Root Finding Method:

1) Bisection Method

2) Method of false position

3) Newton's Raphson's Method

4) fixed point iteration method

Bisection Method: If a function $f(x)$ is continuous between a and b, and $f(a) \cdot f(b) < 0$ (or $f(a)$ and $f(b)$ are of opposite sign) then there exists at least one root between a and b.

Then the first approximation of the root is in

$$x_1 = \frac{a+b}{2}$$

then if $f(x_1) = 0$, x_1 is root of $f(x) = 0$ otherwise the root lies between a and x_1 or x_1 and b according to $f(x_1)$ is negative or positive. Then we bisect the interval as before and continue the process untill the root is fou

Q: Find a root of an equation $f(x) = x^3 + x^2$

between 0 and 1.

→ Step 1: $f(0) = -1 < 0$

$f(1) = 1 > 0$

∴ The root lies between 0 and 1.

$x_0 = \frac{0+1}{2} = 0.5$

$f(x_0) = f(0.5) = -0.625 < 0$

Step 2: $f(0.5) = -0.625 < 0$

$f(1) = 1 > 0$

The root lies between 0.5 and 1.

∴ $x_1 = \frac{0.5+1}{2} = 0.75$

∴ $f(x_1) = f(0.75) = -0.01562 < 0$

Step 3: $f(0.75) = -0.01562 < 0$

$f(1) = 1 > 0$

$\therefore$ The root lies between 0.75 and 1

$\therefore x_2 = \frac{0.75+1}{2} = 0.875$

$f(x_2) = f(0.875) = 0.43555 > 0$

Step 4: $f(0.875) = 0.43555 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.75 and 0.875

$\therefore x_3 = \frac{0.75+0.875}{2} = 0.8125$

$\therefore f(x_3) = f(0.8125) = 0.19653 > 0$

Step 5: $f(0.8125) = 0.19653 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ the root lies between 0.8125 and 0.75

$\therefore x_4 = \frac{0.75+0.8125}{2} = 0.78125$

$\therefore f(x_4) = f(0.78125) = 0.08719 > 0$

Step 6: $f(0.78125) = 0.08719 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.78125 and 0.75

$$\therefore x_5 = \frac{0.75 + 0.78125}{2} = 0.76562$$

$f(x_5) = f(0.76562) = 0.03498 > 0$

Step 7: $f(0.76562) = 0.03498 > 0$

$f(0.75) = -0.01562 < 0$

$\therefore$ The root lies between 0.75 and 0.76562

$$\therefore x_6 = \frac{0.75 + 0.76562}{2} = 0.75781$$

$f(x_6) = f(0.75781) = 0.00948 > 0$

Step 8: $f(0.75781) = 0.00948 > 0$

$f(0.75) = -0.01562 < 0$

$$\therefore x_7 = \frac{0.75 + 0.75781}{2} = 0.75391$$

$\therefore f(0.75391) = -0.00312 < 0$

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Step 9: $f(0.75391) = -0.00312 < 0$

$f(0.75781) = 0.009486 > 0$

$$x_8 = \frac{0.75391 + 0.75781}{2} = 0.75586$$

$f(0.75586) = 0.00316 < 0$

Step 10:

$f(0.75586) = 0.00316 < 0$

$f(0.75781) = 0.00948 > 0$

$$x_9 = \frac{0.75586 + 0.75781}{2} = 0.75488$$

$\therefore f(0.75488) = 0.00002 > 0$

$\therefore$ So, the approximate root is $x_9 = 0.755$

Q: Find a root of an equation:

$$f(x) = x^2 + \sin x - 1 = 0$$

step 1:

Here, $f(0) = -1 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_0 = \frac{0+1}{2} = 0.5$

$f(0.5) = -0.7413 < 0$

Step 2: $f(0.5) = -0.7413 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_1 = \frac{0.5+1}{2} = 0.75$

$f(0.75) = -0.4244 < 0$

Step 3: $f(0.75) = -0.4244 < 0$

$f(1) = 0.0175 > 0$

$\therefore x_2 = \frac{0.75+1}{2} = 0.875$

$\therefore f(0.875) = -0.2191 < 0$

Step 4: $f(0.875) = -0.2191 < 0$

$f(1) = 0.0175 > 0$

$$\therefore x_3 = \frac{0.875+1}{2} = 0.9375$$

$f(0.9375) = -0.1047 < 0$

Step 5: $f(0.9375) = -0.1047 < 0$

$f(1) = 0.0175 > 0$

$$\therefore x_4 = \frac{0.9375+1}{2} = 0.9688$$

$\therefore f(0.9688) = -0.0446 < 0$

Step 6: $f(0.9688) = -0.0446 < 0$

$f(1) = 0.0175 > 0$

$$\therefore x_5 = \frac{0.9688+1}{2} = 0.9844$$

$f(0.9844) = -0.0138 < 0$

Step 7: $f(0.9844) = -0.0138 < 0$

$f(1) = 0.0175 > 0$

$$\therefore x_6 = \frac{0.9844+1}{2} = 0.9922$$

$f(0.9922) = 0.0018 > 0$

Step 8: $f(0.9922) = 0.0018 > 0$

$f(9844) = -0.0138 < 0$

$\therefore x_7 = \dfrac{0.9922 + 0.9844}{2} = 0.9883$

$\therefore f(0.9883) = -0.0061 < 0$

Step 9: $f(0.9883) = -0.0061 < 0$

$f(0.9922) = 0.0018 > 0$

$\therefore x_8 = \dfrac{0.9883 + 0.9922}{2} = 0.9902$

$\therefore f(0.9902) = -0.0022 < 0$

Step 10: $f(0.9902) = -0.0022 < 0$

$f(0.9922) = 0.0018 > 0$

$\therefore x_9 = \dfrac{0.9902 + 0.9922}{2} = 0.9912$

$\therefore f(0.9912) = -0.0002 < 0$

$\therefore$ The approximate root is $x_9 = 0.9912$

False Position Method:

$$\frac{Y - Y_1}{Y_2 - Y_1} = \frac{x - x_1}{x_2 - x_1}$$

In case of A and B point,

$$\frac{Y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}$$

In $(x_0, 0)$ point, $\frac{0 - f(a)}{f(b) - f(a)} = \frac{x_0 - a}{b - a}$

or, $(x_0 - a)(f(b) - f(a)) = af(a) - bf(a)$

or, $x_0 = a + \frac{af(a) - bf(a)}{f(b) - f(a)}$

$$= \frac{af(b) - \cancel{af(a)} + \cancel{af(a)} - bf(a)}{f(b) - f(a)}$$

1st approximation

$$= \boxed{\frac{af(b) - bf(a)}{f(b) - f(a)}}$$

Here, $\boxed{f(x_0)\, f(a) < 0}$ or $\boxed{f(x_0)\, f(b) < 0}$

2 cases about 2^{nd} approximation:

- If $f(x_0) f(b) < 0$

$$x_1 = \frac{x_0 f(b) - b f(x_0)}{f(b) - f(x_0)}$$

- If $f(x_0) f(a) < 0$

$$x_1 = \frac{x_0 f(a) - a f(x_0)}{f(a) - f(x_0)}$$

Example: find a real root using false positition method if the equation $x^3 - 2x^2 - 4 = 0$ between 2 and 3.

Sol^n: $f(x) = x^3 - 2x^2 - 4$

Here, $a = 2$, $b = 3$

1^{st} Step: $f(2) = -4 < 0$ and $f(3) = 5 > 0$

$\therefore$ The root lies between 2 and 3.

$$\therefore x_0 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2.5 - 3(-4)}{5-(-4)} = \frac{10+12}{9} = 2.44444$$

$\therefore f(x_0) = -1.344307274$

2nd Step: $f(\overset{b}{3}) = 5 > 0$ and $f(x_0) < 0$

$\therefore$ The root lies between b and x_0

$$\therefore x_1 = \frac{x_0 f(b) - b f(x_0)}{f(b) - f(x_0)} = 2.56216$$

$f(x_1) = f(2.56216) = -0.3096 < 0$

3rd Step: $f(b) = 5 > 0$ and $f(x_1) = -0.3096 <$

$\therefore$ The root lies between b and x_1

$$\therefore x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

$$= \frac{2.56216 \times 5 - 3(-0.3096)}{5-(-0.3096)} = 2.5877$$

$\therefore f(x_2) = f(2.5877) = -0.0647 < 0$

4th step: $f(x_2) = -0.0647 < 0$ and $f(b) = 5 > 0$

$\therefore$ The root lies between x_2 and b.

$$\therefore x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$

$$= \frac{2.5877 \times 5 - 3(-0.0647)}{5 - (-0.0647)} = 2.59297$$

$$\therefore f(x_3) = f(2.59297) = -0.0133 < 0$$

5th step: $f(x_3) = -0.0133 < 0$ and $f(b) = 5 > 0$

So, the root lies between x_3 and b.

$$\therefore x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$= \frac{2.59297 \times 5 - 3(-0.0133)}{5 - (-0.0133)} = 2.594$$

$$\therefore f(x_4) = \cancel{2}.\ f(2.594) = -0.0027 < 0$$

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6th step: $f(2.594) = -0.0027 < 0$, $f(3) = 5 > 0$

So, the root lies between 2.594 and 3

$$\therefore x_5 = \frac{2.594 \times 5 - 3 \times (-0.0027)}{5 - (-0.0027)} = 2.5943$$

$$\therefore f(x_5) = f(2.5943) = -0.0006 < 0$$

7th step: $f(2.5943) = -0.0006 < 0$

and $f(3) = 5 > 0$

So, the root lies between 2.5943 and 3

$$\therefore x_6 = \frac{2.5943 \times 5 - 3(-0.0006)}{5 - (-0.0006)} = 2.5943$$

$$\therefore f(x_6) = f(2.5943) = -0.0001 < 0$$

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Newton - Raphson Method :

$$f(x_1) = f(x_0 + h)$$

we get by expanding Taylor's series;

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots\cdots = 0$$

$$\rightarrow f(x_0) + hf'(x_0) = 0$$

$$\rightarrow \boxed{h = -\frac{f(x_0)}{f'(x_0)}} \qquad [f'(x_0) \neq 0]$$

$\therefore$ First approximated value,

$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad [n = 0, 1 \ldots]$$

Ex: Let $f(x) = x^3 - 2x^2 - 4$

$a = 2$, $b = 3$

$\therefore x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

$f'(x) = 3x^2 - 4x$

1st iteration: $x_0 = 2.5$

$f(x_0) = f(2.5) = 2.5^3 - 2(2.5)^2 - 4 = -0.875$

$f'(x_0) = f'(2.5) = 3(2.5)^2 - 4(2.5) = 8.75$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 + \frac{0.875}{8.75} = 2.6$

2nd step: $x_1 = 2.6$

$f(2.6) = 2.6^3 - 2(2.6)^2 - 4 = 0.056$

$f'(2.6) = 3(2.6)^2 - 4(2.6) = 9.88$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2.6 - \frac{0.056}{9.88} = 2.59433$