can see the contradiction of *B* and *H*. So *H* cannot be deciding A_{TM}. So A_{TM} is undecidable.

Recursion theorem

Minimal Turing machine:

- If *M* is a Turing machine, then we say that the length of the description <*M*> of *M* is the number of symbols in string describing *M*. Say that *M* is minimal if there is no Turing machine equivalent to *M* that has a shorter description. Let

$$MIN_{TM} = \left\{ \left\langle M \right\rangle \mid M \text{ is a minimal TM} \right\}$$

Recursion theorem

On class exercise:

- Prove MIN_{TM} is not Turing-recognizable.

Recursion theorem

- Proof:
- Assume that some TM E enumerates MIN_{TM} and obtain a contradiction. We construct the following TM C.
- C = “On input w:
  1. Obtain, via the recursion theorem, own description <C>.
  2. Run the enumerator E until a machine D appears with a longer description than that of C.
  3. Simulate D on input w.”

Remark: This is contradict to our assumption that *E* enumerates TM with minimum length

Recursion theorem

- Fixed-point version of recursion theorem:

  A fixed point of a function is a value that isn't changed by computation of the function.

- Let $t:\Sigma^* \rightarrow \Sigma^*$ be a computable function. Then there is a Turing machine F for which t (<F>) describes a Turing machine equivalent to F. If string isn't a proper Turing machine encoding, it describes a Turing machine that always rejects immediately.
- F is the fixed point.

Recursion theorem

- Proof:
- Let *F* be the following Turing machine:
- *F* = "On input *w*:
  1. Obtain, via the recursion theorem, own description <*F*>.
  2. Compute *t* (<*F*>) to obtain the description of a TM *G*.
  3. Simulate *G* on *w*."
- So <*F*> and *t* (<*F*>) = <*G*> describe equivalent Turing machines because *F* simulates *G*.

Turing reducibility

- We have learned one kind of reducibility: mapping reducibility, exist computational function f:

$$w \in A \Leftrightarrow f(w) \in B$$

- But mapping reducibility is not suitable for all cases:
- A_{TM} and $\overline{A_{\mathrm{TM}}}$ are reducible to one another. But there is not a mapping function from $\overline{A_{\mathrm{TM}}}$ to A_{TM}, as is Turing-recognizable but $\overline{A_{\mathrm{TM}}}$ isn’t.

Turing reducibility

- Oracle :
- An oracle for a language B is an external device that is capable of reporting whether any string w is a member of B. An oracle Turing machine is a modified Turing machine that has the additional capability of querying an oracle. Denoted as M^B.
- With oracle a TM could decide more languages.

Turing reducibility

Reduce E_{TM} to A_{TM}.

- Consider an oracle for A_{TM}. The Turing machine with an oracle for A_{TM} can decide A_{TM} itself, by querying the oracle about the input. It can also decide E_{TM} as following:
- $T^{A_{\mathrm{TM}}}$ = "On input <M>, where M is a TM:
  1. Construct the following TM N.

     N = "On any input:
     1. Run M in parallel on all strings in .
     2. If M accepts any of these strings, ***accept***."

Turing reducibility

2. Query the oracle to determine whether $<N,0> \in A_{\text{TM}}$.
3. If the oracle answers NO, ***accept***; if YES, ***reject***.

- So $T^{A_{\text{TM}}}$ decides E_{TM}. We say that E_{TM} is decidable relative to A_{TM}
- Reduce from the current Turing machine to observe the decidability is Turing reducibility.

Turing reducibility

Turing reducible:

- Language A is Turing reducible to language B, written $A \leq_T B$, if A is decidable relative to B.

Theorem:

- If $A \leq_T B$ and B is decidable, then A is decidable.
- Proof: Just replace the oracle for B by and actual procedure that decides B.

Turing reducibility

- Turing reducibility is a generalization of mapping reducibility.

$$A \le_{\mathrm{m}} B \Rightarrow A \le_{\mathrm{T}} B$$

- Turing machine with oracle still can't decides all languages.

Definition of information

- Church-Turing thesis gives a universally applicable definition of algorithm.

  Turing machine algorithms

- Now consider the corresponding definition of information.
  - *A*=0101010101010101010101010101
  - *B*=1110010101110111010111110101
  - We believe that *B* is more informative than *A*.

Definition of information

- Minimal length descriptions: size of object's shortest representation (description) which can be defined as the quantity of information contained in the object.
- Here the description of an object means a precise and unambiguous characterization of the object so that we may recreate it from the description alone.

Definition of information

Why using the shortest description:

- A description that is significantly shorter than the object implies that the information contained within it can be compressed into a small volume, and so the amount of information can not be very large. So the size of the shortest description determines the amount of information.

Minimal length descriptions

Using algorithm to describe strings:

- Method I:
  - Construct a TM M that prints out the string w when it is started on a blank tape.
  - Represent that TM itself as a string $<M>$.
- The length might be longer than $|w| = n$ itself as there might be n states and n rows in the transition function table which results that the description is excessively long.

Minimal length descriptions

Method II <*M*,*w*> = <*M*>*w*:

- We denote the string of which we want to define the length as *x*.
- We use the TM *M* and a binary input *w* to *M* which resulting the string *x*. <*M*>*w*
- When we want to see how *x* is resulted, we have to separate <*M*> and *w* from <*M*>*w*. So we use a method to give the boundary of <*M*> and *w* .

Minimal length descriptions

- We write <*M*> twice by every bit, that is 0 as 00 and 1 as 11.
- At the boundary add the delimiter 01.
- *w* stays.

delimiter

$$x = \langle M \rangle w = \underbrace{11001111001100\cdots1100}_{\langle M \rangle}\ \overbrace{01}\ \underbrace{01101011\cdots010}_{w}$$

TM as the decoder

string as the key

Minimal length descriptions

- Formal definition:
- Let x be a binary string. The minimal description of x, written $d(x)$, is the shortest string $<M,w>$ where TM M on input w halts with x on its tape. If several such strings exist, select the lexicographically first among them. The descriptive complexity of x written $\mathrm{K}(x)$ is

$$\mathrm{K}(x) = |d(x)|$$

Minimal length descriptions

Theorem: $\exists c \forall x \ \left[\mathrm{K}(x) \leq |x| + c \right]$

Proof:

- Prove by construction.
- Construct TM *M* and *M* works as this: halts as soon as it is started. *M* computes the identity function --- its output is the same as its input.
- Let *c* be the length of <*M*>, and the length of description of *x* is simply <*M*>*w*, that is |*x*|+*c*.

Minimal length descriptions

Theorem: $\exists c \forall x \ \left[\mathrm{K}(xx) \leq \mathrm{K}(x) + c \right]$

Proof:

- Prove by construction.
- Construct TM *M* which expects an input of the form <*N*,*w*>, where *N* is a Turing machine and *w* is an input for it.
- *M* = "On input <*N*,*w*> where *N* is a TM and *w* is a string:
  1. Run *N* on *w* until it halts and produces an output string *s*.
  2. Output the string *ss*."
- Let *c* be the length of <*M*>, and the length of description of *xx* is <*M*>*d* (*w*) is |<*M*>|+|*d* (*w*)|= *c* + |K(*x*)|.

Minimal length descriptions

Theorem: $\exists c \forall x, y \quad \left[\mathrm{K}(xy) \leq 2\mathrm{K}(x) + \mathrm{K}(y) + c \right]$

Proof:

- Construct TM *M* that breaks its input w into two separate descriptions. The bits of the first description *d*(*x*) are all doubled and terminated with string 01. the second description *d*(*y*) stays. Run the corresponding machine to output string *xy*.
- Obviously the length of description of *xy* is twice the complexity of *x* plus the complexity of *y* plus a fixed constant for describing *M*.

$$2\mathrm{K}(x) + \mathrm{K}(y) + c$$

Minimal length descriptions

On class exercise:

- Suggest a method find a smaller upper bound for K(*xy*).

Minimal length descriptions

- We can give the length of d(*x*) to indicate the position that separate d(*x*) and d(*y*).

- Using the method described above to find a smaller bound for K(*xy*).

- $2\log_2\left(K\left(x\right)\right)+K\left(x\right)+K\left(y\right)+c$

Optimality

- The minimal description of string K(*x*) has an optimality property among all possible ways of defining descriptive complexity with algorithms.
- Consider a general description language to be any computable function $p : \Sigma^* \to \Sigma^*$ and define K(*x*) with respect to p:
  - ○ written $d_p(x)$ to be the lexicographically shortest string s where $p(s) = x$.

$$K_p(x)=|d_p(x)|$$

Optimality

- For example, consider a programming language such as LISP as the description language, and here we use consider the description all in binary bits.
- The $d_{\mathrm{LISP}}(x)$ would be the minimal LISP program that outputs x, and $\mathrm{K}_{\mathrm{LISP}}(x)$ would be the length of the minimal program.

Optimality

Theorem:

- For any description language *p*, a fixed constant *c* exists that depends only on *p*, where $\forall x \ \left[\mathrm{K}(x) \leq \mathrm{K}_p(x) + c \right]$

Optimality

Proof:

- Take any description language p and consider the following Turing machine M.
- M=“On input w:
  1. Output $p(w)$.”
- Then <M> $d_p(x)$ is a description of x whose length is at most a fixed constant greater than $K_p(x)$. So the constant is the length of <M>.

Incompressible string

- Is there some string that the minimal description is actually as long as the string itself ?
- Answer is YES.
- Let x be a string, say that x is c-compressible if $\mathrm{K}(x) \leq |x| - c$
  - If x is not c-compressible, we say that x is incompressible by c.
  - If x is incompressible by 1, we say that x is incompressible.

Incompressible string

- Notes:
- If x has a description that is c bits shorter than its length, x is c-compressible.
- If x doesn't have any description shorter than itself, x is incompressible.

Incompressible string

Theorem: Incompressible strings of every length exist.

Proof:

- The number of binary strings of length *n* is 2^n .
- Each description is a binary string, and the number of description of length less than *n* is at most the sum of the number of strings of each length up to *n* – 1:

$$\sum_{0 \le i \le n-1} 2^i = 1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$$

So there is at least one string of length *n* is incompressible.

Incompressible string

On class exercise:

- Prove following corollary based on the last theorem:

  At least $2^n - 2^{n-c+1} + 1$ strings of length n are incompressible by c.

Randomness

Theorem:

- Let f be a computable property that holds for almost all strings. Then, for any $b > 0$, the property f is FALSE on only finitely many strings that are incompressible by b.

Proof:

- Let M be the following algorithm.

  M = "On input i, a binary integer:

  1. Find the ith string s where $f(s)$ = FALSE, considering the strings ordered lexicographically.
  2. Output string s."

Randomness

- So M can output short descriptions of strings that fail to have property f as follows:
- For any string x that fail to have property f, let i_x be the position (index) of x on a list of all such strings. Suppose these strings are in length and lexicographically order. The length of the description of x $<M,i_x>$ is $|i_x| + c$.
- Fix any number $b>0$, select n such that at most a $1/2^{b+c+1}$ fraction of strings of length n or less fail to have property f.

Randomness

- As the index of x is small and its description is correspondingly short, and f holds for almost all strings, so all sufficiently large n satisfy this condition.
- Let $|x| = n$, we have 2^{n+1}strings of length n or less, so

$$i_x \leq \frac{2^{n+1}-1}{2^{b+c+1}} \leq 2^{n-b-c}$$

Randomness

- $|i_x|$ is no more than n-b-c, so the length of $<M, i_x>$ is at most n-b.
- Thus every sufficiently long x that fails to have property f is compressible by b.
- Hence only finitely many strings that fail to have property f are incompressible by b.

Randomness

- Incompressible strings have many properties that we expect to find in randomly chosen strings, e.g.:
- Incompressible string of length *n* has roughly an equal number of 0s and 1s
- The length of incompressible strings' longest run of 0s is approximately $\log_2 n$, as we would expect to find in a random string of that length.
- A property holds for almost all strings if the fraction of strings of length *n* on which it is FALSE approaches 0 as n grows large.
  - A randomly chosen long string is likely to satisfy a computable property that holds for almost all strings.

Randomness

Theorem: For some constant *b*, for every string *x*, the minimal description *d*(*x*) of x is incompressible by *b*.

Proof:

- Consider the following TM *M*:
- *M* = "On input <*R*,*y*>, where *R* is a TM and *y* is a string:
  1. Run *R* on *y* and ***reject*** if its output is not of the form <*S*,*z*>.
  2. Run *S* on *z* and halt with its output on the tape."
- Let *b* be |*M*|+1. suppose to the contrary that *d*(*x*) is b-compressible for some string *x*.

Randomness

$$\left|d\left(d\left(x\right)\right)\right| \le \left|d\left(x\right)\right| - b$$

- But <*M*>*d*(*d*(*x*)) is a description of *x* and the length

$$\begin{aligned}\left|\langle M\rangle d\left(d\left(x\right)\right)\ \right| &= \left|\langle M\rangle\right| + \left|d\left(d\left(x\right)\right)\right| \\ &\le \left(b+1\right) + \left(\left|d\left(x\right)\right| - b\right) \\ &\le \left|d\left(x\right)\right| - 1\end{aligned}$$

- This is contradicting the latter's minimality. So the theorem is proved.

Theory of Computation

Lecture 9:
Time Complexity
(Part I)

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Content of Lecture 9

- Measuring Complexity
  - Big-*O* and small-*o* notation
  - Analyzing algorithms
  - Complexity relationships among models
- The class **P**
  - Polynomial time
  - Example of problems in **P**
- The class **NP**

Complexity theory

- ***Complexity theory is 'the art of counting resources'.---- time or memory.***

- Time complexity: How many time steps does the computation of a problem cost?
- Space complexity: How many bits of memory are required for the computation?

Time complexity

- Here we consider the time steps using the Turing machine model.
  - the Turing machine description at a low level
- The number of steps that an algorithm uses on a particular input may depends on several parameters.
- For simplicity we compute the running time of an algorithm as a function of the length of the string representing the input.

Time complexity

- Worst-case analysis vs. average-case analysis
  - Worst-case analysis: the longest running time of all inputs of a particular length.
  - Average-case analysis: the average of all running time of inputs of a particular length.

Time complexity

- Let M be a deterministic Turing machine that halts on all inputs. The running time (time complexity) of M is the function $f : \mathrm{N} \rightarrow \mathrm{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n.
- If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine.
- Customarily we use n to represent the length of the input.

Asymptotic analysis: $O(n)$ and $o(n)$

- Asymptotic analysis
  - Consider only the highest order term of the expression for the running time of the algorithm.
  - Disregard the coefficient of the highest order term and any lower order terms.
- Because the highest order term dominates the other terms on large inputs.

Asymptotic analysis: $O(n)$ and $o(n)$

- Let R^+ be the set of nonnegative real numbers. Let f and g be functions f, g: $N \to R^+$. Say that $f(n) = O(g(n))$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$

$$f(n) \leq cg(n)$$

- We say that $g(n)$ is an asymptotic upper bound for $f(n)$, where the coefficient of $g(n)$ should be 1.

Asymptotic analysis: $O(n)$ and $o(n)$

- Example:

$$f_1(n) = 7n^{15} + 3n^2 + 2$$

$$f_2(n) = 3^{3n} + (3n)^3$$

$$f_3(n) = 3n\log_2 n + 5n\log_2\log_2 n + n + 1$$

$$f_1(n) = O(n^{15})$$

$$f_2(n) = 3^{O(n)}$$

$$f_3(n) = O(n\log_2 n)$$

Asymptotic analysis: $O(n)$ and $o(n)$

On class exercise:

- Give the asymptotic time complexity of the following expression:

$$\log n, n^n, 2^n, n^2, n$$

Asymptotic analysis: $O(n)$ and $o(n)$

- Notice:

$$\text{As} \quad \log_b n = \frac{\log_2 n}{\log_2 b}$$

$$\text{so} \ \log_2 n = O(\log_b n)$$

we can ignore the base and just write $O\left(\log n\right)$

Asymptotic analysis: $O(n)$ and $o(n)$

- Notice:

$$f(n) = 2^{O(\log n)}$$

As $n = 2^{\log_2 n}$

so $n^{O(1)} = n^{c} = 2^{c \log_2 n}$

so $n^{O(1)} = 2^{O(\log n)}$

Asymptotic analysis: $O(n)$ and $o(n)$

On class exercise:

- How about

$$2^n, k^n \text{ where } k \text{ is a constant.}$$

Asymptotic analysis: $O(n)$ and $o(n)$

- The bounds of the form n^k are called polynomial bounds.
- Bounds of the form $2^{\left(n^{\delta}\right)}$ are called exponential bounds when δ is a real number greater than 0.

Asymptotic analysis: $O(n)$ and $o(n)$

- Let *f* and *g* be functions *f*, *g*: $N \rightarrow R^+$. Say that $f(n) = o(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

- $f(n) = o(g(n))$ means that , for any real number $c > 0$, a number n_0 exists, where $f(n) < cg(n)$ for all $n \geq n_0$.

Asymptotic analysis: $O(n)$ and $o(n)$

- Example:

$$\sqrt{n} = o(n).$$

$$n = o(n \log \log n).$$

$$n \log \log n = o(n \log n).$$

$$n \log n = o(n^2).$$

$$n^2 = o(n^3).$$

$$f(n) \text{is never } o(f(n))$$

Analyzing algorithm

Example:

- Analyze the TM algorithm for the language

$$A = \left\{ 0^k 1^k \mid k \geq 0 \right\}$$

Analyzing algorithm

- M_1 = "On input string w:

1. Scan across the tape and reject if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
3. Scan across the tape, crossing off a single 0 and a single 1.
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all te 0s have been crossed off, reject. Otherwise, if neither 0s nor 1s remain on the tape, accept."

Analyzing algorithm

- M_1 = "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s a
3. Scan across th and a single 1.
4. If 0s still remain a off, or if 1s still r crossed off, *reject*. remain on the tape

In stage 1, the machine scans across the tape to verify that the input is of the form 0*1*

Performing this scan uses k ($k \leqslant n$) steps, and repositioning the head at the left-hand use k steps.

Totally use $O(n)$

Analyzing algorithm

- M_1 = "On input s

1. Scan across the the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
3. Scan across the tape, crossing off a single 0 and a single 1.
4. If 0s still remain after all th e been crossed off, or if 1s still remai crossed off, *reject*. Oth remain on the tape, *acc*.

Stage 2 is a judge condition, together with stage 3, the machine repeatedly scan the tape and crossed off a 0 and 1 on each scan.

Each scan uses $O(n)$ steps.

The total time is $O(n) \times O(n) = O(n^2)$

Analyzing algorithm

- M_1 = "On input strin
1. Scan across the tap
the right of a 1.
2. Repeat if both 0s an
3. Scan across the
and a single 1.
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all te 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

In stage 4, the machine makes a single scan to check if there are all cross(×) on tape.

Performing this scan uses $O(n)$

Analyzing algorithm

- M_1 = "On input string w:

1. Scan across the tape and *reject* if a 0 is found to th
2. R pe:
3. single 0 an
4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all te 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

For all, the total time of M_1 on an input of length n is $O(n)+O(n^2)+O(n) = O(n^2)$

Now we complete the analysis of this machine.

Analyzing algorithm

- Let $t: N \rightarrow R^{+}$ be a function. Define the time complexity class, $\mathrm{TIME}(t(n))$, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

Analyzing algorithm

- So language $A=\left\{0^k1^k \mid k \geq 0\right\} \in \mathrm{TIME}\left(t\left(n\right)\right)$ because M_1 decides A in time $O(n^2)$ and $\mathrm{TIME}\left(n^2\right)$ contains all languages that can be decided in $O(n^2)$ time.

Analyzing algorithm

Question:

- Is there a machine that decides A asymptotically more quickly?
- That is to say is $A \in \mathrm{TIME}(t(n))$ for $t(n) = o(n)$ and $t(n) \neq O(n^2)$

Answer:

- The following Turing machine M_2 shows that $A \in \mathrm{TIME}(n \log n)$

Analyzing algorithm

- M_2 = “On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat as long as some 0s and some 1s remain on the tape:
3. Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If it is odd, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain on the tape, *accept*. Otherwise, *reject*.”

Analyzing algorithm

- **M_2 = "On in**

1. **Scan across**
   a 1.
2. **Repeat as lo ome 1s remain on the tape.**
3. **Scan across ape, checking whether the total number of 0s a 1s remaining is even or odd. If it is odd, *reject*.**
4. **Scan again acros the tape, crossing off every other 0 starting with the first 0, an crossing off every other 1 starting with the first 1.**
5. **If no 0s and no 1s remain**
   ***reject*."**

On stage 3, the machine repeatedly scan the tape and check on the agreement of the parity of the 0s with the parity of the 1s.

Each scan uses $O(n)$ steps.

Analyzing algorithm

- M_2 = “On inpu

1. Scan across t of a 1.
2. Repeat as lon :
3. Scan acr al number of 0s and 1s ng is even or odd. If it is odd, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain o tape, *accept*. Otherwise, *reject*.”

On stage 4, the machine crosses off at least half the 0s and 1s each time it is executed. So at most $1+\log_2 n$ iterations of the repeat loop occur before all get crossed off.

Each scan uses $O(n)$ steps.

Analyzing algorithm

- M_2 = "On input string w:

1. Scan a of a 1.
2. Repeat :
3. Sc al numbe d, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain on the tape, *accept*. Otherwise, *reject*."

Each stage takes $O(n)$ time. And stage 1 and 5 executed once. Stage 3 and 4 takes $1+\log_2 n$ iteration. So the running time of M_2 is

$$O(n)+3O(n)(1+\log_2 n)+O(n)= O(n\log n)$$

Analyzing algorithm

- As the running time of Turing machine M_2 is $O(n\log n)$, so we find a better bound

$$A \in \mathrm{TIME}(n \log n)$$

- This result cannot be further improved on single tape Turing machines.
- Theorem: Any language that can be decided in $o(n\log n)$ time on a single-tape Turing machine is regular.

Analyzing algorithm

Question:

- As the single-tape machine can’t decides A asymptotically more quickly, now try multitape Turing machine.

- The following two-tape Turing machine M_3 shows that

$$A \in \mathrm{TIME}(n)$$

Analyzing algorithm

- M_3 = "On input string w in tape 1:

1. Scan across the tape 1 and *reject* if a 0 is found to the right of a 1.
2. Scan across the 0s on tape 1 until the first 1. At the same time, copy the 0s onto tape 2.
3. Scan across the 1s on tape 1 until the end of the input. For each 1 read on tape 1, cross off a 0 on tape 2. If all 0s are crossed off before all the 1s are read, *reject*.
4. If all the 0s have now been crossed off, *accept*. If any 0s remain, *reject*."

Analyzing algorithm

- M_3 = "On input string w in tape 1:

1. Scan across the tape 1 and *reject* if a 0 is found to the right of a 1.
2. Scan acro t 1. At the same t
3. Scan acro of the input. For each 1 read on tape 1, cross off a 0 on tape 2. If all 0s are crossed off before all the 1s are read, *reject*.
4. If all the 0s have now been crossed off, *accept*. If any 0s remain, *reject*."

Each stage uses $O(n)$ steps.
So the total running time is $O(n)$
and thus is linear.

Analyzing algorithm

- As the running time of Turing machine M_3 is $O(n)$, so we find a better bound

$$A \in \mathrm{TIME}\left(n\right)$$

- This result cannot be further improved because n steps are necessary just to read the input.
- We conclude that the complexity of A depends on the model of computation selected.

Analyzing algorithm

- Difference between complexity theory and computability theory:
  - Computability theory: the Church-Turing thesis implies that all reasonable models of computation are equivalent (decide the same class of languages)
  - Complexity theory: the choice of computation model affects the time complexity of languages.

Analyzing algorithm

- As our aim is to present the fundamental properties of computation, rather than properties of Turing machines or any other special model, now the complexity theory shows the relationships among models in order to show that for deterministic model the computation is not sensitive.

Complexity relationship among models

Theorem:

- Let $t(n)$ be a function, where $t(n) \geq n$. Then every $t(n)$ time multitape Turing machine has an equivalent $O\left(t^2(n)\right)$ time single-tape Turing machine.

- Proof idea: the same as we have used to convert any multitape TM into a single-tape TM to simulates it.

Complexity relationship among models

- Let M be a k-tape TM that runs in $t(n)$ time. We construct a single tape TM S that runs in $O\left(t^2(n)\right)$ time.
- Copy content of M's k tapes on S consecutively, with the positions of M's heads marked on the appropriate squares. This will take $O(n)$ time.

Complexity relationship among models

- To simulate each step of *M*
  1. *S* scans all the information stored on its tape on determine the symbols under *M*'s tape head
  2. then *S* make another pass over its tape to update the tape contents and head position.
- Note: if one of *M*'s heads moves rightward onto the previously unread portion of its tape, *S* must increase the amount of space allocated to this tape. We have to shift a portion of its own tape one cell to the right.

Complexity relationship among models

- To finish II, the length of the active portion of *S*'s tape determines how long *S* takes to scan it. We take the sum of the lengths of the active portions of *M*'s *k* tapes as an upper bound.
- *M* use $t(n)$ tape cells in $t(n)$ steps if the head moves rightward at every step and fewer if a head ever moves leftward, so each of these active portions has length at most $t(n)$. So a scan of the active portion of S's tape uses $O(t(n))$

Complexity relationship among models

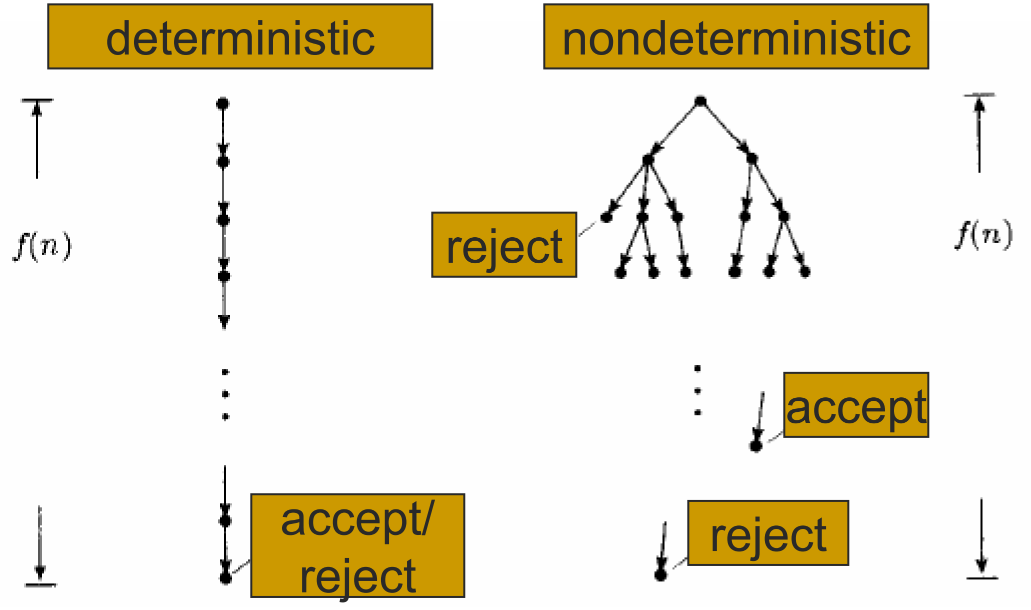
- For each of *M*'s step, uses $O(t(n))$ time for one tape's shift. So the total *k* shifts uses $O(t(n))$ time.
- There are $t(n)$ steps of *M*, so this part of the simulation uses $t(n)\times O(t(n))=O(t^2(n))$ time.
- Together with the initial stage the simulation uses $O(t(n))+O(t^2(n))\overset{t(n)\geq n}{=}O(t^2(n))$ time.

Complexity relationship among models

- Definition of the running time of a nondeterministic Turing machine:
  - Let *N* be a nondeterministic Turing machine that is a decider. The running time of *N* is the function $f : N \rightarrow N$, where *f*(*n*) is the maximum number of steps that *N* uses on any branch of its computation on any input of length *n*.

Complexity relationship among models

- Time complexity of deterministic TM and nondeterministic TM:



Complexity relationship among models

Theorem:

- Let $t(n)$ be a function, where $t(n) \geq n$. Then every $t(n)$ time nondeterministic single-tape Turing machine has an equivalent $2^{O(t(n))}$ time deterministic single-tape Turing machine.

- Proof idea: the same as we have used to convert any nondeterministic single-tape TM into a deterministic single-tape TM to simulates it.

Complexity relationship among models

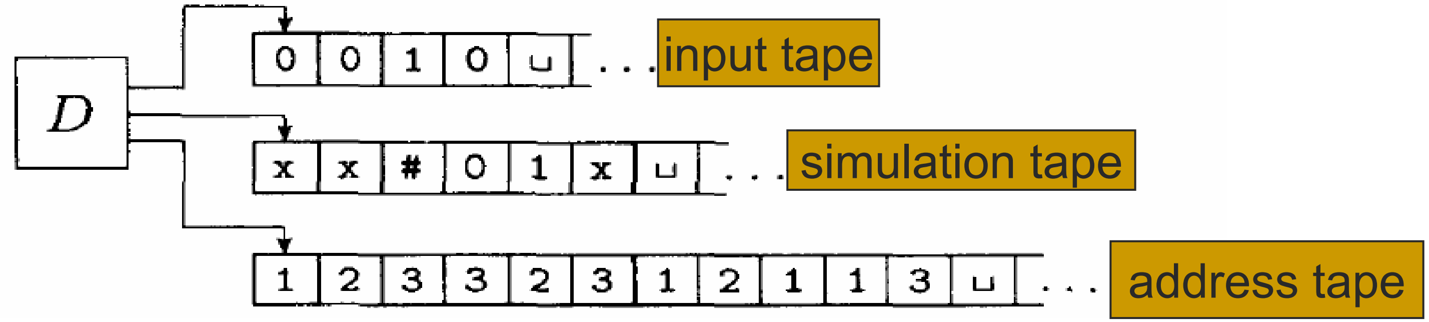
- Proof:
- Let N be a nondeterministic TM running in $t(n)$ time. Construct a deterministic TM D that simulates N by searching N's nondeterministic computation tree.
- Every branch of N's nondeterministic computation tree has a length of at most $t(n)$. Let b be the maximum number of legal choice given by N's transition function. So every node in the tree can have at most b children, and the total number of leaves in the tree is at most $b^{t(n)}$.

Complexity relationship among models

- By the property of tree, the total number of nodes in the tree is less than twice the maximum number of leaves, so the number of nodes are bounded by $O(b^{t(n)})$. Each travel from root to node is $O(t(n))$. There fore the running time of D is $O(t(n)b^{t(n)})=2^{O(t(n))}$.

Complexity relationship among models

- Recall TM *D* is a three-tape TM, by the last theory the running time of the deterministic single-tape TM is $\left(2^{O(t(n))}\right)^2 = 2^{O(2t(n))} = 2^{O(t(n))}$.



input tape

simulation tape

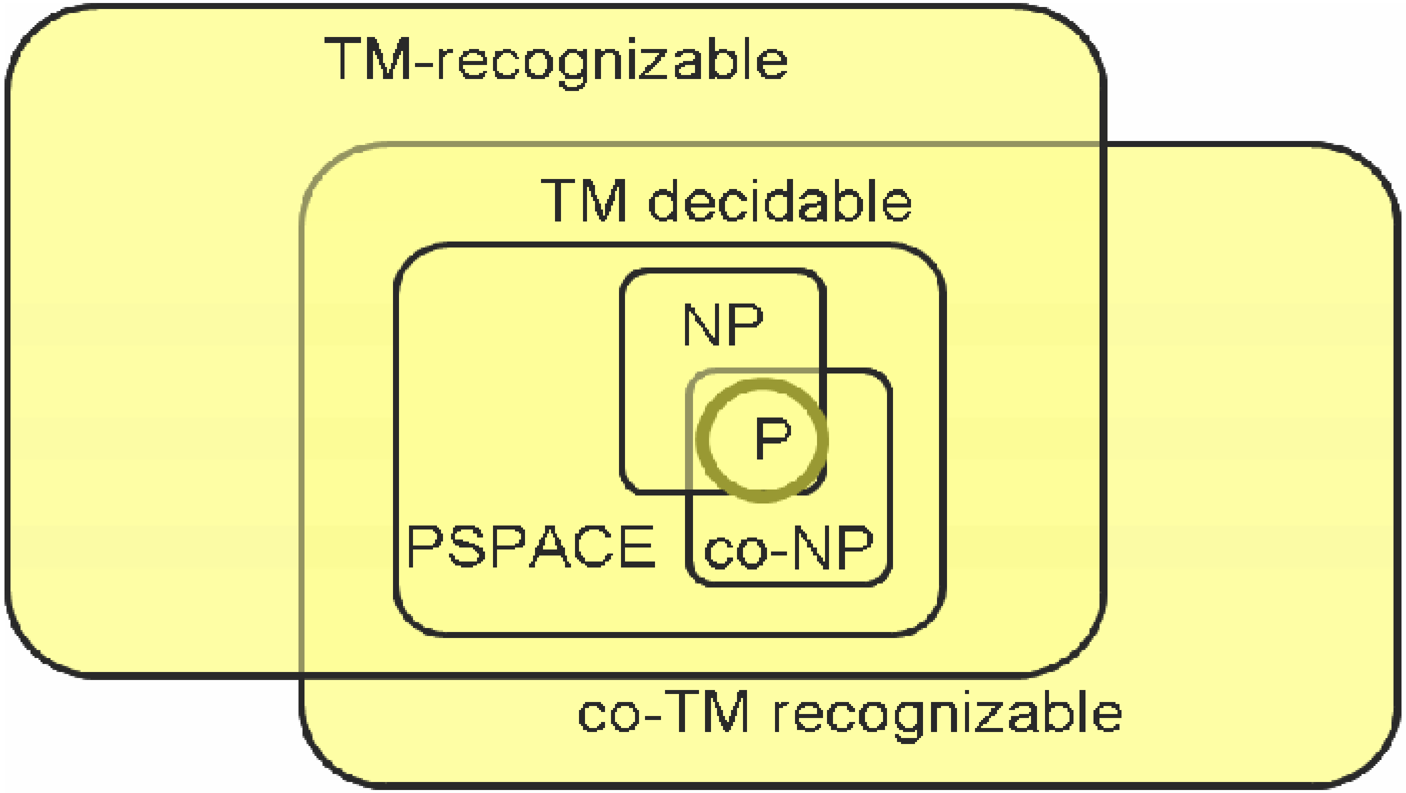
address tape

Complexity relationship among models

- Notice
  - There is at most a square or polynomial difference (which is believed to be the minimal) between the time complexity of problems measured on deterministic single-tape and multitape TM.
  - But there is an exponential difference (which is believed to be the maximal) between the time complexity of problem son deterministic and nondeterministic TM.
  - This introduce the P and NP class problem.

Complexity relationship among models

This is a graph of problems that we have learned and will learned.



The class P

- Polynomial vs exponential:

$$n^3 \text{ vs } 2^n$$

- Let n =1000, a reasonable input to an algorithm.
  - ○ n^3 is 1 billion, a large, but manageable number.
  - ○ 2^n is a number much larger than the number of atoms in the universe.

The class P

- So exponential time algorithms rarely are useful.
- But it is usually easy to design a algorithm to determine a problem in a exponential time.
  - Brute-force search: when we solve problems by exhaustively searching through a space of solutions.

The class P

- polynomials equivalent: any one of the computational modes can simulate another with only a polynomial increase in running time.
- All reasonable deterministic computational models are polynomials equivalent.
- Here reasonable means to contain broad enough models that closely approximate running times on actual computers.

The class P

- From now on we can focus on aspects of time complexity theory that are unaffected by polynomial differences in running time.
- Ignore the selection of the computation model.
- This is the fundamental properties of computation.

The class P

class **P** :

- **P** is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathbf{P} = \bigcup_{k} \mathrm{TIME}\left(n^{k}\right)$$

The class P

- The class **P** plays a central role in the computation theory because:
  1. **P** is invariant for all modles of computation that are polynomials equivalent to the deteministic single-tape Turing machine,
  2. **P** roughly corresponds to the class of problems that are realistically solvable on a computer.

The class P

- Item I means indicates that **P** is a mathematically robust class, and not affected by the particulars of the model of computations.
- Item II indicates that **P** is relevant from a practical standpoint. Once a polynomial time algorithm has been found for a problem that formally appeared to require exponential time, some key insight into it has been gained, and further reductions in its complexity usually follow (often to the point of actual practical utility).

Example of problems in **P**

- From now on we give the high-level description of the polynomial time algorithm.
- So we will not consider the features of a particular computational model. And not need to consider tedious details of tapes and head motions.
- The algorithm analysis in high-level description is the same as low-level description. But the difference is that one step in high-level description in general will require many Turing machine (low-level description) steps.

Example of problems in **P**

- To show an algorithm runs in polynomial time:
  - Give a polynomial upper bound on the number of stages that the algorithm uses when it runs on an input of length n.
  - Examine the individual stages in the description of the algorithm to be sure that each can be implemented in polynomial time on a reasonable deterministic model.

Example of problems in **P**

Note:

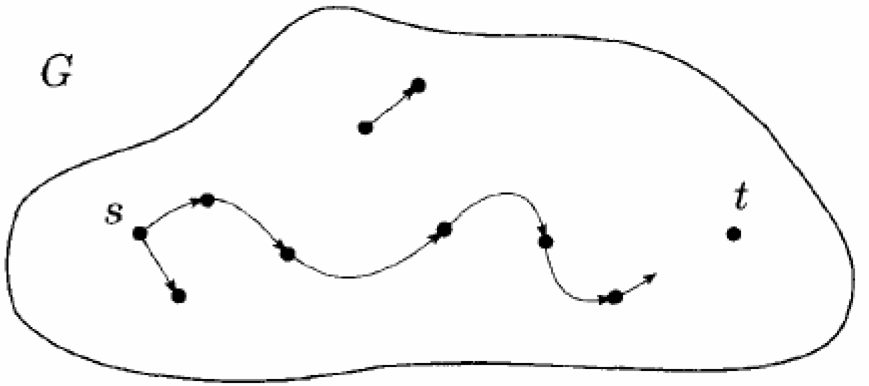
- We still use <•> to demonstrate the encoding of several strings.
- A reasonable method is one that allows for polynomial time encoding and decoding of objects into natural internal representations or into other reasonable encodings. E.g. in binary bits.
- Example: to present a graph $G=<N,E>$

Example of problems in **P**

- Problem 1: The *PATH* problem: Is there a path from *s* to *t*?

$$PATH = \{\langle g, S, T\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$$

- Prove: $PATH \in \mathbf{P}$



Example of problems in **P**

Prove idea:

- Brute-force algorithm for *PATH* needs m^m time where m=|N| demonstrate the number of nodes in G.
- So we use the graph-searching method such as breadth first search.

Example of problems in **P**

Proof:

- A polynomial time algorithm *M* for *PATH* operates as follows:
- *M* = "On input <*G*,*s*,*t*> where *G* is a directed graph with nodes *s* and *t*:

  1. Place a mark on node *s*.
  2. Repeat the following until no additional nodes are marked:
  3. Scan all the edges of *G*. If an edge (*a*,*b*) is found going from a marked node *a* to an unmarked node *b*, mark node *b*.
  4. If *t* is marked, ***accept***. Otherwise, ***reject***."

Example of problems in **P**

- M = "On input <G,s,t> w ed graph with nodes s and t:
  1. Place a mark on node s.
  2. Repeat the following until n dditional nodes are marked:
  3. Scan all the edges of . If an edge (a,b) is found going from a marke node a to an unmarked node b, mark node b.
  4. If t is marked, ***accept***. Otherwise, ***reject***."

Step 1 and 4 executes 1 times.

Example of problems in **P**

- *M* = “On input <*G*,*s*,*t*

  graph with nodes *s* ar

At stage 3, there is a node be marked for one iteration, so step 2 and 3 executes at most m times.

1. Place a mark on node
2. Repeat the following un no additional nodes are marked:
3. Scan all the edges of *G*. If an edge (*a*,*b*) is found going from a marked node *a* to an unmarked node *b*, mark node *b*.
4. If *t* is marked, ***accept***. Otherwise, ***reject***.”

Example of problems in **P**

Simulate by deterministic model:

- Mark node in step 1 can be implemented in polynomial time on any reasonable deterministic model.
- Scan and judge if nodes are marked in step 3 and 4 can be can be implemented in polynomial time.
- Judgment in step 4 can be implemented in polynomial time on any reasonable deterministic model.

Example of problems in **P**

- Problem 2: relatively prime problem, do x and y share the same divisor s that $s \neq 1$.

  $$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$$

- Prove $RELPRIME \in \mathbf{P}$

Example of problems in **P**

Prove idea:

- Brute-force algorithm for *RELPRIME* searches through all possible divisors of both numbers and accepts if none are greater than 1.
- This algorithm has a exponential running time.

Example of problems in **P**

Proof:

- *RELPRIME* problem is equal to greatest common divisor of x and y: gcd $(x, y) = 1$.
- Euclidean algorithm is an algorithm that computing the greatest common divisor..

Example of problems in **P**

The Euclidean algorithm *E* is as follows:

- *E* = "On input <x,y>, where x and y are natural numbers in binary:

```
1. Repeat until $y = 0$:
2.     Assign $x \leftarrow x \bmod y$
3.     Exchange $x$ and $y$.
4. Output $x$."
```

Example of problems in **P**

- The algorithm R solves *RELPRIME*, using *E* as a subroutine.
  - *R* = "On input <*x*,*y*>, where *x* and *y* are natural numbers in binary:
    1. Run *E* on <*x*,*y*>.
    2. If the result is 1, ***accept***. Otherwise, ***reject***."

Example of problems in **P**

- Analyse Euclidean a
- E = "On input $<x,y>$ natural numbers in binar

```
1. Repeat until $y = 0$:
2.     Assign $x \leftarrow x \bmod y$
3.     Exchange $x$ an
4. Output $x$."
```

Except the first executive, step 2 cuts the value of x by at least half.

Prove: After step 2, $x < y$, and after step 3, $x > y$.

$$\begin{cases} x/2 \geq y & \text{thus} \left(x \bmod y \right) < y \leq x/2 \\ x/2 < y & \text{thus} \left(x \bmod y \right) = x - y < x/2 \end{cases}$$

Example of problems in **P**

- Analyse Euclidean a
- E = “On input <x,y> natural numbers in binar

```
1. Repeat until $y = 0$:
2.     Assign $x \leftarrow x \bmod y$
3.     Exchange $x$ an
4. Output $x$.”
```

Except the first executive, step 2 cuts the value of x by at least half.

So step 2 and 3 execute $\min\left(2\log_2 x, 2\log_2 y\right)$ time where $\log_2 x, \log_2 y$ is the length of the binary presents of x and y. So step 2 and 3 run $O(n)$ iteration with each iteration in polynomial times.
Therefore step 2 and 3 takes $O(n)$ time.

Example of problems in **P**

Problem 3:

- Prove: Every context-free language is a member of **P**.
- Prove idea: as we have proved that every CFL is decidable, and gave the algorithm to determine each CFL. Now analysis this algorithm runs in polynomial time.

Example of problems in **P**

- Let L be a CFL generated by CFG *G* that is in Chomsky normal form. So given an input string *w* (|*w*|=*n*), the derivation takes 2*n*-1 steps. The combination of the 2*n*-1 steps will leads the exponential running time.
- We use the dynamic programming methods which store the solution of subproblems to obtain the polynomial running time.
- Table (*i*,*j*) contains the collection of variables that generate the substring $w_i w_{i+1} \ldots w_j$

Example of problems in **P**

Proof:

- Let G be a CFG in Chomsky normal form generating the CFL L. Assume S is the start variable. The following algorithm D running as following:
- D = "On input $w = w_1 w_2 \ldots w_n$:

```
1.  If $w = \varepsilon$ and $S \to \varepsilon$ is a rule, *accept*.
2.  For $i$ = 1 to $n$:
3.      For each variable $A$:
4.          Test whether $A \to b$ is a rule, where $b = w_i$
5.          If so, place $A$ in *table* $(i,i)$
```

Handle $w = \varepsilon$ case.

Examine each substring of length 1.

Example of problems in **P**

```
For $l = 2$ to $n$:
    For $i = 1$ to $n–l+1$:
        Let $j = i+l–1$,
        For $k = i$ to $j-1$:
            For each rule $A \rightarrow BC$
                If *table* $(i,k)$ contains $B$ and *table* $(k+1,j)$ contains $C$, put $A$ in *table* $(i,j)$.
If $S$ is in *table* $(1,n)$, ***accept***. Otherwise, ***reject***.”
```

l is the length of substring.

i is the start positon of substring.

j is the end positon of substring.

k is the split position.

Example of problems in **P**

- D = "On input $w = w_1 w_2 \ldots w_n$:

```
1. If $w = \varepsilon$ and $S \to \varepsilon$ is a rule, accept.
2. For $i$ = 1 to $n$:
3.     For each variable $A$:
4.         Test whether $A \to b$ is a rule, where $b = w_i$
5.          If so, place $A$ in table $(i,i)$
```

Step 4 and 5 run at most *vn* times, where *v* is the number of variables in *G*.

Step 4 and 5 run $O(n)$ times.

Example of problems in P

```
6.  For $l$ = 2 to $n$:
7.      For $i$ = 1 to $n–l+1$:
8.          Let $j = i+l–1$,
9.          For $k = i$ to $j-1$:
10.             For each rule $A \rightarrow BC$
11.                 If table ($i,k$) contains $B$ and table
                    ($k+1,j$) contains $C$, put $A$ in table ($i,j$).
12. If $S$ is in table ($1,n$), accept. Otherwise, reject."
```

Stage 6 runs at most n times.

For each iteration stage 7 runs at most n times.

For each iteration stage 8 and 9 runs at most n times.

For each iteration stage 10 runs r times. Where r is the number of rules of G.

For each iteration stage 11 runs $O(1)$ times.

Example of problems in **P**

```
6.  For $l = 2$ to $n$:
7.      For $i = 1$ to $n–l+1$:
8.          Let $j = i+l–1$,
9.          For $k = i$ to $j-1$:
10.             For each rule $A \rightarrow BC$
11.                 If table $(i,k)$ contains $B$ and table
                    $(k+1,j)$ contains $C$, put $A$ in table $(i,j)$.
12. If $S$ is in table $(1,n)$, accept. Otherwise, reject.”
```

Stage 11, the inner loop of the algorithm runs $O(n^3)$ times.

Totally D executes $O(n^3)$ stages.

- So every CFL can be execute in polynomial times, and is a member of **P**.

The class **NP**

- In practice, attempts to avoid brute-force in certain other problems have not been successful, and polynomial time algorithms that solve them are not known to exist.
- Now we will introduce another class problem: class **NP**.

Theory of Computation

Lecture 10:
Time Complexity
(Part II)

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Content of Lecture 10

- NP-completeness
  - Polynomial time reducibility
  - Definition of NP-completeness
  - The Cook-Levin Theorem
- Additional NP-complete Problems
  - The vertex cover problem
  - The Hamiltonian path problem

Class **NP**

- We have learned the class P problem last week.
- But avoiding brute search (exponential time) is not always be achieved. We can’t design an algorithm to solve the problem in polynomial time.
- One remarkable discovery concerning to this phenomenon shows that the complexities of such problems are linked. This is the theory of class **NP**.

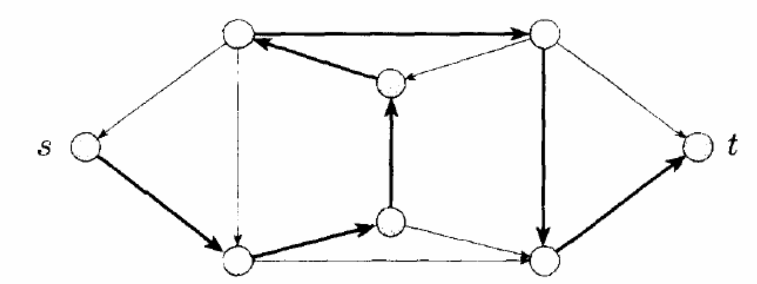
Class **NP**

There are two example:

- Hamiltonian path problem:
- *HAMPATH* = {<*G*,*s*,*t*>| *G* is a directed graph with a Hamiltonian path from *s* to *t*}
- A Hamiltonian path in a directed graph G is a directed path that goes through each node exactly once.

Class **NP**

- A Hamiltonian path:



- We can brute search the path from *s* to *t*, and verify if the potential path is Hamiltonian.
- Nobody knows whether *HAMPATH* is solvable in polynomial time.

Class **NP**

- A *verifier* for a language A is an algorithm V, where

  $$A = \{ w \mid V \text{ accepts } \langle w, c \rangle \text{ for some string } c \}$$

  We measure the time of a verifier only in terms of the length of w. And c is an additional information called a certificate or proof of A.

Class **NP**

- A polynomial time verifier runs in polynomial time in the length of *w*.
- A language *A* is polynomially verifiable if it has a polynomial time verifier.
- For a polynomial time verifier, the certificate has polynomial length ($O(c)=O(w)$).

Class **NP**

- As if there is a path from *s* to *t* that is Hamiltonian, we can verify it in polynomial time. So for the *HAMPATH* problem, a certificate for a string $\langle G, s, t \rangle \in HAMPATH$ is the Hamiltonian path from s to t.
- *HAMPATH* problem is polynomial verifiable.

Class **NP**

Another example:

- *COMPOSITES* problem:

  $COMPOSITES = \{x \mid x = pq,\ \text{for integers } p, q > 1\}$

- A natural number $x \in N$ is composite if it is the product of two integers greater than 1. This is the opposite concept of prime.
- We can solve by exponential time and verify in polynomial time.
- We can solve by polynomial time while the method is hard.

Class **NP**

- Some problems may not be polynomially verifiable.
  - The complement of *HAMPATH* problem: $\overline{HAMPATH}$

    where we will have to check all path to verify that they are not Hamiltonian.

Class **NP**

Definition:

- **NP** is the class of languages that have polynomial time verifiers.

Theorem:

- A language is in **NP** iff it is decided by some nondeterministic polynomial time Turing machine.
  - ○ Term **NP** comes from nondeterministic polynomial time.

Class **NP**

The NTM for *HAMPATH*:

- N_1 = " On input <G,s,t>, where G is a directed graph with nodes s and t:
  1. Write a list of m numbers, $p_1,\ldots p_m$, where m is the number of nodes in G. Each number in the list is nondeterministically selected to between 1 and m.
  2. Check for repetitions in the list. If any are found, ***reject***.
  3. Check whether $s = p_1$ and $t = p_m$. If either fail, ***reject***.
  4. For each i between 1 and m-1, check whether (p_i, p_i+1) is an edge of G. If any are not, ***reject***. Otherwise, all tests have been passed, so ***accept***."

Class **NP**

The NTM for *HAMPATH*:

- N_1 = “ On input <G,s,t
  with nodes s and t:
  1. Write a list of m numb
     of nodes in G. Each nu
     nondeterministically s

All stages runs in polynomial times, so the algorithm runs in nondeterministic polynomial time.

  2. Check for repetitions in the list. If any are found, ***reject***.
  3. Check whether $s = p_1$ and $t = p_m$. If either fail, ***reject***.
  4. For each i between 1 and m-1, check whether (p_i, p_i+1) is an edge of G. If any are not, ***reject***. Otherwise, all tests have been passed, so ***accept***.”

Class **NP**

Now we prove the theorem.

Proof idea:

- Based on the definition of NP, we show how to convert a polynomial time verifier to an equivalent polynomial time NTM and vise versa.
  - The NTM simulates the verifier by guessing the certificate.
  - The verifier simulates the NTM by using the accepting branch as the certificate.

Class **NP**

Proof:

$\Rightarrow$

- Assume that *V* is polynomial time verifier for *A* and *V* is a TM that runs in time n^k and construct *N* as follows:
- *N* = "On input *w* of length *n*:
  1. Nondeterministically select string *c* of length at most n^k.
  2. Run *V* on input <*w*,*c*>.
  3. If *V* accepts, ***accept***; otherwise, ***reject***."

Class **NP**

Proof:

$\Leftarrow$

- Assume A is decided by a polynomial time NTM *N* and construct a polynomial time verifier *V* as following:
- *V* = "On input <*w*,*c*> where *w* and *c* are strings:
  1. Simulate *N* on input *w* as we do in the simulating a NTM by a DTM, treating each symbol of *c* as a description of the nondeterministic choice to make at each step.
  2. If this branch of *N*'s computation accepts, ***accept***; otherwise, ***reject***."

Class **NP**

- Definition:

$$\text{NTIME}(t(n)) = \{L \mid L \text{ is a language decided by a } O(t(n)) \text{ time nondeterministic Turing machine}\}$$

- Corollary:

$$\mathbf{NP} = \bigcup_k \text{NTIME}(n^k)$$

Examples of problems in **NP**

- A clique in an undirected graph is a subgraph, wherein every two nodes are connected by an edge.
- A *k*-clique is a clique that contains *k* nodes.

Examples of problems in **NP**

- $CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

Theorem: *CLIQUE* is in **NP**.

Proof I:

Design a verifier V:

- $V =$ " On input $\langle\langle G, k\rangle, c\rangle$

1. Test whether c is a set of k nodes in G.
2. Test whether G contains all edges connecting nodes in c.
3. If both pass, ***accept***; otherwise, ***reject***."

Examples of problems in **NP**

- Proof II:

Design a NTF N:

- N = “ On input $\langle G, k \rangle$

1. Nondeterministically select a subset c of k nodes of G.
2. Test whether G contains all edges connecting nodes in c.
3. If yes, ***accept***; otherwise, ***reject***.”

Examples of problems in **NP**

- *SUBSET-SUM* problem:
  - We have a collection of numbers $x_1,\ldots,x_k$
  - A target number t.
  - We want to determine whether the collection contains a subcollection that adds up to t.

$$SUBSET\text{-}SUM = \{\langle S,t\rangle \mid S = \{x_1,\ldots,x_k\} \text{ and for some }$$

$$\{y_1,\ldots,y_l\} \subseteq \{x_1,\ldots,x_k\}, \text{ we have } \sum y_i = t\}$$

Examples of problems in **NP**

- Prove *SUBSET-SUM* is in **NP**.
- Proof I:
- Construct a verifier V.
- V = “ On input $\langle\langle S,k\rangle,c\rangle$
  1. Test whether *c* is a collection of numbers that sum to *t*.
  2. Test whether *S* contains all the numbers in *c*.
  3. If both pass, ***accept***; otherwise, ***reject***.”

Examples of problems in **NP**

- Proof II:

Design a NTF N:

- N = “ On input $\langle S, k \rangle$
  1. Nondeterministically select a subset c of the numbers in S.
  2. Test whether c is a collection of numbers that sum to t.
  3. If yes, ***accept***; otherwise, ***reject***.”

Examples of problems in **NP**

- coNP: the languages that are complements of languages in NP.
- Verifying that above languages are coNP is difficulty. $\overline{CLIQUE}$ and $\overline{SUBSET\text{-}SUM}$
- Until now, we don’t know whether coNP is different from NP.

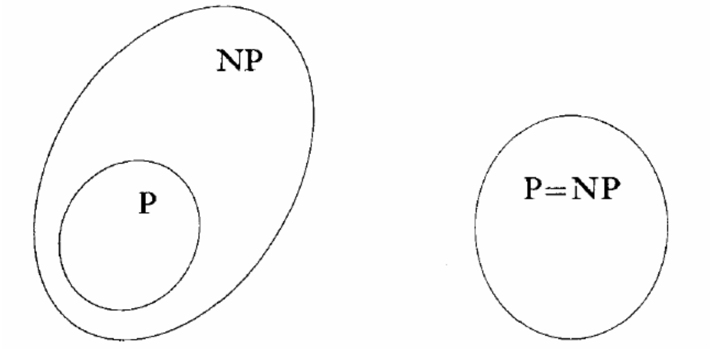
The **P** versus **NP** question

- We loosely refer to polynomial time solvable as solvable "quickly":
  - **P** = the class of languages for which membership can be decided quickly.
  - **NP** = the class of languages for which membership can be verified quickly.

The **P** versus **NP** question

- Now we are unable to prove the existence of a single language in **NP** that is not in **P**.

$$\mathbf{P} \overset{?}{=} \mathbf{NP}$$



The **P** versus **NP** question

- The best known for solving languages in NP deterministically uses exponential time. But can’t improve any more now.

$$\textbf{NP} \subseteq \mathrm{EXPTIME} = \bigcup_k \mathrm{TIME}\left(2^{n^k}\right)$$

NP-completeness

- **NP-completeness problem:**
  - This problem is erected by Stephen Cook and Leonid Levin.
  - Certain problem in **NP** whose individual complexity is related to that of the entire class.
  - So if there is a polynomial time algorithm for any of these problems, all problems in NP would be polynomial time solvable.

NP-completeness

- **NP-completeness problem:**
  - A research trying to show that **P≠NP** may try to show that any problem in NP requires more than polynomial time to achieve this goal.
  - A research trying to prove that **P=NP** may find a polynomial time algorithm for an NP-complete problem to achieve this goal.
  - The phenomenon of **NP**-completeness may prevent wasting time searching for a nonexistent polynomial time algorithm to solve a particular problem.

NP-completeness

Satisfiablility problem:

- Boolean variables: TRUE(1) and FALSE(0).
- Boolean operations: AND($\wedge$), OR($\vee$), and NOT($\neg$). And $\neg x$ is writtern as $\overline{x}$
- Boolean formula is an expression involving Boolean variables and operations.

$$\phi = \left(\overline{x} \wedge y\right) \vee \left(x \wedge \overline{y}\right)$$

NP-completeness

- A Boolean formula is satisfiable if some assignment of 0s and 1s to the variables makes the formula evaluate to 1.
- Satisfiability problem:

$$SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$$

- This is the first **NP**-complete problem we present.

Polynomial time reducibility

Polynomial time computable function:

- A function $f : \Sigma^* \to \Sigma^*$ is a polynomial time computable function if some polynomial time Turing machine *M* exists that halts with just *f*(*w*) on its tape, when started on any input *w*.

Polynomial time reducibility

Polynomial time reducible:

- Language *A* is polynomial time mapping reducible, or simply polynomial time reducible, to language *B*, written $A \leq_P B$, if a polynomial time computable function $f : \Sigma^* \to \Sigma^*$ exists, where for every *w*,

$$w \in A \Leftrightarrow f(w) \in B$$

The function *f* is called the polynomial time reduction of *A* to *B*.

Polynomial time reducibility

Theorem:

If $A \leq_P B$ and $B \in P$, then $A \in P$.

- If one language is polynomial time reducible to a language already known to have a polynomial time solution, we obtain a polynomial time solution to the original language.

Polynomial time reducibility

Proof:

- Let *M* be the polynomial time algorithm deciding *B* and *f* be the polynomial time reduction from *A* to *B*. We describe a polynomial time algorithm *N* deciding *A* as follows.
- *N* = " On input *w*:
  1. Compute *f* (*w*)
  2. Run M on input *f* (w) and output whatever *M* outputs."
- So if $w \in A$,then *M* accept *f* (*w*). And N runs in polynomial time because step 1 runs in polynomial time and stage 2 runs in the composition polynomial times which is still a polynomial.

Polynomial time reducibility

- Literal is a Boolean variable or a negated Boolean variable
  - x or $\overline{x}$.
- Clause is several literals connected with $\vee$s
  - $x_1 \vee x_2 \vee \overline{x}_3$
- A Boolean formula is in conjunctive normal form, called a cnf-formula, if it comprises several clauses connected with $\wedge$s
  - $\left(x_1 \vee x_2 \vee \overline{x}_3\right) \wedge \left(x_3 \vee \overline{x}_4\right) \wedge \left(x_1 \vee \overline{x}_3 \vee x_4 \vee x_2\right)$

Polynomial time reducibility

- It is a 3cnf-formula if all the clauses have three literals.

$$\left(x_1 \vee x_2 \vee \overline{x}_3\right) \wedge \left(x_3 \vee \overline{x}_3 \vee \overline{x}_4\right) \wedge \left(x_1 \vee \overline{x}_3 \vee x_2\right)$$

- *3SAT*:

$$3SAT = \left(\langle \phi \rangle \phi \text{ is a satisfiable 3cnf-formular}\right)$$

- Here, in a satisfiable cnf-formula, each clause must contain at least one literal that is assigned 1.

Polynomial time reducibility

Theorem:

- 3*SAT* is polynomial time reducible to *CLIQUE*.

Proof idea:

- Construct the polynomial time reduction from 3*SAT* to *CLIQUE*.

Polynomial time reducibility

Proof:

- Suppose is ϕ a formula with k clauses such as:

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

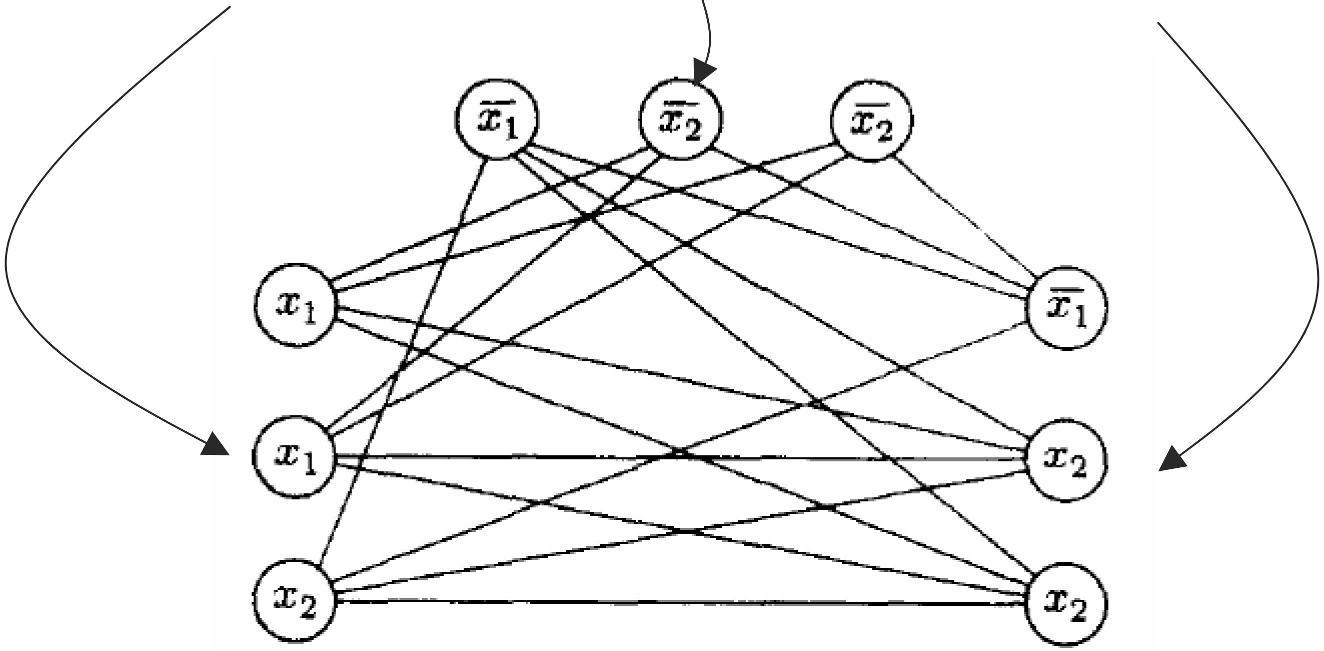
- The reduction *f* generates the string <*G*,*k*>, where *G* is an undirected graph defined as follows.
- The nodes in *G* are organized into *k* groups of three nodes each called the triples, $t_1, \ldots, t_k$. Each triple corresponds to one of the clauses in ϕ , and each node in a triple corresponds to a literal in the associated clause.

Polynomial time reducibility

- The edges of G connect all but two types of pairs of nodes in B.
  - No edge is present between nodes in the same triple
  - No edge is present between two nodes with contradictory labels, as in x or $\overline{x}$.

Polynomial time reducibility

$$(x_1 \vee x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_2)$$



Polynomial time reducibility

- Now we demonstrate that ϕ is satisfiable iff G has a *k*-cluque.
- By the definition of satisfiable, there is at least one literal is true in every clause of the satisfying assignment.
- In each triple of *G*, we select one node corresponding to a true literal in the satisfying assignment.
- By the edge construction rule, we know that each pair of selected nodes is joined by an edge.
- Therefore G contains a *k*-clique.

Polynomial time reducibility

- We assign truth value to the variables of ϕ so that each literal labeling a clique node is made true.
- This assignment to the variables satisfies ϕ because each triple contains a clique node and hence each clause contains a literal that is assigned TRUE.
- So ϕ is satisfiable.

Definition of **NP**-completeness

Definition:

- A language *B* is **NP**-complete if it satisfies two conditions:
  1. *B* is in **NP**, and
  2. Every *A* in **NP** is polynomial time reducible to *B*.

- We can see the usefulness of **NP**-complete:
  - If *B* is **NP**-complete and $B \in \mathbf{P}$, then **P = NP**
  - If *B* is **NP**-complete and $B \leq_P C$ for C in **NP**, then C in **NP**-complete.

Definition of **NP**-completeness

- Cook-Levin theorem:

$$SAT \in \mathbf{P} \text{ iff } \mathbf{P} = \mathbf{NP}$$

- By the definition of NP-completeness, we first prove the following theorem:

SAT is **NP**-complete.

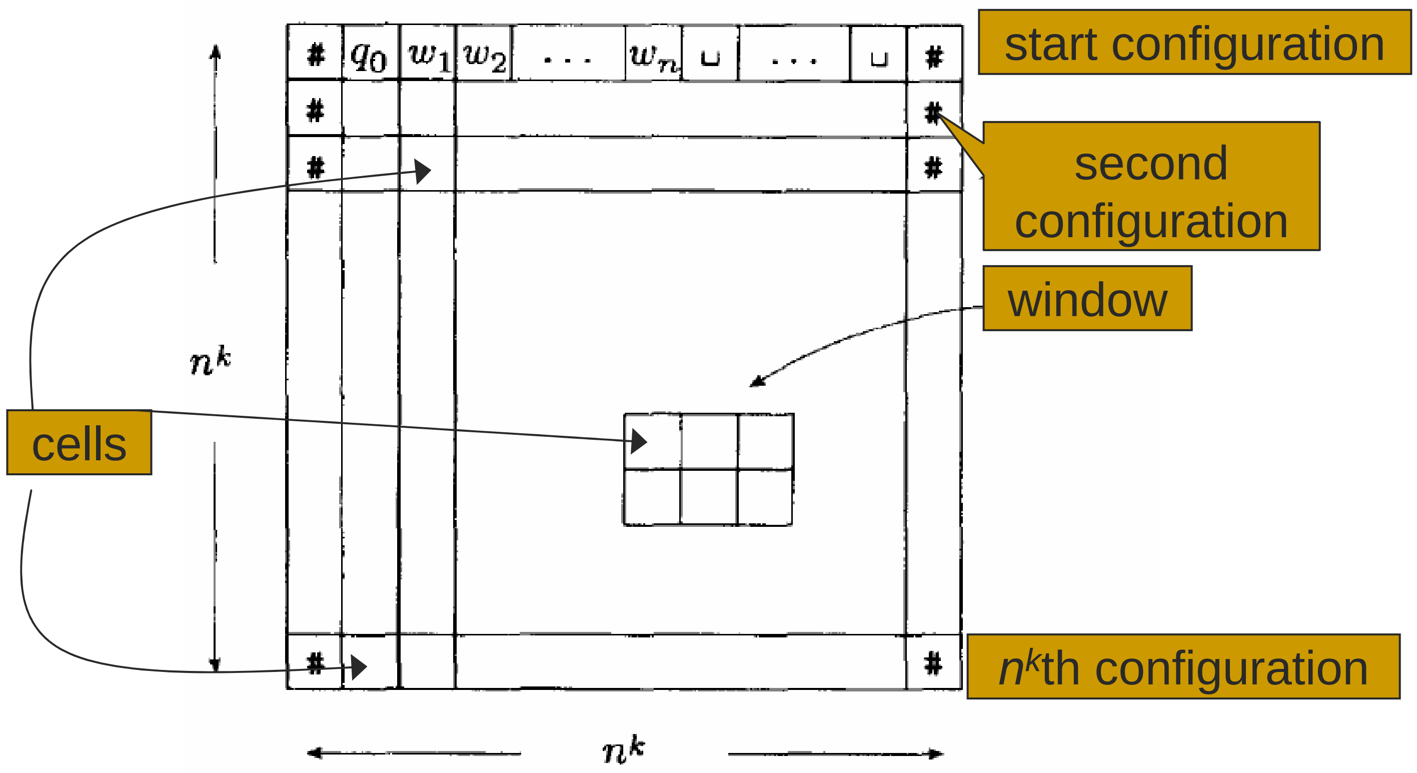
- Proof idea: construct a polynomial time reduction for each language A in **NP** to SAT, that is the reduction for A takes a string w and produces a Boolean formula ϕ that simulates the **NP** machine for A on input w.

Definition of **NP**-completeness

- Proof:
- It is obvious that $SAT \in \mathbf{NP}$.
- Suppose A is a language that belongs to **NP**. Let N be a nondeterministic Turing machine that decides A in n^k time (suppose in form of n^k-3 for some constant k.
- A tableau for N on w is an $n^k \times n^k$ table whose rows are the configurations of a branch of the computation of N on input w.

Definition of **NP**-completeness

- Tableau



Definition of **NP**-completeness

- For convenience we assume that each configuration starts and ends with a # symbol.
- A tableau is accepting if any row of the tableau is an accepting configuration.

Definition of **NP**-completeness

- Construct the reduction *f* which produces a formula ϕ.
- Suppose Q and Γ are the state and tape alphabet of *N*.
- Let $C = Q \cup \Gamma \cup \{\#\}$,for each *i* and *j* between 1 and n^k and for each *s* in *C* we have a variable, $x_{i,j,s}$.
- Demonstrate the cell in row *i* and column *j* as *cell* [*i*,*j*].

Definition of **NP**-completeness

- We know that every cell contains a symbol from *C*. we represent the contents of the cells with the variables of ϕ . If $x_{i,j,s}$. takes on the value 1, it means that the content of *cell* [*i*,*j*] should be *s*.

Definition of **NP**-completeness

- Now design ϕ . ϕ Should satisfy the AND of the four parts $\phi_{cell} \wedge \phi_{start} \wedge \phi_{move} \wedge \phi_{accept}$.
- ϕ_{cell} is defined as:

$$\phi_{cell} = \bigwedge_{1 \le i,j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} \overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right]$$

- We can see that ϕ_{cell} contains a fragment for each cell in the tableau because *i* and *j* range from 1 to n^k.

Definition of **NP**-completeness

- Now design ϕ . ϕ Should satisfy the AND of the four parts $\phi_{cell} \wedge \phi_{start} \wedge \phi_{move} \wedge \phi_{accept}$.
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ϕ_{cell} contai

ableau beca

from 1 to

This part constrains that at least one variable is turned on in the corresponding cell.

This part constrains that no more than one variable is turned on in the corresponding

So any assignment to the variables that satisfies must have exactly one variable on for every cell.

Definition of **NP**-completeness

- ϕ_{start} has to ensures that the first row of the table is the string configuration of N on string w. It is defined as:

$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \wedge x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \ldots \wedge x_{1,n+2,w_n} \wedge$$

$$x_{1,n+3,\sqcup} \wedge \ldots \wedge x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}$$

Definition of **NP**-completeness

- ϕ_{accept} has to guarantee that an accepting configuration occurs in the tableau. *w*. It is defined as:

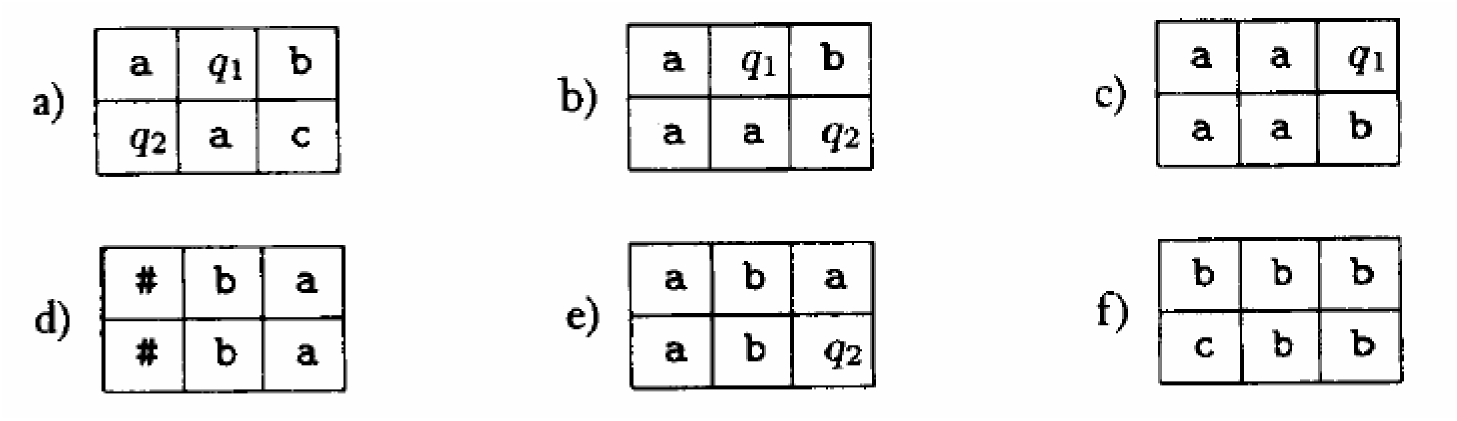
$$\phi_{accept} = \bigvee_{1 \le i, j \le n^k} x_{i,j,q_{accept}}$$

Definition of **NP**-completeness

- ϕ_{move} guarantees that each row of the table corresponds to a configuration that legally follows the preceding row's configuration according to *N*'s rules.
- If each 2×3 window of cells is legal, then the configuration is legal.

Definition of **NP**-completeness

- Suppose *a*, *b*, and *c* are members of the tape alphabet and q_1 and q_2 are states of *N*.
- And $\delta(q_1,a)=((q_1,b,\mathrm{R}))$ and $\delta(q_1,b)=((q_2,c,\mathrm{L}),(q_2,a,\mathrm{R}))$
- The legal window examples are:



a)

a	q_1	b
q_2	a	c

b)

a	q_1	b
a	a	q_2

c)

a	a	q_1
a	a	b

d)

#	b	a
#	b	a

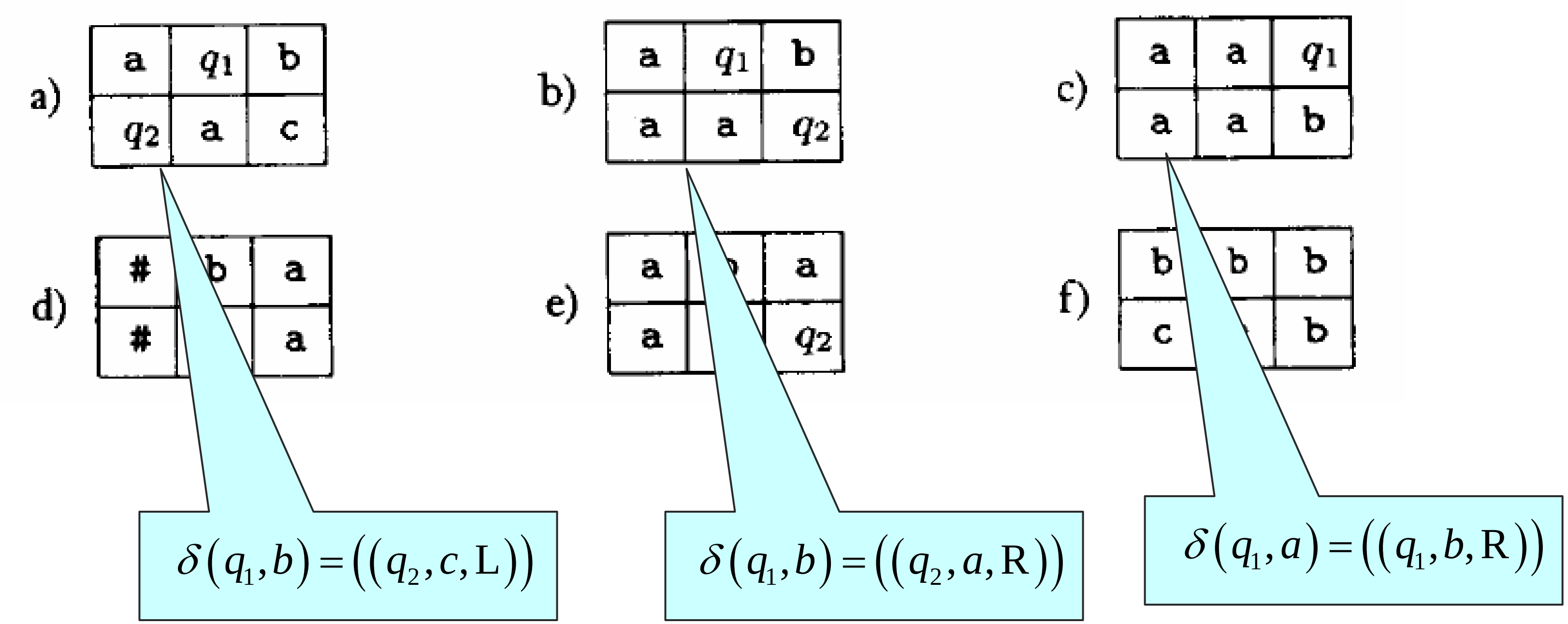
e)

a	b	a
a	b	q_2

f)

b	b	b
c	b	b

Definition of **NP**-completeness



$\delta(q_1, b) = ((q_2, c, \mathrm{L}))$

$\delta(q_1, b) = ((q_2, a, \mathrm{R}))$

$\delta(q_1, a) = ((q_1, b, \mathrm{R}))$

Definition of **NP**-completeness

a)

a	q_1	b
q_2	a	c

b)

a	q_1	b
a	a	q_2

c)

a	a	q_1
a	a	b

d)

#	b	a
#	b	a

e)

a	b	a
a	b	q_2

f)

b	b	b
c	b	b

Legal as the two rows are the same

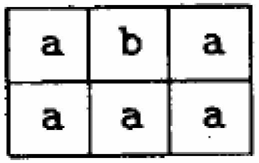
$\delta(q_1,b)=((q_2,c,\mathrm{L}))$

$\delta(q_1,b)=((q_2,c,\mathrm{L}))$

Definition of **NP**-completeness

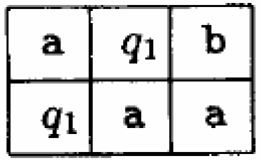
- Example of illegal for *N*:

a)



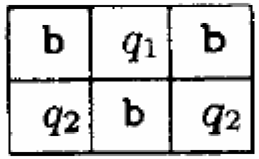
a	b	a
a	a	a

b)



a	q_1	b
q_1	a	a

c)



b	q_1	b
q_2	b	q_2

Definition of **NP**-completeness

Claim:

- If the top row of the table is the start configuration and every window in the table is legal, each row of the table is a configuration that legally follows the preceding one.
- It is obviously true, according the legal definition of 2×3 window.

Definition of **NP**-completeness

- Come to construct .

$$\phi_{move} = \bigwedge_{\substack{1\le i<n^k,\\ 1<j<n^k}} \left(\text{the } (i, j) \text{ window is legal}\right)$$

- We replace the text “the (*i*, *j*) windo is legal” with the following formula. And the contents of six cells of a window is written as $a_1,\ldots,a_6$

$$\bigvee_{\substack{a_1,\ldots,a_6 \text{ is a}\\ \text{legal window}}} \left(x_{i,j-1,a_1} \wedge x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6}\right)$$

Definition of **NP**-completeness

- Now demonstrate the reduction can be finished in polynomial tiem.
- There are $n^k \times n^k$ variables, each is related to $|C|=l$, so the number of variables is $O(n^{2k})$.
- ϕ_{cell} : $O(n^{2k})$.
- ϕ_{start} : $O(n^k)$.
- ϕ_{move} and ϕ_{accept} : $O(n^{2k})$.

Definition of **NP**-completeness

- So we get the polynomial reduction from input string w to ϕ .
- So $SAT \in$ **NP**-complete.

Definition of **NP**-completeness

- Now we can use the theorem $SAT \in \mathbf{NP}\text{-complete}$ to reduce a problem is NP-complete.
- We now prove $3SAT \in \mathbf{NP}\text{-complete}$ which is always be easy to be reduced to.

Definition of **NP**-completeness

Proof:

- Obviously $3SAT \in \mathbf{NP}$
- We modify the last proof, to directly produces a formula in conjunctive normal form with three literals per clause.

Definition of **NP**-completeness

- First we show how to convert a cnf to 3cnf.
- If clause has one or two literals, we replicate one of the literals until the total number is three.
- More than three literals:
  - Split it into several clauses
  - Add additional variables to preserve the satisfiability or nonsatisfiability of the original.

Definition of **NP**-completeness

- Example:

$$\left(a_1 \vee a_2 \vee a_3 \vee a_4\right) = \left(a_1 \vee a_2 \vee z\right) \wedge \left(\overline{z} \vee a_3 \vee a_4\right)$$

$$\left(a_1 \vee a_2 \vee \ldots \vee a_l\right) = \left(a_1 \vee a_2 \vee z_1\right) \wedge \left(\overline{z}_1 \vee a_3 \vee z_2\right) \wedge \ldots \wedge \left(\overline{z}_{l-3} \vee a_{l-1} \vee a_l\right)$$

Definition of **NP**-completeness

- Now we make ϕ to produce a formula that is the conjunctive of cnf.
- ϕ_{cell} is an AND of clauses and so is already in cnf.
- ϕ_{start} is an AND of variables (which can be seen as clauses) and so is already in cnf.
- ϕ_{accept} is an OR of variables and is a single clause.

Definition of **NP**-completeness

- ϕ_{move} is a big AND of subformulas.
- By the theorem $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
- We can convert ϕ_{move} to cnf , with the total size of ϕ_{move} increases by a constant factor.

- We finished the proof.

Additional **NP**-complete problems

- The practice sense of NP-complete is obviously:
  - If you seek a polynomial time algorithm for a new NP-problem, spending part of your effort attempting to prove it NP-complete is sensible because doing so may prevent you from working to find a polynomial time algorithm that doesn’t exist.

Additional **NP**-complete problems

The proof of NP-complete:

- Usually the strategy is to exhibit a polynomial time reduction from 3*SAT* to the language in question.
- In the reduction process, we look for structures (called gadgets) in that language that can simulate the variables and clauses in Boolean formulas.

Additional **NP**-complete problems

Corollary:

- Clique is **NP**-complete.

Proof:

- We can conclude by the reduction from 3*SAT* to *CLIQUE*.

The vertex cover problem

- *VERTEX-COVER*={<*G*,*k*>|*G* is an undirected graph that has a *k*-node vertex cover}

- If *G* is an undirected graph, vertex cover of *G* is a subset of the nodes where every edge of *G* touches one of those nodes.

- Prove: *VERTEX-COVER* is **NP**-complete.

The vertex cover problem

Proof:

- It is easy to prove *VERTEX-COVER* belongs to **NP**.
- We take the certificate as a vertex-cover of length *k*.

The vertex cover problem

- Now reduce from $3SAT$ which reduce a 3cnf-formular ϕ to a graph G and a number k and G has a vertex cover with k nodes.

$$3SAT \leq_P VERTEX\text{-}COVER$$

- That means that for every variable x in , construct a edge connecting two nodes. Then label the node with x and $\overline{x}$, and set x=TRUE.

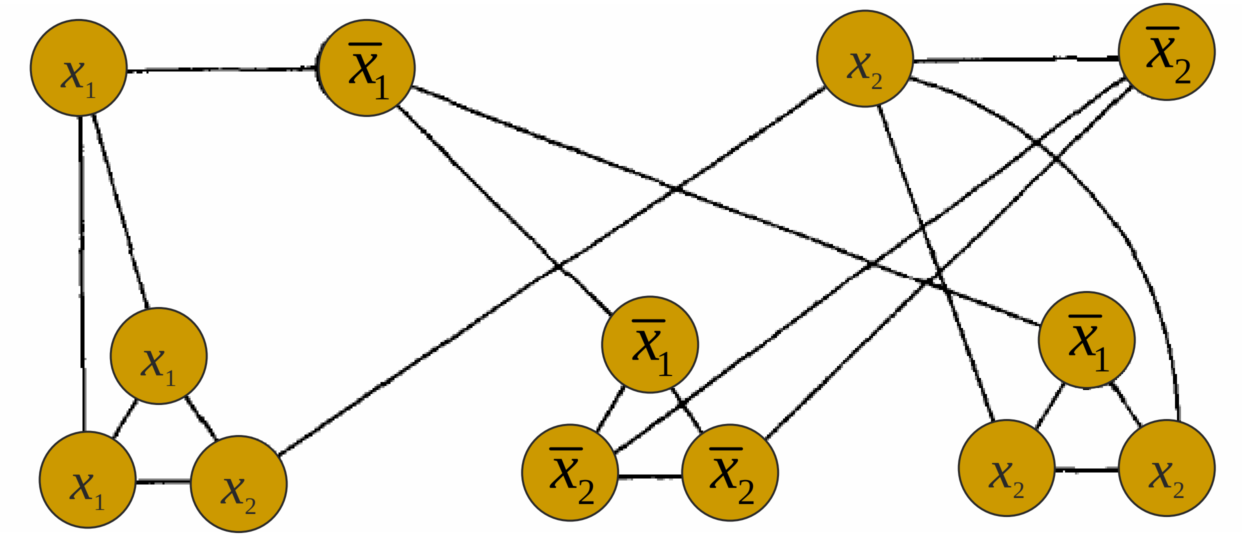
The vertex cover problem

- Each clause gadget is a triple of three nodes that are labeled with the three literals of the clause.
- Connect these three nodes to each other and to the nodes in the variables gadgets that have the identical labels.
- Thus the total number of nodes that appear in G is $2m+3l$, where ϕ has m variables and l clauses. And $k=m+2l$.

The vertex cover problem

- Example:

$$\phi = \left(x_1 \vee x_1 \vee x_2\right) \wedge \left(\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2\right) \wedge \left(\overline{x}_1 \vee x_2 \vee x_2\right)$$



The vertex cover problem

- Now prove that ϕ is satisfiable iff G has a vertex cover with k nodes.

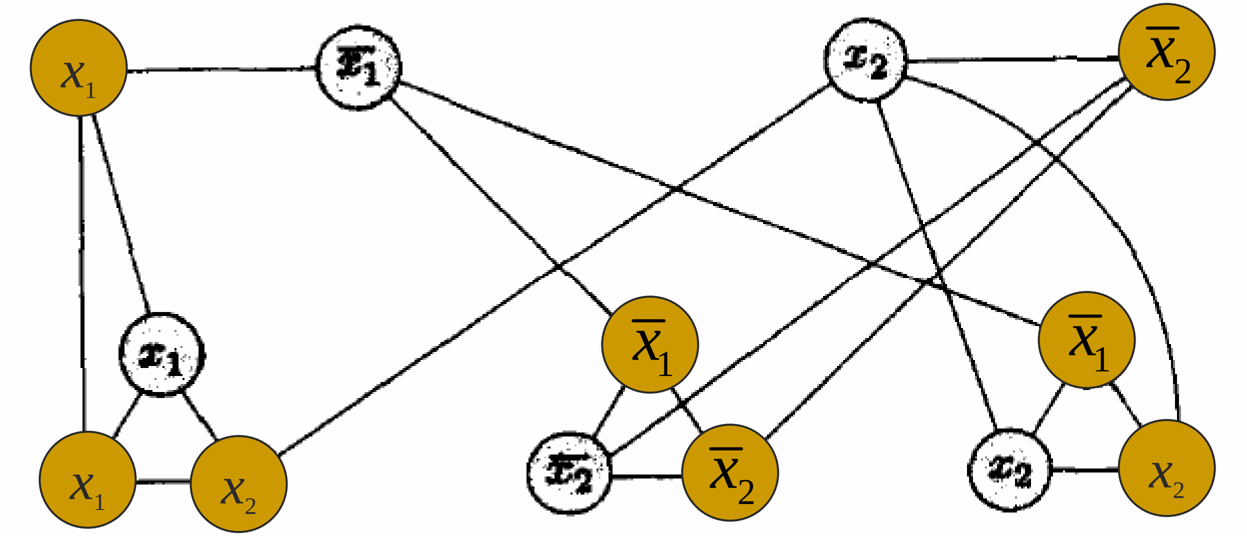
  $\Rightarrow$

- First put the nodes of the variable gadgets that correspond to the true literals in the assignment into the vertex cover.
- We select one true literal in every clause and put the remaining two nodes from every clause gadget into the vertex cover.
- It is obviously that this k nodes cover all edges because
  - Every variable gadget edge is clearly covered
  - All three edges within every clause are covered
  - All edges between variable and clause gadgets are covered.
- Hence G has a vertex cover with k nodes.

The vertex cover problem

- Example: $\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2 \vee x_2)$

  where $x_1 = TRUE, \overline{x}_2 = TRUE.$



The vertex cover problem

$\Leftarrow$

- Now G has a vertex cover with k nodes, we show that ϕ is satisfiable by constricting the satisfying assignment.
- The vertex cover must contain one node in each variable gadget and two in every clause gadget
  - To cover the edges of the variable gadgets
  - To cover the three edges with the clause gadgets.
- We take the nodes of the variable gadgets that are in the vertex cover an assign the corresponding leterals TRUE.

The vertex cover problem

- That assignment satisfies   because each of the three edges connecting the variable gadgets with each clause gadget is covered and only two nodes of the clause gadget are in the vertex cover.
- So one of the edges must be covered by a node from a variable gadget and so we have finished the proof.

The Hamiltonian path problem

Prove *HAMPATH* is **NP**-complete.

- Proof idea:
- We use the same method in proving *VERTEX-COVER* problem.
- Here the variable gadget has the form of diamond, and the clause gadget is a node.

The Hamiltonian path problem

- We have proved *HAMPATH* is **NP**.

$$3SAT \leq_P HAMPATH$$

- Construct a directed graph G, where a Hamiltonian path exists between s and t iff ϕ is satisfiable.

The Hamiltonian path problem

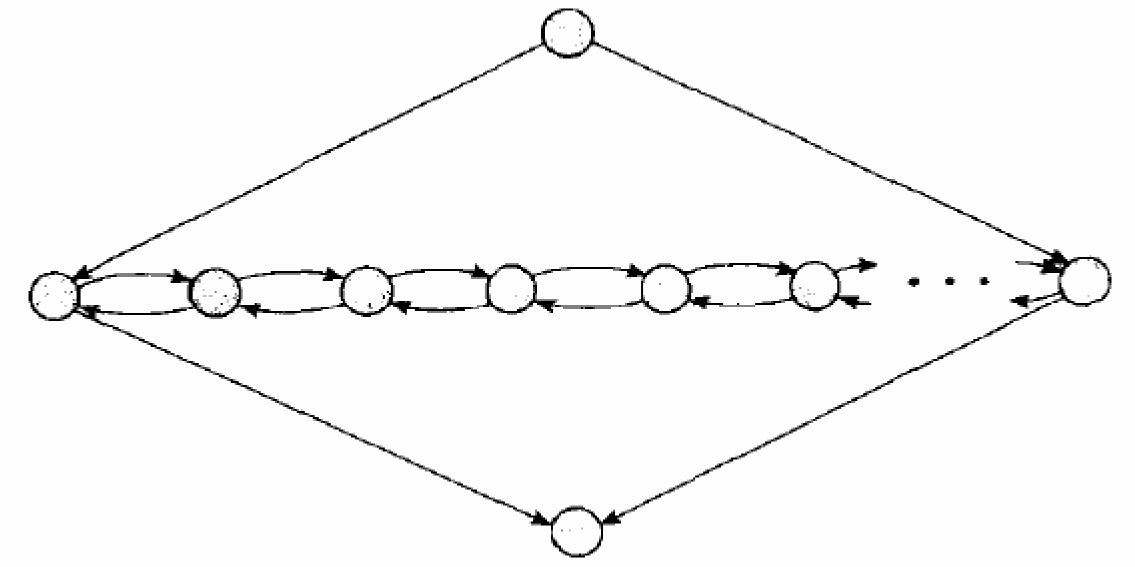
- Suppose

$$\phi = (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

- Where each *a*, *b*, and *c* is a literal x_i or $\overline{x_i}$. Let $x_1, \ldots, x_l$ be the *l* variables of ϕ .
- Represent each variable x_i with a diamond-shaped structure that contains a horizontal row of nodes.

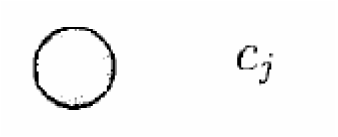
The Hamiltonian path problem

- Diamond-shaped structure to present variable.



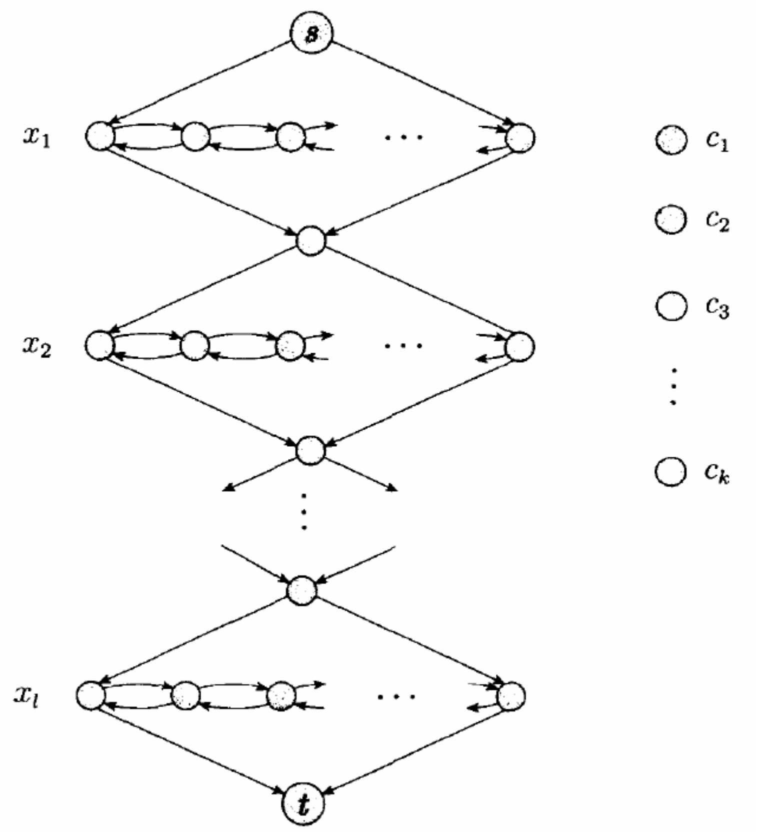
The Hamiltonian path problem

- Represent each clause of ϕ as a single node.



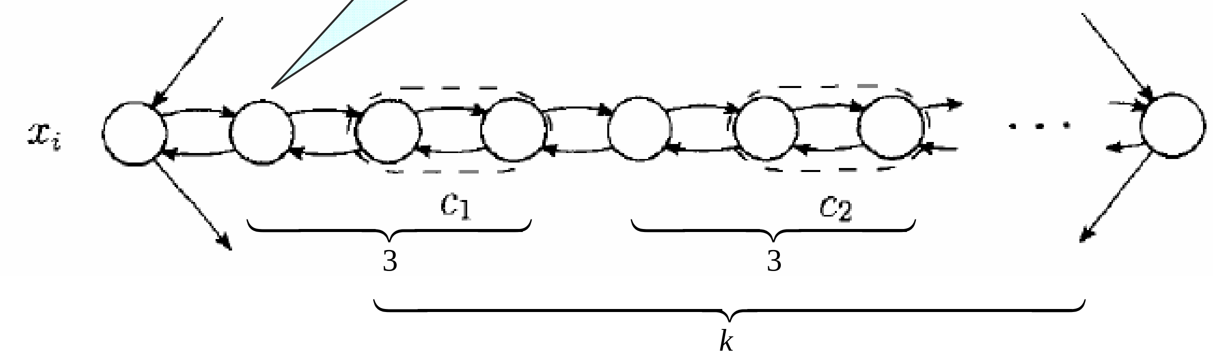
The Hamiltonian path problem

- The global structure of G without the connecting of variables to the clauses.



The Hamiltonian path problem

- The diamond structure



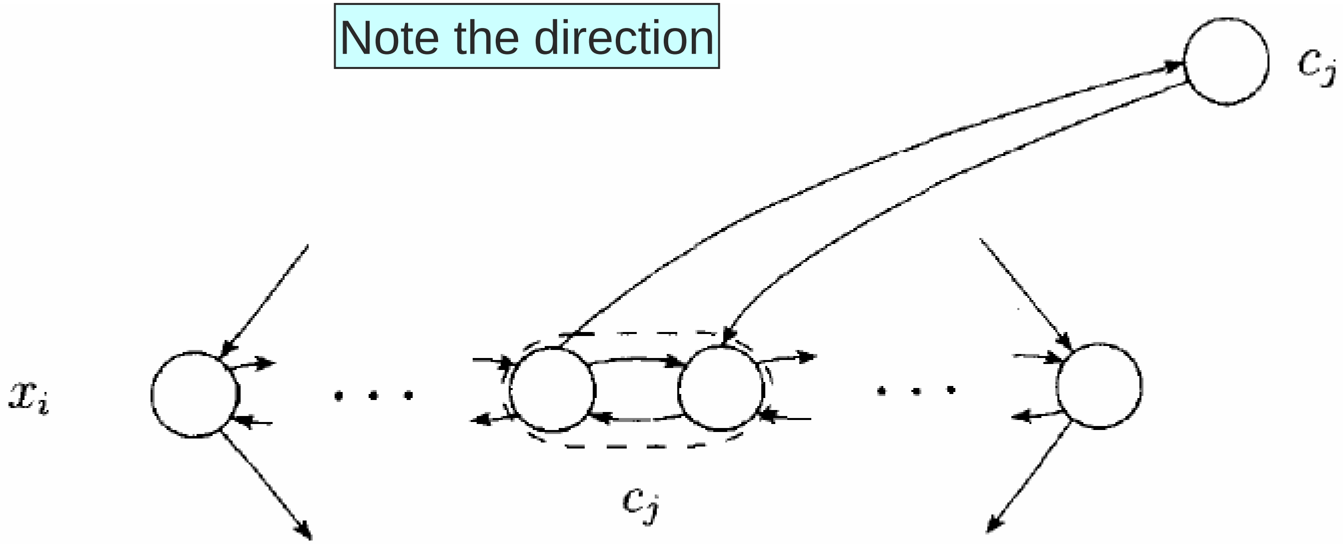
separator node

- The horizontal row contains 3*k*+1 nodes in addition to the two nodes on the ends belonging to the diamond.

The Hamiltonian path problem

- If variable x_i appears in clause c_j, we add two edges from the *j*th pair in the *i*th diamond to the *j*th clause node.

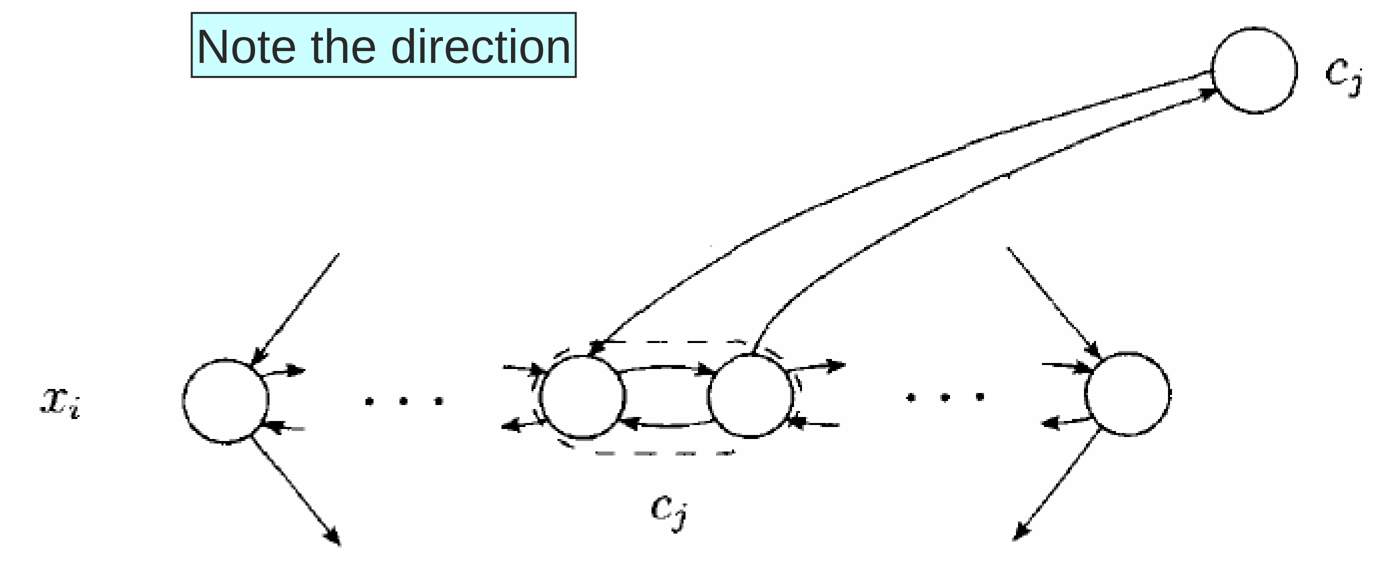
Note the direction



The Hamiltonian path problem

- If variable $\overline{x}_i$ appears in clause c_j, we add two edges from the *j*th pair in the *i*th diamond to the *j*th clause node.

Note the direction



The Hamiltonian path problem

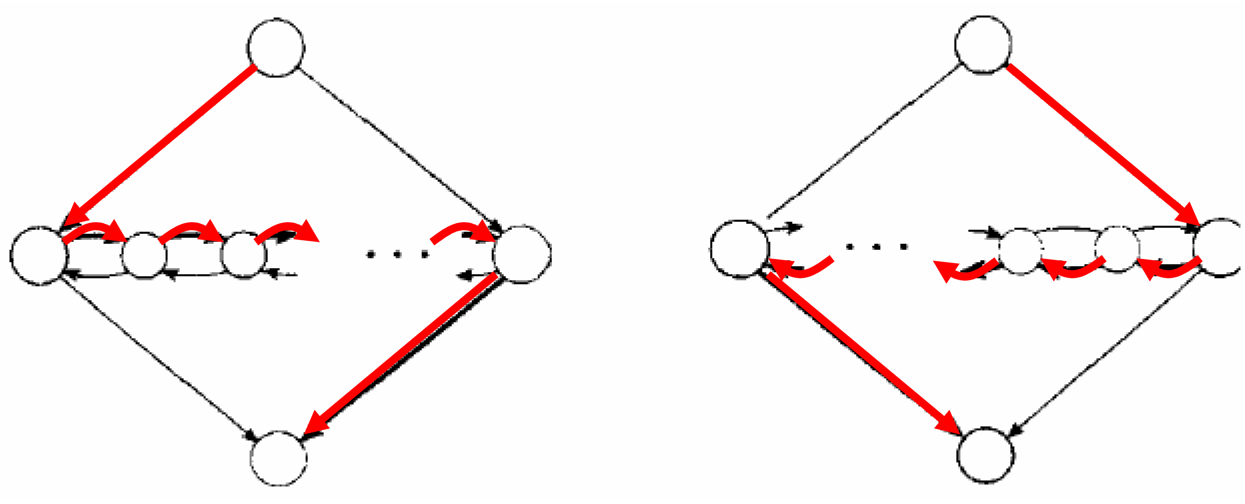
- Now ϕ is satisfiable iff a Hamiltonian path exists from *s* to *t*.

  $\Rightarrow$

- First ignore the clause nodes:
- To hit the horizontal nodes in a diamond, the path
  - zig-zags from left to right if $x_i = \text{TRUE}$
  - zag-zigs from right to left $x_i = \text{FALSE}$

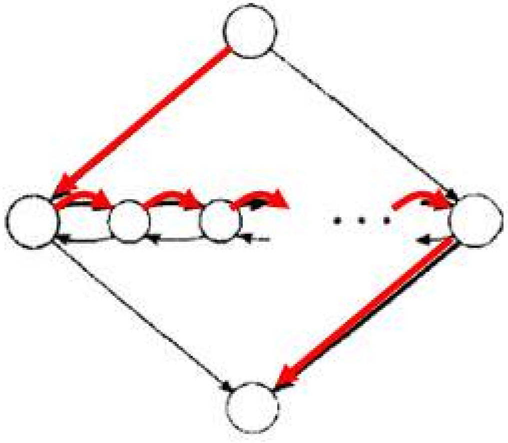
The Hamiltonian path problem

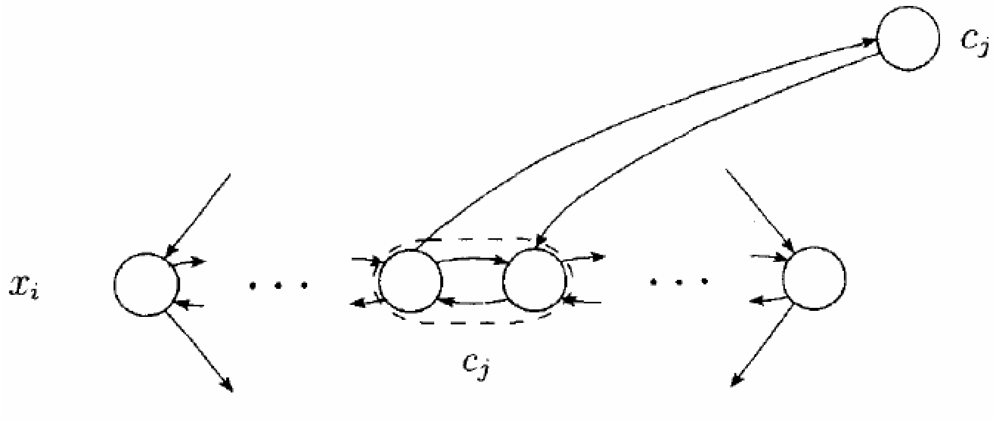
- Zig-zag and zag-zig



The Hamiltonian path problem

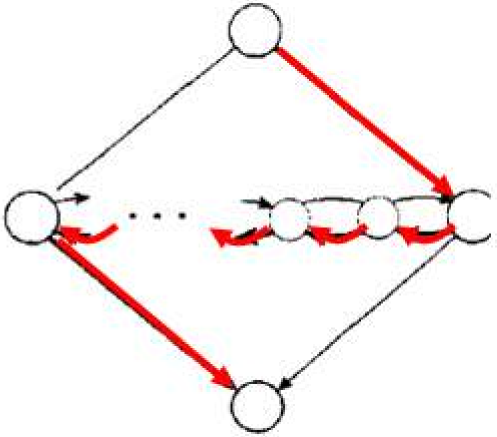
- If we select x_i in clause c_j , we can detour at the *j*th pair in the *i*th diamond.
- This is because x_i =TRUE , so the path zig-zags from left to right, and based on the connection between clause and variable, we can achieve.

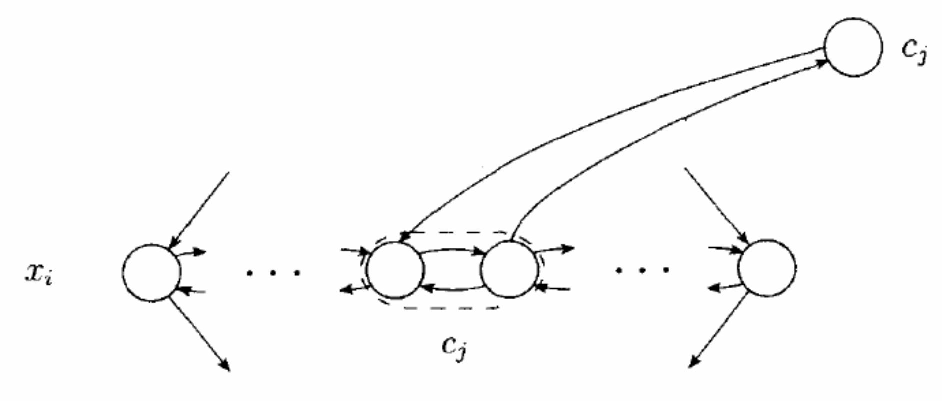




The Hamiltonian path problem

- If we select $\overline{x}_i$ in clause c_j, we can detour at the *j*th pair in the *i*th diamond.
- This is because x_i =FALSE , so the path zag-zigs from right to left, and based on the connection between clause and variable, we can achieve.





The Hamiltonian path problem

- Notice that in clause c_j there may be two or three literal that is TRUE. Just detour once.

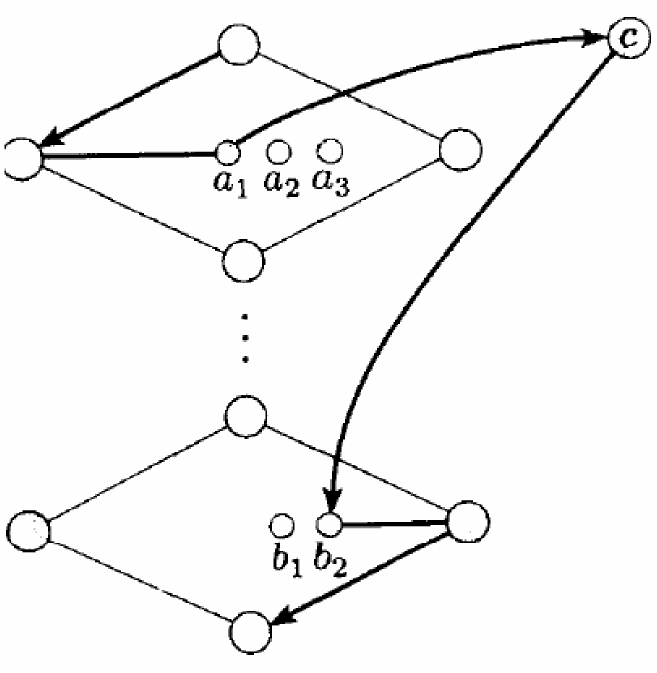
The Hamiltonian path problem

$\Leftarrow$

- If we have a G which is a Hamiltonian path from s to t (suppose in diamond form), then it goes through the diamonds in order from the top one to the bottom one, except fro the dectours to the clause nodes.
  - If the path zig-zags through the diamond, we assign the corresponding variable TRUE.
  - If the path zag-zigs through the diamond, we assign the corresponding variable FALSE.

The Hamiltonian path problem

- Now shown how Hamiltonian path has the diamond form which is called normal.
- Normality may fail only if the path enters a clause from one diamond but returns to another.



The Hamiltonian path problem

- If the situation occurs, either a_2 or a_3 must be a separator node.
- If a_2 were a separator node, the only edges entering a_2 would be from a_1 and a_3.
- If a_3 were a separator node, a_1 and a_2 would be in the same clause pair, and hense the only edges entering a_2 would be from a_1 and a_3.
- So the path could not contain node a_2.

The Hamiltonian path problem

- The path cannot enter a_2 from c or a_1 because the path goes else where from these nodes.
- The path cannot enter a_2 from a_3 because a_3 is the only available node that a_2 points at.
- So there must be path from a_2 to a_3 .
- So the above situation won’t appear. The Hamiltonian path must be normal.

- We finish the proof.

The Hamiltonian path problem

- *UHAMPATH* ={<*G*,*s*,*t*>| *G* is a undirected graph with a Hamiltonian path from *s* to *t*}
- We can conclude it by reduction from *HAMPATH*.

Theory of Computation

Lecture 11:
Space Complexity

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Content of Lecture 11

- Space Complexity
- Savitch’s Theorem
- PSPACE Class
- PSPACE-Completeness
- Examples
- L and NL Classes
- NL-Completeness

Space Complexity

- Instead of time complexity, we discuss complexity in another way to classify the computational difficulty of an algorithm – Space
    - Amount of space or memory required for computation
    - Both time and space are important issue when we want to propose an practical solution
    - Time and Space complexities share many common characteristics

Space Complexity

- Definition of Space Complexity
  - Let *M* be a **DTM** that halts on all inputs
  - The space complexity of M is the function $f : N \rightarrow N$
    - $f(n)$ is the max. no. of tape cells that *M* scans on any input of length n
    - If the space complexity of *M* is $f(n)$, we say that *M* runs in space $f(n)$

Space Complexity

- Definition of Space Complexity
  - Let M be a **NTM** that halts on all inputs
  - The space complexity of M is the function $f : N \rightarrow N$
    - $f(n)$ is the max. no. of tape cells that M scans **on any branch of its computation** for any input of length n

Space Complexity

- Definition of **Space Complexity Classes**
  - Let $f : N \rightarrow R^{+}$ be a function
  - SPACE($f(n)$) = $\{L \mid L$ is a language decided by an $O(f(n))$ space DTM$\}$
  - NSPACE($f(n)$) = $\{L \mid L$ is a language decided by an $O(f(n))$ space NTM$\}$

Space Complexity

- Recall that SAT is an NP-Complete problem, it can not be solved by any polynomial time algorithm. But it can be solved using linear space
  - Space is more powerful than time
    - Because…Space can be reused

Space Complexity

- M_1 = "On input < ϕ >, where ϕ is a Boolean function
  1. For each truth assignment to the variables $x_1, \ldots, x_m$ of ϕ
  2. Evaluate ϕ on that truth assignment
  3. If ϕ ever evaluated to 1, *accept*; if not, *reject*"
- M_1 runs in linear space because each iteration can reuse the space
- So, the space complexity of M_1 is $O(m)$, or $O(n)$

Space Complexity

- Another example:
  - $ALL_{\mathrm{NFA}} = \{<A> \mid A \text{ is a NFA and } L(A) = \Sigma^*\}$
  - This is to determine whether a NFA accept all strings
  - Instead of directly implementing an algorithm to deal with it, we implement a nondeterminstic linear space algorithm to decides the complement of this language, $\overline{ALL_{\mathrm{NFA}}}$

Space Complexity

- N = "On input $<M>$ where M is an NFA
  1. Place a marker on the start state of the NFA
  2. Repeat 2^q times, where q is the number of states of M
     3. Nondeterministically select an input symbol and change the positions of the markers on M's statest to simulate reading that symbol
  4. If a marker was ever placed on an accept state, reject; otherwise accept."

Space Complexity

- If M accepts any strings, it must accept one of length at most 2^q because strings longer than this length consist of repeated substring and markers are placed repeatedly.
- The repetition part in the string could be removed to obtain a shorter accepted string. Hence, N decides ALL_{NFA}.
- This algorithm only needs space to store the location of the markers and therefore it runs in nondeterministic space $O(n)$

Savitch’s Theorem

- One of the earliest results about space complexity
- This theorem shows that deterministic machines can simulate nondeterministic machines by using surprisingly small amount of space
  - For time complexity, Savitchs’ theorem shows that DTM needs a exponential increase in time to simulate NTM
  - For space complexity, any NTM uses $f(n)$ space can be converted to a DTM uses only $f^2(n)$ space

Savitch's Theorem

- Definition of Savitch's Theorem
  - For any function $f : N \rightarrow R+$, where $f(n) \geqslant n$, $\mathrm{NSPACE}(f(n)) \subseteq \mathrm{SPACE}(f^2(n))$

Savitch's Theorem

- Proof Idea:
  - We may simulate an $f(n)$ spce NTM deterministically
    - The simplest way is try out all the branches of the NTM computation, one by one
    - Then, we keep track of the branch's current status to resume this branch later
    - However, each branch uses $f(n)$ space and each branch may produce another nondeterministic branches
    - We need $2^{O(f(n))}$ space for this approach, too bad

Savitch's Theorem

- Instead, we consider a more general problem:
  - We are given 2 configurations of the NTM, c_1 and c_2, and a number t . We need to decide that can the NTM get from c_1 to c_2 within t steps.
  - We call this problem the **Yieldability Problem**
  - If c_1 is the starting configuration and c_2 is the accept configuration and t is the max. no. of the step that the nondeterministic machine can get from c_1 to c_2, then we can determine whether the machine accept its input

Savitch's Theorem

- We give a deterministic, recursive algorithm to solve this problem
  - It searches an intermediate configuration c_m and recursively testing whether:
    1. c_1 can get to c_m within $t/2$ steps
    2. c_m can get to c_2 within $t/2$ steps
  - Each iteration in the loop uses $O(f(n))$ to store the configuration
  - The depth of the recursion is log t, we have $t = 2^{O(f(n))}$ and therefore log $t = O(f(n))$
  - So, the total space needed is $O(f^2(n))$

Savitch’s Theorem

- Proof:
  - Let N be an NTM deciding a language A in space $f(n)$
  - Then, we construct a DTM M to decide language A
  - We use the procedures described in previous slide to build the procedure called CANYIELD
    - Let w be an input string of N and it generates two configurations c_1 and c_2.
    - CANYIELD(c_1, c_2,t) outputs accept if N can go from c_1 to c_2 within t steps along some nondetereministic path

Savitch's Theorem

- CANYIELD = "On input c_1, c_2 and t:

```
1. If $t = 1$, then test directly whether $c_1 = c_2$ or whether $c_1$ yields $c_2$ in one step by $N$. *Accept* if either one test succeeds; *reject* otherwise.
2. If $t > 1$, then for each configuration $c_m$ of $N$ on $w$ using space $f(n)$:
    3. Run CANYIELD($c_1$, $c_m$, $t/2$)
    4. Run CANYIELD($c_m$, $c_2$, $t/2$)
    5. If steps 3 and 4 both accept, then *accept*
6. If have not yet accepted, *reject*"
```

Savitch's Theorem

- Then we define *M* to simulate *N*
  - Modify *N* to clear its tape and move the head to leftmost cell when it accepts (configuration c_{accept})
  - Let c_{start} be the start configuration of *N* on *w*
  - Select a constant *d* such that *N* has no more than $2^{df(n)}$ configurations using $f(n)$ tape and *n* is the length of *w*
    - So, $2^{df(n)}$ is the upper bound of the running time of any branch of *N* on *w*

Savitch's Theorem

- M = "On input w:
  1. Output the result of CANYIELD(c_{start},c_{accept},$2^{df(n)}$)."
- CANYIELD solves the yieldability problem and hence *M* correctly simulate *N*
- Whenever CANYIELD invokes itself recursively, the current stage number and c_1, c_2 and *t* are stored on a stack for the use after returning from the recursion
  - So, each level uses $O(f(n))$ space
  - And the depth of recursion is $O(\log 2^{df(n)}) = O(f(n))$
- Therefore, the total space needed is $O(f^2(n))$

The Class PSPACE

- Definition:
  - PSPACE is the class of language that are decidable in polynomial space on a DTM. In other words,

$$\text{PSPACE} = \bigcup_k \text{SPACE}\left(n^k\right)$$

  - Interestingly, we do not need to define a counterpart for the NP class. In the light of Savitch's theorem:

$$\text{PSPACE} = \text{NPSPACE}$$

The Class PSPACE

- Relationship between classes for time complexity and classes for space complexity:

  $$\mathrm{P} \subseteq \mathrm{PSPACE}$$

  - A machine runs quickly can not use a lot of space
  - More precisely, for $t(n) \geqslant n$, any machine that operates in time $t(n)$ can use at most $t(n)$ space because a machine can only explore one cell at each step of its computation

The Class PSPACE

- Similarly, $\text{NP} \subseteq \text{NPSPACE}$ and so $\text{NP} \subseteq \text{PSPACE}$
- Conversely, for $f(n) \geqslant n$, a TM that uses $f(n)$ space can have at most $f(n)2^{O(f(n))}$ different configurations
  - Remember that a TM that halts may not repeat a configuration
  - So, TM uses $f(n)$ space must run in time $f(n)2^{O(f(n))}$

$$\text{PSPACE} \subseteq \text{EXPTIME} = \bigcup_{k} \text{TIME}\left(2^{n^k}\right)$$

The Class PSPACE

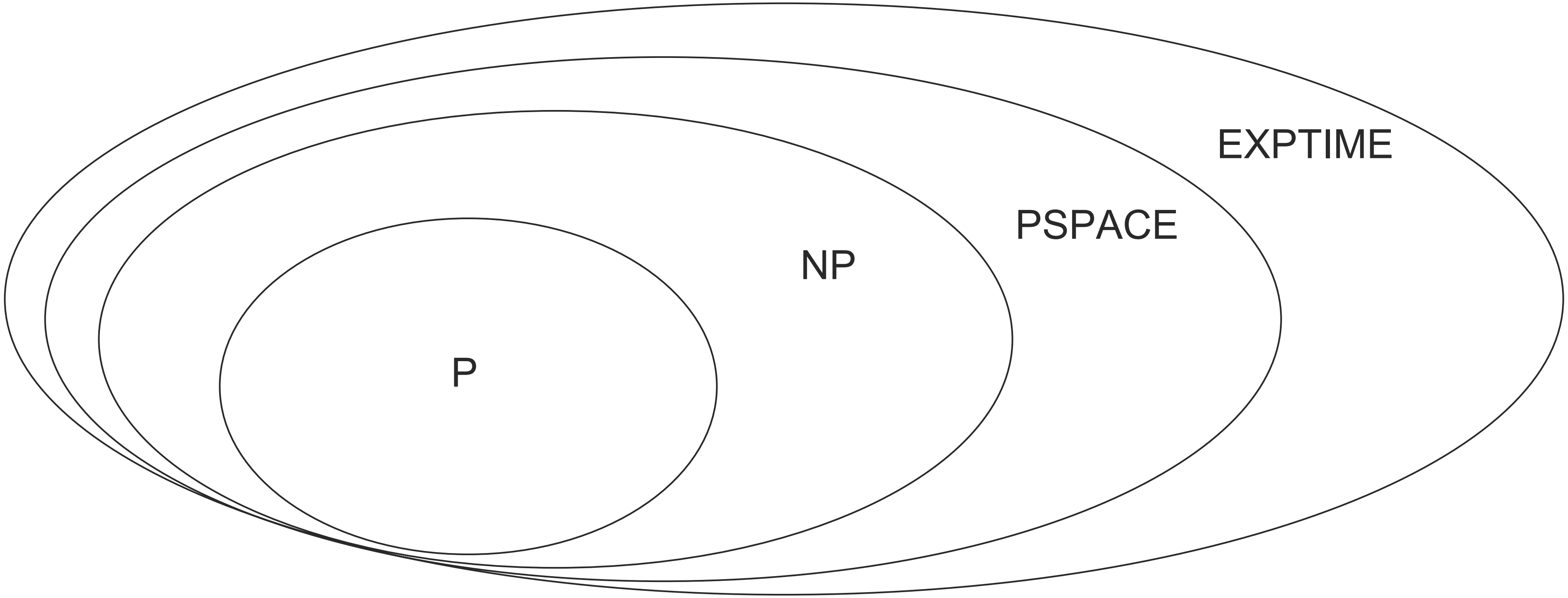
- In summary, we have

$$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE} \subseteq \mathrm{EXPTIME}$$

- Wee do not know whether any these containments is actually an equality
  - Some classes may be merged like the one Savitch's theorem does
  - However, $\mathrm{P} \neq \mathrm{EXPTIME}$, so at least one the preceding containments is proper, but we do not know which one

The Class PSPACE

- Most researchers believe that all containments are proper and have the following relationship



PSPACE-Completeness

- Definition
  - A language *B* is **PSPACE-complete** if it satisfies two conditions:
    1. *B* is in PSPACE and
    2. Every *A* in PSPACE is polynomial time reducible to *B*
  - If *B* merely satisfies condition 2, we say that it is **PSPACE-hard**

PSPACE-Completeness

- Why use polynomial time reducible?
  - Complete problems are important because they are the most difficult problem in the class
  - They are most difficult because all other problems in the class could be reduced to it
  - If one finds a simple solution to it, the solution can also be used to solve entire class’s problems
  - The reduction method must be simple, otherwise the simple solution to a complete problem may not yield a simple solution to other problems in the class

The TQBF Problem

- TQBF = {< ϕ> | ϕ is a true fully quantified Boolean formula}
  - E.g. For the universe of {0, 1}

$$\phi = \forall x \exists y \left[\left(x \vee y \right) \wedge \left(\bar{x} \vee \bar{y} \right) \right]$$

  - E.g. For the universe of Natural Numbers

$$\phi = \forall x \left(x + 1 > x \right) \qquad \phi = \exists y \left(y + y = 3 \right)$$

  - E.g. For the universe of Real Numbers

$$\phi = \exists y \left(y + y = 3 \right)$$

The TQBF Problem

- Theorem:
  - TQBF is PSPACE-complete
- Proof Idea:
  - We give a straightforward algorithm that assigns values to the variables and recursively evaluates the truth of the formula for those values
  - Then, we show every language A in PSPACE reduces to TQBF in polynomial time
    - We begin with a polynomial space-bounded TM for A
  - Then, we give a polynomial time reduction between them
    - The ϕ will be true iff the machine accepts

The TQBF Problem

- Firstly, we may try to imitate the proof of the Cook-Levin Theorem.
  - We construct a formula ϕ that simulates *M* on input *w* by expressing the requirements for an accepting tableau
  - The width of the tableau is $O(n^k)$, the space used by *M*
  - But, its height is exponential in n^k because *M* can run for exponential time
  - So, this is not a good choice
    - A polynomial time reduction can not produce an exponential-size result

The TQBF Problem

- Instead, we use technique related to the proof of Savitch's Theorem to construct the formula.
  - The formula divides the tableau into halves and employs the universal (∀) quantifier to represent each half with the same part of the formula

The TQBF Problem

- Proof:
  - The polynomial space algorithm deciding TQBF
  - T = "On input < ϕ>, a fully quantified Boolean formula

1. If ϕ contains no quantifier, then it is an expression with only constants, so evaluate ϕ and ***accept*** if it is true; otherwise ***reject***
2. If $\exists x \theta$, recursively call T on θ, first with 0 substituted for x and then 1 for x. If **either** result is accept, then ***accept***; otherwise ***reject***
3. If $\forall x \theta$, recursively call T on θ, first with 0 substituted for x and then 1 for x. If **both** result is accept, then ***accept***; otherwise ***reject***"

The TQBF Problem

- Algorithm *T* decides TQBF.
- Total space needed for *T* is $O(m)$, where m is the number of variables
  - The depth of recursion is at most the number of variables
  - In every iteration, only one variable is needed to be stored
  - So, *T* runs in linear space

The TQBF Problem

- Then, we show that TQBF is PSPACE-hard.
- Let A be a language decided by a TM M in space n^k for some constant k
- We give a polynomial time reduction from A to TQBF
  - We map a string w to a quantified Boolean formula ϕ that is true iff M accepts w

The TQBF Problem

- To show how to construct ϕ, we solve a more general problem
  - Using two collections of variables denoted by configurations c_1 and c_2 and a number $t > 0$, then we construct the formula $\phi_{c_1,c_2,t}$
  - If we assign actual configurations to c_1 and c_2, the formula is true iff M can go from c_1 to c_2 in at most t steps.
  - We can let ϕ be the formula $\phi_{c_{start},c_{accept},h}$, where $h = 2^{df(n)}$ for a constant d chosen so that M has no more than $2^{df(n)}$ possible configurations on input with length n
  - Here, let $f(n) = n^k$

The TQBF Problem

- The formula encodes the contents of tape cells as we did in the proof of the Cool-Levin Theorem
  - Each cell has several variables associated with it and one for each tape symbol and state, corresponding to the possible settings of that cell
  - Each configuration has n^k cell, so is encoded by $O(n^k)$ variables
  - If $t = 1$, we design the formula to say that either c_1 equals c_2 or c_2 follows from c_1 in a single step of M
    - **Equality**: Boolean expression for each variables in c_1 contains the same values as corresponding variables in c_2
    - **Follows**: Boolean expressions stating that the contents of each triple of c_1's cell correctly yields the contents of corresponding triple of c_2's

The TQBF Problem

- If $t > 1$, we construct $\phi_{c_1,c_2,t}$ recursively
- Let $$\phi_{c_1,c_2,t} = \exists m_1 \left(\phi_{c_1,m_1,t/2} \wedge \phi_{m_1,c_2,t/2} \right)$$
  - m_1 represent a configuration of M
  - $\exists m_1$ is a shorthand for $\exists x_1, \cdots, x_l$, where $l = O(n^k)$ and $x_1, \cdots, x_l$ are the variables that encodes m_1
  - If M has configuration m_1 which can be yielded from c_1 within t/2 steps and m_1 can go from m_1 to c_2 within t/2 steps, then we construct the formulas $\phi_{c_1,m_1,t/2}$ and $\phi_{m_1,c_2,t/2}$ recursively
  - The formula $\phi_{c_1,c_2,t}$ has the correct value, i.e. it is true whenever M can go from c_1 to c_2 within t steps.
  - But, it is too big because it **doubles the size** of the formula in each recursion

The TQBF Problem

- The method shown in previous slide requires t = 2df(n), which is exponential. To reduce the size of the formula, we use the $\forall$ quantifier in addition to the $\exists$ quantifier
- Let $\phi_{c_1,c_2,t} = \exists m_1 \forall (c_3, c_4) \in \{(c_1, m_1), (m_1, c_2)\} (\phi_{c_3,c_4,t/2})$
  - The 2 new configurations allow us to "fold" the two recursive subformulas into a single subformula while preserving the meaning
  - $\forall (c_3, c_4) \in \{(c_1, m_1), (m_1, c_2)\}$ means that c_3 and c_4 may take the values of the variables of c_1 and m_1 or of m_1 and c_2, respectively, and the resulting formula $\phi_{c_3,c_4,t/2}$ is true in either case

The TQBF Problem

- Then, we analyze the space used by the new algorithm for the formula $\phi_{c_{\text{start}},c_{\text{accept}},h}$, where $h = 2^{df(n)}$
  - At each level of recursion, we add a portion to th eformula which is linear in the size of the configuration and thus the size is $O(f(n))$
  - The number of levels of the recursion is $\log 2^{df(n)}$, or $O(f(n))$
  - So, the total space used is $O(f^2(n))$

Winning Strategies for Games

- **Game** is loosely defined to be a competition in which opposing parties attempt to achieve some goal according to prespecified rules
  - Broad games, chess, economic games, world war…etc..
- Games are closely related to quantifiers
  - One can find correspondence between them
    - This helps us to understand the complexity of the game
    - This also gives us insight into the statement's meaning

Winning Strategies for Games

- **Formula Game**
  - Let $\phi = \exists x_1 \forall x_2 \exists x_3 \cdots Q x_k (\theta)$
    - Q means either $\exists$ or $\forall$
  - Two players A and E, take turn to select variables x_1, …, x_k.
  - Player A selects values for the variables that are bound to $\forall$ quantifiers and E selects for those bound to $\exists$ quantifiers.
  - The order of player is the same as the quantifiers at the beginning of the formula
  - The end of the game, we use the selected variable to verify θ. If θ is TRUE, E wins, otherwise A wins.

Winning Strategies for Games

- **Exmple:**

$$\phi_1 = \exists x_1 \forall x_2 \exists x_3 \left(\left(x_1 \vee x_2\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\bar{x}_2 \vee \bar{x}_3\right)\right)$$

- Start Game:

1. E selects $x_1 = 1$
2. A selects $x_2 = 0$
3. E selects $x_3 = 1$
4. Finally, $\left(\left(x_1 \vee x_2\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\bar{x}_2 \vee \bar{x}_3\right)\right) = 1$

○ So, E has won the game.

Winning Strategies for Games

- **Winning Strategy**
  - In fact, E may always win this game when E selects x_1 = 1 and then selecting x_3 to be the negation of whatever A select for x_2.
- If we change the game as followings, A has a winning strategy:

$$\phi_2 = \exists x_1 \forall x_2 \exists x_3 \left(\left(x_1 \vee x_2 \right) \wedge \left(x_2 \vee x_3 \right) \wedge \left(x_2 \vee \overline{x}_3 \right) \right)$$

  - No matter what E selects for x_1, A may select x_2 = 0 and win the game because the whole formula equals to 0

Winning Strategies for Games

- Then we may ask that how to determine which player has a wining strategy in the formula game associated with a particular formula?

Winning Strategies for Games

- Then we may ask that how to determine which player has a wining strategy in the formula game associated with a particular formula?
- Let

$$\mathrm{FORMULA-GAME} = \left\{ \begin{array}{l} \langle \phi \rangle \mid \text{Player E has a winning strategy in} \\ \qquad \text{the formula game associated with } \phi \end{array} \right\}$$

- Theorem:
  - FORMULA-GAME is PSPACE-complete
  - **Why?**

Winning Strategies for Games

- Theorem:
  - FORMULA-GAME is PSPACE-complete
  - **It is the same as TQBF.**
  - A formula is true in TQBF is exactly when Player E has a winning strategy in the associated formula game

Winning Strategies for Games

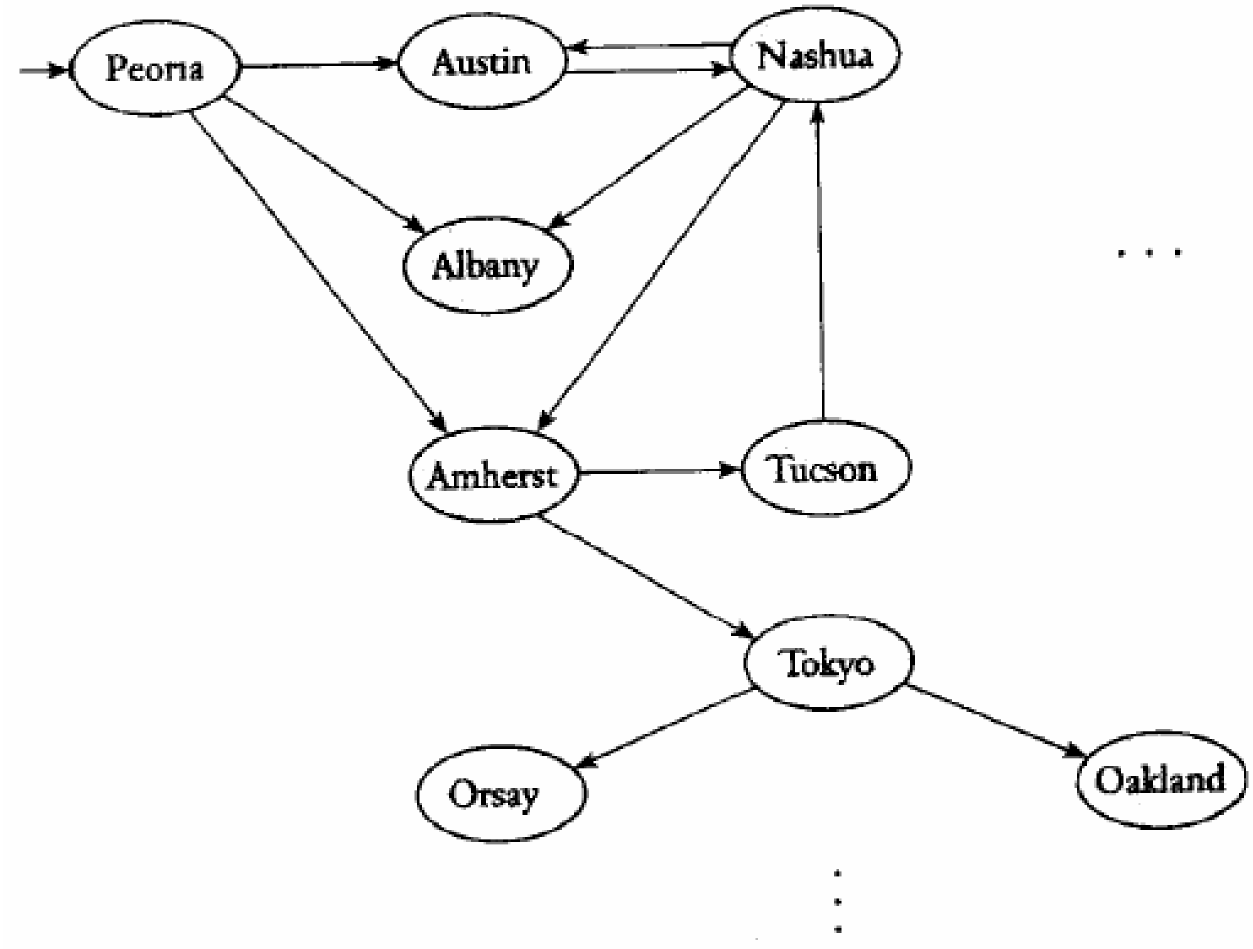
- Proof
  - The formula $\phi = \exists x_1 \forall x_2 \exists x_3 \cdots (\theta)$ is TRUE when
    - some setting for x_1 exists,
    - such that for any setting of x_2,
    - a setting of x_3 exists,
    - and so on…,
    - Where θ is TRUE under the settings of the variables
  - Similarly, Player E can make the same assignments to make E wins the game
    - some setting for x_1 exists,
    - such that for any setting of x_2,
    - a setting of x_3 exists,
    - and so on…,
    - Where θ is TRUE under the settings of the variables

Generalized Geography

- Geography Game
  - A game that players name a city in turn and the beginning character of the city must be equal to the last character in the previous named city and no repetition is allowed
  - E.g.
    1. Shenzhen
    2. Nanjing
    3. Guangzhou
    4. …
  - Player loses the game if he/she can not go on naming a city according to the rule

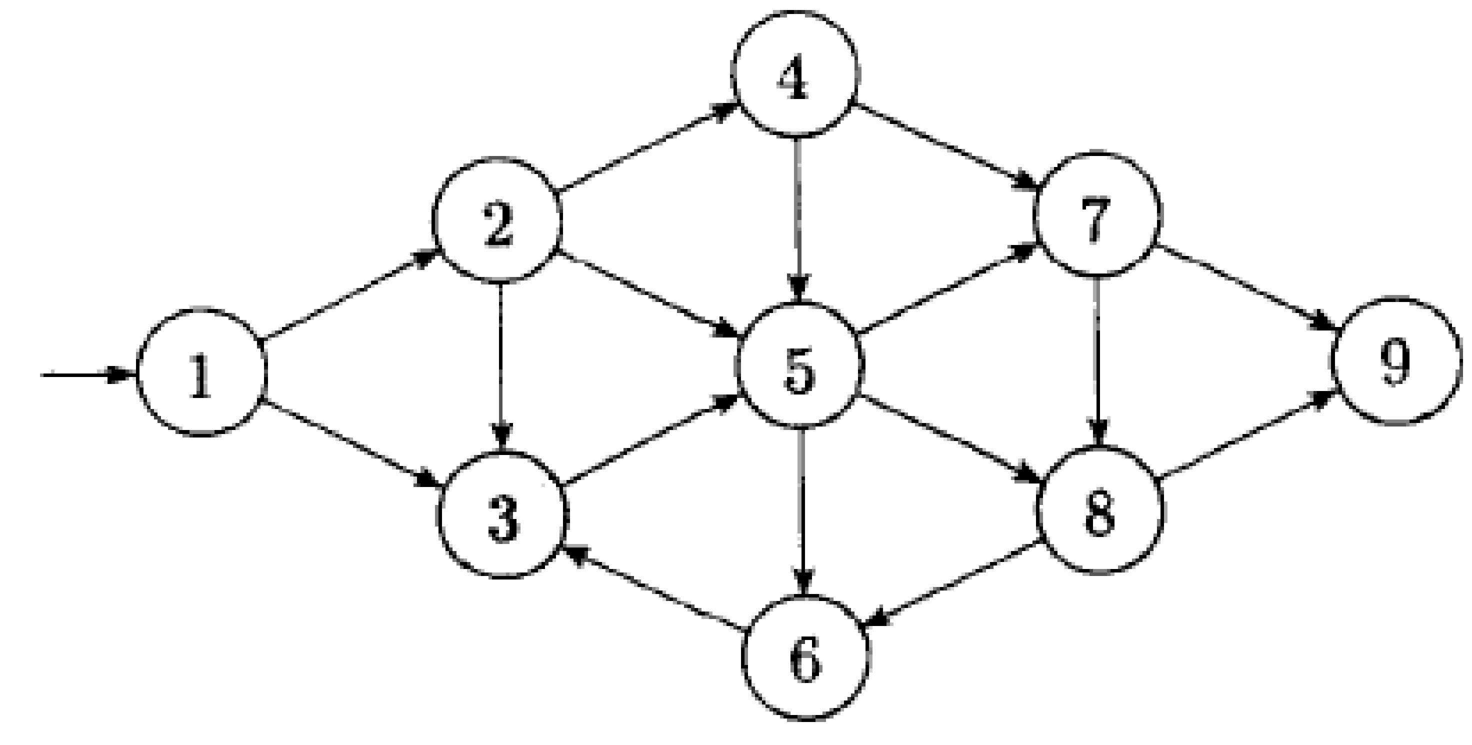
Generalized Geography

- We may summarize the naming in a directed graph



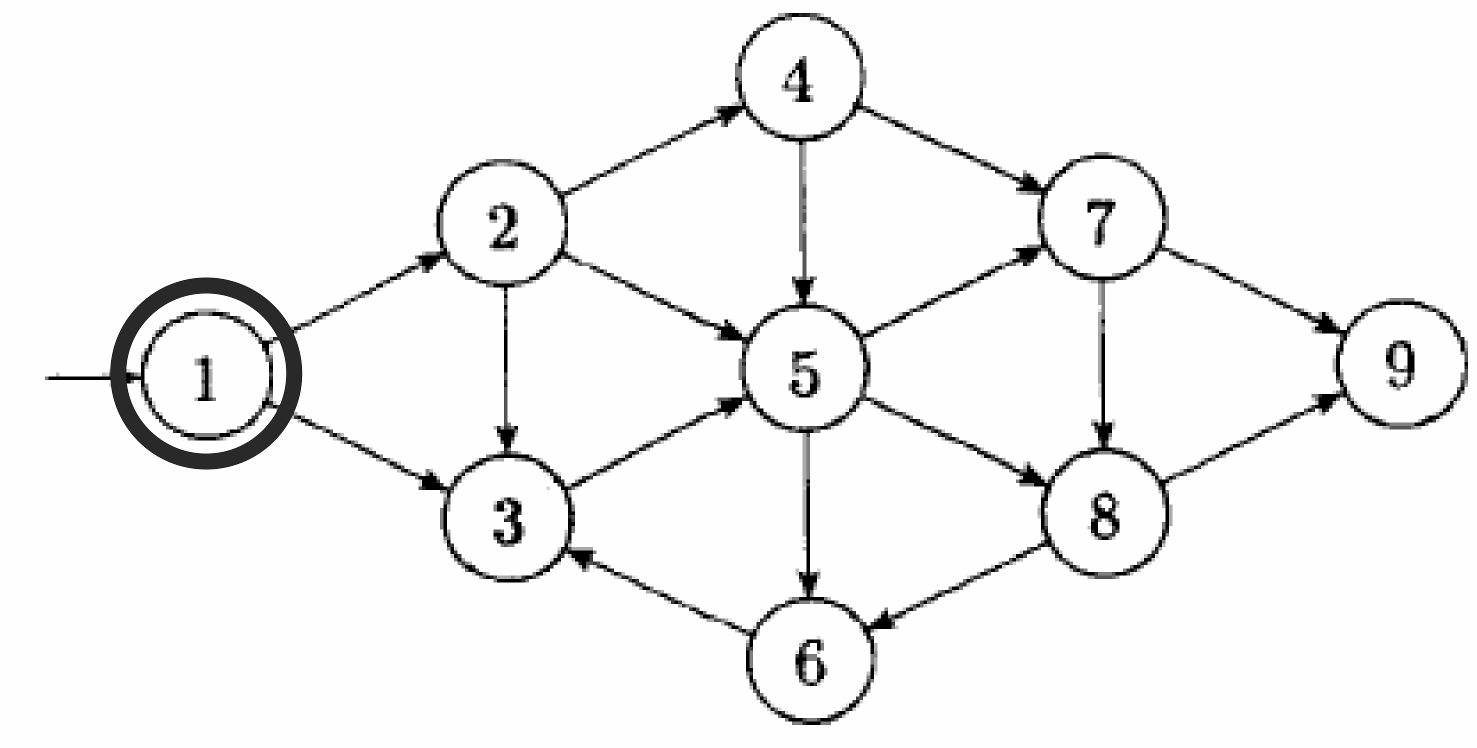
Generalized Geography

- In the **Generalized Geography** Game, we take an arbitrary directed graph with a designated start node instead of city names and associations



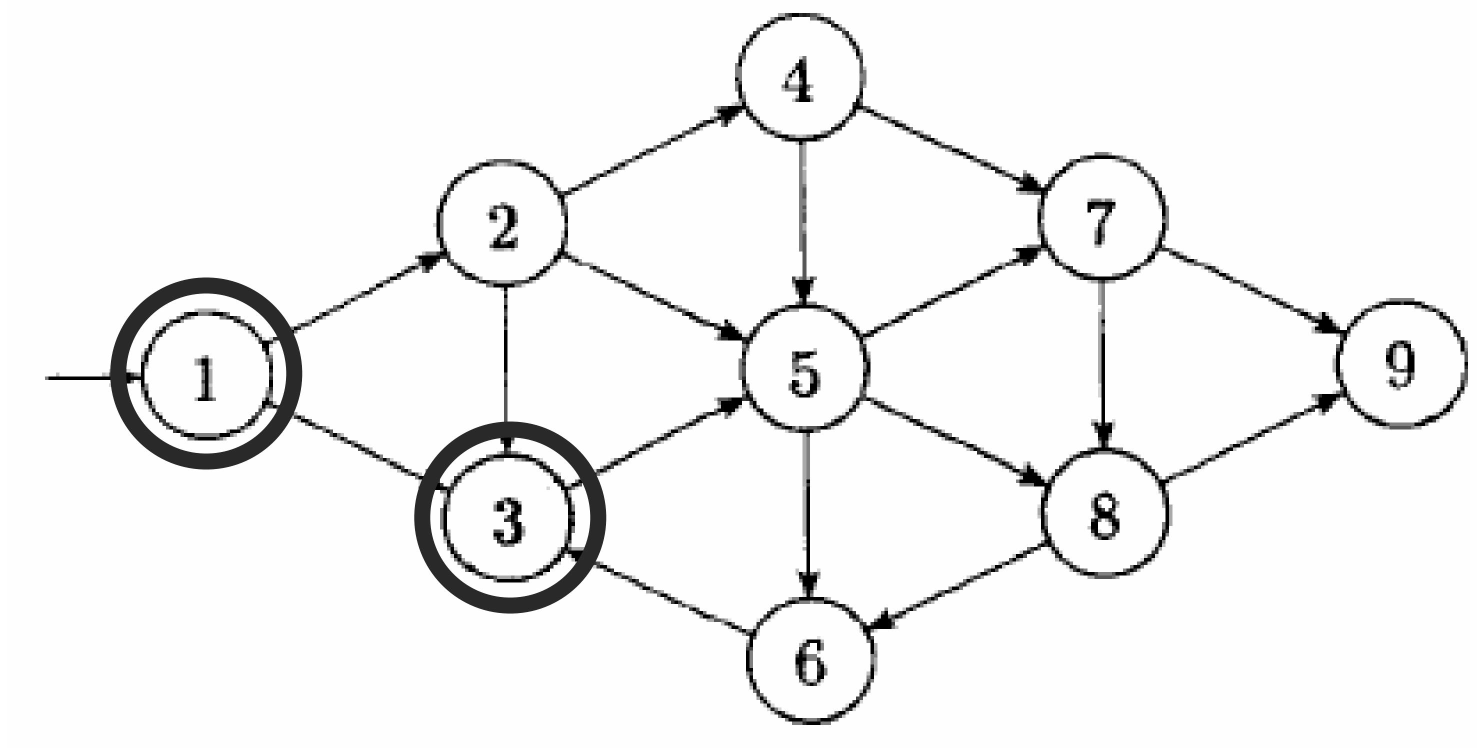
Generalized Geography

- Player I moves first and then Player II
  - Player I selects 3 from choices of 2 and 3



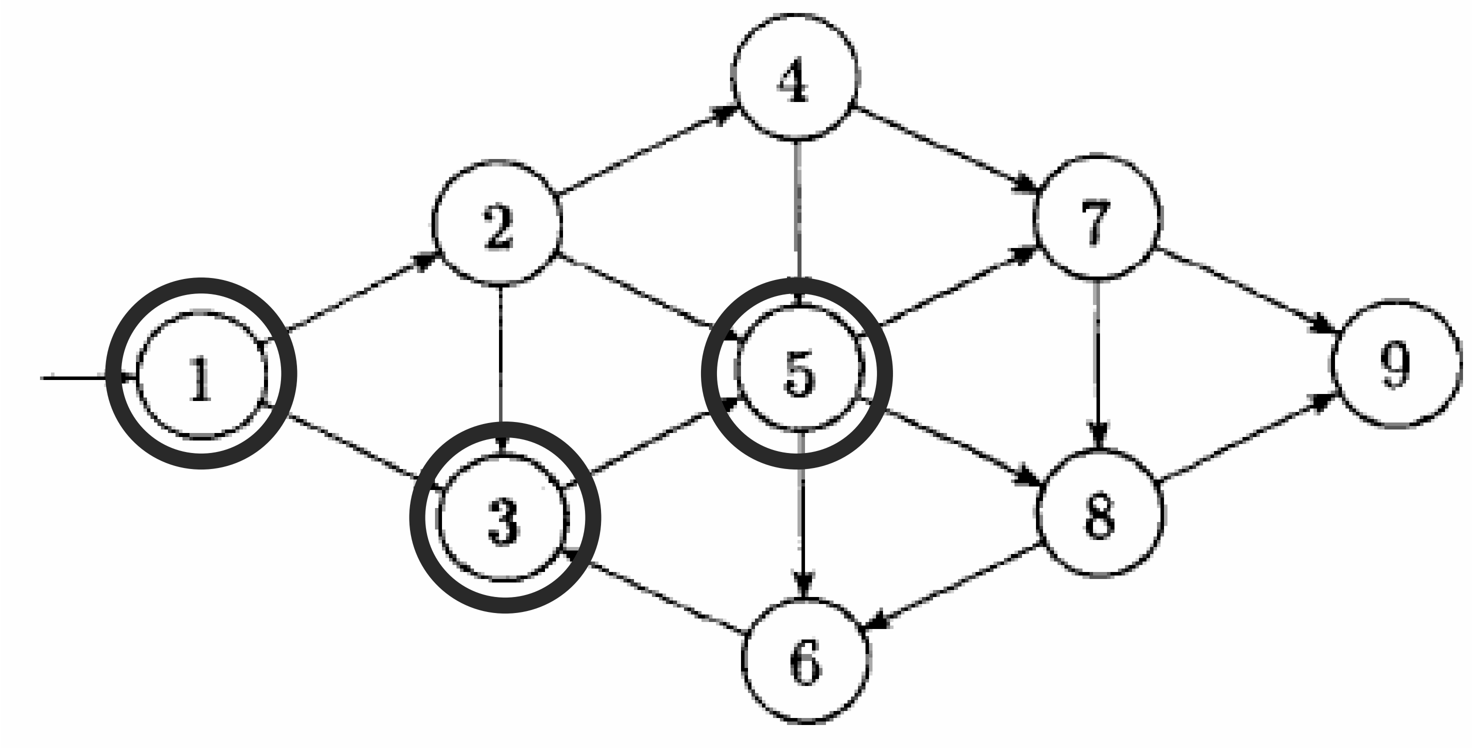
Generalized Geography

- Player II can only move to 5



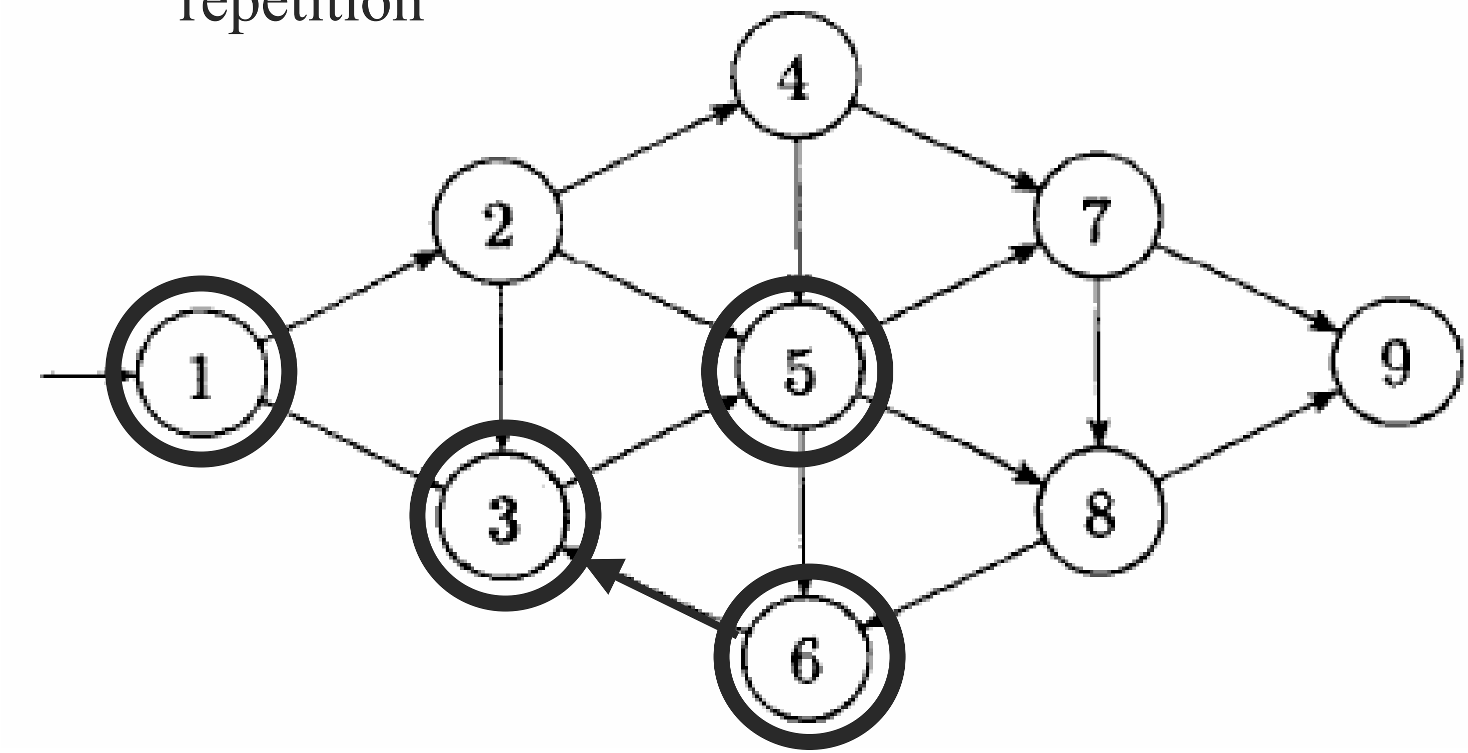
Generalized Geography

- Player I moves to 6 from choices of 6, 7 and 8



Generalized Geography

- Player I moves to 6 from choices of 6, 7 and 8
- Player II must lose because he must choose 3 which is a repetition



Generalized Geography

- The problem of determining which player has a winning strategy in a generalized geography game is PSPACE-Complete. Let

$$GG = \left\{ \begin{array}{r} \langle G, b \rangle \mid \text{Player I has a winning strategy for generalized} \\ \text{geography game on graph } G \text{ starting at node } b \end{array} \right\}$$

Generalized Geography

- Proof Idea:
  - We design a recursive algorithm similar to the one used for TQBF to determine which player has a winning strategy
  - To prove that GG is PSPACE-hard, we give a polynomial time reduction from FORMULA-GAME to GG.
    - This converts formula game to a GG graph, so that the graph mimics play in the formula game
    - The selection of movement on the graph is in fact an encoded form of selecting variable values in formula game

Generalized Geography

- Proof:
  - We give an algorithm that decides whether Player I has a winning strategy in instances of generalized geography, i.e. it decides GG
  - M = "On input <G, b>, where G is a directed graph and b is a node of G
    1. If b has outdegree 0, ***reject***, because Player I loses immediately
    2. Remove node b and all connected arrows to get a new graph G_1
    3. For each of the nodes b_1, b_2,…, b_k that b originally pointed at, recursively call M on <G_1, b_i>
    4. If all of these accept, Player II has a winning strategy in the original game, so ***reject***. Otherwise, Player I has it, so ***accept***"

Generalized Geography

- Proof:
  - We give an algorithm that decides whether Player I has a winning strateg aphy, i.e. it decides C
  - M = “On input h and b is a node of G

Every turn, only 1 node is added to the stack, so the space requirement for each step is $O(1)$

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2. Remove node b and all connected arrows to get a new graph G_1
3. For each of the nodes b_1, b_2,…, b_k that b originally pointed at, recursively call M on $<G_1, b_i>$
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Generalized Geography

- Proof:
  - We give has a winning phy, i.e. it dec
  - M = "On and *b* is a node
    1. If b has ately
    2. Remove node connected arrows to get a new graph G_1
    3. For each of the nodes $b_1, b_2, \ldots, b_k$ that b originally pointed at, recursively call M on $<G_1, b_i>$
    4. If all of these accept, Player II has a winning strategy in the original game, so ***reject***. Otherwise, Player I has it, so ***accept***"

There are at most m recursion level as there are only m nodes in the graph, so the total space required is $O(m)$, or $O(n)$

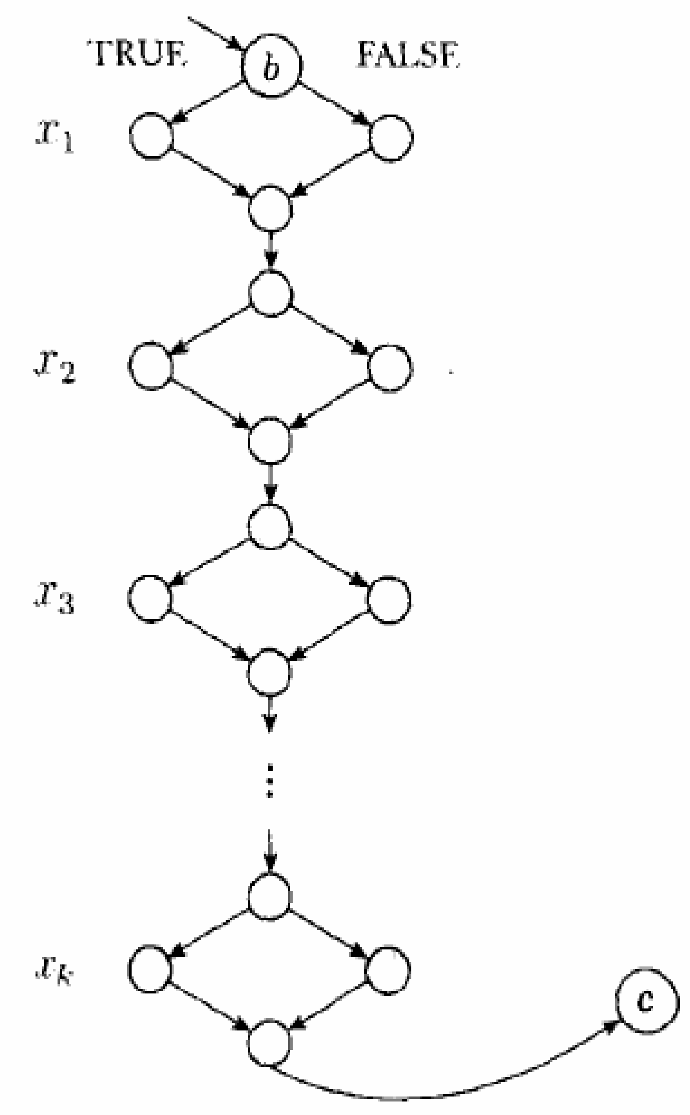
So, we completed the proof that it runs in polynomial space

Generalized Geography

- Proof:
  - Then, we need to find a polynomial time reduction from FORMULA-GAME to GG
    - To map the formula to an instance of Graph *G* with node *b*
    - For simplicity, we assume that the formula begin and end with $\exists$ and is strictly alternate between $\exists$ and $\forall$
      - If the formula is not fulfilling this format, we could convert it by little bit enlarging the formula
    - We also assume that the formula is in cnf
  - The reduction constructs a geography game on G where optimal play mimics optimal play of the formula game on θ
  - Player I takes the role of Player E and II takes that of A

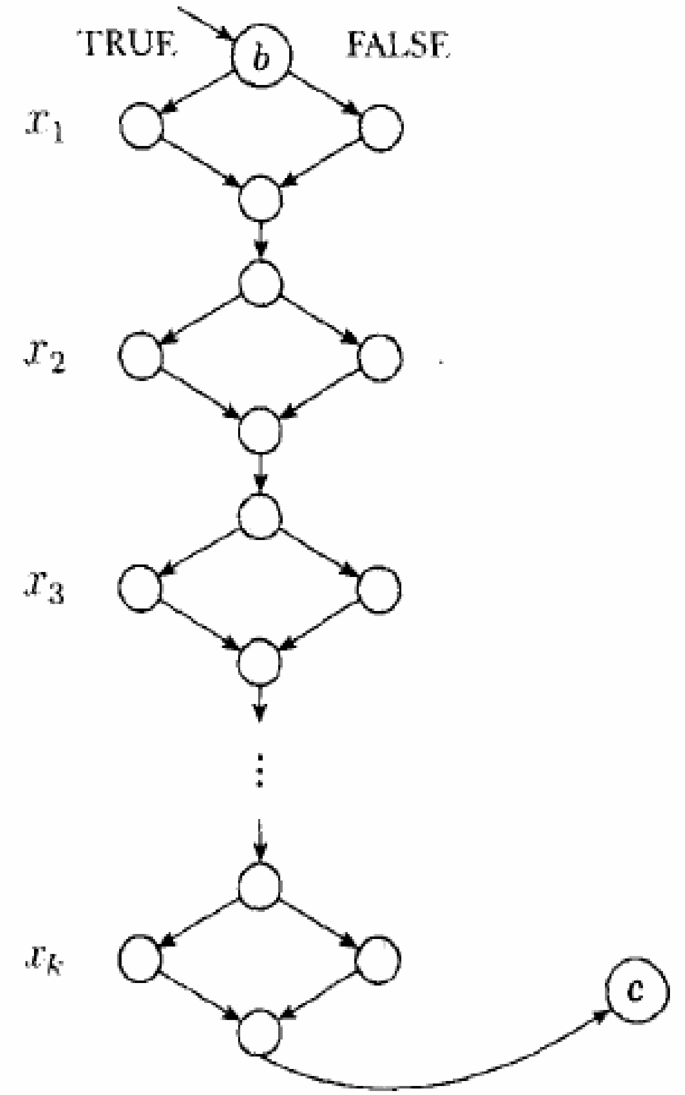
Generalized Geography

- Proof:
  - The structure of left-part of graph *G* is shown on right hand side
  - Each variable in the formula occupy one diamond in the graph
  - Going via left path is corresponding to the choice of TRUE in formula game; and right path for FALSE
  - Then, Player II is forced to move and Player I does too…then we arrive the top of the second diamond



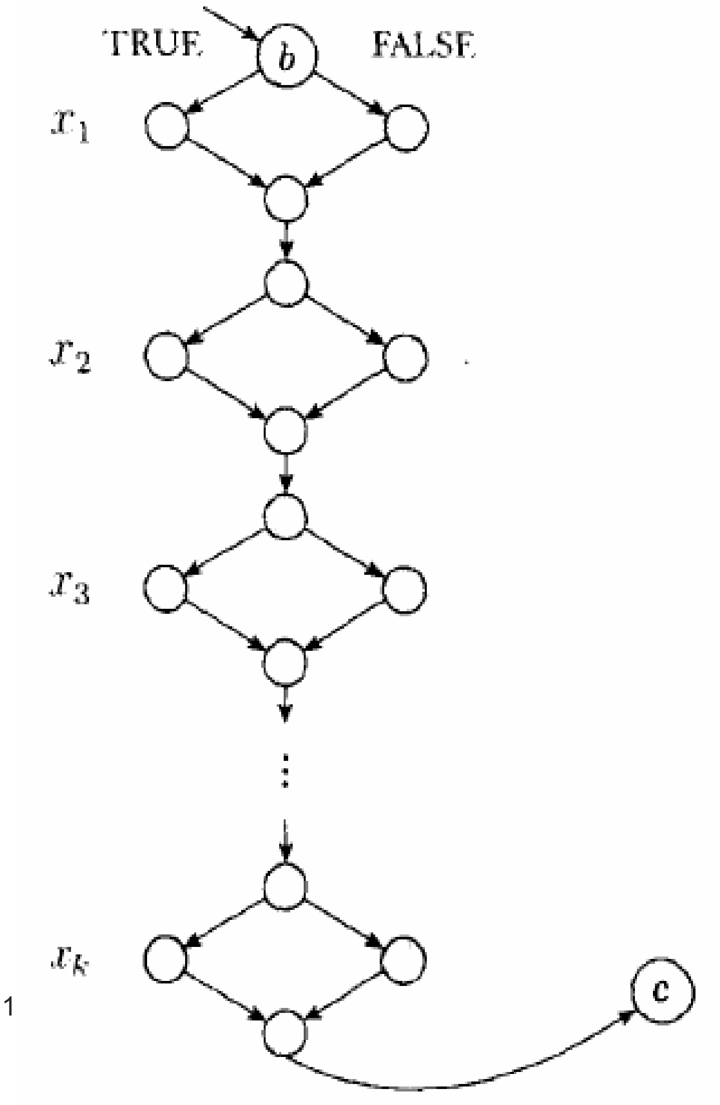
Generalized Geography

- Proof:
  - Now, Player II can make his choice of LEFT or RIGHT now, and this corresponding to the first choice of Player A for the formula game
  - Then, Players I and II continue the game by alternatively make the choice of the path through each diamond
  - After plays passes through all the diamond, the head of path arrives the bottom of the last diamond and Player I must goes to node *c*



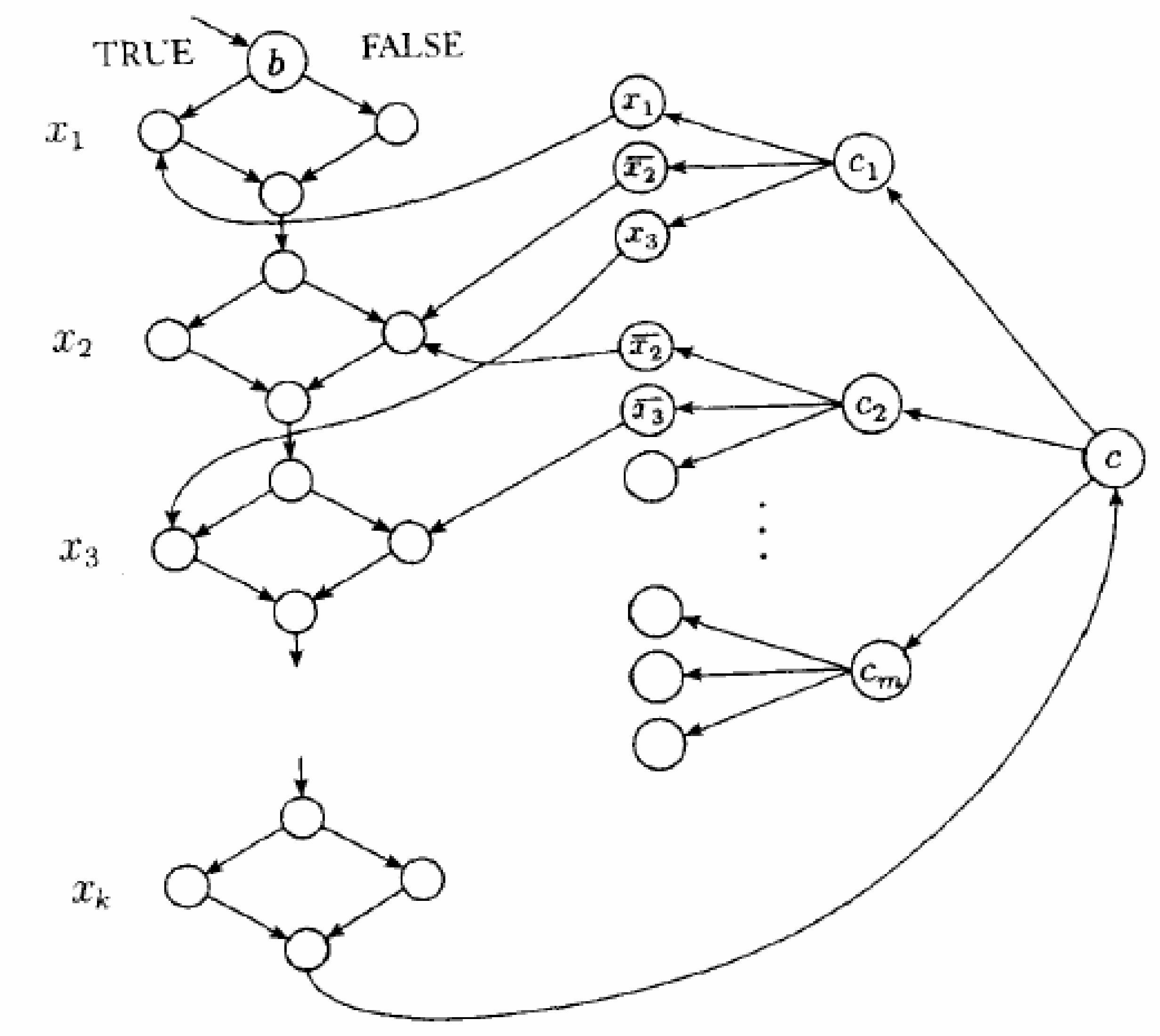
Generalized Geography

- Proof:
  - This point corresponding to the end of the formula game
  - The choice of path corresponding to the assignment of variables
  - If the formula is TRUE, Player E wins
  - If the formula is FALSE, Player A wins



Generalized Geography

- Then, we add the right hand side of the graph and such that:
  - Player I wins if Player E wins
  - Player II wins if Player A wins



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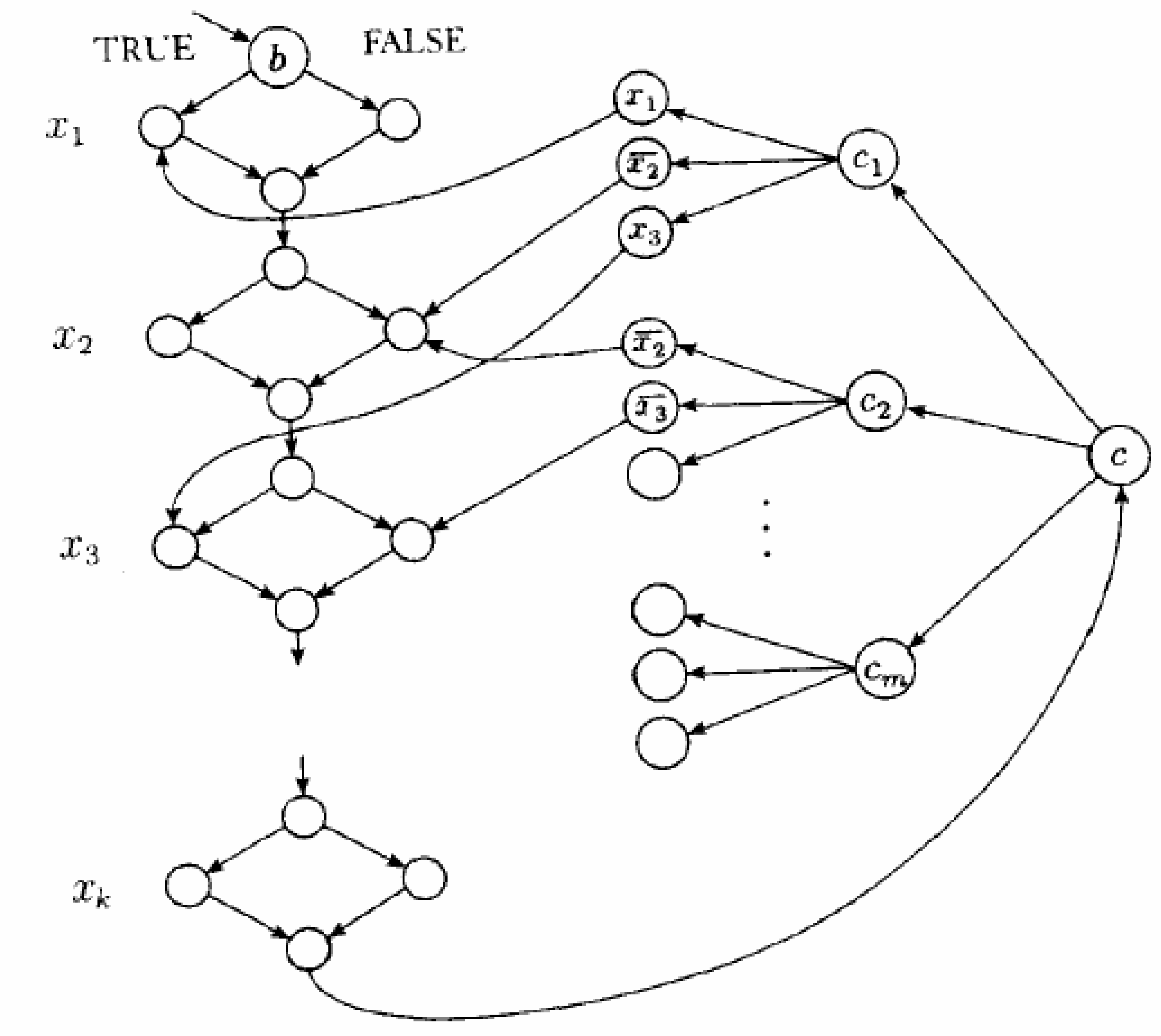
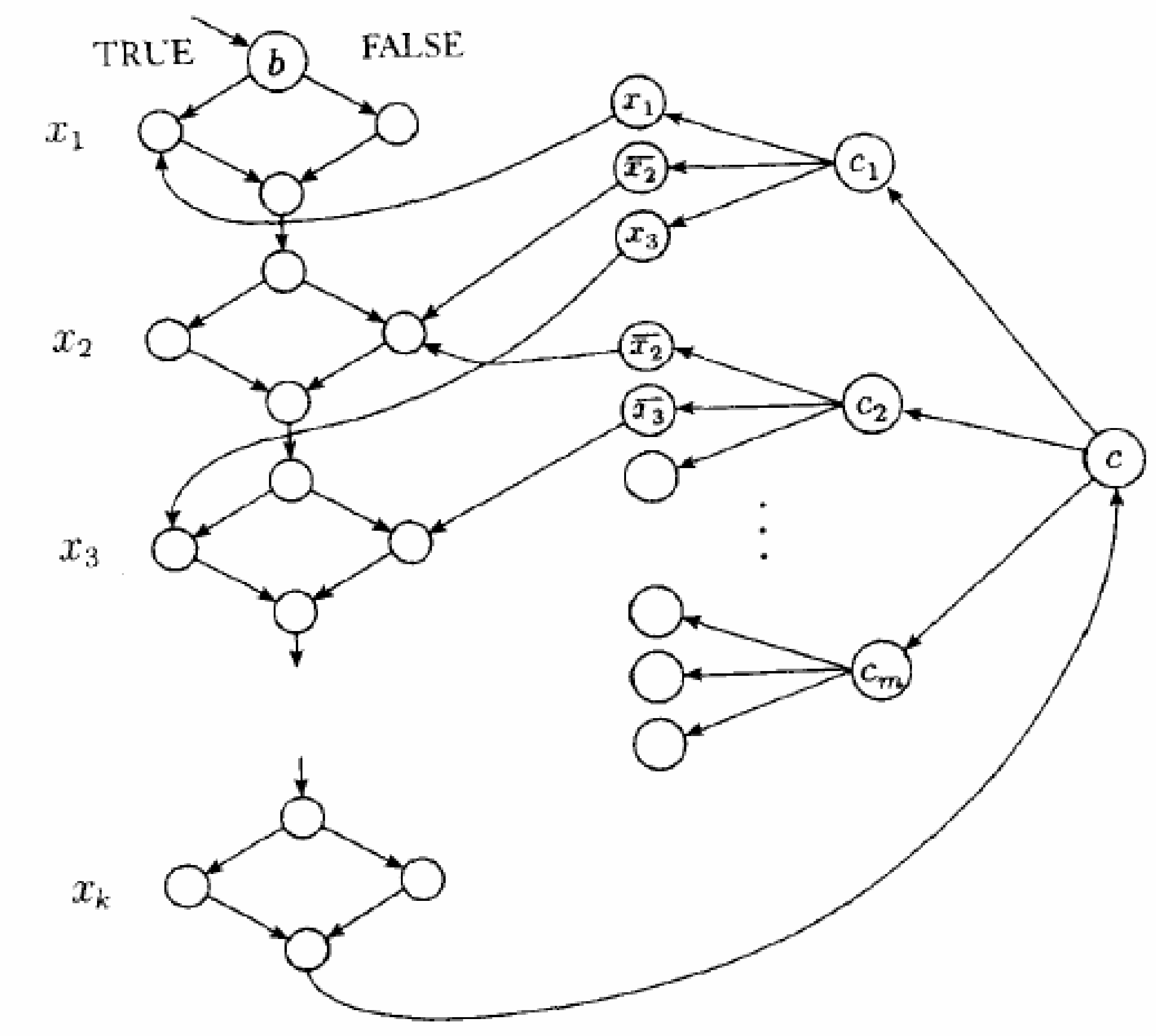
Generalized Geography

- At node c, Player II selects a node corresponding to one of the clause in θ and Player I selects a node corresponding to a literal in that clause



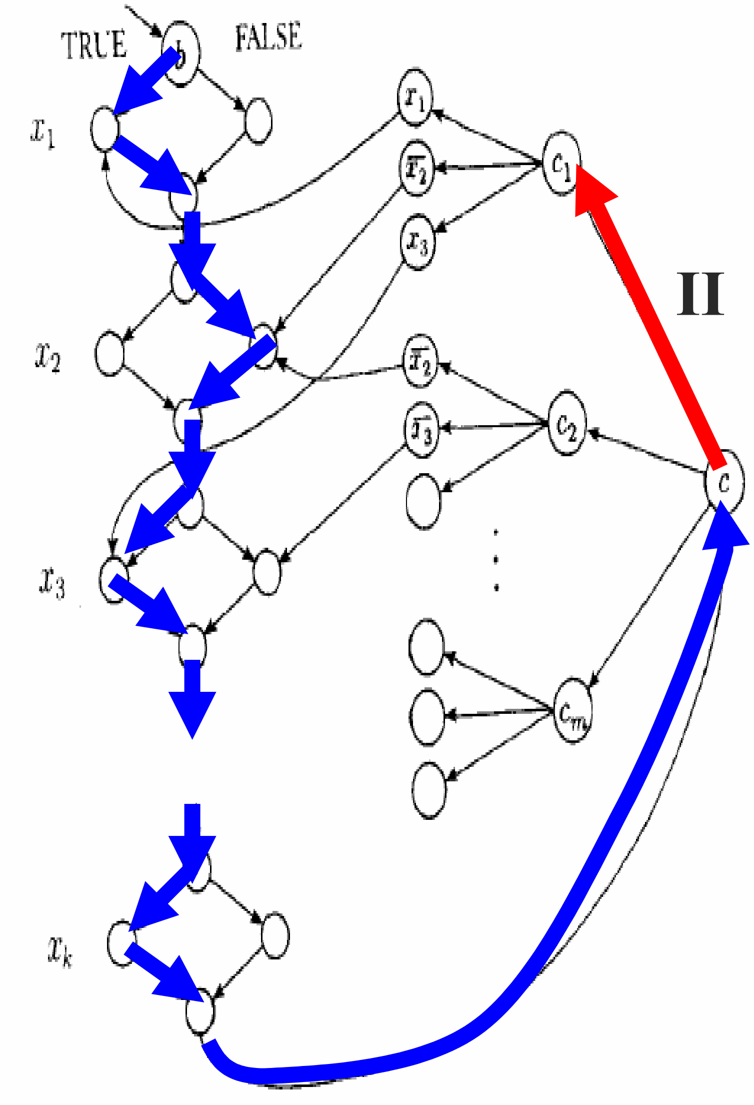
Generalized Geography

- Unnegated literals are connected to the left side of the diamond of the “left hand side” of the graph
- Negated literals are connected to the right side of the diamond of the “left hand side” of the graph



Generalized Geography

- If ϕ is FASLE, Player II may win by selecting the unsatisfied clause,
- then any literal picked by Player I is FALSE
- This is connected to the side that the diamond has not be played
- Then, Player II may select the next node which is not yet selected
- However, the bottom node of this diamond is repeated which Player I forced to select



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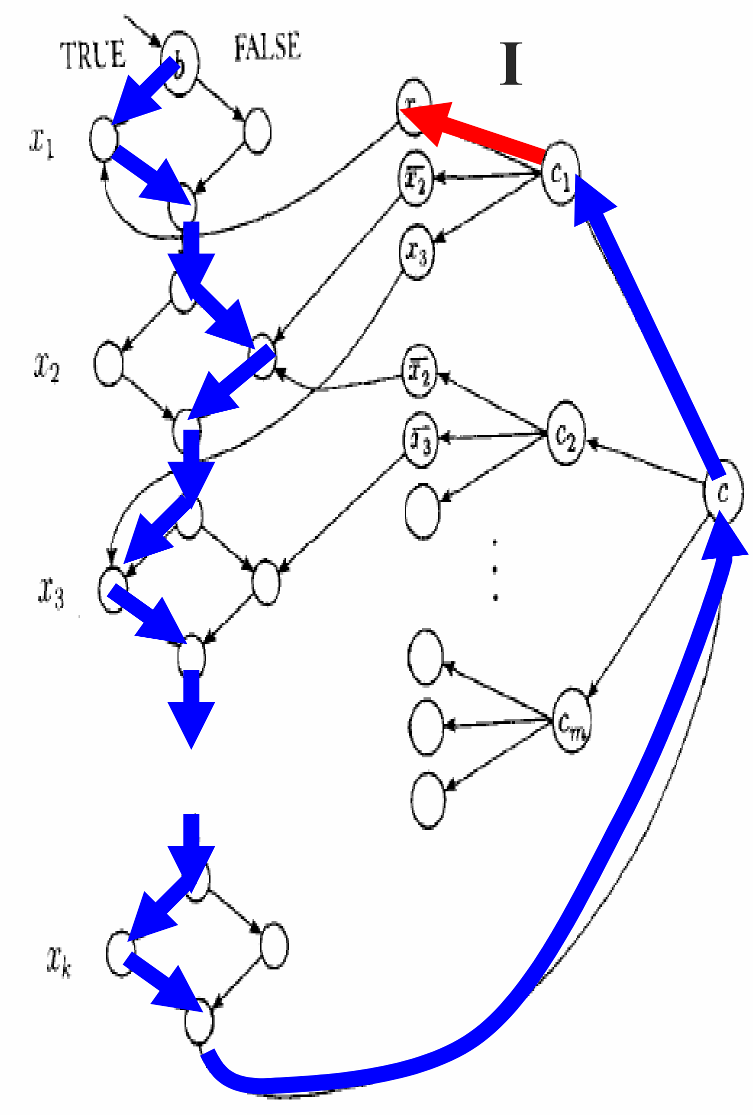

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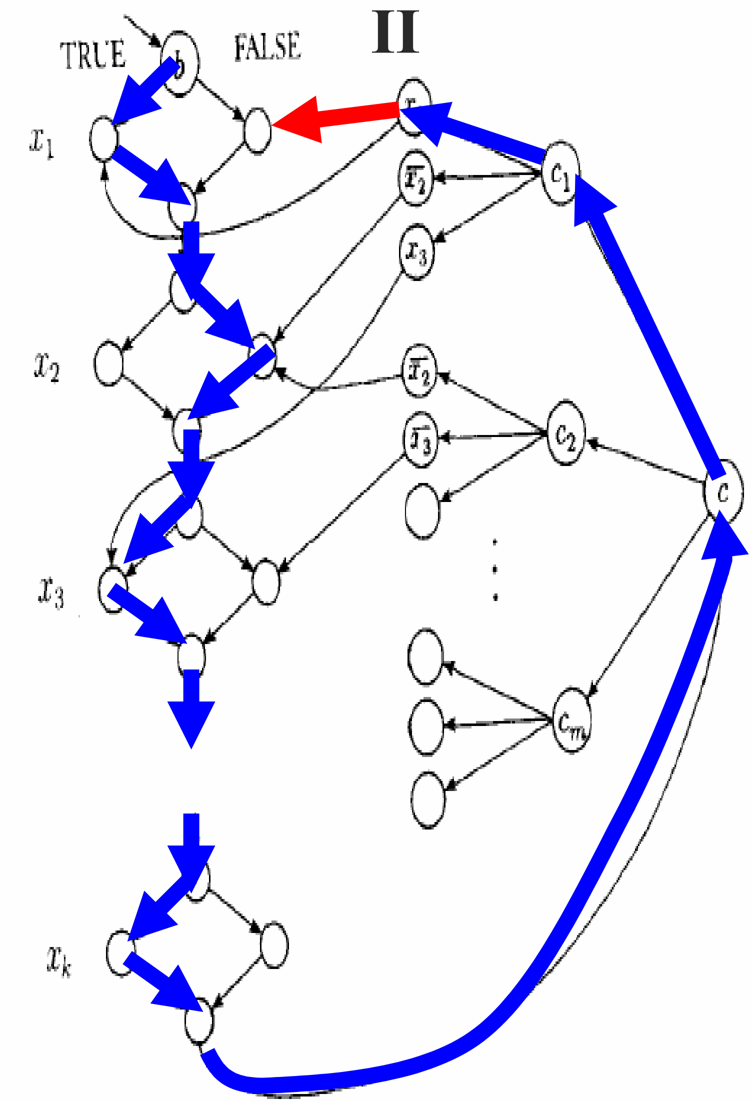

Generalized Geography

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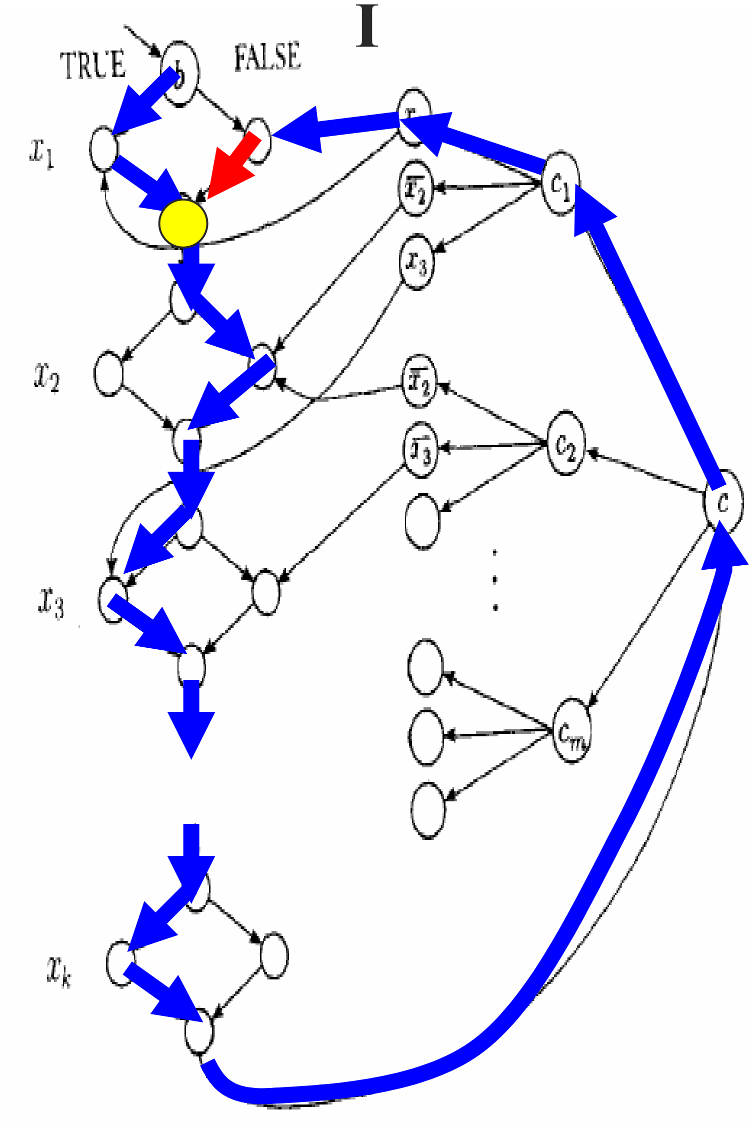
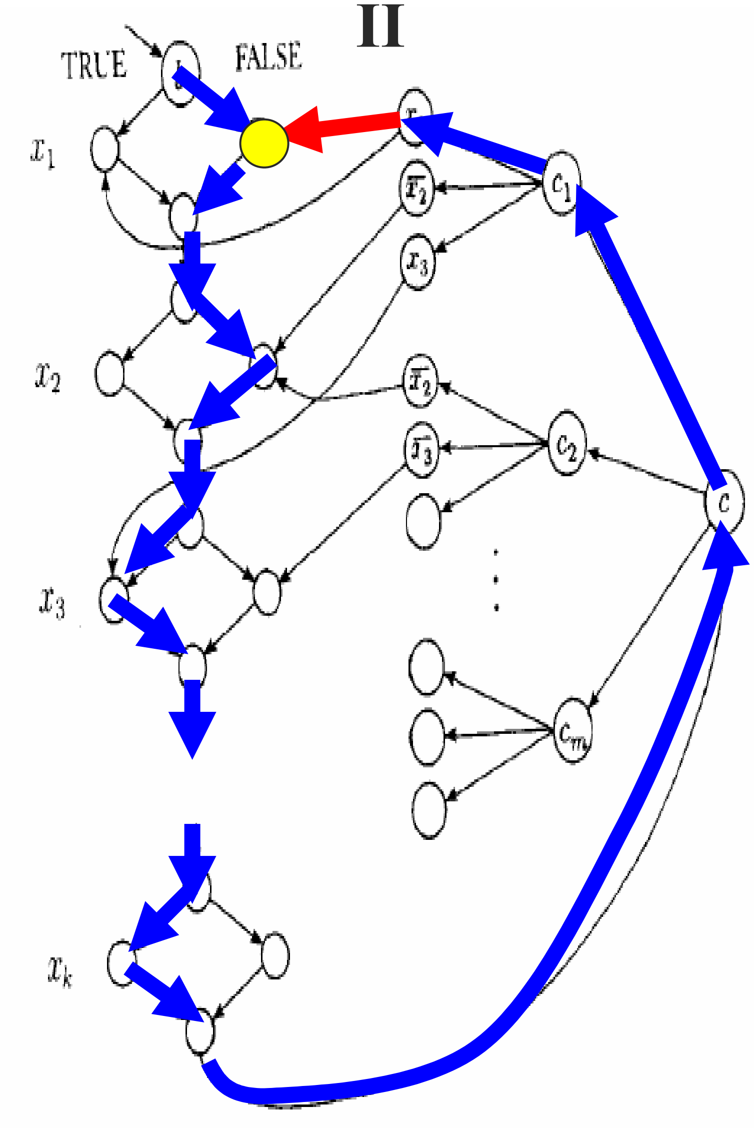
- then any literal picked by Player I is FALSE
- This is connected to the side that the diamond has not be played
- Then, Player II may select the next node which is not yet selected
- However, the bottom node of this diamond is repeated which Player I forced to select, he LOSE!



Generalized Geography

- If ϕ is TRUE, Player II loses in this step



Generalized Geography

- In this theorem, we showed that no polynomial time algorithm exists for optimal play in generalized geography unless P = PSPACE
- E.g. in a 8x8 broad game, computer may remember all the possible move and corresponding best next move in a table
  - Then, it can play optimally in linear time
  - The table may be too large to fit inside our galaxy…
  - But…still finite and therefore…
  - We could store it using a TM or FA

Generalized Geography

- In future, we may able to construct method that quantify the complexity of finite problems
- However, current methods are only asymptotic
  - We only can measure the rate of growth of the complexity when the problem size increases, not to fixed size
- One could generalize many broad games, e.g. chess, checkers and GO, to nxn broad game
- Such generalizations have been shown to be PSPACE-hard or hard for even larger complexity classes

Classes L and NL

- Until now, we consider time and space complexities to be at least linear, i.e. $f(n)$ is at least n
- A smaller one is **sublinear** space bound
  - In time complexity, it is insufficient to scan once on the entire content on the tape
  - In sublinear space complexity, the machine reads the entire content, but it does not have enough space to store it
  - We introduce a new type of TM which has two tapes
    - Read-only input tape – CD-ROM or DVD-ROM
    - Read/write work tape – RAM
      - Only cells scanned on work tape contribute to space complexity

Classes L and NL

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    - Read-only input tape – CD-ROM or DVD-ROM
    - Read/write work tape – RAM
      - Only cells scanned on work tape contribute to space complexity

Classes L and NL

- Usually, CD-ROM or DVD-ROM stores much more data than RAM does.
  - We design the sublinear space algorithm to use very few memory
- Definition
  - L is the class of languages that are decidable in logarithmic space on a DTM

    $$\mathrm{L} = \mathrm{SPACE}(\log n)$$

  - NL is the class of languages that are decidable in logarithmic space on a NTM

    $$\mathrm{NL} = \mathrm{NSPACE}(\log n)$$

Classes L and NL

- We ignore $\sqrt{n}$ and $\log^2 n$, just similar to the reason that we ignore n^2, etc, in polynomial time and space bounds
- Logarithmic space is large enough to solve a number of interesting computational problems
- It has attractive Maths. Properties
  - E.g. robustness even when machine model and input encoding method change
  - Pointers into the input may be represented in logarithmic space
    - The power of log space algorithms is to consider the power of a fixed number of input pointers

Classes L and NL

- Example
  - The language $A = \{0^k 1^k \mid k \geqslant 0\}$ is a member of L
    - Remember that we zigzagging back and forth across the input and crossing off the 0s and 1s as they are matched?
    - This uses linear space to record the positions have been crossed off
    - We may modify it to use only log space
  - Instead, we use our new TM which places the input on the read-only tape
    - The machine counts the number of 0s and, separately, the number of 1s in binary on the work tape
    - Counter uses only logarithmic space
    - So, the algorithm runs in $O(\log n)$ and $A \in L$

Classes L and NL

- Example
  - Recall the PATH problem

$$PATH = \{\langle g, S, T\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$$

  - PATH is in P, but uses only linear space
  - We may construct a NTM to decide PATH with log space
    - The NTM nondeterministically guesses the path from *s* to *t*
    - The NTM only records the current node on the path, instead of the entire path
    - Then, the NTM nondeterministically select the next node connecting to the current node
    - It repeats until it reaches node t and ***accepts*** or it has gone on for *m* (the number of nodes in graph) steps and ***reject***

Classes L and NL

- Example of PATH
  - So, the new algorithm decides PATH using only log space and therefore PATH is in NL

- Remember that we claimed that any $f(n)$ space bounded TM runs in time $2^{O(f(n))}$?
  - This is no longer true for very small space bounds
  - E.g. a TM uses $O(1)$ space may run for n steps

Classes L and NL

- Definition
  - If M is a TM that has a separate read-only input tape and w is an input, a configuration of M on w is a setting of the state, the work tape, and the positions of the two tape heads. The input w is not a part of the configuration of M on w.

  - If M runs in $f(n)$ space and length of w is n, the number of configuration of M on w is $n2^{O(f(n))}$
    - Let M has c states and g tape symbols, the number of possible strings on the work tap is $g^{f(n)}$.
    - Input head can be in one of the n positions and working tape head can be in one of the $f(n)$ position
    - So the number of configuration is $cnf(n)g^{f(n)}$ or $n2^{O(f(n))}$.

Classes L and NL

- We focus almost exclusively on space bounds $f(n)$ which are at least log n.
- So, our claim of a machine is at most exponential in its space complexity remains true for such bounds because $n2^{O(f(n))}$ is $2^{O(f(n))}$ when $f(n) \geqslant n$

Classes L and NL

- Remember that the Savitch's theorem shows that we can convert NTM to DTM and increase the space complexity by only a square?
  - We extend it to hold for sublinear spce bounds down to $f(n) \geqslant \log n$
  - The proof is identical to the old one, except we use the new TM with read-only tape
  - Storing a configuration of N on w uses $\log(n2^{O(f(n))}) = \log n + O(f(n))$ space
  - If $f(n) \geqslant \log n$, *the storage used is* $O(f(n))$ and the remainder of the proof remains the same

NL-Completeness

- As mentioned before, PATH problem is known to be in NL, but not known to be in L
  - We believe that PATH does not belong to L, but know how to prove this conjecture
  - We do not know of problem in NL that can be proven to be outside L
  - Analogous to the question of whether P = NP, we have the question of whether **L = NL**?
  - As a step forward solving this problem, we exhibit certain language that are **NL-Complete**
    - Similar to other complexity classes, NL-complete is the most difficult problem in NL class
    - If L and NL are different, all NL-complete languages do not belong to L

NL-Completeness

- We also define an NL-complete language to be one which is in NL and any other language in NL is reducible to it
- But we do not use polynomial time reducibility here
  - Because ALL problems in NL are solvable in polynomial time
  - Every two problems in NL except NULL and Σ^* are polynomial time reducible to one another
  - So, it is too strong to differentiate problems in NL from one another

NL-Completeness

- Definition
  - A **log space transducer** is a TM with a read-only input tape, a write-only output tape, and a read/write work tape.
  - The work tape may contain $O(\log n)$ symbols.
  - A log space transducer M computes a function f: $\Sigma^* \rightarrow \Sigma^*$, where $f(w)$ is the string remaining on its input tape after M halts when it is stated with w on its input tape.
  - We call f a **log space computable function**.
  - Language A is **log space reducible** to language B, written $A \leqslant_L B$, if A is mapping reducible to B by means of a log space computable function f.

NL-Completeness

- Definition
  - A **language B is NL-complete** if

    1. $B \in NL$, and

    2. Every A in NL is log space reducible to B

NL-Completeness

- Theorem
  - If $A \leq_L B$ and $B \in L$, then $A \in L$
  - Proof
    - We may follow the proof for

      If $A \leq_P B$ and $B \in P$, then $A \in P$
    - M_A = " On input w:
      1. Compute $f(w)$
      2. Run M_B on input $f(w)$ and output whatever M_B outputs."
    - However, the storage required is $f(w)$, which is too large for us

NL-Completeness

- Proof
  - Instead, A's machine M_A computes individual symbols of $f(w)$ as requested by B's machine M_B.
  - In the simulation, M_A keeps track of where M_B's input head would be on $f(w)$
  - Every time M_B moves, M_A restarts the computation of f on w
  - Doing so may require occasional recompilation of parts of $f(w)$ and so is inefficient in time complexity
  - The advantage is that only a single symbol of $f(w)$ needs to be stored at any point
  - **It trade-off time for space**

NL-Completeness

- Corollary
  - **If any NL-complete language is in L, then L = NL**

Searching in Graphs

- PATH is NL-complete
- Proof Idea
  - We had shown that PATH is in NL, so we need to show PATH is in NL-hard by showing that every NL language is log space reducible to PATH
  - The key idea is to build a graph that represents the computation of the nondeterministic log space TM for *A*
  - The reduction maps a *w* to a graph whose node is corresponding to the configuration of the NTM on *w*
  - One node points to another node corresponding to the first configuration yields the second one in a single step of the NTM
  - So, NTM **accept** if a **path** exists from the node corresponding to the **start configuration** leads to the node corresponding to the **accepting configuration**

Searching in Graphs

- Let's NTM M decides A in $O(\log n)$ space
- Given an input w, we construct $<G, s, t>$ in log space
    - G is a directed graph that contains a path from s to t iff M accepts w
    - The nodes of G are the configurations of M on w
    - For configuration c_1 and c_2 of M on w, the pair (c_1, c_2) is an edge of G if c_2 is one of the possible next configuration of M starting from c_1
    - Node s is the start configuration
    - Machine M is modified to have a unique accept configuration and it is corresponding to node t in G

Searching in Graphs

- To show that the reduction operates in log space, we give a log space transducer which, on input *w*, outputs a description of *G*. This description comprises:
  - List of *G*'s nodes
    - The list of nodes is the same to the list of configurations of *M* on *w*
    - Can be represented in *c*log*n* space for some constant *c*
    - The transducer sequentially goes through all possible string of length *c*log*n* and tests whether each is a legal configuration of *M* on *w*.
    - The transducer outputs those configurations that pass the test
  - List of *G*'s edges
    - Similar to the list of *G*'s nodes

Searching in Graphs

- Log space is enough to verify does c_1 yield c_2 of M on w
  - The transducer only need to examine the actual tape contents under the head locations given in c_1 to determine tat M's transition function would give c_2
  - The transducer tries all pair (c_1, c_2) in turn to find which qualify as edges of G. Only those qualified are added to the output tape

Searching in Graphs

- Corollary $\mathrm{NL} \subseteq \mathrm{P}$
- Proof
  - We had shown that NL is log space reducible to PATH
  - Recall that TM uses space $f(n)$ runs in time $n2^{O(f(n))}$, so reducer that runs in log space also runs in polynomial time
  - So, every language in NL is polynomial time reducible to PATH, which in turn is in P
  - Recall that every language that is polynomial time reducible to a language in P is also in P
  - Completed the proof

NL equals to coNL

- Theorem NL = coNL
  - Recall that NP and coNP are generally believed to be different
  - NL = coNL seems to be not reasonable
  - This shows that our intuition about computation has many gaps
- Proof Idea
  - We show that $\overline{\text{PATH}}$ is in NL, thereby establish that every problem in coNL is also in NL
  - The NL algorithm *M* for $\overline{\text{PATH}}$ must have an accepting computation wherever the input graph *G* does not contain a path from *s* to *t*

NL equals to coNL

- Proof Idea
  - Let c be the number of nodes in G that are reachable from s
    - We assume that c is an input to M
  - Given G, s, t and c, the machine M operates as follows.
  - M goes through all the m nodes in G, one by one, and guess whether each one is reachable from s
  - Whenever a node u is guessed to be reachable, m attempts to verify this guess by guessing a path of length m or less from s to u. If a computation branch fails to verify it, ***reject***
  - If a computation branch guesses that t is reachable, ***reject***
  - When branches goes through all nodes and it checks the number of nodes reachable from s is equal to c, ***accept***; otherwise, ***reject***

NL equals to coNL

- Proof Idea
  - We compute c by a nondeterministic log space procedure
    - At least has one computation branch has the correct value of c
    - All other branches reject
  - For each i from 0 to m, we define A_i to be the collection of nodes that are at a distance of i or less from s
  - $A_0 = \{s\}$ and each $A_i \subseteq A_{i+1}$
  - A_m contains all nodes that are reachable from s
  - Let c_i be the number of nodes in A_i
  - To compute c_{i+1} from c_i, we go through all the nodes of G and determines whether each is a member of A_{i+1} and count the members
  - $c = c_m$

NL equals to coNL

- Proof Idea
  - To determine whether a node v is in A_{i+1}, we use an inner loop to go through all the nodes of G and guess whether each node is in A_i
  - Each positive guess is verified by guessing the path of length at most i from s
  - For each node u verified to be in A_i, the algorithm tests whether (u, v) is an edge in G
    - If it is an edge, v is in A_{i+1}
  - The number of nodes verified to be in A_i is counted
  - At the completion of inner loop, if the total number of nodes verified is to be in A_i is not c_i, all A_i have not been found, so this computation branch is rejected
  - If the count equals c_i to and v has not yet been shown to be in A_{i+1}, we conclude that it is not in A_{i+1} and then go on to the next v in the outer loop

NL equals to coNL

- Proof

M = " On input <G,s,t>:

1. Let $c_0 = 1$.

$A_0 = \{s\}$ has 1 node

NL equals to coNL

```
2.  For $i = 0$ to $m$ - 1:
3.      Let $c_{i+1}$=1.
4.      For each node $v \neq s$ in $G$.
5.          Let $d = 0$.
6.          For each node $u$ in $G$:
7.              Nondeterministically either perform or skip
                these steps:
8.                  Nondeterministically follow a path of length at
                    most $i$ from $s$ and reject if it doesn't end at $u$.
9.                  Increment $d$.
10.                 If $(u,v)$ is an edge of $G$, increment $c_{i+1}$ and
                    go to stage 5 with the next $v$.
9.  If $d \neq c_i$ , then ***reject***.
```

compute c_{i+1} from c_i

c_{i+1} counts nodes in A_{i+1}

check if $v \in A_{i+1}$

d re-counts A_i

check if $u \in A_i$

verified that $u \in A_i$

verified that $v \in A_{i+1}$

check whether found all A_i

NL equals to coNL

```
12. Let $d = 0$.
13. For each node $u$ in $G$:
14. Nondeterministically either perform or skip these steps:
15. Nondeterministically follow a path of length at most m from s and reject if it doesn't end at u.
16. If $u = t$, then reject.
17. Increment $d$.
18. If $d \neq c_m$, then reject. Otherwise, accept."
```

c_{i+1} now known; d re-counts A_m

check if $u \in A_m$

found path from s to t

verified that $u \in A_m$

check that found all of A_m

Summary

- To summarize our present knowledge of the relationships among complexity classes:

$$\mathrm{L} \subseteq \mathrm{NL} = \mathrm{coNL} \subseteq \mathrm{P} \subseteq \mathrm{PSPACE}$$

Theory of Computation

Lecture 12:
Intractability

Lecturer: Dr. Ng, Wing Yin
TA: Ms. Sun, Binbin

MiLeS Computing Lab,
Department of Computer Science and Technology,
Harbin Institute of Technology Shenzhen Graduate School.

Content of Lecture 12

- Hierarchy Theorems
  - Space hierarchy
  - Time Hierarchy
  - EXPSPACE-complete
- Relativization
- Circuit Complexity

Intractability

- Intractable
  - We know that some computational problem is solvable in principle…but not in practical
  - We will try out some examples which can be proven to be intractable

Hierarchy Theorems

- Common sense may tell us that a TM with more time and more space could solve more problems
  - Is it true that a TM uses time n^3 should decide more languages than the one uses time n^2 **?**
  - Hierarchy Theorems prove that this intuition is true under some conditions.
    - We will start with the easy one – Space Hierarchy Theorems

Hierarchy Theorems

- Definition
  - A function f: $N \rightarrow N$, where $f(n)$ is at least $O(\log n)$, is called **space constructible** if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in space $O(f(n))$
  - i.e. f is space constructible if some $O(f(n))$ space TM exists that always halts with the binary representation of $f(n)$ on its tape when started on input 1^n

Hierarchy Theorems

- All commonly occuring functions that are at least $O(\log n)$ are space constructible, e.g.
  - $\log_2 n$
  - $n \log_2 n$
  - n^2

Hierarchy Theorems

- Example
  - **n^2 is space constructible** because a machine may take its input 1^n, obtain n in binary by counting the number of 1s and output n^2 by using any standard method for multiplying n by itself.
  - The total space used is $O(n)$ which is certainly $O(n^2)$

Hierarchy Theorems

- Example: ***n^2* is space constructible**
  - When we show f(n) that are o(n) to be space constructible, we use a separate read-only tape
    - Our DVD-ROM in the definition of sublinear space complexity
  - So, it maps 1^n to the binary representation of $\log_2 n$ as follows
    - Count the number of 1s in its input in binary
    - Compute $\log_2 n$ by counting the number of bits in binary representation of n on its work tape

Hierarchy Theorems

- Why we need Space Hierarchy Theorems?
  - If $f(n)$ and $g(n)$ are two space bounds
    - $f(n)$ is asymptotically larger than $g(n)$, TM in $f(n)$ should be able to compute more than the one in $g(n)$
    - If $f(n)$ exceeds $g(n)$ by only a small and hard to compute amount, TM in $f(n)$ may not be able to use extra space profitably
      - Computing the extra space may require more space than it
      - **We avoid this situation by stipulating that $f(n)$ is space constructible**

Space Hierarchy Theorem

- For any space constructible function f: $N \rightarrow N$. a language A exists that is decidable in $O(f(n))$ space but not in $o(f(n))$ space
- Proof Idea:
  - We first demonstrate that A is decidable in $O(f(n))$
  - Then, we demonstrate that A is not decidable in $o(f(n))$

Space Hierarchy Theorem

- Proof Idea:
  - We describe A by an algorithm D that decides it and runs in $O(f(n))$ space
  - Then, D guarantees that A is different from any language that is decidable in $o(f(n))$ space
  - But how?

Space Hierarchy Theorem

- Proof Idea:
  - We design D to implement the diagonalization method that we used to prove the unsolvability of the halting problem A_{TM}
  - If M is a TM that decides a language in $o(f(n))$ space, D guarantees that A differs from M's language in at least one place
    - The different place is…the description of M itself!

Space Hierarchy Theorem

- Proof Idea:
  - Roughly, D takes its input to be the description of a TM M
    - Reject if input is not a description of any TM
  - *D* runs *M* on the same input, <*M*> within the space bound *f*(*n*)
  - If *M* halts within that much space, *D* accepts iff *M* rejects
  - If M does not halt, reject
  - If M runs within the space bound *f*(*n*), *D* has enough space to ensure that its language is different from *M*'s
  - It is okay that if *D* does not have enough space to figure out what *M* does, because *D* is not required to act differently if *M* does not run in $o(f(n))$ space

Space Hierarchy Theorem

- Proof Idea:
  - If M runs in $o(f(n))$ space, D must guarantee that its language is different from M's
    - However, even M runs in $o(f(n))$ space, it may uses more than $f(n)$ space when n is small
      - E.g. $100n^2$
    - Then D will miss the opportunity to avoid M's language
    - To avoid it, instead of running M when D receives input $<M>$, it runs M when it receives input of the form **$<M>10^*$**
    - If M is really runs in $o(f(n))$ space, D will have enough space to run it to completion on input $<M>10^k$ for some large k

Space Hierarchy Theorem

- Proof Idea:
  - We assumed that D is a decider, so we could reject an input if it runs longer than $2^{o(f(n))}$ time

Space Hierarchy Theorem

- Proof:
  - D = "On input *w*:
    1. Let *n* be the length of *w*.
    2. Compute *f*(n) using space constructability and mark off this much tape, If later stages ever attempt to use more space, **reject**
    3. If *w* is not of the form <*M*>10^* for some TM *M*, **reject**
    4. Simulate *M* on *w* while counting the number of steps used in the simulation. If the count ever exceeds $2^{f(n)}$, **reject**
    5. If *M* accepts, **reject**. If *M* rejects, **accept**."

Space Hierarchy Theorem

- In stage 4,
  - The simulated TM *M* has an arbitrary tape alphabet and *D* has a fixed tape alphabet, so we represent each cell of *M*'s tape with several cells on *D*'s tape
    - So, we introduced a constant overhead in the space used
    - If *M* runs in $g(n)$ space , then *D* uses $dg(n)$ space to simulate it
      - So, *d* is a constant depending on *D*
- *D* is a decider because each stage runs in limited time.

Space Hierarchy Theorem

- Let A be the language that D decides, A is decidable in space $O(f(n))$ because D does so
- Then, is it A decidable in $o(f(n))$ space?
  - Assume contrary that some TM M decides A in space $g(n)$ where $g(n)$ is $o(f(n))$
  - As mentioned, D can simulate M using space $dg(n)$ for some constant d
  - Some constant n_0 exists, where $dg(n) < f(n)$ for all $n \geqslant n_0$.

Space Hierarchy Theorem

- So, *D*'s simulation of *M* will run to completion so long as the input has length n_0 or more
- When *D* is run on input $<M>10^{n_0}$. This input is longer than n_0, so simulation in stage 4 will complete
- Therefore, *D* will do the opposite of *M* on the same input
- Hence, *M* does not decide *A*, which contradicts out assumption
- So, *A* is not decidable in $o(f(n))$ space

Space Hierarchy Theorem

- Corollary
  - For any two functions $f_1, f_2 : N \rightarrow N$, where $f_1(n)$ is $o(f_2(n))$ and f_2 is space constructible,

$$\mathrm{SPACE}(f_1(n)) \subset \mathrm{SPACE}(f_2(n))$$

  - This is also true for rational numbers c_1 and c_2

$$\mathrm{SPACE}(f_1(n^{c_1})) \subset \mathrm{SPACE}(f_2(n^{c_2}))$$

Space Hierarchy Theorem

- Corollary
  - Observing that two rational numbers are always exist between any two real numbers,

$$\text{for any two real numbers } 0 \le \varepsilon_1 < \varepsilon_2$$

$$\mathrm{SPACE}\left(f_1\left(n^{\varepsilon_1}\right)\right) \subset \mathrm{SPACE}\left(f_2\left(n^{\varepsilon_2}\right)\right)$$

Space Hierarchy Theorem

- Corollary $\mathrm{NL} \subset \mathrm{PSPACE}$
  - Savitch's theorem shows that $\mathrm{NL} \subseteq \mathrm{SPACE}\left(\log^2 n\right)$,
  - the space hierarchy theorem shows that
  - $\mathrm{SPACE}\left(\log^2 n\right) \subset \mathrm{SPACE}(n)$
  - Hence the corollary follows
  - As we observed, this separation allows us to conclude that $\mathrm{TQBF} \notin \mathrm{NL}$ because TQBF is PSPACE-complete with respect to log space reducibility

Time Hierarchy Theorem

- Then, we prove the existence of problems that are decidable in principle but not in practice
  - i.e. problems that are decidable but intractable!
  - Each of the classes SPACE(n^k) is contained within the class SPACE($n^{\log n}$), which in turn is strictly contained within the class SPACE(2^n).
  - So, we have $\mathrm{EXPSPACE} = \bigcup_k \mathrm{SPACE}\left(2^{n^k}\right)$

$$\mathrm{PSPACE} \subset \mathrm{EXPSPACE}$$

Time Hierarchy Theorem

- $\text{PSPACE} \subset \text{EXPSPACE}$
- The above corollary establishes the existence of decidable problems that are intractable.
  - Their decision procedures must use more than polynomial space
  - The languages themselves are artificial for the interest of separating complexity classes only.

Time Hierarchy Theorem

- Definition
  - A function t: $N \rightarrow N$, where $t(n)$ is at least $O(n \log n)$, is called time constructible if the function that maps the string 1^n to the binary representation of $t(n)$ is computable in time $O(t(n))$
  - That is, t is time constructible if some $O(t(n))$ TM M exists that always halts with the binary representation of $t(n)$ on its tape when started on input 1^n

Time Hierarchy Theorem

- All commonly occuring functions that are at least n log n are time constructible, including the functions $n\log n, n\sqrt{n},\ n^2, 2^n$
- For example, to show that $n\sqrt{n}$ is time constructible, we design a TM to count the number of 1s in binary
  - The TM moves a binary counter along the tape and incremet by 1 for every input position until it reaches the end of input

Time Hierarchy Theorem

- This takes O(n log n) steps because O(log n) steps are used for each of the n input positions. Then we compute $\lfloor n\sqrt{n} \rfloor$ in binary from the binary representation of n

- Any reasonable method of doing so will work in $O(n \log n)$ time because the length of the numbers involved is $O(\log n)$

Time Hierarchy Theorem

- The time hierarchy theorem is an analog for time complexity space hierarchy theorem
- For technical reasons, that will appear in its proof the time hierarchy theorem is slightly weaker than the space one.
    - For space constructible asymptotic increase in space bound enlarges the class of languages so decidable
    - For time, we must increase the time bound by a **logarithmic factor** in order to guarantee that we can obtain additional languages
    - Conceivably, a tighter time hierachy theorem is true, but we just do not know how to prove it at present
    - This issue occur because we measure time complexity with a single-tape TM

Time Hierarchy Theorem

- Time Hierarchy Theorem
  - For any time constructible function t: $N \rightarrow N$, a language A exists that is decidable in $O(t(n))$ time but not decidable in time $o(t(n)/\log t(n))$
- Proof Idea:
  - We construct a TM D that decides a language A in time $O(t(n))$ where A can not be decided in $o(t(n)/\log t(n))$
  - D takes an input w of the form $<M>10^*$ and simulate M on input w, making sure not to use more than tn time. If M halts within that much time, D gives the opposite output

Time Hierarchy Theorem

- Proof Idea:
  - The important difference in the proof concerns the cost of simulating *M* while, at the same time, count the number of steps that the simulation is using.
  - TM *D* must perform this timed simulation efficiently so that *D* runs in $O(t(n))$ while avoiding all languages decidable in $o(t(n)/\log t(n))$ time
  - For space complexity, the simulation introduced a constant factor overhead
  - For time complexity, the simulation introduced a logarithmic factor overhead
    - This is why we have $1/\log t(n)$ factor in the theorem
    - No more efficient simulation is known yet

Time Hierarchy Theorem

- Proof:
  - D = “On input *w*:
    1. Let *n* be the length of *w*.
    2. Compute *t*(n) using time constructability and store the value $\lceil t(n)/\log t(n) \rceil$ in a binary counter. Decrement this counter before each step used to carry out stages 3, 4 and 5. If the counter hit 0, **reject**
    3. If *w* is not of the form <*M*>10^* for some TM *M*, **reject**
    4. Simulate *M* on *w*
    5. If *M* accepts, **reject**. If *M* rejects, **accept**.”

Time Hierarchy Theorem

- Obviously, stages 1, 2 and 3 can be performed within $O(t(n))$ time
- In stage 4, every time *D* simulates one step of *M*, it takes *M*'s current state in its transition function so that it can update *M*'s tape head and looks up *M*'s next action in its transition function to update *M*'s tape appropriately.
  - All state, tape symbol and transition function of *M* are stored somewhere on *D*'s tape
  - This will be very inefficient if they are stored far away from each other

Time Hierarchy Theorem

- We organize *D*'s single tape into tracks
  - We store content of the first track in the odd positions and content of the second track in the even positions
  - Alternatively, we may have this effect by enlarging D's tape alphabet to include each pair of symbols, one from the first track, one from the other
  - This could be extended to multiple tracks, with more than 2 tracks
  - This only introduce a constant factor overhead in time
  - *D* has 3 tracks

Time Hierarchy Theorem

- At stages 3 and 4, *D* must decrement the step counter
  - *D* can do so without adding to much time to the simulation by keeping the counter in binary on a third track
  - And, moving it to keep it near the present head position
  - The magnitude of the count is about $t(n)/\log t(n)$, so its length is $\log (t(n)/\log t(n)) = O(\log t(n))$
  - So, it adds $O(\log t(n))$ to the simulation
  - Totally, the simulation running time is $O(t(n))$.
  - So, A is decibadable in time $O(t(n))$

Time Hierarchy Theorem

- Is it A not decidable in $o(t(n)/\log t(n))$?
  - Assume contrary that TM M decides A in time $g(n)$, where $g(n)$ is $o(t(n)/\log t(n))$.
  - D can simulate M using time $dg(n)$ for some constant d
  - If the total simulation time is at most $t(n)/\log t(n)$, simulation will run to completion (without counting the counter updating)
  - Some constant n_0 exists, where $dg(n) < t(n)/\log t(n)$ for all $n \geqslant n_0$.
  - So, D's simulation of M will run to completion so long as the input has length n_0 or more

Time Hierarchy Theorem

- Is it A not decidable in $o(t(n)/\log t(n))$?
  - When D is run on input $\langle M \rangle 10^{n_0}$. This input is longer than n_0, so simulation in stage 4 will complete
  - Therefore, D will do the opposite of M on the same input
  - Hence, M does not decide A, which contradicts out assumption
  - So, A is not decidable in $o(t(n)/\log t(n))$ time

Time Hierarchy Theorem

- Corollary
  - For any two functions t_1, $t_2 : N \rightarrow N$, where $t_1(n)$ is $o(t_2(n)/\log t_2(n))$ and t_2 is space constructible,

$$\text{TIME}(t_1(n)) \subset \text{TIME}(t_2(n))$$

- Corollary
  - This is also true for real numbers c_1 and c_2

$$\text{TIME}(t_1(n^{c_1})) \subset \text{TIME}(t_2(n^{c_2}))$$

- Corollary

$$\text{P} \subset \text{EXPTIME}$$

Exponential Space Completeness

- We use previous results to demonstrate that a specific language is actually intractable.
  - The hierarchy theorems tell us that a TM can decide more languages in EXPSPACE than it can in PSPACE
  - Then, we show that a particular language concerning generalized regular expressions is complete for EXPSPACE and hence can not be decided in polynomial time or even in polynomial space

Exponential Space Completeness

- Recall that regular expressions are built up from the atomic expressions ϕ, ε and members of the alphabet, by using regular operations union (∪), concatenation (o) and star (*)
- Recall that we can test the equivalence of two regular expressions in polynomial space
- Now, we add one more operation. Let ↑ be the exponentiation operation.
- If R is a regular expression and k is a nonnegative integer, writing $R \uparrow k$ is equivalent to the concatenation of R with itself k times, in shorthand R^k

Exponential Space Completeness

- Obviously, the generalized regular expressions still generate the same class of regular languages as do the standard regular expressions
  - We can eliminate the exponentiation operation by repeating the base expression.
  - Let:

$EQ_{\mathrm{REX}\uparrow}$ ={<Q, R> | Q and R are equivalent regular experssions with exponentiation}

Exponential Space Completeness

- Definition

A lagnuage B is EXPSPACE-complete if

- $B \in \text{EXPSPACE}$
- Every A in EXPSPACE is polynomial time reducible to B

Exponential Space Completeness

- Theorem
  - $EQ_{\text{REX}\uparrow}$ is EXPSPACE-complete
- Proof Idea:
  - When measuring the complexity of deciding $EQ_{\text{REX}\uparrow}$, we assume that all exponents are written as binary integers
    - The length of an expression is the total number of symbols in it
  - To test whether two expressions with exponentiation are equivalent, we first use repetition to eliminate exponentiation, then we convert the resulting expression to NFAs
  - Then, we use an NFA equivalence testing procedure similar to the one used for deciding the complement of ALL_{NFA}

Exponential Space Completeness

- Proof Idea:
  - To show that a language A in EXPSPACE is polynomial time reducible to $EQ_{\text{REX}\uparrow}$, we utilize the technique of reductions via computation histories
  - Given a TM M for A, we design a polynomial time reduction mapping an input w to a pair of expressions R_1 and R_2, that are equivalent exactly when M accepts w
  - The expressions R_1 and R_2 simulate the computation of M on w
  - Expression R_1 generates all stings over the alphabet consisting of symbols that may appear in computation histories
  - Expression R_2 generates all stings that are not rejecting computation histories

Exponential Space Completeness

- Proof Idea:
  - So, if the TM accepts its input, no rejecting computation histories exist and R_1 and R_2 generate the same language
  - The difficulty in this proof is that the size of the expressions constructed must be polynomial in n, such that the reduction can run in polynomial time
    - The exponentiation is useful here to represent the long computation with a relatively short expression

Exponential Space Completeness

- Proof:
  - The following nondeterministic algorithm tests whether two NFAs are inequivalent
  - N = "On input $< N_1, N_2 >$, where N_1 and N_2 are NFAs:
    1. Place a marker on each of the start states of N_1 and N_2
    2. Repeat $2^{q_1+q_2}$ times, where q_1 and q_2 are the numbers of states in N_1 and N_2
    3. Nondeterministically select an input symbol and change the positions of the markers on the states of N_1 and N_2 to simulate reading that symbol
    4. If at any point, a marker was placed on an accept state of one of the FA and not on any accept state of the other FA, **accept**. Otherwise, **reject**"

Exponential Space Completeness

- Proof:
  - If N_1 and N_2 are equivalent, N clearly rejects because it only accepts when one accepts a string that the other does not accept
  - If the FA are not equivalent, some string is accepted by one, but not the other and such string must be of the length at most $2^{q_1+q_2}$
  - A string longer than that consists of repeating portion and marker will be placed repeatedly, so we could remove it without affecting the result
  - Hence, N would guess this string among its nondeterministic choices and would accept
  - Thus, N operate correctly

Exponential Space Completeness

- Proof:
  - N runs in nondeterministic linear space and thus, by applying Savitch's theorem, we obtain a deterministic $O(n^2)$ space algorithm for this problem. Then we use N to design algorithm E that decides $EQ_{\text{REX}\uparrow}$
- E = "on input $<R_1, R_2>$ where R_1 and R_2 are regular expressions with exponentiation:
  1. Convert R_1 and R_2 to equivalent regular expressions B_1 and B_2 that use repetition instead of exponentiation
  2. Convert B_1 and B_2 to equivalent NFAs N_1 and N_2
  3. Use the deterministic version of algorithm N to determine whether N_1 and N_2 are equivalent"

Exponential Space Completeness

- Proof:
  - The use of repetition to replace exponentiation may increase the length of an expression by a factor of 2^l where l is the sum of the lengths of the exponents
  - Expressions B_1 and B_2 have a length of $n2^n$ where n is the input length.
  - Converting regular expression to NFA increase the size linearly and hence NFAs N_1 and N_2 have at most $O(n2^n)$ states
  - So, with input size $O(n2^n)$, the deterministic version of N uses space $O((n2^n)^2) = O(n^2 2^{2n})$
  - **Hence, $\boldsymbol{EQ_{\text{REX}\uparrow}}$ is decidable in exponential space**

Exponential Space Completeness

- Proof of is $EQ_{\mathrm{REX}\uparrow}$ EXPSPACE-hard:
  - Let A be a language that is decided by TM M running in space $2^{(n^k)}$ for some constant k
  - The reduction maps an input w to a pair of regular expressions, R_1 and R_2
  - Expression R_1 is Δ^*
    - where Δ = (M's tape alphabet) **U** (states) **U** $\{\#\}$ is the alphabet consisting of all symbols that may appear in a computation history
  - Expression R_2 generates all strings that are not rejecting computation histories
  - M accepts w iff M on w has no rejecting computation history, so the two expressions are equivalent iff M accepts w

Exponential Space Completeness

- Proof:
  - Rejecting computation history is a sequence of configuration separated by # symbols
  - We use our standard encoding method and the symbol of current state is place to the left of the current head position
  - Every configuration has the length of $2^{(n^k)}$ and is padded on right by blank symbols if it is too short
  - The first configuration is the start configuration
  - The last configuration is a rejecting configuration
  - Each configuration must follow from the preceding one according to thte transition function

Exponential Space Completeness

- Proof:
  - A string may fail to be rejecting computation history if
    - Fail to start
    - Fail to end
    - Incorrect some where in the middle
  - So, expression $R_2 = R_{\text{bad-start}} \cup R_{\text{bad-window}} \cup R_{\text{bad-reject}}$
  - We construct expression $R_{\text{bad-start}}$ to generate all strings that fail to start with the start configuration C_1 of M on w
  - C_1 looks like $q_0 w_1 w_2 \ldots w_n \sqcup\sqcup \#$
  - We write $R_{\text{bad-start}}$ as the union of several subexpression to handle each part of C_1 :

$R_{\text{bad-start}} = S_0 \cup S_1 \cup \ldots\ldots \cup S_n \cup S_b \cup S_\#$

Exponential Space Completeness

- Proof:
  - Expression S_0 generates all strings that don't start with q_0
    - S_0 be the expression $\Delta_{-q_0}\Delta^*$
  - Expression S_i generates all strings that don't contain w_i , for 1 $\leqslant i \leqslant$ n
    - S_i be the expression $\Delta^i\Delta_{-w_i}\Delta^*$
  - Expression S_b generates all strings that fail to contain blank symbol in some position n+2 through $2^{(n^k)}$
    - S_b be the expression $\Delta^{n+1}(\Delta\cup\varepsilon)^{2^{\left(n^k\right)}-n-2}\Delta_{-\sqcup}\Delta^*$
  - Expression $S_\#$ generates all string don't have a # symbol in position $2^{(n^k)}+1$
    - $S_\#$ be the expression $\Delta^{2^{\left(n^k\right)}}\Delta_{-\#}\Delta^*$

Exponential Space Completeness

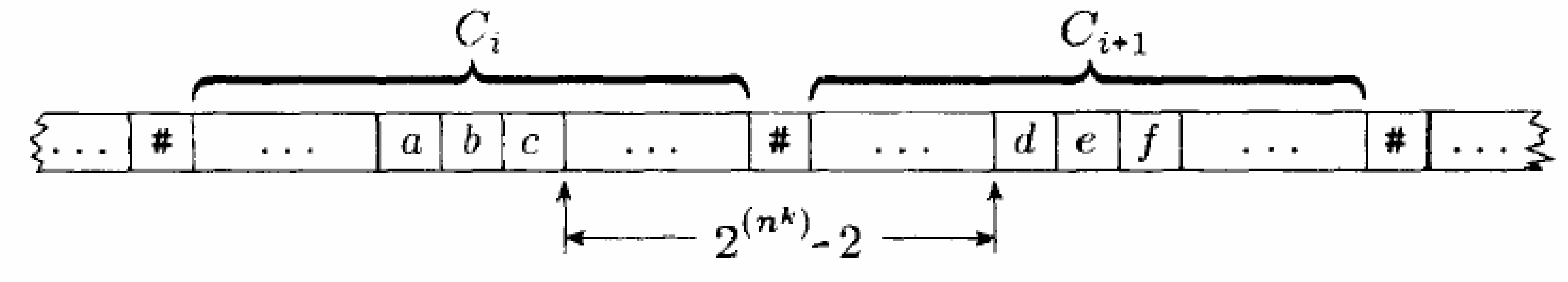
- Proof:
  - We completed the construction of $R_{\text{bad-start}}$
  - $R_{\text{bad-reject}}$ generates all strings that don't end properly
    - strings failed to contain a rejecting configuration, i.e. does not contain rejecting state $R_{\text{bad}-\text{reject}} = \Delta^*_{-q_{\text{reject}}}$

Exponential Space Completeness

- Proof:
  - Then, we construct $R_{\text{bad-window}}$ which generates all strings whereby one configuration does not properly
    - Remember the 2x3 windows to check the transition?

$$R_{\text{bad}-\text{window}} = \bigcup_{\text{bad}(abc,def)} \Delta^{*} abc \Delta^{\left(2^{\left(n^{k}\right)}-2\right)} def \Delta^{*}$$

    - where bad(*abc*, *def*) means that *abc* doesn't yield *def*



Exponential Space Completeness

- Proof:
  - To calculate the length of R, we need to determine the length of the exponents in it
  - Several exponents of magnitude roughly $2^{(n^k)}$ appear
    - The length in binary is $O(n^k)$
  - So, the length of R is polynomial in n

Relativization

- The proof of $EQ_{\mathrm{REX}\uparrow}$ is rests on the diagonalization method
- Why don’t we show that SA is intractable in the same way?
- We introduce **Relativization** to give strong evidence against the possibility of solving the P versus NP question by using diagonalization
- In the relativization method, we modify TM to give it certain information for “free”
  - This makes TM solve problem more easily

Relativization

- For example, we give the TM ability to solve the SAT problem in a single step for any size of Boolean formula
- Never mind how we do it, imagine an attached "black-box", we call this black-box an **oracle**
  - Oracle could correspond to any language
- TM could use the oracle to solve any NP problem in polynomial time, regardless whether P equals NP because every NP problem is polynomial time reducible to the SAT problem
- Such a TM is said to be computing **relative to** the SAT problem

Relativization

- Definition:
  - An oracle for a language A is device that is capable of reporting whether any string w is a member of A.
  - An oracle Turing machine M^A is a modified TM that has the additional capability of querying an oracle.
  - Whenever M^A writes a string on a special oracle tape it is informed whether that string is a member of A, in a single computation step
  - Let P^A be the class of language decidable with a polynomial time oracle TM that uses oracle A.
  - Define NPA similarly

Relativization

- As we mentioned earlier, polynomial time computation relative to the SAT problem contains all of NP
- $\mathrm{NP} \subseteq \mathrm{P}^{\mathrm{SAT}}$
- Furthermore… $\mathrm{coNP} \subseteq \mathrm{P}^{\mathrm{SAT}}$
  - Because P^{SAT} being a deterministic complexity class is closed under complementation

Relativization

- Just as P^{SAT} contains language that we believe are not in P, the class NP^{SAT} contains languages that we believe are not in NP!
- Let say we have two Boolean formulas ϕ and ψ over variables $x_1,\ldots,x_l$ are equivalent if the formulas have the same value on any assignment to the variables
  - A formula is minimal if no smaller formula is equivalent to it
  - NONMIN-FORMULA

$$= \{<\phi> \mid \phi \text{ is not a minimal Boolean formula}\}$$

Relativization

- NONMIN-FORMULA does not seem to be in NP
- However, NONMIN-FORMULA is in NP^{SAT} because a nondeterministic polynomial time oracle TM with a SAT oracle can test whether ϕ is a member
    - First, inequivalence problem for two Boolean formulas is solvable in NP, hence the equivalence problem is in coNP
        - A nondeterministic TM can guess the assignment on which the two formulas have different values
    - Then the nondeterministic oracle TM for NONMIN-FORMULA nondeterministically guesses the smaller equivalent formula
        - Test whether it actually is equivalent using the SAT oracle and accept it

Relativization

- Limitats of the diagonalization method
  - We demonstrate A and B for which P^{A} and NP^{A} are provably different and P^{B} and NP^{B} are provably equal.
  - This is important because their existence indicates that we are unlikely to resolve the P versus NP question by using the diagonalization method.
  - The core of diagonalization method is a simulation of one TM by another.
  - The simulation is done so that the simulating machine can determine the behavior of the other machine and then behave differently

Relativization

- Limitats of the diagonalization method
  - Suppose that both of these TM were given identical oracles.
    - Whenever the simulated machine queries the oracle, so can the simulator
    - So, the simulation can proceed as before
  - Consequently, any theorem proved about TM by using only the diagonalization method would still hold if both machines were given the same oracle
  - If we could prove that P and NP were different by diagonalizing, we could conclude that they are different relative to any oracle as well
  - But P^{B} and NP^{B} are equal, so that conclusion is false

Relativization

- Limitats of the diagonalization method
  - Hence, diagonalization is not sufficient to separate these two classes
  - Similarly, no proof that relies on a simple simulation could show that the two classes are the same because that would show that they are the same relative to any oracle, but in fact P^A and NP^A are different.

Relativization

- Theorem
  - An oracle *A* exists whereby $\mathrm{P}^{A} \neq \mathrm{NP}^{A}$
  - An oracle *B* exists whereby $\mathrm{P}^{B} = \mathrm{NP}^{B}$

Relativization

- Proof Idea:
  - Exhibiting oracle *B* is easy.
  - Let *B* be any PSPACE-complete problem such as TQBF
  - We exhibit oracle *A* by construction. We design *A* so that a certain language L_A in NP^A provably requires brute-force search. So, L_A can not be in P^A
  - Hence, we can conclude that P^A is not equal to NP^A
  - The construction considers every polynomial time oracle machine in turn and ensures that each fails to decide the language L_A

Relativization

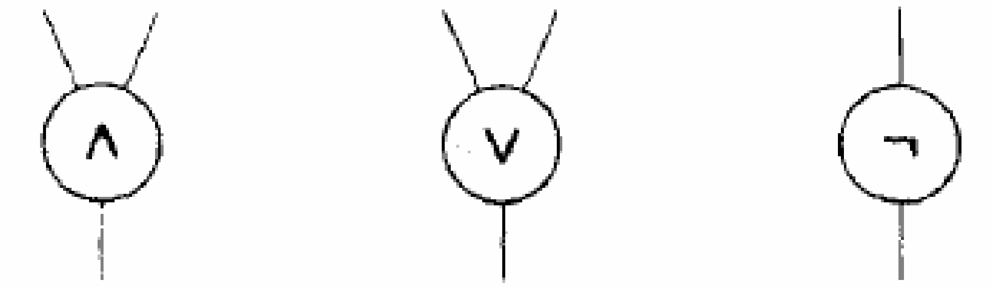
- In summary
  - The relativization method tells us that to solve the P versus NP question, we must analyze computations, not just simulate them
  - Circuit Complexity is one of the approaches

Circuit Complexity

- Computers are built from electronic devices wired together in a design called a **digital circuit**
- We simulate TM with its theoretical counterpart to digital circuit, called **Boolean circuits**
- Researchers believe that circuits provide a convenient computational model for attacking P vs NP and related questions
- Circuits provide an alternative proof of the Cook-Levin theorem that SAT is NP-complete

Circuit Complexity

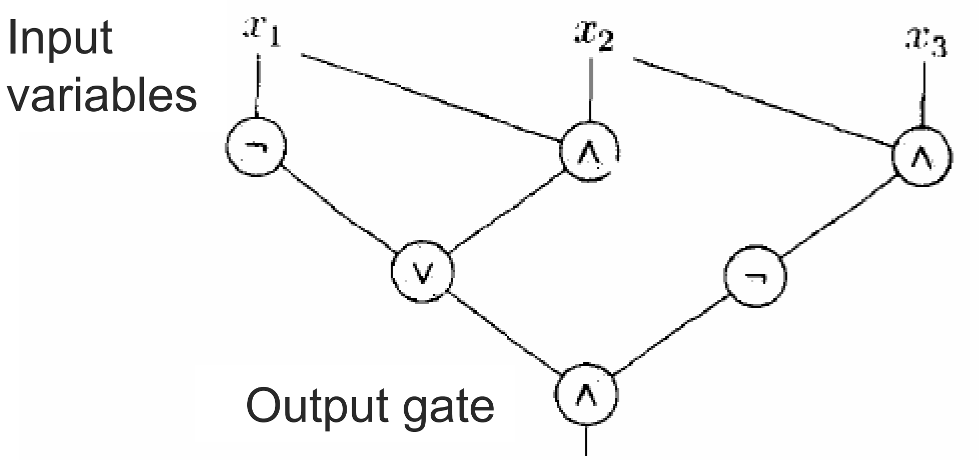
- Definition
  - A Boolean Circuit is a collection of **gates** and **inputs** connected by **wires**.
  - Cycles are not permitted
  - Gates take three forms
    - AND gate
    - OR gate
    - Not gate



AND OR NOT

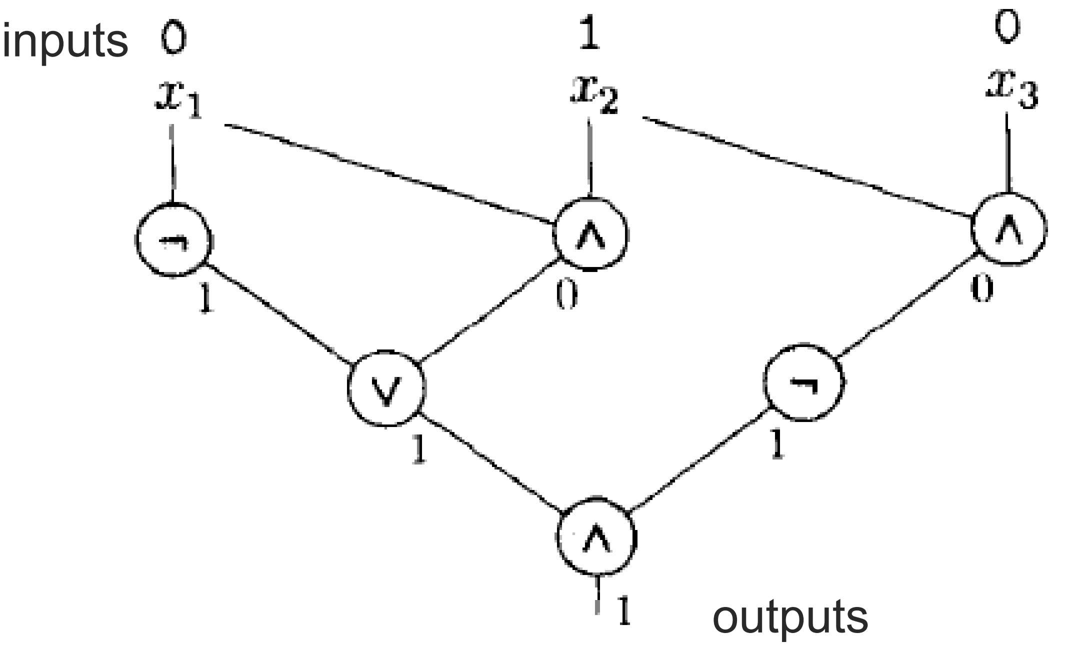
Circuit Complexity

- Wires in a Boolean circuit carry the Boolean values 0 and 1
- The gates are simple processors that compute the Boolean functions AND, OR and NOT.
- Inputs are labeled $x_1,\ldots,x_n$
- One of the gates is designated as the output gate



Circuit Complexity

- A Boolean circuit computes an output value from a setting of the inputs by propagating values along the wires and computing the function associated with the respective gates until the output gate is assigned a value



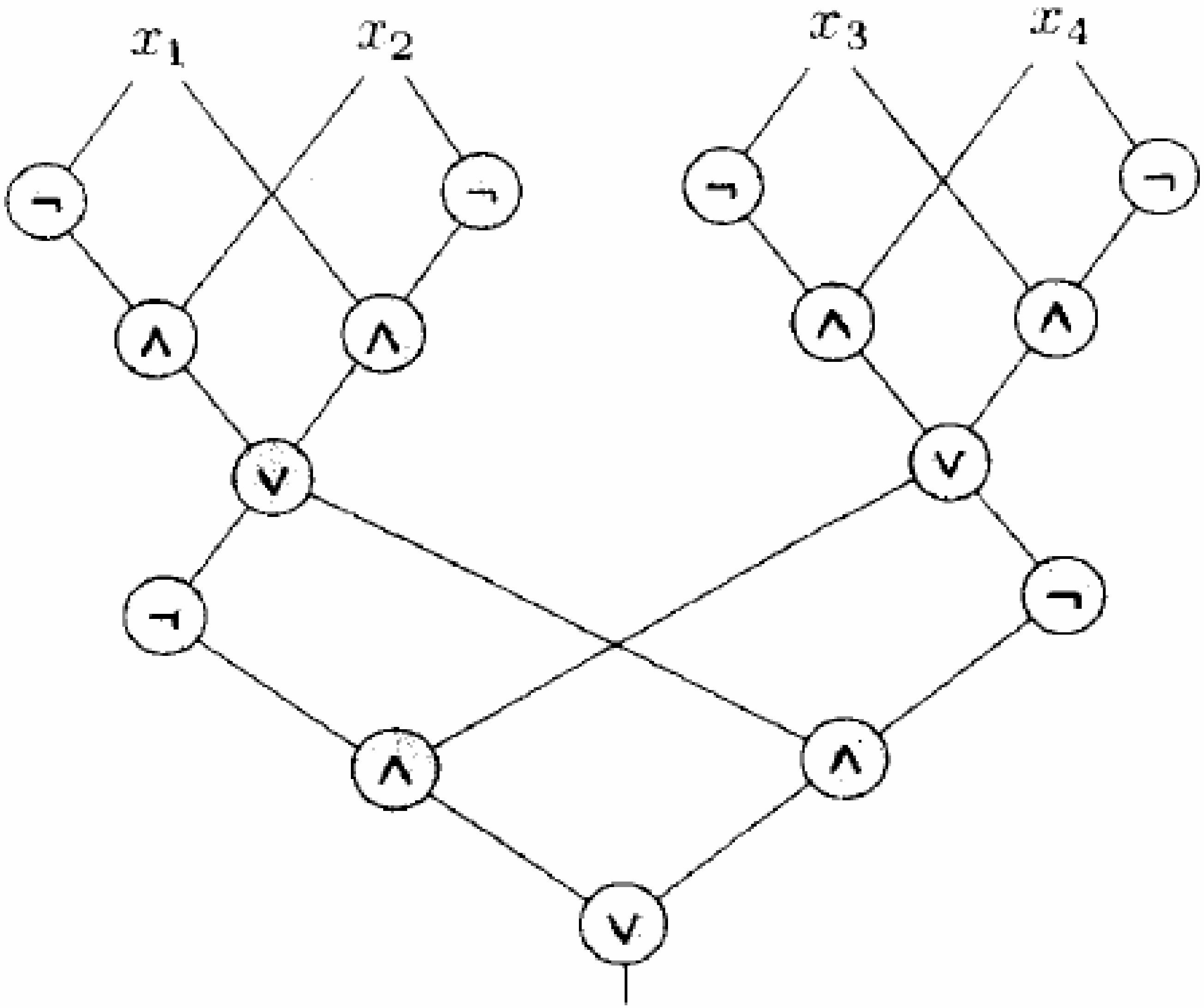
Circuit Complexity

- We use functions to describe the input/output behavior of Boolean circuits. To a Boolean circuit C with n input variables, we associate a function f_C: $\{0,1\}^n \rightarrow \{0,1\}$, where if C outputs b when its inputs $x_1,\ldots,x_n$ are set to $a_1,\ldots,a_n$, we write $f_C\,(a_1,\ldots,a_n) = b$
- C computes the function f_C
- If we have k outputs, the range of f_C is $\{0,1\}^k$

- Then, we look at the n-input parity function $parity_n$: $\{0,1\}^n \rightarrow \{0,1\}$.

Circuit Complexity

- $parity_4$



Circuit Complexity

- One may observe that we need to fix the number of variables beforehand in circuit design
- But TM takes arbitrary length of input string and may different every time
- So, we design a circuit for particular input length for testing the membership of a language
- Then, we have a family of circuits, one for each input length

Circuit Complexity

- Definition
  - A **circuit family** C is an infinite list of circuits, (C_0, C_1, C_2,…), where C_n has n input variables
  - We say that C decides a language A over {0,1} if for every string w

$$w \in A \quad \text{iff} \quad C_n(w) = 1$$

where n is the length of w

Circuit Complexity

- The **size** of a circuit is the number of gates that it contains
- Two circuits are **equivalent** if they have the same input variables and output the same value on every input assignment
- A circuit size is **minimal** if no smaller circuit is equivalent to it
- A circuit family for a language is minimal if every C_i is minimal
- Finding the minimal circuit may not be solvable in N or NP

Circuit Complexity

- **Size complexity** of a circuit family (C_0, C_1, C_2,…) is a function f: $N \rightarrow N$, where $f(n)$ is the size of C_n
- **Depth** of a circuit is the length (number of wires) of the longest path from an input variable to the output gate.
- Definition of **depth minimal** and **depth complexity** are similar to the one for circuit size, but use depth to instead of size
- **Circuit size complexity of a language** is the size complexity of fa minimal circuit family for that language.
- Circuit depth complexity is defined similar to size's

Circuit Complexity

- The circuit complexity of a language is related to its time complexity
- **Theorem**
  - Let t: $N \rightarrow N$ be a function, where $t(n) \geq n$
  - If $A \in \mathrm{TIME}(t(n))$, then A has a circuit complexity $O(t^2(n))$
- This theorem gives an approach to proving that P is not equal to NP whereby we attempt to show that some language in NP has more than polynomial circuit complexity

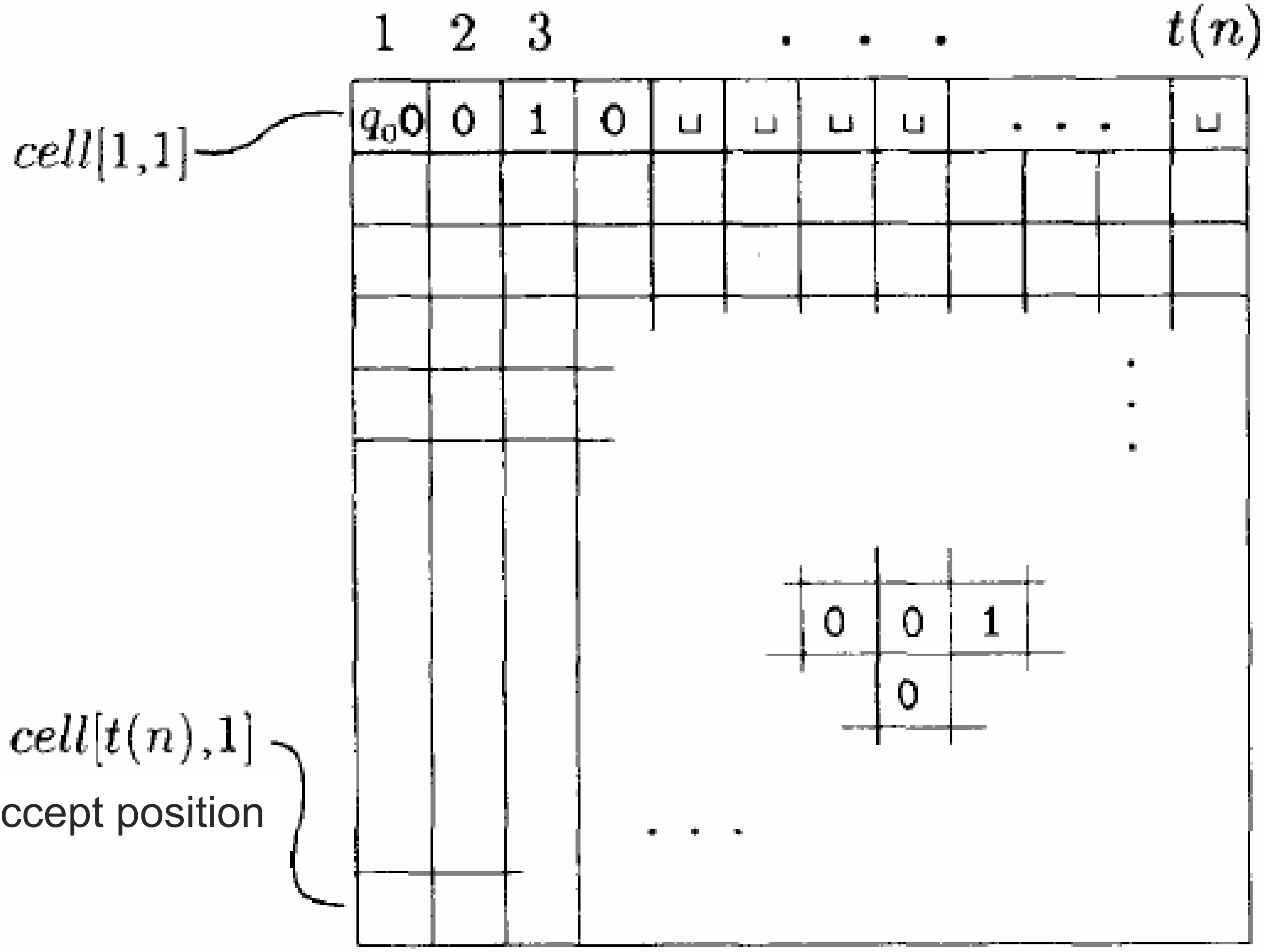
Circuit Complexity

- Proof Idea:
  - Let M be a TM that decides A in time $t(n)$, we ignore $O(t(n))$ for simplicity
  - For each n we construct a circuit C_n that simulates M on inputs of length n
  - The gates of C_n are organized in rows, one for each of the $t(n)$ step in M's computation on an input of length n
  - Each row of gates represents the configuration of M at the corresponding step
  - Each row is wired into the previous row so that it can calculate its configuration from previous row's configuration

Circuit Complexity

- Proof Idea:
  - We modify M so that the input is encoded into $\{0,1\}$
  - When M is about to accept, it moves its head onto the leftmost tape cell and writes the blank symbol on that cell prior to entering the accept state
  - Then, we can designate a gate in the final row of the circuit to be the output gate

Circuit Complexity



Accept position

Start configuration

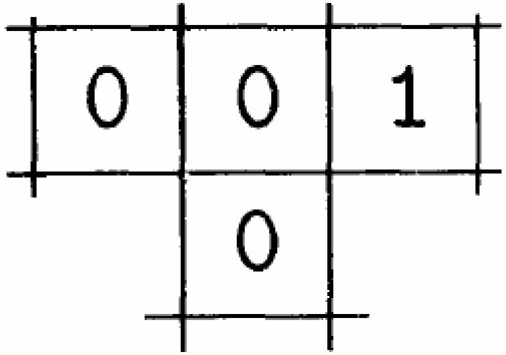
2nd configuration

.
.
.

$T(n)$th configuration

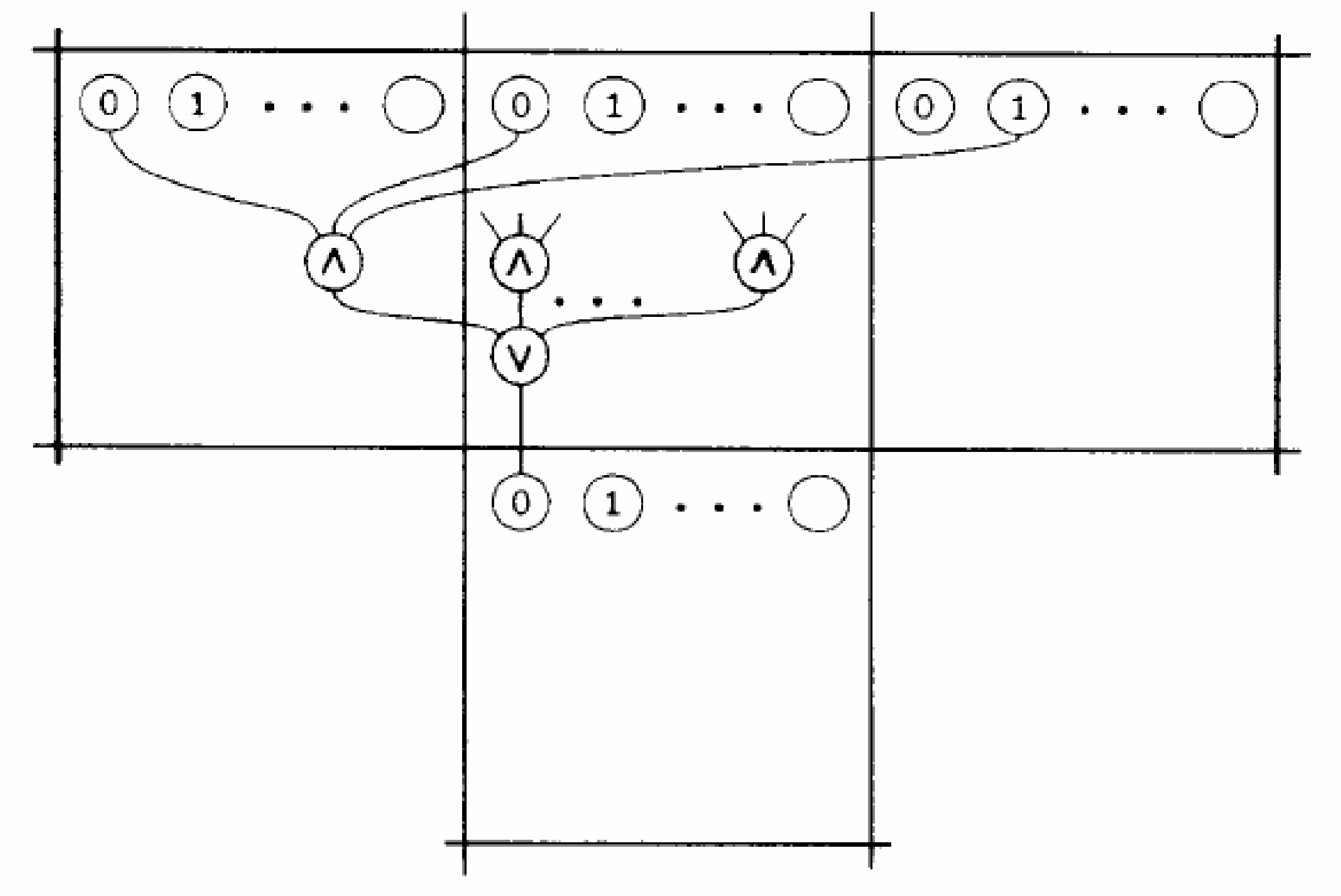
Circuit Complexity

- Remember the 2x3 windows? We use similar technique to obtain values in the tableau



Circuit Complexity

- Then we wire those gates at appropriate positions



Circuit Complexity

- Besides linking circuit complexity and time complexity, we also found an alternative proof of Cook-Levin theorem
- We say that a Boolean circuit is satifiable if some setting of the inputs causes the circuit to output 1.
- The circuit-satisfiability problem tests whether a circuit is satisfiable

CIRCUIT-SAT = {<*C*> | *C* is satisfiable Boolean circuit}

Circuit Complexity

- Theorem
  - CIRCUIT-SAT is NP-complete
- Proof:
  - CIRCUIT-SAT is obviously in NP
  - $f(w)=\langle C\rangle$ implies that

    $$w \in A \Longleftrightarrow \text{Boolean circuit } C \text{ is satisfiable}$$
  - Because *A* is in NP, it has a polynomial time verifier *V* whose input has the form <*x*, *c*>, where c may be the certificate showing that *x* is in *A*
  - To construct *f*, we obtain the circuit simulating *V* using the tableau method in previous slides

Circuit Complexity

- Proof:
  - The only remaining input is the circuit correspond to the certificate *c*, we call this circuit *C* and output it
  - If *C* is satisfiable, a certificate exists, so w is in *A*
  - Or, if w is in *A*, a certificate exists, so *C* is satisfiable
  - Then we need to show the reduction runs in polynomial time
  - In the tableau method, the construction of the circuit can be done in time that is polynomial in n
  - The running time of the verifier is n^k for some k
  - So, the size of the circuit is $O(n^{2k})$
  - So, the running time of the reduction is $O(n^{2k})$

Circuit Complexity

- Theorem:
  - 3SAT is NP-complete
- Proof Idea:
  - 3SAT is obviously in NP
  - Then we concentrate on showing that all languages in NP reduce to 3SAT in polynomial time
    - We reduce CIRCUIT-SAT to 3SAT in polynomial time
  - The reduction converts a circuit C to a formula Φ ,
    - whereby C is staisfiable iff Φ is satisfiable
    - The formula contains one variable for each variable and each gate in the circuit

Circuit Complexity

- Proof Idea:
  - Conceptually, the formula simulates the circuit
  - A satisifying assignment for ϕ contains a satisfying assignment to *C*
  - If also contains the values at each of *C*'s gates in *C*'s computation on its satisfying assignment
  - In effect, ϕ 's satisfying assignment guess C's entire computation on its satisfying assignment
  - ϕ 's cluases check the correctness of that computation
  - In addition, ϕ contains a clause stipulating that *C*'s output is 1

Computational Theory

- ***This is almost the end of this subject!!***

- Remember that we will have a final review on next Monday
- We will have 2.5 hours final examination on next THU

- Lecture notes and suggested answer to assignments and mid-term test could be found in the following website

http://miles.hitsz.edu.cn/course/tc0607

Assignment 1

- All question must be answered **in English**. No mark will be given to assignments answered using other language.
- **10%** of total marks in this subject
- Due day: **2nd April 2007**
- Answer **ALL** of the following questions
  - **1.4(a)**
  - **1.8(a), 1.9(a), 1.10(a)**
  - **1.16(b)**
  - **1.24(a-d)**
  - **1.29(b)**
  - **2.6(b) and the corresponding PDA using state graph.**
  - **2.14**
  - **2.21**

Assignment 2 (April 9th)

- Due on April 16th. Please answer ALL following questions in English.

- 4.3
- 4.7
- 4.12
- 5.1
- 5.3
- 5.4

Assignment 3 (April 26th)

- Due on May 10th. Please answer ALL following questions in English.

- 7.1 (d),(e),(f)
- 7.2 (b),(c),(e)
- 7.4
- 7.8
- 7.27
- 8.2
- 8.17
- 8.23