$\sqrt{i}$ 1 $H(v) = 1 - 2vv^*$, v - unit column vector.

rank(H) - ? prove unitarity.

$\square$ $H(v)$ - unitary $\Leftrightarrow$ $H^{-1} = H^*$ i.e. $H^*H = I$

$$H^*H = (I - 2vv^*)^*(I - 2vv^*) = (I - 2vv^*)(I - 2vv^*) =$$

$$= I - 2vv^* - 2vv^* + 4v\underbrace{v^*v}_{=1}v^* =$$

$$= I - 4vv^* + 4vv^* = I \quad \text{Q.E.D}$$

since $H(v)$ is unitary then rank $H(v) = m$, $v \in \mathbb{C}^m$

$H(v) \in \mathbb{C}^{m \times m}$ $\blacksquare$

$\sqrt{i}$ 2

1) $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$, $x \in \mathbb{C}^m$

$$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \sqrt{x^*x}, \quad \|x\|_\infty = \max_{1 \le i \le m} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \le \left(m \cdot \max_{i=\overline{1,m}} |x_i|^2\right)^{1/2} = \sqrt{m}\,\|x\|_\infty \quad \text{Q.E.D.}$$

if $\forall i \to x_i = a$, $a \in \mathbb{R}$ then $\|x\|_2 = (m a^2)^{1/2} = \sqrt{m}\,\|x\|_\infty$

2) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$, $A \in \mathbb{C}^{m\times n}$

$\|A\|_\infty = \max\limits_{i=\overline{1,m}} \|a_i^*\|_1$ ("max row sum"; a_i - i_{th} row)

$$\|A\|_2 = \sup_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}} \|Ax\|_2 = \sup_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}} \left(\sum_{i=1}^m |(Ax)_i|^2\right)^{1/2} = \sup_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}} \left(\sum_{i=1}^m |a_i x|^2\right) =$$

$$= \sup_{\substack{x\in\mathbb{C}^n\\ \|x\|_2=1}} \left(\sum_{i=1}^m |x^* a_i^*|^2\right)^{1/2} \geq \left(\sum_{i=1}^m \frac{1}{n}\|a_i^*\|_1^2\right)^{1/2}$$

$x^* = \underbrace{(1/\sqrt{n} \;\ldots\; 1/\sqrt{n})}_{1\times n}$, $\|x^*\|_2 = 1$

i.e. $$\sqrt{n}\,\|A\|_2 \geq \left(\sum_{i=1}^m \|a_i^*\|_1^2\right)^{1/2} = \left(\|a_1^*\|_1^2 + \ldots + \|a_m^*\|_1^2\right)^{1/2} \geq$$

$$\geq \max_{i=\overline{1,m}} \|a_i^*\|_1 = \|A\|_\infty \quad \text{Q.E.D.}$$

$\int_3^{-0}$ u, v – m-vectors

$A = I + uv^*$, $\operatorname{rank}(uv^*) = 1$

Can A be singular?

Assuming it is not compute A^{-1}.

□ $A^{-1} = I + \alpha uv^*$

$A \cdot A^{-1} = I \Leftrightarrow (I + uv^*)(I + \alpha uv^*) = I + \alpha uv^* + uv^* +$

$+ \alpha uv^* uv^* = I$

$\alpha uv^* + uv^* + \alpha u \underbrace{v^* u}_{=\beta\in\mathbb{R}} v^* = 0$

Then:

$$\alpha uv^* + uv^* + \alpha\beta uv^* = 0$$

$$\alpha uv^*(I + \beta I) + uv^* = 0$$

$$\alpha \cdot (I + \beta I) = -I$$

$$\alpha I(1+\beta) = -I \Rightarrow \alpha = -\frac{1}{1+\beta} = -\frac{1}{1+v^*u}$$

then $A^{-1} = I - \frac{1}{1+v^*u} uv^*$

$A = uv^* - (-I) \Rightarrow A$ - singular $\Leftrightarrow \det A = 0 \Leftrightarrow$

$\Leftrightarrow \det(uv^* - (-I)) = 0 \Rightarrow$

$\Rightarrow$ A-singular $\Leftrightarrow \lambda = -1$ - eigenvalue of uv^*

Дальше не знаю как

$\sqrt{-4}$ Prove: U - unitary $\Rightarrow \|UA\|_F = \|AU\|_F = \|A\|_F$

$\square$

U - unitary $\Rightarrow U^{-1} = U^*$

$$\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$$

$$\|UA\|_F = \sqrt{tr(A^*\underbrace{U^*U}_{I}A)} = \sqrt{tr(A^*A)} = \|A\|_F$$

$$\|AU\|_F = \sqrt{tr(A^*\underbrace{U^*U}_{I}A)} = \|A\|_F$$

$\blacksquare$