Neoclassical current diffusion solver

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General Atomics

I. TRANSPORT-TIMESCALE CURRENT DIFFUSION

We want to derive an equation for the evolution of the poloidal magnetic field. We begin by assuming a field of the axisymmetric form

$$\mathbf{B} = \nabla\phi \times \nabla\psi + F\nabla\phi, \tag{1}$$

as assumed in GACODE. This field is, of course, advanced by Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2}$$

where the electric field will be given by a generalized Ohm's law of the form

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{R}. \tag{3}$$

Here, $\mathbf{u} = (m_i \mathbf{u}_i + m_e \mathbf{u}_e) / (m_i + m_e) \approx \mathbf{u}_i$ is the single-fluid velocity, while the right-hand side, $\mathbf{R}$, is defined for the transport model. In resistive MHD, we have

$$\mathbf{R}_r = \eta \mathbf{J}, \tag{4}$$

while a common neoclassical model removes the noninductive bootstrap current, $\mathbf{J}_{BS}$, from the Ohm's law and writes

$$\mathbf{R}_{neo} = \eta_{neo} (\mathbf{J} - \mathbf{J}_{BS}). \tag{5}$$

Taking the time derivative of Eq. 1, and denoting $\dot{\alpha} = \partial\alpha/\partial t$, we find that

$$\nabla\dot{\psi} \times \nabla\phi - \dot{F}\nabla\phi = \nabla \times \mathbf{E}. \tag{6}$$

Using a right-handed $\mathbf{x} = (R, Z, \phi)$ coordinate system, we can use the $\hat{R}$ and $\hat{Z}$ components of Eq. 6 to show that the poloidal flux (over 2π) evolves as

$$\dot{\psi} = R^2 \mathbf{E} \cdot \nabla\phi. \tag{7}$$

Substituting in Eq. 3, we get

$$\dot{\psi} + \mathbf{u} \cdot \nabla\psi = R^2 \mathbf{R} \cdot \nabla\phi. \tag{8}$$

Thus far, we've used an Eulerian frame of reference with the time derivatives taken with respect to a stationary coordinate system, $\mathbf{x}$. We will now change to a moving flux-coordinate system, $\mathbf{r} = (r, \theta, \phi)$, with

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} - \mathbf{u}_c \cdot \nabla, \tag{9}$$

where $\mathbf{u}_c$ is the velocity of the coordinates. Redefining $\dot{\alpha} = \left.\frac{\partial \alpha}{\partial t}\right|_{\mathbf{r}}$ as the time derivative in the moving flux-coordinate frame, Eq. 8 becomes

$$\dot{\psi} + \mathbf{u}_r \cdot \nabla\psi = R^2 \mathbf{R} \cdot \nabla\phi, \tag{10}$$

where the relative velocity is defined as

$$\mathbf{u}_r = \mathbf{u} - \mathbf{u}_c. \tag{11}$$

In order to maintain ψ as a flux function over time, we require

$$\mathbf{u}_r \cdot \nabla\psi - R^2 \mathbf{R} \cdot \nabla\phi = f(r), \tag{12}$$

where $f(r)$ is an arbitrary flux function, the specification of which defines the moving flux coordinate system.

We wish to work in a coordinate system such that the toroidal flux, Φ, remains constant throughout time. We note that

$$\Phi = \frac{1}{2\pi}\int_0^r dr \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi\, \mathcal{J}\mathbf{B} \cdot \nabla\phi = \frac{1}{2\pi}\int_0^r dr\, FV' \left\langle R^{-2} \right\rangle, \tag{13}$$

where the flux surface average is defined as

$$\langle \cdots \rangle = \frac{2\pi}{V'} \int_0^{2\pi} d\theta\, \mathcal{J} \cdots \tag{14}$$

with differential volume, $V' = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \mathcal{J}$, and Jacobian, $\mathcal{J} = [\nabla r \times \nabla\theta \cdot \nabla\phi]^{-1}$. Thus the toroidal flux will remain constant if

$$\left.\frac{\partial \Phi}{\partial t}\right|_{\mathbf{r}} = \frac{1}{2\pi}\int_0^r dr\, \left.\frac{\partial}{\partial t}\left(FV' \left\langle R^{-2} \right\rangle\right)\right|_{\mathbf{r}} = 0. \tag{15}$$

Now, taking the toroidal component of Faraday's law, Eq. 6, we know that

$$\frac{\partial}{\partial t}\left(\frac{F}{R^2}\right)\bigg|_{\mathbf{x}} = \nabla\cdot\left(\nabla\phi\times\mathbf{E}\right). \tag{16}$$

Transforming to the moving flux coordinate frame with Eq. 9, and noting that

$$\frac{\partial\mathcal{J}}{\partial t}\bigg|_{\mathbf{r}} = \mathcal{J}\nabla\cdot\mathbf{u}_c, \tag{17}$$

one can show that Eq. 16 becomes

$$\frac{\partial}{\partial t}\left(\mathcal{J}\frac{F}{R^2}\right)\bigg|_{\mathbf{r}} + \mathcal{J}\nabla\cdot\left[\frac{F}{R^2}\mathbf{u}_r - \left(\mathbf{u}_r\cdot\nabla\phi\right)\mathbf{B} - \nabla\phi\times\mathbf{R}\right] = 0. \tag{18}$$

Then, integrating Eq. 18 over all θ, we find that

$$\frac{\partial}{\partial t}\left(FV'\left\langle R^{-2}\right\rangle\right) + \frac{\partial}{\partial r}\left[FV'\left\langle R^{-2}\mathbf{u}_r\cdot\nabla r\right\rangle - V'\left\langle\nabla r\cdot\left(\nabla\phi\times\mathbf{R}\right)\right\rangle\right] = 0, \tag{19}$$

where we've made use of the identity

$$2\pi\int_0^{2\pi} d\theta\,\mathcal{J}\nabla\cdot\mathbf{A} = \frac{\partial}{\partial r}\left[V'\left\langle\mathbf{A}\cdot\nabla r\right\rangle\right]. \tag{20}$$

Thus, we can see from Eqs. 15 and 20 that the toroidal flux will remain constant if

$$F\left\langle R^{-2}\mathbf{u}_r\cdot\nabla r\right\rangle = \left\langle\mathbf{R}\cdot\left(\nabla r\times\nabla\phi\right)\right\rangle. \tag{21}$$

Comparing to Eq. 12, we can enforce constant torodial flux by defining

$$f(r) = -\frac{\langle\mathbf{B}\cdot\mathbf{R}\rangle}{\langle\mathbf{B}\cdot\nabla\phi\rangle}. \tag{22}$$

Using this form of f, Eq. 10 becomes

$$\dot{\psi} = \frac{\langle\mathbf{B}\cdot\mathbf{R}\rangle}{\langle\mathbf{B}\cdot\nabla\phi\rangle}. \tag{23}$$

Taking the derivative of Eq. 23 with respect to Φ, and defining the rotational transform as

$$\iota = 2\pi\frac{d\psi}{d\Phi} \tag{24}$$

and the loop voltage as

$$V_L = 2\pi\frac{\langle\mathbf{B}\cdot\mathbf{R}\rangle}{\langle\mathbf{B}\cdot\nabla\phi\rangle}, \tag{25}$$

we find that

$$\frac{\partial\iota}{\partial t} = \frac{1}{\Phi'}\frac{\partial V_L}{\partial r}, \tag{26}$$

where $\Phi' = d\Phi/dr$.

In the case of the common neoclassical closure to MHD, we can substitute Eq. 5 into Eq. 25. Then, noting that

$$\mu_0 \mathbf{B}\cdot\mathbf{J} = \frac{F^2}{\mathcal{J}}\left[\frac{\partial}{\partial r}\left(\frac{\mathcal{J}\psi'}{FR^2}|\nabla r|^2\right) + \frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}\psi'}{FR^2}\nabla r\cdot\nabla\theta\right)\right], \tag{27}$$

we find that

$$\langle\mathbf{B}\cdot\mathbf{J}\rangle = \frac{F^2}{\mu_0 V'}\frac{\partial}{\partial r}\left[\frac{\Phi' V'}{2\pi F}\left\langle\frac{|\nabla r|^2}{R^2}\right\rangle \iota\right]. \tag{28}$$

Then, noting that $\langle\mathbf{B}\cdot\nabla\phi\rangle = F\langle R^{-2}\rangle$ and substituting Eq. 5 into Eq. 25, we find that the neoclassical loop voltage is

$$V_{neo} = \frac{\eta_{neo}F}{\mu_0 V'\langle R^{-2}\rangle}\frac{\partial}{\partial r}\left[\frac{\Phi' V'}{2\pi F}\left\langle\frac{|\nabla r|^2}{R^2}\right\rangle \iota\right] - \frac{\eta_{neo}}{F\langle R^{-2}\rangle}\langle\mathbf{B}\cdot\mathbf{J}_{BS}\rangle. \tag{29}$$

Eq. 26 then becomes a transport-timescale diffusion equation for the rotational transform with what amounts to a neoclassical source term. In practice, both η_{neo} and $\langle\mathbf{B}\cdot\mathbf{J}_{BS}\rangle$ should be readily provided by a neoclassical model (e.g., NEO, Sauter). In addition, we need certainly equilibrium quantities: F, Φ', and an initial ι. Finally, we need a few flux quantities defined by the geometry, namely:

$$V', \quad \left\langle R^{-2}\right\rangle, \quad \text{and} \quad \left\langle\frac{|\nabla r|^2}{R^2}\right\rangle. \tag{30}$$