$\sqrt{3}$ $A=\begin{pmatrix}-2 & 11\\ -10 & 5\end{pmatrix}$ rg $A=2 \Rightarrow 2$ - the number of nonzero singular values;

$$AA^*=\begin{pmatrix}-2 & 11\\ -10 & 5\end{pmatrix}\begin{pmatrix}-2 & -10\\ 11 & 5\end{pmatrix}=\begin{pmatrix}4+121 & 20+55\\ 20+55 & 100+25\end{pmatrix}=\begin{pmatrix}125 & 75\\ 75 & 125\end{pmatrix}$$

$$\det(AA^*-\lambda I)=(125-\lambda)^2-75^2=0 \Rightarrow (125-\lambda)^2=75^2$$

$$\Rightarrow 125-\lambda=\pm 75$$

$$\lambda=50,200; \Rightarrow \boxed{\begin{matrix}\sigma_1=\sqrt{200}=10\sqrt{2}\\ \sigma_2=\sqrt{50}=5\sqrt{2}\end{matrix}}$$

$$\Sigma=\begin{pmatrix}10\sqrt{2} & 0\\ 0 & 5\sqrt{2}\end{pmatrix}=\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}$$

$$A=U\Sigma V^* \Rightarrow AA^*=U\Sigma \overbrace{V^*V}^{I}\Sigma^* U^*=U\Sigma\Sigma^* U^*$$

$$AA^*=U\Sigma\Sigma^*U^*=(u_1\ u_2)\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=(\sigma_1^2u_1\ \ \sigma_2^2u_2)\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

$$\Longleftrightarrow AA^*(u_1\ u_2)=(\sigma_1^2u_1\ \ \sigma_2^2u_2) \Rightarrow AA^*u_1=\sigma_1^2u_1 \Rightarrow \begin{pmatrix}-75 & 75\\ 75 & -75\end{pmatrix}u_1=0 \Rightarrow -u_1^1+u_1^2=0 \Rightarrow u_1^1=u_1^2 \Rightarrow \boxed{u_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}}$$

$$AA^*u_2=\sigma_2^2u_2 \Rightarrow \begin{pmatrix}75 & 75\\ 75 & 75\end{pmatrix}u_2=0 \Rightarrow -u_2^1=u_2^2 \Rightarrow \boxed{u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}}$$

$$Av_1=\sigma_1u_1 \Rightarrow \left(\begin{array}{cc|c}-2 & 11 & 10\\ -10 & 5 & 10\end{array}\right)\sim\left(\begin{array}{cc|c}-2 & 11 & 10\\ 0 & 1 & \frac{4}{5}\end{array}\right)\sim\left(\begin{array}{cc|c}1 & 0 & \left(10-\frac{4\cdot 11}{5}\right)\left(-\frac{1}{2}\right)\\ 0 & 1 & \frac{4}{5}\end{array}\right)\Rightarrow \boxed{v_1=\begin{pmatrix}-3/5\\ 4/5\end{pmatrix}}$$

$$Av_2=\sigma_2u_2 \Rightarrow \left(\begin{array}{cc|c}-2 & 11 & 5\\ -10 & 5 & -5\end{array}\right)\sim\left(\begin{array}{cc|c}-2 & 11 & 5\\ 0 & 1 & \frac{3}{5}\end{array}\right)\sim\left(\begin{array}{cc|c}1 & 0 & \left(-5+\frac{33}{5}\right)\frac{1}{2}\\ 0 & 1 & \frac{3}{5}\end{array}\right)\sim\left(\begin{array}{cc|c}1 & 0 & 4/5\\ 0 & 1 & 3/5\end{array}\right)$$

$$\Rightarrow \boxed{v_2=\begin{pmatrix}4/5\\ 3/5\end{pmatrix}}$$

$$\Rightarrow \boxed{A=U\Sigma V^*=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}10\sqrt{2} & 0\\ 0 & 5\sqrt{2}\end{pmatrix}\begin{pmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{pmatrix}}$$

$$\|A\|_2=10\sqrt{2} \qquad \|A\|_F=\sqrt{\sigma_1^2+\sigma_2^2}=5\sqrt{10}$$

$$A^{-1}=V\Sigma^{-1}U^T=\begin{pmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}1/10\sqrt{2} & 0\\ 0 & 1/5\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=\boxed{\begin{pmatrix}1/20 & -11/100\\ 1/10 & -1/50\end{pmatrix}=A^{-1}}$$

$\sqrt{2}$ $\underset{m\times n}{A} = \overset{m\times m}{U}\underset{m\times n}{\Sigma}\underset{n\times n}{V^T}$; U, V - unitary

1) $\underset{n\times n}{(A^TA)^{-1}} = (V\Sigma^T \underbrace{U^TU}_{I}\Sigma V^T)^{-1} = (V\Sigma^T\Sigma V^T)^{-1} = V(\Sigma^T\Sigma)^{-1}V^T$

2) $\underset{n\times m}{(A^TA)^{-1}A^T} = V(\Sigma^T\Sigma)^{-1}\underbrace{V^TV}_{I}\Sigma^TU^T = V\left[(\Sigma^T\Sigma)^{-1}\Sigma^T\right]U^T$

3) $\underset{m\times n}{A(A^TA)^{-1}} = U\Sigma \underbrace{V^TV}_{I}(\Sigma^T\Sigma)^{-1}V^T = U\left[\Sigma(\Sigma^T\Sigma)^{-1}\right]V^T$

4) $\underset{m\times m}{A(A^TA)^{-1}A^T} = U\left[\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T\right]U^T$