ODE : Ordinary Differential Equations

① Euler's Method : Given 3 things :

$\rightarrow y(x_0) = y_0$

get x_0, y_0

$\rightarrow \frac{dy}{dx} = f(x, y)$

$\rightarrow h$ (step size)

Find y_n, where

$$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$$

$$= y_{n-1} + h \left.\frac{dy}{dx}\right|_{(x_{n-1}, y_{n-1})}$$

Ex: $\frac{dy}{dx} = x + y$, $y(0) = 1$, for $0 \le x \le 1$

Find $y(1)$ using Euler method.

Sol^n: at $x = 0.1$

$$y_1 = y_0 + h \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$

$$= 1 + 0.1\,(0 + 1) = 1.1$$

$x = 0.2$

$$y_2 = y_1 + h \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$$

$$= 1.1 + 0.1\,(0.1 + 1.1) = 1.22$$

$x = 0.3$ $\quad y_3 = y_2 + h \frac{dy}{dx}_{(x_2, y_2)}$

$= 1.22 + 0.1\left(0.2 + 1.22\right) = 1.362$

$x = 0.4$ $\quad y_4 = y_3 + h \frac{dy}{dx}_{(x_3, y_3)}$

$= 1.362 + 0.1\left(0.3 + 1.362\right) = 1.5282$

$x = 0.5$ $\quad y_5 = y_4 + h \frac{dy}{dx}_{(x_4, y_4)}$

$= 1.5282 + 0.1\left(0.4 + 1.5282\right) = 1.72102$

$x = 0.6$ $\quad y_6 = y_5 + h \frac{dy}{dx}_{(x_5, y_5)}$

$= 1.72102 + 0.1\left(0.5 + 1.72102\right) = 1.943$

$x = 0.7$ $\quad y_7 = y_6 + h \frac{dy}{dx}_{(x_6, y_6)}$

$= 1.943 + 0.1\left(0.6 + 1.943\right) = 2.197$

$x = 0.8$ $\quad y_8 = y_7 + h \frac{dy}{dx}_{(x_7, y_7)}$

$= 2.197 + 0.1\left(0.7 + 2.197\right) = 2.487$

$x = 0.9$

$$y_9 = y_8 + h \frac{dy}{dx}_{(x_8, y_8)}$$

$$= 2.487 + 0.1\,(0.8 + 2.487) = 2.8159$$

$x = 1$

$$y_{10} = y_9 + h \frac{dy}{dx}_{(x_9, y_9)}$$

$$= 2.8159 + 0.1\,(0.9 + 2.8159) = 3.1875$$

$$\therefore \boxed{y(1) = 3.1875}$$

(11) Runge-Kutta 2 Method:

Given: * $y(x_0) = y_0$

$\rightarrow x_0, y_0$

* Range
* $h \rightarrow$ step size
* $dy/dx = f(x, y)$

(*) Find $y(x_n)$ or y_n

$$\boxed{y_n = y_{n-1} + \frac{1}{2}(k_1 + k_2)}$$

where, $k_1 = hf(x_{n-1}, y_{n-1})$

$k_2 = hf(x_{n-1} + h, y_{n-1} + k_1)$

Ex: Use Runge kutta method, solve

$\frac{dy}{dx} = \frac{x^2 + y^2}{10}$, $y(0) = 1$, for $0 \leq x \leq 0.4$;

$h = 0.1$

Sol^n: Step 1: $k_1 = hf(x_0, y_0)$

$= hf(0, 1)$

$= 0.1 \times \frac{0^2 + 1^2}{10} = 0.01$

$$k_2 = h\,f(x_0 + h,\ y_0 + k_1)$$
$$= h\,f(0 + 0.1,\ 1 + 0.01)$$
$$= h\,f(0.1,\ 1.01)$$
$$= 0.1 \times \frac{0.1^2 + 1.01^2}{10} = 0.0103$$

$$\therefore\ y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$
$$= 1 + \frac{1}{2}(0.01 + 0.0103)$$
$$= 1.0102$$
$$\therefore\ \boxed{y(0.1) = 1.0102}$$

Step 2: $k_1 = h\,f(x_1, y_1) = h\,f(0.1, 1.0102)$

$$= 0.1 \times \frac{0.1^2 + 1.0102^2}{10} = 0.0103$$

$$k_2 = h\,f(x_1 + h,\ y_1 + k_1)$$
$$= h\,f(0.1 + 0.1,\ 1.0102 + 0.0103)$$
$$= h\,f(0.2,\ 1.0205)$$
$$= 0.1 \times \frac{0.2^2 + 1.0205^2}{10}$$
$$= 0.0108$$

$\therefore\ y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$= 1.0102 + \frac{1}{2}(0.0103 + 0.0108)$

$= 1.02075$

$\therefore\ \boxed{y(0.2) = 1.02075}$

Step 3: $k_1 = hf(x_2, y_2)$

$= hf(0.2, 1.02075)$

$= 0.1 \times \frac{0.2^2 + 1.02075^2}{10}$

$= 0.01082$

$k_2 = hf(x_2 + h, y_2 + k_1)$

$= hf(0.2 + 0.1, 1.02075 + 0.01082)$

$= hf(0.3, 1.0316)$

$= 0.1 \times \frac{0.3^2 + 1.0316^2}{10}$

$= 0.011542$

$\therefore\ y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$

$= 1.02075 + \frac{1}{2}(0.01082 + 0.011542)$

$= 1.032$ $\therefore\ \boxed{y(0.3) = 1.032}$

Step 4: $k_1 = hf(x_3, y_4)$

$= hf(0.3, 1.032)$

$= 0.1 \times \frac{0.3^2 + 1.032^2}{10}$

$= 0.01155$

$k_2 = hf(x_3 + h, y_3 + k_1)$

$= hf(0.3 + 0.1, 1.032 + 0.01155)$

$= hf(0.4, 1.04355)$

$= 0.1 \times \frac{0.4^2 + 1.04355^2}{10}$

$= 0.0125$

$\therefore y_4 = y_3 + \frac{1}{2}(k_1 + k_2)$

$= 1.032 + \frac{1}{2}(0.01155 + 0.0125)$

$= 1.044$

$\therefore \boxed{y(0.4) = 1.044}$ (Ans)