

Interpolation

Equal Interval

∴ Newton's forward Interpolation formula



$$\therefore y(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \dots$$

$\Delta^2 y_0$

∴ y_0 = Table's most Left side value

$$u = \frac{x - x_0}{h}$$

∴ Newton's backward Interpolation formula

$$\therefore y(x) = y_n + u \Delta y_n + \frac{u(u+1)}{2!} \Delta^2 y_n + \frac{u(u+1)(u+2)}{3!} \Delta^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!} \Delta^4 y_n + \dots$$

$$u = \frac{x - x_n}{h}$$

y_n = Table's most
Right side

① Using Newton formula for interpolation

① estimate the population for year 1905

② the population for year 1925

Year	1891	1901	1911	1921	1931
Population	98,752	1,32,285	1,68,076	1,95,690	2,46,050

Ans ① For 1905 = forward Interpolation

Year x	Population y	Δy $x_2 - x_1$	Δy^2	$\Delta^3 y$	$\Delta^4 y$
1891 (x_1)	98,752	33,533	2,258	-10,435	41,358
1901 (x_2)	1,32,287	35,791	-8,177	30,923	
1911	1,68,076	27,614	22,746		
1921	1,95,690	50,360			
1931	2,46,050				

Here,

$$x = 1905$$

$$x_0 = 1891$$

$$\therefore h = 10$$

$$\therefore u = \frac{x - x_0}{h}$$

$$= \frac{1905 - 1891}{10}$$

$$= 1.4$$

$$\therefore y_0 = 98.752$$

$$\therefore \Delta y_0 = 33.533$$

$$\therefore \Delta^2 y_0 = 2258$$

$$\Delta^3 y_0 = -10.435$$

$$\Delta^4 y_0 = 41.358$$

$$\Delta^2 y_0$$

$$\Delta^3 y_0$$

$$\therefore y(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \dots$$

$$y(1905) = 1,47,841 \quad (\text{Approximately})$$

Ans

$$\frac{u^2 - 14u + 2}{3!} \Delta^3 y_0$$

$$+ \frac{u^2 - 100u}{4!} \Delta^4 y_0$$

② for 1925 \rightarrow backward Interpolation

Year x	Population y	Δy $y_1 - y_0$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	98,752				
1901	1,32,285	33,533			
1911	1,68,076	35,791	2258		
1921	1,95,690	27,614	-8,177	-10,435	
1931	2,46,050	50,0360	22,746	30,923	41,358

Hence $\therefore x' = 1925$

$$x_n = 1931$$

$$h = 10$$

$$\therefore u = \frac{x - x_n}{h}$$

$$= -0.6$$

$$y(1925) = y_n + u \Delta y_n + \frac{u(u-1)}{2!} \Delta^2 y_n + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_n + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_n + \dots$$

$$= 2,25,982 \text{ (approx.)}$$

Ans

$\Delta^2 y_0$

Find the area of circle diameter 52 Given that the area 'A' of a circle of diameter d are as follows :-

$10^4 \dots$

d	50	55	60	65	70
A	1963	2376	2827	3318	3848

$$\frac{u^2 - 14u + 2}{3!} \Delta^3 y_0$$

$$+ \frac{u^2 - 100u}{4!} \Delta^4 y_0$$

Ans:- $d=52$; forward Interpolation (2501)0

Diameter (x)	Area (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	1963	413	38	2	-3
55	2376	451	40	-1	
60	2827	491	39		
65	3318	530			
70	3848				

$$\therefore y(52) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 +$$

$$\frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0$$

$$\therefore u = \frac{x - x_0}{h}$$

$$= \frac{52-50}{5}$$

$$= 0.4$$

$$x = 52$$

$$x_0 = 50$$

$$h = 5$$

$$\begin{aligned}
 y &= 1963 + 0.4 \times 413 + \frac{0.4(0.4-1)}{2!} \times 38 \\
 (52) &+ \frac{0.4(0.4-1)(0.4-2)}{3!} \times 2 \\
 &+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} (-3)
 \end{aligned}$$

$$= 1963 + 165.2 + (-4.56) + 0.128 + 0.1248$$

$\Delta^2 y_0$

$$y(52) = 2123.8928 \text{ (approx.)}$$

$$\text{Ans} \quad \frac{(1x-x)(8x-x)(0x-x)}{(1x-5x)(8x-5x)(0x-5x)} +$$

$$\frac{(1x-8x)(5x-x)(1x-x)(0x-x)}{(1x-5x)(5x-5x)(1x-0x)(0x-5x)} +$$

$$\frac{(8x-x)(5x-x)(1x-x)(0x-x)}{(8x-1x)(5x-1x)(1x-1x)(0x-1x)} +$$

$1_0 + \dots$

$$\frac{4^2 - 14 + 2}{3!} \Delta^3 y_0$$

$$\begin{aligned}
 &+ \\
 &+ 24 \\
 &\frac{4^2 - 1004}{\Delta^5 y_0}
 \end{aligned}$$

Lagrange's Interpolation formula

(for unequal interval)

x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

formula for any x , $y(x)$ is given by

$$\begin{aligned} \therefore y(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 \\ & + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 \\ & + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 \\ & + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 \\ & + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4 \end{aligned}$$

Q

Given values

	x_0	x_1	x_2	x_3	x_4
$x :$	5	7	11	13	17
$f(x) :$	150	392	1452	2366	5202
	y_0	y_1	y_2	y_3	y_4

\therefore find $f(9)$ using Lagrange's formula

$$\begin{aligned}
 \therefore f(9) &= \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\
 &+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\
 &+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\
 &+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\
 &+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202
 \end{aligned}$$

$$y(9) =$$

$$f(9) = 810$$

Ans

$$\Delta^2 y_0$$

$$10^+$$

$$\frac{4^2 - 4 + 2}{3!} \Delta^3 y_0$$

$$+ \frac{4^2 - 10 + 4}{4!} \Delta^4 y_0$$

$$\Delta^5 y_0$$

Q Using Lagrange's Interpolation formula compute the value of $\log_{10} 321.5$

	x_0	x_1	x_2	x_3
x	321.0	322.8	324.2	325.0
$y(x)$	$\log_{10} x$			
	2.50651	2.50893	2.51081	2.51188
	y_0	y_1	y_2	y_3

Ans:- find $y(\log_{10} 321.5)$ using Lagrange formula

$$\begin{aligned}
 y(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \\
 & \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \\
 & \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \\
 & \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3
 \end{aligned}$$

$$\log_{10} 321.5 =$$

$$\frac{-12.28500 \times 2.5065}{-23.0400}$$

$$+ \frac{5.54400}{-23.0400} \times 2.50893$$

$$\boxed{\log_{10} 321.5 = 2.50718}$$

$$\Delta^2 y_0$$

$$= 1.33148 + 2.13829 + (-1.59378)$$

$$+ 0.62619$$

$$= 2.50718$$

Using calculator

$$Y = \frac{(x - A)(x - B)(x - C)}{(D - A)(D - B)(D - C)}$$

$x_0 \quad x_1 \quad x_2 \quad x_3$
 $x_0 \quad x_1 \quad x_2 \quad x_3$

$$\begin{matrix} F \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{matrix}$$

$$\begin{matrix} D = x_1 & A = x_0 & B = x_2 & C = x_3 \\ = x_2 & x_0 & x_1 & x_3 \\ x_3 & x_0 & x_1 & x_2 \end{matrix}$$

$$\frac{4^2 - 14 + 2}{3!} \Delta^3 y_0$$

$$+ \frac{4^2 - 10 + 4}{2!} \Delta^2 y_0$$

sum all

Q) Using Lagrange's formula, find the form of function $f(x)$ following table

	x_0	x_1	x_2	x_3
x	0	1	3	4
$f(x)$	-12	0	12	24
	y_0	y_1	y_2	y_3

\therefore Using Lagrange formula

$$\begin{aligned}
 f(x) = & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 \\
 & + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 \\
 & + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 \\
 & + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3
 \end{aligned}$$

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} x^{-12}$$

$$+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} x^0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} x^{12}$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} x^{24}$$

$$= (x-1)(x-3)(x-4) + 0 + 2x(x-1)(x-4) + 2x(x-1)(x-3)$$

$$= (x-1)(x^2 - 3x - 4x + 12 + 2x^2 - 8x + 2x^2 - 6x)$$

$$= (x-1)(5x^2 - 21x + 12)$$

$$= 5x^3 - 21x^2 + 12x - 5x^2 + 21x - 12$$

$$= 5x^3 - 26x^2 + 33x - 12$$

$$\therefore f(x) = 5x^3 - 26x^2 + 33x - 12$$

Ans/

$$\frac{34^2 - 14 + 2}{3!} \Delta^3 y_0$$

$$10 + 24 + 55x^2 - 100x$$

$$\Delta^5 y_0$$

Newton forward formula and its derivative

\therefore Newton forward formula

$$y = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots$$

$$u = \frac{x - x_0}{h}$$

$$\frac{dy}{dx} = \frac{1}{h} (1 - 0) = \frac{1}{h}$$

1st derivatives

$$\frac{dy}{dx} = 0 + \frac{1}{h} \Delta y_0 + \frac{2u-1}{2!} \times \frac{1}{h} \Delta^2 y_0 + \frac{3u^2-6u+2}{3!h} \Delta^3 y_0 + \dots$$

$$= \frac{1}{h} \left(\Delta y_0 + \frac{2u-1}{2!} \Delta^2 y_0 + \frac{3u^2-6u+2}{3!} \Delta^3 y_0 + \dots \right)$$

2nd derivatives

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2} \left(\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \dots \right)$$

3rd derivatives

$$\frac{d^3 y}{dx^3} = \frac{1}{h^3} \left(\Delta^3 y_0 + \dots \right)$$

Q. Find 1st, 2nd, 3rd derivatives of the function tabulated below at $x = 1.5$

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7.000	13.625	24.000	38.875	59.000

Sol Forward Difference Table

x	y	Δy_0 $y_2 - y_1$	$\Delta^2 y_0$	$\Delta^3 y_0$
1.5	3.375 y_1	3.625	3.000	0.750
2.0	7.000 y_2	6.625	3.750	0.750
2.5	13.625	10.375	4.500	0.750
3.0	24.000	14.875	5.250	
3.5	38.875	20.125		
4.0	59.000			

Hence

$$x = 1.5$$

$$x_0 = 1.5$$

$$h = 0.5$$

$$\therefore u = \frac{x - x_0}{h} = 0$$

1st derivatives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{h} \left[\Delta y_0 + \frac{3u-1}{2!} \Delta^2 y_0 + \frac{3u^2-6u+2}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{0.5} \left[3.675 + \frac{0-1}{2!} (3.000) + \frac{0-0+2}{3!} (.750) \right] \\ &= 4.750\end{aligned}$$

2nd derivatives

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{1}{h^2} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 \right] \\ &= \frac{1}{.5^2} (3.000 + (0-1) \cdot 750) \\ &= 9.000\end{aligned}$$

3rd derivatives

$$\begin{aligned}\frac{d^3 y}{dx^3} &= \frac{1}{h^3} \Delta^3 y_0 \\ &= \frac{1}{.5^3} \times .750 = 6.000\end{aligned}$$

\therefore Thus at $x = 1.5$

$$\frac{dy}{dx} = 4.750$$

$$\frac{d^2 y}{dx^2} = 9.000$$

$$\frac{d^3 y}{dx^3} = 6.000$$

Newton's forward Interpolation formula and its derivatives

Newton forward formula

$$f(x) = f(x_0 + uh)$$

$$y(x) = y(x_0 + uh) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$$u = \frac{x - x_0}{h}$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 +$$

$$\frac{u(u-1)(u-2)(u-3)(u-4)}{5!} \Delta^5 y_0 + \dots$$

1st derivatives

$$y'(x) = \frac{y'}{h}(x_0 + uh) = \Delta y_0 + \frac{2u-1}{2!} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!} \Delta^3 y_0$$

$$+ \frac{4u^3 - 18u^2 + 22u - 6}{4!} \Delta^4 y_0 +$$

$$\frac{5u^4 - 40u^3 + 105u^2 - 100u}{5!} \Delta^5 y_0$$

$$= y'(x_0 + uh)$$

$$y'(x) = \frac{1}{h} \left\{ \Delta y_0 + \frac{2u-1}{2!} \Delta^2 y_0 + \frac{3u^2-6u+2}{6} \Delta^3 y_0 \right. \\ \left. + \frac{4u^3-18u^2+22u-6}{4!} \Delta^4 y_0 \right. \\ \left. + \frac{5u^4-40u^3+105u^2-100u+24}{5!} \Delta^5 y_0 + \dots \right\}$$

2nd derivative

$$y''(x) = y''(x_0 + uh) \cdot h$$

$$= \frac{1}{h} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{12u^2-36u+22}{4!} \Delta^4 y_0 \right. \\ \left. + \frac{2u^3-12u^2+21u-16}{12} \Delta^5 y_0 + \dots \right]$$

$$y''(x) = y''(x_0 + uh)$$

$$= \frac{1}{h^2} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{6u^2-18u+11}{12} \Delta^4 y_0 \right. \\ \left. + \frac{2u^3-12u^2+21u-16}{12} \Delta^5 y_0 + \dots \right]$$

$$= y'(x_0 + uh)$$

$$y'(x) = \frac{1}{h} \left\{ \Delta y_0 + \frac{2u-1}{2!} \Delta^2 y_0 + \frac{3u^2-6u+2}{6} \Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!} \Delta^4 y_0 + \frac{5u^4-40u^3+105u^2-100u+24}{5!} \Delta^5 y_0 + \dots \right\}$$

2nd derivative

$$y''(x) = y''(x_0 + uh) \cdot h$$

$$= \frac{1}{h} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{12u^2-36u+22}{4!} \Delta^4 y_0 + \frac{2u^3-12u^2+21u-16}{12} \Delta^5 y_0 + \dots \right]$$

$$y''(x) = y''(x_0 + uh)$$

$$= \frac{1}{h^2} \left[\Delta^2 y_0 + (u-1) \Delta^3 y_0 + \frac{6u^2-18u+11}{12} \Delta^4 y_0 + \frac{2u^3-12u^2+21u-16}{12} \Delta^5 y_0 + \dots \right]$$

Find the first and 2nd, 3rd derivative at
point $x = 1.5$.

x	1.00	1.05	1.10	1.15 1.15	1.20	1.25	1.30
$y = \sqrt{x}$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017