On graphs that are not PCGs

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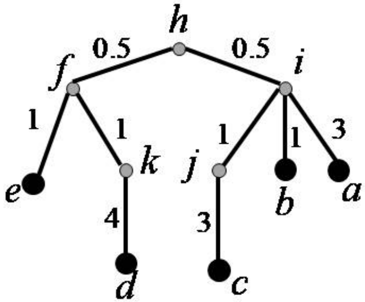
16th March, 2019

Pairwise Compatible Graphs

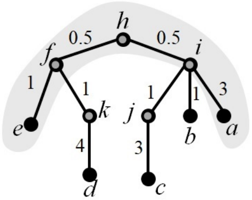


$d_{min} = 4$
$d_{max} = 7$

A pairwise compatible graph of (T, d_{min} ,d_{max})

- Has n vertices that correspond to the n leaves of T
- Two vertices are adjacent in G if and only if their tree distance is in [d_{min} ,d_{max}]

Pairwise Compatible Graphs

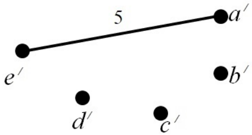


T

$d_{\min} = 4$

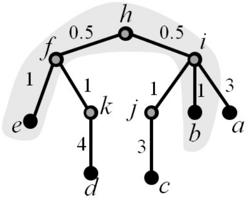
$d_{\max} = 7$

$5 \in [4,7]$



G

Pairwise Compatible Graphs

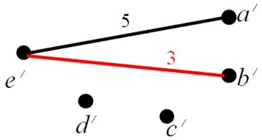


T

$d_{\text{min}} = 4$

$d_{\text{max}} = 7$

$3 \notin [4,7]$



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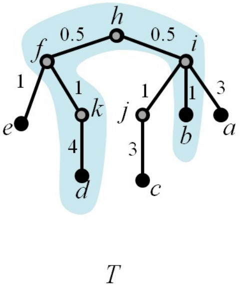
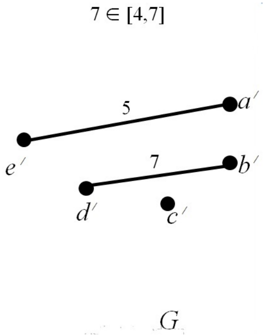
Pairwise Compatible Graphs



$d_{\min} = 4$

$d_{\max} = 7$



Problem Definition

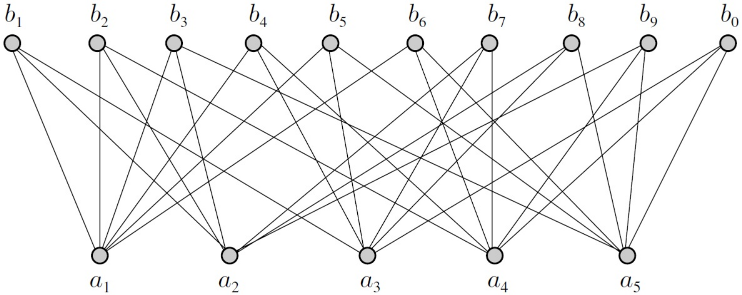
- Characterized graphs that are not PCG
- Characterize a generalized PCG Recognition problem to be NP-hard

Previous Works

- In 2003, **Kearney, Munro and Phillips** introduced the concept of PCG
  - Characterized evolutionary relationship
  - Proved that The clique problem is polynomially solvable for PCG if its pairwise compatibility tree can be constructed in polynomial time.
  - **Conjecture:** All graphs are PCG

Previous Works

- In 2010, **Yanhaona,Bayzid and Saidur Rahman Sir** refuted the conjecture by showing a graph of 15 vertices that is not PCG
  - Identified two restricted classes of bipartite graphs as PCG
  - Showed that the well known tree power graphs and some of their extensions are PCGs.



Previous Works

- In 2008, **Yanhaona,Hossain and Saidur Rahman** established some properties of PCGs
  - proved that graphs having cycles as their maximal biconnected components are PCGs
  - showed that all chordless cycles and single chord cycles are PCGs
- **Salma and Saidur Rahman** proved that every triangle-free maximum-degree-three outerplanar graph is a PCG

Previous Works

- **Calamoneri,Petreschi and Sinaimeri** analyzed the class of PCGs in relation to two particular subclasses resulting from the cases where $d_{min} = 0$(LPG) and $d_{max} = +\infty$ (mLPG)
  - proved that the intersection of these classes is not empty
  - Neither of them is contained in the other
- **Calamoneri, Frascaria, Sinaimeri** showed all graphs with at most seven vertices are pairwise compatibility graphs

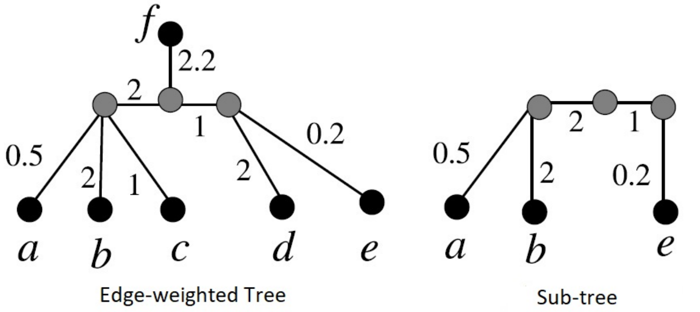
Contribution of this work

- In this paper a graph with 9 vertices was shown to be not PCG
- Then from the 9 vertice graph a 8 vertice graph was constructed which is the smallest graph that is not PCG
- Constructed a 20 vertices planar graph which is the first planar graph known to be not PCG.
- Showed that a variant of the PCG Recognition Problem is NP-hard.

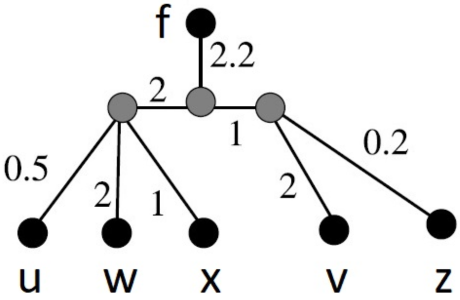
Some Terminologies

Let G =(V, E) be a graph with vertex set V, edge set E and u and v be two leaves of it's pair-wise compatibility tree T

- By P_{uv} we denote the unique path between u and v in T
- By $d_T(u, v)$ we denote the weighted distance between u and v
- By $T_{x_1 x_2 \ldots x_t}$ we denote the subgraph of T induced by the paths $P_{x_i x_j}$, where $1 \leq i, j \leq t$.

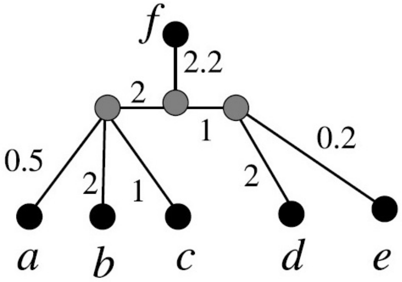


Lemma 1



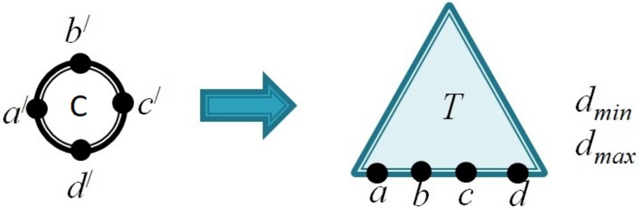
Let T be an edge-weighted tree and let u, v and w be three leaves of T such that P_{uv} is the longest path in T_{uvw}. Let x be a leaf of T other than u, v and w. Then $d_T(w,x) \leq d_T(u,x)$ or $d_T(w,x) \leq d_T(v,x)$

Lemma 2



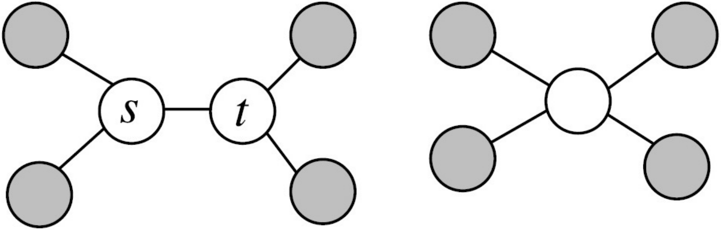
Let $G = PCG(T, d_{min}, d_{max})$. Let a, b, c, d, e be five leaves of T and a', b', c', d', e' be the corresponding vertices of G, respectively. Let P_{ae} and P_{bd} be the longest path in T_{abcde} and T_{bcd}, respectively. Then any vertex x' in G that is adjacent to a', c', e' must be adjacent to at least one vertex in b', d'.

Lemma 3



Let C be the cycle a', b', c', d' of four vertices. If $C = PCG(T, d_{min}, d_{max})$ for some tree T and values d_{min} and d_{max}, where the leaves a,b, c and d of T correspond to the vertices a', b', c' and d' of G, respectively, then $d_T(a, c)$ and $d_T(b, d)$ cannot be both greater than d_{max}.

Lemma 3



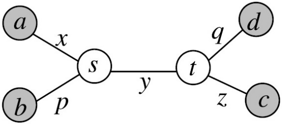
Possible Topologies for T

- Replace each vertex of degree 2
- T can have any of the 2 topologies above

Lemma 3

Case 1: Suppose for a contradict $d_T(a, c)$ and $d_T(b, d)$ are greater than d_{max}

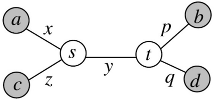
- $d_T(a, c) + d_T(b, d) =$ x+z+p+q+2y $> 2d_{max}$
- a' and d' are adjacent, $d_{min} \leq$ x +y +q $\leq d_{max}$
- b' and c' are adjacent, $d_{min} \leq p + y + z \leq d_{max}$
- $x+z+p+q+2y \leq 2d_{max}$, which contradicts that $d_T(a, c) + d_T(b, d) > 2d_{max}$.



Case 1

Lemma 3

Case 2: Suppose for a contradict $d_T(a,c)$ and $d_T(b,d)$ are greater than d_{max}

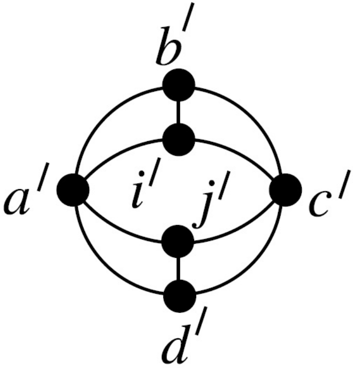


Case 2

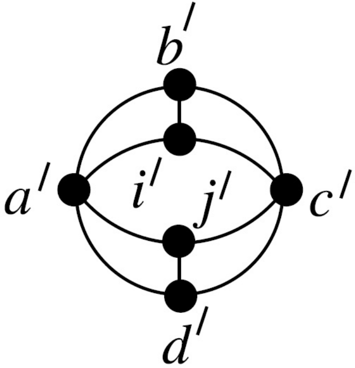
- $d_T(a,c) = x + z > d_{max}$, $d_T(b,d) = p + q > d_{max}$
- either $x > d_{max}/2$ or $z > d_{max}/2$
- either $p > d_{max}/2$ or $q > d_{max}/2$
- $d_T(a,b) \geq x + p$, $d_T(b,c) \geq z + p$, $d_T(c,d) \geq z + q$,$d_T(a,d) \geq x + q$
- one of the four pairs among $(a',b'),(b',c'),(c',d'),(a',d')$ must be non-adjacent which contradicts that a',b',c',d' is a cycle .

Lemma 4

We now construct a graph H with six vertices a/, b/, c/, d/, i/, j/ such that each pair of vertices in H are adjacent except the pairs (a/, c/), (b/, d/), (i/, d/), (j/, b/), (i/, j/),

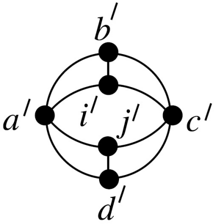


Lemma 4



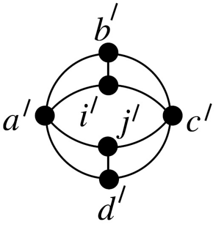
Let $H = PCG(T, d_{min}, d_{max})$. Let a, b, c, d, i, j be the leaves of T that correspond to the vertices a', b', c', d', i', j' of H. Then at least one of $d_T(a, c)$, $d_T(b, d)$, $d_T(i, d)$, $d_T(j, b)$, $d_T(i, j)$ must be greater than d_{max}.

Lemma 4



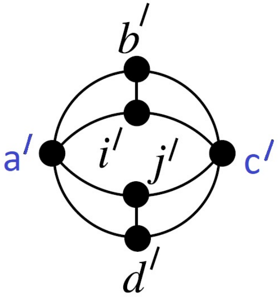
- For each pair $(x', y') \in (a', c'), (b', d'), (i', d'), (j', b'), (i', j')$, x' and y' are non-adjacent in H
- Assume that each of $d_T(a, c), d_T(b, d), d_T(i, d), d_T(j, b)$ is less than d_{min}, and then prove that $d_T(i, j)$ must be greater than d_{max}
- Suppose for a contradiction that $d_T(i, j) < d_{min}$

Lemma 4



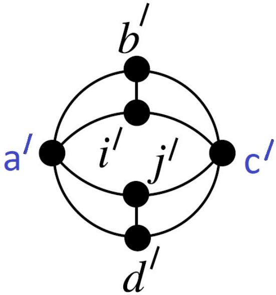
- since i' and b' are adjacent in H , $d_T(i,b) \leq d_T max$
- path P_{ib} must be the longest path T_{ijb},By Lemma 1, $d_T(j,d) \leq d_T(i,d)$ or $d_T(j,d) \leq d_T(b,d)$.
- $d_T(i,d) < d_{min}$ and $d_T(b,d) < d_{min}$, the inequality $d_T(j,d) < d_{min}$
- But this contradicts that $j', d'$are adjacent in G.

Lemma 5



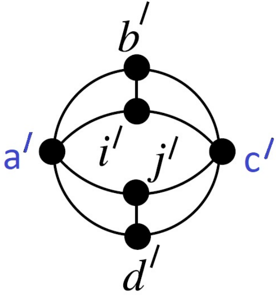
Let $G = PCG(T, d_{min}, d_{max})$ be a graph that contains an induced subgraph G′ isomorphic to H. Let a, b, c, d, i, j be the leaves of T that correspond to the vertices a′, b′, c′, d′, i′, j′o f G′. Let a′ and c′ be the vertices of degree four in G′. Then $d_T(a, c)$ must be less than d_{min}.

Lemma 5



- Suppose for a contradiction that $d_T(a, c) > d_{max}$
- Subgraph induced by a', b', c', d' is a cycle, by Lemma 3, $d_T(b', d') < d_{min}$
- Subgraph induced by a', i', c', d' is a cycle, by Lemma 3, $d_T(i', d') < d_{min}$
- P_{bi} is the longest path in T_{ibd}

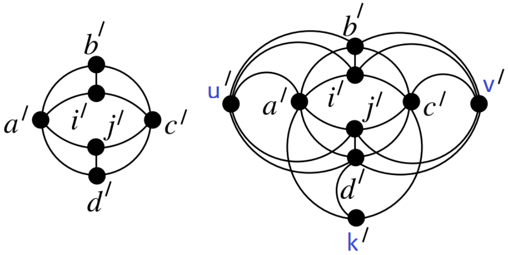
Lemma 5



- For each pair $(x', y') \in (a', b'), (a', d'), (a', i'),$ $(b', d'), (b', c'), (b', i'), (c', d'), (c', i'), (d', i'),$ $d_T(x, y) \leq d_{max}$
- Therefore, P_{ac} is the longest path in T_{abcdi}
- By Lemma 2, any vertex j'in G' that is adjacent to a', c', d' must be adjacent to i' or b'
- But i' nor b' is adjacent to j', a contradiction

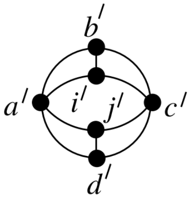
Theorem 1

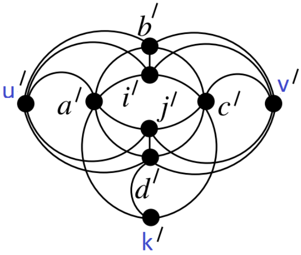
- We add three vertices k‘, u‘, v‘ to H to construct a 9 vertex graph G1
- This theorem proves that G1 is not a PCG



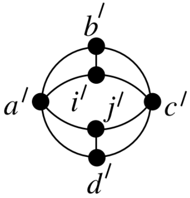
G1 is not a PCG

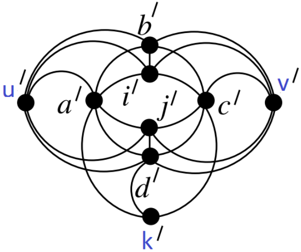
- For each pair $(x', y') \in (a', c'), (b', d'), (i', d'), (j', b'), (i', j')$, x' and y' are non-adjacent in H
- By Lemma 4 at least one of $d_T(a, c)$, $d_T(b, d)$, $d_T(i, d)$, $d_T(j, b)$, $d_T(i, j)$ must be greater than d_{max}

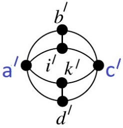


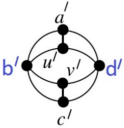


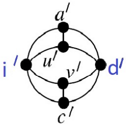
G1 is not a PCG

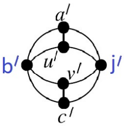


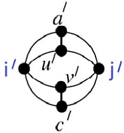




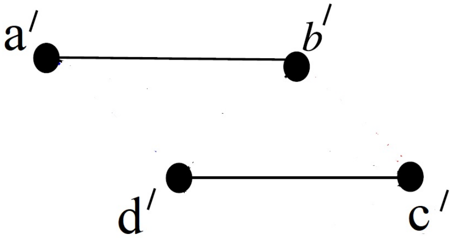








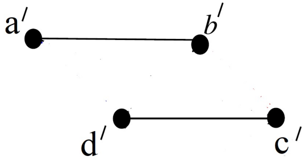
Lemma 6



Let G be a graph of four vertices a', b', c', d' and two edges (a', b') and (c', d'). If $G = PCG(T, d_{min}, d_{max})$ for some tree T and values d_{min} and d_{max}, where the leaves a, b, c, d of T correspond to the vertices a', b', c', d' of G, respectively, then at least one of $d_T(a,d), d_T(b,d), d_T(b,c), d_T(a,c)$ must be greater than d_{max}.

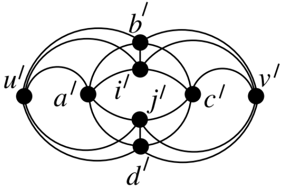
Lemma 6

Each of $d_T(a,d), d_T(b,d), d_T(b,c), d_T(a,c)$ is either greater than d_{max} or less than d_{min}.Suppose for a contradiction that $d_T(a,d), d_T(b,d), d_T(b,c), d_T(a,c)$ are less than d_{min}.



- a', b' are adjacent and $d_T(a,c), d_T(b,c)$ are less than d_{min}. P_{ab} must be the longest path in T_{abc}.
- $d_T(c,d) \leq d_T(a,d)$ or $d_T(c,d) \leq d_T(b,d)$.
- By assumption, both $d_T(a,d)$ and $d_T(b,d)$ are less than d_{min}.
- $d_T(c,d) < d_{min}$, which contradicts that c' and d' are adjacent in G.

Lemma 7

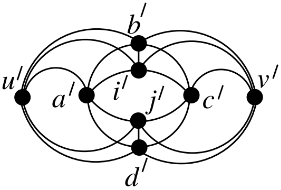


Let $G2 = PCG(T, d_{min}, d_{max})$ and let a, b, c, d, i, j, u, v be the leaves of T that correspond to the vertices $a', b', c', d', i', j', u', v'$ of G2. Then (a) at least one of $d_T(u, v), d_T(a, v), d_T(a, c), d_T(u, c)$ must be greater than d_{max}, and (b) at least one of $d_T(b, j), d_T(b, d), d_T(i, d)$, $d_T(i, j)$ must be greater than d_{max}.

We only prove claim (a), i.e., one of $d_T(u, v), d_T(a, v), d_T(a, c), d_T(u, c)$ must be greater than d_{max}, since the proof for claim (b) is similar.

Lemma 7

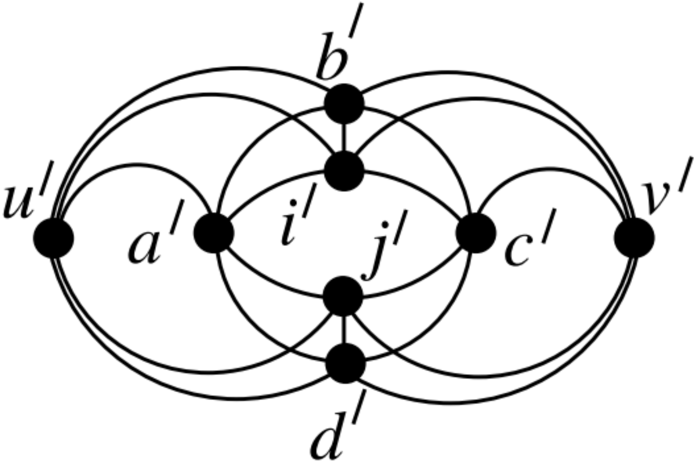
Each of $d_T(u,v), d_T(a,v), d_T(a,c), d_T(u,c)$ is either greater than d_{max} or less than d_{min}. Suppose for a contradiction that $d_T(u,v), d_T(a,v), d_T(a,c), d_T(u,c)$ are less than d_{min}



- u', a' are adjacent and $d_T(u,c), d_T(a,c)$ are less than d_{min}, P_{au} must be the longest path in T_{acu}.
- $d_T(c,v) \leq d_T(a,v)$ or $d_T(c,v) \leq d_T(u,v)$
- $d_T(u,v), d_T(a,v)$ are less than d_{min}. $d_T(c,v) < d_{min}$, which contradicts that c' and v' are adjacent in G2.

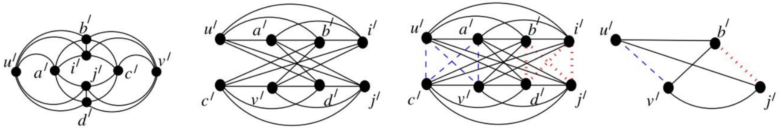
Theorem 2

G2 is not a PCG.



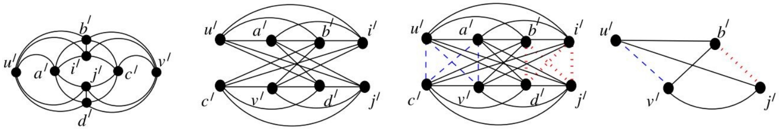
Theorem 2

For a contradiction that $G2 = PCG(T, d_{min}, d_{max})$



- For any $((w', x'), (y', z'))$, where $(w', x') \in (u', v'), (a', v'), (a', c'), (u', c')$ and $(y', z') \in (b', j'), (b', d'), (i', d'), (i', j')$, the vertices w', x', y', z' induce a cycle C such that w', x' and y', z' are non-adjacent in C.

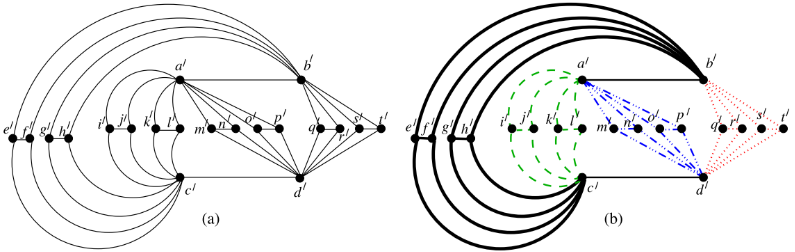
Theorem 2



- For some $((w', x'), (y', z'))$, both $d_T(w, x)$ and $d_T(y, z)$ are greater than d_{max}. This contradicts Lemma 3 since the vertices w', x', y', z' induce a cycle.

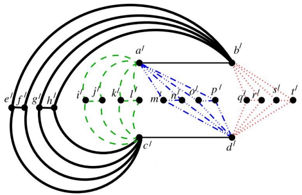
Theorem 3

G_p is not a PCG.



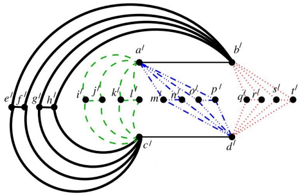
Theorem 3

For a contradiction that $G_p = PCG(T, d_{min}, d_{max})$



- a', b', c', d' induce a graph with two edges (a', b') and (c', d'), by Lemma 6, one of $d_T(a, d), d_T(b, d), d_T(b, c), d_T(a, c)$ must be greater than d_{max}.

Theorem 3



- For any pair $(x', y') \in (a', d'), (b', d'), (b', c'), (a', c')$, there exists an induced sub-graph in G_p that is isomorphic to H. $d_T(x, y) < d_{min}$, which contradicts that at least one of $d_T(a, d), d_T(b, d), d_T(b, c), d_T(a, c)$ must be greater than d_{max}. Consequently, G_p cannot be a PCG.

Generalized PCG Recognition Problem

- Given a graph G and a subset S of edges of its complement graph, determine a PCG G'=(T, d_{min}, d_{max}) that contains G as a subgraph, but does not contain any edge of S
- If S contains all edges of $\bar{G}$, then it is the problem of deciding whether G is a PCG
- The PCG recognition problem is a special case of the generalized PCG recognition problem.

Generalized PCG Recognition Problem

- The generalized PCG recognition problem is NP-hard if we require maximum number of edges of S to have weighted tree distance greater than d_{max} between their corresponding leaves
- Thus comes the Max Generalized PCG Recognition problem
- This helps characterize the complexity of the PCG recognition problem

Max-Generalized-PCG-Recognition

Problem: Max-Generalized-PCG-Recognition
Instance: A graph G, a subset S of the edges of its complement graph, and a positive integer k.
Question: Is there a PCG $G‘$=PCG(T, d_{min}, d_{max})such that $G‘$ contains G as a subgraph but does not contain any edge of S; and at least k edges of S have distance greater than d_{max} between their corresponding leaves in T?

NP-hardness of Max-Generalized-PCG-Recognition

The NP-hardness of Max-Generalized-PCG-Recognition is proved by reduction form Monotone-One-In-Three-3-SAT

Problem: Monotone-One-In-Three-3-SAT

Instance: A set U of variables and a collection C of clauses over U such that each clause consists of exactly three non-negated literals.

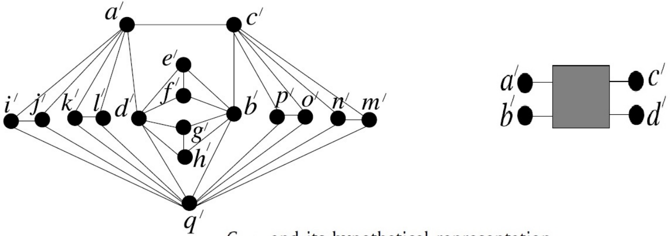
Question: Is there a satisfying truth assignment for U such that each clause in C contains exactly one true literal?

NP-hardness of Max-Generalized-PCG-Recognition

- Let I(U, C) be an instance of Monotone-One-In-Three-3-SAT
- An instance I(G, S,k) of Max-Generalized-PCG-Recognition is Constructed
- I(U, C)has an affirmative answer if and only if I(G, S, k)has an affirmative answer

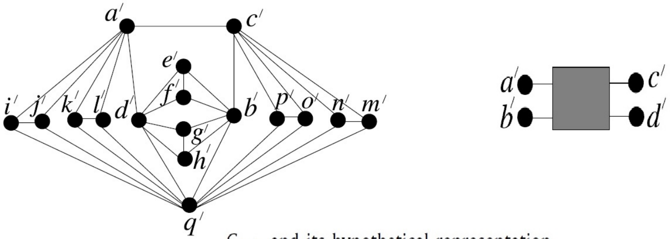
Graph G_{not} as NOT gate

- Let G_{not} be the graph showed below
- We see how to use this graph as a NOT gate



G_{not}, and its hypothetical representation

Lemma 8

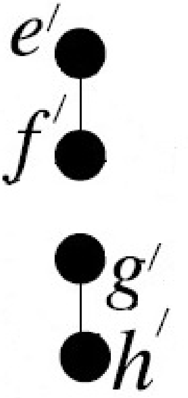


G_{not}, and its hypothetical representation

Assume that $G_{not} = PCG(T, d_{min}, d_{max})$, where a, b, . . . , q are the leaves of T that correspond to the vertices $a', b', ..., q'$ of G_{not}. Then $d_T(c, d) < d_{min}$ if and only if $d_T(a, b) > d_{max}$.

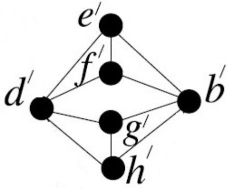
Lemma 8

By Lemma 6, from the subgraph one of $d_T(e, g)$, $d_T(e, h)$, $d_T(f, g)$, $d_T(f, h)$ must be greater than d_{max}



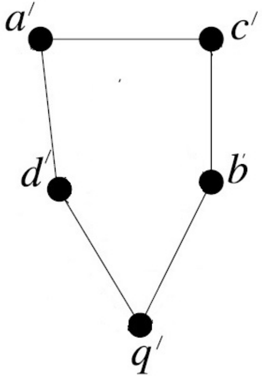
Lemma 8

- Observe that for any pair (x, y) (e', g'), (e', h'), (f', g'), (f', h'), the vertices b', x', d', y' form an induced cycle
- By Lemma 3, $d_T(b, d) < d_{min}$
- Similarly, $d_T(a, q) < d_{min}$ and $d_T(c, q) < d_{min}$



Lemma 8

- Since a', c', b', q', d' induce a cycle of five vertices, one of $d_T(a, b)$, $d_T(c, d)$, $d_T(a, q)$, $d_T(c, q)$, $d_T(b, d)$ is greater than d_{max}
- As $d_T(a, q)$, $d_T(c, q)$, $d_T(b, d)$ are less than d_{min}, one of or both $d_T(a, b)$and $d_T(c, d)$ are greater than d_{max}

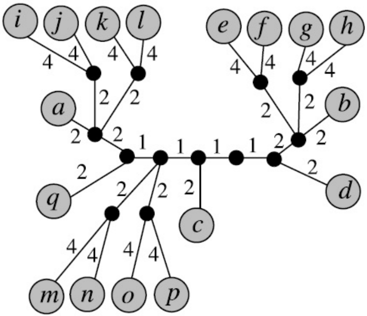


Lemma 8

- Without loss of generality assume that $d_T(a, b) > d_{max}$
- In T_{abc}, P_{ab} is the largest path as a', c' and b', c' adjacent
- By Lemma 1, $d_T(c, d) \leq d_T(a, d)$ or $d_T(c, d) \leq d_T(b, d)$
- Since $d_T(a, d) \leq d_{max}$ and $d_T(b, d) < d_{min}$, $d_T(c, d)$ must be less than d_{min}
- Similarly, we can prove that if $d_T(c, d) > d_{max}$, then $d_T(a, b) < d_{min}$.

Properties of G_{not}

- The vertices a,b and c,d play the role of the input and output of a NOT gate
- The figure below describes a pair-wise compatibility tree T, where G_{not}= PCG(T, 7, 11)and $d_T(a, b) > d_{max}$ and $d_T(c, d) < d_{min}$



Properties of G_{not}

- Observe once we construct the tree T_{abqcd}, it it straightforward to add the trees T_{efgh}, T_{ijkl} and T_{mnop}.
- So in the rest of presentation we only consider the simplified representation for T, as shown below figure

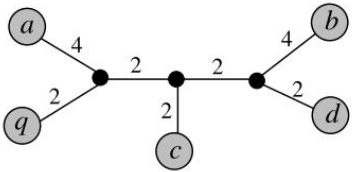
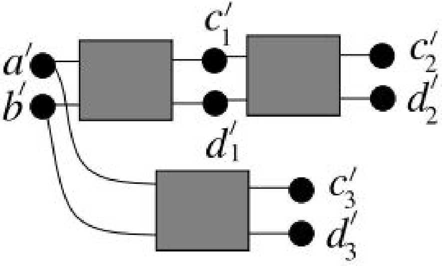


Figure: Simplified representation of T

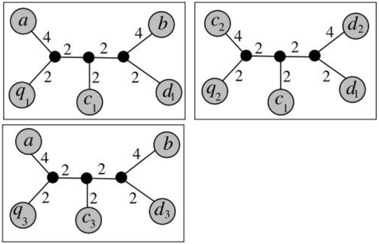
Cascading of NOT gates

- By cascading NOT gates can duplicate or invert the input
- Here in the figure below we illustrate the cascading of NOT gates.



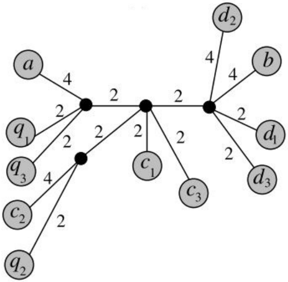
Cascading of NOT gates

- The below figure shows the simplified tree representations for three gates
- If any input pair (respectively, output pair) x,y of the NOT gate is true (respectively, false), then the corresponding unique path in the tree has the weight sequence (4, 2, 2, 4)(respectively, (2, 2, 2)).



Cascading of NOT gates

- Tree merging operation
- The figure below illustrates the tree that corresponds to the cascading of the three NOT gates .



Cascading of NOT gates

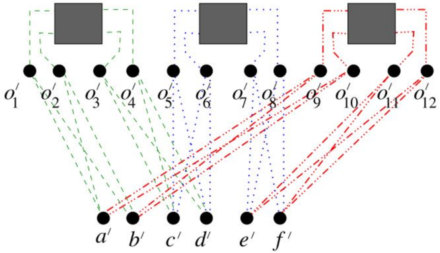
- The PCG G' of the final tree T contains all the edges of G_{not} graphs, and also many redundant edges
- None of these redundant edges can belong to a single G_{not}
- G_{not} has 101 non-adjacent pairs
- By construction, in any pairwise compatibility tree T' of G_{not}, all the distances $d'_T(a,q)$, $d'_T(c,q)$, $d'_T(b,d)$ and one of $d'_T(a,b)$, $d'_T(c,d)$ must be less than d_{min}. Therefore, at most 97 edges of compliment of G_{not} can have distance greater than d_{max}.

Literal gadgets

- Each literal gadget consists of a pair of non-adjacent vertices
- Every edge determined by these two vertices, belongs to S
- A literal $(a', b')$is true if and only if $d_T(a, b) > d_{max}$; else it is false

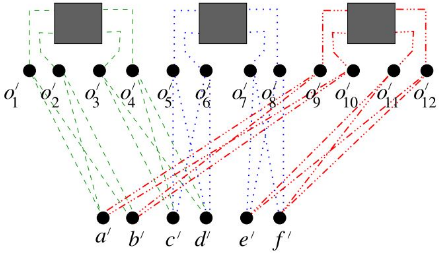
Clause gadgets

- Every clause gadget G_{clause}, as shown in below figure, corresponds to a logic circuit L that is consistent if and only if at most one of its three inputs is true
- The 3 pairs of vertices (a', b'), (c', d'), and (e', f') of G_{clause} play the role of the inputs



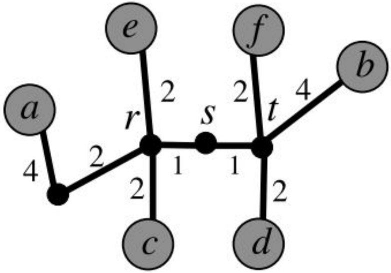
Clause gadgets

- For each pair of inputs, e.g., $((a', b'), (c', d'))$, G_{clause} contains a G_{not} such that the ports o_1‘, o_2‘ of G_{not} form a cycle with a‘, b‘, and the ports o_3‘, o_4‘ of G_{not} form a cycle with c‘, d‘
- L is consistent if and only if at most one input is true.



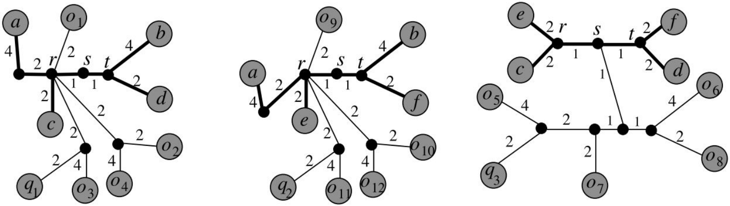
Construction of G_{clause}

- A pairwise compatibility tree T such that the corresponding PCG G'_{clause} contains G_{clause} a subgraph
- r, s, t is the medial path of T



Construction of G_{clause}

- Now we add the subtrees that correspond to the instances of G_{not} to T

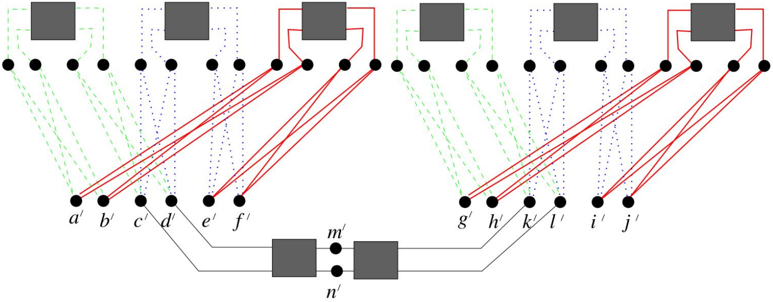


Theorem 4

Max-Generalized-PCG-Recognition is NP-hard

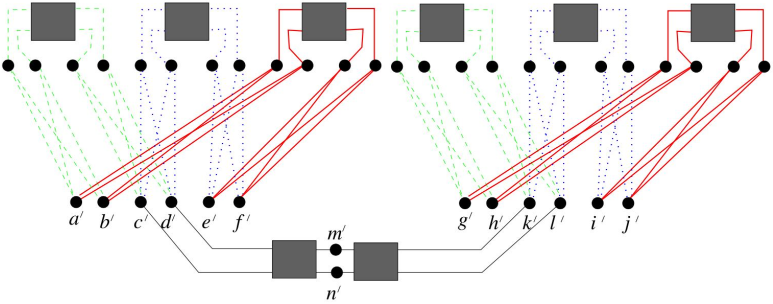
Max-Generalized-PCG-Recognition is NP-hard

- Let an instance of Monotone-One-In-Three-3-SAT be I(U,C)
- U$= x_i, ..., x_t$and C $= c_1, c_2, ..., c_t$'
- If the same literal appears in more than two clauses, we create a copy of by cascading of NOT gates as illustrated below



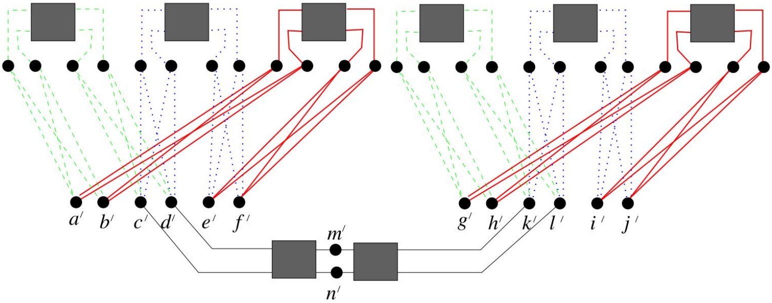
Max-Generalized-PCG-Recognition is NP-hard

- The set S consists of the edges of compliment graph of G_{not} and the edges that are determined by the literal gadgets
- N is the number of instances of G_{not} in G
- Since each G_{not} has 101 non-adjacent pairs, $|S|$=101N+t and $|k|$ =97N+t‘



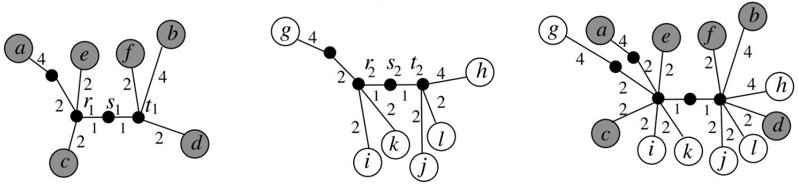
Max-Generalized-PCG-Recognition is NP-hard

- Assume that I(U, C)has an affirmative answer
- Noe we construct a PCG $G' = (T, d_{min}, d_{max})$ such that G' contains G as a subgraph, does not contain any edge of S, and at least k edges of S have distance greater than d_{max} between their corresponding leaves in T



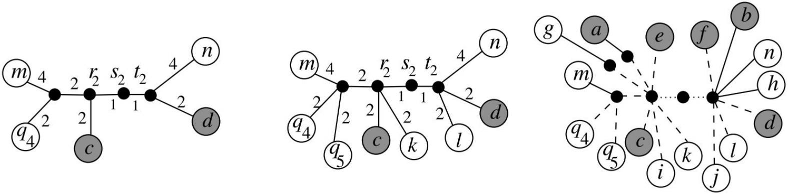
Max-Generalized-PCG-Recognition is NP-hard

- Construction of T(j) for each clause C_j
- Merging of all T(j) and remove any duplicate vertex or multi-edge



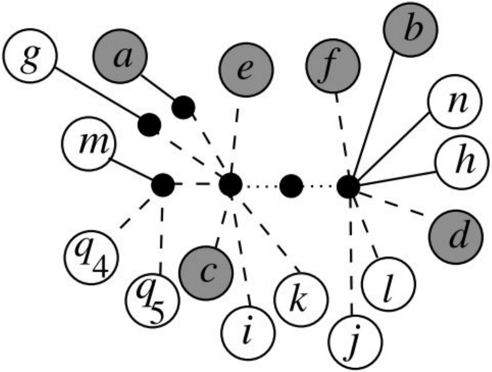
Max-Generalized-PCG-Recognition is NP-hard

- Finally, add the subtrees corresponding to the instances of G_{not} that was used for duplicating the input values



Max-Generalized-PCG-Recognition is NP-hard

- Let the resulting tree be T
- We prove that its corresponding PCG is the required PCG G‘



Max-Generalized-PCG-Recognition is NP-hard

- Assume that I(U, C)does not have any affirmative answer
- We see the prove that in any PCG G‘ that contains G as a subgraph, must have less than k $=$97N$+t$‘ edges of S

On Graphs That Are Not PCG

That was all in our presentation

Thank You