Four Colour Theorem

Advanced Algorithm Project Preseentation

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March 9, 2018

The Map Coloring Problem

Question: How many colors are required to color a map of the United States so that no two adjacent regions are given the same color?

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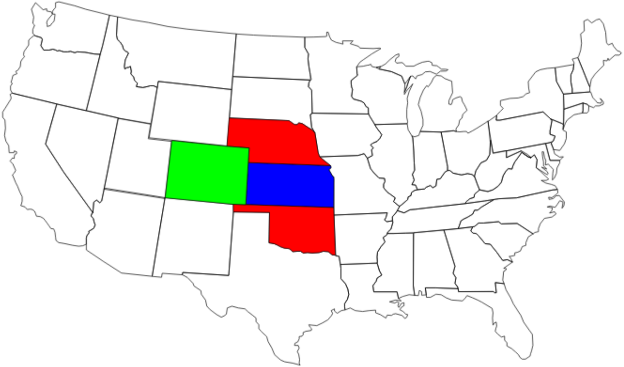
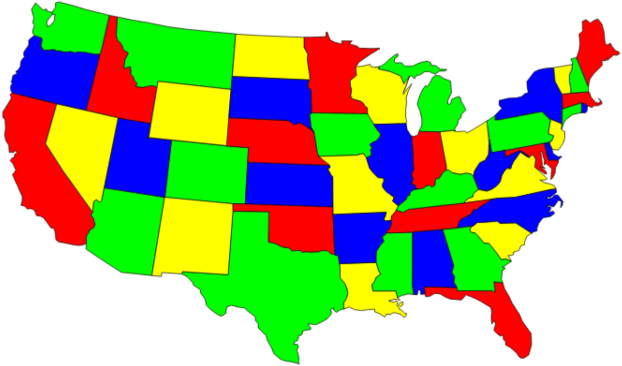


Color the map by Backtracking

The Map Coloring Problem

Answer: Four colors are enough. Three are not enough.

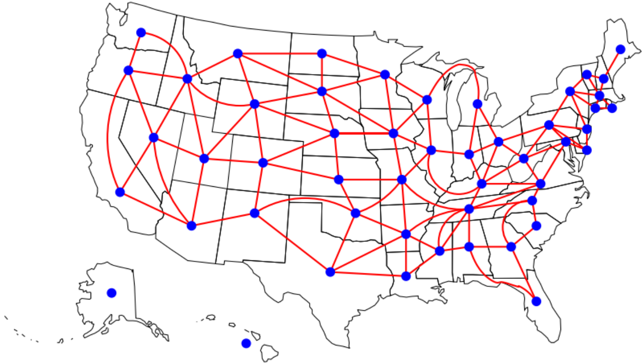


The Map Coloring Problem

Every map can be modeled as a planar graph.

A graph is planar if its vertices and edges **can be drawn** as points and line segments with no crossings.

Vertices represent regions; edges represent common borders.



Formal Definition of Four Color Theorem

The Four-Color Problem: Is there some **map (planar graph)** that requires five colors?
In order to give a negative answer, you have to show that every map no matter how cleverly constructed can be colored with 4 or fewer colors.

History of Four Color Theorem

- 1879: Alfred Kempe proves the **Four-Color Theorem (4CT)**: Four colors suffice to color any map.
- 1880: Peter Tait finds another proof.
- 1890: Percy John Heawood shows that Kempes proof was wrong.
- 1891: Julius Petersen shows that Taits proof was wrong.
- 20th century: Many failed attempts to (dis)prove the **4CT**. Some lead to interesting discoveries; many dont.

History of Four Color Theorem

- 1976: Kenneth Appel and Wolfgang Haken prove the 4CT. Their proof relies on checking a large number of cases by computer, sparking ongoing debate over what a proof really is.
- 1997: N. Robertson, D.P. Sanders, P.D. Seymour, and R. Thomas improve Appel and Hakens methods to reduce the number of cases (but still rely on computer assistance).
- 2005: Georges Gonthier publishes a formal proof (automating not just the case-checking, but the proof process itself).

Appel and Hakens Proof

In 1976, Kenneth Appel and Wolfgang Haken, of the University of Illinois, announced a proof of the Four-Color Theorem.

Suppose there exists a planar graph that requires more than 4 colors the idea is to show that something impossible must then happen.

Observation: If there is at least one such graph, then there is a smallest such graph (i.e., with the minimum number of vertices).

Let G be a non-4-colorable planar graph that is as small as possible (minimal counterexample).

Appel and Hakens Proof (Cont'd)

Step 1: Prove that G contains at least one of 1476 unavoidable configurations. (To do this, assign each vertex a charge. Let the electrons flow around G (according to 487 discharging rules). If a vertex still has electrons that it cannot discharge, the reason must be that there is one of those 1476 configurations nearby.)
Step 2: Prove that each one of those 1476 unavoidable configurations is reducible it can be replaced with something smaller without affecting the chromatic number of G. (This part of the proof was carried out by a computer.)
Conclusion: G was not a minimal counterexample. There is no such thing as a minimal non-4-colorable planar graph. Therefore, there are no non-4-colorable planar graphs!

Appel and Haken's Proof (Cont'd)

Appel and Haken, describing their proof of the 4CT:

This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in the 24 lemmas in the main sections of text. In addition, the reader is told that certain facts have been verified with the use of about twelve hundred hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail.

Reactions to Appel and Haken's Proof

> 4CT is not really a theorem ... [N]o mathematician has seen a proof of the 4CT, nor has any seen a proof that it has a proof. Moreover, it is very unlikely that any mathematician will ever see a proof of the 4CT.
>
> *Philosopher Thomas Tymoczko (1979)*

> I do not find it easy to say what we learned from all that.
>
> *Mathematician Paul Halmos (1990)*

> Few people are aware that it is really a ONE-LINE Proof: The following finite set of reducible configurations, let's call it S, is unavoidable. The set S itself does not have to be actually examined by human eyes, and perhaps should not. The computer would be much more reliable than any human in checking its claim. ... FOUR COLORS SUFFICE BECAUSE THE COMPUTER SAID SO!
>
> *Mathematician Doron Zeilberger (2002)*

Review and Correction to Appel and Haken's Proof

A discussion of errors, their correction, and other potential problems may be found in the above article, in [1], and in F. Bernhart's review of [1].

The main difficulties with the A&H proof:

- Part of the proof uses a computer, and cannot be verified by hand.
- even the part that is supposedly hand-checkable has not, as far as I know, been independently verified in its entirety.

the most comprehensive effort to verify the AH proof was undertaken by Schmidt. According to [1], during the one-year limitation imposed on his master's thesis, Schmidt was able to verify about 40% of part I of the A&H proof.

RSST Proof

Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas tried to verify the Appel & Haken proof, but soon gave up, and decided that it would be more profitable to work out our own proof. Neil Robertson, Daniel Sanders, Paul Seymour and Robin Thomas [RSST] (1995–1997) gave an improved proof. The basic idea of their proof is the same as Appel and Haken's.

Improvements in RSST Proof

- Unavoidable set has size **633 configurations** as opposed to the ~~1476 member set~~ of Appel and Haken.
- Discharging method uses only **32 discharging rules**, instead of the ~~487 discharging rules~~ of Appel & Haken.
- Obtaining a quadratic algorithm to 4- color planar graphs, an **improvement over the quadratic algorithm** of Appel and Haken.
- Their proof, including the computer part, has been independently verified, and the ideas have been and are being used to prove more general results.

Preliminaries

Theorem (Euler 1750)

Let G be a connected plane graph, and let n, m, f denote respectively the numbers of vertices, edges and faces of G. Then

$$n-m+f=2$$

.

maximal planar graph

A maximal planar graph is one to which no edge can be added without losing planarity. Thus in any embedding of a maximal planar graph G with $n \geq 3$, the boundary of every face of G is a triangle,

Preliminaries

Corollary

If G is a planar graph with $n(\geq 3)$ vertices and m edges, then $m \leq 3n - 6$. Moreover the equality holds if G is maximal planar.

Proof

We can assume without loss of generality that G is a maximal planar graph; otherwise add new edges without increasing n so that the resulting graph is maximal planar.
Every face is bounded by exactly three edges, and each edge is on the boundaries of two faces.

$$3f = 2m$$

Applying `Euler Theorem`, we obtain

$$m = 3n - 6$$

Preliminaries

Corollary

Let G be a planar graph. Then G has a vertex of degree at most 5.

Lemma

Every simple planar graph is 6-colorable.

Theorem (`Kuratowski 1930`)

A graph is planar if and only if it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.

Theorem

Every simple planar graph is 5-colorable.

Preliminaries

Internally 6-connected

A graph G is internally 6-connected if for every set X of at most five vertices, either the graph $G \setminus X$ obtained from G by deleting X is connected, or $|X| = 5$ and $G \setminus X$ has exactly two connected components, one of which consists of a single vertex. Every vertex of an internally 6-connected graph has degree at least five.

Theorem (Birkhoff's 1913)

Every minimal counterexample to the Four Color Theorem is an internally 6-connected triangulation.

Configuration

Configurations are technical devices that permit us to capture the structure of a small part of a larger triangulation.

If a reducible configuration appears in a sufficiently connected planar graph G, then one can construct in constant time a smaller planar graph $G\prime$ such that any 4-coloring of $G\prime$ can be converted to a 4-coloring of G in linear time.

If (G, γ) is a configuration, one should think of G as an induced subgraph of an internally 6-connected triangulation T, with $\gamma(v)$ being the degree of v in T.

Configuration

Near-Triangulation

A near-triangulation is a nonnull connected plane graph with one region designated as special such that every region, except possibly the special region, is a triangle

Configuration

Configuration

A configuration K is a pair (G, γ), where G is a near-triangulation, and γ is a mapping from $V(G)$ to the integers with the following properties:

- For every vertex v of G, if v is not incident with the special region of G, then $\gamma(v)$ equals $deg(v)$, the degree of v, and otherwise $\gamma(v) \geq deg(v)$ and in either case $\gamma(v) \geq 5$.
- for every vertex v of G, $G \setminus v$ has at most two components, and if there are two, then the degree of v in G is $\gamma(v) - 2$
- K has ring-size at least 2, where the ring-size of K is defined to be $\sum(\gamma(v) - deg(v) - 1)$, summed over all vertices v incident with the special region of G such that $G \setminus v$ is connected

Configuration

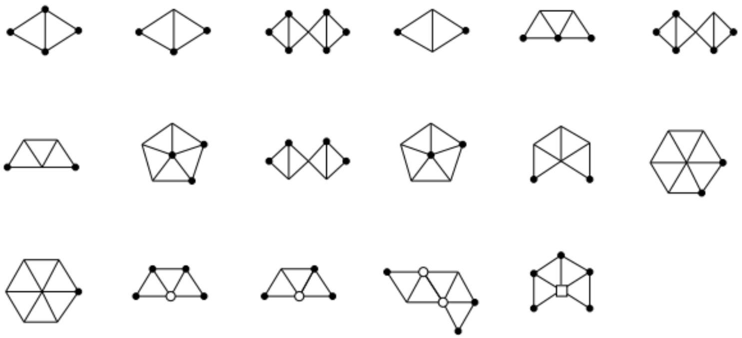
Drawing Configuration

one possibility is to draw the underlying graph, and write the value of γ next to each vertex. There is a more convenient way which is introduced by Heesch. The shapes of vertices indicate the value of $\gamma(v)$

Four colors are enough. Three are not enough.

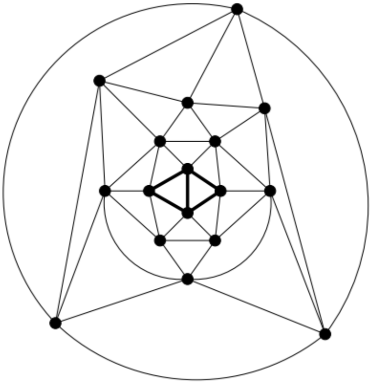
●	$\gamma_K(v) = 5$
·	$\gamma_K(v) = 6$
○	$\gamma_K(v) = 7$
□	$\gamma_K(v) = 8$
▽	$\gamma_K(v) = 9$
⬠	$\gamma_K(v) = 10$

Configuration



Isomorphism Of Configuration

Isomorphism of configurations is defined in the natural way. A configuration (G, γ) appears in triangulation T if G is an induced subgraph of T, every region of G except possibly the special region is a region of T, and $\gamma(v)$ equals the degree of v in T for every vertex v of G.



Configuration

We have exhibited a set U of 633 configurations such that

Theorem (`THEOREM 1`)

No member of U "appear" in a minimal counterexample to the 4CT

Theorem (`THEOREM 2`)

For every internally 6-connected triangulation T, some member of U "appears" in T

REDUCIBILITY

Theorem

No member of U "appear" in a minimal counterexample to the 4CT

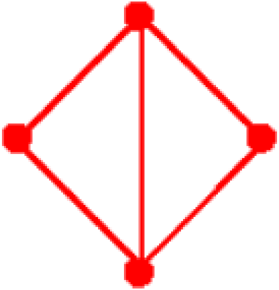
[`THEOREM 1`]

Proof

Suppose one of them does.

A configuration K appears in a triangulation T if K is an induced subgraph of T and for every vertex of K its label equals its degree in T

REDUCIBILITY (contd.)



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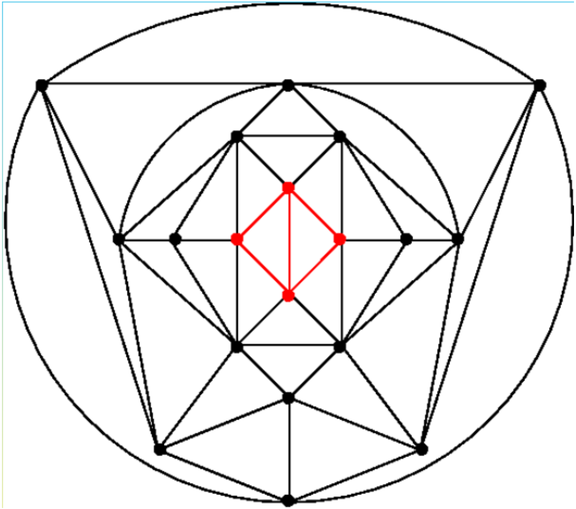

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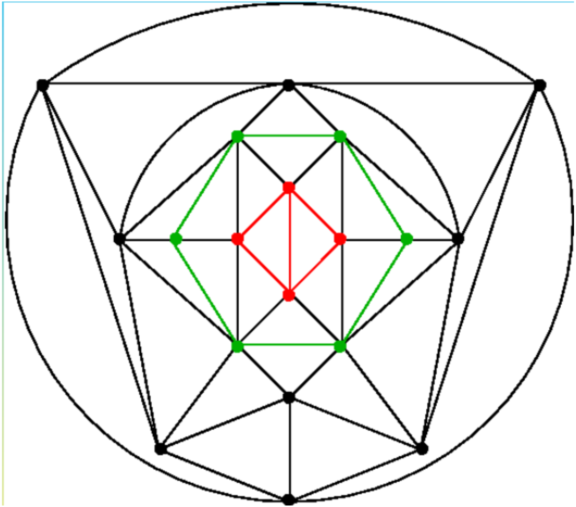

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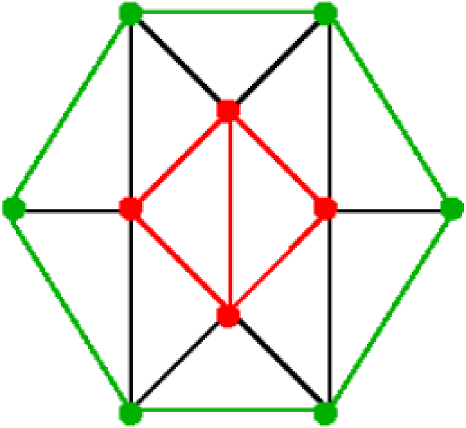

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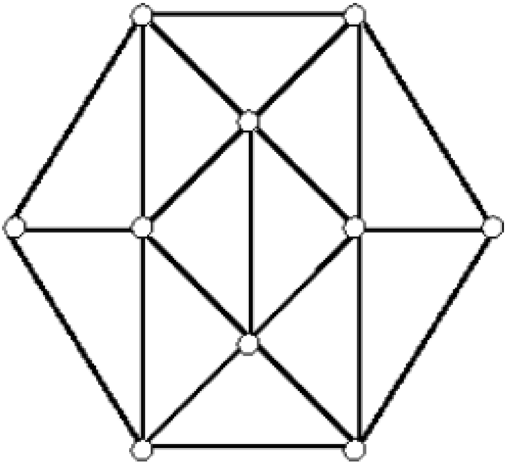

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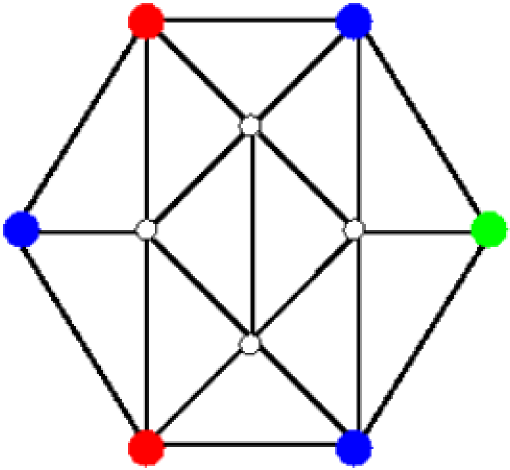

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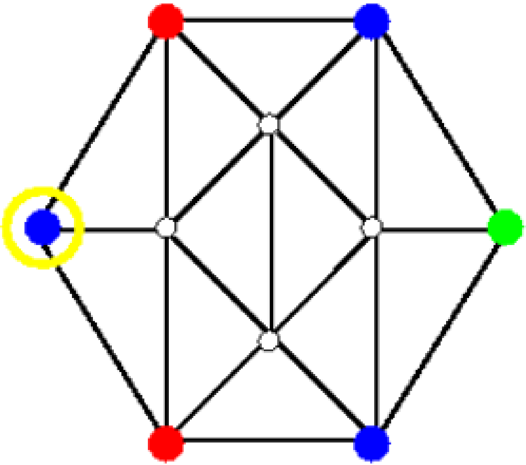

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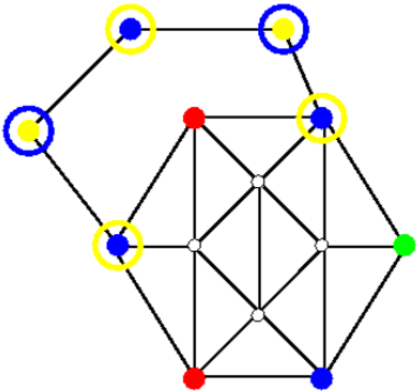

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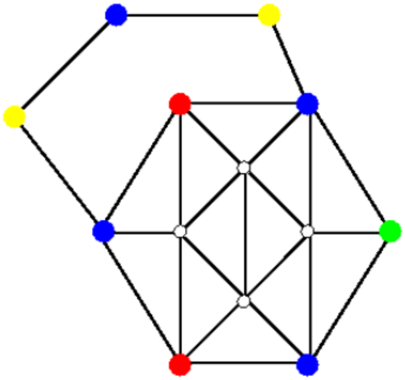

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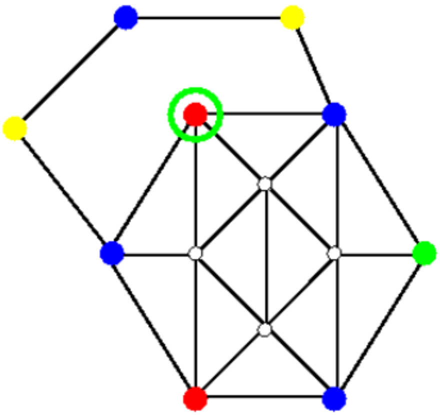

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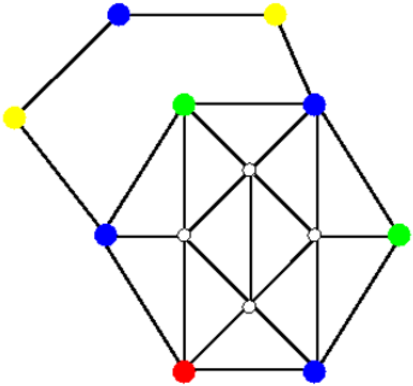

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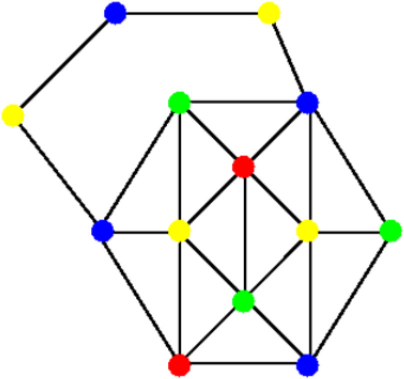
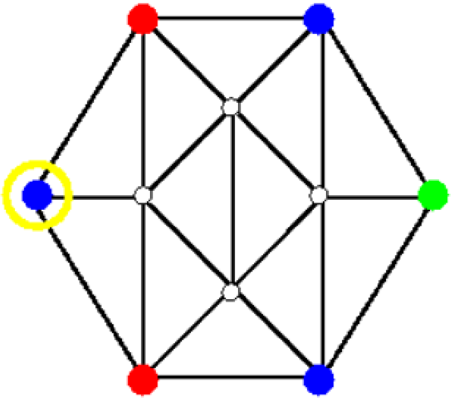


This gives a coloring of the entire graph, a contradiction.

REDUCIBILITY (contd.)



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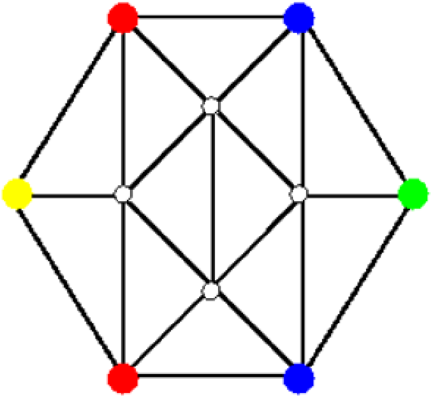

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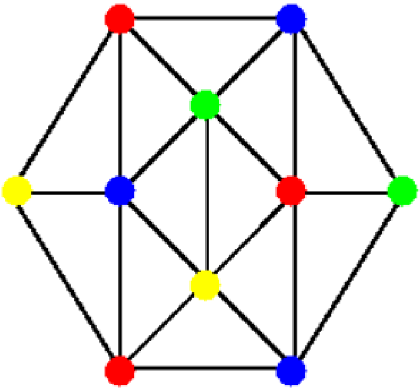


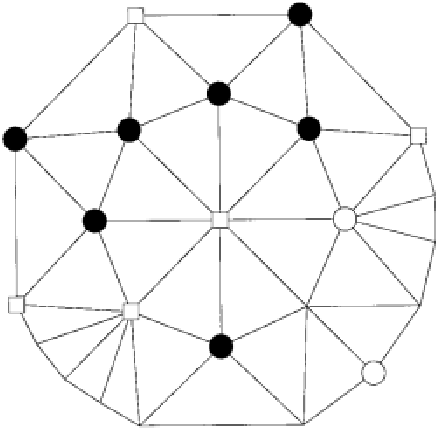
This gives a coloring of the entire graph, a contradiction.

REDUCIBILITY (contd.)

The order in which colorings are handled is important here. Reducibility can be automated and carried out on a computer. In fact, they must be carried out on a computer, because we need to test configurations with ring-size as large as 14, in which case there are almost $200,000$ colorings to be checked.

UNAVOIDABILITY

Cartwheel



UNAVOIDABILITY

Cartwheel

A cartwheel is a configuration W such that there is a vertex w and two circuits C_1, C_2 of $G(W)$ with the following properties:

- $\{w\}, V(C_1), V(C_2)$ are pairwise disjoint and have union $V(G(W))$
- C_1 and C_2 are both induced subgraphs of $G(W)$, and $U(C_2)$ bounds the infinite region of $G(W)$
- w is adjacent to all vertices of C_1 and to no vertices of C_2 .

It follows that the edges of $G(W)$ are of four kinds: edges of C_1 , edges of C_2 , edges between w and $V(C_1)$, and edges between $V(C_1)$ and $V(C_2)$. We call w the hub of the cartwheel.

UNAVOIDABILITY

Theorem (`Birkhoff's 1913`)

Let v be a vertex of an internally 6-connected triangulation T. There is a unique cartwheel appearing in T with hub v.

Path

A path is a walk with no repeated vertex in which every vertex has degree 2. Its length is $|\,E(Q)\,|$. It is a u, v-path if $u, v \in V(Q)$ and u, v are the vertices of Q of degree < 2.

UNAVOIDABILITY

Pass

A pass P is a quadruple (K, r, s, t), where

- K is a configuration
- r is a positive integer
- s and t are distinct adjacent vertices of $G(K)$
- for each $v \in V(G(K))$ there is an s, v-path and a t, v-path in $G(K)$, both of length ≤ 2

We write $r(P) = r, s(P) = s, t(P) = t, K(P) = K$. We call r the value of the pass, s its source, and t its sink.

UNAVOIDABILITY

A pass P appears in a triangulation T if $K(P)$ appears in T. A pass P appears in a cartwheel W if $K(P)$ appears in W.
Let $\mathcal{P}$ be a set of passes. We write $P \sim \mathcal{P}$ to denote that P is a pass isomorphic to a member of $\mathcal{P}$. If W is a cartwheel, we define $N_{\mathcal{P}}(W)$ to be:

$$10(6-\gamma_W(w)) + \sum (r(P) : P \sim \mathscr{P},\, P \text{ appears in } W,\, t(P) = w)$$

$$-\sum (r(P) : P \sim \mathscr{P},\, P \text{ appears in } W,\, s(P) = w),$$

where w is the hub of W.

UNAVOIDABILITY

Theorem

Let T be an internally 6-connected triangulation, and let P be a set of passes. Then the sum of $N_P(W)$, over all cartwheels W appearing in T, equals 120.

Proof

Let P be a pass appearing in T, with source s. We claim that P appears in W_s.

$$\begin{gathered}\sum(r(P) : P \sim \mathcal{P}, \text{ P appears in } T) \\ = \sum_{v \in V(T)} \sum(r(P) : P \sim \mathcal{P}, P \text{ appears in } W_v, s(P) = v)\end{gathered}$$

UNAVOIDABILITY

Proof (contd.)

The same equation holds with $s(P)$ replaced by $t(P)$, and consequently,

$$\sum_{v\in V(T)} N_{\mathcal{P}}(W_v) = \sum_{v\in V(T)} 10(6-\gamma_W(v))$$

Let $\mid V(T) \mid = n$. For each vertex v, $\gamma_W(v) = d_T(v)$, and

$$\sum_{v\in V(T)} d_T(v) = 2 \mid E(T) \mid = 6n-12$$

by the well-known application of Euler's formula. Hence,

$$\sum_{v\in V(T)} N_{\mathcal{P}}(W_v) = 60n - 10(6n-12) = 120$$

UNAVOIDABILITY

Theorem

Let T be an internally 6-connected triangulation, and let P be a set of passes. Then there is a cartwheel W appearing in T with $N_P(W) > 0$

Theorem

For every cartwheel W with $N_P(W) > 0$, some good configuration appears in W

our set P contains infinitely many non-isomorphic passes, but they can be divided conveniently into 32 classes, each described by what we call a “rule”.

UNAVOIDABILITY

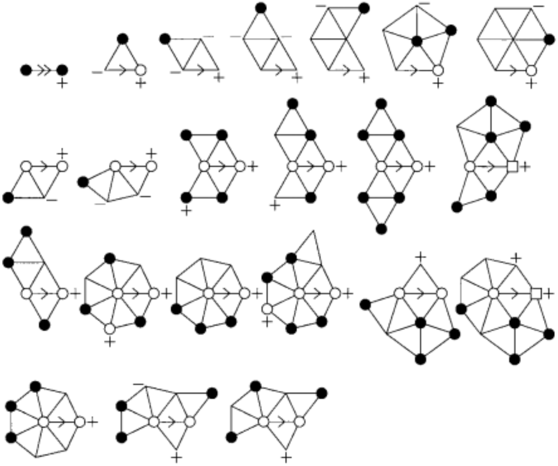
Rule

A rule is a 6-tuple $(G, \beta, \sigma, r, s, t)$, where

- G is a near-triangulation, and $G \setminus v$ is connected for every vertex v
- β is a map from $V(G)$ to $\mathbb{Z}_+$; and is a map from $V(G)$ to $\mathbb{Z}_+ \cap \{\infty\}$ satisfying $\beta(v) \leq \sigma(v)$ for every vertex v
- $r > 0$ is an integer
- s and t are distinct, adjacent vertices of G, and for every $v \in V(G)$ there is a v, s-path and a v, t-path of length ≤ 2

A pass P obeys a rule $(G, \beta, \sigma, r, s, t)$ if P is isomorphic to some (K, r, s, t), where $G(K) = G$ and $\beta(v) \leq \gamma_k(v) \leq$(v) for every vertex $v \in V(G)$

UNAVOIDABILITY



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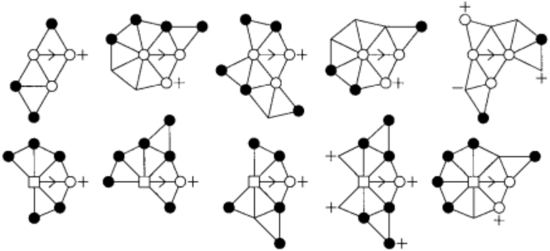

UNAVOIDABILITY

Lemma

Let W be a cartwheel, with hub w of degree 5 or 6. For $k = 1, \ldots, 32$ let p_k (respectively, q_k) be the sum of $r(P)$ over all passes $\mathcal{P}$ obeying rule k and appearing in W with sink (respectively, source) w. Suppose that no good configuration appears in W. Then:

- $p_1 = q_2 + q_3$
- $p_3 = q_4$
- $p_4 = q_5 + q_6$ *and,*
- $p_5 = q_7$

UNAVOIDABILITY

Theorem

Let W be a cartwheel with $N_{\mathcal{P}}(W) > 0$, and with hub of degree 5 or 6. Then a good configuration appears in W

Proof

Let w be the hub, and define p_k and q_k for $k = 1, \ldots, 32$. Suppose that no good configuration appears; we shall show that $N_{\mathcal{P}}(W) = 0$, a contradiction. First, let $\gamma_W(w) = 5$. Then $p_k = 0$ for $k = 2, ..., 32$ and $q_k = 0$ for $k = 4, ..., 32$ and so

$$N_{\mathcal{P}}(W) = 10 + p_1 - q_1 - q_2 - q_3 = 0$$

Now $\gamma_W(w) = 6$. Then,

$$N_{\mathcal{P}}(W) = p_1 + p_3 + p_4 + p_5 - q_2 - q_3 - q_4 - q_5 - q_6 - q_7 = 0$$

In either case, we have a contradiction.

UNAVOIDABILITY

Lemma

Let W be a cartwheel with hub w, and let v be a neighbour of w. If no good configuration appears in W, then the sum of $r(P)$ over all passes $P \in \mathcal{P}$ appearing in W with source v and sink w is at most 5.

UNAVOIDABILITY

Theorem

Let W be a cartwheel with $N_{\mathcal{P}}(W) > 0$, and with hub of degree ≥ 12. Then a good configuration appears in W

Proof

Suppose that no good configuration appears. Let $\gamma_W(w) = d$ and let D be the set of neighbours of w, where w is the hub of W. For each $v \in D$, let $R(v)$ be the sum of $r(P)$ over all passes $P \in \mathcal{P}$ appearing in W with source v and sink w. Then, $\sum_{v \in D} R(v) \leq 5$. Hence,

$$N_{\mathcal{P}}(W) = 10(6-d) + \sum_{v \in D} R(v) \leq= 10(6-d) + 5d = 60 - 5d \leq 0$$

a contradiction. The result follows.

UNAVOIDABILITY

Theorem

Let W be a cartwheel with $N_{\mathcal{P}}(W) > 0$, and with hub of degree $7, 8, 9, 10, 11$. Then a good configuration appears in W

For each of the five cases, we have a proof. Unfortunately they are very long (altogether about 13,000 lines, and a large proportion of the lines take some thought to verify), and so they cannot be given here. Moreover, although any line of the proofs can be checked by hand, the proofs themselves are not "really" checkable by hand because of their length. We therefore wrote the proofs so that they are machine-readable, and in fact, a computer can check these proofs in a few minutes.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal of This and That, 2(1):50–100, 2000.