Coefficient of restitution

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A bouncing basketball captured with a stroboscopic flash at 25 images per second: Ignoring air resistance, the square root of the ratio of the height of one bounce to that of the preceding bounce gives the coefficient of restitution for the ball/surface impact.

The **coefficient of restitution** (**COR**) is a measure of the "restitution" of a collision between two objects: how much of the kinetic energy remains for the objects to rebound from one another vs. how much is lost as heat, or work done deforming the objects.

The coefficient, **e** is defined as the ratio of relative speeds after and before an impact, taken along the line of the impact:

$$\text{Coefficient of restitution } (e) = \frac{\text{Relative velocity after collision}}{\text{Relative velocity before collision}}$$ [1]

Alternatively, this may be expressed as: $\text{Relative velocity of separation} = e \times \text{Relative velocity of approach}$

The mathematics were developed by Sir Isaac Newton[2] in 1687. It is also known as Newton's experimental law.

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Further details

Line of impact- It's the line along which 'e' is defined or in absence of tangential reaction force between colliding surfaces force of impact is shared along this line between bodies. During physical contact between bodies during impact its line along common normal to pair of surfaces in contact of colliding bodies. Hence 'e' is defined as 1-D parameter which is unit-less.

Range of values for e -treated as a constant

e is usually a positive, real number between 0 and 1.0:

e = 0: This is a perfectly 'inelastic' collision. The objects do not move apart after the collision, but instead they coalesce. Kinetic energy is converted to heat or work done in deforming the objects.

0 < e < 1: This is a real-world 'inelastic' collision, in which some kinetic energy is dissipated.

e = 1: This is a perfectly 'elastic' collision, in which no kinetic energy is dissipated, and the objects rebound from one another with the same relative speed with which they approached.

e < 0: A COR less than zero would represent a collision in which the separation velocity of the objects has the same direction (sign) as the closing velocity, implying the objects passed through one another without fully engaging. This may also be thought of as an incomplete transfer of momentum. An example of this might be a small, dense object passing through a large, less dense one – e.g., a bullet passing through a target, or a motorcycle passing through a motor home or a wave tearing through a dam.

e > 1: This would represent a collision in which energy is released, for example, nitrocellulose billiard balls can literally explode at the point of impact. Also, some recent articles have described superelastic collisions in which it is argued that the COR can take a value greater than one in a special case of oblique collisions.[3][4][5] These phenomena are due to the change of rebound trajectory caused by friction. In such collision kinetic energy is increased in a manner energy is released in some sort of explosion. It is possible that $e = \infty$ for a perfect explosion of a rigid system.

Maximum deformation phase-In any collision for 0<e≤1 there is a condition when for short moment along line of impact colliding bodies have same velocity when its condition of kinetic energy is lost in maximum fraction as heat, sound and light with deformation potential energy. for this short duration this collision e=0 and may be referred as inelastic phase.

Paired objects

The COR is a property of a 'pair' of objects in a collision, not a single object. If a given object collides with two different objects, each collision would have its own COR. When an object (singular) is described as having a coefficient of restitution, as if it were an intrinsic property without reference to a second object, the definition is assumed to be with respect to collisions with a perfectly rigid and elastic object.

Generally, the COR is thought to be independent of collision speed. However, in a series of experiments performed at Florida State University in 1955, the COR was shown to vary as the collision speed approached zero, first rising significantly as the speed drops, then dropping significantly as the speed drops to about 1 cm/s and again as the collision speed approaches zero. This effect was observed in slow-speed collisions involving a number of different metals.[6]

Relationship with conservation of energy and momentum

In a one-dimensional collision, the two key principles are: conservation of energy (conservation of kinetic energy if the collision is perfectly elastic) and conservation of (linear) momentum. A third equation can be derived[7] from these two, which is the restitution equation as stated above. When solving problems, any two of the three equations can be used. The advantage of using the restitution equation is that it sometimes provides a more convenient way to approach the problem.

Let m_1, m_2 be the mass of object 1 and object 2 respectively. Let u_1, u_2 be the initial velocity of object 1 and object 2 respectively. Let v_1, v_2 be the final velocity of object 1 and object 2 respectively.

$$\begin{cases} \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \end{cases}$$

From the first equation,

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$$
$$m_1(u_1 + v_1)(u_1 - v_1) = m_2(v_2 + u_2)(v_2 - u_2)$$

From the second equation,

$$m_1(u_1 - v_1) = m_2(v_2 - u_2)$$

After division,

$$u_1 + v_1 = v_2 + u_2$$
$$u_1 - u_2 = -(v_1 - v_2)$$
$$-\frac{v_1 - v_2}{u_1 - u_2} = 1$$

The equation above is the restitution equation, and the coefficient of restitution is 1, which is a perfectly elastic collision.

Example

Q. A cricket ball is bowled at 50 km/h towards a batsman who swings the bat at 30 km/h. How fast, approximately, does the ball move after impact?

Step 1: Speed of separation = e x speed of approach. Speed of approach = relative closing speed of ball and bat = 50 km/h + 30 km/h = 80 km/h.

Step 2: Approximating that the collision is perfectly elastic (e = 1), therefore speed of separation is approximately 80 km/h.

Step 3: Approximating the ball as being of much smaller mass than the bat, the momentum of the bat is (almost) unchanged by the impact, therefore the bat continues to move at (nearly) the same speed (30 km/h) after impact.

Step 4: Therefore, the ball's final speed is (slightly less than) 30 km/h + 80 km/h = 110 km/h.

[This assumes that the ball is struck head-on by the bat, and that the collision is perfectly elastic. To obtain a more accurate answer, a measured value for the coefficient of restitution for cricket-ball-on-bat is needed, and use the equation for conservation of linear momentum simultaneously with the restitution formula.]

Sports equipment

The coefficient of restitution entered the common vocabulary, among golfers at least, when golf club manufacturers began making thin-faced drivers with a so-called "trampoline effect" that creates drives of a greater distance as a result of the flexing and subsequent release of stored energy, imparting greater impulse to the ball. The USGA (America's governing golfing body) has started testing drivers for COR and has placed the upper limit at 0.83. According to one article (addressing COR in tennis racquets), "[f]or the Benchmark Conditions, the coefficient of restitution used is 0.85 for all racquets, eliminating the variables of string tension and frame stiffness which could add or subtract from the coefficient of restitution."[8]

The International Table Tennis Federation specifies that the ball shall bounce up 24–26 cm when dropped from a height of 30.5 cm on to a standard steel block thereby having a COR of 0.89 to 0.92.[9] For a hard linoleum floor with concrete underneath, a leather basketball has a COR around 0.81-0.85.[10]

Equations

In the case of a one-dimensional collision involving two objects, object A and object B, the coefficient of restitution is given by:

$$C_R = \frac{v_b - v_a}{u_a - u_b}$$, where:

v_a is the final velocity of object A after impact
v_b is the final velocity of object B after impact

u_a is the initial velocity of object A before impact
u_b is the initial velocity of object B before impact

Though C_R does not explicitly depend on the masses of the objects, it is important to note that the final velocities are mass-dependent. For two- and three-dimensional collisions of rigid bodies, the velocities used are the components perpendicular to the tangent line/plane at the point of contact, i.e. along the line of impact.

For an object bouncing off a stationary target, C_R is defined as the ratio of the object's speed after the impact to that of prior to impact:

$$C_R = \frac{v}{u}$$, where

v is the speed of the object after impact
u is the speed of the object before impact

In a case where frictional forces can be neglected and the object is dropped from rest onto a horizontal surface, this is equivalent to:

$C_R = \sqrt{\frac{h}{H}}$, where

h is the bounce height
H is the drop height

The coefficient of restitution can be thought of as a measure of the extent to which mechanical energy is conserved when an object bounces off a surface. In the case of an object bouncing off a stationary target, the change in gravitational potential energy, *PE*, during the course of the impact is essentially zero; thus, C_R is a comparison between the kinetic energy, *KE*, of the object immediately before impact with that immediately after impact:

$$C_R = \sqrt{\frac{KE_{\text{(after impact)}}}{KE_{\text{(before impact)}}}} = \sqrt{\frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2}} = \sqrt{\frac{v^2}{u^2}} = \frac{v}{u}$$

In a cases where frictional forces can be neglected (nearly every student laboratory on this subject[11]) and the object is dropped from rest onto a horizontal surface, the above is equivalent to a comparison between the *PE* of the object at the drop height with that at the bounce height. In this case, the change in *KE* is zero (the object is essentially at rest during the course of the impact and is also at rest at the apex of the bounce); thus:

$$C_R = \sqrt{\frac{PE_{\text{(at bounce height)}}}{PE_{\text{(at drop height)}}}} = \sqrt{\frac{mgh}{mgH}} = \sqrt{\frac{h}{H}}$$

Speeds after impact

The equations for collisions between elastic particles can be modified to use the COR, thus becoming applicable to inelastic collisions, as well, and every possibility in between.

$$v_a = \frac{m_a u_a + m_b u_b + m_b C_R (u_b - u_a)}{m_a + m_b}$$

and

$$v_b = \frac{m_a u_a + m_b u_b + m_a C_R (u_a - u_b)}{m_a + m_b}$$

where

v_a is the final velocity of the first object after impact
v_b is the final velocity of the second object after impact
u_a is the initial velocity of the first object before impact
u_b is the initial velocity of the second object before impact
m_a is the mass of the first object
m_b is the mass of the second object

Derivation

The above equations can be derived from the analytical solution to the system of equations formed by the definition of the COR and the law of the conservation of momentum (which holds for all collisions). Using the notation from above where u represents the velocity before the collision and v after, yields:

$$m_a u_a + m_b u_b = m_a v_a + m_b v_b$$
$$C_R = \frac{v_b - v_a}{u_a - u_b}$$

Solving the momentum conservation equation for v_a and the definition of the coefficient of restitution for v_b yields:

$$\frac{m_a u_a + m_b u_b - m_b v_b}{m_a} = v_a$$
$$v_b = C_R(u_a - u_b) + v_a$$

Next, substitution into the first equation for v_b and then resolving for v_a gives:

$$\frac{m_a u_a + m_b u_b - m_b C_R(u_a - u_b) - m_b v_a}{m_a} = v_a$$

$$\frac{m_a u_a + m_b u_b + m_b C_R(u_b - u_a)}{m_a} = v_a \left[1 + \frac{m_b}{m_a}\right]$$

$$\frac{m_a u_a + m_b u_b + m_b C_R(u_b - u_a)}{m_a + m_b} = v_a$$

A similar derivation yields the formula for v_b.

Predicting the coefficient from material properties

This discussion will only address direct impact like a ball dropped on a horizontal surface without any tangential movement.

Many materials are assumed to be perfectly elastic for engineering purposes when their yield strength is not approached during impact. This would result in e=1 and bounce for a long time if it were not for air resistance. In practice a good stainless steel has e ~ 0.85 and an ideal titanium alloy has e ~ 0.90. Amorphous metals can achieve e = 0.95 or higher.

The following will show how the coefficient of restitution can be estimated for some materials under certain conditions, but it shows an estimate of the relative restitution for materials with at a similar velocity and density is:

$$\text{coefficient of restitution is approximately} \propto \sqrt{\frac{\text{yield strength}}{\text{elastic modulus}}}$$

When there is an impact velocity that exceeds the elastic range of the material at the contact point, but does not otherwise substantially deform the sphere, the yield strength in a portion of material at the contact area will be exceeded, reducing its coefficient of restitution. The collision is said to be in the elastic-plastic region. For example, this occurs in metal spheres dropped onto a large block of the same material from distances that are typically at least 1 meter high. Dropping non-spherical objects loses energy to rotation.

Materials that return to their original shape even after high pressure have a yield strength are generally able to store more of the impact energy like a spring. This energy can be returned to a dropped sphere in the form of upward velocity which gives a high coefficient of restitution. If the ability of the material to reversibly store energy during compression is exceeded, the impact energy will be lost to plastic deformation, diminishing the energy available for recoil. So a high yield strength allows the material to stay in the elastic region at higher energies and will result in a higher e. A lower elastic modulus allows a larger surface area of contact during impact so the energy is distributed to a larger volume just beneath the surface which helps prevent the yield strength from being exceeded. A lower velocity at impact will also increase the coefficient of restitution by needing less energy to be absorbed. For the same reason, a lower density will increase the coefficient because mass is proportional to the kinetic energy needing to be absorbed. The density instead of mass is needed because the volume of the sphere cancels out with the volume of the affected elastic-plastic volume at the contact area.

Dropping hard spherical objects onto a softer surface (lower elastic modulus) which also has a lower coefficient of restitution will reduce the apparent coefficient of restitution of the dropped object. For example, most rubber and plastic balls have a lower coefficient than glass and some metal alloys, but when dropped on wood or cement, the softer material will bounce higher. This is because the harder objects distribute the impact energy over a much smaller contact area, losing energy to heat by exceeding the elastic range of the floor.

Combining these four effects, a theoretical estimation of the coefficient of restitution can be made when a ball is dropped onto a surface of the same material.[12] The derivation is based on estimating the amount of potential energy that is stored in elastic compression (the portion not lost to elastic deformation) and dividing it by the initial kinetic energy at impact. It assumes there is no heating.

- e = coefficient of restitution
- S_y = dynamic yield strength

- E' = effective elastic modulus
- ρ = density
- v = velocity at impact
- μ = Poisson's ratio

$$e = 3.1\left(\frac{S_y}{1}\right)^{\frac{5}{8}}\left(\frac{1}{E'}\right)^{\frac{1}{2}}\left(\frac{1}{v}\right)^{\frac{1}{4}}\left(\frac{1}{\rho}\right)^{\frac{1}{8}}$$

$$E' = \frac{E}{1-\mu^2}$$

This applies for a direct impact and when:

$$0.001 < \frac{\rho v^2}{S_y} < 0.1$$

Although the accuracy of this equation is not good, it is easy to calculate and accurately predicts the relative coefficient for many materials, even at velocities above and below its intended range. Plastics and rubbers will give higher values than their actual values because they are not as ideally elastic as metals, glasses, and ceramics because of heating during compression.

Theoretical coefficient of restitution solid spheres dropped 1 meter (v= 4.5 m/s). Values > 1.0 indicates the equation indicates the equation has errors.[13] Yield strength instead of dynamic yield strength was used.

Metals and Ceramics:

- silicon 1.77
- silica glass 1.50
- highest amorphous metal 1.27
- highest magnesium alloy 0.86
- titanium alloy 0.82
- aluminum alloy 0.75
- glass (soda-lime) 0.69
- glass (borosilicate) 0.66
- best stainless steel 0.64
- wood 0.1 to 0.7
- highest nickel alloy 0.60
- highest zinc alloy 0.60
- cast iron 0.3 to 0.6
- highest copper alloy 0.55
- magnesium alloy 0.44
- tungsten 0.37
- titanium 0.27
- tungsten carbide 0.17
- magnesium 0.15
- nickel 0.15
- copper 5% zinc 0.15
- copper 0.15
- aluminum 0.10
- lead 0.08

Polymers (overestimated compared to metals and ceramics):

- polybutadiene (golf balls shell) 11.8
- butyl rubber 6.24
- EVA 4.85
- silicone elastomers 2.80
- polycarbonate 1.46
- nylon 1.28
- polyethylene 1.24
- Teflon 1.21
- polypropylene 1.14
- ABS 1.12
- acrylic 1.06
- PET 0.95

- polystyrene 0.87
- PVC 0.86

For metals the range of speeds to which this theory can apply is about 5 to 100 m/s which is a drop of 1 to 500 meters, provided the sphere is small enough for Hertzian contact theory to apply (see page 366[14]) But the above rankings that it provides remain accurate.

For metals, the theoretically perfect elastic range (the coefficient theoretically equals 1.0 and the above equation does not apply) is when the velocity is less than

$$v = \left(26\frac{S_y}{\rho}\left(\frac{S_y}{E'}\right)^4\right)^{0.5}$$

which is less than 0.1 m/s.

See also

- Bouncing ball
- Collision
- Resilience

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External links

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- Chris Hecker's physics introduction (http://chrishecker.com/images/e/e7/Gdmphys3.pdf)
- "Getting an extra bounce" by Chelsea Wald (http://focus.aps.org/story/v14/st14)
- FIFA Quality Concepts for Footballs – Uniform Rebound (http://footballs.fifa.com/Football-Tests/Rebound)
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