Root Finding Method:

continue finding the value of x untill $f(x) \approx 0$

1) Bisection Method

2) Method of false position

3) Newton's Raphson's Method

4) fixed point iteration method

Bisection Method: If a function $f(x)$ is continuous between a and b, and $f(a) \cdot f(b) < 0$ (or $f(a)$ and $f(b)$ are of opposite sign) then there exists at least one root between a and b.

Then the first approximation of the root is in

$$x_1 = \frac{a+b}{2}$$

then if $f(x_1) = 0$, x_1 is root of $f(x) = 0$

otherwise the root lies between a and x_1 or x_1 and b according to $f(x_1)$ is negative or positive. Then we bisect the interval as before and continue the process untill the root is foun

Q: Find a root of an equation $f(x) = x^3 + x^2$

between 0 and 1.

→ Step 1: $f(0) = -1 < 0$

$f(1) = 1 > 0$

∴ The root lies between 0 and 1.

$x_0 = \frac{0+1}{2} = 0.5$

$f(x_0) = f(0.5) = -0.625 < 0$

Step 2: $f(0.5) = -0.625 < 0$

$f(1) = 1 > 0$

The root lies between 0.5 and 1.

∴ $x_1 = \frac{0.5+1}{2} = 0.75$

∴ $f(x_1) = f(0.75) = -0.01562 < 0$

Step 3: $f(0.75) = -0.01562 < 0$

$f(1) = 1 > 0$

$\therefore$ The root lies between 0.75 and 1

$\therefore x_2 = \frac{0.75+1}{2} = 0.875$

$f(x_2) = f(0.875) = 0.43555 > 0$

step 4: $f(0.875) = 0.43555 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.75 and 0.875

$\therefore x_3 = \frac{0.75+0.875}{2} = 0.8125$

$\therefore f(x_3) = f(0.8125) = 0.19653 > 0$

Step 5: $f(0.8125) = 0.19653 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ the root lies between 0.8125 and 0.75

$\therefore x_4 = \frac{0.75+0.8125}{2} = 0.78125$

$\therefore f(x_4) = f(0.78125) = 0.08719 > 0$

Step 6: $f(0.78125) = 0.08719 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.78125 and 0.75

$$\therefore x_5 = \frac{0.75 + 0.78125}{2} = 0.76562$$

$$f(x_5) = f(0.76562) = 0.03498 > 0$$

Step 7: $f(0.76562) = 0.03498 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.75 and 0.76562

$$\therefore x_6 = \frac{0.75 + 0.76562}{2} = 0.75781$$

$$f(x_6) = f(0.75781) = 0.00948 > 0$$

Step 8: $f(0.75781) = 0.00948 > 0$

$f(0.75) = -0.01562 < 0$

$$\therefore x_7 = \frac{0.75 + 0.75781}{2} = 0.75391$$

$f(0.75391) = -0.00312 < 0$

Step 9: $f(0.75391) = -0.00312 < 0$

$f(0.75781) = 0.009486 > 0$

$$x_8 = \frac{0.75391 + 0.75781}{2} = 0.75586$$

$f(0.75586) = 0.00316 < 0$

Step 10:

$f(0.75586) = 0.00316 > 0$

$f(0.75391) = -0.00312 > 0$

$$x_9 = \frac{0.75586 + 0.75391}{2} = 0.754885$$

$\therefore f(0.754885) = 0.00002 > 0$

$\therefore$ So, the approximate root is $x_9 = 0.755$

Q: Find a root of an equation:

$$f(x) = x^2 + \sin x - 1 = 0$$

Step 1:

Here, $f(0) = -1 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_0 = \frac{0+1}{2} = 0.5$

$f(0.5) = -0.7413 < 0$

Step 2: $f(0.5) = -0.7413 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_1 = \frac{0.5+1}{2} = 0.75$

$f(0.75) = -0.4244 < 0$

Step 3: $f(0.75) = -0.4244 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_2 = \frac{0.75+1}{2} = 0.875$

$\therefore f(0.875) = -0.2191 < 0$

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step 4: $f(0.875) = -0.2191 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_3 = \frac{0.875+1}{2} = 0.9375$

$f(0.9375) = -0.1047 < 0$

step 5: $f(0.9375) = -0.1047 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_4 = \frac{0.9375+1}{2} = 0.9688$

$\therefore f(0.9688) = -0.0446 < 0$

step 6: $f(0.9688) = -0.0446 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_5 = \frac{0.9688+1}{2} = 0.9844$

$f(0.9844) = -0.0138 < 0$

step 7: $f(0.9844) = -0.0138 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_6 = \frac{0.9844+1}{2} = 0.9922$

$f(0.9922) = 0.0018 > 0$

Step 8: $f(0.9922) = 0.0018 > 0$

$f(9844) = -0.0138 < 0$

$\therefore x_7 = \dfrac{0.9922 + 0.9844}{2} = 0.9883$

$\therefore f(0.9883) = -0.0061 < 0$

Step 9: $f(0.9883) = -0.0061 < 0$

$f(0.9922) = 0.0018 > 0$

$\therefore x_8 = \dfrac{0.9883 + 0.9922}{2} = 0.9902$

$\therefore f(0.9902) = -0.0022 < 0$

Step 10: $f(0.9902) = -0.0022 < 0$

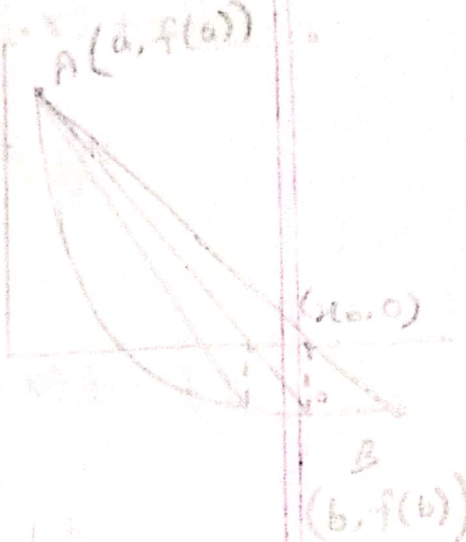
$f(0.9922) = 0.0018 > 0$

$\therefore x_9 = \dfrac{0.9902 + 0.9922}{2} = 0.9912$

$\therefore f(0.9912) = -0.0002 < 0$

$\therefore$ The approximate root is $x_9 = 0.9912$

False Position Method:



$$\frac{Y - Y_1}{Y_2 - Y_1} = \frac{x - x_1}{x_2 - x_1}$$

In case of A and B point,

$$\frac{Y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}$$

In $(x_0, 0)$ point, $\dfrac{0 - f(a)}{f(b) - f(a)} = \dfrac{x_0 - a}{b - a}$

or, $(x_0 - a)(f(b) - f(a)) = af(a) - bf(a)$

or, $x_0 = a + \dfrac{af(a) - bf(a)}{f(b) - f(a)}$

$$= \frac{af(b) - a\cancel{f(a)} + a\cancel{f(a)} - bf(a)}{f(b) - f(a)}$$

1st approximation

$$= \boxed{\frac{af(b) - bf(a)}{f(b) - f(a)}}$$

Here, $\boxed{f(x_0)\, f(a) < 0}$ or $\boxed{f(x_0)\, f(b) < 0}$

2 cases about 2^{nd} approximation :

- If $f(x_0) f(b) < 0$

$$x_1 = \frac{x_0 f(b) - b f(x_0)}{f(b) - f(x_0)}$$

- If $f(x_0) f(a) < 0$

$$x_1 = \frac{x_0 f(a) - a f(x_0)}{f(a) - f(x_0)}$$

Example: Find a real root using false positition method if the equation $x^3 - 2x^2 - 4 = 0$ between 2 and 3.

Soln : $f(x) = x^3 - 2x^2 - 4$

Here, $a = 2$, $b = 3$

1st Step: $f(2) = -4 < 0$ and $f(3) = 5 > 0$

$\therefore$ The root lies between 2 and 3.

$$\therefore x_0 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2.5 - 3(-4)}{5-(-4)} = \frac{10+12}{9} = 2.44444$$

$\therefore f(x_0) = -1.344307274$

<u>2^{nd} Step:</u> $f(\overset{b}{3}) = 5 > 0$ and $f(x_0) < 0$

$\therefore$ The root lies between b and x_0

$$\therefore x_1 = \frac{x_0 f(b) - b f(x_0)}{f(b) - f(x_0)} = 2.56216$$

$f(x_1) = f(2.56216) = -0.3096 < 0$

<u>3^{rd} Step:</u> $f(b) = 5 > 0$ and $f(x_1) = -0.3096 < 0$

$\therefore$ The root lies between b and x_1

$$\therefore x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

$$= \frac{2.56216 \times 5 - 3(-0.3096)}{5-(-0.3096)} = 2.5877$$

$\therefore f(x_2) = f(2.5877) = -0.0647 < 0$

4th step: $f(x_2) = -0.0647 < 0$ and $f(b) = 5 > 0$

$\therefore$ The root lies between x_2 and b.

$$\therefore x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$

$$= \frac{2.5877 \times 5 - 3(-0.0647)}{5 - (-0.0647)} = 2.59297$$

$$\therefore f(x_3) = f(2.59297) = -0.0133 < 0$$

5th step: $f(x_3) = -0.0133 < 0$ and $f(b) = 5 > 0$

So, the root lies between x_3 and b.

$$\therefore x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$= \frac{2.59297 \times 5 - 3(-0.0133)}{5 - (-0.0133)} = 2.594$$

$$\therefore f(x_4) = \cancel{2}\ f(2.594) = -0.0027 < 0$$

6th step: $f(2.594) = -0.0027 < 0$, $f(3) = 5 > 0$

So, the root lies between 2.594 and 3

$$\therefore x_5 = \frac{2.594 \times 5 - 3 \times (-0.0027)}{5 - (-0.0027)} = 2.5943$$

$$\therefore f(x_5) = f(2.5943) = -0.0006 < 0$$

7th step: $f(2.5943) = -0.0006 < 0$

and $f(3) = 5 > 0$

So, the root lies between 2.5943 and 3

$$\therefore x_6 = \frac{2.5943 \times 5 - 3(-0.0006)}{5 - (-0.0006)} = 2.5943$$

$$\therefore f(x_6) = f(2.5943) = -0.0001 < 0$$

Newton - Raphson Method :

$$f(x_1) = f(x_0 + h)$$

we get by expanding Taylor's series,

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

$$\rightarrow f(x_0) + h f'(x_0) = 0$$

$$\rightarrow \boxed{h = -\frac{f(x_0)}{f'(x_0)}} \quad [f'(x_0) \neq 0]$$

$\therefore$ First approximated value,

$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad [n = 0, 1 \ldots]$$

Ex: Let $f(x) = x^3 - 2x^2 - 4$

$a = 2, b = 3$

$$\therefore x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$$f'(x) = 3x^2 - 4x$$

1st iteration: $x_0 = 2.5$

$$f(x_0) = f(2.5) = 2.5^3 - 2(2.5)^2 - 4 = -0.875$$

$$f'(x_0) = f'(2.5) = 3(2.5)^2 - 4(2.5) = 8.75$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 + \frac{0.875}{8.75} = 2.6$$

2nd step: $x_1 = 2.6$

$$f(2.6) = 2.6^3 - 2(2.6)^2 - 4 = 0.056$$

$$f'(2.6) = 3(2.6)^2 - 4(2.6) = 9.88$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

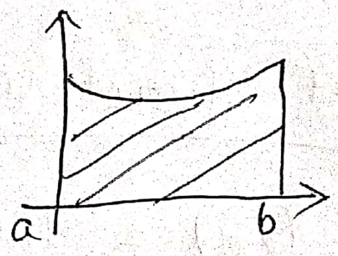
$$= 2.6 - \frac{0.056}{9.88} = 2.59433$$

Numerical Integration :

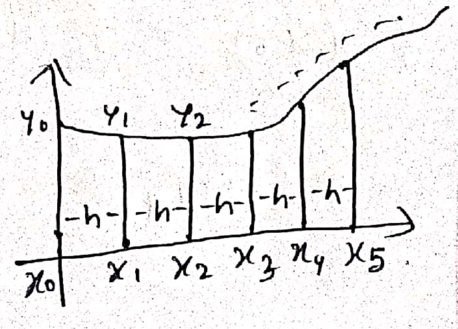
The area bounded by the curve $f(x)$ and x axis between limit a and b is denoted by

$$I = \int_a^b f(x)\,dx \quad \text{- - - (1)}$$

divide the interval (a,b) into n equal ~~in~~ part with h interval.



where, $h = \dfrac{b-a}{n}$



$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\vdots$

$x_n = x_{n-1} + h$

3 rules :

① Trapezoidal Rule

② Simpson's 1/3 rule

③ Simpson's 3/8 rule

Trapezoidal's rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots y_{n-1}) + y_n\right]$$

Simpson's 1/3 rule: (n-even)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots y_{n-1}) + 2(y_2 + y_4 + \cdots y_{n-2}) + y_n\right]$$

Simpson's 3/8 rule: (n-divisible by 3)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5 + \cdots y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \cdots y_{n-3}) + y_n\right]$$

Ex: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

a) trapezoidal formula

b) simpson's 1/3 "

c) " 3/8 "

Given,

Sol^n: $h = \frac{6-0}{6} = 1$; $y = \frac{1}{1+x^2}$

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = 0+1 = 1$	$y_1 = \frac{1}{1+1} = 0.5$
$x_2 = 1+1 = 2$	$y_2 = \frac{1}{1+4} = 0.2$
$x_3 = 3$	$y_3 = \frac{1}{1+9} = 0.1$
$x_4 = 4$	$y_4 = \frac{1}{1+16} = 0.058$
$x_5 = 5$	$y_5 = 0.03846$
$x_6 = 6$	$y_6 = 0.0270270$

$x_1 = x_0 + h$

a) Trapezoidal Rule:

$$\int_0^6 y\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+y_3+y_4+y_5) + y_6\right]$$

$$= \frac{1}{2}\left[1 + 2(0.5+0.2+0.1+0.058+0.03846) + 0.027027\right]$$

$= 1.40997735$

b) Simpson's 1/3 rule:

$$\int_0^6 \cancel{f(x} \, y \, dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6\right]$$

$$= \frac{1}{3}\left[1 + 4(0.5 + 0.1 + 0.03846) + 2(0.2 + 0.058) + 0.027027\right]$$

$$= 1.365622$$

c) Simpson's 3/8 rule:

$$\int_0^6 y \, dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) + y_6\right]$$

$$= \frac{3 \times 1}{8}\left[1 + 3(0.5 + 0.2 + 0.058 + 0.03846) + 2(0.1) + 0.027027\right]$$

$$= 1.356153$$

Interpolation: method of estimating unknown values from given set of observation.

3 formula's :

1. Newton's Forward Formula
2. Newton's Backward formula
3. Lagrange's interpolation formula

equal interval

x	$f(x)$
1971	1000
1981	1025
1991	1080
2001	1120

$\boxed{\text{Forward and Backward used}}$

unequal

x	$f(x)$
75	670
80	685
87	750
90	800
99	812

Lagrange's formula used

Find 1983 - you've to see the progress of forward

1995 - back

* Newton's forward interpolation formula:

$$y(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots$$

here, y_0 = most left side value

$$u = \frac{x - x_0}{h} \quad \therefore \frac{du}{dx} = \frac{1-0}{h} = \frac{1}{h}$$

1^{st}, 2^{nd} and 3^{rd} derivatives:

$$\frac{dy}{dx} = 0 + \frac{1}{h}\Delta y_0 + \frac{(2u-1)}{h.2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{h.3!}\Delta^3 y_0 + \cdots$$

$$= \frac{1}{h}\left(\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \right)$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \cdots\right)$$

$$\frac{d^3 y}{dx^3} = \frac{1}{h^3}\left(\Delta^3 y_0 + \cdots\right)$$

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* Newton's backward interpolation formula:

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

y_n = most right side value

$$u = \frac{x - x_n}{h} \qquad \therefore \frac{du}{dx} = \frac{1-0}{h} = \frac{1}{h}$$

1^{st} derivative,

$$\frac{dy}{dx} = \frac{1}{h}\left(\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \cdots\right)$$

2^{nd}: $$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left(\nabla^2 y_n + (u+1)\nabla^3 y_n + \cdots\right)$$

3^{rd}: $$\frac{d^3 y}{dx^3} = \frac{1}{h^3}\left(\nabla^3 y_n + \cdots\right)$$

Ex: Using Newton's formula for interpolation, estimate the population for the year

1) 1905   2) 1926

Year	1891	1901	1911	1921	1931
Population	98,752	1,32,285	1,68,076	1,95,690	2,46,05

1) Solution: Let's form the forward difference table:

for 1905

Year (x)	Population (y)	Δy $y_2 - y_1$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	98,752	33,533	2,258	-10,435	41,358
1901	1,32,287	35,791	-8,177	30,923	
1911	1,68,076	27,614	22,746		
1921	1,95690	50,360			
1931	2,46,050				

Here, $x = 1905$ $x_0 = 1891$, $h = 10$

$u = \dfrac{x - x_0}{h} = 1.4$, $y_0 = 98,752$,

$\Delta y_0 = 33,533$, $\Delta^2 y_0 = 2258$, $\Delta^3 y_0 = -10,435$,
$\Delta^4 y_0 = 41,358$

Now, $y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots$

$\therefore y(1905) = 98,752 + 1.4 \times 33,533 + \frac{1.4(1.4-1)}{2!} \times 2258$

$+ \frac{1.4(1.4-1)(1.4-2)}{3!}(-10,435) + \dots$

$= 1,47,841$ (approx.)

2) Let's form the backward difference table:

Year (x)	Population (y)	∇y $y_5 - y_4$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	98,752				
1901	132,287	33,533			
1911	1,68,076	35,791	2258		
1921	1,95690	27,614	-8,177	-10,435	
1931	2,46,050	50,360	22,746	30,923	41,358

Here, $x = 1925$, $x_n = 1931$, $h = 10$, $u = \frac{x - x_n}{h} = -0.6$

$\therefore y_n = 2,46,050$, $\nabla y_n = 50,360$, $\nabla^2 y_n = 22,746$, $\nabla^3 y_n = 30,923$

$\nabla^4 y_n = 41,358$

Now, $y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$

$\therefore y(1925) = 2,46,050 + (-0.5)\,50,360 + \frac{(-0.5)(-0.5+1)}{2!} \times 22,746 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!} \times 30,923 + \cdots$

$= 211372$ (approx.)

Ex: Find the area of circle diameter 52. The area 'A' of a circle of diameter d are as follows:

d	50	55	60	65	70
A	1963	2376	2827	3318	3848

Sol^n: $d = 52$; Let's form forward diff. table:

Diameter (x)	Area (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	1963	413	38	2	-3
55	2376	451	40	-1	
60	2827	491	39		
65	3318	530			
70	3848				

$$y(52) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1963 + 0.4 \times 413 + \frac{0.4(0.4-1)}{2!} \times 38 + \frac{0.4(0.4-1)(0.4-2)}{3!} \times 2 + \cdots$$

$$= 2123.8928 \text{ (approx.)}$$

Here, $x = 52$,

$x_0 = 50$, $h = 5$, $u = \dfrac{x - x_0}{h} = 0.4$

$\Delta y = 413$, $\Delta^2 y = 38$, $\Delta^3 y = 2$, $y = 1963$

Ex: find the 1st, 2nd, 3rd derivatives of the function tabulated below:

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7	13.625	24	38.875	59

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.5	3.375	3.625	3.0	0.75
2.0	7.0	6.625	3.75	0.75
2.5	13.625	10.375	4.5	0.75
3.0	24.0	14.875	5.25	
3.5	38.875	20.125		
4.0	59.0			

at the point $x = 1.5$,

$x_0 = 1.5$, $x = 1.5$, $h = 0.5$

$$u = \frac{x - x_0}{h} = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0\right]$$

$$= \frac{1}{0.5}\left[3.625 + \frac{2\times 0-1}{2}(3.0) + \frac{2}{6}(0.75)\right]$$

$$= \boxed{4.750}$$

$$\frac{d^2y}{dx} = \frac{1}{h^2}\left(\Delta^2 y_0 + (u-1)\Delta^3 y_0\right)$$

$$= \frac{1}{0.25}\left(3.0 + (-1)(0.75)\right) = \boxed{9.0}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \ldots\right] = \frac{1}{0.125}(0.75) = \boxed{6}$$

thus at $\boxed{x = 1.5}$

$$\frac{dy}{dx} = 4.75$$

$$\frac{d^2y}{dx^2} = 9.0 \quad \text{and}$$

$$\frac{d^3y}{dx^3} = 6.0$$

Lagrange's Interpolation

Derivation:

Let $y = f(x)$ be a polynomial of nth degree, which takes the value $f(x_0)$, $f(x_1), f(x_2), \ldots f(x_n)$ for any values x_0, $x_1, x_2 \ldots x_n$ of the argument x. The polynomial may be written as:

$$f(x) = a_0 (x-x_1)(x-x_2)\cdots(x-x_n) + a_1(x-x_2)(x-x_3)\cdots(x-x_n) + \cdots a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad \text{---}(1)$$

[a's → constant]

To find the values of a's, we put $x = x_0, x_1 \ldots x_n$ in (1) no. eqn.

$$f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)$$

$$\therefore \quad a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

$$a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$$

$$\vdots$$

$$a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}$$

$$(1) \rightarrow f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} f(x_1) + \cdots$$

$$\frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})} f(y_n) \quad \text{---} \; (2)$$

Ex: Given values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Find $f(9)$ using Lagrange's formula.

Sol^n:

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150$$

$$+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= 810$$

Ex: Using Lagrange's formula find value of $\log_{10} 321.5$

x	321.0	322.8	324.2	325.0
$y(x) = \log_{10} x$	2.50651	2.50893	2.51081	2.51188

Soln:

$$y(321.5) = \frac{(321.5-322.8)(321.5-324.2)(321.5-325)}{(321.0-322.8)(321.0-324.2)(321.0-325)} \times 2.50651$$

$$+ \frac{(321.5-321.0)(321.5-324.2)(321.5-325.0)}{(322.8-321.0)(322.8-324.2)(322.8-325.0)} \times 2.50893$$

$$+ \frac{(321.5-321.0)(321.5-322.8)(321.5-325.0)}{(324.2-321.0)(324.2-322.8)(324.2-325.0)} \times 2.51081$$

$$+ \frac{(321.5-321.0)(321.5-322.8)(321.5-324.2)}{(325-321.0)(325-322.8)(325-324.2)} \times 2.51188$$

$$= 2.50718$$

Ex: Find the form of the function $f(x)$ using the following table.

x	0	1	3	4
$f(x)$	-12	0	12	24

Sol^n:

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \times (-12)$$

$$+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \times 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \times 12$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \times 24$$

$$= (x-1)(x-3)(x-4) - 2x(x-1)(x-4) + 2x(x-1)(x-3)$$

$$= (x-1)(x^2 - 3x - 4x + 12 - \cancel{2x^2} + 8x + \cancel{2x^2} - 6x)$$

$$= (x-1)(x^2 - 5x + 12)$$

$$= x^3 - 6x^2 + 17x - 12$$

SLE: System of Linear Equations.

Direct / Exact Method:

1) Gauss Jordan
2) Gaussian Elimination

Indirect / Iterative:

1) Gauss-Jacobi
2) Gauss-seidal

* Gauss-Jacobi Iterative Method:

$10x - 2y + z = 2 \dashrightarrow a_{11}x + a_{12}y + a_{13}z = b_1$

$-3x + 11y + 2z = 5 \dashrightarrow a_{21}x + a_{22}y + a_{23}z = b_2$

$x - y + 5z = 1 \dashrightarrow a_{31}x + a_{32}y + a_{33}z = b_3$

ignore sign

Application condition:

$$\left.\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned}\right] \text{diagonally dominated}$$

Solution:

Here, $x = (2 + 2y - z)/10$

$y = (5 + 3x - 2z)/11$

$z = (1 - x + y)/5$

Step 1: Let $x = x_0 = 0,\ y = y_0 = 0,\ z = z_0 = 0$

$$\therefore\ x_1 = (2+0-0)/10 = 0.2$$

$$y_1 = 5/11 = 0.4545$$

$$z_1 = 1/5 = 0.2$$

Step 2: $x = x_1,\ y = y_1,\ z = z_1$

$$x_2 = (2+2y_1 - z_1)/10$$

$$= (2 + 2\times 0.4545 - 0.2)/10$$

$$= 0.2709$$

$$y_2 = (5 + 3x_1 - 2z_1)/11$$

$$= (5 + 3\times 0.2 - 2\times 0.2)/11$$

$$= 0.473$$

$$z_2 = (1 - x_1 + y_1)/5$$

$$= (1 - 0.2 + 0.4545)/5$$

$$= 0.2509$$

Def: (an iterative method for determining the solutions for the system of linear equations, which is diagonally dominant.)

Step 3:

$x = x_2 = 0.2709$

$y = y_2 = 0.473$

$z = z_2 = 0.2509$

$x_3 = (2 + 2y_2 - z_2)/10 = 0.269$

$y_3 = (5 + 3x_2 - 2z_2)/11 = 0.4828$

$z_3 = (1 - x_2 + y_2)/5 = 0.24$

Step 4: $x = x_3 = 0.269,\ y = y_3 = 0.4828,\ z = z_3 = 0.24$

$x_4 = (2 + 2y_3 - z_3)/10 = 0.2726$

$y_4 = (5 + 3x_3 - 2z_3)/11 = 0.4843$

$z_4 = (1 - x_3 + y_3)/5 = 0.24276$

Step 5: $x = x_4 = 0.2726,\ y = y_4 = 0.4843,\ z = z_4 = 0.24276$

$x_5 = (2 + 2y_4 - z_4)/10 = 0.2726$

$y_5 = (5 + 3x_4 - 2z_4)/11 = 0.48475$

$z_5 = (1 - x_4 + y_4)/5 = 0.24234$

Step 6: $x = x_5 = 0.2726$, $y = y_5 = 0.48475$

$z = z_5 = 0.24234$

$$x_6 = (2 + 2y_5 - z_5)/10 = 0.2727 \approx \frac{3}{11}$$

$$y_6 = (5 + 3x_5 - 2z_5)/11 = 0.4848 \approx 16/33$$

$$z_6 = (1 - x_5 + y_5)/5 = 0.2424 \approx 8/33$$

— o —

First, $x = g_1(y, z)$

$y = g_2(x, z)$

$z = g_3(x, y)$

Then each step general form:

$x_i = g_1(y_{i-1}, z_{i-1})$

$y_i = g_2(x_{i-1}, z_{i-1})$

$z_i = g_3(x_{i-1}, y_{i-1})$

untill the $\boxed{\text{approx. values}}$ are found.

Gauss - Seidal Iterative method:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Condition:

diagonally dominated

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{33}|$$

First,

$$x = g_1(y, z)$$

$$y = g_2(x, z)$$

$$z = g_3(x, y)$$

Then each step's ~~st~~ general form:

$$x_i = g_1(y_{i-1}, z_{i-1})$$

$$y_i = g_2(x_i, z_{i-1})$$

$$z_i = g_3(x_i, y_i)$$

Found in previous step (difference)

Ex:

$$10x - 2y + z = 2$$
$$-3x + 11y + 2z = 5$$
$$x - y + 5z = 1$$

Sol^n:

$$x = (2 + 2y - z)/10$$
$$y = (5 + 3x - 2z)/11$$
$$z = (1 - x + y)/5$$

Step 1: $\cancel{x}$ $x_0 = 0, \ y_0 = 0, \ z_0 = 0$

$$x_1 = (2 + 2y_0 - z_0)/10 = 0.2$$
$$y_1 = (5 + 3x_1 - 2z_0)/11 = 0.509$$
$$z_1 = (1 - x_1 + y_1)/5 = 0.262$$

Step 2: $x_2 = (2 + 2y_1 - z_1)/10 = 0.2756$

$$y_2 = (5 + 3x_2 - 2z_1)/11 = 0.45825$$
$$z_2 = (1 - x_2 + y_2)/5 = 0.2365$$

Step 3: $x_3 = (2+2y_2 - z_2)/10 = .268$

$y_3 = (5 + 3x_3 - 2z_2)/11 = 0.48464$

$z_3 = (1 - x_3 + y_3)/5 = 0.2433$

Step 4: $x_4 = (2+2y_3 - z_3)/10 = 0.2726$

$y_4 = (5+3x_4 - 2z_3)/11 = 0.4847$

$z_4 = (1 - x_4 + y_4)/5 = 0.2424$

Step 5: $x_5 = (2+2y_4 - z_4)/10 = 0.2727$

$y_5 = (5+3x_5 - 2z_4)/11 = 0.4848$

$z_5 = (1 - x_5 + y_5)/5 = 0.2424$

$\therefore x = 0.2727 \approx 3/11$

$y = 0.4848 \approx 16/33$

$z = 0.2424 \approx 8/33$

Solving Linear Equation

1) Gaussian Elimination Method:

$$\begin{matrix} R_{11} & R_{12} & \cdots & R_{1M} \\ R_{21} & R_{22} & \cdots & R_{2M} \\ \vdots & \vdots & & \\ R_{N1} & \cdots & \cdots & R_{NM} \end{matrix}$$

$\boxed{N \times M \text{ Matrix}}$

Making these part 0

Ex:

$$\left.\begin{aligned} x + 2y &= 1 \\ 2x - 3y &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 7 & 2 \end{array}\right] \qquad R_2 \to 2R_1 - R_2$$

$$\therefore x + 2y = 1 \quad \cdots\cdots (1)$$

$$7y = 2 \quad \cdots\cdots (2)$$

$$\rightarrow y = 2/7$$

$$(1) \rightarrow x + \frac{4}{7} = 1 \qquad \therefore x = \frac{3}{7}$$

$$\therefore (x, y) = \left(\frac{3}{7}, \frac{2}{7}\right)$$

2) Gauss Jordan Elimination method :

- Making all elements 'zero' except one element.

Ex:

$$\left.\begin{aligned} x + 2y &= 1 \\ 2x - 3y &= 0 \end{aligned}\right\}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -3 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 7 & 2 \end{array}\right] \quad R_2 \to 2R_1 - R_2$$

$$= \left[\begin{array}{cc|c} 7 & 0 & 3 \\ 0 & 7 & 2 \end{array}\right] \quad R_1 \to 7R_1 - 2R_2$$

$$= \left[\begin{array}{ccc} 1 & 0 & 3/7 \\ 0 & 1 & 2/7 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{7} R_1 \\ R_2 \to \frac{1}{7} R_2 \end{array}$$

$\therefore$ $x = 3/7$ $\quad\quad$ $\therefore$ $(x, y) = \left(\frac{3}{7}, \frac{2}{7}\right)$

$y = 2/7$

ODE : Ordinary Differential Equations

① **Euler's Method:** Given 3 things :

$\rightarrow y(x_0) = y_0$

get x_0, y_0

$\rightarrow \frac{dy}{dx} = f(x,y)$

$\rightarrow h$ (step size)

Find y_n, where

$$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$$

$$= y_{n-1} + h \left.\frac{dy}{dx}\right|_{(x_{n-1}, y_{n-1})}$$

Ex: $\frac{dy}{dx} = x + y$, $y(0) = 1$, for $0 \le x \le 1$

Find $y(1)$ using Euler method.

Soln: at $x = 0.1$ $y_1 = y_0 + h \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$

$$= 1 + 0.1\,(0+1) = 1.1$$

$x = 0.2$ $y_2 = y_1 + h \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$

$$= 1.1 + 0.1\,(0.1 + 1.1) = 1.22$$

$x = 0.3$

$$y_3 = y_2 + h \frac{dy}{dx}_{(x_2, y_2)}$$

$$= 1.22 + 0.1 (0.2 + 1.22) = 1.362$$

$x = 0.4$

$$y_4 = y_3 + h \frac{dy}{dx}_{(x_3, y_3)}$$

$$= 1.362 + 0.1 (0.3 + 1.362) = 1.5282$$

$x = 0.5$

$$y_5 = y_4 + h \frac{dy}{dx}_{(x_4, y_4)}$$

$$= 1.5282 + 0.1 (0.4 + 1.5282) = 1.72102$$

$x = 0.6$

$$y_6 = y_5 + h \frac{dy}{dx}_{(x_5, y_5)}$$

$$= 1.72102 + 0.1 (0.5 + 1.72102) = 1.943$$

$x = 0.7$

$$y_7 = y_6 + h \frac{dy}{dx}_{(x_6, y_6)}$$

$$= 1.943 + 0.1 (0.6 + 1.943) = 2.197$$

$x = 0.8$

$$y_8 = y_7 + h \frac{dy}{dx}_{(x_7, y_7)}$$

$$= 2.197 + 0.1 (0.7 + 2.197) = 2.487$$

$x = 0.9$

$$y_9 = y_8 + h \frac{dy}{dx}_{(x_8, y_8)}$$

$$= 2.487 + 0.1 (0.8 + 2.487) = 2.8159$$

$x = 1$

$$y_{10} = y_9 + h \frac{dy}{dx}_{(x_9, y_9)}$$

$$= 2.8159 + 0.1 (0.9 + 2.8159) = 3.1875$$

$$\therefore \boxed{y(1) = 3.1875}$$

(iii) Runge-Kutta 2 Method:

Given : * $y(x_0) = y_0$

$\rightarrow x_0, y_0$

* range
* $h \rightarrow$ step size
* $dy/dx = f(x, y)$

* Find $y(x_n)$ or y_n

$$\boxed{y_n = y_{n-1} + \frac{1}{2}(k_1 + k_2)}$$

where, $k_1 = hf(x_{n-1}, y_{n-1})$

$k_2 = hf(x_{n-1} + h, y_{n-1} + k_1)$

Ex: Use Runge kutta method, solve $\frac{dy}{dx} = \frac{x^2 + y^2}{10}$, $y(0) = 1$, for $0 \le x \le 0.4$; $h = 0.1$

Sol^n: Step 1: $k_1 = hf(x_0, y_0)$

$= hf(0, 1)$

$= 0.1 \times \frac{0^2 + 1^2}{10} = 0.01$

$$k_2 = h\,f(x_0 + h,\ y_0 + k_1)$$
$$= h\,f(0 + 0.1,\ 1 + 0.01)$$
$$= h\,f(0.1,\ 1.01)$$
$$= 0.1 \times \frac{0.1^2 + 1.01^2}{10} = 0.0103$$

$$\therefore\ y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$
$$= 1 + \frac{1}{2}(0.01 + 0.0103)$$
$$= 1.0102$$
$$\therefore\ \boxed{y(0.1) = 1.0102}$$

Step 2: $k_1 = h\,f(x_1, y_1) = h\,f(0.1, 1.0102)$

$$= 0.1 \times \frac{0.1^2 + 1.0102^2}{10} = 0.0103$$

$$k_2 = h\,f(x_1 + h,\ y_1 + k_1)$$
$$= h\,f(0.1 + 0.1,\ 1.0102 + 0.0103)$$
$$= h\,f(0.2,\ 1.0205)$$
$$= 0.1 \times \frac{0.2^2 + 1.0205^2}{10}$$
$$= 0.0108$$

$\therefore\ y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$= 1.0102 + \frac{1}{2}(0.0103 + 0.0108)$

$= 1.02075$

$\therefore\ \boxed{y(0.2) = 1.02075}$

Step 3: $k_1 = hf(x_2, y_2)$

$= hf(0.2, 1.02075)$

$= 0.1 \times \frac{0.2^2 + 1.02075^2}{10}$

$= 0.01082$

$k_2 = hf(x_2 + h, y_2 + k_1)$

$= hf(0.2 + 0.1, 1.02075 + 0.01082)$

$= hf(0.3, 1.0316)$

$= 0.1 \times \frac{0.3^2 + 1.0316^2}{10}$

$= 0.011542$

$\therefore\ y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$

$= 1.02075 + \frac{1}{2}(0.01082 + 0.011542)$

$= 1.032$ $\therefore\ \boxed{y(0.3) = 1.032}$

Step 4: $k_1 = h f(x_3, y_4)$

$= h f(0.3, 1.032)$

$= 0.1 \times \frac{0.3^2 + 1.032^2}{10}$

$= 0.01155$

$k_2 = h f(x_3 + h, y_3 + k_1)$

$= h f(0.3 + 0.1, 1.032 + 0.01155)$

$= h f(0.4, 1.04355)$

$= 0.1 \times \frac{0.4^2 + 1.04355^2}{10}$

$= 0.0125$

$\therefore y_4 = y_3 + \frac{1}{2}(k_1 + k_2)$

$= 1.032 + \frac{1}{2}(0.01155 + 0.0125)$

$= 1.044$

$\therefore \boxed{y(0.4) = 1.044}$ (Ans)