

Commutative Algebra

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Introduction

The study and application of commutative rings with identity (CRW1).

1. AC in calculus. $\mathcal{C}(\mathbb{R}) = \{\text{continuous functions } \mathbb{R} \rightarrow \mathbb{R}\}$, $\mathcal{D}(\mathbb{R}) = \{\text{differentiable functions } \mathbb{R} \rightarrow \mathbb{R}\}$ are both CRW1's.
2. AC in graph theory. Let G be a finite simple graph with vertex set $V = \{v_1, \dots, v_d\}$. The *edge ideal* of G is $I(G) = \langle v_i v_j \mid v_i v_j \text{ is an edge in } G \rangle \leq K[v_1, \dots, v_d]$.

algebraic properties of $I(G) \iff$ combinatorial properties of G .

3. AC in CO (combinatorics). A simplicial complex Δ on V . Stanley-Reisner ideal $J(\Delta) \leq K[v_1, \dots, v_d]$.

algebraic properties of $J(\Delta) \iff$ combinatorics properties of Δ .

Let \mathcal{P} be a poset and $\Delta(\mathcal{P}) = \text{"order complex of } \mathcal{P}\text{"} = \{\text{chains in } \mathcal{P}\}$. Study \mathcal{P} via $J(\Delta(\mathcal{P}))$.

4. AC in NT (number theory). NT is the study of solutions of polynomial equations over \mathbb{Z} . Given an intermediate field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, let $R = \{\alpha \in K \mid \exists \text{ monic } f \in \mathbb{Z}[x] \text{ s.t. } f(\alpha) = 0\}$. Then $\mathbb{Z} \subseteq R \subseteq K$ and R is a subring of K . (Chapter 5)
5. AC in AG (algebraic geometry). AG is the study of solution sets for systems of polynomial equations over fields. Let k be a field, $f_1, \dots, f_m \in k[X_1, \dots, X_d]$, $V := V(f_1, \dots, f_m) = \{\underline{x} \in k^d \mid f_i(\underline{x}) = 0, \forall i = 1, \dots, m\}$, where V for "variety", and $I(V) = \{f \in k[X_1, \dots, X_d] \mid f(\underline{x}) = 0, \forall \underline{x} \in V\} \leq k[X_1, \dots, X_d]$.

algebraic properties of $I(V) \iff$ geometric properties of V .

Why modules? b/c in NT, $R = \{\alpha \in K \mid \exists \text{ monic } f \in \mathbb{Z}[x] \text{ s.t. } f(\alpha) = 0\}$ is a subring of K .

Challenge-exercise: prove this by definition. Let $\alpha, \beta \in R$. Then there exist $f, g \in \mathbb{Z}[X]$ monic such that $f(\alpha) = 0 = f(\beta)$. Prove/construct monic polynomials $s, d, p \in \mathbb{Z}[X]$ such that $s(\alpha + \beta) = 0$, $d(\alpha - \beta) = 0$ and $p(\alpha\beta) = 0$.

Proof is a straight forward application of modules.

Why topology? To study geometry, need continuity. Let $V = V(f_1, \dots, f_m)$, $W = V(g_1, \dots, g_n)$ and $\phi : V \rightarrow W$. What does it mean for ϕ to be continuous if $K = \mathbb{F}_3$? Need a notion of open sets in V and W .

Chapter 1

Rings and Ideals

Let R be a CRW1.

Fact 1.1. $R = 0$ iff $1_R = 0_R$.

Fact 1.2. (1) 1_R and 0_R are both unique.

(2) For any $r \in R$, $-r$ is unique.

(3) If $r \in R$ is a unit, i.e., there exists $r^{-1} \in R$ such that $rr^{-1} = 1_R = r^{-1}r$, then r^{-1} is also unique.

Definition 1.3. A homomorphism of CRW1's is a function $\phi : R \rightarrow S$, where R and S are CRW1's, such that

(1) $\phi(r + r') = \phi(r) + \phi(r')$,

(2) $\phi(rr') = \phi(r)\phi(r')$,

(3) $\phi(1_R) = 1_S$.

A.K.A. a ring homomorphism.

Fact 1.4. Let $\phi : R \rightarrow S$ be a ring homomorphism.

(a) $\phi(0_R) = 0_S$.

(b) $\phi(-r) = -\phi(r)$ for any $r \in R$.

(c) $\phi(r - s) = \phi(r) - \phi(s)$ for any $r, s \in R$.

(d) $\phi(\sum_{i=1}^m r_i s_i) = \sum_{i=1}^m \phi(r_i)\phi(s_i)$ for any $r_1, \dots, r_m, s_1, \dots, s_m \in R$.

(e) If r is a unit in R , then $\phi(r)$ is a unit in S and $\phi(r)^{-1} = \phi(r^{-1})$.

(f) A composition of ring homomorphisms is a ring homomorphism.

Definition 1.5. A *subring* of R is a subset $S \subseteq R$ such that S is a CRW1 under the operations for R and such that $1_S = 1_R$, i.e., $1_R \in S$.

Fact 1.6 (Subring test). A subset $S \subseteq R$ is a subring iff it is closed under $+$, \cdot , $-$ and $1_R \in S$.

Example 1.7. Subring test: need $\emptyset \neq S \subseteq R$, S is closed under $+$, \cdot , $-$ and $1_R \in S$.

If S is not closed under $-$, then fail. Let $\mathbb{N}_0 = \{0, 1, 2, \dots\} \subseteq \mathbb{Z}$ not a subring.

If $1_R \notin S$, then fail. $R = \mathbb{F}_3 \times \mathbb{F}_3 \supseteq \{(a, a) \mid a \in \mathbb{F}_3\} =: S$. Then S is a subring of R . Although $S_1 := \{(a, 0) \mid a \in \mathbb{F}_3\} \cong \mathbb{F}_3 \cong \{(0, a) \mid a \in \mathbb{F}_3\} =: S_2$ are rings but not subrings of R since $1_R = (1, 1) \notin S_1$ and $1_R = (1, 1) \notin S_2$.

Fact 1.8. If $S \subseteq R$ is a subring, then the inclusion map $\varepsilon : S \rightarrow R$ given by $\varepsilon(s) = s$ is a ring homomorphism.

Definition 1.9. An *ideal* of R is a non-empty subset $\mathfrak{a} \subseteq R$ and a subgroup under addition such that for any $r \in R$ and any $a \in \mathfrak{a}$, $ra \in \mathfrak{a}$.

An ideal $\mathfrak{a} \leq R$ is *prime* if $\mathfrak{a} \neq R$ and for any $a, b \in R$, if $a, b \notin \mathfrak{a}$, then $ab \notin \mathfrak{a}$, i.e., if $ab \in \mathfrak{a}$, then $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$.

An ideal $\mathfrak{a} \leq R$ is *maximal* if $\mathfrak{a} \neq R$ and for any ideal $\mathfrak{b} \leq R$, if $\mathfrak{a} \subseteq \mathfrak{b} \subseteq R$, then either $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{b} = R$.

Fact 1.10 (Ideal test). If $\mathfrak{a} \neq \emptyset$ and \mathfrak{a} is closed under \cdot , then for any $a \in \mathfrak{a}$, $-a = (-1_R)a \in \mathfrak{a}$, also, since \mathfrak{a} is closed under $+$, it is automatically closed under $-$.

Thus, A subset $\mathfrak{a} \subseteq R$ is an ideal iff $\mathfrak{a} \neq \emptyset$ and \mathfrak{a} is closed under $+$ and \cdot .

Example 1.11. (a) Let $R = \mathbb{Z}$, then ideals of R are $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$.

$n\mathbb{Z}$ is prime iff $n = 0$ or $|n|$ is prime.

$n\mathbb{Z}$ is maximal iff $|n|$ is prime.

(b) If $I_\lambda \leq R$ for any $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} I_\lambda \leq R$.

(c) If $r_1, \dots, r_m \in R$, then

$$\begin{aligned} \langle r_1, \dots, r_m \rangle &= \langle r_1, \dots, r_m \rangle R = (r_1, \dots, r_m) = (r_1, \dots, r_m)R = \bigcap_{r_1, \dots, r_m \in I \leq R} I \\ &= \left\{ \sum_{i=1}^m a_i r_i \mid a_i \in R, \forall i = 1, \dots, m \right\} \leq R. \end{aligned}$$

In particular, for any $r \in R$, $\langle r \rangle = \langle r \rangle R = (r) = (r)R = rR = Rr = \{ar \mid a \in R\} = \bigcap_{r \in I \leq R} I$.

(d) If $A \subseteq R$, then $\langle A \rangle = \bigcap_{A \subseteq I \leq R} I$ and $\langle A \rangle = \{\sum_{a \in A}^{\text{finite}} r_a a \mid r_a \in R, \forall a\}$.

Fact 1.12. For any $r_1, \dots, r_m \in R$, $\langle r_1, \dots, r_m \rangle$ is the smallest ideal of R containing r_1, \dots, r_m , i.e., for any $\mathfrak{a} \leq R$, $r_1, \dots, r_m \in \mathfrak{a}$ iff $\langle r_1, \dots, r_m \rangle \subseteq \mathfrak{a}$. Similarly, $A \subseteq \mathfrak{a}$ iff $\langle A \rangle \subseteq \mathfrak{a}$. For example, if $A \leq R$, then $A = \langle A \rangle$.

Construction 1.13. Let $\mathfrak{a} \leq R$. For any $r \in R$, $r + \mathfrak{a} = \{r + a \mid a \in \mathfrak{a}\} = \bar{r}$. Let $R/\mathfrak{a} := \{r + \mathfrak{a} \mid r \in R\}$. Then R/\mathfrak{a} is a CRW1 with $\bar{r} \pm \bar{s} = \overline{r \pm s}$, $\bar{r}\bar{s} = \overline{rs}$, $0_{R/\mathfrak{a}} = \overline{0_R}$ and $1_{R/\mathfrak{a}} = \overline{1_R}$.

Let $\pi : R \rightarrow R/\mathfrak{a}$ be given by $\pi(r) = \bar{r}$. Then π is a well-defined ring epimorphism.

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow \pi & \nearrow \exists ! \bar{\phi} & \\ R/\mathfrak{a} & & \end{array}$$

For any $\phi : R \rightarrow S$ ring homomorphism, if $\phi(\mathfrak{a}) = 0$, then there exists a unique ring homomorphism $\bar{\phi} : R/\mathfrak{a} \rightarrow S$ making the diagram commute, where $\bar{\phi}(\bar{r}) = \bar{\phi}(\pi(r)) = \phi(r)$.

Note $\phi(\mathfrak{a}) = 0$ iff $\mathfrak{a} \subseteq \text{Ker}(\phi)$. In particular, if $\mathfrak{a} = \langle A \rangle$, then $\mathfrak{a} \subseteq \text{Ker}(\phi)$ iff $A \subseteq \text{Ker}(\phi)$.

Fact 1.14. Let $\mathfrak{a} \leq R$.

- (a) \mathfrak{a} is prime iff R/\mathfrak{a} is an integral domain.
- (b) \mathfrak{a} is maximal iff R/\mathfrak{a} is a field.
- (c) If R is a field, then it is an integral domain. So if \mathfrak{a} is maximal, then \mathfrak{a} is prime.

Fact 1.15 (Ideal correspondence for quotients). Let $\mathfrak{a} \leq R$ and $\pi : R \rightarrow R/\mathfrak{a}$ be the canonical epimorphism.

$$\begin{aligned}
 \{\text{ideals } I \leq R/\mathfrak{a}\} &\rightleftarrows \{\text{ideals } J \leq R \mid \mathfrak{a} \subseteq J\} \\
 I &\mapsto \pi^{-1}(I) = \{r \in R \mid r + \mathfrak{a} \in I\} \\
 J/\mathfrak{a} &\leftarrow J \\
 \{\text{ideals } I \leq R/\mathfrak{a}\} &\rightleftarrows \{\text{ideals } J \leq R \mid \mathfrak{a} \subseteq J\} \\
 \{\text{prime ideals of } R/\mathfrak{a}\} &\rightleftarrows \{\text{prime ideals } \mathfrak{p} \leq R \mid \mathfrak{a} \subseteq \mathfrak{p}\} \\
 \{\text{maximal ideals of } R/\mathfrak{a}\} &\rightleftarrows \{\text{maximal ideals } \mathfrak{m} \leq R \mid \mathfrak{a} \subseteq \mathfrak{m}\}
 \end{aligned}$$

In both R and R/\mathfrak{a} , maximal ideals are a subset of prime ideals and prime ideals are a proper subset of ideals.

Claim $\frac{R/\mathfrak{a}}{J/\mathfrak{a}} \cong \frac{R}{J}$.

$$\begin{array}{ccccc}
 R & \xrightarrow{p} & R/\mathfrak{a} & \xrightarrow{\tau} & \frac{R/\mathfrak{a}}{J/\mathfrak{a}} \\
 \downarrow \pi & & & \nearrow \bar{\phi} & \\
 R/J & & & \exists ! &
 \end{array}$$

Clearly $J \subseteq \text{Ker}(\tau \circ p)$, so we can use the UMP.

Since the diagram commutes, $J = \text{Ker}(\pi) = \text{Ker}(\pi \circ p)$.

So the kernel is “modulo out” by π and hence $\bar{\phi}$ is 1-1.

Since $\tau \circ p$ is onto and the diagram commutes, $\bar{\phi}$ is onto.

Note $\bar{\phi}(\bar{r}) = \bar{r}$, i.e., $\bar{\phi}(r + J) = (r + \mathfrak{a}) + J/\mathfrak{a}$.

Notation. $\text{Spec}(R) = \{\text{prime ideals of } R\}$, called the *prime spectrum* of R .

$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

Fact 1.16. Let $\phi : R \rightarrow S$ be a ring homomorphism. Then $\text{Ker}(\phi) \leq R$, $\text{Im}(\phi) \subseteq S$ is a subring and $\text{Im}(\phi) \cong R/\text{Ker}(\phi)$.

If S is an integral domain, then so is $\text{Im}(\phi)$. Hence $\text{Ker}(\phi)$ is prime.

More generally, for any $\mathfrak{b} \leq S$, we have $\phi^{-1}(\mathfrak{b}) = \{x \in R \mid \phi(x) \in \mathfrak{b}\} \leq R$.

$$\begin{array}{ccccc}
 R & \xrightarrow{\phi} & S & \xrightarrow{\pi} & S/\mathfrak{q} \\
 \downarrow p & & & \nearrow \overline{\pi \circ \phi} & \\
 \frac{R}{\phi^{-1}(\mathfrak{q})} & & & \exists ! &
 \end{array}$$

Let $\mathfrak{q} \in \text{Spec}(S)$. Then S/\mathfrak{q} is an integral domain. Also, since $R/\text{Ker}(\pi \circ \phi) \cong S/\mathfrak{q}$, we have $R/\text{Ker}(\pi \circ \phi)$ is an integral domain and then $\text{Ker}(\pi \circ \phi)$ is prime. Observe $\phi^{-1}(\mathfrak{q}) = \text{Ker}(\pi \circ \phi)$ is then prime, i.e., $\phi^{-1}(\mathfrak{q}) \in \text{Spec}(R)$. Thus, ϕ induces a well-defined map $\phi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$.

Example. Let $\phi : \mathbb{Z} \rightarrow \mathbb{Q}$ be an inclusion map. Note $\mathfrak{q} := (0)\mathbb{Q} \leq \mathbb{Q}$ is maximal, but $\phi^{-1}(\mathfrak{q}) = \phi^{-1}(0) = \text{Ker}(\phi) = 0\mathbb{Z}$, which is not maximal in \mathbb{Z} .

Fact 1.17. (a) If $R \neq 0$, then R has a maximal ideal \mathfrak{m} . So R has a prime ideal. Moreover, for any $\mathfrak{a} \leq R$, there exists a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$. In particular, $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\} \neq \emptyset$.

(b) Let $\mathfrak{a} \leq R$. Then $0 \neq R/\mathfrak{a}$ is a CRW1. So R/\mathfrak{a} has a maximal ideal and this ideal corresponds for quotients, hence it is of the form $\mathfrak{m}/\mathfrak{a}$, where \mathfrak{m} is the maximal ideal of R containing \mathfrak{a} .

Definition 1.18. R is *local* if it has a unique maximal ideal \mathfrak{m} . A.K.A. “*quasi-local*”. The *residue field* of R is R/\mathfrak{m} .

Shorthand, assume (R, \mathfrak{m}, k) is local, where \mathfrak{m} is the unique maximal ideal of R and $k = R/\mathfrak{m}$. Or assume (R, \mathfrak{m}) is local.

Example 1.19. (a) Any field is local with the maximal ideal (0) .

(b) Let $n \in \mathbb{N}$ and p be prime in \mathbb{Z} . Note $0 \neq \mathbb{Z}/\langle p^n \rangle$ has a maximal ideal $\mathfrak{m} = \langle p \rangle / \langle p^n \rangle$, where $\langle p \rangle$ is a maximal ideal of R containing $\langle p^n \rangle$. Assume there is $\mathfrak{m}_1 \leq R$ maximal such that $\mathfrak{m}_1 \supseteq \langle p^n \rangle$. Then \mathfrak{m}_1 is prime, so $p \in \mathfrak{m}_1$ and hence $\langle p \rangle \subseteq \mathfrak{m}_1$. Since $\langle p \rangle$ is prime in \mathbb{Z} and \mathbb{Z} is a PID, $\langle p \rangle$ is maximal. So $\langle p \rangle = \langle \mathfrak{m}_1 \rangle$. Thus, $\langle p \rangle$ is the unique maximal ideal containing $\langle p^n \rangle$ and so $\mathbb{Z}/\langle p^n \rangle$ is local. Similarly, we can show $\langle p \rangle$ is the unique prime ideal containing $\langle p^n \rangle$, so $\text{Spec}(\mathbb{Z}/\langle p^n \rangle) = \{\langle p \rangle / \langle p^n \rangle\}$.

(c) Let k be a field. Then $R = k[X]/\langle X^n \rangle$ is local with $\mathfrak{m} = \langle X \rangle / \langle X^n \rangle$. In fact, $\text{Spec}(R) = \{\langle X \rangle / \langle X^n \rangle\}$.

(d) Let k be a field and $R = \frac{k[X_1, \dots, X_d]}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$, where $a_i \in \mathbb{N}$ for any $i = 1, \dots, d$. Then R is local with $\mathfrak{m} = \langle X_1, \dots, X_d \rangle / \langle X_1^{a_1}, \dots, X_d^{a_d} \rangle$. In fact, $\text{Spec}(R) = \left\{ \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \right\}$.

Fact 1.20. If (R, \mathfrak{m}) is local and $\mathfrak{a} \leq R$, then $(R/\mathfrak{a}, \mathfrak{m}/\mathfrak{a})$ is also local and $\frac{R/\mathfrak{a}}{\mathfrak{m}/\mathfrak{a}} \cong R/\mathfrak{m}$ canonically isomorphic residue fields. Converse fails in general by Example 1.19.

Notation 1.21. Let $R^\times = R^* = \mathcal{U}(R) = \{\text{units of } R\}$.

Proposition 1.22. TFAE.

- (i) R is local.
- (ii) $R \setminus R^\times \leq R$.
- (iii) There exists $\mathfrak{a} \leq R$ such that $R \setminus \mathfrak{a} \subseteq R^\times$.

When these are satisfied, $\mathfrak{m} = R \setminus R^\times = \mathfrak{a}$.

Proof. “(i) \Rightarrow (ii)”. Assume (R, \mathfrak{m}) is local. Claim $\mathfrak{m} = R \setminus R^\times$. It suffices to show $R \setminus \mathfrak{m} = R^\times$.
“ \supseteq ”. Let $u \in R^\times$. Then $\langle u \rangle = R$ and so $u \notin \mathfrak{m} \leq R$, i.e., $u \in R \setminus \mathfrak{m}$. Hence $R^\times \subseteq R \setminus \mathfrak{m}$.
“ \subseteq ”. Let $x \in R \setminus R^\times$. Then $\langle x \rangle \leq R$. Since \mathfrak{m} is the unique maximal ideal in R , $\langle x \rangle \subseteq \mathfrak{m}$, i.e., $x \in \mathfrak{m}$. Thus, $R \setminus R^\times \subseteq \mathfrak{m}$, i.e., $R \setminus \mathfrak{m} \subseteq R^\times$.
“(ii) \Rightarrow (iii)”. Assume $R \setminus R^\times \leq R$. Set $\mathfrak{a} = R \setminus R^\times$. Then $R \setminus \mathfrak{a} = R^\times$.
“(iii) \Rightarrow (i)”. Let $\mathfrak{a} \leq R$ such that $R \setminus \mathfrak{a} \subseteq R^\times$. Claim, $\mathfrak{a} = R \setminus R^\times$. Clearly $\mathfrak{a} \supseteq R \setminus R^\times$. On the other hand (OTOH), let $a \in \mathfrak{a} \leq R$, then $a \notin R^\times$ since $\mathfrak{a} \leq R$, so $a \in R \setminus R^\times$ and hence $\mathfrak{a} \subseteq R \setminus R^\times$. Thus, $\mathfrak{a} = R \setminus R^\times$. Let $\mathfrak{n} \leq R$ be maximal and $y \in \mathfrak{n}$. Then $y \notin R^\times$. So $y \in R \setminus R^\times = \mathfrak{a}$. Thus, $\mathfrak{n} \subseteq \mathfrak{a} \leq R$. Since \mathfrak{n} is maximal, $\mathfrak{n} = \mathfrak{a}$. Thus, \mathfrak{a} is the unique maximal ideal in R and so R is local. \square

Proposition 1.23. Let $\mathfrak{m} \leq R$ be maximal such that $1 + \mathfrak{m} \subseteq R^\times$. Then R is local.

Proof. By previous proposition, it suffices to show $R \setminus \mathfrak{m} \subseteq R^\times$. Let $x \in R \setminus \mathfrak{m}$. Set $\langle x, \mathfrak{m} \rangle = \langle \{x\} \cup \mathfrak{m} \rangle = \{ax + m \mid a \in R, m \in \mathfrak{m}\}$. Since $x \notin \mathfrak{m}$, $\mathfrak{m} \subsetneq \langle x, \mathfrak{m} \rangle \leq R$. Also, since \mathfrak{m} is maximal, $\langle x, \mathfrak{m} \rangle = R$. So $ax + m = 1$ for some $a \in R$ and $m \in \mathfrak{m}$, i.e., $ax = 1 - m \in 1 + \mathfrak{m} \subseteq R^\times$. Thus, $a, x \in R^\times$. \square

Definition 1.24. $x \in R$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $x^n = 0$. Then *nilradical* of R is $\text{Nil}(R) = \text{N}(R) = \mathfrak{N}_R = \mathfrak{N} = \{\text{nilpotent elements of } R\}$.

Example 1.25. In $\mathbb{Z}/\langle p^n \rangle$, \bar{p} is nilpotent. It is similar in $k[x]/\langle x^n \rangle$ and $k[x_1, \dots, x_n]/\langle x_1^{a_1}, \dots, x_d^{a_d} \rangle$, where k is a field.

Proposition 1.26. (a) $\text{Nil}(R) \leq R$.

(b) $\text{Nil}(R/\text{Nil}(R)) = 0$.

(c) $\text{Nil}(R) = R$ iff $R = 0$.

(d) $\text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$.

Proof. (a) Since $0 \in \text{Nil}(R)$, $\text{Nil}(R) \neq \emptyset$. Let $r \in R$ and $a, b \in \text{Nil}(R)$. Then there exists $m, n \in \mathbb{N}$ such that $a^m = 0 = b^n$. Then $(ra)^m = r^m a^m = 0$ and so $ra \in \text{Nil}(R)$. By binomial theorem, $(a + b)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} a^i b^{m+n-i} = 0$. Since for any $0 \leq i \leq m+n$, either $i \geq m$ or $i < m$, i.e., $i \geq m$ or $m+n-i > n$. we have $a^i = 0$ when $i \geq m$, and $b^{m+n-i} = 0$ when $m+n-i > n$. So $(a + b)^{m+n} = 0$ and thus $a + b \in \text{Nil}(R)$.

(b) Let $\bar{x} \in \text{Nil}(R/\text{Nil}(R))$. Then there exists $n \in \mathbb{N}$ such that $\bar{x}^n = \bar{x}^n = 0$, i.e., $x^n \in \text{Nil}(R)$. So there exists $m \in \mathbb{N}$ such that $(x^n)^m = 0$, i.e., $x^{mn} = 0$. Thus, $x \in \text{Nil}(R)$, i.e., $\bar{x} = 0$.

(c) Since $1 \in \text{Nil}(R)$, there exists $n \in \mathbb{N}$ such that $1 = 1^n = 0$. So $R = 0$.

(d) “ \subseteq ”. Let $x \in \text{Nil}(R)$. Then there exists $n \in \mathbb{N}$ such that $x^n = 0 \in \mathfrak{p}$ for any $\mathfrak{p} \in \text{Spec}(R)$. So $x \in \mathfrak{p}$ for any $\mathfrak{p} \in \text{Spec}(R)$. Thus, $x \in \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$.

“ \supseteq ”. Let $x \in R \setminus \text{Nil}(R)$. Need to show $x \notin \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$. It is equivalent to show there exists $\mathfrak{p} \in \text{Spec}(R)$ such that $x \notin \mathfrak{p}$. Let $\Sigma = \{\mathfrak{a} \leq R \mid x, x^2, x^3 \dots \notin \mathfrak{a}\}$. Since $x \notin \text{Nil}(R)$, $x^k \neq 0$ for any $k \in \mathbb{N}$. So $(0) \in \Sigma$ and then $\Sigma \neq \emptyset$. Let $\mathcal{C} \subseteq \Sigma$ be chain. Then we have $\mathfrak{q} := \bigcup_{\mathfrak{a} \in \mathcal{C}} \mathfrak{a} \leq R$. Suppose $x^n \in \mathfrak{q}$ for some $n \in \mathbb{N}$. Then $x^n \in \mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{C} \subseteq \Sigma$, contradicting $\mathfrak{a} \in \Sigma$. So $x^n \notin \mathfrak{q}$ for any $n \in \mathbb{N}$ and hence $\mathfrak{q} \in \Sigma$. Hence \mathfrak{q} is an upper bound for \mathcal{C} in Σ . Since the chain

$\mathcal{C} \subseteq \Sigma$ is arbitrary, by Zorn's lemma, Σ has a maximal element I .

Claim I is prime. Suppose $I = R$. Then $x \in R = I$, contradicting $I \in \Sigma$. So $I \subsetneq R$. Let $r, s \in R \setminus I$. Then $I \subsetneq \langle r, I \rangle \leq R$ and $I \subsetneq \langle s, I \rangle \leq R$. By the maximality of I in Σ , we have $\langle r, I \rangle, \langle s, I \rangle \notin \Sigma$. So there exists $m, n \in \mathbb{N}$ such that $x^m \in \langle r, I \rangle$ and $x^n \in \langle s, I \rangle$. Then $x^m = ar + i$ for some $a \in R$ and $i \in I$, and $x^n = bs + j$ for some $b \in R$ and $j \in I$. So $x^{m+n} = x^m x^n = (ar + i)(bs + j) = abrs + \underbrace{(arj + bsi + ij)}_{\in I} \in \langle rs, I \rangle$. Hence $\langle rs, I \rangle \notin \Sigma$. Also, since $I \in \Sigma$, $rs \notin I$. Thus, $I \in \text{Spec}(R)$ such that $x \notin I$. □

Example. Let k be a field and $R = \frac{k[X_1, \dots, X_d]}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \neq 0$, where $a_i \in \mathbb{N}$ for any $i = 1, \dots, d$. Then $\text{Nil}(R) = \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$.

Proof. Method 1: Since $\text{Spec}(R) = \left\{ \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \right\}$, $\text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p} = \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$.

Method 2: Since $\bar{X}_i \in \text{Nil}(R) \leq R$ for each $i = 1, \dots, d$, we have $\langle X_1, \dots, X_d \rangle = \langle \bar{X}_1, \dots, \bar{X}_d \rangle \subseteq \text{Nil}(R) \subsetneq R$. Also, since $\langle X_1, \dots, X_d \rangle$ is maximal, we have $\text{Nil}(R) = \langle X_1, \dots, X_d \rangle$. □

Fact. If $\mathfrak{a} \leq R$ and $r_1, \dots, r_n \in R$, then $R/\mathfrak{a} \supseteq \langle \bar{r}_1, \dots, \bar{r}_n \rangle = \frac{\langle r_1, \dots, r_n, \mathfrak{a} \rangle}{\mathfrak{a}}$. In particular, if $\langle r_1, \dots, r_n \rangle \supseteq \mathfrak{a}$, then $\langle \bar{r}_1, \dots, \bar{r}_n \rangle = \frac{\langle r_1, \dots, r_n \rangle}{\mathfrak{a}}$.

Definition 1.27. The Jacobson radical of R is $\text{Jac}(R) = \mathfrak{J}(R) = \bigcap_{\mathfrak{m} \leq R \text{ max}} \mathfrak{m}$.

Fact 1.28. $\text{Jac}(R) \supseteq \text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$.

Proposition 1.29. $\mathfrak{J}(R) = \{x \in R \mid 1 - xy \in R^\times, \forall y \in Y\}$.

Proof. “ \subseteq ”. Let $x \in \mathfrak{J}(R)$. By way of contradiction (BWOC), suppose there exists $y \in R$ such that $1 - xy \notin R^\times$. Then there exists $\mathfrak{m} \leq R$ maximal such that $1 - xy \in \mathfrak{m}$. Since $x \in \mathfrak{J}(R) \subseteq \mathfrak{m}$, $xy \in \mathfrak{m}$. So $1 = (1 - xy) + xy \in \mathfrak{m}$, a contradiction.

“ \supseteq ”. Argue by contrapositive. Let $x \in R$ such that $1 - xy \in R^\times$ for any $y \in Y$. Suppose $x \notin \mathfrak{J}(R)$. Then there exists $\mathfrak{m} \leq R$ maximal such that $x \notin \mathfrak{m}$. So $\mathfrak{m} \subsetneq \langle \mathfrak{m}, x \rangle \subseteq R$. Hence $\langle x, \mathfrak{m} \rangle = R$. Then there exists $y \in R$ and $m \in \mathfrak{m}$ such that $xy + m = 1$, i.e., $1 - xy = m \in \mathfrak{m}$. So $1 - xy \notin R^\times$, a contradiction. □

1.1 Operations on Ideals

Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \leq R$, $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$ and $\mathfrak{a}_\lambda \leq R$ for any $\lambda \in \Lambda$, where Λ is an index set.

Definition 1.30. $\mathfrak{a} + \mathfrak{b} = \langle \mathfrak{a} \cup \mathfrak{b} \rangle = \bigcap_{\mathfrak{a} \cup \mathfrak{b} \subseteq I \leq R} I$.

Fact 1.31. (a) $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{c}$ iff $\mathfrak{a} \cup \mathfrak{b} \subseteq \mathfrak{c}$.

(b) $\mathfrak{a} + \mathfrak{b}$ is the (unique) smallest ideal of R that contains $\mathfrak{a} \cup \mathfrak{b}$.

(c) $\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$.

(d) If $\mathfrak{a} = \langle S \rangle$ and $\mathfrak{b} = \langle T \rangle$, then $\mathfrak{a} + \mathfrak{b} = \langle S \cup T \rangle$.

- (e) If $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$ and $\mathfrak{b} = \langle y_1, \dots, y_n \rangle$, then $\mathfrak{a} + \mathfrak{b} = \langle x_1, \dots, x_m, y_1, \dots, y_n \rangle$.
- (f) If $x \in R$, then $\langle x, \mathfrak{a} \rangle = \langle x \rangle + \mathfrak{a}$.
- (g) If $\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}$.

Proof. (c) Set $I = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\} \leq R$ (Check). For any $a \in \mathfrak{a}$, $a = a + 0 \in I$ and for any $b \in \mathfrak{b}$, $b = 0 + b \in I$. So $\mathfrak{a} \cup \mathfrak{b} \subseteq I$. By (a), $\mathfrak{a} + \mathfrak{b} \subseteq I$. OTOH, for any $a + b \in I$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$, $a, b \in \mathfrak{a} \cup \mathfrak{b} \subseteq \mathfrak{a} + \mathfrak{b} \leq R$. So $a + b \in \mathfrak{a} + \mathfrak{b}$.

- (d) Let $I \leq R$. Note $I \supseteq \mathfrak{a} \cup \mathfrak{b}$ iff $I \supseteq \mathfrak{a}, \mathfrak{b}$ iff $I \supseteq \langle S \rangle, \langle T \rangle$ iff $I \supseteq S, T$ iff $I \supseteq S \cup T$. So $\mathfrak{a} + \mathfrak{b} = \bigcap_{\mathfrak{a} \cup \mathfrak{b} \subseteq I \leq R} I = \bigcap_{S \cup T \subseteq I \leq R} I = \langle S \cup T \rangle$.
- (e) By (d).
- (f) By (c).
- (g) The essential point is $\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = \langle \mathfrak{a} \cup (\mathfrak{b} \cup \mathfrak{c}) \rangle = \langle (\mathfrak{a} \cup \mathfrak{b}) \cup \mathfrak{c} \rangle = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}$.

□

Example. $m\mathbb{Z} + n\mathbb{Z} = \langle m, n \rangle \mathbb{Z} = \gcd(m, n)\mathbb{Z}$, where $m \neq 0$ or $n \neq 0$.

Recall. $\text{Spec}(R) = \{\text{prime ideals of } R\}$. For any $S \subseteq R$, $V(S) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq S\}$.

Proposition 1.32. Let $S \subseteq R$.

- (a) $V(S) = V(\langle S \rangle)$.
- (b) $\mathfrak{a} = R$ iff $V(\mathfrak{a}) = \emptyset$.
- (c) $\mathfrak{a} \subseteq \text{Nil}(R)$ iff $V(\mathfrak{a}) = \text{Spec}(R)$.
- (d) If $\mathfrak{a} \subseteq \mathfrak{b}$, then $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$. If $S \subseteq T \subseteq R$, then $V(S) \supseteq V(T)$.

Proof. (d) Since $S \subseteq T \subseteq R$, we have $V(S) \supseteq V(T)$ by definition.

- (a) “ \supseteq ”. Since $S \subseteq \langle S \rangle$, $V(S) \supseteq V(\langle S \rangle)$ by (d).
“ \subseteq ”. Let $\mathfrak{p} \in V(S)$. Then $\mathfrak{p} \supseteq S$. So $\mathfrak{p} \supseteq \langle S \rangle$ and then $\mathfrak{p} \in V(\langle S \rangle)$. Hence $V(S) \subseteq V(\langle S \rangle)$.
- (b) “ \Rightarrow ”. Let $\mathfrak{a} = R$. Then $\mathfrak{p} \not\supseteq \mathfrak{a}$ for any $\mathfrak{p} \in \text{Spec}(R)$. So $V(\mathfrak{a}) = \emptyset$.
“ \Leftarrow ”. Let $V(\mathfrak{a}) = \emptyset$. Suppose $\mathfrak{a} \neq R$, then there exists $\mathfrak{m} \leq R$ maximal such that $\mathfrak{m} \supseteq \mathfrak{a}$. Also, since $\mathfrak{m} \in \text{Spec}(R)$, we have $\mathfrak{m} \in V(\mathfrak{a})$, contradicting $V(\mathfrak{a}) = \emptyset$.
- (c) $\mathfrak{a} \subseteq \text{Nil}(R)$ iff $\mathfrak{p} \supseteq \mathfrak{a}$ for all $\mathfrak{p} \in \text{Spec}(R)$ iff $V(\mathfrak{a}) = \text{Spec}(R)$.

□

Proposition 1.33. (a) $V(\mathfrak{a} + \mathfrak{b}) = V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$.

- (b) $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$ iff $\mathfrak{a} + \mathfrak{b} = R$.

Proof. (a) Since $\mathfrak{a} + \mathfrak{b} = \langle \mathfrak{a} \cup \mathfrak{b} \rangle$, $V(\mathfrak{a} + \mathfrak{b}) = V(\langle \mathfrak{a} \cup \mathfrak{b} \rangle) = V(\mathfrak{a} \cup \mathfrak{b})$.

Let $\mathfrak{p} \in \text{Spec}(R)$. Note $\mathfrak{p} \supseteq \mathfrak{a} \cup \mathfrak{b}$ iff $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \supseteq \mathfrak{b}$. So $V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$.

(b) $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$ iff $V(\mathfrak{a} + \mathfrak{b}) = \emptyset$ iff $\mathfrak{a} + \mathfrak{b} = R$.

□

Remark. You can define $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n$ inductively and same properties as above hold for finite sums.

Definition 1.34. $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \langle \bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \rangle = \bigcap_{\substack{I \subseteq R \\ \bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq I}} I$.

Fact 1.35. (a) $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq \mathfrak{c}$ iff $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq \mathfrak{c}$.

(b) $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}$ is the (unique) smallest ideal of R containing $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}$.

(c) $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \{\sum_{\lambda \in \Lambda}^{\text{finite}} a_{\lambda} \mid a_{\lambda} \in \mathfrak{a}_{\lambda}, \forall \lambda \in \Lambda\}$.

(d) If $\mathfrak{a}_{\lambda} = \langle S_{\lambda} \rangle$ for any $\lambda \in \Lambda$, then $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \langle \bigcup_{\lambda \in \Lambda} S_{\lambda} \rangle$.

Fact 1.36. (a) $V(\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}) = V(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}) = \bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_{\lambda})$.

(b) $\bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_{\lambda}) = \emptyset$ iff $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = R$.

Definition 1.37. $\mathfrak{a}\mathfrak{b} = \langle N \rangle = \bigcap_{N \subseteq I \subseteq R} I$, where $N = \{ab \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$.

Fact 1.38. Let $\mathfrak{a}\mathfrak{b} = \langle N \rangle$.

(a) $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c}$ iff $N \subseteq \mathfrak{c}$.

(b) $\mathfrak{a}\mathfrak{b}$ is the (unique) smallest ideal of R containing N .

(c) $\mathfrak{a}\mathfrak{b} = \{\sum_i^{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}, \forall i\}$.

(d) If $\mathfrak{a} = \langle S \rangle$ and $\mathfrak{b} = \langle T \rangle$, then $\mathfrak{a}\mathfrak{b} = \langle st \mid s \in S, t \in T \rangle$.

(e) If $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$ and $\mathfrak{b} = \langle y_1, \dots, y_n \rangle$, then $\mathfrak{a}\mathfrak{b} = \langle x_i y_j \mid i = 1, \dots, m, j = 1, \dots, n \rangle$.

(f) $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

Proof. (c) Let $I = \{\sum_i^{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\} \subseteq R$. Check $I \leq R$ and $I \subseteq \mathfrak{a}\mathfrak{b} \subseteq I$ like 1.31(c).

(f) To show $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, it suffices to show $ab \in \mathfrak{a} \cap \mathfrak{b}$ for any $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. For any $a \in \mathfrak{a} \leq R$, we have $ab \in \mathfrak{a}$ for any $b \in \mathfrak{b}$. For any $b \in \mathfrak{b} \leq R$, we have $ab \in \mathfrak{b}$ for any $a \in \mathfrak{a}$. So $ab \in \mathfrak{a} \cap \mathfrak{b}$ for any $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$.

□

Proposition 1.39. (a) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

(b) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \text{Spec}(R)$ iff $\mathfrak{a}\mathfrak{b} \subseteq \text{Nil}(R)$ iff $\mathfrak{a} \cap \mathfrak{b} \subseteq \text{Nil}(R)$.

Proof. (a) • Let $\mathfrak{p} \in \text{Spec}(R)$. Claim $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ iff $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. “ \Leftarrow ”. Let $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. Then $\mathfrak{p} = \mathfrak{p}R \supseteq \mathfrak{a}R \supseteq \mathfrak{a}\mathfrak{b}$ or $\mathfrak{p} = \mathfrak{p}R \supseteq R\mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$. “ \Rightarrow ”. Let $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Suppose $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $\mathfrak{p} \not\supseteq \mathfrak{b}$. Then there exists $a \in \mathfrak{a} \setminus \mathfrak{p}$ and exists $b \in \mathfrak{b} \setminus \mathfrak{p}$. Since $\mathfrak{p} \in \text{Spec}(R)$, $ab \notin \mathfrak{p}$, contradicting $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ iff $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq \mathfrak{b}$. So $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

- Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a} \cap \mathfrak{b})$. Let $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$. Then $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$. Let $x \in \mathfrak{a} \cap \mathfrak{b}$. Then $x \in \mathfrak{a}$ and $x \in \mathfrak{b}$. So $x^2 = x \cdot x \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Since \mathfrak{p} is prime, $x \in \mathfrak{p}$. So $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$ and then $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$. Hence $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$. Thus, $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$.

(b) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \text{Spec}(R)$ iff $V(\mathfrak{a}\mathfrak{b}) = \text{Spec}(R)$ iff $\mathfrak{a}\mathfrak{b} \subseteq \text{Nil}(R)$ and similarly for $\mathfrak{a} \cap \mathfrak{b}$. □

Proposition 1.40. (a) $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ and $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$.

(b) $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$.

(c) If $\mathfrak{a} + \mathfrak{b} = R$, i.e., \mathfrak{a} and \mathfrak{b} are “coprime” and “co-maximal”, then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

The converse holds if R is a PID and $\mathfrak{a}, \mathfrak{b} \neq 0$.

Proof. (c) “ \supseteq ”. We always have $\mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$.

“ \subseteq ”. Assume $\mathfrak{a} + \mathfrak{b} = R$.

Method 1: Note $1 = a + b$ for some $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Let $x \in \mathfrak{a} \cap \mathfrak{b}$. Then $x \in \mathfrak{b}$ and $x \in \mathfrak{a}$. So $x = 1 \cdot x = (a + b)x = ax + bx = ax + xb \in \mathfrak{a}\mathfrak{b}$. Hence $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$.

Method 2: Note $\mathfrak{a} \cap \mathfrak{b} = R(\mathfrak{a} \cap \mathfrak{b}) = (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} \cap \mathfrak{b}) = \underbrace{\mathfrak{a}(\mathfrak{a} \cap \mathfrak{b})}_{\subseteq \mathfrak{b}} + \underbrace{\mathfrak{b}(\mathfrak{a} \cap \mathfrak{b})}_{\subseteq \mathfrak{a}} \subseteq \mathfrak{a}\mathfrak{b}$ by (a) and (b).

Conversely, assume R is a PID and $\mathfrak{a}, \mathfrak{b} \neq 0$. Write $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n} R$ and $\mathfrak{b} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_n^{f_n} R$ with $e_i, f_i \geq 0$ for any $i = 1, \dots, n$, and \mathfrak{p}_i 's $\in \text{Spec}(R)$ non-associates. Assume $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. Since R is a PID, $\mathfrak{a} \cap \mathfrak{b} = \text{lcm}(\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}, \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_n^{f_n})R = \mathfrak{p}_1^{\max\{e_1, f_1\}} \cdots \mathfrak{p}_n^{\max\{e_n, f_n\}}$. By the fact 1.38(e), $\mathfrak{a}\mathfrak{b} = \mathfrak{p}_1^{e_1+f_1} \cdots \mathfrak{p}_n^{e_n+f_n}$. So $\max\{e_i, f_i\} = e_i + f_i$, i.e., $e_i = 0$ or $f_i = 0$ for any $i = 1, \dots, n$. In other words, for any $\mathfrak{p} \in \text{Spec}(R)$, either $\mathfrak{a} \not\subseteq \mathfrak{p}R$ or $\mathfrak{b} \not\subseteq \mathfrak{p}R$. So $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$ for any $\mathfrak{p} \in \text{Spec}(R)$. Thus, $\mathfrak{a} + \mathfrak{b} = R$. □

Remark. You can do this for $\mathfrak{a}_1, \dots, \mathfrak{a}_n$, where $n \in \mathbb{Z}^{\geq 3}$.

Example 1.41. Let $R = k[x, y]$, $\mathfrak{a} = \langle x \rangle$ and $\mathfrak{b} = \langle y \rangle$. Then $\mathfrak{a} \cap \mathfrak{b} = \langle xy \rangle = \mathfrak{a}\mathfrak{b}$ by the fact 1.38(e). But $\mathfrak{a} + \mathfrak{b} = \langle x, y \rangle \subsetneq R$. So the converse in (c) fails for any non-PID.

Definition 1.42. Let $n \in \mathbb{N}$. Let $\mathfrak{a}^n = \underbrace{\mathfrak{a} \cdots \mathfrak{a}}_{n \text{ times}}$ and $\mathfrak{a}^0 = R$.

Warning 1.43. \mathfrak{a}^n is **not** generated by $\{a^n \mid a \in \mathfrak{a}\}$. For example, let $R = \mathbb{F}_2[x, y]$ and $\mathfrak{a} = \langle x, y \rangle$, then $\mathfrak{a}^2 = \langle x^2, xy, y^2 \rangle \neq \langle f^2 \mid f \in \mathfrak{a} \rangle \not\supseteq xy$.

Fact 1.44. Let $n \in \mathbb{N}$ and $N = \{a_1 \cdots a_n \mid a_i \in \mathfrak{a}, \forall i = 1, \dots, n\}$.

(a) $\mathfrak{a}^n = \langle N \rangle$ and for any $\mathfrak{b} \leq R$, we have $\mathfrak{a}^n \subseteq \mathfrak{b}$ iff $N \subseteq \mathfrak{b}$.

(b) \mathfrak{a}^n is the (unique) smallest ideal of R containing N .

(c) $\mathfrak{a}^n = \{\sum_i^{\text{finite}} a_{i1} \cdots a_{in} \mid a_{ij} \in \mathfrak{a}, \forall i, \forall j = 1, \dots, n\}$.

(d) If $\mathfrak{a} = \langle S \rangle$, then $\mathfrak{a}^n = \langle s_1 \cdots s_n \mid s_i \in S, \forall i = 1, \dots, n \rangle$.

(e) If $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$, then $\mathfrak{a}^n = \langle x_{i_1} \cdots x_{i_n} \mid i_j = 1, \dots, m, \forall j = 1, \dots, n \rangle$.

Fact 1.45. $V(\mathfrak{a}^n) = V(\mathfrak{a})$.

Proof. By the proposition 1.39, $V(\mathfrak{a}^n) = \bigcup_{i=1}^n V(\mathfrak{a}) = V(\mathfrak{a})$. \square

Proposition 1.46 (CRT). Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$.

- (a) The function $\phi : R \rightarrow (R/\mathfrak{a}_1) \times \dots \times (R/\mathfrak{a}_n)$ given by $\phi(x) = (\bar{x}, \dots, \bar{x}) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n)$ is a well-defined ring homomorphism.
- (b) If $\mathfrak{a}_i + \mathfrak{a}_j = R$ for any $1 \leq i \neq j \leq n$, i.e., $\{\mathfrak{a}_1, \dots, \mathfrak{a}_n\}$ are pairwise co-prime, then $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ and $\mathfrak{a}_i + (\bigcap_{j \neq i} \mathfrak{a}_j)R = R$ for any $i = 1, \dots, n$.
- (c) ϕ is surjective iff $\mathfrak{a}_i + \mathfrak{a}_j = R$ for any $1 \leq i \neq j \leq n$.
- (d) $\text{Ker}(\phi) = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$.
- (e) If $\mathfrak{a}_i + \mathfrak{a}_j = R$ for any $1 \leq i \neq j \leq n$ and $\bigcap_{i=1}^n \mathfrak{a}_i = 0$, then $R \cong (R/\mathfrak{a}_1)R \cap \dots \times (R/\mathfrak{a}_n)R$.

Proof. (b) Let $i \in \{1, \dots, n\}$. To show $\mathfrak{a}_i + (\bigcap_{j \neq i} \mathfrak{a}_j)R = R$. It suffices to show $V(\mathfrak{a}_i) \cap (\bigcup_{j \neq i} V(\mathfrak{a}_j)) =$

$V(\mathfrak{a}_i) \cap V(\bigcap_{j \neq i} \mathfrak{a}_j) = V(\mathfrak{a}_i + \bigcap_{j \neq i} \mathfrak{a}_j) = \emptyset$. Suppose $V(\mathfrak{a}_i) \cap (\bigcup_{j \neq i} V(\mathfrak{a}_j)) \neq \emptyset$. Then there exists $\mathfrak{p} \in V(\mathfrak{a}_i) \cap V(\mathfrak{a}_j) = V(\mathfrak{a}_i + \mathfrak{a}_j) = V(R) = \emptyset$ for some $j \neq i$, a contradiction.

Now for $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_n$, prove by induction on n . Base case $n = 1$: trivial. Base case $n = 2$: by 1.40(c). Induction step: assume $n \in \mathbb{Z}^{\geq 3}$ and $\bigcap_{i=1}^{n-1} \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}$. Then $\mathfrak{a}_n + (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1})R = \mathfrak{a}_n + (\bigcap_{j=1}^{n-1} \mathfrak{a}_j)R = R$. So by proposition 1.40(c), we have $\bigcap_{i=1}^n \mathfrak{a}_i = (\bigcap_{i=1}^{n-1} \mathfrak{a}_i) \cap \mathfrak{a}_n = (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}) \cap \mathfrak{a}_n = (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1})\mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$.

- (c) “ \Rightarrow ”. Assume ϕ is surjective. In particular, there exists $x \in R$ such that $(\bar{1}, \bar{0}, \dots, \bar{0}) = \phi(x) = (\bar{x}, \bar{x}, \dots, \bar{x})$. So $x + \mathfrak{a}_1 = 1 + \mathfrak{a}_1$ and $x + \mathfrak{a}_i = 0 + \mathfrak{a}_i$ for any $2 \leq i \leq n$. Hence $1 - x \in \mathfrak{a}_1$ and $x \in \mathfrak{a}_i$ for any $2 \leq i \leq n$. Also, since $\underset{\in \mathfrak{a}_i}{(x)} + \underset{\in \mathfrak{a}_1}{(1-x)} = 1$, we have $\mathfrak{a}_i + \mathfrak{a}_1 = R$ for any $2 \leq i \leq n$.

Similarly, consider $(\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0}) \rightsquigarrow \underset{\uparrow j\text{th}}{\bar{1}} \rightsquigarrow \mathfrak{a}_i + \mathfrak{a}_j = R$ for any $1 \leq i \neq j \leq n$.

“ \Leftarrow ”. Assume $\mathfrak{a}_i + \mathfrak{a}_j = R$ for any $1 \leq i \neq j \leq n$. By (b), $\mathfrak{a}_1 + (\bigcap_{j=2}^n \mathfrak{a}_j)R = R$. So $a_1 + y = 1$ with $a_1 \in \mathfrak{a}_1$ and $y \in \bigcap_{j=2}^n \mathfrak{a}_j$, i.e., $1 - y = a_1 \in \mathfrak{a}_1$ and $y \in \mathfrak{a}_j$ for any $2 \leq j \leq n$. Then $\phi(y) = (\bar{y}, \bar{y}, \dots, \bar{y}) = (y + \mathfrak{a}_1, y + \mathfrak{a}_2, \dots, y + \mathfrak{a}_n) = (1 + \mathfrak{a}_1, 0 + \mathfrak{a}_2, \dots, 0 + \mathfrak{a}_n) = (\bar{1}, \bar{0}, \dots, \bar{0})$. Similarly, for any $j = 1, \dots, n$, there exists y_j such that $\phi(y_j) = (\bar{0}, \dots, \bar{0}, \underset{\uparrow j\text{th}}{\bar{1}}, \bar{0}, \dots, \bar{0})$. Then for any

$$(\bar{r}_1, \dots, \bar{r}_n) \in \frac{R}{\mathfrak{a}_1} \times \dots \times \frac{R}{\mathfrak{a}_n}, (\bar{r}_1, \dots, \bar{r}_n) = \sum_{j=1}^n r_j (\bar{0}, \dots, \bar{0}, \underset{\uparrow j\text{th}}{\bar{1}}, \bar{0}, \dots, \bar{0}) = \sum_{j=1}^n r_j \phi(y_j) =$$

$\phi(\sum_{j=1}^n r_j y_j)$. So ϕ is surjective. \square

Proposition 1.47. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$ and $\mathfrak{p} \in \text{Spec}(R)$.

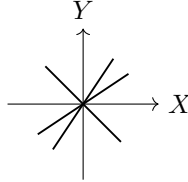
- (a) If $\mathfrak{p} = \mathfrak{a}_1 \cdots \mathfrak{a}_n$, then $\mathfrak{p} = \mathfrak{a}_i$ for some $i \in \{1, \dots, n\}$.
- (b) If $\mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$, then $\mathfrak{p} = \mathfrak{a}_1$ for some $i \in \{1, \dots, n\}$.

Proof. (b) Assume $\mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n \supseteq \mathfrak{a}_1 \cdots \mathfrak{a}_n$. Since $\mathfrak{p} \in \text{Spec}(R)$, there exists some $i \in \{1, \dots, n\}$ such that $\mathfrak{p} \supseteq \mathfrak{a}_i$. Then $\mathfrak{a}_i \subseteq \mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n \subseteq \mathfrak{a}_i$. So $\mathfrak{p} = \mathfrak{a}_i$.

(a) Follow from (b) directly. □

Example. The converses fail in general. Let $R = k[X, Y]$, $\mathfrak{p} = \mathfrak{a}_1 = \langle x \rangle$ and $\mathfrak{a}_2 = \langle y \rangle$. Then $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \langle xy \rangle \neq \langle x \rangle = \mathfrak{p} = \langle x \rangle \neq \langle xy \rangle = \mathfrak{a}_1 \mathfrak{a}_2$.

Lemma 1.48. Let k be an infinite field, $V \neq 0$ a vector space over k , and $V_1, \dots, V_n \subsetneq V$. Then $\bigcup_{i=1}^n V_i \subsetneq V$.



Proof. Induction on n . Base case $n = 1$: trivial.

Inductive step: assume $n \geq 2$ and $\bigcup_{i \neq j} V_i \subsetneq V$ for any $j = 1, \dots, n$. Then there exists $v_j \in V \setminus \{\bigcup_{i \neq j} V_i\}$ for any $j = 1, \dots, n$. BWOC, suppose $\bigcup_{i=1}^n V_i = V \ni v_j$. Since $v_j \notin \bigcup_{i \neq j} V_i$, $v_j \in V_j$. Let $1 \leq i \neq j \leq n$. Consider $v_i + \lambda v_j$ with $\lambda \in k$. Check if $\lambda \neq \mu$ in k , then $v_i + \lambda v_j \neq v_i + \mu v_j$. So there exists l such that V_l contains two distinct elements $v_i + \lambda v_j$ and $v_i + \mu v_j$ with $\lambda, \mu \neq 0$. Note $V_l \ni (v_i + \lambda v_j) - (v_i + \mu v_j) = (\lambda - \mu)v_j$. Since $\lambda \neq \mu$, $v_j \in V_l$ and then $l = j$. Note $V_l \ni \lambda^{-1}(v_i + \lambda v_j) - \mu^{-1}(v_i + \mu v_j) = (\lambda^{-1} - \mu^{-1})v_i$. So $v_i \in V_l$ and then $i = l$. Hence $i = l = j$, a contradiction. □

Example 1.49. If $|k| < \infty$, then fail. Since $|k^2| < \infty$, $k^2 = \bigcup_{v \in k^2} \{v\} = \bigcup_{0 \neq v \in k^2} \text{span}\{v\}$ finite union but $\text{span}(v) \subsetneq k^2$?

Same technique shows that can't replace V_1, \dots, V_n with V_1, V_2, \dots , over \mathbb{Q} .

Theorem 1.50 (More general version). Let $\mathfrak{b}_1, \dots, \mathfrak{b}_n \leq R$. Assume

(1) R contains an infinite field k as a subring, or

(2) $\mathfrak{b}_3, \dots, \mathfrak{b}_n \in \text{Spec}(R)$.

Then if $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{b}_i$, $\mathfrak{a} \subseteq \mathfrak{b}_i$ for some $i \in \{1, \dots, n\}$.

Proof. (1) Assume $\mathfrak{a} \not\subseteq \mathfrak{b}_i$ for any $i = 1, \dots, n$. Then $\mathfrak{a} \cap \mathfrak{b}_i \subsetneq \mathfrak{a}$ for any $i = 1, \dots, n$. Since \mathfrak{a} is a k -vector space and $\mathfrak{a} \cap \mathfrak{b}_i \subsetneq \mathfrak{a}$ for any $i = 1, \dots, n$, by 1.48, $\mathfrak{a} \cap \bigcup_{i=1}^n \mathfrak{b}_i = \bigcup_{i=1}^n (\mathfrak{a} \cap \mathfrak{b}_i) \subsetneq \mathfrak{a}$. So $\mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{b}_i$.

(2) Induct on n . Base case $n = 1$: done. Base case $n = 2$. Let $a_i \in \mathfrak{a} \setminus \mathfrak{b}_i$ for any $i = 1, 2$. Consider $a_1 + a_2 \in \mathfrak{a}$. Suppose $\mathfrak{a} \subseteq \mathfrak{b}_1 \cup \mathfrak{b}_2$. Then $a_1 + a_2 \in \mathfrak{b}_1 \cup \mathfrak{b}_2$, say $a_1 + a_2 \in \mathfrak{b}_2$. Since $a_1 \in \mathfrak{a} \subseteq \mathfrak{b}_1 \cup \mathfrak{b}_2$ and $a_1 \notin \mathfrak{b}_1$, $a_1 \in \mathfrak{b}_2$. So $a_2 = (a_1 + a_2) - a_1 \in \mathfrak{b}_2$, a contradiction.

Induct on $n \geq 3$. Assume $\mathfrak{a} \not\subseteq \mathfrak{b}_i$ for any $i = 1, \dots, n$. Suppose $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{b}_i$. Inductive hypothesis: $\mathfrak{a} \not\subseteq \bigcup_{i \neq j} \mathfrak{b}_i$ for any $j = 1, \dots, n$, there exists $a_j \in \mathfrak{a} \setminus \{\bigcup_{i \neq j} \mathfrak{b}_i\}$. Consider $a_1 \cdot a_{n-1} + a_n \in \mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{b}_i$. Since $a_j \in \mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{b}_i$ and $a_j \notin \bigcup_{i \neq j} \mathfrak{b}_i$, we have $a_j \in \mathfrak{b}_j$. So there exists $l \in \{1, \dots, n\}$ such that $a_1 \cdot a_{n-1} + a_n \in \mathfrak{b}_l$. Suppose $l = n$. Then $a_n \in \mathfrak{b}_n$ and so $a_1 \cdot a_{n-1} \in \mathfrak{b}_n$. Since $n \geq 3$, \mathfrak{b}_n is prime. So $\mathfrak{b}_n \ni a_i$ for some $i < n$, a contradiction. Hence $l < n$. So $a_1 \cdot a_{n-1} \in \mathfrak{b}_l$. □

Theorem 1.51 (Prime Avoidance). *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Spec}(R)$. If $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some $i \in \{1, \dots, n\}$, i.e., if $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for any $i = 1, \dots, n$, then $\mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$.*

Definition 1.52. Let $\mathfrak{a} \leq R$ and $S \subseteq R$. $(\mathfrak{a} : S) = \{r \in R \mid rs \in \mathfrak{a}, \forall s \in S\}$.

Example 1.53. Let $R = k[X, Y]$. $(\langle XY \rangle : \{X, Y\}) = \langle XY \rangle$ and $(\langle X^2, XY \rangle : \{X, Y\}) = \langle X \rangle$.

Fact 1.54. (a) $\mathfrak{a} \subseteq (\mathfrak{a} : S) \leq R$.

(b) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.

(c) If $S \subseteq T$, $(\mathfrak{a} : S) \supseteq (\mathfrak{a} : T)$.

(d)

Proof. (b) Let $r \in (\mathfrak{a} : \mathfrak{b})$ and $b \in \mathfrak{b}$. Then $br \in \mathfrak{a}$. So all generators of $(\mathfrak{a} : \mathfrak{b})\mathfrak{b}$ is in \mathfrak{a} .

(e) Since $S \subseteq \langle S \rangle$, by (c), $(\mathfrak{a} : S) \supseteq (\mathfrak{a} : \langle S \rangle)$. “ \subseteq ”.

(j) “ \subseteq ”.

“ \supseteq ”. Reverse the above proof.

□

Example 1.55. Let $R = k[X, Y]$.

Definition 1.56. The *radical* of $\mathfrak{a} \leq R$ is $\text{rad}(\mathfrak{a}) = \mathfrak{r}(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a}, \forall n \geq 0\} = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$.

Example 1.57. $\text{rad}(\langle X^2Y, XY^2 \rangle) = \langle XY \rangle$ in $k[X, Y]$.