Continuity estimates for Riesz potentials on polygonal boundaries

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Abstract. Riesz potentials are well known objects of study in the theory of singular integrals that have been the subject of recent, increased interest from the numerical analysis community due to their connections with fractional Laplace problems and proposed use in certain domain decomposition methods. While the L^p-mapping properties of Riesz potentials on flat geometries are well-established, comparable results on rougher geometries for Sobolev spaces are very scarce. In this article, we study the continuity properties of the surface Riesz potential generated by the $1/\sqrt{x}$ singular kernel on a polygonal domain $\Omega \subset \mathbb{R}^2$. We prove that this surface Riesz potential maps $L^2(\partial\Omega)$ into $H^{+1/2}(\partial\Omega)$. Our proof is based on a careful analysis of the Riesz potential in the neighbourhood of corners of the domain Ω. The main tool we use for this corner analysis is the Mellin transform which can be seen as a counterpart of the Fourier transform that is adapted to corner geometries.

1. Introduction

Although Riesz potentials and fractional integrals are classical objects of study in harmonic analysis, they have been the subject of recent attention from the perspective of numerical analysis because of a growing interest in the solution to fractional Laplace problems. Riesz potentials have been analysed for a long time in flat or smooth geometries [21] where, if needed, micro-local analysis offers powerful tools to study their properties in detail (see e.g. [12]). In rougher geometries, where Fourier calculus is no longer available, more sophisticated approaches such as Calderon-Zygmund theory (see, e.g., [18, 22]) are required, and in this context most of the results offered by the literature on singular integrals are formulated in the functional framework of either L^p spaces or Besov spaces.

On the other hand, Sobolev spaces appear as the functional setting of reference in numerical analysis because the variational theory of Galerkin discretisations classically relies on regularity properties in these spaces. This state of affairs makes the analysis of singular integrals more delicate when they arise in the context of PDE discretisations by, for instance, finite element schemes. Sobolev regularity results on general Lipschitz manifolds have been established by Costabel [5] for the classical single layer, double layer and hypersingular boundary integral operators but, to the best of our knowledge, comparable results are still not available for Riesz potentials. Having said this, certain natural variational mapping properties for Riesz potentials on Lipschitz surfaces have been derived in [10, 11], and there is admittedly an active literature on the numerical solution to fractional Laplace problems (see, e.g., [1, 15] and the references therein) but these are considered on flat spaces (with a potentially Lipschitz or polyhedral/polygonal boundary).

Recent works on domain decomposition for wave propagation problems by means of Optimized Schwarz methods [3, 4, 14] have also made use of Riesz potentials. In this approach, the computational domain is split according to a non-overlapping subdomain partition and local wave equations with Robin-type boundary condition are solved in each subdomain with the coupling between subdomains being enforced by exchanging Robin traces of the form $\partial_{\boldsymbol{n}} u|_\Gamma \pm iT(u|_\Gamma)$ across interfaces Γ. The precise choice of the impedance factor T that comes into play in these Robin traces plays a crucial role in the convergence properties of these domain decomposition algorithms. In the strategy proposed in [3, 4, 14], the impedance factor takes the form $T = \Lambda^*\Lambda$ where Λ is a Riesz potential supported on polygonal interfaces Γ. The mapping properties of Λ then appear as a cornerstone of the convergence analysis of this algorithm.

The above domain decomposition context is the primary motivation for the present work where we study the surface Riesz potential associated with the singularity $1/\sqrt{x}$. More precisely, we study, in the case of a polygonal domain $\Omega \subset \mathbb{R}^2$, the surface Riesz potential given by

$$\mathscr{A}(u)(\boldsymbol{x}) = \int_{\partial\Omega} \frac{u(\boldsymbol{y})d\sigma(\boldsymbol{y})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \quad \boldsymbol{x} \in \partial\Omega.$$

It is already known from [10, 11] that $\mathscr{A}$ maps $\mathrm{H}^{-1/4}(\partial\Omega)$ into $\mathrm{H}^{+1/4}(\partial\Omega)$. In the present article, we prove that $\mathscr{A}$ maps $\mathrm{L}^2(\partial\Omega)$ into $\mathrm{H}^{+1/2}(\partial\Omega)$ (see Theorem 3.1 below).

The remainder of this article is organised as follows. In Section 2 we establish some notation and state precisely the definitions of trace Sobolev spaces that we require for our analysis. Next, in Section 3, we properly define the Riesz potential on polygonal boundaries and show that the analysis of the mapping properties of this operator reduces to a thorough study in the neighbourhood of corners. Subsequently, in Section 4 we provide a brief recap of the Mellin transform which can be seen as a counterpart of the Fourier transform that is adapted to corner geometries, and we recall how to characterize Sobolev trace spaces by means of the Mellin transform. In Section 5 we perform a detailed study of the Mellin symbols of the Riesz potential following which we deduce, in Section 6, the required continuity estimates for the Riesz potential on boundaries containing corners.

2. Trace spaces

We start with classical considerations related to the functional analysis of trace spaces. For any open, connected set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, and any closed subset $\Gamma \subset \partial\Omega$, we shall consider the space $\mathrm{H}^{1/2}(\Gamma)$ as the completion of the space $\mathscr{C}^\infty(\Gamma) := \{\varphi|_\Gamma, \ \varphi \in \mathscr{C}^\infty(\mathbb{R}^2)\}$ for the norm given by $\|\varphi\|^2_{\mathrm{H}^{1/2}(\Gamma)} := \|\varphi\|^2_{\mathrm{L}^2(\Gamma)} + |\varphi|^2_{\mathrm{H}^{1/2}(\Gamma)}$ where we use the so-called Sobolev-Slobodeckii semi-norm [17, Chap.3] given by the formula

$$|\varphi|^2_{\mathrm{H}^{1/2}(\Gamma)} = \int_{\Gamma\times\Gamma} \frac{|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^2} \, d\boldsymbol{x}d\boldsymbol{y}.$$

We shall also consider the space $\tilde{\mathrm{H}}^{1/2}(\Gamma) := \{v \in \mathrm{H}^{1/2}(\partial\Omega), \ v = 0 \text{ on } \partial\Omega \setminus \Gamma\}$. We emphasize that $\tilde{\mathrm{H}}^{1/2}(\Gamma)$ is a closed subspace of $\mathrm{H}^{1/2}(\partial\Omega)$ under the norm $\|\cdot\|_{\mathrm{H}^{1/2}(\partial\Omega)}$. On the other hand the set $\{\varphi|_\Gamma, \varphi \in \tilde{\mathrm{H}}^{1/2}(\Gamma)\}$ is a subspace of $\mathrm{H}^{1/2}(\Gamma)$ which is *not* closed with respect to the norm $\|\cdot\|_{\mathrm{H}^{1/2}(\Gamma)}$. Of particular interest to us in the sequel will be the case where $\Gamma = \mathbb{R}$ or $\Gamma = \mathbb{R}_+ := [0, +\infty)$. For any $u \in \tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)$, denoting $v := u|_{\mathbb{R}_+}$, a straightforward calculus yields

$$|u|^2_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)} := |u|^2_{\mathrm{H}^{1/2}(\mathbb{R})} = |v|^2_{\mathrm{H}^{1/2}(\mathbb{R}_+)} + 2\int_0^\infty |v(x)|^2 \, \frac{dx}{x},$$

$$\|u\|^2_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)} := |u|^2_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)} + \|u\|^2_{\mathrm{L}^2(\mathbb{R}_+)}. \tag{1}$$

Clearly, we have $\|u\|_{\mathrm{H}^{1/2}(\mathbb{R}_+)} \leq \|u\|_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)}$ for any $u \in \tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)$ by construction. Let us remark in addition that we will take $\mathrm{H}^0(\mathbb{R}_+) = \mathrm{L}^2(\mathbb{R}_+)$ as a convention (and similarly for $\mathrm{H}^0(\Gamma)$ and $\tilde{\mathrm{H}}^0(\mathbb{R}_+)$).

Next, given any open set $\Gamma \subset \partial\Omega$, we will frequently use the notation $\langle \cdot, \cdot \rangle_\Gamma$ to denote the usual L^2 inner product of square-integrable functions defined on Γ, which extends as duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual space $\mathrm{H}^{1/2}(\Gamma)^*$. With Ω and Γ as above, we define $\tilde{\mathrm{H}}^{-1/2}(\Gamma) := \mathrm{H}^{1/2}(\Gamma)^*$ and $\mathrm{H}^{-1/2}(\Gamma) := \tilde{\mathrm{H}}^{1/2}(\Gamma)^*$ and consider the naturally associated dual norms

$$
\begin{aligned}
\|p\|_{\tilde{\mathrm{H}}^{-1/2}(\Gamma)} &:= \sup_{v \in \mathrm{H}^{1/2}(\Gamma) \setminus \{0\}} |\langle p, v \rangle_\Gamma| / \|v\|_{\mathrm{H}^{1/2}(\Gamma)}, \\
\|q\|_{\mathrm{H}^{-1/2}(\Gamma)} &:= \sup_{v \in \tilde{\mathrm{H}}^{1/2}(\Gamma) \setminus \{0\}} |\langle q, v \rangle_\Gamma| / \|v\|_{\tilde{\mathrm{H}}^{1/2}(\Gamma)}.
\end{aligned}
\tag{2}
$$

It is easy to see that any $p \in \tilde{\mathrm{H}}^{-1/2}(\mathbb{R}_+)$ induces an element of $\mathrm{H}^{-1/2}(\mathbb{R}_+)$ with $\|p\|_{\mathrm{H}^{-1/2}(\mathbb{R}_+)} \leq \|p\|_{\tilde{\mathrm{H}}^{-1/2}(\mathbb{R}_+)}$.

Finally, we will make regular use of the space

$$
\mathscr{C}_0^\infty(\mathbb{R}_+) := \{\varphi \in \mathscr{C}^\infty(\mathbb{R}_+) | \text{ with bounded } \operatorname{supp}(\varphi) \subset (0, +\infty) \},
$$

which is dense in each of the spaces $\mathrm{H}^{\pm 1/2}(\mathbb{R}_+), \tilde{\mathrm{H}}^{\pm 1/2}(\mathbb{R}_+)$ equipped with its respective norm (see, e.g., [17]). We will rely on this density result to make calculus more explicit. Occasionally, for any open set $\Omega \subset \mathbb{R}^d, d = 1, 2$, we shall also refer to the space $\mathscr{C}_c^\infty(\Omega) := \{\varphi|_\Omega, \ \varphi \in \mathscr{C}^\infty(\mathbb{R}^d), \ \operatorname{supp}(\varphi) \text{ bounded} \}$.

3. Riesz potentials on polygonal boundaries

We will now reduce the scope of the subsequent analysis by assuming that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain. We define the Riesz potential on $\partial\Omega$ as a map $\mathscr{A} : \mathscr{C}^\infty(\partial\Omega) \to \mathscr{C}^\infty(\partial\Omega)^*$ that satisfies

$$
\langle \mathscr{A}(u), v \rangle_{\partial\Omega} := \int_{\partial\Omega} \int_{\partial\Omega} \frac{u(\boldsymbol{y}) v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y}). \tag{3}
$$

The main topic of the present contribution is a fine analysis of this operator. As mentioned in the introduction, such operators have been studied in detail on flat spaces in [21, Chapter V] where an explicit expression of the Fourier symbol is provided. Analyses on the continuity properties of such operators can also be found in [10, 11] for the case of smooth surfaces. Here, we are specifically interested in the case of polygonal, a priori non-smooth surfaces. The main result of this article is to establish the following theorem.

Theorem 3.1. *If $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, then the map $\mathscr{A}$ defined by (3) extends as a bounded linear operator from $\mathrm{L}^2(\partial\Omega)$ into $\mathrm{H}^{1/2}(\partial\Omega)$, i.e.*

$$
\sup_{u, v \in \mathscr{C}^\infty(\partial\Omega) \setminus \{0\}} \frac{|\langle \mathscr{A}(u), v \rangle_{\partial\Omega}|}{\|u\|_{\mathrm{L}^2(\partial\Omega)} \|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)}} < +\infty. \tag{4}
$$

Of course the main difficulty of the proof lies in the analysis of the mapping properties of $\mathscr{A}$ at corners. We shall thus decompose the proof into two steps. In the first step, we consider a decomposition of the boundary $\partial\Omega$ and define an appropriate partition of unity. This will allow us to study localisations of the operator $\mathscr{A}$, which are simple to analyse in cases where they do not contain a corner point of $\partial\Omega$. In the second step, we analyse localisations of $\mathscr{A}$ in the presence of corner points of $\partial\Omega$. We shall see that the Mellin transform appears as an appropriate tool for this analysis.

3.1. Localisation of the problem

We now propose a simple decomposition of the polygonal boundary $\partial\Omega$ and a partition of unity associated with this decomposition.
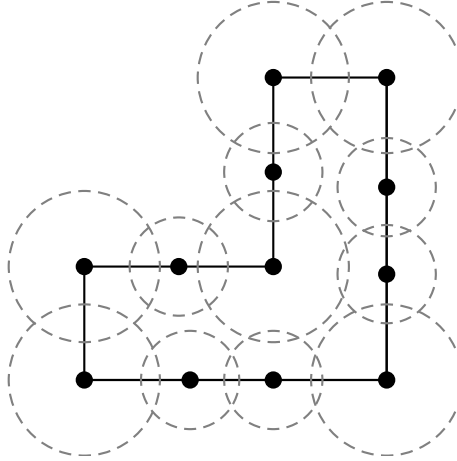
FIGURE 1. An example of the polygonal domain Ω with a decomposition of the boundary $\partial\Omega$ using a finite cover $\mathscr{P}$ of open disks.

Notation 3.1 (Partition of Unity). *We denote by $\mathscr{P}$ a finite collection of open disks $\mathrm{D} \subset \mathbb{R}^2$ with centre $\boldsymbol{c}_{\mathrm{D}} \in \partial\Omega$ such that*

- *$\mathscr{P}$ is an open cover of the boundary $\partial\Omega$, i.e., $\partial\Omega \subset \cup_{\mathrm{D}\in\mathscr{P}}\mathrm{D}$;*

- *Each corner $\boldsymbol{c}$ of $\partial\Omega$ is the centre of a disk, i.e., $\boldsymbol{c} = \boldsymbol{c}_{\mathrm{D}}$ for some $\mathrm{D} \in \mathscr{P}$;*

- *Each such corner belongs to the closure of exactly one disk, i.e., $\boldsymbol{c} = \boldsymbol{c}_{\mathrm{D}} \notin \overline{\mathrm{D}}'$ for $\mathrm{D}' \in \mathscr{P}\backslash\{\mathrm{D}\}$.*

Moreover, given $\mathrm{D} \in \mathscr{P}$, we define $\Gamma_{\mathrm{D}} := \mathrm{D} \cap \partial\Omega$, and we denote by $\chi_{\mathrm{D}} \in \mathscr{C}^\infty(\mathbb{R}^2)$ a smooth function that satisfies $\mathrm{supp}(\chi_{\mathrm{D}}) \subset \mathrm{D}$ and $\sum_{\mathrm{D}\in\mathscr{P}} \chi_{\mathrm{D}}(\boldsymbol{x}) = 1 \; \forall \boldsymbol{x} \in \partial\Omega$. Finally, in case $\mathrm{D} \in \mathscr{P}$ is a 'corner' disk, i.e., if $\boldsymbol{c}_{\mathrm{D}} = \boldsymbol{c}$ for some corner $\boldsymbol{c}$, then we assume without loss of generality that χ_{D} is radially symmetric with respect to $\boldsymbol{c}_{\mathrm{D}}$, i.e., $\chi_{\mathrm{D}}(\boldsymbol{x}) = \chi_{\mathrm{D}}^{\mathscr{R}}(|\boldsymbol{c}_{\mathrm{D}} - \boldsymbol{x}|)$ for some radial function $\chi_{\mathrm{D}}^{\mathscr{R}} \in C_c^\infty(\mathbb{R}^2)$ and all $\boldsymbol{x} \in \mathbb{R}^2$. This last assumption has an important use in Section 3.2.

The finite cover $\mathscr{P}$ introduced above is not uniquely determined. We shall assume that it is fixed once and for all for the remainder of the present article. Equipped with this convention, using the linearity of the Riesz potential we may write for all $u, v \in \mathscr{C}^\infty(\partial\Omega)$

$$\langle \mathscr{A}(u), v \rangle_{\partial\Omega} = \sum_{\mathrm{D},\mathrm{D}'\in\mathscr{P}} \langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi(u), v \rangle_{\partial\Omega} \quad \text{where}$$
$$\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi(u), v \rangle_{\partial\Omega} := \langle \mathscr{A}(\chi_{\mathrm{D}} u), \chi_{\mathrm{D}'} v \rangle_{\partial\Omega}. \tag{5}$$

In Equation (5), each operator $\mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi$ is associated with the kernel $\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^\chi(\boldsymbol{x}, \boldsymbol{y}) := \chi_{\mathrm{D}}(\boldsymbol{x})|\boldsymbol{x}-\boldsymbol{y}|^{-1/2}\chi_{\mathrm{D}'}(\boldsymbol{y})$, i.e., $\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi(u), v \rangle_{\partial\Omega} = \int_{\partial\Omega\times\partial\Omega} \mathfrak{A}_{\mathrm{D},\mathrm{D}'}^\chi(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{x}) v(\boldsymbol{y}) \, d\boldsymbol{x} d\boldsymbol{y}$.

Clearly, in order to prove that the mapping property (4) holds for the Riesz potential $\mathscr{A}$, it suffices to prove that the mapping property (4) holds for each localised operator $\mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi$. More precisely, it suffices to prove that

$$\sup_{u,v\in\mathscr{C}^\infty(\partial\Omega)\backslash\{0\}} \frac{|\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^\chi(u), v \rangle_{\partial\Omega}|}{\|u\|_{\mathrm{L}^2(\partial\Omega)}\|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)}} < +\infty. \tag{6}$$

for each $\mathrm{D}, \mathrm{D}' \in \mathscr{P}$. As the following proposition shows, such an estimate only presents difficulties whenever $\mathrm{D} = \mathrm{D}'$ and D is centred at a corner of the domain.

Proposition 3.1. *Estimate (6) holds if $\mathrm{D} \neq \mathrm{D}'$ or if $\mathrm{D} = \mathrm{D}'$ but D is not centred at a corner of $\partial\Omega$.*

Proof. Take two disks $\mathrm{D}, \mathrm{D}' \in \mathscr{P}$. Estimate (6) is clearly satisfied if $\overline{\mathrm{D}} \cap \overline{\mathrm{D}}' = \emptyset$ since, in this case, the associated kernel satisfies $\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\chi} \in \mathscr{C}^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$. Therefore, it suffices to consider the case

$$\overline{\mathrm{D}} \cap \overline{\mathrm{D}}' \neq \emptyset \text{ and } \overline{\mathrm{D}} \cap \overline{\mathrm{D}}' \text{ does not contain any corner of } \partial\Omega. \tag{7}$$

Condition (7) corresponds either to the case $\mathrm{D} = \mathrm{D}'$ when D is not centred at a corner of $\partial\Omega$, or to the case of two neighbouring disks which may or may not contain a corner. Regardless, we now show that in both cases the estimate (6) follows from the continuity properties of Riesz potentials in flat spaces, as discussed in [21, Chapter V].

To this end, notice that under Condition (7), there exists a straight infinite line $\Sigma \subset \mathbb{R}^2$ such that $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{D}'} \subset \Sigma$. Let $\psi \in \mathscr{C}^{\infty}(\mathbb{R}^2)$ be a cut-off function with the property that $\psi = 1$ on a neighbourhood of $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{D}'}$ and $\psi = 0$ on $\partial\Omega \setminus \Sigma$. We define the functions $\psi_{\mathrm{D}} := \psi\chi_{\mathrm{D}}$ and $\psi_{\mathrm{D}'} := \psi\chi_{\mathrm{D}'}$, and we define the integral kernel $\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(\boldsymbol{x}, \boldsymbol{y}) := \psi_{\mathrm{D}}(\boldsymbol{x})|\boldsymbol{x} - \boldsymbol{y}|^{-1/2}\psi_{\mathrm{D}'}(\boldsymbol{y})$. It follows that for all $u, v \in \mathscr{C}^{\infty}(\partial\Omega)$ we have

$$\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\chi}(u), v \rangle_{\partial\Omega} = \langle (\mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\chi} - \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi})u, v \rangle_{\partial\Omega} + \langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(u), v \rangle_{\partial\Omega},$$
$$\text{where} \quad \langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(u), v \rangle_{\partial\Omega} := \int_{\partial\Omega \times \partial\Omega} \mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{x})v(\boldsymbol{y}) \, d\boldsymbol{x}d\boldsymbol{y}. \tag{8}$$

We now estimate each term of the right hand side above. In order to estimate the first term, we recall that the partition of unity functions χ_{D} and $\chi_{\mathrm{D}'}$ are supported in the disks D and D' respectively so the integral kernel $\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\chi} - \mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}$ can only be singular on the set $\mathrm{D} \cap \mathrm{D}' \times \mathrm{D} \cap \mathrm{D}'$. On the other hand, the definition of the cutoff function $\psi \in \mathscr{C}^{\infty}(\mathbb{R}^2)$ implies that $\psi_{\mathrm{D}}(\boldsymbol{y})\psi_{\mathrm{D}'}(\boldsymbol{x})$ coincides with $\chi_{\mathrm{D}}(\boldsymbol{y})\chi_{\mathrm{D}'}(\boldsymbol{x})$ on a neighbourhood of $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{D}'}$. We can therefore conclude that the integral kernel $\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\chi} - \mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}$ is infinitely smooth on $\mathbb{R}^2 \times \mathbb{R}^2$, and as a consequence there exists a constant $C > 0$ such that for all $u, v \in \mathscr{C}^{\infty}(\partial\Omega)$ it holds that

$$|\langle (\mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\chi} - \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi})u, v \rangle_{\partial\Omega}| \leq C\|u\|_{\mathrm{L}^2(\partial\Omega)}\|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)}. \tag{9}$$

Next, observe that $(\partial\Omega \times \partial\Omega) \cap \mathrm{supp}(\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}) = (\Sigma \times \Sigma) \cap \mathrm{supp}(\mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi})$. Considering therefore $u, v \in \mathscr{C}^{\infty}(\partial\Omega)$, the second term in (8) can be written as

$$\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(u), v \rangle_{\partial\Omega} = \int_{\Sigma \times \Sigma} \mathfrak{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(\boldsymbol{x}, \boldsymbol{y})v(\boldsymbol{x})u(\boldsymbol{y}) \, d\boldsymbol{x}d\boldsymbol{y}.$$

The expression on the right hand side above only depends on the traces u, v restricted to the infinite line Σ. Hence, according to the continuity properties of Riesz potentials stated in [21, Chapter V], we conclude that there exist constants $C, C' > 0$ such that

$$\begin{aligned} |\langle \mathscr{A}_{\mathrm{D},\mathrm{D}'}^{\psi}(u), v \rangle_{\partial\Omega}| &\leq C\|\psi_{\mathrm{D}}u\|_{\mathrm{L}^2(\Sigma)}\|\psi_{\mathrm{D}'}v\|_{\mathrm{H}^{-1/2}(\Sigma)} \\ &= C\|\psi_{\mathrm{D}}u\|_{\mathrm{L}^2(\partial\Omega)}\|\psi_{\mathrm{D}'}v\|_{\mathrm{H}^{-1/2}(\partial\Omega)} \\ &\leq CC'\|u\|_{\mathrm{L}^2(\partial\Omega)}\|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)}. \end{aligned} \tag{10}$$

In the estimate above we have used the fact that $\mathrm{supp}(\psi_{\mathrm{D}}) \cap \Sigma = \mathrm{supp}(\psi_{\mathrm{D}}) \cap \partial\Omega$ (and similarly for $\psi_{\mathrm{D}'}$) so that $\|\psi_{\mathrm{D}}u\|_{\mathrm{L}^2(\Sigma)} = \|\psi_{\mathrm{D}}u\|_{\mathrm{L}^2(\partial\Omega)}$ and $\|\psi_{\mathrm{D}'}v\|_{\mathrm{H}^{-1/2}(\Sigma)} = \|\psi_{\mathrm{D}'}v\|_{\mathrm{H}^{-1/2}(\partial\Omega)}$. Combining the bounds (9)-(10) with Equation (8), we obtain that the estimate (6) indeed holds under Condition (7). $\square$

It therefore remains to prove Estimate (6) in the case where $\mathrm{D} = \mathrm{D}'$ and the disk D contains a corner of $\partial\Omega$. As mentioned previously, this proof is non-trivial and requires a careful study of the Riesz kernel at corners.
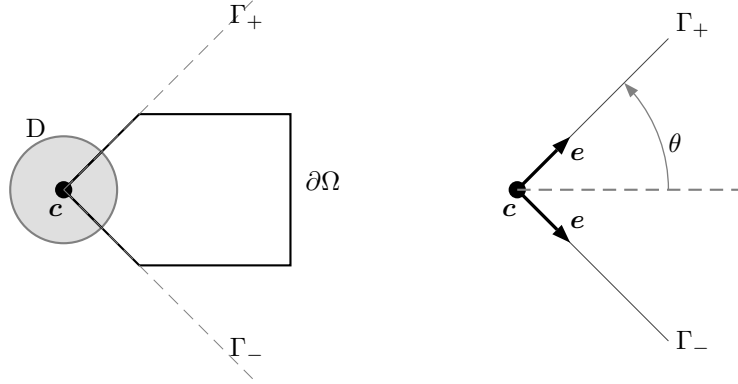
FIGURE 2. An example of a corner $\boldsymbol{c} \in \partial\Omega$ with unit vectors $\boldsymbol{e}_\pm \in \mathbb{R}^2$ describing the neighbouring edges.

3.2. Description of corner operators

Throughout this subsection, we assume that $D \in \mathscr{P}$ is a disk that is centred at a corner $\boldsymbol{c} \in \partial\Omega$. We will now introduce a parameterisation of this corner. To this end, we denote by $\boldsymbol{e}_\pm \in \mathbb{R}^2$ the two unit vectors tangent to $\partial\Omega$ at $\boldsymbol{c}$ with the convention that both unit vectors $\boldsymbol{e}_\pm$ point outwards from the corner $\boldsymbol{c}$. Moreover, we define the rays $\Gamma_\pm \subset \mathbb{R}^2$ and the conic surface $\Gamma \subset \mathbb{R}^2$ as

$$\Gamma_\pm := \{\boldsymbol{c} + t\,\boldsymbol{e}_\pm, \ t > 0\} \qquad \text{and} \qquad \Gamma := \overline{\Gamma}_- \cup \overline{\Gamma}_+.$$

An example of this geometric construction is displayed in Figure 2. Notice that the case where $\boldsymbol{e}_- = -\boldsymbol{e}_+$ corresponds to a dummy corner, i.e., in this situation Γ is obviously flat and $\theta = \pi/2$.

Our goal is now to study in more detail the operator $\mathscr{A}_{D,D}^\chi$ defined through Equation (5). Observe that $D \cap \partial\Omega = D \cap \Gamma$ by construction, so that

$$\langle \mathscr{A}_{D,D}^\chi(u), v \rangle_{\partial\Omega} = \langle \mathscr{A}_\Gamma(\chi_D u), \chi_D v \rangle_\Gamma,$$

$$\text{where} \ \ \langle \mathscr{A}_\Gamma(u), v \rangle_\Gamma := \int_{\Gamma \times \Gamma} \frac{u(\boldsymbol{y})v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\boldsymbol{x} d\boldsymbol{y}, \tag{11}$$

where we have introduced the operator $\mathscr{A}_\Gamma : \mathscr{C}_c^\infty(\Gamma) \to \mathscr{C}_c^\infty(\Gamma)^*$.

To obtain the mapping property at corners we are looking for, we need to prove the finiteness of the following continuity modulus[1] associated with the operator $\mathscr{A}_\Gamma$:

$$\|\mathscr{A}_\Gamma^\chi : L^2(\Gamma) \to H^{1/2}(\Gamma)\| :=$$

$$\sup_{u,v \in \mathscr{C}_c^\infty(\Gamma) \setminus \{0\}} \frac{|\langle \mathscr{A}_\Gamma(\chi_D u), \chi_D v \rangle_\Gamma|}{\|u\|_{L^2(\Gamma)} \|v\|_{H^{-1/2}(\Gamma)}}. \tag{12}$$

As a first remark, we claim that there is a special situation where this continuity modulus is easy to bound. Consider indeed the case where $\theta = \pi/2$ so that $\Gamma = \{0\} \times \mathbb{R}$. This is the previously mentioned situation of a dummy corner. To avoid tedious notation, we denote $\mathscr{A}_\mathbb{R} := \mathscr{A}_{\{0\} \times \mathbb{R}}$. Similarly we also write $H^s(\mathbb{R})$ (resp. $\tilde{H}^s(\mathbb{R})$, $L^2(\mathbb{R})$) instead of $H^s(\{0\} \times \mathbb{R})$ (resp. $\tilde{H}^s(\{0\} \times \mathbb{R})$, $L^2(\{0\} \times \mathbb{R})$), which

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[1]We emphasise here that due to the presence of the cutoff function χ_D, the continuity modulus (12) is *not* the operator norm of $\mathscr{A}_\Gamma$ when viewed as an operator from $L^2(\Gamma) \to H^{1/2}(\Gamma)$. It is rather the operator norm of $\chi_D \mathscr{A}_\Gamma \chi_D : L^2(\Gamma) \to H^{1/2}(\Gamma)$ with χ_D considered a multiplicative operator. A similar remark applies to (13).

should not raise any confusion. Since $\mathbb{R}$ is obviously flat, the operator $\mathscr{A}_{\mathbb{R}}$ is nothing but a classical Riesz potential operator. It follows that

$$\|\mathscr{A}_{\mathbb{R}}^{\chi} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{H}^{1/2}(\mathbb{R})\| :=$$
$$\sup_{u,v \in \mathscr{C}_c^{\infty}(\mathbb{R}) \setminus \{0\}} \frac{|\langle \mathscr{A}_{\mathbb{R}}(\chi_{\mathrm{D}} u), \chi_{\mathrm{D}} v \rangle_{\mathbb{R}}|}{\|u\|_{\mathrm{L}^2(\mathbb{R})} \|v\|_{\mathrm{H}^{-1/2}(\mathbb{R})}} < +\infty. \tag{13}$$

The boundedness property above is a clear consequence of the mapping properties of the Riesz potential operator in flat spaces as can be found in, e.g., [21, Chap.V], together with the fact that χ_{D} is smooth with bounded support. In the sequel, we shall exploit the mapping property (13), comparing the "flat potential" $\mathscr{A}_{\mathbb{R}}$ with the operator $\mathscr{A}_{\Gamma}$ under investigation.

In order to tackle the case of a 'true' corner, i.e., when $\Gamma \neq \{0\} \times \mathbb{R}$, we proceed as follows: For a pair of functions $p, \tilde{p} \in \mathscr{C}_0^{\infty}(\mathbb{R}_+)$, we define a function $\Theta_{\Gamma}(p, \tilde{p}) \in \mathscr{C}^{\infty}(\Gamma)$ by the formula

$$\Theta_{\Gamma}(p, \tilde{p})(\boldsymbol{c} + t\boldsymbol{e}_{\pm}) := \frac{p(t) \pm \tilde{p}(t)}{\sqrt{2}} \quad \forall t > 0, \tag{14}$$

with the convention that we denote $\Theta_{\mathbb{R}} := \Theta_{\{0\} \times \mathbb{R}}$.

Let us now examine how the operator $\mathscr{A}_{\Gamma}$ is transformed under the action of this map. In fact using the map Θ_{Γ}, the operator $\mathscr{A}_{\Gamma}$ on the conic surface Γ can be written as a combination of multiplicative convolution operators on $\mathbb{R}_+$, associated with appropriate kernels.

Lemma 3.1. *Let $2\theta \in (0, 2\pi)$ denote the aperture angle of the conic surface Γ. For any $p, \tilde{p}, q, \tilde{q} \in \mathscr{C}_0^{\infty}(\mathbb{R}_+)$ we have*

$$\langle \mathscr{A}_{\Gamma} \Theta_{\Gamma}(p, \tilde{p}), \Theta_{\Gamma}(q, \tilde{q}) \rangle_{\Gamma} = \langle \mathscr{A}_{\theta}^{+}(p), q \rangle_{\mathbb{R}_+} + \langle \mathscr{A}_{\theta}^{-}(\tilde{p}), \tilde{q} \rangle_{\mathbb{R}_+},$$

where the operators $\mathscr{A}_{\theta}^{\pm} : \mathscr{C}_0^{\infty}(\mathbb{R}_+) \to \mathscr{C}_0^{\infty}(\mathbb{R}_+)^*$ are defined as

$$\langle \mathscr{A}_{\theta}^{\pm}(p), q \rangle_{\mathbb{R}_+} := \int_{\mathbb{R}_+ \times \mathbb{R}_+} \left(\mathfrak{K}_0(t/s) \pm \mathfrak{K}_{\theta}(t/s) \right) p(s) q(t) (st)^{-1/4} \, ds dt \tag{15}$$
$$\text{with} \quad \mathfrak{K}_{\alpha}(\tau) := (4 \sin^2(\alpha) + (\sqrt{\tau} - 1/\sqrt{\tau})^2)^{-1/4}, \qquad \alpha \in [0, 2\pi).$$

Proof. The proof follows from a direct calculation in two steps. First, we simplify the integral kernel associated with the Riesz potential. Pick two points $\boldsymbol{x} \in \Gamma_+$ and $\boldsymbol{y} \in \Gamma_-$. There exist $s, t > 0$ such that $\boldsymbol{x} = \boldsymbol{c} + t\boldsymbol{e}_+$ and $\boldsymbol{y} = \boldsymbol{c} + s\boldsymbol{e}_-$. It follows that $|\boldsymbol{x} - \boldsymbol{y}|^{-1/2} = |t\boldsymbol{e}_+ - s\boldsymbol{e}_-|^{-1/2}$. Obviously, we have $\boldsymbol{e}_- \cdot \boldsymbol{e}_+ = \cos(2\theta)$ and thus $|s\boldsymbol{e}_+ - t\boldsymbol{e}_-|^2 = (s - t)^2 + 4st \sin^2(\theta)$. We therefore obtain

$$\frac{1}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} = \mathfrak{K}_{\theta}(s/t)(st)^{-1/4}. \tag{16}$$

A similar result can be deduced for pairs of points $\boldsymbol{x}', \boldsymbol{y}' \in \Gamma_+$ or $\boldsymbol{x}', \boldsymbol{y}' \in \Gamma_-$ by setting $\theta = 0$.

Next let us set $u = \Theta_{\Gamma}(p, \tilde{p})$, $v = \Theta_{\Gamma}(q, \tilde{q})$, and $u_{\pm} = (p \pm \tilde{p})/\sqrt{2}$ and $v_{\pm} = (q \pm \tilde{q})/\sqrt{2}$. Using Equation (11) we therefore have

$$\langle \mathscr{A}_{\Gamma} \Theta_{\Gamma}(p, \tilde{p}), \Theta_{\Gamma}(q, \tilde{q}) \rangle_{\Gamma} = \langle \mathscr{A}_{\Gamma}(u), v \rangle$$
$$= \int_{\Gamma_+ \times \Gamma_+} \frac{u(\boldsymbol{y}) v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\boldsymbol{x} d\boldsymbol{y} + \int_{\Gamma_- \times \Gamma_-} \frac{u(\boldsymbol{y}) v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\boldsymbol{x} d\boldsymbol{y}$$
$$+ \int_{\Gamma_+ \times \Gamma_-} \frac{u(\boldsymbol{y}) v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\boldsymbol{x} d\boldsymbol{y} + \int_{\Gamma_- \times \Gamma_+} \frac{u(\boldsymbol{y}) v(\boldsymbol{x})}{\sqrt{|\boldsymbol{x} - \boldsymbol{y}|}} \, d\boldsymbol{x} d\boldsymbol{y}.$$

The parameterisations of the rays $\Gamma_\pm$ and (16) then yield

$$
\begin{aligned}
\langle \mathscr{A}_\Gamma(u), v \rangle = \ & \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mathfrak{K}_0(t/s)\big(\, u_+(s)v_+(t) + u_-(s)v_-(t)\,\big) \frac{dsdt}{(st)^{1/4}} \\
& + \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mathfrak{K}_\theta(t/s)\big(\, u_+(s)v_-(t) + u_-(s)v_+(t)\,\big) \frac{dsdt}{(st)^{1/4}}.
\end{aligned}
\tag{17}
$$

The result now follows by plugging the expression of $u_\pm, v_\pm$ with respect to $p, \tilde{p}, q, \tilde{q}$ in the equation above, and rearranging the terms. $\qquad\square$

Lemma 3.1 describes precisely how the operator $\mathscr{A}_\Gamma$ is transformed under the action of the map Θ_Γ. We remind the reader however, that the mapping property we seek, namely Estimate (12), also involves the partition of unity function χ_D. In order to account for this presence of χ_D, we will use the following result, which is a straightforward consequence of Lemma 3.1.

Corollary 3.1. *Consider the setting of Lemma 3.1. For any $p, \tilde{p}, q, \tilde{q} \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ we have*

$$
\langle \mathscr{A}_\Gamma \chi_\mathrm{D} \Theta_\Gamma(p, \tilde{p}), \chi_\mathrm{D} \Theta_\Gamma(q, \tilde{q}) \rangle_\Gamma = \langle \mathscr{A}_\theta^+(\chi p), \chi q \rangle_{\mathbb{R}_+} + \langle \mathscr{A}_\theta^-(\chi \tilde{p}), \chi \tilde{q} \rangle_{\mathbb{R}_+},
$$

where $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ is a cutoff function on $\mathbb{R}_+$ that depends only on the partition of unity function χ_D, and the operators $\mathscr{A}_\theta^\pm$ are defined by Equation (15), exactly as in Lemma 3.1.

Proof. Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ be defined as $\chi(t) := \chi_\mathrm{D}(\boldsymbol{c} + t\boldsymbol{e}_+) \ \forall t \geq 0$. Since p (resp. $\tilde{p}, q, \tilde{q}) \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, it follows that χp (resp. $\chi \tilde{p}, \chi q, \chi \tilde{q}) \in \mathscr{C}_0^\infty(\mathbb{R}_+)$. We may therefore apply Lemma 3.1 to obtain that

$$
\langle \mathscr{A}_\Gamma \Theta_\Gamma(\chi p, \chi \tilde{p}), \Theta_\Gamma(\chi q, \tilde{\chi} q) \rangle_\Gamma = \langle \mathscr{A}_\theta^+(\chi p), \chi q \rangle_{\mathbb{R}_+} + \langle \mathscr{A}_\theta^-(\tilde{\chi} p), \tilde{\chi} q \rangle_{\mathbb{R}_+}.
$$

Next, we recall from Notation 3.1 that the partition of unity function χ_D is radially symmetric with respect to the corner $\boldsymbol{c}$ by assumption. As a consequence, we have $\chi(t) := \chi_\mathrm{D}(\boldsymbol{c} + t\boldsymbol{e}_+) = \chi_\mathrm{D}(\boldsymbol{c} + t\boldsymbol{e}_-) \ \forall t > 0$. It follows that

$$
\Theta_\Gamma(\chi p, \chi \tilde{p}) = \chi_\mathrm{D} \Theta_\Gamma(p, \tilde{p}) \qquad \text{and} \qquad \Theta_\Gamma(\chi q, \chi \tilde{q}) = \chi_\mathrm{D} \Theta_\Gamma(q, \tilde{q}),
$$

which completes the proof. $\qquad\square$

Our motivation for introducing the mapping Θ_Γ is that this map can be used to characterize trace norms in a very explicit manner. Indeed, the following result was established in [6, Lemma 1.12].

Lemma 3.2. *For all $s \in [0, 3/2)$, the map Θ_Γ defined through Equation (14) extends to a continuous isomorphism $\Theta_\Gamma : \mathcal{H}^s(\mathbb{R}_+) \to \mathrm{H}^s(\Gamma)$ where $\mathcal{H}^s(\mathbb{R}_+) := \mathrm{H}^s(\mathbb{R}_+) \times \tilde{\mathrm{H}}^s(\mathbb{R}_+)$ is equipped with the cartesian product norm*

$$
\|(p, \tilde{p})\|_{\mathcal{H}^s(\mathbb{R}_+)}^2 := \|p\|_{\mathrm{H}^s(\mathbb{R}_+)}^2 + \|\tilde{p}\|_{\tilde{\mathrm{H}}^s(\mathbb{R}_+)}^2.
$$

In a dual manner, the map Θ_Γ extends to a continuous isomorphism $\Theta_\Gamma : \mathcal{H}^{-s}(\mathbb{R}_+) \to \mathrm{H}^{-s}(\Gamma)$ where $\mathcal{H}^{-s}(\mathbb{R}_+) = \mathcal{H}^{+s}(\mathbb{R}_+)^* := \tilde{\mathrm{H}}^{-s}(\mathbb{R}_+) \times \mathrm{H}^{-s}(\mathbb{R}_+)$ is equipped with the cartesian product norm

$$
\|(\tilde{q}, q)\|_{\mathcal{H}^{-s}(\mathbb{R}_+)}^2 := \|\tilde{q}\|_{\tilde{\mathrm{H}}^{-s}(\mathbb{R}_+)}^2 + \|q\|_{\mathrm{H}^{-s}(\mathbb{R}_+)}^2.
$$

Lemma 3.2 will be important in the subsequent analysis because it reduces the study of the mapping properties of an operator on Γ (in this case $\mathscr{A}_\Gamma$) to a fine analysis of the mapping properties of an appropriately transformed operator acting on functions defined over $\mathbb{R}_+$ (in this case $\mathscr{A}_\theta^\pm$). The following proposition, which follows by combining both Lemma 3.2 and Corollary 3.1 states these ideas more precisely.

Proposition 3.2. *We have $\mathscr{A}^+_{\pi/2} - \mathscr{A}^+_\theta = \mathscr{A}^-_{\pi/2} - \mathscr{A}^-_\theta$ and there exists a finite constant $C > 0$ such that*

$$C \, \|\mathscr{A}^\chi_\Gamma : \mathrm{L}^2(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)\| \leq \|\mathscr{A}^\chi_\mathbb{R} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{H}^{1/2}(\mathbb{R})\|$$

$$+ \sup_{p,q \in \mathscr{C}^\infty_0(\mathbb{R}_+) \setminus \{0\}} \frac{|\langle (\mathscr{A}^+_{\pi/2} - \mathscr{A}^+_\theta)\chi p, \chi q \rangle_\mathbb{R}|}{\|p\|_{\mathrm{L}^2(\mathbb{R}_+)} \|q\|_{\mathrm{H}^{-1/2}(\mathbb{R}_+)}}, \tag{18}$$

where $\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+)$ is a cutoff function on $\mathbb{R}_+$ that depends only on χ_{D}.

Proof. Let us consider the operator $\Theta^*_\Gamma \mathscr{A}_\Gamma \Theta_\Gamma : \mathscr{C}^\infty_0(\mathbb{R}_+)^2 \to \mathscr{C}^\infty_0(\mathbb{R}_+)^2$. The main idea of the proof is to exploit the isomorphy of Θ_Γ provided by Lemma 3.2. To this end, we first write $\Theta^*_\Gamma \mathscr{A}_\Gamma \Theta_\Gamma = \Theta^*_\mathbb{R} \mathscr{A}_\mathbb{R} \Theta_\mathbb{R} + (\Theta^*_\Gamma \mathscr{A}_\Gamma \Theta_\Gamma - \Theta^*_\mathbb{R} \mathscr{A}_\mathbb{R} \Theta_\mathbb{R})$ and take into account the expression of $\Theta^*_\Gamma \mathscr{A}_\Gamma \Theta_\Gamma$ offered by Corollary 3.1. We thus obtain the estimate

$$\sup_{u,v \in \mathscr{C}^\infty_0(\mathbb{R}_+)^2 \setminus \{0\}} \frac{|\langle \mathscr{A}_\Gamma \chi_{\mathrm{D}} \Theta_\Gamma(u), \chi_{\mathrm{D}} \Theta_\Gamma(v) \rangle_\Gamma|}{\|u\|_{\mathcal{H}^0(\mathbb{R}_+)} \|v\|_{\mathcal{H}^{-1/2}(\mathbb{R}_+)}} \leq$$

$$\sup_{u,v \in \mathscr{C}^\infty_0(\mathbb{R}_+)^2 \setminus \{0\}} \frac{|\langle \mathscr{A}_\mathbb{R} \chi_{\mathrm{D}} \Theta_\mathbb{R}(u), \chi_{\mathrm{D}} \Theta_\mathbb{R}(v) \rangle_\mathbb{R}|}{\|u\|_{\mathcal{H}^0(\mathbb{R}_+)} \|v\|_{\mathcal{H}^{-1/2}(\mathbb{R}_+)}} + \sup_{p,q \in \mathscr{C}^\infty_0(\mathbb{R}_+) \setminus \{0\}} \frac{|\langle (\mathscr{A}^+_{\pi/2} - \mathscr{A}^+_\theta)\chi p, \chi q \rangle_{\mathbb{R}_+}|}{\|p\|_{\mathrm{L}^2(\mathbb{R}_+)} \|q\|_{\tilde{\mathrm{H}}^{-1/2}(\mathbb{R}_+)}} \tag{19}$$

$$+ \sup_{p,q \in \mathscr{C}^\infty_0(\mathbb{R}_+) \setminus \{0\}} \frac{|\langle (\mathscr{A}^-_{\pi/2} - \mathscr{A}^-_\theta)\chi p, \chi q \rangle_{\mathbb{R}_+}|}{\|p\|_{\mathrm{L}^2(\mathbb{R}_+)} \|q\|_{\mathrm{H}^{-1/2}(\mathbb{R}_+)}},$$

where $\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+)$ is defined exactly as in Corollary 3.1, i.e., $\chi(t) := \chi_{\mathrm{D}}(\boldsymbol{c} + t\boldsymbol{e}_+) \; \forall t \geq 0$.

Equation (15) clearly implies $\mathscr{A}^+_{\pi/2} - \mathscr{A}^+_\theta = \mathscr{A}^-_{\pi/2} - \mathscr{A}^-_\theta$. Additionally, $\|q\|_{\mathrm{H}^{-1/2}(\mathbb{R}_+)} \leq \|q\|_{\tilde{\mathrm{H}}^{-1/2}(\mathbb{R}_+)}$ so the second term in the right hand side above is bounded above by the last term. To conclude, it only remains to bound from below the left hand side of (19) and to bound from above the first term of the right hand side.

To this end, we observe that due to the bijectivity of Θ_Γ given by Lemma 3.2 and the density of $\mathscr{C}^\infty_0(\mathbb{R}_+) \times \mathscr{C}^\infty_0(\mathbb{R}_+)$ in $\mathcal{H}^s(\mathbb{R}_+)$ for $s = 0, 1/2$, there exists constants $C^\Gamma_\pm > 0$ such that

$$C^\Gamma_- \|\mathscr{A}^\chi_\Gamma : \mathrm{L}^2(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)\| \leq \sup_{u,v \in \mathscr{C}^\infty_0(\mathbb{R}_+)^2 \setminus \{0\}} \frac{|\langle \mathscr{A}_\Gamma \chi_{\mathrm{D}} \Theta_\Gamma(u), \chi_{\mathrm{D}} \Theta_\Gamma(v) \rangle_\Gamma|}{\|u\|_{\mathcal{H}^0(\mathbb{R}_+)} \|v\|_{\mathcal{H}^{-1/2}(\mathbb{R}_+)}},$$

$$C^\Gamma_+ \|\mathscr{A}^\chi_\Gamma : \mathrm{L}^2(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)\| \geq \sup_{u,v \in \mathscr{C}^\infty_0(\mathbb{R}_+)^2 \setminus \{0\}} \frac{|\langle \mathscr{A}_\Gamma \chi_{\mathrm{D}} \Theta_\Gamma(u), \chi_{\mathrm{D}} \Theta_\Gamma(v) \rangle_\Gamma|}{\|u\|_{\mathcal{H}^0(\mathbb{R}_+)} \|v\|_{\mathcal{H}^{-1/2}(\mathbb{R}_+)}}. \tag{20}$$

Similar estimates obviously hold with Γ replaced by $\{0\} \times \mathbb{R}$ (and a priori different constants $C_\pm > 0$). Plugging (20) into (19) therefore leads to the desired estimate. $\qquad\square$

To summarise the developments of this section, we have first reduced the study of the mapping properties of $\mathscr{A}^\chi_{\mathrm{D,D}} : \mathscr{C}^\infty(\partial\Omega) \to \mathscr{C}^\infty(\partial\Omega)^*$ to that of the corner operator $\mathscr{A}_\Gamma : \mathscr{C}^\infty_c(\Gamma) \to \mathscr{C}^\infty_c(\Gamma)^*$. In view of Proposition 3.2 and Estimate (13), the study of the corner operator $\mathscr{A}_\Gamma$ can in turn be reduced to the study of the operator $\mathscr{A}_\theta : \mathscr{C}^\infty_0(\mathbb{R}_+) \to \mathscr{C}^\infty_0(\mathbb{R}_+)^*$. The later operator is a particular combination of multiplicative convolutions naturally diagonalized by the Mellin transform.

4. Recap on the Mellin Transform

There are only a few references in the literature that provide an overview of the Mellin transform and its use to characterise weighted Sobolev spaces on the positive real line. Additionally, the precise conventions on the definition of the Mellin transform often vary across different references. The goal of the current section is to fix notations, summarise the main properties of the Mellin transform and provide a brief, self-contained and consistent exposition on its connection with weighted Sobolev spaces. Most of the subsequent results are standard and can, for instance, be found

in [8, 16, 19]; we follow the convention of [13]. Let us recall that we denote $\mathscr{C}_0^\infty(\mathbb{R}_+) := \{\varphi \in \mathscr{C}^\infty(\mathbb{R}_+)|$ with bounded $\mathrm{supp}(\varphi) \subset (0, +\infty)\}$. Moreover, for any subset $\Lambda \subseteq \mathbb{C}$, we will frequently denote $\mathcal{H}(\Lambda) := \{v \colon \Lambda \to \mathbb{C}$ such that v is analytic over $\Lambda\}$.

4.1. Definition of the Mellin transform

Definition 4.1 (Mellin Transform). *The Mellin transform of $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, denoted $\hat{u} = \mathscr{M}(u)$, is defined by the formula*

$$\mathscr{M}u(\lambda) = \hat{u}(\lambda) := \int_0^{+\infty} u(r)r^{-\lambda}dr/r \qquad \forall \lambda \in \mathbb{C}. \tag{21}$$

Morera's theorem [19, Chapter 10] implies that for any function $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, the Mellin transform $\hat{u}$ is an entire function, i.e., $\hat{u} \in \mathcal{H}(\mathbb{C})$.

There is a close relationship between Mellin and Fourier transforms. Indeed, denoting by $\mathscr{S}(\mathbb{R})^*$ the space of tempered distributions and by $\mathscr{F} \colon \mathscr{S}(\mathbb{R})^* \to \mathscr{S}(\mathbb{R})^*$ the Fourier transform, it is a simple exercise to show that for all $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ and all $\lambda \in \mathbb{C}$,

$$(\mathscr{M}u)(\imath\lambda) = \mathscr{F}(u \circ \exp)(\lambda). \tag{22}$$

Equation (22) transports all results from classical Fourier theory to the framework of the Mellin transform using the so-called Euler change of variables, i.e., using the map $t \mapsto \exp(t)$. In particular, we have a counterpart to the well-known Parseval theorem.

Lemma 4.1 (Parseval's Theorem for the Mellin Transform). *For all $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ and all $\beta \in \mathbb{R}$ it holds that*

$$\int_0^\infty |u(r)|^2 r^{-2\beta} dr/r = \frac{1}{2\imath\pi} \int_{\beta-\imath\infty}^{\beta+\imath\infty} |\hat{u}(\lambda)|^2 d\lambda.$$

Lemma 4.1 in particular extends the domain of definition of the Mellin transform. Indeed, we have the following result which follows from a classical density argument.

Lemma 4.2. *For every $\beta \in \mathbb{R}$, set $\mathbb{R}_\beta := \{\lambda \in \mathbb{C}, \Re\{\lambda\} = \beta\}$. Then the Mellin transform extends as an isometric isomorphism from $\mathcal{L}_\beta^2(\mathbb{R}_+)$ onto $\mathrm{L}^2(\mathbb{R}_\beta)$, where $\mathcal{L}_\beta^2(\mathbb{R}_+)$ is defined as the completion of $\mathscr{C}_0^\infty(\mathbb{R}_+)$ with respect to the norm*

$$\|u\|_{\mathcal{L}_\beta^2(\mathbb{R}_+)}^2 := \int_0^\infty |u(r)|^2 r^{-2\beta} dr/r. \tag{23}$$

4.2. Inversion formula

Next, we introduce the so-called Hardy spaces which are intimately connected to the Mellin transform. The following result is a direct consequence of the Paley-Wiener theorem (see e.g. [19, Chap.19]) combined with Equation (22).

Lemma 4.3 (Hardy Spaces). *For $\beta \in \mathbb{R}$, define $\mathbb{C}_\beta^+ := \{\lambda \in \mathbb{C}| \Re\{\lambda\} > \beta\}$. The Mellin transform isomorphically maps the subspace $\mathcal{L}_\beta^2(1, \infty) := \{v \in \mathcal{L}_\beta^2(\mathbb{R}_+) : v(x) = 0 \text{ for } x < 1\}$ onto the right Hardy space*

$$\mathfrak{H}^+(\mathbb{R}_\beta) := \{u \in \mathcal{H}(\mathbb{C}_\beta^+), \sup_{\alpha > \beta} \|u\|_{\mathrm{L}^2(\mathbb{R}_\alpha)}^2 < \infty\}.$$

Similarly define $\mathbb{C}_\beta^- := \{\lambda \in \mathbb{C} | \Re e\{\lambda\} < \beta\}$. The Mellin transform isomorphically maps the subspace $\mathcal{L}_\beta^2(0,1) := \{v \in \mathcal{L}_\beta^2(\mathbb{R}_+) : v(x) = 0 \text{ for } x > 1\}$ onto the left Hardy space

$$\mathfrak{H}^-(\mathbb{R}_\beta) := \{u \in \mathcal{H}(\mathbb{C}_\beta^-), \sup_{\alpha < \beta} \|u\|_{L^2(\mathbb{R}_\alpha)}^2 < \infty\}.$$

Remark 4.1. *Using the inverse Fourier transform together with Equation* (22), *we can also deduce an inversion formula for the Mellin transform. Indeed, let $u \in \mathcal{L}_\beta^2(1,\infty)$ and let $\hat{u} = \mathscr{M}(u)$. Then for all $\alpha \geq \beta$ it holds that*

$$u(r) = \frac{1}{2\imath\pi} \int_{\alpha-\imath\infty}^{\alpha+\imath\infty} \hat{u}(\lambda) r^\lambda d\lambda \tag{24}$$

where the integral should be understood in the sense of Fourier (see [19, Theorem 9.13] *or* [9, Proposition 22.1.6]*). Similarly, if $u \in \mathcal{L}_\beta^2(0,1)$ and $\hat{u} = \mathscr{M}(u)$, then the inversion formula* (24) *holds for all $\alpha \leq \beta$.*

Remark 4.2. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. A direct calculation shows that $\mathcal{L}_\beta^2(0,1) \subset \mathcal{L}_\alpha^2(0,1)$ and $\mathcal{L}_\alpha^2(1,\infty) \subset \mathcal{L}_\beta^2(1,\infty)$. We can therefore deduce that $\mathcal{L}_\alpha^2(\mathbb{R}_+) \cap \mathcal{L}_\beta^2(\mathbb{R}_+) = \mathcal{L}_\alpha^2(1,\infty) \oplus \mathcal{L}_\beta^2(0,1)$ and hence, due to Lemma 4.3, the Mellin transform isomorphically maps $\mathcal{L}_\alpha^2(\mathbb{R}_+) \cap \mathcal{L}_\beta^2(\mathbb{R}_+)$ onto $\mathfrak{H}^+(\mathbb{R}_\alpha) \oplus \mathfrak{H}^-(\mathbb{R}_\beta)$ which should be understood as a space of functions that are analytic on the strip $\alpha < \Re e\{\lambda\} < \beta$.*

4.3. Norm characterisation using the Mellin transform

It is well known that the Fourier Transform can be used to derive an alternative characterisation of the classical Sobolev norms in Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$ (see, e.g., [7]). A similar characterisation of both the classical and weighted Sobolev norms on $\mathbb{R}_+$ can be accomplished using the Mellin transform. Indeed, we recall from the Parseval identity for Mellin transforms (Lemma 4.1) that for all $\phi \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ it holds that

$$\|\phi\|_{\mathcal{L}_{\beta-1/2}^2(\mathbb{R}_+)}^2 = \|x^{-\beta}\phi\|_{L^2(\mathbb{R}_+)}^2 = \frac{1}{2\imath\pi} \int_{\beta-1/2-\imath\infty}^{\beta-1/2+\imath\infty} |\widehat{\phi}(\lambda)|^2 \, d\lambda. \tag{25}$$

Of particular interest are the cases $\beta = 0$ and $\beta = 1/2$ (recall the weighted semi-norm introduced in Equation (1)). In addition, we have the following result due to Costabel and Stephan [6].

Lemma 4.4 ([6, Lemma 2.3]). *There exists constants $C, C' > 1$ such that for all $\phi \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ it holds that*

$$\frac{C}{2\imath\pi} \int_{-\imath\infty}^{+\imath\infty} \frac{|\lambda|^2}{1+|\lambda|} |\widehat{\phi}(\lambda)|^2 \, d\lambda \leq |\phi|_{H^{1/2}(\mathbb{R}_+)}^2 \leq \frac{C'}{2\imath\pi} \int_{-\imath\infty}^{+\imath\infty} \frac{|\lambda|^2}{1+|\lambda|} |\widehat{\phi}(\lambda)|^2 \, d\lambda.$$

Lemma 4.4 therefore allows us to obtain characterisations of the $\|\cdot\|_{H^{1/2}(\mathbb{R}_+)}$ and $\|\cdot\|_{\tilde{H}^{1/2}(\mathbb{R}_+)}$ norms introduced in Section 2 in terms of the Mellin transform which will be of use in the sequel.

5. Mellin analysis of Riesz potentials

We now return to our analysis of the Riesz potential on the polygonal boundary $\partial\Omega$. We have shown in detail in Section 3 that establishing the mapping properties of the Riesz potential reduces to the

study of localised 'corner' operators. More precisely, we need to investigate (see Proposition 3.2 and Equation (11)) the multiplicative convolution operator $\mathscr{K}_\theta : \mathscr{C}_0^\infty(\mathbb{R}_+) \to \mathscr{C}_0^\infty(\mathbb{R}_+)^*$ defined by

$$\mathscr{K}_\theta(u) := \int_0^\infty (st)^{-1/4} \mathfrak{K}_\theta(t/s) u(s) ds \tag{26}$$
$$\text{with} \quad \mathfrak{K}_\theta(\tau) := (4\sin^2(\theta) + (\sqrt{\tau} - 1/\sqrt{\tau})^2)^{-1/4}, \quad \theta \in (0, \pi).$$

As claimed in Section 3, the appropriate tool to study the operator $\mathscr{K}_\theta$ is the Mellin transform. As a first step, let us check that $\mathscr{K}_\theta(u)$ belongs to some weighted Lebesgue space so that it lends itself to Mellin calculus.

Lemma 5.1. *For $\theta \in (0, \pi)$ and $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ we have $\mathscr{K}_\theta(u) \in \mathcal{L}^2_{-\beta}(\mathbb{R}_+)$ $\forall \beta \in (0, 1/2)$.*

Proof. Pick $\beta \in (0, 1/2)$ and set $\widetilde{u}(s) = s^{3/4} u(s)$ and $\widetilde{\mathfrak{K}}(\tau) = \tau^{\beta - 1/4} \mathfrak{K}_\theta(\tau)$. It follows that $\widetilde{u} \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ and $\widetilde{\mathfrak{K}} \in \mathscr{C}^\infty(\mathbb{R}_+)$ with $\widetilde{\mathfrak{K}}(\tau) \sim \tau^\beta$ for $\tau \to 0_+$ and $\widetilde{\mathfrak{K}}(\tau) \sim \tau^{\beta - 1/2}$ for $\tau \to +\infty$. We can therefore conclude in particular that $\int_0^\infty \widetilde{\mathfrak{K}}(\tau) d\tau/\tau < +\infty$.

Applying now the definition of the weighted norm (23) to $\mathscr{K}_\theta(u)$ yields

$$\|\mathscr{K}_\theta(u)\|^2_{\mathcal{L}^2_{-\beta}(\mathbb{R}_+)} = \int_0^\infty \left| \int_0^\infty (st)^{-1/4} \mathfrak{K}_\theta(t/s) u(s) ds \right|^2 t^{2\beta} dt/t \tag{27}$$
$$= \int_0^\infty \left| \int_0^\infty \widetilde{\mathfrak{K}}(t/s) \widetilde{u}(s) ds/s \right|^2 dt/t.$$

From here we simply adapt a classical proof on convolution calculus (see e.g [2, Thm.4.15]). Applying the Cauchy-Schwarz inequality and using the change of variable $\xi = t/s$, we obtain the estimate

$$\left| \int_0^\infty \widetilde{\mathfrak{K}}(t/s) \widetilde{u}(s) ds/s \right|^2 \leq \left(\int_0^\infty \widetilde{\mathfrak{K}}(\xi) d\xi/\xi \right) \left(\int_0^\infty \widetilde{\mathfrak{K}}(t/s) |\widetilde{u}(s)|^2 ds/s \right). \tag{28}$$

As $\widetilde{u} \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, we obviously have $\int_0^\infty |\widetilde{u}(s)|^2 ds/s < +\infty$. Consequently, plugging (28) into (27) and applying Fubini's theorem leads to

$$\|\mathscr{K}_\theta(u)\|^2_{\mathcal{L}^2_{-\beta}(\mathbb{R}_+)} \leq \left(\int_0^\infty \widetilde{\mathfrak{K}}(\xi) d\xi/\xi \right) \int_0^\infty \left(\int_0^\infty \widetilde{\mathfrak{K}}(t/s) dt/t \right) |\widetilde{u}(s)|^2 ds/s$$
$$\leq \left(\int_0^\infty \widetilde{\mathfrak{K}}(\xi) d\xi/\xi \right)^2 \int_0^\infty |\widetilde{u}(s)|^2 ds/s < +\infty.$$

$\square$

Lemma 27 implies that for any $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, we have $\mathscr{K}_\theta(u) \in \mathcal{L}^2_\alpha(\mathbb{R}_+) \cap \mathcal{L}^2_\beta(\mathbb{R}_+)$ for $-1/2 < \alpha < \beta < 0$. In particular, according to Remark 4.2, the Mellin transform of $\mathscr{K}_\theta(u)$ is properly defined and analytic in the strip $-1/2 < \Re e\{\lambda\} < 0$. The next lemma provides an expression for the Mellin transform of $\mathscr{K}_\theta(u)$ in this strip.

Lemma 5.2. *For $\theta \in (0, \pi)$, let $\mathscr{K}_\theta : \mathscr{C}_0^\infty(\mathbb{R}_+) \to \mathscr{C}_0^\infty(\mathbb{R}_+)^*$ be defined through Equation (26). Then for each $\lambda \in \mathbb{C}$ such that $-1/2 < \Re e\{\lambda\} < 0$ and all $u \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ we have*

$$\widehat{\mathscr{K}_\theta(u)}(\lambda) = \widehat{\mathfrak{K}}_\theta(\lambda + 1/4) \widehat{u}(\lambda - 1/2)$$
$$\text{where} \quad \widehat{\mathfrak{K}}_\theta(\lambda) := \int_0^{+\infty} \mathfrak{K}_\theta(r) r^{-\lambda} dr/r.$$

Proof. The proof follows by a direct calculation. Indeed, using the definition of the Mellin transform for $\lambda \in \mathbb{C}$ and applying the change of variables $\xi := t/s$ yields

$$\widehat{\mathscr{K}_\theta(u)}(\lambda) = \int_0^\infty \int_0^\infty \mathfrak{K}_\theta(t/s)u(s)s^{3/4}t^{-\lambda-1/4}\frac{dsdt}{st} = \int_0^\infty \left(\int_0^\infty \mathfrak{K}_\theta(t/s)t^{-\lambda-1/4}\frac{dt}{t}\right)u(s)s^{3/4}\frac{ds}{s}$$

$$= \int_0^\infty \left(\int_0^\infty \mathfrak{K}_\theta(\xi)\xi^{-\lambda-1/4}\frac{d\xi}{\xi}\right)u(s)s^{-\lambda+1/2}\frac{ds}{s} = \widehat{\mathfrak{K}}_\theta(\lambda+1/4)\widehat{u}(\lambda-1/2).$$

$\square$

In the remainder of this section, we will investigate the properties of $\widehat{\mathfrak{K}}_\theta(\lambda)$, and understand its regularity properties and asymptotic behaviour in the complex plane. Equipped with this knowledge, we will be able to characterise more precisely the continuity properties of $\mathscr{K}_\theta$ using the Mellin characterisation of Sobolev norms described in Section 4.3. As a first step, we establish that the Mellin symbol $\widehat{\mathfrak{K}}_\theta$ is analytic on a strip in the complex plane.

Proposition 5.1. *For all $\alpha \in (0,\pi)$, the Mellin symbol $\widehat{\mathfrak{K}}_\alpha(\lambda)$ is well defined and analytic in the strip defined by $|\Re e\{\lambda\}| < 1/4$. Additionally, $\widehat{\mathfrak{K}}_{\pi/2}(\lambda) - \widehat{\mathfrak{K}}_\alpha(\lambda)$ is well defined and analytic in the strip $|\Re e\{\lambda\}| < 5/4$.*

Proof. We have by definition (see Equation (15)) that $\mathfrak{K}_\alpha(\tau) = (4\sin^2(\alpha) + (\sqrt{\tau} - 1/\sqrt{\tau})^2)^{-1/4}$. Simple algebra yields that the kernel $\mathfrak{K}_\alpha$ can equivalently be written as $\mathfrak{K}_\alpha(\tau) = \tau^{1/4}(1 - (2\cos(2\alpha)\tau - \tau^2))^{-1/4}$ for $\tau \in (0,+\infty)$. Consequently, the generalised binomial series yields some $\delta_0 > 0$ such that for all $0 < \tau < \delta_0$ the following series expansion converges absolutely

$$\mathfrak{K}_\alpha(\tau) = \tau^{1/4}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\Gamma(n+1/4)}{\Gamma(1/4)}\big(2\cos(2\alpha)\tau - \tau^2\big)^n.$$

Expanding $(2\cos(2\alpha)\tau - \tau^2)^n$ yields a power series expansion with coefficients $\kappa_{\alpha,n} \in \mathbb{R}$, $n \in \mathbb{N}$ that, for clarity can be written as

$$\mathfrak{K}_\alpha(\tau) = \tau^{1/4} + \sum_{n=1}^{+\infty}\kappa_{\alpha,n}\tau^{n+1/4}. \tag{29}$$

The above series also converges absolutely for all $0 < \tau < \delta_0$. Moreover, since $\mathfrak{K}_\alpha(\tau) = \mathfrak{K}_\alpha(1/\tau)$, the series expansion (29) also holds with τ replaced by $1/\tau$ for all $\tau > 1/\delta_0$. We see in particular that $\mathfrak{K}_\alpha(\tau) \sim \tau^{+1/4}$ for $\tau \to 0$, and $\mathfrak{K}_\alpha(\tau) \sim \tau^{-1/4}$ for $\tau \to +\infty$. From this we conclude that $\mathfrak{K}_\alpha \in \mathcal{L}^2_{1/4-\epsilon}(0,1) \oplus \mathcal{L}^2_{-1/4+\epsilon}(1,\infty)$ for all $\epsilon > 0$. According to Remark 4.2, $\widehat{\mathfrak{K}}_\alpha(\lambda)$ is well defined and analytic in the strip $|\Re e\{\lambda\}| < 1/4$.

Finally we observe that the first term in the series expansion appearing in Equation (29) does not depend on α, which shows that $\mathfrak{K}_{\pi/2}(\tau) - \mathfrak{K}_\alpha(\tau) \sim \tau^{5/4}$ for $\tau \to 0$ and, once again using the fact that $\mathfrak{K}_{\pi/2}(\tau) - \mathfrak{K}_\alpha(\tau) = \mathfrak{K}_{\pi/2}(1/\tau) - \mathfrak{K}_\alpha(1/\tau)$, we deduce that $\mathfrak{K}_{\pi/2}(\tau) - \mathfrak{K}_\alpha(\tau) \sim \tau^{-5/4}$ for $\tau \to +\infty$. Following the same arguments as above, we conclude that $\mathfrak{K}_{\pi/2} - \mathfrak{K}_\alpha$ is well defined and analytic in the strip $|\Re e\{\lambda\}| < 5/4$. $\square$

We now demonstrate that the Mellin transform of the integral kernel $\mathfrak{K}_\alpha$ can, in fact, be extended analytically to the entire complex plane $\mathbb{C}$, except at a countable number of points. In order to prove this result, we first require a preparatory lemma.

Lemma 5.3. *Let $\phi \in \mathscr{C}^\infty(\mathbb{R}_+)$ be a cut-off function satisfying $\phi(\tau) = 0$ for $\tau > 1/2$ and $\phi(\tau) = 1$ for $\tau < 1/4$. Then the Mellin transform $\widehat{\phi}(\lambda)$ is analytic on the entire complex plane $\mathbb{C}$ except at $\lambda = 0$ where it admits a simple pole. Furthermore for all $\mu \in \mathbb{R}$ and all $p \geq 0$ it holds that*

$$\lim_{\xi \to \infty} |\xi|^p \widehat{\phi}(\mu \pm \imath \xi) = 0. \tag{30}$$

Proof. The definition of ϕ implies that $\phi \in \mathcal{L}^2_{-\epsilon}(0,1)$ for every $\epsilon > 0$ and Remark 4.2 therefore implies that $\widehat{\phi}(\lambda)$ is well-defined and analytic for $\Re\{\lambda\} < 0$. Furthermore, for any such $\lambda \in \mathbb{C}$ we have, using integration by parts, that

$$\begin{aligned}
\widehat{\phi}(\lambda) &= \int_0^\infty \phi(r) r^{-\lambda} dr/r = \frac{1}{\lambda} \int_0^\infty \Upsilon(r) r^{-\lambda} dr/r \\
&= \widehat{\Upsilon}(\lambda)/\lambda \quad \text{where} \quad \Upsilon(r) := r \partial_r \phi(r).
\end{aligned} \tag{31}$$

Since $\operatorname{supp}(\partial_r \phi) \subset [\frac{1}{4}, \frac{1}{2}]$ by assumption we have that $\Upsilon \in \mathscr{C}_0^\infty(\mathbb{R}_+)$. This implies in particular (see Section 4.1) that the Mellin transform $\widehat{\Upsilon}(\lambda)$ is analytic in the entire complex plane $\mathbb{C}$ and therefore $\widehat{\phi}(\lambda)$ is analytic on the entire complex plane $\mathbb{C}$ except at $\lambda = 0$ where it has a simple pole.

Next we demonstrate the validity of the decay condition (30). To this end, let $g: \mathbb{R} \to \mathbb{R}$ be defined as $g(t) := \Upsilon(\exp(t)) \; \forall t \in \mathbb{R}$, and let $\lambda = \xi + \imath \mu \in \mathbb{C}$. Obviously, $g \in \mathscr{C}_0^\infty(\mathbb{R})$ and thus any order derivative of g is an integrable function on $\mathbb{R}$. The Riemann-Lebesgue lemma (see, e.g., [20, Chapter 5]) therefore implies that for any fixed $\mu \in \mathbb{R}$ and all $p \geq 0$ it holds that

$$\lim_{\xi \to \pm\infty} |\xi|^p (\mathscr{F}g)(\xi + \imath \mu) = 0,$$

where $\mathscr{F}g$ denotes the Fourier transform of g. The decay condition (30) now follows using the correspondence between the Fourier transform and the Mellin transform given by Equation (22). $\qquad \square$

Proposition 5.2. *For all $\alpha \in (0, \pi)$, the Mellin symbol $\widehat{\mathfrak{K}}_\alpha$ can be extended as an analytic function defined on $\mathbb{C} \setminus \mathfrak{S}$ where $\mathfrak{S} := (+1/4 + \mathbb{N}) \cup (-1/4 - \mathbb{N})$. Moreover $\widehat{\mathfrak{K}}_\alpha(\lambda)$ admits a simple pole at each point of $\mathfrak{S}$, and its residue at $\lambda = \pm 1/4$ does not depend on α.*

Proof. Let $\phi \in \mathscr{C}^\infty(\mathbb{R}_+)$ be a cut-off function satisfying $\phi(\tau) = 0$ for $\tau > 1/2$, and $\phi(\tau) = 1$ for $\tau < 1/4$ as described in Lemma 5.3, and let $Q \geq 2$ be a natural number. We define for all $\tau > 0$

$$\begin{aligned}
\Phi_Q(\tau) &:= \left(\tau^{1/4} + \sum_{q=1}^{Q-1} \kappa_{\alpha,q} \tau^{q+1/4} \right) \phi(\tau), \\
\mathrm{R}_Q(\tau) &:= \mathfrak{K}_\alpha(\tau) - \Phi_Q(\tau) - \Phi_Q(1/\tau),
\end{aligned} \tag{32}$$

where the coefficients $\kappa_{\alpha,q}$ appearing in the sum are the same as those appearing in the series expansion of $\mathfrak{K}_\alpha$ given by (29).

It is a simple exercise to prove that for any function $v \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, if we define $v_\sharp(\tau) := v(1/\tau)$ then we have $\hat{v}_\sharp(\lambda) = \hat{v}(-\lambda)$. Consequently, from Equation (32) we obtain that

$$\widehat{\mathfrak{K}}_\alpha(\lambda) = \widehat{\mathrm{R}}_Q(\lambda) + \widehat{\Phi}_Q(\lambda) + \widehat{\Phi}_Q(-\lambda). \tag{33}$$

Next, we analyse the behaviour of the Mellin transforms of the two functions Φ_Q and R_Q. To this end, we define the set $\mathbb{B}_Q := \{\lambda \in \mathbb{C}: \; |\Re e\{\lambda\}| < Q + 1/4\}$. A direct calculation yields

$$\widehat{\Phi}_Q(\lambda) = \widehat{\phi}(\lambda - 1/4) + \sum_{q=1}^{Q-1} \kappa_{\alpha,q} \widehat{\phi}(\lambda - q - 1/4). \tag{34}$$

Using Lemma 5.3, we deduce that the Mellin transform $\widehat{\Phi}_Q(\lambda)$ is well-defined and analytic for all $\lambda \in \mathbb{C} \setminus \mathfrak{S}$ where $\mathfrak{S} = (+1/4 + \mathbb{N}) \cup (-1/4 - \mathbb{N})$. Moreover, $\widehat{\Phi}_Q$ admits a simple pole at each point of $\mathfrak{S} \cap \mathbb{B}_Q$ and its residue at $\lambda = 1/4$ does not depend on α.

Furthermore, using the same arguments involving series expansions in neighbourhoods of $\tau = 0$ and $\tau \to \infty$ that we used in the proof of Proposition 5.1, we obtain that

$$\mathrm{R}_Q \in \mathcal{L}^2_{Q+1/4-\epsilon}(0,1) \oplus \mathcal{L}^2_{-Q-1/4+\epsilon}(1,\infty) \qquad \forall \epsilon > 0.$$

As a consequence the Mellin transform $\widehat{\mathrm{R}}_Q(\lambda)$ is well defined and analytic for all $\lambda \in \mathbb{B}_Q$. In view of Equation (33), we therefore conclude that the Mellin transform $\widehat{\mathfrak{K}}_\alpha(\lambda)$ is analytic for all $\lambda \in \mathbb{B}_Q \setminus \mathfrak{S}$, admits a simple pole for any $\lambda \in \mathfrak{S} \cap \mathbb{B}_Q$ and has residue at $\lambda = \pm 1/4$ that does not depend on α. Since Q can be chosen arbitrarily large, the proof now follows. $\qquad\square$

Proposition 5.2 has a straightforward corollary concerning the decay properties of the Mellin transform of the integral kernel $\mathfrak{K}_\alpha, \alpha \in (0,\pi)$ on vertical lines in the complex plane.

Corollary 5.1. *For all $\alpha \in (0,\pi)$, the Mellin symbol $\widehat{\mathfrak{K}}_\alpha$ satisfies*

$$\lim_{\xi \to +\infty} |\widehat{\mathfrak{K}}_\alpha(\mu \pm \imath\xi)|^2 |\xi|^{2p} = 0 \ \forall \mu \in \mathbb{R}, \ \forall p \geq 0.$$

Proof. Let $p \geq 0$ be a non-negative integer and consider the proof of Proposition 5.2. Due to the decomposition (33), it suffices to show that there exists some $Q \geq 2$ such that

$$\lim_{\xi \to +\infty} |\widehat{\Phi}_Q(\mu \pm \imath\xi)|^2 |\xi|^{2p} = 0 \ \forall \mu \in \mathbb{R}, \ \forall p \geq 0 \quad \text{and} \tag{35a}$$

$$\lim_{\xi \to +\infty} |\widehat{\mathrm{R}}_Q(\mu \pm \imath\xi)|^2 |\xi|^{2p} = 0 \ \forall \mu \in \mathbb{R}, \ \forall p \geq 0. \tag{35b}$$

The decay condition (35a) can be deduced for all $Q \geq 2$ using the decay condition (30) established in Lemma 5.3 together with the expression (34) for the Mellin transform $\widehat{\Phi}_Q$.

In order to establish the decay condition (35b), we recall the earlier argument presented in the proof of Lemma 5.3 involving the Riemann-Lebesgue lemma to prove the decay condition (30). In view of this argument, it is sufficient to establish that there exists some $Q \geq 2$ such that the function $\mathrm{R}_Q \in \mathscr{C}^\infty\big((0,\infty)\big)$ and the pth derivative of R_Q is an integrable function on $\mathbb{R}_+$.

Let $Q \geq 2$. Notice that for all $\alpha \in (0,\pi)$ the kernel $\mathfrak{K}_\alpha$ is by definition in $\mathscr{C}^\infty(\mathbb{R}_+)$ (see, e.g., Equation (15)). Moreover, since the cutoff function $\phi \in \mathscr{C}^\infty(\mathbb{R}_+)$, the relation (32) implies that $\Phi_Q \in \mathscr{C}^\infty\big((0,\infty)\big)$ from which we can deduce that $\mathrm{R}_Q \in \mathscr{C}^\infty\big((0,\infty)\big)$.

It therefore remains to prove that for some choice of $Q \geq 2$, the pth derivative of R_Q is an integrable function on $\mathbb{R}_+$. Thus, it suffices to show that for some choice of $Q \geq 2$ we have

$$\lim_{\tau \to 0} \partial_\tau^p \mathrm{R}_Q(\tau) = \lim_{\tau \to \infty} \partial_\tau^p \mathrm{R}_Q(\tau) = 0.$$

We first consider the limit $\tau \to 0$. Using the relation (32) and the definition of the cutoff-function ϕ we see that for $0 < \tau < \delta_0$ (i.e. τ sufficiently small) we have

$$\mathrm{R}_Q(\tau) = \mathfrak{K}_\alpha(\tau) - \left(\tau^{1/4} + \sum_{q=1}^{Q-1} \kappa_{\alpha,q} \tau^{q+1/4} \right) \phi(\tau) = \sum_{q=Q}^{\infty} \kappa_{\alpha,q} \tau^{q+1/4}.$$

Consequently, if we pick $Q \geq p$, we deduce that $\lim_{\tau \to 0} \partial_\tau^p \mathrm{R}_Q(\tau) = 0$ as required. In a similar fashion, we see that for $\tau > 1/\delta_0$ (i.e. τ sufficiently large) we have

$$\mathrm{R}_Q(\tau) = \mathfrak{K}_\alpha(\tau) - \left(\tau^{-1/4} + \sum_{q=1}^{Q-1} \kappa_{\alpha,q} \tau^{-q-1/4} \right) \phi(\tau) = \sum_{q=Q}^{\infty} \kappa_{\alpha,q} \tau^{-q-1/4},$$

where the second equality follows from the asymptotic expansion of the kernel $\mathfrak{K}_\alpha$ obtained in the proof of Proposition 5.1. It therefore follows that $\lim_{\tau \to \infty} \partial_\tau^p \mathrm{R}_Q(\tau) = 0$. This completes the proof. $\qquad\square$

We conclude this section by stating two corollaries that follow from the results stated above. These corollaries will be used to conclude the analysis of the Riesz potential on corners of the polyhedral domain Ω that we began in Section 3.

Corollary 5.2. *For all $\alpha \in (0, \pi)$, the Mellin symbol $\widehat{\mathfrak{K}}_\alpha$ satisfies*

$$\sup_{\xi \in \mathbb{R}} \frac{|\widehat{\mathfrak{K}}_\alpha(\mu + \imath\xi)|^2 |\xi|^2}{1 + |\xi|} < \infty \qquad \forall \mu \in \mathbb{R}.$$

Proof. Given any $\lambda \in \mathbb{C}$, we write $\lambda = \mu + \imath\xi$. In view of Corollary 5.1, it suffices to show that for any $\mu \in \mathbb{R}$ and any bounded set $K \subset \mathbb{R}$ we have

$$\sup_{\xi \in K} |\widehat{\mathfrak{K}}_\alpha(\mu + \imath\xi)|^2 |\xi|^2 < \infty. \tag{36}$$

We recall from Proposition 5.2 that the Mellin transform $\widehat{\mathfrak{K}}_\alpha$ can be extended as an analytic function defined on $\mathbb{C} \setminus \mathfrak{S}$ where $\mathfrak{S} = (+1/4 + \mathbb{N}) \cup (-1/4 - \mathbb{N})$ and furthermore that $\widehat{\mathfrak{K}}_\alpha(\lambda)$ admits a simple pole at each point of $\mathfrak{S}$. Consequently, the mapping

$$\widehat{\mathfrak{K}}_\alpha^{\mathrm{extend}}(\mu + \imath\xi) := \widehat{\mathfrak{K}}_\alpha(\mu + \imath\xi)\, \xi,$$

can be extended as a continuous function for all $\xi, \mu \in \mathbb{R}$. Estimate (36) therefore follows. $\qquad\square$

Corollary 5.3. *For all $\alpha \in (0, \pi)$, the Mellin symbol $\widehat{\mathfrak{K}}_\alpha$ satisfies*

$$\sup_{\lambda \in \pm 1/4 + \imath\mathbb{R}} |\widehat{\mathfrak{K}}_\alpha(\lambda) - \widehat{\mathfrak{K}}_{\pi/2}(\lambda)| < +\infty \quad and \tag{37a}$$

$$\sup_{\lambda \in \imath\mathbb{R}} |\widehat{\mathfrak{K}}_\alpha(\lambda)| < +\infty \tag{37b}$$

Proof. Estimate (37b) follows by combining Proposition 5.1, which demonstrates that $\widehat{\mathfrak{K}}_\alpha(\lambda)$ is analytic for $\lambda \in \imath\mathbb{R}$, together with Corollary 5.1 which shows that $\lim_{\xi \to \infty} |\widehat{\mathfrak{K}}_\alpha(\pm \imath\xi)| = 0$.

In order to establish Estimate (37a), we recall from Proposition 5.1 that the function $\widehat{\mathfrak{K}}_{\pi/2}(\lambda) - \widehat{\mathfrak{K}}_\alpha(\lambda)$ is analytic in the strip $|\Re e\{\lambda\}| < 5/4$. Using once again Corollary 5.1 to establish the decay behaviour of the Mellin symbols for $\xi \to \infty$ therefore completes the proof. $\qquad\square$

6. Application to corner operators

The goal of this section is to complete the proof of Theorem 3.1 using the tools and results developed thus far. In view of the development of Section 3 and in particular Proposition 3.2 and Estimate (13), it suffices to prove the following lemma.

Lemma 6.1. *For any fixed $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ and any $\theta \in (0, \pi)$ we have the continuity estimate*

$$\sup_{u \in \mathscr{C}_0^\infty(\mathbb{R}_+)\backslash\{0\}} \frac{\|(\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u)\|_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)}}{\|u\|_{\mathrm{L}^2(\mathbb{R}_+)}} < +\infty.$$

Proof. Picking an arbitrary $u \in \mathscr{C}_0^\infty(\mathbb{R}_+) \setminus \{0\}$, according to Equation (1), we need to study and derive an upper bound for the norm

$$\|v\|_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)}^2 = \|v\|_{\mathrm{L}^2(\mathbb{R}_+)}^2 + \|v\|_{\mathcal{L}_0^2(\mathbb{R}_+)}^2 + |v|_{\tilde{\mathrm{H}}^{1/2}(\mathbb{R}_+)}^2$$

$$\text{where} \quad v = (\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u). \tag{38}$$

To do so, we shall make use of the characterisation of the Lebesgue norm and Sobolev semi-norms in terms of Mellin symbols given in Section 4.3. To estimate the first term on the right hand side of Equation (38), we combine Equation (25) together with Lemma 5.2 and Corollary 5.3 to obtain a constant $C > 0$ such that

$$\|(\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u)\|_{\mathrm{L}^2(\mathbb{R}_+)}^2$$

$$= \frac{1}{2\imath\pi} \int_{-1/2-\imath\infty}^{-1/2+\imath\infty} \left|\widehat{\mathfrak{K}}_\theta(\lambda + 1/4) - \widehat{\mathfrak{K}}_{\pi/2}(\lambda + 1/4)\right|^2 |\widehat{\chi u}(\lambda - 1/2)|^2 \, d\lambda$$

$$\leq \frac{C}{2\imath\pi} \int_{-1/2-\imath\infty}^{-1/2+\imath\infty} |\widehat{\chi u}(\lambda - 1/2)|^2 \, d\lambda = \frac{C}{2\pi} \left\|\widehat{\chi u}\right\|_{\mathrm{L}^2(\mathbb{R}_{-1})}^2 = C\|\chi u\|_{\mathcal{L}_{-1}^2(\mathbb{R}_+)}^2,$$

where the last equality follows directly from the Parseval theorem for the Mellin transform (Lemma 4.1). Finally, using the fact that $\chi \in \mathscr{C}_c^\infty(\mathbb{R}_+)$ is fixed, we can deduce the existence of some constant $C'_\chi > 0$ that depends only on χ such that

$$\|(\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u)\|_{\mathrm{L}^2(\mathbb{R}_+)}^2 \leq C\|\chi u\|_{\mathcal{L}_{-1}^2(\mathbb{R}_+)}^2 \leq CC'_\chi\|u\|_{\mathcal{L}_{-1/2}^2(\mathbb{R}_+)}^2 = CC'_\chi\|u\|_{\mathrm{L}^2(\mathbb{R}_+)}^2.$$

The second term in Equation (38) can be estimated in an identical manner using Equation (25), Lemma 5.2 and Corollary 5.3 to yield

$$\|(\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u)\|_{\mathcal{L}_0^2(\mathbb{R}_+)}^2 \leq C\tilde{C}_\chi\|u\|_{\mathrm{L}^2(\mathbb{R}_+)}^2,$$

where the constant $C > 0$, which is independent of u and χ, arises due to the use of Corollary 5.3 and the constant $\tilde{C}_\chi > 0$ depends only on χ.

To estimate the third term, we use the Mellin characterisation of the Sobolev semi-norm given by Lemma 4.4 together with Lemma 5.2 and Corollary 5.2. We thus deduce the existence of constants $C', \bar{C}, C'_\chi > 0$ with $C', \bar{C}$ independent of u and χ and C'_χ dependent only on χ such that

$$|(\mathscr{A}_{\pi/2}^+ - \mathscr{A}_\theta^+)(\chi u)|_{\mathrm{H}^{1/2}(\mathbb{R}_+)}^2$$

$$\leq \frac{C'}{2\imath\pi} \int_{-\imath\infty}^{+\imath\infty} \frac{|\lambda|^2}{1 + |\lambda|} \left|\widehat{\mathfrak{K}}_\theta(\lambda + 1/4) - \widehat{\mathfrak{K}}_{\pi/2}(\lambda + 1/4)\right|^2 |\widehat{\chi u}(\lambda - 1/2)|^2 \, d\lambda,$$

$$\leq \frac{\bar{C}C'}{2\imath\pi} \int_{-\imath\infty}^{+\imath\infty} |\widehat{\chi u}(\lambda - 1/2)|^2 d\lambda = \bar{C}C'\|\chi u\|_{\mathrm{L}^2(\mathbb{R}_+)}^2 \leq \bar{C}C'C'_\chi\|u\|_{\mathrm{L}^2(\mathbb{R}_+)}^2.$$

Combining the estimates obtained for each term in Equation (38) now completes the proof. $\qquad\square$

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