

# Commutative Algebra

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# Contents

# Introduction

The study and application of commutative rings with identity (CRW1).

1. AC in calculus.  $\mathcal{C}(\mathbb{R}) = \{\text{continuous functions } \mathbb{R} \rightarrow \mathbb{R}\}$ ,  $\mathcal{D}(\mathbb{R}) = \{\text{differentiable functions } \mathbb{R} \rightarrow \mathbb{R}\}$  are both CRW1's.
2. AC in graph theory. Let  $G$  be a finite simple graph with vertex set  $V = \{v_1, \dots, v_d\}$ . The *edge ideal* of  $G$  is  $I(G) = \langle v_i v_j \mid v_i v_j \text{ is an edge in } G \rangle \leq K[v_1, \dots, v_d]$ .

algebraic properties of  $I(G) \iff$  combinatorial properties of  $G$ .

3. AC in CO (combinatorics). A simplicial complex  $\Delta$  on  $V$ . Stanley-Reisner ideal  $J(\Delta) \leq K[v_1, \dots, v_d]$ .

algebraic properties of  $J(\Delta) \iff$  combinatorics properties of  $\Delta$ .

Let  $\mathcal{P}$  be a poset and  $\Delta(\mathcal{P}) = \text{"order complex of } \mathcal{P}\text{"} = \{\text{chains in } \mathcal{P}\}$ . Study  $\mathcal{P}$  via  $J(\Delta(\mathcal{P}))$ .

4. AC in NT (number theory). NT is the study of solutions of polynomial equations over  $\mathbb{Z}$ . Given an intermediate field  $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$ , let  $R = \{\alpha \in K \mid \exists \text{ monic } f \in \mathbb{Z}[x] \text{ s.t. } f(\alpha) = 0\}$ . Then  $\mathbb{Z} \subseteq R \subseteq K$  and  $R$  is a subring of  $K$ . (Chapter 5)
5. AC in AG (algebraic geometry). AG is the study of solution sets for systems of polynomial equations over fields. Let  $k$  be a field,  $f_1, \dots, f_m \in k[X_1, \dots, X_d]$ ,  $V := V(f_1, \dots, f_m) = \{\underline{x} \in k^d \mid f_i(\underline{x}) = 0, \forall i = 1, \dots, m\}$ , where  $V$  for "variety", and  $I(V) = \{f \in k[X_1, \dots, X_d] \mid f(\underline{x}) = 0, \forall \underline{x} \in V\} \leq k[X_1, \dots, X_d]$ .

algebraic properties of  $I(V) \iff$  geometric properties of  $V$ .

Why modules? b/c in NT,  $R = \{\alpha \in K \mid \exists \text{ monic } f \in \mathbb{Z}[x] \text{ s.t. } f(\alpha) = 0\}$  is a subring of  $K$ .

**Challenge-exercise:** prove this by definition. Let  $\alpha, \beta \in R$ . Then there exist  $f, g \in \mathbb{Z}[X]$  monic such that  $f(\alpha) = 0 = f(\beta)$ . Prove/construct monic polynomials  $s, d, p \in \mathbb{Z}[X]$  such that  $s(\alpha + \beta) = 0$ ,  $d(\alpha - \beta) = 0$  and  $p(\alpha\beta) = 0$ .

Proof is a straight forward application of modules.

Why topology? To study geometry, need continuity. Let  $V = V(f_1, \dots, f_m)$ ,  $W = V(g_1, \dots, g_n)$  and  $\phi : V \rightarrow W$ . What does it mean for  $\phi$  to be continuous if  $K = \mathbb{F}_3$ ? Need a notion of open sets in  $V$  and  $W$ .



# Chapter 1

## Rings and Ideals

Let  $R$  be a CRW1.

**Fact 1.1.**  $R = 0$  iff  $1_R = 0_R$ .

**Fact 1.2.** (1)  $1_R$  and  $0_R$  are both unique.

(2) For any  $r \in R$ ,  $-r$  is unique.

(3) If  $r \in R$  is a unit, i.e., there exists  $r^{-1} \in R$  such that  $rr^{-1} = 1_R = r^{-1}r$ , then  $r^{-1}$  is also unique.

**Definition 1.3.** A homomorphism of CRW1's is a function  $\phi : R \rightarrow S$ , where  $R$  and  $S$  are CRW1's, such that

(1)  $\phi(r + r') = \phi(r) + \phi(r')$ ,

(2)  $\phi(rr') = \phi(r)\phi(r')$ ,

(3)  $\phi(1_R) = 1_S$ .

A.K.A. a ring homomorphism.

**Fact 1.4.** Let  $\phi : R \rightarrow S$  be a ring homomorphism.

(a)  $\phi(0_R) = 0_S$ .

(b)  $\phi(-r) = -\phi(r)$  for any  $r \in R$ .

(c)  $\phi(r - s) = \phi(r) - \phi(s)$  for any  $r, s \in R$ .

(d)  $\phi(\sum_{i=1}^m r_i s_i) = \sum_{i=1}^m \phi(r_i)\phi(s_i)$  for any  $r_1, \dots, r_m, s_1, \dots, s_m \in R$ .

(e) If  $r$  is a unit in  $R$ , then  $\phi(r)$  is a unit in  $S$  and  $\phi(r)^{-1} = \phi(r^{-1})$ .

(f) A composition of ring homomorphisms is a ring homomorphism.

**Definition 1.5.** A *subring* of  $R$  is a subset  $S \subseteq R$  such that  $S$  is a CRW1 under the operations for  $R$  and such that  $1_S = 1_R$ , i.e.,  $1_R \in S$ .

**Fact 1.6** (Subring test). A subset  $S \subseteq R$  is a subring iff it is closed under  $+$ ,  $\cdot$ ,  $-$  and  $1_R \in S$ .

**Example 1.7.** Subring test: need  $\emptyset \neq S \subseteq R$ ,  $S$  is closed under  $+$ ,  $\cdot$ ,  $-$  and  $1_R \in S$ .

If  $S$  is not closed under  $-$ , then fail. Let  $\mathbb{N}_0 = \{0, 1, 2, \dots\} \subseteq \mathbb{Z}$  not a subring.

If  $1_R \notin S$ , then fail.  $R = \mathbb{F}_3 \times \mathbb{F}_3 \supseteq \{(a, a) \mid a \in \mathbb{F}_3\} =: S$ . Then  $S$  is a subring of  $R$ . Although  $S_1 := \{(a, 0) \mid a \in \mathbb{F}_3\} \cong \mathbb{F}_3 \cong \{(0, a) \mid a \in \mathbb{F}_3\} =: S_2$  are rings but not subrings of  $R$  since  $1_R = (1, 1) \notin S_1$  and  $1_R = (1, 1) \notin S_2$ .

**Fact 1.8.** If  $S \subseteq R$  is a subring, then the inclusion map  $\varepsilon : S \rightarrow R$  given by  $\varepsilon(s) = s$  is a ring homomorphism.

**Definition 1.9.** An *ideal* of  $R$  is a non-empty subset  $\mathfrak{a} \subseteq R$  and a subgroup under addition such that for any  $r \in R$  and any  $a \in \mathfrak{a}$ ,  $ra \in \mathfrak{a}$ .

An ideal  $\mathfrak{a} \leq R$  is *prime* if  $\mathfrak{a} \neq R$  and for any  $a, b \in R$ , if  $a, b \notin \mathfrak{a}$ , then  $ab \notin \mathfrak{a}$ , i.e., if  $ab \in \mathfrak{a}$ , then  $a \in \mathfrak{a}$  or  $b \in \mathfrak{a}$ .

An ideal  $\mathfrak{a} \leq R$  is *maximal* if  $\mathfrak{a} \neq R$  and for any ideal  $\mathfrak{b} \leq R$ , if  $\mathfrak{a} \subseteq \mathfrak{b} \subseteq R$ , then either  $\mathfrak{a} = \mathfrak{b}$  or  $\mathfrak{b} = R$ .

**Fact 1.10** (Ideal test). If  $\mathfrak{a} \neq \emptyset$  and  $\mathfrak{a}$  is closed under  $\cdot$ , then for any  $a \in \mathfrak{a}$ ,  $-a = (-1_R)a \in \mathfrak{a}$ , also, since  $\mathfrak{a}$  is closed under  $+$ , it is automatically closed under  $-$ .

Thus, A subset  $\mathfrak{a} \subseteq R$  is an ideal iff  $\mathfrak{a} \neq \emptyset$  and  $\mathfrak{a}$  is closed under  $+$  and  $\cdot$ .

**Example 1.11.** (a) Let  $R = \mathbb{Z}$ , then ideals of  $R$  are  $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$ , where  $n \in \mathbb{Z}$ .

$n\mathbb{Z}$  is prime iff  $n = 0$  or  $|n|$  is prime.

$n\mathbb{Z}$  is maximal iff  $|n|$  is prime.

(b) If  $I_\lambda \leq R$  for any  $\lambda \in \Lambda$ , then  $\bigcap_{\lambda \in \Lambda} I_\lambda \leq R$ .

(c) If  $r_1, \dots, r_m \in R$ , then

$$\begin{aligned} \langle r_1, \dots, r_m \rangle &= \langle r_1, \dots, r_m \rangle R = (r_1, \dots, r_m) = (r_1, \dots, r_m)R = \bigcap_{r_1, \dots, r_m \in I \leq R} I \\ &= \left\{ \sum_{i=1}^m a_i r_i \mid a_i \in R, \forall i = 1, \dots, m \right\} \leq R. \end{aligned}$$

In particular, for any  $r \in R$ ,  $\langle r \rangle = \langle r \rangle R = (r) = (r)R = rR = Rr = \{ar \mid a \in R\} = \bigcap_{r \in I \leq R} I$ .

(d) If  $A \subseteq R$ , then  $\langle A \rangle = \bigcap_{A \subseteq I \leq R} I$  and  $\langle A \rangle = \{\sum_{a \in A}^{\text{finite}} r_a a \mid r_a \in R, \forall a\}$ .

**Fact 1.12.** For any  $r_1, \dots, r_m \in R$ ,  $\langle r_1, \dots, r_m \rangle$  is the smallest ideal of  $R$  containing  $r_1, \dots, r_m$ , i.e., for any  $\mathfrak{a} \leq R$ ,  $r_1, \dots, r_m \in \mathfrak{a}$  iff  $\langle r_1, \dots, r_m \rangle \subseteq \mathfrak{a}$ . Similarly,  $A \subseteq \mathfrak{a}$  iff  $\langle A \rangle \subseteq \mathfrak{a}$ . For example, if  $A \leq R$ , then  $A = \langle A \rangle$ .

**Construction 1.13.** Let  $\mathfrak{a} \leq R$ . For any  $r \in R$ ,  $r + \mathfrak{a} = \{r + a \mid a \in \mathfrak{a}\} = \bar{r}$ . Let  $R/\mathfrak{a} := \{r + \mathfrak{a} \mid r \in R\}$ . Then  $R/\mathfrak{a}$  is a CRW1 with  $\bar{r} \pm \bar{s} = \overline{r \pm s}$ ,  $\bar{r}\bar{s} = \overline{rs}$ ,  $0_{R/\mathfrak{a}} = \overline{0_R}$  and  $1_{R/\mathfrak{a}} = \overline{1_R}$ .

Let  $\pi : R \rightarrow R/\mathfrak{a}$  be given by  $\pi(r) = \bar{r}$ . Then  $\pi$  is a well-defined ring epimorphism.

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow \pi & \nearrow \exists ! \bar{\phi} & \\ R/\mathfrak{a} & & \end{array}$$

For any  $\phi : R \rightarrow S$  ring homomorphism, if  $\phi(\mathfrak{a}) = 0$ , then there exists a unique ring homomorphism  $\bar{\phi} : R/\mathfrak{a} \rightarrow S$  making the diagram commute, where  $\bar{\phi}(\bar{r}) = \bar{\phi}(\pi(r)) = \phi(r)$ .

Note  $\phi(\mathfrak{a}) = 0$  iff  $\mathfrak{a} \subseteq \text{Ker}(\phi)$ . In particular, if  $\mathfrak{a} = \langle A \rangle$ , then  $\mathfrak{a} \subseteq \text{Ker}(\phi)$  iff  $A \subseteq \text{Ker}(\phi)$ .

**Fact 1.14.** Let  $\mathfrak{a} \leq R$ .

(a)  $\mathfrak{a}$  is prime iff  $R/\mathfrak{a}$  is an integral domain.

(b)  $\mathfrak{a}$  is maximal iff  $R/\mathfrak{a}$  is a field.

(c) If  $R$  is a field, then it is an integral domain. So if  $\mathfrak{a}$  is maximal, then  $\mathfrak{a}$  is prime.

**Fact 1.15** (Ideal correspondence for quotients). Let  $\mathfrak{a} \leq R$  and  $\pi : R \rightarrow R/\mathfrak{a}$  be the canonical epimorphism.

$$\begin{aligned} \{\text{ideals } I \leq R/\mathfrak{a}\} &\rightleftharpoons \{\text{ideals } J \leq R \mid \mathfrak{a} \subseteq J\} \\ I &\mapsto \pi^{-1}(I) = \{r \in R \mid r + \mathfrak{a} \in I\} \\ J/\mathfrak{a} &\leftarrow J \\ \{\text{ideals } I \leq R/\mathfrak{a}\} &\rightleftharpoons \{\text{ideals } J \leq R \mid \mathfrak{a} \subseteq J\} \\ \{\text{prime ideals of } R/\mathfrak{a}\} &\rightleftharpoons \{\text{prime ideals } \mathfrak{p} \leq R \mid \mathfrak{a} \subseteq \mathfrak{p}\} \\ \{\text{maximal ideals of } R/\mathfrak{a}\} &\rightleftharpoons \{\text{maximal ideals } \mathfrak{m} \leq R \mid \mathfrak{a} \subseteq \mathfrak{m}\} \end{aligned}$$

In both  $R$  and  $R/\mathfrak{a}$ , maximal ideals are a subset of prime ideals and prime ideals are a proper subset of ideals.

Claim  $\frac{R/\mathfrak{a}}{J/\mathfrak{a}} \cong \frac{R}{J}$ .

$$\begin{array}{ccccc} R & \xrightarrow{p} & R/\mathfrak{a} & \xrightarrow{\tau} & \frac{R/\mathfrak{a}}{J/\mathfrak{a}} \\ \downarrow \pi & & & \nearrow \bar{\phi} & \\ R/J & & & \exists ! & \end{array}$$

Clearly  $J \subseteq \text{Ker}(\tau \circ p)$ , so we can use the UMP.

Since the diagram commutes,  $J = \text{Ker}(\pi) = \text{Ker}(\pi \circ p)$ .

So the kernel is “modulo out” by  $\pi$  and hence  $\bar{\phi}$  is 1-1.

Since  $\tau \circ p$  is onto and the diagram commutes,  $\bar{\phi}$  is onto.

Note  $\bar{\phi}(\bar{r}) = \bar{r}$ , i.e.,  $\bar{\phi}(r + J) = (r + \mathfrak{a}) + J/\mathfrak{a}$ .

**Notation.**  $\text{Spec}(R) = \{\text{prime ideals of } R\}$ , called the *prime spectrum* of  $R$ .

$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ .

**Fact 1.16.** Let  $\phi : R \rightarrow S$  be a ring homomorphism. Then  $\text{Ker}(\phi) \leq R$ ,  $\text{Im}(\phi) \subseteq S$  is a subring and  $\text{Im}(\phi) \cong R/\text{Ker}(\phi)$ .

If  $S$  is an integral domain, then so is  $\text{Im}(\phi)$ . Hence  $\text{Ker}(\phi)$  is prime.

More generally, for any  $\mathfrak{b} \leq S$ , we have  $\phi^{-1}(\mathfrak{b}) = \{x \in R \mid \phi(x) \in \mathfrak{b}\} \leq R$ .

$$\begin{array}{ccccc} R & \xrightarrow{\phi} & S & \xrightarrow{\pi} & S/\mathfrak{q} \\ \downarrow p & & & \nearrow \overline{\pi \circ \phi} & \\ \frac{R}{\phi^{-1}(\mathfrak{q})} & & & \exists ! & \end{array}$$



Let  $\mathfrak{q} \in \text{Spec}(S)$ . Then  $S/\mathfrak{q}$  is an integral domain. Also, since  $R/\text{Ker}(\pi \circ \phi) \cong S/\mathfrak{q}$ , we have  $R/\text{Ker}(\pi \circ \phi)$  is an integral domain and then  $\text{Ker}(\pi \circ \phi)$  is prime. Observe  $\phi^{-1}(\mathfrak{q}) = \text{Ker}(\pi \circ \phi)$  is then prime, i.e.,  $\phi^{-1}(\mathfrak{q}) \in \text{Spec}(R)$ . Thus,  $\phi$  induces a well-defined map  $\phi^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$ .

**Example.** Let  $\phi : \mathbb{Z} \rightarrow \mathbb{Q}$  be an inclusion map. Note  $\mathfrak{q} := (0)\mathbb{Q} \leq \mathbb{Q}$  is maximal, but  $\phi^{-1}(\mathfrak{q}) = \phi^{-1}(0) = \text{Ker}(\phi) = 0\mathbb{Z}$ , which is not maximal in  $\mathbb{Z}$ .

**Fact 1.17.** (a) If  $R \neq 0$ , then  $R$  has a maximal ideal  $\mathfrak{m}$ . So  $R$  has a prime ideal. Moreover, for any  $\mathfrak{a} \leq R$ , there exists a maximal ideal  $\mathfrak{m} \supseteq \mathfrak{a}$ . In particular,  $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\} \neq \emptyset$ .

(b) Let  $\mathfrak{a} \leq R$ . Then  $0 \neq R/\mathfrak{a}$  is a CRW1. So  $R/\mathfrak{a}$  has a maximal ideal and this ideal corresponds for quotients, hence it is of the form  $\mathfrak{m}/\mathfrak{a}$ , where  $\mathfrak{m}$  is the maximal ideal of  $R$  containing  $\mathfrak{a}$ .

**Definition 1.18.**  $R$  is *local* if it has a unique maximal ideal  $\mathfrak{m}$ . A.K.A. “*quasi-local*”. The *residue field* of  $R$  is  $R/\mathfrak{m}$ .

Shorthand, assume  $(R, \mathfrak{m}, k)$  is local, where  $\mathfrak{m}$  is the unique maximal ideal of  $R$  and  $k = R/\mathfrak{m}$ .

Or assume  $(R, \mathfrak{m})$  is local.

**Example 1.19.** (a) Any field is local with the maximal ideal  $(0)$ .

(b) Let  $n \in \mathbb{N}$  and  $p$  be prime in  $\mathbb{Z}$ . Note  $0 \neq \mathbb{Z}/\langle p^n \rangle$  has a maximal ideal  $\mathfrak{m} = \langle p \rangle / \langle p^n \rangle$ , where  $\langle p \rangle$  is a maximal ideal of  $R$  containing  $\langle p^n \rangle$ . Assume there is  $\mathfrak{m}_1 \leq R$  maximal such that  $\mathfrak{m}_1 \supseteq \langle p^n \rangle$ . Then  $\mathfrak{m}_1$  is prime, so  $p \in \mathfrak{m}_1$  and hence  $\langle p \rangle \subseteq \mathfrak{m}_1$ . Since  $\langle p \rangle$  is prime in  $\mathbb{Z}$  and  $\mathbb{Z}$  is a PID,  $\langle p \rangle$  is maximal. So  $\langle p \rangle = \langle \mathfrak{m}_1 \rangle$ . Thus,  $\langle p \rangle$  is the unique maximal ideal containing  $\langle p^n \rangle$  and so  $\mathbb{Z}/\langle p^n \rangle$  is local. Similarly, we can show  $\langle p \rangle$  is the unique prime ideal containing  $\langle p^n \rangle$ , so  $\text{Spec}(\mathbb{Z}/\langle p^n \rangle) = \{\langle p \rangle / \langle p^n \rangle\}$ .

(c) Let  $k$  be a field. Then  $R = k[X]/\langle X^n \rangle$  is local with  $\mathfrak{m} = \langle X \rangle / \langle X^n \rangle$ . In fact,  $\text{Spec}(R) = \{\langle X \rangle / \langle X^n \rangle\}$ .

(d) Let  $k$  be a field and  $R = \frac{k[X_1, \dots, X_d]}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$ , where  $a_i \in \mathbb{N}$  for any  $i = 1, \dots, d$ . Then  $R$  is local with  $\mathfrak{m} = \langle X_1, \dots, X_d \rangle / \langle X_1^{a_1}, \dots, X_d^{a_d} \rangle$ . In fact,  $\text{Spec}(R) = \left\{ \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \right\}$ .

**Fact 1.20.** If  $(R, \mathfrak{m})$  is local and  $\mathfrak{a} \leq R$ , then  $(R/\mathfrak{a}, \mathfrak{m}/\mathfrak{a})$  is also local and  $\frac{R/\mathfrak{a}}{\mathfrak{m}/\mathfrak{a}} \cong R/\mathfrak{m}$  canonically isomorphic residue fields. Converse fails in general by Example 1.19.

**Notation 1.21.** Let  $R^\times = R^* = \mathcal{U}(R) = \{\text{units of } R\}$ .

**Proposition 1.22.** TFAE.

- (i)  $R$  is local.
- (ii)  $R \setminus R^\times \leq R$ .
- (iii) There exists  $\mathfrak{a} \leq R$  such that  $R \setminus \mathfrak{a} \subseteq R^\times$ .

When these are satisfied,  $\mathfrak{m} = R \setminus R^\times = \mathfrak{a}$ .

*Proof.* “(i) $\Rightarrow$ (ii)”. Assume  $(R, \mathfrak{m})$  is local. Claim  $\mathfrak{m} = R \setminus R^\times$ . It suffices to show  $R \setminus \mathfrak{m} = R^\times$ .  
“ $\supseteq$ ”. Let  $u \in R^\times$ . Then  $\langle u \rangle = R$  and so  $u \notin \mathfrak{m} \leq R$ , i.e.,  $u \in R \setminus \mathfrak{m}$ . Hence  $R^\times \subseteq R \setminus \mathfrak{m}$ .  
“ $\subseteq$ ”. Let  $x \in R \setminus R^\times$ . Then  $\langle x \rangle \leq R$ . Since  $\mathfrak{m}$  is the unique maximal ideal in  $R$ ,  $\langle x \rangle \subseteq \mathfrak{m}$ , i.e.,  $x \in \mathfrak{m}$ . Thus,  $R \setminus R^\times \subseteq \mathfrak{m}$ , i.e.,  $R \setminus \mathfrak{m} \subseteq R^\times$ .  
“(ii) $\Rightarrow$ (iii)”. Assume  $R \setminus R^\times \leq R$ . Set  $\mathfrak{a} = R \setminus R^\times$ . Then  $R \setminus \mathfrak{a} = R^\times$ .  
“(iii) $\Rightarrow$ (i)”. Let  $\mathfrak{a} \leq R$  such that  $R \setminus \mathfrak{a} \subseteq R^\times$ . Claim,  $\mathfrak{a} = R \setminus R^\times$ . Clearly  $\mathfrak{a} \supseteq R \setminus R^\times$ . On the other hand (OTOH), let  $a \in \mathfrak{a} \leq R$ , then  $a \notin R^\times$  since  $\mathfrak{a} \leq R$ , so  $a \in R \setminus R^\times$  and hence  $\mathfrak{a} \subseteq R \setminus R^\times$ . Thus,  $\mathfrak{a} = R \setminus R^\times$ . Let  $\mathfrak{n} \leq R$  be maximal and  $y \in \mathfrak{n}$ . Then  $y \notin R^\times$ . So  $y \in R \setminus R^\times = \mathfrak{a}$ . Thus,  $\mathfrak{n} \subseteq \mathfrak{a} \leq R$ . Since  $\mathfrak{n}$  is maximal,  $\mathfrak{n} = \mathfrak{a}$ . Thus,  $\mathfrak{a}$  is the unique maximal ideal in  $R$  and so  $R$  is local.  $\square$

**Proposition 1.23.** Let  $\mathfrak{m} \leq R$  be maximal such that  $1 + \mathfrak{m} \subseteq R^\times$ . Then  $R$  is local.

*Proof.* By previous proposition, it suffices to show  $R \setminus \mathfrak{m} \subseteq R^\times$ . Let  $x \in R \setminus \mathfrak{m}$ . Set  $\langle x, \mathfrak{m} \rangle = \langle \{x\} \cup \mathfrak{m} \rangle = \{ax + m \mid a \in R, m \in \mathfrak{m}\}$ . Since  $x \notin \mathfrak{m}$ ,  $\mathfrak{m} \subsetneq \langle x, \mathfrak{m} \rangle \leq R$ . Also, since  $\mathfrak{m}$  is maximal,  $\langle x, \mathfrak{m} \rangle = R$ . So  $ax + m = 1$  for some  $a \in R$  and  $m \in \mathfrak{m}$ , i.e.,  $ax = 1 - m \in 1 + \mathfrak{m} \subseteq R^\times$ . Thus,  $a, x \in R^\times$ .  $\square$

**Definition 1.24.**  $x \in R$  is *nilpotent* if there exists  $n \in \mathbb{N}$  such that  $x^n = 0$ . Then *nilradical* of  $R$  is  $\text{Nil}(R) = \text{N}(R) = \mathfrak{N}_R = \mathfrak{N} = \{\text{nilpotent elements of } R\}$ .

**Example 1.25.** In  $\mathbb{Z}/\langle p^n \rangle$ ,  $\bar{p}$  is nilpotent. It is similar in  $k[x]/\langle x^n \rangle$  and  $k[x_1, \dots, x_n]/\langle x_1^{a_1}, \dots, x_d^{a_d} \rangle$ , where  $k$  is a field.

**Proposition 1.26.** (a)  $\text{Nil}(R) \leq R$ .

(b)  $\text{Nil}(R/\text{Nil}(R)) = 0$ .

(c)  $\text{Nil}(R) = R$  iff  $R = 0$ .

(d)  $\text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$ .

*Proof.* (a) Since  $0 \in \text{Nil}(R)$ ,  $\text{Nil}(R) \neq \emptyset$ . Let  $r \in R$  and  $a, b \in \text{Nil}(R)$ . Then there exists  $m, n \in \mathbb{N}$  such that  $a^m = 0 = b^n$ . Then  $(ra)^m = r^m a^m = 0$  and so  $ra \in \text{Nil}(R)$ . By binomial theorem,  $(a + b)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} a^i b^{m+n-i} = 0$ . Since for any  $0 \leq i \leq m+n$ , either  $i \geq m$  or  $i < m$ , i.e.,  $i \geq m$  or  $m+n-i > n$ . we have  $a^i = 0$  when  $i \geq m$ , and  $b^{m+n-i} = 0$  when  $m+n-i > n$ . So  $(a + b)^{m+n} = 0$  and thus  $a + b \in \text{Nil}(R)$ .

(b) Let  $\bar{x} \in \text{Nil}(R/\text{Nil}(R))$ . Then there exists  $n \in \mathbb{N}$  such that  $\bar{x}^n = \bar{x}^n = 0$ , i.e.,  $x^n \in \text{Nil}(R)$ . So there exists  $m \in \mathbb{N}$  such that  $(x^n)^m = 0$ , i.e.,  $x^{mn} = 0$ . Thus,  $x \in \text{Nil}(R)$ , i.e.,  $\bar{x} = 0$ .

(c) Since  $1 \in \text{Nil}(R)$ , there exists  $n \in \mathbb{N}$  such that  $1 = 1^n = 0$ . So  $R = 0$ .

(d) “ $\subseteq$ ”. Let  $x \in \text{Nil}(R)$ . Then there exists  $n \in \mathbb{N}$  such that  $x^n = 0 \in \mathfrak{p}$  for any  $\mathfrak{p} \in \text{Spec}(R)$ . So  $x \in \mathfrak{p}$  for any  $\mathfrak{p} \in \text{Spec}(R)$ . Thus,  $x \in \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$ .

“ $\supseteq$ ”. Let  $x \in R \setminus \text{Nil}(R)$ . Need to show  $x \notin \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$ . It is equivalent to show there exists  $\mathfrak{p} \in \text{Spec}(R)$  such that  $x \notin \mathfrak{p}$ . Let  $\Sigma = \{\mathfrak{a} \leq R \mid x, x^2, x^3, \dots \notin \mathfrak{a}\}$ . Since  $x \notin \text{Nil}(R)$ ,  $x^k \neq 0$  for any  $k \in \mathbb{N}$ . So  $(0) \in \Sigma$  and then  $\Sigma \neq \emptyset$ . Let  $\mathcal{C} \subseteq \Sigma$  be chain. Then we have  $\mathfrak{q} := \bigcup_{\mathfrak{a} \in \mathcal{C}} \mathfrak{a} \leq R$ . Suppose  $x^n \in \mathfrak{q}$  for some  $n \in \mathbb{N}$ . Then  $x^n \in \mathfrak{a}$  for some  $\mathfrak{a} \in \mathcal{C} \subseteq \Sigma$ , contradicting  $\mathfrak{a} \in \Sigma$ . So  $x^n \notin \mathfrak{q}$  for any  $n \in \mathbb{N}$  and hence  $\mathfrak{q} \in \Sigma$ . Hence  $\mathfrak{q}$  is an upper bound for  $\mathcal{C}$  in  $\Sigma$ . Since the chain

$\mathcal{C} \subseteq \Sigma$  is arbitrary, by Zorn's lemma,  $\Sigma$  has a maximal element  $I$ .

Claim  $I$  is prime. Suppose  $I = R$ . Then  $x \in R = I$ , contradicting  $I \in \Sigma$ . So  $I \subsetneq R$ . Let  $r, s \in R \setminus I$ . Then  $I \subsetneq \langle r, I \rangle \leq R$  and  $I \subsetneq \langle s, I \rangle \leq R$ . By the maximality of  $I$  in  $\Sigma$ , we have  $\langle r, I \rangle, \langle s, I \rangle \notin \Sigma$ . So there exists  $m, n \in \mathbb{N}$  such that  $x^m \in \langle r, I \rangle$  and  $x^n \in \langle s, I \rangle$ . Then  $x^m = ar + i$  for some  $a \in R$  and  $i \in I$ , and  $x^n = bs + j$  for some  $b \in R$  and  $j \in I$ . So  $x^{m+n} = x^m x^n = (ar + i)(bs + j) = abrs + \underbrace{(arj + bsi + ij)}_{\in I} \in \langle rs, I \rangle$ . Hence  $\langle rs, I \rangle \notin \Sigma$ . Also, since  $I \in \Sigma$ ,  $rs \notin I$ . Thus,  $I \in \text{Spec}(R)$  such that  $x \notin I$ . □

**Example.** Let  $k$  be a field and  $R = \frac{k[X_1, \dots, X_d]}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \neq 0$ , where  $a_i \in \mathbb{N}$  for any  $i = 1, \dots, d$ . Then  $\text{Nil}(R) = \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$ .

*Proof.* Method 1: Since  $\text{Spec}(R) = \left\{ \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle} \right\}$ ,  $\text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p} = \frac{\langle X_1, \dots, X_d \rangle}{\langle X_1^{a_1}, \dots, X_d^{a_d} \rangle}$ .

Method 2: Since  $\overline{X}_i \in \text{Nil}(R) \leq R$  for each  $i = 1, \dots, d$ , we have  $\langle X_1, \dots, X_d \rangle = \langle \overline{X}_1, \dots, \overline{X}_d \rangle \subseteq \text{Nil}(R) \subsetneq R$ . Also, since  $\langle X_1, \dots, X_d \rangle$  is maximal, we have  $\text{Nil}(R) = \langle X_1, \dots, X_d \rangle$ . □

**Fact.** If  $\mathfrak{a} \leq R$  and  $r_1, \dots, r_n \in R$ , then  $R/\mathfrak{a} \supseteq \langle \bar{r}_1, \dots, \bar{r}_n \rangle = \frac{\langle r_1, \dots, r_n, \mathfrak{a} \rangle}{\mathfrak{a}}$ . In particular, if  $\langle r_1, \dots, r_n \rangle \supseteq \mathfrak{a}$ , then  $\langle \bar{r}_1, \dots, \bar{r}_n \rangle = \frac{\langle r_1, \dots, r_n \rangle}{\mathfrak{a}}$ .

**Definition 1.27.** The Jacobson radical of  $R$  is  $\text{Jac}(R) = \mathfrak{J}(R) = \bigcap_{\mathfrak{m} \leq R \text{ max}} \mathfrak{m}$ .

**Fact 1.28.**  $\text{Jac}(R) \supseteq \text{Nil}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$ .

**Proposition 1.29.**  $\mathfrak{J}(R) = \{x \in R \mid 1 - xy \in R^\times, \forall y \in Y\}$ .

*Proof.* “ $\subseteq$ ”. Let  $x \in \mathfrak{J}(R)$ . By way of contradiction (BWOC), suppose there exists  $y \in R$  such that  $1 - xy \notin R^\times$ . Then there exists  $\mathfrak{m} \leq R$  maximal such that  $1 - xy \in \mathfrak{m}$ . Since  $x \in \mathfrak{J}(R) \subseteq \mathfrak{m}$ ,  $xy \in \mathfrak{m}$ . So  $1 = (1 - xy) + xy \in \mathfrak{m}$ , a contradiction.

“ $\supseteq$ ”. Argue by contrapositive. Let  $x \in R$  such that  $1 - xy \in R^\times$  for any  $y \in Y$ . Suppose  $x \notin \mathfrak{J}(R)$ . Then there exists  $\mathfrak{m} \leq R$  maximal such that  $x \notin \mathfrak{m}$ . So  $\mathfrak{m} \subsetneq \langle \mathfrak{m}, x \rangle \subseteq R$ . Hence  $\langle x, \mathfrak{m} \rangle = R$ . Then there exists  $y \in R$  and  $m \in \mathfrak{m}$  such that  $xy + m = 1$ , i.e.,  $1 - xy = m \in \mathfrak{m}$ . So  $1 - xy \notin R^\times$ , a contradiction. □

## 1.1 Operations on Ideals

Let  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \leq R$ ,  $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$  and  $\mathfrak{a}_\lambda \leq R$  for any  $\lambda \in \Lambda$ , where  $\Lambda$  is an index set.

**Definition 1.30.**  $\mathfrak{a} + \mathfrak{b} = \langle \mathfrak{a} \cup \mathfrak{b} \rangle = \bigcap_{\mathfrak{a} \cup \mathfrak{b} \subseteq I \leq R} I$ .

**Fact 1.31.** (a)  $\mathfrak{a} + \mathfrak{b} \subseteq \mathfrak{c}$  iff  $\mathfrak{a} \cup \mathfrak{b} \subseteq \mathfrak{c}$ .

(b)  $\mathfrak{a} + \mathfrak{b}$  is the (unique) smallest ideal of  $R$  that contains  $\mathfrak{a} \cup \mathfrak{b}$ .

(c)  $\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$ .

(d) If  $\mathfrak{a} = \langle S \rangle$  and  $\mathfrak{b} = \langle T \rangle$ , then  $\mathfrak{a} + \mathfrak{b} = \langle S \cup T \rangle$ .

- (e) If  $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$  and  $\mathfrak{b} = \langle y_1, \dots, y_n \rangle$ , then  $\mathfrak{a} + \mathfrak{b} = \langle x_1, \dots, x_m, y_1, \dots, y_n \rangle$ .
- (f) If  $x \in R$ , then  $\langle x, \mathfrak{a} \rangle = \langle x \rangle + \mathfrak{a}$ .
- (g) If  $\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}$ .

*Proof.* (c) Set  $I = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\} \leq R$  (Check). For any  $a \in \mathfrak{a}$ ,  $a = a + 0 \in I$  and for any  $b \in \mathfrak{b}$ ,  $b = 0 + b \in I$ . So  $\mathfrak{a} \cup \mathfrak{b} \subseteq I$ . By (a),  $\mathfrak{a} + \mathfrak{b} \subseteq I$ . OTOH, for any  $a + b \in I$  with  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$ ,  $a, b \in \mathfrak{a} \cup \mathfrak{b} \subseteq \mathfrak{a} + \mathfrak{b} \leq R$ . So  $a + b \in \mathfrak{a} + \mathfrak{b}$ .

- (d) Let  $I \leq R$ . Note  $I \supseteq \mathfrak{a} \cup \mathfrak{b}$  iff  $I \supseteq \mathfrak{a}, \mathfrak{b}$  iff  $I \supseteq \langle S \rangle, \langle T \rangle$  iff  $I \supseteq S, T$  iff  $I \supseteq S \cup T$ . So  $\mathfrak{a} + \mathfrak{b} = \bigcap_{\mathfrak{a} \cup \mathfrak{b} \subseteq I \leq R} I = \bigcap_{S \cup T \subseteq I \leq R} I = \langle S \cup T \rangle$ .
- (e) By (d).
- (f) By (c).
- (g) The essential point is  $\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = \langle \mathfrak{a} \cup (\mathfrak{b} \cup \mathfrak{c}) \rangle = \langle (\mathfrak{a} \cup \mathfrak{b}) \cup \mathfrak{c} \rangle = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}$ .

□

**Example.**  $m\mathbb{Z} + n\mathbb{Z} = \langle m, n \rangle \mathbb{Z} = \gcd(m, n)\mathbb{Z}$ , where  $m \neq 0$  or  $n \neq 0$ .

**Recall.**  $\text{Spec}(R) = \{\text{prime ideals of } R\}$ . For any  $S \subseteq R$ ,  $V(S) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq S\}$ .

**Proposition 1.32.** Let  $S \subseteq R$ .

- (a)  $V(S) = V(\langle S \rangle)$ .
- (b)  $\mathfrak{a} = R$  iff  $V(\mathfrak{a}) = \emptyset$ .
- (c)  $\mathfrak{a} \subseteq \text{Nil}(R)$  iff  $V(\mathfrak{a}) = \text{Spec}(R)$ .
- (d) If  $\mathfrak{a} \subseteq \mathfrak{b}$ , then  $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$ . If  $S \subseteq T \subseteq R$ , then  $V(S) \supseteq V(T)$ .

*Proof.* (d) Since  $S \subseteq T \subseteq R$ , we have  $V(S) \supseteq V(T)$  by definition.

- (a) “ $\supseteq$ ”. Since  $S \subseteq \langle S \rangle$ ,  $V(S) \supseteq V(\langle S \rangle)$  by (d).  
“ $\subseteq$ ”. Let  $\mathfrak{p} \in V(S)$ . Then  $\mathfrak{p} \supseteq S$ . So  $\mathfrak{p} \supseteq \langle S \rangle$  and then  $\mathfrak{p} \in V(\langle S \rangle)$ . Hence  $V(S) \subseteq V(\langle S \rangle)$ .
- (b) “ $\Rightarrow$ ”. Let  $\mathfrak{a} = R$ . Then  $\mathfrak{p} \not\supseteq \mathfrak{a}$  for any  $\mathfrak{p} \in \text{Spec}(R)$ . So  $V(\mathfrak{a}) = \emptyset$ .  
“ $\Leftarrow$ ”. Let  $V(\mathfrak{a}) = \emptyset$ . Suppose  $\mathfrak{a} \neq R$ , then there exists  $\mathfrak{m} \leq R$  maximal such that  $\mathfrak{m} \supseteq \mathfrak{a}$ . Also, since  $\mathfrak{m} \in \text{Spec}(R)$ , we have  $\mathfrak{m} \in V(\mathfrak{a})$ , contradicting  $V(\mathfrak{a}) = \emptyset$ .
- (c)  $\mathfrak{a} \subseteq \text{Nil}(R)$  iff  $\mathfrak{p} \supseteq \mathfrak{a}$  for all  $\mathfrak{p} \in \text{Spec}(R)$  iff  $V(\mathfrak{a}) = \text{Spec}(R)$ .

□

**Proposition 1.33.** (a)  $V(\mathfrak{a} + \mathfrak{b}) = V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$ .

- (b)  $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$  iff  $\mathfrak{a} + \mathfrak{b} = R$ .

*Proof.* (a) Since  $\mathfrak{a} + \mathfrak{b} = \langle \mathfrak{a} \cup \mathfrak{b} \rangle$ ,  $V(\mathfrak{a} + \mathfrak{b}) = V(\langle \mathfrak{a} \cup \mathfrak{b} \rangle) = V(\mathfrak{a} \cup \mathfrak{b})$ .

Let  $\mathfrak{p} \in \text{Spec}(R)$ . Note  $\mathfrak{p} \supseteq \mathfrak{a} \cup \mathfrak{b}$  iff  $\mathfrak{p} \supseteq \mathfrak{a}$  and  $\mathfrak{p} \supseteq \mathfrak{b}$ . So  $V(\mathfrak{a} \cup \mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$ .

(b)  $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$  iff  $V(\mathfrak{a} + \mathfrak{b}) = \emptyset$  iff  $\mathfrak{a} + \mathfrak{b} = R$ .

□

**Remark.** You can define  $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n$  inductively and same properties as above hold for finite sums.

**Definition 1.34.**  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \langle \bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \rangle = \bigcap_{\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq I \leq R} I$ .

**Fact 1.35.** (a)  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq \mathfrak{c}$  iff  $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} \subseteq \mathfrak{c}$ .

(b)  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}$  is the (unique) smallest ideal of  $R$  containing  $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}$ .

(c)  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \{\sum_{\lambda \in \Lambda}^{\text{finite}} a_{\lambda} \mid a_{\lambda} \in \mathfrak{a}_{\lambda}, \forall \lambda \in \Lambda\}$ .

(d) If  $\mathfrak{a}_{\lambda} = \langle S_{\lambda} \rangle$  for any  $\lambda \in \Lambda$ , then  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = \langle \bigcup_{\lambda \in \Lambda} S_{\lambda} \rangle$ .

**Fact 1.36.** (a)  $V(\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}) = V(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_{\lambda}) = \bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_{\lambda})$ .

(b)  $\bigcap_{\lambda \in \Lambda} V(\mathfrak{a}_{\lambda}) = \emptyset$  iff  $\sum_{\lambda \in \Lambda} \mathfrak{a}_{\lambda} = R$ .

**Definition 1.37.**  $\mathfrak{a}\mathfrak{b} = \langle N \rangle = \bigcap_{N \subseteq I \leq R} I$ , where  $N = \{ab \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$ .

**Fact 1.38.** Let  $\mathfrak{a}\mathfrak{b} = \langle N \rangle$ .

(a)  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{c}$  iff  $N \subseteq \mathfrak{c}$ .

(b)  $\mathfrak{a}\mathfrak{b}$  is the (unique) smallest ideal of  $R$  containing  $N$ .

(c)  $\mathfrak{a}\mathfrak{b} = \{\sum_i^{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}, \forall i\}$ .

(d) If  $\mathfrak{a} = \langle S \rangle$  and  $\mathfrak{b} = \langle T \rangle$ , then  $\mathfrak{a}\mathfrak{b} = \langle st \mid s \in S, t \in T \rangle$ .

(e) If  $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$  and  $\mathfrak{b} = \langle y_1, \dots, y_n \rangle$ , then  $\mathfrak{a}\mathfrak{b} = \langle x_i y_j \mid i = 1, \dots, m, j = 1, \dots, n \rangle$ .

(f)  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ .

*Proof.* (c) Let  $I = \{\sum_i^{\text{finite}} a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\} \leq R$ . Check  $I \leq R$  and  $I \subseteq \mathfrak{a}\mathfrak{b} \subseteq I$  like 1.31(c).

(f) To show  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ , it suffices to show  $ab \in \mathfrak{a} \cap \mathfrak{b}$  for any  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$ . For any  $a \in \mathfrak{a} \leq R$ , we have  $ab \in \mathfrak{a}$  for any  $b \in \mathfrak{b}$ . For any  $b \in \mathfrak{b} \leq R$ , we have  $ab \in \mathfrak{b}$  for any  $a \in \mathfrak{a}$ . So  $ab \in \mathfrak{a} \cap \mathfrak{b}$  for any  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$ .

□

**Proposition 1.39.** (a)  $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ .

(b)  $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \text{Spec}(R)$  iff  $\mathfrak{a}\mathfrak{b} \subseteq \text{Nil}(R)$  iff  $\mathfrak{a} \cap \mathfrak{b} \subseteq \text{Nil}(R)$ .

*Proof.* (a) • Let  $\mathfrak{p} \in \text{Spec}(R)$ . Claim  $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$  iff  $\mathfrak{p} \supseteq \mathfrak{a}$  or  $\mathfrak{p} \supseteq \mathfrak{b}$ . “ $\Leftarrow$ ”. Let  $\mathfrak{p} \supseteq \mathfrak{a}$  or  $\mathfrak{p} \supseteq \mathfrak{b}$ . Then  $\mathfrak{p} = \mathfrak{p}R \supseteq \mathfrak{a}R \supseteq \mathfrak{a}\mathfrak{b}$  or  $\mathfrak{p} = \mathfrak{p}R \supseteq \mathfrak{b}R \supseteq \mathfrak{a}\mathfrak{b}$ . “ $\Rightarrow$ ”. Let  $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ . Suppose  $\mathfrak{p} \not\supseteq \mathfrak{a}$  and  $\mathfrak{p} \not\supseteq \mathfrak{b}$ . Then there exists  $a \in \mathfrak{a} \setminus \mathfrak{p}$  and exists  $b \in \mathfrak{b} \setminus \mathfrak{p}$ . Since  $\mathfrak{p} \in \text{Spec}(R)$ ,  $ab \notin \mathfrak{p}$ , contradicting  $ab \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ . Hence  $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$  iff  $\mathfrak{p} \supseteq \mathfrak{a}$  or  $\mathfrak{p} \supseteq \mathfrak{b}$ . So  $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ .

- Since  $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$ ,  $V(\mathfrak{a}\mathfrak{b}) \supseteq V(\mathfrak{a} \cap \mathfrak{b})$ . Let  $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$ . Then  $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ . Let  $x \in \mathfrak{a} \cap \mathfrak{b}$ . Then  $x \in \mathfrak{a}$  and  $x \in \mathfrak{b}$ . So  $x^2 = x \cdot x \in \mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ . Since  $\mathfrak{p}$  is prime,  $x \in \mathfrak{p}$ . So  $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$  and then  $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ . Hence  $V(\mathfrak{a}\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$ . Thus,  $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$ .

(b)  $V(\mathfrak{a}) \cup V(\mathfrak{b}) = \text{Spec}(R)$  iff  $V(\mathfrak{a}\mathfrak{b}) = \text{Spec}(R)$  iff  $\mathfrak{a}\mathfrak{b} \subseteq \text{Nil}(R)$  and similarly for  $\mathfrak{a} \cap \mathfrak{b}$ . □

**Proposition 1.40.** (a)  $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$  and  $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$ .

(b)  $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$ .

(c) If  $\mathfrak{a} + \mathfrak{b} = R$ , i.e.,  $\mathfrak{a}$  and  $\mathfrak{b}$  are “coprime” and “co-maximal”, then  $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ .

The converse holds if  $R$  is a PID and  $\mathfrak{a}, \mathfrak{b} \neq 0$ .

*Proof.* (c) “ $\supseteq$ ”. We always have  $\mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{a}\mathfrak{b}$ .

“ $\subseteq$ ”. Assume  $\mathfrak{a} + \mathfrak{b} = R$ .

Method 1: Note  $1 = a + b$  for some  $a \in \mathfrak{a}$  and  $b \in \mathfrak{b}$ . Let  $x \in \mathfrak{a} \cap \mathfrak{b}$ . Then  $x \in \mathfrak{b}$  and  $x \in \mathfrak{a}$ . So  $x = 1 \cdot x = (a + b)x = ax + bx = ax + xb \in \mathfrak{a}\mathfrak{b}$ . Hence  $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$ .

Method 2: Note  $\mathfrak{a} \cap \mathfrak{b} = R(\mathfrak{a} \cap \mathfrak{b}) = (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} \cap \mathfrak{b}) = \underbrace{\mathfrak{a}(\mathfrak{a} \cap \mathfrak{b})}_{\subseteq \mathfrak{b}} + \underbrace{\mathfrak{b}(\mathfrak{a} \cap \mathfrak{b})}_{\subseteq \mathfrak{a}} \subseteq \mathfrak{a}\mathfrak{b}$  by (a) and (b).

Conversely, assume  $R$  is a PID and  $\mathfrak{a}, \mathfrak{b} \neq 0$ . Write  $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n} R$  and  $\mathfrak{b} = \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_n^{f_n} R$  with  $e_i, f_i \geq 0$  for any  $i = 1, \dots, n$ , and  $\mathfrak{p}_i$ 's  $\in \text{Spec}(R)$  non-associates. Assume  $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ . Since  $R$  is a PID,  $\mathfrak{a} \cap \mathfrak{b} = \text{lcm}(\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_n^{e_n}, \mathfrak{p}_1^{f_1} \cdots \mathfrak{p}_n^{f_n})R = \mathfrak{p}_1^{\max\{e_1, f_1\}} \cdots \mathfrak{p}_n^{\max\{e_n, f_n\}}$ . By the fact 1.38(e),  $\mathfrak{a}\mathfrak{b} = \mathfrak{p}_1^{e_1+f_1} \cdots \mathfrak{p}_n^{e_n+f_n}$ . So  $\max\{e_i, f_i\} = e_i + f_i$ , i.e.,  $e_i = 0$  or  $f_i = 0$  for any  $i = 1, \dots, n$ . In other words, for any  $\mathfrak{p} \in \text{Spec}(R)$ , either  $\mathfrak{a} \not\subseteq \mathfrak{p}R$  or  $\mathfrak{b} \not\subseteq \mathfrak{p}R$ . So  $V(\mathfrak{a}) \cap V(\mathfrak{b}) = \emptyset$  for any  $\mathfrak{p} \in \text{Spec}(R)$ . Thus,  $\mathfrak{a} + \mathfrak{b} = R$ . □

**Remark.** You can do this for  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ , where  $n \in \mathbb{Z}^{\geq 3}$ .

**Example 1.41.** Let  $R = k[x, y]$ ,  $\mathfrak{a} = \langle x \rangle$  and  $\mathfrak{b} = \langle y \rangle$ . Then  $\mathfrak{a} \cap \mathfrak{b} = \langle xy \rangle = \mathfrak{a}\mathfrak{b}$  by the fact 1.38(e). But  $\mathfrak{a} + \mathfrak{b} = \langle x, y \rangle \subsetneq R$ . So the converse in (c) fails for any non-PID.

**Definition 1.42.** Let  $n \in \mathbb{N}$ . Let  $\mathfrak{a}^n = \underbrace{\mathfrak{a} \cdots \mathfrak{a}}_{n \text{ times}}$  and  $\mathfrak{a}^0 = R$ .

**Warning 1.43.**  $\mathfrak{a}^n$  is **not** generated by  $\{a^n \mid a \in \mathfrak{a}\}$ . For example, let  $R = \mathbb{F}_2[x, y]$  and  $\mathfrak{a} = \langle x, y \rangle$ , then  $\mathfrak{a}^2 = \langle x^2, xy, y^2 \rangle \neq \langle f^2 \mid f \in \mathfrak{a} \rangle \not\supseteq xy$ .

**Fact 1.44.** Let  $n \in \mathbb{N}$  and  $N = \{a_1 \cdots a_n \mid a_i \in \mathfrak{a}, \forall i = 1, \dots, n\}$ .

(a)  $\mathfrak{a}^n = \langle N \rangle$  and for any  $\mathfrak{b} \leq R$ , we have  $\mathfrak{a}^n \subseteq \mathfrak{b}$  iff  $N \subseteq \mathfrak{b}$ .

(b)  $\mathfrak{a}^n$  is the (unique) smallest ideal of  $R$  containing  $N$ .

(c)  $\mathfrak{a}^n = \{\sum_i^{\text{finite}} a_{i1} \cdots a_{in} \mid a_{ij} \in \mathfrak{a}, \forall i, \forall j = 1, \dots, n\}$ .

(d) If  $\mathfrak{a} = \langle S \rangle$ , then  $\mathfrak{a}^n = \langle s_1 \cdots s_n \mid s_i \in S, \forall i = 1, \dots, n \rangle$ .

(e) If  $\mathfrak{a} = \langle x_1, \dots, x_m \rangle$ , then  $\mathfrak{a}^n = \langle x_{i_1} \cdots x_{i_n} \mid i_j = 1, \dots, m, \forall j = 1, \dots, n \rangle$ .

**Fact 1.45.**  $V(\mathfrak{a}^n) = V(\mathfrak{a})$ .

*Proof.* By the proposition 1.39,  $V(\mathfrak{a}^n) = \bigcup_{i=1}^n V(\mathfrak{a}) = V(\mathfrak{a})$ .  $\square$

**Proposition 1.46 (CRT).** Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$ .

- (a) The function  $\phi : R \rightarrow (R/\mathfrak{a}_1) \times \dots \times (R/\mathfrak{a}_n)$  given by  $\phi(x) = (\bar{x}, \dots, \bar{x}) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n)$  is a well-defined ring homomorphism.
- (b) If  $\mathfrak{a}_i + \mathfrak{a}_j = R$  for any  $1 \leq i \neq j \leq n$ , i.e.,  $\{\mathfrak{a}_1, \dots, \mathfrak{a}_n\}$  are pairwise co-prime, then  $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_n$  and  $\mathfrak{a}_i + (\bigcap_{j \neq i} \mathfrak{a}_j)R = R$  for any  $i = 1, \dots, n$ .
- (c)  $\phi$  is surjective iff  $\mathfrak{a}_i + \mathfrak{a}_j = R$  for any  $1 \leq i \neq j \leq n$ .
- (d)  $\text{Ker}(\phi) = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$ .
- (e) If  $\mathfrak{a}_i + \mathfrak{a}_j = R$  for any  $1 \leq i \neq j \leq n$  and  $\bigcap_{i=1}^n \mathfrak{a}_i = 0$ , then  $R \cong (R/\mathfrak{a}_1)R \cap \dots \times (R/\mathfrak{a}_n)R$ .

*Proof.* (b) Let  $i \in \{1, \dots, n\}$ . To show  $\mathfrak{a}_i + (\bigcap_{j \neq i} \mathfrak{a}_j)R = R$ . It suffices to show  $V(\mathfrak{a}_i) \cap (\bigcup_{j \neq i} V(\mathfrak{a}_j)) =$

$V(\mathfrak{a}_i) \cap V(\bigcap_{j \neq i} \mathfrak{a}_j) = V(\mathfrak{a}_i + \bigcap_{j \neq i} \mathfrak{a}_j) = \emptyset$ . Suppose  $V(\mathfrak{a}_i) \cap (\bigcup_{j \neq i} V(\mathfrak{a}_j)) \neq \emptyset$ . Then there exists  $\mathfrak{p} \in V(\mathfrak{a}_i) \cap V(\mathfrak{a}_j) = V(\mathfrak{a}_i + \mathfrak{a}_j) = V(R) = \emptyset$  for some  $j \neq i$ , a contradiction.

Now for  $\bigcap_{i=1}^n \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ , prove by induction on  $n$ . Base case  $n = 1$ : trivial. Base case  $n = 2$ : by 1.40(c). Induction step: assume  $n \in \mathbb{Z}^{\geq 3}$  and  $\bigcap_{i=1}^{n-1} \mathfrak{a}_i = \mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}$ . Then  $\mathfrak{a}_n + (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1})R = \mathfrak{a}_n + (\bigcap_{j=1}^{n-1} \mathfrak{a}_j)R = R$ . So by proposition 1.40(c), we have  $\bigcap_{i=1}^n \mathfrak{a}_i = (\bigcap_{i=1}^{n-1} \mathfrak{a}_i) \cap \mathfrak{a}_n = (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}) \cap \mathfrak{a}_n = (\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1})\mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ .

- (c) “ $\Rightarrow$ ”. Assume  $\phi$  is surjective. In particular, there exists  $x \in R$  such that  $(\bar{1}, \bar{0}, \dots, \bar{0}) = \phi(x) = (\bar{x}, \bar{x}, \dots, \bar{x})$ . So  $x + \mathfrak{a}_1 = 1 + \mathfrak{a}_1$  and  $x + \mathfrak{a}_i = 0 + \mathfrak{a}_i$  for any  $2 \leq i \leq n$ . Hence  $1 - x \in \mathfrak{a}_1$  and  $x \in \mathfrak{a}_i$  for any  $2 \leq i \leq n$ . Also, since  $\underset{\in \mathfrak{a}_i}{x} + \underset{\in \mathfrak{a}_1}{(1-x)} = 1$ , we have  $\mathfrak{a}_i + \mathfrak{a}_1 = R$  for any  $2 \leq i \leq n$ .

Similarly, consider  $(\bar{0}, \dots, \bar{0}, \bar{1}, \bar{0}, \dots, \bar{0}) \rightsquigarrow \mathfrak{a}_i + \mathfrak{a}_j = R$  for any  $1 \leq i \neq j \leq n$ .

“ $\Leftarrow$ ”. Assume  $\mathfrak{a}_i + \mathfrak{a}_j = R$  for any  $1 \leq i \neq j \leq n$ . By (b),  $\mathfrak{a}_1 + (\bigcap_{j=2}^n \mathfrak{a}_j)R = R$ . So  $a_1 + y = 1$  with  $a_1 \in \mathfrak{a}_1$  and  $y \in \bigcap_{j=2}^n \mathfrak{a}_j$ , i.e.,  $1 - y = a_1 \in \mathfrak{a}_1$  and  $y \in \mathfrak{a}_j$  for any  $2 \leq j \leq n$ . Then  $\phi(y) = (\bar{y}, \bar{y}, \dots, \bar{y}) = (y + \mathfrak{a}_1, y + \mathfrak{a}_2, \dots, y + \mathfrak{a}_n) = (1 + \mathfrak{a}_1, 0 + \mathfrak{a}_2, \dots, 0 + \mathfrak{a}_n) = (\bar{1}, \bar{0}, \dots, \bar{0})$ . Similarly, for any  $j = 1, \dots, n$ , there exists  $y_j$  such that  $\phi(y_j) = (\bar{0}, \dots, \bar{0}, \underset{\uparrow j\text{th}}{\bar{1}}, \bar{0}, \dots, \bar{0})$ . Then for any

$$(\bar{r}_1, \dots, \bar{r}_n) \in \frac{R}{\mathfrak{a}_1} \times \dots \times \frac{R}{\mathfrak{a}_n}, (\bar{r}_1, \dots, \bar{r}_n) = \sum_{j=1}^n r_j (\bar{0}, \dots, \bar{0}, \underset{\uparrow j\text{th}}{\bar{1}}, \bar{0}, \dots, \bar{0}) = \sum_{j=1}^n r_j \phi(y_j) =$$

$\phi(\sum_{j=1}^n r_j y_j)$ . So  $\phi$  is surjective.  $\square$

**Proposition 1.47.** Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_n \leq R$  and  $\mathfrak{p} \in \text{Spec}(R)$ .

- (a) If  $\mathfrak{p} = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ , then  $\mathfrak{p} = \mathfrak{a}_i$  for some  $i \in \{1, \dots, n\}$ .
- (b) If  $\mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n$ , then  $\mathfrak{p} = \mathfrak{a}_1$  for some  $i \in \{1, \dots, n\}$ .

*Proof.* (b) Assume  $\mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n \supseteq \mathfrak{a}_1 \cdots \mathfrak{a}_n$ . Since  $\mathfrak{p} \in \text{Spec}(R)$ , there exists some  $i \in \{1, \dots, n\}$  such that  $\mathfrak{p} \supseteq \mathfrak{a}_i$ . Then  $\mathfrak{a}_i \subseteq \mathfrak{p} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n \subseteq \mathfrak{a}_i$ . So  $\mathfrak{p} = \mathfrak{a}_i$ .

(a) Follow from (b) directly.

□

**Example.** The converses fail in general. Let  $R = k[X, Y]$ ,  $\mathfrak{p} = \mathfrak{a}_1 = \langle x \rangle$  and  $\mathfrak{a}_2 = \langle y \rangle$ . Then  $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \langle xy \rangle \neq \langle x \rangle = \mathfrak{p} = \langle x \rangle \neq \langle xy \rangle = \mathfrak{a}_1 \mathfrak{a}_2$ .

**Theorem 1.51** (Prime Avoidance). *Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Spec}(R)$ . If  $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$ , then  $\mathfrak{a} \subseteq \mathfrak{p}_i$  for some  $i \in \{1, \dots, n\}$ , i.e., if  $\mathfrak{a} \not\subseteq \mathfrak{p}_i$  for any  $i = 1, \dots, r$ , then  $\mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i$ .*

**Theorem 1.50** (More general version). *Let  $\mathfrak{b}_1, \dots, \mathfrak{b}_n \leq R$ . Assume*

(1)  *$R$  contains an infinite field  $k$  as a subring, or*

(2)  *$\mathfrak{b}_3, \dots, \mathfrak{b}_n \in \text{Spec}(R)$ .*

*Then if  $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{b}_i$ ,  $\mathfrak{a} \subseteq \mathfrak{b}_i$  for some  $i \in \{1, \dots, n\}$ .*

**Lemma 1.48.** Let  $k$  be an infinite field,  $V \neq 0$  a vector space over  $k$ , and  $V_1, \dots, V_n$  proper subspaces of  $V$ . Then  $V \neq \bigcup_{i=1}^n V_i$ .

