INTERPOLATION

Horner's Rule: Instead of evaluating $P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ [3n flops]

Eval $P(x) = a_0 + x(a_1 + x(a_3 + \dots + x(a_n))\dots))$ [2n flops]

Polynomial Interpolation: Given $S = \{(t_i, y_i)\}_{i=1}^{m}$, find unique $P_{m-1}(t) = c_1 + c_2 t + \dots + c_{m-1} t^{m-1}$ that interpolates S

Solving
$$\begin{cases} c_1 + c_2 t_1 + c_3 t_1^2 + \dots + c_m t_1^{m-1} = y_1 \\ c_1 + c_2 t_2 + c_3 t_2^2 + \dots + c_m t_2^{m-1} = y_2 \\ c_1 + c_2 t_3 + c_3 t_3^2 + \dots + c_m t_3^{m-1} = y_3 \end{cases}$$

is the same as
(Vandermonde matrix)

$$\underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{m-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{m-1} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_m \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y}$$

Using IEEE double precision gets c_s to about 4 digits of accuracy.
– difficult to be accurate.

Another approach.

$P_{m-1}(t) = c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2 + \dots + c_m \cdot t^{m-1}$ is expressed using monomial basis $\{1, t, t^2, \dots, t^{m-1}\}$

Suppose $\{b_i(t)\}_{i=1}^{m}$ is a basis for $\mathbb{P}_{m-1}$

$$\begin{bmatrix} b_1(t_1) & b_2(t_1) & \cdots & b_m(t_1) \\ b_1(t_2) & b_2(t_2) & \cdots & b_m(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(t_m) & b_2(t_m) & \cdots & b_m(t_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$B\alpha = y$$

Choose $B = I$ (identity matrix), then $\alpha = y$ LAGRANGE BASIS

construct basis s.t. $b_i(t_j) = \begin{cases} 1 & \text{when } i=j \\ 0 & \text{when } i \neq j \end{cases}$

We want $b_1(t_2) = 0 \rightarrow$ includes $(t-t_2)$

$b_1(t_m) = 0 \rightarrow$ includes $(t-t_m)$

so far $b_1(t_1) = (t-t_2)(t-t_3)\dots(t-t_m)$

we want $b_1(t_1) = 1$ so scale by $\dfrac{1}{(t_1-t_2)(t_1-t_3)\dots(t_1-t_m)}$

$$\text{so } b_1(t) = \frac{(t-t_2)(t-t_3)\dots(t-t_m)}{(t_1-t_2)(t_1-t_3)\dots(t_1-t_m)}$$

$$b_k(t) = \prod_{\substack{j=1 \\ j \neq k}}^{m} \frac{(t-t_j)}{(t_k-t_j)}$$

$$I\alpha = Y$$

$$\alpha = Y$$

Choose B = Lower triangular Newton Basis

ERROR

$$f(t) - P_{m-1}(t) = \frac{1}{m!} \cdot f^{(m)}(c_t) \cdot w_m(t)$$

- c_t some point in (t_1, t_m) that depends on t
- $w_m(t) = (t-t_1)(t-t_2)\dots(t-t_m) = \prod_{i=1}^{m}(t-t_i)$

$$\max_{t\in[t,t_m]} \left| f(t) - P_{m-1}(t) \right| \leq \frac{1}{m!} \max_{t\in[t,t_m]} |f^{(m)}(t)| \max_{t\in[t,t_m]} |w_m(t)|$$

$\max_i |w_m(t)|$ is messy for $m > 4$, but we can show $\max_{t\in[t_1,t_m]} |w_m(t)| \leq \dfrac{h^m (m-1)!}{4}$

- $h = \max_i |t_{i+1} - t_i|$

PIECEWISE POLYNOMIAL INTERPOLATION

NONLINEAR EQUATIONS

Intermediate Value Theorem (IVT)

Bisection Method

given continuous $f(x)$, L, R s.t. $f(L) \cdot f(R) < 0$, $L < R$

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while ½(R-L) > ABSTOL:
    m = L + (R-L)/2
    if sign(f(L)) == sign(f(M)):
        L = M
    else:
        R = M
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$x^* = L + (R-L)/2$

After n intervals, error is $e_n = |m_n - x^*| \leq \frac{1}{2^n}|R_1 - L_1|$

Fixed Point Iteration

: find p^* s.t. $g(p^*) = p^*$ where $g(x) = x - f(x)$

Compute $x_0 = 0$

then $x_{i+1} = g(x_i)$

until $|x_{i+1} - x_i| < 5 \times 10^{-7}$

$g \in C[a,b]$, $\forall x \in [a,b]$, $g(x) \in [a,b] \Rightarrow g(x)$ has fixed point in $[a,b]$

$g'(x)$ exists on (a,b) $\wedge$ $|g'(x)| \leq K < 1$ $\forall x \in (a,b) \Rightarrow$ fixed point is unique

Newton's Method

$$\begin{cases} x_0 = \text{given initial guess} \\ x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)} \end{cases}$$

Fixed partition with $g(x) = x - \frac{f(x)}{f'(x)}$

Secant Method

$$\begin{cases} x_{i+1} = x_i - \dfrac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \\ \text{where } x_0, x_1 \text{ is given} \end{cases}$$

Rate Of Convergence : $\lim_{x_i \to x^*} \frac{e_{i+1}}{(e_i)^r} = C \neq 0$

Bisection: $e_{i+1} = \frac{1}{2} e_i \longrightarrow \lim_{x_i \to x^*} \frac{e_{i+1}}{e_i} = \frac{1}{2}$ and $r = 1$

$r=1$ means linear convergence

$r>1$ meas super linear convergence

Newton: $e_{i+1} = \left|\frac{f''(\theta)}{2f'(x_i)}\right| e_i^2 \longrightarrow \lim_{x_i \to x^*} \frac{e_{i+1}}{e_i^2} = \left|\frac{f''(\theta)}{2f'(x_i)}\right| = \text{Constant}$

$r=2$ quadratic convergence

if $f'(x^*) = 0$, x^* is a multiple root and convergence is linear

Secant: $\lim_{x_i \to x^*} \frac{e_{i+1}}{e_i e_{i-1}} = \lim_{x_i \to x^*} \frac{e_{i+1}}{e_i^r} = C \neq 0$ for $r = \frac{1+\sqrt{5}}{2} = 1.618$