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A Quillen adjunction between globular and complicial approaches to (∞, n)-categories

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ABSTRACT

We prove the compatibility between the suspension construction and the complicial nerve of ω-categories. As a motivating application, we produce a Quillen pair between the models of (∞, n)-categories given by Rezk's complete Segal Θ_n-spaces and Verity's n-complicial sets.

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Contents

Introduction

Category theory has become a ubiquitous language in mathematics. However, it has its limitations and needs to be extended in two ways. On the one hand, many category-like objects turn out to have further composable structure, which we like to refer to as *higher morphisms*. On the other hand, many category-like structures satisfy the axioms for composition only in a weak sense, which is oftentimes witnessed by the presence of higher coherence morphisms.

As it has become apparent in the last two decades, the right way to capture both phenomena at once is the language of (∞, n)-categories. While this is the nickname for a general idea, several mathematical objects are used to make this idea precise, each with its own advantages and disadvantages. Amongst those, there are Verity's n-complicial sets [23,26,27,34,35], Rezk's complete Segal Θ_n-spaces [25], Bergner–Rezk's categories enriched over an appropriate model of $(\infty, n-1)$-categories [6,7], Ara's n-quasi-categories [1], and Campion–Doherty–Kapulkin–Maehara's n-comical sets [9,10].

Many equivalences amongst those models have been established, combining work by Bergner–Rezk [6,7], Ara [1], Barwick–Schommer–Pries [8], Campion–Doherty–Kapulkin–Maehara [9,10]. Thanks to these contributions, it has been known that there are at most two equivalence classes of models of (∞, n)-categories. This paper is a step towards connecting the two – a priori distinct – connected components, thus unifying the notion of an (∞, n)-category. In fact, using the results from this paper it was recently shown by Loubaton [20] that all the said models are indeed equivalent.

More precisely, in this paper we work towards establishing that the homotopy theory $\mathit{msSet}_{(\infty,n)}$ of n-complicial sets is equivalent to the homotopy theory $\mathit{sSet}^{\Theta_n^{\mathrm{op}}}_{(\infty,n)}$ of complete Segal Θ_n-spaces. The fact that complicial sets should be the appropriate model for weak higher categories, and therefore be equivalent to other established models of (∞, n)-categories, was conjectured more than three decades ago (see e.g. [8,32,35]) and has only been proven [11,21] for $n \leq 2$ in 2019.

The model comparison is important as each model comes with its own advantages and disadvantages. In our specific situation, the model of n-complicial sets has the virtue to be based on simplicial sets with marking, very close in nature to the well-understood world of simplicial sets. By contrast, the model of complete Segal Θ_n-spaces gives a more direct access to globular compositions and other higher categorical operations.

Partially generalizing joint work of the authors with Bergner [5], we construct an adjunction between the model categories $s\mathcal{S}et_{(\infty,n)}^{\Theta_n^{\mathrm{op}}}$ and $ms\mathcal{S}et_{(\infty,n)}$, and show in Theorem 4.16 that it is a Quillen pair, obtaining the following.

Theorem A. *There is an adjunction of ∞-categories between the ∞-category of complete Segal Θ_n-spaces and the ∞-category of n-complicial sets.*

We designed these adjoint functors to translate appropriately the theory of complete Segal Θ_n-spaces into that of n-complicial sets and the resulting adjunction is the first explicitly constructed candidate to implement an equivalence of homotopy theories between complete Segal Θ_n-spaces and n-complicial sets.

In order to prove Theorem A, it is necessary to understand how two basic higher-categorical constructions – the *wedge* and the *two-point suspension* – are implemented in the worlds of n-complicial sets and complete Segal Θ_n-spaces. Both are inherently present in the model of complete Segal Θ_n-spaces, but are more subtle when working with complicial sets.

We have already studied the wedge construction for n-complicial sets in [24], and we refer the reader to [24] for the definition and a detailed treatment. In this paper, we focus on developing the analogous study for the two-point suspension of complicial sets. Roughly speaking, given an n-dimensional category $\mathcal{C}$, its two-point suspension $\Sigma\mathcal{C}$ is an $(n+1)$-dimensional category with two objects and a single interesting hom given precisely by $\mathcal{C}$. The assignment Σ can be considered as a left adjoint to the functor picking from an $(n+1)$-category with distinguished objects x, y the n-category of morphisms from x to y.

The two-point suspension of strict n-categories naturally defines a functor $\Sigma\colon n\mathcal{C}at \to (n+1)\mathcal{C}at$, and there is also a candidate counterpart to the two-point-suspension for complicial sets, incarnated by a functor $\Sigma\colon ms\mathcal{S}et_{(\infty,n)} \to ms\mathcal{S}et_{(\infty,n+1)}$. The crucial ingredient for Theorem A, that we establish as Theorem 3.22, is that the suspension construction is compatible with a natural way of regarding a strict n-category as an (∞,n)-category, given by the *Roberts–Street nerve* $N^{\mathrm{RS}}\colon n\mathcal{C}at \to ms\mathcal{S}et$.

Theorem B. *If $\mathcal{C}$ admits an algebraic model in an appropriate sense, then $N^{\mathrm{RS}}\Sigma\mathcal{C}$ is equivalent to $\Sigma N^{\mathrm{RS}}\mathcal{C}$ in the model structure for n-complicial sets.*

Here, the precise condition on $\mathcal{C}$ requires it to be obtained from an (augmented directed) chain complex via Steiner's functor $\nu\colon ad\mathcal{C}h \to \omega\mathcal{C}at$ (see [3,28]).

The theorem has also impact beyond the construction of the Quillen adjunction above. Most prominently, it is used in work by Loubaton [19,20], who gives a criterion to identify self-equivalences on the ∞-category of n-complicial sets, proves that the Quillen adjunction from Theorem A is indeed a Quillen equivalence, and shows that the homotopy theory of n-complicial sets is equivalent to that of complete Segal n-spaces, as well to that of the other known models of (∞,n)-categories.

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1. Steiner's augmented directed chain complexes

We recall the basic definitions around Steiner's augmented directed chain complexes, as well as some constructions based on augmented directed chain complexes: the suspension, tensor product, and the total dual, as well as the main properties that we use later in the paper and relevant examples. Most of the material is drawn from [28] (see also [3]).

1.1. Augmented directed chain complexes

By a *chain complex* C we will always mean an $\mathbb{N}$-graded chain complex of abelian groups with homological indexing, that is, a family $(C_q)_{q\geq 0}$ of abelian groups, together with maps $\partial_q\colon C_{q+1}\to C_q$ satisfying $\partial_q\partial_{q+1}=0$. We also assume that, whenever occurring, $C_{-1}=0$, and $\partial_{-1}=0$.

Given chain complexes C and $\overline{C}$, a *chain map* or *morphism of chain complexes* $\phi\colon C\to\overline{C}$ consists of a family of homomorphisms $(\phi_q\colon C_q\to\overline{C}_q)_{q\geq 0}$ that commutes with the differentials in the sense that $\overline{\partial}_q\phi_{q+1}=\phi_q\partial_q$ for every $q\geq 0$.

An *augmented chain complex* is a pair (C,ε) of a chain complex C and an augmentation, namely a map $\varepsilon\colon C_0\to\mathbb{Z}$ such that $\varepsilon\partial_0=0$.

An *augmented chain map* $\phi\colon(C,\varepsilon)\to(\overline{C},\overline{\varepsilon})$ between augmented chain complexes (C,ε) and $(\overline{C},\overline{\varepsilon})$ consists of a chain map $\phi\colon C\to\overline{C}$ that is moreover compatible with the augmentations, namely such that $\overline{\varepsilon}\phi_0=\varepsilon$.

We recall the enhancement of augmented chain complexes developed by Steiner [28, §2].

Definition 1.1 *([28, Def. 2.2]).* An *augmented directed chain complex* is a triple (C,C^+,ε) where (C,ε) is an augmented chain complex and $C^+=(C_q^+)_{q\geq 0}$ is a collection of commutative monoids, where C_q^+ is a submonoid of C_q called the *positivity submonoid* of C_q.

A *morphism of augmented directed chain complexes*, or an *augmented directed chain map* $\phi\colon (C, C^+, \varepsilon) \to (\overline{C}, \overline{C}^+, \overline{\varepsilon})$ between augmented directed chain complexes (C, C^+, ε) and $(\overline{C}, \overline{C}^+, \overline{\varepsilon})$ is an augmented chain map $\phi\colon (C, \varepsilon) \to (\overline{C}, \overline{\varepsilon})$ that moreover preserves the positivity submonoids, namely such that

$$\phi_q(C_q^+) \subseteq \overline{C}_q^+$$

for all $q \geq 0$.

We denote by $ad\mathcal{C}h$ the category of augmented directed chain complexes and maps of chain complexes that preserve the augmentation and the positivity submonoids.

Remark 1.2. The category $ad\mathcal{C}h$ is cocomplete, colimits are computed degreewise, and epimorphisms are detected pointwise in the category $\mathcal{A}b$ of abelian groups and the category $c\mathcal{M}on$ of commutative monoids, which are both cocomplete. That is, the forgetful functor

$$ad\mathcal{C}h \to \prod_{q\geq 0}(\mathcal{A}b \times c\mathcal{M}on)$$

given by $C \mapsto (C_q, C_q^+)_{q\geq 0}$ creates colimits (and in particular epimorphisms).

Remark 1.3. Consider the following left adjoint functors.

(1) The free abelian group functor on a set and the free commutative monoid functor on a set,

$$\mathbb{Z}[-]\colon \mathcal{S}et \to \mathcal{A}b \text{ and } \mathbb{N}[-]\colon \mathcal{S}et \to c\mathcal{M}on,$$

given by $X \mapsto \mathbb{Z}[X]$ and $X \mapsto \mathbb{N}[X]$. The right adjoint functors are the forgetful functors.

(2) The free abelian group functor on a pointed set and the free commutative monoid functor on a pointed set,

$$\mathbb{Z}[-]\colon \mathcal{S}et_* \to \mathcal{A}b \text{ and } \mathbb{N}[-]\colon \mathcal{S}et_* \to c\mathcal{M}on,$$

given by $(X, x_0) \mapsto \mathbb{Z}[X \setminus \{x_0\}]$ and $(X, x_0) \mapsto \mathbb{N}[X \setminus \{x_0\}]$. The right adjoint functors are the forgetful functors that retain the identity as a base point.

(3) The functor that freely adds a base point to a set,

$$(-)_+\colon \mathcal{S}et \to \mathcal{S}et_*,$$

given by $X \mapsto (X \amalg \{*\}, *)$. The right adjoint functor is the functor that forgets the base point.

Being left adjoint functors, they all preserve colimits (and in particular epimorphisms).

Notation 1.4. Let $m \geq -1$ and $q \geq -1$. We denote

- by $\Delta[m]_q = \mathcal{C}at([q],[m])$ the set of q-simplices of the standard simplex[1] $\Delta[m]$. A generic q-simplex in $\Delta[m]$ is of the form

$$[\mathbf{a}] = [a_0, a_1, \ldots, a_q]$$

  with $0 \leq a_0 \leq a_1 \leq \ldots \leq a_q \leq m$. We say that q is the *length* $|\mathbf{a}|$ of $[\mathbf{a}]$.
- by $B[m]_q \subseteq \Delta[m]_q$ the set of non-degenerate q-simplices of $\Delta[m]$, namely those simplices for which $0 \leq a_0 < a_1 < \ldots < a_q \leq m$.
- by $O[m]_q = \mathbb{Z}[B[m]_q] \cong \mathbb{Z}^{\binom{[m]}{[q]}}$ the abelian group freely generated by non-degenerate q-simplices of $\Delta[m]$. The generic element of $O[m]_q$ is a formal sum

$$c = \sum_{[\mathbf{a}] \in B[m]_q} c_{[\mathbf{a}]} \cdot [\mathbf{a}]$$

  where $c_{[\mathbf{a}]} \in \mathbb{Z}$.
- by $O[m]_q^+ = \mathbb{N}[B[m]_q] \cong \mathbb{N}^{\binom{[m]}{[q]}}$ the abelian monoid freely generated by non-degenerate q-simplices of $\Delta[m]$. The generic element of $O[m]_q^+$ is one for which $c_{[\mathbf{a}]} \in \mathbb{N}$.

There are canonical inclusions $\Delta[m]_q \supseteq B[m]_q \subseteq O[m]_q^+ \subseteq O[m]_q$.

The augmented directed chain complex $O[m]$ is the algebraic model of the m-oriental $\mathcal{O}[m]$, in a sense that will be made precise in Example 2.10.

Example 1.5 *([28, Ex. 3.8]).* For $m \geq -1$, we construct the augmented directed chain complex $O[m]$ with the following structure.

- For $q \geq 0$ the abelian group of q-chains is given by $O[m]_q$.
- For $q \geq 0$ the commutative monoid of positive q-chains is given by $O[m]_q^+$.
- For $q \geq -1$ the differential $\partial_q \colon O[m]_{q+1} \to O[m]_q$ is given by

$$\begin{aligned} \partial_q[\mathbf{a}] &= \partial_q[a_0, \ldots, a_q, a_{q+1}] \\ &= \sum_{i=0}^{q+1} (-1)^i \cdot [a_0, \ldots, \widehat{a_i}, \ldots, a_{q+1}] \in O[m]_q \end{aligned}$$

[1] We follow the convention that $[-1]$ is the empty category, and $\Delta[-1]$ is the initial simplicial set, which is levelwise empty.

- The augmentation map $\varepsilon\colon O[m]_0 \to \mathbb{Z}$ is given by

$$\varepsilon[a] = 1 \in \mathbb{Z}.$$

Later in the paper, we will make use of the following dual construction for an augmented directed chain complex.

Definition 1.6 *([3, §2.18]).* Let C be an augmented directed chain complex. The *total dual* C° of C is the augmented directed chain complex with the following structure

- For $q \geq 1$ the abelian group of q-chains is given by $C^\circ_q = C_q$;
- For $q \geq 1$ the commutative monoid of positive chains is given by $(C^\circ)^+_q = C^+_q$;
- For $q \geq 1$, the differential $\partial^{C^\circ}_q : C^\circ_q \to C^\circ_{q-1}$ is given by $\partial^{C^\circ}_q(c) = -\partial^C_q(c)$.
- The augmentation $\varepsilon^{C^\circ} : C^\circ_0 \to \mathbb{Z}$ is given by $\varepsilon^{C^\circ}(a) = \varepsilon^C(a)$.

This construction defines an involution $(-)^\circ\colon ad\mathcal{C}h \to ad\mathcal{C}h$.

1.2. Suspension of augmented directed chain complexes

We define a two-point suspension for augmented directed chain complexes. This is the construction that Steiner denotes $V(1, C)$ in [30, §5].[2]

Definition 1.7. Let C be an augmented directed chain complex. The *suspension* of C is the augmented directed chain complex ΣC with the following structure:

- For $q \geq 0$, the abelian group $(\Sigma C)_q$ of q-chains is given by

$$(\Sigma C)_q = \begin{cases} \mathbb{Z}[\perp, \top] & \text{if } q = 0, \\ C_{q-1} & \text{if } q \geq 1. \end{cases}$$

- For $q \geq 0$, the commutative monoid $(\Sigma C)^+_q$ of positive q-chains is given by

$$(\Sigma C)^+_q = \begin{cases} \mathbb{N}[\perp, \top] & \text{if } q = 0, \\ C^+_{q-1} & \text{if } q \geq 1. \end{cases}$$

- For $q \geq 0$, the differential $\partial_q\colon (\Sigma C)_q \to (\Sigma C)_{q-1}$ is given by

$$\partial^{\Sigma C}_q(c) := \begin{cases} \varepsilon^C(c) \cdot (\top - \perp) = -\varepsilon^C(c) \cdot \perp + \varepsilon^C(c) \cdot \top & \text{if } q = 0, \\ \partial^C_{q-1}(c) & \text{if } q \geq 1. \end{cases}$$

[2] This is different from the one-point suspension considered by Ara–Maltsiniotis in [3, §6.3].

- The augmentation $\varepsilon^{\Sigma C}\colon (\Sigma C)_0 \to \mathbb{Z}$ is given by

$$\varepsilon^{\Sigma C}\bot = 1 = \varepsilon^{\Sigma C}\top.$$

The augmented directed chain complex ΣC comes with a map $\Sigma O[-1] \to \Sigma C$ so it can be naturally regarded as an object of ${}^{\Sigma O[-1]/}ad\mathcal{C}h$. The following is a consequence of [30, Theorem 5.6].

Proposition 1.8. *The suspension functor $\Sigma\colon ad\mathcal{C}h \to {}^{\Sigma O[-1]/}ad\mathcal{C}h$ is fully faithful.*

1.3. Tensor product of augmented directed chain complexes

We consider the tensor product of abelian groups $\otimes\colon \mathcal{A}b \times \mathcal{A}b \to \mathcal{A}b$, as well as the (less known) tensor product of commutative monoids $\otimes\colon c\mathcal{M}on \times c\mathcal{M}on \to c\mathcal{M}on$. See e.g. [13, Chapter 16] for more details on this construction. We will mostly use instances of the tensor product of *free* abelian groups and *free* commutative monoids, which is described by the following remark.

Remark 1.9. Recall the functors from Remark 1.3.

(1) The free abelian group functor $\mathbb{Z}[-]\colon (\mathcal{S}et, \times) \to (\mathcal{A}b, \otimes)$ and the free commutative monoid functor $\mathbb{N}[-]\colon (\mathcal{S}et, \times) \to (c\mathcal{M}on, \otimes)$ is strong monoidal, namely, there are natural bijections

$$\mathbb{Z}[X] \otimes \mathbb{Z}[Y] \cong \mathbb{Z}[X \times Y] \quad \text{and} \quad \mathbb{N}[X] \otimes \mathbb{N}[Y] \cong \mathbb{N}[X \times Y],$$

for any sets X and Y. In particular, the tensor product of free abelian groups, resp. commutative monoids, is a free abelian group, resp. free commutative monoid. It follows that also the free abelian group functor $\mathbb{Z}[-]\colon (\mathcal{S}et_*, \times) \to (\mathcal{A}b, \otimes)$ and the free commutative monoid functor $\mathbb{N}[-]\colon (\mathcal{S}et_*, \times) \to (c\mathcal{M}on, \otimes)$ is strong monoidal.

(2) The free abelian group functor $\mathbb{Z}[-]\colon (\mathcal{S}et, \amalg) \to (\mathcal{A}b, \oplus)$ and the free commutative monoid functor $\mathbb{N}[-]\colon (\mathcal{S}et, \amalg) \to (c\mathcal{M}on, \oplus)$ are strong monoidal.

Definition 1.10 *([28, Example 3.10]).* Let C and D be augmented directed chain complexes. The *tensor product* of C and D is the augmented directed chain complex $C \otimes D$ with the following structure:

- For $q \geq 0$, the abelian group $(C \otimes D)_q$ of q-chains is given by

$$(C \otimes D)_q = \bigoplus_{k+\ell=q} C_k \otimes D_\ell.$$

- For $q \geq 0$, the commutative monoid $(C \otimes D)_q^+$ of positive q-chains is given by

$$(C \otimes D)_q^+ = \bigoplus_{k+\ell=q} C_k^+ \otimes D_\ell^+$$

- For $q \geq 0$, the differential $\partial_q^{C \otimes D} \colon (C \otimes D)_q \to (C \otimes D)_{q-1}$ is given by

$$\partial_q^{C \otimes D}(c \otimes d) := \partial^C c \otimes D + (-1)^{|c|} c \otimes \partial^D d$$

- The augmentation $\varepsilon^{C \otimes D} \colon (C \otimes D)_0 \cong C_0 \otimes D_0 \to \mathbb{Z}$ is given by

$$\varepsilon^{C \otimes D}(c \otimes d) = \varepsilon^C c \cdot \varepsilon^D d.$$

The construction defines a functor $\otimes \colon ad\mathcal{C}h \times ad\mathcal{C}h \to ad\mathcal{C}h$.
We now unpack tensor product of orientals.

Example 1.11. Let $k, \ell \geq 0$.

- For $q \geq 0$, the abelian group $(O[k] \otimes O[\ell]^\circ)_q$ of q-chains is given by

$$(O[k] \otimes O[\ell]^\circ)_q = \bigoplus_{i=0}^{q} O[k]_i \otimes O[\ell]_{q-i} \cong \bigoplus_{i=0}^{q} \mathbb{Z}[B[k]_i] \otimes \mathbb{Z}[B[\ell]_{q-i}] \cong \bigoplus_{i=0}^{q} \mathbb{Z}^{\binom{[k]}{[i]}} \otimes \mathbb{Z}^{\binom{[\ell]}{[q-i]}}$$

- For $q \geq 0$, the commutative monoid $(O[k] \otimes O[\ell]^\circ)_q^+$ of positive q-chains is given by

$$(O[k] \otimes O[\ell]^\circ)_q^+ = \bigoplus_{i=0}^{q} O[k]_i^+ \otimes O[\ell]_{q-i}^+ \cong \bigoplus_{i=0}^{q} \mathbb{N}[B[k]_i] \otimes \mathbb{N}[B[\ell]_{q-i}] \cong \bigoplus_{i=0}^{q} \mathbb{N}^{\binom{[k]}{[i]}} \otimes \mathbb{N}^{\binom{[\ell]}{[q-i]}}$$

- For $q > 0$, the differential $\partial_q^{O[k] \otimes O[\ell]^\circ} \colon (O[k] \otimes O[\ell]^\circ)_{q+1} \to (O[k] \otimes O[\ell]^\circ)_q$ is given by

$$\partial_q^{O[k] \otimes O[\ell]^\circ}([\mathbf{a}] \otimes [\mathbf{b}]) := \partial^{O[k]}[\mathbf{a}] \otimes [\mathbf{b}] + (-1)^{|\mathbf{a}|}[\mathbf{a}] \otimes \partial^{O[\ell]^\circ}[\mathbf{b}]$$

- The augmentation $\varepsilon^{O[k] \otimes O[\ell]^\circ} \colon (O[k] \otimes O[\ell]^\circ)_0 \cong O[k]_0 \otimes O[\ell]_0^\circ \to \mathbb{Z}$ is given by

$$\varepsilon^{O[k] \otimes O[\ell]^\circ}([a] \otimes [b]) := \varepsilon^{O[k]}([a]) \cdot \varepsilon^{O[\ell]^\circ}([b]) = 1 \cdot 1 = 1$$

Recall the functor $(-)_+ \colon \mathcal{S}et \to \mathcal{S}et_*$, which is the left adjoint to the forgetful functor.

Remark 1.12. For $q \geq 0$, there is a canonical map of pointed sets

$$\binom{[k+1+\ell]}{[q]}_+ \to \bigvee_{r=0}^{q} \left(\binom{[k]}{[r]} \times \binom{[\ell]}{[q-r]} \right)_+$$

that splits a subset of $[k+1+\ell]$ into its $[k]$-part and $[\ell]$-part, using the base point whenever any of them is empty. This induces a map of abelian groups

$$\phi_q\colon O[k+1+\ell]_q \cong \mathbb{Z}^{\binom{[k+1+\ell]}{[q]}} \to \bigoplus_{r=0}^{q} \mathbb{Z}^{\binom{[k]}{[r]}} \otimes \mathbb{Z}^{\binom{[\ell]}{[q-r]}} \cong \Sigma(O[k]\otimes O[\ell]^\circ)_q$$

and commutative monoids

$$\phi_q\colon O[k+1+\ell]_q^+ \cong \mathbb{N}^{\binom{[k+1+\ell]}{[q]}} \to \bigoplus_{r=0}^{q} \mathbb{N}^{\binom{[k]}{[r]}} \otimes \mathbb{N}^{\binom{[\ell]}{[q-r]}} \cong \Sigma(O[k]\otimes O[\ell]^\circ)_q^+.$$

Explicitly, $\phi_0\colon O[k+1+\ell]_0 \to \Sigma\left(O[k]\otimes O[\ell])^\circ\right)_0$ is given by

$$\phi_0([a'']) := \begin{cases} \perp & \text{if } 0\leq a''\leq k,\\ \top & \text{if } k+1\leq a''\leq k+1+\ell.\end{cases}$$

For $q>0$, the map $\phi_q\colon O[k+1+\ell]_q \to \Sigma\left(O[k]\otimes O[\ell]^\circ\right)_q$ is given by

$$\phi_q([\mathbf{a},\mathbf{a}']) := \begin{cases} [\mathbf{a}]\otimes (s^0)^{k+1}[\mathbf{a}'], & \text{with } \mathbf{a}\subseteq[0,k], \mathbf{a}'\subseteq[k+1,k+1+\ell], |\mathbf{a}|\geq 0, |\mathbf{a}'|\geq 0,\\ 0 & \text{else.}\end{cases}$$

Proposition 1.13. *Let $k,\ell\geq 0$. There is a map of augmented directed chain complexes*

$$\phi\colon O[k+1+\ell]\to\Sigma\left(O[k]\otimes O[\ell]^\circ\right), \tag{1.14}$$

given degreewise by the maps ϕ_q described in Remark 1.12.

Proof. Given $\mathbf{a}\subseteq[0,k]$, and $\mathbf{a}'\subseteq[k+1,k+1+\ell]$, with $|\mathbf{a}|\geq 0$ and $|\mathbf{a}'|\geq 0$, which is the only case of interest, we obtain

$$\begin{aligned}\partial\phi([\mathbf{a},\mathbf{a}']) &= \partial\left([\mathbf{a}]\otimes(s^0)^{k+1}[\mathbf{a}']\right)\\ &= \partial[\mathbf{a}]\otimes(s^0)^{k+1}[\mathbf{a}'] + (-1)^{|\mathbf{a}|+1}[\mathbf{a}]\otimes\partial(s^0)^{k+1}[\mathbf{a}']\\ &= \phi(\partial([\mathbf{a},\mathbf{a}']))\end{aligned}$$

as desired. □

For $k,\ell\geq 0$, the maps from Proposition 1.13, together with other canonical maps, can be used to build a commutative diagram

$$\begin{tikzcd} O[k]\oplus O[\ell]^\circ \arrow[r] \arrow[d] & O[k+1+\ell] \arrow[d] \\ O[0]\oplus O[0]^\circ \arrow[r] & \Sigma(O[k]\otimes O[\ell]^\circ). \end{tikzcd} \tag{1.15}$$

Proposition 1.16. *Let $k, \ell \geq 0$. The diagram (1.15) induces a natural isomorphism of augmented directed chain complexes*

$$\Sigma(O[k] \otimes O[\ell]^\circ) \cong (O[0] \oplus O[0]^\circ) \coprod_{O[k] \oplus O[\ell]^\circ} O[k+1+\ell].$$

Proof. There are pushout squares of abelian groups and of commutative monoids:

$$\begin{tikzcd} \mathbb{Z}^{[k]} \oplus \mathbb{Z}^{[\ell]} \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}^{[k+1+\ell]} \arrow[d] \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\cong"'] & \mathbb{Z}[\perp, \top] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{N}^{[k]} \oplus \mathbb{N}^{[\ell]} \arrow[r, "\cong"] \arrow[d] & \mathbb{N}^{[k+1+\ell]} \arrow[d] \\ \mathbb{N} \oplus \mathbb{N} \arrow[r, "\cong"'] & \mathbb{N}[\perp, \top]. \end{tikzcd}$$

Here, the left vertical maps are the sums of the canonical map that folds the first $k+1$ copies of $\mathbb{Z}$ and the one that folds the last $\ell+1$ copies of $\mathbb{Z}$. This means that (1.15) induces a pushout of abelian groups and of commutative monoids:

$$\begin{tikzcd} (O[k] \oplus O[\ell]^\circ)_0 \arrow[r] \arrow[d] & O[k+1+\ell]_0 \arrow[d] \\ O[0]_0 \oplus (O[0]^\circ)_0 \arrow[r] & \Sigma(O[k] \otimes O[\ell]^\circ)_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (O[k] \oplus O[\ell]^\circ)_0^+ \arrow[r] \arrow[d] & O[k+1+\ell]_0^+ \arrow[d] \\ O[0]_0^+ \oplus (O[0]^\circ)_0^+ \arrow[r] & \Sigma(O[k] \otimes O[\ell]^\circ)_0^+. \end{tikzcd} \tag{1.17}$$

Let $q > 0$. Vandermonde's identity

$$\binom{[k+1+\ell]}{[q]} = \sum_{i=-1}^{q} \binom{[k]}{[i]} \cdot \binom{[\ell]}{[q-1-i]} = \binom{[k]}{[q]} + \binom{[\ell]}{[q]} + \sum_{i=0}^{q-1} \binom{[k]}{[i]} \cdot \binom{[\ell]}{[q-1-i]},$$

can be equivalently expressed as a pushout of pointed sets

$$\begin{tikzcd} \binom{[k]}{[q]}_+ \vee \binom{[\ell]}{[q]}_+ \arrow[r] \arrow[d] & \binom{[k+1+\ell]}{[q]}_+ \arrow[d] \\ \{*\} \arrow[r] & \bigvee_{i=0}^{q-1} \left(\binom{[k]}{[i]} \times \binom{[\ell]}{[q-1-i]} \right)_+ . \end{tikzcd}$$

Here, the cocomponents of the top horizontal map are the canonical, and the right vertical map is the one from Remark 1.12. By Remark 1.2(2), we then obtain pushouts of abelian groups and of commutative monoids:

$$\begin{tikzcd} \mathbb{Z}^{\binom{[k]}{[q]}} \oplus \mathbb{Z}^{\binom{[\ell]}{[q]}} \arrow[r] \arrow[d] & \mathbb{Z}^{\binom{[k+1+\ell]}{[q]}} \arrow[d] \\ 0 \oplus 0 \arrow[r] & \bigoplus_{r=0}^{q} \mathbb{Z}^{\binom{[k]}{[r]}} \otimes \mathbb{Z}^{\binom{[\ell]}{[q-1-r]}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{N}^{\binom{[k]}{[q]}} \oplus \mathbb{N}^{\binom{[\ell]}{[q]}} \arrow[r] \arrow[d] & \mathbb{N}^{\binom{[k+1+\ell]}{[q]}} \arrow[d] \\ 0 \oplus 0 \arrow[r] & \bigoplus_{r=0}^{q} \mathbb{N}^{\binom{[k]}{[r]}} \otimes \mathbb{N}^{\binom{[\ell]}{[q-1-r]}}. \end{tikzcd}$$

This means that (1.15) induces pushouts of abelian groups and of commutative monoids:

$$
\begin{tikzcd}
(O[k] \oplus O[\ell]^\circ)_q \arrow[r] \arrow[d] & O[k+1+\ell]_q \arrow[d] \\
O[0]_q \oplus (O[0]^\circ)_q \arrow[r] & \Sigma(O[k] \otimes O[\ell]^\circ)_q
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(O[k] \oplus O[\ell]^\circ)_q^+ \arrow[r] \arrow[d] & O[k+1+\ell]_q^+ \arrow[d] \\
O[0]_q^+ \oplus (O[0]^\circ)_q^+ \arrow[r] & \Sigma(O[k] \otimes O[\ell]^\circ)_q^+ .
\end{tikzcd}
\tag{1.18}
$$

Combining (1.17) and (1.18), by Remark 1.2 we obtain that the square of augmented directed chain complexes from (1.15) is a pushout, as desired. □

Proposition 1.19. *Let $k, \ell \geq 0$. The map*

$$O[k+1+\ell] \to \Sigma\left(O[k] \otimes O[\ell]^\circ\right)$$

from Proposition 1.13 is an epimorphism of augmented directed chain complexes.

Proof. Using the explicit computations from Proposition 1.16 we see that for every $q \geq 0$ the canonical map

$$(O[k] \oplus O[\ell]^\circ)_q^{(+)} \to (O[0] \oplus O[0]^\circ)_q^{(+)}$$

is an epimorphism of abelian groups (resp. commutative monoids). By Remark 1.3 the canonical map

$$O[k] \oplus O[\ell]^\circ \to O[0] \oplus O[0]^\circ$$

is then an epimorphism of augmented directed chain complexes. Given that epimorphisms are closed under pushout, by Proposition 1.16, the map

$$O[k+1+\ell] \to \Sigma\left(O[k] \otimes O[\ell]^\circ\right)$$

from Proposition 1.13 is then an epimorphism of augmented directed chain complexes, too. □

We can describe how to map orientals into suspensions:

Proposition 1.20. *For $m \geq 1$ and C an augmented directed chain complex, the map from Proposition 1.13 induces a natural bijection*

$$\coprod_{\substack{k+1+\ell=m, \\ k,\ell \geq -1}} ad\mathcal{C}h(O[k] \otimes O[\ell]^\circ, C) \xrightarrow{\cong} ad\mathcal{C}h(O[m], \Sigma C).$$

Proof. For any $x \colon O[m] \to \Sigma C$ we set

$$k := \#\{0 \leq i \leq m \mid x([i]) = \bot\} - 1 \text{ and } \ell := m - 1 - k,$$

and construct a corresponding preimage $\widehat{x}\colon O[k] \otimes O[\ell]^\circ \to C$.

Let $k = -1$ (resp. $k = m$). Then we take

$$\widehat{x}\colon O[-1] \otimes O[m] \cong O[-1] \to C, \text{ resp. } \widehat{x}\colon O[m] \otimes O[-1] \cong O[-1] \to C$$

to be the trivial map.

Let $0 \leq k \leq m-1$. By Proposition 1.19, the function

$$\Sigma\colon \coprod_{\substack{k+1+\ell=m\\ 0\leq k,\ell}} ad\mathcal{C}h(O[k] \otimes O[\ell]^\circ, C) \to \coprod_{\substack{k+1+\ell=m\\ k,\ell\geq 0}} {}^{\Sigma O[-1]/}ad\mathcal{C}h(\Sigma(O[k] \otimes O[\ell]^\circ), \Sigma C)$$

is bijective, so x can be uniquely identified with the suspension map

$$\Sigma x\colon \Sigma(O[k] \otimes O[\ell]^\circ) \to \Sigma C$$

under $\Sigma O[-1]$. By composing with the map

$$\coprod_{\substack{k+1+\ell=m\\ k,\ell\geq 0}} {}^{\Sigma O[-1]/}ad\mathcal{C}h(\Sigma(O[k] \otimes O[\ell]^\circ), \Sigma C) \to ad\mathcal{C}h(O[m], \Sigma C)$$

induced by Proposition 1.13, which is injective by Proposition 1.8, Σx can be uniquely identified with a map of the form

$$\widehat{x}\colon O[k] \otimes O[\ell]^\circ \to C.$$

It is a straightforward verification that the assignment $x \mapsto \widehat{x}$ defines the inverse for the desired bijection. $\square$

2. ω-categories and algebraic models

We recall the basic definitions around ω-categories, as well as some constructions based on ω-categories: the ω-categorical suspension, the tensor product, the total dual, and Steiner's linearization, as well as the main properties that we use later in the paper, and relevant examples.

2.1. ω-categories

While we refer the reader to e.g. [32] for a traditional approach to the definition of an ω-category, we briefly recall the main features here.

The data of an *ω-category* $\mathcal{C}$ consists of a collection of sets $\mathcal{C}_q$, for $q \geq 0$, where $\mathcal{C}_0$ is called the set of *objects* of $\mathcal{C}$ and $\mathcal{C}_q$ for $q > 0$ is the set of *q-cells* or cells of *dimension* q of $\mathcal{C}$, together with:

- *source* and *target* operators $s_q, t_q\colon \mathcal{C}_p \to \mathcal{C}_q$ for all $p > q \geq 0$;

- *identity* operators $\mathrm{id}_q : \mathcal{C}_p \to \mathcal{C}_q$ for all $q \geq p \geq 0$;
- *composition* operators $*_p : \mathcal{C}_q \times_{\mathcal{C}_p} \mathcal{C}_q \to \mathcal{C}_q$ defined for all $q > p \geq 0$ and all pairs of q-cells (g, f) for which $s_p(g) = t_p f$.

We say that $\mathcal{C}$ is an ω-category if for all $r > q > p \geq 0$ the triple $(\mathcal{C}_p, \mathcal{C}_q, \mathcal{C}_r)$ together with all the relevant source, target, identity and composition operators is a 2-category. In particular,

$$s_p s_q f = s_p f \qquad \text{and} \qquad t_p t_q f = t_p f \tag{2.1}$$

for any r-cell f of $\mathcal{C}$ and $r > q > p$.

An *ω-functor* $F : \mathcal{C} \to \mathcal{D}$ between ω-categories $\mathcal{C}$ and $\mathcal{D}$ is a collection of maps $F_q : \mathcal{C}_q \to \mathcal{D}_q$ for $q \geq 0$ that preserves source, target, identity, and composition operators. We denote by $\omega\mathcal{C}at$ the category of (small) ω-categories and ω-functors.

A cell in an ω-category $\mathcal{C}$ is said to be *trivial* if it is the identity of a cell of lower dimension. For $m \geq 0$, an *m-category* is an ω-category in which all q-cells are trivial for $q > n$, and an *n-functor* is an ω-functor between n-categories. We denote by $n\mathcal{C}at$ the (full) subcategory of $\omega\mathcal{C}at$ given by n-categories and n-functors.

Example 2.2. For $m \geq -1$, the *m-oriental* $\mathcal{O}[m]$ from [32], [29, Theorem 3.2] or [3, §7.2] is an m-category, and in particular an ω-category. The reader who is not familiar with the original definition can also take the formula from Example 2.10 as the definition of $\mathcal{O}[m]$.

Definition 2.3 *([3, §1.8]).* Let $\mathcal{C}$ be an ω-category. The *total dual* of $\mathcal{C}$ is the ω-category $\mathcal{C}^\circ$ with the following structure.

- The set of q-cells $\mathcal{C}^\circ_q$ is $\mathcal{C}^\circ_q := \mathcal{C}_q$;
- The source map $s_q : \mathcal{C}^\circ_p \to \mathcal{C}^\circ_q$ is given by $s^{\mathcal{C}^\circ}_q f = t^{\mathcal{C}}_q f$ for all $p > q \geq 0$;
- The target map $t_q : \mathcal{C}^\circ_p \to \mathcal{C}^\circ_q$ is given by $t^{\mathcal{C}^\circ}_q f = s^{\mathcal{C}}_q f$ for all $p > q \geq 0$;
- The composition map $*_p : \mathcal{C}^\circ_q \times_{\mathcal{C}^\circ_p} \mathcal{C}^\circ_q \to \mathcal{C}^\circ_q$ is given by $f *^{\mathcal{C}^\circ}_p g = g *^{\mathcal{C}}_p f$ for all $q > p \geq 0$;
- The identity map $\mathrm{id}_q : \mathcal{C}^\circ_p \to \mathcal{C}^\circ_q$ is given by $\mathrm{id}^{\mathcal{C}^\circ}_q f = \mathrm{id}^{\mathcal{C}}_q f$ for all $q \geq p \geq 0$.

The construction defines a functor $(-)^\circ : \omega\mathcal{C}at \to \omega\mathcal{C}at$.

2.2. Suspension of ω-categories

The following is a variant[3] of the construction treated in [3, §B.6.5]. When the input is an ordinary 1-category $\mathcal{C}$, the suspension agrees with the one that we previously considered in [24].

[3] Precisely, what we present in Definition 2.4 is the composite of the one used in [3, §B.6.5] with the total dual from [3, §1.8].

Definition 2.4. Let $\mathcal{C}$ be an ω-category. The *suspension* of $\mathcal{C}$ is the ω-category $\Sigma\mathcal{C}$ with the following structure.

- The set of q-cells $(\Sigma\mathcal{C})_q$ is
$$(\Sigma\mathcal{C})_q := \{\perp, \top\} \cup \mathcal{C}_{q-1}, \quad \text{with} \quad (\Sigma\mathcal{C})_0 := \{\perp, \top\}$$
- The source map $s_q \colon \Sigma\mathcal{C}_p \to (\Sigma\mathcal{C})_q$ for $q > 1$ is given by
$$s_q^{\Sigma\mathcal{C}} f = s_{q-1}^{\mathcal{C}} f, \quad s_q^{\Sigma\mathcal{C}} \perp = \perp, \quad s_q^{\Sigma\mathcal{C}} \top = \top, \quad \text{with} \quad s_0^{\Sigma\mathcal{C}} f = \perp.$$
- The target map $t_q \colon \Sigma\mathcal{C}_p \to (\Sigma\mathcal{C})_q$ for $q > 1$ is given by
$$t_q^{\Sigma\mathcal{C}} f = t_{q-1}^{\mathcal{C}} f, \quad t_q^{\Sigma\mathcal{C}} \perp = \perp, \quad t_q^{\Sigma\mathcal{C}} \top = \top, \quad \text{with} \quad t_0^{\Sigma\mathcal{C}} f = \top.$$
- The identity map $\mathrm{id}_q \colon \Sigma\mathcal{C}_p \to (\Sigma\mathcal{C})_q$ is given by
$$\mathrm{id}_q^{\Sigma\mathcal{C}} f = \mathrm{id}_q^{\mathcal{C}} f, \quad \mathrm{id}_q^{\Sigma\mathcal{C}} \perp = \perp, \quad \mathrm{id}_q^{\Sigma\mathcal{C}} \top = \top.$$
- The composition map $*_p \colon \Sigma\mathcal{C}_q \times_{(\Sigma\mathcal{C})_p} (\Sigma\mathcal{C})_q \to (\Sigma\mathcal{C})_q$ is given by
$$g *_p^{\Sigma\mathcal{C}} f = g *_{p-1}^{\mathcal{C}} f$$

Regarding $\Sigma\mathcal{C}$ as an ω-category bipointed on $\perp$ and $\top$, the construction defines a functor $\Sigma \colon \omega\mathcal{C}at \to \omega\mathcal{C}at_{*,*}$.

2.3. Steiner's functors

We briefly recall Steiner's adjoint pair that relates ω-categories and augmented directed chain complexes. For a more detailed treatment, see [28, Definition 2.8] or [3, §2.4].

Definition 2.5. Let C be an augmented directed chain complex. A *Steiner table* in C is a matrix
$$x = \begin{pmatrix} x_0^- & \dots & x_{q-1}^- & x_q^- \\ x_0^+ & \dots & x_{q-1}^+ & x_q^+ \end{pmatrix}$$
such that, for $\alpha = +, -$ and $0 \leq p \leq q$, the following hold:

(1) x_p^α belongs to C_p^+;
(2) $\partial(x_p^\alpha) = x_{p-1}^+ - x_{p-1}^-$ for $0 < p \leq q$;
(3) $\varepsilon(x_0^\alpha) = 1$;

(4) $x_q^- = x_q^+$.

Definition 2.6 *([28, Definition 2.8], [3, §2]).* Let C be an augmented directed chain complex. The *ω-categorical realization* of C is the ω-category νC is defined as follows.

- The set $(\nu C)_q$ of q-cells is given by

$$(\nu C)_q \coloneqq \left\{ x = \begin{pmatrix} x_0^- & \dots & x_{q-1}^- & x_q^- \\ x_0^+ & \dots & x_{q-1}^+ & x_q^+ \end{pmatrix} \;\middle|\; x \text{ Steiner table in } C \right\}.$$

- The source map $s_q\colon (\nu C)_p \to (\nu C)_q$ is given by

$$s_q \begin{pmatrix} x_0^- & \dots & x_{p-1}^- & x_p^- \\ x_0^+ & \dots & x_{p-1}^+ & x_p^+ \end{pmatrix} \coloneqq \begin{pmatrix} x_0^- & \dots & x_{q-1}^- & x_q^- \\ x_0^+ & \dots & x_{q-1}^+ & x_q^- \end{pmatrix}$$

- The target map $t_q\colon (\nu C)_p \to (\nu C)_q$ is given by

$$t_q \begin{pmatrix} x_0^- & \dots & x_{p-1}^- & x_p^- \\ x_0^+ & \dots & x_{p-1}^+ & x_p^+ \end{pmatrix} \coloneqq \begin{pmatrix} x_0^- & \dots & x_{q-1}^- & x_q^+ \\ x_0^+ & \dots & x_{q-1}^+ & x_q^+ \end{pmatrix}$$

- The composition map $*_p\colon (\nu C)_q \times_{(\nu C)_p} (\nu C)_q \to (\nu C)_q$ is given by

$$\begin{pmatrix} x_0^- & \dots & x_{q-1}^- & x_q^- \\ x_0^+ & \dots & x_{q-1}^+ & x_q^+ \end{pmatrix} *_p \begin{pmatrix} y_0^- & \dots & y_{q-1}^- & y_q^- \\ y_0^+ & \dots & y_{q-1}^+ & y_q^+ \end{pmatrix} \coloneqq$$
$$\begin{pmatrix} x_0^- & \dots & x_{p-1}^- & y_p^- & x_{p+1}^- + y_{p+1}^- & \dots & x_q^- + y_q^- \\ x_0^+ & \dots & x_{p-1}^+ & x_p^+ & x_{p+1}^+ + y_{p+1}^+ & \dots & x_q^+ + y_q^+ \end{pmatrix}$$

- The identity map $\mathrm{id}_q\colon (\nu C)_p \to (\nu C)_q$ is given by

$$\mathrm{id}_q \begin{pmatrix} x_0^- & \dots & x_{p-1}^- & x_p^- \\ x_0^+ & \dots & x_{p-1}^+ & x_p^+ \end{pmatrix} \coloneqq \underbrace{\begin{pmatrix} x_0^- & \dots & x_{p-1}^- & x_p^- & 0 & 0 & 0 \dots & 0 \\ x_0^+ & \dots & x_{p-1}^+ & x_p^+ & 0 & 0 & 0 \dots & 0 \end{pmatrix}}_{q+1}$$

The construction extends to a functor $\nu\colon ad\mathcal{C}h \to \omega\mathcal{C}at$.

Definition 2.7. Let $\mathcal{C}$ be an ω-category. The *linearization* of $\mathcal{C}$ is the augmented directed chain complex $\lambda\mathcal{C}$ defined as follows.

- The abelian group $(\lambda\mathcal{C})_q$ of q-chains of $\lambda\mathcal{C}$ is the quotient of $\mathbb{Z}[\mathcal{C}_q]$ given by

$$(\lambda\mathcal{C})_q \coloneqq \frac{\mathbb{Z}[\mathcal{C}_q]}{\langle [x *_p y]_q - [x]_q - [y]_q \mid x, y \in \mathcal{C}_q; p < q\rangle}. \tag{2.8}$$

- The positivity submonoid $(\lambda\mathcal{C})_q^+$ is the submonoid of $(\lambda\mathcal{C})_q$ generated by the collection of elements $[f]_q$ for f a q-cell of $\mathcal{C}$.
- The differential map $\partial_q\colon (\lambda\mathcal{C})_{q+1} \to (\lambda\mathcal{C})_q$ is determined by the condition on generators $f \in \mathcal{C}_q$ given by

$$\partial_q([f]_{q+1}) \coloneqq [t_q f]_q - [s_q f]_q,$$

- The augmentation map $\varepsilon\colon (\lambda\mathcal{C})_0 \to \mathbb{Z}$ is determined by the condition on generators $x \in \mathcal{C}_0$ given by

$$\varepsilon([x]_0) \coloneqq 1.$$

The construction extends to a functor $\lambda\colon \omega\mathcal{C}at \to ad\mathcal{C}h$.

Theorem 2.9 *([28, §2]). The functors ν and λ form an adjoint pair*

$$\lambda\colon \omega\mathcal{C}at \rightleftarrows ad\mathcal{C}h\colon \nu.$$

In other words, for any ω-category $\mathcal{C}$ and any augmented directed chain complex $\overline{C}$ there is a natural bijection

$$ad\mathcal{C}h(\lambda\mathcal{C}, \overline{C}) \cong \omega\mathcal{C}at(\mathcal{C}, \nu\overline{C}).$$

Example 2.10 *([29, Theorem 3.2]).* For $m \geq 0$, there is an isomorphism of augmented directed chain complexes

$$\lambda\mathcal{O}[m] \cong O[m]$$

and an isomorphism of ω-categories

$$\mathcal{O}[m] \cong \nu O[m].$$

Lemma 2.11 *([3, Proposition 2.19]). Let C be an augmented directed chain complex and $\mathcal{C}$ an ω-category.*

(1) There is a natural isomorphism of ω-categories

$$\nu(C^\circ) \cong (\nu C)^\circ.$$

(2) There is a natural isomorphism of augmented directed chain complexes

$$\lambda(\mathcal{C}^\circ) \cong (\lambda\mathcal{C})^\circ.$$

2.4. Steiner's functors and suspension

Lemma 2.12. *For an augmented chain complex C, there is a natural isomorphism*

$$\nu\Sigma C \cong \Sigma\nu C.$$

Proof. The prototypical element of both $(\nu\Sigma C)_q$ and $(\Sigma\nu C)_q$ can be expressed as a table of the form

$$x = \begin{pmatrix} \perp & x_0^- & \dots & x_{q-2}^- & x_{q-1}^- \\ \top & x_0^+ & \dots & x_{q-2}^+ & x_{q-1}^+ \end{pmatrix},$$

where

(1) x_p^α belongs to C_p^+;
(2) $\partial(x_p^\alpha) = x_{p-1}^+ - x_{p-1}^-$ for $0 < p < q$;
(3) $\varepsilon(x_0^\alpha) = 1$;
(4) $x_{q-1}^- = x_{q-1}^+$.

One can check that this identification is compatible with source, target, identity and composition operations, and the desired isomorphism of ω-categories follows. □

2.5. Tensor product of ω-categories

The statement of the following theorem relies on the notion of a *strong Steiner ω-category*, a.k.a. *ω-category that admits a strongly loop-free atomic basis*. Those are particularly nice ω-categories that are in a sense "free" and "loop-free" and we refer the reader to [2,3,28] for an account of strong Steiner ω-categories. There is also a notion of a *strong Steiner complex*, a.k.a. *augmented directed chain complex that admits a strongly loop-free and unital basis*, which correspond in a precise sense to strong Steiner ω-categories under the adjunction (λ, ν). For the purpose of this paper, it is sufficient to know the following.

- For every $m \geq 0$ the m-oriental $\mathcal{O}[m]$ is a strong Steiner ω-category (as shown in [28, Example 3.8]), and so it its total dual $\mathcal{O}[m]^\circ$ (which can be verified directly).
- For every $m \geq 0$ the m-cell is a strong Steiner ω-category (as shown in [28, Example 3.9]).
- For any strong Steiner ω-category $\mathcal{C}$, the unit of the adjunction from Theorem 2.9 is an isomorphism of ω-categories $\eta_{\mathcal{C}}\colon \mathcal{C} \cong \nu\lambda\mathcal{C}$ (as shown in [28, Theorem 5.11]).
- For any strong Steiner complex C, the counit of the adjunction from Theorem 2.9 is an isomorphism of augmented directed chain complexes $\epsilon_C\colon \lambda\nu C \cong C$ (as shown in [28, Theorem 5.11]).

- For any strong Steiner ω-category $\mathcal{C}$, the augmented directed chain complex $\lambda\mathcal{C}$ is a strong Steiner complex (as shown in [28, Theorem 5.11]).
- For any strong Steiner complex C, the ω-category νC is a strong Steiner ω-category (as shown in [28, Theorem 5.11]).
- For any strong Steiner complexes C and $\overline{C}$ the augmented directed chain complex $C \otimes \overline{C}$ is a strong Steiner complex (as shown in [3, Proposition A.3]).
- For any strong Steiner complexes C and $\overline{C}$ there is a natural isomorphism of augmented directed chain complexes $\nu C \otimes \nu\overline{C} \cong \nu(C \otimes \overline{C})$ (as shown in [3, Theorem A.15]).

Theorem 2.13 *([3, Theorem A.15]). There exists a unique – up to unique monoidal isomorphism – monoidal structure $\otimes\colon \omega\mathcal{C}at\times\omega\mathcal{C}at \to \omega\mathcal{C}at$ on $\omega\mathcal{C}at$, called the* tensor product *of ω-categories, such that*

- *for any strong Steiner ω-categories $\mathcal{C}$ and $\overline{\mathcal{C}}$ the tensor product $\mathcal{C}\otimes\overline{\mathcal{C}}$ is the ω-category*
$$\mathcal{C}\otimes\overline{\mathcal{C}} \coloneqq \nu(\lambda\mathcal{C}\otimes\lambda\overline{\mathcal{C}});$$
- *the functor $-\otimes-$ commutes with colimits in each variable.*

Proposition 2.14. *The linearization functor defines a strong monoidal functor $\lambda\colon (\omega\mathcal{C}at,\otimes) \to (ad\mathcal{C}h,\otimes)$. That is, for any ω-categories $\mathcal{C}$ and $\overline{\mathcal{C}}$ there is a natural isomorphism of augmented directed chain complexes*
$$\lambda(\mathcal{C}\otimes\overline{\mathcal{C}}) \cong \lambda\mathcal{C}\otimes\lambda\overline{\mathcal{C}}.$$

Proof. First, we observe that for any strong Steiner ω-categories $\mathcal{C}$ and $\overline{\mathcal{C}}$ we have
$$\begin{aligned}\lambda(\mathcal{C}\otimes\overline{\mathcal{C}}) &\cong \lambda(\nu\lambda\mathcal{C}\otimes\nu\lambda\overline{\mathcal{C}})\\ &\cong \lambda\nu(\lambda\mathcal{C}\otimes\lambda\overline{\mathcal{C}})\\ &\cong \lambda\mathcal{C}\otimes\lambda\overline{\mathcal{C}},\end{aligned}$$
so the desired isomorphism holds for strong Steiner ω-categories. Since any ω-category is a colimit of strong Steiner ω-categories (as cells are in particular strong Steiner ω-categories), and the functors $\lambda(-\otimes-), (\lambda-)\otimes(\lambda-)\colon \omega\mathcal{C}at\times\omega\mathcal{C}at \to \omega\mathcal{C}at$ commute with colimits in both variables, the desired isomorphism follows. $\square$

We can describe the tensor product of orientals:

Example 2.15. Let $k,\ell \geq -1$. There are isomorphisms of ω-categories
$$\begin{array}{rcll}\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ & \cong & \nu O[k]\otimes(\nu O[\ell])^\circ & \text{Example 2.10}\\ & \cong & \nu O[k]\otimes\nu(O[\ell]^\circ) & \text{Lemma 2.11(1)}\\ & \cong & \nu(O[k]\otimes O[\ell]^\circ) & \text{Theorem 2.13}\end{array}$$

and of augmented directed chain complexes

$$\begin{array}{rcll} \lambda(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ) & \cong & \lambda\mathcal{O}[k]\otimes\lambda(\mathcal{O}[\ell]^\circ) & \text{Proposition 2.14} \\ & \cong & \lambda\mathcal{O}[k]\otimes(\lambda\mathcal{O}[\ell])^\circ & \text{Lemma 2.11(2)} \\ & \cong & O[k]\otimes O[\ell]^\circ & \text{Example 2.10} \end{array}$$

We can describe the suspension of tensor product of orientals:

Remark 2.16. Let $k,\ell\geq 0$. Applying ν to the square (1.15) – and recalling Examples 2.10 and 2.15 and Lemma 2.12) – we obtain the diagram of ω-categories

$$\begin{tikzcd} \mathcal{O}[k]\oplus\mathcal{O}[\ell]^\circ \arrow[r] \arrow[d] & \mathcal{O}[k+1+\ell] \arrow[d] \\ \mathcal{O}[0]\oplus\mathcal{O}[0] \arrow[r] & \Sigma(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ). \end{tikzcd} \tag{2.17}$$

In particular, the map

$$\mathcal{O}[k+1+\ell]\to\Sigma(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ), \tag{2.18}$$

is induced by (1.13).

Proposition 2.19. *Let $k,\ell\geq -1$. The diagram (2.17) induces a natural isomorphism of ω-categories*

$$\Sigma(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ)\cong(\mathcal{O}[0]\oplus\mathcal{O}[0]^\circ)\underset{\mathcal{O}[k]\oplus\mathcal{O}[\ell]^\circ}{\amalg}\mathcal{O}[k+1+\ell].$$

Proof. Consider the commutative diagram of augmented directed chain complexes on the left, and the induced commutative diagram of ω-categories on the right:

$$\begin{tikzcd} O[k]\oplus O[\ell]^\circ \arrow[r] \arrow[d] & O[k+1+\ell] \arrow[d] \\ O[0]\oplus O[0]^\circ \arrow[r] & \Sigma(O[k]\otimes O[\ell]^\circ) \end{tikzcd} \quad\rightsquigarrow\quad \begin{tikzcd} \mathcal{O}[k]\oplus\mathcal{O}[\ell]^\circ \arrow[r] \arrow[d] & \mathcal{O}[k+1+\ell] \arrow[d] \\ \mathcal{O}[0]\oplus\mathcal{O}[0]^\circ \arrow[r] & \Sigma(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ). \end{tikzcd}$$

The square on the left is a pushout by (1.15) and, as an application of [18, Théorème 3.1.5], so is the pushout on the right. □

We can describe how to map orientals into suspension ω-categories:

Proposition 2.20. *For $m\geq 1$ and $\mathcal{C}$ an ω-category of the form $\mathcal{C}\cong\nu C$, there is a natural bijection*

$$\coprod_{\substack{k+1+\ell=m\\ k,\ell\geq -1}}\omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ,\mathcal{C})\xrightarrow{\cong}\omega\mathcal{C}at(\mathcal{O}[m],\Sigma\mathcal{C}).$$

Proof. There's a natural bijection

$$\begin{array}{rcll}
\coprod_{k,\ell} \omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ},\mathcal{C}) & \cong & \coprod_{k,\ell} \omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ},\nu C) & \\
 & \cong & \coprod_{k,\ell} ad\mathcal{C}h(\lambda(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ}),C) & \text{Theorem 2.9} \\
 & \cong & \coprod_{k,\ell} ad\mathcal{C}h(O[k]\otimes O[\ell]^{\circ},C) & \text{Example 2.10}
\end{array}$$

and a natural bijection

$$\begin{array}{rcll}
\omega\mathcal{C}at(\mathcal{O}[m],\Sigma\mathcal{C}) & \cong & \omega\mathcal{C}at(\mathcal{O}[m],\Sigma\nu C) & \\
 & \cong & \omega\mathcal{C}at(\mathcal{O}[m],\nu\Sigma C) & \text{Lemma 2.12} \\
 & \cong & ad\mathcal{C}h(\lambda\mathcal{O}[m],\Sigma C) & \text{Theorem 2.9} \\
 & \cong & ad\mathcal{C}h(O[m],\Sigma C) & \text{Example 2.10}
\end{array}$$

They fit into a commutative diagram of sets

$$\begin{tikzcd}
\omega\mathcal{C}at(\mathcal{O}[m],\Sigma\mathcal{C}) & \coprod_{k,\ell}\omega\mathcal{C}at(\Sigma(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ}),\Sigma\mathcal{C}) \arrow[l] & \coprod_{k,\ell}\omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ},\mathcal{C}) \arrow[l, "\Sigma"'] \arrow[ll, bend right] \\
ad\mathcal{C}h(O[m],\Sigma C) \arrow[u, "\cong"'] & \coprod_{k,\ell} ad\mathcal{C}h(\Sigma(O[k]\otimes O[\ell]^{\circ}),\Sigma C) \arrow[l] \arrow[u] & \coprod_{k,\ell} ad\mathcal{C}h(O[k]\otimes O[\ell]^{\circ},C) \arrow[l, "\Sigma"'] \arrow[u, "\cong"] \arrow[ll, bend left]
\end{tikzcd}$$

The lower curved arrow in the final diagram is bijection by Proposition 1.20, so the upper curved arrow is bijective, as desired. □

3. Complicial sets and complicial nerve of ω-categories

We recall the basic definitions around simplicial sets with marking and n-complicial sets, as well as some constructions based on simplicial sets with marking: the suspension and the complicial nerve, as well as the main properties that we use later in the paper, and relevant examples. The study of the homotopy theory of complicial sets originated with [34,35], and continued with [23,24,26,27].

3.1. Complicial sets

We recall the main facts about complicial sets that will be used in this paper.

Definition 3.1. A *simplicial set with marking* is a pair (X,tX) where X is a simplicial set, and $tX=\coprod_{m\geq 1} tX_m \subseteq \coprod_{m\geq 1} X_m$ is a collection of subsets $tX_m\subseteq X_m$ of simplices of X for $m\geq 1$, called *marked* simplices, which contain all degenerate simplices of X.

We denote by *msSet* the category of simplicial sets with marking and marking-preserving simplicial maps.

Remark 3.2. The category *msSet* is cocomplete, and colimits are computed degreewise (a simplex is marked in a colimit if it admits a marked representative).

Definition 3.3. A sub-simplicial set with marking X of a simplicial set with marking Y is *regular* if a simplex of X is marked in X precisely when it is marked in Y.

Notation 3.4. We denote

- by $\Delta^k[m]$, for $0 \leq k \leq m$, the standard m-simplex in which a non-degenerate simplex is marked if and only if it contains the vertices $\{k-1, k, k+1\} \cap [m]$;
- by $\Delta^k[m]'$, for $0 \leq k \leq m$, the standard m-simplex with marking obtained from $\Delta^k[m]$ by additionally marking the $(k-1)$-st and $(k+1)$-st $(m-1)$-dimensional face of $\Delta[m]$, whenever defined;
- by $\Delta^k[m]''$, for $0 \leq k \leq m$, the standard m-simplex with marking obtained from $\Delta^k[m]'$ by additionally marking the k-th face of $\Delta[m]$;
- by $\Lambda^k[m]$, for $0 \leq k \leq m$, the regular sub-simplicial set of $\Delta^k[m]$ with marking whose simplicial set is the k-horn $\Lambda^k[m]$;
- by $\Delta[m]^\sharp$ the standard m-simplex with the maximal marking;
- by $\Delta[m]_t$ the standard m-simplex with the in which the top m-dimensional simplex is marked, as well as all degenerate simplices;
- by $\Delta[3]_{eq}$ the standard 3-simplex in which the 1-simplices $[0, 2]$ and $[1, 3]$ are marked, as well as all degenerate 1-simplices and all simplices in dimension 2 or higher.

The following class of maps plays a role in the model structure on *msSet* for (∞, n)-categories, with $n \in \mathbb{N} \cup \{\infty\}$.

Definition 3.5. Let $n \in \mathbb{N} \cup \{\infty\}$.

(1) For $m > 1$ and $0 < k < m$, the *complicial inner horn extension* is the inclusion

$$\Lambda^k[m] \to \Delta^k[m].$$

(2) For $m \geq 2$ and $0 < k < m$, the *complicial thinness extension* is the inclusion

$$\Delta^k[m]' \to \Delta^k[m]''.$$

(3) For $m > n$, the *triviality extension* is the inclusion

$$\Delta[m] \to \Delta[m]_t.$$

(4) For $m \geq -1$, the *complicial saturation extension* is the inclusion[4]

$$\Delta[3]_{\mathrm{eq}} \star \Delta[m] \to \Delta[3]^{\sharp} \star \Delta[m].$$

We fix the following terminology (cf. [34, Def. 15]).

Definition 3.6. A map of simplicial sets with marking $X \to Y$ is a *complicial inner anodyne extension* if it can be written as a retract of a transfinite composition of pushouts of maps of type (1) and (2) from Definition 3.5.

Remark 3.7. One can prove with standard model categorical techniques the following formal properties of complicial inner anodyne extensions.

(1) The underlying simplicial map of a complicial inner anodyne extension is an inner anodyne extension of simplicial sets.
(2) The class of complicial inner anodyne extensions is closed under transfinite composition and pushouts.

Lemma 3.8 *([24, Lemma 1.12]). For $m \geq 2$ and $0 < k < m$, let $\Lambda^k[m]'$ denote the regular subset of $\Delta^k[m]'$ whose underlying simplicial set is given by the k-horn $\Lambda^k[m]$. The inclusion*

$$\Lambda^k[m]' \to \Delta^k[m]''$$

is a complicial inner anodyne extension.

The category *msSet* hosts a model for (∞, n)-categories.

Definition 3.9. A *saturated n-complicial set* is a simplicial set that has the right lifting property with respect to all maps of type (1)-(4) from Definition 3.5.

Theorem 3.10 *([23, Theorem 1.25]). Let $n \in \mathbb{N} \cup \{\infty\}$. The category msSet supports a cartesian closed model structure $msSet_{(\infty,n)}$, that we call the* model structure for (∞, n)-categories*, where*

- *the fibrant objects are precisely the* saturated n-complicial sets*,*
- *the cofibrations are precisely the monomorphisms (of underlying simplicial sets),*
- *all complicial inner anodyne extensions are weak equivalences.*

[4] The reader can find the join of simplicial sets with marking $\star\colon msSet \times msSet \to msSet$ in [34, §3.1], but it will not be needed explicitly in this paper.

Remark 3.11. Other model structures are sometimes considered on $m s\mathcal{S}et$, for instance those from [34, Theorem 100] and [26, Examples 3.33–3.36]. But in all the aforementioned model structures complicial inner anodyne extensions are weak equivalences.

3.2. Suspension of complicial sets

We now define the suspension of simplicial sets with marking.

Definition 3.12 *([24, Definition 2.6]).* Let X be a simplicial set with marking. The *suspension* ΣX of X is the simplicial set with marking defined as follows.

- The set $(\Sigma X)_m$ of m-simplices is given by

$$(\Sigma X)_m = \begin{cases} \{\bot, \top\} & \text{if } m = 0, \\ \{\bot, \top\} \cup \coprod_{k=0}^{m-1} X_k & \text{if } m > 0. \end{cases}$$

- The face map $d_i \colon (\Sigma X)_m \to (\Sigma X)_{m-1}$ satisfies

$$d_i \bot = \bot \text{ and } d_i \top = \top,$$

  and restricts to the map $X_k \subseteq (\Sigma X)_m \to (\Sigma X)_{m-1}$ given by

$$d_i(x) = \begin{cases} d_i x \in X_{k-1} \subseteq (\Sigma X)_{m-1} & \text{if } 0 \leq i \leq k, \\ x \in X_k \subseteq (\Sigma X)_m & \text{if } k+1 \leq i \leq m, \end{cases}$$

- The degeneracy map $s_i \colon (\Sigma X)_m \to (\Sigma X)_{m+1}$ satisfies

$$s_i \bot = \bot \text{ and } s_i \top = \top,$$

  and restricts to the map $X_k \subseteq (\Sigma X)_m \to (\Sigma X)_{m+1}$ given by

$$s_i(x) = \begin{cases} s_i x \in X_{k+1} \subseteq (\Sigma X)_{m+1} & \text{if } 0 \leq i \leq k, \\ x \in X_k \subseteq (\Sigma X)_m & \text{if } k+1 \leq i \leq m. \end{cases}$$

- The set $t(\Sigma X)_m$ of marked k-simplices is given by

$$t(\Sigma X)_m = \coprod_{k=0}^{m} tX_k.$$

Regarding ΣX as a simplicial set with marking bipointed on $\bot$ and $\top$, the construction defines a functor $\Sigma \colon m s\mathcal{S}et \to m s\mathcal{S}et_{*,*}$.

Remark 3.13. The set $(\Sigma X)^{\mathrm{nd}}_m$ of non-degenerate m-simplices of ΣX for $m > 0$ is contained in the set of the non-degenerate $(m-1)$-simplices of X, namely

$$(\Sigma X)^{\mathrm{nd}}_m \subseteq X_{m-1}.$$

As a special case of the slice model structures, constructed e.g. in [15], we also obtain a model structure $(ms\mathcal{S}et_{(\infty,n+1)})_{*,*}$ on the category $ms\mathcal{S}et_{*,*}$ of bipointed simplicial sets with marking.

Lemma 3.14 *([24, Lemma 2.7]). The marked suspension defines a left Quillen functor*

$$\Sigma\colon ms\mathcal{S}et_{(\infty,n)} \to (ms\mathcal{S}et_{(\infty,n+1)})_{*,*}.$$

3.3. Complicial nerve of ω-categories

The geometry of orientals is such that the construction $m \mapsto \mathcal{O}[m]$ defines a cosimplicial object $\mathcal{O}[\bullet]$ in $\omega\mathcal{C}at$, and in particular it makes sense to define the nerve construction $N\colon \omega\mathcal{C}at \to s\mathcal{S}et$ originally due to Street [32]. The Street nerve can be endowed with the following marking, originally considered by Roberts in unpublished work and Street in [32], further studied by Verity in [33], and later discussed by Riehl in [26], obtaining a functor $N^{\mathrm{RS}}\colon \omega\mathcal{C}at \to ms\mathcal{S}et$.

Definition 3.15. Let $\mathcal{C}$ be an ω-category. The *Roberts–Street nerve* of $\mathcal{C}$ is the simplicial set with marking defined as follows:

- The set of m-simplices is the set of ω-functors $\mathcal{O}[m] \to \mathcal{C}$, namely

$$N_m\mathcal{C} = \omega\mathcal{C}at(\mathcal{O}[m], \mathcal{C}),$$

- the simplicial structure is induced by the geometry of orientals.
- an m-simplex of $N\mathcal{C}$ is marked in $N^{\mathrm{RS}}\mathcal{C}$ if and only if the corresponding ω-functor $\mathcal{O}[m] \to \mathcal{C}$ sends the unique non-trivial m-cell $\langle[0,1,\dots,m]\rangle$ of $\mathcal{O}[m]$ to a trivial m-cell of $\mathcal{C}$, namely

$$x \in t(N\mathcal{C})_m \quad \Leftrightarrow \quad x(\langle[0,1,\dots,m]\rangle) = \mathrm{id},$$

  where $\langle[0,1,\dots,m]\rangle$ denotes the top non-identity m-cell of $\mathcal{O}[m]$.

The construction extends to a functor $N^{\mathrm{RS}}\colon \omega\mathcal{C}at \to ms\mathcal{S}et$.

In particular, in the Street nerve of an n-category $\mathcal{C}$ all simplices in dimension at least $n+1$ are marked.

3.4. Complicial nerve and suspension

We describe a comparison map between $\Sigma N^{\mathrm{RS}}\mathcal{C}$ and $N^{\mathrm{RS}}\Sigma\mathcal{C}$, which we will show to be furthermore a weak equivalence for an ω-category of the form $\mathcal{C} \cong \nu C$.

The simplicial sets have the same sets of 0-simplices, namely

$$(\Sigma N^{\mathrm{RS}}\mathcal{C})_0 = \{\bot, \top\} = (N^{\mathrm{RS}}\Sigma\mathcal{C})_0,$$

and we now analyze the set of m-simplices for $m > 0$.

Remark 3.16. We have the following description for the set $(N\Sigma\mathcal{C})_m$ of m-simplices of $N\Sigma\mathcal{C}$:

$$(N\Sigma\mathcal{C})_m = \omega\mathcal{C}at(\mathcal{O}[m], \Sigma\mathcal{C}).$$

Moreover,

$$x \in t(N\Sigma\mathcal{C})_m \qquad \Leftrightarrow \qquad x(\langle[0, 1, \ldots, m]\rangle) = \mathrm{id}\,.$$

If furthermore $\mathcal{C} \cong \nu C$, by Theorem 2.9 and Propositions 1.13 and 2.20 we also obtain the "algebraic" descriptions of $(N\Sigma\mathcal{C})_m$:

$$\begin{tikzcd}
(N\Sigma\mathcal{C})_m = \omega\mathcal{C}at(\mathcal{O}[m], \Sigma\mathcal{C}) \arrow[r, "\cong"] & \coprod\limits_{\substack{k+1+\ell=m,\\ k,\ell\geq -1}} \omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ}, \mathcal{C}) \\
ad\mathcal{C}h(O[m], \Sigma C) \arrow[u, "\cong"'] \arrow[r, "\cong"] & \coprod\limits_{\substack{k+1+\ell=m,\\ k,\ell\geq -1}} ad\mathcal{C}h(O[k]\otimes O[\ell]^{\circ}, C) \arrow[u, "\cong"]
\end{tikzcd}$$

Moreover,

$$x \in t(N\Sigma\mathcal{C})_m \qquad \Leftrightarrow \qquad x([0, \ldots, k]\otimes[0, \ldots \ell]) = 0.$$

Recall the bijection from Proposition 2.20 which we use in the following definition.

Definition 3.17. Let $\mathcal{C}$ be an ω-category and $x \in (N\Sigma\mathcal{C})_m$. We say that the m-simplex x is of *type* k if

$$x \in \omega\mathcal{C}at(\mathcal{O}[k]\otimes\mathcal{O}[\ell]^{\circ}, \mathcal{C}) \subseteq (N\Sigma\mathcal{C})_m.$$

Remark 3.18. Let $\mathcal{C}$ be an ω-category and $x \in (N\Sigma\mathcal{C})_m$. The following are equivalent.

(1) The simplex x is the degeneracy of a 0-simplex, namely $x = s_0^m\bot$ or $x = s_0^m\bot$.

(2) The simplex x has type -1 or m.

If the equivalent conditions are met, we say that x is *totally degenerate*.

Remark 3.19. Combining results from previous sections, we have the following equivalent descriptions for the set of m-simplices of $\Sigma N\mathcal{C}$:

$$(\Sigma N\mathcal{C})_m \cong \{s_0^m\bot, s_0^m\top\} \amalg \coprod_{k=0}^{m-1} (N\mathcal{C})_k \cong \{s_0^m\bot, s_0^m\top\} \amalg \coprod_{k=0}^{m-1} \omega\mathcal{C}at(\mathcal{O}[k], \mathcal{C}).$$

Moreover, for a non-totally degenerate simplex x, we have

$$x \in t(\Sigma N_m\mathcal{C}) \qquad \Leftrightarrow \qquad k < m-1 \text{ or } x(\langle[0,1,\ldots,k]\rangle) = \mathrm{id}\,.$$

If furthermore $\mathcal{C} \cong \nu C$, by Theorem 2.9 also get the "algebraic" descriptions:

$$\begin{aligned}(\Sigma N\mathcal{C})_m \cong \{s_0^m\bot, s_0^m\top\} \amalg \coprod_{k=0}^{m-1} (N\mathcal{C})_k \quad &\cong \{s_0^m\bot, s_0^m\top\} \amalg \coprod_{k=0}^{m-1} \omega\mathcal{C}at(\mathcal{O}[k], \mathcal{C}) \\ &\cong \{s_0^m\bot, s_0^m\top\} \amalg \coprod_{k=0}^{m-1} ad\mathcal{C}h(O[k], C).\end{aligned}$$

Moreover,

$$x \in t(\Sigma N\mathcal{C})_m \qquad \Leftrightarrow \qquad k < m-1 \text{ or } x([0,\ldots,k]) = 0.$$

The canonical map(s) from either (1.14) or (2.17) then induce a canonical natural map $(\Sigma N\mathcal{C})_m \to (N\Sigma\mathcal{C})_m$ which assembles into a map $\Sigma N\mathcal{C} \to N\Sigma\mathcal{C}$.

Proposition 3.20. *For any ω-category of the form $\mathcal{C} \cong \nu C$, either of the maps Proposition 1.13 or Proposition 2.20 induces:*

(1) a natural inclusion of simplicial sets

$$\Sigma N\mathcal{C} \to N\Sigma\mathcal{C};$$

(2) a natural regular inclusion of simplicial sets with marking

$$\Sigma N^{\mathrm{RS}}\mathcal{C} \to N^{\mathrm{RS}}\Sigma\mathcal{C}.$$

Proof. At the level of simplicial sets, both inclusions act as follows:

- they are identities on 0-simplices, namely they send the 0-simplex $\bot$ to $\bot$ and the 0-simplex $\top$ to $\top$, and

- referring to the identifications Remarks 3.16 and 3.19, they act on an $(m+1)$-simplex $y\colon \mathcal{O}[k] \to \mathcal{C}$ of $\Sigma(N\mathcal{C})$ as

$$[y\colon \mathcal{O}[k] \to \mathcal{C}] \mapsto [\mathcal{O}[m] \to \Sigma(\mathcal{O}[k] \otimes \mathcal{O}[\ell]^\circ) \to \Sigma(\mathcal{O}[k] \otimes \mathcal{O}[0]) \cong \Sigma(\mathcal{O}[k]) \xrightarrow{\Sigma y} \Sigma\mathcal{C}].$$

From this explicit description, we see that both maps are inclusions, and that the second one is regular, namely, a non-degenerate $(m+1)$-simplex of $\Sigma N\mathcal{C}$ is marked in $\Sigma N^{\mathrm{RS}}\mathcal{C}$ if and only if the corresponding m-simplex of $N\mathcal{C}$ is marked in $N^{\mathrm{RS}}\mathcal{C}$. $\square$

Remark 3.21. Given an ω-category of the form $\mathcal{C} \cong \nu C$, the map from Proposition 3.20 can be seen as induced by the canonical map

$$N^{\mathrm{RS}}\mathcal{C} \star \Delta[0] \to N^{\mathrm{RS}}(\mathcal{C} \star \Delta[0])$$

from [12, Theorem 5.2].

$$\begin{tikzcd}
N^{\mathrm{RS}}\mathcal{C} \star \Delta[0] \arrow[r] \arrow[d] & N^{\mathrm{RS}}(\mathcal{C} \star \Delta[0]) \arrow[d] \\
\Sigma N^{\mathrm{RS}}\mathcal{C} \arrow[r, dashed] & N^{\mathrm{RS}}\Sigma\mathcal{C}
\end{tikzcd}$$

We now prove that the comparison map is a weak equivalence if $\mathcal{C} \cong \nu C$.

Theorem 3.22. *Let $\mathcal{C}$ be an ω-category of the form $\mathcal{C} \cong \nu C$.*

(1) The inclusion from Proposition 3.20(1) is an inner anodyne extension, and in particular a weak equivalence in the Joyal model structure $sSet_{(\infty,1)}$

$$\Sigma N\mathcal{C} \xhookrightarrow{\simeq} N\Sigma\mathcal{C}.$$

(2) The inclusion from Proposition 3.20(2) is a complicial inner anodyne extension, and in particular a weak equivalence in the model structure[5] $msSet_{(\infty,n)}$.

$$\Sigma N^{\mathrm{RS}}\mathcal{C} \xhookrightarrow{\simeq} N^{\mathrm{RS}}\Sigma\mathcal{C}.$$

The proof is given in the coming subsections.

3.5. Proof of Theorem 3.22

Given an ω-category $\mathcal{C} \cong \nu C$, by Remark 3.18, we know that the non-degenerate m-simplices of $N\Sigma\mathcal{C}$ for $m \geq 1$ are all of the form

[5] The same map is also a weak equivalence in the model structures mentioned in Remark 3.11.

$$x\colon \mathcal{O}[k]\otimes\mathcal{O}[\ell]^\circ\to\mathcal{C}\qquad\leftrightarrow\qquad x\colon O[k]\otimes O[\ell]^\circ\to C,$$

for some $k,\ell\geq 0$ with $m=k+1+\ell$. Each such simplex has the following features:

- the dimension $m=k+1+\ell$;
- the type k, which was defined in Definition 3.17;
- the suspect index r, which will be defined in Definition 3.23.

The goal is to filter the inclusion(s) from Proposition 3.20 by a sequence of anodyne extensions $\Sigma N^{\mathrm{RS}}\mathcal{C}=: X_0\subseteq X_1\subseteq\ldots\subseteq X_m\subseteq\ldots\subseteq N^{\mathrm{RS}}\Sigma\mathcal{C}$, where the inclusions are regular and the underlying simplicial set of X_m contains X_0, all the simplices of $N^{\mathrm{RS}}\Sigma\mathcal{C}$ of dimension less than m as well as all $m+1$-simplices that are *suspect*, a notion that we'll give in Lemma 3.24 in term of the *suspect index*

$$k:=\#\{0\leq i\leq m\mid x([i])=\perp\}-1.$$

To this end, we will filter the inclusion $X_{m-1}\subseteq X_m$ again via a sequence of anodyne extensions, as $X_{m-1}=: Y_{m-1}\subseteq Y_{m-2}\subseteq\ldots\subseteq Y_1\subseteq Y_0=X_m$, where Y_k contains X_{m-1}, all the simplices of $N^{\mathrm{RS}}\Sigma\mathcal{C}$ of dimension m and type at least k as well as all suspect simplices of dimension m and type at least $k-1$. Perhaps surprisingly, to show that the inclusions $Y_{k+1}\subseteq Y_k$ are anodyne, we will construct a further filtration $Y_{k+1}=: W_0\subseteq W_1\subseteq\ldots\subseteq W_{k-1}\subseteq W_k=Y_k$ and show then that all the inclusions $W_{r-1}\subseteq W_r$ are anodyne by exhibiting them as a pushout of a sum of anodyne extensions.

Definition 3.23. Let $\ell>0$, $k\geq 0$ and $m=k+1+\ell$. Let $x\colon O[k]\otimes O[\ell]^\circ\to C$ be a non-totally degenerate $(k+1+\ell)$-simplex of $N\Sigma\mathcal{C}$. The *suspect index* of x is the minimal integer $0\leq r\leq k$ – if existing – so that the following two conditions hold.

(SuspInd 1) Whenever $\mathbf{a}\subseteq[r,k]$, $\mathbf{a}'\subseteq[0,r-1]$, $\mathbf{b}\subseteq[0,\ell]$, with $|\mathbf{a}|\geq 0$, $|\mathbf{a}'|\geq 0$, $|\mathbf{b}|\geq 0$, we have

$$x([\mathbf{a}',\mathbf{a}]\otimes[0,\mathbf{b}])=0$$

(SuspInd 2) Whenever $\mathbf{a}\subseteq[r,k]$, $\mathbf{b}\subseteq[0,\ell]$, with $|\mathbf{b}|\geq 1$, we have

$$x([\mathbf{a}]\otimes[\mathbf{b}])=0.$$

If there is no integer r for which the conditions hold, we say that the *suspect index* of x is $k+1$.

The following are direct consequences of [31, Proposition 3.4].

Lemma 3.24. *Let $x\colon O[k]\otimes O[\ell]^\circ\to C$ be a non-totally degenerate $(k+1+\ell)$-simplex of $N\Sigma\mathcal{C}$. For $0\leq i\leq k-1$ the following are equivalent:*

(1) Whenever $\mathbf{a}' \subseteq [0, i-1]$, $\mathbf{a} \subseteq [i+2, k]$, *with* $|\mathbf{a}| \geq -1$, $|\mathbf{a}'| \geq -1$, $|\mathbf{b}| \geq 0$ *we have* $x([\mathbf{a}', i, i+1, \mathbf{a}] \otimes [\mathbf{b}]) = 0$.
(2) The $(k+1+\ell)$*-simplex* $x\colon O[k] \otimes O[\ell]^\circ \to C$ *of* $(N\Sigma\mathcal{C})_{k+1+\ell}$ *is degenerate at* i.

For $k+1 \leq i \leq m-1 = k+\ell$ *the following are equivalent:*

(1) Whenever $\mathbf{a} \subseteq [0, k]$, $\mathbf{b}' \subseteq [0, i-(k+1)-1]$, $\mathbf{b} \subseteq [i-(k+1)+2, \ell]$, *with* $|\mathbf{a}| \geq 0$, $|\mathbf{b}'| \geq -1$, $|\mathbf{b}| \geq -1$, *we have* $x([\mathbf{a}] \otimes [\mathbf{b}', i-(k+1), i-(k+1)+1, \mathbf{b}]) = 0$.
(2) The $(k+1+\ell)$*-simplex* $x\colon O[k] \otimes O[\ell]^\circ \to C$ *of* $(N\Sigma\mathcal{C})_{k+1+\ell}$ *is degenerate at* i.

Lemma 3.25. *Let* $y\colon O[k+1] \otimes O[\ell]^\circ \to C$ *be a non-totally degenerate* $(k+2+\ell)$*-simplex of* $N\Sigma\mathcal{C}$ *of suspect index* r*. The following are equivalent:*

(1) Whenever $\mathbf{a}' \subseteq [0, r-2]$, $\mathbf{a} \subseteq [r+1, m]$, *with* $|\mathbf{a}| \geq -1$, $|\mathbf{a}'| \geq -1$, *we have* $y([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0]) = 0$.
(2) The $(k+1)$*-simplex* $y([-] \otimes [0])\colon O[k+1] \to \mathcal{C}$ *of* $(N\mathcal{C})_{k+1}$ *is degenerate at* $r-1$.

If the equivalent conditions are satisfied, we say that y *is a* suspect *simplex.*

We analyze the faces of a suspect simplex.

Lemma 3.26. *Let* $y\colon O[k+1] \otimes O[\ell]^\circ \to C$ *be a suspect simplex of* $N\Sigma\mathcal{C}$ *suspect index* r*. Then, if* $d_i y$ *is not degenerate, we have that* $d_i y$ *is*

(Face 1) *of suspect index at most* $r-1$ *if* $0 \leq i \leq r-2$,
(Face 2) *of suspect index at most* $r-1$ *if* $i = r-1$,
(Face 3) *either of suspect index at most* $r-1$ *or suspect of dimension* m *and suspect index* r *if* $r+1 \leq i \leq k+1$,
(Face 4) *of type* $k+1$ *if* $k+2 \leq i \leq (k+1)+1+l = m+1$.

Proof. We distinguish several cases, which correspond to the different cases appearing in the statement.

(Face 1) Let $0 \leq i \leq r-2$. Whenever $\mathbf{a}' \subseteq [0, r-2], \mathbf{a} \subseteq [r-1, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{a}'| \geq 0$, $|\mathbf{b}| \geq 0$, we have $d^i[\mathbf{a}'] \subseteq [0, r-1]$, $d^i[\mathbf{a}] \subseteq [r, k+1]$ so that

$$(d_i y)([\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = y(d^i[\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = 0,$$

yielding that (SuspInd 1) holds for $r-1$ and $d_i y$. Whenever $\mathbf{a} \subseteq [r-1, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{b}| \geq 1$, we have $d^i[\mathbf{a}] \subseteq [r, k+1]$ and so

$$(d_i y)([\mathbf{a}] \otimes [\mathbf{b}]) = y(d^i[\mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

yielding that (SuspInd 2) holds for $r-1$ and $d_i y$. So the suspect index of $d_i y$ is at most $r-1$.

(Face 2) Let $i = r-1$. Whenever $\mathbf{a}' \subseteq [0, r-2], \mathbf{a} \subseteq [r-1, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{a}'| \geq 0$, $|\mathbf{b}| \geq 0$, we have $d^{r-1}[\mathbf{a}'] \subseteq [0, r-1]$, $d^{r-1}[\mathbf{a}] \subseteq [r, k+1]$ so that

$$(d_i y)([\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = y(d^i[\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = 0,$$

yielding that (SuspInd 1) holds for $r-1$ and $d_i y$. Whenever $\mathbf{a} \subseteq [r-1, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{b}| \geq 1$, we have $d^{r-1}\mathbf{a} \subseteq [r, k+1]$ and so

$$(d_i y)([\mathbf{a}] \otimes [\mathbf{b}]) = y(d^i[\mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

yielding that (SuspInd 2) holds for $r-1$ and $d_i y$. So the suspect index of $d_i y$ is at most $r-1$.

(Face 3) Let $r+1 \leq i \leq k+1$. Whenever $\mathbf{a}' \subseteq [0, r-1], \mathbf{a} \subseteq [r, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{a}'| \geq 0$, $|\mathbf{b}| \geq 0$, we have

$$(d_i y)([\mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

yielding that (SuspInd 1) holds for r and $d_i y$.

Moreover, whenever $|\mathbf{a}| \geq 0$, $\mathbf{a} \subseteq [r, k]$, with $|\mathbf{b}| \geq 1$, we have

$$(d_i y)([\mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

yielding that (SuspInd 2) holds for r and $d_i y$. So the suspect index of $d_i y$ is at most r.

To see that the simplex is suspect in the case that the suspect index is exactly r, we observe that for any $\mathbf{a}' \subseteq [0, r-2], \mathbf{a} \subseteq [r+1, k]$, with $|\mathbf{a}| \geq -1$, $|\mathbf{a}'| \geq -1$ we have

$$(d_i y)([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0]) = y(d^i[\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0]) = 0.$$

(Face 4) The last case is clear since the face operator acts on the second coordinate. $\square$

Lemma 3.27. *Let x be a non-degenerate non-suspect simplex of suspect index r. There is a simplex $\widetilde{x} \colon O[k+1] \otimes O[\ell]^\circ \to C$ defined by the following formulas:*

(P1)	$\widetilde{x}(d^r[\mathbf{a}] \otimes [\mathbf{b}]) = x([\mathbf{a}] \otimes [\mathbf{b}])$	*if* $|\mathbf{a}| \geq -1, |\mathbf{b}| \geq -1,$
(P2)	$\widetilde{x}([r, \mathbf{a}] \otimes [b]) = x(s^{r-1}[r, \mathbf{a}] \otimes [0])$	*if* $|\mathbf{a}| \geq -1;$
(P3)	$\widetilde{x}([\mathbf{a}', r] \otimes [\mathbf{b}]) = x([\mathbf{a}'] \otimes [0, \mathbf{b}])$	*if* $|\mathbf{a}'| \geq 0, |\mathbf{b}| \geq 1;$
(P4)	$\widetilde{x}([\mathbf{a}', r] \otimes [b]) = x([\mathbf{a}'] \otimes [0, b]) + x([\mathbf{a}', r-1] \otimes [0])$	*if* $|\mathbf{a}'| \geq 0;$
(P5)	$\widetilde{x}([\mathbf{a}', r, \mathbf{a}] \otimes [b]) = x(s^{r-1}[\mathbf{a}', r, \mathbf{a}] \otimes [0])$	*if* $|\mathbf{a}| \geq 0, |\mathbf{a}'| \geq 0;$
(P6)	$\widetilde{x}[\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}] = 0$	*if* $|\mathbf{a}| \geq 0, |\mathbf{a}'| \geq -1, |\mathbf{b}| \geq 1;$
(P7)	$\widetilde{x}([r] \otimes [\mathbf{b}]) = 0$	*if* $|\mathbf{b}| \geq 1.$

The proof consists of a careful (and tedious) analysis of all possible cases, and is postponed until Appendix A.

Remark 3.28. Given a non-degenerate non-suspect simplex x of suspect index r, by construction we have $d_r(\widetilde{x}) = x$.

We record the following features of $\widetilde{x}$.

Lemma 3.29. *If x is a non-suspect simplex of $N\Sigma\mathcal{C}$ with dimension $m = k + 1 + \ell$, type k and suspect index r, then the simplex $\widetilde{x}$ is a suspect simplex of dimension $m + 1$, type $k + 1$ and suspect index r.*

Proof. By construction, the simplex $\widetilde{x}$ is a suspect simplex of dimension $m + 1$, type $k + 1$ and suspect index at most r. We now prove the suspect index of $\widetilde{x}$ is exactly r.

Assume by contradiction that the suspect index of $\widetilde{x}$ is at most $r-1$. Then, whenever $\mathbf{a}' \subseteq [0, r-2]$, $\mathbf{a} \subseteq [r-1, k]$, we have $d^r[\mathbf{a}] \subseteq [r-1, k]$, so that

$$x([\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = \widetilde{x}(d^r[\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}])) = 0,$$

yielding that (SuspInd 1) holds for $r - 1$ and x. Also, whenever $\mathbf{a} \subseteq [r - 1, k]$, with $|\mathbf{a}| \geq 0$, and $|\mathbf{b}| \geq 1$, we have

$$x([\mathbf{a}] \otimes [\mathbf{b}]) = \widetilde{x}(d^r[\mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

yielding that (SuspInd 2) holds for $r - 1$ and x. This would thus imply that x is also of suspect index at most $r - 1$, contrary to the assumption. $\square$

Lemma 3.30. *Let x be a non-suspect simplex of $N\Sigma\mathcal{C}$. If x is non-degenerate, then $\widetilde{x}$ is non-degenerate.*

Proof. Let x be a simplex of dimension m, type k and suspect index r. Assume that $\widetilde{x} = s_i z$ is degenerate at some $0 \leq i \leq m$, and deduce a contradiction by distinguishing several cases.

- If $i = r-1$, we can prove that x would be of suspect index at most $r-1$, contradicting the assumption. Since $\widetilde{x}$ is degenerate at $r - 1$, we have $x = d_r\widetilde{x} = d_{r-1}\widetilde{x}$.

  We check first that then the condition (SuspInd 1) holds for $r - 1$ and x. Assume $\mathbf{a}' \subseteq [0, r-2], \mathbf{a} \subseteq [r-1, k]$, with $|\mathbf{a}| \geq 0$, $|\mathbf{a}'| \geq 0$, $|\mathbf{b}| \geq 0$. If $\mathbf{a}$ does not contain $r-1$, then we have

  $$x([\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) = 0$$

  since x has suspect index r. If $\mathbf{a}$ contains $r - 1$ and at least one further element, then we have

$$x([\mathbf{a}', r, \mathbf{a}] \otimes [0, \mathbf{b}]) = 0$$

since x has suspect index $r > r-1$. Finally, if $\mathbf{a} = [r-1]$, we have

$$\begin{aligned} x([\mathbf{a}', r-1] \otimes [0, \mathbf{b}]) &= d_{r-1}\widetilde{x}([\mathbf{a}', r-1] \otimes [0, \mathbf{b}]) \\ &= \widetilde{x}([\mathbf{a}', r] \otimes [0, \mathbf{b}]) = 0. \end{aligned}$$

This yields the condition (SuspInd 1) for $r-1$ and x.

For (SuspInd 2), we observe that, whenever $\mathbf{a} \subseteq [r, k]$, we have

$$x([r-1, \mathbf{a}] \otimes [\mathbf{b}]) = \widetilde{x}([r, \mathbf{a}] \otimes [\mathbf{b}]) = 0.$$

This yields the condition (SuspInd 2) for $r-1$ and x.

- If $i = r$, we can prove that x would be suspect, contradicting the assumption. Indeed, we observe that

$$\begin{aligned} x([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0]) &= x(s^{r-1}d^{r-1}([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0])) & \\ &= \widetilde{x}(d^{r-1}([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0])) & \text{(P5), (P2)} \\ &= (s_r z)(d^{r-1}([\mathbf{a}', r-1, r, \mathbf{a}] \otimes [0])) & \text{Assumption} \\ &= 0 & \text{Lemma 3.24} \end{aligned}$$

  so that x would be indeed suspect. This yields the desired contradiction.
- If $i \neq r-1, r$, we can prove that x would be degenerate, contradicting the assumption. Indeed, we would obtain

$$x = d_r\widetilde{x} = d_r s_i z = \begin{cases} s_{i-1}d_r z & \text{if } r+1 \leq i \leq k+1+l \\ s_i d_r z, & \text{if } 0 \leq i \leq r-2 \end{cases}$$

  contrary to the assumption that x is non-degenerate.

This concludes the proof. □

The following shows that the values of a suspect simplex y are enforced by those of its face $d_r y$.

Lemma 3.31. *Let $y\colon O[k+1] \otimes O[\ell]^\circ \to C$ be a suspect simplex of $N\Sigma\mathcal{C}$ with suspect index r.*

(S1) $y([d^r\mathbf{a}] \otimes [\mathbf{b}]) = d_r y([\mathbf{a}] \otimes [\mathbf{b}])$ *if* $|\mathbf{a}| \geq -1, |\mathbf{b}| \geq -1$,
(S2) $y([r, \mathbf{a}] \otimes [b]) = d_r y(s^{r-1}[r, \mathbf{a}] \otimes [0])$ *if* $|\mathbf{a}| \geq -1$;
(S3) $y([\mathbf{a}', r] \otimes [\mathbf{b}]) = d_r y([\mathbf{a}'] \otimes [0, \mathbf{b}])$ *if* $|\mathbf{a}'| \geq 0, |\mathbf{b}| \geq 1$;
(S4) $y([\mathbf{a}', r] \otimes [b]) = d_r y([\mathbf{a}'] \otimes [0, b]) + d_r y([\mathbf{a}', r-1] \otimes [0])$ *if* $|\mathbf{a}'| \geq 0$;
(S5) $y([\mathbf{a}', r, \mathbf{a}] \otimes [b]) = d_r y(s^{r-1}[\mathbf{a}', r, \mathbf{a}] \otimes [0])$ *if* $|\mathbf{a}| \geq 0, |\mathbf{a}'| \geq 0$;

(S6) $y([\mathbf{a}', r, \mathbf{a}]\otimes[\mathbf{b}]) = 0$ *if* $|\mathbf{a}| \geq 0, |\mathbf{a}'| \geq -1, |\mathbf{b}| \geq 1$;
(S7) $y([r] \otimes [\mathbf{b}]) = 0$ *if* $|\mathbf{b}| \geq 1$.

Proof. We now show that a suspect m-simplex $y\colon O[k+1]\otimes O[\ell]^\circ \to C$ of suspect index r is already completely specified by $d_r y$, in the way described more precisely by the statement.

(S1) The value of y in this case is by definition of simplicial structure of $N\Sigma\mathcal{C}$.
(S2) We prove the formula for $y([r, \mathbf{a}] \otimes [b])$ in this case. We observe that

$$\begin{aligned}
0 &= \partial y([r, \mathbf{a}] \otimes [0, b]) && \text{(SuspInd 2)}\\
&= y([\mathbf{a}] \otimes [0, b]) - y([r, \partial\mathbf{a}] \otimes [0, b])\\
&\quad +(-1)^{|\mathbf{a}|+1} y([r, \mathbf{a}] \otimes [0]) - (-1)^{|\mathbf{a}|+1} y([r, \mathbf{a}] \otimes [b])\\
&= (-1)^{|\mathbf{a}|+1} y([r, \mathbf{a}] \otimes [0]) - (-1)^{|\mathbf{a}|+1} y([r, \mathbf{a}] \otimes [b]) && \text{(SuspInd 2)}\\
&= y([r, \mathbf{a}] \otimes [0]) - y([r, \mathbf{a}] \otimes [b])\\
&= y(d^r s^{r-1}[r, \mathbf{a}] \otimes [0]) - y([r, \mathbf{a}] \otimes [b]) && \text{Lemma 3.24}
\end{aligned}$$

The desired formula follows.
(S3) We prove the formula for $y([\mathbf{a}', r] \otimes [\mathbf{b}])$ in this case. If $\mathbf{b}$ contains 0, the term vanishes and the formula follows by (SuspInd 1). If $\mathbf{b}$ does not contain 0, we have

$$\begin{aligned}
0 &= \partial y([\mathbf{a}', r] \otimes [0, \mathbf{b}]) && \text{(SuspInd 1)}\\
&= (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}'] \otimes [0, \mathbf{b}]) + y([\partial\mathbf{a}', r] \otimes [0, \mathbf{b}])\\
&\quad +(-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r] \otimes [\mathbf{b}]) + (-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r] \otimes [0, \partial^\circ\mathbf{b}])\\
&= (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}'] \otimes [0, \mathbf{b}]) + (-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r] \otimes [\mathbf{b}]) && \text{(SuspInd 1),(SuspInd 2)}\\
&= y([\mathbf{a}'] \otimes [0, \mathbf{b}]) - y([\mathbf{a}', r] \otimes [\mathbf{b}])
\end{aligned}$$

The desired formula follows.
(S4) We prove the formula for $y([\mathbf{a}', r] \otimes [b])$ in this case. If $b = 0$, the formula follows from Lemma 3.24. If $b > 0$, we have

$$\begin{aligned}
0 &= \partial y([\mathbf{a}', r] \otimes [0, b]) && \text{(SuspInd 1)}\\
&= (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}'] \otimes [0, b]) + y([\partial\mathbf{a}', r] \otimes [0, b])\\
&\quad +(-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r] \otimes [b]) + (-1)^{|\mathbf{a}'|+3} y([\mathbf{a}', r] \otimes [0])\\
&= (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}'] \otimes [0, b])\\
&\quad +(-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r] \otimes [b]) + (-1)^{|\mathbf{a}'|+3} y([\mathbf{a}', r] \otimes [0]) && \text{(SuspInd 1),(SuspInd 2)}\\
&= y([\mathbf{a}'] \otimes [0, b]) - y([\mathbf{a}', r] \otimes [b]) + y([\mathbf{a}', r] \otimes [0])\\
&= y([\mathbf{a}'] \otimes [0, b]) - y([\mathbf{a}', r] \otimes [b]) + y([\mathbf{a}', r-1] \otimes [0]) && \text{Lemma 3.24.}
\end{aligned}$$

The desired formula follows.
(S5) We prove the formula for $y([\mathbf{a}', r, \mathbf{a}] \otimes [b])$ in this case. If $b = 0$, then the equality follows from Lemma 3.24. If $b > 0$, we have

$$
\begin{aligned}
0 &= \partial y([\mathbf{a}', r, \mathbf{a}] \otimes [0, b]) && \text{(SuspInd 1)}\\
&= y([\partial \mathbf{a}', r, \mathbf{a}] \otimes [0, b]) + (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}', \mathbf{a}] \otimes [0, b])\\
&\quad + (-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r, \partial \mathbf{a}] \otimes [0, b])\\
&\quad + (-1)^{|\mathbf{a}'|+1+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [0])\\
&\quad - (-1)^{|\mathbf{a}'|+1+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [b])\\
&= (-1)^{|\mathbf{a}'|+1+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [0])\\
&\quad - (-1)^{|\mathbf{a}'|+1+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [b]) && \text{(SuspInd 1),(SuspInd 2)}\\
&= y([\mathbf{a}', r, \mathbf{a}] \otimes [0]) - y([\mathbf{a}', r, \mathbf{a}] \otimes [b])\\
&= y([\mathbf{a}', r, \mathbf{a}] \otimes [0]) - d_r y(s^{r-1}[\mathbf{a}', r, \mathbf{a}] \otimes [0]) && \text{Lemma 3.24.}
\end{aligned}
$$

The desired formula follows.
(S6) We prove that $y([\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}])$ necessarily vanishes. If $\mathbf{b}$ contains 0, this is a special case of (SuspInd 1). If $\mathbf{b}$ does not contain 0, we have

$$
\begin{aligned}
0 &= \partial y([\mathbf{a}', r, \mathbf{a}] \otimes [0, \mathbf{b}]) && \text{(SuspInd 1)}\\
&= y([\partial \mathbf{a}', r, \mathbf{a}] \otimes [0, \mathbf{b}]) + (-1)^{|\mathbf{a}'|+1} y([\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}])\\
&\quad + (-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r, \partial \mathbf{a}] \otimes [0, \mathbf{b}])\\
&\quad + (-1)^{|\mathbf{a}'|+2+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}])\\
&\quad + (-1)^{|\mathbf{a}'|+2+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [0, \partial^\circ \mathbf{b}])\\
&= (-1)^{|\mathbf{a}'|+2} y([\mathbf{a}', r, \partial \mathbf{a}] \otimes [0, \mathbf{b}])\\
&\quad + (-1)^{|\mathbf{a}'|+2+|\mathbf{a}|} y([\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}]) && \text{(SuspInd 1), (SuspInd 2)}
\end{aligned}
$$

If $|\mathbf{a}| > 0$, the desired vanishing follows directly from (SuspInd 1). If $|\mathbf{a}| = 0$, the desired vanishing follows from (S3) which we have treated before.
(S7) The fact that $y([r] \otimes [\mathbf{b}])$ vanishes in this case can be seen applying (SuspInd 2) for y and r.

This concludes the proof. □

We record the following features of $d_r y$.

Lemma 3.32. *Let $y\colon O[k+1] \otimes O[\ell]^\circ \to C$ be a non-degenerate suspect simplex of $N\Sigma C$ with suspect index r. Then the face $d_r y$ is a simplex of dimension $k+1+\ell$, type k and suspect index r.*

Proof. The value of the dimension and type of $d_r y$ is immediate from the definitions.

We now show that $d_r y$ is of suspect index r. It is straightforward from the construction that the suspect index of $d_r y$ is at most r. Assuming for contradiction the suspect index of $d_r y$ to be at most $r-1$, we show that the suspect index of y would be also at most $r-1$, contrary to the assumptions.

One can show that the conditions (SuspInd 1) and (SuspInd 2) hold for y and $r-1$. If $[\mathbf{a}]$ contains r, this follows from (SuspInd 1) and (SuspInd 2). If $[\mathbf{a}]$ does not contain r, this follows from (S3), (S6), (S7) of Lemma 3.31.

Finally, we show that $d_r y$ is a non-suspect simplex. Assuming for contradiction that $d_r y$ is suspect, we show using the characterization from Lemma 3.24 that then y is degenerate at r, contrary to the assumptions. To this end, we need to show that $y([\mathbf{a}', r, r+1, \mathbf{a}] \otimes [\mathbf{b}])$ vanishes. If $|\mathbf{b}| \geq 1$, this term vanishes by (S6). If $|\mathbf{b}| = 0$ and $|\mathbf{a}'| = -1$, then this term vanishes by (S2) using the assumption that $d_r y$ is suspect with suspect index r. If $|\mathbf{b}| = 0$ and $|\mathbf{a}'| \geq 0$, this term vanishes by (S5) using the assumption that $d_r y$ is suspect. $\square$

Lemma 3.33. *Let y be a suspect simplex of $N\Sigma\mathcal{C}$ with suspect index r. If y is non-degenerate, then the face $d_r y$ is a non-degenerate simplex.*

Proof. Let $k+1$ be the type of y. Assuming for contradiction that $d_r y$ is degenerate, we show that y itself has to be degenerate at some $0 \leq i \leq k+\ell$. Notice that the case $i = k$ cannot happen, because it would violate the type property from Lemma 3.32.

- If $0 \leq i < r-1$, we use Lemma 3.24 to show that y is degenerate at i. The fact that
$$y([\mathbf{a}', i, i+1, \mathbf{a}] \otimes [\mathbf{b}])$$
vanishes is by definition when r does not occur in $[\mathbf{a}]$. Otherwise, it can be deduced from the formulas (S3), (S4), (S5), (S6), together with the assumption that $d_r y$ is degenerate at i.
- If $i = r-1$, we use Lemma 3.24 to show that y is degenerate at r. The fact that
$$y([\mathbf{a}', r, r+1, \mathbf{a}] \otimes [\mathbf{b}])$$
vanishes can be deduced from the formulas (S2), (S5), (S6), together with the assumption that $d_r y$ is degenerate at $r-1$ in the formulation from Lemma 3.24.
- If $r \leq i < k$, we use Lemma 3.24 to show that y is degenerate at $i+1$. The fact that
$$y([\mathbf{a}', i+1, i+2, \mathbf{a}] \otimes [\mathbf{b}]),$$
vanishes can be deduced from the formulas (S2), (S5), (S6), together with the assumption that $d_r y$ is degenerate at i in the formulation from Lemma 3.24.
- If $k+1 \leq i \leq k+\ell$, we use Lemma 3.24 to show that y is degenerate at $i+1$. The fact that
$$y([\mathbf{a}] \otimes [\mathbf{b}])$$
vanishes follows from the formulas from Lemma 3.31, together with the fact that $d_r y$ is degenerate at i in the formulation from Lemma 3.24.

This concludes the proof. $\square$

We can now establish a correspondence between the suspect and non-suspect simplices of $N\Sigma\mathcal{C}$.

Proposition 3.34. *Let $\mathcal{C}$ be an ω-category of the form $\mathcal{C} \cong \nu C$ and let $m \geq 0$. Recall the inclusion $\Sigma N\mathcal{C} \hookrightarrow N\Sigma\mathcal{C}$ from Proposition 3.20.*

(i) The non-degenerate $(m+1)$-simplices in $\Sigma N\mathcal{C}$, regarded as a simplicial subset of $N\Sigma\mathcal{C}$, are contained in the $(m+1)$-simplices of type m.
(ii) The non-degenerate $(m+1)$-simplices in $N\Sigma\mathcal{C}$ that do not belong to $\Sigma N\mathcal{C}$ are non-degenerate $(m+1)$-simplices of type $0 \leq k \leq m-1$ and suspect index $1 \leq r \leq k+1$.
(iii) The r-th face map gives a bijective correspondence between the non-degenerate suspect $(m+1)$-simplices $\widetilde{x}$ in $N\Sigma\mathcal{C} \setminus \Sigma N\mathcal{C}$ of type $1 \leq k \leq m-1$ and suspect index $1 \leq r \leq k+1$ and the non-degenerate non-suspect m-simplices x of type $0 \leq k-1 \leq m-2$ and suspect index $1 \leq r \leq k+1$.

Proof. The first two statements can be verified by direct inspection. For the third statement, we observe that the assignment $\widetilde{(-)}$ from Lemma 3.27 is an inverse for the function d_r with the given domain and codomain. Indeed, Lemmas 3.32 and 3.33 show that $y \mapsto d_r y$ is a well-defined function, Lemmas 3.29 and 3.30 show that $x \mapsto \widetilde{x}$ is a well-defined function, Remark 3.28 shows that $d_r\widetilde{x} = x$, and the formulas from Lemmas 3.27 and 3.31 together imply that $\widetilde{d_r y} = y$. $\square$

We now prove the theorem.

Proof of Theorem 3.22. We prove (2), and (1) follows then by applying the forgetful functor from marked simplicial sets to simplicial sets. In order to show that the inclusion $\Sigma N^{\mathrm{RS}}\mathcal{C} \to N^{\mathrm{RS}}\Sigma\mathcal{C}$ is a complicial inner anodyne extension, we will realize it as a transfinite composite of intermediate complicial inner anodyne extensions

$$\Sigma N^{\mathrm{RS}}\mathcal{C} =: X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{m-1} \hookrightarrow X_m \hookrightarrow \cdots \hookrightarrow N^{\mathrm{RS}}\Sigma\mathcal{C}.$$

For $m \geq 2$, we let X_m be the smallest regular subsimplicial set of $N^{\mathrm{RS}}\Sigma\mathcal{C}$ containing X_{m-1}, all m-simplices of $N\Sigma\mathcal{C}$, as well as the suspect $(m+1)$-simplices of $N\Sigma\mathcal{C}$. Note that X_1 already contains all non-degenerate 1-simplices of $N\Sigma\mathcal{C}$ and that there are no non-degenerate suspect 2-simplices. Moreover, by Proposition 3.34, the subsimplicial set X_1 contains all non-degenerate $(m+1)$-simplices of type m. We see that the difference between X_{m-1} and X_m are the non-degenerate non-suspect m-simplices of type at most $m-2$ and the non-degenerate suspect $(m+1)$-simplices of type at most $m-1$.

In order to show that the inclusion $X_{m-1} \hookrightarrow X_m$ is a complicial inner anodyne extension for all $d \geq 2$, we realize it as a composite of intermediate complicial inner anodyne extensions

$$X_{m-1} =: Y_m \hookrightarrow Y_{m-1} \hookrightarrow \ldots \hookrightarrow Y_{k+1} \hookrightarrow Y_k \hookrightarrow \ldots \hookrightarrow Y_1 = X_m.$$

For $1 \leq k < m$, let Y_k be the smallest regular subset of X_m containing Y_{k+1} as well as all non-degenerate suspect $(m+1)$-simplices $\widetilde{x}$ of $N\Sigma\mathcal{C}$ of type k and all non-degenerate non-suspect m-simplices of type $k-1$. Note that Y_m already contains all non-degenerate m-simplices of type $m-1$ and that any suspect $(m+1)$-simplex of type m is necessarily a degeneracy of an m-simplex of type $m-1$ and thus can be checked to be also already in Y_m. We see using Lemma 3.26 and Proposition 3.34 that the difference between Y_k and Y_{k+1} are the non-degenerate suspect $(m+1)$-simplices of type k and possibly some of their faces (precisely those that are neither suspect nor of type k or higher).

In order to show that the inclusion $Y_{k+1} \hookrightarrow Y_k$ is a complicial inner anodyne extension for $1 \leq k \leq m-1$, we realize it as a filtration made by intermediate complicial inner anodyne extensions

$$Y_{k+1} =: W_0 \hookrightarrow W_1 \hookrightarrow \ldots \hookrightarrow W_{r-1} \hookrightarrow W_r \hookrightarrow \ldots \hookrightarrow W_k = Y_k.$$

For $0 < r \leq k$, we let W_r be the smallest regular simplicial subset of Y_k containing W_{r-1} as well as all suspect $(m+1)$-simplices of $N^{\mathrm{RS}}\Sigma\mathcal{C}$ of type k and suspect index r, namely those $\widetilde{x}$ for which each i-th row constant for $r \leq i \leq k$. Note that any m-simplex of suspect index 0 is either degenerate or of type $m-1$ and thus can be checked to be already in $X_1 \subseteq W_0$. We see using Lemma 3.26 and Proposition 3.34 that the difference between W_{r-1} and W_r are the non-degenerate suspect $(m+1)$-simplices $\widetilde{x}$ of type k and suspect index r and the non-degenerate non-suspect m-simplices x of type $k-1$ and suspect index r. There is a bijective correspondence between the $(m+1)$- and m-simplices mentioned above, as shown in Proposition 3.34.

We now record some relevant properties of the $(m+1)$-simplices $\widetilde{x}$ as above.

- We argue by induction and using Lemma 3.26 that the r-horn of $\widetilde{x}$ belongs to W_{r-1}; in particular, the r-horn defines a map of (underlying) simplicial sets

  $$\Lambda^r[m+1] \to W_{r-1}.$$

  Indeed, using Lemma 3.26 we see that the a-th face of $\widetilde{x}$ is already in W_{r-1} for $a \neq r$ since it is either a degeneracy of a simplex of smaller dimension or:

  - ♢ if $0 \leq a \leq r-2$, the face $d_a\widetilde{x}$ is of suspect index at most $r-1$, and in particular it belongs to W_{r-1}.
  - ♢ if $a = r-1$, the face $d_a\widetilde{x}$ has suspect index at most $r-1$, and in particular it belongs to W_{r-1}.
  - ♢ if $r+1 \leq a \leq k$, the face $d_a\widetilde{x}$ is either of suspect index at most $r-1$ or suspect of dimension m and suspect index r; in either case, it belongs to W_{r-1}.
  - ♢ if $k+1 \leq a \leq m+1$, the face $d_a\widetilde{x}$ is of type $k+1$, and in particular it belongs to $Y_{k+1} \subseteq W_{r-1}$.

- We argue that the r-th horn of $\widetilde{x}$ defines a map of simplicial sets

$$\Lambda^r[m+1] \to W_{r-1}$$

with marking.

Let $[\mathbf{a}'']$ be a marked p-simplex of $\Lambda^r[m+1]$, namely $[\mathbf{a}'']$ contains the vertices $\{r-1,r,r+1\}\cap[m+1]$. If the simplex $\widetilde{x}([\mathbf{a}''])$ is totally degenerate, it is in particular marked, so we will exclude this case for the rest of the discussion. By definition of the suspect index, we have $0 \leq r \leq k+1$. Note that $r=0$ would imply $x = s_{k+1}d_{k+1}x$ using that $\ell > 0$, the characterization Lemma 3.24 and (SuspInd 1). Thus, we can assume $0 < r \leq k+1 < k+1+\ell = m+1$. In particular, $\{r-1,r,r+1\} \subseteq [m+1]$.

If $r \leq k$, using (P6) or (P5) we have

$$\widetilde{x}([\mathbf{a}'']) = \widetilde{x}([\mathbf{a}', r-1, r, r+1, \mathbf{a}] \otimes [\mathbf{b}]) = 0,$$

so $\widetilde{x}([\mathbf{a}''])$ is marked by Remark 3.16. If instead $r = k+1$, then using (SuspInd 1) we obtain

$$\widetilde{x}([\mathbf{a}', r-1, r] \otimes [0, \mathbf{b}]) = 0,$$

so $\widetilde{x}([\mathbf{a}''])$ is marked by Remark 3.16. In total, we see that such a face is necessarily marked. Since this holds for all faces as above, we indeed obtain a map of simplicial sets with marking

$$\Lambda^r[m+1] \to W_{r-1}.$$

- If furthermore x is marked, we argue that the r-th horn of $\widetilde{x}$ defines a map of simplicial sets with marking

$$\Lambda^r[m+1]' \to W_{r-1},$$

with the simplicial set with marking $\Lambda^r[m+1]'$ defined in Lemma 3.8.

We show that the $(r-1)$-st face is marked using Remark 3.16. If $r \leq k$, since $\ell \geq 1$ we can use (P5) and obtain

$$\widetilde{x}([0, \dots, \widehat{r-1}, \dots, k+1] \otimes [0, \dots, \ell]) = 0,$$

so the $(r-1)$-st face is marked in this case. If $r = k+1$, by (P7) or (P3) we obtain

$$\widetilde{x}([0, \dots, \widehat{k}, k+1] \otimes [0, \dots, \ell]) = 0$$

so the $(r-1)$-st face is marked in this case.

We show that the $(r+1)$-st face is marked using Remark 3.16. If $r \leq k$, since $\ell \geq 1$, we can use (P3) or (P6) to obtain that

$$\widetilde{x}([0,\ldots,\widehat{r+1},\ldots,k+1]\otimes[0,\ldots,\ell])=0,$$

so the $(r+1)$-st face is marked in this case. If $r=k+1$, using (P3) and the fact that x is marked we obtain

$$\widetilde{x}([0,\ldots,k+1]\otimes[1,\ldots,\ell])=x([0,\ldots,k]\otimes[0,1,\ldots,\ell])=0.$$

So the $(r+1)$-st face is marked in this case.

By filling all r-horns of suspect $(m+1)$-simplices $\widetilde{x}$ of W_r, we then obtain their r-th face x, which was missing in W_{r-1}, as well as the suspect $(m+1)$-simplex $\widetilde{x}$ itself. This can be rephrased by saying that there is a pushout square

$$\begin{tikzcd}
\coprod\limits_{\substack{x\\ \text{non-marked}}}\Lambda^r[m+1]\amalg\coprod\limits_{\substack{x\\ \text{marked}}}\Lambda^r[m+1]' \arrow[r] \arrow[d] & \coprod\limits_{\substack{x\\ \text{non-marked}}}\Delta^r[m+1]\amalg\coprod\limits_{\substack{x\\ \text{marked}}}\Delta^r[m+1]'' \arrow[d] \\
W_{r-1} \arrow[r] & W_r.
\end{tikzcd}$$

Since the involved horn inclusions are in fact inner horn inclusions, the inclusions of simplicial sets with marking $\Lambda^r[m+1]\hookrightarrow\Delta^r[m+1]$ and $\Lambda^r[m+1]'\hookrightarrow\Delta^r[m+1]''$ are complicial inner anodyne extensions by Lemma 3.8.

It follows that the inclusion $W_{r-1}\hookrightarrow W_r$ for any $1\leq r\leq m-j$, the inclusion $Y_{j-1}\hookrightarrow Y_j$ for any $1\leq j\leq m$, the inclusion $Y_{j-1}\hookrightarrow Y_j$ for any $1\leq j\leq m$, the inclusion $X_{m-1}\hookrightarrow X_m$ for any $m\geq 1$, and the inclusion $\Sigma N^{\mathrm{RS}}\mathcal{C}\to N^{\mathrm{RS}}\Sigma\mathcal{C}$ are complicial inner anodyne extensions, as desired. □

4. Θ_n-spaces and Quillen pair with complicial sets

In this section we apply Theorem 3.22 to produce an explicit Quillen adjunction between the model structure of n-complicial sets, and the model structure for complete Segal Θ_n-spaces, which we first recall.

4.1. Θ_n-spaces

We recall the main facts about Θ_n-spaces that will be used in this paper.

Remark 4.1. Let $n\geq 0$. The suspension functor $\Sigma\colon\omega\mathcal{C}at\to\omega\mathcal{C}at_{*,*}$ restricts and corestricts to a functor $\Sigma\colon(n-1)\mathcal{C}at\to n\mathcal{C}at_{*,*}$.

Let Θ_n denote Joyal's cell category from [16], which is by [4,22] a full subcategory of $n\mathcal{C}at$. By definition, Θ_0 is the terminal category, Θ_1 is the ordinal category Δ, and Θ_n

is for $n > 0$ the full subcategory of $n\mathcal{C}at$ whose generic object is obtained as a pushout of n-categories

$$\theta = [k|\theta_1, \dots, \theta_k] = \Sigma\theta_1 \coprod_{[0]} \Sigma\theta_2 \coprod_{[0]} \dots \coprod_{[0]} \Sigma\theta_k$$

for $k \geq 0$ and $\theta_1, \dots, \theta_k \in \Theta_{n-1}$.

For $n > 0$, there is a full inclusion $\Theta_{n-1} \hookrightarrow \Theta_n$, and whenever needed we will regard any object of Θ_{n-1} as an object of Θ_n without further specification.

Definition 4.2. Let ≥ 0. A Θ_n*-space* (resp. Θ_n*-set*) is a presheaf $W\colon \Theta_n^{\mathrm{op}} \to s\mathcal{S}et$ (resp. $W\colon \Theta_n^{\mathrm{op}} \to \mathcal{S}et$).

For $n \geq 0$, we denote by $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ (resp. $\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$) the category of Θ_n-spaces (resp. Θ_n-sets).

Remark 4.3. The categories $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ and $\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ are cocomplete, and colimits are computed componentwise.

For $n \geq 0$ the canonical inclusion $\mathcal{S}et \hookrightarrow s\mathcal{S}et$ of sets as discrete simplicial sets induces a canonical inclusion $\mathcal{S}et^{\Theta_n^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$, which preserves limits and colimits. In particular, we often regard Θ_n-sets as discrete Θ_n-spaces without further specification.

For any object θ in Θ_n, we denote by $\Theta_n[\theta]$ the Θ_n-set represented by θ via the Yoneda embedding $\Theta_n \hookrightarrow \mathcal{S}et^{\Theta_n^{\mathrm{op}}}$.

Remark 4.4. As a special case of [1, §3.1], given any Θ_n-set A and any space B one can consider the Θ_n-space $A \boxtimes B$, which is defined levelwise as the simplicial set

$$(A \boxtimes B)_\theta := A_\theta \times B.$$

The construction extends to a bifunctor

$$\boxtimes\colon \mathcal{S}et^{\Theta_n^{\mathrm{op}}} \times s\mathcal{S}et \to s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$$

that preserves colimits in each variable.

4.2. Suspension of Θ_n-spaces

Remark 4.5. Let $n \geq 0$. The suspension functor $\Sigma\colon \omega\mathcal{C}at \to \omega\mathcal{C}at$ restricts and corestricts to a functor $\Sigma\colon \Theta_{n-1} \to \Theta_n$.

Definition 4.6. Let $n > 0$, and $\theta \in \Theta_{n-1}$. The *suspension* of the representable presheaf $\Theta_{n-1}[\theta]$ is the (discrete) Θ_n-space

$$\Sigma\Theta_n[\theta] \coloneqq \Theta_n[\Sigma\theta].$$

The enriched left Kan extension of the functor $\Theta_{n-1} \xrightarrow{\Sigma} \Theta_n \hookrightarrow s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ defines a functor $\Sigma\colon s\mathcal{S}et^{\Theta_{n-1}^{\mathrm{op}}} \subseteq s\mathcal{S}et_{*,*}^{\Theta_n^{\mathrm{op}}}$.

As discussed in [25, Remark 4.5] and in [25, Lemma 11.10], this functor agrees with the functor $V[1]$ from [25, § 4.4].

4.3. The adjunction

Let us begin by defining the functor that we use to make our comparison.

Construction 4.7. Let $n \geq 0$. The functor $\Theta_n \times \Delta \subseteq s\mathcal{S}et^{\Theta_n^{\mathrm{op}}} \to ms\mathcal{S}et$ given by

$$(\theta, [\ell]) \mapsto (\Theta_n \times \Delta)[\theta, \ell] = \Theta_n[\theta] \boxtimes \Delta[\ell] \mapsto N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$$

induces an adjunction

$$L_n\colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}} \rightleftarrows ms\mathcal{S}et\colon R_n.$$

4.4. The Quillen pair before localizing

Notation 4.8. Let $n \geq 0$. The category $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ supports the projective model structure $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{\mathrm{proj}}$ where the fibrant objects are precisely the projectively fibrant presheaves and the cofibrations are precisely the projective cofibrations.

Remark 4.9. Let $n \geq 0$. Combining [14, Theorem 11.6.1, Definition. 11.5.33, Definition 11.5.25], we know that

(1) a set of generating cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_n[\theta] \boxtimes \partial\Delta[\ell] \to \Theta_n[\theta] \boxtimes \Delta[\ell] \quad \text{ for } \theta \in \Theta_n \text{ and } \ell \geq 0;$$

and
(2) a set of generating acyclic cofibrations for the projective model structure on $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ is given by all maps of the form

$$\Theta_n[\theta] \boxtimes \Lambda^k[\ell] \to \Theta_n[\theta] \boxtimes \Delta[\ell] \quad \text{ for } \theta \in \Theta_n \text{ and } 0 \leq k \leq \ell.$$

The following can be proven similarly to [5, Lemma 1.27].

Proposition 4.10. *Let $n \geq 0$. The functor*

$$(-)^\sharp \colon s\mathcal{S}et_{(\infty,0)} \to ms\mathcal{S}et_{(\infty,n)},$$

which endows a simplicial set with its maximal marking, is left Quillen.

Proposition 4.11. *Let $n \geq 0$. The functor*

$$L_n \colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{\mathrm{proj}} \to ms\mathcal{S}et_{(\infty,n)}$$

is left Quillen.

We include the proof for the reader's convenience, but the argument is the evident generalization of the 2-dimensional case treated in [5, Proposition 2.2].

Proof. We want to show that the functor L_n preserves cofibrations and acyclic cofibrations. Using the facts that $(-)^\sharp$ commutes with colimits, which is a consequence of Proposition 4.10, and that the box product $\boxtimes$ preserves colimits in each variable, which was recalled in Remark 4.4, we see that

(1) the image of the generic generating cofibration via L_n is the map

$$N^{\mathrm{RS}}\theta \times \partial\Delta[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp \quad \text{for } \theta \in \Theta_n \text{ and } \ell \geq 0;$$

(2) the image of the generic generating acyclic cofibration via L_n is the map

$$N^{\mathrm{RS}}\theta \times \Lambda^k[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp \quad \text{for } \theta \in \Theta_n \text{ and } 0 \leq k \leq \ell.$$

Since the model structure $ms\mathcal{S}et_{(\infty,n)}$ is cartesian closed by Theorem 3.10 and $(-)^\sharp$ is a left Quillen functor by Proposition 4.10, we conclude that

(1) the map $N^{\mathrm{RS}}\theta \times \partial\Delta[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$ is a cofibration and
(2) the map $N^{\mathrm{RS}}\theta \times \Lambda^k[\ell]^\sharp \to N^{\mathrm{RS}}\theta \times \Delta[\ell]^\sharp$ is an acyclic cofibration

It follows that L_n preserves cofibrations and acyclic cofibrations, so it is a left Quillen functor, as desired. □

4.5. The Quillen pair before localizing

We construct a variant of Rezk's model structure from [25, §11.4] (see Remark 4.15).

Definition 4.12. Let $n \geq 0$. A map of (discrete) Θ_n-spaces is an *elementary acyclic cofibration* if it is of one of the following kinds:

(1) For $0 \leq j \leq n-1$, $k \geq 1$ and objects $\theta_1, \ldots, \theta_k$ of Θ_{n-j}, the *j-fold k-Segality extension*

$$\Sigma^j\Theta_n[\theta_1] \underset{\Sigma^j\Theta_n[0]}{\amalg} \cdots \underset{\Sigma^j\Theta_n[0]}{\amalg} \Sigma^j\Theta_n[\theta_k] \hookrightarrow \Sigma^j\Theta_n[\theta_1 \underset{[0]}{\amalg} \cdots \underset{[0]}{\amalg} \theta_k]$$

(2) For $0 \leq j \leq n-1$, the *j-fold completeness extension*

$$\Sigma^j\Theta_n[0] \hookrightarrow \Sigma^j\Theta_n[0] \underset{\Sigma^j\Theta_n[1]}{\amalg} \Sigma^j\Theta_n[3] \underset{\Sigma^j\Theta_n[1]}{\amalg} \Sigma^j\Theta_n[0].$$

Definition 4.13. A *complete Segal Θ_n-space* is a Θ_n-space that is local with respect to all maps of type (1) and (2) from Definition 4.12.

By localizing the projective model structure $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{\mathrm{proj}}$ at the class of maps from Definition 4.12, we obtain the following.

Theorem 4.14. *Let $n \geq 0$. The category $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$ supports a cartesian closed model structure $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{(\infty,n)}$ where the fibrant objects are precisely the projectively fibrant complete Segal Θ_n-spaces and the cofibrations are precisely the projective cofibrations.*

Remark 4.15. The model structure from Theorem 4.14 differs from Rezk's from [25, §11.4] in the following aspects:

(1) We work with localizations of the projective model structure, instead of the injective model structure.
(2) To express the completeness extension, we use $\Theta_n[0] \amalg_{\Theta_n[1]} \Theta_n[3] \amalg_{\Theta_n[1]} \Theta_n[0]$ instead of the Θ_n-nerve of the walking isomorphism.

However, the two model structures are Quillen equivalent (see [25, §2.5–2.13,§10]).

We now show that we still have a Quillen pair after localizing the projective model structure on $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}$.

Theorem 4.16. *Let $n \geq 0$. The functor*

$$L_n \colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{(\infty,n)} \to ms\mathcal{S}et_{(\infty,n)}$$

is left Quillen.

Proof. We prove this by induction on $n \geq 2$. The basis of the induction $n = 0, 1, 2$ is treated in Proposition 4.10, the combination of [34, §6.5] with [17], and [5, Theorem 2.4], respectively. We now assume $n > 2$ and that the statement is true for $n-1$. By [14, Proposition 3.3.18] applied to the Quillen adjunction from Proposition 4.11, it suffices to show that the derived functor L^h_n of L_n sends all elementary acyclic cofibrations from Definition 4.12 to acyclic cofibrations. By Lemma 4.17, this is equivalent to proving that L_n sends all elementary acyclic cofibrations from Definition 4.12 to acyclic cofibrations. We do so by proving that L_n preserves all elementary acyclic cofibrations from Definition 4.12, which we do in Propositions 4.19 and 4.20. $\square$

The proof makes use of the following technical lemma, that discusses the values of the derived functor L^h_n of the left Quillen functor $L_n\colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{\mathrm{proj}} \to ms\mathcal{S}et_{(\infty,n)}$ on certain (homotopy) pushouts in $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{(\infty,n)}$.

Lemma 4.17. *Suppose we are given maps $B \leftarrow A \to C$ in $s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{(\infty,n)}$, such that $A \to B$ is a monomorphism, $L_nA \to L_nB$ is a cofibration, and there is an equivalence $L^h_nX \simeq L_nX$ in $ms\mathcal{S}et_{(\infty,n)}$ for $X = A, B, C$. Then there is a canonical natural weak equivalence in $ms\mathcal{S}et_{(\infty,n)}$*

$$L^h_n(A \coprod_B C) \simeq L_n(A \coprod_B C).$$

Proof. We have a chain of natural weak equivalences in $ms\mathcal{S}et_{(\infty,n)}$

$$L^h_n(B \coprod_A C) \simeq L^h_n(B \overset{h}{\coprod_A} C) \simeq L^h_nB \overset{h}{\coprod_{L^h_nA}} L^h_nC \simeq L_nB \coprod_{L_nA} L_nC \cong L_n(B \coprod_A C),$$

as desired. $\square$

The following result by Steiner will allow us to apply Theorem 3.22 to the case $\mathcal{C} = \theta$, for some $\theta \in \Theta_n$.

Theorem 4.18 *([30]). Let $n \geq 0$ and $\theta \in \Theta_n$. There is an isomorphism of ω-categories*

$$\theta \cong \nu T$$

for some augmented directed chain complex T.

We analyze the action of L_n on the Segality extensions.

Proposition 4.19. *Let $n > 0$. If the functor $L_{n-1}\colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}}_{(\infty,n-1)} \to ms\mathcal{S}et_{(\infty,n-1)}$ is left Quillen, the functor $L_n\colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}} \to ms\mathcal{S}et$ sends the j-fold Segal acyclic cofibration from Definition 4.12,*

$$\Sigma^j\Theta_n[\theta_1] \coprod_{\Sigma^j\Theta_n[0]} \cdots \coprod_{\Sigma^j\Theta_n[0]} \Sigma^j\Theta_n[\theta_k] \hookrightarrow \Sigma^j\Theta_n[\theta_1 \coprod_{[0]} \cdots \coprod_{[0]} \theta_k]$$

for $0 \leq j \leq n-1$, $k \geq 1$ and objects $\theta_1, \ldots, \theta_k$ of Θ_{n-j}, to a weak equivalence in $ms\mathcal{S}et_{(\infty,n)}$.

Proof. We prove the statement by induction on $j \geq 0$, and fixed $k \geq 1$. The basis case(s) $j = 0$ is a direct consequence of [24, Theorem 4.9], and we now assume $j > 1$ for the inductive step. By definition, the map

$$\Sigma^{j-1}\Theta_n[\theta_1] \underset{\Sigma^{j-1}\Theta_n[0]}{\amalg} \cdots \underset{\Sigma^{j-1}\Theta_n[0]}{\amalg} \Sigma^{j-1}\Theta_n[\theta_k] \hookrightarrow \Sigma^{j-1}\Theta_n[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k]$$

is an acyclic cofibration in $s\mathcal{S}et_{(\infty,n-1)}^{\Theta_{n-1}^{\mathrm{op}}}$. This acyclic cofibration can be rewritten as

$$\Theta_n\Sigma^{j-1}[\theta_1] \underset{\Theta_n\Sigma^{j-1}[0]}{\amalg} \cdots \underset{\Theta_n\Sigma^{j-1}[0]}{\amalg} \Theta_n\Sigma^{j-1}[\theta_k] \hookrightarrow \Theta_n\Sigma^{j-1}[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k].$$

By applying to it the left Quillen functor $L_{n-1} \colon s\mathcal{S}et_{(\infty,n-1)}^{\Theta_n^{\mathrm{op}}} \to ms\mathcal{S}et_{(\infty,n-1)}$ we obtain an acyclic cofibration in $ms\mathcal{S}et_{(\infty,n-1)}$

$$\begin{aligned} &N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_1] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]}{\amalg} \cdots \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_k] \\ &\quad \hookrightarrow N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k]. \end{aligned}$$

By applying to it the left Quillen functor $\Sigma \colon ms\mathcal{S}et_{(\infty,n-1)} \to (ms\mathcal{S}et_{(\infty,n)})_{*,*}$ from Lemma 3.14 we obtain an acyclic cofibration in $ms\mathcal{S}et_{(\infty,n)}$

$$\begin{aligned} &\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_1] \underset{\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]}{\amalg} \cdots \underset{\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]}{\amalg} \Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_k] \\ &\quad \hookrightarrow \Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k]. \end{aligned}$$

Since Σ commutes with nerve by Theorems 3.22 and 4.18, we also get an acylic cofibration

$$N^{\mathrm{RS}}\Theta_n\Sigma^j[\theta_1] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^j[0]}{\amalg} \cdots \underset{N^{\mathrm{RS}}\Theta_n\Sigma^j[0]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^j[\theta_k] \hookrightarrow N^{\mathrm{RS}}\Theta_n\Sigma^j[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k],$$

which is

$$L_n\Theta_n\Sigma^j[\theta_1] \underset{L_n\Theta_n\Sigma^j[0]}{\amalg} \cdots \underset{L_n\Theta_n\Sigma^j[0]}{\amalg} L_n\Theta_n\Sigma^j[\theta_k] \hookrightarrow L_n\Theta_n\Sigma^j[\theta_1 \underset{[0]}{\amalg} \dots \underset{[0]}{\amalg} \theta_k].$$

This concludes the proof. $\square$

We analyze the action of L_n on the completeness extensions.

Proposition 4.20. *Let $n > 0$. If the functor $L_{n-1} \colon s\mathcal{S}et_{(\infty,n-1)}^{\Theta_n^{\mathrm{op}}} \to ms\mathcal{S}et_{(\infty,n-1)}$ is left Quillen, the functor $L_n \colon s\mathcal{S}et^{\Theta_n^{\mathrm{op}}} \to ms\mathcal{S}et$ sends the j-fold Segal acyclic cofibration from Definition 4.12,*

$$\Sigma^j\Theta_n[0] \hookrightarrow \Sigma^j\Theta_n[0] \underset{\Sigma^j\Theta_n[1]}{\amalg} \Sigma^j\Theta_n[3] \underset{\Sigma^j\Theta_n[1]}{\amalg} \Sigma^j\Theta_n[0]$$

for $0 \leq j \leq n-1$, *to a weak equivalence in* $msSet_{(\infty,n)}$.

Proof. We prove the statement by induction on $j \geq 0$. The basis case(s) $j = 0, 1$ are proven in [5, Propositions 2.7, 2.9], and we now assume $j > 0$ for the inductive step. By definition, the map

$$\Sigma^{j-1}\Theta_n[0] \hookrightarrow \Sigma^{j-1}\Theta_n[0] \underset{\Sigma^{j-1}\Theta_n[1]}{\amalg} \Sigma^{j-1}\Theta_n[3] \underset{\Sigma^{j-1}\Theta_n[1]}{\amalg} \Sigma^{j-1}\Theta_n[0]$$

is an acyclic cofibration in $sSet_{(\infty,n-1)}^{\Theta_{n-1}^{\mathrm{op}}}$. This acyclic cofibration can be rewritten as

$$\Theta_n\Sigma^{j-1}[0] \hookrightarrow \Theta_n\Sigma^{j-1}[0] \underset{\Theta_n\Sigma^{j-1}[1]}{\amalg} \Theta_n\Sigma^{j-1}[3] \underset{\Theta_n\Sigma^{j-1}[1]}{\amalg} \Theta_n\Sigma^{j-1}[0].$$

By applying to it the left Quillen functor $L_{n-1}\colon sSet_{(\infty,n-1)}^{\Theta_n^{\mathrm{op}}} \to msSet_{(\infty,n-1)}$ we obtain an acyclic cofibration in $msSet_{(\infty,n-1)}$

$$\begin{aligned} &N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0] \\ &\quad\hookrightarrow N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[1]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[3] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[1]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]. \end{aligned}$$

By applying to it the left Quillen functor $\Sigma\colon msSet_{(\infty,n-1)} \to (msSet_{(\infty,n)})_{*,*}$ from Lemma 3.14 we obtain an acyclic cofibration in $(msSet_{(\infty,n)})_{(*,*)}$

$$\begin{aligned} &\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0] \\ &\quad\hookrightarrow \Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0] \underset{\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[1]}{\amalg} \Sigma N^{\mathrm{RS}}\Sigma^{j-1}\Theta_n[3] \underset{\Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[1]}{\amalg} \Sigma N^{\mathrm{RS}}\Theta_n\Sigma^{j-1}[0]. \end{aligned}$$

Since Σ commutes with nerve by Theorems 3.22 and 4.18 we also get an acyclic cofibration

$$N^{\mathrm{RS}}\Theta_n\Sigma^{j}[0] \hookrightarrow N^{\mathrm{RS}}\Theta_n\Sigma^{j}[0] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j}[1]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^{j}[3] \underset{N^{\mathrm{RS}}\Theta_n\Sigma^{j}[1]}{\amalg} N^{\mathrm{RS}}\Theta_n\Sigma^{j}[0],$$

which is

$$L_n\Theta_n\Sigma^{j}[0] \hookrightarrow L_n\Theta_n\Sigma^{j}[0] \underset{L\Theta_n\Sigma^{j}[1]}{\amalg} L_n\Theta_n\Sigma^{j}[3] \underset{L_n\Theta_n\Sigma^{j}[1]}{\amalg} L_n\Theta_n\Sigma^{j}[0].$$

This concludes the proof. $\square$

With the establishment of Propositions 4.19 and 4.20, the proof of Theorem 4.16 is now complete.

Appendix A. Proof of Lemma 3.27

We now prove Lemma 3.27.

Proof of Lemma 3.27. Since these cases are mutually exclusive and cover all possibilities, this at least defines a map, and by construction we will have $d_r\widetilde{x} = x$.

It is immediate that the map is directed. Observe that neither the case (P4) nor the case (P5) can apply to a chain of dimension 0, proving that $\widetilde{x}$ is augmented since x is augmented. What we need to check is that $\widetilde{x}$ is a chain map.

(P1) This case is immediate since x is a chain map (and 'not containing r in the first component' is preserved by the differential).

(P2) For $|\mathbf{a}| = -1$, there is nothing to check. For $|\mathbf{a}| \geq 0$, on the one hand, we have

$$\begin{array}{rcl}\widetilde{x}(\partial[r,\mathbf{a}]\otimes[b]) & = & \widetilde{x}([\mathbf{a}]\otimes[b]-[r,\partial\mathbf{a}]\otimes[b])\\ & = & x(s^{r-1}[\mathbf{a}]\otimes[b])-x(s^{r-1}[r,\partial\mathbf{a}]\otimes[0]);\end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([r,\mathbf{a}]\otimes[b]) & = & \partial x(s^{r-1}[r,\mathbf{a}]\otimes[0])\\ & = & x(s^{r-1}[\mathbf{a}]\otimes[0])-x(s^{r-1}[r,\partial\mathbf{a}]\otimes[0]).\end{array}$$

From (SuspInd 2), we obtain

$$\begin{array}{rcl}0=\partial x(s^{r-1}[\mathbf{a}]\otimes[0,b]) & = & x(s^{r-1}[\partial\mathbf{a}]\otimes[0,b])\\ & & +(-1)^{|\mathbf{a}|}x(s^{r-1}[\mathbf{a}]\otimes[0])\\ & & -(-1)^{|\mathbf{a}|}x(s^{r-1}[\mathbf{a}]\otimes[b]).\end{array}$$

Using (SuspInd 2) again, the first summand vanishes, yielding the equality of the other two. This shows the desired equality.

(P3) For $|\mathbf{a}'| \geq 1$, $|\mathbf{b}| \geq 2$, on the one hand we have

$$\begin{array}{rcl}\widetilde{x}(\partial([\mathbf{a}',r]\otimes[\mathbf{b}])) & = & \widetilde{x}([\partial\mathbf{a}',r]\otimes[\mathbf{b}]\\ & & +(-1)^{|\mathbf{a}'|+1}[\mathbf{a}']\otimes[\mathbf{b}]+(-1)^{|\mathbf{a}'|+1}[\mathbf{a}',r]\otimes\partial^{\circ}[\mathbf{b}])\\ & = & x([\partial\mathbf{a}']\otimes[0,\mathbf{b}])+(-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[\mathbf{b}])+\\ & & +(-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[0,\partial^{\circ}\mathbf{b}]);\end{array}$$

on the other hand, we have

$$\begin{array}{rcl}\partial\widetilde{x}([\mathbf{a}',r]\otimes[\mathbf{b}]) & = & \partial(x([\mathbf{a}']\otimes[0,\mathbf{b}]))\\ & = & x([\partial\mathbf{a}']\otimes[0,\mathbf{b}])\\ & & +(-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[\mathbf{b}])\\ & & +(-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[0,\partial^{\circ}\mathbf{b}]),\end{array}$$

so the two expressions coincide.

For $|\mathbf{a}'| = 0$, $|\mathbf{b}| \geq 2$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([a', r] \otimes [\mathbf{b}])) & = & \widetilde{x}([r] \otimes [\mathbf{b}] - [a'] \otimes [\mathbf{b}] - [a', r] \otimes \partial^\circ [\mathbf{b}]) \\ & = & -x([a'] \otimes [\mathbf{b}]) - x([a'] \otimes [0, \partial^\circ \mathbf{b}]); \end{array}$$

On the other hand we have

$$\begin{array}{rcl} \partial\widetilde{x}([a', r] \otimes [\mathbf{b}]) & = & \partial(x([a'] \otimes [0, \mathbf{b}])) \\ & = & -x([a'] \otimes [\mathbf{b}]) - x([a'] \otimes [0, \partial^\circ \mathbf{b}]), \end{array}$$

so the two expressions coincide.

For $|\mathbf{a}'| \geq 1$, $|\mathbf{b}| = 1$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([\mathbf{a}', r] \otimes [b_0, b_1])) & = & \widetilde{x}([\partial \mathbf{a}', r] \otimes [b_0, b_1] \\ & & +(-1)^{|\mathbf{a}'|+1} [\mathbf{a}'] \otimes [b_0, b_1] \\ & & +(-1)^{|\mathbf{a}'|+1} [\mathbf{a}', r] \otimes [b_0] \\ & & -(-1)^{|\mathbf{a}'|+1} [\mathbf{a}', r] \otimes [b_1]) \\ & = & x([\partial \mathbf{a}'] \otimes [0, b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+1} x([\mathbf{a}'] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+1} x([\mathbf{a}'] \otimes [0, b_0]) \\ & & +x([\mathbf{a}', r-1] \otimes [0]) \\ & & -(-1)^{|\mathbf{a}'|+1} x([\mathbf{a}'] \otimes [0, b_1]) \\ & & -x([\mathbf{a}', r-1] \otimes [0]); \end{array}$$

on the other hand we have

$$\begin{array}{rcl} \partial\widetilde{x}([\mathbf{a}', r] \otimes [b_0, b_1]) & = & \partial(x([\mathbf{a}'] \otimes [0, b_0, b_1])) \\ & = & x([\partial \mathbf{a}'] \otimes [0, b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+1} x([\mathbf{a}'] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|} x([\mathbf{a}'] \otimes [0, b_1]) \\ & & -(-1)^{|\mathbf{a}'|} x([\mathbf{a}'] \otimes [0, b_0]), \end{array}$$

so the two expressions coincide.

For $|\mathbf{a}'| = 0$, $|\mathbf{b}| = 1$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([a', r] \otimes [b_0, b_1])) & = & \widetilde{x}([r] \otimes [b_0, b_1] - [a'] \otimes [b_0, b_1] \\ & & -[a', r] \otimes [b_0] + [a', r] \otimes [b_1]) \\ & = & -x[a'] \otimes [b_0, b_1] - x([a'] \otimes [0, b_0]) \\ & & -x([a', r-1] \otimes [0]) + x([a'] \otimes [0, b_1]) \\ & & +x([a', r-1] \otimes [0]); \end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([\mathbf{a}',r]\otimes[b_0,b_1]) & = & \partial(x([a']\otimes[0,b_0,b_1]))\\ & = & -x([a']\otimes[b_0,b_1]-x([a']\otimes[0,b_0])\\ & & +x([a']\otimes[0,b_1]),\end{array}$$

so the two expressions coincide.

(P4) For $|\mathbf{a}'|\geq 1$, on the one hand we have

$$\begin{array}{rcl}\widetilde{x}(\partial([\mathbf{a}',r]\otimes[b])) & = & \widetilde{x}((-1)^{|\mathbf{a}'|+1}[\mathbf{a}']\otimes[b]+[(\partial\mathbf{a}'),r]\otimes[b])\\ & = & (-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[b])+x([(\partial\mathbf{a}')]\otimes[0,b])\\ & & +x([(\partial\mathbf{a}'),r-1]\otimes[0]);\end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([\mathbf{a}',r]\otimes[b]) & = & \partial(x([\mathbf{a}')]\otimes[0,b])+x([\mathbf{a}',r-1]\otimes[0]))\\ & = & x([(\partial\mathbf{a}')]\otimes[0,b])-(-1)^{|\mathbf{a}'|}x([\mathbf{a}']\otimes[b])\\ & & +(-1)^{|\mathbf{a}'|}x([\mathbf{a}']\otimes[0])+x([(\partial\mathbf{a}'),r-1]\otimes[0])\\ & & +(-1)^{|\mathbf{a}'|+1}x([\mathbf{a}']\otimes[0]);\end{array}$$

so the two expressions coincide.

For $|\mathbf{a}'|=0$, on the one hand we have

$$\begin{array}{rcl}\widetilde{x}(\partial([a',r]\otimes[b])) & = & \widetilde{x}([r]\otimes[b]-[a']\otimes[b])\\ & = & x([r-1]\otimes[0])-x([a']\otimes[b]);\end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([a',r]\otimes[b]) & = & \partial(x([a']\otimes[0,b])+F([a',r-1]\otimes[0]))\\ & = & x([a']\otimes[0])-x([a']\otimes[b])\\ & & +x([r-1]\otimes[0])-x([a']\otimes[0]);\end{array}$$

so the two expressions coincide.

(P5) For $|\mathbf{a}|\geq 1,|\mathbf{a}'|\geq 1$, on the one hand we have

$$\begin{array}{rcl}\widetilde{x}(\partial([\mathbf{a}',r,\mathbf{a}]\otimes[b])) & = & \widetilde{x}([\partial\mathbf{a}',r,\mathbf{a}]\otimes[b])\\ & & +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}',\mathbf{a}]\otimes[b])\\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}',r,\partial\mathbf{a}]\otimes[b])\\ & = & x(s^{r-1}[\partial\mathbf{a}',r,\mathbf{a}]\otimes[0])\\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',\mathbf{a}]\otimes[b])\\ & & +(-1)^{|\mathbf{a}'|+2}x(s^{r-1}[\mathbf{a}',r,\partial\mathbf{a}]\otimes[0]);\end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([\mathbf{a}',r,\mathbf{a}]\otimes[b]) & = & \partial x(s^{r-1}[\mathbf{a}',r,\mathbf{a}]\otimes[0])\\ & = & x(s^{r-1}[\partial\mathbf{a}',r,\mathbf{a}]\otimes[0])\\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',\mathbf{a}]\otimes[0])\\ & & +(-1)^{|\mathbf{a}'|+2}x(s^{r-1}[\mathbf{a}',r,\partial\mathbf{a}]\otimes[0]).\end{array}$$

Observe that from (SuspInd 1), we obtain

$$\begin{array}{rcl} 0 & = & \partial x(s^{r-1}[\mathbf{a}',\mathbf{a}]\otimes[0,b]) \\ & = & x(s^{r-1}[\partial\mathbf{a}',\mathbf{a}]\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',\partial\mathbf{a}]\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}'|+|\mathbf{a}|}x(s^{r-1}[\mathbf{a}',\mathbf{a}]\otimes[0]) \\ & & -(-1)^{|\mathbf{a}'|+|\mathbf{a}|}x(s^{r-1}[\mathbf{a}',\mathbf{a}]\otimes[b]). \end{array}$$

The first two summands vanish by (SuspInd 1) again, yielding the equality of the other two summands. This shows the desired equality.

For $|\mathbf{a}|=0, |\mathbf{a}'|\geq 1$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([\mathbf{a}',r,a]\otimes[b])) & = & \widetilde{x}([\partial\mathbf{a}',r,a]\otimes[b]) \\ & & +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}',a]\otimes[b]) \\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}',r]\otimes[b]) \\ & = & x(s^{r-1}[\partial\mathbf{a}',r,a]\otimes[0]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',a]\otimes[b]) \\ & & +(-1)^{|\mathbf{a}'|+2}x([\mathbf{a}']\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}'|+2}x([\mathbf{a}',r-1]\otimes[0]); \end{array}$$

on the other hand we have

$$\begin{array}{rcl} \partial\widetilde{x}([\mathbf{a}',r,a]\otimes[b]) & = & \partial x(s^{r-1}[\mathbf{a}',r,a]\otimes[0]) \\ & = & x(s^{r-1}[\partial\mathbf{a}',r,a]\otimes[0]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',a]\otimes[0]) \\ & & +(-1)^{|\mathbf{a}'|+2}x(s^{r-1}[\mathbf{a}',r]\otimes[0]). \end{array}$$

Observe that from (SuspInd 1), we obtain

$$\begin{array}{rcl} 0 & = & \partial x(s^{r-1}[\mathbf{a}',a]\otimes[0,b]) \\ & = & x(s^{r-1}[\partial\mathbf{a}',a]\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}']\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',a]\otimes[0]) \\ & & -(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}',a]\otimes[b]). \end{array}$$

The first summand vanishes again by (SuspInd 1). This implies the desired equality.

For $|\mathbf{a}|\geq 1, |\mathbf{a}'|=0$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([a',r,\mathbf{a}]\otimes[b])) & = & \widetilde{x}([r,\mathbf{a}]\otimes[b])-\widetilde{x}([a',\mathbf{a}]\otimes[b]) \\ & & +\widetilde{x}([a',r,\partial\mathbf{a}]\otimes[b]) \\ & = & x(s^{r-1}[r,\mathbf{a}]\otimes[0]) \\ & & -x(s^{r-1}[a',\mathbf{a}]\otimes[b]) \\ & & +x(s^{r-1}[a',r,\partial\mathbf{a}]\otimes[0]); \end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([a',r,\mathbf{a}]\otimes[b]) & = & \partial x(s^{r-1}[a',r,\mathbf{a}]\otimes[0]) \\ & = & x(s^{r-1}[r,\mathbf{a}]\otimes[0]) - x(s^{r-1}[a',\mathbf{a}]\otimes[0]) \\ & & +x(s^{r-1}[a',r,\partial\mathbf{a}]\otimes[0]).\end{array}$$

Observe that from (SuspInd 1), we obtain

$$\begin{array}{rcl}0 & = & \partial x(s^{r-1}[a',\mathbf{a}]\otimes[0,b]) \\ & = & x(s^{r-1}[\mathbf{a}]\otimes[0,b]) \\ & & -x(s^{r-1}[a',\partial\mathbf{a}]\otimes[0,b]) \\ & & +(-1)^{|\mathbf{a}|+1}x(s^{r-1}[a',\mathbf{a}]\otimes[0]) \\ & & -(-1)^{|\mathbf{a}|+1}x(s^{r-1}[a',\mathbf{a}]\otimes[b]).\end{array}$$

The first summand vanishes again by (SuspInd 2) and the second by (SuspInd 1), so that the other two summands are equal. This implies the desired equality.

For $|\mathbf{a}|=|\mathbf{a}'|=0$, on the one hand we have

$$\begin{array}{rcl}\widetilde{x}(\partial([a',r,a]\otimes[b])) & = & \widetilde{x}([r,a]\otimes[b]) \\ & & -\widetilde{x}([a',a]\otimes[b]) \\ & & +\widetilde{x}([a',r]\otimes[b]) \\ & = & x([s^{r-1}(r,a)]\otimes[0]) \\ & & -x(s^{r-1}[a',a]\otimes[b]) \\ & & +x(s^{r-1}[a',r]\otimes[0]) \\ & & +x([a']\otimes[0,b]);\end{array}$$

on the other hand we have

$$\begin{array}{rcl}\partial\widetilde{x}([a',r,a]\otimes[b]) & = & \partial x(s^{r-1}[a',r,a]\otimes[0]) \\ & = & x(s^{r-1}[r,a]\otimes[0]) \\ & & -x(s^{r-1}[a',a]\otimes[0]) \\ & & +x(s^{r-1}[a',r]\otimes[0]).\end{array}$$

Using (SuspInd 1), we obtain

$$\begin{array}{rcl}0 & = & \partial x([a',a-1]\otimes[0,b]) \\ & = & x([a-1]\otimes[0,b]) \\ & & -x([a']\otimes[0,b]) \\ & & +x([a',a-1]\otimes[b]) - x([a',a-1]\otimes[0]).\end{array}$$

Now the first summand vanishes by (SuspInd 2). This yields the desired equality.

(P6) For $|\mathbf{a}| \geq 1, |\mathbf{a}'| \geq 0$ and $|\mathbf{b}| \geq 2$, on the one hand we have

$$\begin{aligned}
\widetilde{x}(\partial([\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}])) &= \widetilde{x}([\partial\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}', \mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r, \partial\mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad +(-1)^{|\mathbf{a}'|+2+|\mathbf{a}|}\widetilde{x}([\mathbf{a}', r, \mathbf{a}] \otimes \partial^\circ[\mathbf{b}]) \\
&= (-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [\mathbf{b}]);
\end{aligned}$$

on the other hand we have

$$\partial\widetilde{x}([\mathbf{a}', r, \mathbf{a}] \otimes [\mathbf{b}]) = 0.$$

Observe that from (SuspInd 1), we obtain

$$\begin{aligned}
0 &= \partial x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) \\
&= x(s^{r-1}[\partial\mathbf{a}', \mathbf{a}] \otimes [0, \mathbf{b}]) \\
&\quad +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', \partial\mathbf{a}] \otimes [0, \mathbf{b}]) \\
&\quad -(-1)^{|\mathbf{a}'|+|\mathbf{a}|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad +(-1)^{|\mathbf{a}'|+|\mathbf{a}|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [0, \partial^\circ\mathbf{b}]).
\end{aligned}$$

The first two summands as well as the last one vanish again by (SuspInd 1), so the third summand also needs to vanish. This yields the desired equality.

For $|\mathbf{a}| \geq 1, |\mathbf{a}'| \geq 0$ and $|\mathbf{b}| = 1$, on the one hand

$$\begin{aligned}
\widetilde{x}\partial([\mathbf{a}', r, \mathbf{a}] \otimes [b_0, b_1]) &= \widetilde{x}([\partial\mathbf{a}', r, \mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}', \mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r, \partial\mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+2+|\mathbf{a}|}\widetilde{x}([\mathbf{a}', r, \mathbf{a}] \otimes [b_0 - b_1]) \\
&= (-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +x(s^{r-1}[\mathbf{a}', r, \mathbf{a}] \otimes [0]) \\
&\quad -x(s^{r-1}[\mathbf{a}', r, \mathbf{a}] \otimes [0]);
\end{aligned}$$

on the other hand, we have

$$\partial\widetilde{x}([\mathbf{a}', r, \mathbf{a}] \otimes [b_0, b_1]) = 0.$$

Observe that from (SuspInd 1), we obtain

$$\begin{aligned}
0 &= \partial x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [0, b_0, b_1]) \\
&= x(s^{r-1}[\partial\mathbf{a}', \mathbf{a}] \otimes [0, b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', \partial\mathbf{a}] \otimes [0, b_0, b_1]) \\
&\quad -(-1)^{|\mathbf{a}'|+|\mathbf{a}|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+|\mathbf{a}|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [0, b_0]) \\
&\quad -(-1)^{|\mathbf{a}'|+|\mathbf{a}|+1}x(s^{r-1}[\mathbf{a}', \mathbf{a}] \otimes [0, b_1]).
\end{aligned}$$

The first two summands as well as the last two vanish again by (SuspInd 1), so the third summand also needs to vanish. This yields the desired equality.

For $|\mathbf{a}| = 0, |\mathbf{a}'| \geq 0$ and $|\mathbf{b}| \geq 2$; on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([\mathbf{a}', r, a] \otimes [\mathbf{b}])) & = & \widetilde{x}([\partial\mathbf{a}', r, a] \otimes [\mathbf{b}]) \\ & & +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}', a] \otimes [\mathbf{b}]) \\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r] \otimes [\mathbf{b}]) \\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r, a] \otimes \partial^{\circ}[\mathbf{b}]) \\ & = & (-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', a] \otimes [\mathbf{b}]) \\ & & +(-1)^{|\mathbf{a}'|+2}x([\mathbf{a}'] \otimes [0, \mathbf{b}]); \end{array}$$

on the other hand we have

$$\partial\widetilde{x}([\mathbf{a}', r, a] \otimes [\mathbf{b}]) \quad = \quad 0.$$

Observe that by (SuspInd 1), we have

$$\begin{array}{rcl} 0 & = & \partial x(s^{r-1}[\mathbf{a}', a] \otimes [0, \mathbf{b}]) \\ & = & x(s^{r-1}[\partial\mathbf{a}', a] \otimes [0, \mathbf{b}]) \\ & & +(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}'] \otimes [0, \mathbf{b}]) \\ & & -(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', a] \otimes [\mathbf{b}]) \\ & & -(-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', a] \otimes [0, \partial^{\circ}\mathbf{b}]). \end{array}$$

The first and the last summands vanish using (SuspInd 1) once again. This yields the desired equality.

For $|\mathbf{a}| = 0, |\mathbf{a}'| \geq 0$, $|\mathbf{b}| = 1$, on the one hand we have

$$\begin{array}{rcl} \widetilde{x}(\partial([\mathbf{a}', r, a] \otimes [b_0, b_1])) & = & \widetilde{x}([\partial\mathbf{a}', r, a] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+1}\widetilde{x}([\mathbf{a}', a] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+2}\widetilde{x}([\mathbf{a}', r, a] \otimes [b_0 - b_1]) \\ & = & (-1)^{|\mathbf{a}'|+1}x(s^{r-1}[\mathbf{a}', a] \otimes [b_0, b_1]) \\ & & +(-1)^{|\mathbf{a}'|+2}x([\mathbf{a}'] \otimes [0, b_0, b_1]) \\ & & +x(s^{r-1}[\mathbf{a}', r, a] \otimes [0]) \\ & & -x(s^{r-1}[\mathbf{a}', r, a] \otimes [0]); \end{array}$$

on the other hand we have

$$\partial\widetilde{x}([\mathbf{a}', r, a] \otimes [b_0, b_1]) \quad = \quad 0.$$

Observe that by (SuspInd 1), we have

$$\begin{aligned}
0 &= \partial x(s^{r-1}[\mathbf{a}', a] \otimes [0, b_0, b_1]) \\
&= x(s^{r-1}[\partial \mathbf{a}', a] \otimes [0, b_0, b_1]) \\
&\quad +(-1)^{|\mathbf{a}'|+1} x(s^{r-1}[\mathbf{a}'] \otimes [0, b_0, b_1]) \\
&\quad -(-1)^{|\mathbf{a}'|+1} x(s^{r-1}[\mathbf{a}', a] \otimes [b_0, b_1]) \\
&\quad -(-1)^{|\mathbf{a}'|+1} x(s^{r-1}[\mathbf{a}', a] \otimes [0, b_0]) \\
&\quad +(-1)^{|\mathbf{a}'|+1} x(s^{r-1}[\mathbf{a}', a] \otimes [0, b_1]).
\end{aligned}$$

The first and the last two summands vanish using (SuspInd 1) once again. This yields the desired equality.

For $|\mathbf{a}| \geq 1, |\mathbf{a}'| = -1$, $|\mathbf{b}| \geq 2$, on the one hand we have

$$\begin{aligned}
\widetilde{x}(\partial([r, \mathbf{a}] \otimes [\mathbf{b}])) &= \widetilde{x}([\mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad -\widetilde{x}([r, \partial \mathbf{a}] \otimes [\mathbf{b}]) \\
&\quad +(-1)^{1+|\mathbf{a}|} \widetilde{x}([r, \mathbf{a}] \otimes \partial^\circ [\mathbf{b}]) \\
&= x(s^{r-1}[\mathbf{a}] \otimes [\mathbf{b}]);
\end{aligned}$$

on the other hand we have

$$\partial \widetilde{x}([r, \mathbf{a}] \otimes [\mathbf{b}]) = 0.$$

Observe that by (SuspInd 2), the two results coincide.

For $|\mathbf{a}| \geq 1, |\mathbf{a}'| = -1$, $|\mathbf{b}| = 1$, on the one hand we have

$$\begin{aligned}
\widetilde{x}(\partial([r, \mathbf{a}] \otimes [b_0, b_1])) &= \widetilde{x}([\mathbf{a}] \otimes [b_0, b_1]) \\
&\quad -\widetilde{x}([r, \partial \mathbf{a}] \otimes [b_0, b_1]) \\
&\quad +(-1)^{1+|\mathbf{a}|} \widetilde{x}(([r, \mathbf{a}] \otimes [b_0])) \\
&\quad -(-1)^{1+|\mathbf{a}|} \widetilde{x}(\partial([r, \mathbf{a}] \otimes [b_1])) \\
&= x([s^{r-1}(\mathbf{a})] \otimes [b_0, b_1]) \\
&\quad +(-1)^{1+|\mathbf{a}|} x(s^{r-1}[r, \mathbf{a}] \otimes [0]) \\
&\quad -(-1)^{1+|\mathbf{a}|} x(s^{r-1}[r, \mathbf{a}] \otimes [0]);
\end{aligned}$$

on the other hand we have

$$\partial \widetilde{x}([r, \mathbf{a}] \otimes [b_0, b_1]) = 0.$$

Then (SuspInd 2) yields the desired equality. For $|\mathbf{a}| = 0$, $|\mathbf{a}'| = -1$, $|\mathbf{b}| \geq 2$, on the one hand we have

$$\begin{aligned}
\widetilde{x}(\partial([r, a] \otimes [\mathbf{b}])) &= \widetilde{x}([a] \otimes [\mathbf{b}]) - \widetilde{x}([r] \otimes [\mathbf{b}]) \\
&\quad +(-1)^1 \widetilde{x}([r, a] \otimes \partial^\circ [\mathbf{b}]) \\
&= x(s^{r-1}[a] \otimes [\mathbf{b}]);
\end{aligned}$$

on the other hand we have

$$\partial\widetilde{x}([r,a]\otimes[\mathbf{b}]) \quad = \quad 0.$$

Using (SuspInd 2) once again, we obtain the desired equality.

For $|\mathbf{a}|=0$, $|\mathbf{a}'|=-1$, $|\mathbf{b}|=1$; on the one hand

$$\begin{aligned}\widetilde{x}(\partial([r,a]\otimes[b_0,b_1])) &= \widetilde{x}([a]\otimes[b_0,b_1])-\widetilde{x}([r]\otimes[b_0,b_1])\\ &\quad +\widetilde{x}([r,a]\otimes[b_1])-\widetilde{x}([r,a]\otimes[b_0])\\ &= x([s^{r-1}(a)]\otimes[b_0,b_1])\\ &\quad +x(s^{r-1}[r,a]\otimes[0])\\ &\quad -x(s^{r-1}[r,a]\otimes[0]);\end{aligned}$$

on the other hand we have

$$\partial\widetilde{x}([r,a]\otimes[b_0,b_1]) \quad = \quad 0.$$

Using (SuspInd 2) once again yields the desired equality.

(P7) In this case, all constituents can be seen to be 0 in a straightforward manner.

Since we treated all possible cases, we conclude that $\widetilde{x}$ is indeed a chain map. □

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