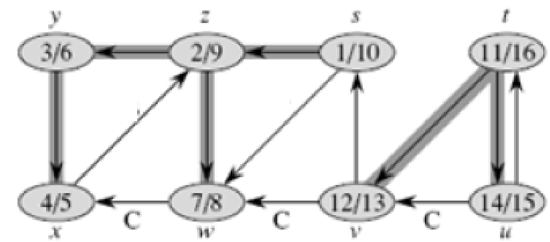
Topological sorting using DFS

Depth-First Search

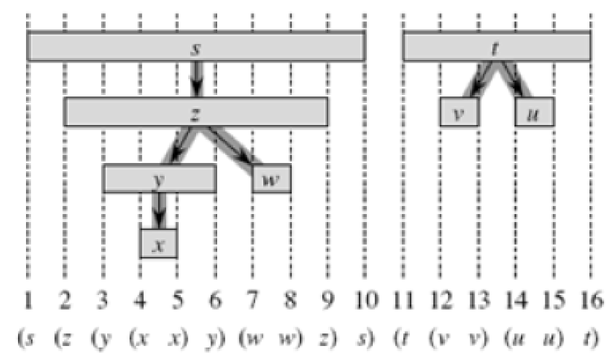
- When one vertex is a descendant of another in the forest that was constructed by DFS?
  - Parenthesis Theorem
  - White-path Theorem

Parenthesis Theorem

- **Theorem** (Parenthesis theorem):
  - For all u, v, exactly one of the following holds:
    1. u.d < u.f < v.d < v.f **or** v.d < v.f < u.d < u.f and neither of u and v is a descendant of the other.
    2. u.d < v.d < v.f < u.f **and** v is a descendant of u.
    3. v.d < u.d < u.f < v.f **and** u is a descendant of v.
  - So u.d < v.d < u.f < v.f cannot happen.



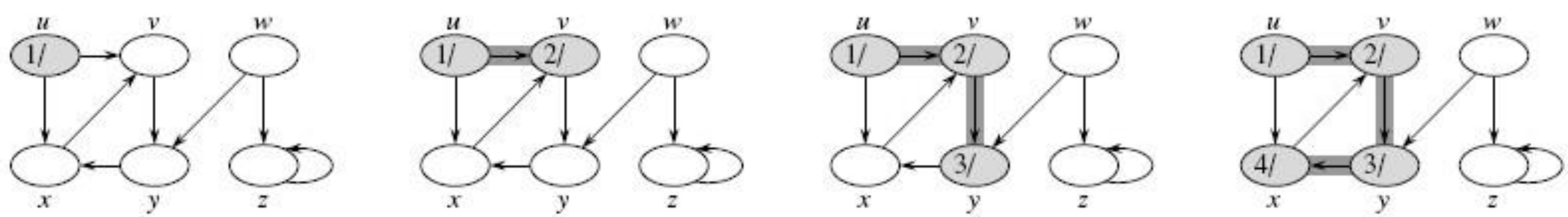
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])



- **Corollary (Nesting of descendants' intervals)**:
  - v is a proper descendant of u if and only if u.d < v.d < v.f < u.f.

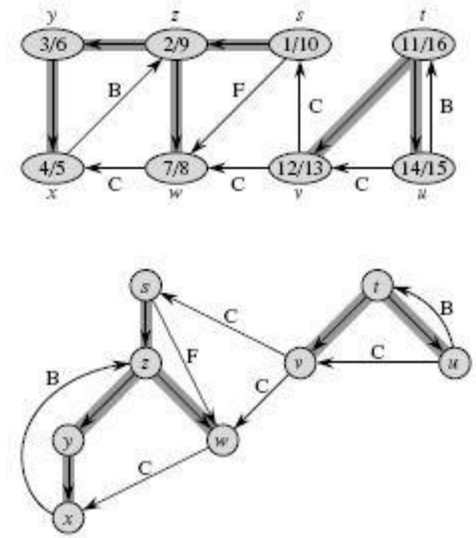
White-path Theorem

- Theorem (White-path theorem):
  - v is a descendant of u if and only if at time u.d, there is a path u ⇝ v consisting of only white vertices (except for u, which was just colored gray)



Classification of edges in directed graph

- **Tree edge**:
  - In the constructed forest.
  - Found by exploring (u, v).
- **Back edge**:
  - (u, v), where u is a descendant of v.
- **Forward edge**:
  - (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge**:
  - any other edge.
  - Can go between vertices in same tree or in different trees.

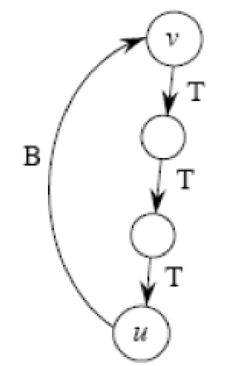


Classification of Edges

- Edge (u, v) can be classified by the color of v when the edge is first explored:
  - WHITE - indicates a **tree edge**
  - GRAY - indicates a **back edge**
  - BLACK indicates a **forward or cross edge**.
    - (u, v) is a **forward edge** if $u.d < v.d$
    - (u, v) is a **cross edge** if $u.d > v.d$.

Detection of Cycles

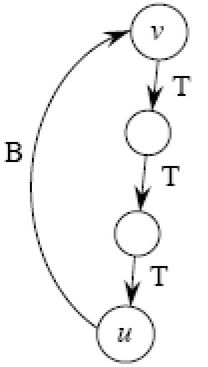
- Lemma:
  - A directed graph G is acyclic if and only if a DFS of G yields no back edges.



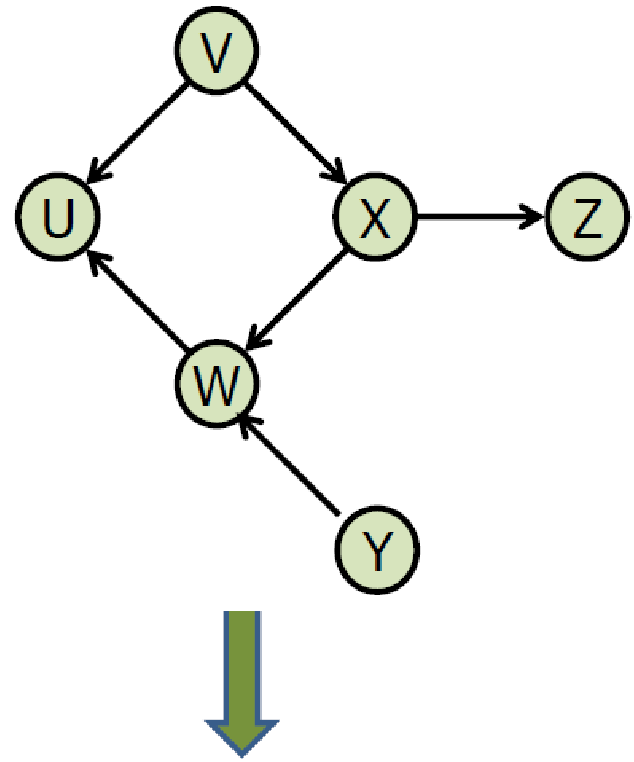
- Proof:
  - Back edge $\Rightarrow$ Cycle
    - Suppose there is a back edge (u, v)
    - Then v is ancestor of u in the constructed forest
    - Therefore, there is a path $v \rightsquigarrow u$, so $v \rightsquigarrow u \rightarrow v$ is a cycle

Detection of Cycles

- Cycle => back edge.
  - Suppose G contains cycle C.
  - Let v be the first vertex discovered in C, and let (u, v) be the preceding edge in C.
  - At time v.d, vertices of C form a white path v ⇝ u
    - since v is the first vertex discovered in c.
  - By white-path theorem, u is descendant of v in depth-first forest.
  - Therefore, (u, v) is a back edge.

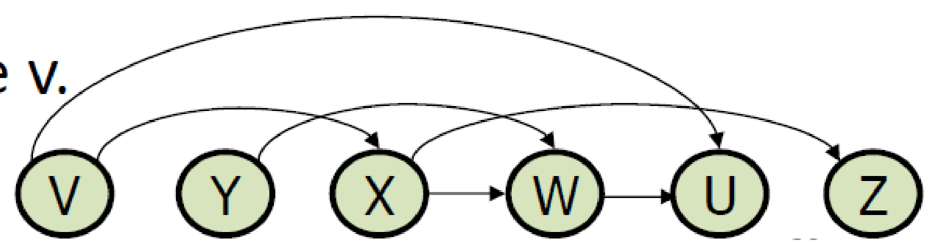


Topological Sort



- **Directed acyclic graph** (DAG) is a directed graph with no cycles

- **Topological sort of a DAG**: a linear ordering of vertices such that if $(u, v) \in E$, then u appears somewhere before v.



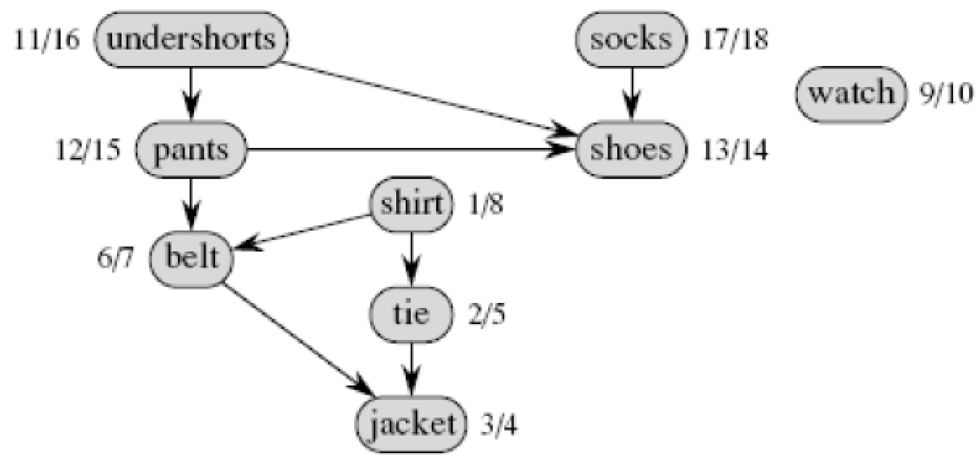
Topological Sort

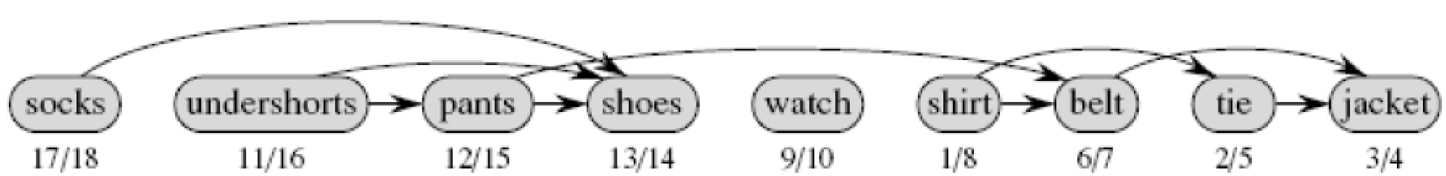
topologicalSort (G) // Assume G is a DAG **Complexity: Θ(V + E)**

Call DFS(G) to compute finishing times v.f for all v ∈ V

as each vertex is finished, insert it onto the front of a linked list

return the linked list of vertices





Topological Sort

- **Correctness**:
  - Just need to show if $(u, v) \in E$, then $v.f < u.f$.
- When we explore (u, v), what are the colors of u and v?
  - u is gray.
  - v can't be gray.
    - Because then (u, v) is a back edge, but G is a dag.
  - If v is white.
    - By parenthesis theorem, $u.d < v.d < v.f < u.f$.
  - If v is black.
    - Then v is already finished, but u doesn't, therefore, $v.f < u.f$.