Chapter 04: Dynamic Programming Algorithm
Design and Analysis of Computer Algorithms

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Outline

An example of task scheduling

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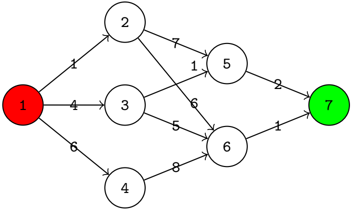

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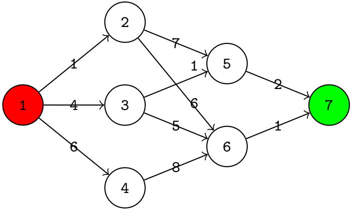

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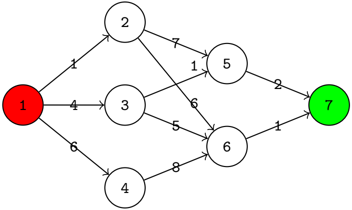

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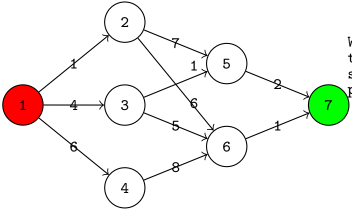

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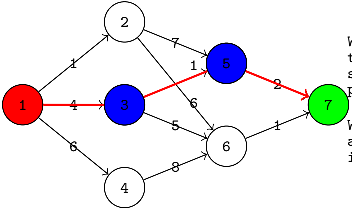

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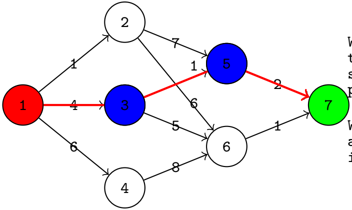

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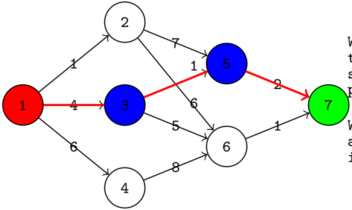

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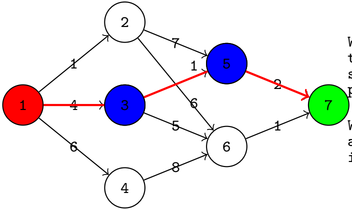

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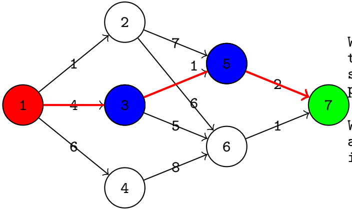

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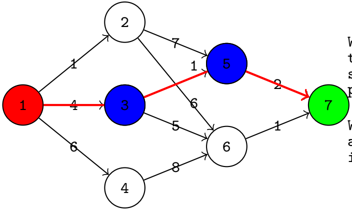

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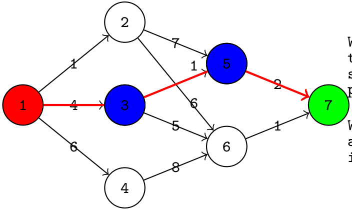

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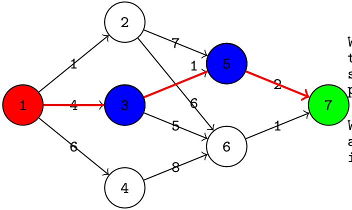

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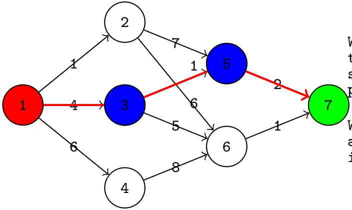

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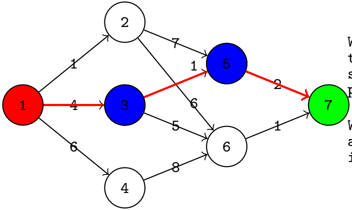

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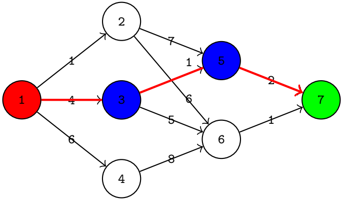

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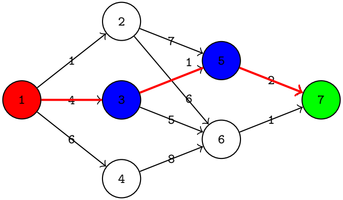

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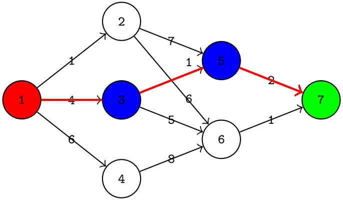

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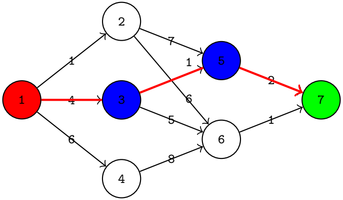

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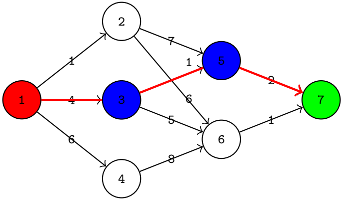

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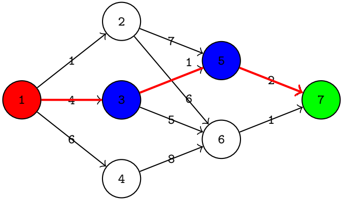

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6	Tracebacking the optimal solution	P(1) - P(3) -P(5) -P(7)

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  - choose items maximizing value ?
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  - ...
- **Dynamic programming** is one of great choice

Step 1. Identifying subproblems

n=5, c=10	●	●	●	●	●
p=	6	3	5	4	6
w=	2	2	6	5	4
X	x_1	x_2	x_3	x_4	x_5

- By fewer items: exclude an item i , what changed?

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  - however, we do not know which i,
  - $f(c) = \max_{i=1, w_i<c}^{n} \{f(c-w_i) + v_i\}$

Step 2. Validating Principle of Optimization

n=5, c=10

p=	6	3	5	4	6
w=	2	2	6	5	4
X	x_1	x_2	x_3	x_4	x_5

[Optimal substructure] A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems.

- Supposed that $(x_1, x_2, \cdots, x_n)$ is the optimal solution of original problem P^0.

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- We should show that $(y_1, y_2, \cdots, y_{n-1}, x_n)$ is no worse than $(x_1, x_2, \cdots, x_n)$ to P^0

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- How? using a "cut-and-paste" technique (homework)

Step 3. Defining an optimal value function

n=5, c=10					
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We have known that a subproblem is constrained by two parameters: number of items and capacities.

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- For a given subproblem with items $i, i+1, \cdots, n$ and capacities y, we define its optimal value by $f(i, y)$.

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- For a given subproblem with items $i, i+1, \cdots, n$ and capacities y, we define its optimal value by $f(i, y)$.
- $f(n, 0) = 0$, and $f(n, w_n) = p_n$ if $w_n < c$ else 0

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- How to get $f(1, c)$ from $f(n, y)$?

Step 4. Deriving the recursive equation

n=5, c=10					
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To calculate $f(i, y)$ from $f(i+1, y)$, we only need know <u>if item i could be included</u>

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  - if so, $f(i, y) = f(i+1, y - w_i) + p_i$
  - else, $f(i, y) = f(i+1, y)$, why?
  - finally, we use the larger one.

Step 5. Solve recursive equation

Now we have the recursive equation

$$f(i,y) = \begin{cases} f(i+1,y) & \text{if } 0 \leq y < w_i \\ \max \begin{cases} f(i+1, y-w_i) + p_i \\ f(i+1,y) \end{cases} & \text{if } y \geq w_j \end{cases} \tag{1}$$

and the initial condition

$$f(n,y) = \begin{cases} p_n & \text{if } y \geq w_n \\ 0 & \text{if } 0 \leq y < w_n \end{cases} \tag{2}$$

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Now we have the recursive equation

$$f(i,y) = \begin{cases} f(i+1,y) & \text{if } 0 \le y < w_i \\ \max \begin{cases} f(i+1,y-w_i)+p_i \\ f(i+1,y) \end{cases} & \text{if } y \ge w_i \end{cases} \tag{1}$$

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How to solve them?

- Recursive version
- Non-recursive version
- Tuple version

Recursive version

Algorithm 1: RKnapsack

Input: n,c,p[1..n],w[1..n]

Output: the optimal value $f(1, c)$

```
Function f(int i, int y)
    if (i==n) then
        return (y < w[n]?0 : p[n]);
    if (y < w[i]) then
        return f(i+1,y) ;
    return max(f(i+1,y),f(i+1,y-w[i])+p[i]) ;
```

Recursive version

Algorithm 2: RKnapsack

Input: n,c,p[1..n],w[1..n]

Output: the optimal value $f(1, c)$

```
1 Function f(int i, int y)
2     if (i==n) then
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4     if (y < w[i]) then
5         return f(i+1,y) ;
6     return max(f(i+1,y),f(i+1,y-w[i])+p[i]) ;
```

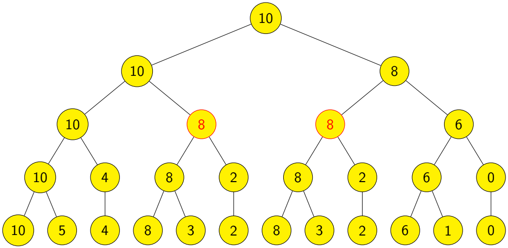
Algorithm complexity:
We use $t(n)$ to denote the algorithm time complexity with n items

- $t(1) = a$
- At the best case(step4), $t(n) = t(n-1) + b$, so $t(n) = \Theta(n)$
- At the worst case(step6), $t(n) = 2t(n-1) + b$, so $t(n) = \Theta(2^n)$

Recursive version: an example

n=5, c=10					
p=	6	3	5	4	6
w=	2	2	6	5	4
X	x_1	x_2	x_3	x_4	x_5

Recursive call relation tree



The number in the node is y value and the layer order corresponds to i, there are total 26 nodes;

If these repetitive calls are saved, the algorithm complexity should be reduced!

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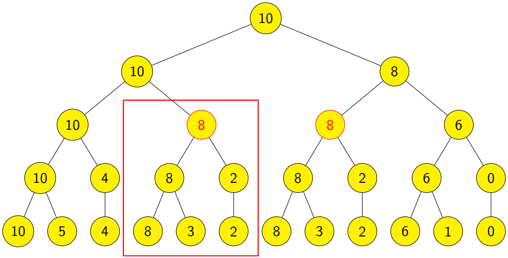

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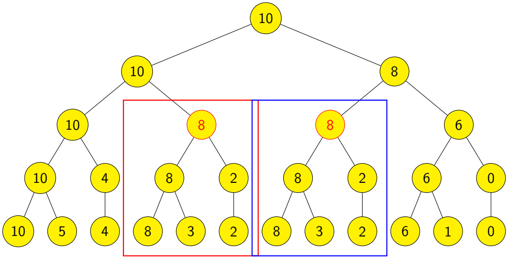

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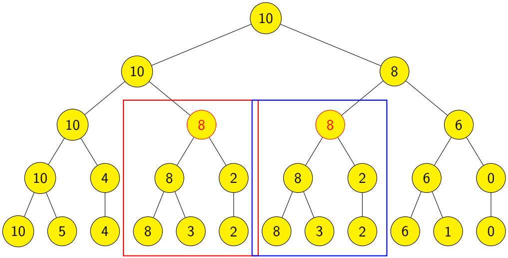

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If these repetitive calls are saved, the algorithm complexity should be reduced!

First attempt: using array

```
template<class T>
void Knapsack(T p[], int w[], int c, int n, T** f)
{// Compute f[i][y] for all i and y.

   // initialize f[n][]
   int yMax = min(w[n]-1,c);
   for (int y = 0; y <= yMax; y++)
      f[n][y] = 0;
   for (int y = w[n]; y <= c; y++)
      f[n][y] = p[n];

   // compute remaining f's
   for (int i = n - 1; i > 1; i--) {
      yMax = min(w[i]-1,c);
      for (int y = 0; y <= yMax; y++)
         f[i][y] = f[i+1][y];
      for (int y = w[i]; y <= c; y++)
         f[i][y] = max(f[i+1][y],
                       f[i+1][y-w[i]] + p[i]);
      }
   f[1][c] = f[2][c];
   if (c >= w[1])
      f[1][c] = max(f[1][c], f[2][c-w[1]] + p[1]);
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                       f[i+1][y-w[i]] + p[i]);
      }
   f[1][c] = f[2][c];
   if (c >= w[1])
      f[1][c] = max(f[1][c], f[2][c-w[1]] + p[1]);
}
```

- The algorithm saves all possible values using an array f, so that every $f(i, y)$ is calculated only once.
- It needs $\Theta(nc)$ extra space.
- Its time complexity is $\Theta(nc)$.
  - It is not polynomial: to describe c need $log_2 c$ bits
  - but pseudo-polynomial: exponential dependence on numerical inputs
- Its disadvantage
  - The capacity c must an integer
  - The complexity might still be very hight when c is large enough, for instance $c = 2^n$.

Second attempt: using a tuple

n=5, c=10					
p=	6	3	5	4	6
w=	2	2	6	5	4
X	x_1	x_2	x_3	x_4	x_5

- Calculate f values using array

Second attempt: using a tuple

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X	x_1	x_2	x_3	x_4	x_5

- Calculate f values using array
  - $f(5,0) = 0, \cdots, f(5,4) = 6, \cdots, f(5,10) = 6$

Second attempt: using a tuple

n=5, c=10	●	●	●	●	●
p=	6	3	5	4	6
w=	2	2	6	5	4
X	x_1	x_2	x_3	x_4	x_5

- Calculate f values using array
  - $f(5,0) = 0, \cdots, f(5,4) = 6, \cdots, f(5,10) = 6$
  - $f(4,0) = 0, \cdots, f(4,4) = 6, \cdots, f(4,9) = 10), f(4,10) = 10$

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  - $\cdots$

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  - $\cdots$
- Save step points only for each i

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  - $\cdots$
- Save step points only for each i
  - Define a tuple (a, b), where $a = y$ and $b = f(i, y)$

Second attempt: using a tuple

n=5, c=10	●	●	●	●	●
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- Calculate f values using array
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  - P_1 contains our goal! Why?

Tuple method: principle

- Let $Q = \{(s,t) | w_i \leq s < c, (s - w_i, t - p_i) \in P_{i+1}\}$

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- The complexity is still $O(2^n)$. The number of elements in P_i increase exponentially at worst case

Tuple method:an example

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The optimal value is 15 and the optimal solution is [1,1,0,0,1] by tracebacking

Outline

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Identifying subproblems

- The number of orders of multiplications equal to the number of parenthesization

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  - **MPC satisfies the principle of optimization**

Define the optimal function

Define $c(i, j)$ is the cost of $M_i \times M_{i+1} \times \cdots \times M_j$

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$$c(i,j) = min_{i \leq k < j}\{c(i,k) + c(k+1,j) + r_i * r_{k+1} * r_{j+1}\} \quad (3)$$

- Let kay(i,j) is the number minimizing above equation.
  - kay(i,i)=0
  - kay(i,i+1)=i

An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

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An exmple

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c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}

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c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}
c(3,5)=min	{c(3,3)+c(4,5)+100, c(3,4)+c(5,5)+20} min{300,40} =40

An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

c(1,5)=min	{c(1,1)+c(2,5)+500, c(1,2)+c(3,5)+100, c(1,3)+c(4,5)+1000, c(1,4)+c(5,5)+200}
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c(3,5)=min	{c(3,3)+c(4,5)+100, c(3,4)+c(5,5)+20} min{300,40} =40
c(2,4)=min	{c(2,2)+c(3,4)+10, c(2,3)+c(4,4)+100} min{30,150} =30

An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

c(1,5)=min	{c(1,1)+c(2,5)+500, c(1,2)+c(3,5)+100, c(1,3)+c(4,5)+1000, c(1,4)+c(5,5)+200}	
c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	
c(3,5)=min	{c(3,3)+c(4,5)+100, c(3,4)+c(5,5)+20} min{300,40} =40	
c(2,4)=min	{c(2,2)+c(3,4)+10, c(2,3)+c(4,4)+100} min{30,150} =30	c(2,4)=30,kay(2,4)=2

An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

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c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	
c(3,5)=min	{c(3,3)+c(4,5)+100, c(3,4)+c(5,5)+20} min{300,40} =40	c(3,5)=40,kay(3,5)=4
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Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

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c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	c(2,5)=90, kay(2,5)=2
c(3,5)=min	{c(3,3)+c(4,5)+100, c(3,4)+c(5,5)+20} min{300,40} =40	c(3,5)=40,kay(3,5)=4
c(2,4)=min	{c(2,2)+c(3,4)+10, c(2,3)+c(4,4)+100} min{30,150} =30	c(2,4)=30,kay(2,4)=2

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Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

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c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	c(2,5)=90, kay(2,5)=2
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An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

c(1,5)=min	{c(1,1)+c(2,5)+500, c(1,2)+c(3,5)+100, c(1,3)+c(4,5)+1000, c(1,4)+c(5,5)+200}	c(1,5)=190, kay(1,5)=2 c(1,3)=150, kay(1,3)=2 c(1,4)=90, kay(1,4)=2
c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	c(2,5)=90, kay(2,5)=2
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An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

	$M(1,5) = M(1,2) \times M(3,5)$	
c(1,5)=min	{c(1,1)+c(2,5)+500, c(1,2)+c(3,5)+100, c(1,3)+c(4,5)+1000, c(1,4)+c(5,5)+200}	c(1,5)=190, kay(1,5)=2 c(1,3)=150, kay(1,3)=2 c(1,4)=90, kay(1,4)=2
c(2,5)=min	{c(2,2)+c(3,5)+50, c(2,3)+c(4,5)+500, c(2,4)+c(5,5)+100}	c(2,5)=90, kay(2,5)=2
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An exmple

Suppose that q =5 and r =(10 , 5 , 1 , 10 , 2 , 10), find the optimal orders of multiplications

	$M(1,5) = M(1,2) \times M(3,5)$	$M(3,5) = M(3,4) \times M(5,5)$
c(1,5)=min	{c(1,1)+c(2,5)+500, c(1,2)+c(3,5)+100, c(1,3)+c(4,5)+1000, c(1,4)+c(5,5)+200}	c(1,5)=190, kay(1,5)=2 c(1,3)=150, kay(1,3)=2 c(1,4)=90, kay(1,4)=2
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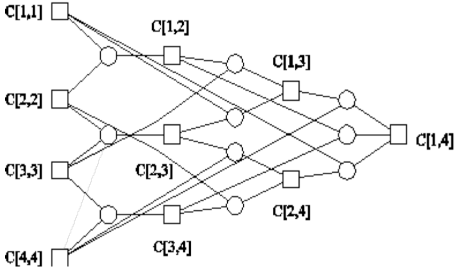
Recursive Algorithm for MPC

```
int RC(int i, int j)
{// Return c(i,j) and compute kay(i,j)=kay[i][
      j].
// Avoid recomputations,check if already
      computed
    if (c[i][j] > 0) return c[i][j];
// c[i][j] not computed before, compute now
    if (i == j) return 0;  // one matrix
    if (i == j - 1) {// two matrices
                    kay[i][i+1] = i;
                    c[i][j] = r[i]*r[i+1]*r[i
      +2];
                    return c[i][j];}
// more than two matrices
// set u to mini term for k = i
    int u = RC(i,i) + RC(i+1,j) + r[i]*r[i+1]*r
      [j+1];
    kay[i][j] = i;
// compute remaining min terms and update u
    for (int k = i+1; k < j; k++) {
       int t = RC(i,k) + RC(k+1,j) + r[i]*r[k
      +1]*r[j+1];
       if (t < u) {// smaller min term
                  u = t;
                  kay[i][j] = k;}
       }
    c[i][j] = u;
    return u;
}
```

```
void Traceback(int i, int j, int
      **kay)
{
   if (i == j) return;
   Traceback(i, kay[i][j], kay);
   Traceback(kay[i][j]+1, j, kay);
   cout << "Multiply M " << i << "
     , " << kay[i][j];
   cout << " and M " << (kay[i][j
     ]+1) << ", " << j
       << endl;
}
```

Revision recursive algorithm



The above figure suggest that $c(i,j)$ can be calculated in an iterative miner

$$c(i, i+s) = \min_{i \le k < s} \{c(i,k) + c(k,j) + r_i * r_k * r_{i+s+1}\}$$

$$s = 1, 2, \cdots, q$$

Iterative algorithm for MPC

```
void MatrixChain(int r[], int q, int **c, int **kay)
{//Compute costs and kay for all Mij's.
//initialize c[i][i],c[i][i+1],and kay[i][i+1]
   for (int i = 1; i < q; i++) {
      c[i][i] = 0;
      c[i][i+1] = r[i]*r[i+1]*r[i+2];
      kay[i][i+1] = i;
      }
   c[q][q] = 0;
//compute remaining c's and kay's
   for (int s = 2; s < q; s++)
      for (int i = 1; i <= q - s; i++) {
         // min term for k = i
         c[i][i+s] = c[i][i] + c[i+1][i+s]
                     + r[i]*r[i+1]*r[i+s+1];
         kay[i][i+s] = i;
         // remaining mini terms
         for (int k = i+1; k < i + s; k++) {
            int t = c[i][k] + c[k+1][i+s]
                     + r[i]*r[k+1]*r[i+s+1];
            if (t < c[i][i+s]) {// smaller mini term
               c[i][i+s] = t;
               kay[i][i+s] = k;}
            }
         }
}
```

Outline

Problem statement

- **Input**: Given a directed graph $G = (V, E)$ and a matrix (a_{ij}) where

$$V = \{1, 2, \cdots, n\} \text{with edge weight function} W : E \rightarrow R$$

$$a_{ij} = \begin{cases} w(i,j) & \text{if } (i,j) \in E \\ 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

- **Output**: A $n \times n$ matrix of shortest-path lengths $c(i,j)$
- **Assumption**: No negative-weight cycles

Subprobem identification by edges

- Define d_{ij}^{m} = weight of a shortest path from i to j that only uses at most m edges

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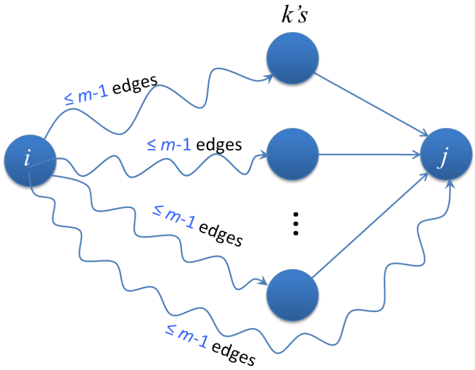
Theorem 7

For m $= 1, 2, \cdots, n-1$, we have

$$d_{ij}^{m} = \min_{k}\{d_{ik}^{m-1} + a_{kj}\}$$

Proof



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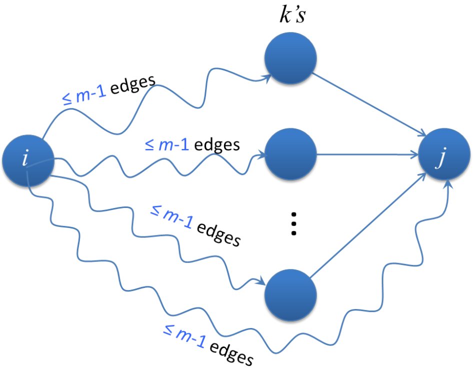


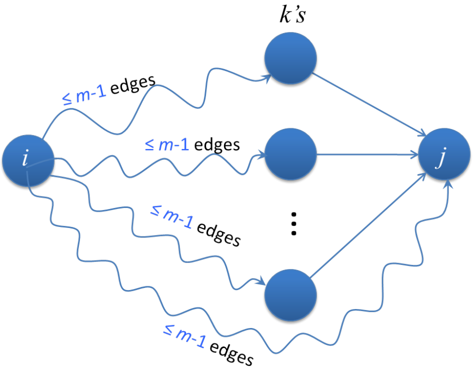
Proof

$$d_{ij}^{m} = \min_{k}\{d_{ik}^{m-1} + a_{kj}\}$$

for k=1 to n

if $d_{ij} > d_{ik} + a_{kj}$

$d_{ij} = d_{ik} + a_{kj}$



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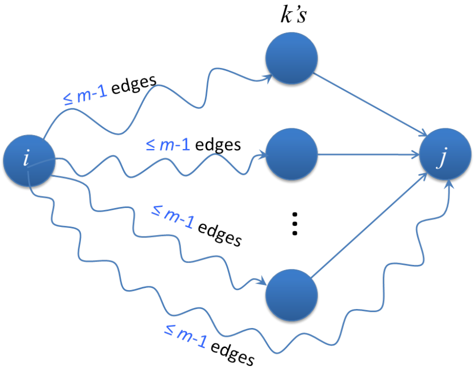
for k=1 to n

$\quad$ *if* $d_{ij} > d_{ik} + a_{kj}$

$\quad\quad d_{ij} = d_{ik} + a_{kj}$



Running time $O(n^4)$ - similar to n runs of Bellman-Ford algorithm
The Bellman–Ford algorithm is an algorithm that computes shortest paths from a single source vertex to all of the other vertices in a weighted digraph.

Subprolems identification by intermediate vertices

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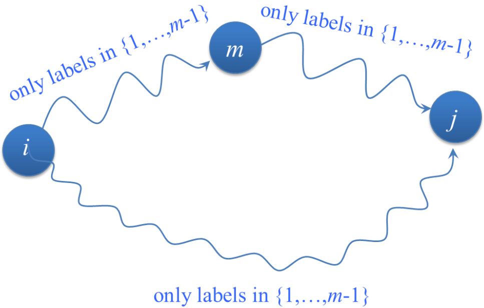
  - $d_{kj}^{k-1} = d_{kj}^{k}$

Theorem 16

For m = 1, 2, $\cdots$, n-1, we have

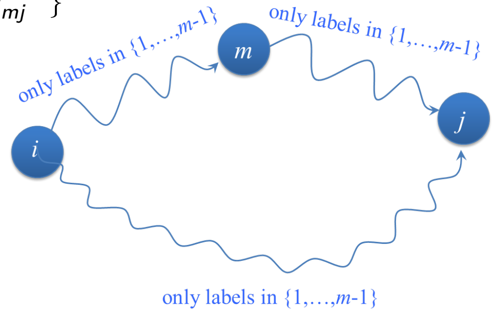
$$d_{ij}^{m} = \min\{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$$

Proof



Proof

$$d_{ij}^{m} = \min\{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$$



Running time $O(n^3)$ Known as Floyd-Warshall algorithm

Iterative algorithm for ASAP

```
template<class T>
void AdjacencyWDigraph<T>::AllPairs(T **c, int **kay)
{// All pairs shortest paths.
 // Compute c[i][j] and kay[i][j] for all i and j.
   // initialize c[i][j] = c(i,j,0)
   for (int i = 1; i <= n; i++)
      for (int j = 1; j <= n; j++) {
         c[i][j] = a[i][j];
         kay[i][j] = 0;
         }
   for (int i = 1; i <= n; i++)
      c[i][i] = 0;

   // compute c[i][j] = c(i,j,k)
   for (int k = 1; k <= n; k++)
      for (int i = 1; i <= n; i++)
         for (int j = 1; j <= n; j++) {
            T t1 = c[i][k];
            T t2 = c[k][j];
            T t3 = c[i][j];
            if (t1 != NoEdge && t2 != NoEdge &&
                (t3 == NoEdge || t1 + t2 < t3)) {
                  c[i][j] = t1 + t2;
                  kay[i][j] = k;}
            }
}
```

Kay[i][j] is used to store the largest vertex in the shortest path from i to j, so that traceback the shortest path

```
void outputPath(int **kay, int i, int j)
{// Actual code to output i to j path.
   if (i == j) return;
   if (kay[i][j] == 0) cout << j << ' ';
   else {outputPath(kay, i, kay[i][j]);
         outputPath(kay, kay[i][j], j);}
}

template<class T>
void OutputPath(T **c, int **kay, T NoEdge,
                                int i, int j)
{// Output shortest path from i to j.
   if (c[i][j] == NoEdge) {
      cout << "There is no path from " << i << " to "
           << j << endl;
      return;}
   cout << "The path is" << endl;
   cout << i << ' ';
   outputPath(kay,i,j);
   cout << endl;
}
```

An example

Adjacent matrix

0	1	4	4	8
3	0	1	5	9
2	2	0	1	8
8	8	9	0	1
8	8	2	9	0

Distance matrix of shortest path

0	1	2	3	4
3	0	1	2	3
2	2	0	1	2
5	5	3	0	1
4	4	2	3	0

matrix kay_{ij}

0	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$
0	0	$\boxed{0}$	3	4
0	0	0	$\boxed{0}$	4
5	5	5	0	$\boxed{0}$
3	3	0	3	0

- we can traceback the shortest paths from matrix (kay_{ij}), for example, from 1 to 5
  - kay(1,5)=4 , kay(4,5) =0 , $4 \rightarrow 5$ is a sub-path

An example

Adjacent matrix

0	1	4	4	8
3	0	1	5	9
2	2	0	1	8
8	8	9	0	1
8	8	2	9	0

Distance matrix of shortest path

0	1	2	3	4
3	0	1	2	3
2	2	0	1	2
5	5	3	0	1
4	4	2	3	0

matrix kay_{ij}

0	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$
0	0	$\boxed{0}$	3	4
0	0	0	$\boxed{0}$	4
5	5	5	0	$\boxed{0}$
3	3	0	3	0

- we can traceback the shortest paths from matrix (kay_{ij}), for example, from 1 to 5
  - kay(1,5)=4 , kay(4,5) =0 , 4 $\rightarrow$ 5 is a sub-path
  - kay(1,4)=3, kay(3,4)=0, 3 $\rightarrow$ 4 is a sub-path

An example

Adjacent matrix

0	1	4	4	8
3	0	1	5	9
2	2	0	1	8
8	8	9	0	1
8	8	2	9	0

Distance matrix of shortest path

0	1	2	3	4
3	0	1	2	3
2	2	0	1	2
5	5	3	0	1
4	4	2	3	0

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An example

`Adjacent matrix`

$$\begin{matrix} 0 & 1 & 4 & 4 & 8 \\ 3 & 0 & 1 & 5 & 9 \\ 2 & 2 & 0 & 1 & 8 \\ 8 & 8 & 9 & 0 & 1 \\ 8 & 8 & 2 & 9 & 0 \end{matrix}$$

`Distance matrix of shortest path`

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`matrix` kay_{ij}

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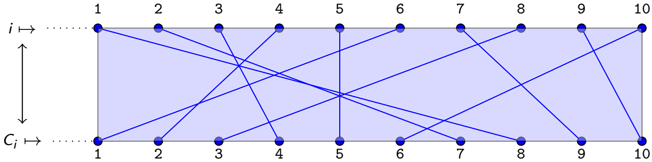
  - The final shortest path is $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$

Outline

Problem statement



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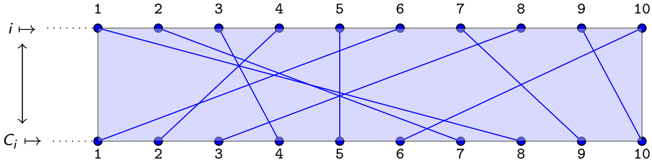

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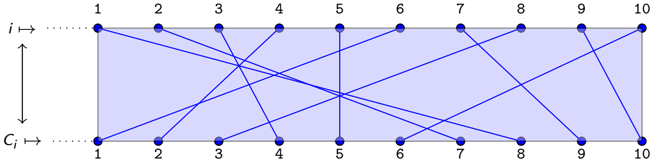

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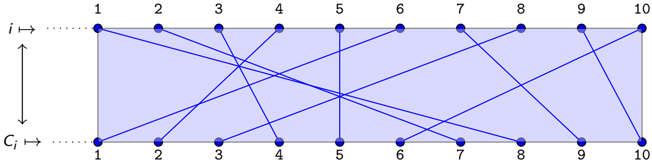

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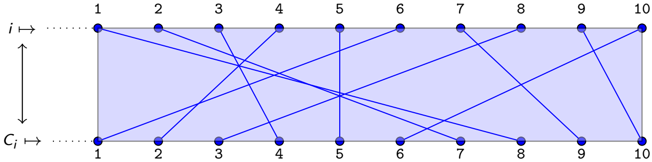

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- Define size(i,j) $= |MNS(i, j)|$

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    - finally, we choose the maximum of them!

$$size(i,j) = \begin{cases} size(i-1,j) & \text{if } j < c_i \\ \max\{size(i-1,j), size(i-1, c_i - 1) + 1\} & \text{if } j \geq c_i \end{cases}$$

Outline

Problem statements

Definition 1: Subsequence

Given a sequence X=$x_1x_2\cdots x_m$, another sequence Z=$z_1z_2\cdots z_k$ is a subsequence of X if there exists a strictly increasing sequence= $i_1x_2\cdots i_k$ of indices of X such that for all j=1,2,...k, we have $x_{i_j} = z_j$.

Example 1: If X=abcdefg, Z=abdg is a subsequence of X.

Definition 2: Common subsequence

Given two sequences X and Y, a sequence Z is a common subsequence of X and Y if Z is a subsequence of both X and Y.

Example 2: X=abcdefg and Y=aaadgfd. Z=adf is a common subsequence of X and Y

Definitions

Definition 3: Longest common subsequence:LCS

A longest common subsequence of X and Y is a common subsequence of X and Y with the longest length.

- Longest common subsequence may not be unique, for example, strings both acd and abd are LCS of abcd and acbd

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  - DP approach: O(nm)

DP approach for LCS

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3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y[1..m-1].

Recursive equation

By the theorem, we can easily get the recursive equation

$$c[i,j] = \begin{cases} 0 & \text{if i=0 or j=0} \\ c[i-1,j-1]+1 & \text{if x[i]=y[j]} \\ \max\{c[i-1,j], c[i,j-1]\} & \text{otherwise} \end{cases} \tag{6}$$

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```
LCS(X,Y,m,n,b)
for i=1 to m do
    c[i,0]=0;
for j=0 to n do
    c[0,j]=0;
for i=1 to m  do
    for j=1 to n do
    {//b[i,j] stores the directions.
    if x[i] ==y[j]  then
        c[i,j]=c[i-1,j-1]+1;
        b[i,j]=1;  //1-diagonal,
    else if c[i-1,j]>=c[i,j-1] then
            c[i,j]=c[i-1,j]
            b[i,j]=2;//2-up,
        else c[i,j]=c[i,j-1]
             b[i,j]=3; //3-forward.
    }
```

LCS algorithm

```
1   PrintLCS(b,X,i,j)
2   i=m
3   j=n;
4   if i==0 or j==0 then exit;
5   if b[i,j]==1 then
6   {
7         i=i-1;
8         j=j-1;
9         print x[i];
10   }
11  if b[i,j]==2  i=i-1
12  if b[i,j]==3  j=j-1
13  Goto Step 3.
```

Print LCS algorithm

Coming up: Backtracking Algorithm

Chapter 05: Backtracking Algorithm
Design and Analysis of Computer Algorithms

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School of Computer Science and Technology, Tianjin University

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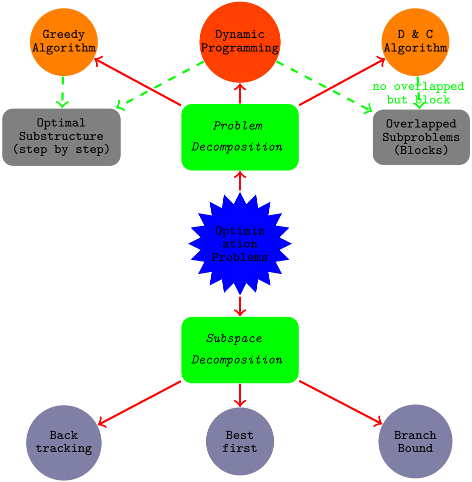
October 29, 2019

Outline

Motivations



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    2. For VLT: $X = \{X^1, X^2, \cdots, X^k\}$ where $X^j = \{(x_{j1}, x_{j2}, \cdots, x_{jk}), k \in [1..n]\}$

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  1. Define solution representations
  2. Formulate solution space
  3. Search solution space

Define solution representations

Eight queens puzzle

Using a regular chess board, the challenge is to place eight queens on the board such that no queen is attacking any of the others.

	1	2	3	4	5	6	7	8
1				Q				
2						Q		
3								Q
4		Q						
5							Q	
6	Q							
7			Q					
8					Q			

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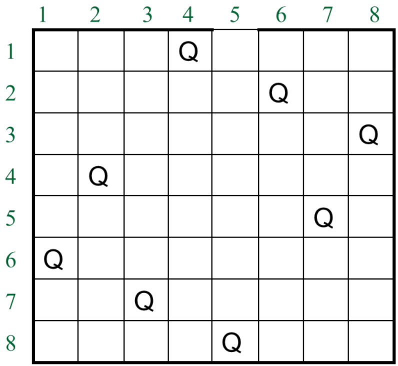

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- The size of solution space is 8^8
- Constrains: $x_i \neq x_j$ and $|x_i - x_j| \neq |j - i|$ for all i, j

Define solution representations

Subset sum problem

Finding what subset of a set of positive integers $S = \{w_1, w_2, \cdots, w_n\}$ has a given sum M,

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VLT$\mapsto$	$(j_1, j_2, \cdots, j_k)$ where j_i is the order number of i-th integer chosen in S and k is the total number chosen	(1,2,4) (3,4)	$O(2^n)$

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Definition

The state space of a problem is a 4-tuple (N, A, S, G) where:

- N is a set of problem states
- A is a set of arcs connecting the states
- S is a nonempty subset of N that contains start states
- G is a nonempty subset of N that contains the goal states.
- Our goal is to find paths states from S to G

State space tree

State space tree is the representation of state space in the form of tree structures

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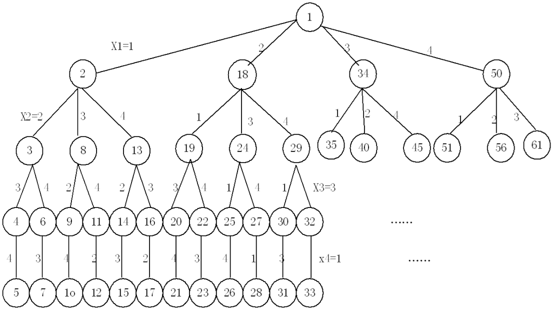
4-queens puzzle

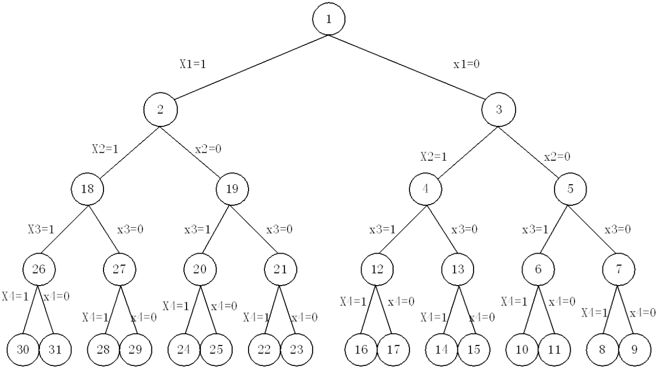


- Number inside a node is the order of depth first searching the tree;
- Edge label is x_i and i is the depth of tree
- This kind of tree is called Permutation Tree.

State space tree

State space tree is the representation of state space in the form of tree structures

Subset sum using FLT representation



M=31, n=4, and W=(11,13,24,7)
subset tree

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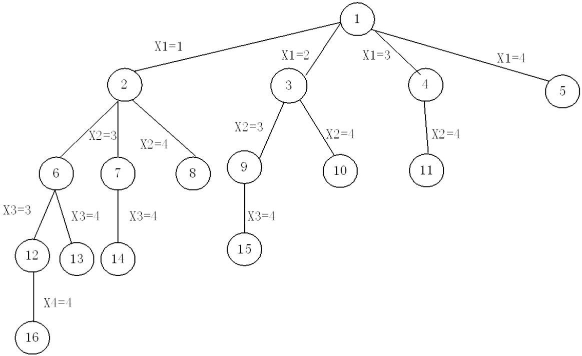
Subset sum using VLT representation



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Bounding is a boolean function to kill a live node

Backtrack algorithm

Algorithm 1: Backtrack Algorithm

```
Function backtrack(int n)
    k=1;
    while k > 0 do
        forall x[k] ∈ T(X(1),··· ,X(k-1)) do
            if not B(X(1), ··· , X(k)) then
                if (X(1), ··· , X(k)) is an answer then
                    print (X(1), ··· , X(k)) ;
                k=k+1 /*loop next */
            else
                k =k-1 /*backtrack */
```

- T(X(1),$\cdots$,X(k-1)) is a set containing all possible values x(k), given X(1),$\cdots$,X(k-1)
- B(X(1),$\cdots$, X(k)) judge whether X(k) satisfies constrains
- Solution is store in X(1:n) , once it is decided, output it

Bounding function for Subset sum problem

- Simple bounding: B(X(1),$\cdots$,X(k))=true iff

$$\sum_{i=1}^{k} W(i)X(i) + \sum_{i=k+1}^{n} W(i) < M \tag{1}$$

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- Tighter bounding: B(X(1),$\cdots$,X(k))=true iff (1) and

$$\sum_{i=1}^{k} W(i)X(i) + W(k+1) > M \tag{2}$$

when sorting W(i) by non-decreasing order

Subset sum algorithm with bounding

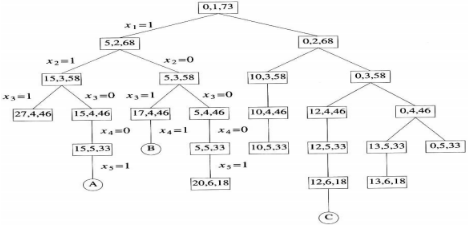
Algorithm 2: Subset sum problem: pseudo code

```
Let s=w(1)x(1) + ··· + w(k−1)x(k−1) ;
    r = w(k) + ··· + w(n), assumed s+r ≥ M ;
Expanding left child node ;
if S + W(k) > M then
    stop expanding ;
    r=r-w(k) ;
    Expanding right child node ;
else
    x(k)=1 ;
    s=s+w(k);
    r=r-w(k) ;
    let (x(1), ··· , x(k)) be E-Node ;
Expanding right child node;
if s+r <M or s+w(k+1) >M then
    stop expanding
else
    x(k)=0
```

State space tree with bounding

Subset sum problem

M=30,n=6 and w=(5,10,12,13,15,18)
UPDATED



Numbers in a rectangle node corresponds to s, k and r values respectively, Circle nodes correspond to answer states, There are only 23 nodes, but $2^7 - 1 = 63$ nodes without bounding

Outline

Problem statement

- Given two ships with capacities c_1 and c_2, and n containers with weights $(w_1, \cdots, w_n)$, such that

$$\sum_{i=1}^{n} w_i \leq c_1 + c_2 \tag{3}$$

- We wish to determine whether there is a way to load all n containers into shipws without sinking.

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Note that

- When $\sum_{i=1}^{n} w_i = c_1 + c_2$, it is equivalent to the sum-of-subset problem
- When $c_1 = c_2$, it is equivalent to the partition problem

Solution

1. load the first ship as close to its capacity as possible and
2. put the remaining containers into the second ship.

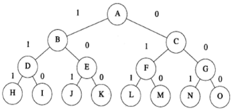
To load the first ship as close to capacity as possible, we need to select a subset of containers with total weight as close to c_1 as possible.

Using fixed length tuple $x = (x_1, \cdots, x_n)$ ($x_i = 1$, if container i is loaded) as solution space representation, just need

$$\max \sum_{i=1}^{n} w_i x_i \tag{4}$$

An example of state space tree is shown below.

n= 4,w= [8 , 6 , 2 , 3], c1 = 12



Bound function

- Suppose node i-1 is the E-node, let

$$cw = \sum_{j=1}^{i-1} x_j w_j$$ total weight of containers loaded already

$$r = \sum_{j=i}^{n} w_j$$ total weight of unloaded containers

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3. Above two bounding functions can be used at same time

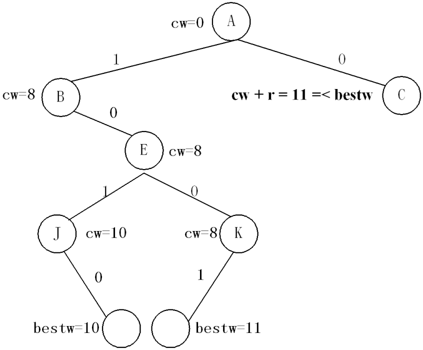
Backtrack algorithm for TSLP

```
template<class T>
T MaxLoading(T w[], T c, int n, int
      bestx[])
{// Return best loading and its value.
   Loading<T> X;
   // initialize X
   X.x = new int [n+1];
   X.w = w;
   X.c = c;
   X.n = n;
   X.bestx = bestx;
   X.bestw = 0;
   X.cw = 0;
   // initial r is sum of all weights
   X.r = 0;
   for (int i = 1; i <= n; i++)
      X.r += w[i];
   X.maxLoading(1);
   delete [] X.x;
   return X.bestw;
}
```

```
template<class T>
void Loading<T>::maxLoading(int i)
{// Search from level i node.
   if (i > n) {// at a leaf
      for (int j = 1; j <= n; j++)
         bestx[j] = x[j];
      bestw = cw; return;}
   // check subtrees
   r -= w[i];
   if (cw + w[i] <= c) {// try x[i] =
      1
      x[i] = 1;
      cw += w[i];
      maxLoading(i+1);
      cw -= w[i];}
   if (cw + r > bestw) {// try x[i] =
      0
      x[i] = 0;
      maxLoading(i+1);}
   r += w[i];
}
```

State space tree with bounding

n= 4, w= [8 , 6 , 2 , 3], c = 12



Outline

Bound function for 0/1 Knapsack

- Suppose that items are sorted in non-decreasing miner of p/w and node k-1 is the E-node, let

$$cp = \sum_{j=1}^{k-1} x_j p_j$$ profit of current packing

$$rp = \sum_{j=k}^{n} p_j$$ total profit of remain items

$$bestp =$$ max profit so far

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$bestp =$	max profit so far

- Bound(x) =true if cp+rp $\leq$ bestp

Backtrack algorithm for Knapsack

```
template<class Tw, class Tp>
void Knap<Tw, Tp>::Knapsack(int i)
{// Search from level i node.
   if (i > n) {// at a leaf
      bestp = cp;
      return;}
   // check subtrees
   if (cw + w[i] <= c) {// try x[i] = 1
      cw += w[i];
      cp += p[i];
      Knapsack(i+1);
      cw -= w[i];
      cp -= p[i];}
   if (Bound(i+1) > bestp) // try x[i] = 0
      Knapsack(i+1);
}
```

```
template<class Tw, class Tp>
Tp Knap<Tw, Tp>::Bound(int i)
{// Return upper bound on value of
 // best leaf in subtree.
   Tw cleft = c - cw;  // remaining capacity
   Tp b = cp;          // profit bound
   // fill remaining capacity
   // in order of profit density
   while (i <= n && w[i] <= cleft) {
      cleft -= w[i];
      b += p[i];
      i++;
      }

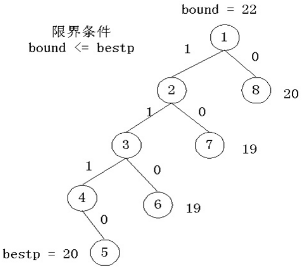
   // take fraction of next object
   if (i <= n) b += p[i]/w[i] * cleft;
   return b;
}
```

NEW

Assumed that items have been sorted by the profit density $\left(\frac{p[i]}{w[i]}\right)$

State space tree with bounding

$$n = 4,\ c = 7,\ p = [\,9\ ,\ 10\ ,\ 7\ ,\ 4\,], w = [\,3\ ,\ 5\ ,\ 2\ ,\ 1\,]$$



Outline

Max Clique: problem statements

A subgraph $G' =< V', E' >$ is a complete subgraph of an undirected graph $G =< V, E >$, if and only if $V' \subset V$ and for $\forall u \in V', \forall v \in V'$, $(u, v) \in E' \subset E$.

A clique is a complete subgraph of G if no larger inclusion of other complete subgraphs.

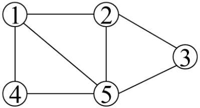
A indepedent vertex set is a subgraph of G with empty edges if no larger inclusion of other independent vertex sets.

A max clique is a clique of the largest possible size in a given graph.

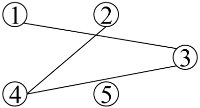
A max indepedent vertex set is a independent vertex set of the largest possible size in a given graph.

An example

{1,2} is a compete subgraph, but not a clique
{1,2,5} {1,4,5} {2,3,5} are max cliques
{2,4} is a max independent vertex set



{1,2} is a empty subgraph, but not a independent vertex set
~~{2,3}~~ {1,2,5} are independent vertex sets
{1,2,5} is also a max independent vertex sets



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    1. B(x)=true if the vertexes from root to i can not form a complete subgraph
    2. B(x)=true if the number of vertexes from root to i plus remained vertexes is no larger than bestn

Backtrack algorithm for maxclique

```
int AdjacencyGraph::MaxClique(int
      v[])
{// Return size of largest clique.
 // Return clique vertices in v[1:
      n].
   // initialize for maxClique
   x = new int [n+1];
   cn = 0;
   bestn = 0;
   bestx = v;

  // find max clique
   maxClique(1);

   delete [] x;
   return bestn;
}
```

```
void AdjacencyGraph::maxClique(int i)
{// Backtracking code to compute largest clique.
   if (i > n) {// at leaf
      // found a larger clique, update
      for (int j = 1; j <= n; j++)
         bestx[j] = x[j];
      bestn = cn;
      return;}
   // see if vertex i connected to others
   // in current clique
   int OK = 1;
   for (int j = 1; j < i; j++)
      if (x[j] && a[i][j] == NoEdge) {
         // i not connected to j
         OK = 0;
         break;}

   if (OK) {// try x[i] = 1
      x[i] = 1;  // add i to clique
      cn++;
      maxClique(i+1);
      x[i] = 0;
      cn--;}

   if (cn + n - i > bestn) {// try x[i] = 0
      x[i] = 0;
      maxClique(i+1);}
}
```

Outline

TSP

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- It is a minimization problem starting and finishing at a specified vertex after having visited each other vertex exactly once.

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  1. B(i)=true if no edge connection between x[i] and x[i-1]
  2. B(i)=true if route distance from root to x[i] is larger than bestc (bestc is the shortest route distance so far)

Backtrack algorithm for maxclique

```
void AdjacencyWDigraph<T>::tSP(int i)
{// Backtracking code for traveling salesperson.
   if (i == n) {// at parent of a leaf
      // complete tour by adding last two edges
      if (a[x[n-1]][x[n]] != NoEdge &&
         a[x[n]][1] != NoEdge &&
         (cc + a[x[n-1]][x[n]] + a[x[n]][1] < bestc
      ||
         bestc == NoEdge)) {// better tour found
         for (int j = 1; j <= n; j++)
            bestx[j] = x[j];
         bestc = cc + a[x[n-1]][x[n]] + a[x[n
      ]][1];}
      }
   else {// try out subtrees
      for (int j = i; j <= n; j++)
         // is move to subtree labeled x[j]
      possible?
         if (a[x[i-1]][x[j]] != NoEdge &&
               (cc + a[x[i-1]][x[i]] < bestc ||
                bestc == NoEdge)) {// yes
            // search this subtree
            Swap(x[i], x[j]);
            cc += a[x[i-1]][x[i]];
            tSP(i+1);
            cc -= a[x[i-1]][x[i]];
            Swap(x[i], x[j]);}
      }
}
```

```
template<class T>
T AdjacencyWDigraph<T>::TSP(int
      [])
{// Traveling salesperson by
      backtracking.
 // Return cost of best tour,
      return tour in v[1:n].
   // initialize for tSP
   x = new int [n+1];
   // x is identity permutation
   for (int i = 1; i <= n; i++)
      x[i] = i;
   bestc = NoEdge;
   bestx = v;  // use array v t
      store best tour
   cc = 0;

   // search permutations of x[
      ]
   tSP(2);

   delete [] x;
   return bestc;
}
```

Coming up: Branch & Bound Algorithm

Chapter 06: Branch & Bound Algorithm
Design and Analysis of Computer Algorithms

GONG Xiu-Jun

School of Computer Science and Technology, Tianjin University

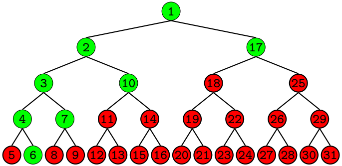
Email: gongxj@tju.edu.cn

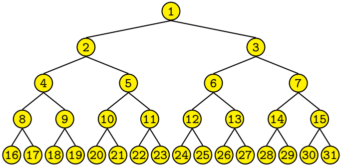
November 5, 2019

Outline

Motivations

Let's consider the 0/1 knapsack problem, its state space can be expanded in two ways using fixed length tuple representation of solution space: (n=4,c=7,p=[4,7,9,10], w=[1,2,3,5])





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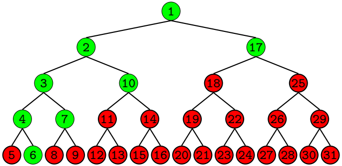

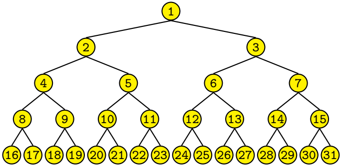

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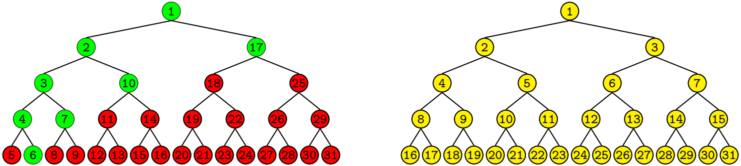

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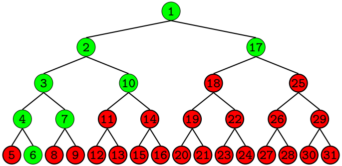

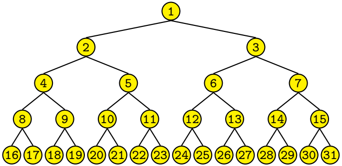

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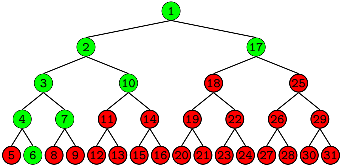

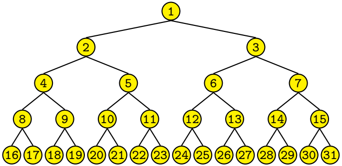

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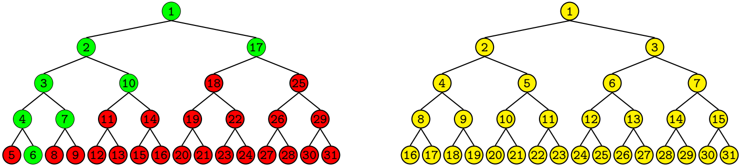

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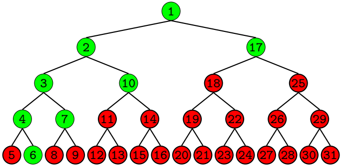

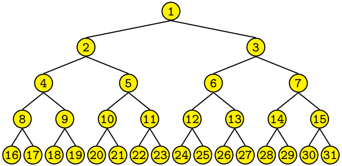

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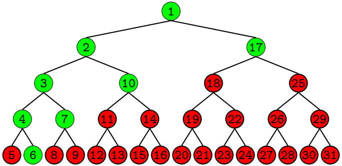

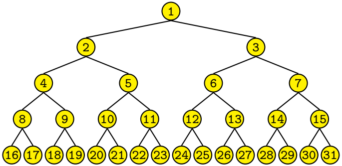

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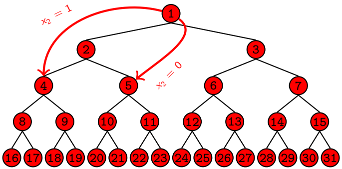
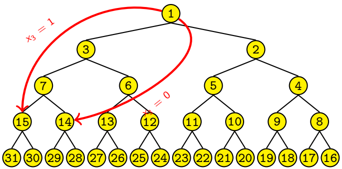
- How to manage the live nodes and How to bound?

Managing live nodes





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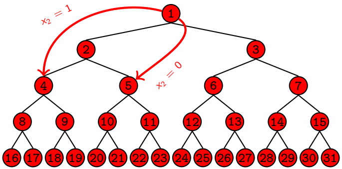

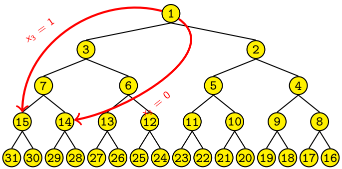

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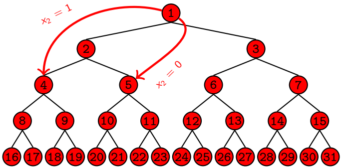

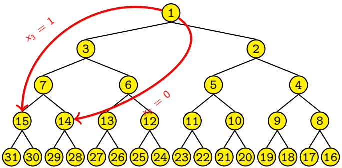

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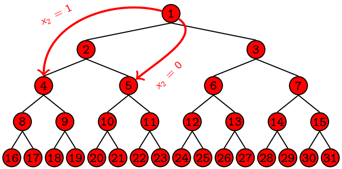

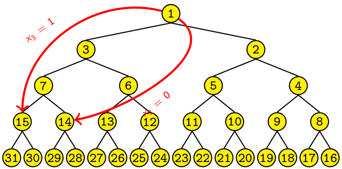

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we need bound solution states so that the best possible solutions can be reached and fruitless candidates are discarded quickly .
How to?

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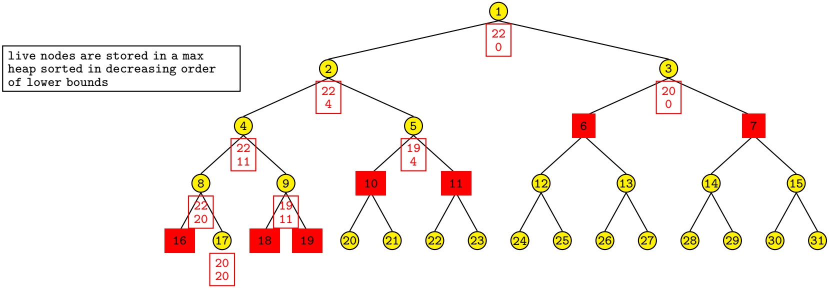
    2. for two states A and B, if A's upper bound is no larger than B's lower bound, then A should be discarded

State space tree with lower/upper bounds

(n=4,c=7,p=[4,7,9,10], w=[1,2,3,5])



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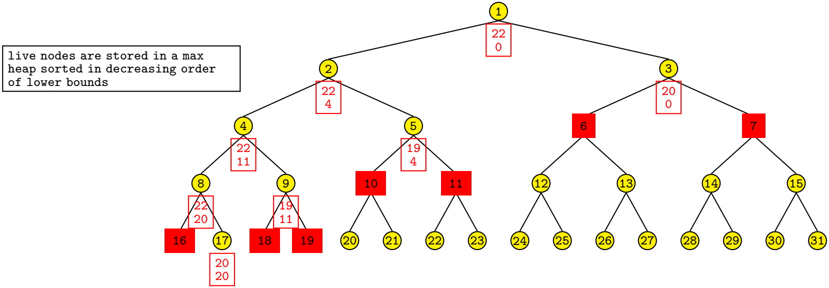

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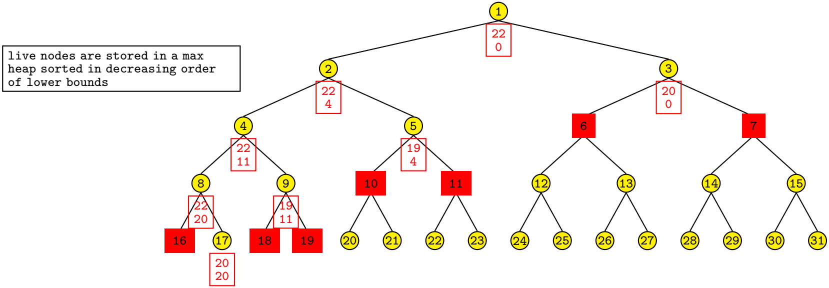

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Problem statements

- Branch & Bound (BB) is an enhancement of backtracking for searching solution space with bounding, respect to a global optimization problem(min/max, for simplicity, just consider the min problem)

$$\min_{x \in S} f(x) \tag{1}$$

where x=$(x_1, x_2, \cdots, x_n)$, $x \in S_i$, $S = S_1 \cup S_2, \cdots, \cup S_n$, and each S_i is a limited set. Usually , x is constrained with

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  3. The recursion stops when the current candidate set S is reduced to a single element, or when the upper bound for set S matches the lower bound.

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Hard point to B&B

I have always wanted to prove a lower bound about the behavior of branch and bound, but I never could.

–One of the mysteries of computational theory

–George Nemhauser, DECEMBER 19, 2012

Outline

Problem statements

Given n jobs and 1 processor, each job j_i has a 3-tuple $(p_i; d_i; t_i)$ associated with it, where t_i is the number of units of processing time for job j_i, p_i is a penalty if processing is not completed by deadline d_i.

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The problem can be formulated as

$$\min_{x[1..n]} \sum_{i=1}^{n} (1 - x_i) * p_i \tag{4}$$

$$\textbf{Subject to:}\ t_i \leq d_i \tag{5}$$

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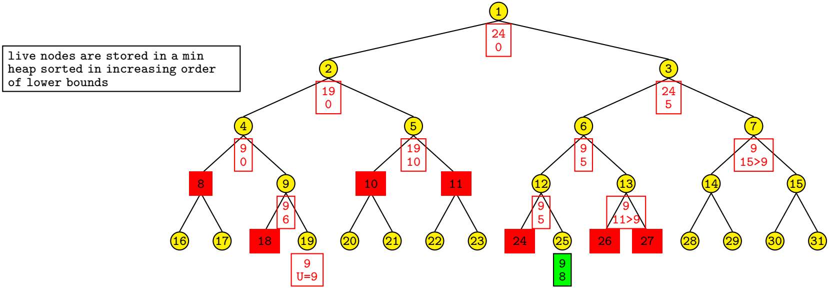
  3. Enhanced upper bound: using $U(i)$ to update a global U such that keeping it as the smallest upper bound

An example

$$(\text{n}{=}4,\ (p,d,t)_1^4 = \{(5,1,1),(10,3,2),(6,2,1),(3,1,1)\})$$



Every time the machine is freed or a new job is released, pick the uncompleted job with minimum due date.

Outline

The basic idea

Problem statement of TSP can refer to previous lecture.
For B & B solutions of TSP, we need consider:

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  2. the lower bound can be obtained using reduction matrix approach

Matrix reduction

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Row reduction: Let $r_i = \min_{j=1}^{n}\{A(i,j)\}$ and $r = \sum_{i=1}^{n} r_i$, then $R = (A(i,j)) - (r_i\_)$ is the row reduced matrix of A, r is the row reduced number.

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∞	20	30	10	11	-10
15	∞	16	4	2	-2
3	5	∞	2	4	-2
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16	4	7	16	∞	-4
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$\Rightarrow$

row reduced matrix

∞	10	20	0	1	
13	∞	14	2	0	
1	3	∞	0	2	
16	3	15	∞	0	
12	0	3	12	∞	
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12 & \infty & 11 & 2 & 0 \\
0 & 3 & \infty & 0 & 2 \\
15 & 3 & 12 & \infty & 0 \\
11 & 0 & 0 & 12 & \infty \\
 & & & & \boxed{25}
\end{array}
$$

Lower bound estimation

Let $f = (e_1, e_2, \cdots, e_n)$ is a TSP tour of the undirected weighted graph $G =< V, E >$, A= $(A(i,j))$ is its cost matrix, and e_i is the edge from i-th row.

Theorem

if A is reduced to B by row number r ***and B is reduced to C by column number*** c***, then***
cost(f)= cost(f,C)+ $r + c \geq r + c$

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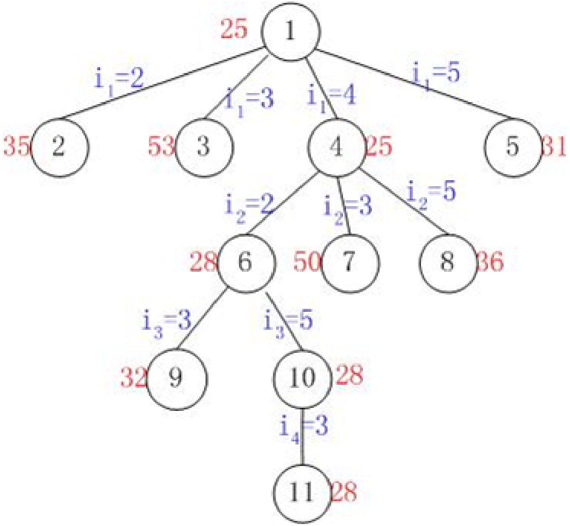
We use the reduced number $r + c$ as the estimation of lower bounds.

Let S be a sub node of R and (i,j) a edge from i to j:

- If S is non-leaf node, then $LB(S) = LB(R) + R(i,j) + rn$, where rn is the reduced number of S by
  1. Let all numbers in i-th row and j-th column of R be ∞
  2. Let R(i,j) be ∞
  3. Let S=R
- If S is leaf node, $LB(S) = L(S) + A(S, root)$

An example of state space tree



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  1. In B&B, the node with the best (shortest) path that you've found so far is expanded first.
  2. In A-star, the node with the shortest estimated total length from start to goal, where the total length is estimated as length so far plus a heuristic estimate of the remaining distance to the goal, is expanded first.

Properties of A-star

In practice, how to estimate h(x) to its real value $h^*(x)$ is a hard point

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Theorem

A-star is the optimal if h holds admissibility and monotonicity.

An example: 8 puzzle problem

Problems statements: The 8 puzzle is a simple game which consists of eight sliding tiles, numbered by digits from 1 to 8, placed in a 3×3 squared board of nine cells.

- One of the cells is always empty
- Any adjacent (horizontally and vertically) tile can be moved into the empty cell
- The objective of the game is to start from an initial configuration and end up in a configuration which the tiles are placed in ascending number order.

2	8	3
1	6	4
7		5

$\Longrightarrow$

1	2	3
8		4
7	6	5

Define solution representation

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$\Longrightarrow$

1	2	3
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- Encoding a configuration with 8-dimension vector $(x_1, x_2, \cdots, x_8)$.
- If number i is in its right position, $x_i = 0$, else $x_i = 1$
- The initial state is $(1, 1, 0, 0, 0, 1, 0, 1)$
- The goal state is $(0, 0, 0, 0, 0, 0, 0, 0)$
- The size of solution space is 2^8

Design a heuristic cost function

2	8	3
1	6	4
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$\Longrightarrow$

1	2	3
8		4
7	6	5

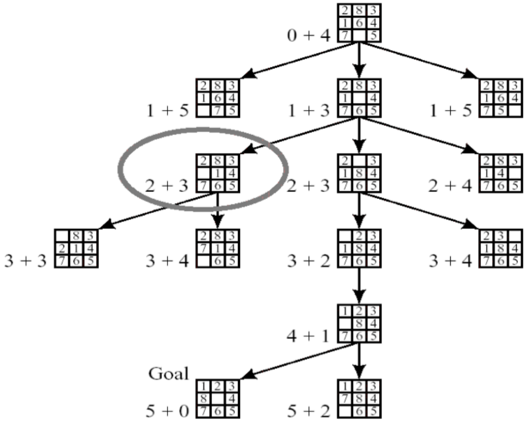
For a given state x, we need a heuristic cost function

$$\hat{f}(x) = \hat{g}(x) + \hat{h}(x) \tag{6}$$

Where

- $\hat{g}(x)$: the number of moves from initial state to state x.
- $\hat{h}(x)$: the number of tiles out of place (compared with the goal state) (Hamming priority function)

State space tree



Pseudo code of A-star algorithm

```
function A*(start,goal)
    closedset := the empty set    // The set of nodes already evaluated.
    openset := {start}    // The set of tentative nodes to be evaluated, initially containing the start node
    came_from := the empty map    // The map of navigated nodes.
    g_score[start] := 0    // Cost from start along best known path.
    // Estimated total cost from start to goal through y.
    f_score[start] := g_score[start] + heuristic_cost_estimate(start, goal)
    while openset is not empty
        current := the node in openset having the lowest f_score[] value
        if current = goal
            return reconstruct_path(came_from, goal)
        remove current from openset
        add current to closedset
        for each neighbor in neighbor_nodes(current)
            if neighbor in closedset
                continue
            tentative_g_score := g_score[current] + dist_between(current, neighbor)

            if neighbor not in openset or tentative_g_score < g_score[neighbor]
                came_from[neighbor] := current
                g_score[neighbor] := tentative_g_score
                f_score[neighbor] := g_score[neighbor] + heuristic_cost_estimate(neighbor, goal)
                if neighbor not in openset
                    add neighbor to openset
    return failure
function reconstruct_path(came_from, current_node)
    if current_node in came_from
        p := reconstruct_path(came_from, came_from[current_node])
        return (p + current_node)
    else
        return current_node
```

Coming up: NP Complete Problems

Chapter 07: Non-deterministic Polynomial Complete Problems

Design and Analysis of Computer Algorithms

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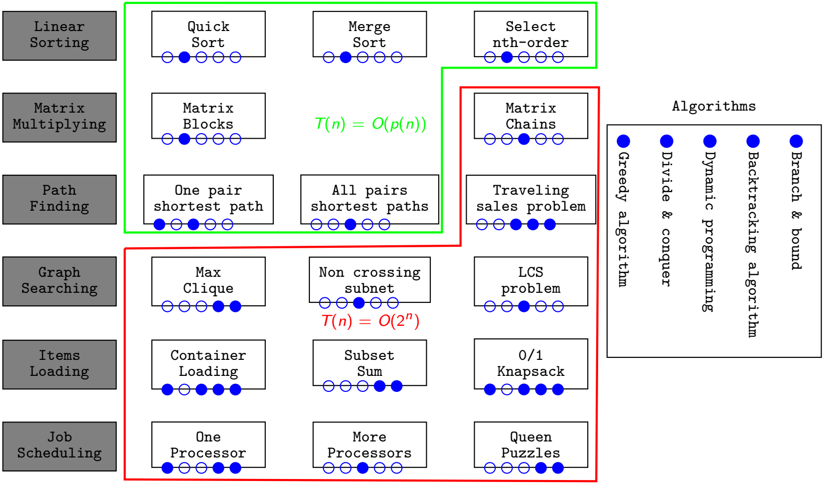
May 26, 2019

Outline

1. Are problems easy or hard?
   - Problem stacks
   - tractable problems
   - intractable problems
   - unsolvable problems
   - Decision vs Optimization problems
2. What makes a problem hard ?
3. Hardness equivalence of problems


Linear Sorting
Matrix Multiplying
Path Finding
Graph Searching
Items Loading
Job Scheduling
Quick Sort
Merge Sort
Select nth-order
Matrix Blocks
$T(n) = O(p(n))$
Matrix Chains
One pair shortest path
All pairs shortest paths
Traveling sales problem
Max Clique
Non crossing subnet
$T(n) = O(2^n)$
LCS problem
Container Loading
Subset Sum
0/1 Knapsack
One Processor
More Processors
Queen Puzzles
Algorithms
Greedy algorithm
Divide & conquer
Dynamic programming
Backtracking algorithm
Branch & bound

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- THEOREM RECOGNITION PROBLEM - Given a mathematical statement, test whether or not it is a theorem (i.e. whether or not it is "true"). For example, test a first order logical formula to see if it is valid (in all models), or test a statement of number theory to see if it holds.

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- Clearly, if one can solve an optimization problem (in polynomial time), then one can answer the decision version (in polynomial time)
- Conversely, by doing binary search on the bound b, one can transform a polynomial time answer to a decision version into a polynomial time algorithm for the corresponding optimization problem
- In that sense, these are essentially equivalent. We will then restrict ourselves to decision problems

Outline

1. Are problems easy or hard?
2. What makes a problem hard ?
   - Input size
   - Non-deterministic
3. Hardness equivalence of problems

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Proof.

Supposed that $p_i = O(c)$ and $w_i = O(c)$, we have $m = O(n \log_2 c)$
$c = O(2^m)$
$T = \Theta(nc) = O(n2^m)$

$\square$

Example 2. Composites numbers: Are there integers $k > 1$ and $p > 1$ such that $n = kp$?

```
int ComNum(int n)
{
factor=0;
for (j=2;j<n;j++)
      if ((n mod j)==0)
            factor=j;
            break;
return factor
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Time complexity: $T(n) = \Theta(n \log^2 n)$ is pseudo polynomial

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Proof.

Input length $m = \log_2 n$

$n = 2^m$

$T(n) = \Theta(n \log^2 n) = \Theta(m^2 * 2^m)$

□

Definition

a nondeterministic algorithm is an algorithm that, even for the same input, can exhibit different behaviors on different runs

There are several ways an algorithm may behave differently from run to run.

- A probabilistic algorithm's behaviors depends on a random number generator.
- A concurrent algorithm can perform differently on different runs due to a race condition(竞态条件).

```
Void nondetA(String w)
     String c=genCertif();
     boolean checkOK=verifyA(w,c)
     if (checkOK) Output “ yes “ return; //SUCCESS
     FAILURE
```

Comments on Non-deterministic algorithms

It consists of two procedures for a given input string w

- Guessing: get a "certificated" string c in a non-deterministic way.
- Checking: verify whether c is the solution. if so, return "yes", else FAILURE

Comments:

- c is a form of feasible solutions.
- "non-deterministic" means that the algorithm uses many different c to verify its soultions for a given w.

Outline

1. Are problems easy or hard?
2. What makes a problem hard ?
3. Hardness equivalence of problems
   - Class P
   - class NP
   - NP Complete

Definition

Class P contains all decision problems that can be solved by a deterministic Turing machine using a polynomial amount of computation time, or polynomial time.

- Polynomial-time algorithms are closed under composition, addition and multiplication.
- Any problem solved by composition, addition and multiplication of polynomial-time algorithms is also in class P.
- Notable problems in P include greatest common divisor, prime and linear programming problems.

Definition

A decision problem is said to be in class NP (Non-deterministic Polynomial)if there exists a verifier V for the problem. Given any instance w of the problem, where the answer is "yes", there must exist a certificate (also called a witness) c such that, given the ordered pair (w, c) as input, V returns the answer "yes" in polynomial time.

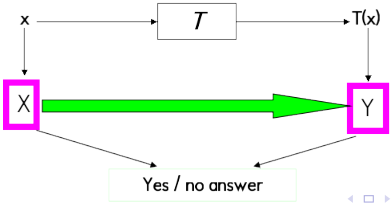
- if the answer to w is "no", the verifier will return "no" with input (w, c) for all possible c.
- V could return the answer "No" even if the answer to w is "yes", if c is not a valid witness.
- Notable problems in class NP include al problems in P, graph isomorphism problem, travelling salesman problem, and boolean satisfiability problem.

Polynomial reduction

Definition

If problem X can be reduced to problem Y ($X \leq_P Y$)polynomially if and only if there exists a polynomial deterministic algorithm T ,such that

- for each input string x, T generates a string $T(x)$
- x is an admissible input and corresponds a "yes" answer for X if and only $T(x)$ is an admissible input and corresponds a "yes" answer for Q.



Theorem 1

if $X \leq_P Y$ and Y is in class P, then X is also in class P.

Proof.

- there exists polynomial algorithm with complexity q for Y
- let the complexity of T be polynomial p, then the length of $T(x)$ is $O(p(|x|))$
- for input $T(x)$, complexity for Q is $O(q(p(|x|)))$
- so, complexity for soloving X is $O(p(x) + q(p(|x|)))$

$\square$

$X \leq_P Y$ means that Y is at least as hard as X.

Definition

Problem Y is said to be a NP hard probelm if and prolem X in class NP can polynomial reduces to problem Y.

Definition

Problem Y is said to be a NP Complete(NPC) probelm if Y is in class NP and Y is a NP hard problem.

- all NPC problems consist of a closed set respect to polynomial reduction(reflexive, symmetric, transitive).
- if there exists a polynomial algorithm for a NPC problem, then $P = NP$.
- we known that there are some NP hard problems, but not clear that whether they are in class NP (K^{th} largest subset problem: 第 K 大子集问题).

K^{th} largest subset problem

Instance:

- A finite set A of positive integers,
- Two nonnegative integers $K \le 2^{|A|}$ and $B \le \sum_{a \in A} a$.

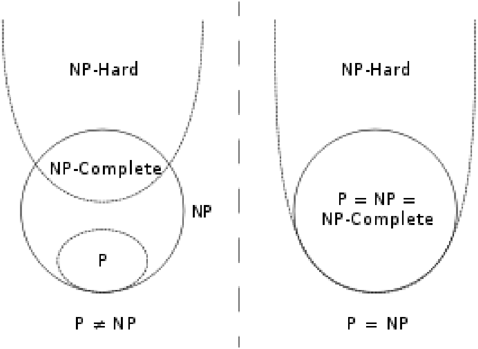
Question: Are there at least K distinct subsets $Y \subseteq A$ such that

$$\sum_{a \in Y} a \le B \tag{1}$$

or

$$\left\| Y \subseteq A : \sum_{a \in Y} a \le B \right\| \ge K \tag{2}$$

Euler diagram



Lists of NP-Complete problems

- Boolean satisfiability problem (SAT)
- N-puzzle
- Knapsack problem
- Hamiltonian path problem
- Traveling salesman problem
- Subgraph isomorphism problem
- Subset sum problem
- Clique problem
- Vertex cover problem
- Independent set problem
- Graph coloring problem

sumary

N- Nondeterministic Algorithms

- Deterministic algorithm: Given a particular input, it will always produce the same correct output
- Non-deterministic algorithm: with one or more choice points where multiple different continuations are possible, without any specification of which one will be taken

P- Polynomial Time complexity

- Computable
- Polynomial time is assumed the lowest complexity

C- Complete transform closed

- Reducible

Coming up: The End.

Thank You