THE ALGEBRA OF CATEGORICAL SPECTRA

by

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Abstract

This thesis develops fundamental theories of the algebra of categorical spectra. A categorical spectrum is a sequence (X_n) of pointed (∞,∞)-categories with identifications $X_n \xrightarrow{\sim} X_{n+1}$. After developing background materials on (∞,∞)-categories and categorical spectra, we will define the tensor product of categorical spectra. This tensor product is analogous to the lax Gray tensor product of (∞,∞)-categories, just as the tensor product of spectra is to the cartesian product of ∞-groupoids. The tensor product will be applied to studying the stability of categorical spectra, i.e., the coincidence of certain finite weighted colimits and limits. This will further be applied to topological quantum field theories. In particular, we make precise Lurie's sketch of the deduction of the cobordism hypothesis with singularities from the plain cobordism hypothesis.

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Table of Contents

List of Tables

Chapter 1

Introduction

This thesis focuses on developing the fundamental theory of *categorical spectra*, a higher-categorical generalization of the notion of spectra. Let S denote the $(\infty, 1)$-category of ∞-groupoids (a.k.a. anima, weak homotopy types, spaces) and S_* be that of pointed ∞-groupoids. Recall that the category of *spectra* is defined as the limit along the loop functor on S_*:

$$\mathsf{Sp} := \lim(\cdots \xrightarrow{\Omega} \mathsf{S}_* \xrightarrow{\Omega} \mathsf{S}_*).$$

In other words, a spectrum X is a sequence of pointed ∞-groupoids $(X_n)_{n \in \mathbb{N}}$ equipped with the identifications $X_n \xrightarrow{\sim} \Omega X_{n+1}$. For $0 \leq n \leq \infty$, let $n\mathsf{Cat}$ denote the $(\infty, 1)$-category of (small) (∞, n)-categories. That is, we set $0\mathsf{Cat} = \mathsf{S}$ and inductively define $(n+1)\mathsf{Cat} = (n\mathsf{Cat})\text{-}\mathsf{Cat}$ as the $(\infty, 1)$-category of $n\mathsf{Cat}$-enriched categories. We also set $\infty\mathsf{Cat} = \lim_n n\mathsf{Cat}$, i.e., an (∞, ∞)-category X is a compatible collection of the underlying (∞, n)-categories $X^{\leq n}$ given by discarding higher noninvertible cells. An essential feature of $\infty\mathsf{Cat}$ is that it is a fixed point of enrichment: $(\infty\mathsf{Cat})\text{-}\mathsf{Cat} \simeq \infty\mathsf{Cat}$. The $(\infty, 1)$-category CatSp of categorical spectra is defined by replacing S in the definition of Sp by $\infty\mathsf{Cat}$:

$$\mathsf{CatSp} := \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*),$$

so a categorical spectrum X is a sequence $(X_n)_{n\in\mathbb{N}}$ of pointed (∞,∞)-categories with identifications $X_n \xrightarrow{\sim} \Omega X_{n+1}$. Here, the loop of a pointed (∞,∞)-category means the (∞,∞)-category of *endomorphisms* of the base object: $\Omega(X,x) = \mathrm{Hom}_X(x,x)$. A spectrum is now an example of a categorical spectrum where every component is an ∞-groupoid: $\mathsf{Sp} \subset \mathsf{CatSp}$.

Notice $X_0 \xrightarrow{\sim} \Omega X_1 \xrightarrow{\sim} \Omega^2 X_2 \xrightarrow{\sim} \cdots$; just as for ∞-groupoids, n-fold loop object is canonically an $\mathbb{E}_n$-monoid object. In the limit, every component X_n of a categorical spectrum gets a structure of a symmetric monoidal (∞,∞)-category. Conversely, given a symmetric monoidal (∞,∞)-category X_0, there is a minimal choice for X_n with $X_0 \xrightarrow{\sim} \Omega^n X_n$, that is the *(connective) delooping* $X_n = \mathrm{B}^n X_0$ of X_0. A categorical spectrum is *connective* if it is determined by X_0 in this way. They form another class of examples equivalent to symmetric monoidal (∞,∞)-categories: $\mathrm{B}^\infty : \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{\simeq} \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$. Note that a non-grouplike $\mathbb{E}_\infty$-space is a special case of this example.

Of course, this is a naive categorification that anyone could try (in fact, it was considered independently by many groups of people: Remark 3.3.6). Categorification is somewhat of an art than a recipe and a naive one is sometimes not the right thing to do. Therefore, it is fair to question how fruitful the notion is. As usual for an abstract mathematical structure, there are two layers of affirmative answers to this question. On the surface layer, we wish to have enough examples, old and new, to show that it is a good organizational language for the relevant structure it abstracts. This is mostly achieved in the thesis of Stefanich [Ste21]. The notion of categorical spectrum is a great language to bundle iterated categorifications. The $(\infty,1)$-category CatSp contains Sp and $\mathsf{CMon}(\infty\mathsf{Cat})$ as full subcategories, as well as more interesting examples such as the categorical spectra of iterated spans and iterated modules. The functoriality of the n-quasi-coherent sheaves and the compatibility across different n is formulated as being a map of categorical spectra from that of iterated spans of the (pre)stacks to that of iterated modules.

In the deeper layer, however, one asks if they are legitimate mathematical *objects* beyond a language. The utility of spectra partially comes from the fact that they admit various interrelated interpretations (the following list is roughly taken from [Lur18, §0.2.3]):

- Spectra are infinite loop spaces; this is the definition we chose.

- Spectra are generalized abelian groups; there is an equivalence between infinite loop spaces $\Omega^\infty X$ for a spectrum X and grouplike $\mathbb{E}_\infty$-spaces. In particular, we have $\pi_0 : \mathsf{Sp} \to \mathsf{Ab}$ and an inclusion $\mathsf{Ab} \to \mathsf{Sp}$ as the heart of the Postnikov t-structure. Moreover, there is a symmetric monoidal structure $\otimes = \otimes_{\mathbb{S}}$ deriving the tensor product of abelian groups. This viewpoint allows us to develop the whole "brave new" versions of classical algebra and algebraic geometry, called *higher algebra* and *spectral algebraic geometry*.

- Spectra are stable homotopy types; for instance, this is apparent in the definition of finite spectra using the Spanier-Whitehead category. Namely, for two finite pointed CW-complexes X, Y, the space of maps between their suspension spectra is computed as

$$\mathrm{Map}_{\mathsf{Sp}}(\Sigma^\infty X, \Sigma^\infty Y) \simeq \mathrm{colim}(\mathrm{Map}_{\mathsf{S}_*}(X, Y) \to \mathrm{Map}_{\mathsf{S}_*}(\Sigma X, \Sigma Y) \to \cdots).$$

  A finite spectrum is a shift $\Sigma^{\infty - n} X$ of a suspension spectrum of a finite CW complex, and general spectra are filtered colimits of finite spectra: $\mathsf{Sp} = \mathrm{Ind}(\mathsf{Sp}^{\mathrm{fin}})$. Moreover, the Freudenthal suspension theorem implies that the above colimit sequence stabilizes after finite suspensions.

- Spectra are generalized (co)homology theory; more precisely, given a spectrum E, one defines the associated cohomology theory $E^n(X) = \pi_0 \mathrm{Map}_{\mathsf{S}}(X, \Omega^{\infty - n} E)$ and homology theory $E_n(X) = \pi_n(\Sigma^\infty X \otimes E)$. The Brown representability theorem says this gives a bijection between equivalence classes of cohomology theories and those of spectra. As for homology, it leads to another neat definition of spectra as reduced excisive copresheaves on finite pointed ∞-groupoids (or CW complexes): $\mathsf{Sp} \simeq \mathsf{Exc}(\mathsf{S}_*^{\mathrm{fin}}, \mathsf{S})$. This perspective is essential in Goodwillie's calculus of functors; spectra play the role of the linearization of ∞-groupoids.

- Spectra is the universal stable $(\infty, 1)$-category: more precisely, Sp is the free stable presentable $(\infty, 1)$-category generated by a single object (the sphere spectrum) $\mathbb{S} \in \mathsf{Sp}$, i.e., $\mathrm{ev}_{\mathbb{S}} : \mathsf{LFun}(\mathsf{Sp}, \mathcal{C}) \to \mathcal{C}$ is an equivalence for a stable presentable $(\infty, 1)$-category $\mathcal{C}$. It implies that Sp is the tensor unit of the symmetric monoidal $(\infty, 1)$-category $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ of presentable stable $(\infty, 1)$-categories for the Lurie tensor product of presentable categories, which provides an elegant way to define the tensor product of spectra. This machinery is also essential for categorified sheaf theory, including the recent development on six functor formalism [LZ17][Man22].

Heuristically, we would like categorical spectra to admit similar interpretations but with everything directed or lax:

- Categorical spectra are infinite loop higher categories; again, this is essentially our definition.

- Categorical spectra are generalized symmetric monoidal categories and commutative monoids; we have already seen that connective spectra are equivalent to symmetric monoidal (∞, ∞)-categories and a general categorical spectrum is a compatible sequence (X_n) of possibly nonconnective deloopings of X_0. In particular, a robust theory of categorical spectra leads to a robust derived algebra of commutative monoids. We wish categorical spectra to play a similarly fundamental role in categorified algebra as spectra do in higher algebra. In particular, we need a tensor product of categorical spectra. In many examples,

X_0 is a classical object (of categorical level 0 or 1), and X_n is obtained by iterated categorifications. In other words, the new interesting information grows to the *cohomological* direction. For this reason, Johnson-Freyd [Joh23] proposes to call the new categorified algebra the *deeper algebra*, in contrast to *higher algebra* whose derived information grows to the homological direction.

- Categorical spectra are stable directed homotopy types; this is based on the interpretation that (∞, ∞)-categories are directed homotopy types. That is, (∞, ∞)-categories are like cell complexes, but with directions. Just as spectra are like CW complexs with negative dimensional cells, categorical spectra are like (∞, ∞)-categories with negative dimensional cells. Additionally, a Spanier-Whitehead style definition and a form of Freudenthal suspension theorem are desiable.

- Categorical spectra are "(co)homology theory" of higher categories; in other words, we would like an axiomatization of the functors on (∞, ∞)-categories that categorical spectra represent. In particular, we must understand the correct analog of excision properties. This is more or less equivalent to understanding which higher categorical colimit in CatSp is also a limit, i.e., we would like to classify the *absolute colimits* in CatSp.

- Categorical spectra are universal stable presentable Gray-bimodules; as we will see below, the natural replacement for the cartesian product in the higher categorical world is the (lax) Gray tensor product. Accordingly, the ambient categorical setting will in general only have compositions of higher cells up to some noninvertible higher cells. Once we overcome the relevant difficulty around the Gray tensor product, it is more or less clear that CatSp is the universal presentable Gray-bimodule with invertible loop-suspension. The tensor product of categorical spectra falls out of it. However, stability for $(\infty, 1)$-categories can be defined in many other ways; understanding the appropriate notion of stability boils down to understanding the excision properties, or the absolute colimits.

In essence, the theme of this project is to explore this list. In particular, we will pave the way to the tensor product of categorical spectra based on the last viewpoint, and commence the study of absolute colimits of CatSp —since the notion of weighted colimits uses the tensoring by the enriching category, we already need the tensor product to study the notion of stability. Now we expand the contents of the paper, after the following table of analogies:

Table 1.1: Analogies between the algebra based on sets, homotopy types, and higher categories

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\mathsf{Set} = (0,0)\mathsf{Cat}$	spaces/∞-groupoids $\mathsf{S} = 0\mathsf{Cat}$	(∞,∞)-categories $\infty\mathsf{Cat}$
—	homotopy n-type	(∞,n)-category
Cartesian product $\times$	Cartesian product $\times$	lax Gray tensor product $\otimes$
$(1,1)$-categories $(1,1)\mathsf{Cat}$	$(\infty,1)$-categories $1\mathsf{Cat}$	Gray-categories $(\infty\mathsf{Cat}^{\otimes})$-$\mathsf{Cat}$
abelian groups Ab	spectra Sp	categorical spectra CatSp
	grouplike $\mathbb{E}_\infty$-spaces $\simeq \mathsf{Sp}^{\mathrm{cn}}$	$\mathsf{CMon}(\infty\mathsf{Cat}) \simeq \mathsf{CatSp}^{\mathrm{cn}}$
—	loop $\Omega(X,x) = \mathrm{Aut}_X(x)$	$\Omega(X,x) = \mathsf{End}_X(x)$
—	suspension $\Sigma = \mathrm{B}\mathbb{Z} \wedge (-)$	$\Sigma = \mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1} = \mathrm{B}\mathbb{N} \oslash (-)$
free functor $\mathsf{Set} \to \mathsf{Ab}$	suspension spectra $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$	$\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$
integers $\mathbb{Z}$	sphere spectrum $\mathbb{S}$	finite set categorical spectrum $\mathbb{F}$
tensor product $\otimes_{\mathbb{Z}}$	tensor (smash) product $\otimes_{\mathbb{S}}$	tensor product $\otimes_{\mathbb{F}}$
abelian categories	(pre)stable categories	stable Gray-bimodules

Higher category theory

The first two chapters of the body of the thesis, as well as the appendix, are spent on preparation. In Chapter 2, we will give a model-independent exposition of the theory of (∞,∞)-categories. Our main focus will be on the comparison between enriched-categorically defined notions and the notions that are native and specific to ∞-category theory. In doing so, we will pass between weak and strict notions of higher categories; we make a computation on the strict objects and left Kan-extend to the weak world. The Steiner theory of the Appendix A provides an algebraic language for computations of strict ∞-categories.

The central character of the (∞,∞)-category theory specifics side of the story is the *(lax) Gray tensor product.* It is a biclosed monoidal structure on $\infty\mathsf{Cat}$ that acts *additively* on category levels (compare to the fact that $n\mathsf{Cat} \subset \infty\mathsf{Cat}$ is closed under the cartesian product (in fact, all limits and colimits)). More precisely, it is characterized by $\square^m \otimes \square^n \simeq \square^{m+n}$, where $\square^n$ is the n-category called the (fully lax) n-cube (we refer the reader to Example 2.4.4 for some pictures). The main technical complication comes from the *non-commutativity*, essentially coming from the choice of the direction of the 2-cell in the cube category $\square^2 = \square^1 \otimes \square^1$. For us, the most important result will be Lemma 2.5.1 that compares the enriched-categorical suspension and the suspension as a quotient of the Gray cylinder, i.e., the Gray tensor product with the interval.

Most of the contents in the section are well-known to experts and appear in the literature such as [Cam23b][Lou23] (see also [AM20] for the strict case). However, we will pay special attention to not making any mistake with duality involutions (this corresponds to the sign issue in the presence of negatives, e.g. in Sp), as it will be crucial in the later chapters.

Categorical spectra

Chapter 3 is another preparation chapter. We begin with the study of loop-suspension adjunctions and the delooping hypothesis. In particular in Proposition 3.1.9, we observe that the loop-suspension is the tensor-hom adjunction for *lax smash product* with the directed circle $\vec{S}^1 = \mathrm{B}\mathbb{N}$, so $\Sigma X \simeq \vec{S}^1 \oslash X$ (where $\oslash$ is to $\otimes$ as $\wedge$ is to $\times$). We will then define categorical spectra precisely and include some examples that are universally produced by considering a levelwise property of categorical spectra. We close the section with a formal study of the finiteness properties of categorical spectra. More than half of this chapter is a summary of [Ste21, Chapter 13], simplified by quickly specializing to the situation of our interest, i.e., from enriched $(\infty, 1)$-categories to (∞, ∞)-categories.

Tensor product of categorical spectra

Chapter 4 is devoted to the proof of our first main theorem: the construction and the universal properties of the tensor product of categorical spectra. The strategy was mentioned above, but let us describe the method and the obstacle in more detail. Recall that the tensor product (always assumed to be biclosed) of abelian groups is characterized by the fact that the free abelian group functor Free : $\mathsf{Set} \to \mathsf{Ab}$ promotes to a symmetric monoidal functor. The tensor product of spectra is similarly characterized by the fact that $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ promotes to a symmetric monoidal functor. However, the homotopical nature of the object makes it impossible to even formulate a correct universal property without a solid foundation of $(\infty, 1)$-category theory. Lurie first constructed a symmetric monoidal structure $\otimes$ on Pr^L promoting the presheaf functor $\mathsf{PSh} : \mathsf{Cat} \to \mathsf{Pr}^\mathsf{L}$ to a symmetric monoidal functor[1]. A (commutative) algebra object in Pr^L is precisely a presentably (symmetric) monoidal category.

Theorem ([Lur17])**.** $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{Sp}$ *is an idempotent* $\mathbb{E}_0$*-algebra in* Pr^L*, i.e.,* $\Sigma^\infty_+ \otimes \mathrm{id} : \mathsf{Sp} \simeq \mathsf{S} \otimes \mathsf{Sp} \to \mathsf{Sp} \otimes \mathsf{Sp}$ *is an equivalence. Since the forgetful functor* $\mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^\mathsf{L}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^\mathsf{L})$ *is an equivalence,* Sp *uniquely promotes to an object of* $\mathsf{CAlg}(\mathsf{Pr}^\mathsf{L})$ *whose unit is the sphere spectrum* $\mathbb{S}$*.*

This method is robust and elegant, so we wish to play the same game with categorical spectra. However, it turns out to be trickier than it may seem. First, one cannot expect $\Sigma^\infty_+ : \mathsf{S} \hookrightarrow \infty\mathsf{Cat} \to \mathsf{CatSp}$ to be an idempotent $\mathbb{E}_0$-algebra in Pr^L, as the category $\infty\mathsf{Cat}$ is already not idempotent[2] over S. One can more reasonably ask

[1] This is an example of the microcosm principle: to talk about an object with a certain structure (e.g. a commutative monoid), we must first equip the ambient category with the corresponding structure (e.g. a symmetric monoidal structure).

[2] In fact, the equivalences $\mathsf{CAlg}^{\mathrm{idem}}(\mathsf{Pr}^\mathsf{L}) \xrightarrow{\sim} \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_1}(\mathsf{Pr}^\mathsf{L}) \xrightarrow{\sim} \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathsf{Pr}^\mathsf{L})$ and the fact that

if $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ (or equivalently, $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$) is idempotent, but to make sense of it, we must choose a monoidal structure on $\infty\mathsf{Cat}$. The obvious choice would be the Cartesian product, but the suspension is far from being a module map over it : if X, Y are m, n-categories respectively, then $X \wedge \Sigma Y$ is a $\max\{m, n+1\}$-category, while $\Sigma(X \wedge Y)$ is a $\max\{m, n\} + 1$-category, so $X \wedge \Sigma Y$ and $\Sigma(X \wedge Y)$ does not even have the same category level in general. This is the same problem as $\Sigma X \not\simeq \vec{S}^1 \wedge X$.

This observation suggests that we should use a monoidal structure that adds the category level; fortunately, we have already proven that the Gray tensor product satisfies $\Sigma X \simeq \vec{S}^1 \oslash X$. In particular, $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ is a morphism of right $\infty\mathsf{Cat}_*$-modules. However, now the noncommutativity is a serious obstacle; it makes no sense to ask the idempotence of a right module over a noncommutative algebra because there is no relative tensor product. It turns out that one can save the situation by lifting Σ to a *bimodule* morphism $\infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$. In fact, we will prove that there is a unique such lift if we twist the bimodule structure on one of $\infty\mathsf{Cat}$ by the *total dual* involution. In terms of the coefficient object $\vec{S}^1$, this means that we may commute objects with $\vec{S}^1$ in a completely canonical manner provided we twist the object by the total dual involution: $\vec{S}^1 \oslash X \simeq X^\circ \oslash \vec{S}^1$. We call this the *half-central structure* of $\vec{S}^1$. This is of fundamental importance beyond this application; it is the higher categorical incarnation of the *Koszul sign rule*: since we have no negatives to express the sign, we must twist one side of the equivalence by an appropriate duality. Once the half-central structure of $\vec{S}^1$ is understood, the rest of the work to show the following is rather formal:

Theorem A (=Theorem 4.2.1)**.** *The functor* $\Sigma^\infty_+ : \infty\mathsf{Cat} \to \mathsf{CatSp}$ *is an idempotent* $\mathbb{E}_0$*-algebra in* $\mathsf{BMod}_{\infty\mathsf{Cat}^\otimes}(\mathsf{Pr}^\mathsf{L})$*. In particular,* CatSp *admits a unique biclosed* $\mathbb{E}_1$ *structure* $\otimes$ *underlying an* $\infty\mathsf{Cat}^\otimes$*-algebra structure (in* Pr^L*) promoting* Σ^∞_+ *to an*

$(\infty\mathsf{Cat}, \times) \in \mathsf{CAlg}(\mathsf{Pr}^\mathsf{L})$, $(\infty\mathsf{Cat}, \otimes^{(\mathrm{op})\mathrm{lax}}) \in \mathsf{Alg}(\mathsf{Pr}^\mathsf{L})$ has the same underlying $\mathbb{E}_0$-algebra (i.e, the unit is terminal) disproves the idempotence.

$\infty\mathsf{Cat}^{\otimes}$*-algebra morphism. Moreover, the monoidal category* $\mathsf{CatSp}^{\otimes}$ *can be obtained by universally inverting* $\vec{S}^1 \in \infty\mathsf{Cat}_*^{\oslash}$.

The reader who finds the noncommutativity of the tensor product disturbing is invited to check Remark 4.2.5 for a few ideas that potentially remedy the situation. However, the Gray tensor product seems unavoidable in higher category theory, at least from the viewpoint that (∞,∞)-categories are directed homotopy types. Also, we expect that any useful commutative variant of the tensor product receives a lax monoidal functor from our tensor product. In other words, the results we prove using our tensor product are universal and can be transferred to a preferred situation if required. We hope the following chapters justify the richness and naturality of the theory unlocked by this tensor product of categorical spectra.

Duality, absolute colimits and stability

Recall that one of the standard motivations to introduce spectra is the *Spanier-Whitehead duality*. That is, unlike the symmetric monoidal categories S_* or $\mathsf{Sp}^{\mathrm{cn}}$ (where the only dualizable objects are the free ones on finite sets), Sp has many interesting dualizable objects: any finite (i.e., compact) spectrum is dualizable, which is an upper bound of dualizable objects (as in any closed monoidal category with a compact unit object, cf. Proposition 3.5.10). This reflects the fact that the category of spectra has better exactness properties: by $\mathsf{Sp} \simeq \mathsf{LFun}(\mathsf{Sp},\mathsf{Sp}) \simeq \mathsf{RFun}(\mathsf{Sp},\mathsf{Sp})^{\mathrm{op}}$, an object $X \in \mathsf{Sp}$ is dualizable if and only if $X \otimes (-) \in \mathsf{RFun}(\mathsf{Sp},\mathsf{Sp})$, i.e. if it preserves limits. By stability, limits commute with finite colimits, so if X is built out of finite colimits (and shifts) of $\mathbb{F}$, X is dualizable. More generally, weighted colimits in Sp for a finite weight (i.e., a diagram $J \to \mathsf{Sp}^{\mathrm{fin}}$ for a finite category J) is limit-preserving, or *absolute*: it is preserved by any Sp-enriched functors. This captures the essence of stability in a way free of an ad-hoc choice of the shape of relevant diagrams.

In the same spirit, we expect a good supply of dualizable objects and absolute colimits in CatSp. Although we are not yet able to classify all of them, we will show that many fundamental finite categorical spectra and "finite weights" are dualizable (or absolute). For the following, let $X \leftarrow Y \rightarrow Z$ be a span of categorical spectra and $X \overrightarrow{\amalg}_Y Z$ denote the directed pushout $X \amalg_{\{0\}\otimes Y} (\square^1 \otimes Y) \amalg_{\{1\}\otimes Y} Z$.

Theorem B (=Theorem 5.2.7, Corollary 5.2.12)**.** *The functor* $\overrightarrow{\amalg} : \mathsf{Fun}((\bullet \leftarrow \bullet \rightarrow \bullet), \mathsf{CatSp}) \rightarrow \mathsf{CatSp}$ *admits a right adjoint given by* $X \mapsto (\Sigma^{\infty-1} I^{\mathrm{op}} \otimes X \leftarrow \Sigma^{-1} X \rightarrow \Sigma^{\infty-1} I \otimes X)$, *where* I *is the interval category* $0 \rightarrow 1$ *with the basepoint at* 0. *It follows that suspension spectra of the cubes, orientals, and objects of Joyal's* Θ *category are all dualizable.*

We expect that proving the absoluteness of these basic examples and discussion on closure properties of absolute weights will lead to the complete classification of absolute weights. Note, however, that the ordinary pushouts are not absolute. In fact, it forces the invertibility of $(\infty, 1)$-categorical loop-suspension, i.e., the $(\infty, 1)$-categorical stability. Therefore, to classify absolute colimits, we must define the notion of "finite weights" in a way that excludes such "too invertible" examples. A particular case that is worth spelling out is the following.

Theorem C (=Theorem 5.3.1)**.** *Let* $f : X \rightarrow Y$ *be a morphism of categorical spectra. There is a canonical equivalence between the lax cofiber* $\overrightarrow{\mathrm{cof}}(f) = 0 \overrightarrow{\amalg}_X Y$ *and the lax fiber of the suspension* $\overrightarrow{\mathrm{fib}}(\Sigma f) = 0 \overrightarrow{\times}_{\Sigma Y} \Sigma X$. *Moreover, the canonical triangle* $Y \rightarrow (\overrightarrow{\mathrm{cof}}(f) \simeq \overrightarrow{\mathrm{fib}}(\Sigma f)) \rightarrow \Sigma X$ *is a (non-lax) bifiber sequence.*

In this situation, we say $Z = \overrightarrow{\mathrm{cof}}(f)$ is an *extension* of ΣX by Y (and similarly for a *coextension* $\overleftarrow{\mathrm{cof}}(f)$). The theorem is a lift of Barratt-Puppe sequence in the category of spectra. This is obvious in stable $(\infty, 1)$-categories, but the result here is surprising because a naive extension of the standard facts in stable $(\infty, 1)$-categories tend to fail, e.g., pasting law of directed pushouts fails and lax fiber sequences do not coincide with lax cofiber sequences. This suggests that the classical notion of (co)fiber sequences in a stable $(\infty,1)$-categories splits into three different classes of sequences, i.e. lax cofiber sequences, bifiber sequences, and lax fiber sequences, and they appear in a three-periodic pattern.

Categorical spectra with adjoints and applications to TQFT

The knowledge of absolute colimits in categorical spectra is not only theoretically important but also a useful tool for computation. One can often draw a strong consequence out of a coincidence of limits and colimits[3]. In particular, even if one is only interested in symmetric monoidal (∞,n)-categories, studying categorical spectra allows a better understanding of their (pre)stability. We will apply this principle to the study of TQFTs. We say a categorical spectrum (X_n) is *d-adjointful* if for $k < d+n$, any k-cell of X_n has left and right adjoints. For instance, a (∞,d)-category $\mathcal{C}$ *has duals* in the sense of [Lur09c] (i.e., its objects are fully dualizable) if the corresponding connective categorical spectrum $\mathrm{B}^\infty\mathcal{C}$ is d-adjointful (note that objects of X_n are dualizable if 1-morphisms of X_{n+1} has adjoints). We denote the full sub $(\infty,1)$-category of d-adjointful categorical spectra by $\mathsf{CatSp}^{d\text{-adj}} \subset \mathsf{CatSp}$. We also let $d\mathsf{CatSp} \subset \mathsf{CatSp}$ denote the full subcategory of d-categorical(ly truncated) spectrum and $d\mathsf{CatSp}^{\mathrm{adj}} = d\mathsf{CatSp} \cap \mathsf{CatSp}^{d\text{-adj}}$.

Theorem D (=Theorem 6.1.8, Corollary 6.1.10)**.** *The subcategories* $\mathsf{CatSp}^{d\text{-adj}} \subset \mathsf{CatSp}$ *are closed under extensions. Moreover, the tensor product of categorical spectra localizes to categorical spectra with adjoints. More precisely, there is a (unique) tensor product functor making the following diagram commute (where the vertical arrows are the localizations):*

$$\begin{tikzcd}
k\mathsf{CatSp} \otimes l\mathsf{CatSp} \arrow[r, "\otimes"] \arrow[d, "L^{k\text{-adj}} \otimes L^{l\text{-adj}}"] & (k+l)\mathsf{CatSp} \arrow[d, "L^{(k+l)\text{-adj}}"] \\
k\mathsf{CatSp}^{\mathrm{adj}} \otimes l\mathsf{CatSp}^{\mathrm{adj}} \arrow[r, dashed, "\otimes^{\mathrm{adj}}"] & (k+l)\mathsf{CatSp}^{\mathrm{adj}}
\end{tikzcd}$$

[3]For instance, ambidexterity is a technology of this flavor, which has recently opened some new directions in chromatic homotopy theory.

In particular, there are unique tensor products on $0\mathsf{CatSp}^{\mathrm{adj}}$ *and* $\infty\mathsf{CatSp}^{\mathrm{adj}}$ *promoting the localizations* $\mathsf{CatSp} \twoheadrightarrow 0\mathsf{CatSp}^{\mathrm{adj}}, \infty\mathsf{CatSp}^{\mathrm{adj}}$ *to monoidal functors.*

The proof reduces to a certain pushout formula for $\square^1 \otimes \mathsf{Adj}$, where Adj is the walking adjunction 2-category. The proof takes up the bulk of Section 6.1. Using this result, we can formally pack the framed cobordism hypothesis of different dimensions to cooperate in a multiplicatively structured way[4]:

Theorem E (=Corollary 6.2.8,Theorem 6.4.1)**.** *Assume the framed cobordism hypothesis. Then*

- $\bigoplus_{n\geq 0} \mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ *is a tensor algebra in* $\mathsf{CatSp}^{0\text{-adj}}$ *freely generated by a* (-1)*-cell.*

- *The stably framed bordism* (∞,∞)*-category* $\mathrm{B}^\infty\mathsf{Bord}^{\mathrm{sfr}}$ *is the tensor unit of* $\mathsf{CatSp}^{\infty\text{-adj}}$.

In both cases (at least intuitively) multiplication in the algebra structure is given by the cartesian product of manifolds. Notice that it is not possible to formulate the above structure without the lax tensor product because the cartesian product adds the dimension of the manifolds[5]. Note, however, that everything up to here could have been done in the world of connective categorical spectra, i.e., symmetric monoidal (∞,∞)-categories, by appropriate reindexing. The true power of our theory lies in the stability results such as Theorem C, which we now use. A *cobordism category with singularities* is a cell-complex-like object built up from the ordinary cobordism categories (with tangential structures), i.e., B^k in the following form of

[4]it is convenient to normalize by shifting $\mathrm{B}^\infty\mathsf{Bord}_n \in \mathsf{CatSp}^{n\text{-adj}}$ down to $\mathrm{B}^{\infty-n}\mathsf{Bord}_n \in \mathsf{CatSp}^{0\text{-adj}}$ so that we may avoid the use of graded monoidal structure. It also aligns with the philosophy of deeper algebra to use the codimensional indexing and puts the *partition function* in the zeroth degree.

[5]This theorem was roughly mentioned in [Yua]; however, the discussion is not made rigorous due to the lack of correct background setup.

iterated (co)extensions of categorical spectra (in $\mathsf{CatSp}^{\text{0-adj}}$):

$$\begin{tikzcd}
0 \arrow[r, tail] & B^d \arrow[r, tail] \arrow[d, two heads] & B^{d-1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & B^1 \arrow[r, tail] \arrow[d, two heads] & B^0 \arrow[d, two heads] \\
& \mathrm{B}^{\infty-d}\mathsf{Bord}_d^{\tilde{X}^d} & \mathrm{B}^{\infty-(d-1)}\mathsf{Bord}_{d-1}^{\tilde{X}^{d-1}} & & \mathrm{B}^{\infty-1}\mathsf{Bord}_1^{\tilde{X}^1} & \mathrm{B}^{\infty}\mathsf{Bord}_0^{\tilde{X}^0}
\end{tikzcd}$$

where $\tilde{X}^k$ are $\mathrm{O}(k)$-spaces classifying tangential structures. Each extension B^k is classified by an $\mathrm{O}(k)$-equivariant morphism $E^k : \tilde{X}^k \to (\Omega^{\infty-k-1}B^{k+1})^{\leq 0}$. Recall that an extension of categorical spectra can either be described as a lax fiber or a lax cofiber. The former describes maps *into* B^k, allowing us to interpret B^k geometrically. The latter describes maps *out* of it, i.e., TQFTs. This dual description is precisely the claim of the cobordism hypothesis with singularities.

Theorem F (=Theorem 6.3.5 [Lur09c, Theorem 4.3.11])**.** *The categorical spectrum B^k admits a description as the cobordism category of $\vec{X}$-manifolds as in [Lur09c, Definition Sketch 4.3.2], with the singularity datum $(\tilde{X}^d, \tilde{X}^{d-1}, E^{d-1}, \cdots, \tilde{X}^k, E^k)$. Moreover, for any 0-adjointful categorical spectrum A, there is a cartesian square*

$$\begin{tikzcd}
\mathrm{Map}(B^k, A) \arrow[r] \arrow[d] & \mathrm{Map}(B^{k+1}, A) \arrow[d] & \ni & Z_0 \arrow[d, mapsto] \\
\mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, (\mathsf{Alg}_{\mathbb{E}_0}(A_{k+1}))^{\leq 0}) \arrow[r] & \mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, (A_{k+1})^{\leq 0}) & \ni & \Omega^{\infty-k-1}Z_0 \circ E^k.
\end{tikzcd}$$

Note that this formulation is in some sense dimension independent: B^d can be 0 if $\tilde{X}^d = \emptyset$, so the value of d does not matter, as far as the sequence is bounded below. This is somewhat curious because, in the study of TQFTs, dimension is usually a fixed parameter.

Notations and Terminologies

- We mainly follow the standard notations of [Lur09b][Lur17], with a few exceptions below. In particular, for a type of categorical object, we use $\widehat{(-)}$ to denote the (very large) category of the large variant of that object.

- From now on (except in Appendix A), we will assume the objects are homotopical by default; in particular, we leave "∞-" in (∞,n)-categories or ∞-category/groupoid implicit and call them n-categories or category/groupoid. The $((\infty,1)$-$)$category of n-categories will be denoted by $n\mathsf{Cat}$. Similarly, ∞-operads in the sense of [Lur17] will be called operads. We still denote the category of (∞-)groupoids by $\mathsf{S} \coloneqq 0\mathsf{Cat}$, even though we will avoid calling them *spaces* unless they are supposed to have additional structures of topological spaces or CW complexes. Presheaves takes value in groupoids by default: $\mathsf{PSh}_{\mathcal{C}}(\mathcal{D}) \coloneqq \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ and $\mathsf{PSh}(\mathcal{D}) = \mathsf{PSh}_{\mathsf{S}}(\mathcal{D}) = \mathsf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathsf{S})$.

- We let $\mathsf{Pr}^{\mathsf{L}} \subset \widehat{\mathsf{Cat}} \supset \mathsf{Pr}^{\mathsf{R}}$ denote the (non-full) subcategories whose objects are presentable categories and the morphisms are left and right adjoint functors, respectively. We let $\mathsf{LFun}(\mathcal{C},\mathcal{D}) \subset \mathsf{Fun}(\mathcal{C},\mathcal{D}) \supset \mathsf{RFun}(\mathcal{C},\mathcal{D})$ denote the full subcategory spanned by the left adjoints and the right adjoints. We see Pr^{L} as a symmetric monoidal category by Lurie tensor product $\otimes$ (the internal hom is LFun). Also, $\mathsf{Pr}^{\mathsf{L}}_{\omega} \subset \mathsf{Pr}^{\mathsf{L}}$ will denote the (non-full) symmetric monoidal subcategory of compactly generated categories and compact-object preserving left adjoints. Its opposite category, $\mathsf{Pr}^{\mathsf{R}}_{\omega} \subset \mathsf{Pr}^{\mathsf{R}}$ is the category of presentable categories and filtered colimit preserving right adjoints.

- The $(1,1)$-category of strict n-categories will be denoted by $n\mathsf{Cat}^{\mathrm{str}}$ (the notation $(n,n)\mathsf{Cat}$ will mean the $(n+1,1)$-category of weak (n,n)-categories).

- We write Map for the hom groupoid of a (possibly underlying) $(\infty,1)$-category and Hom for a generic hom object of an algebroid or an enriched category, in particular, the cartesian internal hom for $\infty\mathsf{Algbrd}$. The notation $[-,-]$ (resp. $[\![-,-]\!]$) will mean the *left* (resp. *right*) internal hom for the Gray-type tensor product, i.e., $\mathrm{Map}(X \otimes Y, Z) \simeq \mathrm{Map}(Y, [X,Z]) \simeq \mathrm{Map}(X, [\![Y,Z]\!])$. As above, Fun will mean the functor $((\infty,1)$-$)$category between category-type objects.

- We let $*$ denote generically a terminal object (most often the contractible category), while 1 denote the unit of the monoidal structure under consideration. We use $\mathcal{C}_*$ and $\mathcal{C}_{**}$ to denote the category of pointed and bipointed objects, i.e., the category of objects under $*$ and $* \amalg *$.

- We tend to use $\sqcup$ to denote a disjoint union (i.e., a coproduct that is disjoint) whereas $\amalg$ will mean a generic coproduct. The wedge sum $\vee$ either means the coproduct in $\mathcal{C}_*$ or the *bipointed wedge sum* (typically *sink-source*), i.e., $(X, x_0, x_1) \vee (Y, y_0, y_1) = (X \amalg Y/(x_1 = y_0), x_0, y_1)$. We will not be very strict about the distinction of the notation.

- $\times$, $\wedge$ will mean the cartesian product and the corresponding smash product. $\otimes$ denotes the lax Gray tensor product of unpointed ∞-categories or the tensor product of categorical spectra, whereas $\oslash$ denotes the Gray smash product of pointed ∞-categories.

- We σ to denote the unreduced suspensions (of ∞-algebroids/categories and augmented directed complexes), reserving Σ for the (categorical) reduced suspension. Note that this does not agree with the usual suspension for ∞-groupoids. We use B to denote (the univalent completion of) the delooping of a monoidal category. For categorical spectra, we may both use Σ and B for the shift functor (also denoted by [1] as usual); we will be flexible with the notation here.

- $\mathbb{G}$, $\boldsymbol{\Delta}$, $\square$, Θ denote the $(1,1)$-category of globes, orientals, (fully lax) cubes, and Joyal's theta. C_n, Δ^n, $\square^n$ denote the n-cell, the n-oriental (i.e., the fully lax simplex) and the n-cube. Note the somewhat nonstandard notation for the orientals. We use Θ_1 to mean the usual simplex category, whose objects (1-categorical simplexes) are denoted by $[n]$.

- For $\mathcal{C} \in \widehat{\mathsf{Cat}}$, we denote the full subcategory of (homotopically) n-truncated objects by $\mathcal{C}_{\leq n} \subset \mathcal{C}$. In contrast, for $X \in \infty\mathsf{Algbrd}$, we denote the n-categorical truncation by $X^{\leq n} \subset X$, i.e., the right adjoint to the inclusion $n\mathsf{Algbrd} \hookrightarrow \infty\mathsf{Algbrd}$. There is also a left adjoint to the inclusion, which we denote by $X \mapsto {}^{\leq n}X$, but the notation is ambiguous and it can mean the further localization to $n\mathsf{Cat}$, $n\mathsf{Cat}^{\mathrm{str}}$ or $n\mathsf{Gaunt}$. See Section 2.1 for details.

- There seems to be no consensus about whether our Gray tensor product should be called the lax or oplax Gray tensor product. We will follow [AM20]. That is, the one making the linearization functor $\infty\mathsf{Cat}^{\mathrm{str}} \to \mathsf{adCh}$ strong monoidal with respect to the usual Koszul sign rule is the *oplax* Gray tensor product, and we take its opposite, the *lax* tensor product as default. This choice is more convenient for us, ultimately because of the choice we made about the way a monoidal functor is identified with a bimodule in Chapter 4.

Chapter 2

Preliminaries on category theory

The goal of this chapter is to equip the reader with background knowledge on n-category theory (i.e., (∞, n)-category theory) including the case $n = \infty$. It is largely meant to be expository, but we also provide some proof of the folklore results that the author could not find in the literature. Steiner's theory on strict ∞-categories is at the core of techniques, but it is separated in the appendix so that the reader can avoid getting too distracted by the combinatorics of strict ∞-categories.

We start with a general introduction in Section 2.1. Our focus is to provide (without proofs) various natively $(\infty, 1)$-categorical treatments of n-categories. They are roughly divided into two flavors: one is as *enriched categories*: n-categories are categories enriched in $(n-1)$-categories. This is inductive by nature. Another is by *presentation*: the category $n\mathsf{Cat}$ of n-categories is a localization of presheaves $\mathsf{PSh}(\mathcal{C})$, where $\mathcal{C} \subset n\mathsf{Cat}$ is some full sub $(1,1)$-category of combinatorial shapes and the localization is prescribed by gluings that exist in $\mathcal{C}$. Heuristically, $\mathcal{C}$ is a lax version of test categories, i.e., we probe the structure of n-categories by mapping combinatorial shapes into it. This approach separates combinatorial complexities from the homotopical complexity and makes combinatorial calculation possible. Another advantage is its flexibility, offering different options for $\mathcal{C}$ depending on our purpose.

Section 2.2 and Section 2.3 introduces two of the fundamental operations: the *(unreduced) suspension* and *duality involutions.* Roughly speaking, the suspension takes $X \in n\mathsf{Cat}$ to $\sigma X \in (n+1)\mathsf{Cat}$ which is generated by two objects $\perp, \top$ and $\mathrm{Hom}_{\sigma X}(\perp, \top) = X$. This is treated most naturally from enriched category theory. Duality involutions generalize the operation $(-)^{\mathrm{op}} : \mathsf{Cat} \to \mathsf{Cat}$ of taking the opposite categories. There are $(\mathbb{Z}/2)^n$-worth of duality involutions on $n\mathsf{Cat}$ by flipping morphism of specified dimensions. In fact, these are all automorphisms of $n\mathsf{Cat}$.

Section 2.4 introduces another fundamental operation of the *Gray tensor product* of ∞-categories. It is a noncommutative monoidal structure that will replace the role of cartesian products in many situations. Heuristically, the Gray tensor product of two ∞-categories looks like the cartesian product except that every product cell is filled with a non-invertible arrow of a coherently chosen direction. The tensor product is naturally characterized on the dense full subcategory $\square \subset \infty\mathsf{Cat}$ of the *cubes* and as such it is rooted in the presentation over the cube category. Using the Gray tensor product, we can define two suspension-like operations by tensoring the interval either from left or right and collapsing the top and bottom faces of the cylinder. We will show in Section 2.5 that these agree with the suspension (up to a duality involution in one case). This will be a crucial ingredient in the later chapters.

2.1 n-categories and n-algebroids

There are a few different attitudes when working with (∞, n)-categories. In this thesis, we take the *model-independent* approach; we assume and work natively in $(\infty, 1)$-category theory (as developed in [Lur09b]) and let $n\mathsf{Cat}$ be *the* (large) category (recall the "implicit ∞-" convention) of (small) (∞, n)-categories without choosing a point-set presentation. Thus, our treatment is similar in spirit to [BS21] (see also [Cam23a][Cam23b]): we use certain kinds of *strict* ∞-categories as the combinatorial blueprint of weak ∞-categories. We will also use the larger category $n\mathsf{Algbrd} = n\mathsf{Cat}^{\mathrm{f}} \supset n\mathsf{Cat}$ of n-algebroids (a.k.a. flagged (∞, n)-categories) to encompass both strict and weak categories. We begin with some definitions on the strict side.

Definition 2.1.1. A *strict* 0*-category* is a set: $0\mathsf{Cat}^{\mathrm{str}} := \mathsf{Set}$. Let $n \geq 0$ and suppose inductively that the $(1,1)$-category $n\mathsf{Cat}^{\mathrm{str}}$ of strict n-categories is already defined. Then a *strict* $(n+1)$*-category* is a strictly $n\mathsf{Cat}^{\mathrm{str}}$-enriched category: $(n+1)\mathsf{Cat}^{\mathrm{str}} := (n\mathsf{Cat}^{\mathrm{str}})\text{-}\mathsf{Cat}^{\mathrm{str}}$. These categories are presentable and the inclusion $n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow (n+1)\mathsf{Cat}^{\mathrm{str}}$ admits both left and right adjoint, denoted by ${}^{\leq n}(-)$ (or ${}^{\leq n,\mathrm{str}}(-)$ if there is a risk of confusion) and $(-)^{\leq n}$, respectively. Let $\infty\mathsf{Cat}^{\mathrm{str}}$ be the colimit

$$\mathrm{colim}(0\mathsf{Cat}^{\mathrm{str}} \hookrightarrow 1\mathsf{Cat}^{\mathrm{str}} \hookrightarrow \cdots \hookrightarrow n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow \cdots) \in \mathsf{Pr}^{\mathsf{L}},$$

or equivalently, the limit in Pr^{R} or $\widehat{\mathsf{Cat}}$ along the *truncations* $(-)^{\leq n}$.

Definition 2.1.2. Let X be a strict n-category. The *suspension* σX is a strict $(n+1)$-category with two objects $\{\bot, \top\}$ and the hom categories

$$\mathrm{Hom}_{\sigma X}(\bot, \top) = X, \quad \mathrm{Hom}_{\sigma X}(\bot, \bot) = * = \mathrm{Hom}_{\sigma X}(\top, \top), \quad \mathrm{Hom}_{\sigma X}(\top, \bot) = \emptyset$$

equipped with uniquely determined compositions. A suspension has a canonical *source-sink* bipointing $* \sqcup * = \sigma\emptyset \to \sigma X$. The functor $\sigma : n\mathsf{Cat}^{\mathrm{str}} \to (n+1)\mathsf{Cat}^{\mathrm{str}}_{**}$ is colimit-preserving and the right adjoint is $(X, x_0, x_1) \mapsto \mathrm{Hom}_X(x_0, x_1)$.

Definition 2.1.3. Let (X, x_0, x_1), (Y, y_0, y_1) be bipointed strict ∞-categories. The *wedge sum* $X \vee Y$ is the quotient $(X \sqcup Y)/(x_1 = y_0)$ equipped with the induced bipointing (x_0, y_1).

Definition 2.1.4.

- The *n-cell* C_n is the strict n-category $\sigma^n(*)$. We define the *(reflexive) globe category* $\mathbb{G}_n$ as the full subcategory $\{C_k \mid 0 \leq k \leq n\} \subset n\mathsf{Cat}^{\mathrm{str}}$ and $\mathbb{G} := \mathbb{G}_\infty$.
- The *canonically bipointed theta category* $\Theta^{\mathrm{can}}_{**} \subset \infty\mathsf{Cat}^{\mathrm{str}}_{**}$ is the smallest full subcategory containing the terminal object C_0 and closed under suspension and wedge sum operations. The (Joyal's) *theta category* $\Theta \subset \infty\mathsf{Cat}^{\mathrm{str}}$ is the image of $\Theta^{\mathrm{can}}_{**}$ under the forgetful functor. Let $\Theta_n = \Theta \cap n\mathsf{Cat}^{\mathrm{str}}$. The bipointing of

$\theta \in \Theta$ will always be the canonical one, i.e., given by the source and the sink vertices.

Example 2.1.5. Let us give a generic example of an object of Θ. The following diagram depicts the generating cells of $(\sigma^2 C_0) \vee \sigma(\sigma C_0 \vee \sigma^2 C_0) \vee \sigma C_0 = C_2 \sqcup_{C_0} (C_2 \sqcup_{C_1} C_3) \sqcup_{C_0} C_0)$:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow\Rrightarrow\Downarrow"] & \bullet \arrow[r] & \bullet \end{tikzcd} .$$

Remark 2.1.6. The full subcategory $\Theta_n \subset n\mathsf{Cat}^{\mathrm{str}}$ is (already $(1,1)$-categorically) dense, i.e., the restricted Yoneda embedding $n\mathsf{Cat}^{\mathrm{str}} \to \mathsf{PSh}_{\mathsf{Set}}(\Theta_n)$ is fully faithful. The image is characterized by the so-called Segal condition, a certain locality that can be stated as follows: any object $\theta \in \Theta$ admits a canonical colimit representation as the maximal cells (under the inclusion) glued along their shared boundary: $\operatorname{colim}_i C_{k_i} \xrightarrow{\sim} \theta$. Now a presheaf $P : \Theta_n^{\mathrm{op}} \to \mathcal{C}$ valued in an arbitrary category $\mathcal{C}$ satisfies the *Segal condition* if for any $\theta \in \Theta_n$, the induced map $P(\theta) \to \lim_i P(C_{k_i})$ is an equivalence. In contrast, the full subcategory $\mathbb{G}_n \subset n\mathsf{Cat}^{\mathrm{str}}$ is a set of colimit generators but not dense, i.e., the further restricted Yoneda embedding $n\mathsf{Cat}^{\mathrm{str}} \to \mathsf{PSh}_{\mathsf{Set}}(\mathbb{G}_n)$ is conservative (in fact, monadic) but not fully faithful; it fails to remember compositions (cf. Remark A.0.1).

Now we describe the category $n\mathsf{Algbrd} = n\mathsf{Cat}^{\mathrm{f}}$ of n-algebroids, a.k.a. flagged n-categories in a few different ways. They play the role of a minimal common generalization of strict n-categories and (∞, n)-categories:

- (iterated enrichment, [Lur09a][GH15][Hin20][Ste21]) For a given groupoid $X \in \mathsf{S}$ of objects, one can functorially assign the "X-worth-of-objects" version of the associative (nonsymmetric) operad Assoc_X, which agrees with Assoc when $X = *$. For example, when X is a set, Assoc_X is equivalent to the multicategory with objects (x,y) for $x,y \in X$ and multimorphism

$$\mathrm{Map}((x_0,y_1),(x_1,y_2),\dots,(x_{k-1},y_k);(y_0,x_k)) \coloneqq \prod_{i=0}^{k} \delta_{x_i,y_i}, \quad k \geq 0,$$

where $\delta_{x,y} = *$ if $x = y$ and $\emptyset$ if $x \neq y$. For a monoidal category V, we define the category $\mathsf{Algbrd}_X(\mathsf{V}) \coloneqq \mathsf{Alg}_{\mathrm{Assoc}_X}(\mathsf{V})$ of V-*algebroid*[1] *with the groupoid of objects* X. Roughly speaking, $\mathcal{A} \in \mathsf{Algbrd}_X(\mathsf{V})$ assigns $\mathcal{A}(x_0,x_1) \in \mathsf{V}$ for a pair of points $(x_0,x_1) \in X^2$ and a composition morphism $\mathcal{A}(x_0,x_1) \otimes \cdots \otimes \mathcal{A}(x_{k-1},x_k) \to \mathcal{A}(x_0,x_k)$ for $(x_0,\dots,x_k) \in X^k$, $k \geq 0$ in a coherently unital and associative way. We define the category of V-algebroids as the domain of the cartesian fibration $\mathrm{ob} : \mathsf{Algbrd}(\mathsf{V}) \to \mathsf{S}$ classifying the contravariant functor $X \mapsto \mathsf{Algbrd}_X(\mathsf{V}) \in \widehat{\mathsf{Cat}}$. When X is a set and V is a $(1,1)$-category, we recover the notion of stirctly V-enriched categories with the set of objects X, so we have $\mathsf{Algbrd}(\mathsf{V})|_{\mathsf{Set}} = \mathsf{V}\text{-}\mathsf{Cat}^{\mathrm{str}}$.

Let $n\mathsf{Algbrd} \coloneqq \mathsf{Algbrd}(\dots(\mathsf{Algbrd}(\mathsf{S}))\dots)$ be the n-fold iteration (with the cartesian monoidal structure). The inclusion $* \xrightarrow{\emptyset} \mathsf{S}$ in Pr^{L} induces the inclusion $\mathsf{S} \simeq \mathsf{Algbrd}(*) \hookrightarrow \mathsf{Algbrd}(\mathsf{S})$ (given by the initial section of the object fibration) and inductively $(n-1)\mathsf{Algbrd}(\mathsf{S}) \hookrightarrow n\mathsf{Algbrd}(\mathsf{S})$ in Pr^{L}. We let $\infty\mathsf{Algbrd}$ be the colimit as $n \to \infty$.

- (Θ-presheaves, [Rez10]) Iteratedly applying the inclusion

$$\mathsf{Algbrd}(\mathsf{V})|_{\mathsf{Set}} \hookrightarrow \mathsf{Algbrd}(\mathsf{V})$$

starting $\mathsf{V} = *$, we have the inclusion $n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow n\mathsf{Algbrd}$. Its restriction $\Theta_n \hookrightarrow n\mathsf{Algbrd}$ is dense, i.e., the restricted Yoneda embedding $n\mathsf{Algbrd} \to \mathsf{PSh}(\Theta_n)$ is fully faithful. The essential image is characterized by the same Segal condition

[1] These are called *categorical algebras* in [GH15]. In [Ste21], where the author took the terminology, a more general setting with *categories* of objects is considered. It is also worth mentioning that since the relevant symmetric monoidal structure on V is cartesian and V contains S, V-algebroids can also be defined as simplicial objects $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{V}$ satisfying $X_0 \in \mathsf{S}$ and the Segal condition $X_n \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

as in 2.1.6.

- (presheaves on a suitable site) More generally, if $\mathcal{S} \subset n\mathsf{Cat}^{\mathrm{str}} \hookrightarrow n\mathsf{Algbrd}$ is dense, then one can study $n\mathsf{Algbrd}$ by describing the localization $\mathsf{PSh}(\mathcal{S}) \to n\mathsf{Algbrd}$ and the combinatorics of $\mathcal{S}$. The localization admits an explicit description by cell attachments of torsion-free complexes when $\mathcal{S}$ is *suitable* in the sense of [Cam23b, Theorem B].

- If we take the notion of n-categories (see below) as the primary one, the notion of n-algebroids is equivalent to that of *flagged* n-categories of [AF18]: $n\mathsf{Algbrd} \simeq n\mathsf{Cat}^{\mathrm{f}}$. Flagging is an extra structure of an n-category. Roughly speaking, it keeps track of the choices of the groupoids of objects (X above) at each stage of enrichment, which is not invariant under categorical equivalence.

Our official definition is the first one; various presheaf presentations will replace the traditional use of models. Notice n-algebroids are "evil" as a notion of categories, taken up to isomorphisms instead of equivalences. This is why it contains the $(1,1)$-category of strict n-categories, constructed out of strict enrichment. We define a localization $n\mathsf{Cat} \subset n\mathsf{Algbrd}$ by fixing this:

- The category of V-categories $\mathsf{V}\text{-}\mathsf{Cat} \subset \mathsf{Algbrd}(\mathsf{V})$ is the localization by *categorical equivalences*, i.e., fully faithful and essentially surjective maps of algebroids. It suffices to invert $E \to *$, where E is (the base change to V of) the contractible groupoid with two objects. Let $\mathsf{S} := 0\mathsf{Cat}$, $(n{+}1)\mathsf{Cat} := (n\mathsf{Cat})\text{-}\mathsf{Cat}$ and $\infty\mathsf{Cat} := \operatorname{colim}(\cdots \hookrightarrow n\mathsf{Cat} \hookrightarrow (n{+}1)\mathsf{Cat} \hookrightarrow \cdots)$ in Pr^{L}. The localization $n\mathsf{Cat} \subset n\mathsf{Algbrd}$ are generated by the *Rezk maps* $\sigma^k(E \to *)$, $0 \leq k < n$. The local objects are also called *univalent* or *Rezk-complete*, meaning that the prescribed groupoid of objects of the algebroid is in fact the maximal subgroupoid so it can be recovered from the notion of equivalences *internal* to the algebroid.

- The Rezk maps considered as maps in $\mathsf{PSh}(\Theta)$ yields the localization $\mathsf{PSh}(\Theta_n) \to n\mathsf{Algbrd} \to n\mathsf{Cat}$. This is the original context by Rezk.

- In terms of flagged n-categories, the univalent complete objects are those with the *maximal* flags.

Note that a strict n-category is usually not univalent. For instance, the delooping $\mathrm{B}'\mathbb{Z}$ of $\mathbb{Z}$ as an algebroid with a single object $*$ is a strict 1-category, but its univalent completion $\mathrm{B}\mathbb{Z}$ is the circle S^1; the automorphism of $*$ must be already in the groupoid of objects to be univalent. In fact, a strict n-category is univalent if and only if it is *gaunt*, i.e., if no cell has a nonidentity automorphism [BS21, §3]. Summarizing the discussion, we obtain the following diagram:

$$\begin{tikzcd}
n\mathsf{Gaunt} \arrow[r, hook] \arrow[d, hook] & n\mathsf{Cat}^{\mathrm{str}} \arrow[r, hook] \arrow[d, hook] & \mathsf{PSh}_{\mathsf{Set}}(\Theta_n) \arrow[d, hook] \\
n\mathsf{Cat} \arrow[r, hook] & n\mathsf{Algbrd} \arrow[r, hook] & \mathsf{PSh}(\Theta_n)
\end{tikzcd} \tag{2.1}$$

Each inclusion in the diagram is right adjoint to an ω-accessible localization. In the top row are $(1,1)$-categories and are the 0-truncated parts of the bottom row. The middle (resp. left) column is the part of the right (resp. middle) satisfying the Segal conditions (resp. univalence). In particular, these are compactly generated with a set of compact generators $\mathbb{G}_n$. The distinction between $n\mathsf{Cat}$ and $n\mathsf{Cat}^{\mathrm{f}} = n\mathsf{Algbrd}$ is often irrelevant, but one is more appropriate in some cases. It is usually clear if the discussion works in both settings or just one, but we will clarify when necessary.

We write ${}^{\leq n}(-)$, $(-)^{\leq n}$ for the left and right adjoints of the inclusion $n\mathsf{Algbrd} \hookrightarrow \infty\mathsf{Algbrd}$. The right adjoint $(-)^{\leq n}$ preserves the univalence and homotopically k-truncated objects (in particular strictness or gauntness). We call $X^{\leq n}$ the *underlying n-algebroid (or category)* or *n-(categorical) truncation* (there is potential confusion with homotopical truncation, but the relevant notion is usually clear from the context). On the other hand, the left adjoint ${}^{\leq n}(-)$ (the *n-categorical localization*) does not preserve univalence nor strictness in general. For example, $\mathrm{B}\mathbb{N}$ is a gaunt 1-category freely generated by an object and an endomorphism, but its 0-categorical localization depends on the ambient setting; in $0\mathsf{Algbrd} = 0\mathsf{Cat} = \mathsf{S}$ we have ${}^{\leq 0}(\mathrm{B}\mathbb{N}) \simeq \mathrm{B}\mathbb{Z} \simeq S^1$, whose image in $0\mathsf{Cat}^{\mathrm{str}}$ and $0\mathsf{Gaunt}$ are $\pi_0 S^1 = *$. Moreover, $\mathrm{B}^2\mathbb{N}$ is a gaunt 2-category but ${}^{\leq 1,\mathrm{algbrd}}(\mathrm{B}^2\mathbb{N}) = \mathrm{B}' S^1$ is an algebroid generated by a point and a S^1 worth of automorphisms, whereas ${}^{\leq 1,\mathrm{cat}}(\mathrm{B}^2\mathbb{N}) = \mathrm{B}S^1 \simeq \mathbb{C}P^\infty$. The localization $\infty\mathsf{Cat} \twoheadrightarrow n\mathsf{Cat}$ preserves finite products [Ste21, Proposition 3.6.13].

Note that $\mathsf{Algbrd}_X(-)$ preserves limits of operads and thus of symmetric monoidal categories. Integrating the equivalence $\mathsf{Algbrd}_X(\infty\mathsf{Algbrd}) \simeq \lim_n \mathsf{Algbrd}_X(n\mathsf{Algbrd})$ and $\mathsf{Algbrd}_X(\infty\mathsf{Cat}) \simeq \lim_n \mathsf{Algbrd}_X(n\mathsf{Cat})$ over X and with some univalence consideration, one sees that $\infty\mathsf{Algbrd}$ and $\infty\mathsf{Cat}$ are fixed points of the corresponding constructions [Ste21, Remark 3.6.12]:

$$\infty\mathsf{Cat} \simeq (\infty\mathsf{Cat})\text{-}\mathsf{Cat}, \quad \infty\mathsf{Algbrd} \simeq \mathsf{Algbrd}(\infty\mathsf{Algbrd}).$$

Moreover, [Gol23] shows that these are universal among such fixed points of enrichment in Pr^{L}.

Remark 2.1.7. There are also a few combinatorial presentations of the categories $n\mathsf{Cat}$ and $n\mathsf{Algbrd}$ using marked simplicial and cubical sets [Ver08][CKM20]. These are convenient for many purposes; not only are they *set-valued* presheaves on combinatorial shapes, but they also handle relative categories by design. However, the representable presheaves are not fibrant. To compute the correct mapping groupoid one must perform a fibrant replacement, so the calculation is not as straightforward as in the $n = 1$ case of (naturally marked) quasicategories. We will not rely on these combinatorial approaches because passing between those and ours is not very straightforward. See [Lou22] for proof that the model category for n-complicial sets indeed model $n\mathsf{Cat} \in \widehat{\mathsf{Cat}}$.

2.2 Suspension

The suspension functor on $n\mathsf{Cat}^{\mathrm{str}}$ extends to $n\mathsf{Algbrd}$. There are many possible equivalent definitions; we first give one based on enriched category theory.

Let $\mathrm{Assoc}_{\{\perp,\top\}}$ be the nonsymmetric operad for two-object algebroids. There is a morphism of nonsymmetric operads $\mathrm{Triv} = \boldsymbol{\Delta}_{\mathrm{inert}} \to \mathrm{Assoc}_{\{\perp,\top\}}$ characterized by $[1] \mapsto (\perp, \top)$, which by definition corepresents $\mathrm{Hom}_{(-)}(\perp, \top) : \mathsf{Algbrd}_{\{\perp,\top\}}(\mathsf{V}) \to \mathsf{V}$. When V has an initial object that is compatible with the monoidal structure, there is a left adjoint given by the operadic left Kan extension (for nonsymmetric operads, see [GH15, §A.4]). Also notice $\mathsf{Algbrd}_{\{\perp,\top\}} \subset \mathsf{Algbrd}(\mathsf{V})_{**}$ is the fiber of the cartesian fibration $\mathrm{ob} : \mathsf{Algbrd}(\mathsf{V})_{**} \to \mathsf{S}_{**}$ over the initial object, so the inclusion admits a right adjoint that sends X to its full subalgebroid spanned by the base objects.

Definition 2.2.1. ([Ste21, Example 3.3.6]) The (unreduced) suspension functor is the composition of the left adjoints

$$\sigma : \mathsf{V} \to \mathsf{Algbrd}_{\{\perp,\top\}}(\mathsf{V}) \hookrightarrow \mathsf{Algbrd}(\mathsf{V})_{**},$$

whose right adjoint is $(X, x_0, x_1) \mapsto \mathrm{Hom}_X(x_0, x_1)$. In particular we have

$$\sigma : n\mathsf{Algbrd} \to (n+1)\mathsf{Algbrd}_{**}.$$

We denote the limit case $\infty\mathsf{Algbrd} \to \mathsf{Algbrd}(\infty\mathsf{Algbrd})_{**} \simeq \infty\mathsf{Algbrd}_{**}$ also by σ.

Remark 2.2.2. (1) If V is a 1-category, σ factors through $\mathsf{V} \to \mathsf{Algbrd}(\mathsf{V})|_{\mathsf{Set},**} = \mathsf{V}\text{-}\mathsf{Cat}^{\mathrm{str}}_{**}$. This agrees with the suspension of strictly V-enriched categories. More generally, the suspension functor is fully faithful and has the same description of hom objects as 2.1.2 (use the description of the operadic left Kan extension). In particular, σ on $\infty\mathsf{Algbrd}$ restricts to the σ on $\infty\mathsf{Cat}^{\mathrm{str}}$ previously defined.

(2) The suspension σ lands in $\mathsf{V}\text{-}\mathsf{Cat}_{**}$ unless $\mathsf{V} = *$. To see this, assume there is a nontrivial map $E \to \sigma X$, so $f : 1_{\mathsf{V}} \to X \simeq \mathrm{Hom}_{\sigma X}(\perp, \top)$, $g : 1_{\mathsf{V}} \to \emptyset \simeq$

$\mathrm{Hom}_{\sigma X}(\top, \perp)$ is inverse to each other. Then

$$\mathrm{Hom}_{\sigma X}(\top, \top) \xrightarrow{\mathrm{id} \otimes g \otimes f} \mathrm{Hom}_{\sigma X}(\top, \top) \otimes \mathrm{Hom}_{\sigma X}(\top, \perp) \otimes \mathrm{Hom}_{\sigma X}(\perp, \top)$$

$$\to \mathrm{Hom}_{\sigma X}(\top, \top)$$

is an equivalence, so $1_{\mathsf{V}} \simeq \mathrm{Hom}_{\sigma X}(\top, \top)$ is a retract of $\emptyset$. It follows that $1_{\mathsf{V}} \simeq \emptyset$, inducing natural equivalence $\mathrm{id}_{\mathsf{V}} \simeq \mathrm{const}_{\emptyset}$, i.e., V must be trivial.

(3) In particular, when X is an ∞-category (i.e., univalent), σX is also univalent, so we will use the same notation for the restricted functors.

(4) We may also give a definition based on Θ-presheaves [Cam23a, Theorem 2.25]. Notice by our definition of Θ, the suspension of Definition 2.1.2 restricts to a functor $\sigma : \Theta_n \to (\Theta^{\mathrm{can}}_{**}) \cap (n+1)\mathsf{Cat}_{**}$. Let $\tilde{\sigma} : \mathsf{PSh}(\Theta_n) \to \mathsf{PSh}(\Theta_{n+1})_{**}$ be the unique colimit-preserving extension. This restricts to a colimit-preserving functor $\sigma : n\mathsf{Algbrd} \to (n+1)\mathsf{Algbrd}_{**}$. The two definitions of σ given above are equivalent, as both are colimit-preserving and agree on Θ_n.

In Lemma 2.5.1, we will give another description using the Gray tensor product.

2.3 Duality

All categories in the diagram 2.1 have the same group of automorphisms:

Proposition 2.3.1. *Let* C *denote one of* Gaunt, $\mathsf{Cat}^{\mathrm{str}}$, Cat, Algbrd *and* $0 \leq n \leq \infty$. *Then any automorphism of* $n\mathsf{C}$ *preserves the subcategories* $\mathbb{G}_n$, Θ_n *and the restriction* $\mathrm{Aut}(n\mathsf{C}) \to \mathrm{Aut}(\mathbb{G}_n) \simeq (\mathbb{Z}/2)^n$ *is an equivalence.*

When n is finite, the proposition is [BS21, Theorem 4.13, Lemma 10.2] for $n\mathsf{Gaunt}$, $n\mathsf{Cat}$ and the same argument works for $n\mathsf{Cat}^{\mathrm{str}}$, $n\mathsf{Algbrd}$. The key idea is to characterize the n-cell C_n as an object of the abstract category $n\mathsf{Gaunt}$, $n\mathsf{Cat}$, etc. as the smallest generator (i.e., an object corepresenting a conservative functor) with respect to the retract relation. This shows that any automorphism restricts to an automorphism of $\mathbb{G}_n$ (which is identity on objects). Each copy of $\mathbb{Z}/2$ in $\mathrm{Aut}(\mathbb{G}_n) \simeq (\mathbb{Z}/2)^n$ corresponds to flipping the cosource and the cotarget maps $s, t : C_{k-1} \to C_k$ for $0 < k \leq n$. These automorphisms uniquely extend to $\mathsf{PSh}(\Theta_n)$ and fix all relevant subcategories.

The same idea does not apply directly when $n = \infty$ because the infinite cell C_∞ is a proper retract of itself. However, the following lemma inductively reconstructs the subcategories $n\mathsf{C} \subset \infty\mathsf{C}$ from abstract category $\infty\mathsf{C}$[2]. Consequently, any automorphism of $\infty\mathsf{C}$ preserves the subcategory $n\mathsf{C}$ and therefore restricts to $\mathbb{G}_n$ for any $n \geq 0$, and the proposition follows.

Lemma 2.3.2. (1) $0\mathsf{C} \subset \infty\mathsf{C}$ *is the colimit-closure of the terminal object.*

(2) *Suppose we have already identified the full subcategory $(n-1)\mathsf{C} \subset \infty\mathsf{C}$ and in particular the right adjoint $(-)^{\leq n-1}$ to the inclusion. Let $n\mathsf{C}' \subset \infty\mathsf{C}$ be the full subcategory spanned by the objects X satsifying the following condition:*

> *for any (homotopically) 0-truncated object $Y \in (\infty\mathsf{C})_{\leq 0}$, the counit $Y^{\leq n-1} \to Y$ induces a monomorphism $\mathrm{Map}(Y, X) \to \mathrm{Map}(Y^{\leq n-1}, X)$ in S.*

Then $n\mathsf{C}$ is the colimit-closure of $n\mathsf{C}' \subset \infty\mathsf{C}$.

Proof. The first point is clear. For the second point, it suffices to prove $\mathbb{G}_n \subset n\mathsf{C}' \subset n\mathsf{C}$. First, we show $\mathbb{G}_n \subset n\mathsf{C}'$. Consider the (-1)-truncation functor $\mathsf{Set} \to \{\emptyset, *\}$ in Pr^L. Since it is a product-preserving localization, it induces the *underlying n-preordered set* functor $\mathsf{Set}(\text{-}\mathsf{Cat}^{\mathrm{str}})^n \to \{\emptyset, *\}(\text{-}\mathsf{Cat}^{\mathrm{str}})^n$. Note that the latter contains $\mathbb{G}_n$. For any $X \in \{\emptyset, *\}(\text{-}\mathsf{Cat}^{\mathrm{str}})^n$, the map $(s_{n-1}, t_{n-1}) : X_n \to X_{n-1} \times X_{n-1}$ is mono and $X_{\geq n}$ is constant. It follows that for any strict ∞-category Y,

[2] This idea is based on the post [Cama] characterizing the posets inside $\mathrm{ho}\,\mathsf{Cat}_{(1,1)}$.

the map $\mathrm{Map}_{\mathsf{PSh}_{\mathsf{Set}}(\mathbb{G})}(Y_\bullet, X_\bullet) \to \mathrm{Map}_{\mathsf{PSh}_{\mathsf{Set}}(\mathbb{G})}(Y_\bullet^{\leq n-1}, X_\bullet)$ is a monomorphism. Since $\infty\mathsf{Cat}^{\mathrm{str}} \to \mathsf{PSh}_{\mathsf{Set}}(\mathbb{G})$ is conservative, $\mathrm{Map}_{\infty\mathsf{Cat}^{\mathrm{str}}}(Y, X) \to \mathrm{Map}_{\infty\mathsf{Cat}}(Y^{\leq n-1}, X)$ is also a monomorphism.

Next assume $X \in n\mathsf{C}'$. We must show that any k-cell of X is degenerate for $k > n$, i.e., the map $\mathrm{Map}(C_k, X) \to \mathrm{Map}(C_{k+1}, X)$ induced by the projection $C_{k+1} \twoheadrightarrow C_k$ is an isomorphism. We claim that the following conditions are equivalent when $k \geq n$:

(1) The map $\mathrm{Map}(C_k, X) \to \mathrm{Map}(\partial C_k, X)$ induced by $\partial C_k \hookrightarrow C_k$ is mono.

(2) The map $\mathrm{Map}(C_k, X) \to \mathrm{Map}(\partial C_{k+1}, X)$ induced by $\partial C_{k+1} \twoheadrightarrow C_k$ is an isomorphism.

(3) The maps

$$\mathrm{Map}(C_k, X) \to \mathrm{Map}(C_{k+1}, X) \to \mathrm{Map}(\partial C_{k+1}, X)$$

induced by $\partial C_{k+1} \hookrightarrow C_{k+1} \twoheadrightarrow C_k$ are isomorphisms.

(1) and (2) are equivalent because the second map is the diagonal of the first. (2) and (3) are equivalent because the maps of (3) are mono by assumption: they are maps between subobjects of $\mathrm{Map}(\partial C_n, X)$ because $(\partial C_{k+1})^{\leq n-1} = (C_{k+1})^{\leq n-1} = (C_k)^{\leq n-1} = \partial C_n$ when $k \geq n$. Now, again by assumption, (1) is true for $k = n$. By induction using the equivalence, we see that (3) is true for all $k \geq n$, so we have $X \in n\mathsf{C}$. □

Remark 2.3.3. The category $n\mathsf{Gaunt}'$ is the category of so-called n-posets; by convention, a (-1)-poset is the terminal object, and n-poset are those enriched in $(n-1)$-posets.

Definition 2.3.4. For a function $\tau : \mathbb{Z}_{\geq 1} \to \mathbb{Z}/2$, we let D_τ denote the corresponding involution of the categories of n-categories. That is, if we let $s_k, t_k : C_k \to C_{k-1}$ be the k-th co-source and co-target map, D_τ is characterized by

$$D_\tau(s_k) = \begin{cases} s_k & \text{if } \tau(k) = 0, \\ t_k & \text{if } \tau(k) = 1. \end{cases}$$

One can think of τ as the indicator function of the dimensions of the cells that get flipped.

The following copy of $\mathbb{Z}/2 \times \mathbb{Z}/2 \subset \mathrm{Aut}(\infty\mathsf{Algbrd})$ is of dimension-independent importance (one reason is Proposition 2.4.17; the proof does not work for a general τ):

Definition 2.3.5. The *odd dual* (resp. *even dual*) flips $s, t : C_{k-1} \to C_k$ for k odd (resp. even), i.e., they are the duality involution D_τ when τ is the indicator function of the odd (resp. even) numbers. We denote the odd and even duals by $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{co}}$, respectively. The *total dual* flips the cells of all dimensions, i.e. $(-)^{\mathrm{coop}}$, which we denote by $(-)^{\circ}$ or D.

2.4 The cubes and the Gray tensor product

The (lax) Gray tensor product is a monoidal structure on $\infty\mathsf{Algbrd}$ and its various localizations. It differs from the cartesian product in a few important ways:

- When X, Y are m, n-categories respectively, the Gray tensor product $X \otimes Y$ is a $(m+n)$-category while the cartesian product $X \times Y$ is a $\max\{m, n\}$-category.
- The cartesian product is symmetric, but the Gray tensor product is not. It behaves like a star-algebra with respect to the odd (or even) dual.
- Both the cartesian product and Gray tensor product are closed with the terminal unit. However, the Gray internal hom classifies the functor category with *(op)lax* natural transformations, while the cartesian internal hom classifies the functor category with strong natural transformations.

Remark 2.4.1. In the classical 2-categorical literature, sometimes the Gray tensor product refers to the *pseudo*-Gray tensor product (as opposed to the *lax* or oplax Gray tensor product). We will never use this language because the pseudo-Gray tensor product is just a cartesian product in our natively homotopical setting.

We will freely use Steiner's theory (see Appendix A for a review). Here, let us only recall the existence of an adjunction

$$\infty\mathsf{Cat}^{\mathrm{str}} \underset{\nu}{\overset{\lambda}{\underset{\perp}{\rightleftarrows}}} \mathsf{adCh}$$

that restricts to an equivalence $\infty\mathsf{Gaunt}^{\mathrm{Ste}} \simeq \mathsf{adCh}^{\mathrm{Ste}}$ of the full subcategories of *strong Steiner objects* of the both sides (for $\infty\mathsf{Cat}^{\mathrm{str}}$ side they are automatically gaunt). adCh is the category of *augmented directed complexes* (homologically-graded augmented chain complex with an additional data of *positivity submonoid*). Strong-Steinerness is a reasonably checkable "free and loop-free" condition and is satisfied by many simple and combinatorially important gaunt ∞-categories (an important non-example is the walking adjunction category Adj, however). Steiner's theory gives a neat way to define the (lax) Gray tensor product of strict ∞-categories; it corresponds to the tensor product of chain complexes. We endow the category adCh with a monoidal structure by the (reversed) Koszul sign rule (i.e., $\partial x \otimes y = (-1)^{\deg(y)} \partial x \otimes y + x \otimes \partial y$). One can check that $\mathsf{adCh}^{\mathrm{Ste}} \subset \mathsf{adCh}$ is a monoidal subcategory.

Definition 2.4.2. (1) There exists a unique biclosed monoidal structure, called the *(strict) Gray tensor product* on $\infty\mathsf{Cat}^{\mathrm{str}}$ such that $\mathsf{adCh}^{\mathrm{Ste}} \simeq \infty\mathsf{Gaunt}^{\mathrm{Ste}} \hookrightarrow \infty\mathsf{Cat}^{\mathrm{str}}$ promotes to a monoidal functor.

(2) The *cube category* $\square \subset \infty\mathsf{Gaunt}^{\mathrm{Ste}} \subset \infty\mathsf{Cat}^{\mathrm{str}}$ is the monoidal full subcategory generated by the interval $\square^1 := C_1$. The objects of $\square$ are the n-cubes $\square^n := (\square^1)^{\otimes n} \in n\mathsf{Gaunt}$.

Remark 2.4.3. Campion [Cam22] shows that Θ is contained in the idempotent completion of $\square$ (also see Corollary A.3.6 for another proof with the same idea). In particular, $\square \subset \infty\mathsf{Cat}^{\mathrm{str}}$ is dense, i.e., the left Kan extension $\mathsf{PSh}_{\mathsf{Set}}(\square) \to \infty\mathsf{Cat}^{\mathrm{str}}$ is a localization. This shows the uniqueness part of (1); even stronger, it is characterized as the unique biclosed monoidal structure promoting $\square \hookrightarrow \infty\mathsf{Cat}^{\mathrm{str}}$ to a monoidal functor.

Example 2.4.4. The n-cube $\square^n$, or a strong Steiner category in general, is an example of *computads* (a.k.a. *polygraphs*). It is the relevant notion of the freeness for strict n-categories, defined similarly to the notion of CW-complexes Definition A.1.6. By definition, we have $\square^n = \lambda\left((\cdots \to 0 \to \underline{?}\mathbb{Z} \xrightarrow{(1,-1)} \underline{0}\mathbb{Z} \oplus \underline{1}\mathbb{Z})^{\otimes n}\right)$. An polygraphic basis element (i.e., an atomic cell) of $\square^n$ is a string of the letters $0, 1, ?$. The number of $?$ is the dimension of the cell. The differential $\partial(\underline{?}) = \underline{1} - \underline{0}$ together with the Koszul sign rule describes what the domain and the codomain of these cells are. The following depicts the atomic cells of the first three cubes $\square^1, \square^2, \square^3$:

$$0 \longrightarrow 1, \qquad \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & 01 \arrow[d] \\ 10 \arrow[r] & 11, \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & 001 \arrow[dr] \arrow[dd, phantom, "\Uparrow"] & & & & 001 \arrow[dr] \arrow[dd] & \\ 000 \arrow[ur] \arrow[dr] \arrow[dd] & & 011 \arrow[dd] & & 000 \arrow[ur] \arrow[dd] \arrow[dr, phantom, "\Nearrow"] & & 011 \arrow[dd] \\ & 010 \arrow[ur] \arrow[dd] \arrow[dr, phantom, "\Nearrow"] & & \Rrightarrow & & 101 \arrow[dr] \arrow[ur, phantom, "\Nearrow"] \arrow[dd, phantom, "\Uparrow"] & \\ 100 \arrow[dr] \arrow[ur, phantom, "\Nearrow"] & & 111 & & 100 \arrow[ur] \arrow[dr] & & 111. \\ & 110 \arrow[ur] & & & & 110 \arrow[ur] & \end{tikzcd}$$

Remark 2.4.5. A monoidal structure $\otimes'$ on $\square$ such that $\square^n \otimes' \square^m = \square^{n+m}$ is completely characterized by the bifunctor $\otimes' : \square \times \square \to \square$; since $\square$ is a 0-truncated object of Cat (see Remark A.3.4), any $\mathbb{A}_\infty$-structure is strictly associative and associativity is a property of the underlying $\mathbb{A}_2$-structure.

Next, we explain the Gray tensor product for weak categories.

Theorem 2.4.6 ([Cam23b])**.** *There exist a unique closed $\mathbb{E}_1$-monoidal structure, called the* Gray tensor product *on $\infty\mathsf{Algbrd}$ promoting the inclusion $\infty\mathsf{Gaunt}^{\mathrm{Ste}} \hookrightarrow \infty\mathsf{Algbrd}$ to a strong monoidal functor. Moreover, the reflective subcategories $\infty\mathsf{Cat}$,*

$\infty\mathsf{Cat}^{\mathrm{str}}$*, and* $n\mathsf{Algbrd}$ *are all exponential ideals (see the remark below). In particular, the tensor product localizes to these categories and their intersections.*

Definition 2.4.7. We will denote the Gray tensor product of $\infty\mathsf{Algbrd}$ and $\infty\mathsf{Cat}$ by $\otimes$ and the left internal hom by $[-,-]$; that is, we have a natural equivalence $\mathrm{Map}_{\infty\mathsf{Cat}}(X,[Y,Z]) \simeq \mathrm{Map}_{\infty\mathsf{Cat}}(Y\otimes X, Z)$ and similarly for $\infty\mathsf{Algbrd}$. Beware that $X\otimes Y$ can be ambiguous up to univalent completion when X, Y are ∞-categories; it is usually clear from the context if the tensor product is localized or not (there is no such ambiguity for internal hom). We will occasionally use $[\![X,Y]\!]$ to denote the right internal hom, i.e., $\mathrm{Map}(Z,[\![X,Y]\!]) \simeq \mathrm{Map}(Z\otimes X, Y)$.

Remark 2.4.8. By definition, a full subcategory $\mathcal{C}\subset\infty\mathsf{Algbrd}$ is an *exponential ideal* if for any $Y\in\mathcal{C}$ and $X\in\infty\mathsf{Algbrd}$, we have $[X,Y], [\![X,Y]\!]\in\mathcal{C}$. When $L:\infty\mathsf{Algbrd}\to\mathcal{C}$ is a localization, this is precisely when L-equivalences are preserved by tensoring objects from both sides, or when there is a (necessarily unique) monoidal structure on $\mathcal{C}$ promoting L to a monoidal functor [Lur17, Proposition 2.2.1.9]. If we denote the localized tensor product by $X\otimes^L Y\simeq L(X\otimes Y)$, there is a canonical comparison map $X\otimes Y\to X\otimes^L Y$ but this is not necessarily an equivalence.

Remark 2.4.9. Because $\square\subset\infty\mathsf{Algbrd}$ is dense, promoting this inclusion to a strong monoidal functor is enough to characterize the biclosed monoidal structure; if it exists, it must be induced from the localization $\mathsf{PSh}(\square)\to\infty\mathsf{Algbrd}$, where $\mathsf{PSh}(\square)$ is endowed with the Day convolution monoidal structure. Similar characterization is also true for various localizations. Since $\square$ is a set of compact generators, the Gray tensor product is compactly generated, i.e., $\infty\mathsf{Cat}^{\otimes}\in\mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\omega})$.

Remark 2.4.10. The theorem in particular states that the weak and strict tensor product agrees on the strong Steiner categories, i.e., the comparison map $X\otimes Y\to\tau_{\leq 0}(X\otimes Y)$ for $X,Y\in\infty\mathsf{Algbrd}$ (or $\infty\mathsf{Cat}$) is an equivalence when X,Y are strong Steiner. The monoidal category $\infty\mathsf{Gaunt}^{\mathrm{Ste}}$ in the theorem can be replaced by a larger category of *torsion-free complexes*. The author does not know if the gauntness is preserved in general by the Gray tensor product. However, Loubaton [Lou23, Theorem 4.3.3.26] shows that the tensor product with the cubes preserves gauntness[3].

Remark 2.4.11. The Gray tensor product is *additive* on category levels, i.e., if X, Y are k, l-categories respectively, then $X \otimes Y$ is an $(k+l)$-category. To see this, consider the subcategory $\mathcal{C}$ of pairs $(X, Y) \in \infty\mathsf{Cat} \otimes \infty\mathsf{Cat}$ (beware the use of $\otimes$ for presentable categories) such that $X \otimes Y$ is an n-category. $\mathcal{C}$ is closed under colimits and contains $(\square^k, \square^l)$ for $k + l \leq n$, so it contains $\bigcup_{k+l\leq n} k\mathsf{Cat} \otimes l\mathsf{Cat}$.

Remark 2.4.12. [Lou22], [Lou23] also shows the existence of the Gray tensor product by showing the equivalence between ∞-categories and complicial sets and allowing one to transfer the Gray tensor product of complicial sets [VRO23].

The Gray tensor product is *semicartesian*, that is, its unit object is terminal. In this case, there is a natural transformation $X \otimes Y \to X \times Y$. The following lemma shows that this extends to a lax monoidal functor. This fact seems standard, but we include the proof because the author does not know a proof in the literature:

Lemma 2.4.13. *Let $\mathcal{C}^\otimes \to \Delta^{\mathrm{op}}$ be a semicartesian monoidal category. Then there is a lax monoidal functor $\mathcal{C}^\times \to \mathcal{C}^\otimes$ from the cartesian monoidal structure whose underlying functor is* $\mathrm{id}_\mathcal{C}$*.*

Proof. One can construct the opposite monoidal category $(\mathcal{C}^{\mathrm{op}})^\otimes$ by postcomposing the involution $(-)^{\mathrm{op}} : \mathsf{Cat} \to \mathsf{Cat}$ to the Segal object $\Delta^{\mathrm{op}} \to \mathsf{Cat}$ it classifies. Then $\mathcal{C}^\otimes$ is semicartesian if and only if $(\mathcal{C}^{\mathrm{op}})^\otimes$ is *unital* in the sense of [Lur17, Definition 2.3.1.1]. Let $(-)^{\mathrm{sym}} : \mathsf{Mon}(\mathsf{Cat}) \to \mathsf{CMon}(\mathsf{Cat})$ be the left adjoint to the forgetful functor. Then constructing a lax monoidal functor $\mathcal{C}^\times \to \mathcal{C}^\otimes$ which is the identity

[3]Technically speaking, the theorem is proven for the monoidal structure transferred from Verity's monoidal structure on complicial sets, which is previously not known to be equivalent to Campion's monoidal structure. However, the theorem in particular proves that the transferred monoidal structure agrees with the strict one on $\square$, so by the characterization, the equivalence of the two monoidal structures follows.

on the underlying categories is equivalent to constructing a lax monoidal functor $((\mathcal{C}^{\mathrm{op}})^{\otimes})^{\mathrm{sym}} \to (\mathcal{C}^{\mathrm{op}})^{\amalg}$ which is the identity on underlying categories. Now [Lur17, Proposition 2.4.3.9] says that a lax monoidal functor from an unital operad to a cocartesian operad is determined by the underlying functor. □

Remark 2.4.14. Because $\otimes$ is closed and semicartesian, the comparison morphism $X \otimes Y \to X \times Y$ is an isomorphism when X or Y is a 0-category. By adjunction, one sees that $[X, Y]^{\leq 0} \simeq \mathrm{Map}_{\infty\mathsf{Cat}}(X, Y) \simeq [\![X, Y]\!]^{\leq 0}$.

Remark 2.4.15. In particular, the identity functor promotes to a lax monoidal functor $(\infty\mathsf{Algbrd}, \times) \to (\infty\mathsf{Algbrd}, \otimes)$ and $(\infty\mathsf{Cat}, \times) \to (\infty\mathsf{Cat}, \otimes)$. This induces a fully faithful change-of-enrichment functor $\mathsf{Algbrd}(\infty\mathsf{Algbrd}) \to \mathsf{Algbrd}(\infty\mathsf{Algbrd}^{\otimes})$ and $\iota : \infty\mathsf{Cat} \simeq (\infty\mathsf{Cat})\text{-}\mathsf{Cat} \to (\infty\mathsf{Cat}^{\otimes})\text{-}\mathsf{Cat}$.

Definition 2.4.16. A *Gray category* is a category enriched in the Gray tensor product of $\infty\mathsf{Cat}$, i.e., an object of $\infty\mathsf{Cat}^{\otimes}\text{-}\mathsf{Cat}$.

We close the section by showing that some of the duality functors interact well with the Gray tensor product.

Proposition 2.4.17. *The total dual functor promotes to a monoidal endofunctor $\infty\mathsf{Algbrd}^{\otimes} \to \infty\mathsf{Algbrd}^{\otimes}$. The odd and even dual functors promote to antimonoidal endofunctors $\infty\mathsf{Algbrd}^{\otimes} \to \infty\mathsf{Algbrd}^{\otimes^{\mathrm{oplax}}}$. Similar statements are true for $\infty\mathsf{Cat}$, $\infty\mathsf{Cat}^{\mathrm{str}}$, $\infty\mathsf{Gaunt}$.*

Proof. By the universal property of the Gray tensor product, it suffices to show the following:

(1) The total dual $(-)^{\circ}$ restricts to $\infty\mathsf{Gaunt}^{\mathrm{Ste}}$ and promotes to a monoidal functor.

(2) The odd and even dual $(-)^{\mathrm{op}}$, $(-)^{\mathrm{co}}$ restrict to $\infty\mathsf{Gaunt}^{\mathrm{Ste}}$ and promote to antimonoidal functors.

These follow from the corresponding claims in the category adCh (see Appendix A.2). The part "restricts to" follows from the fact that strong Steiner-ness is preserved by the duality functors. To check the monoidal property in the first claim, note that $(A \otimes B)^\circ$ and $A^\circ \otimes B^\circ$ have the identical underlying graded abelian group. The identification clearly respects the augmentation and the positive parts. For differentials, observe

$$\begin{aligned}\partial^{(A\otimes B)^\circ}(a \otimes b) &= -\partial^{A\otimes B}(a \otimes b) = -((-1)^{\deg b}(\partial^A a) \otimes b + a \otimes (\partial^B b)) \\ &= (-1)^{\deg b}(\partial^{A^\circ} a) \otimes b + a \otimes (\partial^{B^\circ} b) = \partial^{A^\circ \otimes B^\circ}(a \otimes b).\end{aligned}$$

The proof of the second claim is similar. □

2.5 The Gray suspension is the suspension

In this section, we prove the key lemma that connects the Gray tensor product and the self-enrichment $\infty\mathsf{Algbrd} \simeq \mathsf{Algbrd}(\infty\mathsf{Algbrd})$. This will feed into Proposition 3.1.9 and ultimately become one of the main ingredients in the construction of the tensor product of categorical spectra. Another important byproduct is the pushout formula for $(\sigma X) \otimes \square^1$, see Corollary 2.5.3.

Lemma 2.5.1. *There are functors*

$$P, P^\circ : \infty\mathsf{Algbrd} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})$$

whose values $P(X), P^\circ(X)$ are pushout squares with the following specification (only the bottom arrows and the 2-cells are not predetermined):

$$\begin{tikzcd} \partial\square^1 \otimes X \arrow[r, "\partial\square^1\otimes(X\to *)"] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] & X \otimes \partial\square^1 \arrow[r] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] \\ \square^1 \otimes X \arrow[r] & \sigma X, & X \otimes \square^1 \arrow[r] & \sigma(X^\circ). \end{tikzcd} \tag{2.2}$$

Remark 2.5.2. The diagrams restrict to $X \in \infty\mathsf{Cat}, \infty\mathsf{Cat}^{\mathrm{str}}, \mathsf{Gaunt}$. For the strict categories, we could run the same proof with the strict tensor product, but by Remark 2.4.10 this coincides with the restriction of the lemma. The left pushout formula is [Cam23b, Lemma 3.8] (based on the strict ∞-category case [AM20, Cor. B.6.6]). The right pushout formula seems new and requires some care regarding the duality introduced, so we spell out the proof in great detail.

Proof. We first analyze P and P° assuming such functors exist. First, $P(\emptyset)$, $P^\circ(\emptyset)$ both must be the identity square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \partial\Box^1 \arrow[d] \\ \emptyset \arrow[r] & \partial\Box^1, \end{tikzcd}$$

so P, P° must lift to functors $\infty\mathsf{Algbrd} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$. These are necessarily colimit-preserving because the colimit in the codomain is computed componentwise (in each coslice category)[4]. Let $\mathsf{Fun}^{\mathrm{po}}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$ denote the (colimit-closed) full subcategory of the codomain consisting of pushout squares. Notice that, once we have functors $P^{(\circ)}|_\Box : \Box \to \mathsf{Fun}^{\mathrm{po}}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$ with the components as specified, their unique colimit-preserving extensions to $\mathsf{PSh}(\Box)$ automatically descend to $\infty\mathsf{Algbrd}$ and meet the specifications:

$$\begin{aligned} \mathsf{Fun}(\Box, \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}) &\simeq \mathsf{LFun}(\mathsf{PSh}(\Box), \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}) \\ &\hookleftarrow \mathsf{LFun}(\infty\mathsf{Algbrd}, \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}). \end{aligned}$$

They descend to $\infty\mathsf{Algbrd}$ because each component of the square does (which is a consequence of the existence of Gray tensor product), and the extended functor lands in pushout squares if the original functor does. We provide $P|_\Box$ and $P^\circ|_\Box$ in two steps: (1) construct the commutative squares valued in $\infty\mathsf{Gaunt}^{\mathrm{Ste}} \simeq \mathsf{adCh}^{\mathrm{Ste}}$ and (2) check that they are pushouts in $\infty\mathsf{Algbrd}$. Note that the (weak) tensor product of

[4]Recall that the colimit of $p : X \to Y_{y/}$ is (almost by definition) the colimit of the corresponding cone $\bar{p} : X^{\triangleleft} \to Y$.

objects in $\infty\mathsf{Gaunt}^{\mathrm{Ste}}$, as well as the suspension and the total dual, agrees with the strict notion.

(1) Since our definition of the Gray tensor product involves Steiner theory, so does the construction of the squares $P(\square^n), P^\circ(\square^n)$. We define the functors $\tilde{P}, \tilde{P}^\circ : \mathsf{adCh}^{\mathrm{Ste}} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{adCh}^{\mathrm{Ste}})$ with

$$\tilde{P}(A) = \begin{array}{ccc} \lambda(\partial\square^1) \otimes A & \longrightarrow & \lambda(\partial\square^1) \\ \downarrow & & \downarrow \\ \lambda\square^1 \otimes A & \longrightarrow & \sigma(A), \end{array} \qquad \tilde{P}^\circ(A) = \begin{array}{ccc} A \otimes \lambda(\partial\square^1) & \longrightarrow & \lambda(\partial\square^1) \\ \downarrow & & \downarrow \\ A \otimes \lambda\square^1 & \longrightarrow & \sigma(A^\circ). \end{array}$$

The two has identical commutative squares of underlying graded $\mathbb{Z}$-modules with the bottom map (under the notation $\lambda\square^1 = (e\mathbb{Z} \xrightarrow{\binom{-1}{1}} \bot\mathbb{Z} \oplus \top\mathbb{Z})$)

$$A_q \otimes \bot\mathbb{Z} \oplus A_q \otimes \top\mathbb{Z} \oplus A_{q-1} \otimes e\mathbb{Z} \xrightarrow{(0,0,1)} A_{q-1} \quad (q > 0),$$

$$A_0 \otimes \bot\mathbb{Z} \oplus A_0 \otimes \top\mathbb{Z} \xrightarrow{(\varepsilon,\varepsilon)} \bot\mathbb{Z} \oplus \bot\mathbb{Z} \quad (q = 0).$$

In both diagrams, it is straightforward to check that it defines a map of augmented directed complexes functorial in A, so they indeed define (strictly) commutative squares $\tilde{P}, \tilde{P}^\circ : \mathsf{adCh}^{\mathrm{Ste}} \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{adCh}^{\mathrm{Ste}})$. Through the equivalence $\mathsf{Gaunt}^{\mathrm{Ste}} \simeq \mathsf{adCh}^{\mathrm{Ste}}$ and restricting to $\square \cup \{\emptyset\}$, we obtain $P^{(\circ)} : \square \to \mathsf{Fun}(\Delta^1 \times \Delta^1, \mathsf{Gaunt}^{\mathrm{Ste}})_{P(\emptyset)/} \subset \mathsf{Fun}(\Delta^1 \times \Delta^1, \infty\mathsf{Algbrd})_{P(\emptyset)/}$ (and its colimit-extension to $\infty\mathsf{Algbrd}$).

(2) It remains to show that $P^{(\circ)}(\square^n)$ are pushout squares. We say $X \in \infty\mathsf{Algbrd}$ is good (resp.$\circ$-good) if $P(X)$ (resp. $P^\circ(X)$) is a pushout. The terminal category $\square^0$ is clearly ($\circ$-)good, so it suffices to show that if a strong Steiner ∞-category X is ($\circ$-)good, then so is $X \otimes \square^1$ (resp. $\square^1 \otimes X$). By assumption that $P^{(\circ)}(X)$

is a pushout (which is preserved by tensoring $\square^1$), we have the following factorizations of $P(X \otimes \square^1)$ and $P^\circ(\square^1 \otimes X)$:

$$\begin{tikzcd}
\partial\square^1 \otimes X \otimes \square^1 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & \partial\square^1 \otimes \square^1 \arrow[r] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] \\
\square^1 \otimes X \otimes \square^1 \arrow[r] & (\sigma X) \otimes \square^1 \arrow[r] & \sigma(X \otimes \square^1),
\end{tikzcd}$$

$$\begin{tikzcd}
\square^1 \otimes X \otimes \partial\square^1 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & \square^1 \otimes \partial\square^1 \arrow[r] \arrow[d, hook] & \partial\square^1 \arrow[d, hook] \\
\square^1 \otimes X \otimes \square^1 \arrow[r] & \square^1 \otimes \sigma(X^\circ) \arrow[r] & \sigma((\square^1 \otimes X)^\circ).
\end{tikzcd}$$

The composite rectangles are pushouts if and only if the right squares are. To ease the notation, we precompose the right diagram with the total dual so that X°, $(\square^1 \otimes X)^\circ$ are replaced with X, $(\square^1 \otimes X^\circ)^\circ \simeq (\square^1)^\circ \otimes X$. Here $(\square^1)^\circ$ is isomorphic to $\square^1$, but it indicates that if we use the labeling of the vertices compatible with the other entries of the diagram, the 1-morphism goes from 1 to 0. Factor the right squares into $(a) + (b)$ of the next diagrams by factoring the left inclusions as follows ($0, 1$ (resp. $\perp, \top$) are the source and the sink of $\square^1$ (resp. σX)):

$$\begin{aligned}
\partial\square^1 \otimes \square^1 &\hookrightarrow \Big((\sigma X \otimes \{0\}) \vee (\{\top\} \otimes \square^1)\Big) \sqcup \Big((\{\perp\} \otimes \square^1) \vee (\sigma X \otimes \{1\})\Big) \\
&\to (\sigma X) \otimes \square^1, \\
\square^1 \otimes \partial\square^1 &\hookrightarrow \Big((\square^1 \otimes \{\perp\}) \vee (\{1\} \otimes \sigma X)\Big) \sqcup \Big((\{0\} \otimes \sigma X) \otimes (\square^1 \otimes \{\top\})\Big) \\
&\to \square^1 \otimes (\sigma X).
\end{aligned}$$

(these wedge sums are both strict and weak by [Cam23a, Theorem 2.31]).

$$\begin{tikzcd}
& \partial\square^1\otimes\square^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & *\sqcup * \arrow[d] \\
\sigma X\sqcup\sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & \sigma X\vee\square^1\sqcup\square^1\vee\sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \sigma X\sqcup\sigma X \arrow[d] \\
\sigma(X\otimes\square^1) \arrow[r] & (\sigma X)\otimes\square^1 \arrow[r] & \sigma(X\otimes\square^1)
\end{tikzcd}$$

$$\begin{tikzcd}
& \square^1\otimes\partial\square^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & *\sqcup * \arrow[d] \\
\sigma X\sqcup\sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(c)"] & \sigma X\vee\square^1\sqcup\square^1\vee\sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \sigma X\sqcup\sigma X \arrow[d] \\
\sigma((\square^1)^\circ\otimes X) \arrow[r] & \square^1\otimes\sigma X \arrow[r] & \sigma((\square^1)^\circ\otimes X)
\end{tikzcd}$$

Since (a) is a pushout, it suffices to show that there exist pushout squares (c) such that the horizontal compositions of (b)+(c) are the identities. Heuristically, the horizontal maps of (c) are the maps from suspensions that pick up the "long" hom of the wedge sums and the "diagonal" hom of the tensor product. To construct these maps rigorously, one can write down the sections of (b) in adCh; there are obvious sections of graded abelian groups realizing the above heuristics (which are identities in degrees greater than 1) and one just need to check that those maps commute with differentials (which is not completely trivial only up to degree 1).

To show that the squares (c) are pushouts, it suffices to show that these are in fact pushouts in $\mathsf{PSh}(\Theta)$, i.e., for $\theta\in\Theta$, the squares $\mathrm{Map}(\theta,(c))$ are pushout in S (these squares are Set-valued, however). Separating the cases based on the map on vertices, it reduces to checking that $\mathrm{Hom}_{**}(\sigma\theta',\sigma(X\otimes\square^1))\to\mathrm{Hom}_{**}(\sigma\theta',(\sigma X)\otimes\square^1)$ is a bijection; since $\mathbb{G}$ is colimit-generating, we may assume $\theta'=C_n\in\mathbb{G}$. In this case, one can explicitly check (in Steiner complexes side) that any bipointed map $C_n\to(\sigma X)\otimes\square^1$ must factor through $\sigma(C_{n-1}\to X\otimes\square^1)$.

$\square$

By colimit-extending diagrams (c) in the same fashion as the first part of the proof (i.e., in the category under the value of $X = \emptyset$), we get the following pushout formula as a byproduct:

Corollary 2.5.3. *There are the following pushout squares in* $\infty\mathsf{Algbrd}$*, functorial in* $X \in \infty\mathsf{Algbrd}$*:*

$$\begin{tikzcd} \sigma X \sqcup \sigma X \arrow[r] \arrow[d] & \sigma X \vee \square^1 \sqcup \square^1 \vee \sigma X \arrow[d] \\ \sigma(X \otimes \square^1) \arrow[r] & (\sigma X) \otimes \square^1, \end{tikzcd} \qquad \begin{tikzcd} \sigma X \sqcup \sigma X \arrow[r] \arrow[d] & \sigma X \vee \square^1 \sqcup \square^1 \vee \sigma X \arrow[d] \\ \sigma((\square^1)^\circ \otimes X) \arrow[r] & \square^1 \otimes (\sigma X) \end{tikzcd}$$

where the top arrows pick up the hom category between the source and the sink objects of the wedge sum and the bottom arrows pick up the "diagonal" hom category.

For later use, we record a generalization of the lemma for higher suspensions and a consequence on iterated hom categories.

Corollary 2.5.4. *Let $k \geq 0$ be an integer. There is the following pushout square functorial in* $X \in \infty\mathsf{Algbrd}$.

$$\begin{tikzcd} \partial C_k \otimes X \arrow[r, two heads] \arrow[d] & \partial C_k \arrow[d] \\ C_k \otimes X \arrow[r, two heads] \arrow[ur, phantom, "\ulcorner", very near end] & \sigma^k X \end{tikzcd}$$

Proof. The case $k = 0$ is obvious and $k = 1$ is the first diagram of the lemma. We proceed by induction, so assume the conclusion for some $k \geq 1$. Plugging $X = \sigma Y = (\square^1 \otimes Y) \cup_{\partial\square^1 \otimes Y} \partial\square^1$ into the induction hypothesis, we see that $\sigma^{k+1} Y$ is the colimit of the following punctured cube diagram:

$$\begin{tikzcd}[row sep=small, column sep=small]
& \partial C_k \otimes \partial\square^1 \arrow[rr] \arrow[dd] & & \partial C_k \arrow[dd, dashed] \\
\partial C_k \otimes \partial\square^1 \otimes Y \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & \partial C_k \otimes \square^1 \otimes Y \arrow[ur] & \\
& C_k \otimes \partial\square^1 \arrow[rr, dashed] & & \sigma^{k+1} Y \\
C_k \otimes \partial\square^1 \otimes Y \arrow[ur] \arrow[rr] & & C_k \otimes \square^1 \otimes Y \arrow[ur, dashed] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The pushout of the span in the back face is $(C_k \sqcup C_k) \cup_{(\partial C_k \sqcup \partial C_k)} \partial C_k \simeq \partial C_{k+1}$, so we have the following pushout diagram:

$$\begin{tikzcd}
\Big((C_k \otimes \partial\square^1) \cup_{(\partial C_k \otimes \partial\square^1)} (\partial C_k \otimes \square^1)\Big) \otimes Y \arrow[r] \arrow[d] & \partial C_{k+1} \arrow[d] \\
C_k \otimes \square^1 \otimes Y \arrow[r] & \sigma^{k+1} Y
\end{tikzcd}$$

The top arrow factors through $\partial C_{k+1} \otimes Y$ by pushing out along $(\partial C_k \otimes \square^1 \twoheadrightarrow \partial C_k) \otimes Y$ (note that the spans in the front and the back face is, up to the factor of Y, only differs by the $\square^1$ in the entry $\partial C_k \otimes \square^1 \otimes Y$). Since $(C_k \otimes \square^1) \cup_{\partial C_k \otimes \square^1} \partial C_k \simeq C_{k+1}$, the pushout along the same map factors the bottom map as $C_k \otimes \square^1 \otimes Y \to C_{k+1} \otimes Y \to \sigma^{k+1} Y$. $\square$

Since σ preserves contractible colimits, for any $A \in \infty\mathsf{Algbrd}$, we have an adjunction $\infty\mathsf{Algbrd}_{A/} \underset{\omega}{\overset{\sigma}{\underset{\perp}{\rightleftarrows}}} \infty\mathsf{Algbrd}_{\sigma(A)/}$. By iteration we have

$$\sigma^k : \ \infty\mathsf{Algbrd} \underset{\mathrm{Hom}}{\overset{\sigma}{\underset{\perp}{\rightleftarrows}}} \infty\mathsf{Algbrd}_{\partial C_1/} \underset{\mathrm{Hom}}{\overset{\sigma}{\underset{\perp}{\rightleftarrows}}} \cdots \underset{\mathrm{Hom}}{\overset{\sigma}{\underset{\perp}{\rightleftarrows}}} \infty\mathsf{Algbrd}_{\partial C_k/} \ : \omega^k.$$

For a parallel pair of $(k-1)$-morphisms $(s_{k-1}, t_{k-1}) : \partial C_k \to X$, let $X(s_{k-1}, t_{k-1})$ denote the mapping category $\omega^k(X, s_{k-1}, t_{k-1})$. By adjunction, we have the following pullback square:

$$\begin{tikzcd}
\mathrm{Hom}(Y, X(s_{k-1}, t_{k-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(\sigma^k Y, X) \arrow[d] \\
* \arrow[r] & \mathrm{Hom}(\partial C_k, X)
\end{tikzcd} .$$

Composing with the pullback square of the corollary (after $\mathrm{Hom}(-, X)$), we obtain the following:

Corollary 2.5.5. *Let $X \in \infty\mathsf{Algbrd}$ and $(s_{k-1}, t_{k-1}) : \partial C_k \to X$ be parallel $(k-1)$-morphisms. Then we have the pullback square*

$$\begin{tikzcd}
X(s_{k-1}, t_{k-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[C_k, X]} \arrow[d] \\
* \arrow[r, "{(s_{k-1}, t_{k-1})}"] & {[\partial C_k, X].}
\end{tikzcd}$$

Chapter 3

Categorical spectra

In this chapter, we introduce our main object of study: *categorical spectra*. We will define the category CatSp of categorical spectra in a completely analogous way to that of spectra, i.e., the limit along the sequence of the *loop* endofunctors on $\infty\mathsf{Cat}_*$, instead of S_*:

$$\mathsf{CatSp} \coloneqq \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*).$$

Roughly speaking, Ω takes (X, x) to the endomorphism category $\mathsf{End}_X(x)$ of the basepoint. In Section 3.1, we start by studying the loop-suspension adjunction $\Sigma \dashv \Omega$ for pointed ∞-categories. This is the *reduced* version of $\sigma \dashv \mathrm{Hom}$ from the previous chapter, i.e., $\Sigma X = (\sigma X)/(\sigma *)$. Unlike in the classical algebraic topology, Σ is not equivalent to σ; the latter is simpler because of the absence of morphisms in one direction. Collapsing the basepoints breaks this feature and in particular, does not preserve gauntness in general. In other words, it is essentially an operation for a weak ∞-categories. As before, we first define the suspension using enriched category theory and connect it to the Gray tensor product, observing that the loop-hom adjunction is equivalent to the tensor-hom adjunction for $\vec{S}^1$ for the *Gray smah product*, where $\vec{S}^1$ is the *directed circle*, the category freely generated by a loop on a basepoint. Analogously to May's recognition principle, the delooping hypothesis states that the n-fold loop construction provides an equivalence between $\mathbb{E}_n$-monoidal ∞-categories and *n-connective* ∞-categories, i.e., those ∞-categories trivial up through $(n-1)$-th categorical level. This will be made precise in Section 3.2.

We will then define categorical spectra in Section 3.3. By a similar argument for infinite loop objects, we will show that it naturally fills the following pullback square:

$$\begin{tikzcd} \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S}) \arrow[r, hook, "\mathrm{B}^\infty"] \arrow[d, hook] & \mathsf{Sp} \arrow[d, hook] \\ \mathsf{CMon}(\infty\mathsf{Cat}) \arrow[r, hook, "\mathrm{B}^\infty"] & \mathsf{CatSp}. \end{tikzcd}$$

The left column can be thought of as the *connective part* of the right. We will see that the horizontal arrows have both left and right adjoints: the right adjoint takes the maximal Picard subgroupoid and the left adjoint inverts all the stable cells. These offer ways to extract information from categorical spectra in a classical form. Iteration of categorification often provides an example of categorical spectra. We try to give a unified description in Section 3.4. We end the chapter by studying finiteness properties of categorical spectra in Section 3.5. We can make analogous definitions to various finiteness properties of spectra. They are all equivalent for spectra, but at this point, we will only treat the formal implications between them.

This chapter largely overlaps [Ste21, Chapter 13]. We refer the reader there for some detail and different perspectives.

3.1 Loop and suspension

Let V be a presentable cartesian closed category (whose default monoidal structure is the cartesian one). We denote the category of pointed objects $* \xrightarrow{x} X$ (which we write (X, x), $x \in X$) by V_*. Notice $\mathsf{V}_* := \mathsf{V}_{*/} \simeq \mathsf{Alg}_{\mathbb{E}_0}(\mathsf{V}) \simeq \mathsf{S}_* \otimes \mathsf{V}$. We will omit the basepoint from the notation when there is no risk of confusion.

Definition 3.1.1. Notice the underlying groupoid functor

$$\mathrm{ob} : \mathsf{Algbrd}(\mathsf{V})_* \simeq \mathsf{Algbrd}(\mathsf{V}) \times_{\mathsf{S}} \mathsf{S}_* \to \mathsf{S}_*$$

is a cartesian fibration. The *algebroid delooping* functor is the inclusion of the fiber over the initial object $*$:

$$\mathrm{B}'_{\mathsf{V}} : \mathsf{Mon}(\mathsf{V}) \simeq \mathsf{Alg}(\mathsf{V}) = \mathsf{Algbrd}_*(\mathsf{V}) \hookrightarrow \mathsf{Algbrd}(\mathsf{V})_*,$$

and the *delooping* functor is its univalent completion $\mathrm{B}_{\mathsf{V}} : \mathsf{Mon}(\mathsf{V}) \xrightarrow{\mathrm{B}'_{\mathsf{V}}} \mathsf{Algbrd}(\mathsf{V})_* \xrightarrow{L^{\mathrm{uni}}} \mathsf{V}\text{-}\mathsf{Cat}_*$. The *loop* functor is their right adjoints (i.e., the cartesian transport of ob along the basepoint map $* \to X$) $\Omega_{\mathsf{V}} : \mathsf{Algbrd}(\mathsf{V})_* \to \mathsf{Alg}(\mathsf{V})$ or its restriction to $\mathsf{V}\text{-}\mathsf{Cat}_*$. We will omit V in the subscript when it is not confusing.

Definition 3.1.2. By abuse of notation, we continue to denote by Ω_{V} the functor that returns the underlying pointed (resp. unpointed) object of the loop:

$$(\mathsf{V}\text{-}\mathsf{Cat}_* \hookrightarrow)\mathsf{Algbrd}(\mathsf{V})_* \xrightarrow{\Omega_{\mathsf{V}}} \mathsf{Alg}(\mathsf{V}) \to \mathsf{Alg}_{\mathbb{E}_0}(\mathsf{V}) \simeq \mathsf{V}_*(\to \mathsf{V}).$$

These functors have left adjoints, called the *suspension*, given by the (partial) composite of

$$\Big(\mathsf{V} \xrightarrow{(-)_+}\Big)\mathsf{V}_* \xrightarrow{\mathrm{Free}_{\mathbb{E}_1/\mathbb{E}_0}} \mathsf{Alg}(\mathsf{V}) \xrightarrow{\mathrm{B}'} \mathsf{Algbrd}(\mathsf{V})_*\Big(\xrightarrow{L^{\mathrm{uni}}} \mathsf{V}\text{-}\mathsf{Cat}_*\Big);$$

we will use the notation $\Sigma' = \mathrm{B}' \circ \mathrm{Free}$, $\Sigma = \mathrm{B} \circ \mathrm{Free}$ and Σ'_+, Σ_+ for their unpointed versions.

Remark 3.1.3. $\Omega_{\mathsf{V}} : \mathsf{V}\text{-}\mathsf{Cat}_* \to \mathsf{Alg}(\mathsf{V})_*$ depends functorially on the monoidal category V via change-of-enrichment [Ste21, Remark 13.1.7]. The loop of a pointed V-algebroid (X, x) is the object of endomorphisms $\mathsf{End}_X(x)$ endowed with the monoid structure by composition (so the basepoint is id_x). In particular, the loop functor $\Omega_{\mathsf{V}} : \mathsf{Algbrd}(\mathsf{V})_* \to \mathsf{Alg}(\mathsf{V})_*$ inverts fully faithful morphisms so it factors through the univalent completion $\mathsf{Algbrd}(\mathsf{V})_* \to \mathsf{V}\text{-}\mathsf{Cat}_*$. It follows that B is fully faithful: the unit $\mathrm{id} \to \Omega\mathrm{B} \simeq \Omega L^{\mathrm{uni}}\mathrm{B}' \simeq \Omega\mathrm{B}'$ is an equivalence because B' is fully faithful.

The following lemma relates the reduced and unreduced suspension:

Lemma 3.1.4. *The composite* $\mathsf{V}_* \xrightarrow{\sigma} \mathsf{Algbrd}(\mathsf{V})_{\sigma(*)/} \xrightarrow{\mathrm{cofib}} \mathsf{Algbrd}(\mathsf{V})_*\Big(\xrightarrow{L^{\mathrm{uni}}} \mathsf{V}\text{-}\mathsf{Cat}_*\Big)$, *where the second arrow is the cobase change along $\sigma(*) \to *$, is equivalent to the* *reduced suspension* Σ' *(resp.* Σ*).*

Proof. Observe that the right adjoint is given by

$$\mathsf{Algbrd}(\mathsf{V})_* \to \mathsf{Algbrd}(\mathsf{V})_{\sigma(*)/} \xrightarrow{\mathrm{Hom}_{(-)}(*,*)} \mathsf{V}_*$$

which is equivalent to Ω_{V}. $\square$

In other words, we have a cofiber sequence $\sigma(*) \xrightarrow{\sigma(x)} \sigma(X) \to \Sigma(X)$. Now we restrict attention to the most interesting case when $\mathsf{V} = \infty\mathsf{Cat}$ or $\infty\mathsf{Algbrd}$. We let Σ (resp. Σ') denote the *endofunctor* $\infty\mathsf{Cat}_* \xrightarrow{\Sigma_{\infty\mathsf{Cat}}} (\infty\mathsf{Cat})\text{-}\mathsf{Cat}_* \simeq \infty\mathsf{Cat}_*$ (resp. its algebroid version) and similarly for $\Omega : \infty\mathsf{Algbrd}_* \to \mathsf{Mon}(\infty\mathsf{Algbrd})$.

Remark 3.1.5. The loop functor on $\infty\mathsf{Algbrd}_*$ restricts to $\infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ and $\mathsf{S}_* = 0\mathsf{Cat}_* \to \mathsf{S}_*$. The latter agrees with the classical loop that takes the groupoid of *automorphisms* of the basepoint. In contrast, the suspension functor $\Sigma : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ does not restrict to the classical suspension functor $\mathsf{S}_* \to \mathsf{S}_*$. Instead, the classical suspension is the delooping of the *group completion* of the free monoid.

Now we combine the above lemma and Lemma 2.5.1 to deduce the formula relating the Gray smash product (which we define now) and the suspension. This will later be the first level of the half-central structure of $\vec{S}^1$.

Definition 3.1.6. Since $\mathsf{S} \to \mathsf{S}_*$ is an idempotent algebra (with the smash product) in Pr^{L}, the base-change promotes to a (symmetric) monoidal localization $\mathsf{S}_* \otimes (-) : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ and in particular a functor $(-)_* : \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$, sending a presentably monoidal category to the category of its pointed objects with "smashed" monoidal product. Applying this construction to $\infty\mathsf{Algbrd}$ and $\infty\mathsf{Cat}$ with the Gray tensor product, we get the *Gray smash product* on the pointed objects, which we will denote by $\oslash$. Explicitly, we have $X \oslash Y \simeq \mathrm{cofib}(X \otimes * \sqcup_{*\otimes *} * \otimes Y \to X \otimes Y)$.

Definition 3.1.7. Let $\vec{S}^1 = \mathrm{B}\mathbb{N} = \Sigma_+(*)$ (note that $\mathbb{N}$ is the free $\mathbb{E}_1$ algebra on a point). This is the *directed circle*. Also let $\vec{S}^n := (\vec{S}^1)^{\oslash n}$ be the *directed n-sphere*.

Remark 3.1.8. The directed circle $\vec{S}^1$ is a gaunt 1-category and it is the free category on the graph $\Delta^1/\partial\Delta^1$. In contrast, $\vec{S}^n$ is not strict for $n > 1$ because by the next proposition $\vec{S}^n = \Sigma^n S^0 = \mathrm{B}^n\,\mathrm{Free}_{\mathbb{E}_n}(*)$ and $\mathrm{Free}_{\mathbb{E}_n}(*) \simeq \bigsqcup_{k\geq 0}(\mathrm{Conf}_k(\mathbb{R}^n)_{h\Sigma_k}$ is not a 0-truncated homotopy type.

Proposition 3.1.9. *There exist canonical identifications functorial in X:*

$$\Sigma X \simeq \vec{S}^1 \oslash X, \quad X \oslash \vec{S}^1 \simeq \Sigma X^\circ.$$

In particular, this provides a natural isomorphism

$$\tau_X : \vec{S}^1 \oslash X \simeq \Sigma X \simeq X^\circ \oslash \vec{S}^1.$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & * \otimes * \arrow[dl] \arrow[dd] & & * \otimes \partial\square^1 \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] & & \partial\square^1 \arrow[dl] \arrow[dd] \\
X \otimes * \arrow[dd] \arrow[dr, phantom, "(1)"] & & X \otimes \partial\square^1 \arrow[ll] \arrow[rr] \arrow[dd] \arrow[dr, phantom, "(2)"] & & \partial\square^1 \arrow[dd] \arrow[dr, phantom, "(3)"] & \\
 & * \otimes \vec{S}^1 \arrow[dl] & & * \otimes \square^1 \arrow[ll] \arrow[rr] \arrow[dl] & & S* \arrow[dl] \\
X \otimes \vec{S}^1 & & X \otimes \square^1 \arrow[ll] \arrow[rr] & & SX^\circ &
\end{tikzcd}$$

All the arrows from back to front are induced by the basepoint $* \to X$. The front-right and the back-right faces are the pushout diagrams of Lemma 2.5.1, whereas front-left and back-left faces are pushouts by $\vec{S}^1 = \square^1/\partial\square^1$. Since all the faces in front and back are pushouts, we have the induced equivalences on the total cofibers

$$\vec{S}^1 \oslash X \simeq \mathrm{cofib}(1) \leftarrow \mathrm{cofib}(2) \rightarrow \mathrm{cofib}(3) \simeq \Sigma X^\circ.$$

The other equivalence $\Sigma X \simeq X \oslash \vec{S}^1$ follows from the other pushout diagram of Lemma 2.5.1. $\square$

Remark 3.1.10. The first equivalence the proposition is equivalent to that the suspension $\infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ is a morphism in $\mathsf{RMod}_{\infty\mathsf{Cat}_*^{\oslash}}(\mathsf{Pr}^{\mathsf{L}})$. The similar statement for the usual smash product instead of the Gray smash product fails fundamentally: if X is an n-category for $n \geq 1$, then $\vec{S}^1 \wedge X$ is still an n-category, while ΣX is an $(n+1)$-category.

Remark 3.1.11. The loop $\Omega : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ preserves filtered colimits, or equivalently, the suspension Σ preserves compact objects. It follows from the fact that the filtered colimit commutes with taking the hom object in the enriched categories but also from the pushout formula and Remark 2.4.9.

3.2 Connectivity of ∞-categories and the delooping hypothesis

In the literature, the delooping hypothesis is often phrased as "an $(n+k)$-category with a single $0, 1, \cdots, (k-1)$-cells is equivalent to a $\mathbb{E}_k$-monoidal n-category;" this is literally true in the flagged/algebroid setting, but such "k-connective $(n+k)$-algebroids" are usually not univalent. The goal of this section is to give a precise treatment of the notion of connectivity in the univalent setting.

Let $\mathrm{Map}(X, Y)$ denote $\mathrm{Hom}(X, Y)^{\leq 0} \simeq [X, Y]^{\leq 0}$ (so the cartesian and Gray enrichment are the same after 0-truncation). Recall that a functor $f : X \to Y$ in $\infty\mathsf{Cat}$ is essentially surjective if the map $f^{\leq 0} : X^{\leq 0} \to Y^{\leq 0}$ is an effective epimorphism, i.e., induces a surjection on π_0.

Definition 3.2.1. [Lur09c] Let $n \geq -1$ be an integer. We define the *n-connectivity* of a map $f : X \to Y$ inductively as follows:

- By convention, any map f is (-1)-connective.
- If $n \geq 0$, a map f is n-connective if f is essentially surjective and for any pair of object $x, x' \in X$, the induced map $X(x, x') \to Y(fx, fx')$ is $(n-1)$-connective.

We say f is *∞-connective* if f is n-connective for any n.

The notion of connectivity is closely related to that of surjectivity:

Definition 3.2.2 (cf. [BS10])**.** Let $n \geq 0$. A morphism $f : X \to Y$ is *n-surjective* if the natural map $\mathrm{Map}(C_n, X) \to \mathrm{Map}(\partial C_n, X) \times_{\mathrm{Map}(\partial C_n, Y)} \mathrm{Map}(C_n, Y)$ is an effective epimorphism of groupoids (note the convention $\partial C_0 = \emptyset$).

Remark 3.2.3. A map is 0-surjective iff 0-connective iff essentially surjective. When $n \geq 1$, a map $f : X \to Y$ is n-surjective if and only if for any parallel pair of $(n-1)$-morphisms $(s_{n-1}, t_{n-1}) : \partial C_n \to X$, the inducded map $X(s_{n-1}, t_{n-1}) \to Y(fs_{n-1}, ft_{n-1})$ is essentially surjective. In fact, taking the 0-truncation of Corollary 2.5.5, the latter statement is equivalent to that the induced map

$$\mathrm{fib}(\mathrm{Map}(C_n, X) \to \mathrm{Map}(\partial C_n, X)) \to \mathrm{fib}(\mathrm{Map}(C_n, Y) \to \mathrm{Map}(\partial C_n, Y))$$

is an effective epimorphism for any choice of basepoint of $\mathrm{Map}(\partial C_n, X)$. This is equivalent for $\mathrm{Map}(C_n, X) \to \mathrm{Map}(\partial C_n, X) \times_{\mathrm{Map}(\partial C_n, Y)} \mathrm{Map}(C_n, Y)$ to be effective epi because it can be checked fiberwise over $\mathrm{Map}(\partial C_n, X)$.

Example 3.2.4. A morphism f is n-surjective if and only if the n-truncation $f^{\leq n} : X^{\leq n} \to Y^{\leq n}$ is n-surjective. In particular, the inclusion $X^{\leq n} \hookrightarrow X$ is k-surjective for $k \leq n$.

Proposition 3.2.5. *The following are equivalent:*

(1) $f : X \to Y$ *is n-connective.*

(2) $f : X \to Y$ *is k-surjective for* $0 \leq k \leq n$.

(3) $f^{\leq n} : X^{\leq n} \to Y^{\leq n}$ *is n-connective.*

Proof. Unpacking the induction, the n-connectivity of f is equivalent to the essential surjectivity in addition to that for any $0 \leq k \leq n-1$ and any parallel pair of k-morphisms $(s_k, t_k) : \partial C_{k+1} \to X$, the map $X(s_k, t_k) \to Y(fs_k, ft_k)$ is $(n-k-1)$-connective. By downward induction k, the latter condition can be weakened to that

$X(s_k, t_k) \to Y(fs_k, ft_k)$ is essentially surjective, so the equivalence of (1) and (2) follows from the remark. Since the k-surjectivity of a map only depends on the k-truncation, (1) is equivalent to (3). $\square$

Remark 3.2.6. The class of n-connecetd maps are closed under pullbacks, filtered colimits, and disjoint unions. Classically, in an $(\infty, 1)$-topos, there is a inductive characterization of n-connected maps [Lur09b, Proposition 6.5.1.18]. The same statement or an obvious analog using the directed diagonals (see Chapter 5 for the directed pullbacks) seems to fail and the author does not know if there is a version of it.

Definition 3.2.7. Let $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A pointed ∞-category X is said to be *n-connective* if the structure map $* \to X$ is $(n-1)$-connective.

Remark 3.2.8. The n-connectivity of a pointed category does not depend on the choice of the basepoint, so we may also define the notion of connecitivity for unpointed ∞-categories together with the convention that a ∞-category is 0-connective when it is nonempty.

Proposition 3.2.9. *Let (X, x) be a pointed ∞-category and $n \geq 0$. The following are equivalent:*

(1) *X is n-connective, i.e., the structure map $* \xrightarrow{x} X$ is $(n-1)$-connective.*

(2) *$\Omega^k X$ is connected (i.e., $(\Omega^k X)^{\leq 0}$ is connected) for $0 \leq k \leq n-1$.*

(3) *The counit map of $\mathrm{B}^n \Omega^n X \to X$ is an equivalence.*

(4) *X belongs to the essential image of the functor $\mathrm{B}^n : \mathsf{Mon}_{\mathbb{E}_n}(\infty\mathsf{Cat}) \to \infty\mathsf{Cat}_*$.*

(5) *There exists an ∞-algebroid $\tilde{X}$ whose underlying $(n-1)$-algebroid is trivial and a univalent completion $\tilde{X} \xrightarrow{\simeq} X$.*

Proof. When $n = 0$, all the conditions are empty, so we may assume $n \geq 1$ and proceed by induction. When $n = 1$, the equivalence of the first two conditions is clear from the definition. Since $\mathrm{B}\Omega X \to X$ is always fully faithful, it is an equivalence if and only if X is connected. This is also equivalent to (4) because B is fully faithful by Remark 3.1.3. For (5), we may take $\tilde{X} = \mathrm{B}'\Omega X$. Now assume $n \geq 1$. From the case $n = 1$, we know that all the conditions imply $X = \mathrm{B}Y$ for some $Y \in \mathsf{Mon}(\infty\mathsf{Cat})$. By induction, it suffices to observe that each condition is equivalent to that for Y with n replaced by $n-1$. This is clear for (1) and (2). For (3), observe that the counit map $\mathrm{B}^n\Omega^n\mathrm{B}Y \to \mathrm{B}Y$ equivalent to $\mathrm{B}(\mathrm{B}^{n-1}\Omega^{n-1}Y \to Y)$ and that $\mathrm{B} : \infty\mathsf{Cat}_* \xrightarrow{\mathrm{B}} \mathsf{Mon}(\infty\mathsf{Cat}) \xrightarrow{\text{forget}} \infty\mathsf{Cat}_*$ is conservative. The equivalence of (3) and (4) is easier directly because B^n is fully faithful with the right adjoint Ω^n (the existence of the functor B^n follows from that B is product preserving and Dunn additivity; also see the argument for Proposition 3.3.7). For (5), if either of $\tilde{Y} \xrightarrow{\sim} Y$ or $\tilde{X} \xrightarrow{\sim} X$ is given, we may take $\tilde{X} = \mathrm{B}'\tilde{Y}$ and $\tilde{Y} = \Omega\tilde{X}$ for the other. □

Proposition 3.2.10. *If (X, x) is a n-connective pointed $(n-1)$-category. Then X is in fact a groupoid.*

Proof. $\Omega^{n-1}X$ is a $\mathbb{E}_{n-1}$-groupoid if X is an $(n-1)$-category, so $\Omega^n X$ is a grouplike $\mathbb{E}_n$-groupoid and $\mathrm{B}^n\Omega^n X$ is a groupoid. If n is connective, the counit map $\mathrm{B}^n\Omega^n X \to X$ is an equivalence, so X is also a groupoid. □

Corollary 3.2.11. *A pointed ∞-category $(X, x) \in \mathsf{Cat}_*$ is ∞-connected if and only if X is terminal.*

Proof. The structure map $* \to X$ is $(n-1)$-connective if and only if $* \to X^{\leq n-1}$ is, so $X^{\leq n-1}$ is terminal by the proposition. Applying Proposition 3.5.1, we have $X = \mathrm{colim}_{n\in\mathbb{N}} X^{\leq n}$, so $X \simeq *$. □

Remark 3.2.12. A natural question is the obvious analog of the Freudenthal suspension theorem, i.e., if the unit map $X \to \Omega\Sigma X$ is $2n$-connective if X is n-connective. The author does not know the proof or a counterexample. Notice a potential simplification due to the lack of group completion: $\Omega\Sigma X \simeq \Omega\mathrm{B}\,\mathrm{Free}_{\mathbb{E}_1/\mathbb{E}_0} X \simeq \bigvee_{k\geq 0} X^{\wedge k}$. In particular, the cofiber of the unit map is $\bigvee_{k\geq 2} X^{\wedge k}$. Various classical splitting results, as discussed in [DH21], are also worth exploring in our context.

3.3 Categorical spectra

Our goal here is to give the definitions and some examples of categorical spectra, roughly summarizing [Ste21, Chapter 13].

Definition 3.3.1. The category of *categorical spectra* is the limit of right adjoints

$$\mathsf{CatSp} \coloneqq \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Cat}_* \xrightarrow{\Omega} \infty\mathsf{Cat}_*)$$

in Pr^{R} (or $\widehat{\mathsf{Cat}}$). Its object, a categorical spectrum $X \in \mathsf{CatSp}$, is a sequence (X_n, x_n) of pointed ∞-categories equipped with identifications

$$f_n : (X_n, x_n) \xrightarrow{\sim} (\mathsf{End}_{X_{n+1}}(x_{n+1}), \mathrm{id}_{x_{n+1}}).$$

We will often suppress x_n and f_n in the notation. We write $\Omega^{\infty-n}$ for the projection to the n-th component, so $\Omega^{\infty-n}X = X_n$ for $X = (X_n)$. We let $\Sigma^{\infty-n}$ denote the left adjoint of $\Omega^{\infty-n}$. One can also define an obvious variant of flagged categorical spectra: $\mathsf{CatSp}^{\mathrm{f}} \coloneqq \lim(\cdots \xrightarrow{\Omega} \infty\mathsf{Algbrd}_* \xrightarrow{\Omega} \infty\mathsf{Algbrd}_*)$.

Example 3.3.2. Recall that the following diagram commutes:

$$\begin{tikzcd} \mathsf{S}_* \arrow[r, "\Omega"] \arrow[d, hook] & \mathsf{S}_* \arrow[d, hook] \\ \infty\mathsf{Cat}_* \arrow[r, "\Omega"] & \infty\mathsf{Cat}_* \end{tikzcd}$$

Therefore the vertical maps induce a fully faithful functor $\mathsf{Sp} \hookrightarrow \mathsf{CatSp}$. We will see in Remark 3.4.10 that this inclusion admits both left and right adjoints, and in the next chapter we will see that the sphere spectrum $\mathbb{S}$ is an idempotent algebra and Sp is precisely the category of $\mathbb{S}$-modules.

Remark 3.3.3. The endomorphism Ω on $\infty\mathsf{Cat}_*$ commutes with the limit diagram, so it induces an endomorphism $\Omega : \mathsf{CatSp} \to \mathsf{CatSp}$. This is actually an automorphism by the coinitiality of $(\mathbb{N}_{\geq 1})^{\mathrm{op}} \hookrightarrow \mathbb{N}^{\mathrm{op}}$ and its adjoint inverse Σ is induced by taking the colimit of $\Sigma : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ in Pr^{L}. This is an example of the general fact that one can universally invert an endomorphism on a category by passing to the sequential limit along the endomorphism. More precisely, the left adjoint to the inclusion $\mathsf{Fun}(\mathrm{B}\mathbb{Z}, \widehat{\mathsf{Cat}}) \hookrightarrow \mathsf{Fun}(\mathrm{B}\mathbb{N}, \widehat{\mathsf{Cat}})$ (resp. $\mathsf{Fun}(\mathrm{B}\mathbb{Z}, \mathsf{Pr}^{\mathsf{L}}) \hookrightarrow \mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathsf{Pr}^{\mathsf{L}})$) sends the pair $(\infty\mathsf{Cat}_*, \Omega)$ (resp. $(\infty\mathsf{Cat}_*, \Sigma)$) of a category and an endomorphism to (CatSp, Ω) (resp. (CatSp, Σ)). Moreover, it sends the morphism $(\mathsf{S}_*, \Omega) \to (\infty\mathsf{Cat}_*, \Omega)$ of pairs to $(\mathsf{Sp}, \Omega) \to (\mathsf{CatSp}, \Omega)$.

Warning 3.3.4. We will occasionally use the standard notation $X[n] \coloneqq \Sigma^n X$ for $n \in \mathbb{Z}$, we warn that this is the *left* action by $\vec{S}^n \in \infty\mathsf{Cat}_*^{\otimes}$, so for instance, in the later chapters, $X \otimes (Y[n])$ and $(X \otimes Y)[n]$ are not necessarily equivalent.

Remark 3.3.5. The reader may wonder what happens if we choose to work with a fixed finite category level and make the same definition. It turns out that we gain nothing new: we end up with the category Sp. We may regard $\Omega : n\mathsf{Cat}_* \to (n-1)\mathsf{Cat}_*$ as an endomorphism of $n\mathsf{Cat}_*$ by composing with the inclusion. Now note that Ω^n factors through S_*:

$$\begin{tikzcd}
\cdots \arrow[r, "\Omega^n"] & \mathsf{S}_* \arrow[r, "\Omega^n"] \arrow[d, hook] & \mathsf{S}_* \arrow[d, hook] \\
\cdots \arrow[r, "\Omega^n"] \arrow[ur, dashed, "\Omega^n"] & n\mathsf{Cat}_* \arrow[r, "\Omega^n"] \arrow[ur, dashed, "\Omega^n"] & n\mathsf{Cat}_*
\end{tikzcd}$$

Taking the limit horizontally, we observe that the dashed arrows induce the inverse to the inclusion by the coinitiality of $(\mathbb{N}_{\geq n})^{\mathrm{op}} \hookrightarrow \mathbb{N}^{\mathrm{op}}$, so if we universally invert $\Omega : n\mathsf{Cat}_* \to n\mathsf{Cat}_*$, we recover the category Sp. If we allow the category levels to vary appropriately over the limiting diagram, there is a meaningful notion of categorical levels of categorical spectra (see Example 3.4.9).

Remark 3.3.6. The definition is not new and was made or indicated independently by many authors, including Horiuchi [Hor18], the author [Mas21] (informally, inspired by the works of Connes and Consani around $\mathbb{F}_1$ [CC20]), Stefanich [Ste21] (who attributes the notion to Constantin Teleman), Johnson-Freyd [Joh23] (who attributes the notion to Claudia Scheimbauer).

Recall that the category of spectra is also the stabilization of the category of connective spectra:

$$\mathsf{Sp} \coloneqq \lim(\cdots \xrightarrow{\Omega} \mathsf{Sp}^{\mathrm{cn}} \xrightarrow{\Omega} \mathsf{Sp}^{\mathrm{cn}}).$$

Here $\mathsf{Sp}^{\mathrm{cn}} \simeq \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S})$ is the category of infinite loop groupoids. We now give an analogous description of CatSp as the "stabilization" of $\infty\mathsf{SMCat}$, following [Ste21, §13.4]. Note that $\Omega : \infty\mathsf{Cat}_* \to \mathsf{Mon}(\infty\mathsf{Cat})$ preserves the cartesian product because it is a right adjoint. In particular, it induces

$$\Omega : \mathsf{Mon}_{\mathbb{E}_n}(\infty\mathsf{Cat}) \to \mathsf{Mon}_{\mathbb{E}_n}(\mathsf{Mon}(\infty\mathsf{Cat})) \simeq \mathsf{Mon}_{\mathbb{E}_{n+1}}(\infty\mathsf{Cat})$$

for $0 \leq n \leq \infty$ that are compatible along forgetful functors for different values of n. Now consider the following diagram in $\mathsf{Pr}^{\mathsf{R}}_{\omega}$:

$$\begin{tikzcd}
\vdots \arrow[d, equal] & \cdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathsf{CatSp} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \mathsf{Mon}(\infty\mathsf{Cat}) \arrow[r, "\Omega"] \arrow[d] & \mathsf{Mon}_{\mathbb{E}_2}(\infty\mathsf{Cat}) \arrow[r, "\Omega"] \arrow[d] & \mathsf{Mon}_{\mathbb{E}_3}(\infty\mathsf{Cat}) \arrow[d] \\
\mathsf{CatSp} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] \arrow[d, equal] & \mathsf{Mon}(\infty\mathsf{Cat}) \arrow[r, "\Omega"] \arrow[d] & \mathsf{Mon}_{\mathbb{E}_2}(\infty\mathsf{Cat}) \arrow[d] \\
\mathsf{CatSp} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] \arrow[d, equal] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] \arrow[d, equal] & \mathsf{Mon}(\infty\mathsf{Cat}) \arrow[d] \\
\mathsf{CatSp} \arrow[r] & \cdots \arrow[r] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] & \infty\mathsf{Cat}_*.
\end{tikzcd}$$

Taking the vertical limit (along the forgetful functors), one obtains the following:

Proposition 3.3.7. *The diagram defining* CatSp *lifts to the following diagram in* $\mathsf{Pr}^{\mathsf{R}}_{\omega}$*:*

$$\mathsf{CatSp} \xrightarrow{\sim} \lim(\cdots \xrightarrow{\Omega} \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{\Omega} \mathsf{CMon}(\infty\mathsf{Cat})).$$

In terms of left adjoints, we have

$$\begin{tikzcd}
\infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0}"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0}"] & \cdots \arrow[r] & \mathsf{CatSp} \arrow[d, equal] \\
\mathsf{CMon}(\infty\mathsf{Cat}) \arrow[r, hook, "\mathrm{B}"] & \mathsf{CMon}(\infty\mathsf{Cat}) \arrow[r, hook, "\mathrm{B}"] & \cdots \arrow[r, hook] & \mathsf{CatSp}.
\end{tikzcd}$$

The bottom row lies in $\mathsf{Mod}_{\mathsf{CMon}}(\mathsf{Pr}^{\mathsf{L}})$, *so* CatSp *is semiadditive, i.e., it has a zero object* 0 *and biproducts* $\oplus$. *Also, we have an equivalence* $\Sigma^\infty \simeq \mathrm{B}^\infty \circ \mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0}$.

We continue to denote the projection to the n-th component by $\Omega^{\infty-n}$. The left adjoint of $\Omega^{\infty-n}$ will be denoted by $\mathrm{B}^{\infty-n} : \mathsf{CMon}(\infty\mathsf{Cat}) \to \mathsf{CatSp}$. By cofinality, we may 1-periodically extend the above $\mathbb{N}^{(\mathrm{op})}$-indexed diagrams to $\mathbb{Z}$-indexed diagrams without chaning the (co)limits; so we use the notations $\mathrm{B}^{\infty-n}$, etc. for $n \in \mathbb{Z}$.

Definition 3.3.8. We say that a categorical spectrum is *connective* if it lies in the essential image of B^∞. We let $\mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ denote the category of connective categorical spectra.

Warning 3.3.9. Even though we use the term "connective," we do not yet know an appropriate definition of an analog of a t-structure on a stable $(\infty, 1)$-category.

Example 3.3.10. For $\mathcal{C} \in \mathsf{CMon}(\infty\mathsf{Cat})$, the categorical spectra $\mathrm{B}^{\infty-n}\mathcal{C}$ is a sequence of ∞-categories

$$(\mathrm{B}^{\infty-n}\mathcal{C})_k = \begin{cases} \Omega^{n-k}\mathcal{C} & (k \leq n) \\ \mathcal{C} & (k = n) \\ \mathrm{B}^{k-n}\mathcal{C} & (k \geq n) \end{cases}$$

For the next definition, recall a free $\mathbb{E}_\infty$-algebra on $X \in \infty\mathsf{Cat}$ (resp. $\infty\mathsf{Cat}_*$) is given by the symmetric algebra $\coprod_{n\geq 0} X^{\times n}/\Sigma_n$ (resp. $\bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n$).

Definition 3.3.11. A categorical spectrum of the form $\Sigma^\infty X$ for $X \in \omega\mathsf{Cat}_*$ (in particular, $\Sigma^\infty_+ X = \mathrm{B}^\infty \mathrm{Free}_{\mathbb{E}_\infty} X$ for $X \in \infty\mathsf{Cat}$) is called a *suspension spectrum*. We let

$$\mathbb{F} \coloneqq \Sigma^\infty_+(*) = \Sigma^\infty S^0 = \mathrm{B}^\infty \mathsf{Fin}^{\simeq}$$

be the suspension spectrum on a point. We will call it the *finite set spectrum*, the *unit*, or the *directed sphere spectrum*.

Remark 3.3.12. Since Ω preserves filtered colimits by Remark 3.1.11, CatSp is compactly generated and $\Omega^\infty : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ preserves filtered colimits. It follows that $\mathbb{F}$ is a compact object of CatSp.

We close this section with some remarks on the duality involutions.

Remark 3.3.13. ([Ste21, Definition 13.2.12]) Just as a spectrum can be seen as a "CW complex with possibly negative dimensional cells," one can regard a categorical spectrum as a "∞-category with negative dimensional cells." To make this precise, let (X_n) be a categorical spectrum and consider the map $\mathrm{Map}(C_m, X_n) \xrightarrow{\sim} \mathrm{Map}(C_m, \Omega X_{n+1}) \xrightarrow{\sim} \mathrm{Map}(\Sigma C_m, X_{n+1}) \to \mathrm{Map}(C_{m+1}, X_{n+1})$ induced by $C_{m+1} = \sigma C_m \twoheadrightarrow \Sigma C_m$. We define the pro-representable globular presheaf $\mathrm{cell}_\bullet(-) : \mathsf{CatSp} \to \mathsf{PSh}(\mathbb{G})$ as the colimit of the following diagram of monomorphisms:

$$\mathrm{Map}(C_\bullet, X_0) \rightarrowtail \mathrm{Map}(C_{\bullet+1}, X_1) \rightarrowtail \mathrm{Map}(C_{\bullet+2}, X_2) \rightarrowtail \cdots .$$

This can be moreover extended to a presheaf over $\mathbb{G}_{(-\infty,\infty)} = \mathrm{colim}(\mathbb{G} \xrightarrow{\sigma} \mathbb{G} \xrightarrow{\sigma} \cdots)$ and given a structure of compositions.

Remark 3.3.14. Let $\tau : \mathbb{Z}_{\geq 1} \to \mathbb{Z}/2$ be a function and let $[n]$ (for $n \leq 0$) be the shift operator, i.e., $\tau[n]$ is the function $\tau[n](k) = \tau(k-n)$. Then the following diagram commutes (to construct the commuting homotopy, observe that two compositions are equal on Θ):

$$\begin{array}{ccc} \infty\mathsf{Cat} & \xrightarrow{\sigma} & \infty\mathsf{Cat} \\ \downarrow{\scriptstyle D_{\tau[-1]}} & & \downarrow{\scriptstyle D_\tau} \\ \infty\mathsf{Cat} & \xrightarrow{\sigma} & \infty\mathsf{Cat}. \end{array}$$

Because $\Sigma X = \mathrm{cof}(\sigma(*) \to \sigma X)$, it follows that the following diagrams also commute (the right one is by taking the right adjoints; the truncated information of $\tau(1) = \tau[-1](0)$ acts by reversal of the monoidal structure, which is invisible on the underlying category):

$$\begin{tikzcd}
\infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "D_{\tau[-1]}"] & \infty\mathsf{Cat}_* \arrow[d, "D_\tau"] \\
\infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_*,
\end{tikzcd}
\qquad
\begin{tikzcd}
\infty\mathsf{Cat}_* \arrow[r, "\Omega"] \arrow[d, "D_\tau"] & \infty\mathsf{Cat}_* \arrow[d, "D_{\tau[-1]}"] \\
\infty\mathsf{Cat}_* \arrow[r, "\Omega"] & \infty\mathsf{Cat}_*
\end{tikzcd}.$$

Consequently, the category CatSp admits an action of

$$\prod_{\mathbb{Z}} \mathbb{Z}/2 \simeq \lim(\cdots \to \prod_{\mathbb{Z}_{\geq 1}} \mathbb{Z}/2 \xrightarrow{[-1]} \prod_{\mathbb{Z}_{\geq 1}} \mathbb{Z}/2).$$

We continue to denote by D_τ the involution corresponding to $\tau : \mathbb{Z} \to \mathbb{Z}/2$. By definition, this is the involution satisfying $D_{\tau[n]} \circ \Omega^{\infty-n} = \Omega^{\infty-n} \circ D_\tau$ and $D_\tau \circ \Sigma^{\infty-n} = \Sigma^{\infty-n} \circ D_{\tau[n]}$ i.e., it sends (X_n) to $(D_{\tau[n]} X_n)$ in components. One may think of τ as the indicator function of the dimensions of stable cells that get flipped.

Definition 3.3.15. The *total dual* (resp. *odd dual*, *even dual*) is the duality involution D_τ corresponding to when τ is the indicator function of $\mathbb{Z}$ (resp. odd numbers, even numbers). We continue to denote the total dual by $D = (-)^\circ$ and the odd and even dual by $(-)^{\mathrm{op}}$, $(-)^{\mathrm{co}}$, respectively.

Explicitly, the total dual of $X = (X_n)$ is given by $X^\circ = (X_n^\circ)$, and the odd dual X^{op} is given by the "alternating" sequence $(X_0^{\mathrm{op}}, X_1^{\mathrm{co}}, X_2^{\mathrm{op}}, \cdots)$ and similarly for the even dual.

3.4 Levelwise properties of categorical spectra

Many properties and structures of categorical spectra are defined levelwise. We will list some examples and formulate typical ways to universally impose such properties.

Definition 3.4.1. Let $P = \{P(n)\}_n$ be a sequence of properties of symmetric monoidal (flagged) ∞-categories such that if X_{n+1} satisfies $P(n+1)$, then ΩX_{n+1}

satisfies $P(n)$. We say a categorical spectrum $X = (X_n)_n$ satisifes P (or is a P-categorical spectrum) if X_n satisfies $P(n)$. We let the full subcategory of the P-categorical spectrum by

$$\mathsf{CatSp}^P \coloneqq \lim_n(\cdots \to \infty\mathsf{SMCat}^{P(n)} \xrightarrow{\Omega} \infty\mathsf{SMCat}^{P(n+1)} \to \cdots) \subset \mathsf{CatSp}.$$

If the property only depends on the underlying pointed ∞-categories, one can also define

$$\mathsf{CatSp}^P \coloneqq \lim_n(\cdots \to \infty\mathsf{Cat}_*^{P(n)} \xrightarrow{\Omega} \infty\mathsf{Cat}_*^{P(n+1)} \to \cdots) \subset \mathsf{CatSp}.$$

More generally, if we have a sequence of categories $\{i_n : \mathcal{C}_n \to \infty\mathsf{SMCat}\}$ equipped with the lift of the loop, i.e., functors $\tilde{\Omega} : \mathcal{C}_{n+1} \to \mathcal{C}_n$ with the following commutative square $(*_n)$, one can define $\mathsf{CatSp}^{\mathcal{C}}$ as the limit $\lim_n \mathcal{C}_n$ in $\widehat{\mathsf{Cat}}$:

$$\begin{tikzcd}
\mathsf{CatSp}^{\mathcal{C}} \arrow[r] \arrow[d, "i"] & \cdots \arrow[r] & \mathcal{C}_{n+1} \arrow[r, "\tilde{\Omega}"] \arrow[d, "i_{n+1}"] \arrow[dr, phantom, "(*_n)"] & \mathcal{C}_n \arrow[d, "i_n"] \\
\mathsf{CatSp} \arrow[r] & \cdots \arrow[r] & \infty\mathsf{SMCat} \arrow[r, "\Omega"] & \infty\mathsf{SMCat},
\end{tikzcd}$$

and similarly using $\infty\mathsf{Cat}_*$ or $\infty\mathsf{Cat}^{\mathrm{f}}$ instead of $\infty\mathsf{SMCat}$.

Remark 3.4.2. An index-shift of a levelwise property is again a levelwise property. The corresponding full subcategory is some shift $\mathsf{CatSp}^P[n]$ of CatSp^P.

Remark 3.4.3. If i_n and $\tilde{\Omega}$ admit left adjoints $L_n \dashv i_n$ and $\tilde{\mathrm{B}} \dashv \tilde{\Omega}$, then the above diagram lives in Pr^{R}, so i also admits a left adjoint L. If $\mathcal{C}_n$ is a localization, i.e., i_n is fully faithful with a left adjoint L_n, the existence of $\tilde{\mathrm{B}}$ is automatic. In fact, we take $\tilde{\mathrm{B}} = L_{n+1} \circ B \circ i_n$ with the unit and counit

$$\tilde{\mathrm{B}}\tilde{\Omega} = L_{n+1}\mathrm{B}i_n\tilde{\Omega} \simeq L_{n+1}\mathrm{B}\Omega i_{n+1} \to L_{n+1}i_{n+1} \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_{n+1}},$$

$$\mathrm{id}_{\mathcal{C}_n} \xleftarrow{\sim} L_n i_n \to L_n\Omega\mathrm{B}i_n \xrightarrow{\sim} L_n\Omega i_{n+1}L_{n+1}\mathrm{B}i_n \simeq L_n i_n\tilde{\Omega}L_{n+1}\mathrm{B}i_n \xrightarrow{\sim} \tilde{\Omega}\tilde{\mathrm{B}},$$

and moreover $\tilde{\mathrm{B}}$ is fully faithful. In other words, for a levelwise property $P = \{P(n)\}$, we may levelwise perform P-delooping B^P by localizing the connective delooping B, and the P-*envelope* $\mathrm{B}^{\infty,P}$ of $\mathrm{B}^{\infty}X$ is (as a categorical spectrum) the colimit of these deloopings: $\mathrm{B}^{\infty,P}X \simeq \mathrm{colim}_n\, \mathrm{B}^{\infty-n}(\mathrm{B}^P)^n X$. An analogous claim except the fully faithfulness of $\tilde{B}$ is true when $\infty\mathsf{SMCat}$, B is replaced by $\infty\mathsf{Cat}_*$ and Σ. Note that the left adjoint is in general not given levelwise by L_n.

Let Adj be the free adjunction 2-category and $l : C_1 \to \mathsf{Adj}$ be the inclusion of the universal left adjoint 1-morphism (see Section 6.1 for more detail); note that this is an epimorphism. Recall the following consequece of [Ste21, Proposition 5.3.17]:

Proposition 3.4.4. *Let I be a category, $\mathcal{D}$ be a 2-category (e.g. $\mathcal{D} = \widehat{\mathsf{Cat}}$) and $X : I \to \mathsf{Fun}(\mathsf{Adj}, \mathcal{D})$ be a diagram whose restriction $Y : I \to \mathsf{Fun}(\mathsf{Adj}, \mathcal{D}) \xrightarrow{l^*} \mathsf{Fun}(C_1, \mathcal{D})$ admits a limit $Y^{\triangleleft} : I^{\triangleleft} \to \mathsf{Fun}(C_1, \mathcal{D})$ (which is computed pointwise). Then there is a limit diagram $X^{\triangleleft} : I^{\triangleleft} \to \mathsf{Fun}(\mathsf{Adj}, \mathcal{D})$ making the followig diagram commute:*

$$\begin{tikzcd} I \arrow[r, "X"] \arrow[d, hook] & \mathsf{Fun}(\mathsf{Adj}, \mathcal{D}) \arrow[d, hook, "l^*"] \\ I^{\triangleleft} \arrow[r, "Y^{\triangleleft}"'] \arrow[ur, "X^{\triangleleft}"] & \mathsf{Fun}(C_1, \mathcal{D}) \end{tikzcd}$$

Remark 3.4.5. By replacing $\mathcal{D}$ by $\mathcal{D}^{\mathrm{op}}$ or $\mathcal{D}^{\mathrm{co}}$, similar consequences with "limit" replaced by "colimit" or "left adjoint" replaced by "right adjoint" hold.

Corollary 3.4.6. *Suppose the square $(*_n)$ is vertically right (resp. left) adjointable for every n with $i_n \dashv R_n$ (resp. $L_n \dashv i_n$). Then the morphism i admits a right adjoint R levelwise given by R_n (resp. left adjoint L levelwise given by L_n) , i.e., the natural map $\tilde{\Omega}^{\infty-n} \circ R \to R_n \circ \Omega^{\infty-n}$ is an equivalence.*

Remark 3.4.7. For the assumption of the lemma (for R_n), it suffices to check the horizontal left adjointability of $(*)_n$, i.e., that $\tilde{\mathrm{B}} \dashv \tilde{\Omega}$ exists and commute with i_n, and the existence of the right adjoints R_n.

Example 3.4.8 (Univalence)**.** Being a categorical spectrum is a levelwise property of a flagged categorical spectrum; if $X \in \infty\mathsf{Algbrd}_*$ is an ∞-category, then ΩX is also an ∞-category. There is a univalent completion functor $L^{\mathrm{uni}} : \mathsf{CatSp}^{\mathrm{f}} \to \mathsf{CatSp}$

which levelwise is $L^{\mathrm{uni}} : \infty\mathsf{Cat}^{\mathrm{f}} \to \infty\mathsf{Cat}$ because $(*_n)$ is vertically left adjointable by Remark 3.1.3. As we have seen, the infinite delooping spectrum of a symmetric monoidal ∞-category $\mathcal{C}$ is the univalent completion of a more naive algebroid delooping: $\mathrm{B}^\infty\mathcal{C} = L^{\mathrm{uni}}\mathrm{B}'^\infty\mathcal{C}$.

Example 3.4.9 (category levels, [Ste21, Definition 13.2.17, Proposition 13.2.20])**.** Let $-\infty \le k \le \infty$. We say X is a *k-categorical spectrum* if X_n is $\max\{n+k, 0\}$-category. We let $k\mathsf{CatSp} \subset \mathsf{CatSp}$ be the full subcategory of k-categorical spectra. On one extreme, we have $\infty\mathsf{CatSp} = \mathsf{CatSp}$, while on the other we have $-\infty\mathsf{CatSp} = \mathsf{Sp}$. For finite k, $k\mathsf{CatSp}$ are shifts of one another. Many interesting examples of categorical spectra live in $0\mathsf{CatSp}$ or $1\mathsf{CatSp}$. It defines the categorical hierarchy that interpolates between spectra and categorical spectra:

$$\mathsf{Sp} = -\infty\mathsf{CatSp} \subset \cdots \subset (-1)\mathsf{CatSp} \subset 0\mathsf{CatSp} \subset 1\mathsf{CatSp} \subset \cdots \subset \infty\mathsf{CatSp} = \mathsf{CatSp}.$$

Recall that the inclusion $n\mathsf{Cat} \hookrightarrow \infty\mathsf{Cat}$ admits both left and right adjoints, denoted by ${}^{\le n}(-)$ and $(-)^{\le n}$. In particular, $k\mathsf{CatSp} \subset \mathsf{CatSp}$ is closed under limits and colimits, so the inclusion has both left and right adjoints, again denoted by ${}^{\le k}(-)$ and $(-)^{\le k}$. In the following diagram, the square $(*_n)$ is horizontally left adjointable, so in this range, the right adjoint is given levelwise, i.e., $\Omega^{\infty-n}X^{\le k} \simeq (\Omega^{\infty-n}X)^{\le(n+k)}$ if $n \ge -k$.

$$\begin{tikzcd}
k\mathsf{CatSp} \arrow[r] \arrow[d, hook, "i"] & \cdots \arrow[r] & (n+k+1)\mathsf{Cat}_* \arrow[r, "\Omega"] \arrow[d, hook, "i_{n+1}"] \arrow[dr, phantom, "(*_n)"] & (n+k)\mathsf{Cat}_* \arrow[d, hook, "i_n"] \\
\mathsf{CatSp} \arrow[r] & \cdots \arrow[r] & \infty\mathsf{Cat}_* \arrow[r, "\Omega"] & \infty\mathsf{Cat}_*.
\end{tikzcd}$$

However, the left adjoint $\Sigma : \mathsf{S}_* \to \mathsf{S}_*$ to Ω is not the restriction of $\Sigma : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$, so we must take the monoidal structure into account, as the next remark shows.

Remark 3.4.10. The condition that X_n is a groupoid for $n \le -k$ in fact forces that

X_n is *grouplike* for $n < -k$, so $k\mathsf{CatSp} = \lim_n \mathcal{C}_n$ for $\mathcal{C}_n \subset \infty\mathsf{SMCat}$ defined by

$$\mathcal{C}_n = \begin{cases} \mathsf{CMon}((n+k)\mathsf{Cat}) & (n \geq -k) \\ \mathsf{CMon}^{\mathrm{gp}}(\mathsf{S}) & (n < -k). \end{cases}$$

The left (and right) adjoints of $n\mathsf{Cat} \hookrightarrow \infty\mathsf{Cat}$ preserve products, so they induce left and right adjoints of $\mathsf{CMon}(n\mathsf{Cat}) \hookrightarrow \mathsf{CMon}(\infty\mathsf{Cat})$. Also note that $\mathsf{CMon}^{\mathrm{gp}}(\mathsf{S}) \subset \mathsf{CMon}(\mathsf{S})$ admits left adjoint $(-)^{\mathrm{gp}}$ given by the group completion and the right adjoint $(-)^{\times}$ that takes the maximal Picard subgroupoid, i.e. the components of invertible objects. Composing these, $i_n : \mathcal{C}_n \hookrightarrow \infty\mathsf{Cat}$ admits left and right adjoints. The square $(*_n)$ is vertically right adjointable, so in particular, the underlying spectrum functor $(-)^{\leq -\infty} : \mathsf{CatSp} \to \mathsf{Sp}$ is levelwise given by $(X_n) \mapsto ((X_n)^{\leq 0, \times})$. Notice, however, that $(*_n)$ is still not vertically left adjointable [Ste21, Remark 13.4.21].

Example 3.4.11 (Connectivity)**.** Let $-\infty \leq k \leq \infty$ and consider the property $P(n)$ of being $(n+k)$-connective and say X is k*-connective* when it is satisfied. Denote the corresponding full subcategory by $\mathsf{CatSp}^{k\text{-cn}}$. When k is finite, it is the essential image of the fully faithful functor $\mathrm{B}^{\infty+k} : \infty\mathsf{SMCat} \to \mathsf{CatSp}$ whose right adjoint is $\Omega^{\infty+k}$. The k-connective cover of X is $X^{k\text{-cn}} = \mathrm{B}^{\infty+k}\Omega^{\infty+k}X$, i.e., the terminal k-connective categorical spectra with a map to X.

Example 3.4.12 (adjoints and duals)**.** In Chapter 6, we will discuss the levelwise property of being n-adjointful. The cobordism hypothesis gives a geometric description of the adjointful envelope for some categorical spectra.

Remark 3.4.13. Many categorical spectra in nature arise as the P-envelope construction for some property P. We expect that the examples include the following, but we will not pursue the details here (because requires extra work and some are not yet in the literature).

(1) (n-semiadditivity) For finite n and an (∞, n)-category $\mathcal{C}$, [Lur09c, §3.2] outlines the definition of the n-category $\mathrm{Fam}_n^k(\mathcal{C})$. Roughly speaking, it is the n-category

of spans of k-truncated π-finite groupoids coherently decorated by cells of $\mathcal{C}$. There is a morphism $\mathcal{C} \to \mathrm{Fam}_n^k(\mathcal{C})$ exhibiting $\mathrm{Fam}_n^k(\mathcal{C})$ as the universal k-semiadditive n-category under $\mathcal{C}$, as proven by [Har20] in the $n = 1$ case and the general case (including the definition of k-semiadditive n-category) is announced by Scheimbauer–Walde [Sch23]. If $\mathcal{C}$ itself is k-semiadditive, it gives the *finite path integral* functor[1] $\int : \mathrm{Fam}_n^k(\mathcal{C}) \to \mathcal{C}$. If $X = (X_n)$ is a categorical spectrum, almost by definition $\mathrm{Fam}^k(X) := \{\mathrm{Fam}_n^k(X_n)\}_{n\geq 0}$ forms a categorical spectrum. One can define the k-semiadditivity of categorical spectra so that $\mathrm{Fam}^k(X)$ is the k-semiadditive envelope of X.

(2) (n-stability) For a ring spectrum R, the spectrally enriched symmetric monoidal category $\mathrm{B}R$ admits a stable envelope Perf_R and stable presentable envelope LMod_R. In [Ste20], Stefanich defined the categorical spectra $\underline{R} = \{n\mathsf{Mod}_R\}$ and the notion of n-presentable (stable) n-categories. We expect that the construction $\mathrm{B}^\infty R \mapsto \underline{R}$ can be realized as the 0-presentable stable envelope and similarly for a finitary version of it.

(3) (separable closure: [Joh23]) With an appropriate finitary version of $\underline{R}$ as above, for a ring of characteristic 0, Johnson-Freyd and Reutter defined a notion of higher categorical separable closure (either characterized by having a trivial etale homotopy type or by Nullstellensatz-like condition). For complex numbers, it constructs the categorical spectrum of super-vector space, super-algebra, and so on. This is likely another example of an envelope construction with an appropriate property of algebraic closedness.

Combining these envelopes with the right adjoint $\mathbb{G}_m := (-)^{\leq -\infty} : \mathsf{CatSp} \to \mathsf{Sp}$, we can extract a spectrum containing an interesting new information. For instance,

[1] The importance of this functor is explained in [Fre+09]. $\mathrm{Fam}_n^k(\mathcal{C})$ classifies classical field theories, and the composition with $\int$ gives the *quantization*. An important example is the Dijkgraaf-Witten theory.

$\mathbb{G}_m(\underline{R})$ gives an infinite sequence of nontrivial deloopings extending the classically well-known $R^\times, \mathrm{Pic}(R), \mathrm{Br}(R)$, i.e., the units, the Picard space and the Brauer space of a spectrum. A part of the characterizing properties of the separable closure of $\underline{\mathbb{C}}$ is that $\mathbb{G}_m(\underline{\mathbb{C}}^{\mathrm{sep}}) = I_{\mathbb{C}^\times}$, the Brown-Comenetz dualizing spectrum of $\mathbb{C}^\times$. This recovers the Freed–Hopkins' proposal that $I_{\mathbb{C}^\times}$ is the universal target of a "physical" invertible TQFT.

3.5 Finiteness properties of categorical spectra

For a spectrum $X = (X_n)$, a fundamental observation is that X is the colimit $\mathrm{colim}_n\, \Sigma^{\infty-n} X_n$. The formula remains valid for categorical spectra for the same formal reason:

Proposition 3.5.1. *Let $\{\mathcal{C}_n\} \in \mathsf{Fun}(\mathbb{N}^\triangleright, \mathsf{Pr}^{\mathsf{L}}_\omega)$ be a colimit diargram of compactly generated categories and let $L_n \dashv R_n$ denote the structure morphisms $\mathcal{C}_n \to \mathcal{C} := \mathcal{C}_\infty$. Then there is a colimit diagram $L_0R_0 \to L_1R_1 \to \cdots \to \mathrm{id}_{\mathcal{C}}$ in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$ induced by the counit maps.*

Proof. Let $\tilde{\mathcal{C}} \to \mathbb{N}^\triangleright$ be the cartesian and cocartesian fibration classifying the diagram $\{\mathcal{C}_n\}$. Because $\{\mathcal{C} \xrightarrow{R_n} \mathcal{C}_n\}$ is a limit cone, $\mathcal{C}$ is equivalent to the categoary of cartesian sections on $\mathbb{N}$ [Lur09b, Prop. 3.3.3.1] via Cartesian transport: $\mathcal{C} \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{cart}}_{/\mathbb{N}^\triangleright}(\mathbb{N}^\triangleright, \tilde{\mathcal{C}}) \xrightarrow{\sim} \mathsf{Fun}^{\mathrm{cart}}_{/\mathbb{N}}(\mathbb{N}, \tilde{\mathcal{C}}|_{\mathbb{N}})$. Composing with the cocartesian transport $\tilde{\mathcal{C}} \to \mathcal{C}$ (i.e., the left adjoint of the inclusion), we get the diagram in question $\mathcal{C} \to \mathsf{Fun}(\mathbb{N}^\triangleright, \tilde{\mathcal{C}}) \to \mathsf{Fun}(\mathbb{N}^\triangleright, \mathcal{C})$. Let $f : \mathrm{colim}(L_0R_0 \to L_1R_1 \to \cdots) \to \mathrm{id}_{\mathcal{C}}$ be the comparison map. The above equivalence also implies $R_n : \mathcal{C} \to \mathcal{C}_n$ are jointly conservative, so to show that f is an equivalence, it suffices to check $R_n f$ is an equivalence for each $n \in \mathbb{N}$. This follows from that R_m preserves sequential colimits and that for each $m \geq n$, the map $R_n\varepsilon_m : R_nL_mR_m \to R_n$ induced by the counit map is equivalent to $R_{n,m}R_m\varepsilon_m \simeq \mathrm{id}_{R_n}$. $\square$

Corollary 3.5.2. *Let $X = (X_n)$ be a categorical spectrum. Then there are canonical equivalences* $\operatorname{colim}_n \Sigma^{\infty-n} X_n \xrightarrow{\sim} \operatorname{colim}_n \mathrm{B}^{\infty-n} X_n \xrightarrow{\sim} X$.

We will use this corollary to deduce that any categorical spectrum is a filtered colimit of finite categorical spectra. We must first define the notion of finite categorical spectra. Recall that for a spectrum $X \in \mathsf{Sp}$, the following conditions are equivalent:

(1) X is *finite*, i.e., $X \simeq \Sigma^{\infty-n} Y$ for some natural number n and a finite pointed CW complex Y.

(2) X is *perfect*, i.e., X belongs to the smallest stable subcategory which contains $\mathbb{S}$ and is closed under retracts.

(3) X is *compact*, i.e., $\mathrm{Map}_{\mathsf{Sp}}(X, -) : \mathsf{Sp} \to \mathsf{S}$ preserves filtered colimits.

(4) X is *dualizable*, i.e., the functor $X \otimes (-) : \mathsf{Sp} \to \mathsf{Sp}$ admits left or right (equivalently, both) adjoints.

Ideally, we wish to modify each definition for categorical spectra and prove that they are all equivalent. For now, we only work out the formal part of it. To define perfectness of categorical spectra, we must understand the notion of *stability* first. This is still a work in progress (see the introduction to Chapter 5). What is clear is that the perfect categorical spectra should not be closed under 1-categorically finite colimits; instead, they must be replaced by some lax analogs.

Definition 3.5.3. A (pointed) ∞-category is *finite* if it belongs to the smallest subcategory $\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)} \subset \infty\mathsf{Cat}_{(*)}$ that contains the (pointed) cells $\mathbb{G} = \{C_{n(,+)}\}_{n \geq 0}$ and closed under finite colimits. A categorical spectrum is *finite* if it is of the form $\Sigma^{\infty-n} X$ for some integer n and a finite pointed ∞-category X. We write $\mathsf{CatSp}^{\mathrm{fin}} \subset \mathsf{CatSp}$ for the full subcategory of finite categorical spectra.

Remark 3.5.4. Finite ∞-categories are compact and the inclusion $\infty\mathsf{Cat}^{\mathrm{fin}} \to \infty\mathsf{Cat}$ preserves finite colimits. By [Lur09b, Proposition 5.3.5.11, Example 5.3.6.8], the left Kan extension $\mathrm{Ind}(\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)}) \to \infty\mathsf{Cat}_{(*)}$ is fully faithful and colimit preserving. Since the cells generate $\infty\mathsf{Cat}$ under colimits, we have $\mathrm{Ind}(\infty\mathsf{Cat}^{\mathrm{fin}}_{(*)}) \xrightarrow{\sim} \infty\mathsf{Cat}_{(*)}$. In particular, every (pointed) ∞-category is canonically a filtered colimit of finite ones.

Example 3.5.5. Any finite torsion-free complex is a finite ∞-category by [Cam23a, Theorem B]. In particular, any strong Steiner ∞-category corresponding to a finite-dimensional strong Steiner complex is finite; examples include the objects of Θ, lax cubes, and orientals. The author does not know if any finite computad is a finite ∞-category.

Corollary 3.5.6. *Any categorical spectrum is a filtered colimit of finite spectra. More precisely, the inclusion* $\mathsf{CatSp}^{\mathrm{fin}} \subset \mathsf{CatSp}$ *induces an equivalence* $\mathrm{Ind}(\mathsf{CatSp}^{\mathrm{fin}}) \xrightarrow{\sim} \mathsf{CatSp}$.

Proof. Since $\Omega^{\infty-n} : \mathsf{CatSp} \to \infty\mathsf{Cat}_*$ preserves filtered colimits, finite categorical spectra are compact. By [Lur09b, Proposition 5.3.5.11], the left Kan extension $\mathrm{Ind}(\mathsf{CatSp}^{\mathrm{fin}}) \to \mathsf{CatSp}$ is fully faithful. To show that it is an equivalence, we must check that the smallest full subcategory of CatSp containing $\mathsf{CatSp}^{\mathrm{fin}}$ and closed under filtered colimits is CatSp itself, which follows from Corollary 3.5.2 and Remark 3.5.4. $\square$

Corollary 3.5.7. *A categorical spectrum X is compact if and only if it is a retract of a finite categorical spectrum.*

Remark 3.5.8. In the classical case of spectra, we can remove the retraction from the statement. The standard argument uses homology theory and Hurewicz's theorem: homology theory precisely tells us the recipe to approximate a space by cells, and stably it is conservative enough that the recipe can completely recover the spectrum in question. We do not know if any compact categorical spectrum is a finite suspension categorical spectrum (also note that the analogous statement for equivariant spectra is known to be false). In any case, it is desirable to have an algebraic invariant that extracts the data of cells required to build an ∞-category or a categorical spectrum.

We briefly discuss the dualizability of a categorical spectrum. Recall the following definition:

Definition 3.5.9. Let $\mathcal{C}$ be a monoidal category. An object $X \in \mathcal{C}$ is *left- (resp. right-)dualizable* if the functor $X \otimes (-) : \mathcal{C} \to \mathcal{C}$ admits a left (resp. right) adjoint.

The dualizability depends on the monoidal structure constructed in the next chapter. However, note the following general fact in a closed monoidal category that a dualizable object gets as much compactness property as the unit object:

Proposition 3.5.10. *Suppose we are given a closed monoidal structure $\otimes$ on CatSp whose unit is $\mathbb{F}$. Then any left- or right-dualizable object is compact.*

Proof. Suppose X has a right dual X^R. By assumption, the tensor product admits an internal hom: $\mathrm{Map}(Z, [X, Y]) \simeq \mathrm{Map}(X \otimes Z, Y) \simeq \mathrm{Map}(Z, X^R \otimes Y)$, so, in general, we have $[X, Y] \simeq X^R \otimes Y$ (similarly, $[\![X, Y]\!] \simeq Y \otimes X^L$ for the left dual X^L and the right internal hom $[\![-, -]\!]$). Plugging $Z = \mathbb{F}$, we see that $\mathrm{Map}(X, Y) \simeq \mathrm{Map}(\mathbb{F}, X^R \otimes Y)$ and since $\mathbb{F}$ is compact, this functor is colimit-preserving in Y. $\square$

The assumption is clearly satisfied once the monoidal structure is constructed. In particular, any dualizable categorical spectrum is compact. We do not know if all finite categorical spectra are dualizable, but we will give some examples of dualizable categorical spectra in Chapter 5.

Chapter 4

Tensor product of categorical spectra

The main goal of this chapter is to define the tensor product of categorical spectra through its universal property. We have already explained the strategy in detail in the introduction. Here we motivate our approach slightly differently, following the history of the corresponding problem for spectra.

The first *homotopy* category of spectra hSp with the "smash product" symmetric monoidal structure was defined in [Boa65] and later a more handcrafted approach in [Ada95] got popular. However, for higher algebra, it had serious deficiencies: the homotopy category has bad formal properties (e.g. it does not have most limits and colimits), fails to encode homotopically nuanced algebraic structures, and does not work well in families.

The first batch of successful attempts in defining the symmetric monoidal structure remembering all homotopical data came around model-categorically (e.g. [Elm+07], [Man+01]). The trick was a reversed microcosm principle: if we could define a symmetric tensor product, we would have the symmetry of the unit, so instead of defining spectra as merely $\mathbb{N}$-indexed families of spaces, we build a model that by design takes symmetries of the spheres into account. The minimalistic choice is to consider the symmetric group action on the spheres encoding the Koszul sign rule, i.e., we define spectra as a $\mathsf{Fin}^{\simeq} := \bigsqcup_{n\geq 0} \mathrm{B}\Sigma_n$-indexed family of spaces. This leads to the definition of symmetric spectra. They were good enough for many purposes, but model categories were too rigid to behave well in families, and the choice of a model was arbitrary, contrary to the canonicity of the stable homotopy category.

The truly universal object was in between—the $(\infty, 1)$-category Sp of spectra (Boardman's original definition was close to the modern one, except that the language was missing back then). After thoroughly developing $(\infty, 1)$-category theory, Lurie characterized the symmetric monoidal ($(\infty, 1)$-)category of spectra as the unit of the symmetric monoidal category $\mathsf{Pr}^{\mathsf{L}}_{\mathrm{st}}$ of presentable stable categories ([Lur17, §4.8]). Note that once we pass to the $(\infty, 1)$-category land, sequential spectra work perfectly; being natively enriched over homotopy types and not sets, the suspension functors carry the automorphisms equivalent to that of the spheres.

The lesson is that we should take the symmetries of the spheres and suspension functors into account. Coming back to our problem, we have already seen that the suspension $\Sigma : \infty\mathsf{Cat}_* \to \infty\mathsf{Cat}_*$ is naturally equivalent to $\vec{S}^1 \oslash (-)$ and $D(-) \oslash \vec{S}^1$. One can think of this as the *twisted* symmetry of the suspension functor, where the twist comes from the total dual D. Classically, the sphere commutes with other CW complexes by *Koszul sign rule*, but since we do not have negatives and instead directions of the cells, we must express the sign rule *externally* by switching the directions of the cells. This allows us to expect the formula

$$\begin{aligned}\Sigma^{\infty - m} X \otimes \Sigma^{\infty - n} Y &= \vec{S}^{-m} \otimes \Sigma^{\infty} X \otimes \vec{S}^{-n} \otimes \Sigma^{\infty} Y \\ &= \vec{S}^{-(m+n)} \otimes \Sigma^{\infty} D^n X \otimes \Sigma^{\infty} Y \\ &= \Sigma^{\infty - (m+n)}((D^n X) \otimes Y)\end{aligned}$$

for the tensor product of suspension spectra. Based on $\infty\mathsf{Cat}_*^{\oslash}$, we can hope at most an $\mathbb{E}_1$-monoidal structure, or some $*$-algebra strucutre on CatSp. However, even to prove the associativity (and its higher coherence), we must be able to move suspensions around *canonically*; this information is packaged conveniently in the *half-central structure* on $\vec{S}^1$, which we will establish in Section 4.1. In fact, we will show the existence of a *unique* half-central structure on $\vec{S}^1$, which deserves to be called the *categorical Koszul sign rule*.

The rest of the work to define the tensor product is in Section 4.2 and is rather formal, modifying the previously available techniques to the $\mathbb{E}_1$-setting. This will moreover prove the monoidal universal property $\mathsf{CatSp}^{\otimes} = \infty\mathsf{Cat}_*^{\otimes}[\vec{S}^{-1}]$. Note that although CatSp is formally obtained by inverting the endomorphism $\vec{S}^1 \otimes (-)$, it does not immediately imply that this procedure inverts $\vec{S}^1$ monoidally; this problem already exists in the commutative setting (see the references in the introduction of the section). We close the chapter by Section 4.3, establishing some basic results on the tensor product.

4.1 Half-central structure of $\vec{S}^1$

This section contains perhaps the most important technical ingredient of this thesis: the half-central structure on the directed circle $\vec{S}^1 = \mathrm{B}\mathbb{N}$. This is the higher categorical incarnation of the Koszul sign rule in the usual homotopy theory. We will define the half-center of $\infty\mathsf{Cat}_*$ and prove that the directed circle $\vec{S}^1 = \mathrm{B}\mathbb{N}$ admits a unique half-central structure (Theorem 4.1.10). This will be the key technical input for the construction of the tensor product of categorical spectra in section 4.2[1].

4.1.1 Half-center

We begin by recalling the notion of the center of a monoidal category. A good reference for this and the next subsection is [BFN10]. In the following definition, the

[1] When inverting a set of elements S of a (noncommutative) monoid M, one only requires a condition on S weaker than $S \subset Z(M)$, called the Ore condition, to have good control over $S^{-1}M$. For the definition of monoidal structure, it might be possible to define a categorified Ore condition instead. However, it will become a routine to commute the directed spheres with other objects, so it is independently useful to know that such a maneuver is completely canonical and harmless.

classical case is when A is a monoidal $(1,1)$-category, i.e., when $\mathsf{V} = (1,1)\mathsf{Cat}$ with cartesian monoidal structure. The example that we will specialize later is $A = \infty\mathsf{Cat}_*^{\oslash}$ and $\mathsf{V} = \mathsf{Pr}_\omega^{\mathsf{L},\otimes}$.

Definition 4.1.1. Let A be an $\mathbb{E}_1$-algebra object of $\mathsf{V} \in \mathsf{CAlg}(\mathsf{Pr}^\mathsf{L})$. The *center* $\mathfrak{Z}_\mathsf{V}(A)$ of A (in V) is the object $\mathsf{End}_{\mathsf{BMod}_A(\mathsf{V})}(A)$. A *central structure* on $s \in A$ (i.e. $s : 1_\mathsf{V} \to A$) is a lift of s along the forgetful functor[2] $\mathfrak{Z}_\mathsf{V}(A) = \mathsf{End}_{\mathsf{BMod}_A}(A) \to \mathsf{End}_{\mathsf{RMod}_A}(A) \simeq A$. We will omit V from the notation when it is not confusing or relevant.

To understand the meaning of the definition, suppose $s \in A$ admits a central structure. The object s is identified with a right A-module morphism $s \otimes (-) : A \to A$. A bimodule homomorphism structure promoting this includes the isomorphism $s \otimes (t \otimes (-)) \simeq t \otimes (s \otimes (-))$ naturally in t, i.e., $\tau : s \otimes (-) \simeq (-) \otimes s$. This is all we need when V is a $(1,1)$-category (e.g. $\mathsf{V} = \mathsf{Set}$) and being central is a *property*, but in general this is the first piece of the *structure* with infinitely many coherence data.

Remark 4.1.2. As we will see below, the center is just another name for the Hochschild cohomology of A, seen as a bimodule over itself. The notion of center defined above is sometimes called the $\mathbb{E}_2$-center because it admits a canonical structure of an $\mathbb{E}_2$-algebra object of V, with which the center is characterized by a universal property (see [Lur17, §5.3]). This $\mathbb{E}_2$-structure on the center can be naturally understood from Morita theory. One expects that Morita 2-V-category $\mathfrak{Alg}(\mathsf{V})$ of $\mathbb{E}_1$-algebras and bimodules in V can be defined in a similar manner as (the easiest case of) [JS17]. In this category, A as an algebra is an object, and A as a bimodule is the identity morphism of the object A, so the center admits a description as the double-loop object

$$\mathfrak{Z}_\mathsf{V}(A) \simeq \Omega(\Omega(\mathfrak{Alg}(\mathsf{V}), A), \mathrm{id}_A),$$

from which the $\mathbb{E}_2$-structure is clear.

[2]Not to be confused with another forgetful functor $\mathsf{End}_{\mathsf{BMod}_A}(A) \to \mathsf{End}_{\mathsf{LMod}_A}(A) \simeq A^{\mathrm{rev}}$.

According to Proposition 3.1.9, it seems natural to expect that $\vec{S}^2 = \vec{S}^1 \oslash \vec{S}^1$ lifts to the center, which would imply that $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ lifts to an $\infty\mathsf{Cat}_*^{\oslash}$-bimodule homomorphism. However, defining a central structure can be difficult in general, a priori requiring infinitely many coherence data. It turns out to be easier to directly formulate the coherence data extending Proposition 3.1.9, namely the "half-central" structure of $\vec{S}^1$; gauntness of $\vec{S}^1$ makes it homotopy-theoretically more tractable than $\vec{S}^2$.

To define the notion of the half-center (with respect to an involution D), let $D : A \to A$ be a monoidal endofunctor equipped with an equivalence $D \circ D \simeq \mathrm{id}_A$. There is a locally fully faithful functor $\mathsf{Alg}(\mathsf{V}) \to \mathfrak{Alg}(\mathsf{V})$ which is the identity on objects and regards algebra homomorphism as bimodules (we do not need a precise construction of $\mathfrak{Alg}(\mathsf{V})$, however). Explicitly, an algebra homomorphism $f : A \to B$ can be seen as an (A, B)-bimodule ${}^f B$, whose underlying right B-module is B itself and the left action of A is provided by f. By abuse of notation, we denote the (A, A)-bimodule ${}^D A$ also by D.

Definition 4.1.3. The *half-center* of A with respect to an involution D is $\mathfrak{Z}_{\mathsf{V}}(A, D) \coloneqq \mathrm{Hom}_{\mathsf{BMod}_A}(A, D)$[3].

Remark 4.1.4. The following diagram commutes (note $A = D = D \otimes_A D$ after forgetting to RMod_A):

$$\begin{tikzcd}
\mathfrak{Z}(A, D) \simeq \mathrm{Hom}_{\mathsf{BMod}_A}(A, D) \arrow[r, "D\otimes_A(-)", "\sim"'] \arrow[d, "\text{forget}"'] & \mathrm{Hom}_{\mathsf{BMod}_A}(D, D \otimes D) \arrow[d, "\simeq"] \\
A \simeq \mathsf{End}_{\mathsf{RMod}_A}(A, A) & \mathrm{Hom}_{\mathsf{BMod}_A}(D, A) \arrow[l, "\text{forget}"']
\end{tikzcd}$$

Thus lifting $x \in A$ to $\mathfrak{Z}(A, D)$ in fact gives a simultaneous lift to $\mathrm{Hom}(A, D)$ and $\mathrm{Hom}(D, A)$. In particular, a half-central structure on x induces a central structure on $x \otimes x$ by composition $\mathrm{Hom}(A, D) \times \mathrm{Hom}(D, A) \to \mathrm{Hom}(A, A) = \mathfrak{Z}(A)$.

[3]optimally, this is an object of V, but for our purposes the underlying object in S suffices.

4.1.2 Cyclic bar construction and the Hochschild cohomology

Here we review the standard resolution of a bimodule into free ones, called the *cyclic bar construction* and the resulting description of the half-center $\mathfrak{Z}_{\mathsf{V}}(A, D)$ as the Hochschild cohomology of the (A, A)-bimodule D.

Let A, B be $\mathbb{E}_1$-algebras in V and let M, N be (A, B)-bimodules. Our goal is find a convenient description of $\mathrm{Hom}_{{}_A\mathsf{BMod}_B(\mathsf{V})}(M, N)$. Using the equivalence ${}_A\mathsf{BMod}_B(\mathsf{V}) \simeq \mathsf{LMod}_A(\mathsf{RMod}_B(\mathsf{V}))$, we have an adjunction

$$\mathsf{LMod}_A(\mathsf{RMod}_B(\mathsf{V})) \underset{\perp}{\overset{A\otimes(-)}{\longleftarrow\!\!\!\longrightarrow}} \mathsf{RMod}_B(\mathsf{V})$$

with the comonad $T = A \otimes (-)$. Its associated resolution $M \simeq \mathrm{colim}_{[n]\in \mathbf{\Delta}^{\mathrm{op}}}(T^{n+1}M)$ gives

$$\begin{aligned}\mathrm{Hom}_{{}_A\mathsf{BMod}_B}(M, N) &\simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{{}_A\mathsf{BMod}_B}(A^{\otimes(n+1)} \otimes M, N)\\ &\simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{RMod}_B}(A^{\otimes n} \otimes M, N).\end{aligned}$$

Now we apply this to the case $A = B$, $M = A$, $N = D = {}^{D}A$, where $D : A \to A$ be an involution as before. As the right A-module structure on D is identical to A, we see

$$\mathfrak{Z}_{\mathsf{V}}(A, D) \coloneqq \mathrm{Hom}_{\mathsf{BMod}_A}(A, D) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{RMod}_A}(A^{\otimes n} \otimes A, A) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A).$$

Remark 4.1.5. On the right-hand side, the data of the involution D is encoded in the cosimplicial structure. Explicitly, the coface map $d^i : \mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A) \to \mathrm{Hom}_{\mathsf{V}}(A^{\otimes(n+1)}, A)$ sends $f : A^{\otimes n} \to A$ to

$$d^i f : x_0 \otimes \cdots \otimes x_n \mapsto \begin{cases} D(x_0) f(x_1 \otimes \cdots \otimes x_n) \quad (i = 0),\\ f(x_0 \otimes \cdots \otimes x_{i-1}x_i \otimes \cdots \otimes x_n) \quad (1 \le i \le n),\\ f(x_0 \otimes \cdots \otimes x_{n-1})x_n \quad (i = n+1).\end{cases}$$

4.1.3 Digression: obstruction theory for totalization of cosimplicial spaces

We digress a bit and try to explicate the coherence of (half)-central structures on an object. As we saw in the last section, the half-center is the totalization of a cosimplicial object, so one can try to slice the cosimplicial diagram in skeletal layers to write down the obstructions to the existence of a half-central structure. In our case of interest, the obstructions turn out to live in contractible spaces, in which case this section is subsumed by a simpler argument in Lemma 4.1.12. Nevertheless, we are including the exposition to give the intuitive description of the half-central structure in 4.1.4. The material is standard since [Bou89] but we provide a concise, model-independent account. See also [MS15] for a similar treatment of some related material.

Notation 4.1.6. For a functor $f : \mathcal{C} \to \mathcal{D}$, we denote the restriction $\mathsf{Fun}(\mathcal{D}, \mathsf{S}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{S})$ by f^* and the right Kan extension by f_*, so we have an adjunction $f^* \dashv f_*$. Recall the equivalence $\lim f_* F \xrightarrow{\sim} \lim F$.

Let $X = X^\bullet$ be a cosimplicial object in S. Our goal is to understand when a point $x_0 \in X^0$ lifts to the totalization $\operatorname{Tot} X := \lim X^\bullet$. Let $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ denote the full subcategory spanned by $[k]$ for $k \leq n$, let $X_{\leq n} : \mathbf{\Delta}_{\leq n} \to \mathsf{S}$ be the restriction of X and $\operatorname{Tot}_n X := \lim X_{\leq n}$. We have the following tower of canonical maps:

$$\operatorname{Tot} X \to \cdots \to \operatorname{Tot}_2 X \to \operatorname{Tot}_1 X \to \operatorname{Tot}_0 X = X^0.$$

This is a limit diagram because the natural map $\operatorname{colim}_n \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ is an equivalence, and the limit of an S-valued functor is the space of cocartesian sections of its unstraightening (cf. [Hau]). Now our task is to understand the fiber of $\operatorname{Tot}_{n+1} X \to \operatorname{Tot}_n X$ at a given point $x_n \in \operatorname{Tot}_n X$.

Let $R^n X : \mathbf{\Delta}_{\leq n+1} \to \mathsf{S}$ be the right Kan extension of $X|_{\mathbf{\Delta}_{\leq n}}$ and $M^n X := (R^n X)^{n+1}$ (called the *n-th matching object* of X). The limit of the unit natural transformation $\alpha : X_{\leq n+1} \to R^n X$ is the map in question: $\mathrm{Tot}_{n+1} X \to \lim R^n X \xrightarrow{\sim} \mathrm{Tot}_n X$.

Let $\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1} \subset \mathbf{\Delta}_{\leq n+1}$ denote the subcategory with the same objects but only with injective morphisms. The forgetful functor from the category of sub-simplices $u : (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1})_{/[n+1]} \to \mathbf{\Delta}_{\leq n+1}$ is coinitial (see e.g. [Lur17, p. 1.2.4.17]), so $\lim X \simeq \lim u^* X$. Let $\mathcal{C} = (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n})_{/[n+1]}$ so that $\mathcal{C}^{\triangleright} = (\mathbf{\Delta}^{\mathrm{inj}}_{\leq n+1})_{/[n+1]}$, and let $\mathcal{C} \xrightarrow{i} \mathcal{C}^{\triangleright} \xleftarrow{j} \{[n+1]\}$ be the inclusions. For any functor $F : \mathcal{C}^{\triangleright} \to \mathsf{S}$, the functor $j_* j^* F$ is constant at $F([n+1])$ and the counit $F \to i_* i^* F$ replaces the value $F[n+1]$ by the point $*$. It follows that the following square in $\mathsf{Fun}(\mathcal{C}^{\triangleright}, \mathsf{S})$ is cartesian:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\eta"] & j_* j^* F \arrow[d, "\eta'"] \\ i_* i^* F \arrow[r] & i_* i^* j_* j^* F. \end{tikzcd}$$

Recall that the limit of a constant diagram is given by the cotensoring with the geometric realization (groupoidification) of the diagram shape. Also note that the geometric realization of $\mathcal{C}$ and $\mathcal{C}^{\triangleright}$ are S^n and $*$ because, as a simplicial set, $\mathcal{C}$ is the barycentric subdivision of $\partial \Delta^{n+1}$. As a result, the limit of the above square over $\mathcal{C}^{\triangleright}$ is the following cartesian square in S:

$$\begin{tikzcd} \lim F \arrow[r] \arrow[d, "\eta"] & F[n+1] \arrow[d, "\eta'"] \\ \lim i^* F \arrow[r] & (F[n+1])^{S^n}. \end{tikzcd}$$

Plugging $\alpha : X_{\leq n+1} \to R^n X$ into F, we get a cartesian cube (i.e. the cube is a limit diagram, cf. [Lur17, section 6.1.1]) $\eta(\alpha) \Rightarrow \eta'(\alpha)$. Comparing the initial vertices of $\eta(\alpha)$, $\eta'(\alpha)$ with the pullback of the rest of the squares (and since $i^*\alpha$ is an equivalence), we see that the following square in S is cartesian:

$$\begin{tikzcd} \mathrm{Tot}_{n+1} X \arrow[r] \arrow[d] & X^{n+1} \arrow[d] \\ \mathrm{Tot}_n X \arrow[r] & (X^{n+1})^{S^n} \times_{(M^n X)^{S^n}} M^n X \end{tikzcd}$$

Tracing the construction, one sees that the map $S^n \to X^{n+1}$ corresponding to an element $x_n \in \operatorname{Tot}_n X$ sends the basepoint of S^n (coming from $0 \in \partial\Delta^{n+1}$) to $(d^0)^{n+1}(x_0)$, where x_0 is the element of X^0 that underlies x_n. Now we can show the following result:

Proposition 4.1.7. *A given point $x_n \in \operatorname{Tot}_n X$ lifts to $x_{n+1} \in \operatorname{Tot}_{n+1} X$ if and only if the induced map $o(x_n) : S^n \to X^{n+1}$ is trivial in $\pi_n(X^{n+1}, (d^0)^{n+1}(x_0))$. When a lift exists, the space of lifts is equivalent to $\Omega^{n+1}(N^n(X), (d^0)^{n+1}(x_0))$, where $N^n(X)$ is the fiber of $X^{n+1} \to M^n(X)$ at the image of x_n.*

Proof. The image of x_n in $(X^{n+1})^{S^n} \times_{(M^n X)^{S^n}} M^n X$ is equivalent to the data of the following commutative square

$$\begin{tikzcd}
S^n \arrow[r, "o(x_n)"] \arrow[d] & X^{n+1} \arrow[d] \\
* \arrow[r] \arrow[ur, dashed, "x_{n+1}"] & M^n X
\end{tikzcd}$$

and the data of the lift x_{n+1} is equivalent to the dashed arrow with two homotopies filling the triangles. The lower-right triangle can always be filled by composing some $* \to S^n$ with the square, so the triviality of $[o(x_n)] \in \pi_n(X^{n+1})$ suffices. Now assume the triviality of $[o(x_n)]$. The space of fillers is equivalent to the space of nullhomotopies of $S^n \to N^n X$. This is $\Omega^{n+1}(N^n X)$. □

Remark 4.1.8. By a similar argument one can show that, for any $X^\bullet$ satisfying $X^m = *$ for $m \neq n$, the totalization is either empty or $\Omega^n X^n$ (cf. [Lur17, Corollary 1.2.4.18] for the stable variant).

4.1.4 Half-central structure on $\vec{S}^1$

Now we specialize to our case of interest:

Notation 4.1.9. Let $\mathsf{V} = \mathsf{Pr}^{\mathsf{L}}_{\omega}$, $A = \infty\mathsf{Cat}^{\oslash}_{*}$, $D = (-)^\circ : A \to A$ be the total dual (monoidal) functor, which flips the cells of all dimensions (Proposition 2.4.17). We continue to denote the associated bimodule ${}^{D}A$ by D.

The goal of this section is to prove the following:

Theorem 4.1.10. *$\vec{S}^1 \in A$ and $\Sigma \simeq \vec{S}^1 \owedge (-) \in \mathsf{End}_{\mathsf{RMod}_A}(A)$ uniquely lifts along the forgetful functor $\mathfrak{Z}_{\mathsf{V}}(A, D) \to \mathsf{End}_{\mathsf{RMod}_A}(A) \simeq A$.*

Corollary 4.1.11. *The category CatSp and the functor $\Sigma^\infty : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ lifts to $\mathsf{BMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}^{\mathsf{L}}_\omega)$.*

Unpacking the obstruction theory of the totalization of cosimplicial objects, we see that the data of half-central structure on $\vec{S}^1$ amounts to the following:

- An object $\vec{S}^1 \in \omega\mathsf{Cat}_*$,
- A natural isomorphism $\tau_X : \vec{S}^1 \owedge X \xrightarrow{\sim} X^\circ \owedge \vec{S}^1$,
- A natural homotopy $\theta_{X,Y}$ filling the triangle

$$\begin{tikzcd}
 & X^\circ \owedge \vec{S}^1 \owedge Y \arrow[dr, "X^\circ \owedge (\tau_Y)"] & \\
\vec{S}^1 \owedge X \owedge Y \arrow[ur, "(\tau_X) \owedge Y"] \arrow[rr, "\tau_{X \owedge Y}"] & & X^\circ \owedge Y^\circ \owedge \vec{S}^1
\end{tikzcd}$$

- A natural homotopy filling the 3-simplex

$$\begin{tikzcd}
\vec{S}^1 \owedge X \owedge Y \owedge Z \arrow[r] \arrow[d] \arrow[dr] & X^\circ \owedge Y^\circ \owedge Z^\circ \owedge \vec{S}^1 \\
X^\circ \owedge \vec{S}^1 \owedge Y \owedge Z \arrow[r] \arrow[ur] & X^\circ \owedge Y^\circ \owedge \vec{S}^1 \owedge Z \arrow[u]
\end{tikzcd}$$

  whose boundary is filled by homotopies τ and θ.

- and so on.

It turns out that all the vertices live in contractible components, so there is no room for any nontrivial choice of the coherence data at each step after providing the natural equivalence τ. With this in mind, we have the following more direct argument:

Lemma 4.1.12. *Let $X = X^\bullet$ be a cosimplicial groupoid and $x \in X^0$ be a point. Suppose that*

(1) *the connected component of* $(d^0)^n(x) \in X^n$ *is contractible for all* $n \geq 0$*, and*

(2) *there is a path* $d^0 x \simeq d^1 x \in X^1$*.*

Then the fiber of $\mathrm{Tot}(X^\bullet) \to X^0$ *over* x *is contractible.*

Proof. By left Kan extension along $\{[0]\} \hookrightarrow \mathbf{\Delta}$, the data of $* \xrightarrow{x} X^0$ is equivalent to a natural transformation $x : I \to X$, where I is the tautological cosimplicial set $\mathbf{\Delta} \hookrightarrow \mathsf{Set} \hookrightarrow \mathsf{S}$. In this way, the map $\mathrm{Tot}(X^\bullet) \to X^0$ is corepresented by the map $I \to *$ to the terminal cosimplicial groupoid, i.e., the groupoid of the lifts of x is equivalent to that of factorizations $I \to * \to X$ of x. Consider the image factorization $I \twoheadrightarrow Y^\bullet \hookrightarrow X^\bullet$, so $Y^n \subset X^n$ is the union of connected components of the images of x by the structure maps $X^0 \to X^n$. The conditions (1), (2) ensures $Y^\bullet \xrightarrow{\sim} *$, so the factorization $I \to * \simeq Y^\bullet \to X^\bullet$ gives the canonical point in the fiber of $\mathrm{Tot}(X^\bullet) \to X^0$ over x. Conversely, any factorization $I \twoheadrightarrow * \to X$ uniquely factors through Y and is an image factorization, so the groupoid of such factorizations is contractible. $\square$

In the following, let μ_n generically denote the n-ary multiplication map of algebra objects (in particular, monoidal categories).

Lemma 4.1.13. *For* $n \geq 0$*, the connected component of* $\Sigma \circ \mu_n$ *in the underlying groupoid* $\mathrm{Map}_{\mathsf{V}}(A^{\otimes n}, A)$ *of* $\mathrm{Hom}_{\mathsf{V}}(A^{\otimes n}, A) = \mathsf{LFun}_\omega(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*)$ *is contractible.*

We will prove the lemma later; let us first assume it and finish proving the theorem.

Lemma 4.1.14. *There exists a (unique) equivalence of endofunctors* $\tau : (-) \oslash \vec{S}^1 \xrightarrow{\sim} \vec{S}^1 \oslash (-)^\circ$*.*

Proof. The uniqueness follows from the $n = 1$ case of Lemma 4.1.13. The existence is Proposition 3.1.9. $\square$

Proof of Theorem 4.1.10. Apply 4.1.12 to $X^{\bullet} = \mathrm{Map}_{\mathsf{V}}(A^{\otimes \bullet}, A)$ with the cosimplicial structure described in Remark 4.1.5. The conditions (1) and (2) are Lemma 4.1.13 and Lemma 4.1.14, respectively. $\square$

Proof of Lemma 4.1.13. Consider the composition

$$j : \square^{\times n} \to \mathsf{PSh}(\square)^{\otimes n} \to \infty\mathsf{Cat}^{\otimes n} \xrightarrow{(-)_+} \infty\mathsf{Cat}_*^{\otimes n}.$$

The first functor is the Yoneda embedding, the second is the tensor power of the localization $\mathsf{PSh}(\square) \to \omega\mathsf{Cat}$ and the last is the base change along $\mathsf{S} \to \mathsf{S}_*$ in Pr^{L}. From the universal property of each functor, j induces a fully faithful embedding

$$\begin{aligned} \mathsf{LFun}_{\omega}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*) \subset \mathsf{LFun}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*) & \\ \simeq \mathsf{LFun}(\infty\mathsf{Cat}^{\otimes n}, \infty\mathsf{Cat}_*) \hookrightarrow \mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*). & \end{aligned}$$

In particular, the connected component of $F \coloneqq \Sigma \circ \mu_n \in \mathsf{LFun}(\infty\mathsf{Cat}_*^{\otimes n}, \infty\mathsf{Cat}_*)$ is equivalent to that of $\Sigma \circ \mu_n \circ j \in \mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*)$. Now we have the following commutative diagram:

$$\begin{tikzcd}
\infty\mathsf{Cat}_* \otimes \cdots \otimes \infty\mathsf{Cat}_* \arrow[r, "\mu_n"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \\
\infty\mathsf{Cat} \otimes \cdots \otimes \infty\mathsf{Cat} \arrow[r, "\mu_n"] \arrow[u, "(-)_+"] & \infty\mathsf{Cat} \arrow[r, "\mathrm{Free}_{\mathbb{E}_1}"] \arrow[u] & \mathsf{Mon}(\infty\mathsf{Cat}) \arrow[u, hook, "\mathrm{B}"'] \\
\square \times \cdots \times \square \arrow[r, "\mu_n"] \arrow[u, hook] \arrow[uu, bend left, "j"] & \square \arrow[r, "\mathrm{Free}_{\mathbb{E}_1}"] \arrow[u, hook] & \mathsf{Mon}(\mathsf{Gaunt}) \arrow[u, hook]
\end{tikzcd}$$

The lower-left square commutes by the characterization of the Gray tensor product. Free $\mathbb{E}_1$-algebra functors restricted to $\square$ lands in the gaunt monoidal categories by the explicit formula $\mathrm{Free}_{\mathbb{E}_1}(\square^n) \simeq \coprod_{k \geq 0} \square^{kn}$. Therefore the connected component of F in $\mathsf{Fun}(\square^{\times n}, \infty\mathsf{Cat}_*)$ is equivalent to that of $\mathrm{Free}_{\mathbb{E}_1} \circ \mu_n \in \mathsf{Fun}(\square^{\times n}, \mathsf{Mon}(\mathsf{Gaunt}))$. The latter is a $(1,1)$-category, so the connected component is equivalent to a delooping of a monoid in Set. We must show that the (ordinary) group $\mathrm{Aut}(F)$ of the invertible object of that monoid is trivial. We have the following equalizer diagram of sets:

$$\mathrm{Aut}(F) \longrightarrow \prod_{x\in\square^{\times n}} \mathrm{Aut}_{\mathsf{Mon}(\mathsf{Gaunt})}(Fx) \rightrightarrows \prod_{x\to y\in\square^{\times n}} \mathrm{Hom}_{\mathsf{Mon}(\mathsf{Gaunt})}(Fx, Fy).$$

We claim that for any $x = (\square^{k_1}, \dots, \square^{k_n})$ the group

$$\mathrm{Aut}(Fx) = \mathrm{Aut}(\mathrm{Free}_{\mathbb{E}_1}(\square^{k_1+\dots+k_n}))$$

is trivial. Note that the natural map $\mathrm{Aut}_{\mathsf{Gaunt}}(\square^m) \to \mathrm{Aut}_{\mathsf{Mon}(\mathsf{Gaunt})}(\mathrm{Free}(\square^m))$ is a bijection; the inverse is given by the restiction to the indecomposable part[4]. Now the lemma is reduced to Lemma A.3.3. $\square$

4.2 The tensor product of categorical spectra

We are now ready to prove the main theorem, namely the existence and the universal property of the tensor product of categorical spectra. With the half-central structure of $\vec{S}^1$ at hand, it is now a special case of the general result on the monoidal inversion of a central object. The part of its proof relying on even more general facts on idempotent algebras is separated in Section 4.2.2. The commutative case of the results of this section is well-known [Voe98][Rob15][Nik17][Lur17, §4.8.2][CS, Lecture V], but the proof requires not completely obvious modification (e.g. see the footnote of Proposition 4.2.2), so we will spend some pages spelling out the detail.

4.2.1 The main theorem

Recall that Theorem 4.1.10 allows us to lift the defining colimit diagram

$$\infty\mathsf{Cat}_* \xrightarrow{\Sigma} \infty\mathsf{Cat}_* \xrightarrow{\Sigma} \cdots \to \mathsf{CatSp} \in \mathsf{Pr}^{\mathsf{L}}_{\omega}$$

[4]For a monoidal ∞-category $\mathcal{M}$, its indecomposable part can be defined as the pullback of $\mathrm{indec}(\pi_0({}^{\leq 0}\mathcal{M})) \hookrightarrow \pi_0({}^{\leq 0}\mathcal{M}) \leftarrow \mathcal{M}$, where $\infty\mathsf{Cat} \xrightarrow{\leq 0(-)} \mathsf{S} \xrightarrow{\pi_0} \mathsf{Set}$ are (product-preserving) left adjoints to the inclusions. Indecomposables are only functorial in monoidal equivalences.

to a telescope in the category of $(\infty\mathsf{Cat}_*, \infty\mathsf{Cat}_*)$-bimodules in $\mathsf{Pr}^{\mathsf{L}}_{\omega}$:

$$A \xrightarrow{\Sigma} D \xrightarrow{\Sigma} A \xrightarrow{\Sigma} D \xrightarrow{\Sigma} \cdots \to A_{\Sigma} = \mathsf{CatSp}.$$

In particular, $\Sigma_{\infty} : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ canonically lifts to a map of $\infty\mathsf{Cat}_*^{\oslash}$-bimodules. We denote $\mathsf{Alg}(\mathsf{BMod}_A(\mathsf{V}))$ by $\mathsf{Alg}_A(\mathsf{V})$. We can now state and prove the main theorem:

Theorem 4.2.1. (1) *$\vec{S}^1 \in \infty\mathsf{Cat}_*$ acts invertibly (from left and right) on the bimodule CatSp.*

(2) *The map $\Sigma^{\infty} : \infty\mathsf{Cat}_* \to \mathsf{CatSp}$ exhibits CatSp as idempotent $\mathbb{E}_0$-algebra of $\mathsf{BMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}^{\mathsf{L}}_{\omega})$.*

(3) *CatSp admits a unique lift to $\mathsf{Alg}_A(\mathsf{V})$. The lax monoidal forgetful functor $\mathsf{BMod}_A(\mathsf{V}) \to \mathsf{V}$ induces the underlying presentably monoidal structure on CatSp.*

(4) *The monoidal category $\mathsf{CatSp}^{\otimes}$ is the monoidal inversion $\infty\mathsf{Cat}_*^{\oslash}[\vec{S}^{-1}]$. That is, $\mathsf{CatSp}^{\otimes}$ is the initial $\infty\mathsf{Cat}_*^{\oslash}$-algebra on which $\vec{S}^{-1}$ acts invertibly.*

The theorem is a consequence of the following more general observations:

Proposition 4.2.2. *Let $\mathsf{V} = \mathsf{Pr}^{\mathsf{L}}_{\omega}$*[5] *and let $A^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\omega})$ be a monoidal category (not necessarily $\infty\mathsf{Cat}_*$). Let $s \in \mathfrak{Z}(A)$ be an object with a central structure (in particular $\tau : l_s = s \otimes (-) \xrightarrow{\sim} r_s = (-) \otimes s$), with which we regard $l_s : A \to A$ as a morphism in $\mathsf{BMod}_A(\mathsf{V})$. Let $A_s \coloneqq \mathrm{colim}(A \xrightarrow{l_s} A \xrightarrow{l_s} \cdots) \in \mathsf{BMod}_A(\mathsf{V})$ be its telescope. Assume moreover that $\tau_s : s \otimes s \to s \otimes s$ is equivalent to $\mathrm{id}_{s\otimes s}$*[6]*. Then:*

[5] We will only need that the monoidal structure on A commute with sequential colimits.

[6] This is a noncommutative variant of an argument usually attributed to Voevodsky but the way of complication is quite different. The main point of Voevodsky's argument was that when we want to invert an object $s \in A$ by telescoping (in a symmetric monoidal category), it suffices to check that the cyclic permutation (123) on s^3 is homotopic to the identity. In our argument, this condition is trivially satisfied (we use double suspension to begin with, and there is no nontrivial automorphism of s^2). We use the telescope of *left* multiplications, which would intuitively invert the left action of

(1) *The left- and right- actions of $s \in A$ on A_s are equivalent and both invertible.*

(2) *The canonical functor $\eta : A \to A_s$ exhibits A_s as an idempotent $\mathbb{E}_0$-algebra object of $\mathsf{BMod}_A(\mathsf{V})$. In particular, it uniquely lifts to an idempotent $\mathbb{E}_1$-algebra object in $\mathsf{BMod}_A(\mathsf{V})$.*

(3) *For any $B \in \mathsf{Alg}(\mathsf{V})$, the forgetful functor ${}_{A_s}\mathsf{BMod}_B(\mathsf{V}) \to {}_A\mathsf{BMod}_B$ (resp. ${}_B\mathsf{BMod}_{A_s}(\mathsf{V}) \to {}_B\mathsf{BMod}_A(\mathsf{V})$) is fully faithful with the essential image consisting of bimodules M on which s acts invertibly from the left (resp. right).*

(4) *There is an inclusion $\mathsf{Alg}_{A_s}(\mathsf{V}) \hookrightarrow \mathsf{Alg}_A(\mathsf{V})$ whose image is the A-algebras on which $s \in A$ acts invertibly from the both sides. A_s is initial among such A-algebras.*

Remark 4.2.3. The author hopes that the following variants of the last claim is true.

(1) A_s is initial among objects of $\mathsf{Alg}_{A/}$ on which the image of $s \in A$ acts invertibly.

(2) For any $B \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}_{\omega})$, the induced map $\mathsf{Fun}^{\otimes}(A_s, B) \to \mathsf{Fun}^{\otimes}(A, B)$ is a full subcategory inclusion. The image consists of those functors $f : A \to B$ with $f(s) \in B$ invertible.

For now, we do not attempt to prove these claims.

Proof of 4.2.1. Apply 4.2.2 for $A = \infty\mathsf{Cat}_*^{\oslash}$, $s = \vec{S}^2$. To check $\tau_{\vec{S}^2} : \vec{S}^2 \oslash \vec{S}^2 \xrightarrow{\sim} \vec{S}^2 \oslash \vec{S}^2$ is homotopic to the identity, it suffices to observe that the monoid of endomorphisms of $\vec{S}^4 = \mathrm{B}^4 \operatorname{Free}_{\mathbb{E}_4}$ is $\operatorname{Free}_{\mathbb{E}_4} = \coprod_{n \in \mathbb{N}} \mathbb{E}_4(n)/\Sigma_n$, so $\operatorname{Aut}(\vec{S}^4) \simeq *$. □

Proof. (1) Its left (resp. right) action on A_s is given as the colimit of the telescope

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x, except that we need a central structure to make sense of it. The right action is inverted essentially because it can be identified with the left action through the central structure, and the triviality of transposition is used here again.

(in the horizontal direction) of the following commutative squares:

$$\begin{tikzcd} A \arrow[r, "l_s"] \arrow[d, "l_s"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "l_s"] \\ A \arrow[r, "l_s"] & A, \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} A \arrow[r, "l_s"] \arrow[d, "r_s"'] \arrow[dr, phantom, "\overset{\beta}{\swarrow}"] & A \arrow[d, "r_s"] \\ A \arrow[r, "l_s"] & A \end{tikzcd}.$$

Here the natural equivalences α, β are provided by regarding l_s as left (resp. right) module morphisms, so β is just the associator $\beta_x : (s \otimes x) \otimes s \xrightarrow{\sim} s \otimes (x \otimes s)$, whereas α is the composition of the central structure of the horizontal morphisms with the associator, i.e.,

$$\alpha_x : s_b \otimes (s_a \otimes x) \xrightarrow{s_b \otimes \tau_x} s_b \otimes (x \otimes s_a) \xrightarrow{\sim} (s_b \otimes x) \otimes s_a \xleftarrow{\tau_{s_b} \otimes x} s_a \otimes (s_b \otimes x),$$

where we wrote s_a, s_b to avoid confusion between s corresponding to the horizontal and vertical arrows. Notice $\alpha_x \simeq (\tau_{s_b} \otimes x)^{-1} \simeq \mathrm{id}$; the second is by assumption and the first via $\theta_{s_b^2, x}$ (as in 4.1.4, but for s). So we have the factorization of the 2-cell:

$$\begin{tikzcd} A \arrow[r, "l_s"] \arrow[d, "l_s"'] \arrow[dr, phantom, "\shortparallel" near start] & A \arrow[d, "l_s"] \\ A \arrow[r, "l_s"'] \arrow[ur, "f = \mathrm{id}" description] \arrow[ur, phantom, "\shortparallel" near end, shift right=3] & A. \end{tikzcd}$$

By the cofinality of $\mathbb{N} \xrightarrow{+1} \mathbb{N}$, the map $A_s \to A_s$ that f induces is inverse to the map induced by l_s, so the left action of s is invertible.

It remains to verify that the right action is invertible. It suffices to show the existence of the invertible 3-cell filling the following cylinder in V (subscript s of l, r are omitted) because then its telescope exhibits that τ induces the equivalence between left- and right- action on A_s:

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, bend right, "r"'] \arrow[d, bend left, "l"] \arrow[d, phantom, "\overset{\Leftarrow}{\tau}"] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & A \arrow[d, "l"] \\ A \arrow[r, "l"'] & A, \end{tikzcd} \quad \simeq \quad \begin{tikzcd} A \arrow[r, "l"] \arrow[d, "r"'] \arrow[dr, phantom, "\overset{\beta}{\swarrow}"] & A \arrow[d, bend right, "r"'] \arrow[d, bend left, "l"] \arrow[d, phantom, "\overset{\Leftarrow}{\tau}"] \\ A \arrow[r, "l"'] & A \end{tikzcd}$$

The existence of the 3-cell can be verified as follows: unpacking the description of α as above, it reduces to providing the following equivalence of 2-cells (unmarked equivalences are associators):

$$(ll \xrightarrow{l\tau} lr \simeq rl \xrightarrow{r\tau} ll) \simeq (ll \xrightarrow{\tau l} rl \simeq lr \xrightarrow{\tau r} rr).$$

Applying the coherence of the center and the assumption on the right-hand side, we have $\tau l(x) = \tau_{s\otimes x} \xrightarrow[\theta_{s,x}]{\sim} (s \otimes s \otimes x \xrightarrow{\tau_s \otimes x} s \otimes s \otimes x \xrightarrow{s\otimes\tau_x} s \otimes x \otimes s) \simeq s \otimes \tau_x \simeq l\tau_x$ and similarly $\tau r(x) \simeq \tau_x \otimes s \simeq r\tau_x$, so we are done.

(2) The morphisms $A \otimes_A A_s \xrightarrow{\eta\otimes A_s} A_s \otimes A_s$ and $A_s \otimes_A A \xrightarrow{A_s\otimes\eta} A_s \otimes A_s$ are the colimits of the telescope along the endomorphism $l_s \otimes A_s$ and $A_s \otimes l_s$, which are the left and right action of s on A_s, so they are invertible. Any idempotent $\mathbb{E}_0$-algebra lifts uniquely to an idempotent $\mathbb{E}_1$-algebra by Proposition 4.2.7.

(3) The unit transformation $M \to A_s \otimes_A M$ is idempotent, so the free-forgetful adjunction is a localization. The unit is an equivalence if and only if s acts invertibly on M from the left.

□

Remark 4.2.4. If we start from $\infty\mathsf{Cat}_*$ with the oplax smash product $\oslash^{\mathrm{rev}}$, the directed circle $\vec{S}^1 \in \infty\mathsf{Cat}_*$ is still half-central. To see this, note that we have an equivalence op : $\infty\mathsf{Cat}^{\otimes} \xrightarrow{\sim} \infty\mathsf{Cat}^{\otimes^{\mathrm{rev}}}$ and similarly for $\infty\mathsf{Cat}_*$ as objects of $\mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$. Along this equivalence, $\vec{S}^1$ and D are preserved because $(\vec{S}^1)^{\mathrm{op}} = \vec{S}^1$ and $D \circ \mathrm{op} = \mathrm{co} = \mathrm{op} \circ D$. Consequently, they get the transferred structure of monoidal involution and a half-central object, so can also form a monoidal structure on CatSp by inverting $\vec{S}^1$ of $\infty\mathsf{Cat}_*^{\oslash^{\mathrm{rev}}}$. Using the universal property, it is the image of $\infty\mathsf{Cat}_*^{\oslash} \to \mathsf{CatSp}^{\otimes} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$ under the reversal involution on $\mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$.

Remark 4.2.5. The non-commutativity is somewhat unfortunate, given that many constructions in algebraic geometry rely on the commutativity of algebra. The author does not know if there is a reasonably undestructive localization to a commutative (or at least $\mathbb{E}_2$) tensor product. Conjecturally, the tensor product of categorical spectra with adjoints considered in Chapter 6 promotes to a (framed) $\mathbb{E}_2$-structure[7]. The variant of the theorem for $\infty\mathsf{Cat}$ with the cartesian product and $\mathsf{CatSp}^{\mathrm{cn}}$ instead of CatSp is true for a formal reason (namely, CMon is idempotent in Pr^{L} [GGN16]).

Also, note that the noncommutativity of the Gray tensor product is not too uncontrolled; there is a duality involution that reverses the tensor: $X^{\mathrm{op}} \otimes Y^{\mathrm{op}} \simeq (Y \otimes X)^{\mathrm{op}}$, in other words, it is a $*$-*algebra* in Pr^{L}. One sees that CatSp has the induced structure of $*$-algebra. Therefore, to fully exploit the multiplicative structure of categorical spectra, the theory of categorical $*$-algebras seems to be a good language to develop.

4.2.2 Idempotent $\mathbb{E}_1$-algebras

If $\mathcal{C}$ is a symmetric monoidal category, [Lur17, Proposition 4.8.2.9] shows that the forgetful functor $\mathsf{CAlg}^{\mathrm{idem}}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathcal{C})$ is an equivalence. We spell out (straightforward but not completely obvious) verifications of the $\mathbb{E}_1$-variant of the related statements. Let $\mathcal{C}^{\otimes}$ be a presentably $\mathbb{E}_1$-monoidal category with the unit 1.

Definition 4.2.6. An $\mathbb{E}_0$-algebra $\eta : 1 \to E$ in $\mathcal{C}$ is said to be *idempotent* if the maps $E \simeq 1 \otimes E \xrightarrow{\eta \otimes E} E \otimes E$ and $E \simeq E \otimes 1 \xrightarrow{E \otimes \eta} E \otimes E$ are equivalences. An $\mathbb{E}_1$-algebra E is *idempotent* if the multiplication map $E \otimes E \to E$ is an equivalence, or equivalently, the underlying $\mathbb{E}_0$-algebra is idempotent.

By definition, an $\mathbb{E}_0$-algebra $\eta : 1 \to C$ is idempotent if and only if the functor $L^l_E = E \otimes (-) : \mathcal{C} \to \mathcal{C}$ is a localization (as in [Lur09b, Prop. 5.2.7.4]). Notice the monoidal equivalence $\mathcal{C} \simeq \mathsf{End}_{\mathsf{RMod}_{\mathcal{C}}(\mathsf{Pr}^{\mathsf{L}})}(\mathcal{C})$; this implies that $\mathsf{Alg}(\mathcal{C})$ is equivalent to the category of (right-)$\mathcal{C}$-linear monads on $\mathcal{C}$. Since the latter is equivalent to the category of idempotent $\mathbb{E}_0$-algebras in $\mathcal{C}$ by $L^l_E \leftrightarrow E$, we obtain the following equivalence:

[7] The author thank Mayuko Yamashita and Thomas Blom for suggestions.

Proposition 4.2.7. *The forgetful functor* $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_1}(\mathcal{C}) \to \mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_0}(\mathcal{C})$ *is an equivalence.*

Note that we automatically have $E \otimes \eta \simeq \eta \otimes E$ because both are inverse to the multiplication map $E \otimes E \to E$.

Remark 4.2.8. If $\mathcal{C}$ is given a symmetric monoidal structure, any idempotent $\mathbb{E}_1$-algebra automatically upgrades to an $\mathbb{E}_\infty$-algebra, so this section is only relevant if $\mathcal{C}$ itself is noncommutative. This observation also implies that $\mathsf{S} \hookrightarrow n\mathsf{Cat}$ is not idempotent in Pr^{L} for $n \geq 0$ because they have noncommutative monoidal structures (namely, the lax Gray tensor products) on these categories, and similarly for $\Sigma^\infty_+ : \mathsf{S} \to \mathsf{CatSp}$.

Remark 4.2.9. Similarly to the commutative case, the map $\mathsf{Alg}^{\mathrm{idem}}_{\mathbb{E}_1}(\mathcal{C}) \to \mathsf{Alg}_{\mathbb{E}_0}(\mathcal{C})$ is an equivalence of posets. To see this, suppose there exists an $\mathbb{E}_0$-algebra map $f : A \to B$ between idempotent $\mathbb{E}_1$-algebras. We wish to show $\mathrm{Hom}_{\mathbb{E}_0}(A, B)$, $\mathrm{Hom}_{\mathbb{E}_1}(A, B)$ are both contractible. Since $\eta_B \otimes B \simeq (1 \otimes B \xrightarrow{\eta_A \otimes B} A \otimes B \xrightarrow{f \otimes B} B \otimes B)$ is an equivalence, B is a retract of $A \otimes B$ and therefore L^l_A-local, i.e., $B \simeq A \otimes B'$ for some B'. Moreover, $(B \xrightarrow{\eta_A \otimes B} A \otimes B) \simeq (A \otimes B' \xrightarrow{\eta_A \otimes A \otimes B'} A \otimes A \otimes B')$ is an equivalence. Now consider the square $(\eta_A : 1 \to A) \otimes (f : A \to B)$; the diagonal is $(\eta_A \otimes B) \circ f \simeq (A \otimes f) \circ (\eta_A \otimes A) \simeq (A \otimes f) \circ (A \otimes \eta_A) \simeq A \otimes \eta_B$, so we have a canonical contraction $f \simeq (\eta_A \otimes B)^{-1} \circ (A \otimes \eta_B)$, i.e., $\mathrm{Hom}_{\mathbb{E}_0}(A, B) \simeq *$. The symmetric argument implies $B \simeq B \otimes A$, so B lies in $\mathsf{Alg}_{\mathbb{E}_1}(A\mathcal{C}A) \subset \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})$ (notice the inclusion $(A\mathcal{C}A)^{\otimes} \subset \mathcal{C}^{\otimes}$ is a map of (nonsymmetric) operads and is nonunital monoidal functor). It follows that $\mathrm{Hom}_{\mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})}(A, B)$ is contractible because A is initial (tensor unit) in $\mathsf{Alg}_{\mathbb{E}_1}(A\mathcal{C}A)$.

Proposition 4.2.10. *Let E be an idempotent $\mathbb{E}_1$-algebra. Then*

(1) *The forgetful functor* $\mathsf{LMod}_E(\mathcal{C}) \to \mathcal{C}$ *(resp.* $\mathsf{RMod}_E(\mathcal{C}) \to \mathcal{C}$*) is an equivalence to the full subcategory $E\mathcal{C}$ (resp. $\mathcal{C}E$).*

(2) *the forgetful functor* $\mathsf{BMod}_E(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ *is an equivalence to the full suboperad* $(E\mathcal{C}E)^{\otimes} \subset \mathcal{C}^{\otimes}$.

Proof. We spell out the left module case. The unit law $M \simeq 1 \otimes M \xrightarrow{\eta_E \otimes M} E \otimes M \xrightarrow{a} M$ implies M is a retract of $E \otimes M$, so M is L^l_E-local and therefore $\eta_E \otimes M = \eta_E \otimes E \otimes M'$ (for some M') is an equivalence. This means a, the counit map of the free-forgetful adjunction, is also an equivalence. Also, on these local objects, the unit map $E \otimes X \xrightarrow{\eta \otimes E \otimes X} E \otimes E \otimes X$ of the free-forgetful adjunction is equivalence, so the adjunction induces the stated equivalence. A similar argument for bimodule performed componentwise proves the second claim. □

4.3 Basic properties of the tensor product

The goal of this short section is to discuss explicit descriptions of the tensor product and internal homs. We will also show that the tensor product behaves additively on the category level and connectivity and obtain comparison results of the tensor product with those of spectra and symmetric monoidal categories.

4.3.1 Monoidal involutions and the half-central structure of $\mathbb{F}[1]$

Unpacking the proof in the previous section, the following confirms that the directed circle $\vec{S}^1$ acts on categorical spectra in an expected way.

Proposition 4.3.1. *The left action of* $\vec{S}^1 \in \infty\mathsf{Cat}_*$ *on* CatSp *(which is also the left action of* $\mathbb{F}[1]$*) is canonically isomorphic to the shift functor.*

Proof. As the limit of left modules, the left action $l_X : \mathsf{CatSp} \to \mathsf{CatSp}$ of $X \in \infty\mathsf{Cat}_*^{\oslash}$ is given by the limit of the following diagram:

$$\begin{array}{ccccccc}
\infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \cdots \\
{\scriptstyle l_X}\downarrow & & {\scriptstyle l_X}\downarrow & & {\scriptstyle l_X}\downarrow & & \\
\infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \infty\mathsf{Cat}_* & \xrightarrow{l_{\vec{S}^1}} & \cdots ,
\end{array}$$

where the commuting 2-cell is given by the half-central structure $\tau_X : \vec{S}^1 \oslash X \xrightarrow{\sim} (X)^\circ \oslash \vec{S}^1$. When $X = \vec{S}^1$, this morphism is in fact an identity since $(\vec{S}^1)^\circ \simeq (\vec{S}^1)$ and $\mathrm{Aut}(\vec{S}^1 \oslash \vec{S}^1) \simeq (\mathrm{Free}_{\mathbb{E}_2}(*))^\times \simeq *$. Therefore the diagram is equivalent to

$$\begin{tikzcd}
\infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\sslash"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\sslash"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & \cdots \\
\infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \infty\mathsf{Cat}_* \arrow[r, "\Sigma"] & \cdots ,
\end{tikzcd}$$

which induces the shift functor $\Sigma = [1] : \mathsf{CatSp} \to \mathsf{CatSp}$ by definition. □

Proposition 4.3.2. *The total dual functor* $D = (-)^\circ : \mathsf{CatSp} \to \mathsf{CatSp}$ *uniquely promotes to an automorphism in* $\mathsf{Alg}(\mathsf{BMod}_{\infty\mathsf{Cat}}(\mathsf{Pr}^\mathsf{L}))$.

Proof. By the universal property of $\mathsf{CatSp} = \infty\mathsf{Cat}_*^{\oslash}[\vec{S}^{-1}]$, there is a unique involution $\mathsf{CatSp} \to \mathsf{CatSp}$ of $\mathsf{Alg}_{\infty\mathsf{Cat}^\otimes}(\mathsf{Pr}^\mathsf{L})$ making the following diagram commute:

$$\begin{tikzcd}
\infty\mathsf{Cat}_*^{\oslash} \arrow[r, "D"] \arrow[d, "\Sigma^\infty"] & \infty\mathsf{Cat}_*^{\oslash} \arrow[d, "\Sigma^\infty"] \\
\mathsf{CatSp}^{\otimes} \arrow[r, dashed, "\tilde{D}"] & \mathsf{CatSp}^{\otimes}.
\end{tikzcd}$$

To see that the underlying functor of $\tilde{D}$ is the total dual functor defined in Definition 3.3.15, note that $\tilde{D}$ is a $\infty\mathsf{Cat}_*^{\otimes}$-bimodule homomorphism, it must commute with the shift operators. Composing it with the squares of the right adjoints to the above, we have the following commutative diagram:

$$\begin{tikzcd}
\mathsf{CatSp} \arrow[r, "\tilde{D}"] \arrow[d, "\Sigma^n"] \arrow[dd, bend right, "\Omega^{\infty-n}"'] & \mathsf{CatSp}, \arrow[d, "\Sigma^n"'] \arrow[dd, bend left, "\Omega^{\infty-n}"] \\
\mathsf{CatSp} \arrow[r, "\tilde{D}"] \arrow[d, "\Omega^\infty"] & \mathsf{CatSp} \arrow[d, "\Omega^\infty"'] \\
\infty\mathsf{Cat}_* \arrow[r, "D"] & \infty\mathsf{Cat}_*.
\end{tikzcd}$$

This canonically factors through the same square for $n = 0$, so $\tilde{D}$ is the map induced by D in each degree compatibly along the loops, i.e., it agrees with the total dual functor.

$\square$

Proposition 4.3.3. *The half-central structure on* $\vec{S}^1 \in \infty\mathsf{Cat}_*^{\oslash}$ *induces a canonical half-central structure on* $\mathbb{F}[1] \in \mathsf{CatSp}^{\otimes}$*. In particular, we have* $X \otimes \mathbb{F}[1] \simeq \Sigma(X^\circ)$*.*

Proof. Note that the diagram we considered in the proof of Proposition 4.3.1 lifts to the following diagram of $A = \infty\mathsf{Cat}_*^{\oslash}$-bimodules (i.e, $A \otimes_A (-) \xrightarrow{\Sigma} D \otimes_A (-)$ applied to the definitional diagram of $\mathsf{CatSp} \in \mathsf{BMod}_A$):

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "l_{\vec{S}^1}"] & D \arrow[r] \arrow[d, "l_{\vec{S}^1}"] & A \arrow[r] \arrow[d, "l_{\vec{S}^1}"] & \cdots \arrow[r] & \mathsf{CatSp} \arrow[d, "l_{\mathbb{F}[1]}"] \\
D \arrow[r] & A \arrow[r] & D \arrow[r] & \cdots \arrow[r] & D \otimes_A \mathsf{CatSp}.
\end{tikzcd}$$

To show that $D \otimes_A \mathsf{CatSp} \simeq {}^{D}\mathsf{CatSp}$, note that by $\mathsf{BMod}_B = \mathsf{LMod}_B(\mathsf{RMod}_B)$ and so the bimodule structure on $B \in \mathsf{RMod}_B$ is given by an algebra morphism $B \to B$; that is, since $D \otimes_A \mathsf{CatSp}$ forgets to CatSp as the right module over itself, it must be of the form $(-)^{D'}\mathsf{CatSp}$ for some $D' : \mathsf{CatSp} \to \mathsf{CatSp}$. To show $D = D'$, observe that it is an A-algebra homomorphism satisfying $\Sigma^\infty \circ D' \simeq \Sigma^\infty \circ D$. $\square$

Remark 4.3.4. Similarly, one can show that co, op-duals induce $\mathsf{CatSp}^{\otimes^{\mathrm{rev}}} \to \mathsf{CatSp}^{\otimes}$. Tracking the duality of the base and bimodules is somewhat confusing and we do not attempt to precisely write it down here.

4.3.2 Additivity on categorical levels

Just as the Gray tensor product of ∞-categories, the tensor product of categorical spectra behaves additively on category levels.

Proposition 4.3.5. *For* $m, n \in \mathbb{Z} \cup \{\pm\infty\}$*, the essential image of* $m\mathsf{CatSp} \otimes n\mathsf{CatSp}$ *under the tensor product is* $(m+n)\mathsf{CatSp}$*. Here the convention is* $\infty + (-\infty) = -\infty$*. In particular,* $0\mathsf{CatSp} \subset \mathsf{CatSp}$ *is a monoidal subcategory and* $\mathsf{Sp} \subset \mathsf{CatSp}$ *is a* $\otimes$*-ideal.*

Proof. The category $n\mathsf{CatSp}$ is the colimit-closure of the set $\{\Sigma_+^{\infty - i}\square^j \mid 0 \leq j \leq n+i \text{ or } j = 0\}$. Since $D\square^j \simeq \square^j$, we have $\Sigma_+^{\infty-i}\square^j \otimes \Sigma_+^{\infty-k}\square^l \simeq \Sigma_+^{\infty-i-k}(D^k\square^j) \otimes \square^l \simeq$

$\Sigma^{\infty-i-k}_{+}\square^{j+l} \in (n+m)\mathsf{CatSp}$. To see that it is exactly the essential image, observe that $\mathbb{F}[n] \in n\mathsf{CatSp}$ and $(-) \otimes \mathbb{F}[n] : m\mathsf{CatSp} \to (m+n)\mathsf{CatSp}$ is an equivalence for any finite n. $\square$

Proposition 4.3.6. $\mathsf{Sp} \subset \mathsf{CatSp}$ *is an exponential ideal. In particular, the spectral localization (or group completion)* $L : \mathsf{CatSp} \to \mathsf{Sp}$ *uniquely promotes to a monoidal structure. The induced monoidal structure agrees with the usual tensor product of spectra, so* $\mathbb{S}$ *is idempotent and* L *is a smashing localization by* $\mathbb{S}$.

Proof. Let $X \in \mathsf{Sp}$ and $Y \in \mathsf{CatSp}$. To show that $[Y, X]$ is a spectrum, it suffices to check that it is local for $f_{i,j} : \Sigma^{\infty-i}_{+}(C_{j+1} \to C_j)$ for any $i, j \geq 0$. As $\mathsf{Sp} \subset \mathsf{CatSp}$ is closed under limits, we may assume that Y is of the form $\Sigma^{\infty-k}_{+}Y'$, so $Y \otimes f_{i,j}$ is of the form $\Sigma^{\infty-i-k}_{+}D^i(Y') \otimes (C_{j+1} \to C_j)$ which is an L-equivalence because $\mathsf{S} \subset \infty\mathsf{Cat}$ is an exponential ideal [Cam23b, §3.2]. Similarly, $[\![Y, X]\!]$ is a spectrum, so Sp is an exponential ideal. It follows that there exists a unique monoidal structure on Sp promoting L to a monoidal localization. Since $\Sigma^{\infty}_{\mathsf{Sp}} : \mathsf{S}_* \hookrightarrow \infty\mathsf{Cat}_* \xrightarrow{\Sigma^\infty} \mathsf{CatSp} \xrightarrow{L} \mathsf{Sp}$ is monoidal, the induced monoidal structure on Sp is the usual tensor product. Now observe $\mathrm{Hom}(L(Z \otimes Y), X) \simeq \mathrm{Hom}_{\mathsf{CatSp}}(Z, [Y, X]) \simeq \mathrm{Hom}_{\mathsf{Sp}}(LZ \otimes Y, X)$ for $X \in \mathsf{Sp}$, so $L(Z \otimes Y) \simeq LZ \otimes Y$. Plugging $Z = \mathbb{F}$, we get $LY \simeq \mathbb{S} \otimes Y$. $\square$

Warning 4.3.7. One should not expect that $n\mathsf{CatSp} \subset \mathsf{CatSp}$ is an exponential ideal unless $n = \pm\infty$. In fact, since $[\mathbb{F}[-n], X] \simeq \mathbb{F}[n] \otimes X$, this internal hom increases the category level if n is negative.

4.3.3 Additivity on connectivity

Recall we denoted the essential image of the embedding $\mathrm{B}^{\infty-n} : \mathsf{CMon}(\infty\mathsf{Cat}) \hookrightarrow \mathsf{CatSp}$ by $\mathsf{CatSp}^{n\text{-cn}}$ and said its objects are *n-connective*. The following proposition shows that the tensor product respects connectivity and in particular it restricts to what should be called the lax tensor product of symmetric monoidal ∞-categories.

Proposition 4.3.8. *The inclusion* $\mathrm{B}^\infty : \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{\sim} \mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ *exhibits* $\mathsf{CMon}(\infty\mathsf{Cat})$ *as a monoidal subcategory. The monoidal structure is characterized by the fact that the functor* $\mathrm{Free}_{\mathbb{E}_\infty} : \infty\mathsf{Cat} \to \mathsf{CMon}(\infty\mathsf{Cat})$ *promotes to a monoidal functor with respect to the Gray tensor product of the domain. Moreover, the image of* $\mathsf{CatSp}^{m\text{-cn}} \otimes \mathsf{CatSp}^{n\text{-cn}} \subset \mathsf{CatSp} \otimes \mathsf{CatSp}$ *under the tensor product is* $\mathsf{CatSp}^{(m+n)\text{-cn}}$*.*

Proof. The tensor unit $\mathbb{F}$ is connective. To prove the first statement, by [Lur17, Proposition 2.2.1.1] it suffices to check that for any $X, Y \in \mathsf{CMon}(\infty\mathsf{Cat})$, the tensor product $(\mathrm{B}^\infty X) \otimes (\mathrm{B}^\infty Y)$ lies in the image of B^∞. Write $X \simeq \mathrm{colim}_i \mathrm{Free}_{\mathbb{E}_\infty}(\mathcal{C}_i)$, $Y \simeq \mathrm{colim}_j \mathrm{Free}_{\mathbb{E}_\infty}(\mathcal{D}_j)$. Now we have

$$
\begin{aligned}
(\mathrm{B}^\infty X) \otimes (\mathrm{B}^\infty Y) &\simeq (\operatorname*{colim}_i \Sigma^\infty_+(\mathcal{C}_i)) \otimes (\operatorname*{colim}_j \Sigma^\infty_+(\mathcal{D}_j)) \\
&\simeq \operatorname*{colim}_{i,j}(\Sigma^\infty_+(\mathcal{C}_i) \otimes \Sigma^\infty_+(\mathcal{D}_j)) \simeq \operatorname*{colim}_{i,j} \Sigma^\infty_+(\mathcal{C}_i \otimes \mathcal{D}_j).
\end{aligned}
$$

Since $\mathsf{CatSp}^{\mathrm{cn}} \subset \mathsf{CatSp}$ is closed under colimits, the last colimit stays inside $\mathsf{CatSp}^{\mathrm{cn}}$. The characterization is also clear from this computation. $\square$

Recall that by [GGN16] $\mathsf{CMon} \in \mathsf{Pr}^{\mathsf{L}}$ is an idempotent algebra and the associated symmetric monoidal localization $\mathsf{CMon}(-) = \mathsf{CMon} \otimes (-) : \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ universally turns a presentable category to a *semiadditive* presentable categories. In particular, it induces $\mathsf{CMon} : \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{CAlg}(\mathsf{Pr}^{\mathsf{L}})$ and $\mathsf{CMon} : \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$. We denote the symmetric monoidal structure on $\mathsf{CMon}(\infty\mathsf{Cat}, \times)$ by $\circledast$ and the monoidal structure on $\mathsf{CMon}(\infty\mathsf{Cat}, \otimes)$ by $\otimes$. The lax monoidal functor $\mathrm{id} : (\infty\mathsf{Cat}, \times) \to (\infty\mathsf{Cat}, \otimes)$ of Remark 2.4.15 induces a lax monoidal structure on $\mathsf{Mon}(\infty\mathsf{Cat})^{\circledast} \to \mathsf{Mon}(\infty\mathsf{Cat})^{\otimes}$.

Corollary 4.3.9. *The functor* B^∞ *induces a functor* $\mathsf{Rig}_{\mathbb{E}_1}(\infty\mathsf{Cat}) \to \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{CatSp})$*.*

4.3.4 Fomulas for the tensor product and the internal hom

As an immediate consequence of Corollary 3.5.2, we have the following:

Corollary 4.3.10. *Let $X = (X_n)$, $Y = (Y_n)$ be categorical spectra. Then we have*

$$X \otimes Y \simeq \operatorname*{colim}_{i,j}(\Sigma^{\infty-i-j}(D^j X_i) \oslash Y_j) \simeq \operatorname*{colim}_{n}(\Sigma^{\infty-4n}(X_{2n} \oslash Y_{2n})).$$

and similarly for $\mathrm{B}^{\infty-i}$ in place of $\Sigma^{\infty-i}$, regarding X_i, Y_j as symmetric monoidal categories.

Proof. Recall that $X \otimes Y \simeq (\operatorname{colim}_i \Sigma^{\infty-i} X_i) \otimes (\operatorname{colim}_j \Sigma^{\infty-j} Y_j) \simeq \operatorname{colim}_{i,j} \Sigma^{\infty-i} X_i \otimes \Sigma^{\infty-j} Y_j \simeq \operatorname{colim}_{i,j} \Sigma^{-j} D^j(\Sigma^{\infty-i} X_i) \otimes \Sigma^{\infty} Y_j \simeq \operatorname{colim}_{i,j} \Sigma^{\infty-i-j}(D^j X_i) \oslash (Y_j)$. The second part results from restricting to the cofinal diagram of $2\mathbb{N} \xrightarrow{\text{diag}} \mathbb{N} \times \mathbb{N}$. $\square$

Corollary 4.3.11. *We have the corresponding formula for the internal hom of categorical spectra: $\Omega^{\infty-n}[X, Y] \simeq \lim_{k\to\infty}[D^n X_k, Y_{n+k}]$, where the internal hom on the right-hand side is taken in either $\infty\mathsf{Cat}_*^{\oslash}$ or $\infty\mathsf{SMCat}^{\otimes}$.*

Proof. We have the following natural equivalences for $Z \in \infty\mathsf{Cat}_*$:

$$\begin{aligned}
\operatorname{Map}_{\infty\mathsf{Cat}_*}(Z, \Omega^{\infty-n}[X, Y]) &\simeq \operatorname{Map}_{\mathsf{CatSp}}(\Sigma^{\infty-n} Z, [X, Y]) \\
&\simeq \operatorname{Map}_{\mathsf{CatSp}}(\operatorname*{colim}_k \Sigma^{\infty-k} X_k \otimes \Sigma^{\infty-n} Z, Y) \\
&\simeq \lim_k \operatorname{Map}_{\mathsf{CatSp}}(\Sigma^{\infty-k-n}(D^n X_k \oslash Z), Y) \\
&\simeq \lim_k \operatorname{Map}_{\mathsf{CatSp}}(D^n X_k \oslash Z, Y_{n+k}) \\
&\simeq \operatorname{Map}_{\infty\mathsf{Cat}_*}(Z, \lim_k [D^n X_k, Y_{n+k}]),
\end{aligned}$$

and similarly for $Z \in \infty\mathsf{SMCat}$. $\square$

Chapter 5

Absolute colimits in categorical spectra

In this chapter, we begin our study of *stability* in categorical spectra. Understanding stability is clearly important—the category CatSp should not exist in isolation, but it should be put into a larger context of *stable Gray-categories*, just as the category Sp of spectra is a (universal) example of stable (1-)categories. Recall that stability for categories had many equivalent characterizations:

Proposition 5.0.1 ([Lur17, Chapter 1]). *The following conditions on $\mathcal{C} \in \mathsf{Cat}$ are equivalent:*

(1) $\mathcal{C}$ *is pointed, has cofibers and the suspension functor is an equivalence.*

(2) $\mathcal{C}$ *is pointed, has fibers and cofibers, and a triangle is a fiber sequence if and only if it is a cofiber sequence.*

(3) $\mathcal{C}$ *has finite limits and colimits, and a commutative square is a pushout if and only if it is a pullback.*

(4) $\mathcal{C}$ *has finite limits and* $\mathrm{ev}_{S^0} : \mathsf{Exc}_*(\mathsf{S}^{\mathrm{fin}}, \mathcal{C}) \to \mathcal{C}$ *is an equivalence. The domain* $\mathsf{Exc}_*(\mathcal{A}, \mathcal{B}) \subset \mathsf{Fun}(\mathcal{A}, \mathcal{B})$ *is the full subcategory of reduced excisive functors, i.e., functors that preserve terminal objects and send a pushout square to a pullback square.*

If these are satisfied, we say $\mathcal{C}$ is a stable 1-category. Moreover, if $\mathcal{C}$ is presentable, the stability is equivalent to having a (necessarily unique) module structure over $\mathsf{Sp}^{\otimes}$.

Knowing that all of these are equivalent is extremely useful; for instance, one checks the minimalistic condition (1) and knows immediately that it has finite limits and colimits, and they preserve both limits and colimits. We wish to put CatSp into a similar context. This can be separated into two (interrelated) questions:

(1) What is a structure on a category $\mathcal{C}$ that puts it on the same stage as the category CatSp of categorical spectra, even to discuss the stability?

(2) What are the appropriate analogs of the conditions listed above? Which generalizes well and which does not?

In the discussion of (2), it seems reasonable that pointedness is kept as is and suspension is replaced by its directed analog. This indicates that for (1), the underlying 1-category is not enough; we do not know any way to recover the suspension functor out of bare 1-category structure on $\infty\mathsf{Cat}_*$ or CatSp. We used either the fixed-point property under enrichment or the Gray-module structure, i.e., $\Sigma = \vec{S}^1 \oslash (-)$. The former is not too common compared to the latter, so we choose the latter; at least it is more handy in the presentable situation. Also, note that we forego the monoidal structure because we would like to think of stability as a kind of linearity. So, in the presentable context, either left modules or bimodules over $\infty\mathsf{Cat}^{\otimes}$ seems to be a good option[1]. For general categories, a reasonable guess for the relevant structure is that of Gray-categories/algebroids (i.e., $\infty\mathsf{Cat}^{\otimes}$-enriched categories); two are related via the functor $\theta'_{\mathcal{A}} : \mathsf{LMod}_{\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}}) \to \widehat{\mathcal{A}\text{-}\mathsf{Cat}}$ of [Ste21, §4.2] (when $\mathcal{A} = \mathsf{S}$, it is the forgetful functor $\mathsf{Pr}^{\mathsf{L}} \hookrightarrow \widehat{Cat}$).

[1] In the bimodule case, we would not want $\vec{S}^1$ to act randomly from the right, compared to the left action. This indicates that we should ask for a $*$-bimodule-like compatibility. In terms of Gray-category, this should imply that left-hom and right-hom enrichments (one is oplax) pass to each other by an appropriate duality. We will not discuss this point further here.

Let us take this as an answer to (1) and move on to (2); from Theorem 4.2.1, we know that CatSp is the universal module where $\vec{S}^1$ acts invertibly, and a presentable $\infty\mathsf{Cat}^{\otimes}$-module admits a (necessarily unique) CatSp-module structure if and only if the $\vec{S}^1$-action is invertible. We moreover have the notion of tensoring for a general enriched category (which are particular instances of *weighted colimits*), so the suspension is (if it exists) a well-defined operation for any pointed Gray-categories. Summarizing the argument, we make the following preliminary definition:

Pre-definition 5.0.2. A Gray-category $\mathcal{C}$ is *stable* if it has a zero object and "finite weighted colimits," and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence.

This has two problems: the first problem is that we do not know what finiteness is appropriate for the weight; our prototypical example CatSp has all weighted colimits, so it will not help us much. The second problem is that this is not a very useful form of stability. Even for the classical stable categories, it is not so easy to show that other useful characterizations follow, and naive imitation of argument in our case fails (mainly because of the failure of pasting law for the directed pullbacks and pushouts). A fundamental problem with Proposition 5.0.1 is that it depends on some clever choice of a generating class of finite colimits that behaves too nicely for us to generalize. In fact, the naive analogs of (3)(4) (replacing (co)fiber by lax (co)fiber and pushout/pullback by directed pushout/pullback) are false in CatSp, so the characterization of stability using diagrams that happens to be both limit and colimit should be regarded as a happy 1-categorical accident.

This suggests that, instead of searching for a minimalistic definition, we should find *all* exactness property enjoyed by CatSp. For stable 1-categories, this question is asked in [Camb]. We take the viewpoint that stability allows us to recognize a finite colimit as a finite limit over another (Sp-weighted) diagram: if J is a finite category, the colimit functor $\mathsf{Fun}(J, \mathsf{Sp}) \to \mathsf{Sp}$ is both limit and colimit-preserving, so it has both left and right adjoints. This left adjoint is necessarily of the form $X \mapsto W \otimes X$

for a diagram $W : J \to \mathsf{Sp}$, so the colimit functor is also the limit functor weighted by W. In this case, we say the colimit is *absolute*; a J-indexed colimit is preserved by any Sp-enriched functor between Sp-enriched categories. We may put the limits and colimits in a symmetric situation by considering *weighted colimits*. Tensoring by an enriching object is an example, in which case the absoluteness is equivalent to the dualizability of the object. The abundance of dualizable objects, i.e., the *Spanier-Whitehead duality* is an important motivation to introduce spectra. We may roughly summarize the circle of ideas by saying that we may approach the stability by studying absolute colimits in CatSp. The appropriate finiteness condition for the weight should be understood along the way, but for now, it seems too early to conclude (note that we do not even know if all finite categorical spectra are dualizable).

Instead, we start by collecting basic examples of absolute colimits. In Section 5.2, we will show the absoluteness of directed pushouts. In this case, the weight is simple and strict enough to allow explicit diagrammatic calculation. We will spend the first section spelling out basic knowledge and examples of directed pushouts. We regard the directed pushout as a bootstrap for constructing other absolute colimits. We will already observe some interesting consequences, but the full strength of this result is yet to be explored.

5.1 Some weighted (co)limits in Gray-categories

5.1.1 Bimodules and weighted colimits

Let $\mathcal{A} \in \mathsf{Alg}(\mathsf{Pr}^{\mathsf{L}})$ be a presentably monoidal category. Our main example of interest is $\mathcal{A} = \infty\mathsf{Cat}^{\otimes}$ and $\mathsf{CatSp}^{\otimes}$. For completeness, we treat weighted colimits of $\mathcal{A}$-enriched categories, but we work in a minimalistic generality; we only treat the case when the weights live in a presheaf category base-changed from S, i.e., when they are indexed over unenriched categories. See e.g. [Ste21, Chapter 5] and [Hin20] for more general discussion (the latter does not mention weighted colimits, but treats the basic setup of Morita theory over a noncommutative base). See also [Str83] for the original treatment of absolute colimits. Let $I, J \in \mathsf{Cat}$ be small categories. We regard $\mathsf{PSh}_{\mathcal{A}}(I) = \mathsf{Fun}(I^{\mathrm{op}}, \mathcal{A}) \simeq \mathsf{PSh}(I) \otimes_{\mathsf{S}} \mathcal{A}$ as a $\mathcal{A}$-bimodule in Pr^{L}. Let $\mathsf{RMod}_{\mathcal{A}}^{\mathrm{free}}(\mathsf{Pr}^{\mathsf{L}}) \subset \mathsf{RMod}_{\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}})$ be the full sub 2-category spanned by the presheaf categories. The hom category is given by

$$\begin{aligned}\mathsf{LFun}_{\mathcal{A}}(\mathsf{PSh}_{\mathcal{A}}(I), \mathsf{PSh}_{\mathcal{A}}(J)) &\simeq \mathsf{LFun}(\mathsf{PSh}(I), \mathsf{PSh}_{\mathcal{A}}(J)) \\ &\simeq \mathsf{Fun}(I, \mathsf{PSh}_{\mathcal{A}}(J)) \simeq \mathsf{Fun}(I \times J^{\mathrm{op}}, \mathcal{A}).\end{aligned}$$

Notice that this functor category admits a left $\mathcal{A}$-module structure induced by that of $\mathsf{PSh}_{\mathcal{A}}(J)$, i.e., by pointwise tensoring on the value.

Definition 5.1.1. An (I, J)-bimodule, or a profunctor $I \rightarrow\!\!\!\!\!\rightarrow J$ is a functor $W : I \times J^{\mathrm{op}} \to \mathcal{A}$. Let $\mathsf{Prof}_{\mathcal{A}} := \mathsf{RMod}_{\mathcal{A}}^{\mathrm{free}}(\mathsf{Pr}^{\mathsf{L}})$ denote the 2-category of (unenriched) categories and profunctors.

We also have an equivalence

$$\mathsf{Fun}(I \times J^{\mathrm{op}}, \mathcal{A}) \simeq \mathsf{Fun}(J^{\mathrm{op}}, \mathsf{PSh}_{\mathcal{A}}(I^{\mathrm{op}})) \simeq \mathsf{LFun}_{\mathcal{A}}(\mathsf{PSh}_{\mathcal{A}}(J^{\mathrm{op}}), \mathsf{PSh}_{\mathcal{A}}(I^{\mathrm{op}})).$$

We warn that $\mathsf{PSh}_{\mathcal{A}}(I) \leftrightarrow \mathsf{PSh}_{\mathcal{A}}(I^{\mathrm{op}})$ induces an equivalence $\mathsf{Prof} \simeq \mathsf{Prof}^{\mathrm{op}}$, so to avoid confusion, we talk about adjunction in this category only after embedding Prof into $\mathsf{RMod}_{\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}})$.

Let $\mathcal{C}$ be a left $\mathcal{A}$-module. Consider the functor

$$I \times I^{\mathrm{op}} \times J \times \mathsf{Fun}(J, \mathcal{C}) \xrightarrow{I(-,-)\times \mathrm{ev}} \mathsf{S} \times \mathcal{C} \to \mathcal{A} \times \mathcal{C} \to \mathcal{C}.$$

By passing to adjoint, we have a functor $I^{\mathrm{op}} \times J \to \mathsf{Fun}(\mathsf{Fun}(J, \mathcal{C}), \mathsf{Fun}(I, \mathcal{C}))$, which factors through $\mathsf{LFun}(\mathsf{Fun}(J, \mathcal{C}), \mathsf{Fun}(I, \mathcal{C}))$. When $\mathcal{C}$ is an $\mathcal{A}$-bimodule (resp. $\mathcal{A}$-algebra), it moreover factors through the hom category $\mathsf{LFun}_{\mathcal{A}}(\mathsf{Fun}(J, \mathcal{C}), \mathsf{Fun}(I, \mathcal{C}))$ in $\mathsf{RMod}_{\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}})$ (resp. $\mathsf{LFun}_{\mathcal{C}}(\mathsf{Fun}(J, \mathcal{C}), \mathsf{Fun}(I, \mathcal{C}))$ in $\mathsf{RMod}_{\mathcal{C}}(\mathsf{Pr}^{\mathsf{L}})$). In any of these cases, the left $\mathcal{A}$-module structure on $\mathcal{C}$ induces that of $\mathsf{LFun}(\mathsf{Fun}(J,\mathcal{C}),\mathsf{Fun}(I,\mathcal{C}))$ by pointwise tensoring on the value, so we may uniquely extend the functor to that of left $\mathcal{A}$-modules[2]:

$$\mathsf{Fun}(I\times J^{\mathrm{op}},\mathcal{A})\to \mathsf{LFun}(\mathsf{Fun}(J,\mathcal{C}),\mathsf{Fun}(I,\mathcal{C})).$$

The image of a bimodule $W : I \twoheadrightarrow J$ is denoted by $W\otimes(-)$ or $\mathrm{colim}^W(-)$ and call it the *colimit weighted by* W. This functor adimts a right adjoint $\mathsf{Fun}(I,\mathcal{C})\to \mathsf{Fun}(J,\mathcal{C})$, which we denote by $[W,-]$ or $\lim^W$ and call the *limit weighted by* W. By adjunction and the Yoneda lemma (we only need it for plain $(\infty,1)$-categories), this is a unique left $\mathcal{A}$-module functor $\mathsf{Fun}(I\times J^{\mathrm{op}},\mathcal{A})\to \mathsf{RFun}(\mathsf{Fun}(I,\mathcal{C}),\mathsf{Fun}(J,\mathcal{C}))^{\mathrm{op}}$ extending $(i,j)\mapsto [J(j,-),\mathrm{ev}_i(-)] : I\times J\to \mathsf{RFun}(\mathsf{Fun}(I,\mathcal{C}),\mathsf{Fun}(J,\mathcal{C}))$. Here we give the transferred left $\mathcal{A}$-module structure on the codomain (by adjunction, an object a of $\mathcal{A}$ acts by precomposing the pointwise cotensor $[a,-]$ on $\mathsf{Fun}(I,\mathcal{C})$).

For colimit, we often take $I=*$ so $W : J^{\mathrm{op}}\to\mathcal{A}$ is a presheaf. In this case, $\mathrm{colim}^W : \mathsf{Fun}(J^{\mathrm{op}},\mathcal{A})\to \mathsf{LFun}(\mathsf{Fun}(J,\mathcal{C}),\mathcal{C})$ is the unique left $\mathcal{A}$-module morphism with the natural equivalence $よ(j)\otimes F\simeq F(j)$; this is the so-called *Yoneda reduction*. [Ste21, Example 5.5.18] implies that this definition of weighted colimits agrees with that of Stefanich. In this case, the weighted limit $\mathcal{C}\to\mathsf{Fun}(J,\mathcal{C})$ is simlpy the pointwise cotensor $X\mapsto (j\mapsto [W(j),X])$. The other extreme case $J=*$ switches the role of limits and colimits: the functor $\mathsf{Fun}(I,\mathcal{A})\to \mathsf{LFun}_{\mathcal{A}}(\mathcal{C},\mathsf{Fun}(I,\mathcal{C}))$ takes W to $X\mapsto (i\mapsto W(i)\otimes X)$. The weighted limit $[W,-] : \mathsf{Fun}(I,\mathcal{C})\to\mathcal{C}$ is characterized by the Yoneda reduction $[I(i,-),F]=F(i)$ and the fact that it turns colimits into limits, tensor into cotensor ($[a\otimes W,F]\simeq [W,[a,F]]$). In particular, when $I=J=*$, the weighted limit and colimit are tensor and cotensor by the left action of $\mathcal{A}$ on $\mathcal{C}$.

For the study of stability, we are interested in the case when a weighted colimit

[2]Notice the close analogy with the matrix calculus; a profunctor corresponds to a matrix $W_{ij} : I\times J\to A$, where A is a base ring and I, J are some sets. For a left A-module M, we can define a multiplication $M^{\oplus J}\to M^{\oplus I}$ by a matrix W, by extending the action of the elementary matrix $I\times J\to \mathrm{Hom}(M^{\oplus I},M^{\oplus J}); (i,j)\mapsto \iota_j\circ \mathrm{pr}_i$ to a left A-module homomorphism.

$W \otimes (-)$ is also a weighted limit $[W', -]$:

Definition 5.1.2. A weight $W : I \nrightarrow J$ is *absolute* if the weighted colimit functor $W \otimes (-) : \mathsf{Fun}(J, \mathcal{A}) \to \mathsf{Fun}(I, \mathcal{A})$ admits a left adjoint in $\mathsf{RMod}_{\mathcal{A}}(\mathsf{Pr}^{\mathsf{L}})$. In this case, the left adjoint is given by $W^L \otimes (-)$ for some W^L, which we call the *left dual* of the W.

In this case, we have an equivalence $W \otimes (-) \simeq [W^L, -]$.

Remark 5.1.3. Enriched categories and profunctors are algebras and bimodules in a 2-category spanned by $\mathsf{Fun}(X, \mathcal{A})$ for $X \in \mathsf{S}$, or the category of enriched quivers in [Hin20] (the author thank Stefanich for this suggestion). In this context, the absoluteness of the weight is equivalent to the left dualizability as a bimodule.

Example 5.1.4. A profunctor $* \nrightarrow *$ is given by an object $a \in \mathcal{A}$. This weight is absolute precisely when a is left dualizable.

5.1.2 Directed pushouts and pullbacks

The goal of this section is to discuss a particular example of (partially lax) (co)limits of our interest: directed pushouts and directed pullbacks[3]. We will also address some subtleties around Gray-categories as we encounter them. Let us start with a minimalistic working definition of directed pushout and pullbacks:

Definition 5.1.5. Let $\mathcal{C}$ be a presentable category with the action map $\otimes : \mathsf{Cat} \otimes \mathcal{C} \to \mathcal{C}$ in Pr^{L} and let $[X, -] : \mathcal{C} \to \mathcal{C}$ denote the right adjoint to $X \otimes (-)$. The *directed pushout* of a span $B \leftarrow A \to C$, denoted by $B \overrightarrow{\amalg}_A C$, is the colimit of the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr, "0"] & & A \arrow[dl, "1"'] \arrow[dr] & \\
B & & \square^1 \otimes A & & C
\end{tikzcd}$$

[3]The traditional names are cocomma and comma constructions. Other popular names are lax pushouts and lax pullbacks. The latter is somewhat more descriptive, but it has a problem of crashing the general notion of (fully) lax limits and colimits. Lurie uses the term oriented fiber product for our directed pullback in [Lur, Tag 01KE].

Dually, the *directed pullback* of a cospan $B \to D \leftarrow C$, denoted by $B \overrightarrow{\times}_D C$, is the limit of the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
B \arrow[dr] & & {[\square^1, D]} \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "\mathrm{ev}_1"] & & C \arrow[dl] \\
& D & & D &
\end{tikzcd}$$

A slightly more sophisticated definition is given by weighted (co)limits. Note that the directed pushout and pullback naturally fit into a cospan $B \to B \overrightarrow{\amalg}_A \leftarrow C$ and a span $B \leftarrow B \overrightarrow{\times}_D C \to C$. Let $\mathcal{C} \in \mathsf{LMod}_{\infty\mathsf{Cat}^{\otimes}}(\mathsf{Pr}^{\mathsf{L}})$. When J is an 1-category, $\mathsf{Fun}(J, \mathcal{C}) \in \mathsf{Pr}^{\mathsf{L}}$ will mean the functor (1-)category to the underlying category $\mathcal{C} \in \mathsf{Pr}^{\mathsf{L}}$. Let $J = \Lambda^2_0 = \{1 \leftarrow 0 \to 2\}$ be the walking cospan category and $W_{\bullet\bullet} : J^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathsf{Cat}$ be the weight given by the commutative diagram

$$\begin{tikzcd}
\square^0 \arrow[r, "="] \arrow[d, "="] & \square^0 \arrow[d, "0"] & \emptyset \arrow[l] \arrow[d] \\
\square^0 \arrow[r, "0"] & \square^1 & \square^0 \arrow[l, "1"'] \\
\emptyset \arrow[r] \arrow[u] & \square^0 \arrow[u, "1"] & \square^0, \arrow[l, "="] \arrow[u, "="']
\end{tikzcd}$$

Note that $\square^0 \xrightarrow{0} \square^1 \xleftarrow{1} \square^0$ is canonically expressed as the colimit of the following diagram of presheaves over Λ^2_0:

$$\square^0 \otimes \text{よ}(1) \leftarrow \square^0 \otimes \text{よ}(0) \xrightarrow{0} \square^1 \otimes \text{よ}(0) \xleftarrow{1} \square^0 \otimes \text{よ}(0) \to \square^0 \otimes \text{よ}(2).$$

By the characterization of the weighted (colimits) using the Yoneda reduction and preservation of colimits and tensoring, we see that $W_{\bullet\bullet}$ is the weight for the directed pushouts and pullbacks. More precisely, the weight induces the following adjunction between the category of spans and cospans:

$$\begin{tikzcd}
\overrightarrow{\amalg} : \mathsf{Fun}(J, \mathcal{C}) \arrow[r, shift left, "\mathrm{colim}^W"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(J^{\mathrm{op}}, \mathcal{C}) : \overrightarrow{\times} \arrow[l, shift left, "\lim^W"]
\end{tikzcd}$$

Example 5.1.6. Directed pushouts and directed pullbacks specialize to many important constructions.

(1) The *suspension* functor $\Sigma : \mathcal{C} \to \mathcal{C}_{**}$ is defined by $\Sigma X := * \overrightarrow{\amalg}_X *$. When $\mathcal{C} = \infty\mathsf{Cat}^{\otimes}$, it agrees with the original definition of the unreduced suspension thanks to Lemma 2.5.1. When $\mathcal{C} = \infty\mathsf{Cat}_*^{\oslash}$ (where $X \in \infty\mathsf{Cat}$ acts by $X_+ \oslash (-)$), we have $\mathcal{C} = \mathcal{C}_{**}$ and recover the reduced suspension by Proposition 3.1.9 and

$$\vec{S}^1 \oslash X \simeq \operatorname{colim}(0 \leftarrow S^0 \xrightarrow{0} \Box^1_+ \xleftarrow{1} S^0 \to 0) \oslash X$$

$$\simeq \operatorname{colim}(0 \leftarrow X \xrightarrow{0} \Box^1_+ \oslash X \xleftarrow{1} X \to 0).$$

(2) The *hom* functor $\mathcal{C}_{**} \to \mathcal{C}$ is defined by $(X; x_0, x_1) \mapsto X(x_0, x_1) := \{x_0\} \overrightarrow{\times}_X \{x_1\}$. Again, by Corollary 2.5.5 it recovers the self-enrichment of $\infty\mathsf{Cat}$.

(3) The *loop* functor $\Omega : \mathcal{C}_* \to \mathcal{C}_*$ is $(X, x) \mapsto \Omega(X, x) := X(x, x)$. It is equivalent to the pointed internal hom $[\vec{S}^1, -]$ and recovers the loop functors on $\infty\mathsf{Cat}_*$ and CatSp.

(4) The *lax cofiber* (resp. *oplax cofiber*) of $f : X \to Y \in \mathcal{C}$ is $\overrightarrow{\operatorname{cof}} f := * \overrightarrow{\amalg}_X Y$ (resp. $\overleftarrow{\operatorname{cof}} f := Y \overrightarrow{\amalg}_X *$).

(5) Dually, the *lax fiber* (resp. *oplax fiber*) of a morphism $f : X \to Y \in \mathcal{C}$ over $* \xrightarrow{y} Y$ is $\overrightarrow{\operatorname{fib}} f := \{y\} \overrightarrow{\times}_Y X$ (resp. $\overleftarrow{\operatorname{fib}} f := X \overrightarrow{\times}_Y \{y\}$). For $X \in \mathcal{C}_*$, we let $\overrightarrow{\operatorname{path}}(X) = \overrightarrow{\operatorname{fib}}(X \xrightarrow{\mathrm{id}} X)$ and $\overleftarrow{\operatorname{path}}(X) = \overleftarrow{\operatorname{fib}}(X \xrightarrow{\mathrm{id}} X)$.

(6) Additionally, we will use the following notations: $\overrightarrow{\operatorname{cone}}(X) := \overrightarrow{\operatorname{cof}}(X \xrightarrow{\mathrm{id}} X)$, $\overleftarrow{\operatorname{cone}}(X) := \overleftarrow{\operatorname{cof}}(X \xrightarrow{\mathrm{id}} X)$, $\operatorname{cyl}(X) := X \overrightarrow{\amalg}_X X \simeq \Box^1 \otimes X$, $\overrightarrow{\operatorname{cyl}}(X \to Y) = X \overrightarrow{\amalg}_X Y$, $\overleftarrow{\operatorname{cyl}}(X \to Y) = Y \overrightarrow{\amalg}_X X$. $I := \overrightarrow{\operatorname{cone}}(S^0) \in \infty\mathsf{Cat}_*$; it is the interval $0 \to 1$ with the basepoint 0. Notice $I^{\mathrm{op}} = I^{\circ} = \overleftarrow{\operatorname{cone}}(S^0)$. If $\mathcal{C}$ is pointed, we have $\overrightarrow{\operatorname{cone}}(X) \simeq I \oslash X$ and $\overleftarrow{\operatorname{cone}}(X) \simeq I^{\mathrm{op}} \oslash X$.

For more specific examples, we give a few sample calculations of lax (co)fibers.

Example 5.1.7. (1) $\Sigma^\infty I = \mathrm{B}^\infty \operatorname{Free}_{\mathbb{E}_\infty}(I)$ is the symmetric monoidal category freely generated by an object c and a morphism $1 \to c$, or in other words, a single $\mathbb{E}_0$-algebra. In particular, we have an explicit description $\operatorname{Free}_{\mathbb{E}_\infty}(I) = \operatorname{Env}(\mathbb{E}_0) = \mathsf{Fin}^{\mathrm{inj}}$ (where Env is the monoidal envelope of [Lur17, §2.2.4]). The cofiber map $\vec{S}^0 \to I$ induces the inclusion $\mathsf{Fin}^{\simeq} = \operatorname{Env}(\mathrm{Triv}) \to \operatorname{Env}(\mathbb{E}_0) = \mathsf{Fin}^{\mathrm{inj}}$.

(2) The Proposition A.3.5 shows $\overleftarrow{\mathrm{cone}}(\Delta^n) = \Delta^{n+1}$ strictly, and [GHon] proves that this pushout is weak. One could define orientals inductively by the cone construction starting Δ^0. More generally, one can think of $X \overrightarrow{\amalg}_{X \otimes Y} Y$ as the (lax) join of X and Y.

(3) Let $\mathsf{Fin}^{\simeq} \to \mathbb{N}^{\simeq}$ be the 0-truncation map in $\mathsf{CMon}(\mathsf{S})$ and let us (heuristically) compute the lax cofiber $0 \overrightarrow{\amalg}_{\mathsf{Fin}^{\simeq}} \mathbb{N}^{\simeq}$ in $\infty\mathsf{SMCat}$ or equivalently in CatSp. It is a symmetric monoidal category that corepresents a morphism $1 \to c$, where c *strictly commutes* with itself with respect to the symmetric monoidal structure. As we saw above, without strictness, this is $\mathsf{Fin}^{\mathrm{inj}}$ with objects $c^{\otimes n}$ for $n \geq 0$, but we force the Σ_n-action on $c^{\otimes n}$ to be trivial, so the hom groupoid $\operatorname{Hom}(c^{\otimes m}, c^{\otimes n})$ is the quotient $\operatorname{Hom}^{\mathrm{inj}}(\langle m \rangle, \langle n \rangle)/\Sigma_n \simeq \mathrm{B}\Sigma_{n-m}$.

(4) Next, let us compute the lax fiber $0 \vec{\times}_{\mathrm{B}\mathbb{N}^{\simeq}} \mathrm{B}\mathsf{Fin}^{\simeq}$. Note that this is equivalent to the pullback of $\mathrm{B}\mathbb{N}^{\simeq}_{*//} \to \mathrm{B}\mathbb{N}^{\simeq} \leftarrow \mathrm{B}\mathsf{Fin}^{\simeq}$, where $\mathrm{B}\mathbb{N}^{\simeq}_{*//}$ is the fiber of $\mathrm{ev}_0 : [\Delta^1, \mathrm{B}\mathbb{N}] \to \mathrm{B}\mathbb{N}$, or the *lax undercategory* of $\mathrm{B}\mathbb{N}^{\simeq}$ under the basepoint. However, $\mathrm{B}\mathbb{N}$ is just a 1-category, so the lax under category is the same as the undercategory, i.e., $\mathrm{B}\mathbb{N}^{\simeq}_{*//} = \mathrm{B}\mathbb{N}^{\simeq}_{*/} = \mathbb{N}$; we get the *ordered set* of natural numbers because $* \xrightarrow{n} * \in \mathrm{B}\mathbb{N}^{\simeq}_{*/}$ factors through another object $* \xrightarrow{m} *$ if and only if there is k such that $* \xrightarrow{m} * \xrightarrow{k} * = * \xrightarrow{n} *$, i.e., $m+k=n$, or $m \leq n$. The upshot is that the lax fiber in question is the fiber of $\mathbb{N} \to \mathrm{B}\mathbb{N}^{\simeq} \leftarrow \mathrm{B}\mathsf{Fin}^{\simeq}$. This can be

explicitly computed as a $(2,1)$-category: the objects are the natural numbers and the morphisms are given by $\mathrm{Hom}(m,n) = \mathrm{B}\Sigma_{n-m}$. Note that we obtained $\overrightarrow{\mathrm{cof}}(\mathrm{B}^\infty\mathsf{Fin}^\simeq \to \mathrm{B}^\infty\mathbb{N}^\simeq) \simeq \overrightarrow{\mathrm{fib}}(\mathrm{B}^{\infty+1}\mathsf{Fin}^\simeq \to \mathrm{B}^{\infty+1}\mathbb{N}^\simeq)$ by computing the both hands sides independently.

We intuitively know that the directed pushouts and pullbacks are universal lax squares extending spans and cospans. The next goal is to make this precise; since our target category $\mathcal{C}$ (e.g. CatSp) is only a Gray-category and not an honest 2-category, we must be careful about what we even mean by a lax square in $\mathcal{C}$.

Remark 5.1.8. Recall the forgetful functor $\iota : \infty\mathsf{Cat} \simeq \infty\mathsf{Cat}^\times\text{-}\mathsf{Cat} \to \infty\mathsf{Cat}^\otimes\text{-}\mathsf{Cat}$ induced by the lax monoidal functor $\mathrm{id} : \infty\mathsf{Cat}^\times \to \infty\mathsf{Cat}^\otimes$. It does not preserve colimits in general; for instance, the "long" hom category of $\iota(C_2 \vee C_2)$ is $C_1 \times C_1$, whereas that of $(\iota C_2) \vee (\iota C_2)$ is $\square^2$. However, the following colimit-decomposition remains true in $\infty\mathsf{Cat}^\otimes\text{-}\mathsf{Cat}$.

$$\iota\square^2 = \iota\Delta^{\{00,10,11\}} \cup_{\iota\Delta^{\{00,11\}}=\sigma(\{0\})} \iota\sigma\square^1 \cup_{\iota\sigma(\{1\})=\Delta^{\{00,11\}}} \iota\Delta^{\{00,01,11\}}.$$

Definition 5.1.9. A *lax square* in $\mathcal{C}$ is a functor $\iota\square^2 \to \mathcal{C}$ in $\infty\mathsf{Cat}^\otimes\text{-}\mathsf{Cat}$.

Using the colimit-decomposition of $\iota\square^2$, we see that the data of the lax square $\iota\square^2 \to \mathcal{C}$ which we depict as

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\alpha"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is (functorially) equivalent to either of the following commutative diagrams:

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "i_0"] & B \arrow[dd] \\ A \arrow[r, "i_1"] \arrow[d] & \square^1 \otimes A \arrow[dr, "\alpha"] & \\ C \arrow[rr] & & D, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr] \arrow[dr, "\alpha"] \arrow[dd] & & B \arrow[d] \\ & {[\square^1, D]} \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] & D \\ C \arrow[r] & D & \end{tikzcd}$$

Remark 5.1.10. When we write a diagram with more than one cell of dimension at least 2, e.g.,

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\alpha}"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\beta}"] & E \arrow[d] \\ C \arrow[r] & D \arrow[r] & F, \end{tikzcd}$$

in $\mathcal{C}$, it should be interpreted as a functor $\iota\Box^2 \sqcup_{\iota\Box^1} \iota\Box^2 \to \mathcal{C}$, not $\iota(\Box^2 \sqcup_{\Box^1} \Box^2) \to \mathcal{C}$. The latter would mean the existence of a well-defined pasting 2-cell $\beta*\alpha$, but in Gray-category, one would only get a 2-cell up to a natural transformation, i.e., a 3-cell. The author expects the existence of the localization $\infty\mathsf{Cat}^{\otimes}\text{-}\mathsf{Cat} \to \infty\mathsf{Cat}$ left adjoint to ι; one can think of this as the localization by isomorphisms of various pasting theorems. If $\mathcal{C}$ is a ∞-category, then there is a well-defined pasting $\beta * \alpha$.

The pasting law of the pullback and pushout squares is rather restrictive:

Proposition 5.1.11. *In the following diagram, suppose that α is a directed pushout square.*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow_{\alpha}"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq\nearrow\!\!\!\!\nearrow_{\beta}"] & E \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq\nearrow\!\!\!\!\nearrow_{\gamma}"] & D \arrow[r] \arrow[d] & F \\ G \arrow[r] & H & \end{tikzcd}$$

Suppose moreover that β, γ are invertible. Then

(1) *$\beta*\alpha$ is well-defined and is a directed pushout square if and only if β is a pushout square.*

(2) *$\gamma*\alpha$ is well-defined and is a directed pushout square if and only if γ is a pushout square.*

Similar assertions hold for directed pullback and pullback squares.

Proof. It follows from the pasting law of pullback squares together with the fact that the colimit of $C \leftarrow A \to \Box^1 \otimes A \leftarrow A \to B$ can be computed by forming two pushout squares. $\Box$

Notice that even when $\mathcal{C} = \infty\mathsf{Cat}$, the formation of directed pullback and directed pushout squares is not 2-functorial. However, the half-central structure of $\vec{S}^1$ gives rise to an exceptional functoriality of suspensions. Note that when the underlying category of $\mathcal{C}$ is pointed (so $\mathcal{C} \simeq \mathsf{S}_* \otimes \mathcal{C}$), the action of $\infty\mathsf{Cat}$ canonically factors through $\infty\mathsf{Cat}_*$ by $X \otimes (-) = X_+ \oslash (-)$.

Proposition 5.1.12. *Let $\mathcal{C}$ be an object of $\mathsf{LMod}_{\infty\mathsf{Cat}_*}(\mathsf{Pr}^\mathsf{L})$. The the suspension functor induces $\Sigma : \mathcal{C} \to \mathcal{C}$ induces $\mathrm{Map}(\iota\square^2, \mathcal{C}) \to \mathrm{Map}(\iota(\square^2)^{2\text{-op}}, \mathcal{C})$, which is depicted by*

$$\left(\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow_{\alpha}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} \Sigma A \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow_{\Sigma\alpha}"] & \Sigma C \arrow[d] \\ \Sigma B \arrow[r] & \Sigma D \end{tikzcd}\right)$$

Proof. Using the half-central structure of $\vec{S}^1$, we have $\Sigma(\square^1_+ \oslash A) \simeq (\vec{S}^1 \oslash \square^1_+) \oslash A \xrightarrow{\sim} ((\square^1)^{\mathrm{op}}_+ \oslash \vec{S}^1) \oslash A$. This filps the weight $*_+ \xrightarrow{0} \square^1_+ \xleftarrow{1} *_+$ to $*_+ \xrightarrow{1} \square^1_+ \xleftarrow{0} *_+$. $\square$

The proposition is reflected already in the classical definition of triangulated categories. Namely, this is the negative sign introduced when we rotate the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ to $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$; the suspension converts a lax fiber sequence to an oplax fiber sequence, so one flips the direction of homotopy back to lax fiber sequence by negating the maps. Since we do not have negatives in our categorical setting, we must distinguish lax and oplax fiber sequences.

5.2 Absoluteness of directed pushouts

The goal of this section is to prove the absoluteness of directed pushouts and begin the study of absolute colimits in CatSp. Our basic strategy is to guess the dual weight and exhibit the adjunction of weighted colimits directly. We will first prove the case of the cone construction, i.e., $0 \vec{\amalg}_X X = I \oslash X$. In this case, absoluteness is equivalent to the dualizability of $\Sigma^\infty I$. Even though this is a special case of the main theorem, verification of the dualizability of I nicely packages a part of its proof, so we give a separate treatment. After the main theorem, we will list some immediate consequences. A corollary of particular importance is the equivalence $\overrightarrow{\mathrm{cof}}\, f \simeq \overrightarrow{\mathrm{fib}}\, \Sigma f$ for a morphism of categorical spectra $f : X \to Y$ (as observed for a particular case in Example 5.1.7). Curious as it is, indicating both similarities and differences from the stable 1-categories, it is also useful for computations; we will apply this to the study of TQFT in the next chapter.

5.2.1 $\Sigma^\infty I$ is dualizable

Recall the notation $I := \overrightarrow{\mathrm{cof}}(S^0 \to S^0)$. In this section, we will show that $\Sigma^\infty I$ is dualizable in categorical spectra. To guess the dual object $(\Sigma^\infty I)^L$, recall that we hoped to show that the cone and the path category are equivalent up to shift: $\overrightarrow{\mathrm{cone}}(X) \simeq \overrightarrow{\mathrm{path}}(X[1])$. This is equivalent to $I \otimes X \simeq [I, \vec{S}^1 \oslash X] \simeq [\Sigma^{\infty-1} I, X]$, which suggests the following:

Proposition 5.2.1. *$\Sigma^{\infty-1} I$ is the left dual of $\Sigma^\infty I$ with respect to the tensor product of categorical spectra.*

Remark 5.2.2. The proposition should be eventually understood as an example of *caetgorical Atiyah duality* and the degree of shift 1 accounts for the dimension of the interval. Observe that the proof even resembles the usual construction of the Spanier-Whitehead duality by embedding a space into a sphere and contracting the neighborhood, although it is unclear how the idea generalizes. This is a subject of the ongoing extension of this project.

Remark 5.2.3. Since the localization $\mathsf{CatSp} \to \mathsf{Sp}$ is monoidal, any monoidal duality in CatSp gives rise to one in Sp. The duality of the proposition localizes to the trivial duality $0 \dashv 0$.

Remark 5.2.4. Suppose L is the left dual of R in CatSp (so that $L \otimes (-) \dashv [L, -] \simeq R \otimes (-)$). By composing the adjoint inverses $[1]$ and $[-1]$, one sees $\Sigma^{\pm 1} L^\circ \dashv \Sigma^{\mp 1} R$

and $\Sigma^{\pm 1} L \dashv \Sigma^{\mp 1} R^\circ$. In particular, we have the four-periodic cycle

$$\Sigma^\infty I \dashv \Sigma^{\infty-1} I^{\mathrm{op}} \dashv \Sigma^\infty I^{\mathrm{op}} \dashv \Sigma^{\infty-1} I \dashv \Sigma^\infty I,$$

so $\Sigma^\infty I$ is also right-dualizable.

Remark 5.2.5. Another clue for the dual object comes from the Steiner's theory: λI is the augmented chain complex $\mathbb{Z}\underline{e} \to \mathbb{Z}\underline{0} \oplus \mathbb{Z}\underline{1}$ with the basepoint $\underline{0}$. The dual complex of the reduced complex is $\mathbb{Z}\underline{1}^\vee \to \mathbb{Z}\underline{e}^\vee$ in degrees $[-1, 0]$. The dexterity of the dual determines the natural positive part, giving the desuspension of the reduced augmented directed complex of $\Sigma^\infty I^{(\mathrm{op})}$. It seems possible to have a nonconnective version of Steiner theory relating pointed augmented complexes to a reasonably strict part of categorical spectra. A part of the difficulty would be that our version of suspension is essentially not strict, let alone loop-free. In analogy to the case of spectra, it should be rectified by passing to strictly commutative monoid objects, or $\mathrm{H}\mathbb{N}$-modules. The strictification process seems moreover likely to have a Dold-Thom style interpretation.

Proof. Let us first construct the unit and the counit of the duality.

(1) To define the counit $\varepsilon : \Sigma^{\infty-1} I \otimes \Sigma^\infty I \to \mathbb{F}$, it suffices to define $\vec{S}^1 \otimes \varepsilon : \Sigma^\infty(I \oslash I) \to \mathbb{F}[1]$. We define it as the Σ^∞ of the map $I \oslash I \to \vec{S}^1 \simeq \mathrm{B}\mathbb{N}$ depicted as follows:

$$\begin{tikzcd} 00 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow"] & 10 \arrow[d] & & * \arrow[r, equal, "0"] \arrow[d, equal, "0"'] \arrow[dr, phantom, "/\!\!/"] & * \arrow[d, "1"] \\ 01 \arrow[r] & 11 & \mapsto & * \arrow[r, "1"] & *. \end{tikzcd}$$

The diagram shows the image of the atomic cells of $\square^2$ under the composition $\square^2 \twoheadrightarrow I \oslash I \to \mathrm{B}\mathbb{N}$, which descends to the quotient $I \oslash I$ because the restriction to $\{0\} \otimes \square^1 \cup \square^1 \otimes \{0\}$ is trivial.

(2) To define the unit map $\eta : \mathbb{F} \to \Sigma^\infty I \otimes \Sigma^{\infty-1} I = \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \Sigma^\infty I$, we use the half-central structure on $\vec{S}^1$ to pull out the desuspension to the left. Namely,

we define $\vec{S}^1 \otimes \eta$ so that

$$\vec{S}^1 \otimes \mathbb{F} \xrightarrow{\vec{S}^1 \otimes \eta} \vec{S}^1 \otimes \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \xrightarrow[\sim]{\tau_I \otimes \mathrm{id}} \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^1 \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \simeq \Sigma^\infty (I^{\mathrm{op}} \owedge I).$$

is induced by the map $\vec{S}^1 \to I^{\mathrm{op}} \owedge I; l \mapsto rs$, where l is the generating loop of $\vec{S}^1$ and r, s are the 1-cells depicted as follows (we identify I^{op} with the interval $0 \to 1$ with the vertex 1 marked, so $I^{\mathrm{op}} \owedge I$ is the quotient of $\square^2$ by $\square^1 \otimes \{0\} \cup \{1\} \otimes \square^1$):

$$I^{\mathrm{op}} \owedge I = \begin{tikzcd} 00 \arrow[r, equal] \arrow[d, "s"] & 10 \arrow[d, equal] \arrow[dl, phantom, "\Swarrow"] \\ 01 \arrow[r, "r"] & 11. \end{tikzcd}$$

We could have defined η by pulling out the desuspension to the *right*. The coherence data of the half-central structure of $\vec{S}^1$ verifies that two possible definitions are the same. To see this, notice the following diagram canonically commutes:

$$\begin{tikzcd} \vec{S}^1 \otimes \mathbb{F} \arrow[r, "\vec{S}^1 \otimes \eta"] \arrow[d, equal, "\tau_{\mathbb{F}}"] & \vec{S}^1 \otimes \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \arrow[r, "\tau_I"] \arrow[d, "\tau_{I \otimes \vec{S}^{-1} \otimes I}"] & \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^1 \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \arrow[r] \arrow[d, "\tau_{\vec{S}^{-1}}"] & \Sigma^\infty (I^{\mathrm{op}} \owedge I) \arrow[d, equal] \\ \mathbb{F}^\circ \otimes \vec{S}^1 \arrow[r, "\eta^\circ \otimes \vec{S}^1"] & \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^{-1} \otimes \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^1 \arrow[r, "\tau_I^{-1}"] & \Sigma^\infty I^{\mathrm{op}} \otimes \vec{S}^{-1} \otimes \vec{S}^1 \otimes \Sigma^\infty I \arrow[r] & \Sigma^\infty (I^{\mathrm{op}} \owedge I). \end{tikzcd}$$

Taking the total dual of the bottom composite, we see that η is also characterized by that

$$\mathbb{F} \otimes \vec{S}^1 \xrightarrow{\eta \otimes \vec{S}^1} \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \Sigma^\infty I \otimes \vec{S}^1 \to \Sigma^\infty I \otimes \vec{S}^{-1} \otimes \vec{S}^1 \otimes \Sigma^\infty I^{\mathrm{op}} \to \Sigma^\infty (I \owedge I^{\mathrm{op}})$$

is the Σ^∞ of the loop $r's' : \vec{S}^1 \to I \owedge I^{\mathrm{op}}$, where r', s' are the 1-cells depicted as

$$I \owedge I^{\mathrm{op}} = \begin{tikzcd} 00 \arrow[r, "s'"] \arrow[d, equal] & 10 \arrow[d, "r'"] \arrow[dl, phantom, "\Swarrow"] \\ 01 \arrow[r, equal] & 11. \end{tikzcd}$$

Now we check the triangle identities, i.e., that the following compositions are equivalent to the identities:

(1) $\Sigma^\infty I \xrightarrow{\eta \otimes \Sigma^\infty I} \Sigma^\infty I \otimes \Sigma^{\infty - 1} I \otimes \Sigma^\infty I \xrightarrow{\Sigma^\infty I \otimes \eta} \Sigma^\infty I,$

(2) $\Sigma^{\infty-1}I \xrightarrow{\Sigma^{\infty-1}I\otimes\eta} \Sigma^{\infty-1}I\otimes\Sigma^{\infty}I\otimes\Sigma^{\infty-1}I \xrightarrow{\varepsilon\otimes\Sigma^{\infty-1}I} \Sigma^{\infty-1}I.$

For (1), consider the following diagram:

$$\begin{tikzcd}
\vec{S}^1\otimes\Sigma^\infty I \arrow[r, "\vec{S}^1\otimes\eta\otimes I"] \arrow[dr, equal] & \vec{S}^1\otimes\Sigma^\infty I\otimes\Sigma^{\infty-1}I\otimes\Sigma^\infty I \arrow[r, "\tau_I\otimes\mathrm{id}", "\sim"'] \arrow[d, "\vec{S}^1\otimes I\otimes\varepsilon"] & \Sigma^\infty I^{\mathrm{op}}\otimes\Sigma^\infty I\otimes\Sigma^\infty I \arrow[d, "I^{\mathrm{op}}\otimes\vec{S}^1\otimes\varepsilon"] \\
& \vec{S}^1\otimes\Sigma^\infty I \arrow[r, "\tau_I", "\sim"'] & \Sigma^\infty I^{\mathrm{op}}\otimes\vec{S}^1
\end{tikzcd}$$

The right square commutes, so the commutativity of the left triangle reduces to that of the outer compositions. By definition, these are the Σ^∞ of unstable maps $\vec{S}^1 \oslash I \to I^{\mathrm{op}} \oslash I \oslash I \to I^{\mathrm{op}} \oslash \vec{S}^1$, which in turn induced from the maps of cubes $\square^2 \to \square^3 \to \square^2$ by passing to quotients. Therefore we may compute the compositions by tracing the assignments of the relevant atomic cells of the cubes:

$$\begin{tikzcd}
00 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\theta\Nearrow"] & 01 \arrow[d, "l"] \\
10 \arrow[r, equal] & 11
\end{tikzcd}
\to
\begin{tikzcd}[column sep=small]
& & 001 \arrow[dd, "s"] \arrow[drr, equal] & & \\
000 \arrow[urr, equal] \arrow[dd, equal] & \beta\Nearrow & & \gamma\Swarrow & 101 \arrow[dd, equal] \\
& & 011 \arrow[drr, "r"] & & \\
010 \arrow[urr, "t"] \arrow[drr, equal] & & \alpha\Uparrow & & 111 \\
& & 110 \arrow[urr, equal] & &
\end{tikzcd}
\to
\begin{tikzcd}
00 \arrow[r, "l'"] \arrow[d, equal] \arrow[dr, phantom, "\theta'\Nearrow"] & 01 \arrow[d, equal] \\
10 \arrow[r, equal] & 11
\end{tikzcd}$$

The first map assigns $l \mapsto rs$, $l \mapsto \alpha * \beta$ and the second map assigns $s, t \mapsto l'$, $r \mapsto \mathrm{id}_*$, $\beta \mapsto \mathrm{id}_{l'}$, $\alpha, \gamma \mapsto \theta'$. The composition therefore assigns $l \mapsto l'$, $\theta \mapsto \theta'$ so it is equal to $\tau_I : \vec{S}^1 \oslash I \to I \oslash I^{\mathrm{op}} \oslash \vec{S}^1$. Verification of (2) is similar. Tensoring $\vec{S}^1$ from both left and right, (2) is equivalent to identity if and only if the outer compositions of the following diagram commute:

$$\begin{tikzcd}
\Sigma^\infty I\otimes\vec{S}^1 \arrow[r, "I\otimes\eta\otimes\vec{S}^1"] \arrow[dr, equal] & \Sigma^\infty I\otimes\Sigma^\infty I\otimes\Sigma^{\infty-1}I\otimes\vec{S}^1 \arrow[r, "I\otimes I\otimes\tau_I^{-1}", "\sim"'] \arrow[d, "\vec{S}^1\otimes\varepsilon\otimes\Sigma^{\infty-1}I\otimes\vec{S}^1"] & \Sigma^\infty I\otimes\Sigma^\infty I\otimes\Sigma^\infty I^{\mathrm{op}} \arrow[d, "\vec{S}^1\otimes\varepsilon\otimes\mathrm{id}"] \\
& \Sigma^\infty I\otimes\vec{S}^1 \arrow[r, "\tau_I^{-1}", "\sim"'] & \vec{S}^1\otimes\Sigma^\infty I^{\mathrm{op}}.
\end{tikzcd}$$

Using the second description of η, the top-right composition computes as Σ^∞ of the unstable map $I \oslash \vec{S}^1 \to I \oslash I \oslash I^{\mathrm{op}} \to \vec{S}^1 \oslash I^{\mathrm{op}}$ induced from the $\square^2 \to \square^3 \to \square^2$, depicted as:

$$
\begin{tikzcd}
00 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\theta\,\Nearrow"] & 01 \arrow[d, equal] \\
10 \arrow[r, "l"] & 11
\end{tikzcd}
\quad \to \quad
\begin{tikzcd}[column sep=small]
 & & 010 \arrow[drr, equal] \arrow[dd, "t"] & & \\
000 \arrow[urr, equal] \arrow[dd, equal] \arrow[r, phantom, "\alpha\Nearrow"] & & & \arrow[l, phantom, "\beta\Nearrow"] & 011 \arrow[dd, equal] \\
 & & 110 \arrow[drr, "r"] \arrow[d, phantom, "\Downarrow\gamma"] & & \\
100 \arrow[urr, "s"] \arrow[drr, equal] & & {} & & 111 \\
 & & 101 \arrow[urr, equal] & &
\end{tikzcd}
\quad \to \quad
\begin{tikzcd}
00 \arrow[r, equal] \arrow[d, "l'"'] \arrow[dr, phantom, "\theta'\,\Nearrow"] & 01 \arrow[d, equal] \\
10 \arrow[r, equal] & 11
\end{tikzcd}
.
$$

The assignment of relevant cells are $l \mapsto rs$, $\theta \mapsto \alpha * \beta$, $t, s \mapsto l'$, $r \mapsto \mathrm{id}_*$, $\alpha \mapsto \mathrm{id}_l$, $\beta, \gamma \mapsto \theta$, so they compose to $\tau_I^{-1} : l \mapsto l', \theta \mapsto \theta'$. $\square$

5.2.2 Directed pushouts are absolute

As before, we let $J = \Lambda_0^2 = (1 \leftarrow 0 \rightarrow 2)$ be the walking cospan category and $W : (\Lambda_0^2)^{\mathrm{op}} \to \infty\mathsf{Cat}_*$ be the weight $S^0 \to \square_+^1 \leftarrow S^0$ for directed pushouts. Our goal is to show the absoluteness of directed pushouts $\overrightarrow{\amalg} = \mathrm{colim}^W : \mathsf{Fun}(J, \mathsf{CatSp}) \to \mathsf{CatSp}$. Let us start with guessing the dual weight by assuming it has a left adjoint as the right CatSp-modules. The following lemma is useful:

Lemma 5.2.6. *Let J be a $(1,1)$-category enriched in finite sets. Then there is an adjunction*

$$
\begin{tikzcd}
\mathsf{Fun}(J, \mathsf{CatSp}) \arrow[r, shift left, "\mathrm{ev}_i"] \arrow[r, phantom, "\perp"] & \mathsf{CatSp} \arrow[l, shift left, "\text{よ}(i) \otimes (-)"]
\end{tikzcd}
$$

in the category $\mathsf{RMod}_{\mathsf{CatSp}}(\mathsf{Pr}^{\mathsf{L}})$ (or even in $\mathsf{BMod}_{\mathsf{CatSp}}(\mathsf{Pr}^{\mathsf{L}})$).

Proof. In general, the evaluation functor $\mathrm{ev}_i : \mathsf{Fun}(J, \mathsf{CMon}) \to \mathsf{CMon}$ admits a right adjoint given by the right Kan extension $X \mapsto (j \mapsto [\mathrm{Map}_J(j, i), X])$. If $\mathrm{Map}_J(j, i)$ are finite sets, we have $[\mathrm{Map}_J(j, i), X] \simeq \mathrm{Map}_J(j, i) \otimes X$ by semiadditivity (with the opposite functoriality in $\mathrm{Map}_J(j, i)$ via transposing matrices), so the right adjoint is colimit-preserving. The adjunction of the lemma is obtained by tensoring CatSp from the right (note that CatSp is semiadditive so it is an algebra over CMon). $\square$

Therefore, if there is a morphism $L : \mathsf{CatSp} \to \mathsf{Fun}(J, \mathsf{CatSp})$ of right CatSp-modules that is left adjoint to a functor $\operatorname{colim}^W : \mathsf{Fun}(J, \mathsf{CatSp}) \to \mathsf{CatSp}$, by composing with the adjunction in the above lemma, we see that $\mathrm{ev}_i \circ L$ must be given by tensoring the left dual of $\operatorname{colim}^W \text{よ}(i)$. Specializing to $J = \Lambda^2_0$ and $W = S^0 \xrightarrow{0} \square^1_+ \xleftarrow{1} S^0$, we see that $\mathrm{ev}_0 \circ L = (0 \overrightarrow{\amalg}_{\mathbb{F}} 0)^L \simeq \mathbb{F}[-1]$, $\mathrm{ev}_1 \circ L = (\mathbb{F} \overrightarrow{\amalg}_{\mathbb{F}} 0)^L = \Sigma^{\infty-1} I^{\mathrm{op}}$, $\mathrm{ev}_2 \circ L = (0 \overrightarrow{\amalg}_{\mathbb{F}} \mathbb{F})^L \simeq \Sigma^{\infty-1} I$ (see Remark 5.2.4). Therefore, in order to prove the absoluteness of directed pushouts, we must prove the following:

Theorem 5.2.7. *There is an adjunction*

$$\begin{tikzcd} \mathsf{CatSp} \arrow[r, shift left, "\Sigma^{-1}(I^{\mathrm{op}} \leftarrow \mathbb{F} \to I) \otimes (-)"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}(\Lambda^2_0, \mathsf{CatSp}) \arrow[l, shift left, "\operatorname{colim}^W"] \end{tikzcd} .$$

in $\mathsf{RMod}_{\mathsf{CatSp}}(\mathsf{Pr}^{\mathsf{L}})$. *In particular, the directed pushout* $(X \leftarrow Y \to Z) \mapsto X \overrightarrow{\amalg}_Y Z$ *is absolute.*

Remark 5.2.8. One can also guess the right dual weight in a similar (but easier) manner. Namely, we already know that $X \mapsto [W(-), X]$ is the right adjoint to colim^W, even though we do not know if it lies in $\mathsf{RMod}_{\mathsf{CatSp}}(\mathsf{Pr}^{\mathsf{L}})$. Since the (co)presheaf categories are free modules on (co)representable functors, we can always cook up the best approximation of a functor by a right CatSp-module morphism. In our case, it is simply $X \mapsto [W(-), \mathbb{F}] \otimes X$, so we only need to know the right dual of each component of the weight. In our case, we will see that the right dual of $\Sigma^\infty_+ \square^1$ is $\Sigma^{\infty-1}(I \cup_{S^0} I^{\mathrm{op}})$.

Remark 5.2.9. It seems reasonable that the absoluteness follows formally from the above consideration (because, roughly speaking, the right adjoint of the prospective left adjoint is an absolute left Kan extension and should be computed pointwise), more generally for the colimit over a finite poset with a weight whose colimits of corepresentable presheaves are dualizable. We will not pursue this idea here and instead give a direct proof.

Proof. We exhibit the adjunction by spelling out the unit, counit, and checking the triangle identities.

(1) To define the unit $\eta : \mathrm{id}_{\mathsf{CatSp}} \to \operatorname{colim}^W(\Sigma^{-1}(I^{\mathrm{op}} \leftarrow \mathbb{F} \to I) \otimes (-))$, it suffices to define $\eta_{\mathbb{F}}$; for general X, we must define $\eta_X = \eta_{\mathbb{F}} \otimes X$. Note that the codomain is computed as

$$\Sigma^{-1} I^{\mathrm{op}} \overrightarrow{\amalg}_{\Sigma^{-1}\mathbb{F}} \Sigma^{-1} I \simeq \Sigma^{\infty-1}(I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}),$$

where the category $I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}} \simeq I \amalg_{S^0} \square^1_+ \amalg_{S^0} I^{\mathrm{op}} \in \mathsf{Cat}_*$ is the free category on the graph

$$\begin{tikzcd} 0 \arrow[rr, bend left, "\square^1"] & & 1 \arrow[dl, bend left, "I"] \\ & * \arrow[ul, bend left, "I^{\mathrm{op}}"] & \end{tikzcd}.$$

We define $\eta_{\mathbb{F}} : \mathbb{F} \to \Sigma^{\infty-1}(I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}})$ as the map classifying the loop $* \to 0 \to 1 \to * : \vec{S}^1 \to I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}$.

(2) Let $X \leftarrow Y \to Z \in \mathsf{Fun}(\Lambda^2_0, \mathsf{CatSp})$. We define the counit map $\varepsilon_{X \leftarrow Y \to Z}$ as the vertical composite of the following diagram:

$$\begin{tikzcd} \Sigma^{-1} I^{\mathrm{op}} \otimes (X \overrightarrow{\amalg}_Y Z) \arrow[d] & \Sigma^{-1}\mathbb{F} \otimes (X \overrightarrow{\amalg}_Y Z) \arrow[l] \arrow[r] \arrow[d] & \Sigma^{-1} I \otimes (X \overrightarrow{\amalg}_Y Z) \arrow[d] \\ \Sigma^{-1} I^{\mathrm{op}} \otimes (X \overrightarrow{\amalg}_X 0) \arrow[d, "\simeq"] & \Sigma^{-1}(0 \overrightarrow{\amalg}_Y 0) \arrow[l] \arrow[r] \arrow[d, "\simeq"] & \Sigma^{-1} I \otimes (0 \overrightarrow{\amalg}_Z Z) \arrow[d, "\simeq"] \\ \Sigma^{-1}(I^{\mathrm{op}} \oslash I^{\mathrm{op}}) \otimes X \arrow[d, "\varepsilon_{I^{\mathrm{op}}}"] & Y \arrow[l] \arrow[r] \arrow[d, equal] & \Sigma^{-1}(I \oslash I) \otimes Z \arrow[d, "\varepsilon_I"] \\ X & Y \arrow[l] \arrow[r] & Z. \end{tikzcd}$$

(3) Now we must check the triangle identities. Since both our prospective left and right adjoints are morphisms of right CatSp-modules, it suffices to check them for generators.

(i) One of the triangle identities, evaluated at $\mathbb{F} \in \mathsf{CatSp}$, is that (after a suspension) the vertical compositions of the following diagram are the identities:

$$\begin{tikzcd}
I^{\mathrm{op}} \arrow[d] & \mathbb{F} \arrow[l] \arrow[r] \arrow[d] & I \arrow[d] \\
I^{\mathrm{op}} \otimes (\Sigma^{-1} I^{\mathrm{op}} \overrightarrow{\amalg}_{\Sigma^{-1}\mathbb{F}} \Sigma^{-1} I) \arrow[d] & (\Sigma^{-1} I^{\mathrm{op}} \overrightarrow{\amalg}_{\Sigma^{-1}\mathbb{F}} \Sigma^{-1} I) \arrow[l] \arrow[r] \arrow[d] & I \otimes (\Sigma^{-1} I^{\mathrm{op}} \overrightarrow{\amalg}_{\Sigma^{-1}\mathbb{F}} \Sigma^{-1} I) \arrow[d] \\
I^{\mathrm{op}} \otimes (\Sigma^{-1} I^{\mathrm{op}} \overrightarrow{\amalg}_{\Sigma^{-1} I^{\mathrm{op}}} 0) \arrow[d] & (0 \overrightarrow{\amalg}_{\Sigma^{-1}\mathbb{F}} 0) \arrow[l] \arrow[r] \arrow[d] & I \otimes (0 \overrightarrow{\amalg}_{\Sigma^{-1} I} \Sigma^{-1} I) \arrow[d] \\
I^{\mathrm{op}} \otimes I^{\mathrm{op}} \otimes (\Sigma^{-1} I^{\mathrm{op}}) \arrow[d] & \mathbb{F} \arrow[l] \arrow[r] \arrow[d] & I \otimes I \otimes (\Sigma^{-1} I) \arrow[d] \\
I^{\mathrm{op}} & \mathbb{F} \arrow[l] \arrow[r] & I
\end{tikzcd}$$

After another suspension and canceling the desuspension using the half-central structure of $\vec{S}^1$, the compositions can be computed in $\infty\mathsf{Cat}_*$:

$$\begin{tikzcd}
\vec{S}^1 \oslash I^{\mathrm{op}} \arrow[d] & \vec{S}^1 \arrow[l] \arrow[r] \arrow[d] & \vec{S}^1 \oslash I \arrow[d] \\
I \oslash (I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}) \arrow[d] & (I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}) \arrow[l] \arrow[r] \arrow[d] & I^{\mathrm{op}} \oslash (I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}) \arrow[d] \\
I \oslash (0 \overrightarrow{\amalg}_{I^{\mathrm{op}}} I^{\mathrm{op}}) \arrow[d] & (0 \overrightarrow{\amalg}_{S^0} 0) \arrow[l] \arrow[r] \arrow[d] & I^{\mathrm{op}} \oslash (I \overrightarrow{\amalg}_{I} 0) \arrow[d] \\
I \oslash I \oslash I^{\mathrm{op}} \arrow[d] & \vec{S}^1 \arrow[l] \arrow[r] \arrow[d] & I^{\mathrm{op}} \oslash I^{\mathrm{op}} \oslash I \arrow[d] \\
\vec{S}^1 \oslash I^{\mathrm{op}} & \vec{S}^1 \arrow[l] \arrow[r] & \vec{S}^1 \oslash I.
\end{tikzcd}$$

The middle column unpacks to $\vec{S}^1 \to (I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}) \simeq I \amalg_{S^0} \Box^1_+ \amalg_{S^0} I^{\mathrm{op}} \to * \amalg_{S^0} \Box^1_+ \amalg_{S^0} * \simeq \vec{S}^1$, which is evidently an identity. The left and right columns are equivalent to the identity by the triangle identities of the duality of I and I^{op}.

(ii) The other triangle identity is that for any $X \leftarrow Y \to Z$, the following composition is the identity:

$$X \overrightarrow{\amalg}_Y Z \to \Sigma^{-1}(I \overrightarrow{\amalg}_{S^0} I^{\mathrm{op}}) \otimes (X \overrightarrow{\amalg}_Y Z) \to X \overrightarrow{\amalg}_Y Z,$$

where the second map is the W-weighted colimit of the diagram in (2). Since $\mathsf{Fun}(\Lambda^2_0, \mathsf{CatSp})$ is generated by $0 \leftarrow 0 \to \mathbb{F}$, $\mathbb{F} \leftarrow 0 \to 0$, $\mathbb{F} \leftarrow \mathbb{F} \to \mathbb{F}$ as a right CatSp-module, it suffices to check the identity for these three diagrams. The first case unpacks (after a suspension) to that the composition

$$\vec{S}^1 \to I \amalg_{S^0} \square^1_+ \amalg_{S^0} I^{\mathrm{op}} \to (I \oslash I) \amalg_* * \amalg_* * \simeq I \oslash I \to \vec{S}^1,$$

or equivalently $\vec{S}^1 \to I \amalg_{S^0} \square^1_+ \amalg_{S^0} I^{\mathrm{op}} \to I \amalg_{S^0} * \amalg_* * \simeq \vec{S}^1$ is an identity, which is clear, and the second case is similar. The third case unpacks to, after a suspension, the following composition is an identity:

$$\vec{S}^1 \oslash \square^1_+ \to (I \amalg_{S^0} \square^1_+ \amalg_{S^0} I^{\mathrm{op}}) \oslash \square^1_+ \to \square^1_+ \oslash \vec{S}^1 \xrightarrow{\tau} \vec{S}^1 \oslash \square^1_+.$$

We can compute these maps explicitly by presenting these as (weak) quotients of grid-shaped (gaunt) categories (vertical and horizontal directions are the first and the second tensor component, respectively):

$$\left(\begin{tikzcd} * \arrow[r, equal] \arrow[d, "l_0"] \arrow[dr, phantom, "\nearrow\alpha"] & * \arrow[d, "l_1"] \\ * \arrow[r, equal] & * \end{tikzcd}\right) \to \left(\begin{tikzcd} * \arrow[r, equal] \arrow[d, "a_0"] \arrow[dr, phantom, "\nearrow\beta"] & * \arrow[d, "a_1"] \\ 00 \arrow[r, "d"'] \arrow[d, "b_0"] \arrow[dr, phantom, "\nearrow\gamma"] & 01 \arrow[d, "b_1"] \\ 10 \arrow[r, "e"'] \arrow[d, "c_0"] \arrow[dr, phantom, "\nearrow\delta"] & 11 \arrow[d, "c_1"] \\ * \arrow[r, equal] & * \end{tikzcd}\right) \to \left(\begin{tikzcd} * \arrow[r, "l'_1"] \arrow[d, equal] \arrow[dr, phantom, "\alpha'\nearrow"] & * \arrow[d, equal] \\ * \arrow[r, "l'_0"'] & * \end{tikzcd}\right).$$

The first maps is $l_i \mapsto c_i \circ b_i \circ a_i$ $(i = 0, 1)$ and $\alpha \mapsto \delta * \gamma * \beta$. The second map is $(a_0, b_0, b_1, c_1 \mapsto \mathrm{id}_*)$, $(a_1, d \mapsto l'_1)$, $(e, c_1 \mapsto l'_0)$, $(\beta \mapsto \mathrm{id}_{l'_1})$, $(\gamma \mapsto \alpha')$, $(\delta \mapsto l'_0)$, so these two maps and the half-central structure map composes to the identity.

$\square$

Let us now spell out some special cases of the theorem:

Corollary 5.2.10. *There are adjunctions*

$$\mathsf{CatSp} \underset{\overrightarrow{\mathrm{cof}}}{\overset{\Sigma^{\infty-1}(S^0\to I)\otimes(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Fun}(\square^1, \mathsf{CatSp}) \ .$$

$$\mathsf{CatSp} \underset{\overrightarrow{\mathrm{cyl}}}{\overset{\Sigma^{\infty-1}(I^{\mathrm{op}}\to I\amalg_{S^0} I^{\mathrm{op}})\otimes(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Fun}(\square^1, \mathsf{CatSp}) \ .$$

In particular, lax cofibers and lax cylinders are absolute colimits.

Proof. The inclusion $i : \square^1 \simeq \{0 \to 2\} \hookrightarrow \{1 \leftarrow 0 \to 2\} = \Lambda^2_0$ induces the following adjunction quadruple:

$$i_!^L \dashv i_! \dashv i^* \dashv i_* : \ \mathsf{Fun}(\Lambda^2_0, \mathsf{CatSp}) \ \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} \ \mathsf{Fun}(\square^1, \mathsf{CatSp}).$$

The leftmost adjoint $i_!^L$ is the cobase-change functor $(X \leftarrow Y \to Z) \mapsto (X \to X \amalg_Y Z)$; the other three are restriction, left and right Kan extension. The left Kan extension takes $X \to Y$ to $X \xleftarrow{=} X \to Y$ and the right Kan extension takes $X \to Y$ to $0 \leftarrow X \to Y$. Now, the first claim follows by composing the adjunction of the theorm with $i^* \dashv i_*$, whereas the second follws by composing that with $i_!^L \dashv i_!$. $\square$

Corollary 5.2.11. *The* 1*-cube* $\Sigma^\infty_+\square^1$ *is dualizable. The left and the right duals are both* $\Sigma^{\infty-1}(I \cup_{S^1} I^{\mathrm{op}})$*, i.e., the desuspension of the free spectrum on the pointed graph*

$$* \rightleftarrows \bullet \ .$$

Proof. Compose the adjunction of the theorem with the adjunction

$$\mathsf{Fun}(\Lambda^2_0, \mathsf{CatSp}) \underset{\Delta}{\overset{\mathrm{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{CatSp}.$$

$\square$

Corollary 5.2.12. *Left and right dualizable objects are closed under shifts, tensor products, and retracts. In particular, $\Sigma^{\infty-n}X$ is dualizable for any $X \in \widetilde{\square}$. This includes when $X \in \square, \mathbf{\Delta}, \Theta$ by Corollary A.3.6.*

Proof. By composition of adjunctions, if X^L, Y^L (resp. X^R, Y^R) are the left (resp. right) dual of X, Y, then $Y^L \otimes X^L$ (resp. $Y^R \otimes X^R$) is the left (resp. right) dual of $X \otimes Y$. Note that X is right (resp. left) dualizable if and only if $[X, \mathbb{F}] \otimes Y \to [X, Y]$ (resp. $Y \otimes [\![X, \mathbb{F}]\!] \to [\![X, Y]\!]$) induced by the counit $X \otimes [X, \mathbb{F}] \to \mathbb{F}$ (resp. $[\![X, \mathbb{F}]\!] \otimes X \to \mathbb{F}$) is an equivalence. This condition is stable under retracts. Explicitly, the right and left dual of a retract of X is the corresponding retract of $X^R = [X, \mathbb{F}]$ and $X^L = [\![X, \mathbb{F}]\!]$. $\square$

5.3 Extensions of categorical spectra

As an important corollary, we can define a lift of Barratt-Puppe sequence in Sp.

Theorem 5.3.1. *There is the followinng diagram that depends functorially on $f \in \mathsf{Fun}(\square^1, \mathsf{CatSp})$:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\simeq"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\Nearrow"] & Z \arrow[r, "h"] \arrow[d] & \Sigma X \arrow[d, "\Sigma f"] \\
 & 0 \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \Sigma Y,
\end{tikzcd}$$

where the left and bottom squares exhibits $\overrightarrow{\mathrm{cof}}(f) \xrightarrow{\sim} Z \xrightarrow{\sim} \overrightarrow{\mathrm{fib}}(\Sigma f)$ and the top-right square is commutative and bicartesian.

Definition 5.3.2. In the situation of the theorem, we say that Z is an *extension* of ΣX by Y classified by f (or Σf). As usual, we let $\mathrm{Ext}(X, Y)$ denote $\mathrm{Map}(X, \Sigma Y) \simeq \mathrm{Map}(\Omega X, Y) \in \mathsf{CMon}$, the (commutative monoid) groupoid of the extensions of X by Y.

Remark 5.3.3. Given a map $f : X \to Y \in \mathrm{Ext}(\Sigma X, Y)$, there is the transposed diagram of the above with $\overleftarrow{\mathrm{cof}}(f) \simeq Z \simeq \overleftarrow{\mathrm{fib}}(\Sigma f)$. We may say that Z is the *coextension* classified by f. There is no reason to choose one over another; if $Y \to Z \to \Sigma X$ is an extension of categorical spectra, $\Sigma Y \to \Sigma Z \to \Sigma^2 X$ is a coextension.

Proof. Let $g : Y \to Z$ be the lax cofiber of f. Note that the right adjoint $(\Sigma^{-1}\mathbb{F} \to \Sigma^{\infty-1} I) \otimes X$ of $\overrightarrow{\mathrm{cof}} : \mathsf{Fun}(\square^1, \mathsf{CatSp}) \to \mathsf{Fun}(\square^1, \mathsf{CatSp})$ is also the weight of the lax fiber of the shift: $\overrightarrow{\mathrm{fib}}(\Sigma f) \simeq \lim([I, \Sigma Y] \to [\mathbb{F}, \Sigma Y] \leftarrow [\mathbb{F}, \Sigma Y])$, so there is a canonical equivalence $Z \xrightarrow{\sim} \overrightarrow{\mathrm{fib}}(\Sigma f)$. By funtoriality applied to $Y \to 0$, we see that $Z \xrightarrow{\sim} \overrightarrow{\mathrm{fib}}(\Sigma f) \to \Sigma X$ is the same map as the one induced by $(X \xrightarrow{f} Y) \to (X \to 0)$, so by Proposition 5.1.11, the sequence $Y \to Z \to \Sigma X$ is a bifiber sequence. $\square$

Remark 5.3.4. The theorem is surprising given the failure of the pasting law of directed pullbacks; it suggests that the classical notion of fiber sequences splits into those of lax fiber, lax cofiber, and bifiber sequences, and they appear 3-periodically when we rotate a triangle.

Chapter 6

Applications to TQFT

One of the central motivations in the study of n-categories is the theory of functorial quantum field theories, especially topological quantum field theories (TQFTs). The goal of this chapter is to give a few sample applications of the theory of categorical spectra in this direction. We refer the reader to [Lur09c] for a mathematical introduction. Let us only recall a rough definition of TQFTs here. Let $d \geq n \geq 0$ be integers. An *n-category of cobordisms* or an *$(n-1)$-th extended cobordism category* roughly is the univalent completion of an n-algebroid consisting of the following data:

- An object is a $(d-n)$-dimensional manifold (possibly with an extra structure).
- A 1-morphism from M_0 to M_1 is a $(d-n+1)$-dimensional manifold W with boundary equipped with the identification $\partial W = \overline{M_0} \sqcup M_1$, where $\overline{M_0}$ denotes the manifold M_0 with the "reversed" structure. We call W the *cobordism* from M_0 to M_1.
- More generally, for $k \leq n$, a k-morphism is a cobordism between $(k-1)$-morphisms, i.e., a $(d-n+k)$-dimensional manifold with corners equipped with an identification of the boundary with $(d-n+k-1)$-dimensional manifolds corresponding to the source and target $(k-1)$-morphisms.
- An $(n+1)$-morphism is a diffeomorphsim between the n-morphisms. More generally, k-morphisms for $k > n$ are diffeomorphisms and isotopies, or equivalently

trivial cobordisms.

- The composition is given by gluing cobordisms along the shared boundary.
- The symmetric monoidal structure is given by the disjoint union of manifolds.

We often denote the cobordism category by $\mathsf{Bord}^{\mathcal{S}}_{d-n,\dots,d}$ where $\mathcal{S}$ indicates the relevant structure on the manifolds. We will focus on the *topological* case, i.e., when $\mathcal{S}$ is a tangential structure $X \in \mathsf{S}_{/\mathrm{BO}(n)}$, although it is possible to allow more elaborate geometric structures by working over the site of manifolds. We note, however, that to match a structure on a manifold and that of a boundary, we need to identify the boundary with its collar or germ (or at least its first jet). A *TQFT* is a symmetric monoidal functor $Z : \mathsf{Bord}^{\mathcal{S}}_{d-n,\dots,d} \to A$ into another symmetric monoidal n-category, typically of algebraic flavor such as "the n-category of n-vector spaces" over $\mathbb{C}$.

An important feature of the cobordism category is the existence of duals and adjoints: every object M admits a dual $\overline{M}$, every morphism $W : M_0 \to M_1$ admits a left and right adjoint $\overline{W} : M_1 \to M_0$, and so on, up through $(n-1)$-morphisms (note that it is not reasonable to ask for adjointability of all n-morphisms in an n-category, as the top-level morphism is adjointable if and only if it is invertible). Consequently, a TQFT only lands on the sufficiently adjointable cells of the target.

Let us say in general that a symmetric monoidal ∞-category is *n-adjointful* if every k-morphisms for $0 \leq k < n$ is both left and right adjointable (dualizable when $k = 0$). The cobordism hypothesis states, in the largest generality, that the cobordism n-category is a free n-adjointful symmetric monoidal n-category generated by certain generating data. This is, of course, equivalent to classifying TQFTs. Only $d = n$ case is systematically understood; this is when the objects are 0-manifolds. Such cases are called *fully extended*, or *fully local*. We will focus on the fully local cases, but note that the other cobordism categories can be described as $(d-n)$-th loop of a fully extended cobordism category (with a modified structure $\mathcal{S}$). The full locality refers to the heuristic that we can cut global information on higher dimensional manifolds (or a cobordism) into trivial pieces (0-morphisms). This suggests that the TQFT is determined by the value at the (codimension d germ of) points. (The framed version of) the cobordism hypothesis makes this intuition precise and tells us that the cobordism category with d-framing is generated by a codimension d point. When $d = 1$, this translates to the fact that one can uniquely interpret a string diagram; $d > 1$ can be thought of as a higher dimensional analog of this fact.

We start this section with the study of n-adjointful categorical spectra and in particular n-adjointful symmetric monoidal categories. It is a levelwise property $\mathsf{CatSp}^{n\text{-adj}} \subset \mathsf{CatSp}$ (in the sense of Section 3.4) with a localization $L^{n\text{-adj}}$ and it satisfies the following three properties:

(1) X is n-adjointful if and only if ΣX is $(n+1)$-adjointful.

(2) If X is an m-categorical spectrum and $f : Y \to Z$ is $L^{n\text{-adj}}$-equivalence, then $X \otimes f$ is an $L^{(m+n)\text{-adj}}$-equivalence.

(3) If X, Y are n-adjointful and $X \rightarrowtail Z \twoheadrightarrow Y$ is a (co)extension, Z is also n-adjointful.

The first property is immediate, and the second and the third reduces to a formula expressing $\square^1 \otimes (C_1 \to \mathsf{Adj})$ as a pushout of two copies of $C_1 \to \mathsf{Adj}$ and a copy of its suspension, where $C_1 \to \mathsf{Adj}$ is the inclusion of the right adjoint into a walking adjunction category. A reader may safely skip the proof if preferred.

Once this formal property is established, we will spend Section 6.2 translating the usual cobordism hypothesis into the language of categorical spectra. This translation is formal and superficial, but we point out that because 0-adjoint categorical spectra form a monoidal subcategory, it is more convenient to shift the n-adjointful symmetric monoidal n-category $\mathcal{C}$ to a 0-adjointful 0-categorical spectra $\mathrm{B}^{\infty-n}\mathcal{C}$. Philosophically, this means we use the codimensional indexing, putting the top level (i.e., the partition function) at the origin.

A real application of our theory is in Section 6.3. Here we apply our theory to the study of successive extensions of the cobordism categories, which we see as a cobordism category with singularities (a.k.a. defects). The equivalence in Theorem 5.3.1 is precisely the categorical crux in generalizing the cobordism hypothesis to this setting.

We end with the short discussion of the cobordism hypothesis with stable tangential structures in Section 6.4, i.e., when points have infinite codimension. In this case, we will see that the stably framed bordism ∞-category is the tensor unit of the ∞-adjointful categorical spectra.

6.1 Categorical spectra with adjoints

Let us start by recalling the definition of adjunction in higher categories. Let $r : x \to y$ $l : y \to x$ be morphisms in a $(2,2)$-category $\mathcal{K}$. We say a 2-cell $\eta : \mathrm{id}_y \to rl$ is the *unit* of an adjunction if there is another 2-cell $\varepsilon : lr \to \mathrm{id}_x$ satisfying the properties $\mathrm{id}_r \simeq (r\varepsilon)(\eta r)$, $\mathrm{id}_l \simeq (\varepsilon l)(l\eta)$. In this case, we say l, r are *left* and *right adjoint* of the adjunction, and ε is the *counit* of the adjunction. Now let X be a general ∞-algebroid. A 1-morphism $r : x \to y$ in X admits a left adjoint if it admits a left adjoint in the homotopy 2-category of X. If $k \geq 2$ and $x : C_k \to X$ is a k-morphsim with the $(k-2)$-source s_{k-2} and $(k-2)$-target t_{k-2}, then x is said to admit a left adjoint if it admits a left adjoint as a 1-moprhism in $\mathrm{Hom}_X(s_{k-2}, t_{k-2})$. In this section, we study categorical spectra where cells of dimensions in a range admit adjoints.

Let Adj denote the strict 2-category of walking adjunction [SS86]; it is a theorem of Riehl–Verity that it is gaunt and corepresents a homotopy-coherent adjunction in $(\infty, 2)$-categories; we refer to [RV16, §3.1] for a detailed description as a simplicial computad. Here we only name the atomic cells for reference; Adj has two objects $0, 1$ and two atomic 1-cells $r : 0 \to 1$, $l : 1 \to 0$ (denoted $-, +, u, f$, respectively in the reference). There are two atomic 2-cells $\eta : \mathrm{id}_1 \to rl$ and $\varepsilon : lr \to \mathrm{id}_0$ corepresenting the unit and the counit, two atomic 2-cells $\alpha : \mathrm{id}_r \xrightarrow{\sim} (r\varepsilon)(\eta r)$ and $\beta : \mathrm{id}_l \xrightarrow{\sim} (\varepsilon l)(l\eta)$ corepresenting the triangle identities, and the pattern continues, i.e., there are two atomic n-cells $\alpha^{(n)}$, $\beta^{(n)}$ corepresenting the "higher triangle identities." The following is the model-independent translation of the main results of [RV16, §4]:

Theorem 6.1.1. *Consider the following subcategories generated by the atomic cells (generation indicated by overline):*

$$C_1 = \overline{\{r\}} \hookrightarrow \overline{\{r, l, \varepsilon\}} \hookrightarrow \overline{\{r, l, \varepsilon, \eta, \alpha\}} \hookrightarrow \overline{\{r, l, \varepsilon, \eta, \alpha, \beta, \alpha^{(3)}\}} \hookrightarrow \mathsf{Adj}.$$

Then the inclusion of the first three categories into Adj *are epimorphisms and the last one is an equivalence.*

We denote the maps $C_1 \to \mathsf{Adj}$ classifying the morphisms r, l by the same name. The theorem says $r, l : C_1 \to \mathsf{Adj}$ are epi, i.e., a homotopy coherent adjunction extending a prospective right or left adjoint is unique if it exists.

Definition 6.1.2. Let $d \in \bar{\mathbb{Z}} := \mathbb{Z} \cup \{\pm\infty\}$. Let $S_d^{\mathrm{adj}} := \{\Sigma_+^{\infty-i}\sigma^j l, \Sigma_+^{\infty-i}\sigma^j r \mid 0 \leq j \leq d+i-2\}$. A categorical spectrum X is *d-adjointful* if it is S_d^{adj}-local, i.e., for any $f \in S_d^{\mathrm{adj}}$, the induced map $\mathrm{Map}(f, X)$ is an isomorphism. We let $\mathsf{CatSp}^{d\text{-adj}} \subset \mathsf{CatSp}$ denote the category of d-adjointful categorical spectra. We also let $d\mathsf{CatSp}^{\mathrm{adj}}$ denote the intersection $d\mathsf{CatSp} \cap \mathsf{CatSp}^{d\text{-adj}}$.

In other words, a categorical spectrum $X = (X_n)$ is d-adjointful if and only if any $(j+1)$-morphism of X_n has both left and right adjoints for $j = 0, \ldots, d+n-2$. Also note that, as a symmetric monoidal category, $X_n = \Omega X_{n+1}$ automatically has duals for objects if $d + n - 1 \geq 0$.

Example 6.1.3. By definition, we have $0\mathsf{CatSp}^{\mathrm{adj}} \xrightarrow{\sim} \lim \mathsf{CMon}(n\mathsf{Cat})^{\mathrm{dual}}$, where $\mathsf{CMon}(n\mathsf{Cat})^{\mathrm{dual}} \subset \mathsf{CMon}(n\mathsf{Cat})$ denotes the full subcategory of symmetric monoidal n-categories *with duals* in the sense of [Lur09c]. Also, the functor B^∞ restricts to the equivalence $\mathsf{CMon}(d\mathsf{Cat})^{\mathrm{dual}} \xrightarrow{\sim} \mathsf{CatSp}^{\mathrm{cn}} \cap d\mathsf{CatSp}^{d\text{-adj}}$. For any $n < d$, the intersection $n\mathsf{CatSp} \cap \mathsf{CatSp}^{d\text{-adj}}$ is Sp.

Remark 6.1.4. Let $d \in \bar{\mathbb{Z}}$. The inclusion $\mathsf{CatSp}^{d\text{-adj}} \hookrightarrow \mathsf{CatSp}$ admits a left adjoint L_d^{adj}, which freely adds left and right adjoints to stable k-cells with $k \leq d-1$.

The following lemma plays a fundamental role in the study of adjunctions in higher categories. The proof will be deferred to the end of the section.

Lemma 6.1.5. *There is a pushout diagram of (both weak and strict)* 3*-categories:*

$$\begin{tikzcd}
\sigma C_1 \sqcup \partial\square^1 \otimes C_1 \arrow[r, "\phi\sqcup(r_0\sqcup r_1)"] \arrow[d, "\sigma r\sqcup\partial\square^1\otimes r"'] & \square^1 \otimes C_1 \arrow[d, "\square^1\otimes r"] \\
\sigma\mathsf{Adj} \sqcup \partial\square^1 \otimes \mathsf{Adj} \arrow[r] & \square^1 \otimes \mathsf{Adj},
\end{tikzcd}$$

where we label the arrows of $\square^1 \otimes C_1$ *as*

$$\begin{tikzcd}
00 \arrow[r, "r_0"] \arrow[d, "a"] & 01 \arrow[d, "b"] \\
10 \arrow[r, "r_1"'] \arrow[ur, Rightarrow, "\phi"] & 11
\end{tikzcd}$$

and the map $r : C_1 \to \mathsf{Adj}$ *picks up the universal right adjoint.*

Remark 6.1.6. Informally speaking, the lemma states that a morphism in the functor category with lax natural transformation has a left adjoint if and only if each of the component cells has a left adjoint. Applying ${}^{\leq 2}(-)$, we recover [Hau21, Theorem 4.6]. The proof is also similar (some mate calculus) but naturally, our full 3-categorical version is more complicated.

Recall that the class of maps inverted by a localization is characterized by being *strongly saturated*: a class S of morphisms in a presentable category $\mathcal{C}$ is strongly saturated if it satisfies the following three conditions [Lur09b, Definition 5.5.4.5]:

(1) S satisfies 2-out-of-3 property, i.e., if two of $f, g, f \circ g$ belongs to S, then so is the third.

(2) S is closed under cobase change, i.e., if $f : X \to Y \in S$ and

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "f'"] & Y \amalg_X Z \end{tikzcd}$$

is a pushout diagram, then f' belongs to S as well.

(3) S is closed under colimits in $\mathsf{Fun}(\square^1, \mathcal{C})$.

Notice the last condition implies $\mathrm{id}_\emptyset \in S$, so by the second condition any isomorphism belongs to S. For a set S, let $\overline{S}$ denote the smallest strongly saturated class containing S. These are precisely the maps that get inverted by the localization $\mathcal{C} \to \mathcal{C}[S^{-1}]$.

Lemma 6.1.7. *Let S be a class of morphisms in $\infty\mathsf{Cat}$ and $\sigma : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ be the (unpointed) suspension endofunctor. Then we have the following:*

(1) *$\sigma\overline{S} \subset \overline{\sigma S}$,*

(2) *If $f : X \to Y \in S$, then $\sigma f \vee \square^1 : \sigma X \vee \square^1 \to \sigma Y \vee \square^1 \in \overline{\sigma S}$ and similarly for $\square^1 \vee \sigma f$.*

(3) *Let $f : X \to Y$ be a morphism in $\infty\mathsf{Cat}$ such that $\square^1 \otimes f \in \overline{\{f, \sigma f\}}$. Then for any $n \geq 0$, the morphsim $\square^1 \otimes \sigma^n f$ belongs to $\overline{\{\sigma^n f, \sigma^{n+1} f\}}$.*

Proof. (1) We wish to show $\overline{S} \subset T = \{f \mid \sigma f \in \overline{\sigma S}\}$. We have $S \subset T$, so it suffices to show that T is strongly saturated. The 2-out-of-3 property is clear. Note that $\sigma : \infty\mathsf{Cat} \to \infty\mathsf{Cat}$ takes a colimit diagram to a colimit diagram under $\sigma\emptyset$, i.e., $\sigma(\mathrm{colim}_{\lambda \in \Lambda} X_\lambda) \simeq \mathrm{colim}_{\lambda \in \Lambda^\triangleleft}(\sigma X_\lambda)$, where if $\lambda = -\infty \in \Lambda^\triangleleft$ is the cone point we set $X_{-\infty} = \emptyset$. In particular, σ preserves weakly contractible colimits, so if f' is a cobase change of $f \in T$, then $\sigma f'$ is also a cobase change of $\sigma f \in \overline{\sigma S}$, so $f' \in T$. Lastly, if $f' = \mathrm{colim}_\lambda f_\lambda$ in $\mathsf{Fun}(\square^1, \infty\mathsf{Cat})$ and $f_\lambda \in T$, the suspension $\sigma f'$ is computed by $\mathrm{colim}_{\lambda \in \Lambda^\triangleleft}(\sigma f_\lambda)$ with $f_{-\infty} := \mathrm{id}_\emptyset$. As $\sigma f_\lambda, \sigma \,\mathrm{id}_\emptyset \in \overline{\sigma S}$, it follows that $\sigma f' \in \overline{\sigma S}$, i.e., $f' \in T$.

(2) This is immediate from the pushout $f \vee \square^1 \coloneqq \sigma f \amalg_{\mathrm{id}_*} \mathrm{id}_{\square^1}$ in $\overline{\sigma S}$.

(3) Recall the pushout formula

$$\square^1 \otimes \sigma^n f \simeq \sigma(\square^1 \otimes \sigma^{n-1} f) \sqcup_{\sigma^n f \sqcup \sigma^n f} ((\sigma^n f \vee \square^1) \sqcup (\square^1 \vee \sigma^n f))$$

of Corollary 2.5.3. By (2) we have $\sigma^n f \vee \square^1, \square^1 \vee \sigma^n f \in \overline{\{\sigma^n f\}}$, so it suffices to check $\sigma(\square^1 \otimes \sigma^{n-1} f) \in \overline{\{\sigma^n f, \sigma^{n+1} f\}}$. This follows by induction on n and (1) for $S = \{\sigma^{n-1} f, \sigma^n f\}$.

$\square$

Combining Lemma 6.1.5 and Corollary 3.5.2, we obtain $\square^1 \otimes \sigma^n r \in \overline{\{\sigma^n r, \sigma^{n+1} r\}}$, so $\Sigma_+^{\infty-i} \square^1 \otimes \sigma_+^{\infty-j} \sigma^n r \simeq \Sigma_+^{\infty-(i+j)}(\square^1 \otimes \sigma^n r) \in \overline{\{\Sigma_+^{\infty-(i+j)}(\sigma^n r, \sigma^{n+1} r)\}}$.

Theorem 6.1.8. *The graded monoidal structure $\{n\mathsf{CatSp} \subset \mathsf{CatSp}^{\otimes}\}_{n \in \bar{\mathbb{Z}}}$ is compatible with the localizations $L_n^{\mathrm{adj}} : n\mathsf{CatSp} \to n\mathsf{CatSp}^{\mathrm{adj}}$. More precisely, if f is a L_n^{adj}-equivalence and X is a m-categorical spectrum, then $f \otimes X$ and $X \otimes f$ are L_{m+n}^{adj}-equivalence. In particular, the tensor product of categorical spectra localizes to $\otimes^{\mathrm{adj}} : m\mathsf{CatSp}^{\mathrm{adj}} \otimes n\mathsf{CatSp}^{\mathrm{adj}} \to (m+n)\mathsf{CatSp}^{\mathrm{adj}}$.*

Proof. We must show that $f \in S_n^{\mathrm{adj}}$ and $X \in m\mathsf{CatSp}$ implies $f \otimes X, X \otimes f \in \overline{S_{m+n}^{\mathrm{adj}}}$. Since for finite m, n, we have $\Sigma^{-n} S_n^{\mathrm{adj}} = S_0^{\mathrm{adj}}$, $\Sigma^{-m} m\mathsf{CatSp} = 0\mathsf{CatSp}$, so after shifts we may assume $(m,n) = (0,0)$. Other cases reduce to $(m,n) = (0,\infty), (\infty,0), (\infty,\infty)$; the first two are special cases of the third. Now asume $(m,n) = (0,0)$; the case $(m,n) = (\infty,\infty)$ is similar. Since S_n^{adj} and $m\mathsf{CatSp}$ are closed under duality involutions, using $(A \otimes B) \simeq (B^{\mathrm{op}} \otimes A^{\mathrm{op}})^{\mathrm{op}}$, the theorem reduces to proving $X \otimes \Sigma^{\infty-i} \sigma^k r \in \overline{S_0^{\mathrm{adj}}}$. Such X is closed under tensor products and colimits. Note that $0\mathsf{CatSp}$ is generated under colimits by $\Sigma_+^{\infty-n} \square^n$ because $\{\Omega^{\infty-n} : 0\mathsf{CatSp} \to n\mathsf{Cat} \xrightarrow{\mathrm{Map}(\square^n,-)} \mathsf{S}\}_{n \in \mathbb{N}}$ is jointly conservative (note that C_n is a retract of $\square^n$). Consequently, we may only check the case $X = \Sigma^{\infty-1} \square^1 \otimes \Sigma^{\infty-i} \sigma^k r \in \overline{S_0^{\mathrm{adj}}}$ for $0 \leq k \leq i-2$, which follows from $\Sigma^{\infty-(i+1)} \square^1 \otimes \sigma^k r \in \overline{\{\Sigma_+^{\infty-(i+1)} \sigma^k r, \Sigma_+^{\infty-(i+1)} \sigma^{k+1} r\}}$. $\square$

Remark 6.1.9. We expect that the localized monoidal product is more commutative than the original: if a category admits adjoints, passage to adjoints allows us to identify the category with its various duality involutions. In particular, we likely have a natural isomorphism $X \otimes^L Y \simeq (X \otimes^L Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \otimes^L X^{\mathrm{op}} \simeq Y \otimes^L X$. Therefore, we conjecture that the localized tensor product on $0\mathsf{CatSp}^{\mathrm{adj}}$ and $\mathsf{CatSp}^{\mathrm{adj}}$ promotes to a (framed) $\mathbb{E}_2$-structure. In the latter case, there may be some chances that it even promotes to a $\mathbb{E}_\infty$-structure.

Corollary 6.1.10. *n-adjointful categorical spectra are closed under (co)extensions.*

Proof. Let X, Y be n-adjointful categorical spectra and $f : Y \to \Sigma X$ be a morphism and $Z = \overrightarrow{\mathrm{fib}}(f) = \lim(0 \to \Sigma X \xleftarrow{\mathrm{ev}_0} [\square^1, \Sigma X] \xrightarrow{\mathrm{ev}_1} \Sigma X \leftarrow Y)$, so that $X \to Z \to Y$ is an extension of categorical spectra. Since $\mathsf{CatSp}^{n\text{-adj}} \subset \mathsf{CatSp}$ is closed under limits, it suffices to show that ΣX, $[\square^1_+, \Sigma X]$ are also n-adjointful. ΣX is $(n+1)$-adjointful so in particular, n-adjointful. Moreover, $[\square^1_+, \Sigma X]$ is S_n^{adj}-local because $\Sigma^\infty_+ \square^1 \otimes \overline{S_n^{\mathrm{adj}}} \subset \overline{S_{n+1}^{\mathrm{adj}}}$ and ΣX is S_{n+1}^{adj}-local. $\square$

proof of Lemma 6.1.5. Let P denote the pushout $(\square^1 \otimes C_1) \sqcup_{\sigma C_1 \sqcup \partial\square^1 \otimes C_1} (\sigma \mathsf{Adj} \sqcup \partial\square^1 \otimes \mathsf{Adj})$. Since $r : C_1 \to \mathsf{Adj}$ is an epimorphism, $\square^1 \otimes C_1 \to \square^1 \otimes \mathsf{Adj}$ and $\square^1 \otimes C_1 \to P$ are both epimorphisms. Since the epimorphisms from $\square^1 \otimes C_1$ form a poset (they are subobjects in the opposite category), to show $P \simeq \square^1 \otimes \mathsf{Adj}$ in $\infty\mathsf{Cat}_{(\square^1 \otimes C_1)/}$, it suffices to give a map in both directions.

We first define $P \to \square^1 \otimes \mathsf{Adj}$ or equivalently the commutative square in the lemma. The second component $\partial\square^1 \otimes \mathsf{Adj} \to \square^1 \otimes \mathsf{Adj}$ is induced from $\partial\square^1 \hookrightarrow \square^1$. To define the first component $\sigma\mathsf{Adj} \to \square^1 \otimes \mathsf{Adj}$, we must show that the 2-cell $\phi : r_1 \circ a \to b \circ r_0$ given by $\sigma C_1 \to \square^1 \otimes C_1 \to \square^1 \otimes \mathsf{Adj}$ admits a left adjoint.

- Let ψ denote the 2-cell $C_2 \hookrightarrow \square^1 \otimes C_1 \xrightarrow{\mathrm{id} \otimes l} \square^1 \otimes \mathsf{Adj}$ and ϕ^L be the 2-cell of the mate square of $\mathrm{id} \otimes l$:

$$
\phi = \begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "a"'] \arrow[dr, "r_0"'] \arrow[dr, phantom, shift left=3, "="] & 00 \arrow[d, "r_0"] \\
10 \arrow[r, Rightarrow, "\phi"] \arrow[d, "r_1"'] \arrow[dr, "r_1"] \arrow[dr, phantom, shift right=3, "="] & 01 \arrow[d, "b"] \\
11 \arrow[r, "="'] & 11.
\end{tikzcd}
\qquad
\phi^L := \begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "r_0"'] \arrow[d, phantom, shift left=4, "\varepsilon_0 \Nearrow"] & 00 \arrow[d, "a"] \\
01 \arrow[ur, "l_0"] \arrow[r, Rightarrow, "\psi"] \arrow[d, "b"'] & 10 \arrow[d, "r_1"] \arrow[d, phantom, shift right=4, "\Nearrow \eta_1"] \\
11 \arrow[ur, "l_1"] \arrow[r, "="'] & 11.
\end{tikzcd}
$$

- We will exhibit $\phi^L \dashv \phi$ by defining $E : \phi^L \circ \phi \to \mathrm{id}_{r_1 \circ a}$ and $H : \mathrm{id}_{b \circ r_0} \to \phi \circ \phi^L$ and checking the triangle identities. Let $\mathrm{id} \otimes \varepsilon, \mathrm{id} \otimes \eta : \square^1 \otimes C_2 \to \square^1 \otimes \mathsf{Adj}$ also denote the corresponding 3-cell. We define

$$
E : \begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "a"'] \arrow[dr, "r_0"'] & 00 \arrow[r, "="] \arrow[d, phantom, "\Uparrow \varepsilon_0"] & 00 \arrow[d, "a"] \\
10 \arrow[r, Rightarrow, "\phi"] \arrow[d, "r_1"'] \arrow[dr, "r_1"] \arrow[dr, phantom, shift right=3, "="] & 01 \arrow[ur, "l_0"] \arrow[r, Rightarrow, "\psi"] \arrow[d, "b"] & 10 \arrow[d, "r_1"] \arrow[d, phantom, shift right=4, "\Nearrow \eta_1"] \\
11 \arrow[r, "="'] & 11 \arrow[ur, "l_1"] \arrow[r, "="'] & 11.
\end{tikzcd}
\overset{\eta_1 * (r_1 * (\mathrm{id} \otimes \varepsilon))}{\Rrightarrow}
\begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "a"'] & 00 \arrow[r, "="] \arrow[d, phantom, "="] & 00 \arrow[d, "a"] \\
10 \arrow[rr, "="] \arrow[d, "r_1"'] \arrow[dr, "r_1"] \arrow[dr, phantom, shift right=3, "="] & \arrow[d, phantom, "\Uparrow \varepsilon_1"] & 10 \arrow[d, "r_1"] \arrow[d, phantom, shift right=4, "\Nearrow \eta_1"] \\
11 \arrow[r, "="'] & 11 \arrow[ur, "l_1"] \arrow[r, "="'] & 11.
\end{tikzcd}
\underset{\alpha^{-1} * a}{\overset{\sim}{\Rrightarrow}} \mathrm{id}_{r_1 \circ a},
$$

$$
H : \mathrm{id}_{b \circ r_0} \underset{b * \beta}{\overset{\sim}{\Rrightarrow}}
\begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "r_0"'] \arrow[d, phantom, shift left=4, "\varepsilon_0 \Nearrow"] & 00 \arrow[r, "="] \arrow[dr, "r_0"'] \arrow[d, phantom, "\eta_0 \Uparrow"] \arrow[dr, phantom, shift left=3, "="] & 00 \arrow[d, "r_0"] \\
01 \arrow[ur, "l_0"] \arrow[rr, "="] \arrow[d, "b"'] & \arrow[d, phantom, "="] & 01 \arrow[d, "b"] \\
11 \arrow[r, "="'] & 11 \arrow[r, "="'] & 11.
\end{tikzcd}
\overset{((\mathrm{id} \otimes \eta) * r_0) * \varepsilon_0}{\Rrightarrow}
\begin{tikzcd}
00 \arrow[r, "="] \arrow[d, "r_0"'] \arrow[d, phantom, shift left=4, "\varepsilon_0 \Nearrow"] & 00 \arrow[r, "="] \arrow[d, "a"] \arrow[dr, "r_0"'] \arrow[dr, phantom, shift left=3, "="] & 00 \arrow[d, "r_0"] \\
01 \arrow[ur, "l_0"] \arrow[r, Rightarrow, "\psi"] \arrow[d, "b"'] & 10 \arrow[r, Rightarrow, "\phi"] \arrow[dr, "r_1"] \arrow[d, phantom, "\Uparrow \eta_1"] & 01 \arrow[d, "b"] \\
11 \arrow[ur, "l_1"] \arrow[r, "="'] & 11 \arrow[r, "="'] & 11.
\end{tikzcd}
$$

  The triangle identities are given similarly by the 4-cells corresponding to $\mathrm{id} \otimes \alpha$, $\mathrm{id} \otimes \beta : \square^1 \otimes C_3 \to \square^1 \otimes \mathsf{Adj}$ composed with α_0, α_1, β_0, β_1.

Therefore there is an essentially unique map $\sigma\mathsf{Adj} \to \square^1 \otimes \mathsf{Adj}$ extending ϕ, which defines $P \to \square^1 \otimes \mathsf{Adj}$.

Next we define $\square^1 \otimes \mathsf{Adj} \to P$. We must show that the canonical map $\tilde{r} : a \to b$ in $[\square^1, P]$ corresponding to the square $\mathrm{id} \otimes r : \square^1 \otimes C_1 \to P$ admits a left adjoint. We define $\tilde{l} : b \to a$ as follows. Let $\phi : r_1 \circ a \to b \circ r_0$ denote again the 2-cell corresponding to $\tilde{r}$, i.e., the composition $\sigma C_1 \xrightarrow{\sigma r} \sigma\mathsf{Adj} \to P$. Let $\phi^L : b \circ r_0 \to r_1 \circ a$ denote the composition $\sigma C_1 \xrightarrow{\sigma l} \sigma\mathsf{Adj} \to P$, which is left adjoint to ϕ, exhibited by the adjunction data $\sigma\mathsf{Adj} \to P$. Now we let $\tilde{l}$: be the mate square of ϕ_L:

$$\tilde{l} := \begin{tikzcd} 01 \arrow[r, "l_0"] \arrow[d, "="'] \arrow[dr, phantom, "\Rightarrow\eta_0"] & 00 \arrow[d, "a"] \arrow[dl, "r_0"] \\ 01 \arrow[d, "b"'] \arrow[r, phantom, "\xRightarrow{\phi^L}"] & 10 \arrow[d, "="] \arrow[dl, "r_1"'] \\ 11 \arrow[r, "l_1"'] & 10. \arrow[ul, phantom, "\Rightarrow\varepsilon_1"] \end{tikzcd}$$

We wish to show that $\tilde{l}$ is left adjoint to $\tilde{r}$. The unit map $\tilde{\eta} : \mathrm{id}_b \to \tilde{r} \circ \tilde{l}$ and the counit map $\tilde{\varepsilon} : \tilde{l} \circ \tilde{r} \to \mathrm{id}_a$ are defined as follows. We first define $\tilde{\eta}$. We must define a 2-cell $C_2 \to \mathsf{Fun}^{\mathrm{lax}}(\square^1, P)$, or equivalently a map $\square^1 \otimes C_2 \to P$, extending the prescribed boundary $(\mathrm{id}_b, \tilde{r} \circ \tilde{l}) : \square^1 \otimes \partial C_2 \to P$. Now recall that we have a pushout of 3-algebroids $\square^1 \otimes C_2 \simeq (\partial\square^1 \otimes C_2 \sqcup_{\partial\square^1 \otimes \partial C_2} \square^1 \otimes \partial C_2) \sqcup_{\partial C_3} C_3$. We already know $\tilde{\eta}|_{\partial\square^1 \otimes C_2}$ must be $\eta_0 \sqcup \eta_1$, which agrees with the given map on $\square^1 \otimes \partial C_2$. It remains to define $C_3 \to P$ whose boundary is prescribed as follows (notice we reindexed the vertices: $a_0 = 00$, $a_1 = 10$, $b_0 = 01$, $b_1 = 11$):

$$\tilde{\eta} : \begin{tikzcd} & a_0 \arrow[dr, "r_0"] & \\ b_0 \arrow[ur, "l_0"] \arrow[rr, "="] \arrow[d, "b"'] \arrow[rr, phantom, bend left=20, "\eta_0\Uparrow"] & & b_0 \arrow[d, "b"] \\ b_1 \arrow[rr, "="] \arrow[urr, phantom, "="] & & b_1 \end{tikzcd} \Rrightarrow \begin{tikzcd} & a_0 \arrow[d, "a"] \arrow[dr, "r_0"] \arrow[ddl] & \\ b_0 \arrow[ur, "l_0"] \arrow[d, equal] \arrow[r, phantom, "\underset{\eta_0}{\Rightarrow}"] & a_1 \arrow[d, equal] \arrow[ddl] \arrow[r, phantom, "\overset{\phi}{\Rrightarrow}"] & b_0 \arrow[dd, "b"] \\ b_0 \arrow[d, "b"'] \arrow[r, phantom, "\underset{\varepsilon_1}{\Rightarrow}"] \arrow[ur, phantom, "\underset{\phi^L}{\Rightarrow}"] & a_1 \arrow[dr, "r_1"] & \\ b_1 \arrow[ur, "l_1"] \arrow[rr, "="'] \arrow[urr, phantom, "\eta_1\Uparrow"] & & b_1, \end{tikzcd}$$

This is given by the following 2-cell in $\mathrm{Hom}_P(b_0, b_1)$:

$$\begin{tikzcd} b \arrow[r, "b\eta_0"] & br_0l_0 \arrow[r, "\phi^L l_0"] \arrow[ddrr, equal] \arrow[drr, phantom, "\nearrow\eta_\phi l_0"] & r_1al_0 \arrow[r, "\eta_1 r_1 a l_0"] \arrow[dr, equal] \arrow[r, phantom, shift right=12, "\nearrow\alpha_1 a l_0"] & r_1l_1r_1al_0 \arrow[d, "r_1\varepsilon_1 a l_0"] \\ & & & r_1al_0 \arrow[d, "\phi l_0"] \\ & & & br_0l_0 \end{tikzcd}$$

The definition of the counit $\tilde{\varepsilon}$ is similar: the corresponding 3-cell $C_3 \hookrightarrow \square^1 \otimes C_2 \xrightarrow{\tilde{\varepsilon}} P$

is given by the following 2-cell in $\mathrm{Hom}_P(a_0, a_1)$:

$$
\begin{tikzcd}
l_1r_1a \arrow[r, "l_1\phi"] \arrow[ddrr, equal] & l_1br_0 \arrow[r, "l_1b\eta_0r_0"] \arrow[dr, equal] \arrow[d, phantom, "\Swarrow l_1\varepsilon_\phi"] & l_1br_0l_0r_0 \arrow[d, "l_1br_0\varepsilon_0"] \arrow[dl, phantom, "\Swarrow l_1b\alpha_0"] \\
 & & l_1br_0 \arrow[d, "l_1\phi^L"] \\
 & & l_1r_1a \arrow[r, "\varepsilon_1a"] & a
\end{tikzcd}
$$

To verify the triangle identity, we must provide 3-cells $\tilde{\alpha} : \mathrm{id}_{\tilde{r}} \xrightarrow{\sim} (\tilde{r}\tilde{\varepsilon})(\tilde{\eta}\tilde{r})$, $\tilde{\beta} : \mathrm{id}_{\tilde{l}} \to (\tilde{\varepsilon}\tilde{l})(\tilde{l}\tilde{\eta})$ in $\mathsf{Fun}^{\mathrm{lax}}(\Box^1, P)$. Similarly to the construction of unit and counit, the construction of $\tilde{\alpha}$ reduces to defining a 4-cell $C_4 \to \Box^1 \otimes C_3 \to P$ with the domain

$$C_3 \to \Box^1 \otimes C_2^- \cup_{\{1\}\otimes C_2^-} \{1\} \otimes C_3 \xrightarrow{\mathrm{id}_{\tilde{r}} \cup \alpha_1} P$$

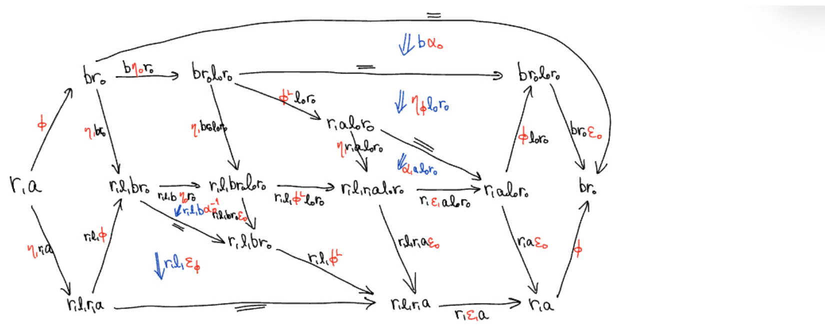
and the codomain

$$C_3 \to \Box^1 \otimes C_2^+ \cup_{\{0\}\otimes C_2^+} \{0\} \otimes C_3 \xrightarrow{(\tilde{r}\tilde{\varepsilon})(\tilde{\eta}\tilde{r}) \cup \alpha_0} P,$$

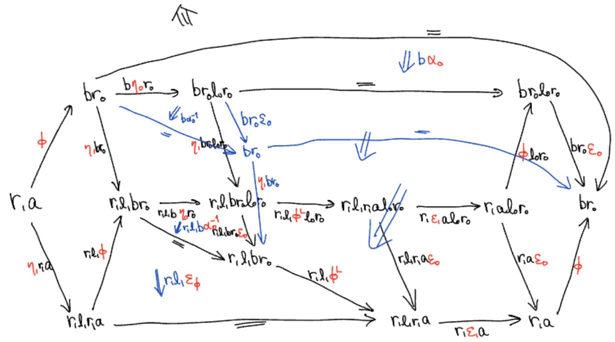
The codomain unpacks to the following 2-cells in $\mathrm{Hom}_P(a_0, b_1)$:

$$
\begin{tikzcd}
 & br_0 \arrow[r, "b\eta_0r_0"] \arrow[dd, "\eta_1br_0"] \arrow[ddrrrr, bend left=40, equal] & br_0l_0r_0 \arrow[rr, equal] \arrow[dr, "\phi^Ll_0r_0"] \arrow[dd, "\eta_1br_0l_0r_0"] \arrow[rr, phantom, bend left=30, "\Downarrow b\alpha_0"] & & br_0l_0r_0 \arrow[ddr, "br_0\varepsilon_0"] \arrow[dl, phantom, "\Downarrow\eta_\phi l_0r_0"] & \\
 & & & r_1al_0r_0 \arrow[d, "r_1\eta_1al_0r_0"] \arrow[dr, equal] \arrow[dr, phantom, "\Downarrow\alpha_1al_0r_0"] & & \\
r_1a \arrow[uur, "\phi"] \arrow[ddr, "\eta_1r_1a"'] & r_1l_1br_0 \arrow[r, "r_1l_1b\eta_0r_0"] \arrow[dr, equal] \arrow[dr, phantom, "r_1l_1b\alpha_0\Downarrow"] & r_1l_1br_0l_0r_0 \arrow[r, "r_1l_1\phi^Ll_0r_0"] \arrow[d, "r_1l_1br_0\varepsilon_0"] & r_1l_1r_1al_0r_0 \arrow[r, "r_1\varepsilon_1al_0r_0"'] \arrow[dd, "r_1l_1r_1a\varepsilon_0"] & r_1al_0r_0 \arrow[uu, "\phi l_0r_0"] \arrow[dd, "r_1a\varepsilon_0"] & br_0 \\
 & & r_1l_1br_0 \arrow[dr, "r_1l_1\phi^L"] \arrow[dl, phantom, "r_1l_1\varepsilon_\phi\Downarrow"] & & & \\
 & r_1l_1r_1a \arrow[rr, equal] \arrow[uu, "r_1l_1\phi"] & & r_1l_1r_1a \arrow[r, "r_1\varepsilon_1a"'] & r_1a \arrow[uur, "\phi"'] &
\end{tikzcd}
$$

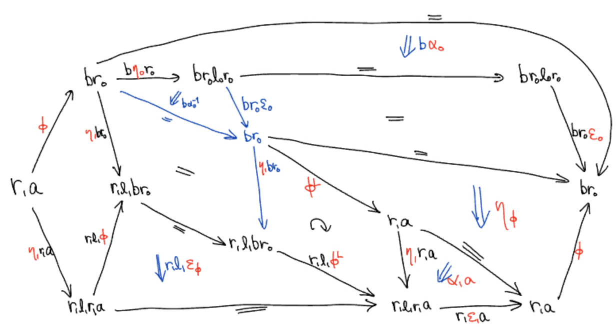
Then the 3-cell corresponding α is the following:



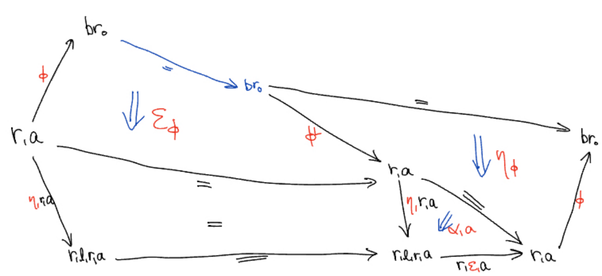
⤊



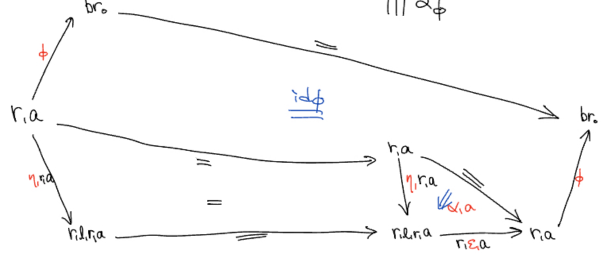
⤊



⤊ $\simeq$



⤊ α_ϕ



The construction of $\tilde{\beta}$ is similar: it reduces to defining a 4-cell $C_4 \to \square^1 \otimes C_3 \to P$ with the domain

$$C_3 \to \square^1 \otimes C_2^- \cup_{\{1\}\otimes C_2^-} \{1\} \otimes C_3 \xrightarrow{\mathrm{id}_{\tilde{l}} \cup \beta_1} P$$

and the codomain

$$C_3 \to \square^1 \otimes C_2^+ \cup_{\{0\}\otimes C_2^+} \{0\} \otimes C_3 \xrightarrow{(\tilde{\varepsilon}\tilde{l})(\tilde{l}\tilde{\eta})\cup\beta_0} P.$$

One can unpack these into 2-cells in $\mathrm{Hom}_P(b_0, a_1)$ and explicitly construct the 3-cell as before. $\square$

6.2 The cobordism hypothesis

The goal of this section is to review the basic form of the cobordism hypothesis and translate it to the context of categorical spectra.

Notation 6.2.1. Recall the straightening-unstraightening equivalence

$$\mathrm{colim} : \mathsf{Fun}(\mathrm{BO}(n), \mathsf{S}) \xrightarrow{\sim} \mathsf{S}_{/\mathrm{BO}(n)}$$

which takes a groupoid $\tilde{X}$ with an $\mathrm{O}(n)$-action to its quotient $X \simeq \tilde{X}/\mathrm{O}(n) = \tilde{X}_{h\mathrm{O}(n)}$. This inverse is given by pulling back the universal $\mathrm{O}(n)$-torsor $* \to \mathrm{BO}(n)$ along $X \to \mathrm{BO}(n)$. The data is also equivalent to the pair of a CW complex representing the groupoid X and the real vector bundle $\zeta \simeq \mathbb{R}^n \times_{\mathrm{O}(n)} \tilde{X}$ of rank n with inner product, so we sometimes use the notation $\tilde{X}$ and (X, ζ) interchangeably or let ζ denote the classifying map $X \to \mathrm{BO}(n)$ itself. These equivalent data will be called *tangential structures*. Despite the notation $\mathrm{O}(n)$, unless we specifically refer to a vector bundle on a topological space, we will not need a topological group structure on $\mathrm{O}(n)$; we only regard it as a group object in S.

Let $m \leq n$ and let M be a smooth m-dimensional manifold and $\tilde{X}$ be a tangential structure. Let $\tau_M : M \to \mathrm{BO}(m)$ be the classifying map of the tangent bundle. A $\tilde{X}$-structure (or (X,ζ)-structure) on M is a commutative diagram (in S) lifting the tangent classifier

$$\begin{tikzcd} & X \arrow[d, "\zeta"] \\ M \arrow[ur, dashed] \arrow[r, "\tau_M \oplus \mathbb{R}^{n-k}"'] & \mathrm{BO}(n). \end{tikzcd}$$

Example 6.2.2. The most basic tangential structure is when $X \to \mathrm{BO}(n) = \mathrm{id}$, $\tilde{X} = *$. In this case, $\tilde{X}$-structure is no additional structure, i.e., that of *unoriented manifolds*. Another fundamental case is $X = * \to \mathrm{BO}(n)$, $\tilde{X} = \mathrm{O}(n)$. In this case, $\tilde{X}$-structure gives a *framing*, i.e., an identification of the tangent bundle with the constant bundle of the reference point of $\mathrm{BO}(n)$. We will use superscript fr to indicate the framing write un (or omit the notation for tangential structure) for unoriented manifolds.

Note that the groupoidification $|C_k|$ of the cell C_k admits a canonical stratified manifold structure as a quotient of the cube $[0,1]^k$ by cylindrically collapsing the appropriate faces. In particular, it comes equipped with a natural framing on every stratum in a compatible manner. Such manifold realization extends to Θ in a functorial way. In the following, it is helpful to think of them as secretly being extended to the "germ" beyond the boundary, so that the objects restricted to the boundary come equipped with some "glue." Roughly speaking, the *n-dimensional bordism category* $\mathsf{Bord}_n^{\tilde{X}}$ *with $\tilde{X}$-structure* is an n-category whose value on $\theta \in \Theta_n$ classifies finite submersive bundle of $\tilde{X}$-manifolds on $|\theta|$ whose $\tilde{X}$-structure is compatible across stratifications. In particular, we have the following description of cells:

(0) an object of $\mathsf{Bord}_n^{\tilde{X}}$ is a finite set of points $x = \{x_1, \ldots, x_n\}$ of X equipped with identification $(\zeta(x_i) \simeq \mathbb{R}^n) \in \mathrm{O}(n)$.

(1) A k-cell for $1 \leq k \leq n$ from M_0 to M_1 is a k-dimensional $\tilde{X}$-manifold W together with the identification of the boundaries $(|C_{k-1}| \xrightarrow{s} |C_k|)^* W \simeq M_0$,

$(|C_{k-1}| \xrightarrow{t} |C_k|)^*W \simeq M_1$, i.e., a cobordism from M_0 to M_1 with $\tilde{X}$-structures.

(2) When $k > n$, one may think of a k-morphism as a bundle on $|C_k|$ which is induced from that on $|C_n|$ via the projection $C_n \to C_k$; these are the trivial cobordism given by cylinders (note however that the identification with the boundary has the freedom of diffeomorphisms).

Moreover, we equip a symmetric monoidal structure on $\mathsf{Bord}_n^{\tilde{X}}$ by disjoint unions of manifolds. Giving a precise definition of the symmetric monoidal (∞, n)-category of cobordisms is a nontrivial task. At least one has to encode the compatibility of tangential structures across different strata systematically, which may be done by considering collars or developing the theory of stratified manifolds. We refer the reader to [CS19][AF17][GP23] for some constructions in the literature. One should also note that the above naive definition usually ends up in a non-univalent category; the underlying groupoid consists of trivial cobordisms (i.e., cylinders), but there are nontrivial invertible cobordisms (e.g. nontrivial h-cobordism), so we usually apply the univalent completion (this inverts the difference of smooth and PL cobordism category, for instance).

Remark 6.2.3. One should define the cobordism categories in a way that it clearly depends functorially on $X \in \mathsf{S}_{/\mathrm{BO}(n)}$. In particular, the framed bordism n-category $\mathsf{Bord}_n^{\mathrm{fr}}$ admits an action of $\mathrm{Aut}(* \to \mathrm{BO}(n)) = \mathrm{O}(n)$. It is a folklore result that $\mathrm{Aut}(\mathsf{Bord}_n^{\mathrm{fr}}) = \mathrm{PL}(n)$ except the unknown case $n = 4$, but we will stick to the $\mathrm{O}(n)$-action for simplicity.

Modulo the problem of definition, it is not difficult to show that the objects of cobordism categories are fully dualizable; for $k < n$, the dual of a k-morphism is the same manifold with the opposite $\tilde{X}$-structure and the unit and counits are given by a bent cylinder. The cobordism hypothesis [Bae96] states the universal property of the cobordism category as the free symmetric monoidal n-category with duals.

Hypothesis 6.2.4. *The forgetful functor* $\mathsf{CMon}(n\mathsf{Cat})^{\mathrm{dual}} \hookrightarrow \mathsf{CMon}(n\mathsf{Cat}) \xrightarrow{(-)^{\leq 0}} \mathsf{S}$ *is corepresented by the* framed Bordism category $\mathsf{Bord}_n^{\mathrm{fr}}$. *In particular, the functor admits an action of* $\mathrm{O}(n)$, *so the above functor canonically factors through the category of groupoids with* $\mathrm{O}(n)$*-action:*

$$\begin{tikzcd}
n\mathsf{SMCat}^{\mathrm{dual}} \arrow[r, dashed] \arrow[d, hook] & \mathsf{S}_{/\mathrm{BO}(n)} \arrow[r, "X\mapsto\tilde{X}", "\sim"'] & \mathsf{Fun}(\mathrm{BO}(n), \mathsf{S}) \arrow[d, "\mathrm{ev}_*"] \\
n\mathsf{SMCat} \arrow[rr, "(-)^{\leq 0}"] & & \mathsf{S}.
\end{tikzcd}$$

Moreover, the left adjoint of $\mathsf{CMon}(n\mathsf{Cat})^{\mathrm{dual}} \to \mathsf{Fun}(\mathrm{BO}(n), \mathsf{S})$ *sends the object* $\tilde{X}$ *to the n-dimensional cobordism category* $\mathsf{Bord}_n^{\tilde{X}}$, *i.e.,* $\mathsf{Bord}_n^{\tilde{X}} \simeq \tilde{X} \otimes_{\mathrm{O}(n)} \mathsf{Bord}_n^{\mathrm{fr}}$.

Remark 6.2.5. The second statement is equivalent to that $\tilde{X} \to \mathsf{Bord}_n^{\tilde{X}}$ is colimit-preserving, i.e., satisfies *descent*. This is an expectation from the *locality* of the fully extended cobordism categories; one can cut a manifold into small bordisms and assemble information on the original manifold from the restricted pieces. In the current situation, it essentially means that one can assemble a tangential structure on a manifold from the local trivializations of the tangent bundle by asking for some properties/structures on the transition functions (and cocycles). This perspective is central in [GP23]. The unoriented case is in some sense universal: by unstraightening the framed bordism category as an $\mathrm{O}(n)$-equivariant object $\mathsf{Bord}_n^{\mathrm{fr}} : \mathrm{BO}(n) \to n\mathsf{SMCat}$, one gets a cocartesian fibration $\mathsf{Bord}_n \coloneqq \mathsf{Bord}_n^{\mathrm{un}} = \int \mathsf{Bord}_n^{\mathrm{fr}} \to \mathrm{BO}(n)$; intuitively, one obtains an unoriented bordism category from the framed bordism category by taking the orbits under the change of framings. All the other cases are base changes of this fibration along the structure map $X \to \mathrm{BO}(n)$. One may think of this construction as the *categorical Madsen-Tillman spectrum.*

Remark 6.2.6. The cobordism hypothesis is widely believed but there is no consensus on the rigorous proof yet, as far as the author knows ([Lur09c] gives a sketch of a proof and [AF17] gives a proof assuming a fundamental conjecture on factorization homology of solidly framed manifolds. [GP22] claims a rigorous proof and generalization to the geometric context, parametrized by the cite of manifolds, but peer-reviewing is still in progress). **From now on, we will work conditionally on this formulation of the cobordism hypothesis.**

Let us restate the cobordism hypothesis using categorical spectra. As the functor $(\Omega^\infty)^{\leq d} : \mathsf{CatSp}^{d\text{-adj}} \to \mathsf{CMon}(d\mathsf{Cat})$ lands in $\mathsf{CMon}(d\mathsf{Cat})^{\mathrm{dual}}$, the underlying groupoid functor $(\Omega^\infty)^{\leq 0} : \mathsf{CatSp} \to 0\mathsf{CatSp}^{\mathrm{cn}} \simeq \mathsf{CMon}(\mathsf{S}) \to \mathsf{S}$ lifts to $\mathsf{Fun}(\mathrm{BO}(n), \mathsf{S})$.

Corollary 6.2.7. *The functor* $\underline{\Omega}^{\infty-d} : \mathsf{CatSp}^{0\text{-adj}} \xrightarrow{\Omega^{\infty-d}} \mathsf{CMon}(\infty\mathsf{Cat}) \xrightarrow{(-)^{\leq 0}} \mathsf{S}$ *is represented by* $\mathrm{B}^{\infty-d}\mathsf{Bord}_d^{\mathrm{fr}}$ *and factors throught* $\mathsf{S}_{/\mathrm{BO}(n)}$*. The left adjoint* $\mathsf{S}_{/\mathrm{BO}(n)} \to \mathsf{CatSp}^{0\text{-adj}}$*, formally given by* $X \mapsto \tilde{X} \otimes_{\mathrm{O}(n)} L_0^{\mathrm{adj}}\mathbb{F}[-n]$*, admits a description as the cobordism categorical spectrum* $\mathrm{B}^{\infty-d}\mathsf{Bord}_d^X$*.*

The translation is superficial, but in a sense, this is closer to the actual situation of the cobordism hypothesis. Via the cobordism hypothesis, the framed cobordism n-category is often described as the free symmetric monoidal category with duals generated by a point. However, this point secretly looks like a germ of $\mathbb{R}^n$ and there are $O(n)$-worth of them. In the above, this generator is placed in dimension $-n$. For example, targets of TQFTs become dimension-independent in this way: for any d, one can write $\{\mathrm{B}^{\infty-d}\mathsf{Bord}_d^X \to \underline{\mathbb{C}}\}$ to mean $\mathsf{Bord}_d \to d\mathsf{Vect}_{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is the categorical spectrum $\{n\mathsf{Mod}_{\mathbb{C}}\}_n$ of [Ste21]. Moreover, we can organize the framed cobordism categories of all dimensions into a single algebra in $0\mathsf{CatSp}^{0\text{-adj}}$.

Corollary 6.2.8. *The categorical spectrum* $\bigoplus_{n\geq 0} \mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ *admits the structure of a tensor algebra on* $\mathrm{B}^{\infty-1}\mathsf{Bord}_1^{\mathrm{fr}}$ *in the monoidal category* $\mathsf{CatSp}^{0\text{-adj}}$*.*

Proof. The first claim is equivalent to the cobordism hypothesis by applying the monoidal localization $L^{0\text{-adj}} : \mathsf{CatSp} \to \mathsf{CatSp}^{0\text{-adj}}$ to the tensor algebra $\mathrm{Tens}(\mathbb{F}[-1]) = \bigoplus \mathbb{F}[-n]$. □

Remark 6.2.9. Intuitively, the $\mathbb{E}_1$-rig structure on $\bigoplus_{n\geq 0} \mathrm{B}^{\infty-n}\mathsf{Bord}_n^{\mathrm{fr}}$ is given by the disjoint union and the cartesian product of manifolds. As soon as such cartesian product functor $\mathsf{Bord}_m^{\mathrm{fr}} \otimes \mathsf{Bord}_n^{\mathrm{fr}} \to \mathsf{Bord}_{m+n}^{\mathrm{fr}}$ is shown to be well-defined, the universal property shows that it is the unique functor that sends the pair of reference codimension m and n point to the reference codimension $(m+n)$, but at the moment of writing the author is not aware of the way that avoids the use of models of cobordism categories.

Remark 6.2.10. It is important to note that the cobordism hypothesis and its suggested proofs have purely categorical consequences that can be stated without mentioning manifolds. One is the $\mathrm{O}(n)$-action on the underlying groupoid of a symmetric monoidal n-category with duals. This action remains elusive without the framed cobordism hypothesis, which forces us to state the cobordism hypothesis in two steps. Combining this with Theorem 6.1.8, we see that the groupoid of k-cells in a symmetric monoidal n-categories with duals admits an $\mathrm{O}(n-k)$-action. This fact was mentioned in [Lur09c, §4.3] without proof and used to formulate the cobordism hypothesis with singularities.

Another categorical takeaway from Lurie's proof sketch is an explicit finite-step pushout description of $L^{(-1)\text{-adj}}\mathbb{F}[-n] \to L^{0\text{-adj}}\mathbb{F}[n]$. Without using manifolds, it is not even clear if the localization can be reached after finitely many pushouts. More precisely, in [Lur09c, §3.4] Lurie describes the inclusion $\mathrm{B}^{\infty-n}\mathsf{Bord}_{n-1} \to \mathrm{B}^{\infty-n}\mathsf{Bord}_n$ by introducing the *index filtration*: $\mathrm{B}^{\infty-n}\mathsf{Bord}_{n-1} = \mathcal{F}_{-1} \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n = \mathrm{B}^{\infty-n}\mathsf{Bord}_n$. Note that $\mathcal{F}_{-1} \hookrightarrow \mathcal{F}_n$ is the reflection under the inclusion $0\mathsf{CatSp}^{0\text{-adj}} \hookrightarrow 0\mathsf{CatSp}^{(-1)\text{-adj}}$. Roughly speaking (at least non-univalently), the category $\mathcal{F}_k$ has the same $0, 1, \ldots, (n-1)$-cells as Bord_{n-1} and allows the n-cells (i.e., cobordisms) of Bord_n that can be built up from handles of index at most k; here we see the time coordinate of the cobordism as a (generalized) Morse function. Each step $\mathcal{F}_{k-1} \hookrightarrow \mathcal{F}_k$ is generated by the single $\mathrm{O}(n-k)$-equivariant n-cell corresponding to the k-handle, subject to the relation of cancellation of a $(k-1)$-handle and a k-handle, as well as the "triangle identity" of cancellation. The case $k=0$ is particularly simple and encoded as the lax cofiber sequence:

$$\begin{tikzcd}
\Sigma^{\infty-1}_{+}\mathrm{BO}(n) \arrow[r, "S^{n-1}"] \arrow[d] & \mathrm{B}^{\infty-n}\mathsf{Bord}_{n-1} \arrow[d] \\
0 \arrow[r] \arrow[ur, Rightarrow, "D^n"] & \mathcal{F}_0
\end{tikzcd}$$

In other words, $\mathcal{F}_0$ is an extension of $\Sigma^{\infty}_{+}\mathrm{BO}(n)$ by $\mathrm{B}^{\infty-n}\mathsf{Bord}_{n-1}$ classified by the standard $\mathrm{O}(n)$-action on the $(n-1)$-sphere $S^{n-1} \in \mathrm{Map}_{\mathsf{S}}(\mathrm{BO}(n), \Omega^{n-1}\mathsf{Bord}_{n-1}) \simeq \mathrm{Ext}(\Sigma^{\infty}_{+}\mathrm{BO}(n), \mathrm{B}^{\infty-n}\mathsf{Bord}_{n-1})$. Corollary 6.1.10 ensures that it does not mess up the previous levels of adjointfulness.

6.3 Cobordism hypothesis with singularlities

As we saw in the last section, the cobordism hypothesis gives a geometric description for the categorical spectra with adjoints freely generated by a ($\mathrm{O}(n)$-equivariant) groupoid of codimension $(-n)$-cells. In this section, we study a generalization called the *cobordism hypothesis with singularities* sketched by Lurie in [Lur09c, §4.3]; we describe the cobordism category with certain kinds of conical singularities (aka. defects) allowed. It turns out that such cobordism categories/categorical spectra arise as *cell complexes* in categorical spectra with adjoints, i.e., an iterated extension of cobordism categorical spectra of different codimensions, and the singularity types precisely classifies the extensions. The goal of this section is to make Lurie's sketch of the argument precise, filling the unproven categorical claims using our previous results. We first make the following definition and later check that it qualifies for its name.

Definition 6.3.1. A *cobordism categorical spectrum with singularities* is a categorical spectrum B^0 that fits into the following sequence, where $d \geq 0$, $X^k \in \mathsf{S}_{/\mathrm{BO}(k)}$, and B^k is an extension (resp. coextension) of $\mathrm{B}^{\infty-k}\mathsf{Bord}_k^{\tilde{X}^k}$ by B^{k+1} when k is odd (resp.

even):

$$\begin{tikzcd}
0 = B^{d+1} \arrow[r, tail] & B^{d} \arrow[r, tail] \arrow[d, two heads] & B^{d-1} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & B^{1} \arrow[r, tail] \arrow[d, two heads] & B^{0} \arrow[d, two heads] \\
& \mathrm{B}^{\infty-d}\mathsf{Bord}_{d}^{\tilde{X}^{d}} & \mathrm{B}^{\infty-(d-1)}\mathsf{Bord}_{d-1}^{\tilde{X}^{d-1}} & & \mathrm{B}^{\infty-1}\mathsf{Bord}_{1}^{\tilde{X}^{1}} & \mathrm{B}^{\infty}\mathsf{Bord}_{0}^{\tilde{X}^{0}}
\end{tikzcd}$$

A *singularity datum* for B^0 is the sequence $\vec{X} = (X^0, E^0, \dots, X^{d-1}, E^{d-1}, X^d)$ where B^k is classified by $E^k \in \mathrm{Ext}(\mathrm{B}^{\infty-k}\mathsf{Bord}_k^{\tilde{X}^k}, B^{k+1})$. In this case, we denote $\mathsf{Bord}_d^{\vec{X}} := \Omega^{\infty-d}B^0$ and call it the *cobordism category of* $\vec{X}$*-manifolds*.

Remark 6.3.2. The distinction between extensions and coextensions are not essential under the existence of adjoints. The above definition is designed to make the statement of Theorem 6.3.5 cleaner.

Remark 6.3.3. In the above definition, if $X^d = \emptyset$, we automatically have $B^d = 0$, $E^{d-1} = 0$, so deleting E^{d-1}, X^d from $\vec{X}$ does not change the resulting B^0. Also, since E^k only depends on the entries on the right, it makes sense to consider a singularity datum $X^{\geq k}$ where X^i, E^i for $i < k$ are replaced by $\emptyset$ and 0, respectively. The resulting sequence of extensions is $0 \rightarrowtail B^d \rightarrowtail \cdots B^k \xrightarrow{=} \cdots \xrightarrow{=} B^k$, so B^k is also a cobordism categorical spectrum with singularities. Combining the both consideration, when $X^i = \emptyset$ unless $k \leq i \leq l$, we may simply denote the singularity datum $\vec{X} := \vec{X}^{[k,l]} := (X^k, E^k, \dots, E^{l-1}, X^l)$ without a risk of confusion. Also note that the categorical spectrum B^0 is independent of the choice of $d(\geq l)$, whereas $\mathsf{Bord}_d^{\vec{X}} = \Omega^{\infty-d}B^0$ does depend on d.

Remark 6.3.4. By Corollary 6.1.10, B^k is 0-adjointful. Similarly we can show that it is 0-categorical and $(-d)$-connective. In particular, $B^k = \mathrm{B}^{\infty-d}\mathsf{Bord}_d^{\vec{X}^{\geq k}}$. By the cobordism hypothesis, one computes the Ext monoid as follows:

$$\begin{aligned}
\mathrm{Ext}(\mathrm{B}^{\infty-k}\mathsf{Bord}_k^{\tilde{X}^k}, B^{k+1}) &\simeq \mathrm{Map}(\tilde{X}^k \otimes_{\mathrm{O}(k)} L^{0\text{-adj}}\mathbb{F}[-k], \Sigma B^{k+1}) \\
&\simeq \mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, \underline{\Omega}^{\infty-k-1}B^{k+1}).
\end{aligned}$$

where the $\mathrm{O}(k+1)$-action on the codomain is restricted to $\mathrm{O}(k)$ by the canonical inclusion $\mathrm{O}(k) \subset \mathrm{O}(k+1)$. By definition, this classifies a $\mathrm{O}(k)$-equivariant local system of $(k+1)$-dimensional $\vec{X}^{\geq k+1}$-manifolds. By abuse of notation, we continue to denote by E^k the corresponding local system. If one gives a topological model of $\tilde{X}^k$, this gives a bundle which continuously and equivariantly assigns an $X^{\geq k+1}$-manifold $E^k(\tilde{x})$ to a point $\tilde{x} \in \tilde{X}^k$.

Theorem 6.3.5 (Cobordism hypothesis with singularities [Lur09c, Theorem 4.3.11])**.** *Let $d > k \geq 0$, $\vec{X} = (X^k, E^k, \ldots, X^d)$ be a singularity datum and $\vec{X}' = \vec{X}^{\geq k+1}$. for any 0-adjointful categorical spectrum $A = (A_n)$, there is a cartesian square*

$$\begin{tikzcd}
\mathrm{Map}_{\mathsf{CatSp}}(\mathrm{B}^{\infty-d}\mathsf{Bord}_d^{\vec{X}}, A) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathsf{CatSp}}(\mathrm{B}^{\infty-d}\mathsf{Bord}_d^{\vec{X}'}, A) \arrow[d, "(E^k)^*"] & \ni & Z_0 \arrow[d, mapsto] \\
\mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, \mathsf{Alg}_{\mathbb{E}_0}(A_{k+1})^{\leq 0}) \arrow[r] & \mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, A_{k+1}^{\leq 0}) & \ni & \Omega^{\infty-k-1} Z_0 \circ E^k.
\end{tikzcd}$$

Proof. By Theorem 5.3.1, there is a (op)lax cofiber sequence

$$\mathrm{B}^{\infty-k-1}\mathsf{Bord}_k^{\tilde{X}^k} \to B^{k+1} \to B^k,$$

or a pushout diagram

$$\begin{tikzcd}
\mathrm{B}^{\infty-k-1}\mathsf{Bord}_k^{\tilde{X}^k} \arrow[r] \arrow[d] & B^{k+1} \arrow[d] \\
\Sigma^\infty D^{k+1} I \otimes \mathrm{B}^{\infty-k-1}\mathsf{Bord}_k^{\tilde{X}^k} \arrow[r] & B^k.
\end{tikzcd}$$

Mapping into A and applying the cobordism hypothesis, one obtains

$$\begin{tikzcd}
\mathrm{Map}(B^k, A) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \mathrm{Map}(B^{k+1}, A) \arrow[d] \\
\mathrm{Map}_{k\mathsf{SMCat}^{\mathrm{dual}}}(\mathsf{Bord}_k^{\tilde{X}^k}, \Omega^{\infty-k-1}[\Sigma^\infty I, A]) \arrow[rr] \arrow[d, "\simeq"] & & \mathrm{Map}_{k\mathsf{SMCat}^{\mathrm{dual}}}(\mathsf{Bord}_k^{\tilde{X}^k}, \Omega^{\infty-k-1} A) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, \underline{\Omega}^{\infty-k-1}[\Sigma^\infty I, A]) \arrow[rr] & & \mathrm{Map}_{\mathrm{O}(k)}(\tilde{X}^k, \underline{\Omega}^{\infty-k-1} A)
\end{tikzcd}$$

Now observe $\Omega^{\infty-k-1}[\Sigma^\infty D^{k+1}I, A] \simeq \Omega^\infty[\Sigma^\infty I, \Sigma^{k+1}A]$ so the underlying groupoid is $\mathrm{Map}_*(I, \underline{\Omega}^{\infty-k-1}A) \simeq \mathsf{Alg}_{\mathbb{E}_0}(A_{k+1})^{\leq 0}$. $\square$

6.4 Stable tangential structures

In this short section, we investigate $\mathsf{CatSp}^{\infty\text{-adj}}$. This category exhibits behavior closer to the category of spectra. Note that there is a sequence of cobordism categories along $\mathsf{Bord}_n^{(X,\zeta)} \to \mathsf{Bord}_{n+1}^{(X,\zeta\oplus\mathbb{R})}$. The colimit is the cobordism category $\mathsf{Bord}_{\mathrm{st}}^X$ of *stably X-manifolds*. This is a natural example of an (∞,∞)-category that is not truncated at any finite level. We denote the stably framed case (when $X = *$) by $\mathsf{Bord}_{\mathrm{st}}^{\mathrm{fr}}$.

Theorem 6.4.1. *Assume the cobordism hypothesis. Then $L^{\infty\text{-adj}}\mathbb{F}$ is the infinite cobordism category $\mathrm{B}^\infty\mathsf{Bord}_{\mathrm{st}}^{\mathrm{fr}}$ of stably framed categories. It is the tensor unit of the monoidal category $\mathsf{CatSp}^{\infty\text{-adj}}$, so for any $X \in \mathsf{CatSp}^{\infty\text{-adj}}$, we have*

$$\mathrm{ev}_* : \mathrm{B}^\infty\mathsf{Bord}_{\mathrm{st}}^{\mathrm{fr}}, X \xrightarrow{\sim} X.$$

Remark 6.4.2. The last statement is an improved version of the stable cobordism hypothesis; we usually recover only the underlying groupoid, but using the lax internal hom we can now recover the whole category (or the categorical spectrum). An important consequence is that the group $\mathrm{O}(\infty)$ (or $\mathrm{PL}(\infty)$, which is reasonably close to $\pi_*(\mathbb{S})$) acts on the category X itself. In particular, one can form a quotient under this action, which should be thought of as *categorical Thom spectra.*

Appendix A

Steiner's theory for strict ∞-categories

In this appendix, we give a summary of Steiner's theory. It provides an equivalence between a class of strict ∞-categories and a class of chain complexes with a kind of positivity structure. It is a powerful computational tool in combinatorics of strict ∞-categories, especially when the dualities involved in making a construction functorial are confusing.

Only in this appendix, a *category* without specification will mean a $(1,1)$-category, not an $(\infty,1)$-category. References include [Ste04], [Ara+23], [AM20], [OR23].

Remark A.0.1. A strict ∞-category X, as defined in Definition 2.1.1, admits the following more explicit description (cf. Remark 2.1.6):

(1) for each integer $n \geq 0$ a set X_n, called the set of n-cells,

(2) for each $p > q \geq 0$ a structure of a category with objects X_q and morphisms X_p, i.e.,

 (i) the (q-)source and the (q-)target maps $s_q, t_q : X_p \to X_q$,

 (ii) the identity map $i_p : X_q \to X_p$,

(iii) the composition map $*_q : X_p {}_{s_q}\times_{X_q} {}_{t_q}X_p \to X_p$ satisfying associativity and unitality,

which are compatible in the sense that for $p > q > r \geq 0$ the above data defines a 2-category structure on (X_p, X_q, X_r), i.e., they satisfy the globularity conditions $s_p s_q = s_p = s_p t_q$, $t_p t_q = t_p = t_p s_q$ and the "interchange law" $(f *_q g) *_r (h *_q k) = (f *_r h) *_q (g *_r k)$ for $f, g, h, k \in C_p$. The data (1) and (2)(i)(ii) with the globularity condition are precisely the data of reflexive globular sets (i.e., set-valued presheaves on $\mathbb{G}$) and (iii) is the structure required to extend it to presheaves on Θ satisfying the Segal conditions.

Remark A.0.2. For any category $\mathcal{C}$ with finite limits, the obvious modification of the above defines the notion of *strict* ∞*-category objects in* $\mathcal{C}$, which describes $\mathcal{C}$-valued presheaves on Θ satisfying the Segal conditions.

A.1 Steiner's adjunction

Steiner's adjunction is between the category of augmented directed complexes and the category of strict ω-categories. The main theorem of Steiner's theory states that it restricts to an adjoint equivalence on the full subcategory of *Strong Steiner objects*. First, we compile relevant definitions on the chain complex side:

Definition A.1.1. (1) An *augmented directed chain complex* (ADC for short) is a triple $(A, A^+, \epsilon) = ((A_\bullet, \partial_\bullet), A^+_\bullet, \epsilon)$, where $A \in \mathsf{Ch}_{\geq 0}(\mathbb{Z})$ is a nonnegatively (homologically) graded chain complex, $\epsilon : A_0 \to \mathbb{Z}$ is an augmentation and $A^+_n \subset A_n$ is a sub-$\mathbb{N}$-module (a.k.a. submonoid) for each n (called the *positivity submonoid*; we do **not** ask $\partial_n(A^+_n) \subset A^+_{n-1}$). We often omit $\partial_\bullet$, A^+, and ϵ when it is not confusing or clear from the context.

(2) A map $A \to B$ of ADCs is a chain map $f : A \to B$ that commutes with augmentations and satisfies $f(A^+) \subset B^+$. Let adCh denote the category of

ADCs.

(3) A *basis* of an ADC $(A_\bullet, A_\bullet^+)$ is a graded subset $\{B_q \subset A_q^+\}_{q\geq 0}$ which is both a $\mathbb{Z}$-basis of A and a $\mathbb{N}$-basis of A^+. This is unique if it exists[1], in which case we call it *the* basis of A and moreover make the following definitions:

(i) Any element $a = \sum_{b\in B_q} \lambda_b b \in A_q$ is uniquely a difference $a = a^+ - a^-$ with $a^+, a^- \in A^+$. We write $\partial_q^\pm(a) := (\partial_q(a))^\pm$. Also, define $\mathrm{supp}(a) \subset B_q$ as the set of $b \in B_q$ with $\lambda_b \neq 0$.

(ii) The basis $\{B_q\}$ is *unital* if for every $q \geq 0$ and $b \in B_q$ we have $\epsilon \circ \partial_0^+ \circ \cdots \circ \partial_{q-1}^+(b) = 1 = \epsilon \circ \partial_0^- \circ \cdots \circ \partial_{q-1}^-(b)$.

(iii) Consider the preorder on $\bigsqcup_{q\geq 0} B_q$ generated by the relation

$$\{(a,b) \mid b \in B_q, a \in \mathrm{supp}(\partial_{q-1}^- b)\} \cup \{(a,b) \mid a \in B_q, b \in \mathrm{supp}(\partial_{q-1}^+ a)\}.$$

The basis is *strongly loop-free* if this preorder is a partial order.

(4) A *strong Steiner complex* is an ADC with a strongly loop-free unital basis.

Remark A.1.2. The category adCh is cocomplete and colimits can be computed degreewise. More precisely, the forgetful functor $\mathsf{adCh} \to \mathsf{grMod}_\mathbb{Z} \times \mathsf{grMod}_\mathbb{N}$ creates colimits.

Example A.1.3. For any CW complex X with chosen orientations of cells, we regard the augmented cellular chain complex $C_\bullet(X) \to \mathbb{Z}$ as an ADC with the basis consisting of the cells. In particular, let $D^p \subset \mathbb{R}^p \subset \mathbb{R}^\infty$ be the unit n-disk with the CW structure $D^p = \left(\bigsqcup_{q\leq p-1}(e_+^q \sqcup e_-^q)\right) \sqcup e^p$, where $e_\pm^q = \{(x_0, \ldots, x_{q-1}, x_q, 0, \ldots, 0) \mid x_0^2 + \cdots + x_q^2 = 1, \pm x_q > 0\} \subset \mathbb{R}^p$. The cellular chain $\{C_\bullet(D^q)\}_{q\geq 0}$ extends to an ω-category object in $\mathsf{adCh}^{\mathrm{op}}$; for $p > q \geq 0$,

[1] This is true in general for a $\mathbb{N}$-module. In fact, any isomorphism $f : \mathbb{N}^{\oplus I} \to \mathbb{N}^{\oplus J}$ is induced by a bijection $I \to J$. To show this, note that both f and f^{-1} are injective, so they do not decrease the sum of the coefficients in the standard basis presentation.

- the co-source and the co-target map $s^q, t^q : C_\bullet(D^q) \to C_\bullet(D^p)$ are induced by the inclusions $D^q \hookrightarrow D^p$ with the image e^q_- and e^q_+,
- the co-identity map $i^q : C_\bullet(D^p) \to C_\bullet(D^q)$ is induced by the projection $D^p \to D^q$,
- the co-composition map $*^p : C_\bullet(D^p) \to \operatorname{colim}(C_\bullet(D^p) \leftarrow C_\bullet(D^q) \to C_\bullet(D^p)) \cong C_\bullet(D^p \sqcup_{D^q} D^p)$ is induced by the q-fold unreduced suspension of the pinch map $D^{p-q} \to D^{p-q} \vee D^{p-q}$. These moreover satisfy the interchange law.

In other words, $\{C_\bullet(D^q)\} : \mathbb{G} \to \mathsf{adCh}$ extends to a functor $\Theta \to \mathsf{adCh}$ satisfying the Segal conditions, so the restricted Yoneda embedding $\mathsf{adCh} \to \mathsf{PSh}_{\mathsf{Set}}(\Theta)$, which is right adjoint to the Yoneda extension $\mathsf{PSh}_{\mathsf{Set}}(\Theta) \to \mathsf{adCh}$, factors through the subcategory $\infty\mathsf{Cat}^{\mathrm{str}}$.

Definition A.1.4. Steiner's adjunction

$$\begin{tikzcd} \infty\mathsf{Cat}^{\mathrm{str}} \arrow[r, shift left, "\lambda"] \arrow[r, phantom, "\perp"] & \mathsf{adCh} \arrow[l, shift left, "\nu"] \end{tikzcd}$$

is the restricted Yoneda extension adjunction of $\{C_\bullet(D^q)\}$.

Remark A.1.5. One should think of λ as the linearization functor; q-cells of a strict ∞-category X generates $(\lambda X)_q$ and the relations are given by splitting a composition into a sum. The functor ν is a sort of cellular nerve: for $A \in \mathsf{adCh}$, a q-cell of νA corresponds to a map $\lambda X_q \to A$. For a detailed explicit description of these functors, see e.g. [OR23, §2.3].

We now explain the notion corresponding to strong Steiner complexes on the category side following [Ara+23]:

Definition A.1.6. Let X be a strict ω-category.

(1) A set of cells $E = \coprod_{n\geq 0} E_n$ of X, where $E_n \subset X_n$, is a *polygraphic basis* if the following diagram is a pushout for any n:

$$\begin{tikzcd} \coprod_{E_q} \partial C_q \arrow[r] \arrow[d, hook] & X^{\leq q-1} \arrow[d, hook] \\ \coprod_{E_q} C_q \arrow[r] & X^{\leq q} \end{tikzcd}$$

C is called a *polygraph* or a *computad* if it admits a polygraphic basis. It is shown in [Mak05][Ara+23, Proposition 2.4] that if a basis exists, it must be the set of nondegenerate indecomposables.

(2) Let E be a polygraphic basis. For $c \in X_q$, define $\mathrm{supp}(c) \subset E_q$ be the set of factors of c.

(3) Consider the preorder on E generated by the relation

$$\bigcup_{p<q} \{(a,b) \in E_p \times E_q \mid a \in \mathrm{supp}(s_p b)\} \cup \bigcup_{p>q} \{(a,b) \in E_p \times E_q \mid b \in \mathrm{supp}(t_q a)\}.$$

A polygraphic basis E is *strongly loop-free* if the preorder is a partial order.

(4) A strict ω-category is *strong Steiner* if it admits a strongly loop-free polygraphic basis.

Remark A.1.7. Any polygraph is Gaunt.

Theorem A.1.8 ([Ste04], [Ara+23]). *The adjunction $\lambda \dashv \nu$ restricts to an equivalence between the category of strong Steiner categories and the category of strong Steiner complexes. Moreover, λX is strong Steiner if and only if X is, and similarly for ν.*

A.2 Operations on augmented directed complexes

Corresponding to the operations on strict ∞-categories, the category adCh has suspension, duality involutions, and the tensor product. A good reference is [OR23, §1, 2].

Definition A.2.1. The *suspension* functor $\sigma : \mathsf{adCh} \to \mathsf{adCh}$ sends an object

$$A : \quad \cdots \to A_n \xrightarrow{\partial_n^A} A_{n-1} \to \cdots \to A_0 \xrightarrow{\varepsilon^A} \mathbb{Z}$$

to

$$\begin{tikzcd} \cdots \arrow[r] & (\sigma A)_n \arrow[r, "\partial_n^{\sigma A}"] \arrow[d, equal] & (\sigma A)_{n-1} \arrow[r, "\partial_{n-1}^{\sigma A}"] \arrow[d, equal] & \cdots \arrow[r] & (\sigma A)_1 \arrow[r, "\partial_1^{\sigma A}"] \arrow[d, equal] & (\sigma A)_0 \arrow[r, "\varepsilon^{\sigma A}"] \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\ \cdots \arrow[r] & A_{n-1} \arrow[r, "\partial_{n-1}^A"] & A_{n-2} \arrow[r, "\partial_{n-2}^A"] & \cdots \arrow[r] & A_0 \arrow[r, "\binom{-\varepsilon^A}{\varepsilon^A}"] & \bot\mathbb{Z} \oplus \top\mathbb{Z} \arrow[r, "{(1,1)}"] & \mathbb{Z} \end{tikzcd}$$

with the positivity submonoids $(\sigma A)_n^+ = A_{n-1}^+$ for $n \geq 1$ and $(\sigma A)_0^+ = \bot\mathbb{N} \oplus \top\mathbb{N}$. On morphsims, σ assigns $f : A \to B$ to its obvious degree shift $\sigma f : \sigma A \to \sigma B$. The functor σ clearly lifts to a colimit-preserving, fully faithful functor $\sigma : \mathsf{adCh} \to \mathsf{adCh}_{\sigma 0/}$.

Definition A.2.2. For $\tau \in (\mathbb{Z}/2)^{\mathbb{Z}_{\geq 1}}$, define the τ-*dual* functor $D_\tau : \mathsf{adCh} \to \mathsf{adCh}$

$$(A_n, \partial_n : A_n \to A_{n-1}, \varepsilon, A^+) \mapsto (A_n, (-1)^{\tau(n)}\partial_n, \varepsilon, A^+)$$

on objects and the identity on morphisms (seen as those of graded abelian groups). When τ is constantly 1, we call the τ-dual the *total dual* and denote by $(-)^\circ$. Similarly, when $\tau(n) \equiv n \pmod 2$ (resp. $\tau(n) \equiv n+1 \pmod 2$) then we call the τ-dual the *odd dual* (resp. *even dual*) and denote by $(-)^{\mathrm{op}}$ (resp. $(-)^{\mathrm{co}}$).

In the following, in order to match with the *lax* Gray tensor product of ∞-categories, we use the *reverse* of the monoidal structure on adCh that is standard in the references [Ste04][OR23]. One can think of the reversed Koszul sign rule as the rule for differential acting from the right.

Definition A.2.3. (1) The usual symmetric monoidal structure on $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$ with Koszul sign rule is equivalent to its *reverse*, i.e., the differential on a tensor product may as well be defined by $\partial(x \otimes y) = (-1)^{\deg y}(\partial x) \otimes y + (x \otimes \partial y)$. We pick this *reversed Koszul sign rule* convention for the tensor product.

(2) We equip $\mathsf{grMod}_{\mathbb{N}}$ with the Day convolution symmetric monoidal structure, i.e., $(A^+_\bullet \otimes_{\mathbb{N}} B^+_\bullet)_n = \bigoplus_{i+j=n} A^+_i \otimes_{\mathbb{N}} B^+_j$.

(3) The symmetric monoidal structure on $\mathsf{Ch}_{\geq 0}(\mathbb{Z})$ induces a symmetric monoidal structure on the category $\mathsf{Ch}_{\geq 0}(\mathbb{Z})_{/\mathbb{Z}}$ of *augmented* complexes by $\epsilon_{A\otimes B} : (A \otimes B)_0 \simeq A_0 \otimes B_0 \xrightarrow{\epsilon_A \otimes \epsilon_B} \mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z}$.

(4) Let $A = (A, A^+, \epsilon_A)$ and $B = (B, B^+, \epsilon_B)$ be ADCs. We define the tensor product $A \otimes B$ as $(A \otimes_{\mathbb{Z}} B, A^+ \otimes_{\mathbb{N}} B^+, \epsilon_{A\otimes B})$. This tensor product canonically extends to a monoidal structure.

Remark A.2.4. The positivity structure in adCh breaks the symmetry of the tensor product: if A and B are both free ADCs on a single basis element in degree 1, say a and b, then the symmetry morphism $a \otimes b \mapsto -b \otimes a$ does not preserve the positivity submonoid.

Compatibility with the operations on $\infty\mathsf{Cat}^{\mathrm{str}}$ through the Steiner's adjunction is summarized as follows:

Remark A.2.5. (1) ([AM20, Proposition A.3]) The functors σ, D_τ, $\otimes$ preserves strong Steiner complexes.

(2) ([OR23, Proposition 2.12]) There is a natural isomorphism $\nu\sigma \cong \sigma\nu : \mathsf{adCh} \to \infty\mathsf{Cat}^{\mathrm{str}}$

(3) ([AM20, Proposition 2.19]) For any τ, the τ-dual functor naturally commutes with λ and ν.

(4) ([AM20, Theorem A.15], [OR23, Proposition 2.14]) There exists a unique bi-closed monoidal structure on $\infty\mathsf{Cat}^{\mathrm{str}}$ satisfying one of the following two equivalent conditions:

- $\nu|_{\mathsf{adCh}^{\mathrm{Ste}}} : \mathsf{adCh}^{\mathrm{Ste}} \to \infty\mathsf{Cat}^{\mathrm{str}}$ promotes to a monoidal functor (so $X \otimes X' := \nu((\lambda X) \otimes (\lambda X'))$ for strong Steiner categories X, X'), or

- $\lambda : \infty\mathsf{Cat} \to \mathsf{adCh}$ promotes to a monoidal functor, so $(\lambda X) \otimes (\lambda X') \simeq \lambda(X \otimes X')$ for strict ∞-categories X, X'.

The monoidal structure is called the *lax Gray tensor product of strict ∞-categories* and denoted by $\otimes$ or $\otimes^{\mathrm{lax}}$.

A.3 The cubes and The orientals

In this section, we define important families of strong Steiner categories and investigate their basic combinatorics. Let $C_\bullet(\Delta^n_{\mathrm{comb}})$ be the cellular chain complex of the combinatorial n-simplex, i.e., $C_k(\Delta^n_{\mathrm{comb}}) = \bigoplus_{\alpha:[k]\rightarrowtail[n]} \mathbb{Z}\alpha$ with the differential $\partial\alpha = \sum_{i=0}^{k}(-1)^i \alpha \circ \delta^i$ (where $\delta^i : [k-1] \to [k]$ skips the value i). We give it the structure of an ADC so that $\{\alpha : [k] \rightarrowtail [n]\}$ is a basis. In the following, we will notationally identify the nondegenerate simplex $\alpha : [k] \rightarrowtail [n]$ with the subset $\operatorname{Im}\alpha \subset [n]$ and the i-th vertex is denoted by $\underline{i}$. We let $\square^1 := C_1$ be the interval category and denote the degree 1 positive generator of $\lambda\square^1$ by $\underline{?}$, so the complex is $\mathbb{Z}\underline{?} \to \mathbb{Z}\underline{0} \oplus \mathbb{Z}\underline{1}$ with $\partial(\underline{?}) = \underline{1} - \underline{0}$, $\varepsilon(\underline{0}) = \varepsilon(\underline{1}) = 1$. We also denote the basis elements of $\lambda\square^n = (\lambda\square^1)^{\otimes n}$ by "(partially undetermined) binary strings" $\underline{a_n a_{n-1} \cdots a_1} := \underline{a_n} \otimes \underline{a_{n-1}} \otimes \cdots \otimes \underline{a_1}$ where $a_i \in \{?, 0, 1\}$.

Definition A.3.1. The *n-oriental* is the strict n-category $\Delta^n := \nu C_\bullet(\Delta^n_{\mathrm{comb}})$. The *$n$-cube* is the strict n-category $\square^n := (\square^1)^{\otimes n} = \nu(C_\bullet(\Delta^1_{\mathrm{comb}})^{\otimes n})$. These are strong Steiner (see below).

The nontrivial part of checking the Strong Steinerness of the ADCs above is to show that the basis is loop-free. It turns out that the preorder on the basis is not only a partial order but in fact a total order in these cases:

Proposition A.3.2. *Let $n_1, \ldots, n_k \geq 0$ be integers. The augmented directed chain complex $C_\bullet(\Delta^{n_1}_{\mathrm{comb}}) \otimes \cdots \otimes C_\bullet(\Delta^{n_k}_{\mathrm{comb}})$ has a basis and the preorder of Definition A.1.1 is a total order.*

Proof. [Ste04, example 3.8, 3.10] explicates the preorder on the lax cone construction and the tensor product. The preorder on the tensor product is essentially the lexicographic order, twisted by the degree according to the Koszul sign rule. The preorder on the lax (left) cone construction is similar and for a subset of $[n]$, it is a twisted lexicographic order on the indicator function, read as a binary string. So both are total orders. $\square$

To give an idea of "the lexicographic order twisted by the degree" in the proof, let us work out the case that we will use in Lemma 4.1.13.

Lemma A.3.3. *$\square^n$ is a strong Steiner category and the preorder of Definition A.1.1 on its polygraphic basis is a linear order. In particular,* $\mathrm{Aut}(\square^n)$ *is trivial.*

Proof. The second part follows from the first because any automorphism must preserve the order of the basis. When $n = 1$, the order on the basis is the total order $\underline{0} < \underline{?} < \underline{1}$. In general, Let $\underline{a_n \cdots a_1}$, $\underline{b_n \cdots b_1}$ be two basis elements such that $a_i = b_i$ for $i < k$ and $a_k \neq b_k$. Unwinding the definition, we see that the order on the basis is the "signed lexicographic order," i.e., $\underline{a_n \cdots a_1} < \underline{b_n \cdots b_1}$ if and only if either

- $a_1, \ldots, a_{k-1}$ contains even number of ? and $a_k < b_k$ (in the totally ordered set $\{0 <? < 1\}$), or
- $a_1, \ldots, a_{k-1}$ contains odd number of ? and $a_k > b_k$.

For example, if $a_1, \ldots, a_{k-1}$ contains even number of ? and $a_k = 0$, $b_k =?$, then

$$\underline{a_n \cdots a_{k+1} 0 a_{k-1} \cdots a_1} \leq \underline{1 \cdots 1 0 a_{k-1} \cdots a_1}$$
$$< \underline{1 \cdots 1 ? a_{k-1} \cdots a_1} \leq \underline{b_n \cdots b_{k+1} ? a_{k-1} \cdots a_1}$$

by the Koszul sign rule (observe that without the sign $\underline{1\cdots 1}$ is maximal and $\underline{0\cdots 0}$ is minimal) and $\underline{1\cdots 10a_{k-1}\cdots a_1} \in \operatorname{supp}(\partial^-(\underline{1\cdots 1?a_{k-1}\cdots a_1}))$. In particular, two basis elements are always comparable by checking the rightmost different entries, so the preorder is linear. $\square$

Remark A.3.4. The subcategory $\mathsf{Gaunt} \subset \infty\mathsf{Cat}$ is an exponential ideal for the cartesian product, so in particular it is self-enriched by the functor category. Using the suspension-hom adjunction, one sees that positive dimensional cells of a Gaunt-enriched (∞,∞)-category have a trivial ∞-groupoid of automorphisms. Therefore, a skeletal subcategory of the category $\mathsf{Gaunt}^{\mathrm{triv-aut}} \subset \mathsf{Gaunt}$ of gaunt ∞-categories with trivial automorphisms is gaunt. It follows that $\square, \mathbf{\Delta}, \Theta$ (both as underlying 1-categories or as (∞,∞)-categories) are gaunt.

We end this section by proving some retract relations of these fundamental gaunt ∞-categories.

Proposition A.3.5. (1) *The quotient by the full subcategory $\{1\}\otimes\Delta^n \subset \square^1\otimes\Delta^n$ is isomorphic to Δ^{n+1} and the quotient map admits a section sending the vertex $\underline{n+1}$ to the vertex $\underline{1}\otimes\underline{n}$.*

(2) *Further quotient by $\Delta^{\{0,\dots n\}} \subset \Delta^{n+1}$ is isomorphic to the (unreduced) suspension $\sigma\Delta^n$ and the quotient map admits a retraction.*

(3) *The sink-source wedge sum $\Delta^n \vee \Delta^m$ is a retract of Δ^{n+m}.*

Proof. (1) We define the map Steiner complexes corresponding to $q : \square^1\otimes\Delta^n \to \Delta^{n+1}$. As a map of graded abelian groups, let

$$\underline{1}\otimes\alpha \mapsto \begin{cases} \underline{n+1} & \text{if } \deg(\alpha)=0 \\ 0 & \text{if } \deg(\alpha)>0 \end{cases}, \quad \underline{0}\otimes\alpha\mapsto\alpha, \quad \underline{?}\otimes\alpha\mapsto\alpha\sqcup\{n+1\}.$$

It is straightforward to check that the map defined is indeed a map of augmented directed chain complexes. Note that the following square commutes and is bicartesian in $\mathsf{Ch}(\mathbb{Z})$ (i.e., it is a short exact sequence in $\mathsf{Ch}(\mathbb{Z})_{\mathbb{Z}//\mathbb{Z}}$ with the basepoint given by the sink vertex):

$$\begin{tikzcd} \lambda(\{1\} \otimes \Delta^n) \arrow[r, hook] \arrow[d] & \lambda(\square^1 \otimes \Delta^n) \arrow[d, two heads] \\ \mathbb{Z} \arrow[r, "\underline{n+1}"] & \lambda(\Delta^{n+1}) \end{tikzcd}$$

So we have a short exact sequence of the reduced complexes

$$\tilde{\lambda}(\{1\} \otimes \Delta^n) \to \tilde{\lambda}(\square^1 \otimes \Delta^n) \to \tilde{\lambda}(\Delta^{n+1}).$$

Now the first map admits an obvious retraction, so the second map admits a section; this splitting is given by the disjoint union decomposition of the standard basis, so the section is again a map of (reduced) ADCs. Adding back the basepoint factor we obtain the section $\lambda(\Delta^{n+1}) \to \lambda(\square^1 \otimes \Delta^n)$.

(2) The argument is similar to the first part; define the map $\lambda(\Delta^{n+1}) \twoheadrightarrow \lambda(\sigma\Delta^n)$ of ADCs by $\underline{0}, \ldots, \underline{n} \mapsto \perp$, $\underline{n+1} \mapsto \top$, and

$$\alpha \mapsto \begin{cases} \sigma(\alpha \setminus \{n+1\}) & \text{if } \deg(\alpha) > 0,\ n+1 \in \alpha \\ 0 & \text{if } \deg(\alpha) > 0,\ n+1 \notin \alpha. \end{cases}$$

Then the following is a short exact sequence in $\mathsf{Ch}(\mathbb{Z})_{\mathbb{Z}//\mathbb{Z}}$ with the basepoint given by the source vertex:

$$\begin{tikzcd} \lambda(\Delta^{\{0,\ldots,n\}}) \arrow[r, hook] \arrow[d] & \lambda(\Delta^{n+1}) \arrow[d, two heads] \\ \mathbb{Z} \arrow[r, "\perp"] & \lambda(\sigma\Delta^n) \end{tikzcd}$$

The rest goes the same as (1).

(3) The degree-k part of the ADC $\lambda(\Delta^n \vee \Delta^m)$ is generated by injections $[k] \to [n+m]$ with the image contained either in $\{0, \ldots, n\}$ or $\{n, \ldots, n+m\}$, so there is an obvious inclusion $s : \lambda(\Delta^n \vee \Delta^m) \hookrightarrow \lambda(\Delta^{n+m})$. The retract r is the one which exhibits $\Delta^n_{\mathrm{comb}} \vee \Delta^m_{\mathrm{comb}}$ as a simple deformation retract of $\Delta^{n+m}_{\mathrm{comb}}$;

explicitly, $r(\alpha)$ for $\alpha : [k] \rightarrowtail [n]$ is defined as follows:

- If $\alpha \subset \{0, \dots, n\}$ or $\alpha \subset \{n, \dots, n+m\}$, then $r(\alpha) = \alpha$. Otherwise,
- If $k = 1$ so $\alpha(0) < n < \alpha(1)$, then $r(\alpha) = \{\alpha(0), n\} + \{n, \alpha(1)\}$.
- If $k \geq 2$ and $\alpha(0) < n < \alpha(1)$, then $r(\alpha) = \{n, \alpha(1), \dots, \alpha(k)\}$.
- If $k \geq 2$ and $\alpha(k-1) < n < \alpha(k)$, then $r(\alpha) = \{\alpha(0), \dots, \alpha(k-1), n\}$
- If $k \geq 2$ and $\alpha(0) < \alpha(1) \leq n \leq \alpha(k-1) < \alpha(k)$, then $r(\alpha) = 0$.

Checking the equation $r(\partial\alpha) = \partial(r\alpha)$ is tedious but a straightforward casework:

- It is clear if $\alpha \subset \{0, \dots, n\}$ or $\{n, \dots, n+m\}$.
- If $\alpha(2) \leq n \leq \alpha(k-2)$, one has $r(\alpha \circ \delta^i) = 0$ for all i.
- If $k \geq 4$ and $\alpha(0) < \alpha(1) \leq n < \alpha(2)$, we have $r(\alpha \circ \delta^0) = r(\alpha \circ \delta^1)$ and $r(\alpha \circ \delta^{\geq 2}) = 0$, s, $r(\partial\alpha) = 0 = \partial(r\alpha)$.
- If $k \geq 3$ and $\alpha(0) < n < \alpha(1)$, we have $r(\alpha \circ \delta^i) = r(\alpha) \circ \delta^i$.
- These and the symmetric cases cover except when $k = 1, 2$ and $k = 3$, $\alpha(1) \leq n \leq \alpha(2)$, which can be checked one by one.

It is clear that r is a map of ADCs and that it is a retract.

□

The following density results are proven by [Cam22] (for the cubes) and [GHon] (for the orientals).

Corollary A.3.6. (1) *The n-oriental Δ^n is a retract of the n-cube $\square^n$.*

(2) *We have the inclusion of the idempotent completions $\Theta \subset \widetilde{\mathbf{\Delta}} \subset \widetilde{\square}$. In particular, the orientals and the cubes are dense in ∞Algbrd.*

Proof. (1) This follows by inductively applying (1) of the proposition. Although it is notationally heavier, it is also not too hard to directly provide the section and retraction: the retraction collapses each face of the form $\underline{0\cdots01?\cdots?}$ (possibly with no $\underline{0}$). The section $\Delta^n \hookrightarrow \square^n$ sends the simplex $\{i_0 < \cdots < i_k\} : [k] \rightarrowtail [n]$ to the composition of the faces of $\square^n$ of the form $\underline{a_n \ldots a_1}$ where:

- $a_i = 0$ if $i > i_k$ and $a_i = 1$ if $i \leq i_0$.
- For each $0 \leq j < k$, exactly one among $\underline{a_{i_{j+1}} \cdots a_{i_j+1}}$ is $\underline{?}$ and on the left (resp. right) of $\underline{?}$ is all $\underline{1}$ (resp. $\underline{0}$).

When $n = 3$, the picture of "coarsening" the cube to the oriental is the following (observe that the directions of 2- and 3-cells also align):

$$\begin{tikzcd}
000 \arrow[rr] \arrow[dr, bend right, red] \arrow[dd, bend right, blue] & & 001 \arrow[dr] \arrow[dd, bend right, purple] & \\
& 010 \arrow[rr, bend right, red] \arrow[dd, dashed] & & 011 \arrow[dd] \\
100 \arrow[rr, dotted] \arrow[dr, bend right, blue] & & 101 \arrow[dr, bend right, purple] & \\
& 110 \arrow[rr, bend right, blue] & & 111
\end{tikzcd}$$

(2) The second inclusion follows from (1). The first inclusion uses the strategy of [Cam22]; by our definition of Θ (Definition 2.1.4), it reduces to (2)(3) of the proposition.

$\square$

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