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# Lesson 1

Talking cryptography is usually done in the “confidentiality” realm, where two characteristics in a communication channel are desirable: it must be *secret*, and *authentic*.

Modern confidentiality/authentication systems are forged under *Kerckhoffs’s principle*, which states that a secure system shall only rely on the encryption keys, and not on the underlying algorithm’s secrecy; in short, “*No security by obscurity*”. However sharing the key between two parties without the risk of eavesdropping is a costly operation.

## 1.1 Secret communication

TODO 1: Image of Alice,Bob, Eve in ske

The typical objects defined and used in secret communication topics are:

- $\mathcal{K}$ : Key space
- $\mathcal{M}$ : Message space
- $\mathcal{C}$ : Ciphertext space
- $\text{Enc} \in \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$ : Encryption routine
- $\text{Dec} \in \mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$ : Decryption routine

$\text{Enc}$  and  $\text{Dec}$  form a *cryptographic secrecy scheme*, or just *secrecy scheme* for brevity, and as such, it must abide by the rule:

$$\forall m \in \mathcal{M}, \forall k \in \mathcal{K} \implies \text{Dec}(k, \text{Enc}(k, m)) = m$$

**Definition 1.** (*Shannon’s “Perfect secrecy”*): Let  $M$  be any distribution over the message space  $\mathcal{M}$ , and  $K$  a uniform distribution over  $\mathcal{K}$ . Then, the cryptosystem  $\Xi : (\text{Enc}, \text{Dec})$  is deemed *perfectly secret* iff the ciphertext obtained by applying the encryption routine to a message sampled by  $M$  using the key in  $K$  is effectively useless in retrieving any info about the message itself, apart from the key-supplied decryption routine. Formally:

$$\forall M \sim \text{Rand}(\mathcal{M}), \forall C \sim \text{Rand}(\mathcal{C}) \implies \Pr[M = m] = \Pr[M = m \mid C \xrightarrow{\text{Dec}} c]$$

◇

Note how this definition doesn't involve the encryption key.

The perfect secrecy definition can be rephrased in different ways, bringing more details to light:

1.  $\Pr[M = m] = \Pr[M = m \mid C \text{ } \S \rightarrow c]$
2.  $M \perp\!\!\!\perp C$
3.  $\forall m_1, m_2 \in \mathcal{M}, c \in \mathcal{C} \implies \Pr[\text{Enc}(K, m_1) = c] = \Pr[\text{Enc}(K, m_2) = c]$

A remark has to be done here:

$$\begin{aligned} & \Pr[\text{Enc}(K, m) = c] \\ &= \Pr[\text{Enc}(K, M) = c \mid M \text{ } \S \rightarrow m] \\ &\neq \Pr[\text{Enc}(K, M) = c] \end{aligned}$$

which is exactly the difference between picking a specific message  $m$  (as in: *choosing*  $m$ ), and picking it at random, just as  $M$  describes.

**Proposition 1.** *All the previous statements are equivalent.*  $\diamond$

*Proof.* The proof is structured as a cyclic implication between the three definitions:

- (1)  $\implies$  (2): Start from one side of the independency definition, and work through the other:

$$\begin{aligned} & \Pr[C = c \wedge M = m] \\ &= \Pr[C = c] \Pr[M = m \mid C \text{ } \S \rightarrow c] && \text{(Conditioned prob.)} \\ &= \Pr[C = c] \Pr[M = m] && \text{(Using 1.)} \end{aligned}$$

This proves that  $M$  and  $C$  are independent distributions.

- (2)  $\implies$  (3): For the proof's purposes, let  $M$  be an arbitrary distribution over  $\mathcal{M}$ ; recall that, by definition:  $C := \text{Enc}(K, M)$ :

$$\begin{aligned} & \Pr[\text{Enc}(K, m_1) = c] \\ &= \Pr[\text{Enc}(K, M) = c \mid M \text{ } \S \rightarrow m_1] && \text{(Introducing } M) \\ &= \Pr[C = c \mid M \text{ } \S \rightarrow m_1] && \text{(} C \text{ definition)} \\ &= \Pr[C = c] && \text{(Using 2.)} \\ &= \dots && \text{(Same steps reversed, where } m_1 \mapsto m_2) \\ &= \Pr[\text{Enc}(K, m_2) = c] \end{aligned}$$

- (3)  $\implies$  (1):

$$\begin{aligned}
& \Pr[C = c] \\
&= \sum_m \Pr[C = c \wedge M = m] && \text{(Total prob.)} \\
&= \sum_m \Pr[\text{Enc}(K, M) = c \wedge M = m] && \text{(C definition)} \\
&= \sum_m \Pr[\text{Enc}(K, M) = c \mid M \text{ } \S \rightarrow m] \Pr[M = m] && \text{(Cond. prob. def.)} \\
&= \sum_m \Pr[\text{Enc}(K, m) = c] \Pr[M = m] && \text{(Cond. collapse)} \\
&= \Pr[\text{Enc}(K, \bar{m}) = c] \sum_m \Pr[M = m] && \text{(Using 3.)} \\
&= \Pr[\text{Enc}(K, \bar{m}) = c] && \text{(Total prob.)} \\
&= \Pr[\text{Enc}(K, M) = c \mid M \text{ } \S \rightarrow \bar{m}] && \text{(Introducing } M) \\
&= \Pr[C = c \mid M \text{ } \S \rightarrow \bar{m}] && \text{(C definition)}
\end{aligned}$$

Knowing this, and applying Bayes' theorem, we get back to the first definition:

$$\begin{aligned}
& \Pr[C = c] = \Pr[C = c \mid M \text{ } \S \rightarrow m] \\
&\implies \Pr[C = c] = \Pr[M = m \mid C \text{ } \S \rightarrow c] \frac{\Pr[C = c]}{\Pr[M = m]} && \text{(Bayes' theorem)} \\
&\implies \Pr[M = m] = \Pr[M = m \mid C \text{ } \S \rightarrow c]
\end{aligned}$$

Thus, we conclude that all three definitions for perfect secrecy are equivalent.  $\square$

### 1.1.1 One Time Pad (OTP)

Let  $\mathcal{K} = \mathcal{M} = \mathcal{C} = \mathbb{Z}^l$ , and define the following cryptographic scheme:

- $\text{Enc}(k, m) = k \oplus m = c$
- $\text{Dec}(k, c) = k \oplus c = m$

Proof of correctness goes like:  $\text{Dec}(k, \text{Enc}(k, m)) = \text{Dec}(k, k \oplus m) = k \oplus k \oplus m = m$ .

**Theorem 1.** *The One-time pad scheme is perfectly secret.*  $\diamond$

*Proof.* Let  $K \sim \text{Unif}(\mathcal{K})$ . Then,  $\forall m_1, m_2, c \in 2^l$ :

$$\begin{aligned}
& \Pr[\text{Enc}(K, m_1) = c] \\
&= \Pr[K \oplus m_1 = c] \\
&= \Pr[K = c \oplus m_1] \\
&= |\mathcal{K}|^{-1} && (\text{K is uniform}) \\
&= \dots && (\text{Same steps reversed, where } m_1 \mapsto m_2) \\
&= \Pr[\text{Enc}(K, m_2) = c]
\end{aligned}$$

This satisfies the third definition of perfect secrecy.  $\square$

By observing our recent proof, some insights (and problems) arise:

1. The key and the message's lengths must always match ( $|k| = |m|$ );
2. As the name suggests, keys are useful just for one encryption. Otherwise, given two encryptions with the same key, an attacker may exploit the XOR's idempotency to extract valuable information from both ciphertexts<sup>1</sup>:

$$c_1 = k \oplus m_1 \wedge c_2 = k \oplus m_2 \implies c_1 \oplus c_2 = m_1 \oplus m_2$$

Combined with the fact that keys must be preemptively shared in a secure fashion, these problems make for a quite impractical cryptographic scheme. One can further this analysis and generalize it to all “perfect” schemes, giving a rather delusive conclusion:

**Theorem 2.** *For a secrecy scheme to be perfectly secret:  $|\mathcal{K}| \geq |\mathcal{M}|$*   $\diamond$

*Proof.* Perfection will be disproved by breaking the first definition. Let  $M \sim \text{Unif}(\mathcal{M})$ , and take  $c \in \mathcal{C} : \Pr[C = c] > 0$ . Consider  $D = \{\text{Dec}(k, c) : k \in \mathcal{K}\}$  as the set of all images of the decryption routine with all keys in  $\mathcal{K}$ . The offending assumption is that  $|\mathcal{K}| < |\mathcal{M}|$ . Then, by how  $D$  is defined:

$$|D| \leq |\mathcal{K}| < |\mathcal{M}| \implies \exists m \in \mathcal{M} \setminus D$$

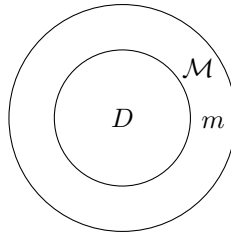


Figure 1.1: Where the messages stand

Fix this message  $m$ , and remember that by  $M$ 's definition,  $\Pr[M = m] = |\mathcal{M}|^{-1}$ . Since  $m \notin D$ , there can be no key in  $\mathcal{K}$  such that  $\text{Dec}(k, c) = m$ . By

---

<sup>1</sup>This vulnerability of applying a function on a ciphertext, and expecting as a result the image of the original message by the same function, is called *malleability*, and is explored further in the notes.

observing that  $C$  strictly distributes over  $D$ , where  $m$  is not present,  $\Pr[M = m \mid C \not\rightarrow c] = 0$ . Putting everything together:

$$0 = \Pr[M = m \mid C \not\rightarrow c] \neq \Pr[M = m] = |\mathcal{M}|^{-1}$$

This clearly violates the first definition of perfect secrecy. □



# Lesson 2

## 2.1 Authentic communication

TODO 2: Msg auth image (bob, eve, tag)

### 2.1.1 Message Authentication Code scheme

Syntax:

- $\mathcal{K}$ : Key space
- $\mathcal{M}$ : Message space
- $\mathcal{C}$ : Ciphertext space
- $\Phi$ : Tag (or signature) space
- $\text{Tag} \in \mathcal{K} \times \mathcal{M} \rightarrow \Phi$ : Tagging (or signing) function
- $\text{Ver} \in \mathcal{K} \times \mathcal{M} \times \Phi \rightarrow \mathbb{2}$ : Verifying (or validating) function

$\text{Tag}$  and  $\text{Ver}$  form a *cryptographic authentication scheme*, or just *authentication scheme*, and it must abide by the rule:

$$\forall m \in \mathcal{M}, \forall k \in \mathcal{K} \implies \text{Ver}(k, m, \text{Tag}(k, m)) = 1$$

In the usual case (deterministic), the verifier consists only of an equality check, reusing the  $\text{Tag}$  routine:  $\text{Tag}(k, m) = \phi$

The security aspect to consider in these schemes is the signatures' *unforgeability*. Suppose that an attacker chooses a message  $m$  and obtains its tag  $\phi$  without knowing the key  $k$  used in the process; then there shall be no better means of declaring a couple  $(m', \phi') : \text{Ver}(k, m', \phi') = 1$  other than random guessing, or knowing the key, of course.

Some more formal definitions of unforgeability follow:

**Definition 2.** ( *$\varepsilon$ -statistical one-time security*): A given authentication scheme has  $\varepsilon$ -statistical one-time security iff, given a valid couple  $(m_1, \phi_1)$ , any adversary cannot *forge* a fresh valid couple  $(m_2, \phi_2)$  without knowing the signature key  $k$ . Formally:

$$\begin{aligned} & \forall m_1, m_2 \in \mathcal{M} : m_1 \neq m_2, \forall \phi_1, \phi_2 \in \Phi \\ & \quad \Downarrow \\ & \Pr[\text{Tag}(K, m_2) = \phi_2 \mid \text{Tag}(K, m_1) \neq \phi_1] \leq \varepsilon \end{aligned}$$

◇

### Pairwise-independent hashing

**Definition 3.** Let  $\mathcal{S}$  be a seeding space; define a family of *hash functions* to be the following object<sup>2</sup>:

$$H \in \mathcal{S} \rightarrow (\mathcal{M} \rightarrow \Phi) : s \mapsto h_s$$

Let  $S$  be a random variable in the seed space; the hash functions are deemed *pairwise-independent*<sup>3</sup> iff, for any two distinct messages  $m$  and  $m'$ , the pair  $(h_S(m), h_S(m'))$  distributes evenly in  $\Phi^2$ . In other words:

$$\forall m, m' \in \mathcal{M} : m \neq m', \forall \phi, \phi' \in \Phi \implies \Pr[h_S(m) = \phi \wedge h_S(m') = \phi'] = \frac{1}{|\Phi|^2}$$

◇

As an example of such a family, consider the additive group of integers modulo  $p$ :  $(\mathbb{Z}_p, +)$ , where  $p$  is a prime integer. Define the family:

$$h_{(a,b)}(x) = ax + b \bmod p$$

where  $\mathcal{S} = \mathbb{Z}_p^2$ , and  $\mathcal{M} = \Phi = \mathbb{Z}_p$ .

**Theorem 3.** *The functions in the family  $H$  are pairwise-independent.* ◇

*Proof.* Let  $S = (a, b)$  be a random seed for  $H$ ; for any distinct messages  $m, m'$ , and for any tags  $\phi, \phi'$ :

$$\begin{aligned} & \Pr[h_S(m) = \phi \wedge h_S(m') = \phi'] \\ &= \Pr[am + b = \phi \wedge am' + b = \phi'] \\ &= \Pr \left[ \begin{pmatrix} m & 1 \\ m' & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} \right] \\ &= \Pr \left[ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m & 1 \\ m' & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \phi \\ \phi' \end{pmatrix} \right] \\ &= \frac{1}{|\mathbb{Z}_p|^2} \end{aligned}$$

which is exactly the definition of pairwise-independency. □

**Theorem 4.** *Define an authentication scheme to be such that its tagging routine is a hash function family:  $\text{Tag}(k, m) = h_k(m)$ . Let this function family be pairwise-independent. Then the authentication scheme is  $\frac{1}{|\Phi|}$ -statistical one-time secure.* ◇

<sup>2</sup>This kind of notation consisting in putting an argument as a subscript to a generic, typically of higher-order function, is also called “currying”; it will be used extensively throughout the lessons.

<sup>3</sup>Care should be taken to not confuse *pairwise* independency with *mutual* independency: while the former acts only on pairs, the latter considers all possible subsets. The two notions are not necessarily equivalent.

*Proof.* For all our purposes, let  $K \sim \text{Unif}(\mathcal{K})$ ; then:

$$\begin{aligned} \forall m \in \mathcal{M}, \phi \in \Phi &\implies \Pr[\text{Tag}(K, m) = \phi] = \Pr[h_K(m) = \phi] = \frac{1}{|\Phi|} \\ \forall m \neq m' \in \mathcal{M}, \phi, \phi' \in \Phi &\implies \Pr[\text{Tag}(K, m) = \phi \wedge \text{Tag}(K, m') = \phi'] = \frac{1}{|\Phi|^2} \end{aligned}$$

Therefore:

$$\begin{aligned} &\Pr[\text{Tag}(K, m') = \phi' \mid \text{Tag}(K, m) \neq \phi] \\ &= \frac{\Pr[\text{Tag}(K, m') = \phi' \wedge \text{Tag}(K, m) \neq \phi]}{\Pr[\text{Tag}(K, m) \neq \phi]} \\ &= \frac{|\Phi|}{|\Phi|^2} = \frac{1}{|\Phi|} \end{aligned}$$

□

**Theorem 5.** Any  $(2^{-\lambda})$ -statistical  $t$ -time secure authentication scheme has a key of size  $(t+1)\lambda$  for any  $\lambda > 0$ . ◇

*Proof.* None given.

Idea: For each “time”, use one “subkey”

□

# Lesson 3

## 3.1 Randomness Extraction

In most of our discourse, the subject of uniformly random variables is much recurrent; this chapter/lesson delves deeper into the topic. For starters, we devise some attempts to extract uniform randomness from “non-uniform” randomness sources.

Suppose to have a biased coin  $B \sim \text{Ber}(p) : p \neq \frac{1}{2}$ . How to craft a fair coin out of it? In his time, Von Neumann devised a simple algorithm, which is now known as the *Von Neumann extractor*:

1. Let  $B \sim \text{Ber}(p)$  be a random variable
2. Sample  $b_1 \leftarrow B$
3. Sample  $b_2 \leftarrow B$
4. If  $b_1 = b_2$  go to step 2
5. Else:
  - If  $b_1 = 0 \wedge b_2 = 1$  output 1
  - If  $b_1 = 1 \wedge b_2 = 0$  output 0

Some considerations can be made: The probability of both single cases in step 5 is  $p(1-p)$ , therefore the probability to reach it is  $2p(1-p)$ . Also, it is apparent that the number of possible failures in reaching step 5 follow a geometric distribution in  $p$ , thus the probability of increased number of failures decrease exponentially.

We now get back to our ultimate goal. Let  $X$  be any random variable over a space  $\Omega$ , we wish to design an “extraction” algorithm  $\text{Ext}$  such that  $U = \text{Ext}(X)$  distributes uniformly over  $\Omega$ . To help ourselves, we will deal with probability spaces of binary strings ( $2^n$ ), and define a measure of “how much” a distribution is uniform over its space:

**Definition 4.** Let  $X$  be a random variable from a given probability distribution. Its *min-entropy* is defined as follows:

$$H_\infty(X) = -\log_2(\max(\Pr[X = x]))$$

◇

Using this measure, we can already see an interesting case, which involves “constant” random variables:

$$\begin{aligned} X \sim \text{Const}(\bar{x}) &\implies \Pr[X = \bar{x}] = 1 \\ &\implies \Pr[X \neq \bar{x}] = 0 \\ &\implies H_\infty(X) = -\log_2(\Pr[X = \bar{x}]) = -\log_2(1) = 0 \end{aligned}$$

And in fact, a constant variable is useless in creating a uniform distribution: it always gives the same outcome, making everything deterministic. Therefore, such variables must be excluded in our search for a “universal extractor”. On the other hand, looking at a uniform distribution:

$$\begin{aligned} X \sim \text{Unif}(\Omega) &\implies \forall x \Pr[X = x] = \frac{1}{|\Omega|} \\ &\implies H_\infty(X) = -\log_2\left(\frac{1}{|\Omega|}\right) \end{aligned}$$

Knowing that  $\Omega$  is be our usual domain choice of binary strings of a given length  $2^n$ , the min-entropy becomes exactly  $n^4$ . Using this measure, we can actually seek how much min-entropy we require in the original distribution  $X$  in order for the extractor to return a uniform distribution. Ideally, we would like a value as close to 0 as possible, because a min-entropy of zero leads to constant variables, which have been excluded beforehand. Alas, it turns out that:

**Claim 1.** *There is no such universal Ext algorithm that returns a uniform distribution from random variables  $X$  with min-entropy  $H_\infty(X) \leq n - 1$*   $\diamond$

*Proof.* TODO 3: Help with the proof, things don’t look good

□

So this approach is doomed, unless we factor in a preemptive small amount of true randomness in the algorithm. This is what a *seeded extractor* does:

$$\text{Ext} : \underbrace{\mathcal{P}^d}_{\text{seed(public)}} \times \underbrace{\mathcal{P}^n}_{\text{input}} \rightarrow \underbrace{\mathcal{P}^l}_{\text{output}}$$

Before giving a formal definition of such an extractor, we require another notion of measure related to probability distributions:

**Definition 5.** *Statistical Distance:* Let  $X$  and  $Y$  be two random variables on the same probability space. Their *statistical distance* is defined as follows:

$$SD(X, Y) = \frac{1}{2} \sum_{x \in \Omega} |\Pr[X = x] - \Pr[Y = x]|$$

◇

---

<sup>4</sup>This also sheds some light in how string length is a frequent topic in the cryptography realm, as it usually expresses a cryptosystem’s strength: the greater its min-entropy, the harder it is to find the right key from scratch for a ciphertext.

In an intuitive stance, this distance amounts to half the area delimited by the two distributions.

TODO 4: Image of the statistical distance

**Definition 6.** Let  $\text{Ext} \in \mathcal{2}^d \times \mathcal{2}^n \rightarrow \mathcal{2}^l$  be a seeded extractor, and  $S \sim \text{Unif}(\mathcal{2}^d)$ . Then it is a  $(k, \varepsilon)$ -extractor iff:

$$\forall X : H_\infty(X) \geq k \implies SD((S, \text{Ext}(S, X)), (S, \text{Unif}(\mathcal{2}^l))) \leq \varepsilon$$

◇

Do note that  $S$  takes part in both sides of the statistical distance: this is to be interpreted that the seed is known at the time of extraction.

### 3.1.1 Universal hash functions

Getting back to our hash function families, we see that they too use an argument as a random seed, and attempt to be as uniform as possible; thus they behave in most ways as seeded extractors. Let's further develop the idea:

**Definition 7.** Let  $S$  be a uniform seed. A hash function family  $H$  is deemed *universal* iff:

$$\forall a \neq b \in \Omega \implies \Pr[h_S(a) = h_S(b)] = \frac{1}{2^l}$$

◇

**Definition 8.** Let  $X$  and  $Y$  be two IID random variables; a *collision* is the event of both evaluating to the same outcome. The probability of such an event is:

$$\text{Col}(X) = \text{Col}(Y) = \Pr[X = Y] = \sum_{x \in \Omega} \Pr[X = x \wedge Y = x] = \sum_{x \in \Omega} \Pr[X = x]^2$$

◇

#### Leftover hash lemma

This lemma has been proven by Russell Impagliazzo, Leonid Levin and Michael Luby:

**Lemma 1** (Leftover hash). *Let  $h_s$  be a pairwise-independent hash function, and define  $\text{Ext} \in \mathcal{2}^d \times \mathcal{2}^n \rightarrow \mathcal{2}^l$  to be a seeded randomness extractor. Then if  $\text{Ext}(s, x) = h_s(x)$ , and  $x$  follows a distribution  $X$  such that:*

$$H_\infty(X) = k \geq l + 2 \log_2 \frac{1}{\varepsilon} - 2$$

*then  $\text{Ext}$  is a  $(k, \varepsilon)$ -seeded randomness extractor.*

◇

*Proof.* TODO 5: —

□

# Lesson 4

## 4.1 Negligible function

What is exactly a negligible function? Below here there is a possible interpretation of this notion, taken from an answer to a question in the Cryptography Stack Exchange website:

“[...] in modern cryptographic schemes, we generally do not try to achieve perfect secrecy [...]. Instead, we define security against a specific set of adversaries whose computational power is bounded. Generally, we assume an adversary that is bounded to run in time polynomial to  $n$ , where  $n$  is the security parameter given to the key generation algorithm [...].

So consider a scheme  $\Pi$  where the only attack against it is brute-force attack. We consider  $\Pi$  to be secure if it cannot be broken by a brute-force attack in polynomial time.

The idea of *negligible probability* encompasses this exact notion. In  $\Pi$ , let's say that we have a polynomial-bounded adversary. Brute force attack is not an option. But instead of brute force, the adversary can try (a polynomial number of) random values and hope to guess the right one. In this case, we define security using negligible functions: The probability of success has to be smaller than the reciprocal of any polynomial function.

And this makes a lot of sense: if the success probability for an individual guess is a reciprocal of a polynomial function, then the adversary can try a polynomial amount of guesses and succeed with high probability. If the overall success rate is  $\frac{1}{\text{poly}(n)}$  then we consider this attempt a feasible attack to the scheme, which makes the latter insecure.

So, we require that the success probability must be less than the reciprocal of every polynomial function. This way, even if the adversary tries  $\text{poly}(n)$  guesses, it will not be significant since it will only have tried:

$$\frac{\text{poly}(n)}{\text{superpoly}(n)}$$

As  $n$  grows, the denominator grows far faster than the numerator and the success probability will not be significant.”<sup>5</sup>

---

<sup>5</sup>> “What exactly is a negligible (and non-negligible) function?” — Cryptography Stack Exchange

**Definition 9.** Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function. Then it is deemed *polynomial*, and denoted as  $f \in \text{Poly}(\lambda)$ , iff:

$$\exists c \in \mathbb{N} : f(\lambda) \in O(\lambda^c) \quad \diamond$$

**Definition 10.** Let  $\nu : \mathbb{N} \rightarrow \mathbb{R}$  be a function. Then it is deemed *negligible*, and denoted as  $\nu \in \text{Negl}(\lambda)$ , iff:

$$\forall f \in \text{Poly}(\lambda) \implies \nu(\lambda) \in O\left(\frac{1}{f(\lambda)}\right) \quad \diamond$$

Note that these actually represent upper bounds for functions: a negligible function adheres to the polynomial function definition, whereas the opposite isn't true. To sum it up:  $\text{Negl} \subset \text{Poly}$ .

**Exercise 6.** Let  $p(\lambda), p'(\lambda) \in \text{Poly}(\lambda)$  and  $\nu(\lambda), \nu'(\lambda) \in \text{Negl}(\lambda)$ . Then prove the following:

1.  $p(\lambda) \cdot p'(\lambda) \in \text{Poly}(\lambda)$
2.  $\nu(\lambda) + \nu'(\lambda) \in \text{Negl}(\lambda)$

**Solution 1 (2).** TODO 6: Questa soluzione usa disuguaglianze deboli; per essere negligibile una funzione dev'essere strettamente minore di un polinomiale inverso. Da approfondire

We need to show that for any  $c \in \mathbb{N}$ , then there is  $n_0$  such that  $\forall n > n_0 \implies \nu(n) + \nu'(n) < \frac{1}{n^c}$ .

Consider an arbitrary  $c \in \mathbb{N}$ . Then, since  $c + 1 \in \mathbb{N}$ , and both  $\nu$  and  $\nu'$  are negligible, there exist  $n_\nu$  and  $n_{\nu'}$  such that:

$$\begin{aligned} \forall n \geq n_\nu &\implies \nu(n) \leq n^{-(c+1)} \\ \forall n \geq n_{\nu'} &\implies \nu'(n) \leq n^{-(c+1)} \end{aligned}$$

Fix  $n_0 = \max(n_\nu, n_{\nu'})$ . Then, since  $n_0 \geq 2$ ,  $\forall n \geq n_0$  we have:

$$\begin{aligned} &\nu(n) + \nu'(n) \\ &\leq n^{-(c+1)} + n^{-(c+1)} \\ &= 2n^{-(c+1)} \\ &\leq n^{-c} \end{aligned}$$

Therefore, we conclude that  $\nu(n) + \nu'(n) \in \text{Negl}(\lambda)$ .

## 4.2 One-way functions

From here, we start defining an object that is fundamental to everyday cryptography: the *one-way* function, or OWF in short. Colloquially, a one-way function is a function that is “easy to compute”, while being “hard to invert” at the same time, the concept of hardness being borrowed by complexity theory.



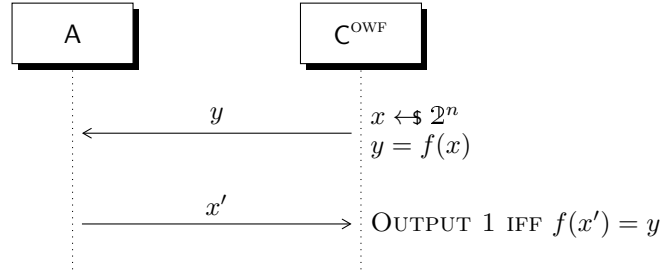


Figure 4.2: One-Way Function hardness

**Definition 11.** Let  $f : \mathcal{D}^{n(\lambda)} \rightarrow \mathcal{D}^{n(\lambda)}$  be a function. Then it is a OWF iff:

$$\forall A \in \text{PPT} \exists \nu(\lambda) \in \text{Negl}(\lambda) : \Pr [\text{GAME}_{f,A}^{\text{OWF}}(\lambda) = 1] \leq \nu(\lambda) \quad \diamond$$

The structure of the “game” appearing in the definition is depicted in figure 4.2. Do note that the game does not check for  $x = x'$ , but rather for  $f(x) = f(x')$ ; in a sense, the adversary is not trying to guess what the original  $x$  was: its goal is to find any value such that its image is  $y$  according to  $f$ , and such value may very well not be unique.

**Exercise 7.** Prove the following claims:

1. There exists an inefficient adversary that wins  $\text{GAME}_{f,A}^{\text{OWF}}$  with probability 1
2. There exists an efficient adversary that wins  $\text{GAME}_{f,A}^{\text{OWF}}$  with probability  $2^{-n}$

**Solution 2 (7).**

1. Adversary uses a brute-force attack.
2. Adversary makes a random guess.

A one-way function can be thought as a function which is very efficient in generating “puzzles” that are very hard to solve from scratch. Furthermore, given a candidate solution, one can efficiently verify its validity. In a twist of perspective, for a given couple  $(\mathcal{P}_{\text{GEN}}, \mathcal{P}_{\text{VER}})$  of a puzzle generator and a puzzle verifier, another “game” can be drawn as in figure 4.3.

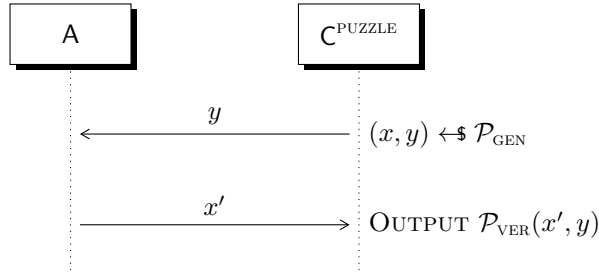


Figure 4.3: The puzzle game

It can also be said that the one-way puzzle problem is in NP, because witness checking is easy, but not in P because finding a solution to begin with is hard.

### Impagliazzo's Worlds

Suppose to have Gauss, a genius child, and his professor. The professor gives to Gauss some mathematical problems, and Gauss wants to solve them all.

Imagine now that, if using one-way functions, the problem is  $f(x)$ , and its solution is  $x$ . According to Impagliazzo, we live in one of these possible worlds:

- *Algorithmica*:  $P = NP$ , meaning all efficiently verifiable problems are also efficiently solvable.

The professor can try as hard as possible to create a hard scheme, but he won't succeed because Gauss will always be able to efficiently break it using the verification procedure to compute the solution

- *Heuristica*: NP problems are hard to solve in the worst case but easy on average.

The professor, with some effort, can create a game difficult enough, but Gauss will solve it anyway; here there are some problems that the professor cannot find a solution to

- *Pessiland*: NP problems are hard on average but no one-way functions exist
- *Minicrypt*: One-way functions exist but public-key cryptography is impractical
- *Cryptomania*: Public-key cryptography is possible: two parties can exchange secret messages over open channels

## 4.3 Computational Indistinguishability

Distribution ensembles  $X = \{X_{\lambda \in \mathbb{N}}\}$  and  $Y = \{Y_{\lambda \in \mathbb{N}}\}$  are distribution sequences.

**Definition 12** (*Comp. indist.*). Let  $X$  and  $Y$  be two distribution sequences; they are deemed *computationally indistinguishable*, written as " $X \approx_c Y$ " iff:

$$\forall A \in \text{PPT} \exists \nu(\lambda) \in \text{Negl}(\lambda) : |\Pr[A(X_{\lambda}) = 1] - \Pr[A(Y_{\lambda}) = 1]| \leq \nu(\lambda) \quad \diamond$$

In words: any *efficient* adversary attempting to distinguish outputs between the two ensembles will succeed with a probability that is negligibly different than randomly guessing. Note the emphasis on "efficient", which makes this relationship between ensembles weaker than what would be a purely statistical one.

With the purpose of making these new concepts clearer, it is presented this mental game.

TODO 7: AP181129-2344: There may be room for improvement, but I like how it's worded: it puts some unusual perspective into the cryptographic game, and it could be a good thing since it closely precedes our first reduction, and the whole hybrid argument mish-mash.

A *challenger*  $C$  chooses a value  $z$  among  $X_\lambda$  and  $Y_\lambda$ , and gives it to a *distinguisher*  $D$ . In turn,  $D$  has to correctly guess which was the source of  $z$ : either  $X_\lambda$  or  $Y_\lambda$ .

If we let  $X_\lambda$  and  $Y_\lambda$  to be *computationally indistinguishable*, then, fixed 1 as one of the sources, the probability that  $D$  says “1!” when  $C$  picks  $z$  from  $X_\lambda$  is *not so far* from the probability that  $D$  says “1!” when  $C$  picks  $z$  from  $Y_\lambda$ .

So, this means that, when this property is verified by two random variables, there isn’t too much *difference* between the two variables in terms of information available to  $D$ , otherwise the distance between the two probabilities should be much more than a negligible quantity.

**Lemma 2.** *Let  $f$  be a function that has polynomial time-complexity. Then, for any two ensembles  $X$  and  $Y$ :*

$$X \approx_c Y \implies f(x) \approx_c f(y) \quad \diamond$$

*Proof.* This proof is by contradiction and uses a reduction. Let  $X \approx_c Y$  be two indistinguishable ensembles, and  $f \in \text{PPT}$  an arbitrary poly-time complex function. Assume there exists an adversary  $A$  to the challenge of distinguishing the ensembles’ images  $f(X)$  from  $f(Y)$  that does efficiently succeed, as shown in figure 4.4.

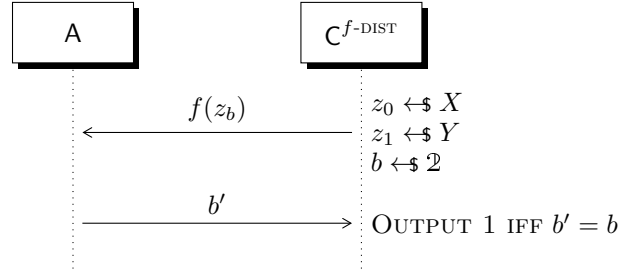


Figure 4.4: A distinguisher for  $f$

Fix this adversary to be the distinguisher  $D_f$ . From here, another adversary  $A \in \text{PPT}$  can use  $D_f$  to effectively distinguish the original ensembles, as depicted in figure 4.5:

1.  $A$  asks for the original sample from the challenger
2.  $A$  applies  $f$  on the sample
3.  $A$  relays the resulting image to  $D_f$
4.  $D_f$  replies with his outcome
5.  $A$  relays the outcome to the challenger

All of this is done in polynomial time, since all functions and machines involved in the process operate in PPT. This contradicts the computational indistinguishability of  $X$  and  $Y$ .

□

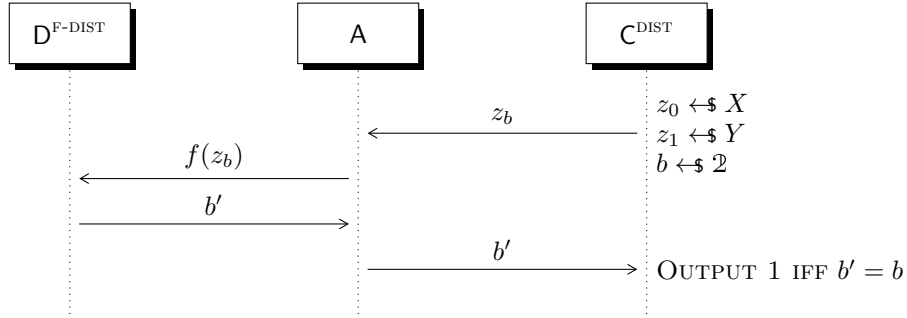


Figure 4.5: Distinguisher reduction

## 4.4 Pseudo-random generators

A deterministic function  $G \in 2^\lambda \rightarrow 2^{\lambda+l(\lambda)}$  is called a *pseudo-random generator*, or PRG in short, iff:

- $G \in \text{PPT}(\lambda)$
- $|G(s)| = \lambda + l(\lambda)$
- Given  $U_n$  to be a distribution ensemble of  $n$  uniform random variables:

$$G(U_\lambda) \approx_c U_{\lambda+l(\lambda)}$$

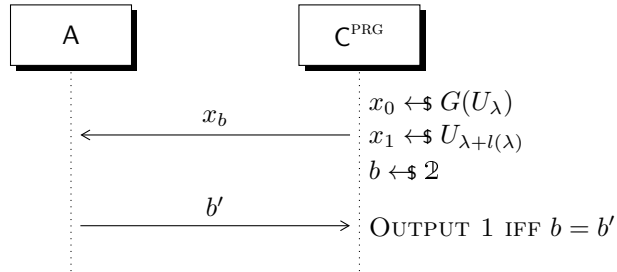


Figure 4.6: The pseudorandom game

So, if we take  $s \leftarrow U_\lambda$ , the output of  $G$  will be indistinguishable from a random pick from  $U_{\lambda+l(\lambda)}$ .

# Lesson 5

This chapter/lesson is devoted in constructing PRGs. We begin by assuming to have already a PRG  $G \in \mathcal{2}^\lambda \rightarrow \mathcal{2}^{\lambda+1}$ , that extends the string length by one bit, and prove that it is possible to extend such string by an indefinite amount while preserving pseudo-randomness.

## 5.1 Stretching a PRG

Consider this algorithm that uses  $G$  to construct  $G^l$ , as depicted in figure 5.7:

1. Let  $s_0 \leftarrow \mathcal{2}^\lambda$
2.  $\forall i \in [l(\lambda)]$ 
  - (a) let  $G(s_{i-1}) = (s_i, b_i)$ , where  $b_i$  is the extra bit generated by a single use of  $G$
3. Compose  $(b_1, b_2, \dots, b_{l(\lambda)}, s_{l(\lambda)})$ . This will be the returned string, which is  $\lambda + l(\lambda)$  bits long

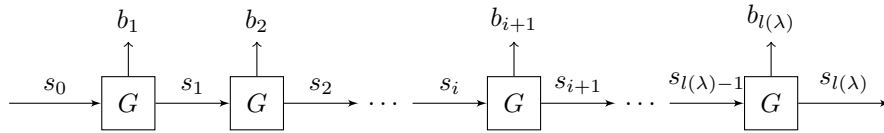


Figure 5.7: Constructing  $G^{l(\lambda)}$  from  $G(\lambda)$

To prove that this construct is a valid PRG, we will make use of a known technique for proving many other results, which relies heavily on reductions like the one employed back in the OWF topic, and is commonly called the “*hybrid argument*”.

**Lemma 3** (*Hybrid argument*). *Let  $X$ ,  $Y$  and  $Z$  be three any distribution ensembles of the same length. The following is true:*

$$X \approx_c Y \wedge Y \approx_c Z \implies X \approx_c Z \quad \diamond$$

*Proof.*  $\forall D \in \text{PPT}$ , by using the triangle inequality:

$$\begin{aligned}
& |\Pr[D(X) = 1] - \Pr[D(Z) = 1]| \\
&= |\Pr[D(X) = 1] - \Pr[D(Y) = 1] + \Pr[D(Y) = 1] - \Pr[D(Z) = 1]| \\
&\leq |\Pr[D(X) = 1] - \Pr[D(Y) = 1]| + |\Pr[D(Y) = 1] - \Pr[D(Z) = 1]| \\
&\leq \nu(n) + \nu'(n)
\end{aligned}$$

where  $\nu, \nu' \in \mathcal{Negl}(n)$ . By the sum property of negligible functions, the result is still negligible, proving the lemma.  $\square$

In essence, the hybrid argument proves that computational indistinguishability is a transitive relationship, which enables us to design “hybrid” games in order to bridge differences two arbitrary ones. This property will be very useful in all future proofs, as it will be shown for the coming theorem:

**Theorem 8.** *If there exists a PRG  $G(\lambda)$  with one bit stretch, then there exists a PRG  $G^{l(\lambda)}$  with polynomial stretch relative to its input length:*

$$G : 2^\lambda \rightarrow 2^{\lambda+1} \implies \forall l(\lambda) \in \text{Poly}(\lambda) \exists G^l \in 2^\lambda \rightarrow 2^{\lambda+l(\lambda)} \quad \diamond$$

*Proof.* First off, do observe that, since both  $G$  and  $l$  are polynomial in  $\lambda$ , then so is  $G^{l(\lambda)}$ , because it combines  $G$   $l(\lambda)$ -many times. To prove that  $G^{l(\lambda)}$  is indeed a PRG, we will apply the hybrid argument. The hybrids are defined as:

- $H_\lambda^0 := G^{l(\lambda)}(U_\lambda)$ , which is the original construct
- $H_\lambda^i := \begin{cases} b_1, \dots, b_i \leftarrow \mathbb{2} \\ s_i \leftarrow 2^{\lambda+i} \\ (b_{i+1}, \dots, b_{l(\lambda)}, s_{l(\lambda)}) := G^{l(\lambda)-i}(s_i) \end{cases}$
- $H_\lambda^{l(\lambda)} := U_{\lambda+l}$

Focusing on two subsequent generic hybrids, as shown in figure 5.8, it can be observed that the only difference between the two resides in how  $b_{i+1}$  is generated: in  $H^i$  it comes from an instance of  $G$ , whereas in  $H^{i+1}$  is chosen at random.  $H_\lambda^0$  is the starting point where all bits are pseudorandom, which coincides with the  $G^{l(\lambda)}$ , and  $H_\lambda^{l(\lambda)}$  will generate a totally random string.

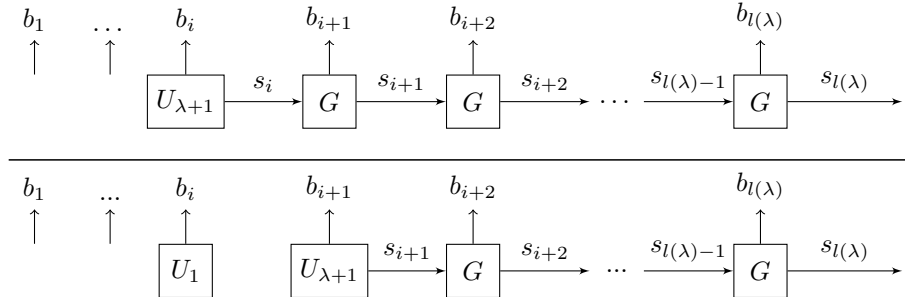


Figure 5.8:  $H_\lambda^i$  and  $H_\lambda^{i+1}$

So let's fix a step  $i$  in the gradual substitution, and define the following function  $f_i$ :

$$f_i(s_{i+1}, b_{i+1}) = (b_1, \dots, b_i, b_{i+1}, b_{i+2}, \dots, b_{l(\lambda)}, s_{l(\lambda)})$$

where the first  $i$  bits are chosen uniformly at random, and the remaining ones are obtained by subsequent applications of  $G$ . It can be observed that:

- $f_i(G(U_\lambda))$  has the exact same distribution of  $H_\lambda^i$
- $f_i(U_{\lambda+1})$  has the exact same distribution of  $H_\lambda^{i+1}$

Since by PRG definition  $G(U_\lambda) \approx_c U_{\lambda+1}$ , by using the lemma 2 we can deduce that  $f_i(G(U_\lambda)) \approx_c f_i(U_{\lambda+1})$ , which in turn, by how  $f$  is defined, implies  $H^i \approx_c H^{i+1}$ . This holds for an arbitrary choice of  $i$ , so by extension:

$$G^{l(\lambda)}(U_\lambda) = H_\lambda^0 \approx_c H_\lambda^1 \approx_c \dots \approx_c H_\lambda^{l(\lambda)} = U_{\lambda+l(\lambda)}$$

which proves that  $G^l$  is indeed a PRG.  $\square$

*Proof. (Contradiction):* This is an alternate proof that, instead of looking for a function  $f$  to model hybrid transitioning, aims for a contradiction.

Suppose  $G^l$  is not a PRG; then there must be a point in the hybrid chain  $H_\lambda^0 \approx_c \dots \approx_c H_\lambda^l$  where  $H_\lambda^i \not\approx_c H_\lambda^{i+1}$ . Thus there exists a distinguisher  $D^{l\text{-TH}}$  able to tell apart  $H_\lambda^i$  from  $H_\lambda^{i+1}$ , as shown in figure 5.9:

$$\exists i \in [0, l], \exists D^{l\text{-TH}} \in \text{PPT} : |\Pr[D^{l\text{-TH}}(H_\lambda^i) = 1] - \Pr[D^{l\text{-TH}}(H_\lambda^{i+1}) = 1]| \notin \text{Negl}(\lambda)$$

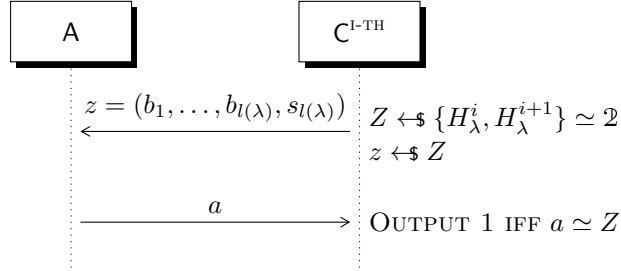


Figure 5.9: Distinguisher for  $H_\lambda^i$  and  $H_\lambda^{i+1}$

If such a distinguisher exists, it can be also used to distinguish an output of  $G$  from a  $\lambda + 1$  uniform string by “crafting” a suitable bit sequence, which will distribute exactly as the hybrids in question, as shown in the reduction in figure 5.10. This contradicts the hypothesis of  $f$  being a PRG, which by definition is to be indistinguishable from a truly random distribution. Therefore,  $G^l$  is indeed a PRG.  $\square$

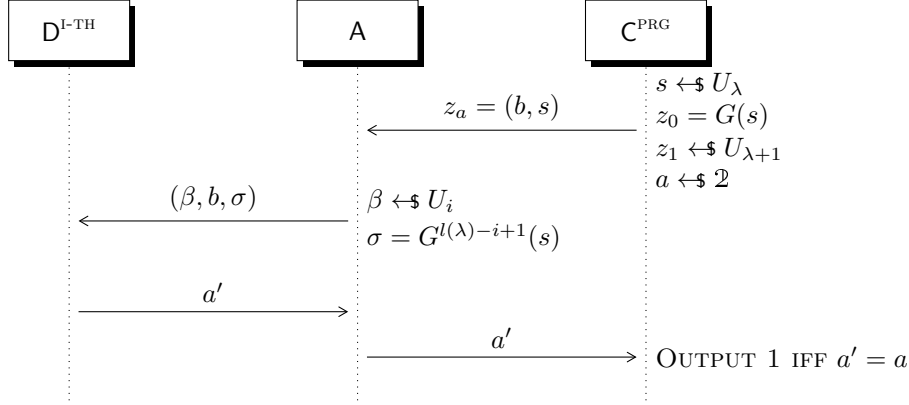


Figure 5.10: Reducing to a distinguisher for  $G$ , where  $\beta = (b_1, \dots, b_{i-1})$  and  $\sigma = (b_{i+1}, \dots, b_{l(\lambda)}, s_{l(\lambda)})$

## 5.2 Hardcore predicates

Now that we've seen how to reuse a one-bit stretch PRG in order to obtain an arbitrary length of pseudorandom bits, we turn to the problem of constructing a 1-bit stretch PRG itself. Let  $f$  be a OWF, and consider the following questions:

- Given an image  $f(x)$ , which bits of the input  $x$  are hard to extract?
- Is it always true that, given  $f$ , the first bit of  $f(x)$  is hard to compute for any choice of  $x$ ?

**Example 1.** Given a OWF  $f$ ,  $f'(x) = x_0 || f(x)$  is a OWF too.

Two definitions for hardcore predicates are given:

**Definition 13.** Let  $f : \mathbb{2}^n \rightarrow \mathbb{2}^n$  be a poly-time complex function. A poly-time complex predicate  $\mathfrak{hc} : \mathbb{2}^n \rightarrow \mathbb{2}$  is said to be *hard-core* for  $f$  iff:

$$\forall A \in \text{PPT} \implies \Pr(A(f(x)) = \mathfrak{hc}(x) \mid x \leftarrow \mathbb{2}^n) \in \text{Negl}(\lambda)$$

◇

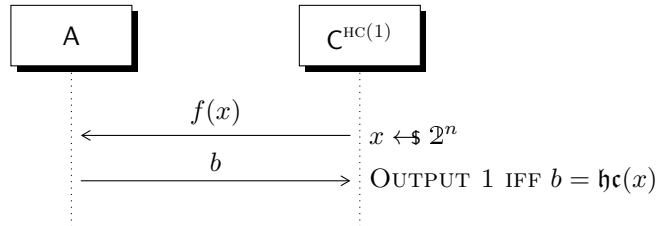


Figure 5.11: The hardcore game,  $f$  and  $\mathfrak{hc}$  are known

**Definition 14.** A polynomial time function  $\mathfrak{hc} : \mathbb{2}^n \rightarrow \mathbb{2}$  is hard-core for a function  $f$  iff:

$$(f(X), h(X)) \approx_c (f(X), U_1)$$

where  $X$  is a uniform distribution ensemble over  $\mathbb{2}^n$ , and  $U_1 \sim \text{Unif}(\mathbb{2})$ .

◇



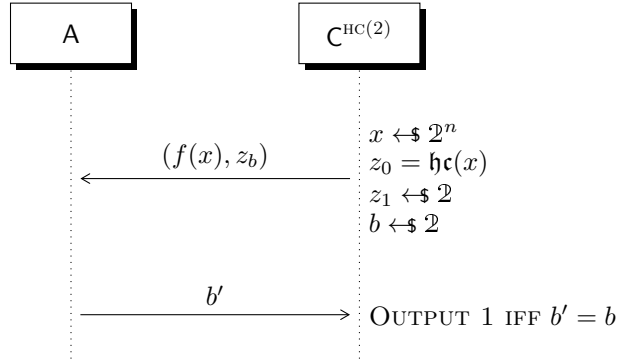


Figure 5.12: Another hardcore game,  $f$  and  $\mathfrak{hc}$  are known

Having made this definition, some observations are in order: we're going to rule out a cheesy solution

**Claim 2.** *There is no universal hardcore predicate  $\mathfrak{HC}$  for all functions.*  $\diamond$

*Proof.* Suppose there exists such a predicate  $\mathfrak{HC}$ . Let  $f'(x) = \mathfrak{HC}(x) || f(x)$  for a given function  $f$ . Then  $\mathfrak{HC}$  cannot be a hardcore predicate of  $f'$ , because any image obtained by  $f$  reveals the predicate's image itself. This contradicts the universality of  $\mathfrak{HC}$ .  $\square$

However, it is always possible to construct a hardcore predicate for a OWF, from another OWF:

**Theorem 9** (Goldreich-Levin, '99). *Let  $f$  be a OWF and consider  $g(x, r) = (f(x), r)$  for  $r \in \mathbb{Z}^n$ . Then  $g$  is a OWF, and:*

$$h(x, r) = \langle x, r \rangle = \bigoplus_{i=1}^n x_i \oplus r_i = \sum_{i=1}^n x_i \oplus r_i \bmod 2$$

*is hardcore for  $g$ .*  $\diamond$

*Proof.* TODO 8: TO BE COMPLETED (...did we actually do this? è una bella menata dimostrare questo)

$\square$

**Exercise 10.** Prove that  $f \in \text{OWF} \implies g \in \text{OWF}$  (Hint: do a reduction).

**Solution 3 (10).** Let  $D^{\text{G-OWF}}$  be a machine that is efficient in inverting  $g$ , and consider the reduction shown in figure 5.13. By how  $g$  is defined,  $r'$  must be equal to  $r$ ; therefore  $x'$  must be a valid pre-image of  $y$  in  $f$ . This contradicts the property of  $f$  being a OWF.

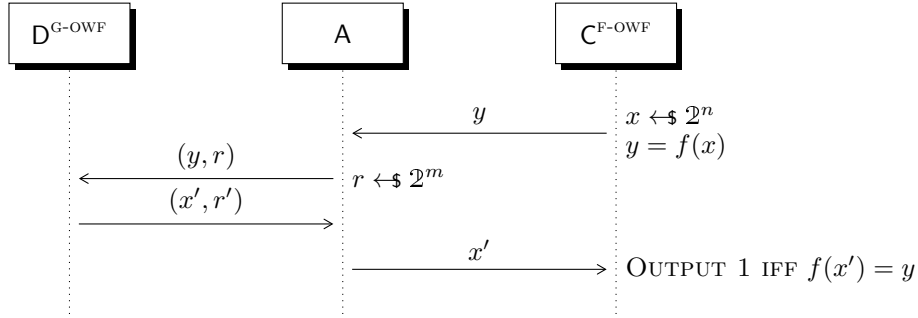


Figure 5.13: Efficiently inverting  $f$

### 5.2.1 One-way permutations

A *one-way permutation*, or OWP in short, is defined exactly as the name itself suggests: a bijective OWF.

$$f \in \mathcal{2}^n \leftrightarrow \mathcal{2}^n \wedge f \in \text{OWF} \implies f \in \text{OWP}$$

**Corollary 1.** *If  $f \in \mathcal{2}^n \rightarrow \mathcal{2}^n$  is a OWP then, by the theorem of Goldreich-Levin defining  $g(), h()$ :*

$$G(s) = (g(s), h(s))$$

*is a PRG.*

◇

*Proof.* The aforementioned theorem states that if  $f$  is an OWF, then so is  $g$ . It's trivial to prove the analogue for OWPs. Moreover,  $h$  is hardcore for  $g$ , thus:

$$\begin{aligned} G(U_{2n}) &\equiv (g(U_{2n}), h(U_{2n})) \\ &\equiv (f(U_n), U_n, h(U_{2n})) \\ &\approx_c (f(U_n), U_n, U_1) \quad (\text{definition 1 of hardcore predicate}) \\ &\equiv U_{2n+1} \end{aligned}$$

□

We've been successful in constructing a 1-bit stretch PRG; from here, by using the results in the previous section, we can construct a PRG that returns binary strings of an arbitrary length that are also pseudo-random.

# Lesson 6

## 6.1 Computationally secure encryption

Having a better idea of what can and can't be accomplished in the cryptographic world, by means of theorems and proofs, we can focus now on the goal of defining a cryptographic system that meets our requirements. In this lesson, we focus specifically on the secrecy-oriented schemes, thus dealing with encryption and decryption of messages.

The requirements of a “good” encryption scheme are collectively called those of *computationally secure encryption*: the characterizing requirement is to design a task, or routine, that is *computationally hard* for an attacker to revert. In detail: this task usually involves a secret key<sup>6</sup>, and is accomplished in polynomial time, and any attacker who wishes to revert it has no efficient means of doing it without knowing such key. Other properties include:

1. *one-wayness* with respect to the encryption key: given  $c = \text{Enc}(k, m)$ , it should be hard to recover  $k$
2. *one-wayness* with respect to the original message: given  $c = \text{Enc}(k, m)$ , it should be hard to recover  $m$
3. In a stricter sense: no information whatsoever must “leak” from the message

To start visualizing these concepts, let  $\Pi = (\text{Enc}, \text{Dec})$  be a secrecy scheme, and consider the game depicted in figure 6.14 where the adversary “wins” the game when the challenger outputs 1.

**Definition 15.** The scheme  $\Pi$  is said to be *computationally one-time secure* iff:

$$\forall A \in \text{PPT} \implies \text{GAME}_{\Pi, A}^{\text{IND}}(\lambda, 0) \approx_c \text{GAME}_{\Pi, A}^{\text{IND}}(\lambda, 1)^7$$

or, rephrased in probability terms:

$$\forall A \in \text{PPT} \implies |\Pr[\text{GAME}_{\Pi, A}^{\text{IND}}(\lambda, 0) = 1] - \Pr[\text{GAME}_{\Pi, A}^{\text{IND}}(\lambda, 1) = 1]| \in \text{Negl}(\lambda) \diamond$$

This last definition shows how such a scheme is compliant with the three properties exposed beforehand. In particular:

<sup>6</sup>This is the case for symmetric-key schemes, though many other kinds exist: some involving “public” keys, some others not having any key at all

<sup>7</sup> $\text{GAME}_{\Pi, A}^{\text{IND}}$  refers to the indistinguishability of the messages sent by  $A$  during the game

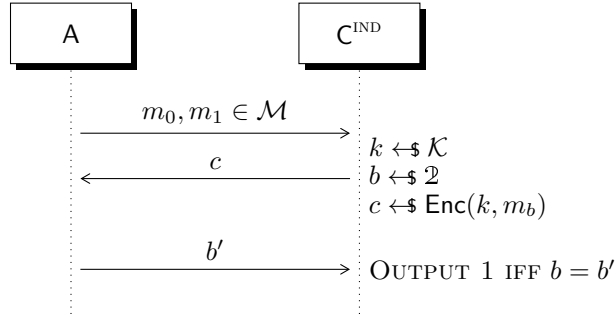


Figure 6.14:  $\text{GAME}_{\Pi, A}^{\text{IND}}(\lambda, b)$

1. *It is hard to recover the key.* If not, then an adversary **A** can efficiently recover the key and use it to decrypt the ciphertext, which in turn enables him to perfectly distinguish  $m_0$  from  $m_1$  on any instance;
2. *It is hard to recover the message.* This is analogous, and even more obvious than the preceding point. Nevertheless, this is a necessary condition for a secrecy scheme to be “good”, and it mustn’t be forgotten;
3. *No information about the message whatsoever may leak from the ciphertext.* This may seem subtler than the previous point, but it warrants caution. Observe how an adversary **A**, if it has the ability to extract even a tiny bit of information of the original message from the ciphertext, then it is actually able to make an educated guess on which message was encrypted in the first place, putting him at an advantage. This leads the probabilities described in the definition to be sensibly more unbalanced than negligible, forfeiting the desired secrecy.

By extension, we may ask ourselves what scheme may or may not be *computationally two-time secure*. For instance, let  $\Pi_{\oplus} = (\text{Enc}, \text{Dec})$  be a secrecy scheme using a PRG  $G : \mathbb{2}^{\lambda} \rightarrow \mathbb{2}^n$ , structured as follows:

- $\mathcal{K} = \mathbb{2}^{\lambda}$ ,  $\mathcal{M} = \mathcal{C} = \mathbb{2}^n$
- $\text{Enc}(k, m) = G(k) \oplus m$
- $\text{Dec}(k, c) = c \oplus G(k) = m$

To be two-time secure means that, even if an adversary **A** gets hold of a valid plaintext-ciphertext couple  $(\bar{m}, \bar{c})$ , he is unable to decrypt any future ciphertexts<sup>8</sup>, apart from the obvious  $\bar{c}$ . However, observe that **A** is now able to extract valuable information for decrypting future ciphertexts:

$$\bar{c} = \text{Enc}(k, \bar{m}) = G(k) \oplus \bar{m} \implies \bar{c} \oplus \bar{m} = G(k)$$

so now, for any second ciphertext **A** receives, he can “mimick” the decryption routine, and efficiently uncover the underlying plaintext. This proves that  $\Pi_{\oplus}$  is not two time-secure; nevertheless, it is still one-time secure:

<sup>8</sup>This example models a technique called “Chosen Plaintext Attack”, which will be discussed in depth later

**Theorem 11.** *If  $G$  is a PRG, then  $\Pi_{\oplus}$  is computationally one-time secure*  $\diamond$

*Proof.* This proof is another example that showcases the use of hybrid games. Recalling the one-time security definition, we need to show that:

$$\forall A \in \text{PPT} \implies \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 0) \approx_c \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 1)$$

Consider the hybrid game in figure 6.15, where the original encryption routine is changed to use a completely random value, instead of using  $G(k)$ <sup>9</sup>. As an exercise, compare it with the original one-time secure definition in figure 6.14, to check that it perfectly matches.

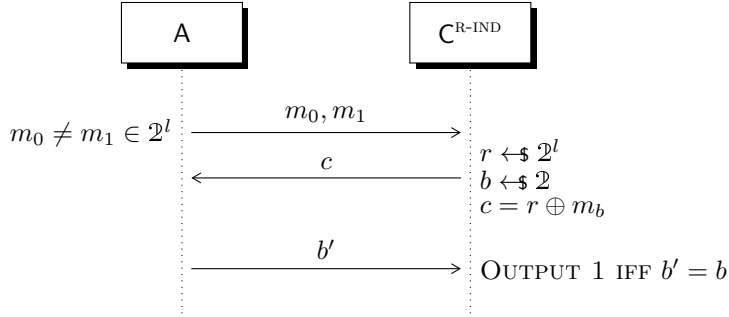


Figure 6.15:  $\text{HYB}_{\Pi_{\oplus}, A}(\lambda, b)$

The proof begins by affirming that:

**Claim 3.**

$$\forall A \in \text{PPT} \implies \text{HYB}_{\Pi_{\oplus}, A}(\lambda, 0) \equiv \text{HYB}_{\Pi_{\oplus}, A}(\lambda, 1)$$

$\diamond$

To prove it, notice that  $r$  is chosen uniformly at random, and independently of  $b$ . Thus, no matter how the messages  $m_0$  and  $m_1$  are structured,  $r$  will effectively make the chosen message completely unrecognizable. In formal terms, let  $B$  and  $R$  be the random variables for  $C$ 's picks of  $b$  and  $r$  respectively; then:

$$\begin{aligned} & |\Pr(B = 0 \mid C = c) - \Pr(B = 1 \mid C \not\rightarrow c)| \\ &= |\Pr(B = 0 \mid R \oplus m_B \not\rightarrow c) - \Pr(B = 1 \mid R \oplus m_B \not\rightarrow c)| \quad (\text{C definition}) \\ &= \frac{|\Pr(R \oplus m_B = c \mid B \not\rightarrow 0) \Pr(B = 0) - \Pr(R \oplus m_B = c \mid B \not\rightarrow 1) \Pr(B = 1)|}{\Pr(R \oplus m_B = c)} \\ & \quad \quad \quad (\text{Bayes' theorem}) \\ &= \frac{|\Pr(R \oplus m_0 = c) \Pr(B = 0) - \Pr(R \oplus m_1 = c) \Pr(B = 1)|}{\Pr(R \oplus m_B = c)} \quad (\text{Cond. collapse}) \\ &= \frac{|\frac{1}{2^l} \frac{1}{2} - \frac{1}{2^l} \frac{1}{2}|}{\Pr(R \oplus m_B = c)} = 0 \end{aligned}$$

Having proven that  $A$ 's success is equivalent to straight guessing in  $\text{HYB}_{\Pi_{\oplus}, A}$ , we now relate the hybrid game to the original one, affirming that:

<sup>9</sup>The observant student may recognize that this modification yields exactly the “one-time pad” secrecy scheme discussed in lesson 1

**Claim 4.**

$$\forall A \in \text{PPT}, \forall b \in \mathbb{2} \implies \text{HYB}_{\Pi_{\oplus}, A}(\lambda, b) \approx_c \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, b) \quad \diamond$$

The proof proceeds by reduction as depicted in figure 6.16, by assuming the existence of a distinguisher  $D^{\text{IND}}$  for  $c = G(k) \oplus m_b$  and  $c = r \oplus m_b$ , and using it to break  $G$ 's pseudo-random generation property.

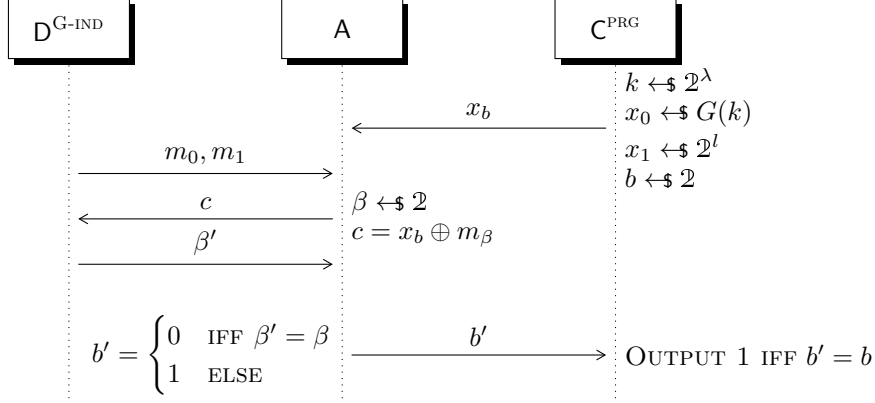


Figure 6.16: Reducing to breaking a PRG

Again, notice how the games of indistinguishability and pseudo-random generation are reliably reproduced on their respective sides. This construct's value centers on how  $D^{\text{IND}}$  will perform in its own challenge:

- if  $x_b$  is a random value, then  $D^{\text{IND}}$  will perform as depicted in the hybrid game, thus giving right or wrong answers at random;
- if  $b$  is the result of  $G$ , then  $D^{\text{IND}}$  has a better chance in finding the right answer by its own design;

From these observations, especially the second point, the adversary  $A$  has a better chance of winning the PRG game by asserting that  $x_b$  comes from  $G$  whenever  $D^{\text{IND}}$  makes a correct guess; conversely,  $A$  will preferably declare that  $x_b$  is truly random whenever  $D^{\text{IND}}$  fails a guess, as the probability of the latter getting fooled by a random value is sensibly greater than by a pseudo-random one.

Either way, by the existence of  $D^{\text{IND}}$ ,  $A$  gains an edge in efficiently recognizing  $G$ , which cannot happen by  $G$ 's definition; the claim is proven. The theorem's proof can be completed by putting the pieces together, as is usual in the hybrid argument:

$$\text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 0) \approx_c \text{HYB}_{\Pi_{\oplus}, A}(\lambda, 0) \equiv \text{HYB}_{\Pi_{\oplus}, A}(\lambda, 1) \approx_c \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 1)$$

$$\text{which finally states that } \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 0) \approx_c \text{GAME}_{\Pi_{\oplus}, A}^{\text{IND}}(\lambda, 1). \quad \square$$

## 6.2 Pseudorandom functions

PRGs are used in practice as a stepping stone for building *pseudo-random functions*, PRF henceforth, which are the principal construct in several cryptographic

schemes. Before formally introducing what a PRF is, we begin instead by defining what a *truly random function* is:

**Definition 16.** A random function  $R : \mathcal{X} \rightarrow \mathcal{Y}$  is a function that, depending on what is known about its previous applications:

- if  $x$  is “fresh” (in formal terms,  $R$  has never been applied to  $x$  beforehand), then a value  $y$  is chosen UAR from  $R$ ’s codomain, and it is permanently associated as the image of  $x$  in  $R$ <sup>10</sup>;
- if  $x$  is not fresh, then  $R(x)$  is directly returned instead.  $\diamond$

It should be noted that, if such functions are to be implemented in computers, they would occupy too much space in memory. Suppose all the possible outputs of  $R$  have been generated and stored as an array in memory; then its total size in bits will be  $l \cdot 2^n$ :



Such a function becomes cumbersome and difficult to maintain in practice; therefore, it is desirable to find a kind of function which looks as a random function possible, but does not require to be wholly memorized, while not forgetting to maintain poly-time complexity. Pseudo-randomness comes to the rescue here:

**Definition 17.** Let  $f$  be a function, then it is deemed pseudo-random (therefore,  $f$  is a PRF) iff it is computationally indistinguishable from a true random function.  $\diamond$

In detail, PRFs are actually designed as function families  $f_k$ <sup>11</sup>, where  $k$  is a parameter that indexes the functions inside the family. To model the PRFs’ indistinguishability from random functions, let  $F \in \mathcal{X} \rightarrow (\mathcal{Y}^{n(\lambda)} \rightarrow \mathcal{Y}^{l(\lambda)})$ , usually denoted simply by  $f_k$ , be a PRF, and define  $\mathcal{R}(n, l)$  to be the domain that collects the random functions from  $\mathcal{X}^{n(\lambda)}$  to  $\mathcal{Y}^{l(\lambda)}$ .

Consider the indistinguishability game drawn in figure 6.17; although one may think that PRGs and PRFs aren’t much different, the game tells a different story, which is best put by an introductory paragraph about PRFs in their Wikipedia page:

“Pseudorandom functions are not to be confused with pseudorandom generators (PRGs). The guarantee of a PRG is that a single output appears random if the input was chosen at random. On the other hand, the guarantee of a PRF is that all its outputs appear random, regardless of how the corresponding inputs were chosen, as long as the function was drawn at random from the PRF family.”<sup>12</sup>

In this game,  $A$  is allowed to make multiple queries to  $C^{\text{PRF}}$ , as opposed to the PRG indistinguishability game where it can perform just one query before making its guess. To reiterate the PRF definition in terms of this game:

<sup>10</sup>This property is also called *lazy sampling*.

<sup>11</sup>Does this remind you of something else? If not, look back in lesson 2 and 3.

<sup>12</sup>➤ [Pseudorandom function family — Wikipedia](#)

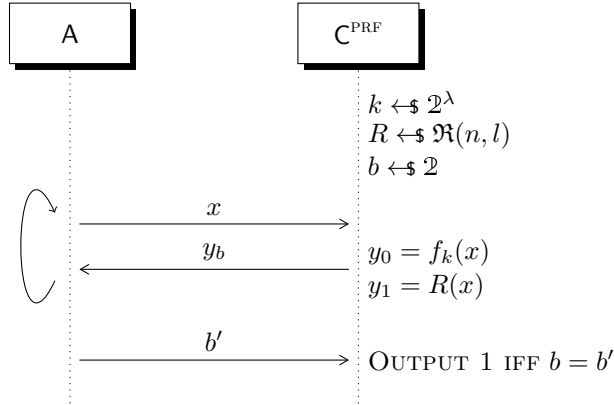


Figure 6.17: The PRF indistinguishability game

**Definition 18.** A function family  $F = f_k$  is a PRF iff:

$$\text{GAME}_{f_k, A}^{\text{PRF}}(\lambda, 0) \approx_c \text{GAME}_{f_k, A}^{\text{PRF}}(\lambda, 1) \quad \diamond$$

**Exercise 12.** Prove the following statements:

- No PRG is secure against unbounded attackers;
- No PRF is secure against unbounded attackers.

### 6.2.1 GGM-tree

This section is dedicated to a concrete example of a PRF which is built from the ground up using a PRG. This construct has been designed from Oded Goldreich, Shafi Goldwasser and Silvio Micali, and its structure is akin to a binary tree, hence its name: *GGM-tree*.

**Construction 1.** Let  $G \in 2^\lambda \rightarrow 2^{2\lambda}$  be a PRG such that it doubles the length of its argument, and denote the images' first and second halves as  $G_0(k)$  and  $G_1(k)$  respectively, so that:

$$k \mapsto (G_0(k), G_1(k))$$

Since the principal mechanism makes use of the halves being the same length of the argument, in the same spirit, we will denote the action of using one half of an image of  $G$  as argument of  $G$  itself in a shorter fashion, as demonstrated in the following example:

$$G_a(G_b(G_c(k))) =: G_{abc}(k)$$

This leads to the final step: let  $f_k$  be a function family, where  $k \in 2^\lambda$ , such that:

$$f_k(r) = G_r(k)$$

This is our candidate PRF. To visualize it, consider the tree structure depicted in figure 6.18: at each level of the tree, a single bit of  $r$  is used to decide which half of  $G$ 's image will be used in the next level. For example,  $f_k(01 \dots 10)$  would evaluate as  $G_0(G_1(\dots G_1(G_0(k))))$ .

◇



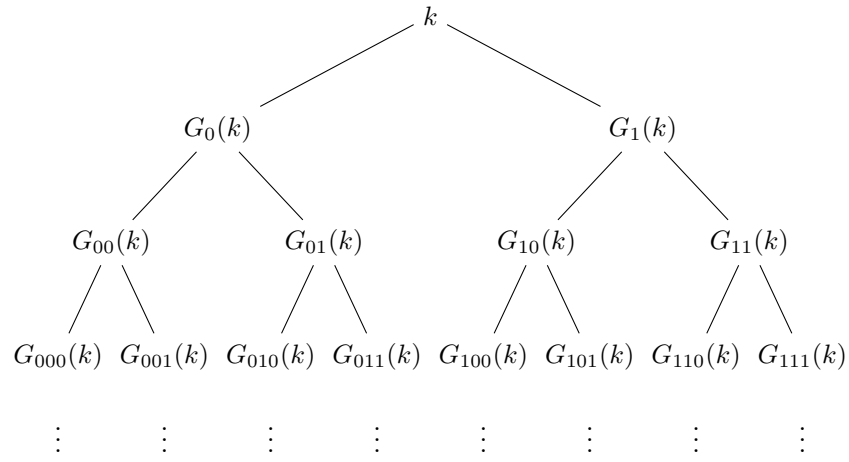


Figure 6.18: The GGM-tree for  $G$

The proof of PRF-ness will be discussed in the next lesson.

# Lesson 7

## GGM-tree (cont'd)

As stated in the previous lesson, given a PRG  $G \in \mathcal{2}^\lambda \rightarrow \mathcal{2}^{2\lambda}$ , we can build a function family  $f_k$  by repeatedly taking halves of  $G$ 's images, and plugging them back into  $G$ . Our goal is to prove the following theorem:

**Theorem 13.** *If  $G$  is a PRG, then  $f_k$  is a PRF.*  $\diamond$

*Proof.* Before starting to prove the PRF-ness of  $f_k$ , we make a brief consideration about its time complexity: computing  $f_k(x)$  consists in computing  $G$  and taking half of the resulting image as many times as is the length of  $x$ . Since the length of  $x$  is polynomial in  $\lambda$ , so is the number of  $G$ 's iterations; combine this with the fact that  $G$  is itself polynomial by definition, and we conclude that  $f_k$  is polynomial too.

Having cleared any doubts about  $f_k$ 's time complexity, we now turn to the essential point of interest: its pseudo-randomness. The proof will proceed by induction over the length of  $x$ , which is also the height of the tree-like structure modeling the algorithm.

**Base case** ( $n = 1$ ):  $f_k$ 's domain is restricted to  $\mathcal{2}$ , meaning that its images will be respectively the two halves on a single iteration of  $G(k)$ ; they are, of course, pseudorandom by  $G$ 's definition:

$$(f_k(0), f_k(1)) = (G_0(k), G_1(k)) \approx_c U_{2\lambda}$$

therefore, in this case,  $f_k$  is pseudorandom.

**Inductive step:** Let  $f'_k : \mathcal{2}^{n-1} \rightarrow \mathcal{2}^\lambda$  be a PRF. Define  $f_k$  as follows:

$$f_k : \mathcal{2}^n \rightarrow \mathcal{2}^\lambda : (b, x) \in \mathcal{2} \times \mathcal{2}^{n-1} \mapsto G_b(f'_k(x))$$

It must be proven that if  $f'_k$  is a PRF, then so is  $f_k$ . To help ourselves, we'll define some hybrid games as usual. These are depicted in figures 7.19 and 7.20.

We are going to prove, in the scope of a single induction step, that these hybrids bridge indistinguishability for the original game. It is all that's needed to complete the proof.

**Lemma 4.**  $\text{HYB}_{f_k, A}^1(\lambda, 0) \approx_c \text{HYB}_{f_k, A}^1(\lambda, 1)$   $\diamond$

*Proof.* Assume  $\exists \text{D}^{\text{N-TH}} \in \text{PPT}$  that can distinguish  $f_k$  from  $\overline{R} \circ G$  at the  $n$ -th step; then an adversary  $A$  can use it to break in turn  $f'_k$ 's PRF-ness, and distinguish it from  $\overline{R} \circ G$  as shown in figure 7.21:  $\square$

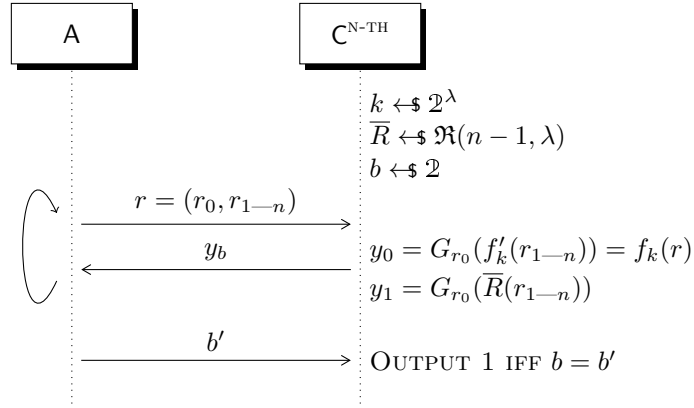


Figure 7.19:  $\text{HYB}_{f_k, A}^1(\lambda, b)$ : The GGM construct is put against randomly driven PRG

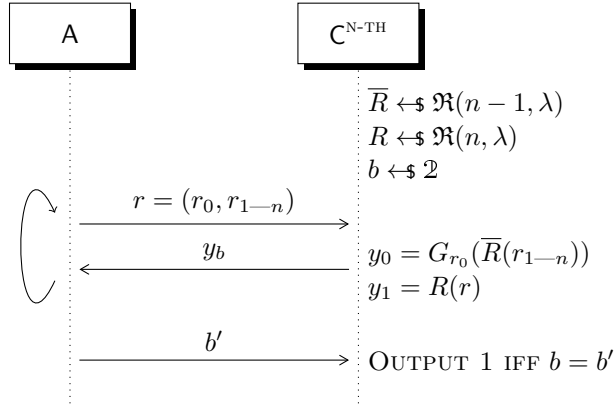


Figure 7.20:  $\text{HYB}_{R \circ G, A}^2(\lambda, b)$ : The randomly driven PRG against a true random function

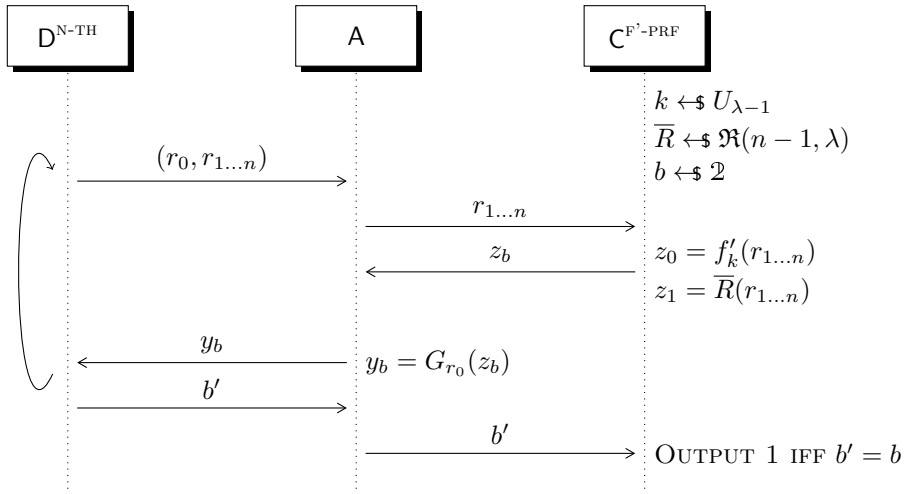


Figure 7.21: Using  $D^{N-TH}$  to break  $f'_k$

Before tackling  $\text{HYB}_{\bar{R},A}^2(\lambda, b)$ , it is best to introduce another lemma:

**Lemma 5.** *If  $G : \mathbb{2}^\lambda \rightarrow \mathbb{2}^{2\lambda}$  is a PRG, then:*

$$\forall K_i \sim \text{Unif}(\lambda) \text{IID} \implies (G(K_1), \dots, G(K_t)) \approx_c (U_{2\lambda}, \dots, U_{2\lambda}) \quad \diamond$$

*Proof.* TODO 9: Idea: all values are independent and pseudorandom on their own, hybridize progressively...

□

Now for the final lemma:

**Lemma 6.**  $\text{HYB}_{\bar{R} \circ G, A}^2(\lambda, 0) \approx_c \text{HYB}_{\bar{R} \circ G, A}^2(\lambda, 1)$  ◇

*Proof.* TODO 10: Need figures here

This proof is trickier: the problem lies in the inherent difference between the task of detecting a PRG from that of detecting a PRF. The aforementioned lemma helps us with the aspect of polynomial queries on a PRG, bringing it in line with the PRF game; yet, we're far from done.

Let's delve into the details: from A's perspective, the game of distinguishing  $\bar{R} \circ G$  from  $R$ , where D performs a number of queries  $q$  polynomial in  $\lambda$ , is perfectly modeled by the act of distinguishing a sequence of evaluations  $(G(K_1), \dots, G(K_q))$  from  $(U_{2\lambda}, \dots, U_{2\lambda})$ . Therefore, the game can be twisted in a way that A receives either one of the two whole sequences beforehand, and respects the rules by sending to D the right piece of information on each of its queries, in order to simulate the  $\bar{R} \circ G/R$  game correctly.

At this point, the simple way for A to send the right info on each query  $i$  would be to check whether the query's first bit  $r_0$  would be 0 or 1, and give back to D either the first or the second half of the  $i$ -th value of the sequence he received from C; this is exactly how GGM operates. However, there's a catch: D might make two queries where the  $r_1 \dots r_n$  parts are the same, the only difference is in the first bit: essentially, D is asking for both parts of  $G(r)$  from two distinct queries. In this case, if A just sends the  $i$ -th value's corresponding half, the simulation would break, because the halves come from different samples.

Thus, the adversary must keep track of which suffixes he has been already queried about, and make sure to send the value's other half (there are only two) whenever he is queried on a suffix for a second time. This final touch solves the problem of simulating the game for D, and now the task of A is changed to an equivalent one of breaking the lemma stated at the beginning.

TODO 11: Prata's notes:

Let T1, T2 be empty tables.

Given query  $x$  input  $x_1 \dots x_n$  let  $xbar = x_1 \dots x_i$

if  $xbar$  not in T1 then  $x_{i+1} \dots x_n$  binary $\lambda$  and add  $k_{xbar}$  to T2

if  $xbar$  in T2 let  $k_{xbar} = T2[T1[xbar]]$

output  $y = G_{x,n}(G_{x,n-1}(\dots G_{x,i+1}(k_{xbar})))$

H.0: GGM tree  
H.n: random function  
exploit lemma  $\text{prg} =_i \text{prf}$

□

In the end the hybrids are proven to mutually indistinguishable, therefore the inductive step is correct, proving the theorem.

□

## 7.1 CPA-security

Now it's time to define a stronger notion of security, which is widely used in cryptology for first assessments of cryptographic strength. Let  $\Pi := (\text{Enc}, \text{Dec})$  be a SKE scheme, and consider the game depicted in figure 7.22. Observe that this time, the adversary can “query” the challenger for the ciphertexts of any messages of his choice, with the only reasonable restriction that the query amount must be polynomially bound by  $\lambda$ . This kind of game/attack is called the *Chosen Plaintext Attack*, because of the adversary's capability of obtaining ciphertexts from messages. The usual victory conditions found in n-time security games, which are based on ciphertext distinguishability, apply.

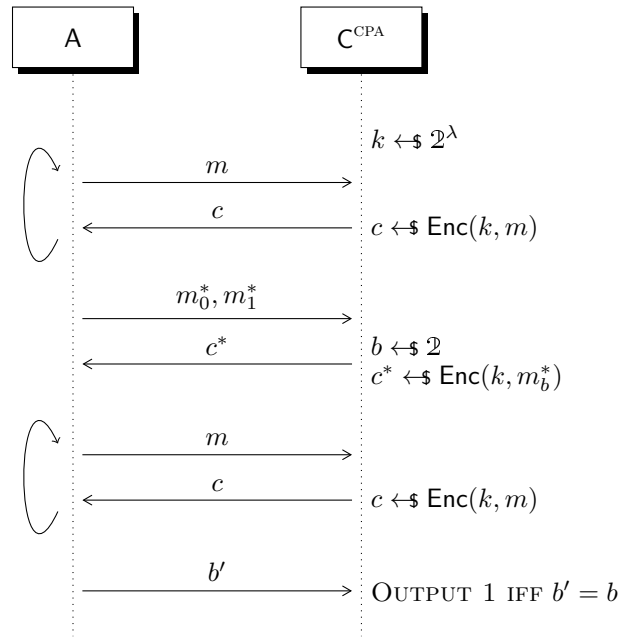


Figure 7.22: The CPA-security game:  $\text{GAME}_{\Pi, A}^{\text{CPA}}(\lambda, b)$

**Definition 19.** A scheme is CPA-secure if  $\text{GAME}_{\Pi, A}^{\text{CPA}}(\lambda, 0) \approx_c \text{GAME}_{\Pi, A}^{\text{CPA}}(\lambda, 1)$   $\diamond$

Having given this definition of security, recall the  $\Pi_{\oplus}$  scheme defined in the previous lesson. It is easy to see that  $\Pi_{\oplus}$  is not CPA-secure for the same reasons

that it is not computationally 2-time secure; however this example sheds some new light about a deeper problem:

**Observation 1.** *No deterministic scheme can achieve CPA-security.*  $\diamond$

This is true, because nothing prevents the adversary from asking the challenger to encrypt either  $m_0$  or  $m_1$ , or even both, before starting the actual challenge; just as in the 2-time case for  $\Pi_{\oplus}$ , he will know the messages' ciphertexts in advance, so he will be able to tell which message the challenger has encrypted every time. The solution for obtaining a CPA-secure encryption scheme consists of returning different ciphertexts for the same message, even better if they look random. This can be achieved by using PRFs.

Consider the following SKE scheme  $\Pi_{f_k}$ , where  $f_k$  is a PRF structured as follows:

- $\text{Enc}(k, m) = (c_1, c_2) = (r, f_k(r) \oplus m)$ , where  $k \leftarrow \mathfrak{K} 2^\lambda$  and  $r \leftarrow \mathfrak{K} 2^n$
- $\text{Dec}(k, (c_1, c_2)) = f_k(c_1) \oplus c_2$

Observe that the random value  $r$  is part of the ciphertext, making it long  $n + l$  bits; also more importantly, the adversary can and will always see  $r$ . The key  $k$  though, which gives a *flavour* to the PRF, is still secret.

**Theorem 14.** *If  $f_k$  is a PRF, then  $\Pi_{f_k}$  is CPA-secure.*  $\diamond$

*Proof.* We have to prove that  $\text{GAME}_{\Pi_{f_k}, \mathcal{A}}^{\text{CPA}}(\lambda, 0) \approx_c \text{GAME}_{\Pi_{f_k}, \mathcal{A}}^{\text{CPA}}(\lambda, 1)$ ; to this end, the hybrid argument will be used. Let the first hybrid  $\text{HYB}_{\Pi, \mathcal{A}}^0$  be the original game, the second hybrid  $\text{HYB}_{\Pi, \mathcal{A}}^1$  will have a different encryption routine:

- $r \leftarrow \mathfrak{K} 2^n$
- $R \leftarrow \mathfrak{R}(n, l)$
- $c = (r, R(r) \oplus m)$ , where  $m$  is the plaintext to be encrypted

and then the last hybrid  $\text{HYB}_{\Pi, \mathcal{A}}^2$  will simply output  $(r_1, r_2) \leftarrow \mathfrak{K} U_{n+l}$ .

**Lemma 7.**  $\forall b \in \mathbb{Z} \implies \text{HYB}_{\Pi, \mathcal{A}}^0(\lambda, b) \approx_c \text{HYB}_{\Pi, \mathcal{A}}^1(\lambda, b)$ .  $\diamond$

*Proof.* As usual, the proof is by reduction: suppose there exists a distinguisher  $D$  capable of telling the two hybrids apart; then  $D$  can be used to break  $f_k$ 's property of being a PRF. The way to use  $D$  is to make it play a CPA-like game, as shown in figure 7.23<sup>13</sup>, where the adversary attempting to break  $f_k$  decides which message to encrypt between  $m_0$  and  $m_1$  beforehand, and checks whether it guesses which message has been encrypted. Either way, the adversary can get a sensible probability gain in guessing if the received values from the challenger were random, or generated by  $f_k$ . Thus, assuming such  $D$  exists,  $\mathcal{A}$  can efficiently break  $f_k$ , contradicting its PRF-ness.  $\square$

**Lemma 8.**  $\forall b \in \mathbb{Z} \implies \text{HYB}_{\Pi, \mathcal{A}}^1(\lambda, b) \approx_c \text{HYB}_{\Pi, \mathcal{A}}^2(\lambda, b)$ .  $\diamond$

<sup>13</sup>An observant student may notice a striking similarity with a previously exposed reduction in figure 6.16

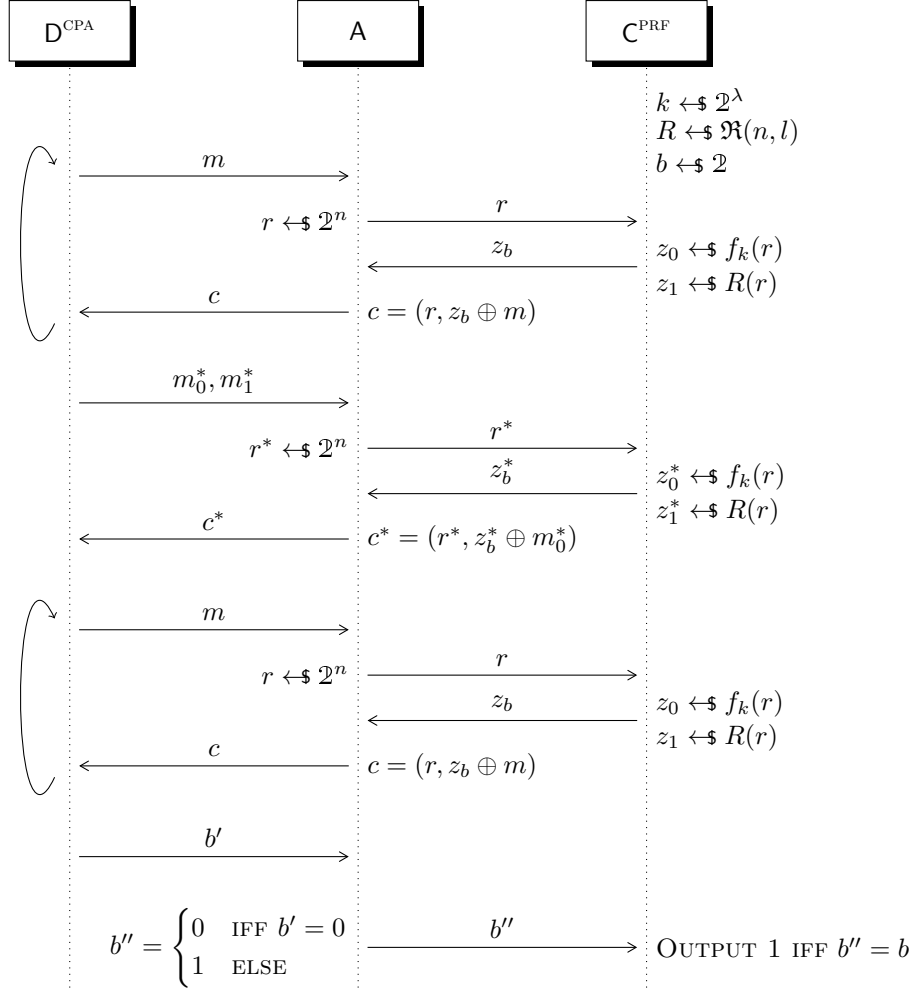


Figure 7.23: Breaking a PRF, for fixed message choice of  $m_0$

*Proof.* Firstly, it can be safely assumed that any ciphertext  $(r_i, R(r_i) \oplus m_b)$  distributes equivalently with its own sub-value  $R(r_i)$ , because of  $R$ 's true randomness, and independency from  $m_b$ .

Having said that, the two hybrids apparently distribute uniformly, making them perfectly equivalent; however there is a caveat: if both games are run and one value  $\bar{r}$  is queried twice in both runs, then on the second query the adversary will receive the same image in  $\text{HYB}_{\Pi, A}^1$ , but almost certainly a different one in  $\text{HYB}_{\Pi, A}^2$ . This is because the first hybrid uses a function, which is deterministic by its nature, whereas the image in the second hybrid is picked completely randomly from the codomain. Nevertheless, this sneaky issue about "collisions" can be proven to happen with negligible probability.

Call REPEAT this collision event on  $\bar{r}$  between 2 consecutive games. Then:

$$\begin{aligned}
\Pr[\text{REPEAT}] &= \Pr[\exists i, j \in q \text{ such that } r_i = r_j] \\
&\leq \sum_{i \neq j} \Pr[r_i = r_j] \\
&= \text{Col}(U_n) \\
&= \sum_{i \neq j} \sum_{e \in \mathcal{D}^n} \Pr[r_1 = r_2 = e] \\
&= \sum_{i \neq j} \sum_{e \in \mathcal{D}^n} \Pr[r = e]^2 \\
&= \binom{q}{2} 2^n \frac{1}{2^{2n}} \\
&= \binom{q}{2} 2^{-n} \\
&\leq q^2 2^{-n} \in \mathcal{Negl}\lambda
\end{aligned}$$

which proves that the REPEAT influences negligibly on the two hybrids' equivalence. Thus  $\text{HYB}_{\Pi, A}^1(\lambda, b) \approx_c \text{HYB}_{\Pi, A}^2(\lambda, b)$ <sup>14</sup>.

□

With the above lemmas, and observing that  $\text{HYB}_{\Pi, A}^2(\lambda, 0) \equiv \text{HYB}_{\Pi, A}^2(\lambda, 1)$ , we can reach the conclusion that  $\text{HYB}_{\Pi, A}^0(\lambda, 0) \approx_c \text{HYB}_{\Pi, A}^0(\lambda, 1)$ , which is what we wanted to demonstrate.

□

---

<sup>14</sup>Do note that the hybrids lose their originally supposed perfect equivalence ( $\text{HYB}_{\Pi, A}^1(\lambda, b) \equiv \text{HYB}_{\Pi, A}^2(\lambda, b)$ ) because of the REPEAT event. Nevertheless, the lemma is still proven because it includes computational bounds into  $A$



# Lesson 8

## 8.1 Domain extension

Up until now, encryption has been dealt with messages of fixed size around a polynomial function to  $\lambda$ . How to deal with messages with arbitrary size? Setting a maximum bound to message length seems impractical, both for waste reasons when messages are too short, and for practicality when messages eventually get too long. The solution takes the form of a “block-cipher”, where a message of a given size is split into equally-sized blocks, and then encrypted using a fixed-size encryption scheme. Various instances of this technique, called *modes*, have been devised.

### 8.1.1 Electronic Codebook mode

The operation of ECB-mode is straightforward: Given a message split into blocks  $(m_1, \dots, m_t)$ , apply the scheme’s encryption routine to each block, as shown in figure 8.24:

$$c_i = F_k(r) \oplus m_i \quad \forall i \in \{0, \dots, t\}$$

Decryption is trivially implemented by XOR-ing the ciphered blocks with  $F_k(r)$ .

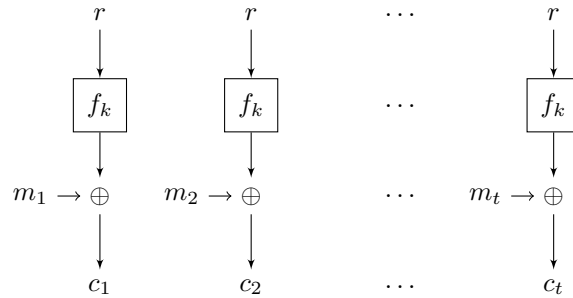


Figure 8.24: ECB-mode block-cipher in action, using a PRF as the encryption routine

This approach has the advantage of being completely parallelizable, as each block can clearly be encrypted separately; however there is a dangerous flaw in being not CPA-secure, even when using a PRF-based encryption scheme. To understand why, observe that random nonces for ciphertext randomization are chosen per-message; this means the encryption of message blocks become deterministic in the message scope, enabling an adversary to attack the scheme

within a single plaintext. It is sufficient to choose an all-0 or all-1 message to realize that all its blocks would encrypt to the same ciphered block.

### 8.1.2 Cipher block chaining mode (CBC)

This mode serializes block encryption by using the preceding ciphered block in the formula:

$$c_i = P_k(r) \oplus m_i \quad \forall i \in \{0, \dots, t\}$$

This time, a *pseudorandom permutation*(PRP) is used instead of a PRF; they will be discussed later on. The diagram in figure 8.25 shows a general view of CBC-mode's operation. The decryption process is analogous but in a reversed fashion, by computing the preimage of a ciphered block and XOR-ing it with the preceding ciphered block:

$$m_i = P_k^{-1}(c_i) \oplus c_{i-1}$$

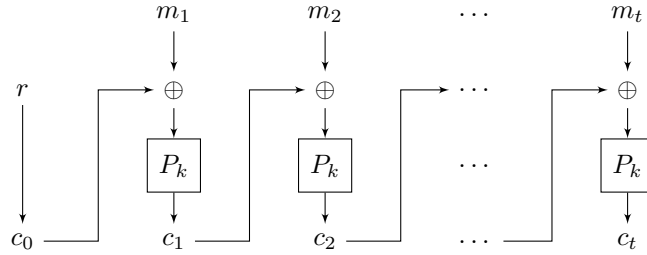


Figure 8.25: CBC-mode block-cipher in action, using a PRP as the encryption routine

### 8.1.3 Counter mode

Also denoted as CTR in short, this mode closely resembles ECB-mode but uses a “rolling” nonce instead of a static one, as shown in figure 8.26. At each successive block, the nonce is incremented by 1 and then used in a single block encryption. Since the nonce is in  $2^n$ , the increment is done modulo  $2^n$  so that the value will wrap around to 0 if it ever overflows. Decryption is analogous.

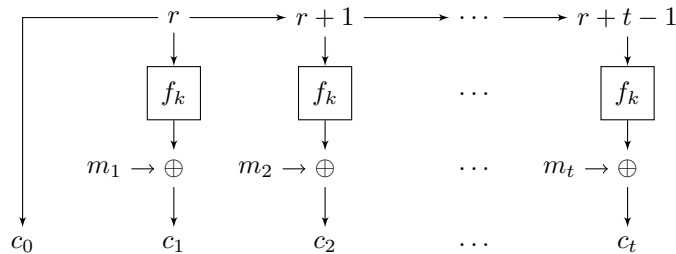


Figure 8.26: Counter-mode block-cipher in action, using a PRF as the encryption routine

This apparently innocuous change to EBC is enough to ensure CPA-security, at the cost of perfect parallelization.

**Theorem 15.** Assume  $f_k$  is a PRF, then the counter-mode block cipher is CPA-secure for variable length messages<sup>15</sup>.  $\diamond$

*Proof.* Figure 8.27 models a CPA attack to a counter-mode block-cipher. The proof will proceed by hybrid argument starting from this game, therefore the statement to verify will be  $\text{GAME}_{\text{CTR},A}^{\text{CPA}}(\lambda, 0) \approx_c \text{GAME}_{\text{CTR},A}^{\text{CPA}}(\lambda, 1)$ .

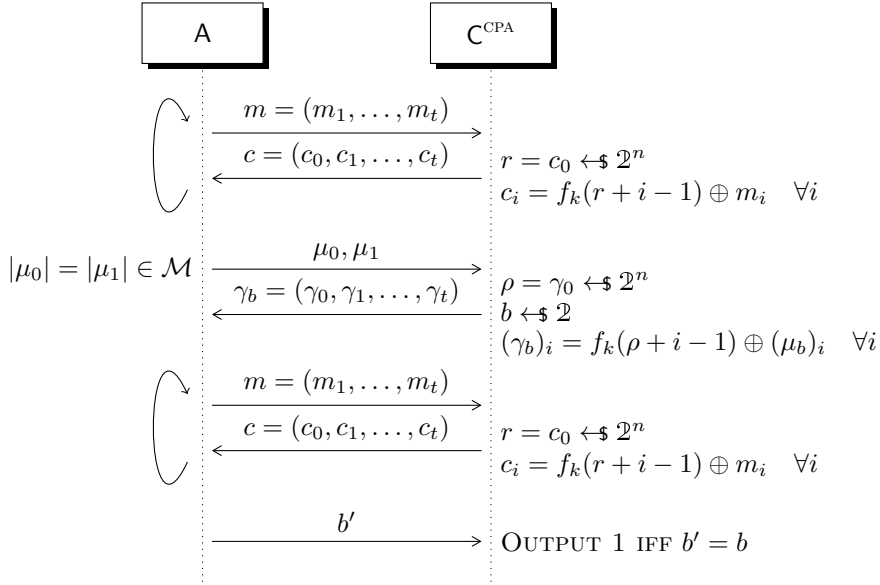


Figure 8.27: A chosen plaintext attack to counter-mode block-cipher

Define the two hybrid games from the original CPA game as follows:

- $\text{HYB}_{\text{CTR},A}^1(\lambda, b)$ : A random function  $R$  is chosen UAR from  $\mathfrak{R}(n, n)$  at the beginning of the game, and is used in place of  $F_k$  in all block encryptions;
- $\text{HYB}_{\text{CTR},A}^2(\lambda, b)$ : The challenger will pick random values from  $\mathbb{Z}^n$  as ciphered blocks, disregarding any encryption routine.

**Lemma 9.**  $\text{GAME}_{\text{CTR},A}^{\text{CPA}}(\lambda, b) \approx_c \text{HYB}_{\text{CTR},A}^1(\lambda, b) \quad \forall b \in \mathbb{Z}$   $\diamond$

*Proof.* The proof is left as exercise.

*Hint:* Since the original game and the first hybrid are very similar, we can use a distinguisher which plays the CPA-game; since this is a lemma, our goal in the reduction is to break the precondition contained in the theorem statement.  $\square$

**Lemma 10.**  $\text{HYB}_{\text{CTR},A}^1(\lambda, b) \approx_c \text{HYB}_{\text{CTR},A}^2(\lambda, b) \quad \forall b \in \mathbb{Z}$   $\diamond$

<sup>15</sup> Variable length messages exactly means every message  $m = (m_1, \dots, m_t)$  is made of  $t$  blocks, and  $t$  can change from any message to a different one.

*Proof.* Since  $m_i$  doesn't affect the distribution of the result at all, for any  $i$ , if  $R(r^*)$  behaves like a true random extractor, then the two hybrids are indistinguishable in the general case ( $R(r+i) \oplus m_i \approx R(r+i)$ ). However, there is a sneaky issue: if in both games it happens that a given nonce  $r_i$  is used in both one query encryption and the challenge message encryption at any step, the subsequent encrypted blocks will be completely random in the second hybrid, whereas in the first hybrid the function's images, albeit random, become predictable, enabling a CPA.

Nevertheless, it can be proved that these "collisions" happen with negligible probability within  $\text{HYB}_{\text{CTR},A}^1$ . Let:

- $q$  = number of encryption queries in a game run
- $t_i$  = number of blocks for the  $i$ -th query
- $\tau$  = number of blocks for the challenge ciphertext
- OVERLAP event:  $\exists i, j, \iota : r_i + j = \rho + \iota$

The OVERLAP event exactly models our problematic scenario. Now it suffices to show that it occurs negligibly. For simplicity, assume the involved messages are of same length, that is  $t_i = \tau =: t$ . Denote with  $\text{OVERLAP}_i$  to be the event that the  $i$ -th query overlaps the challenge sequence as specified above.

Fix some  $\rho$ . One can see that  $\text{OVERLAP}_i$  happens if:

$$\rho - t + 1 \leq r_i \leq \rho + t - 1$$

which means that  $r_i$  should be chosen *at least* in a way that:

- the sequence  $\rho, \dots, \rho + t - 1$  comes before the sequence  $r_i, \dots, r_i + t - 1$ , and they overlap just for the last element  $\rho + t - 1 = r_i$  or
- the sequence  $r_i, \dots, r_i + t - 1$  comes before the sequence the sequence  $\rho, \dots, \rho + t - 1$ , and they overlap just for the last element  $r_i + t - 1 = \rho$ .

Then:

$$\begin{aligned} \Pr[\text{OVERLAP}_i] &= \frac{(\rho + t - 1) - (\rho - t + 1) + 1}{2^n} \\ &= \frac{2t - 1}{2^n} \\ \Pr[\text{OVERLAP}] &\leq \sum_{i=1}^t \Pr[\text{OVERLAP}_i] \\ &\leq 2 \frac{t^2}{2^n} \in \text{Negl} \lambda \end{aligned}$$

which proves that our collision scenario happens with negligible probability, thus the two hybrids are indistinguishable. □

Having proven the indistinguishability between the hybrids, the conclusion is reached:

$$\text{GAME}_{\text{CTR},A}^{\text{CPA}}(\lambda, 0) \approx_c \text{GAME}_{\text{CTR},A}^{\text{CPA}}(\lambda, 1)$$

□

# Lesson 9

## 9.1 Message Authentication Codes and unforgeability

After having explored the security concerns and challenges of the SKE realm, it is time to turn the attention to symmetric authentication schemes, or MAC schemes. Recall that a MAC scheme is a couple  $(Tag, Verify)$ , with the purpose of authenticating the message's source. In this chapter, the tagging function will be denoted as  $Tag_k$ , akin to a PRF.

The desirable property that a MAC scheme should hold is to prevent any attacker from generating a valid couple  $(m^*, \phi^*)$ , even after querying a tagging oracle polynomially many times<sup>16</sup>. The act of generating a valid couple from scratch is called *forging*, and the aforementioned property is defined as *unforgeability against chosen-message attacks* (or UF-CMA, in short); its game diagram is shown in figure 9.28. Do note that  $m^*$  is stated to be outside the query set  $M$ , expressing the “freshness” of the forged couple<sup>17</sup>. In formal terms:

**Definition 20.** A MAC scheme  $\Pi$  is UFCMA-secure iff:

$$\forall A \in \text{PPT} \implies \Pr[\text{GAME}_{\Pi, A}^{\text{UFCMA}}(\lambda) = 1] \in \text{Negl}(\lambda) \quad \diamond$$

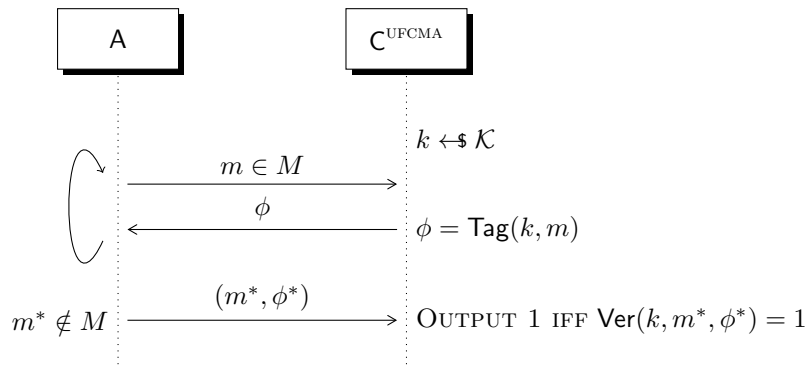


Figure 9.28:  $\text{GAME}_{\Pi, A}^{\text{UFCMA}}(\lambda)$

<sup>16</sup>Do note the similarities (and differences) between unforgeability in this setting and CPA-security in the secrecy setting

<sup>17</sup>Observe how this setup bears a striking resemblance to the PRF game. We're onto something...

Having defined a good notion of security in the MAC scheme domain, we turn our attention to a somewhat trivial scheme, and find out that it is indeed UF-CMA-secure:

**Theorem 16.** *Let  $f_k$  be a PRF, and define the following MAC scheme:*

$$\Pi = (\text{Tag}, \text{Ver}) := (f_k(m), [f_k(m) = \phi])$$

*Then  $\Pi$  is UF-CMA-secure.*  $\diamond$

*Proof.* The usual proof by hybridization to random functions entails. The original game is identical to the UF-CMA game, where the tagging function is the PRF, whereas the hybrid game will have it replaced with a truly random function, as shown in figure 9.29.

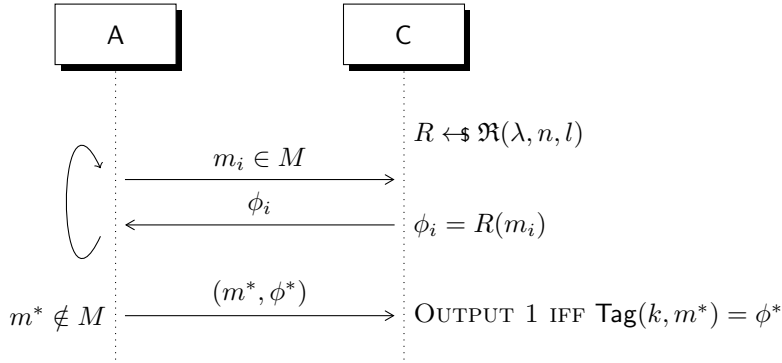


Figure 9.29:  $\text{HYB}^1_{\Pi, A}(\lambda)$

**Lemma 11.**  $\text{GAME}^{\text{UF-CMA}}_{\Pi, A}(\lambda) \approx_c \text{HYB}^1_{\Pi, A}(\lambda)$   $\diamond$

*Proof.* By assuming there is a distinguisher  $D^{\text{UF-CMA}}$  capable of disproving the lemma, it can be used to distinguish the PRF itself, as depicted by the reduction in figure 9.30  $\square$

**Lemma 12.**  $\forall A \in \text{PPT} \implies \Pr[\text{HYB}^1_{\Pi, A}(\lambda) = 1] \leq 2^{-l}$   $\diamond$

*Proof.* This is true because attacker has to predict the output  $R(m^*)$  on a fresh input  $m^*$  to win the game, which can happen at most with probability  $2^{-l}$ .  $\square$

Thus, the conclusion is that  $\Pi$  is UF-CMA-secure.  $\square$

## 9.2 Domain extension for MAC schemes

The previous scheme works on fixed length messages; as in the encryption domain, there are techniques for tagging variable length messages which are UF-CMA-secure. However, before showing them, some other apparently secure modes are described here to give some possible insights on how to tackle the problem.

Assume the message  $m = (m_1, \dots, m_t) \in \mathcal{M}^{n \cdot t}$  for some  $t \geq 1$ . Given the tagging function  $\text{Tag}_k \in \mathcal{M}^n \rightarrow \mathcal{M}^l$ , an attempt to tag the whole message may be to:

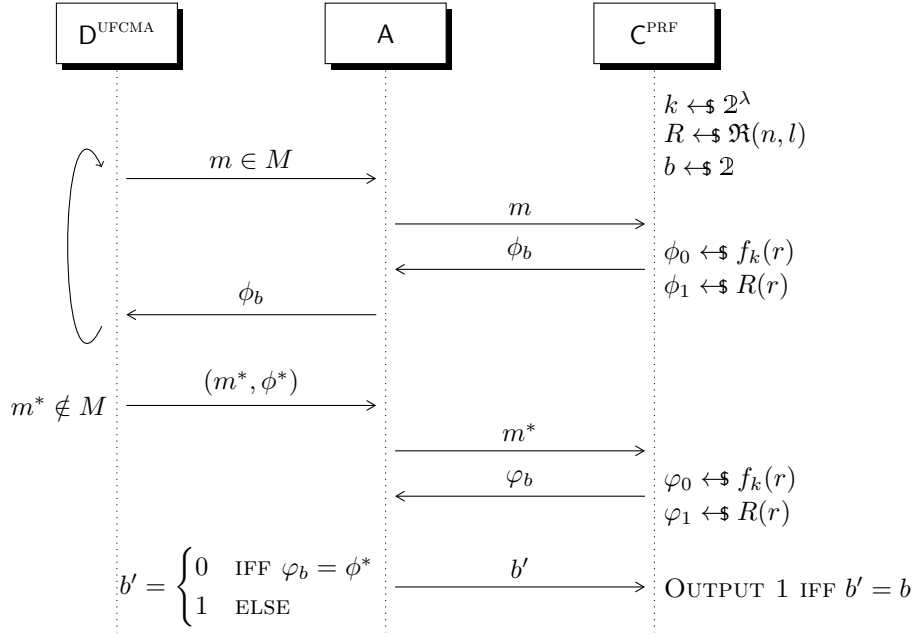


Figure 9.30: Distinguishing a PRF by using  $D^{\text{UFCMA}}$

- XOR all the message blocks, and then tag:  $\phi = \text{Tag}(k, \bigoplus_{i=1}^t m_i)$ . This approach opens to some easy forgeries: given an authenticated message  $(m, \phi)$ , an adversary can create a valid couple  $(m', \phi)$ , where  $m'$  is the original message with two flipped bits in two distinct blocks at the same offset; the resulting XOR would be the same.
- define the tag to be a  $t$ -sequence of tags, one for each message block. Again, there is an easy way to forge authenticated messages: an adversary can just flip the position of two arbitrary distinct message blocks along with their relative tags, resulting in a different, yet authentic message.
- attempt a variant of the above approach, by adding the block number to the block itself to avoid the previous forging. Yet again, this is not UF-CMA-secure: the adversary may just make two queries on two distinct messages, obtain the two tag sequences, and then forge an authenticated message by choosing at each position  $i$  whether to pick the message-tag blocks from the first or second query. Possibilities are endless...

### 9.2.1 Universal hash functions

A devised solution which has been proven to be secure relies on the following definition: a function family  $H$  which can be used to “shrink” variable length messages and then composed with a PRF  $f_k$ :

$$H \in 2^\lambda \rightarrow (2^{n \cdot t} \rightarrow 2^n) : s \mapsto h_s$$

$$\text{Tag}((k, s), m) = f_k(h_s(m))$$

So what are the properties of the such a function family  $H \circ F$ ? The main problem are *collisions*, since for each  $m \in \mathcal{D}^{n \cdot t}$  it should be hard to find  $m' \neq m$  such that  $h_s(m) = h_s(m')$ . But collisions do exist for functions in  $H \circ F$ , because they map elements from  $\mathcal{D}^{n \cdot t}$  to  $\mathcal{D}^t$ , and since the codomain is smaller than the domain, the functions cannot be injective in any way.

To overcome this problem, we can consider two options:

- assume collisions are hard to find, even when  $s \in \mathcal{D}^\lambda$  is known; in this case we have a family of *collision-resistant hash functions*;
- let  $s$  be secret, and assume collisions are hard to find because it is hard to know how  $h_s$  works.

**Definition 21.** A family of hash functions  $h_s$  is deemed  $\varepsilon$ -universal iff:

$$\forall x \neq x' \in \mathcal{D}^{n \cdot t}, S \sim \text{Unif}(\mathcal{D}^\lambda) \implies \Pr[h_S(x) = h_S(x')] \leq \varepsilon \quad \diamond$$

If  $\varepsilon = 2^{-n}$ , meaning the collision probability is minimized, then the family is also called *perfectly universal*; in the case where  $\varepsilon \in \text{Negl}(\lambda)$  instead, it is defined as *almost universal* (AU). Care should be taken in telling the difference between universality and pairwise independence, which states:

$$(h_S(x), h_S(x')) \equiv U_{2n}$$

**Lemma 13.** Any pairwise independent hash function is perfectly universal.  $\diamond$

*Proof.* The proof is left as exercise.

TODO 12: (should I use *Col* for solving this? What is the difference and when should I use *Col* instead of one-shot-probability?)

#### ASK FOR SOLVING PROPERLY

(Thoughts: when I ask *what's the probability that, chosen 2 distinct  $x$ -es, their hashes are the same on a certain value?*, maybe I have to use one-shot, because one-shot refers to the prob. that the two inputs collide on a specific value, even if not specified.

Instead, if I consider *what's the prob. that, chosen 2 distinct  $x$ -es, their hashes are the same?*, maybe I have to calculate all the possible collisions, because I want to know if the 2 inputs can collide in general. )

□

**Theorem 17.** Assuming  $f_k$  is a PRF with  $n$ -bit domain and  $h_s$  is an AU hash function family, then  $H \circ F$  is a PRF on  $(n \cdot t)$ -bit domain, for  $t \geq 1$ .  $\diamond$

*Proof.* This proof too will proceed by hybridizing the original game up to the ideal random one. Consider the three sequences depicted in figures 9.31, 9.32 and 9.33:

**Lemma 14.**

$$\text{Real}_{F,A}(\lambda) \approx_c \text{HYB}_{R,A}(\lambda)$$

◇



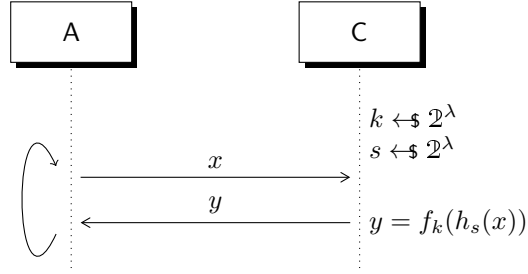


Figure 9.31:  $Real_{F,A}(\lambda)$

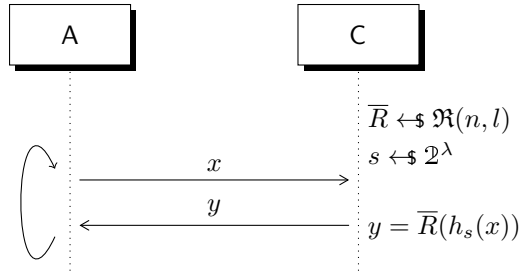


Figure 9.32:  $HYB_{R,A}(\lambda)$

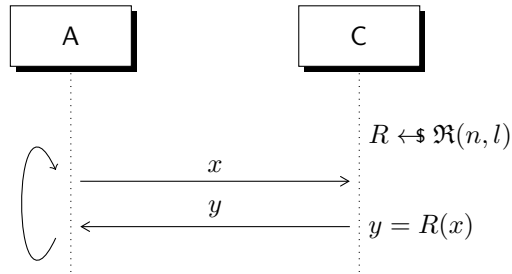


Figure 9.33:  $Rand_{R',A}(\lambda)$

*Proof.* The proof is left as exercise.  $\square$

**Lemma 15.**

$$\text{HYB}_{R,A}(\lambda) \approx_c \text{Rand}_{R',A}(\lambda)$$

$\diamond$

*Proof.* Again, collisions come into play there, but there are multiple cases that complicate things. Given two queries with arguments  $x_1, x_2$  returning the same image  $y$ , the random game can model two scenarios:

- the arguments are equal, but with negligible probability
- the arguments are distinct

while the hybrid can model three of them:

- the arguments are equal, again with negligible probability
- the arguments are distinct, *and so are their hashes*
- the arguments are distinct, *but not their hashes*

We want to show that the collision at hash level is negligible: as long as they don't happen, the random function  $\bar{R}$  is run over a sequence of distinct points, and behaves just as the random game's function  $R$  does. So let BAD be the event:

$$\exists i \neq j \in [q] : h_s(x_i) = h_s(x_j)$$

where  $q$  denotes the adversary's query count. It suffices to show that  $\Pr[\text{BAD}] \in \text{Negl}(\lambda)$ .

Since we don't care what happens after a collision, we can alternatively consider a mental experiment where we answer all queries at random, and only at the end sample  $s \leftarrow \mathcal{S}$  and check for collisions: this does not change the value of  $\Pr(\text{BAD})$ . Now queries are independent of  $s$ , and this eases our proof:

$$\begin{aligned} \Pr[\text{BAD}] &= \Pr[\exists x_i \neq x_j, h_S(x_i) = h_S(x_j)] \\ &\leq \sum_{i \neq j} \Pr[h_S(x_i) = h_S(x_j)] && h_s \text{ is AU} \\ &\leq \binom{q}{2} \text{Negl}(\lambda) \in \text{Negl}(\lambda) \end{aligned}$$

By ruling out this event, the lemma is proven.  $\square$

So now we have  $\text{Real} \approx_c \text{HYB}_{\Pi,A} \approx_c \text{Rand}$   $\square$

**Corollary 2.** *Let  $\Pi = (\text{Tag}, \text{Ver})$  be a variable length message MAC scheme where, given a PRF  $f_k$  and an AU hash function family  $h_s$ , the tagging function is defined as  $f_k(h_s)$ . Then  $\Pi$  is UF-CMA-secure.*  $\diamond$

### 9.2.2 Hash function families from finite fields

A generic  $2^n$ -order finite field has very useful properties: adding two of its elements is equal to XOR-ing their binary representations, while multiplying them is done modulo  $2^n$ . It is possible to define a hash function family that makes good use of these properties, and is suitable for a UF-CMA-secure MAC scheme.

**Construction 2.** Let  $\mathbb{F} = GF(2^n)$  be a *finite field* (or “Galois field”) of  $2^n$  elements, and let  $m = (m_1, \dots, m_t) \in \mathbb{F}^t$  and  $s = (s_1, \dots, s_t) \in \mathbb{F}^t$ . The desired hash function family will have this form:

$$h_s(m) = \sum_{i=1}^t s_i m_i = \langle s, m \rangle = q_m(s)$$

◇

**Lemma 16.** *The above function family  $h_s$  is almost universal.*

◇

*Proof.* In order for  $h_s$  to be almost universal, collisions must happen negligibly. Suppose we have a collision with two distinct messages  $m$  and  $m'$ :

$$\sum_{i=1}^t m_i s_i = \sum_{i=1}^t m'_i s_i$$

Let  $\delta_i = m_i - m'_i$  and assume, without loss of generality, that  $\delta \neq 0$ . Then, by using the previous equation, when a collision happens:

$$0 = \sum_{i=1}^t m_i s_i - \sum_{i=1}^t m'_i s_i = \sum_{i=1}^t \delta_i s_i$$

Since the messages are different from each other, there is at least some  $i$ -th block that contains some of the differences. Assume, without loss of generality, that some of the differences are contained in the first block ( $i = 1$ ); the sum can then be split between the first block itself  $\delta_1 s_1$  and the rest:

$$\begin{aligned} \sum_{i=1}^t \delta_i s_i &= 0 \\ \delta_1 s_1 + \sum_{i=2}^t \delta_i s_i &= 0 \\ \delta_1 s_1 &= - \sum_{i=2}^t \delta_i s_i \\ s_1 &= \frac{- \sum_{i=2}^t \delta_i s_i}{\delta_1} \end{aligned}$$

which means when a collision happens,  $s_1$  must be exactly equal to the sum of the other blocks, which is another element of  $\mathbb{F}$ . But since every seed is

chosen at random among  $\mathbb{F}$ , the probability of picking the element  $s_1$  satisfying the above equation is just  $|\mathbb{F}|^{-1} = 2^{-n} \in \text{Negl}(\lambda)$ . By repeating this reasoning for every difference-block, a sum of negligible probabilities is obtained, which is in turn negligible; therefore the hash function family  $h_s$  is almost universal.  $\square$

### **H with Galois fields elements and polynomials**

**Exercise 18.** Let  $\mathbb{F} = GF(2^n)$  be a finite field of  $2^n$  elements, and let  $m = (m_1, \dots, m_t) \in \mathbb{F}^t$  and  $s \leftarrow \mathbb{F}^t$ . Construct the following hash function family:

$$h_s(m) = \sum_{i=1}^t s^{i-1} m_i$$

Prove that this construction is AU.

**Solution 4.** (possible proof: to be almost universal, looking at the definition, collisions with  $m \neq m'$  must be negligible.

So consider a collision as above: it must be true that

$$\begin{aligned} \sum_{i=1}^t m_i s^{i-1} &= \sum_{i=1}^t m'_i s^{i-1} \\ \sum_{i=1}^t m_i s^{i-1} - \sum_{i=1}^t m'_i s^{i-1} &= 0 \\ q_{m-m'}(s) &= 0 \end{aligned}$$

How can we make a polynomial equal to 0? We have to find the **roots** of the polynomial, which we know are at most the polynomial's rank. The rank is  $t - 1$ , and the probability of picking a root from  $\mathbb{F}$  as seed of  $h_s$  is:

$$\Pr[s = \text{root}] = \frac{t-1}{2^n} \in \text{Negl}(\lambda)$$

# Lesson 10

## 10.1 Domain extension for PRF-based authentication schemes

### 10.1.1 Hash function families from PRFs

Another way to obtain domain extension for a MAC scheme, using the  $H \circ F$  approach, is to construct the hash function family from another PRF. We expect to have:

- $\forall A \in \text{PPT}, S \sim \text{Unif}(\mathbb{Z}^\lambda) \implies \Pr[h_S(m) = h_S(m')] \in \text{Negl}(\lambda);$
- We need two PRFs: one is  $f_k$ , and the other is  $f_s$

### 10.1.2 XOR-mode

Let  $f_s$  be a PRF, and assume that we have this function

$$h_s(m) = f_s(m_1 || 1) \oplus \dots \oplus f_s(m_t || t)$$

so that the input to  $f_s$  is  $n + \log_2(t)$  bytes long.

**Lemma 17.** *Above  $H$  is computational AU.* ◇

*Proof.* The proof is left as exercise.

(Hint: The pseudorandom functions can be defined as  $f_s = f'_k(0, \dots)$  and  $f_k = f'_k(1, \dots)$ ).

Possible proof: we have to show that

$$\forall m \neq m' \implies \Pr[h_s(m) = h_s(m')] \in \text{Negl}(\lambda)$$

This means that:

$$\begin{aligned} \Pr \left[ \left( \bigoplus_{i=1}^t f_s(m_i, i) \right) = \left( \bigoplus_{i=1}^t f_s(m'_i, i) \right) \right] \\ = \Pr[\forall i \quad f_s(m_i, i) \oplus f_s(m'_i, i) = \bigoplus_{j=1, j \neq i}^t f_s(m_j, j) \oplus f_s(m'_j, j) = \alpha] \end{aligned}$$

for each  $i \in [1, t]$ . But  $\alpha$  is one unique random number chosen over  $2^n$  possible candidates, so the collision probability is negligible. □

### 10.1.3 CBC-mode MAC scheme

This mode is used in practice by the TLS standard. It's used with a PRF  $f_s$ , setting the starting vector as  $IV = 0^n = c_0$  and running this PRF as part of CBC. The last block obtained by the whole process is the message's signature:

$$h_s(m) = f_s(m_t \oplus f_s(m_{t-1} \oplus f_s(\dots f_s(m_2 \oplus f_s(m_1 \oplus IV)) \dots)))$$

**Lemma 18.** CBC MAC defines completely an AU family. ◇

*Proof.* (not proven) □

We can use this function to create an *encrypted* CBC, or E-CBC:

$$\text{E-CBC}_{K,S}(m) = f_k(h_s^{\text{CBC}}(m))$$

**Theorem 19.** If  $f_k$  is a PRF, CBC MAC is already a MAC scheme with domain  $n \cdot t$  for arbitrarily fixed  $t \in \mathbb{N}$ . ◇

*Proof.* (not proven) □

### 10.1.4 XOR MAC

Instead of  $H \circ F$ , now the **Tag** routine outputs  $\phi = (\eta, f_k(\eta) \oplus h_s(m))$  where  $\eta \leftarrow \mathbb{Z}^n$  is random and it's called *nonce*. Authentication is done as:

$$(m, (\eta, f_k(\eta) \oplus h_s(m)))$$

When I want to verify a message and I get the couple  $(m, (\eta, v))$ , I just check that  $v = f_k(\eta) \oplus h_s(m)$ . It should be hard to find a value called  $a$  such that, given  $m \neq m'$ ,

$$h_s(m) \oplus a = h_s(m')$$

In fact, since an adversary who wants to break this scheme has to send a valid couple  $(m^*, \phi^*)$  after some queries, he could:

- ask for message  $m$  and store the tag  $(\eta, f_k(\eta) \oplus h_s(m))$
- try to find  $a = h_s(m) \oplus h_s(m')$  and modify the previous stored tag adding  $v \oplus a$ ,

so now he could send the authenticated message

$$(m', (\eta, f_k(\eta) \oplus h_s(m')))$$

which is a valid message.

**Definition 22.** Let  $S$  be a uniform seed; a hash function  $h_s$  is deemed *Almost XOR-universal*, or AXU in short, iff:

$$\forall x_1 \neq x_2, a \implies \Pr[h_S(x_1) \oplus h_S(x_2) = a] \leq \epsilon \quad \diamond$$

Note that, if  $a = 0$ , the AXU property becomes the AU property.

**Lemma 19.** XOR mode gives a computational AXU hash function. ◇

*Proof.* (not proven) □

**Theorem 20.** If  $F$  is a PRF and  $H$  is computational AXU, then XOR-MAC is a MAC. ◇

*Proof.* (not proven) □

## Summary

TODO 13: not sure what to do with this bullet list...

With variable input length:

- AXU based XOR mode is secure;
- $H \circ F$  is insecure with polynomial construction  $h_s(m) = q_m(s)$ , but can be fixed;
- CBC-MAC is not secure, left as exercise;
- E-CBC is secure.

## 10.2 CCA-security

Going back to the encryption realm, a new definition of attack to a SKE scheme will be introduced. Now the adversary can query a decryption oracle, along with the CPA-related encryption oracle, for polynomially many queries. This attack is called the *Chosen Ciphertext Attack*<sup>18</sup>, and schemes that are proven to be CCA-secure are also defined as *non-malleable*, on the reasoning that an attacker cannot craft fresh valid ciphertexts from other valid ones.

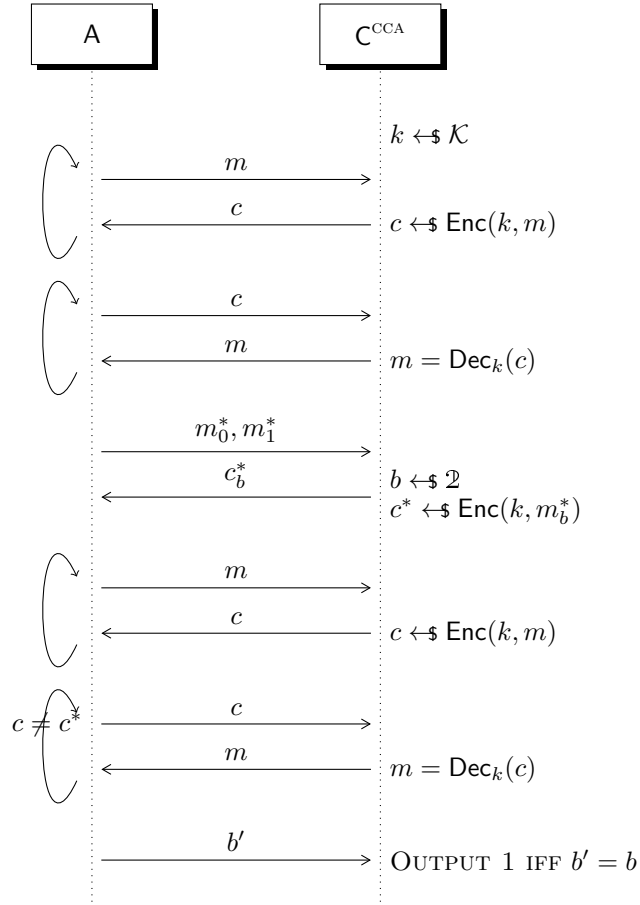


Figure 10.34: The chosen ciphertext attack, on top of CPA

**Exercise 21.** Show that the scheme  $\Pi_F$  defined in theorem 14, while CPA-secure, is not CCA-secure.

*Proof.* Let  $m_0$  and  $m_1$  be the messages the adversary sends to the challenger as the challenge plaintexts; on receiving the ciphertext  $c_b = (r, f_k(r \oplus m_b)) : b \leftarrow \mathbb{Z}$ , the adversary crafts another ciphertext with an arbitrary value  $\alpha$ :

$$\hat{c}_b = (r, f_k(r) \oplus m_b \oplus \alpha)$$

<sup>18</sup>Different versions of the CCA notion exist. The one defined here is also called CCA2, or *adaptive Chosen Ciphertext Attack*



and queries the decryption oracle on it. The latter will decrypt the new ciphertext and return a plaintext, which can be easily manipulated by the adversary to reveal exactly which message was encrypted during the challenge:

$$\begin{aligned}\text{Dec}(k, (\hat{c}_b)) &= \text{Dec}(k, (r, f_k(r) \oplus m_b \oplus \alpha)) \\ &= f_k(r) \oplus f_k(r) \oplus m_b \oplus \alpha \\ &= m_b \oplus \alpha\end{aligned}$$

Therefore, the adversary certainly wins after just one decryption query, proving the scheme's vulnerability to CCA attacks.  $\square$

### 10.3 Authenticated encryption

Trying to create a CCA-secure scheme from scratch may seem a daunting task. Instead, we will start looking for a scheme that achieves both secrecy and authentication. As one might expect, such a scheme is usually a combination of a valid secrecy scheme with a valid authentication scheme; the result would be a scheme composed by four primitives  $\Pi = (\text{Enc}, \text{Dec}, \text{Tag}, \text{Ver})$ . If this scheme turns out to be CPA-secure, and has an additional property AUTH denoting the scheme's resistance to forgeries, much like its MAC cousins, then it is proven that the scheme is also CCA-secure.

The game shown in figure 10.35 models this AUTH property of a scheme  $\Pi = (\text{Enc}, \text{Dec})$ , with an additional quirk to the decryption routine:

$$\text{Dec} \in \mathcal{K} \times \mathcal{C} \rightarrow M \cup \{\perp\}$$

where the  $\perp$  value is returned whenever the decryption algorithm is supplied an invalid or malformed ciphertext.

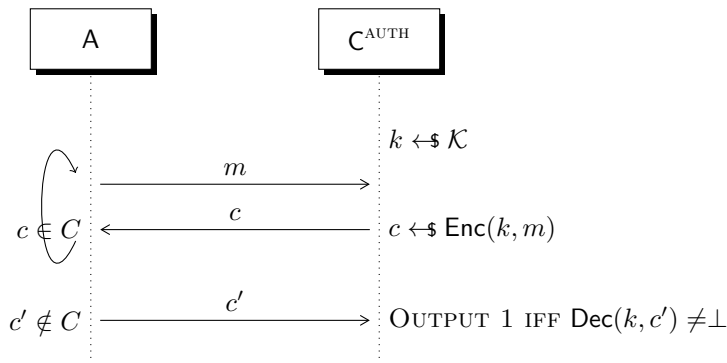


Figure 10.35:  $\text{GAME}_{\Pi, A}^{\text{AUTH}}(\lambda)$

**Theorem 22.** *Let  $\Pi$  be a SKE scheme. If it is CPA-secure, and has the AUTH property, then it is also CCA-secure.*  $\diamond$

*Proof.* The proof is left as an exercise.

*Hint:* consider the experiment where  $\text{Dec}(k, c)$ :

- if  $c$  is not fresh (i.e. output of previous encryption query  $m$ ), then output  $m$
- else output  $\perp$

The approach would be to reduce CCA to CPA; given  $D^{\text{CCA}}$ , we can build  $D^{\text{CPA}}$ .  $D^{\text{CCA}}$  will ask decryption queries, but  $D^{\text{CPA}}$  can answer just with these two properties shown above, so it can reply just if he asked these  $(c, m)$  before to its challenger  $C$ . □

### 10.3.1 Combining SKE & MAC schemes

Let  $\Pi_1 = (\text{Enc}, \text{Dec})$  be a SKE scheme, and  $\Pi_2 = (\text{Tag}, \text{Ver})$  be a MAC scheme; there are 3 ways to combine them into an authenticated encryption scheme:

- *Encrypt-and-Tag:*
  1.  $c \leftarrow \text{Enc}(k_1, m)$
  2.  $\phi \leftarrow \text{Tag}(k_2, m)$
  3.  $c^* = (c, \phi)$
- *Tag-then-encrypt:*
  1.  $\phi \leftarrow \text{Tag}(k_2, m)$
  2.  $c \leftarrow \text{Enc}(k_1, (\phi, m))$
  3.  $c^* = c$
- *Encrypt-then-Tag:*
  1.  $c \leftarrow \text{Enc}(k_1, m)$
  2.  $\phi \leftarrow \text{Tag}(k_2, c)$
  3.  $c^* = (c, \phi)$

Of the three options, only the last one is proven to be CCA-secure for arbitrary scheme choices; the other approaches are not secure “a-priori”, with some couples proven to be secure by themselves. Notable examples are the *Transport Layer Security* (TLS) protocol, which employs the second strategy, and has been proven to be secure because of the chosen encryption scheme; *Secure shell* (SSH) instead uses the first strategy.

**Theorem 23.** *If an authenticated encryption scheme  $\Pi$  is made by combining a CPA-secure SKE scheme  $\Pi_1$  with a strongly unforgeable MAC scheme  $\Pi_2$  in the Encrypt-then-tag method. Then  $\Pi$  is CPA-secure and AUTH-secure.* ◇

*Proof.* Assume that  $\Pi$  is not CPA-secure; then an adversary can use the resulting distinguisher  $D^{\text{CPA}}$  to direct a successful CPA against  $\Pi_1$ , as shown in figure 10.36: the point is to run the two components of  $\Pi$  separately.

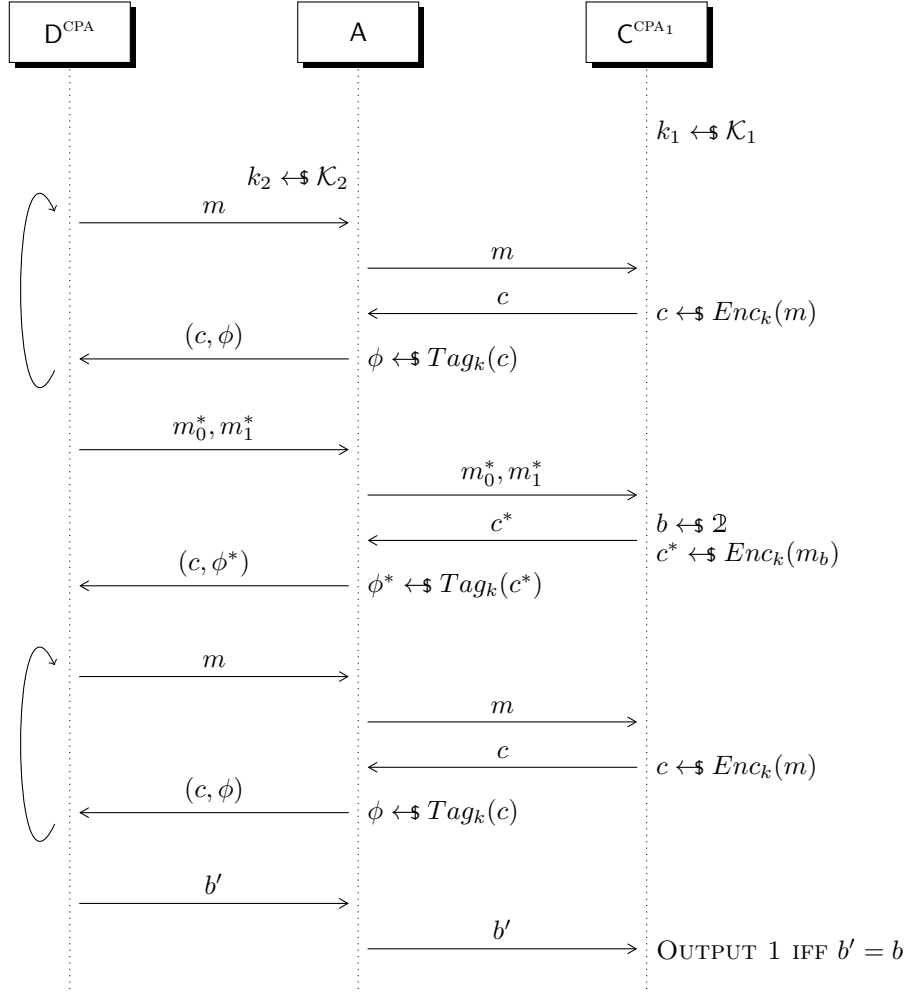


Figure 10.36

TODO 14: Professor says that we have to show that  $\text{GAME}_{\Pi, \mathcal{A}}^{\text{CPA}}(\lambda, 0) \approx_c \text{GAME}_{\Pi, \mathcal{A}}^{\text{CPA}}(\lambda, 1)$ , but why? Isn't this proof enough?

□

Proved for the CPA-security property, now we have to prove, in a similar way, that the AUTH property must hold by  $\Pi$  if  $\Pi_2$  is an AUTH-secure scheme.

**Exercise 24.** Prove it! Solution is in next lesson.

# Lesson 11

## 11.1 Authenticated encryption (continued)

Having proven that an authenticated encryption scheme in an *Encrypt-then-Tag* mode is CPA-secure, it remains to prove that it has the AUTH property. Before this, a new unforgeability definition is needed:

**Definition 23.** Let  $\Pi = (\text{Tag}, \text{Ver})$  be a MAC scheme. Then  $\Pi$  is *strongly* resistant to *existential forgery under chosen message attack*, or SEF-CMA-secure in short, iff:

$$\Pr[\text{GAME}_{\Pi, \mathcal{A}}^{\text{UFCMA}}(\lambda) = 1] \in \text{Negl}(\lambda)$$

i.e. it is UF-CMA, but with the additional restriction that the tag  $\phi^*$  of the forged message must be “fresh” itself.  $\diamond$

Note the small difference in security between UF-CMA and SEF-CMA.

**Theorem 25.** Let  $\Pi = (\text{Enc}, \text{Dec}, \text{Tag}, \text{Ver})$  be an authenticated encryption scheme, composed by a SKE scheme  $\Pi_1$  and a MAC scheme  $\Pi_2$ . If  $\Pi_2$  is SEF-CMA, then  $\Pi$  has the AUTH property.  $\diamond$

*Proof.* The proof is analogous to the previous proof regarding the scheme’s CPA security. Suppose that  $\Pi$  has not the AUTH property; then an adversary can use the distinguisher  $\mathcal{D}^{\text{AUTH}}$  to successfully forge authenticated messages with fresh signatures against  $\Pi_2$ , as depicted in figure 11.37.

From  $\mathcal{D}^{\text{AUTH}}$ ’s perspective, all the couples  $(c_i, \phi_i)$  received are made from  $m$  as follows:

$$c_i \leftarrow \text{Enc}(k_1, m) \wedge \phi_i \leftarrow \text{Tag}(k_2, c_i)$$

Since  $\mathcal{D}^{\text{AUTH}}$  is able to break the AUTH property of  $\Pi$ , the couple it outputs  $(c^*, \phi^*)$  will be such that:

$$\text{Dec}(k_1, c^*) = \widehat{m^*} \wedge \text{Ver}(k_2, c^*, \phi^*) = 1$$

Note that it is not important that the ciphertext decodes to the original plaintext; the matter at hand is that  $\mathcal{A}$  can now use the same couple to break  $\Pi_2$ ’s UF-CMA property, which is a contradiction.

It could happen that, for  $c^* = c$  previously seen,  $\phi^*$  is a fresh tag instead. Just in this case the AUTH game would be valid because  $(c^*, \phi^*)$  would have never been seen before, but not the SEF-CMA game, because  $c^*$  was previously sent to the challenger.  $\square$

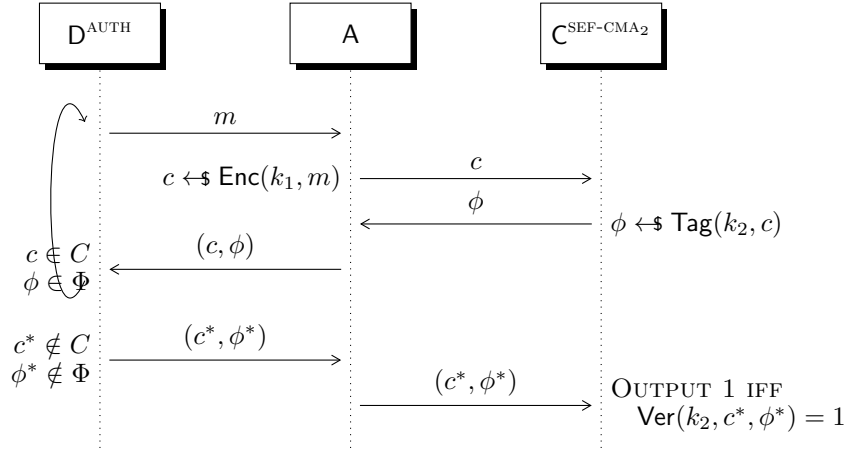


Figure 11.37: Breaking authenticity of  $\Pi_2$

Now we want an *ufcma* secure scheme that is able to resist against message-tag challenge couples, where the tag is fresh but the message has been already requested to the challenger.

## 11.2 Pseudorandom permutations

Nothing prevents a PRF  $g_k$  to be bijective; in this case, it is referred to as a *pseudorandom permutation*, or PRP in short. Their definition is analogous to a generic PRF, as shown in figure 11.38: PRPs are computationally indistinguishable from a random permutation.

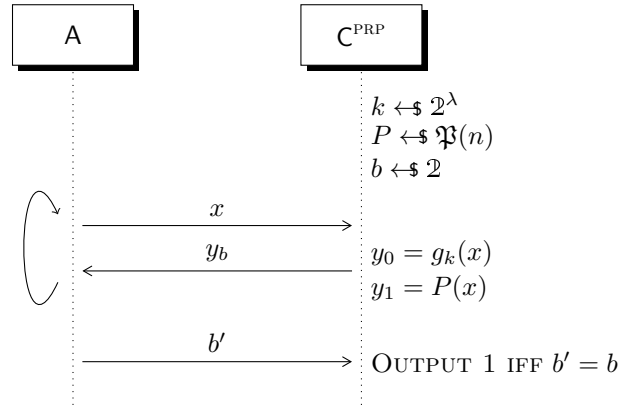


Figure 11.38:  $\text{GAME}_{g_k, A}^{\text{PRP}}(\lambda, b)$

$$\text{GAME}_{g_k, A}^{\text{PRP}}(\lambda, 0) \approx_c \text{GAME}_{g_k, A}^{\text{PRP}}(\lambda, 1)$$

An important difference is that  $g_k$  is efficiently invertible, although knowledge of  $k$  is required in order to do so.

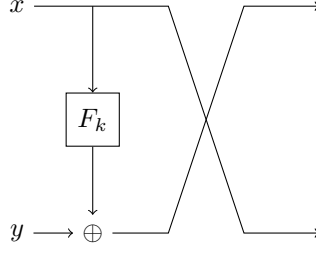


Figure 11.39: A single-round Feistel network

### 11.2.1 Feistel network

PRPs have been successfully constructed by using existing PRFs into what is called a *Feistel network*. As a starting point, let  $f_k \in \mathbb{2}^n \rightarrow \mathbb{2}^n$  be a PRF, and define the function  $\psi_f$  as follows:

$$\begin{aligned}\psi_f(x, y) &= (y, x \oplus F(y)) = (x', y') \\ \psi_f^{-1}(x', y') &= (F(x') \oplus y', x') = (F(y) \oplus x \oplus F(y), y) = (x, y)\end{aligned}$$

While this construct is invertible and uses a PRF, it is not pseudorandom itself, because the first  $n$  bits of  $\psi_F$ 's image are always equal to  $y$ , and thus visible to any adversary. A first attempt at fixing this vulnerability would be to apply the construct two times on two different PRFs  $\psi_{F, F'}^2$ , in an attempt to “hide”  $y$ . Yet, this approach still leaks valuable information:

$$\psi_{F, F'}(x, z) \oplus \psi_{F, F'}(y, z) = (x \oplus y, \dots)$$

However, this example with additional restrictions will be useful very soon, so it is reworded as the following lemma:

**Lemma 20.** *For any unbounded adversary making  $q \in \text{Poly}(\lambda)$  queries, he is unable to consistently win the game depicted in 11.40:*

$$\text{GAME}_{\Pi, \mathcal{A}}^{\text{R-FEIS}}(\lambda, 0) \approx_s \text{GAME}_{\Pi, \mathcal{A}}^{\text{R-FEIS}}(\lambda, 1)$$

as long as  $y_1, \dots, y_q$  are mutually distinct. ◇

*Proof.* TODO 15: Idea: Hybridize over the queries before the challenge, from PR to random; prove that the stat distance between  $i$  and  $i+1$  is negligible

□

Going back to the Feistel networks in general, it should be easy to see that they can be made of an arbitrary number of rounds, by simply chaining output with input. The  $l$ -th iteration is denoted as:

$$\psi_{\Phi}^l(x, y) = \psi_F(\psi_{F''}(\dots \psi_{F^{(l)}}(x, y) \dots))$$

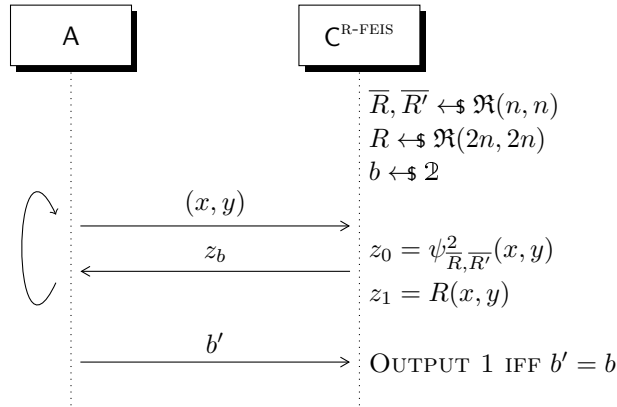


Figure 11.40: The randomic-Feistel distinguishing game:  $\text{GAME}_{\Pi, A}^{\text{R-FEIS}}$

where  $\Phi$  is the sequence of PRFs used at each single step. It can be shown that the rounds needed to obtain a network that is indeed pseudorandom is just 3; also, the same PRF can be used, it is sufficient to change the seed on each iteration<sup>19</sup>:

**Theorem 26** (Michael Luby - Charles Rackoff). *Let  $F_i, F_j, F_k$  be a PRF over three seeds. Then  $\psi_{i,j,k}^3$  is a PRP.*  $\diamond$

*Proof.* TODO 16: Idea: Four total games: original, swap prfs with random functions, swap the three functions with a single one (use previous lemma), swap random function with random permutation (avoid bad events generated by injection property)

□

<sup>19</sup>That is actually the purpose of using a PRF



# Lesson 12

## 12.1 Hashing

Remember one solution to domain-extension for PRFs, as a composition of a PRF  $F$  with an almost universal hash function  $H$ . Hash functions compress their arguments to some “fingerprint”, which is assumed to be unique. However, since this compression in this context inherently introduces information loss, it is not guaranteed that every message gets its own unique fingerprint; indeed, there will be some instances where two messages yield the same hash value, or in other words, the hashes *collide*. It is desirable for a hash function to be *resistant* to these events, meaning that it is hard to reproduce such collisions.

**Definition 24.** A hash function family  $H$  is deemed *collision-resistant*, denoted as  $H \in \text{CRH}$  iff the probability of finding a collision is negligible, even when knowing the seed  $s$ . Formally:

$$\forall A \in \text{PPT} \implies \Pr(\text{GAME}_{\Pi, A}^{\text{CRH}}(\lambda) = 1) \in \text{Negl}(\lambda)$$

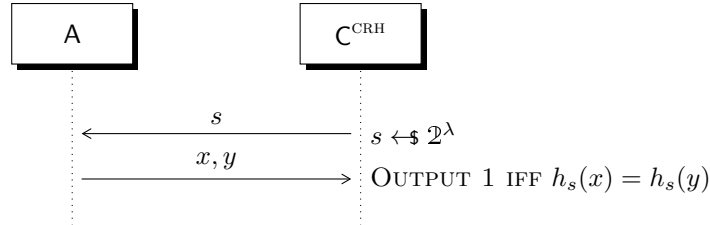


Figure 12.41: The *collision-resistance* game

◇

A note: before, we were dealing with unbounded adversaries, and the key was hidden. Now the key is public, but the adversary must be efficient.

**Exercise 27.** Let  $\Pi$  be a UF-CMA authentication scheme over the message space  $\mathcal{M}$ . Show that  $\Pi' = (\text{Tag}', \text{Ver}') : \text{Tag}'_{k,s}(m) = \text{Tag}_k(h_s(m))$  is UF-CMA-secure over  $\mathcal{M}$ , where  $l \in \text{Poly}(n)$ , as long as  $h_s$  itself is CRH.

### 12.1.1 Merkle-Damgård construction

A construct for starters has been defined by Ralph Merkle and Ivan Damgård, which revolves around the use of a CRH function  $h_s \in \mathcal{M}^{n+c} \rightarrow \mathcal{M}^n$ , where  $c$  is

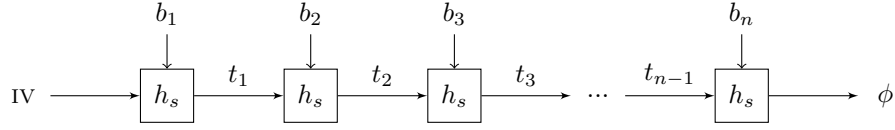


Figure 12.42: Basic outline of a Merkle-Damgård construction

the size of one “block” of a message. For now, consider the messages to be of an arbitrarily fixed length  $l$ , and let  $c$  be 1. The steps to follow are:

1. Let  $IV \in \mathbb{2}^b$  be an *initialization vector*
2. Initialize  $t_0 := IV$
3. For each bit  $b_i$  of the message to hash  $m$ :
  - (a) Compute  $t_i := h_s(b_i, t_{i-1})$
4. Return  $\phi = t_n$

Figure 12.42 depicts a general view of the algorithm. Let it be denoted as another hash function  $h'_s \in \mathbb{2}^l \rightarrow \mathbb{2}^n$ .

**Theorem 28.** *The construction  $H'$  obtained by Merkle-Damgård is a CRH function.*  $\diamond$

*Proof.* Assume  $H'$  can be broken efficiently by a distinguisher  $D^{\text{CRH}}$ , meaning that finding two distinct block sequences that give the same hash is easy. Consider the reduction to  $H$ 's CRH-ness in figure 12.43

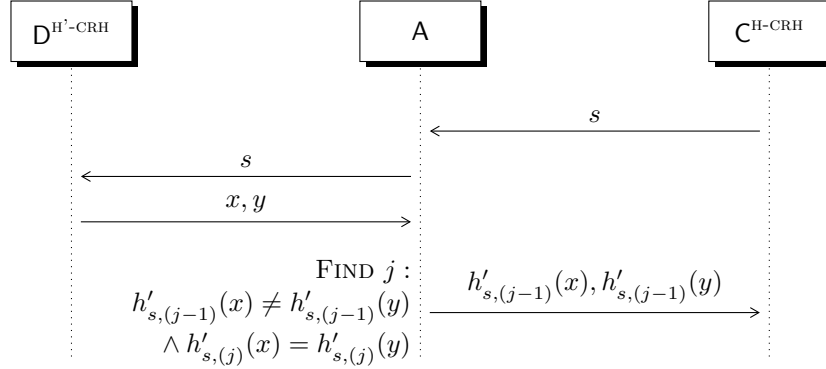


Figure 12.43: Breaking the underlying CRH-ness of  $H$

Ignore same blocks: find largest  $j$  such that:

$$(b_j, x_{j-1}) \neq (b'_j, y_{j-1}) \wedge h_s(b_j, x_{j-1}) = h_s(b'_j, y_{j-1})$$

this implies the rest of the message is equal, then the resulting final hash will be equal, thus for  $j > 0$  we have a collision.  $\square$

### Domain extension

In order to adapt the MD-construct to messages of variable length, some sort of “strengthening” is required:

**Lemma 21** (Length padding). *Let  $H_s \in \mathbb{2}^{n+l} \rightarrow \mathbb{2}^n$ , then:*

$$H'_s = H_s(\langle l' \rangle, H_s(x_{l'}, \dots, H_s(x_1, 0^n) \dots)), |l'|, |x_i| \in \mathbb{2}^c$$

◇

This is an example of padding which encodes the message length in itself.

**Theorem 29.** *The strengthened construct is collision resistant for variable-length messages.*

◇

*Proof.* Hint: similar as above, case by case

□

### Merkle trees

This is an alternative construction to the “linear” approach used beforehand: it starts from the message’s blocks acting as leaves of a complete binary tree, and using a halving compression function on each node from the bottom up, and outputting the final tag as the image of the root invocation. It also has nice properties, such as easy verification of the presence of a single specific block in the message.

#### 12.1.2 Compression functions

The compression functions are the central point of these kinds of “fingerprinting” tag schemes. They are, essentially, hash functions with a smaller codomain than its domain, thereby forcing the existence of collisions.

Let  $(\text{Gen} : 0 \mapsto (\mathbf{pk}, \mathbf{sk}), f, g)$  be a PKE scheme, where the functions  $f$  and  $g$  are keyed PRPs. A *claw* is a couple of values  $(x, x')$  such that:

$$f(\mathbf{pk}, x) = g(\mathbf{pk}, x')$$

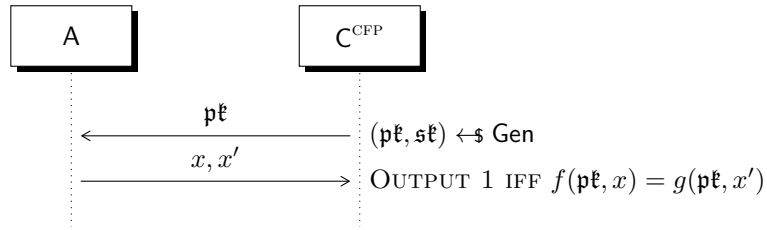


Figure 12.44: The game of claw-free permutations

**Theorem 30.** *Assuming  $\mathcal{F}$  is claw-free, then  $h_{\mathbf{pk}}$  is CRH from  $n + l$  bits to  $n$ .*

◇

### Davies-Meyer construct

**Definition 25.**  $x_{i+1} = E_k(x_i) \oplus x_i$ , maps  $n + \lambda$  to  $n$ .  $E$  is AES

◇

# Lesson 13

## 13.1 Number theory

**Theorem 31** (Fermat's last theorem).

$$\forall x, y, z \in \mathbb{Z}, n > 2 \implies x^n + y^n \neq z^n$$

◇

**Lemma 22.**

$$\forall a \in \mathbb{Z}_n : \gcd(a, n) > 1 \implies a \notin \mathbb{Z}_n^\times$$

◇

*Proof.* Assume there exists  $b \in \mathbb{Z}_n$  such that  $ab \equiv_n 1$ . Then, there exists a quotient for the division  $by$  between  $a$  and  $b$  with remainder 1. Observe that  $\gcd(a, n)$  divides  $ab + qn$ , which is equal to 1. It entails that  $\gcd(a, n) = 1$ , which is a contradiction. □

**Lemma 23.**

$$\forall a, b \in \mathbb{N} : a \geq b \neq 0 \implies \gcd(a, b) = \gcd(b, a \bmod b)$$

◇

*Proof.* Not given. □

**Theorem 32.** *Given two integers  $a$  and  $b$ , their greatest common divisor can be computed efficiently with respect to their lengths. Additionally, two other numbers  $u$  and  $v$  can be efficiently computed in order to satisfy Bézout's identity:  $\gcd(a, b) = au + bv$*  ◇

*Proof.* Hint: Use previous lemma recursively; we stop at  $r_{t+1} = 0$  for some  $t$ , thus:

$$\gcd(a, b) = \dots = \gcd(r_t, r_{t+1}) = r_t$$

□

**Claim 5.**  $r_{i+2} \leq r_i/2 \forall 0 \leq i \leq t-2 \implies \#steps = \lambda - 1$  if  $|b| \in 2^\lambda$  ◇

The hypothesis is a natural consequence of the repeated modulo operations.

**Observation 2** (Exponentiation mod  $n$ : Square and multiply). Let  $b \in \mathbb{Z}^l$ , where by writing  $b_i$  we denote  $b$ 's  $i$ -th bit. Then:

$$a^b \equiv_n a^{\sum_{i=0}^l 2^i b_i} \equiv_n \prod_{i=0}^l a^{2^i b_i}$$

◇

**Theorem 33.** The number of primes lesser than or equal to  $x$  is a number greater than or equal to  $\frac{x}{3 \log_2 x}$  ◇

There exist many algorithms that solve the problem of primality testing, with time complexity polynomial in the length of their numerical representation; of most relevance are those of Miller-Rabin, which is probabilistic, but consistently used in practice, and the completely deterministic Agrawal-Kayal-Saxena (AKS), which has a much greater polynomial rank, and has been deemed impractical for most uses.

What remained as a conjecture is the intractability of determining the factors of a  $\lambda$ -bit composite number, which is widely known as the *factorization hardness*; this in turn would imply that integer multiplication of two  $\lambda$ -bit primes is a OWF.

**Definition 26.** Given a group  $G$ , its order is the least  $i$  such that  $a^i \equiv_n 1$  ◇

**Corollary 3.**

$$\forall a \in \mathbb{Z}_m^\times \implies a^{\phi(n)} \equiv_n 1 \wedge a^b \equiv_n a^{b \bmod \phi(n)} \wedge a^{p-1} \equiv_p 1$$

◇

*Proof.*  $(\mathbb{Z}_n^\times, \cdot)$  is a group with  $\phi(n)$  elements. Take  $\langle a \rangle = \{a^0, \dots, a^{d+1}\}$ , where  $d$  is the order. We must have  $\phi(n) = kd$  for some  $k$ . Therefore,  $a^{\phi(n)} = a^{dk} \equiv_n 1$ .

Also,  $a^b \equiv_n a^{b\phi(n) + b \bmod \phi(n)} \equiv_n 1 \cdot a^{b \bmod \phi(n)}$  ◇

**Theorem 34.** Let  $G, H$  be two groups such that  $H < G$ , meaning the order of  $H$  divides the order of  $G$  ◇

## 13.2 Standard model assumptions

We now turn our attention to some conjectures that form the basis for most of the cryptographic schemes that will follow. We will start from the weakest, and go up to the strongest. For all our purposes, let  $\mathcal{GG}(1^\lambda)$  be a “group generator” with security parameter  $\lambda$ , and let a random sample  $(G, g, q) \leftarrow \mathcal{GG}(1^\lambda)$  be a triplet composed of the group itself  $G$ , one of its generators  $g$ , and its order  $q$ .

### Discrete logarithm

Given  $g$  and  $g^x$  in a  $\lambda$ -bit group, there is no efficient algorithm for computing  $y$  such that  $g^y = g^x$  without knowing  $x$  beforehand:

$$\forall A \in \text{PPT} \implies \Pr[\text{GAME}_{\Pi, A}^{\text{DL}}(\lambda) = 1] \in \text{Negl}(\lambda)$$

This means that the DL for a generic group  $G$  yields a OWF, whereas in a multiplicative group  $\mathbb{Z}_p^\times$ , we obtain a OWP.

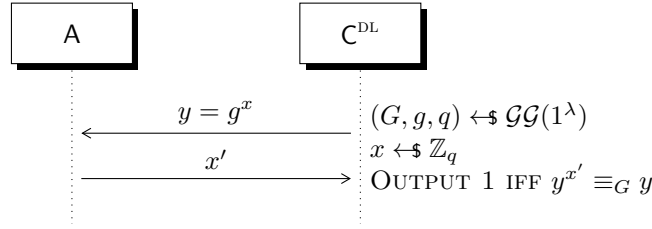


Figure 13.45:  $\text{GAME}_{\Pi, A}^{DL}(\lambda)$

### Computational Diffie-Hellman

Given a group  $G$  and two elements in it  $g^x, g^y$ , it is impractical to compute  $g^{xy}$  without knowing  $x$  or  $y$ .

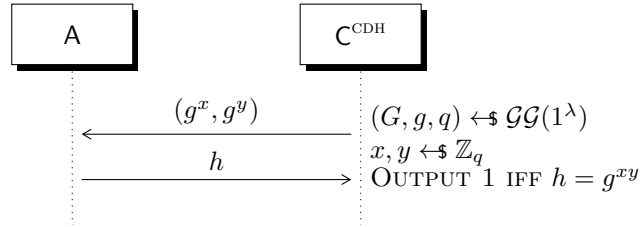


Figure 13.46:  $\text{GAME}_{\Pi, A}^{CDH}(\lambda)$

### Decisional Diffie-Hellman

Given a group  $G$  and three elements in it  $g^x, g^y, g^z$ , it is impractical to distinguish  $g^{xy}$  from  $g^z$  by only knowing  $g^x$  and  $g^y$ , along with the originating triad  $(G, g, q)$ .

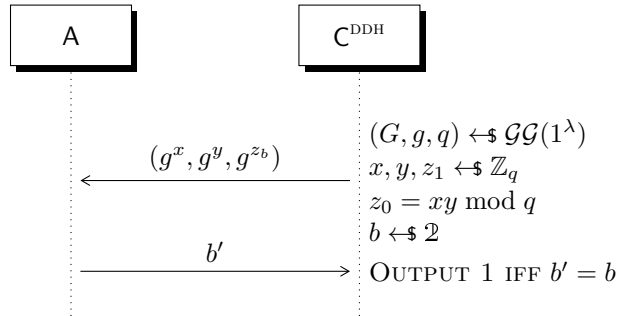


Figure 13.47:  $\text{GAME}_{\Pi, A}^{DDH}(\lambda)$

All these assumptions helped in constructing the *Diffie-Hellman key exchange* protocol, which is a way to establish a SKE channel from an unsafe channel, with any adversary unable to efficiently break the channel's secrecy. Do note that authentication is left out of the picture here.

Some relationships have been established between these assumptions: it is known that  $\text{DDH} \implies \text{CDH} \implies \text{DL}$ ; and that  $\text{CDH} \not\implies \text{DDH}$ .

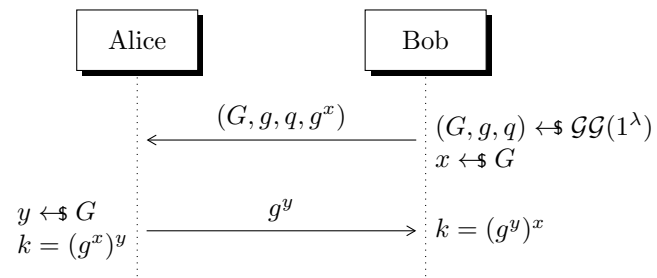


Figure 13.48: The Diffie-Hellman Key Exchange protocol

# Lesson 14

## Decisional Diffie-Hellman (cont'd)

**Claim 6.** DDH is not hard for groups  $\mathbb{Z}_p^\times$  ◇

*Proof.* Let  $Quad_p$  be the group of quadratic residues modulo  $p$ , where the group operation is multiplication:

$$Quad(p) = \{y : \exists x \in \mathbb{Z}_p \implies y \equiv_p x^2\} = \{g^z : \forall z\}$$

where  $g$  generates  $\mathbb{Z}_p^\times$ . Then, we can test if a given number  $y$  is in  $Quad_p$  by checking if  $y^{(p-1)/2} \equiv_p 1$ , because:

$$\exists z : y = g^{2z} \implies y^{(p-1)/2} = g^{\frac{2z(p-1)}{2}} = g^{z(p-1)} \equiv_p 1$$

Otherwise:

$$\nexists z : y = g^{2z} \implies y^{(p-1)/2} \equiv_p g^{z(p-1)} \cdot g^{(p-1)/2} \not\equiv_p 1$$

Furthermore:  $g^{xy} \in Quad_p \implies x \equiv_2 0 \vee y \equiv_2 0$ . With this in mind, given a random choice of  $x$  and  $y$ , the probability that  $g^{xy}$  falls in  $Quad(p)$  is  $\frac{3}{4}$ , and the probability of it being outside the group is  $\frac{1}{4}$ . This is an advantage available to a polynomial adversary. □

Nevertheless, some other groups are believed to harden quadratic residue membership; examples of such groups are  $Quad_p$  itself for  $p = 2q + 1$ , where  $q$  is prime, or the elliptic curve groups.

## Extended DDH

This is a construction that takes the DDH assumption to an extreme using groups. The DDH assumption is reported here:

$$(G, g, q) \leftarrow \mathcal{GG}(1^\lambda), \forall i X_i, Y_i, Z_i \sim \text{Unif}(G) \implies (g^X, g^Y, g^{XY}) \approx_c (g^X, g^Y, g^Z)$$

where  $X, Y$  and  $Z$  are distribution ensembles.

The construction extends this concept of hardness in detecting  $xy$  by replicating it  $n$ -times:

**Theorem 35.** Let  $X, Y_{1-n}$  and  $Z_{1-n}$  be distribution ensembles over a group  $(G, g, q)$ . Then:

$$(g^X, g^{Y_1}, g^{XY_1}, \dots, g^{Y_n}, g^{XY_n}) \approx_c (g^X, g^{Y_1}, g^{Z_1}, \dots, g^{Y_n}, g^{Z_n})$$

◇



*Proof.* TODO 17: The game is modified a bit to ease analysis of the reduction, must take care...

Also, this kind of proof is “tight”; the hybridization would have introduced a negligible difference instead

The proof can be structured by progressive hybridization over the sequences, and breaking the DDH assumption at any step  $i$ . Here instead, we will simulate the EXT sequence directly in the distinguishing game, and assume there is  $D^{\text{EXT}}$  capable of telling them apart; figure 14.49 shows the steps to take.

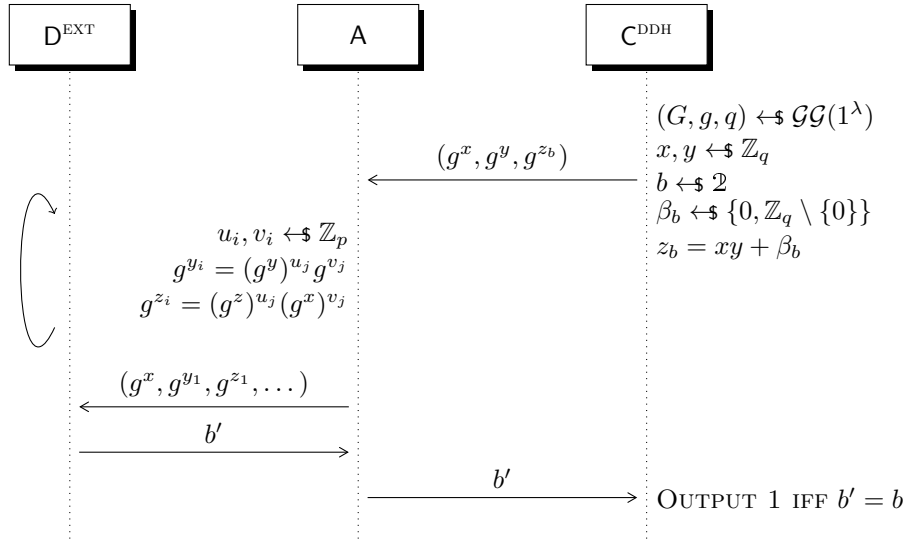


Figure 14.49: Breaking DDH by distinguishing the DDH sequence

□

Notice that  $(g^y)^{u_j} g^{v_j} = g^{yu_j+v_j}$ , and  $(g^z)^{u_j} (g^x)^{v_j} = g^{zu_j+xv_j}$ , which depending on what  $z$  is, becomes either  $g^{x(yu_j+v_j)}$ , matching perfectly the  $y_i$  case, or remains as a random linear combination, matching the  $z_i$  case.

### Naor-Reingold PRF

This is an alternative to the extended DDH seen above, designed by Moni Naor and Omer Reingold. It constructs a PRF as follows:

$$F^{\text{NR}} \in \mathbb{Z}_q^{(n+1)} \times \mathbb{Z}^n \rightarrow (G, g, q) : f_k^{\text{NR}}(x_{1..n}) \mapsto g^{k_0} \prod_{i=1}^n k_i^{x_i}$$

TODO 18: Transcribing notes directly: { This is GGM with  $G^{g,q,a}(g^b) = G_0(g^b) \parallel G_1(g^b) = (g^b, g^{ab})$   
 E.g.:  $011 \mapsto g^{a_0 a_1^0 a_2^1 a_3^1}$   
 }

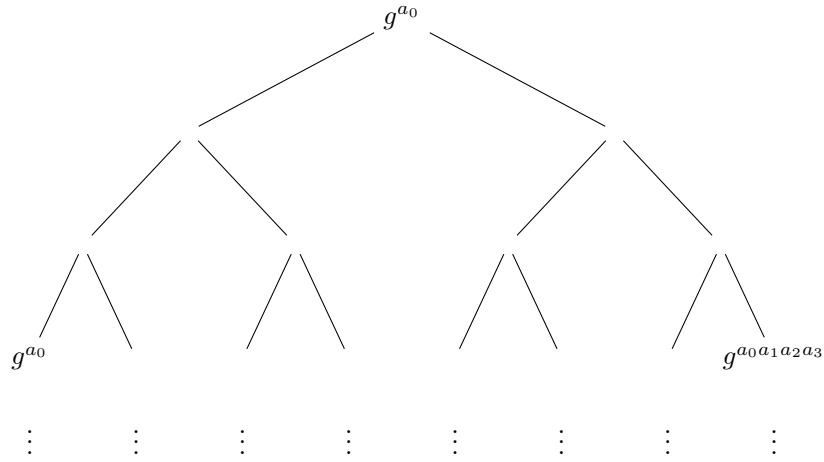


Figure 14.50: Depicting Naor-Reingold method as a tree-like structure

Notes about claw-free permutations follow; also, hash functions make an appearance

## 14.1 Public key encryption schemes

Now we discuss a substantially different way of encrypting messages, which involves two separate keys per party instead of a single, shared key; hence the names *Public key scheme*, or PKE, or *Asymmetric key scheme*.

### CPA-security, revisited

In a PKE setting, the adversary knows the public key by design, which in turn is all that he needs to perform encryptions; therefore, the queries in a hypothetical CPA game are moot, because  $\mathbf{A}$  can make the encryptions on its own; figure 14.51 shows the changed game for a PKE.

TODO 19: Another verbatim transcription:  
 Abstract construction (inefficient) for trapdoor permutation  
 $(Gen, f, f')$ , where  $gen = \text{Keygen} : \mathbb{1} \rightarrow \Xi_{\text{pt}}, f(\text{pt}, \cdot) \in \Xi_{\text{pt}} \rightarrow \Xi_{\text{pt}}, f' = f^{-1}(\text{st}, \cdot) \in \Xi_{\text{pt}} \rightarrow \Xi_{\text{pt}}$   
 Correctness:  $f^{-1}(\text{st}, f(\text{pt}, m)) = m$   
 $TDP \implies PKE$ , Easy but not trivial  
 $\text{GAME}_{\Pi, \mathbf{A}}^{\text{TDP}}$ ... same as CPA2?  
 Why  $f, f'$  is not cpa-secure? Because they are deterministic  
 Use  $f$ 's hardcore predicate

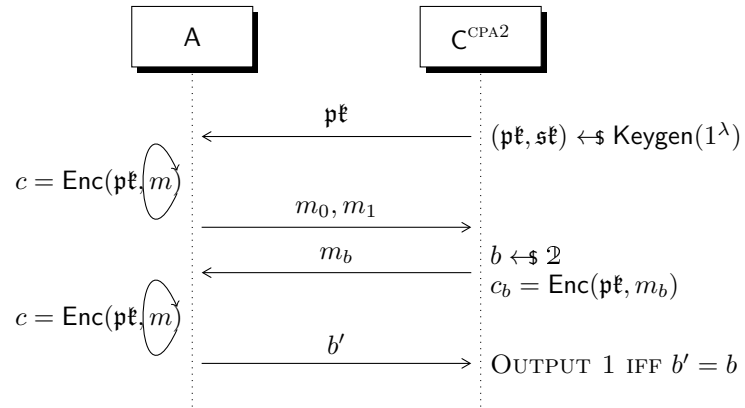


Figure 14.51: Chosen plaintext attacks, revisited for PKE schemes

# Lesson 15

## 15.1 Public key encryption (cont'd)

CCA-security, revisited

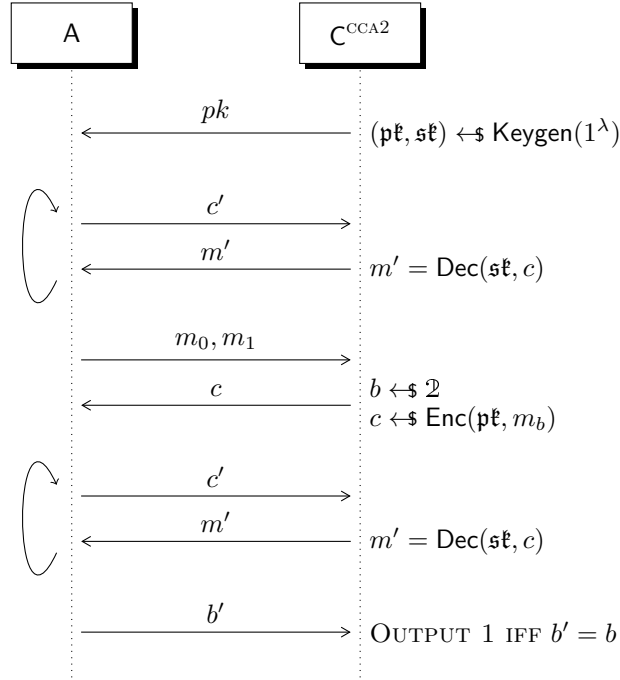


Figure 15.52: Chosen ciphertext attacks, revisited for PKE schemes

### 15.1.1 Trapdoor permutations

Let us consider the following scheme  $\Pi$ , which is an adaption of the old CPA-secure SKE to public keys:

- $\text{Enc}(pk, m) = (r, f_{pk}(m \oplus r))$ , where  $r \sim \text{Unif}(\mathbb{Z}^\lambda)$
- $\text{Dec}(sk, (c_0, c_1)) = g_{sk}(c_1) \oplus c_0$
- Correctness:  $g_{sk}(c_1) \oplus c_0 = g_{sk}(f_{pk}(m \oplus r)) \oplus r = m \oplus r \oplus r = m$

A *trapdoor permutation* (or TDP) is a OWP having the following features:

- A key pair is chosen UAR by a key generator algorithm:

$$(\mathbf{pk}, \mathbf{sk}) \leftarrow \text{Keygen}(1^\lambda)$$

- There is a function family  $F \in \Xi_{\mathbf{pk}} \rightarrow (V_{\mathbf{pk}} \rightarrow V_{\mathbf{pk}})$  such that:
  - Computing  $f_{\mathbf{pk}}$  is efficient
  - Domain sampling ( $x \leftarrow V_{\mathbf{pk}}$ ) is efficient
- There is an efficient function  $g_{\mathbf{sk}}$  that efficiently *inverts*  $f_{\mathbf{pk}}$ , where  $sk$  is the “trapdoor”:

$$g(\mathbf{sk}, f(\mathbf{pk}, x)) = x$$

- No efficient adversary is able to invert  $f_{\mathbf{pk}}$  without knowing  $\mathbf{sk}$

Note that because  $\mathbf{pk}$  is public, an adversary can perform any polynomial number of encryptions with  $\mathbf{pk}$ , and see the corresponding ciphertext. This is the same characteristic of PKE schemes we described just a while ago. It entails that, if left deterministic, a TDP is not CPA-secure.

Also, the scheme described above is not CPA-secure from the start: the adversary, by choosing two messages for the challenge, and receiving the ciphertext ( $c_0 = r, c_1$ ), along with the public key, has everything needed to reconstruct the encryption and check whichever message was encrypted, much like the problem of UF-CMA against a deterministic MAC scheme.

Here, in this scheme, we combine randomness and the notion of hardcore predicate  $\mathbf{hc}$ :

- $(pk, sk) \leftarrow \text{Keygen}(1^\lambda)$
- $r \leftarrow \Xi_{pk}$
- $c := \text{Enc}(\mathbf{pk}, m) = (f_{\mathbf{pk}}(r), \mathbf{hc}(r) \oplus m)$
- Correctness:  $\text{Dec}(\mathbf{sk}, c) = \mathbf{hc}(g_{\mathbf{sk}}(c_1)) \oplus c_2$

**Theorem 36.** *If  $F$  is a TDP and  $\mathbf{hc}$  is hardcore for  $f$ , then the above scheme is CPA-secure.*  $\diamond$

*Proof.* The proof is left as exercise

TODO 20: Apparently, the reduction here is not easy at all, some hints are needed.

□

### 15.1.2 TDP examples

One example stems from the factoring problem: let's look again at  $\mathbb{Z}_n^\times$ , where  $n$  is the product of two prime numbers  $p$ , and  $q$ :

**Theorem 37** (Chinese remainder theorem). *The following isomorphisms to  $\mathbb{Z}_n^\times$  are true:*

- $\mathbb{Z}_n \simeq \mathbb{Z}_p \times \mathbb{Z}_q$
- $\mathbb{Z}_n^\times \simeq \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times$

*Note that the theorem is more general, and holds for any two numbers  $p$  and  $q$  that are coprime.*  $\diamond$

How to use this theorem for constructing a PKE scheme:

**Theorem 38** (Euler's totient theorem). *Let  $x, n$  be two coprime numbers. Then  $x^{\varphi(n)} \equiv_n 1$*   $\diamond$

Also, remember that  $\forall p, q \in \mathbb{P} \implies \varphi(pq) = (p-1)(q-1)$

So let  $a$  be the public key such that  $\gcd(a, \varphi(n)) = 1$ , then  $\exists! b \in \mathbb{Z}_n : ab \equiv_{\varphi(n)} 1$ ;  $b$  will be our private key. Define encryption as  $f(a, m) = m^a \bmod n$ , and then decryption as  $g(b, c) = c^b \bmod n$ . Observe that

$$g(b, f(a, m)) = (m^a)^b = m^{ab} = m^{k\varphi(n)+1} = (m^{\varphi(n)})^k m \equiv_n m$$

because  $ab \equiv 1 \bmod \varphi(n)$ .

So we conjecture that the above is a valid TDP-based PKE scheme. This is actually referred to as the *RSA assumption*, and is depicted in figure 15.53

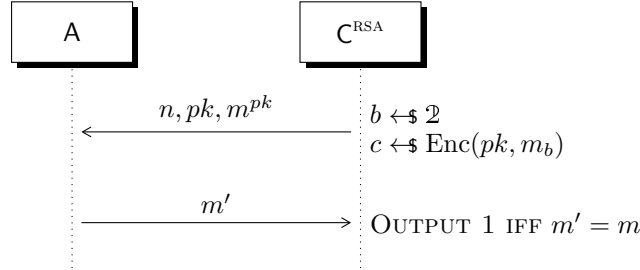


Figure 15.53: The RSA assumption

Relation to the factoring problem:  $\text{RSA} \implies \text{FACT}$

Proof: Given  $p, q$ , an adversary can compute  $\varphi(n) = (p-1)(q-1)$ , and then find the inverse of the public key in  $\mathbb{Z}_{pq}^\times$ .

It hasn't been proven that  $\text{FACT} \implies \text{RSA}$

## 15.2 Textbook RSA

This is an insecure toy example of the more complex RSA (Rivest Shamir Adleman) scheme:

- Setup:  $(pk, sk) \leftarrow \text{Keygen}(\mathbb{Z}_n^\times) : sk \equiv_{\varphi(n)} pk^{-1}$

- Encryption:  $\text{Enc}(\mathbf{pk}, m) = m^{\mathbf{pk}} \bmod n$
- Decryption:  $\text{Dec}(\mathbf{sk}, c) = c^{\mathbf{sk}} \bmod n$
- Correctness:  $\text{Enc}(\mathbf{pk}, \text{Dec}(\mathbf{sk}, m)) = m^{\mathbf{pk} \cdot \mathbf{sk}} \equiv_n m$

Again, since the encryption routine is deterministic, the scheme is not CPA-secure. However, a hardcore predicate can be inserted to the routine:  $\hat{m} = r || m$ , where  $r \leftarrow \mathbb{Z}^{2^l}$ . Now the encryption is pseudorandom.

Some interesting facts:

1.  $l \in \omega(\log(\lambda))$  otherwise a brute-force attack becomes viable.
2. If  $m \in \mathbb{Z}$ , then the scheme is CPA-secure under RSA assumption, just use the standard TDP
3. If  $m$  is “in the middle”:  $2 \leq m \leq 2^l$ ; then RSA is believed to be secure (standard PKCS#1, 5)
4. However, this construct is not CCA-secure.

TODO 21: Counterexample?

### 15.2.1 Trapdoor Permutation from Factoring

Here is an attempt to build a TDP over the group  $\text{Quad}(n)$ : let's look at  $f(x) \equiv_n x^2$  where  $f \in \mathbb{Z}_n^\times \rightarrow \text{Quad}_n(\subset \mathbb{Z}_n^\times)$ . Notice that this is not a permutation in general, so let's consider their CRT's representation<sup>20</sup>, in order to restrict  $f$ 's domain:

$$x = (x_p \equiv_p x, x_q \equiv_q x), f(x) \equiv_p x^2, x \in \mathbb{Z}_p^\times$$

Since  $\mathbb{Z}_p^\times$  is cyclic:

$$\begin{aligned} \mathbb{Z}_p^\times &= \{g^0, g^1, g^2, \dots, g^{\frac{p-1}{2}-1}, g^{\frac{(p-1)}{2}}, \dots, g^{p-2}\} \\ \text{Quad}_p &= \{g^0, g^2, g^4, \dots, g^{2(\frac{p-1}{2}-1)=p-3}, g^{2(\frac{p-1}{2})=p-1 \equiv 0}, \dots, g^{2(p-2)}\} \end{aligned}$$

Because of this,  $|\text{Quad}_p| = \frac{p-1}{2}$ . Moreover, since  $g^{2\frac{p-1}{2}} \equiv_p 1$  and  $g^{\frac{p-1}{2}}$  cannot be 1 (since  $g^0 \neq g^{\frac{p-1}{2}} \neq g^{p-1}$ ) but must be one of the  $p-1$  elements of  $\mathbb{Z}_p^\times$ , then  $g^{\frac{p-1}{2}} \equiv_p -1$ .

Now it's possible to show that  $f : \text{Quad}_p \rightarrow \text{Quad}_p$  is a permutation, and we are going to show how to find  $f^{-1}$ .

Assume  $p \equiv_4 3$ , meaning  $\exists t : p = 4t + 3 \implies t = \frac{p-3}{4}$ ; then squaring modulo  $p$  becomes a permutation. Given  $y \equiv_p x^2$ , observe the following:

<sup>20</sup>Those well-versed in number representation in computers may know this representation as the *residue number system*

$$\begin{aligned}
(y^{t+1})^2 &= y^{2t+2} = y^{2^{\frac{p-3}{4}}+2} = y^{\frac{p-1}{2}+1} = (x^2)^{\frac{p-1}{2}+1} = x^{p-1}x^2 \equiv_p x^2 \\
&\Downarrow \\
x &= \pm y^{t+1}
\end{aligned}$$

But only one among  $\pm y^{t+1}$  is a square: the positive one. Therefore:

$$p = 4t + 3 \implies \frac{p-1}{2} = \frac{4t+2}{2} = 2t+1$$

so  $\frac{p-1}{2}$  is odd.

Now, since we are considering just the elements of  $Quad_p$ , and we can write each  $x \in \mathbb{Z}_p^\times$  as  $g^z$  for a  $z \in \mathbb{Z}_p$ :

$$y = x^2 = (g^z)^2$$

So,  $y = g^{z'} \in Quad_p \Leftrightarrow z'$  is even. If  $z'$  is odd, then  $y \notin Quad_p$ .

Since  $\frac{p-1}{2}$  is odd, then  $g^{\frac{p-1}{2}} \notin Quad_p$ , and since it is possible to generate all of the other numbers with odd exponents

$$g^{odd} = g^{\frac{p-1}{2} \pm even} = g^{\frac{p-1}{2}} g^{\pm even} \implies -1(g^{\pm even})$$

and  $g$  powered to odd exponents will have this form. From here, it's possible to state the following:

**Lemma 24.**  $\forall z \in Quad_p \implies -z \notin Quad_p$   $\diamond$

### 15.2.2 Rabin's Trapdoor permutation

Now we study a one way function built on previous deductions about number theory and modular arithmetic. The *Rabin trapdoor permutation* is defined as:

$$f(x) = x^2 \bmod n$$

where  $n = pq$  for primes  $p, q \equiv_4 3$ .

We can observe that the image of this function is  $Quad_n$ , a subset of  $\mathbb{Z}_n^\times$ .

Because of CRT, it is possible to state that  $f$  maps as follows:

$$x = (x_p, x_q) \implies x^2 = (x_p^2, x_q^2)$$

since each element of  $\mathbb{Z}_n$  has always two different forms, in  $\mathbb{Z}_p$  and in  $\mathbb{Z}_q$ . So

$$y \in Quad_n \Leftrightarrow y_p \in Quad_p \wedge y_q \in Quad_q$$

As before, the image of  $f$  is exactly

$$Quad_n = \{y : \exists x : y \equiv_n x^2\}$$

If we try to invert the function  $f$ , even without applying the previous inversion algorithm, we easily note that among the 4 possible values:

$$f^{-1}(y) = \{(x_p, x_q), (-x_p, x_q), (x_p, -x_q), (-x_p, -x_q)\}$$



only 1 is a quadratic residue since we said, in the last lemma, that only one out of  $-x_k, x_k$  is a quadratic residue for  $k = q, p$ .

Therefore, we have that the Rabin's TDP is a permutation in  $Quad_n$ , and that the cardinality of  $Quad_n$  is  $\frac{|\mathbb{Z}_n^\times|}{4}$ . Furthermore, with the following claim we can state that the Rabin cryptosystem is a OWF thanks to the FACT assumption.

**Claim 7.** *Given  $x$  and  $z$  such that  $x^2 \equiv_n z^2 \equiv_n y$ :*

$$x \neq \pm z \implies n \text{ is factorizable}$$

◇

*Proof.* Since  $f^{-1}(y)$  has only one value out of four,  $x \neq \pm z$  and  $z$  is either  $\{(x_p, x_q), (-x_p, -x_q)\}$ , then:

$$x \in \{(x_p, -x_q), (-x_p, x_q)\} \implies x + z \in \{(0, 2x_q), (2x_p, 0)\}$$

Now assume  $x + z = (2x_p, 0)$  without loss of generality, since the proof for the other case is the same. We have that  $x + z \equiv_q 0$  and  $x + z \not\equiv_p 0$ . But then  $\gcd(x + z, n) = q$ , and we obtain  $q$ . □

**Theorem 39.** *Squaring mod  $n$ , where  $n$  is a Blum integer<sup>21</sup> is a trapdoor permutation under the factoring assumption.* ◇

Since we have already shown that Rabin's function is a permutation since it is invertible, we have to show that Rabin's function is also OWF:

**Proposition 2.**

$$\text{FACT} \implies f(x) \in \text{OWF} \quad \diamond$$

*Proof.* The proof is by contradiction. Assume there is an adversary **A** who, given  $y \equiv_n x^2$ , can find an integer  $z \in \mathbb{Z}_n$  such that  $z^2 \bmod n = y$  while  $z \neq \pm x$ . We can build a reduction as the one in figure 15.54 to show that **A** chooses  $x$ , here *Blum* is a sampler for Blum integers:

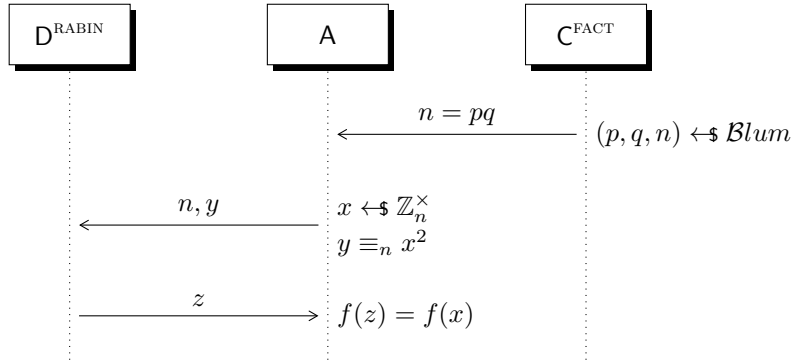


Figure 15.54: —

Once obtained  $z \neq \pm x$  which  $z^2 = y$  we can use **Claim 1** (just summing  $x$  and  $z$  and analyzing the result) to factorize  $n$  in polytime. But factorizing  $n$  in polytime is not possible. □

<sup>21</sup>a Blum integer  $n$  is the product of two numbers  $p$  and  $q$  such that  $p, q \equiv_4 3$ , as the definition of Rabin's TDP

# Lesson 16

## 16.1 PKE schemes over DDH assumption

### 16.1.1 El Gamal scheme

Let's define a new  $\Pi = (\text{Keygen}, \text{Enc}, \text{Dec})$ . Generate the needed public parameters  $(G, g, q) \leftarrow \mathcal{G}(1^\lambda)^{22}$ , then:

- Key generation:  $(\mathbf{pk}, \mathbf{sk}) = (g^x, x)$ , where  $x \leftarrow \mathbb{Z}_q$
- Encryption:  $\text{Enc}(\mathbf{pk}, m) = (g^r, \mathbf{pk}^r \cdot m)$ , where  $r \leftarrow \mathbb{Z}_q^{23}$
- Decryption:  $\text{Dec}(\mathbf{sk}, (c_1, c_2)) = c_1^{-\mathbf{sk}} \cdot c_2$

Correctness of this scheme follows from some algebraic steps:

$$\begin{aligned} \hat{m} &= \text{Dec}(\mathbf{sk}, \text{Enc}(\mathbf{pk}, m)) \\ &= \text{Dec}(x, \text{Enc}(g^x, m)) \\ &= \text{Dec}(x, (g^r, (g^x)^r \cdot m)) \\ &= (g^r)^{-x} \cdot (g^x)^r \cdot m \\ &= m \end{aligned}$$

**Theorem 40.** Assuming DDH, the El Gamal scheme is CPA-secure.  $\diamond$

*Proof.* Consider the two following games  $\text{HYB}_{\Pi, A}^0(\lambda, b)$  and  $\text{HYB}_{\Pi, A}^1(\lambda, b)$  defined as follows. Observe that  $b$  can be fixed without loss of generality.

**Note:** it is important to note that we can measure the advantage of  $A$ , so

fixed its output  $\text{Adv}_A(\lambda) = \underbrace{|\Pr[\overset{b'=1}{A=1} \mid b=0]|}_{A \text{ loses}} - \underbrace{|\Pr[\overset{b'=1}{A=1} \mid b=1]|}_{A \text{ wins}}$ . Since  $b$  is fixed the above formula will give a value  $\lambda \in \text{negl}$ , generally the advantage of an adversary is:  $\frac{1}{2} + \lambda$  (random guessing + a negligible factor).

TODO 22: Here the  $1/2$  value may refer to how the Katz-Lindell book exposes their proofs by leaving arbitrary choice of  $b$  to the challengers, thus limiting the probability of the adversaries' success by  $1/2$ .

<sup>22</sup> $G$  could be any valid group such as  $\text{Quad}_p$ , or an elliptic curve group

<sup>23</sup>We need  $r$  because we want to re-randomize  $c$

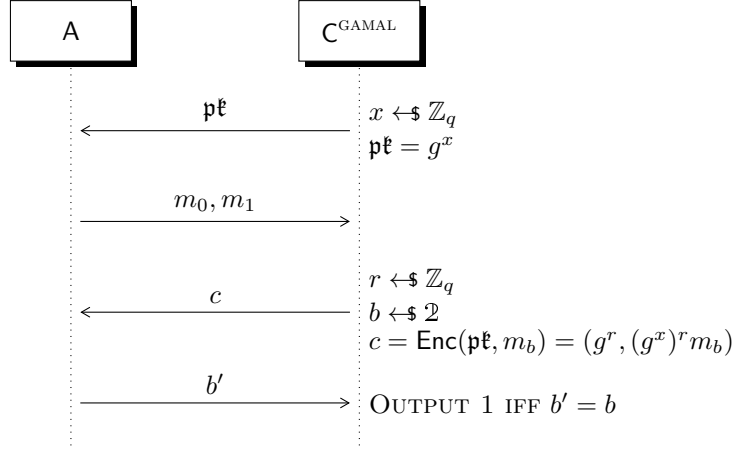


Figure 16.55:  $\text{HYB}_{\Pi, A}^0(\lambda, b) = \text{GAME}_{\Pi, A}^{\text{GAMAL}}(\lambda, b)$

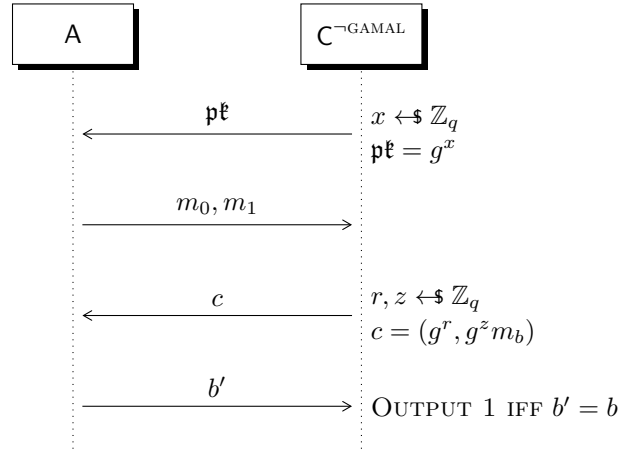


Figure 16.56:  $\text{HYB}_{\Pi, A}^1(\lambda, b)$

Venturi prefers to fix  $b$  beforehand, meaning the whole proof can be stated by setting  $b$  to either 0 or 1, and maintain its validity without changing anything else. In this way, he somehow bounds the probability of success to be  $\leq \text{Negl}(\lambda)$

We will prove that:

$$\text{HYB}_{\Pi, A}^0(\lambda, 0) \approx_c \text{HYB}_{\Pi, A}^1(\lambda, 0) \equiv \text{HYB}_{\Pi, A}^1(\lambda, 1) \approx_c \text{HYB}_{\Pi, A}^0(\lambda, 1)$$

**Claim 8.**  $\forall b \in \mathbb{Z} \implies \text{HYB}_{\Pi, A}^0(\lambda, b) \approx_c \text{HYB}_{\Pi, A}^1(\lambda, b)$   $\diamond$

*Proof.* This proof reduces to disproving DDH. Fix  $b$  without loss of generality, and assume there exists a distinguisher  $D$  able to distinguish  $H_0(\lambda, b)$  and  $H_1(\lambda, b)$ . Consider the game in figure 16.57:

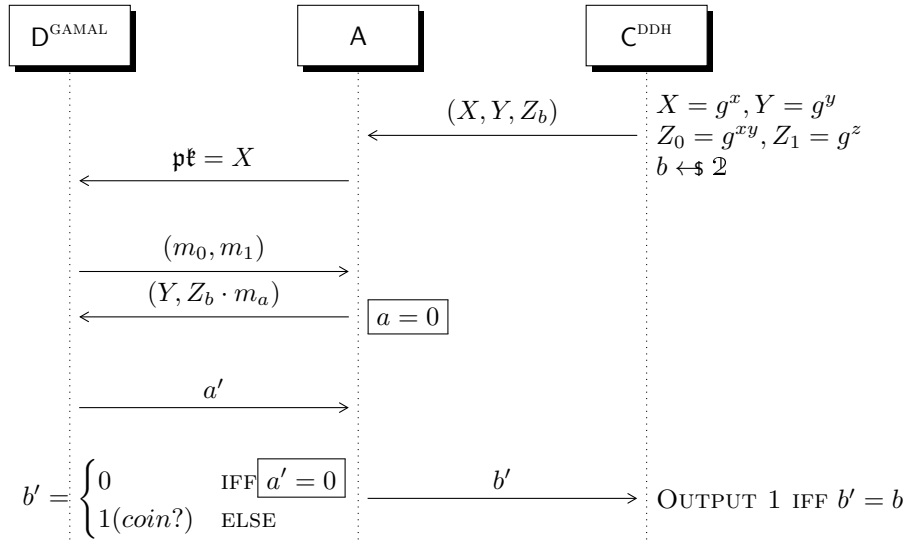


Figure 16.57: —

If  $D^{\text{GAMAL}}$  is able to break the cipher, then  $A$  is in turn able to distinguish a DH triple from a random one, falsifying DDH.  $\square$

**Claim 9.**  $\text{HYB}_{\Pi, A}^1(\lambda, 0) \equiv \text{HYB}_{\Pi, A}^1(\lambda, 1)$   $\diamond$

*Proof.* This follows from the fact that:

$$(g^x, (g^r, g^z m_0)) \equiv (g^x, (g^r, U_\lambda)) \equiv (g^x, (g^r, g^z m_1))$$

$\square$

Composing the two claims, the proof is complete.  $\square$

### Properties of of El Gamal PKE scheme

Some useful observations can be made about this scheme:

- It is **homomorphic**: Given two ciphertexts  $(c_1, c_2)$  and  $(c'_1, c'_2)$ , then doing the product between them yields another valid ciphertext:

$$\begin{aligned} & (c_1 \cdot c'_1, c_2 \cdot c'_2) \\ &= (g^{r+r'}, h^{r+r'}(m \cdot m')) \end{aligned}$$

thus, decrypting  $c \cdot c'$ , gives  $m \cdot m'$ .

- It is **re-randomizable**: Given a ciphertext  $(c_1, c_2)$ , and  $r' \leftarrow \mathbb{Z}_q$ , then computing  $(g^{r'} \cdot c_1, h^{r'} \cdot c_2)$  results in a “fresh” encryption for the same message: the random value used at the encryption step will change from the original  $r$  to  $r + r'$

These observations lead to the conclusion that this scheme is not CCA-secure. However, such properties can be desirable in some use cases, where a message must be kept secret to the second party. In fact, there are some PKE schemes which are designed to be **fully homomorphic**, i.e. they are homomorphic for any kind of function.

Consider the following use case: a client  $C$  has an object  $x$  and wants to apply a function  $f$  over it, but it lacks the computational power to execute it. There is another subject  $S$ , which is able to efficiently compute  $f$ , so the goal is to let it compute  $f(x)$  but the client wishes to keep  $x$  secret from him. This can be achieved using a FH-PKE scheme as follows:

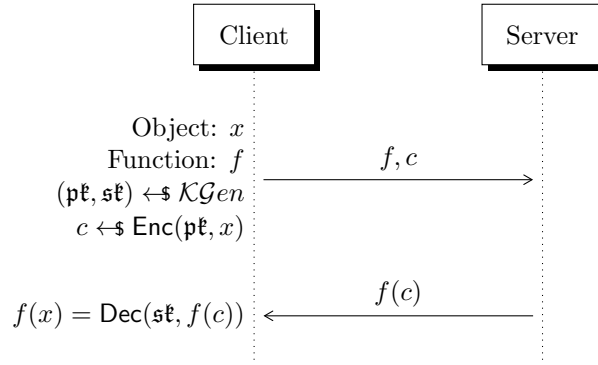


Figure 16.58: Delegated secret computation

However one important consideration must be made: All these useful characteristics expose an inherent malleability of any fully homomorphic scheme: any attacker can manipulate ciphertexts efficiently, and with some predictable results. This compromises even CPA security of such schemes.

## 16.2 Proof systems

This new scheme assumes DDH like its El Gamal cousin, and has the advantage of being CCA-secure. A powerful tool, called *Designated Verifier Non-Interactive*

*Zero-Knowledge* (DV-NIZK in short), or alternatively *Hash-Proof System*, is used here.

TODO 23: the Ali Baba example would be useful here. Also, the coloured balls is a good example too (long live Wikipedia)

Let  $L \subseteq Y$  be a Turing-recognizable language in NP, and a predicate  $V \in X \times Y \rightarrow \mathbb{2}$  such that:

$$L := \{y \in Y : \exists x \in X \implies V(x, y) = 1\}$$

where  $x$  is called a *witness* of  $y$ .

TODO 24: to review and understand/better

In our case, let  $x = (p, q)$ ,  $y = pq$ , and define a scheme  $\Pi$  as follows:

$$\Pi = (\textit{Setup}, \textit{Prove}, \textit{Verify})$$

- Setting up:  $(\omega, \tau) \leftarrow \textit{Setup}(1^\lambda)$ , where  $\omega$  is the *common reference string*, and  $\tau$  is the *trapdoor*
- Proving a statement:  $\pi = \textit{Prove}(\omega, y, x)$
- Verifying:  $\hat{\pi} = \textit{Verify}(\tau, y)$

Correctness is defined as  $\textit{Prove}(\omega, y, x) = \textit{Verify}(\tau, y)$ . Some more observations:

- $\omega$  is public ( $= pk$ )
- $\tau$  is part of the secret key
- $\tau = (x, y) : V(x, y) = 1$
- There is presumably a common third-party, which samples from the setup and publishes  $\omega$ , while giving  $\tau$  to only B.

TODO 25: Why? Can't the verifier do the setup and publish omega directly? Is honesty an issue here?

Purpose of a proof system is to give a way to convince someone (the “verifier”) that someone else (the “prover”) knows something (the “statement”  $y$ ), and nothing more, to no one else. The proof can be computed in two different ways, this is the core notion of *zero-knowledge*,  $\neg\tau \implies \text{ZK}$

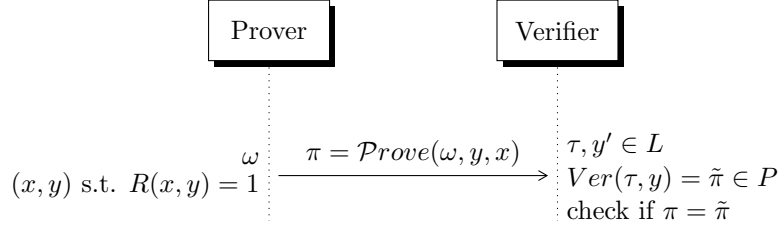


Figure 16.59: Overview of Cramer-Shoup operation

### Properties

- *honest people* **Completeness**:

$$\forall y \in L, \forall (\omega, \tau) \leftarrow \text{Setup}(1^\lambda) \implies \text{Prove}(\omega, y, x) = \text{Verify}(\tau, y)$$

- (*stronger, against malicious prover*) **Soundness**: It is hard to produce a valid proof for any  $y \notin L$
- (*characteristic, against malicious verifier*) **Zero-knowledge**: Proof for  $x$  can be simulated without knowing  $x$  itself

**Definition 27** ( $t$ -universality, or  $t - 1$  simulation soundness). Let  $\Pi$  be a DV-NIZK scheme; it is  $t$ -universal iff for any distinct  $y_{1-t} \notin L$  we have:

$$(\omega, \text{Verify}(\tau, y_1), \dots, \text{Verify}(\tau, y_t)) = (\omega, v_1, \dots, v_t)$$

where  $(\omega, \tau) \leftarrow \text{Setup}(1^\lambda)$  and  $v_{1-t} \leftarrow \text{Proof}$ , where  $\text{Proof}$  is the proofs' space.  $\diamond$

### Enriching a proof system

We can “enrich” a DV-NIZK scheme with labels  $l \in \mathcal{L}^*$ . Suppose to have the following:

$$\mathcal{L}' = \mathcal{L} \parallel \mathcal{L}^* = \{(y, l) \in \mathcal{L} \times \mathcal{L}^*\}$$

Then our scheme changes, because the statements are now compositions of a label  $l$  along with the actual statement  $y$ ; for the  $t$ -universality property, we can now consider two distinct  $(y_i, l_i)$ .

### Membership-hard language

**Definition 28.** Language  $L$  is *membership-hard* iff there exists a language  $\overline{L}$  such that:

1.  $L \cap \overline{L} = \emptyset$
2.  $\exists \text{Sample} \in \text{PPT}$  outputting  $y \leftarrow \mathcal{Y}$  together with  $x \in \overline{X}$  such that  $R(y, x) = 1$   
(therefore  $(y, x) \leftarrow \text{Sample}(1^\lambda)$ )
3.  $\exists \overline{\text{Sample}} \in \text{PPT}$  outputting  $y \leftarrow \overline{L}$
4.  $\{y : (y, x) \leftarrow \text{Sample}(1^\lambda)\} \approx_c \{y : y \leftarrow \overline{\text{Sample}}(1^\lambda)\}$

$\diamond$

# Lesson 17

## 17.1 Construction of a CCA-secure PKE

This section exposes a construction of a CCA-secure PKE scheme, using hash-proof systems, membership-hardness, and the  $t$ -universality property.

Let  $\Pi_1, \Pi_2$  be two distinct hash-proof systems for some NP language  $L$  and the range of  $Prove_2$  supports labels ( $L' = L || 2^\ell$ ).

Construct the CCA scheme as follows:  $\Pi := (\mathcal{KGen}, Enc, Dec)$

- $(\overbrace{(\omega_1, \omega_2)}^{pk}, \overbrace{(\tau_1, \tau_2)}^{sk}) \leftarrow \mathcal{KGen}(1^\lambda)$  ,  $(\omega_1, \tau_1) \leftarrow \mathcal{Setup}_1(1^\lambda), (\omega_2, \tau_2) \leftarrow \mathcal{Setup}_2(1^\lambda)$
- Encryption routine:  $Enc((\omega_1, \omega_2), m)$ 
  - $(y, x) \leftarrow \text{Sample}_1(1^\lambda)$
  - $\pi_1 \leftarrow \text{Prove}_1(\omega_1, y, x)$
  - $l := \pi_1 \cdot m$
  - $\pi_2 \leftarrow \text{Prove}_2(\omega_2, (y, l), x)$
  - $c := (c_1, c_2) = ((y, l), \pi_2)$
- Decryption routine:  $Dec((\tau_1, \tau_2), (c_1, c_2))$ 
  - $\hat{\pi}_2 = \text{Verify}_2(\tau_2, c_1)$
  - IF  $\hat{\pi}_2 \neq c_2$  THEN OUTPUT FALSE
  - Recall:  $c_1 = (y, l)$
  - $\hat{\pi}_1 = \text{Verify}_1(\tau_1, y)$
  - OUTPUT  $l \cdot \hat{\pi}_1^{-1}$

Correctness (assume  $\hat{\pi}_i = \pi_i \forall i$ ):

$$\begin{aligned}
 \hat{m} &= Dec(sk, Enc(pk, m)) \\
 &= Dec((\tau_1, \tau_2), Enc((\omega_1, \omega_2), m)) \\
 &= Dec((\tau_1, \tau_2), ((y, l), \pi_2)) \\
 &= l \cdot \hat{\pi}_1^{-1} \\
 &= \pi_1 \cdot m \cdot \hat{\pi}_1^{-1} \\
 &= m
 \end{aligned}$$

Some additional notes (may be incorrect):



- The message space of the second prover is the range of the first prover.
- The message space of the first prover is a multiplicative group
- The message space of the second prover is a polylogarithmic language in  $(\lambda)$

**Theorem 41.** *Assuming  $\pi_1$  is 1-universal,  $\pi_2$  is 2-universal and  $L$  is a membership-hard language; then the above scheme is CCA-secure.*  $\diamond$

*Proof.* Five different games will be defined, from  $\text{GAME}_{\Pi, A}^0$  up to  $\text{GAME}_{\Pi, A}^4$ ; the first game will be an analogous formalization of how the above PKE scheme works. It shall be proven that, for arbitrarily fixed  $b$  in  $\mathbb{2}$ :

$$\begin{aligned} \text{GAME}_{\Pi, A}^0(\lambda, b) &\equiv \text{GAME}_{\Pi, A}^1(\lambda, b) \\ &\approx_c \text{GAME}_{\Pi, A}^2(\lambda, b) \\ &\approx_s \text{GAME}_{\Pi, A}^3(\lambda, b) \\ &\equiv \text{GAME}_{\Pi, A}^4(\lambda, b) \end{aligned}$$

and finally that  $\text{GAME}_{\Pi, A}^4(\lambda, 0) = \text{GAME}_{\Pi, A}^4(\lambda, 1)$ , therefore concluding that  $\text{GAME}_{\Pi, A}^0(\lambda, 0) \approx_c \text{GAME}_{\Pi, A}^0(\lambda, 1)$ , and proving this scheme is CCA-secure.

TODO 26: Non sono per niente sicuro riguardo all'origine di  $x$  ed  $y$ , né tantomeno dove sia definito il sampler per essi

The games are defined as follows:

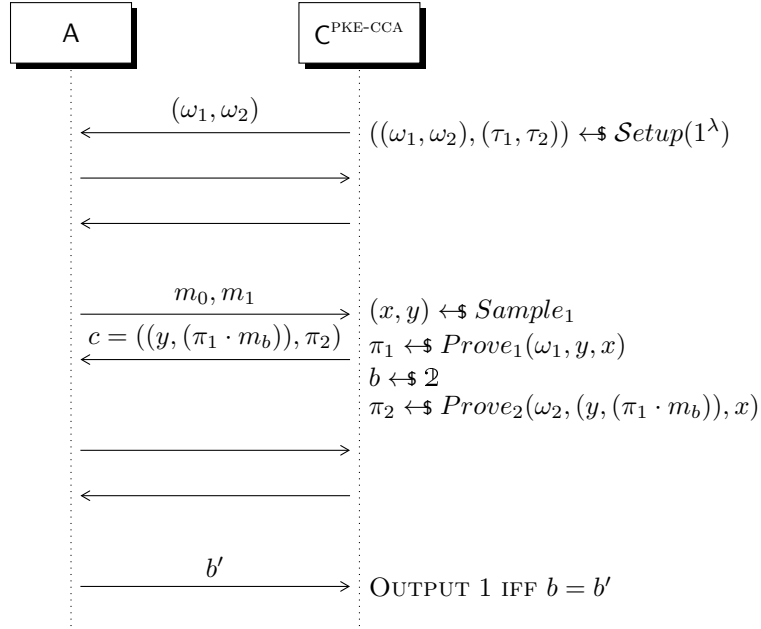


Figure 17.60: Original CCA game

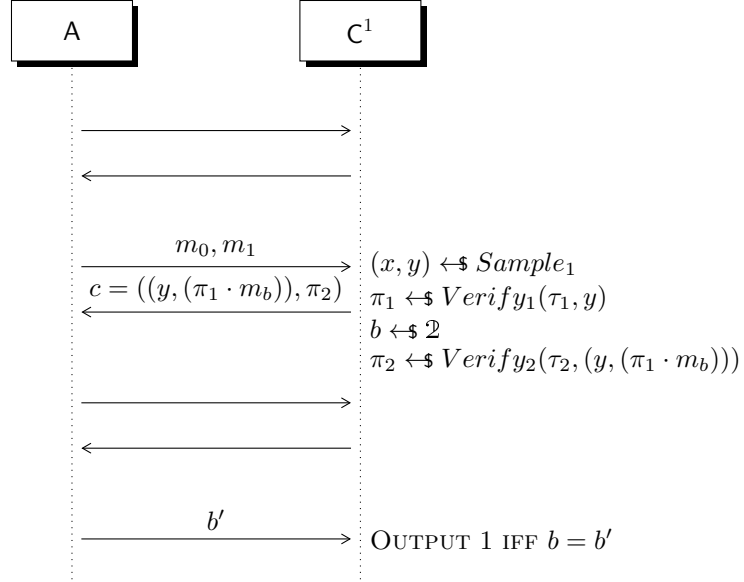


Figure 17.61: Use verifiers

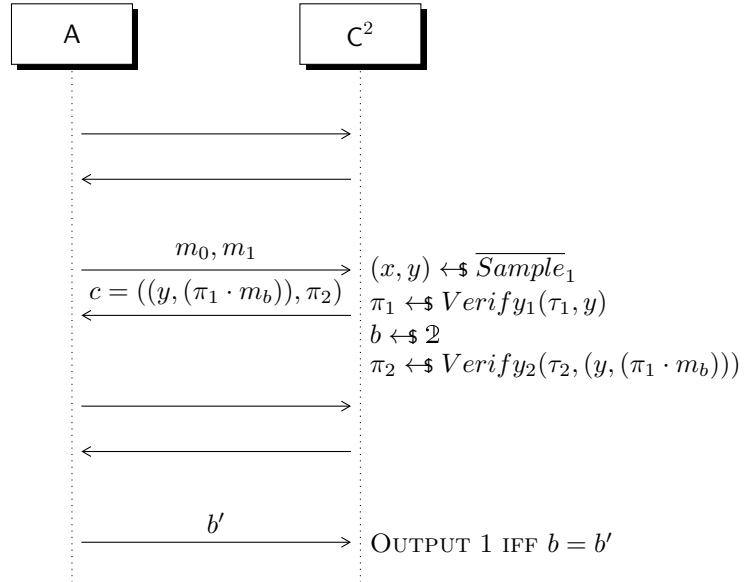


Figure 17.62: Sample statements outside the language

TODO 27: Non è stato chiaro sull'origine di  $x$  ed  $y$ , inoltre mi manca da scrivere le query di decifratura

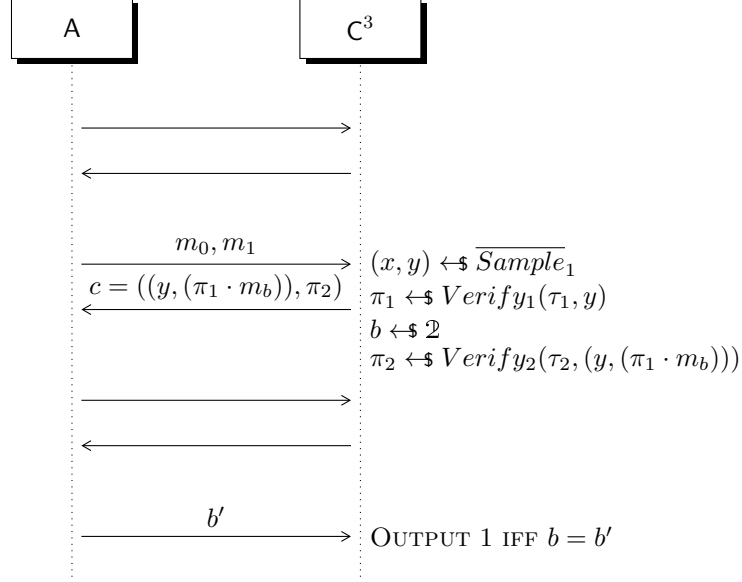


Figure 17.63: Modify decryption queries

□

**Lemma 25.**

$$\forall b, G_0(\lambda, b) \equiv G_1(\lambda, b)$$

◇

*Proof.* This follows by the correctness of  $\pi_1$  and  $\pi_2$

$$\pi_1 = \tilde{\pi}_1 = \text{Ver}_1(\tau_1, y)$$

$$\pi_2 = \tilde{\pi}_2 = \text{Ver}_2(\tau_2, y)$$

with probability 1 over the choice of  $((\omega_1, \omega_2), (\tau_1, \tau_2)) \leftarrow \text{Setup}(1^\lambda)$

$$\begin{aligned} \pi_1 &\leftarrow \text{Prove}_1(\omega, y, x) \\ (\omega_2, \tau_1) &\leftarrow \text{Setup}_2(1^\lambda) \\ \pi_2 &\leftarrow \text{Prove}_2(\omega_2, (y, l), x) \quad \forall y \in L, L \in \text{Proof}_1 \end{aligned}$$

□

**Lemma 26.**

$$\forall b, G_1(\lambda, b) \approx_c G_2(\lambda, b)$$

◇

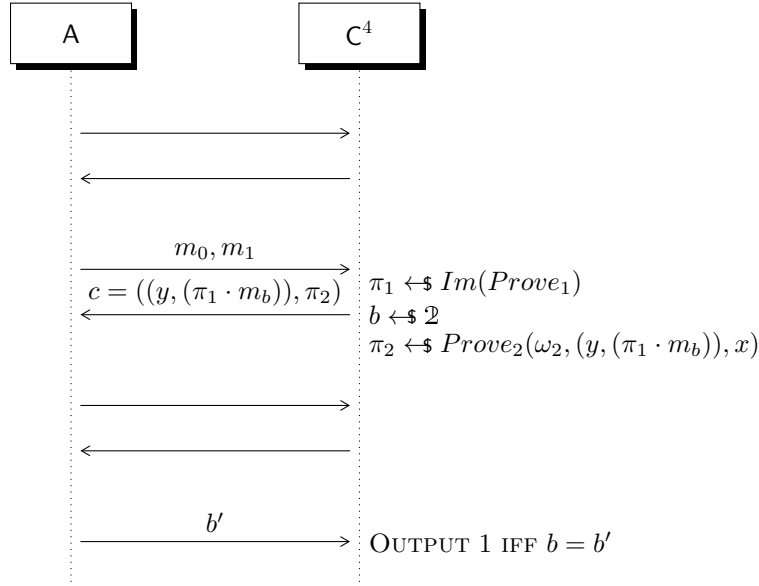


Figure 17.64:  $\pi_1$  is chosen UAR

*Proof.* Straightforward reduction from membership hardness.

**TODO 28:** Transcription: Recall we're using verifiers, so  $y$  can be outside  $L$ :  
 $y \leftarrow \text{Sample}$

□

**Lemma 27.**

$$\forall b, G_2(\lambda, b) \approx_c G_3(\lambda, b)$$

◇

*Proof.* **TODO 29:** I'm not completely sure this proof is complete

Use 2-universality of  $\pi_2$ . Recall that the difference between  $G_2$  and  $G_3$  is that

$$(g^{(i)}, l^{(i)}, \Pi_2^{(i)}) \text{ such that } y^{(i)} \notin L$$

are answered  $\perp$  in  $G_3$ , instead in  $G_2$

$$\perp \text{ comes out as output} \Leftrightarrow \tilde{\Pi}_2^{(i)} = Ver(\tau, y^{(i)} \neq \Pi_2^{(i)})$$

It's possible to distinguish two cases, looking at  $c = (y, l, \Pi_2)$ :

1. if  $(y^{(i)}, l^{(i)}) = (y, l)$  and  $\tilde{\Pi}_2^{(i)} = \Pi_2^{(i)}$ , it outputs  $\perp$  if in the decryption scheme  $(\Pi_2^{(i)} \neq \tilde{\Pi}_2^{(i)})$  it outputs  $\perp$ .

2. <sup>24</sup> otherwise  $(y^{(i)}, l^{(i)}) \neq (y, l)$  if  $y^{(i)} \notin L$  we want that  $\Pi_2^{(i)}$  doesn't output exactly  $Ver_2(\tau, (y^{(i)}, l^{(i)}))$ , but it should output  $\perp$ .

**EVENT BAD:** From **A**'s perspective, the only information he knows is:

$$(\omega_2, \tilde{\Pi}_2 = Ver_2(\tau_2, y))$$

for  $y \notin L$ . The value

$$Ver_2(\tau_2, (y^{(i)}, x^{(i)}))$$

for  $y^{(i)} \in L$  and  $(y^{(i)}, l^{(i)}) \neq (y, l)$  is random. Therefore:

$$\Pr[BAD] = 2^{-|\mathcal{P}_2|}$$

□

**Lemma 28.**

$$\forall b, G_3(\lambda, b) \equiv G_4(\lambda, b)$$

◇

*Proof.* If we look at the view of **A**, the only information known about  $\tau_1$  is  $\omega_1$ , since the decryption oracle only computes for  $y^{(i)} \in L$ :

$$Ver_1(\tau, y^{(i)}) = Prove_1(\omega_1, y^{(i)}, x^{(i)})$$

By 1-universality,  $\Pi = Ver_1(\tau, y)$  for any  $y \in L$  is random.

□

**Lemma 29.**

$$G_4(\lambda, 0) \equiv G_4(\lambda, 1)$$

◇

*Proof.* The challenge ciphertext is independent of  $b$ .

□

TODO 30: referencing something from another part of lesson 17

### 17.1.1 Instantiation of u-HPS (Universal Hash Proof System)

We will define a membership-hard language using the DDH assumption

**Definition 29.** A language  $L_{\text{DDH}}$  is deemed a DDH-tuple language iff:

$$L_{\text{DDH}} = \{(c_1, c_2) : \exists r \implies c_1 = g_1^r, c_2 = g_2^r\}$$

◇

Observe that, given a group  $G$  of order  $q$  with  $(g_1, g_2)$  as generators, we will have  $(g_1, g_2, c_1, c_2)$  but if we impose  $g_1 = g$  and  $g_2 = g^a$  then the previous construction becomes  $(g, g^a, c_1, c_2)$ . By definition,  $c_1 = g_1^{r_1}$  and  $c_2 = g_2^{r_2}$ , thus I can write  $(g, g^a, g^r, g^{ar})$ .

Now we can define our system  $\Pi := (\text{Setup}, \text{Prove}, \text{Verify})$ :

<sup>24</sup>when decryption oracle doesn't output the challenge

- *Setup*( $1^\lambda$ ): Pick  $x_1, x_2 \leftarrow \mathbb{Z}_q$  and define:

$$\begin{aligned}\omega &= h_1 = (g_1^{x_1}, g_2^{x_2}) \\ \tau &= (x_1, x_2)\end{aligned}$$

- *Prove*( $\omega, \underbrace{(c_1, c_2)}_y, r$ ) will output  $\pi = \omega^2$
- *Verify*( $\tau, \underbrace{(c_1, c_2)}_y$ ) will output  $\tilde{\pi} = c_1^{x_1} c_2^{x_2}$

**Correctness:**  $\pi = \omega^2 = (g_1^{x_1} g_2^{x_2})^r = g_1^{rx_1} g_2^{rx_2} = c_1^{x_1} c_2^{x_2} = \tilde{\pi}$

**Theorem 42.** *Above construction defines a 1-universal DV-NIZK scheme for  $L_{\text{DDH}}$ .*  $\diamond$

*Proof.* We want to prove that if we take any  $(c_1, c_2) \notin L_{\text{DDH}}$ , the distribution  $(\omega = h_1, \tilde{\Pi} = \text{Verify}(\tau, (c_1, c_2)))$  is uniform.

Define a map  $\mu(x_1, x_2) \mapsto (\omega, \pi) = (g_1^{x_1}, g_2^{x_2}, c_1^{x_1}, c_2^{x_2})$  it suffices to prove that  $\mu$  is injective. This can easily be done with some constrains:

$$\mu'(x_1, x_2) = \log_{g_1}(\mu(x_1, x_2)) = (\log_{g_1}(\omega), \log_{g_1}(\pi))$$

For  $r_1 \neq r_2$  then  $c_1 = g_1^{r_1}, c_2 = g_2^{r_2} = g^{\alpha r_2}$ . For  $\alpha = \log_{g_2} g_1$  then  $\Pi = c_1^{x_1} c_2^{x_2} = g_1^{r_1 x_1} g_2^{r_1 x_1 + \alpha r_2 x_2}$ .

$$\mu'(x_1, x_2) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ r_1 & r_2 \alpha \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since  $\text{Det} \begin{pmatrix} 1 & \alpha \\ r_1 & r_2 \alpha \end{pmatrix} = \alpha(r_2 - r_1) \neq 0$  the map is injective.

$\square$

- *Setup*( $1^\lambda$ ):
  - Pick  $x_3, x_4, x_5, x_6 \leftarrow \mathbb{Z}_q$  and define:
    - \*  $\omega = (h_2, h_3, s) = (g_1^{x_3}, g_2^{x_4}, g_1^{x_5}, g_2^{x_6}, s)$  where  $s$  is a **seed** for a  $\text{CRH} \rightarrow H = \{H_s\}$
- *Prove*( $\omega, (c_1, c_2, l), r$ )
  - Compute  $\beta = H_s(c_1, c_2, l) \in \mathbb{Z}_p$
  - Output  $\Pi = h_2^r h_3^{r\beta}$
- *Verify*( $\tau, (c_1, c_2, l)$ )
  - Compute  $\beta = H_s(c_1, c_2, l) \in \mathbb{Z}_q$
  - Output  $\tilde{\Pi} = c_1^{x_3 + \beta x_5} c_2^{x_4 + \beta x_6}$

**Correctness:**

$$\Pi = h_2^r h_3^{r\beta} = (g_1^{x_3} g_2^{x_4})^r (g_1^{x_5} g_2^{x_6})^{r\beta} = c_1^{x_3} c_2^{x_4} c_1^{\beta x_5} c_2^{\beta x_6} = c_1^{x_3 + \beta x_5} c_2^{x_4 + \beta x_6} = \tilde{\Pi}$$

**Theorem 43.** *The above construction define a 2-universal DVNIZK for  $L_{\text{DDH}}$*   $\diamond$

*Proof. Same goal and procedure as before*

- Take any  $(c_1, c_2) \notin L_{DDH}$

- Fix  $(c_1, c_2, l) \neq (c'_1, c'_2, l')$  s.t.  $(c_1, c_2), (c'_1, c'_2) \notin L_{DDH}$  which means:

$$\begin{aligned} & \bullet (c_1, c_2) = (g_1^{r_1} g_2^{r_2}) \\ & \bullet (c'_1, c'_2) = (g_1^{r'_1} g_2^{r'_2}) \end{aligned} \left\{ \begin{array}{l} r_1 \neq r_2 \text{ and } r'_1 \neq r'_2 \end{array} \right.$$

$$\begin{aligned} & \bullet \beta = H_s(c_1, c_2, l) \\ & \bullet \beta' = H_s(c'_1, c'_2, l') \end{aligned}$$

- Let's define a MAP

$$\begin{aligned} \mu'(x_3, x_4, x_5, x_6) &= (\omega, \widetilde{\Pi} = Ver(\tau, (c_1, c_2, l)), \widetilde{\Pi}' = Ver(\tau, (c'_1, c'_2, l'))) = \\ &= (\underbrace{(h_2, h_3)}_{\omega}, c_1^{x_3+\beta x_5} c_2^{x_4+\beta x_6}, c'_1{}^{x_3+\beta' x_5} c'_2{}^{x_4+\beta' x_6} = \\ &= ((g_1^{x_3} g_2^{x_4}, g_1^{x_5} g_2^{x_6}), g_1^{r_1 x_3 + \beta r_1 x_5} g_2^{r_2 x_4 + \beta r_2 x_6}, g_1^{r'_1 x_3 + \beta' r'_1 x_5} g_2^{r'_2 x_4 + \beta' r'_2 x_6}) \end{aligned}$$

But I can rewrite  $g_2$  as  $g_2 = g_1^\alpha$  since they are generators. So:

$$\begin{aligned} & ((g_1^{x_3+\alpha x_4}, g_1^{x_5+\alpha x_6}), g_1^{r_1 x_3 + \beta r_1 x_5} g_1^{\alpha(r_2 x_4 + \beta r_2 x_6)}, g_1^{r'_1 x_3 + \beta' r'_1 x_5} g_1^{\alpha(r'_2 x_4 + \beta' r'_2 x_6)}) = \\ &= ((g_1^{x_3+\alpha x_4}, g_1^{x_5+\alpha x_6}), g_1^{r_1 x_3 + \alpha r_2 x_4 + \beta r_1 x_5 + \alpha \beta r_2 x_6}, g_1^{r'_1 x_3 + \alpha r'_2 x_4 + \beta' r'_1 x_5 + \alpha \beta' r'_2 x_6}) = \end{aligned}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ r_1 & \alpha r_2 & \beta r_1 & \alpha \beta r_2 \\ r'_1 & \alpha r'_2 & \beta' r'_1 & \alpha \beta' r'_2 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$$\text{Since Det} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \alpha^2(r_2 - r_1)(r'_2 - r'_1)(\beta - \beta') \neq 0$$

**IFF:**

$\overbrace{r_2 \neq r_1, r'_2 \neq r'_1}^{\text{for construction}}, \beta \neq \beta' \rightarrow$  this last condition is true because we picked  $H_s$  collision resistant.

Otherwise  $H_s$  computed on different elements  $((c_1, c_2, l)$  and  $(c'_1, c'_2, l')$ ) will have to output the same  $\beta(= \beta')$ .  $\square$

# Lesson 18

## Cramer-Shoup scheme construction

From the above two proof systems we can construct a PKE scheme, which is attributed to Cramer and Shoup:

TODO 31: split definition from correctness

- $(\mathbf{pk}, \mathbf{sk}) \leftarrow \text{Keygen}$ , where:
  - $\mathbf{pk} := (h_1, h_2, h_3) = (g_1^{x_1} g_2^{x_2}, g_1^{x_3} g_2^{x_4}, g_1^{x_5} g_2^{x_6})$
  - $\mathbf{sk} := (x_1, x_2, x_3, x_4, x_5, x_6)$
- Encryption procedure:
  - $r \leftarrow \mathbb{Z}_q$
  - $\beta = h_s(c_1, c_2, c_3) = (g_1^r, g_2^r, h_1^r m)$
  - $\text{Enc}(\mathbf{pk}, m) = (c_1, c_2, c_3, (h_2 h_3^\beta)^r)$
- Decryption procedure:
  - Check that  $c_1^{x_3 + \beta x_5} c_2^{x_4 + \beta x_6} = c_4$ . If not, output  $\perp$ .
  - Else, output  $\hat{m} = c_3 c_1^{-x_1} c_2^{-x_2}$

TODO 32: All the proofs here...

## 18.1 Digital signatures

In this section we explore the solutions to the problem of authentication using an asymmetric key method. Some observations are in order:

- In a symmetric setting, a verifier routine could be banally implemented as recomputing the signature using the shared secret key and the message. Here, Bob cannot recompute  $\sigma$  as he's missing Alice's secret key (and for good reasons too...). Thus, the verifying routine must be defined otherwise;



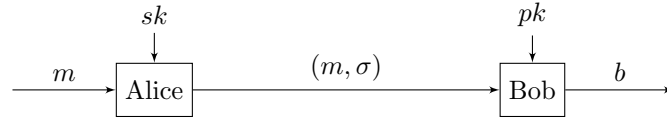


Figure 18.65: Asymmetric authentication

- In a vaguely similar manner to how an attacker could encrypt messages by itself in the asymmetric scenario, because the public key is known to everyone, any attacker can verify any signed messages, for free.

Nevertheless, proving that a DS scheme is secure is largely defined in the same way as in the symmetric scenario, with the UF-CMA property:

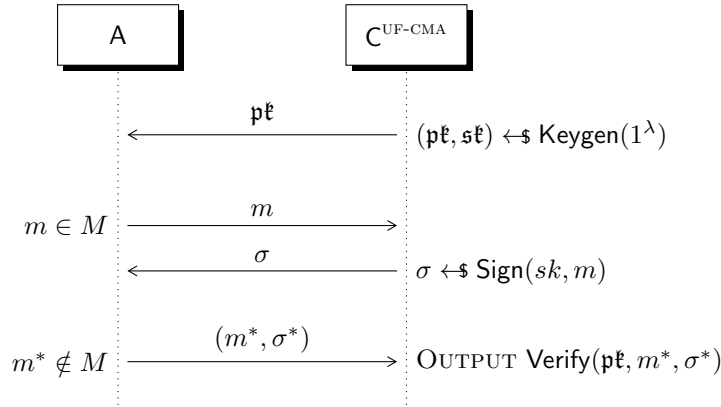


Figure 18.66: Unforgeable digital signatures

### 18.1.1 Public Key Infrastructure

The problem now is that Alice has a public key, but she wants some sort of “certificate of validity” for it, so that Bob will be sure that whenever he receives Alice’s public key, he can be sure it’s the right one by checking such certificate.

For certificates to be useful, the parties need an universally-trusted third party, called *Certification Authority*. It will provide a special *signature* to Alice for proving her identity to Bob, as exemplified by the sequence in figure 18.67.

Whenever Bob wants to check the validity of the Alice’s public key, he can query the authority for the certificate, and verify the public key he just received, as shown in figure 18.68

How can Bob recognize a valid certificate from an expired/invalid one? The infrastructure provides some servers which contain the lists of the currently valid certificates, such as  $CERT_A$ , in the case of Alice.

**Theorem 44.** *Signatures are in **Minicrypt**.* ◇

This is a counterintuitive result, not proven during the lesson, but very interesting because it implies that we can create valid signatures only with hash functions, without considering at all public key encryption.

Up next:

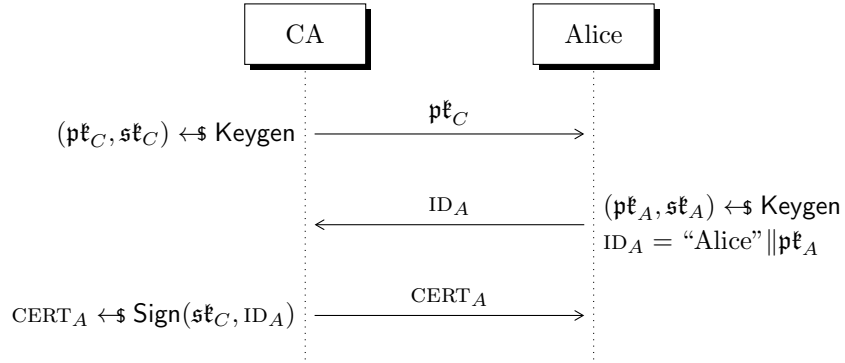


Figure 18.67

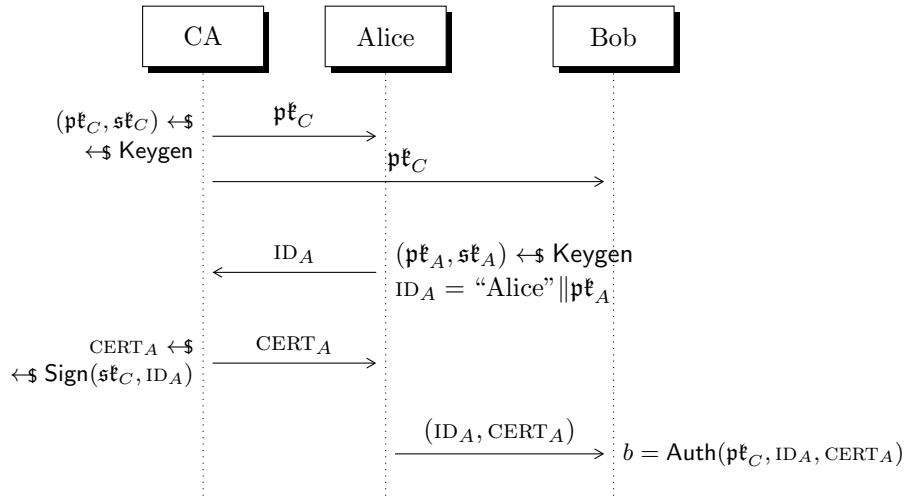


Figure 18.68

- Digital Signatures from TDP\*
- Digital Signatures from ID Scheme\*
- Digital Signatures from CDH

Where \* appears, something called *Random Oracle Model* is used in the proof. Briefly, this model assumes the existence of an ideal hash function which behaves like a truly random function (outputs a random  $y$  as long as  $x$  was never queried, otherwise gives back the already taken  $y$ ).

# Lesson 19

TODO 33: E' venuto fuori un casino in questa lezione, ho cercato di riordinare le cose

## Bilinear Map

**Definition 30.** Let's define a *bilinear group* as  $(G, G_t, q, g, \hat{e}) \leftarrow \mathcal{Bilin}(1^\lambda)$ , where:

- $G, G_t$  are prime order groups (order  $q$ ).
- $g$  is a generator of  $G$  picked UAR.
- $(G, \cdot)$  is a multiplicative group and  $G_t$  is the “target” group.
- $\hat{e}$  is an efficiently computable *bilinear* map:  $G \times G \rightarrow G_t$  defined as such:

$$\forall h \in G \forall a, b \in \mathbb{Z}_q \implies \hat{e}(g^a, h^b) = \hat{e}(g, h)^{ab} = \hat{e}(g^{ab}, h)$$

provided that  $\hat{e}(g, g) \neq 1$ , that is,  $\hat{e}$  is *non-degenerative*

To put it in simple words, the exponents can move. ◇

TODO 34: Venturi said something here related to Weil pairing over an elliptic curve. I found [this](#). Interesting but not useful.

It must be noted that the DDH assumption is easy for the group  $G$ : suffice to see that  $\hat{e}(g^a, g^b) = \hat{e}(g, g^c)$  is true iff  $c = ab$ . On the other hand:

**Proposition 3.** CDH is hard for  $G$ . ◇

*Proof.* Now  $\text{Keygen}(1^\lambda)$  will:

- Generate some params:  $(G, G_t, g, q, \hat{e}) \leftarrow \mathcal{Bilin}(1^\lambda)$
- $a \leftarrow \mathbb{Z}_q$ , then  $g_1 = g^a$
- Pick  $g_2 = g^b$  and  $g_2, u_{0 \dots k} \leftarrow G$ .
- Then output:

$$\text{pk} = (\text{params}, g_1, g_2, u_0, \dots, u_k)$$

$$- \text{sk} = g_2^a = g^{ab}$$

Sign(sk, m):

- Divide the message  $m$  of length  $k$  in single bits as follows:  $m = (m_1 \dots m_k)$
- Now define  $\alpha(m) = u_0 \prod_{i=1}^k u_i^{m_i}$
- Pick  $r \leftarrow \mathbb{Z}_q$  and output the signature  $\sigma = (\text{sk} \cdot \alpha(m)^r, g^r) = (\sigma_1, \sigma_2)$

Ver(pk, m, ( $\sigma_1, \sigma_2$ )):

- Check  $\hat{e}(g, \sigma_1) = \hat{e}(\sigma_2, \alpha(m)) = \hat{e}(g_1, g_2)$

The scheme's correctness is proven by "moving" the exponents:

$$\begin{aligned} \hat{e}(g, \sigma_1) &= \hat{e}(g, g_2^a \cdot \alpha(m)^r) \\ &= \hat{e}(g, g_2^a) \cdot \hat{e}(g, \alpha(m)^r) && \text{(Bilinearity)} \\ &= \hat{e}(g^a, g_2) \cdot \hat{e}(g^r, \alpha(m)) \\ &= \hat{e}(g_1, g_2) \cdot \hat{e}(\sigma_2, \alpha(m)) \end{aligned}$$

We can say that we are moving the exponents from the "private domain" to the "public domain".

□

## 19.1 Waters signatures

**Theorem 45.** *The Waters' signature scheme is UF-CMA-secure*

◇

*Proof.* The trick is to "program the  $u$ 's" (Venturi)

TODO 35: Sequence is incomplete/incorrect, have to study it more...

TODO 36: The following explanation is roundabout, will rectify later

The following describes how the Waters' challenger constructs the  $u$  string. The main idea is, given  $k$  as the message length, to choose each single bit of  $u$  from 1 up to  $k$  such that:

$$\alpha(m) = g_2^{\beta(m)} g^{\gamma(m)}, \quad \beta(m) = x_0 + \sum_{i=1}^k m_i x_i, \quad \gamma(m) = y_0 + \sum_{i=1}^k m_i y_i$$

where  $x_0 \leftarrow \{-kl, \dots, 0\}$ ,  $x_{1-k} \leftarrow \{0, \dots, l\}$ ,  $y_{0-k} \leftarrow \mathbb{Z}_q$

In particular:  $l = 2q_s$ , where  $q_s$  is the number of sign queries made by the adversary.

Therefore, let  $u_i = g_2^{x_i} g^{y_i} \quad \forall i \in [0, k]$ . Then:

$$\alpha(m) = g_2^{x_0} g^{y_0} \prod_{i=1}^k (g_2^{x_i} g^{y_i})^{m_i} = g_2^{x_0 + \sum_{i=1}^k m_i x_i} g^{y_0 + \sum_{i=1}^k m_i y_i} = g_2^{\beta(m)} g^{\gamma(m)}$$

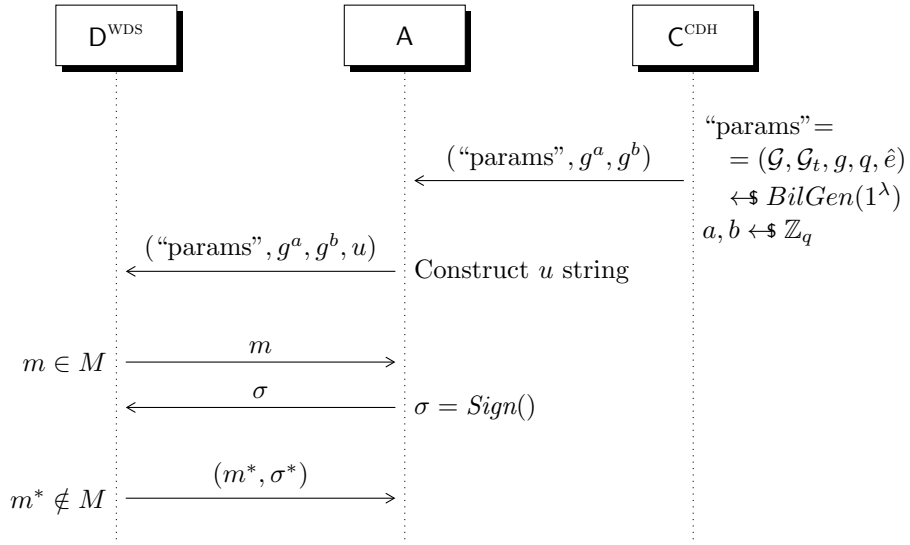


Figure 19.69: Reducing Waters' scheme to CDH

TODO 37: Partizioni, doppi if... qui non ci ho capito 'na mazza

**Step 1:**  $\sigma = (\sigma_1, \sigma_2) = (g_2^a \alpha(m)^{\bar{r}}, g^{\bar{r}})$ , for  $\bar{r} \leftarrow \mathfrak{Z}_q$   $\bar{r} = r - a\beta^{-1}$

$$\begin{aligned} \sigma_1 &= g_2^a \alpha(m)^{\bar{r}} \\ &= g_2^a \alpha(m)^{r - a\beta^{-1}} \\ &= g_2^a (g_2^{\beta(m)} g^{\gamma(m)})^{r - a\beta^{-1}} \\ &= g_2^a g_2^{\beta(m)r - a} g^{\gamma(m)r - \gamma(m)a\beta^{-1}} \\ &= g_2^{\beta(m)r} g^{\gamma(m)r} g^{-\gamma(m)\beta^{-1}} \end{aligned}$$

□

# Lesson 20

## 20.1 Random Oracle Model (ROM)

The Random Oracle Model treats a given hash function  $H$  as a truly random function. As a reminder: a truly random function  $R$  is defined to have a specific evaluation behaviour. They do act, in fact, as truth tables<sup>25</sup>:

- if the argument hasn't been submitted to the function beforehand, then a value is chosen UAR from the codomain, and assigned as the image of said argument in the function;
- otherwise, the function will return the image as assigned in the corresponding previous evaluation.

### 20.1.1 Full domain hashing

Let  $(f, f^{-1}, \text{Gen})$  be a TDP scheme over some domain  $\mathcal{X}_{\text{pt}}$

Take RSA:

- $(m, \text{pk}, \text{sk}) \leftarrow \text{GenRSA}(1^\lambda)$
- $f(\text{pk}, x) = x^{\text{pk}} \bmod n$
- $f^{-1}(\text{sk}, y) = y^{\text{sk}} \bmod n$

Build a similar asymmetric-authentication scheme as such:

- $(m, \text{pk}, \text{sk}) \leftarrow \text{GenRSA}(1^\lambda)$
- $\text{Sign}_{\text{sk}, H}(x) : \sigma = f^{-1}(\text{sk}, H(m))$
- $\text{Verify}_{\text{pk}, H} : H(m) = f(\text{pk}, \sigma)$

**Exercise 46.** Show RSA-sign is not secure without  $H$ . (Hint: The scheme becomes malleable)

**Theorem 47.** *If the above scheme (full-domain hash) uses a TDP for  $f, f^{-1}$ , then it is (asymmetric)-UF-CMA under the random oracle model.*  $\diamond$

*Proof.* Idea: Reduce to TDP, program the random oracle.

Notes: this is a loose reduction

Some assumptions are made:

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<sup>25</sup>Such tables are also aptly called *rainbow tables*.

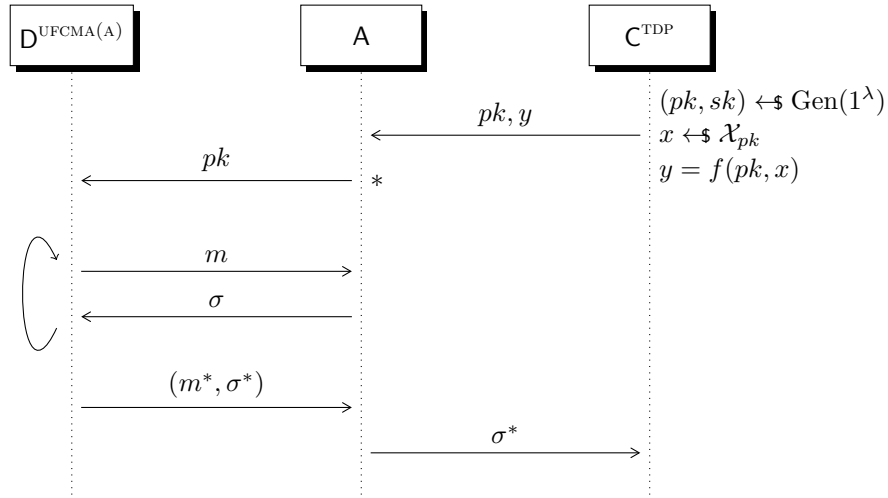


Figure 20.70

- The adversary makes the same number of RO queries as the number of signing queries done by the distinguisher (without loss of generality)
- The RO query must be done *before* the corresponding sign query, otherwise the adversary cannot sign the messages, as specified by the scheme

The RO queries are actually an analogue of the definition of a random function, and it is the *programming* step of the oracle itself; then if the signing queries do not correspond to any RO query, abort the game.

□

## 20.2 ID Scheme

“Sigma” protocol

### 20.2.1 Fiat-Shamir scheme

Honest Verifier Zero Knowledge (HVZK)

Special Soundness (SS)



# Lesson 21

## 21.1 Full domain hashing

# Lesson 22

## 22.1 Examples of ID schemes

# Lesson 23

## 23.1 Bilinear DDH assumption

# Lesson 24

## 24.1 CCA proof for ???