№1 $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = (a_1 | a_2)$ $\quad a_1 \perp a_2$;

projector on range A: $P = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

(full)

$\underset{m\times n}{A} = \underset{m\times m}{Q} \underset{m\times n}{R}$ $\quad$ R - upper-triangular

Q - unitary

$Q = (q_1 | q_2 | q_3)$, (q_1, q_2, q_3) - orthonormal basis in $\mathbb{R}^3$

$A = (a_1 | a_2)$ $\quad a_1 \perp a_2 \Rightarrow$ $\boxed{q_1 = \frac{a_1}{\sqrt{a_1^* a_1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$ $\boxed{q_2 = \frac{a_2}{\sqrt{a_2^* a_2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

$q_3 = q_1 \times q_2 = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} = \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = q_3}$

$\Rightarrow \boxed{A = QR = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$

$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$;

$P = B(B^* B)^{-1} B^* = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5/6 & -2/6 \\ -2/6 & 2/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \boxed{\frac{1}{6} \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{pmatrix} = P}$

(reduced)

$B = \underset{m\times n}{\hat{Q}} \underset{n\times n}{\hat{R}}$

$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\qquad a_2 = r_{12} \cdot q_1 + r_{22} \cdot q_2$ $\qquad q_2 = \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}{r_{22}} = \frac{1}{r_{22}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$(a_2, q_1) = r_{12} = \frac{1}{\sqrt{2}} \cdot 2 = \sqrt{2}$

$(a_2, q_2) = r_{22} = \frac{1}{r_{22}}(2+1) = \frac{3}{r_{22}} \Rightarrow r_{22} = \sqrt{3}$

$\Rightarrow \boxed{B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix} = QR}$

№5 minimize $\|Ax-b\|_2$ subject to $Cx=0$

A^TA - invertible

□ $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $C \in \mathbb{R}^{p\times n}$;

$$f(x) = \frac{1}{2}\|Ax-b\|_2^2 = \frac{1}{2}(Ax-b)^T(Ax-b) = \frac{1}{2}(x^TA^T-b^T)(Ax-b) = \frac{1}{2}(x^TA^TAx - x^TA^Tb - b^TAx + b^Tb) =$$

$$= \frac{1}{2}(x^TA^TAx - 2x^TA^Tb + b^Tb)$$

minimize $\|Ax-b\|_2$ subject to $Cx=0$ $\iff$ minimize $\frac{1}{2}\|Ax-b\|_2^2$ subject to $Cx=0$

$$\mathcal{L}(x) = f(x) + \lambda^TCx, \quad \lambda = \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_p\end{pmatrix}$$

$$\mathcal{L}(x,\lambda) = \frac{x^TA^TAx}{2} - x^TA^Tb + \frac{b^Tb}{2} + \lambda^TCx$$

$$\begin{cases}\nabla_x \mathcal{L} = A^TAx - A^Tb + C^T\lambda = 0\\ \nabla_\lambda \mathcal{L} = Cx = 0\end{cases} \iff \underset{(n+p)\times(n+p)}{\begin{pmatrix}A^TA & C^T\\ C & 0\end{pmatrix}}\underset{(n+p)\times 1}{\begin{pmatrix}x\\ \lambda\end{pmatrix}} = \underset{(n+p)\times 1}{\begin{pmatrix}A^Tb\\ 0\end{pmatrix}} \quad (*)$$

if $\begin{pmatrix}A^TA & C^T\\ C & 0\end{pmatrix}$ is invertible then $(*)$ has unique solution.

$$\boxed{\begin{pmatrix}x\\ \lambda\end{pmatrix} = \begin{pmatrix}A^TA & C^T\\ C & 0\end{pmatrix}^{-1}\begin{pmatrix}A^Tb\\ 0\end{pmatrix}}$$

$p<n$

$\operatorname{rank}(C) = p$, $\operatorname{rank}(A^TA) = n$ $\Rightarrow \operatorname{rank}\begin{pmatrix}A^TA\\ C\end{pmatrix} = n$

$\operatorname{rank}\begin{pmatrix}C^T\\ 0\end{pmatrix} = p \Rightarrow \operatorname{rank}\begin{pmatrix}A^TA & C^T\\ C & 0\end{pmatrix} = n+p \Rightarrow$

$\Rightarrow \exists \begin{pmatrix}A^TA & C^T\\ C & 0\end{pmatrix}^{-1}$

$$\begin{pmatrix}\underset{n\times n}{A^TA} & \underset{n\times p}{C^T}\\ \underset{p\times n}{C} & \underset{p\times p}{0}\end{pmatrix}\begin{pmatrix}\alpha & \gamma\\ \beta & \Omega\end{pmatrix} = \begin{pmatrix}E_n & 0\\ 0 & E_p\end{pmatrix}$$

$$\underset{n\times n}{A^TA}\cdot\underset{n\times n}{\alpha} + \underset{n\times p}{C^T}\underset{p\times n}{\beta} = E_n$$

$$\underset{n\times n}{A^TA}\cdot\underset{n\times p}{\gamma} + \underset{n\times p}{C^T}\cdot\underset{p\times p}{\Omega} = \underset{n\times p}{0}$$

$$\underset{p\times n}{C}\cdot\underset{n\times n}{\alpha} + \underset{p\times p}{0}\cdot\underset{p\times n}{\beta} = \underset{p\times n}{0}$$

$$\underset{p\times n}{C}\cdot\underset{n\times p}{\gamma} + \underset{p\times p}{0}\cdot\underset{p\times p}{\Omega} = E_p$$

A^TA - invertible

$C(A^TA)^{-1}C^T$ - invertible

$$\alpha + (A^TA)^{-1}C^T\beta = (A^TA)^{-1}$$

$$\alpha = (A^TA)^{-1} - (A^TA)^{-1}C^T\beta$$

$$C(A^TA)^{-1} - C(A^TA)^{-1}C^T\beta = 0$$

$$\boxed{\beta = [C(A^TA)^{-1}C^T]^{-1}C(A^TA)^{-1}}$$

$$\boxed{\alpha = (A^TA)^{-1} - (A^TA)^{-1}C^T[C(A^TA)^{-1}C^T]^{-1}C(A^TA)^{-1}}$$

$$\gamma + (A^TA)^{-1}C^T\Omega = 0$$

$$-C(A^TA)^{-1}C^T\Omega = E_p \Rightarrow \boxed{\Omega = -\left[C(A^TA)^{-1}C^T\right]^{-1}}$$

$$\boxed{\gamma = (A^TA)^{-1}C^T\left[C(A^TA)^{-1}C^T\right]^{-1}}$$

$$\begin{pmatrix} (A^TA)^{-1} & C^T \\ C & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \gamma \\ \beta & \Omega \end{pmatrix} \qquad \begin{pmatrix} \underset{n\times n}{\alpha} & \underset{n\times p}{\gamma} \\ \underset{p\times n}{\beta} & \underset{p\times p}{\Omega} \end{pmatrix}\begin{pmatrix} \underset{n\times 1}{A^Tb} \\ \underset{p\times 1}{0} \end{pmatrix} = \begin{pmatrix} \underset{n\times 1}{\alpha A^Tb} \\ \underset{p\times 1}{\beta A^Tb} \end{pmatrix} \Rightarrow X = \alpha A^Tb$$

$$\boxed{X = \left[(A^TA)^{-1} - (A^TA)^{-1}C^T\left[C(A^TA)^{-1}C^T\right]^{-1}C(A^TA)^{-1}\right]A^Tb}$$