Cohen-Macaulay Type of Weighted Edge Ideals and Path Ideals

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Abstract

Type is an important invariant of a Cohen-Macaulay homogeneous ideal I in a polynomial ring $A[X_1, \ldots, X_d]$, where A is a field. In Chapter 2, we recall the algebraic definition of type using Ext modules and depth. First we recall how the Cohen-Macaulay property is defined, how Ext is defined via projective resolutions and how depth is defined through regular sequences or vanishing of Ext. Chapter 2 also contains other necessary background information.

We mainly work with monomial ideals I in the ring $R = A[X_1, \ldots, X_d]$ and the case where the Krull dimension of R/I is zero, implying that I is Cohen-Macaulay and has an irredundant parametric decomposition. In this case, the type of R/I has a computational-friendly formula:

$$\mathrm{type}(R/I) = \sharp\,\{\text{parameter ideals occuring in an irredundant parametric decomposition of } I\}.$$

The goal of this thesis is to use this to derive formulas for the type of other Cohen-Macaulay quotients. We focus on ideals coming from (finite simple) graphs G; our formulas are in terms of graph-theoretical data about G. This falls in the general area of combinatorial commutative algebra, where one uses natural connections between the algebraic properties of a given monomial ideal I in $A[X_1, \ldots, X_d]$ and combinatorics.

Let G be a graph with vertex set $\{v_1, \ldots, v_d\}$. Let ΣG be the suspension of G (see Definition 3.12). In Chapter 3, we define the edge ideal $I(\Sigma G)$ in $R' = A[X_1, \ldots, X_d, Y_1, \ldots, Y_d]$ (see Definition 3.2), and we compute the type of the quotient $R'/I(\Sigma G)$ combinatorially, which is found to be exactly the number of minimal vertex covers of G:

$$\mathrm{type}\big(R'/I(\Sigma G)\big) = \sharp\,\{\text{minimal vertex covers of } G\}. \tag{*}$$

We prove this in Theorem 3.20. In particular, this computes $\mathrm{type}(R/I(G))$ for all trees such that $R/I(G)$ is Cohen-Macaulay (see Fact 3.18(b)).

Next, given a weighted graph G_ω, a weighted suspension $(\Sigma G)_\lambda$ (see Definition 3.39) with λ satisfing the conditions in Fact 3.40, and its weighted edge ideal of $I((\Sigma G)_\lambda)$ (see Definition 3.25), we go further to explore the type of the quotient $R'/I((\Sigma G)_\lambda)$, defined and studied by Paulsen and Sather-Wagstaff [11]. As with Formula (*), we find that the type of $R'/I((\Sigma G)_\lambda)$ is exactly the number of minimal weighted vertex covers of G_ω:

$$\mathrm{type}\big(R'/I((\Sigma G)_\lambda)\big) = \sharp\,\{\text{minimal weighted vertex covers of } G_\omega\}. \tag{**}$$

We prove this in Theorem 3.43. In particular, this computes $\mathrm{type}(R/I(G_\omega))$ for all weighted trees such that $R/I(G_\omega)$ is Cohen-Macaulay (see Fact 3.41(b)).

Finally, with $\Sigma_r G$ being the r-path suspension of G (see Definition 3.45), and $I_r(\Sigma_r G)$ the r-path ideal of $\Sigma_r G$(see Definition 3.47), we determine the type of the quotient $R'/I_r(\Sigma_r G)$, which is given by the number of "$\wp$-minimal r-path vertex covers of $\Sigma_{r-1}G$", in terms of an order on the minimal r-path vertex covers of $\Sigma_r G$. Using similar techniques, plus some extra effort, we deduce the formula

$$\mathrm{type}\big(R'/I_r(\Sigma_r G)\big) = \sharp\,\{\wp\text{-minimal } r\text{-path vertex covers of } \Sigma_{r-1}G\}. \tag{***}$$

We prove this in Theorem 3.74. In particular, this computes $\mathrm{type}(R/I_r(G))$ for all trees such that $R/I_r(G)$ is Cohen-Macaulay (see Fact 3.60(b)).

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Contents

Chapter 1

Introduction

Combinatorial commutative algebra is a branch of mathematics that uses combinatorics and graph theory to understand certain algebraic constructions; it also uses algebra to understand certain objects in combinatorics and graph theory. In this thesis, we explore aspects of this area via edge ideals and path ideals of graphs and weighted graphs.

Let G be a (finite simple) graph with vertex set $V = V(G) = \{v_1, \ldots, v_d\}$ and edge set $E = E(G)$. Let A be a field, and consider the polynomial ring $R = A[X_1, \ldots, X_d]$. The *edge ideal* of G is the ideal $I(G)$ of R that is "generated by the edges of G."

$$I(G) = (X_i X_j \mid v_i v_j \in E)R.$$

Some research in combinatorial commutative algebra uses graph-theoretic properties of G to understand algebraic properties of $I(G)$, and vice versa.

An important concept in commutative algebra is the "Cohen-Macaulay" property; see Definition 2.94. The definition is somewhat technical. For now, the reader should understand that Cohen-Macaulay ideals in polynomial rings are particularly nice. If I is a Cohen-Macaulay ideal in R, the "type" of R/I roughly measures how nice the ideal is. For instance, some of the nicest Cohen-Macaulay ideals are the "Gorenstein" ideals, which end up being the Cohen-Macaulay ideals of type 1.

If G is a tree, a theorem of Villarreal [14] characterizes when $I(G)$ is Cohen-Macaulay; see Fact 3.18. This characterization is purely graph-theoretical. One of the first goals of this thesis is to compute the type of $R/I(G)$ for arbitrary Cohen-Macaulay trees. We accomplish this in Theorem 3.20. As with Villarreal's result, this computation is purely graph-theoretical. In subsequent results of this thesis, we compute the type for other graph-theoretic algebra constructions the edge ideal of a weighted tree and the path ideal of a tree when they are Cohen-Macaulay. These results are in Theorems 3.43 and 3.74. These are the main results of this thesis. They form the bulk of Chapter 3. Necessary background information is collected in Chapter 2 and Section 3.2.

Chapter 2

Definitions and Background

Convention. In this chapter, let R be a commutative ring with identity, M an R-module and $I \subseteq R$ an ideal.

2.1 Commutative Rings with Identity

Definition 2.1. We say R is *local* if it has a unique maximal ideal $\mathfrak{m}$, also known as *"quasi-local"*. The *residue field* of R is $R/\mathfrak{m}$.

"Assume $(R, \mathfrak{m}, k)$ is local" or "assume $(R, \mathfrak{m})$ is local", means that $\mathfrak{m}$ is the unique maximal ideal of R and $k = R/\mathfrak{m}$.

Example 2.2. Let $\mathfrak{k}$ be a field.

(a) $\mathfrak{k}$ is local with the maximal ideal (0).

(b) $R = \mathfrak{k}[X]/(X^n)$ is local with $\mathfrak{m} = (X)/(X^n)$.

(c) Let $R = \mathfrak{k}[X_1, \ldots, X_d]/(X_1^{a_1}, \ldots, X_d^{a_d})$, where $a_i \geq 1$ for $i = 1, \ldots, d$. Then R is local with $\mathfrak{m} = (X_1, \ldots, X_d)/(X_1^{a_1}, \ldots, X_d^{a_d})$.

Definition 2.3. Let $U \subseteq R$ be multiplicatively closed and $1 \in U$. The *localization of M with respect to U* is defined to be

$$U^{-1}M = \{\text{equivalence classes from } M \times U \text{ under } \sim\},$$

where $(m,u) \sim (n,u)$ if there exists $w \in U$ such that $w(vm - un) = 0$. Denote the equivalence class of (m,u) as $\frac{m}{u}$ or m/u.

Notation 2.4. Set $M_{\mathfrak{p}} = (R \smallsetminus \mathfrak{p})^{-1}M$ for any prime ideal $\mathfrak{p} \subseteq R$.

Definition 2.5. The *radical* of I is defined to be

$$\operatorname{rad}(I) = \operatorname{r}(I) = \sqrt{I} = \{x \in R \mid x^n \in I,\ \forall\ n \gg 0\} = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \geq 1\}.$$

Definition 2.6. I is *reducible* if there exist ideals $J, K \subseteq R$ such that $I = J \cap K$ and $J \neq I$ and $K \neq I$. I is *irreducible* if it is not reducible and $I \neq R$.

Definition 2.7. An *irreducible decomposition* of I is an expression $I = \bigcap_{i=1}^{n} J_i$ with $n \geq 1$ such that ideals $J_1, \ldots, J_n \subseteq R$ are irreducible.

Definition 2.8. An irreducible decomposition $I = \bigcap_{i=1}^{n} J_i$ is *redundant* if $I = \bigcap_{i \neq k} J_i$ for some $k \in \{1, \ldots, n\}$. An irreducible decomposition $I = \bigcap_{i=1}^{n} J_i$ is *irredundant* if it is not redundant, that is, if every $k \in \{1, \ldots, n\}$ satisfies $I \neq \bigcap_{i \neq k} J_i$. As $I = \bigcap_{i=1}^{n} J_i \subseteq \bigcap_{i \neq k} J_i$ holds automatically, the given decomposition is irredundant if and only if every $k \in \{1, \ldots, d\}$ satisfies $J \subsetneq \bigcap_{i \neq k} J_i$.

Fact 2.9. [10, Corollaries 1.4.6 and 3.4.8] Let $d \geq 1$. If A is a Noetherian ring, then every proper ideal in A or $A[X_1, \ldots, X_d]$ has an irredundant irreducible decomposition.

2.2 Regular Sequences

Depth is an important invariant of rings and modules in commutative and homological algebra. It is defined in terms of the vanishing of Ext modules and it characterizes the length of maximal regular sequences.

Definition 2.10. An element $x \in R$ is a *non-zero divisor* on M if the multiplication by x map $M \xrightarrow{\cdot x} M$ is 1-1; equivalently, for $m \in M$, if $xm = 0$, then $m = 0$. Set

$$\operatorname{NZD}_R(M) = \{a \in R \mid a \text{ is a nonzero divisor on } M\}.$$

Definition 2.11. An element $x \in R$ is *M-regular* if

(a) $x \in \operatorname{NZD}_R(M)$ and

(b) $xM \neq M$.

Definition 2.12. A sequence $a_1, \ldots, a_n \in I$ is *M-regular* if

(a) a_1 is M-regular, and

(b) a_i is $\frac{M}{(a_1,\ldots,a_{i-1})M}$-regular for $i = 2, \ldots, n$,

Remark. Note that for $a_1, \ldots, a_i \in R$, we have

$$\frac{M}{(a_1,\ldots,a_i)M} \overset{①}{\cong} \frac{M/(a_1,\ldots,a_{i-1})M}{(a_1,\ldots,a_i)M/(a_1,\ldots a_{i-1})M} \cong \frac{M/(a_1,\ldots,a_{i-1})M}{a_iM/(a_1,\ldots,a_{i-1})M},$$

where ① is from the third isomorphism theorem for modules. Thus, we have $a_iM/(a_1,\ldots,a_{i-1})M \neq M/(a_1,\ldots,a_{i-1})M$ if and only if $M/(a_1,\ldots,a_i)M \neq 0$. This observation justifies the following equivalent definition for M-regular sequences.

Definition 2.13. A sequence $a_1, \ldots, a_n \in I$ is *M-regular* if

(a) $a_1 \in \mathrm{NZD}_R(M)$,

(b) $a_i \in \mathrm{NZD}_R(M/(a_1,\ldots,a_{i-1})M)$ for $i = 2, \ldots, n$, and

(c) $(a_1, \ldots, a_n)M \neq M$.

Remark. Condition (c) in Definition 2.13 is sometimes automatic, e.g., if $(R, \mathfrak{m})$ is local with $a_1, \ldots, a_n \in \mathfrak{m}$ and M is nonzero and finitely generated, then by Nakayama's lemma, we have $(a_1, \ldots, a_n)M \subseteq \mathfrak{m}M \subsetneq M$.

Definition 2.14. A sequence $a_1, \ldots, a_n \in I$ is a *maximal M-regular sequence in I* if $a_1, \ldots, a_n$ is an M-regular sequence in I such that for all $b \in I$, the longer sequence $a_1, \ldots, a_n, b$ is not M-regular.

Example 2.15. A list of variables $X_1, \ldots, X_n$ is $A[X_1, \ldots, X_n]$-regular for any commutative ring A.

2.3 Ext via Projective Resolutions

In this section, let N be another R-module. We present some definitions and facts from homological algebra leading to the definition of Ext.

Definition 2.16. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-module homomorphism is *exact* (at B) if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Note that $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ if and only if $g \circ f = 0$.

More generally, a sequence of R-module homomorphism

$$\cdots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is *exact* if $\mathrm{Im}(d_{i+1}) = \mathrm{Ker}(d_i)$ for all $i \in \mathbb{Z}$.

Fact 2.17. We have the following facts:

(a) The sequence $0 \to A \xrightarrow{f} A'$ of R-module homomorphisms is exact (at A) if and only if f is 1-1.

(b) The sequence $B' \xrightarrow{g} B \to 0$ of R-module homomorphisms is exact (at B) if and only if g is onto.

(c) The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-module homomorphisms is exact if f is 1-1, g is onto and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Definition 2.18. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0.$$

Definition 2.19. A *homomorphism of short exact sequences* is a triple (α, β, γ) of R-module homomorphisms such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0.
\end{tikzcd}$$

Fact 2.20 (The Short Five Lemma)**.** [5, Proposition 10.24] Let (α, β, γ) be a homomorphisms of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0.
\end{tikzcd}$$

(a) If α and γ are 1-1, then so is β.

(b) If α and γ are onto, then so is β.

(c) If α and γ are isomorphisms, then so is β.

Definition 2.21. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is *split* if and only if it is equivalent to the canonical exact sequence $0 \to A \xrightarrow{\epsilon} A \oplus C \xrightarrow{\rho} C \to 0$, i.e., if and only if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "="'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & A \arrow[r, "\epsilon"'] & A \oplus C \arrow[r, "\rho"'] & C \arrow[r] & 0.
\end{tikzcd}$$

In this event, β is an isomorphism by the short five lemma, and then β is an R-module isomorphism, so $B \cong A \oplus C$.

Notation 2.22.

$$\mathrm{Hom}_R(M, N) := \{R\text{-module homomorphisms } f : M \to N\},$$

which is an R-module because R is commutative.

Let A, B be R-modules. For each $f \in \mathrm{Hom}_R(A, B)$, define

$$\begin{aligned} f^* = \mathrm{Hom}_R(f, N) : \mathrm{Hom}_R(B, N) &\longrightarrow \mathrm{Hom}_R(A, N) \\ \phi &\longmapsto \phi \circ f. \end{aligned}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed, "f^*(\phi)"'] & B \arrow[d, "\phi"] \\
& N
\end{tikzcd}$$

Then f^* is an R-module homomorphism.

Fact 2.23. $\mathrm{Hom}_R(-, N)$ is a *contravariant functor*, i.e.,

(a) it respects identity maps: $\mathrm{Hom}_R(\mathrm{id}_M, N) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$, and

(b) it respects compositions: for all R-module homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$,

$$\mathrm{Hom}_R(\beta \circ \alpha, N) = \mathrm{Hom}_R(\alpha, N) \circ \mathrm{Hom}_R(\beta, N).$$

Or equivalently, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$, i.e., the following diagram commutes:

$$
\mathrm{Hom}(-,N): \quad
\begin{tikzcd}
\mathrm{Hom}_R(A,N) & \mathrm{Hom}_R(B,N) \arrow[l, "\mathrm{Hom}_R(\alpha,N)"'] \\
 & \mathrm{Hom}_R(C,N). \arrow[u, "\mathrm{Hom}_R(\beta,N)"'] \arrow[ul, "\mathrm{Hom}_R(\beta\circ\alpha,N)"]
\end{tikzcd}
$$

Fact 2.24 (Left Exactness of $\mathrm{Hom}(-,N)$)**.** [5, Theorem 10.33] Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be exact. Then the induced sequence $0 \to \mathrm{Hom}(C,N) \xrightarrow{\beta^*} \mathrm{Hom}(B,N) \xrightarrow{\alpha^*} \mathrm{Hom}(A,N)$ is exact.

Remark. The functor $\mathrm{Hom}(N,-)$ is defined similarly with notation $f_* = \mathrm{Hom}(N,f)$. This functor is covariant and left exact.

Fact 2.25. [5, Theorem 10.30] The following conditions are equivalent.

(i) $\mathrm{Hom}_R(N,-)$ transforms epimorphisms into epimorphisms.

(ii) $\mathrm{Hom}_R(N,-)$ transforms short exact sequences into short exact sequences.

(iii) $\mathrm{Hom}_R(N,-)$ transforms exact sequences into exact sequences.

(iv) Every short exact sequence $0 \to A \to B \to N \to 0$ splits.

(v) For R-modules B and C, if $B \xrightarrow{\beta} C \to 0$ is exact, then every R-module homomorphism from N to C lifts to an R-module homomorphism into B, i.e., given $\phi \in \mathrm{Hom}_R(N,C)$, there is a map $\psi \in \mathrm{Hom}_R(N,B)$ making the following diagram commute:

$$
\begin{tikzcd}
 & N \arrow[dl, dashed, "\exists\ \psi"'] \arrow[d, "\phi"] & \\
B \arrow[r, "\beta"] & C \arrow[r] & 0.
\end{tikzcd}
$$

(vi) There exists an R-module N' such that $N \oplus N'$ is free, i.e., N is a summand of a free R-module.

Definition 2.26. An R-module P is called *projective* if it satisfies any of the equivalent conditions of Fact 2.25.

Definition 2.27. A *chain complex* or *R-complex* is a sequence of R-module homomorphsims

$$M_\bullet = \cdots \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots$$

such that $\partial_{i-1}^M \circ \partial_i^M = 0$ for all $i \in \mathbb{Z}$. We say M_i is the module in *degree* i in the R-complex $M_\bullet$.

The i^{th} *homology module* of an R-complex $M_\bullet$ is the R-module

$$\mathrm{H}_i(M_\bullet) = \operatorname{Ker}(\partial_i^M)/\operatorname{Im}(\partial_{i+1}^M).$$

Definition 2.28. A *projective resolution* of M over R or an *R-projective resolution* of M is an exact sequence of R-module homomorphisms

$$P_\bullet^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \to 0$$

such that each P_i is a projective R-module.

The *truncated projective resolution* of M associated to $P_\bullet^+$ is the R-complex

$$P_\bullet = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \to 0.$$

By convention, we have $P_i = 0$ for all $i \leq -1$ and $\partial_i^P = 0$ for all $i \leq 0$. Define the R-complex $\operatorname{Hom}(P_\bullet^+, N)$ as follows:

$$\operatorname{Hom}(P_\bullet^+, N) = 0 \to M^* \xrightarrow{\tau^*} P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots,$$

where we set $P_i^* = \operatorname{Hom}(P_i, N)$ and $(\partial_i^P)^* = \operatorname{Hom}(\partial_i^P, N)$ for $i \geq 0$. Define the R-complex $P_\bullet^*$ as follows:

$$P_\bullet^* = \operatorname{Hom}(P_\bullet, N) = 0 \to P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots.$$

Let P_i^* be in degree $-i$, i.e., $P_i^* = (P^*)_{-i}$ for $i \in \mathbb{Z}$. Then

$$\begin{array}{ccccccccccccc}
P_\bullet^* = 0 \longrightarrow & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* & \xrightarrow{(\partial_2^P)^*} & \cdots & \xrightarrow{(\partial_{i-1}^P)^*} & P_{i-1}^* & \xrightarrow{(\partial_i^P)^*} & P_i^* & \xrightarrow{(\partial_{i+1}^P)^*} & \cdots \\
 & \| & & \| & & & & \| & & \| & & \\
P_\bullet^* = 0 \to & (P^*)_0 & \xrightarrow{\partial_0^{P^*}} & (P^*)_{-1} & \xrightarrow{\partial_{-1}^{P^*}} & \cdots & \xrightarrow{\partial_{-i+2}^{P^*}} & (P^*)_{-i+1} & \xrightarrow{\partial_{-i+1}^{P^*}} & (P^*)_{-i} & \xrightarrow{\partial_{-i}^{P^*}} & \cdots .
\end{array}$$

So

$$\partial_i^{P^*} = \left(\partial_{-i+1}^P\right)^*, \ \forall\, i \in \mathbb{Z}.$$

By convention, we have $(P^*)_i = P_{-i}^* = 0^* = 0$ and $\partial_i^{P^*} = \left(\partial_{-i+1}^P\right)^* = 0^* = 0$ for all $i \geq 1$.

Remark. Because of the condition $\partial_i^P \circ \partial_{i+1}^P = 0$ for $i \geq 1$, by Fact 2.23, we have

$$\left(\partial_{i+1}^P\right)^* \circ \left(\partial_i^P\right)^* = \left(\partial_i^P \circ \partial_{i+1}^P\right)^* = 0^* = 0, \ \forall \ i \geq 1.$$

Thus, $\mathrm{Hom}(P_\bullet, N)$ and similarly $\mathrm{Hom}(P_\bullet^+, N)$ are R-complexes.

Definition 2.29 (Ext via projective resolutions)**.** Let $P_\bullet^+$ be a projective resolution of M. Define the Ext module by

$$\mathrm{Ext}_R^i(M,N) := \mathrm{H}_{-i}(P_\bullet^*) = \mathrm{Ker}\big(\partial_{-i}^{P^*}\big)/\,\mathrm{Im}\big(\partial_{-i+1}^{P^*}\big) = \mathrm{Ker}\big((\partial_{i+1}^P)^*\big)/\,\mathrm{Im}\big((\partial_i^P)^*\big).$$

Fact 2.30. Let $P_\bullet^+$ be a projective resolution of M. By the left exactness of Hom, we have an exact sequence:

$$0 \longrightarrow M^* \xrightarrow{\tau^*} P_0^* \xrightarrow{(\partial_1^P)^*} P_1^*.$$

Then we have

$$\mathrm{Ext}_R^0(M,N) = \mathrm{Ker}\big((\partial_1^P)^*\big)/\,\mathrm{Im}(0) \cong \mathrm{Ker}\big((\partial_1^P)^*\big) = \mathrm{Im}(\tau^*) \cong M^* = \mathrm{Hom}_R(M,N),$$

$$\mathrm{Ext}_R^i(M,N) = \mathrm{Ker}(\partial_{-i}^{P^*})/\,\mathrm{Im}(\partial_{-i+1}^{P^*}) = \mathrm{Ker}(0 \to (P^*)_{-i-1})/\,\mathrm{Im}(\partial_{-i+1}^{P^*}) = 0, \ \forall \ i \leq -1.$$

Remark. $\mathrm{Ext}_R^i(M,N)$ is well-defined, i.e., independent of the choices of projective resolution of M, by [13, Theorem VIII.5.2].

Remark. We can also define the Ext module via *injective modules*, but this is not needed for this thesis.

2.4 Krull Dimension, Depth and Type

In this section, we define Krull dimension of M and depth of M in I, then we define type of M when $(R, \mathfrak{m})$ is local.

Definition 2.31. The *prime spectrum* of R is

$$\mathrm{Spec}(R) = \{\text{prime ideals of } R\}.$$

Let $\mathrm{V}(I)$ denote the *set of prime ideals in R containing I*:

$$\mathrm{V}(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

The *support* of M is the set

$$\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

Fact 2.32. It is straightforward to show that

$$\mathrm{Supp}_R(R) = \mathrm{Spec}(R),$$

and

$$\mathrm{Supp}_R(R/I) = \mathrm{V}(I).$$

Definition 2.33. The *Krull dimension* of M is

$$\dim_R(M) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{Supp}_R(M)\}.$$

Set $\dim(R) = \dim_R(R)$.

Based on Fact 2.32, we have the following Krull dimension definitions for rings and quotient rings.

Definition 2.34. (a) The *Krull dimension* of R is

$$\dim(R) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{Spec}(R)\}.$$

(b) The *Krull dimension* of R/I is

$$\dim(R/I) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{V}(I)\}.$$

Assumption. For the remainder of this section, we assume R is Noetherian and M is finitely generated.

Fact 2.35. [13, Corollary V.5.12] If $IM \neq M$, then each maximal M-regular sequence in I has the same length, namely

$$\inf\{i \geq 0 \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

Through Fact 2.35, we have the following definition for depth:

Definition 2.36. If $IM \neq M$, we define *depth* of M in I by

$$\operatorname{depth}_R(I; M) = \inf\{i \geq 0 \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

If $IM = M$, then set $\operatorname{depth}_R(I; M) = \infty$.

Remark. (a) By Fact 2.30 we also have $\operatorname{depth}_R(I; M) = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}$.

(b) If $(R, \mathfrak{m})$ is local and $M \neq 0$, then by Nakayama's lemma, $IM \subseteq \mathfrak{m}M \subsetneq M$, so $IM \neq M$.

Notation 2.37. If $(R, \mathfrak{m}, k)$ is local, set $\operatorname{depth}_R(M) = \operatorname{depth}_R(\mathfrak{m}; M)$.

Definition 2.38. Let $(R, \mathfrak{m}, k)$ be local and $M \neq 0$. Assume $\operatorname{depth}_R(M) = n$. The *type* of M is the positive integer

$$\operatorname{type}_R(M) = \dim_k\big(\operatorname{Ext}_R^n(k, M)\big).$$

2.5 Monomial Ideals

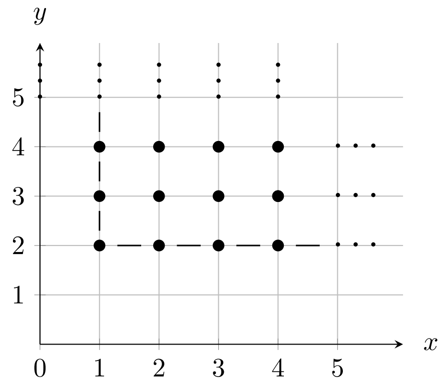
In this section, we introduce monomial ideals and a way of understanding them combinatorially. Let A be a commutative ring with identity, $R = A[X_1, \ldots, X_d]$ unless otherwise stated. Set $\mathfrak{X} = (X_1, \ldots, X_d)R$, the ideal generated by all variables in R and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 2.39. A *monomial* in elements $X_1, \ldots, X_d \in R$ is an element of the form $X_1^{n_1} \ldots X_d^{n_d}$ in R, where $n_1, \ldots, n_d \in \mathbb{N}_0$. For short, we write $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $\underline{X}^{\underline{n}} = X_1^{n_1} \ldots X_d^{n_d}$.

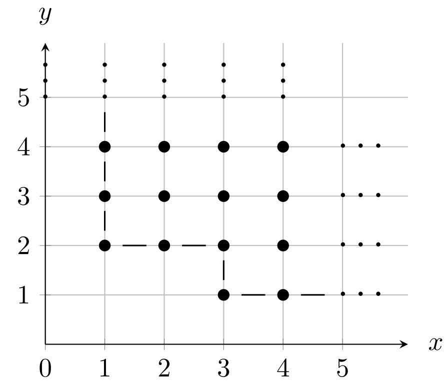
Notation 2.40. Let $\underline{m}, \underline{n} \in \mathbb{N}_0^d$. Write $\underline{m} \succcurlyeq \underline{n}$ when $m_i \geq n_i$ for $i = 1, \ldots, d$. For each $\underline{n} \in \mathbb{N}_0^d$, set

$$\langle \underline{n} \rangle = \{\underline{m} \in \mathbb{N}_0^d \mid \underline{m} \succcurlyeq \underline{n}\} = \underline{n} + \mathbb{N}_0^d.$$

Example 2.41. We describe the two sets $\langle(1,2)\rangle$ and $\langle(1,2)\rangle \cup \langle(3,1)\rangle$ in the following graphs.



$\langle(1,2)\rangle$



$\langle(1,2) \cup (3,1)\rangle$

where bullets represent points in the appropriate set.

Definition 2.42. Denote the set of monomials in R by

$$[\![R]\!] = \left\{ \underline{X}^{\underline{n}} \mid \underline{n} \in \mathbb{N}_0^d \right\}.$$

Definition 2.43. A *monomial ideal* I in R is an ideal generated by monomials in $X_1, \ldots, X_d$, i.e., elements of the form $\underline{X}^{\underline{n}}$ with $\underline{n} \in \mathbb{N}_0^d$.

Remark. The trivial ideals 0 and R are monomial ideals since $0 = (\emptyset)R$ and $R = (1)R = (X_1^0 \cdots X_d^0)R$.

Assumption. For the remainder of this section, let $I \subseteq R$ be a monomial ideal.

Fact 2.44 (Dickson's Lemma)**.** [10, Theorem 1.3.1] I is finitely generated by a set of monomials.

Definition 2.45. Denote the set of monomials in I by

$$[\![I]\!] = \left\{ \underline{X}^{\underline{n}} \in I \mid \underline{n} \in \mathbb{N}_0^d \right\} = I \cap [\![R]\!].$$

Fact 2.46. [10, Lemma 1.1.10] For each $f \in I$, each monomial occurring in f is in I.

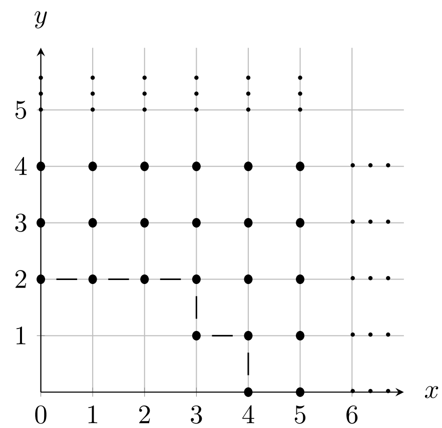
Definition 2.47. The *graph* of I is

$$\Gamma(I) = \left\{ \underline{n} \in \mathbb{N}_0^d \mid \underline{X}^{\underline{n}} \in I \right\} \subseteq \mathbb{N}_0^d.$$

Fact 2.48. [10, Theorem 1.1.12] Let $I = \left(\underline{X}^{\underline{n}_1}, \ldots, \underline{X}^{\underline{n}_t}\right)$ with $\underline{n}_1, \ldots, \underline{n}_t \in \mathbb{N}_0^d$. Then

$$\Gamma(I) = \langle \underline{n}_1 \rangle \cup \cdots \cup \langle \underline{n}_t \rangle.$$

Example 2.49. Let $R = A[X, Y]$ and $I = (X^4, X^3Y, Y^2)R$. Then $\Gamma(I) = \langle(4, 0)\rangle \cup \langle(3, 1)\rangle \cup \langle(0, 2)\rangle \subseteq \mathbb{N}_0^2$. We draw $\Gamma(I)$ in the following graph.



$\Gamma(I)$

Definition 2.50. Define the *monomial radical* of I by

$$\text{m-rad}(I) = \left(\text{rad}(I) \cap [\![R]\!]\right) R,$$

where $\text{rad}(I)$ is the radical of I.

Example 2.55 shows that $\text{rad}(I)$ may not be a monomial ideal.

Fact 2.51. [10, Proposition 2.3.2] We have the following facts:

(a) $\text{m-rad}(I) \subseteq \text{rad}(I)$.

(b) $\text{m-rad}(I) = \text{rad}(I)$ if and only if $\text{rad}(I)$ is a monomial ideal.

(c) If A is a field, then $\text{m-rad}(I) = \text{rad}(I)$.

Definition 2.52. Let $f = \underline{X}^{\underline{n}} \in [\![R]\!]$. The *support* of f is the set of variables that appear in f:

$$\text{Supp}(f) = \left\{i \in \{1, \ldots, d\} : n_i \geq 1\right\} = \left\{i \in \{1, \ldots, d\} : X_i \mid f\right\}.$$

The *reduction* of f is the monomial achieved by reducing all non-zero exponents down to 1:

$$\operatorname{red}(f) = \prod_{i \in \operatorname{Supp}(f)} X_i = \prod_{X_i | f} X_i.$$

Example 2.53. $\operatorname{Supp}(X_1^5 X_3^4) = \{1, 3\}$ and $\operatorname{red}(X_1^5 X_3^4) = X_1 X_3$.

Fact 2.54. [10, Theorem 2.3.7] Assume $I = (S)R$ for some $S \subseteq [\![R]\!]$, then we have $\text{m-rad}(I) = \big(\operatorname{red}(s) \mid s \in S\big)R$.

Example 2.55. The monomial ideal $I := (X^3Y^2, XY^3, Y^5)R$ in $R := A[X, Y]$ has

$$\text{m-rad}(I) = \big(\operatorname{red}(X^3Y^2), \operatorname{red}(XY^3), \operatorname{red}(Y^5)\big)R = (XY, XY, Y)R = (Y)R.$$

If $A = \mathbb{Z}/4\mathbb{Z}$, then $\operatorname{rad}(I) = (2, Y)R \neq \text{m-rad}(I)$.

Definition 2.56. I is *m-reducible* if there exist monomial ideals $J, K \subseteq R$ such that $I = J \cap K$ and $J \neq I$ and $K \neq I$. I is *m-irreducible* if it is not m-reducible and $I \neq R$.

Remark. Fact 2.60 shows when A is a field and $I \neq 0$, we have I is irreducible if and only if I is m-irreducible.

Example 2.57. The monomial ideal $(X^3, X^2Y^2, Y^4)R$ in $R = A[X, Y]$ is m-reducible because we have $(X^3, Y^2)R \cap (X^2, Y^4)R = (X^3, X^2Y^2, Y^4)R$, $Y^2 \in (X^3, Y^2)R \smallsetminus (X^3, X^2Y^2, Y^4)R$ and $X^2 \in (X^2, Y^4)R \smallsetminus (X^3, X^2Y^2, Y^4)R$.

Fact 2.58. I is m-irreducible if and only if $I \neq R$ and for any monomial ideals $J, K \subseteq R$, if $I = J \cap K$, then $I = J$ or $I = K$.

Fact 2.59. [10, Theorem 3.1.4] A nonzero I is m-irreducible if and only if it is generated by "pure powers", i.e., if and only if $I = (X_{i_1}^{a_1}, \ldots, X_{i_t}^{a_t})R$ for some $t \geq 1$ and $a_i \geq 1$ for $i = 1, \ldots, t$.

Fact 2.60. [10, Theorem 3.2.4] Assume A is a field and $I \neq 0$. Then I is irreducible if and only if it is m-irreducible.

Definition 2.61. An *m-irreducible decomposition* of I is an expression $I = \bigcap_{i=1}^n J_i$ with $n \geq 1$ such that monomial ideals $J_1, \ldots, J_n \subseteq R$ are m-irreducible.

Example 2.62. The monomial ideal $I = (X^2, XY, Y^3)R$ in $R = A[X, Y]$ has an m-irreducible decomposition $I = (X, Y^3)R \cap (X^2, Y)R$.

Fact 2.63. [10, Theorem 3.3.3] If $I \neq R$, then I has an m-irreducible decomposition.

Definition 2.64. An m-irreducible decomposition $I = \bigcap_{i=1}^{n} J_i$ is *redundant* if $I = \bigcap_{i \neq k} J_i$ for some $k \in \{1, \ldots, n\}$. An m-irreducible decomposition $I = \bigcap_{i=1}^{n} J_i$ is *irredundant* if it is not redundant, that is, if every $k \in \{1, \ldots, n\}$ satisfies $J \neq \bigcap_{i \neq k} J_i$. As $J = \bigcap_{i=1}^{n} J_i \subseteq \bigcap_{i \neq k} J_i$ holds automatically, the given decomposition is irredundant if and only if every $k \in \{1, \ldots, d\}$ satisfies $J \subsetneq \bigcap_{i \neq k} J_i$.

Example 2.65. The m-irreducible decomposition in Example 2.62 is irredundant.

Fact 2.66. [10, Corollary 3.3.8] If $I \neq R$, then I has an irredundant m-irreducible decomposition.

Fact 2.67. [10, Theorem 3.3.9] Assume I has two irredundant m-irreducible decompositions $I = \bigcap_{i=1}^{n} I_i = \bigcap_{j=1}^{m} J_j$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $I_t = J_{\sigma(t)}$ for $t = 1, \ldots, n$.

Definition 2.68. A monomial $\underline{X}^{\underline{n}}$ with $\underline{n} \in \mathbb{N}_0^d$ is *square-free* if $n_i = 0$ or 1 for $i = 1, \ldots, d$. A monomial ideal I of R is *square-free* if it is generated by square-free monomials.

Fact 2.69. [10, Theorem 3.3.9] If I has two irredundant m-irreducible decompositions $I = \bigcap_{i=1}^{n} I_i$ and $I = \bigcap_{j=1}^{m} I_j$, then $n = m$ and there exists $\sigma \in S_m$ such that $I_i = J_{\sigma(i)}$ for $i = 1, \ldots, n$, where S_n is the permutation group.

Fact 2.70. [10, Theorem 5.1.2] Let A be a field and I have an m-irreducible decomposition $I = \bigcap_{i=1}^{m} J_i$. Then $\dim(R/I) = d - n$, where n is the smallest number of generators needed for one of the J_i.

Definition 2.71. Let I have an irredundant m-irreducible decomposition $I = \bigcap_{i=1}^{m} J_i$. Let n_i be the smallest number of generators needed for J_i for $i = 1, \ldots, m$. We say that I is *m-unmixed* if $n_1 = \cdots = n_m$. We say that I is *m-mixed* if it is not m-unmixed, i.e., there exist $i, j \in \{1, \ldots, m\}$ with $i \neq j$ such that $n_i \neq n_j$.

Definition 2.72. A *parameter ideal* in R is an ideal of the form $(X_1^{a_1}, \ldots, X_d^{a_d})$ with $a_1, \ldots, a_d \geq 1$. For $\underline{X}^{\underline{n}} = X_1^{n_1} \ldots X_d^{n_d} \in [\![R]\!]$ with $\underline{n} \in \mathbb{N}_0^d$, set

$$\mathrm{P}_R(\underline{X}^{\underline{n}}) = \left(X_1^{n_1+1}, \ldots, X_d^{n_d+1}\right)R.$$

Fact 2.73. The parameter ideals of R are exactly the ideals of the form $P_R(f)$.

Definition 2.74. A *parametric decomposition* of I is an m-irreducible decomposition of I of the form $I = \bigcap_{i=1}^{n} \mathrm{P}_R(f_i)$.

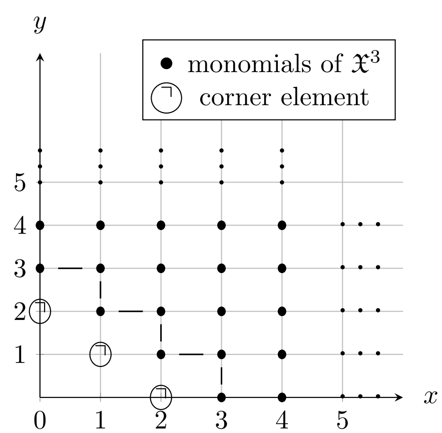
A parametric decomposition $I = \bigcap_{i=1}^{n} \mathrm{P}_R(f_i)$ is *irredundant* if $I \neq \bigcap_{i \neq j} \mathrm{P}_R(f_i)$ for any $j \in \{1, \ldots, n\}$.

A parametric decomposition $I = \bigcap_{i=1}^{n} \mathrm{P}_R(f_i)$ is *redundant* if $I = \bigcap_{i \neq j} \mathrm{P}_R(f_i)$ for some $j \in \{1, \ldots, n\}$.

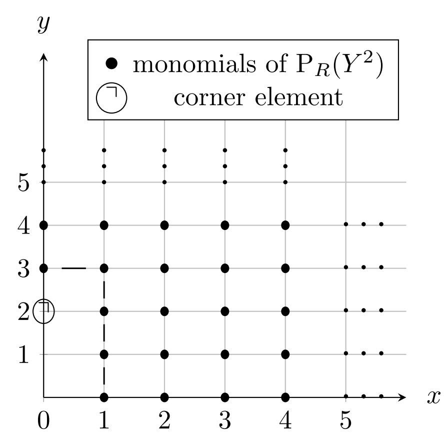
Fact 2.75. [10, Theorem 6.1.5 and Exercise 5.1.7] I has a parametric decomposition if and only if $\text{m-rad}(I) = \mathfrak{X}$. Furthermore, if A is a field, then they are equivalent to $\dim(R/I) = 0$.

Definition 2.76. $f \in [\![R]\!]$ is an *I-corner element* if $f \notin I$ and $X_1 f, \ldots, X_d f \in I$, i.e., $f \notin I$ and $f\mathfrak{X} \subseteq I$. The *set of I-corner elements* in $[\![R]\!]$ is denoted $\mathrm{C_R}(I)$.

Example 2.77. Let $R = A[X, Y]$. Then $\mathrm{C_R}(\mathfrak{X}^3) = \{X^2, XY, Y^2\}$ and $\mathrm{C_R}\big(\mathrm{P}_R(Y^2)\big) = \{Y^2\}$.



$\Gamma(\mathfrak{X}^3)$



$\Gamma(\mathrm{P}_R(Y^2))$

Remark. Corner elements may remind a reader of a border basis, but they are different.

Definition 2.78. Let $J \subseteq R$ be an ideal. Define the *ideal quotient* or *colon ideal* $(I :_R J)$ by

$$(I :_R J) = \{r \in R \mid rJ \subseteq I\} = \{r \in R \mid rj \in I, \ \forall\, j \in J\}.$$

Fact 2.79. [10, Theorem 2.5.1] Let $J \subseteq R$ be an ideal. Then $(I :_R J)$ is an ideal of R. Furthermore, if J is also a monomial ideal, then so is $(I :_R J)$.

Fact 2.80. [10, Proposition 6.2.3] We have the following facts:

(a) $\mathrm{C_R}(I) = [\![(I :_R \mathfrak{X})]\!] \smallsetminus [\![I]\!]$.

(b) If $f, f' \in \mathrm{C_R}(I)$ are distinct, then $f \notin (f')R$ and $f' \notin (f)R$.

(c) $\mathrm{C_R}(I)$ is finite.

Proposition 2.81.

$$(I : \mathfrak{X}) = I + \big(\mathrm{C}_R(I)\big)R.$$

Proof. "$\supseteq$". By definition, $\mathfrak{X}\,\mathrm{C_R}(I) \subseteq I$, i.e., $(I : \mathfrak{X}) \supseteq \mathrm{C_R}(I)$, i.e., $(I : \mathfrak{X}) \supseteq (\mathrm{C}_R(I))R$. Also, $(I : \mathfrak{X}) \supseteq I$. So $(I : \mathfrak{X}) \supseteq I + (\mathrm{C}_R(I))R$.

"$\subseteq$". Let $f \in [\![(I : \mathfrak{X})]\!]$. Then $f\mathfrak{X} \in I$. If $f \in I$, we are done. Assume $f \notin I$. Since $f\mathfrak{X} \in I$, $f \in \mathrm{C}_R(I) \subseteq I + (\mathrm{C}_R(I))R$. $\square$

Proposition 2.82. Assume A is a field. If $f_1, \ldots, f_t$ are the distinct I-corner elements in R, then $\overline{f_1}, \ldots, \overline{f_t}$ is an $A \cong R/\mathfrak{X}$-basis of $(I : \mathfrak{X})/I$.

Proof. Since $(I : \mathfrak{X}) \subseteq R$ is an ideal, $(I : \mathfrak{X})$ is an R-module. Also, since $\mathfrak{X} \cdot \frac{(I:\mathfrak{X})}{I} = 0$ in R/I, $(I : \mathfrak{X})/I$ is an $R/\mathfrak{X} \cong A$-vector space.

Let $\overline{f} \in (I : \mathfrak{X})/I$ with $f \in (I : \mathfrak{X})$. Since $\mathrm{C}_R(I) = \{f_1, \ldots, f_t\}$, by Proposition 2.81, we have $(I : \mathfrak{X}) = I + (f_1, \ldots, f_t)R$. So $\overline{f} \in (I : \mathfrak{X})/I = \frac{I+(f_1,\ldots,f_t)R}{I} = (f_1, \ldots, f_t)\frac{R}{I} = (\overline{f_1}, \ldots, \overline{f_t})\frac{R}{I}$. Also, since $f_1, \ldots, f_t \in \mathrm{C}_R(I)$, $f_1, \ldots, f_t \in (I : \mathfrak{X})$ and then $\overline{f_1}, \ldots, \overline{f_t} \in (I : \mathfrak{X})/I$. So there exist $\overline{r_1}, \ldots, \overline{r_t} \in R/I$ with $r_1, \ldots, r_t \in R$ such that $\overline{f} = \overline{r_1}\,\overline{f_1} + \cdots + \overline{r_t}\,\overline{f_t}$. So $\overline{f_1}, \ldots, \overline{f_t}$ linearly span $(I : \mathfrak{X})/I$ over A.

Assume there exist $\overline{b_1}, \ldots, \overline{b_t} \in R/\mathfrak{X}$ with $b_1, \ldots, b_t \in R$ such that $\overline{b_1}\,\overline{f_1} + \cdots + \overline{b_t}\,\overline{f_t} = 0$ in $(I : \mathfrak{X})/I$. If $\overline{b_i} = 0$, assume without loss of generality that $b_i = 0$. If $\overline{b_i} \neq 0$, assume without loss of generality that b_i is a constant. This is allowable because $R/\mathfrak{X} \cong A$. Then in $(I : \mathfrak{X})/I$, $0 = \overline{b_1}\,\overline{f_1} + \cdots + \overline{b_t}\,\overline{f_t} = \overline{b_1 f_1 + \cdots + b_t f_t}$. So $b_1 f_1 + \cdots + b_t f_t \in I$. Hence $f_i \in I$ for all $i \in \{1, \ldots, t\}$ such that $b_i \neq 0$ by Fact 2.46. By definition, though, we have $f_i \notin I$ for $i = 1, \ldots, t$. Therefore, $b_i = 0$ for $i = 1, \ldots, t$. Thus, $\overline{f_1}, \ldots, \overline{f_t} \in (I : \mathfrak{X})/I$ are linearly independent over A. $\square$

Fact 2.83. [10, Theorem 6.2.9] Let $\mathrm{C_R}(I) = \{f_1, \ldots, f_m\}$. Then $I = \bigcap_{j=1}^{m} \mathrm{P_R}(f_j)$ is an irredundant parametric decomposition.

Fact 2.84. [10, Proposition 6.2.11] Let $f_1, \ldots, f_m \in [\![R]\!]$. Assume $I = \bigcap_{i=1}^{m} \mathrm{P_R}(f_i)$ is an irredundant parametric decomposition of I. Then $\mathrm{C_R}(I) = \{f_1, \ldots, f_m\}$.

Fact 2.85. [10, Theorem 7.5.1] Let $I = \left(\underline{X}^{\underline{a}_1}, \ldots, \underline{X}^{\underline{a}_n}\right)R$ with $\underline{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}_0^d$ for $i = 1, \ldots, n$. Then

$$I = \bigcap_{i_1=1}^{d} \cdots \bigcap_{i_n=1}^{d} \left(X_{i_1}^{a_{1,i_1}}, \ldots, X_{i_n}^{a_{n,i_n}}\right).$$

Example 2.86. Let $R = A[X_1, X_2]$ and $I = (X_1^2 X_2, X_1 X_3)R$. Then by Fact 2.85,

$$\begin{aligned}
I &= (X_1^2, X_1)R \cap (X_1^2, X_2^0)R \cap (X_1^2, X_3)R \\
&\quad \cap (X_2, X_1)R \cap (X_2, X_2^0)R \cap (X_2, X_3)R \\
&\quad \cap (X_3^0, X_1)R \cap (X_3^0, X_2^0)R \cap (X_3^0, X_3)R \\
&= (X_1)R \cap R \cap (X_1^2, X_3)R \cap (X_1, X_2)R \cap R \cap (X_2, X_3)R \cap R \cap R \cap R \\
&= (X_1)R \cap (X_1^2, X_3)R \cap (X_2, X_3)R.
\end{aligned}$$

The polarization of a monomial is a square-free monomial ideal in a new set of variables obtained by turning powers of variables into products of distinct variables. It is constructed as follows.

Definition 2.87. Define the *polarization* of $\underline{X}^{\underline{a}} = X_1^{a_1} \cdots X_d^{a_d} \in [\![R]\!]$ to be the square-free monomial

$$\mathscr{P}\mathscr{O}(\underline{X}^{\underline{a}}) = X_{1,0} \cdots X_{1,a_1-1} \cdots X_{d,1} \cdots X_{d,a_d-1}$$

in the polynomial ring $R' = A[X_{i,j} \mid 1 \leq i \leq d, 0 \leq j \leq a_i - 1]$. Let $I = \left(\underline{X}^{\underline{a}_1}, \ldots, \underline{X}^{\underline{a}_n}\right)R$ with $\underline{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}_0^d$ for $i = 1, \ldots, n$. Define the *polarization* of I by

$$\mathscr{P}\mathscr{O}(I) = \Big(\mathscr{P}\mathscr{O}\big(\underline{X}^{\underline{a}_1}\big), \ldots, \mathscr{P}\mathscr{O}\big(\underline{X}^{\underline{a}_n}\big)\Big)R',$$

where R' is the smallest polynomial ring containing $\mathscr{P}\mathscr{O}(\underline{X}^{\underline{a}_1}), \ldots, \mathscr{P}\mathscr{O}(\underline{X}^{\underline{a}_n})$.

Remark. Note that by identifying each X_i with $X_{i,0}$, one can regard

$$R' = A[X_{1,0}, \ldots, X_{d,0}][X_{i,j} \mid 1 \leq i \leq d, 1 \leq j \leq a_i - 1] = R[X_{i,j} \mid 1 \leq i \leq d, 1 \leq j \leq a_i - 1],$$

which is a ring extension of R.

Example 2.88. Let $R = A[X_1, X_2, X_3]$ and $I = (X_1^2, X_1X_2, X_2^3)$. Then

$$\mathscr{P}\mathscr{O}(I) = (X_{1,0}X_{1,1}, X_{1,0}X_{2,0}, X_{2,0}X_{2,1}X_{2,2})R',$$

with $R' = A[X_{1,0}, X_{1,1}, X_{2,0}, X_{2,1}, X_{2,2}]$.

Fact 2.89. [6] Let $I = \left(\underline{X}^{\underline{a}_1}, \ldots, \underline{X}^{\underline{a}_n}\right)R$. Let $m_j = \max_{1 \le i \le n}\{a_{i,j}\}$ for $j = 1, \ldots, d$. Then the sequence of elements $Z = \{X_{i,0} - X_{i,k} \mid 1 \le i \le d, 1 \le k \le m_i - 1\}$ forms a regular sequence on $R'/\mathscr{P}\mathscr{O}(I)$ where R' is the smallest polynomial ring containing $\mathscr{P}\mathscr{O}(\underline{X}^{\underline{a}^1}), \ldots, \mathscr{P}\mathscr{O}(\underline{X}^{\underline{a}^n})$, and

$$\frac{R}{I} \cong \frac{R'/\mathscr{P}\mathscr{O}(I)}{(Z)R'/\mathscr{P}\mathscr{O}(I)} \cong \frac{R'}{(\mathscr{P}\mathscr{O}(I) + (Z))R'}.$$

Example 2.90. We have

$$\frac{A[X_1, X_2, X_3]}{(X_1^2, X_2^2, X_3^2)} \cong \frac{A[X_1, X_2, X_3, X_{1,1}, X_{2,1}, X_{3,1}]}{(X_1X_{1,1}, X_2X_{2,1}, X_3X_{3,1}) + (X_1 - X_{1,1}, X_2 - X_{2,1}, X_3 - X_{3,1})}.$$

2.6 Homogeneous Cohen-Macaulay Rings

Let A be a field, set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$ and let $I \subsetneq R$ be an ideal generated by homogeneous polynomials. In this section, we define Cohen-Macaulayness and we see how to compute the type of R/I, when R/I is Cohen-Macaulay.

Remark. The quotient ring R/I behaves analogously with local rings, e.g., every maximal homogeneous regular sequence on R/I has the same length (See Fact 2.92).

We have already defined depth and type in the local setting. Now we define them in the homogeneous setting.

Definition 2.91. The *depth* of R/I is

$$\operatorname{depth}(R/I) = \text{the length of a maximal homogeneous } (R/I)\text{-regular sequence in } \mathfrak{X}.$$

The *type* of R/I is

$$\operatorname{type}(R/I) = \dim_A(\operatorname{Ext}_R^n(A, R/I)),$$

where $n = \operatorname{depth}(R/I)$.

Fact 2.92. [2, Proposition 1.5.15] We have

$$\operatorname{depth}(R/I) = \operatorname{depth}(R_{\mathfrak{X}}/I_{\mathfrak{X}}),$$

and

$$\operatorname{type}(R/I) = \operatorname{type}(R_{\mathfrak{X}}/I_{\mathfrak{X}}).$$

Fact 2.93. [2, Theorems 1.2.10 and 2.1.2] If $f_1, \ldots, f_r \in R$ is a homogeneous regular sequence for R/I, then

$$\operatorname{depth}(R/(I + (f_1, \ldots, f_r)R)) = \operatorname{depth}(R/I) - r,$$

and

$$\dim(R/(I + (f_1, \ldots, f_r)R)) = \dim(R/I) - r.$$

Cohen-Macaulay rings, defined next, have been shown over and over again in the literature to be extremely nice. See the discussion in [2, p.57] for more about this.

Definition 2.94. The quotient R/I is *Cohen-Macaulay* if $\operatorname{depth}(R/I) = \dim(R/I)$. We say that I is *Cohen-Macaulay* if the quotient R/I is Cohen-Macaulay.

Fact 2.95. We have the following facts:

(a) Let R/I be Cohen-Macaulay. If $f_1, \ldots, f_n$ is a maximal homogeneous regular sequence for R/I, then $\dim(R/(I + (f_1, \ldots, f_n)R)) = 0$ and $\operatorname{type}(R/I) = \operatorname{type}(R/(I + (f_1, \ldots, f_n)R))$.

(b) If I is a monomial ideal and has an irredundant parametric decomposition $I = \bigcap_{i=1}^{t} Q_i$, then $\operatorname{type}(R/I) = t$.

Proof. (a) Since R/I is Cohen-Macaulay, by Fact 2.93, we have

$$\dim(R/(I + (f_1, \ldots, f_n)R)) = \dim(R/I) - n = \dim(R/I) - \operatorname{depth}(R/I) = 0.$$

By [8, Proposition A.6.2], $\operatorname{type}(R/I) = \operatorname{type}(R/(I + (f_1, \ldots, f_n)R))$.

(b) By Facts 2.83 and 2.84, we have $|\mathrm{C}_R(I)| = t$. Also, $\operatorname{type}(R/I) = \dim_A(\operatorname{Ext}_R^0(A; R/I)) = \dim_A(\operatorname{Hom}_R(A, R/I))$ with $A \cong R/\mathfrak{X}$. So to show $\operatorname{type}(R/I) = t$, it is equivalent to verify $|\mathrm{C}_R(I)| =$

$\dim_A(\mathrm{Hom}_R(A, R/I))$. Assume $\mathrm{C}_R(I) = \{\underline{X}^{\underline{n}_1}, \ldots, \underline{X}^{\underline{n}_t}\}$. Define

$$\begin{aligned}
\Phi : \mathrm{C}_R(I) &\longrightarrow \mathrm{Hom}_R(R/\mathfrak{X}, R/I) \\
\underline{X}^{\underline{n}_i} &\longmapsto \varphi_i : R/\mathfrak{X} \longrightarrow R/I \\
&\qquad\qquad \bar{r} \longmapsto \overline{r\underline{X}^{\underline{n}_i}}
\end{aligned}$$

We first show that φ_i is well-defined. Let $\overline{r_1} = \overline{r_2}$ in $R/\mathfrak{X}$. Then $r_1 - r_2 \in \mathfrak{X}$. Since $\underline{X}^{\underline{n}_i} \in \mathrm{C}_R(I)$, $(r_1 - r_2)\underline{X}^{\underline{n}_i} \in I$. So $\overline{r_1\underline{X}^{\underline{n}_i}} - \overline{r_2\underline{X}^{\underline{n}_i}} = \overline{(r_1 - r_2)\underline{X}^{\underline{n}_i}} = 0$ in R/I. Hence φ_i is well-defined, and it follows readily that Φ is well-defined as well.

Let $e_1, \ldots, e_t$ be the standard basis of the vector space $A^t \cong (R/\mathfrak{X})^t$. Define

$$\begin{aligned}
\hat{\Phi} : A^t &\longrightarrow \mathrm{Hom}_R(A, R/I) \\
\overline{e_i} &\longmapsto \Phi\big(\underline{X}^{\underline{n}_i}\big) = \varphi_i, \ \forall\ i = 1, \ldots, t \\
\sum_{i=1}^{t} \overline{a_i}\,\overline{e_i} &\longmapsto \sum_{i=1}^{t} \overline{a_i}\Phi\big(\underline{X}^{\underline{n}_i}\big) = \sum_{i=1}^{t} \overline{a_i}\varphi_i.
\end{aligned}$$

By [13, Remark IX.3.4], $\mathrm{Hom}_R(A, R/I)$ is a finite dimensional A-vector space. So by the universal mapping property for A-vector space, $\hat{\Phi}$ is a well-defined A-linear transformation.

Let $x = \sum_{i=1}^{t} a_i e_i \in A^t$ with $a_1, \ldots, a_t \in A$. Then $x \in \mathrm{Ker}(\hat{\Phi})$ if and only if $0 = \hat{\Phi}(x) = \hat{\Phi}\big(\sum_{i=1}^{t} a_i e_i\big) = \sum_{i=1}^{t} a_i\varphi_i$ if and only if in R/I, $0 = \big(\sum_{i=1}^{t} a_i\varphi_i\big)(\bar{1}) = \sum_{i=1}^{t} a_i\varphi_i(\bar{1}) = \sum_{i=1}^{t} a_i\overline{\underline{X}^{\underline{n}_i}}$ if and only if $a_1 = \cdots = a_t = 0$ by Proposition 2.82 if and only if $x = 0$. So $\hat{\Phi}$ is 1-1.

Let $\psi \in \mathrm{Hom}_R(A, R/I)$. If $\psi = 0$, then $\hat{\Phi}(0) = 0 = \psi$ since $\hat{\Phi}$ is a linear transformation. By Dickson's Lemma, we have $I = (f_1, \ldots, f_m)R$ for some $f_1, \ldots, f_m \in [\![R]\!]$. Assume $\psi \neq 0$. Since A is a cyclic R-module, there exists $s \in R$ such that

$$\begin{aligned}
\psi : A &\longrightarrow R/I \\
\bar{1} &\longmapsto \bar{s}.
\end{aligned}$$

Note that in R/I, $0 = \psi(\mathfrak{X}/\mathfrak{X}) = (s\mathfrak{X} + I)/I$, i.e., $s\mathfrak{X} + I \subseteq I$, so $s\mathfrak{X} \subseteq I$. Hence by Proposition 2.81, $s \in (I : \mathfrak{X}) = I + (\mathrm{C}_R(I))R = I + \big(\underline{X}^{\underline{n}_1}, \ldots, \underline{X}^{\underline{n}_t}\big)R$. So in R/I, $\bar{s} = \sum\limits_{i=1}^{t} \overline{b_i\underline{X}^{\underline{n}_i}}$ for some $b_1, \ldots, b_t \in R$.

Since $\hat{\Phi}$ is A-linear, for all $\bar{r} \in A$,

$$\psi(\bar{r}) = \overline{sr} = \bar{s}\bar{r} = \left(\sum_{i=1}^{t} \overline{b_i \underline{X}^{\underline{n}_i}}\right)\bar{r} = \left(\sum_{i=1}^{t} \varphi_i(\overline{b_i})\right)\bar{r} = \left(\sum_{i=1}^{t} \Phi(\underline{X}^{\underline{n}_i})(\overline{b_i})\right)\bar{r} = \left(\sum_{i=1}^{t} \hat{\Phi}(\overline{e_i})(\overline{b_i})\right)\bar{r}$$
$$= \left(\sum_{i=1}^{t} \hat{\Phi}\big(\overline{b_i}\overline{e_i}\big)(\bar{1})\right)\bar{r} = \left(\hat{\Phi}\left(\sum_{i=1}^{t} \overline{b_i}\overline{e_i}\right)(\bar{1})\right)\bar{r} = \hat{\Phi}\left(\sum_{i=1}^{t} \overline{b_i}\overline{e_i}\right)(\bar{r}).$$

So $\hat{\Phi}\Big(\sum_{i=1}^{t} \overline{b_i}\overline{e_i}\Big) = \psi$. Thus, $\hat{\Phi}$ is onto. $\square$

Fact 2.96. [10, Theorem 5.3.16] Let I be a monomial ideal. If R/I is Cohen-Macaulay, then I is m-unmixed.

Chapter 3

Cohen-Macaulay Type of Weighted Edge Ideals and f-weighted Path Ideals

In Sections 3.2-3.4, we present the main results of this thesis. These results give formulas to compute the type of the edge ideal of a suspension of a graph, the type of the weighted edge ideal of a weighted suspension and the type of the r-path ideal of an r-path suspension of a graph. Section 3.1 contains a little more background needed for these results. Let A be a field, $d \geq 2$ and $R = A[X_1, \ldots, X_d]$. Set $\mathfrak{m} = (X_1, \ldots, X_d)R$. Let $G = (V, E)$ be a (finite simple) graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. An edge between vertices v_i and v_j is denoted v_iv_j.

3.1 Connections Between Combinatorics and Monomial Ideals

In this section, we list some combinatorial facts about square-free monomial ideals.

We first list the definitions for paths and cycles from Diestel [4].

Definition 3.1. An *r-path* is a non-empty graph $P = (V', E')$ of the form $V' = \{x_1, \ldots, x_{r+1}\}$ and $E' = \{x_1x_2, x_2x_3, \ldots, x_rx_{r+1}\}$, where x_i are all distinct. We denote an r-path by $P_r =$

$\left(x_1 \text{ —— } x_2 \text{ —— } \cdots \text{ —— } x_{r+1} \right)$ or $x_1 \dots x_{r+1}$ for simplicity. Note that there are $r+1$ vertices and r edges in P_r.

If $P_r = \left(x_1 \text{ —— } x_2 \text{ —— } \cdots \text{ —— } x_r \right)$ is an $(r-1)$-path, then the graph $C_r := P_{r-1} + x_r x_1$ is called an *r-cycle*. Note that there are r vertices and r edges in C_r.

Definition 3.2. We have the following definitions:

(a) The *edge ideal* associated to G is the ideal $I(G) \subseteq R$ that is "generated by the edges of G":

$$I(G) = (X_i X_j \mid v_i v_j \in E)R.$$

(b) For each $V' \subseteq V$, let $P_{V'} \subseteq R$ be the ideal "generated by the elements of V'":

$$P_{V'} = (X_i \mid v_i \in V')R.$$

For instance, $P_V = \mathfrak{m} = (X_1, \dots, X_d)R$.

Example 3.3. Consider the following 3-cycle $C_3 = \left(v_1 \text{ —— } v_2 \text{ —— } v_3 \text{ —— } v_1 \right)$.

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[rr, dash, bend left] & v_2 \arrow[r, dash] & v_3 \end{tikzcd}$$

The edge ideal $I(C_3) = (X_1X_2, X_2X_3, X_1X_3)$ in $R = A[X_1, X_2, X_3]$. We have $P_{V'} = (X_1, X_3)R$ for $V' = \{v_1, v_3\}$.

Definition 3.4. A *vertex cover* of G is a subset $V' \subseteq V$ such that for each edge $v_i v_j \in E$ we have $v_i \in V'$ or $v_j \in V'$. A vertex cover V' is *minimal* if it does not properly contain another vertex cover of G.

Example 3.5. The minimal vertex covers for the 2-path $P_2 = \left(v_1 \text{ —— } v_2 \text{ —— } v_3 \right)$ are depicted in the following sketches.

$$\begin{tikzcd} v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 \end{tikzcd}$$

Fact 3.6. [10, Theorem 4.3.6] The ideal $I(G)$ is a square-free monomial ideal. It has the following m-irreducible decomposition

$$I(G) = \bigcap_{V' \text{ v. cover}} P_{V'} = \bigcap_{V' \text{ min. v. cover}} P_{V'},$$

where the first intersection is taken over all vertex covers of G, and the second intersection is taken over all minimal vertex covers of G. The second intersection is irredundant.

Definition 3.7. For $V' \subseteq V$, set

$$\underline{X}^{V'} = \prod_{v_i \in V'} X_i.$$

Definition 3.8. A *simplicial complex* on V is a nonempty collection Δ of subsets of V that is closed under subsets. An element of Δ is called a *face* of Δ. A face of the form $\{v_i\}$ is called a *vertex* of Δ. A face of the form $\{v_j, v_k\}$ with $j \neq k$ is called an *edge* of Δ. A maximal element of Δ with respect to containment is a *facet* of Δ. The $(d-1)$-*simplex* consists of all the subsets of V and is denoted Δ_{d-1}.

Definition 3.9. Let Δ be a simplicial complex on V. The *Stanley-Reisner ideal* of R associated to Δ is the ideal "generated by the non-faces of Δ":

$$J_\Delta = \left(\underline{X}^{V'} \mid V' \subseteq V \text{ and } V' \notin \Delta\right)R.$$

Definition 3.10. $W \subseteq V$ is *independent* in G if for any distinct $x_i, x_j \in W$: x_i is not adjacent to x_j in G. An independent subset in G is *maximal* if it is maximal with respect to containment. Let Δ_G denote the set of independent subsets of G. This is the *independence complex* of G.

Fact 3.11. [10, Theorem 4.4.9] Δ_G is a simplicial complex such that

$$I_G = J_{\Delta_G}.$$

3.2 Edge Ideals $I(G)$ and the Type of $R'/I(\Sigma G)$

In this section, we compute the type of $R'/I(\Sigma G)$ naturally using some known facts. See Formula (*) from the abstract and Theorem 3.20. Let $R' = A[X_1, \ldots, X_d, Y_1, \ldots, Y_d]$.

Definition 3.12. The *suspension* of G is the graph ΣG with vertex set

$$V(\Sigma G) = V \sqcup \{w_1, \ldots, w_d\} = \{v_1, \ldots, v_d, w_1, \ldots, w_d\}$$

and edge set

$$E(\Sigma G) = E(G) \sqcup \{v_1 w_1, \ldots, v_d w_d\}.$$

This is also known as the K_1-*corona* of G.

Remark. The term "suspension" is due to Villarreal [14]. It is not related to the suspension of a topological space.

Example 3.13. The suspension ΣP_2 of the 2-path $G = P_2 = \left(v_1 \text{ ---\!--- } v_2 \text{ ---\!--- } v_3 \right)$ is

$$\begin{tikzcd}
w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] \\
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3.
\end{tikzcd}$$

Fact 3.14. The minimal vertex covers of ΣG are of the form $V' \sqcup W'$, where V' is a vertex cover of G and $W' = \{w_i \mid v_i \notin V'\}$. So the size of each minimal vertex cover of ΣG is d.

Example 3.15. Consider the following graph ΣP_2 as in Example 3.13.

$$\begin{tikzcd}
w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] \\
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3
\end{tikzcd}$$

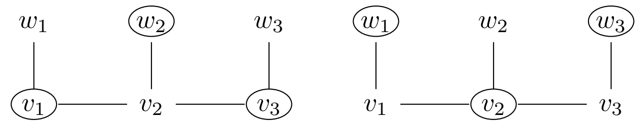
We depict the vertex covers of P_2 in the following sketches.

$$\begin{tikzcd}
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3
\end{tikzcd}$$

$$\begin{tikzcd}
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3
\end{tikzcd}$$

So by Fact 3.14, we present the minimal vertex covers of ΣP_2 in the following sketches.

$$\begin{tikzcd}
w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] & w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] & w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] \\
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 & v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3
\end{tikzcd}$$



Then by Fact 3.6, the irredundant m-irreducible decomposition of $I(\Sigma P_2)$ is given by

$$
\begin{aligned}
I(\Sigma P_2) &= (X_1X_2, X_2X_3, X_1Y_1, X_2Y_2, X_3Y_3)R' \\
&= (X_1, X_2, X_3)R' \cap (X_1, X_2, Y_3)R' \cap (Y_1, X_2, X_3)R' \cap (X_1, Y_2, X_3)R' \cap (Y_1, X_2, Y_3)R'.
\end{aligned}
$$

Fact 3.16.

$$\dim\left(\frac{R'}{I(\Sigma G)}\right) = d.$$

Proof. By Facts 3.6, 3.14 and 2.70, $\dim\left(\frac{R'}{I(\Sigma G)}\right) = 2d - d = d$. $\square$

Fact 3.17. Note that $I(\Sigma G)$ is the polarization of $I(G) + (X_1^2, \ldots, X_d^2)R$. So by Fact 2.89, the list $X_1 - Y_1, \ldots, X_d - Y_d$ is a maximal homogeneous regular sequence for $\frac{R'}{I(\Sigma G)}$ and

$$\frac{R}{I(G) + (X_1^2, \ldots, X_d^2)R} \cong \frac{R'}{I(\Sigma G) + (X_1 - Y_1, \ldots, X_d - Y_d)R'}.$$

Because of the following fact, the main result of this section gives a formula to compute $\operatorname{type}(R/I(G))$ for all trees such that $R/I(G)$ is Cohen-Macaulay.

Fact 3.18. We have the following facts.

(a) $R'/I(\Sigma G)$ is Cohen-Macaulay.

(b) If Γ is a tree and $R/I(\Gamma)$ is Cohen-Macaulay, then $\Gamma = \Sigma H$ for some subtree H, in fact, H is the subgraph induced by vertices of degree ≥ 2.

Proof. (a) By Fact 3.17, $\operatorname{depth}(R'/I(\Sigma G)) = d$. Also, by Fact 3.16, $\dim\bigl(R'/I(\Sigma G)\bigr) = d$. So $R'/I(\Sigma G)$ is Cohen-Macaulay.

(b) By [14, Theorem 2.4]. $\square$

Example 3.19. We have the following two quotients $R/I(G)$, one of which is Cohen-Macaulay, and the other is not.

(a) Consider the 2-path

$$G := P_2 = (\ v_1 \ \text{———}\ v_2 \ \text{———}\ v_3\ ).$$

Then $I(G) = (X_1X_2, X_2X_3)R = (X_1, X_3)R \cap (X_2)R$ by Fact 3.6. So by the fourth isomorphism theorem, we have in $R/I(G)$, $0 = \overline{I(G)} = \overline{(X_1, X_3)R} \cap \overline{(X_2)R}$, which is a minimal primary decomposition. So the set of associated primes $\operatorname{Ass}_{R/I(G)}(0) = \{\operatorname{rad}(\overline{(X_1, X_3)R}), \operatorname{rad}(\overline{(X_2)R})\} = \{\overline{(X_1, X_3)R}, \overline{(X_2)R}\}$. Hence by [1, Proposition 4.7] the set of zero divisors $\operatorname{ZD}(R/I(G))$ of $R/I(G)$ is $\overline{(X_1, X_3)R} \cup \overline{(X_2)R}$. So $\overline{X_2 - X_3}$ is regular in $R/I(G)$. We simplify the quotient $R/(I(G) + (X_2 - X_3)R) \cong R_1/(X_1X_2, X_2^2)R_1$, where $R_1 = A[X_1, X_2]$.

By Fact 2.85, $J := (X_1X_2, X_2^2)R_1 = (X_1, X_2^2)R_1 \cap (X_2)R_1$. As before, $\operatorname{ZD}(R_1/J) = \overline{(X_1, X_2)R_1}$. Since $\overline{(X_1, X_2)R_1}$ is a maximal ideal of R_1/J, $\operatorname{depth}(R_1/J) = 0$. So $\operatorname{depth}(R/I(G)) = \operatorname{depth}(R_1/J) + 1 = 1$ by Fact 2.93. On the other hand, by Fact 2.70, $\dim(R/I(G)) = 3 - 1 = 2$. Hence $R/I(G)$ is not Cohen-Macaulay.

Observe that P_2 is not a suspension of any subtree.

(b) Consider the 3-path

$$G := P_3 = \left(v_1 \;\text{——}\; v_2 \;\text{——}\; v_3 \;\text{——}\; v_4 \right).$$

Then $I(G) = (X_1X_2, X_2X_3, X_3X_4)R = (X_1, X_3)R \cap (X_2, X_3)R \cap (X_2, X_4)R$ by Fact 3.6. So by the fourth isomorphism theorem, we have in $R/I(G)$, $0 = \overline{I(G)} = \overline{(X_1, X_3)R} \cap \overline{(X_2, X_3)R} \cap \overline{(X_2, X_4)R}$, which is a minimal primary decomposition. As in part (a), we have $\overline{X_3 - X_4}$ is regular in $R/I(G)$. We simplify the quotient $R/(I(G) + (X_3 - X_4)R) \cong R_1/(X_1X_2, X_2X_3, X_3^2)R_1$, where $R_1 = A[X_1, X_2, X_3]$.

By Fact 2.85, $J := (X_1X_2, X_2X_3, X_3^2)R_1 = (X_2, X_3^2)R_1 \cap (X_1, X_3)R_1$. As before, $\overline{X_1 - X_2}$ is regular in R_1/J. We simplify the quotient

$$R/(I(G) + (X_3 - X_4, X_1 - X_2)R) \cong R_1/(J + (X_1 - X_2)R_1) \cong R_2/(X_2^2, X_2X_3, X_3^2)R_2,$$

where $R_2 = A[X_2, X_3]$. Let $K = (X_2^2, X_2X_3, X_3^2)R_2$. Then $\operatorname{depth}(R_2/K) = 0$ as before and so $\operatorname{depth}(R/I(G)) = \operatorname{depth}(R_2/K) + 2 = 2$ by Fact 2.93. On the other hand, by Fact 2.70, $\dim(R/I(G)) = 4 - 2 = 2$. Hence $R/I(G)$ is Cohen-Macaulay.

Observe that P_3 is a suspension of the subtree 1-path $\left(v_2 \;\text{——}\; v_3 \right)$.

The following theorem is the first main result of this thesis. It is Formula (*) from the abstract.

Theorem 3.20.

$$\operatorname{type}\left(\frac{R'}{I(\Sigma G)}\right) = \sharp\,\{\textit{minimal vertex covers of } G\}.$$

Proof. We compute

$$\begin{aligned}
\operatorname{type}\left(\frac{R'}{I(\Sigma G)}\right) &= \operatorname{type}\left(\frac{R'}{I(\Sigma G)+(X_1-Y_1,\ldots,X_d-Y_d)R'}\right)\\
&= \operatorname{type}\left(\frac{R}{I(G)+(X_1^2,\ldots,X_d^2)R}\right)\\
&= \sharp\,\{\text{ideals in an irredundant parametric decomposition of } I(G)+(X_1^2,\ldots,X_d^2)\}\\
&= \sharp\,\{\text{ideals in irredundant m-irrededucible decomposition of } I(G)\}\\
&= \sharp\,\{\text{minimal vertex covers of } G\},
\end{aligned}$$

where the first equality is from Facts 2.95(a), 3.18(a) and 3.17, the second equality is from Fact 3.17, the third equality is from Fact 2.95(b) since $\dim\left(\frac{R}{I(G)+(X_1^2,\ldots,X_d^2)R}\right) = 0$, the fourth equality is from [10, Theorem 7.5.3], and the last equlity is from Fact 3.6. $\square$

Remark. Because of Fact 3.18, we can use Theorem 3.20 to compute $\operatorname{type}(R/I(G))$ for all trees G such that $I(G)$ is Cohen-Macaulay.

Example 3.21. Consider the following graph ΣP_2 as in Example 3.13.

$$\begin{tikzcd} w_1 \arrow[d, dash] & w_2 \arrow[d, dash] & w_3 \arrow[d, dash] \\ v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 \end{tikzcd}$$

We depict the minimal vertex covers of P_2 in the following sketches.

$$\begin{tikzcd} v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 \end{tikzcd} \qquad \begin{tikzcd} v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & v_3 \end{tikzcd}$$

By Theorem 3.20,

$$\operatorname{type}\big(R'/I(\Sigma P_2)\big) = \sharp\,\{\text{minimal vertex covers of } P_2\} = 2.$$

Since $R'/I(\Sigma P_2)$ is Cohen-Macaulay by Fact 3.18(a), $\operatorname{depth}\big(R'/I(\Sigma P_2)\big) = \dim\big(R'/I(\Sigma P_2)\big) = 3$ by

Fact 3.16. Hence

$$\operatorname{Ext}^3_{R'}\left(A, R'/I(\Sigma P_2)\right) \cong A^2.$$

3.3 Weighted Edge Ideals $I(G_\omega)$ and the Type of $R'/I((\Sigma G)_\lambda)$

In this section, we prove a weighted version of Theorem 3.20 based on results from [11]. See Formula (**) from the abstract and Theorem 3.43. Let $R' = A[X_1, \ldots, X_d, Y_1, \ldots, Y_d]$. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers.

Definition 3.22. A *weight function* on a graph G is a function $\omega : E \to \mathbb{N}$ that assigns a *weight* to each edge. A *weighted graph* G_ω is a graph G equipped with a weight function ω.

Example 3.23. Let $G := P_2 = (\ v_1 \ \text{——} \ v_2 \ \text{——} \ v_3 \ )$. We assign a weight to each edge of ΣG, then we get, e.g., the following weighted graph $(\Sigma G)_\omega$.

$$\begin{tikzcd} w_1 \arrow[d, dash, "5"] & w_2 \arrow[d, dash, "3"] & w_3 \arrow[d, dash, "4"] \\ v_1 \arrow[r, dash, "2"] & v_2 \arrow[r, dash, "3"] & v_3 \end{tikzcd}$$

Definition 3.24. Let Ω consist of the pairs (V', δ') with the set $V' \subseteq V$ and $\delta' : V' \to \mathbb{N}$.

Definition 3.25. We have the following definitions:

(a) The *weighted edge ideal* associated to G_ω is the ideal $I(G_\omega) \subseteq R$ that is "generated by the weighted edges of G":

$$I(G_\omega) = \left(X_i^{\omega(v_i v_j)} X_j^{\omega(v_i v_j)} \mid v_i v_j \in E \right) R.$$

(b) Let $P(V', \delta') \subseteq R$ be the ideal "generated by the elements of (V', δ')":

$$P(V', \delta') = \left(X_i^{\delta'(v_i)} \mid v_i \in V' \right) R.$$

Remark. If $1 : E \to \mathbb{N}$ is the constant function defined by $1(e) = 1$ for $e \in E$, then $I(G_1) = I(G)$.

Example 3.26. Consider the following graph $(\Sigma P_2)_\omega$ as in Example 3.23.

$$\begin{tikzcd} w_1 \arrow[d, dash, "5"] & w_2 \arrow[d, dash, "3"] & w_3 \arrow[d, dash, "4"] \\ v_1 \arrow[r, dash, "2"] & v_2 \arrow[r, dash, "3"] & v_3 \end{tikzcd}$$

The weighted edge ideal associated to $(\Sigma P_2)_\omega$ is

$$I\big((\Sigma P_2)_\omega\big) = \big(X_1^2X_2^2, X_2^3X_3^3, X_1^5Y_1^5, X_2^3Y_2^3, X_3^4Y_3^4\big)R'.$$

Let $V' = \{v_1, w_2, v_3\} \subseteq V(\Sigma P_2)$ and $\delta' : V' \longrightarrow \mathbb{N}$ be defined by $v_1 \longmapsto 3$, $w_2 \longmapsto 2$ and $v_3 \longmapsto 4$. Then

$$P(V', \delta') = \big(X_1^3, Y_2^2, X_3^4\big)R'.$$

Definition 3.27. A *weighted vertex cover* of G_ω is an ordered pair $(V', \delta') \in \Omega$ such that the set V' is a vertex cover of G and for each edge $v_iv_j \in E$, we have

(a) $v_i \in V'$ and $\delta'(v_i) \leq \omega(v_iv_j)$, or

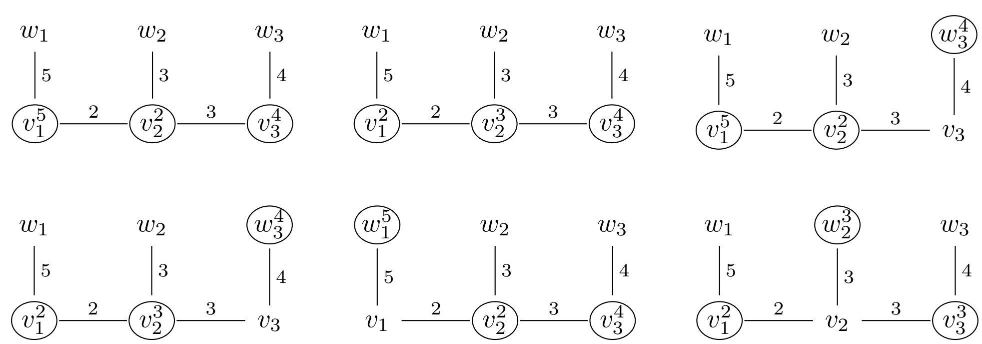
(b) $v_j \in V'$ and $\delta'(v_j) \leq \omega(v_iv_j)$.

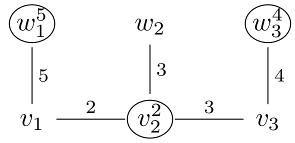
The number $\delta'(v_i)$ is the *weight* of v_i.

Remark. For each weighted vertex cover (V', δ') of G_ω, we also use $\big\{v_i^{\delta'(v_i)} \mid v_i \in V'\big\}$ to denote it, especially when we depict weighted vertex covers of G_ω in sketches.

Definition 3.28. Given two weighted vertex covers (V_1', δ_1') and (V_2', δ_2') of G_ω, we write $(V_2', \delta_2') \leq (V_1', \delta_1')$ if $V_2' \subseteq V_1'$ and $\delta_2'(v_i) \geq \delta_1'(v_i)$ for all $v_i \in V_2'$. A weighted vertex cover (V', δ') is *minimal* if there does not exist another weighted vertex cover (V'', δ'') such that $(V'', \delta'') < (V', \delta')$. We define $|(V', \delta')| = |V'|$.

Example 3.29. The minimal weighted vertex covers of $(\Sigma P_2)_\omega$ as in Example 3.23 are displayed in the following sketches.



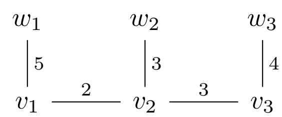


Fact 3.30. [11, Theorem 3.5]

$$I(G_\omega) = \bigcap_{(V',\delta')\text{ w. v. cover}} P(V',\delta') = \bigcap_{(V',\delta')\text{ min. w. v. cover}} P(V',\delta'),$$

where the first intersection is taken over all weighted vertex covers of G_ω, and the second intersection is taken over all minimal weighted vertex covers of G_ω. The second intersection is irredundant.

Example 3.31. Consider the following graph $(\Sigma P_2)_\omega$ as in Example 3.23.



Then by Fact 3.30 and Example 3.29, the irredundant m-irreducible decomposition of $I\big((\Sigma P_2)_\omega\big)$ is given by

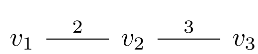
$$\begin{aligned} I\big((\Sigma P_2)_\omega\big) = &\big(X_1^5, X_2^2, X_3^4\big)R' \cap \big(X_1^2, X_2^3, X_3^4\big)R' \cap \big(X_1^5, X_2^2, Y_3^4\big)R' \cap \big(X_1^2, X_2^3, Y_3^4\big)R' \\ &\cap \big(Y_1^5, X_2^2, X_3^4\big)R' \cap \big(X_1^2, Y_2^3, X_3^3\big)R' \cap \big(Y_1^5, X_2^2, Y_3^4\big)R'. \end{aligned}$$

Notation 3.32. For $(V',\delta') \in \Omega$, set

$$\underline{X}^{(V',\delta')} = \prod_{v_i \in V'} X_i^{\delta'(v_i)}.$$

Definition 3.33. $(V',\delta') \in \Omega$ is said to be *weighted-independent* in G_ω if V' is independent in G, or for any adjacent $v_i, v_j \in V'$, we have $\delta'(v_i) < \omega(v_iv_j)$ or $\delta'(v_j) < \omega(v_iv_j)$. Let Δ_{G_ω} denote the set of weighted-independent subsets $(V',\delta') \in \Omega$ in G_ω.

Example 3.34. Consider the following weighted 2-path $(P_2)_\omega$.



Then $\{v_1^{10}, v_3^{10}\}$ is weighted-independent in $(P_2)_\omega$ since $\{v_1, v_3\}$ is independent in P_2, and $\{v_1^1, v_2^2\}$

is weighted-independent in $(P_2)_\omega$ since $\delta'(v_1) = 1 < 2 = \omega(v_1v_2)$ and $\delta'(v_2) = 2 < 3 = \omega(v_2v_3)$.

Lemma 3.35. Let $(V', \delta') \in \Omega$. If $\underline{X}^{(V',\delta')} \in I(G_\omega)$, then $\underline{X}^{V'} \in I(G)$.

Proof. Since $\underline{X}^{(V',\delta')} \in I(G_\omega)$, there exists $v_iv_j \in E$ such that $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \mid \underline{X}^{(V',\delta')}$. Since $v_iv_j \in E$, we have $X_iX_j \in I(G)$. Since $\omega(v_iv_j) \geq 1$, we have $v_i, v_j \in V'$, i.e., $X_iX_j \mid \underline{X}^{V'}$. So $\underline{X}^{V'} \in I(G)$. $\square$

Theorem 3.36. *Let $(V', \delta') \in \Omega$. Then $(V', \delta') \in \Delta_{G_\omega}$ if and only if $\underline{X}^{(V',\delta')} \notin I(G_\omega)$.*

Proof. "$\Rightarrow$". Assume $(V', \delta') \in \Delta_{G_\omega}$. First, we assume V' is independent in G, i.e., $V' \in \Delta_G$. Suppose $\underline{X}^{V'} \in J_{\Delta_G}$, where J_{Δ_G} is the Stanley-Reisner ideal of R associated to the independence complex Δ_G. Then there exists $V'' \subseteq V$ and $V'' \notin \Delta_G$ such that $\underline{X}^{V''} \mid \underline{X}^{V'}$, so $V'' \subseteq V'$, contradicting the fact that Δ_G is a simplicial complex and $V' \in \Delta_G$. Hence $\underline{X}^{V'} \notin J_{\Delta_G} = I(G)$ by Fact 3.11. Thus, $\underline{X}^{(V',\delta')} \notin I(G_\omega)$ by Lemma 3.35. Assume now V' is dependent in G. Fix an adjacent $v_i, v_j \in V'$. Then $\delta'(v_i) < \omega(v_iv_j)$ or $\delta'(v_j) < \omega(v_iv_j)$, so $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \nmid X_i^{\delta'(v_i)} X_j^{\delta'(v_j)}$, hence $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \nmid \underline{X}^{(V',\delta')}$. Since the adjacent $v_i, v_j \in V'$ are arbitrary, $\underline{X}^{(V',\delta')} \notin I(G_\omega)$.

"$\Leftarrow$". Assume $\underline{X}^{(V',\delta')} \notin I(G_\omega)$. If V' is independent in G, we are done. Assume V' is dependent in G. Then we can fix adjacent $v_i, v_j \in V'$. Since $\underline{X}^{(V',\delta')} \notin I(G_\omega)$, we have $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \nmid \underline{X}^{(V',\delta')}$, i.e., $\delta'(v_i) < \omega(v_iv_j)$ or $\delta'(v_j) < \omega(v_iv_j)$. Since the adjacent $v_i, v_j \in V'$ are arbitrary, (V', δ') is weighted-independent in G_ω, i.e., $(V', \delta') \in \Delta_{G_\omega}$. $\square$

Definition 3.37.

$$J_{\Delta_{G_\omega}} = \left(\underline{X}^{(V',\delta')} \mid (V', \delta') \in \Omega \smallsetminus \Delta_{G_\omega}\right) R.$$

Theorem 3.38.

$$I(G_\omega) = J_{\Delta_{G_\omega}}.$$

Proof. "$\supseteq$". Let $\underline{X}^{(V',\delta')} \in J_{\Delta_{G_\omega}}$ be a generator. Then $(V', \delta') \in \Omega \smallsetminus \Delta_{G_\omega}$. So $\underline{X}^{(V',\delta')} \in I(G_\omega)$ by Theorem 3.36.

"$\subseteq$". Let $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \in I(G_\omega)$ with $v_iv_j \in E$. Let $V' = \{v_i, v_j\}$ and define $\delta' : V' \to \mathbb{N}$ by $\delta'(v_i) = \omega(v_iv_j) = \delta'(v_j)$. Then $(V', \delta') \in \Omega$ and $\underline{X}^{(V',\delta')} = X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \in I(G_\omega)$. So $(V', \delta') \in \Omega \smallsetminus \Delta_{G_\omega}$ by Theorem 3.36. Hence $X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} = \underline{X}^{(V',\delta')} \in J_{\Delta_G}$. $\square$

Definition 3.39. A *weighted suspension* of G_ω is a weighted graph $(\Sigma G)_\lambda$ with weight function $\lambda : \Sigma G \to \mathbb{N}$ such that the underlying graph ΣG is a suspension of G and $\lambda(v_iv_j) = \omega(v_iv_j)$ for all

$v_iv_j \in E(G)$, i.e., $\lambda|_{E(G)} = \omega$. Graphically, $(\Sigma G)_\lambda$ has the form

$$\begin{array}{ccccccccc} \cdots & & w_i & & w_j & & w_k & & \cdots \\ & & \Big| \lambda(v_iw_i) & & \Big| \lambda(v_jw_j) & & \Big| \lambda(v_kw_k) & & \\ \cdots & \text{———} & v_i & \underset{\omega(v_iv_j)}{\text{———}} & v_j & \underset{\omega(v_jv_k)}{\text{———}} & v_k & \text{———} & \cdots . \end{array}$$

Fact 3.40. Let $(\Sigma G)_\lambda$ be a weighted suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iw_i)$ and $\lambda(v_iv_j) \leq \lambda(w_jv_j)$ for each $v_iv_j \in E$. Then by [11, Lemma 5.3], $I\big((\Sigma G)_\lambda\big)$ is the polarization of $I(G_\omega) + \big(X_1^{2\lambda(v_1w_1)}, \ldots, X_d^{2\lambda(v_dw_d)}\big)R$. So by Fact 2.89, the list $X_1 - Y_1, \ldots, X_d - Y_d$ is a maximal regular sequence for $\frac{R'}{I((\Sigma G)_\lambda)}$ and

$$\frac{R}{I(G_\omega) + \big(X_1^{2\lambda(v_1w_1)}, \ldots, X_d^{2\lambda(v_dw_d)}\big)R} \cong \frac{R'}{I((\Sigma G)_\lambda) + (X_1 - Y_1, \ldots, X_d - Y_d)R'}.$$

Because of the following fact, the main result of this section gives a formula to compute $\operatorname{type}\big(R/I(G_\omega)\big)$ for all weighted trees such that $R/I(G_\omega)$ is Cohen-Macaulay.

Fact 3.41. [11, Theorems 5.7 and 5.10] Let $(\Sigma G)_\lambda$ be a weighted suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iw_i)$ and $\lambda(v_iv_j) \leq \lambda(w_jv_j)$ for each $v_iv_j \in E$.

(a) $R'/I((\Sigma G)_\lambda)$ is Cohen-Macaulay.

(b) If $\Gamma_{\lambda'}$ is a weighted tree and $R/I(\Gamma_{\lambda'})$ is Cohen-Macaulay, then $\Gamma_{\lambda'} = (\Sigma H)_{\lambda'}$ for some weighted subtree $H_{\omega'}$ and the weight function λ' satisfies the above condition.

Example 3.42. Consider the following weighted 3-path.

$$G_\lambda := (P_3)_\lambda = \Big(\ v_1 \overset{1}{\text{———}}\ v_2 \overset{2}{\text{———}}\ v_3 \overset{2}{\text{———}}\ v_4\ \Big)$$

Then $I(G_\lambda) = (X_1X_2, X_2^2X_3^2, X_3^2X_4^2)R = (X_1, X_3^2)R \cap (X_2, X_3^2)R \cap (X_1, X_2^2, X_4^2)R \cap (X_2, X_4^2)R$ by Fact 3.30. As in Example 3.19, $\overline{X_3 - X_4}$ is regular in $R/I(G_\lambda)$. We simplify the quotient $R/(I(G_\lambda) + (X_3 - X_4)R) \cong R_1/(X_1X_2, X_2^2X_3^2, X_3^4)R_1$, where $R_1 = A[X_1, X_2, X_3]$.

By Fact 2.85, $J := (X_1X_3, X_2^2X_3^2, X_3^4)R_1 = (X_1, X_2^2, X_3^4)R_1 \cap (X_2, X_3^4) \cap (X_1, X_3^2)R_1$. Since $\overline{(X_1, X_2, X_3)R_1}$ is a maximal ideal of R_1/J, $\operatorname{depth}(R/I(G_\lambda)) = 1$ as in Example 3.19. On the other hand, by Fact 2.70, $\dim(R/I(G_\lambda)) = 4 - 2 = 2$. Hence $R/I(G_\lambda)$ is not Cohen-Macaulay.

Observe that $(G)_\lambda$ is a weighted suspension of $P_1 = \big(\ v_2 \text{———}\ v_3\ \big)$, i.e., $(P_3)_\lambda = (\Sigma P_1)_\lambda$ with $(P_1)_\omega = \big(\ v_2 \overset{2}{\text{———}}\ v_3\ \big)$, but we have $\lambda(v_2v_3) > \lambda(v_1v_2)$.

The following theorem is the second main result of this thesis. It is Formula (**) from the abstract.

Theorem 3.43. *Let $(\Sigma G)_\lambda$ be a weighted suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iw_i)$ and $\lambda(v_iv_j) \leq \lambda(w_jv_j)$ for each $v_iv_j \in E$. Then*

$$\operatorname{type}\left(\frac{R'}{I((\Sigma G)_\lambda)}\right) = \sharp\,\{\textit{minimal weighted vertex covers of } G_\omega\}.$$

Proof. We compute

$$\begin{aligned}
\operatorname{type}\left(\frac{R'}{I((\Sigma G)_\lambda)}\right) &= \operatorname{type}\left(\frac{R'}{I((\Sigma G)_\lambda) + (X_1 - Y_1, \ldots, X_d - Y_d)R'}\right) \\
&= \operatorname{type}\left(\frac{R}{I(G_\omega) + \left(X_1^{2\lambda(v_1w_1)}, \ldots, X_d^{2\lambda(v_dw_d)}\right)R}\right) \\
&= \sharp\,\{\text{ideals in an irredundant parametric decomposition of} \\
&\qquad I(G_\omega) + \left(X_1^{2\lambda(v_1w_1)}, \ldots, X_d^{2\lambda(v_dw_d)}\right)\} \\
&= \sharp\,\{\text{ideals in an irredundant m-irreducible decomposition of } I(G_\omega)\} \\
&= \sharp\,\{\text{minimal weighted vertex covers of } G_\omega\},
\end{aligned}$$

where the first equality is from Facts 2.95(a), 3.41(a) and 3.40, the second equality is from Fact 3.40, the third equality is from Fact 2.95(b) since $\dim\left(\frac{R}{I(G_\omega)+(X_1^{2\lambda(v_1w_1)},\ldots,X_d^{2\lambda(v_dw_d)})R}\right) = 0$, the fourth equality is from [10, Exercise 7.5.10], and the last equality is from Fact 3.30. $\square$

Remark. Because of Fact 3.41, we can use Theorem 3.43 to compute $\operatorname{type}\big(R/I(G_\omega)\big)$ for all weighted trees G_ω such that $I(G_\omega)$ is Cohen-Macaulay.

Example 3.44. Consider the following weighted graph $(\Sigma P_2)_\lambda$ as in Example 3.23.

$$\begin{tikzcd}
w_1 \arrow[d, dash, "5"] & w_2 \arrow[d, dash, "3"] & w_3 \arrow[d, dash, "4"] \\
v_1 \arrow[r, dash, "2"] & v_2 \arrow[r, dash, "3"] & v_3
\end{tikzcd}$$

The minimal weighted vertex covers of $(P_2)_\omega = \left(v_1 \overset{2}{\text{———}} v_2 \overset{3}{\text{———}} v_3 \right)$ are displayed in the following sketches.

$$v_1 \overset{2}{\text{———}} v_2^2 \overset{3}{\text{———}} v_3 \qquad v_1^2 \overset{2}{\text{———}} v_2^3 \overset{3}{\text{———}} v_3 \qquad v_1^2 \overset{2}{\text{———}} v_2 \overset{3}{\text{———}} v_3^3$$

Then by Theorem 3.43,

$$\operatorname{type}\bigl(R'/I((\Sigma P_2)_\lambda)\bigr) = \sharp\,\{\text{minimal weighted vertex covers of } (P_2)_\omega\} = 3.$$

We observe that the smallest number of vertices for one of the weighted vertex covers of $(\Sigma P_2)_\lambda$ is 3. Then by Facts 3.30 and 2.70, $\dim\bigl((R'/I((\Sigma P_2)_\lambda)\bigr) = 6 - 3 = 3$. Since $R'/I((\Sigma P_2)_\lambda)$ is Cohen-Macaulay by Fact 3.41(a), $\operatorname{depth}(R'/I((\Sigma P_2)_\lambda)) = \dim\bigl(R'/I((\Sigma P_2)_\lambda)\bigr) = 3$. Hence

$$\operatorname{Ext}^3_{R'}\bigl(A, R'/I((\Sigma P_2)_\lambda)\bigr) \cong A^3.$$

3.4 Path Ideals and the Type of $R'/I_r(\Sigma_r G)$

In this section, we prove a path-version of Theorem 3.20. See Formula (***) from the abstract and Theorem 3.74. Let r be a positive integer and $R' = A\bigl[\{X_{i,j} \mid i = 1,\dots,d, j = 0,\dots,r\}\bigr]$. Let $\mathbb{N} = \{1,2,3,\dots\}$ be the set of positive integers.

Definition 3.45. The *r-path suspension* of G is the graph $\Sigma_r G$ obtained by adding a new path of length r to each vertex of G such that the vertex set

$$V(\Sigma_r G) = \{v_{i,j} \mid i = 1,\dots,d, j = 0,\dots,r\} \text{ with } v_{i,0} = v_i,\ \forall\, i = 1,\dots,d.$$

The new r-paths are called *r-whiskers*.

Example 3.46. The 2-path suspension $\Sigma_2 P_2$ of the 2-path $G = P_2 = \bigl(\ v_1 \ \text{——}\ v_2 \ \text{——}\ v_3\ \bigr)$ is

$$\begin{array}{ccccc} v_1 & \text{——} & v_{1,1} & \text{——} & v_{1,2} \\ | & & & & \\ v_2 & \text{——} & v_{2,1} & \text{——} & v_{2,2} \\ | & & & & \\ v_3 & \text{——} & v_{3,1} & \text{——} & v_{3,2}. \end{array}$$

Definition 3.47. The *r-path ideal* associated to G is the ideal $I_r(G) \subseteq R$ that is "generated by the paths in G of length r":

$$I_r(G) = (X_{i_1} \dots X_{i_{r+1}} \mid v_{i_1} \dots v_{i_{r+1}} \text{ is a path in } G)R'.$$

Remark. $I_1(G) = I(G)$.

Example 3.48. Consider the following graph $\Sigma_2 P_2$ as in Example 3.46.

$$\begin{array}{ccccc} v_1 & \text{---} & v_{1,1} & \text{---} & v_{1,2} \\ \vert & & & & \\ v_2 & \text{---} & v_{2,1} & \text{---} & v_{2,2} \\ \vert & & & & \\ v_3 & \text{---} & v_{3,1} & \text{---} & v_{3,2} \end{array}$$

Then the 2-path ideal of $\Sigma_2 P_2$ is

$$I_2(\Sigma_2 P_2) = \big(X_{1,2}X_{1,1}X_1, X_{1,1}X_1X_2, X_1X_2X_{2,1}, X_1X_2X_3, X_{2,2}X_{2,1}X_2,$$
$${}_{2,1}X_2X_3, X_2X_3X_{3,1}, X_{3,2}X_{3,1}X_3\big)R'.$$

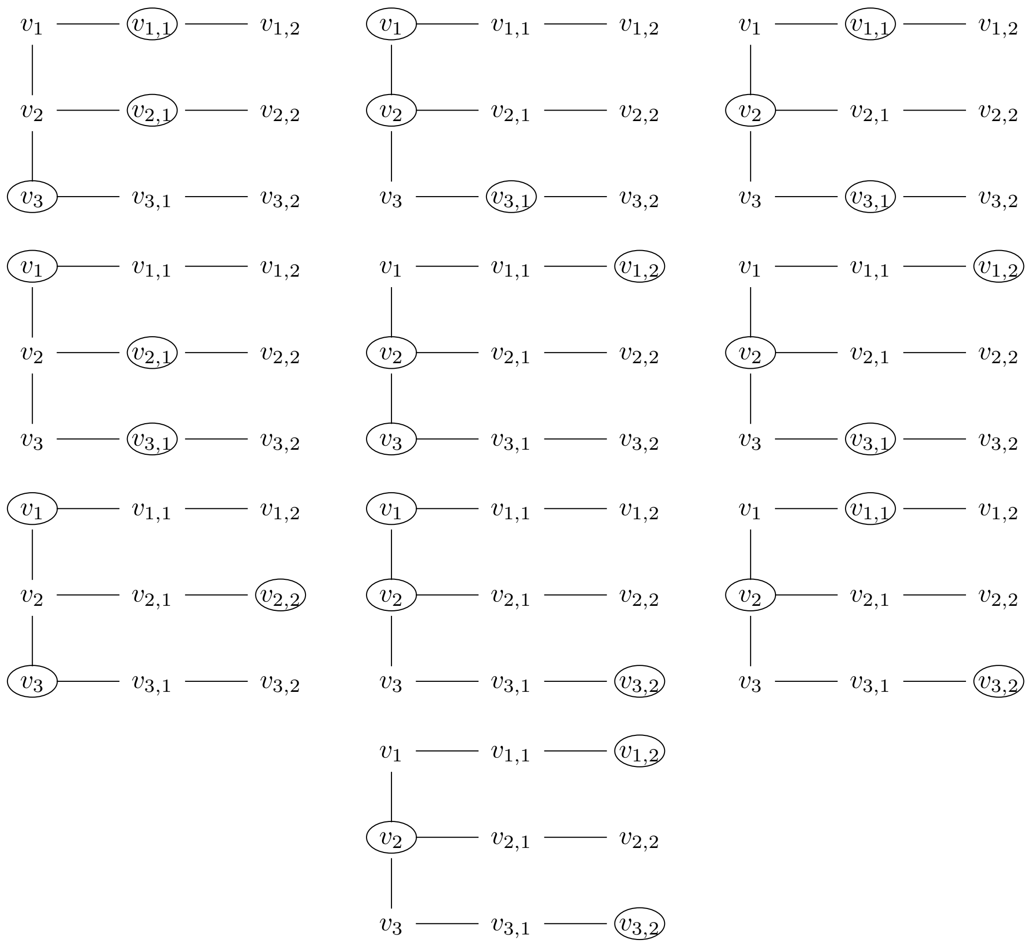
Definition 3.49. An *r-path vertex cover* of G is a subset $V' \subseteq V$ such that for any r-path $v_{i_1} \dots v_{i_{r+1}}$ in G, we have $v_{i_j} \in V'$ for some $j \in \{1, \dots, r+1\}$. An r-path vertex cover V' is *minimal* if it does not properly contain another r-path vertex cover of G.

Example 3.50. Consider the following graph $\Sigma_2 P_2$ as in Example 3.46.

$$\begin{array}{ccccc} v_1 & \text{---} & v_{1,1} & \text{---} & v_{1,2} \\ \vert & & & & \\ v_2 & \text{---} & v_{2,1} & \text{---} & v_{2,2} \\ \vert & & & & \\ v_3 & \text{---} & v_{3,1} & \text{---} & v_{3,2} \end{array}$$

We depict the minimal 2-path vertex covers of $\Sigma_2 P_2$ in the following sketches.

$$\begin{array}{ccccc} v_1 & \text{---} & v_{1,1} & \text{---} & v_{1,2} \\ \vert & & & & \\ v_2 & \text{---} & v_{2,1} & \text{---} & v_{2,2} \\ \vert & & & & \\ v_3 & \text{---} & v_{3,1} & \text{---} & v_{3,2} \end{array} \qquad \begin{array}{ccccc} v_1 & \text{---} & v_{1,1} & \text{---} & v_{1,2} \\ \vert & & & & \\ v_2 & \text{---} & v_{2,1} & \text{---} & v_{2,2} \\ \vert & & & & \\ v_3 & \text{---} & v_{3,1} & \text{---} & v_{3,2} \end{array} \qquad \begin{array}{ccccc} v_1 & \text{---} & v_{1,1} & \text{---} & v_{1,2} \\ \vert & & & & \\ v_2 & \text{---} & v_{2,1} & \text{---} & v_{2,2} \\ \vert & & & & \\ v_3 & \text{---} & v_{3,1} & \text{---} & v_{3,2} \end{array}$$

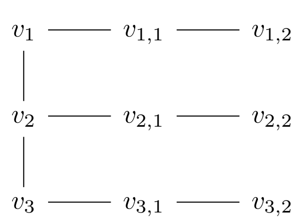


Fact 3.51. [9, Theorem 2.7]

$$I_r(G) = \bigcap_{V' \ r\text{-path v. cover}} P_{V'} = \bigcap_{V' \ \text{min. } r\text{-path v. cover}} P_{V'},$$

where the first intersection is taken over all r-path vertex covers of G, and the second intersection is taken over all minimal r-path vertex covers of G. The second intersection is irredundant.

Example 3.52. Consider the following graph $\Sigma_2 P_2$ as in Example 3.46.



Then by Fact 3.51 and Example 3.50, the irredundant m-irreducible decomposition of $I_2(\Sigma_2 P_2)$ is given by

$$\begin{aligned} I_2(\Sigma_2 P_2) = {} & (X_1, X_2, X_3)R' \cap (X_{1,1}, X_2, X_3)R' \cap (X_1, X_{2,1}, X_3)R' \cap (X_{1,1}, X_{2,1}, X_3)R' \\ & \cap (X_1, X_2, X_{3,1})R' \cap (X_{1,1}, X_2, X_{3,1})R' \cap (X_1, X_{2,1}, X_{3,1})R' \cap (X_{1,2}, X_2, X_3)R' \\ & \cap (X_{1,2}, X_2, X_{3,1})R' \cap (X_1, X_{2,2}, X_3)R' \cap (X_1, X_2, X_{3,2})R' \cap (X_{1,1}, X_2, X_{3,2})R' \\ & \cap (X_{1,2}, X_2, X_{3,2})R'. \end{aligned}$$

Based on the convention that $v_{i,0} = v_i$ for $i = 1, \ldots, d$, we have $X_{i,0} = X_i$ for $i = 1, \ldots, d$.

Definition 3.53. Define a ring homomorphism p by

$$\begin{aligned} p : R' &\longrightarrow R \\ a &\longrightarrow a, \ \forall \ a \in A, \\ X_{ij} &\longmapsto X_i \ \forall \ i = 1, \ldots, d, \ j = 0, \ldots, r. \end{aligned}$$

Let $I \subseteq R'$ be a monomial ideal and set

$$IR = p(I)R = \left(X_{i_1}^{a_1} \ldots X_{i_n}^{a_n} \in R \mid \exists \ X_{i_1,j_1}^{a_1} \cdots X_{i_n,j_n}^{a_n} \in [\![I]\!]\right)R.$$

In words, IR is the monomial ideal of R obtained from I by setting $X_{i,j} = X_i$ for all i, j. It is straightforward to show that if $f_1, \ldots, f_m$ is a monomial generating sequence for I, then $p(f_1), \ldots, p(f_m)$ is a monomial generating sequence for IR.

Example 3.54. Consider the 2-path ideal $I_2(\Sigma_2 P_2)$ from Example 3.48. Then

$$\begin{aligned} I_2(\Sigma_2 P_2)R &= \big(X_1X_1X_1, X_1X_1X_2, X_1X_2X_2, X_1X_2X_3, X_2X_2X_2, \\ & \qquad X_2X_2X_3, X_2X_3X_3, X_3X_3X_3\big)R \\ &= \big(X_1^3, X_1^2X_2, X_1X_2^2, X_1X_2X_3, X_2^3, X_2^3, X_2^2X_3, X_2X_3^2, X_3^3\big)R. \end{aligned}$$

Definition 3.55. Let $I \subseteq R$ be an ideal. For $k = 1, 2, \ldots$, the k^{th} *bracket power* of I is the ideal $I^{[k]} = (T_k)R$, where $T_k = \left\{f^k \mid f \in [\![I]\!]\right\}$.

Fact 3.56. We have $I_r(\Sigma_r G)$ is the polarization of $I_r(\Sigma_r G)R$ by e.g., [9, Proposition 3.7]. So by Fact 2.89, the list $X_i - X_{i,k}, 1 \leq i \leq d, 1 \leq k \leq r$ is a maximal homogeneous regular sequence for $\frac{R'}{I_r(\Sigma_r G)}$ and

$$\frac{R}{I_r(\Sigma_r G)R} \cong \frac{R'}{I_r(\Sigma_r G) + (X_i - X_{i,k} \mid 1 \leq i \leq d, 1 \leq k \leq r)R'}.$$

Furthermore, it is straightforward to show that

$$I_r(\Sigma_r G)R = I_r(\Sigma_{r-1} G)R + \mathfrak{m}^{[r+1]},$$

where $\mathfrak{m} = (X_1, \ldots, X_d)R$.

Example 3.57. By definition, the polarization of $I_2(\Sigma_2 P_2)R$ from Example 3.54 is

$$\begin{aligned}
\mathscr{P}\mathscr{O}(I_2(\Sigma_2 P_2)R) &= \mathscr{P}\mathscr{O}\big((X_1^3, X_1^2 X_2, X_1 X_2^2, X_1 X_2 X_3, X_2^3, X_2^2 X_3, X_2 X_3^2, X_3^3)R\big) \\
&= (X_1 X_{1,1} X_{1,2}, X_1 X_{1,1} X_2, X_1 X_2 X_{2,1}, X_1 X_2 X_3, X_2 X_{2,1} X_{2,2}, \\
&\qquad X_2 X_{2,1} X_3, X_2 X_3 X_{3,1}, X_3 X_{3,1} X_{3,2})\, R' \\
&= I_2(\Sigma_2 P_2),
\end{aligned}$$

where the last equality is from Example 3.48. Note that

$$\begin{aligned}
I_2(\Sigma P_2)R + \mathfrak{m}^{[r+1]} &= \big((X_{1,1} X_1 X_2, X_1 X_2 X_{2,1}, X_1 X_2 X_3, X_{2,1} X_2 X_3, X_2 X_3 X_{3,1})R'\big)R \\
&\quad + (X_1^3, X_2^3, X_3^3)R \\
&= (X_1^2 X_2, X_1 X_2^2, X_1 X_2 X_3, X_2^2 X_3, X_2 X_3^2)R + (X_1^3, X_2^3, X_3^3)R \\
&= (X_1^2 X_2, X_1 X_2^2, X_1 X_2 X_3, X_2^2 X_3, X_2 X_3^2, X_1^3, X_2^3, X_3^3)R \\
&= I_2(\Sigma_2 P_2)R,
\end{aligned}$$

where the last equality is from Example 3.54.

Definition 3.58. Let v_i be a vertex of degree 1 in G that is not a part of any r-path in G. We write that v_i is an *r-pathless leaf* of G. Let H be the subgraph of G induced by the vertex subset $V \smallsetminus \{v_i\}$. We write that H is obtained by *pruning* an r-pathless leaf from G. A subgraph Γ of G is obtained by *pruning a sequence of r-pathless leaves* from G if there exists a sequence of graphs $G = G_0, G_1, \ldots, G_l = \Gamma$ such that each G_{i+1} is obtained by pruning an r-pathless leaf from G_i.

Example 3.59. The vertex v_6 in the following tree G is an 4-pathless leaf, because v_6 is not part of any 4-path in G.

$$\begin{tikzcd}[arrows=dash] v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] \arrow[d] & v_4 \arrow[r] & v_5 \\ & & v_6 & & \end{tikzcd}$$

Pruning this leaf yields the following 4-path, which has no 4-pathless leaves.

$$\begin{tikzcd}[arrows=dash] v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & v_4 \arrow[r] & v_5 \end{tikzcd}$$

Because of the following fact, the main result of this section gives a formula to compute $\operatorname{type}(R/I_r(G))$ for all trees such that $R/I_r(G)$ is Cohen-Macaulay.

Fact 3.60. [9, Proposition 3.7 and Theorem 3.11] We have the following facts:

(a) $R'/I_r(\Sigma_r G)$ is Cohen-Macaulay.

(b) If Γ is a tree and $R/I_r(\Gamma)$ is Cohen-Macaulay, then there exists a subtree H such that $\Sigma_r H$ is obtained by pruning a sequence of r-pathless leaves from Γ.

Example 3.61. Consider the following tree G as in Example 3.59.

$$\begin{tikzcd}[arrows=dash] v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] \arrow[d] & v_4 \arrow[r] & v_5 \\ & & v_6 & & \end{tikzcd}$$

Then $I_4(G) = (X_1X_2X_3X_4X_5)R$. As in Example 3.19, $\overline{X_4 - X_5}, \overline{X_3 - X_4}, \overline{X_2 - X_3}, \overline{X_1 - X_2}$ is a (maximal) regular sequence in $R/I_4(G)$. We simplify the quotient

$$R/(I_4(G) + (X_4 - X_5, X_3 - X_4, X_2 - X_3, X_1 - X_2)R) \cong R_1/(X_1^5)R_1,$$

where $R_1 = A[X_1]$. Since $\overline{X_1}$ is a maximal ideal of $R_1/(X_1^5)R$, $\operatorname{depth}(R/I_4(G)) = 4$. On the other hand, by Fact 2.70, $\dim(R/I(G)) = 5 - 1 = 4$. Hence $R/I(G)$ is Cohen-Macaulay.

Observe that there exists a subtree v_1 of G such that $\Sigma_4 v_1$ is obtained by pruning a 4-pathless leaf v_6 from G:

$$\Sigma_4 v_1 : \begin{tikzcd}[arrows=dash] v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & v_4 \arrow[r] & v_5. \end{tikzcd}$$

Fact 3.62. [9, Lemma 1.11] For every r-path vertex cover V' of G, there is a minimal r-path vertex cover W' of G such that $W' \subseteq V'$.

The main result of this section gives formulas to compute the type of $R'/I_r(\Sigma_r G)$ in terms of minimal r-path vertex covers of $\Sigma_{r-1}G$. Compare this to Theorem 3.20 (the case $r = 1$). Thus, we study $\Sigma_{r-1}G$ before our main theorem.

Definition 3.63. Define $q : V(\Sigma_{r-1}G) \to V(G)$ as $q(v_{i,j}) = v_i$. Let $V'' \subseteq V(\Sigma_{r-1}G)$. Then

$$q(V'') = \{v_i \mid \exists\ v_{i,j} \in V''\},$$

and we set

$$\begin{aligned} \gamma_{V''} : q(V'') &\longrightarrow \mathbb{N} \\ v_i &\longmapsto 1 + \min\{j \mid v_{i,j} \in V''\}. \end{aligned}$$

Example 3.64. Consider the following graph $\Sigma_2 P_2$ as in Example 3.46.

$$\begin{tikzcd}[arrows=-] v_1 \arrow[r] \arrow[d] & v_{1,1} \arrow[r] & v_{1,2} \\ v_2 \arrow[r] \arrow[d] & v_{2,1} \arrow[r] & v_{2,2} \\ v_3 \arrow[r] & v_{3,1} \arrow[r] & v_{3,2} \end{tikzcd}$$

Then $V'' = \{v_1, v_{2,1}, v_{3,2}, v_3\}$ is a 2-path vertex cover of $\Sigma_2 P_2$. We have $q(V'') = \{v_1, v_2, v_3\}$, $\gamma_{V''}(v_1) = 1$, $\gamma_{V''}(v_2) = 2$ and $\gamma_{V''}(v_3) = 1$.

The following theorem is a key for decomposing $I_r(\Sigma_{r-1}G)R$ and hence $I_r(\Sigma_r G)R$. The proof is somewhat technical. The reader may wish to follow the argument with the preceding example.

Theorem 3.65. *Let $V'' \subseteq V(\Sigma_{r-1}G)$. Then $I_r(\Sigma_{r-1}G)R \subseteq P(q(V''), \gamma_{V''})$ if and only if V'' is an r-path vertex cover of $\Sigma_{r-1}G$.*

Proof. "$\Rightarrow$". Assume $I_r(\Sigma_{r-1}G)R \subseteq P(q(V''), \gamma_{V''})$. Let $P_r := v_{p_1,q_1} \ldots v_{p_{r+1},q_{r+1}}$ be an r-path in $\Sigma_{r-1}G$. Then $X_{p_1} \ldots X_{p_{r+1}} \in [\![I_r(\Sigma_{r-1}G)R]\!] \subseteq [\![P(q(V''), \gamma_{V''})]\!]$. So

$$X_{i_0}^{\gamma_{V''}(v_{i_0})} \mid X_{p_1} \ldots X_{p_{r+1}} \text{ for some } v_{i_0} \in q(V'').$$

Hence $v_{i_0} = v_{p_l}$ for some $l \in \{1, \ldots, r+1\}$ and

$$\gamma_{V''}(v_{i_0}) \leq \sharp\{\text{times } i_0 \text{ occurs in the list } p_1, \ldots, p_{r+1}\}.$$

So $v_{p_l} = v_{i_0} \in q(V'')$. Hence $j_0 := \min\{t \mid v_{i_0,t} \in V''\}$ is well-defined. Note that $j_0 \in \{0, \ldots, r-1\}$, $v_{i_0,j_0} \in V''$ and

$$1 + j_0 = \gamma_{V''}(v_{i_0}) \leq \sharp\{\text{times } i_0 \text{ occurs in the list } p_1, \ldots, p_{r+1}\}. \tag{3.65.1}$$

Since P_r is an r-path in $\Sigma_{r-1}G$, P_r is of the following form.

$$\begin{array}{ccccccc}
v_{p_{1+q_1},0} & & & & & & \\
\| & & & & & & \\
v_{p_1,0} & \text{------} & v_{p_1,1} & \text{------} & \cdots & \text{------} & v_{p_1,q_1} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
v_{p_{r+1},0} & \text{------} & v_{p_{r+1},1} & \text{------} & \cdots & \text{------} & v_{p_{r+1},q_{r+1}} \\
\| & & & & & & \\
v_{p_{1+r-q_{r+1}},0} & & & & & &
\end{array}$$

where q_1 or q_{r+1} may be 0. Let $M_0 := \max_{1\leq k\leq r+1}\{q_k \mid i_0 = p_k\}$. Then

$$M_0 = \begin{cases} q_1 & \text{if } i_0 = p_1, \\ q_{r+1} & \text{if } i_0 = p_{r+1}. \end{cases}$$

Observe that

$$1 + j_0 \leq \sharp\{\text{times } i_0 \text{ occurs in the list } p_1, \ldots, p_{r+1}\} = 1 + M_0, \text{ i.e., } j_0 \leq M_0,$$

and there must exist a sub-path of P_r of the form

$$v_{i_0,0} \text{ ------ } v_{i_0,1} \text{ ------ } \cdots \text{ ------ } v_{i_0,M_0}.$$

Since $0 \leq j_0 \leq M_0$, there exists a vertex in this path of the form $v_{i_0,j_0} = v_{p_k,q_k}$ for some k in

$\{1, \ldots, r+1\}$. So $v_{p_k,q_k} = v_{i_0,j_0} \in V''$. Thus, V'' is an r-path vertex cover of $\Sigma_{r-1}G$.

"$\Leftarrow$". Assume V'' is an r-path vertex cover of $\Sigma_{r-1}G$. We need to show every monomial generator of $I_r(\Sigma_{r-1}G)R$ is in $P\big(q(V''), \gamma_{V''}\big)$. Let $\underline{X}^{\underline{b}} := X_{i_1} \cdots X_{i_{r+1}}$ be such a generator corresponding to an r-path $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $\Sigma_{r-1}G$. We need to show $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{V''}\big)$. Note that $X_{i_1,j_1} \cdots X_{i_{r+1},j_{r+1}}$ is of the following form.

$$
\begin{array}{ccccccc}
X_{i_{1+j_1},0} & & & & & & \\
\| & & & & & & \\
X_{i_1,0} & \text{———} & X_{i_1,1} & \text{———} & \cdots & \text{———} & X_{i_1,j_1} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
X_{i_{r+1},0} & \text{———} & X_{i_{r+1},1} & \text{———} & \cdots & \text{———} & X_{i_{r+1},j_{r+1}} \\
\| & & & & & & \\
X_{i_{1+r-j_{r+1}},0} & & & & & &
\end{array}
$$

where j_1 or j_{r+1} may be 0. So

$$j_k + 1 \leq \sharp\,\{\text{times } i_k \text{ occurs in the list } i_1, \ldots, i_{r+1}\} = b_{i_k}, \ \forall\ k = 1, \ldots, r+1.$$

Since $v_{i_1,j_1} \ldots v_{i_{r+1},j_{r+1}}$ is an r-path in $\Sigma_{r-1}G$ and V'' is an r-path vertex cover of $\Sigma_{r-1}G$, we have $v_{i_l,j_l} \in V''$ for some $l \in \{1, \ldots, r+1\}$. Since $\underline{X}^{\underline{b}} = X_{i_1} \ldots X_{i_{r+1}}$,

$$\gamma_{V''}(v_{i_l}) = 1 + \min\{j \mid v_{i_l,j} \in V''\} \leq 1 + j_l \leq \sharp\,\{\text{times } i_l \text{ occurs in the list } i_1, \ldots, i_{r+1}\} = b_{i_l}.$$

So $X_{i_l}^{\gamma_{V''}(v_{i_l})} \mid \underline{X}^{\underline{b}}$. Hence $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{V''}\big)$. $\square$

The next result gives our first decomposition needed for computing $\operatorname{type}\big(R'/I_r(\Sigma_r G)\big)$.

Theorem 3.66. *One has*

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V''\ r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big),$$

and

$$I_r(\Sigma_r G)R = \bigcap_{V''\ r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) + \mathfrak{m}^{[r+1]}.$$

Proof. Since $I_r(\Sigma_r G)R = I_r(\Sigma_{r-1}G)R + \mathfrak{m}^{[r+1]}$ by Fact 3.56 and the power of each variable in each generator of $I_r(\Sigma_{r-1}G)R$ is $\leq r$, by [10, Theorem 7.5.3], it is enough to show that

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V'' \text{ } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big).$$

By Fact 2.85, the monomial ideal $I_r(\Sigma_{r-1}G)R$ can be written as a finite intersection of m-irreducible ideals, i.e., ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{V''}\big)$ with $V'' \subseteq V(\Sigma_{r-1}G)$ such that $\gamma_{V''}(v_{i_j}) = 1 + \min\{k \mid v_{i_j,k} \in V''\}$ for $j = 1, \ldots, t$. Then by Theorem 3.65,

$$\begin{aligned} I_r(\Sigma_{r-1}G)R &\subseteq \bigcap_{V'' \text{ } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) \\ &\subseteq \bigcap_{V'' \text{ } r\text{-path v. cover of } \Sigma_{r-1}G \text{ in the decomp. of } I_r(\Sigma_{r-1}G)R} P\big(q(V''), \gamma_{V''}\big) \\ &= I_r(\Sigma_{r-1}G)R. \end{aligned}$$

So

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V'' \text{ } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big). \qquad \square$$

The next result is key for our second decomposition result, Corollary 3.68.

Lemma 3.67. Let $V_1'', V_2'' \subseteq V(\Sigma_{r-1}G)$. If $V_1'' \subseteq V_2''$, then $P\big(q(V_1''), \gamma_{V_1''}\big) \subseteq P\big(q(V_2''), \gamma_{V_2''}\big)$.

Proof. Let $X_i^{\gamma_{V_1''}(v_i)} \in P\big(q(V_1''), \gamma_{V_1''}\big)$. Then $X_i^{\gamma_{V_2''}(v_i)} \in P\big(q(V_2''), \gamma_{V_2''}\big)$ and $\gamma_{V_1''}(v_i) = \min\{j \mid v_{i,j} \in V_1''\} \geq \min\{j \mid v_{i,j} \in V_2''\} = \gamma_{V_2''}(v_i)$. So $X_i^{\gamma_{V_2''}(v_i)} \mid X_i^{\gamma_{V_1''}(v_i)}$. Hence $P\big(q(V_1''), \gamma_{V_1''}\big) \subseteq P\big(q(V_2''), \gamma_{V_2''}\big)$. $\square$

Here is our second decomposition result for computing $\operatorname{type}\big(R'/I_r(\Sigma_r G)\big)$.

Corollary 3.68. One has

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V'' \text{ min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big),$$

and

$$I_r(\Sigma_r G)R = \bigcap_{V'' \text{ min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) + \mathfrak{m}^{[r+1]}.$$

Proof. By Fact 3.56 and [10, Theorem 7.5.3], it is enough to prove that

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V'' \text{ min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big).$$

By Theorem 3.66, it is enough to show that

$$\bigcap_{V'' \ r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) = \bigcap_{V'' \text{ min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big).$$

"$\subseteq$" follows because every minimal r-path vertex cover is an r-path vertex cover.

"$\supseteq$" follows from Fact 3.62 and Lemma 3.67. □

The decomposition in Corollary 3.68 may be redundant (See Example 3.76). So we define another order from which we can produce an irredundant decomposition. Lemma 3.72 is the key for understanding how this ordering helps with irredundancy.

Definition 3.69. Given two minimal r-path vertex covers V_1', V_2' of $\Sigma_{r-1}G$, we write $V_1' \leq_{\not p} V_2'$ if $q(V_1') \subseteq q(V_2')$ and $\gamma_{V_1'} \geq \gamma_{V_2'}|_{q(V_1')}$. A minimal r-path vertex cover V' is *$\not p$-minimal* if there is not another r-path vertex cover W' such that $W' <_{\not p} V'$.

Lemma 3.70. Let W', W'' be two minimal r-path vertex covers of $\Sigma_{r-1}G$ such that $W'' \leq_{\not p} W'$, then $|W''| = |W'|$ and $q(W'') = q(W')$.

Proof. Since W' is a minimal r-path vertex cover of $\Sigma_{r-1}G$, for distinct $v_{i_1,j_1}, v_{i_2,j_2} \in W'$, we have $i_1 \neq i_2$. Also, since $q(W'') \subseteq q(W')$, $|W''| = |q(W'')| \leq |q(W')| = |W'|$. Suppose $|W''| < |W'|$. Then there exists $v_{i,j} \in W'$ such that $v_i \notin q(W'')$. Since W' is a minimal r-path vertex cover of $\Sigma_{r-1}G$, there is an r-path P_r in $\Sigma_{r-1}G$ that can only be covered by $v_{i,j}$. By assumption, P_r can be covered by some $v_{k,l} \in W''$, so $v_k \in q(W'')$. Also, since $v_i \notin q(W'')$, we have $k \neq i$. Since $\gamma_{W'}(v_k) \leq \gamma_{W''}(v_k)$, we have $v_{k,t} \in W'$ for some $t = \gamma_{W'}(v_k) \leq \gamma_{W''}(v_k) = l$. Note that P_r can also be covered by $v_{k,t} \in W'$, satisfies the above condition.a contradiction. Hence $|W''| = |W'|$ and thus $|q(W'')| = |q(W')|$. Since $q(W'') \subseteq q(W')$, we have $q(W'') \subseteq q(W')$. □

The next two results are key for our third and final decomposition result.

Proposition 3.71. For every minimal r-path vertex cover W' of $\Sigma_{r-1}G$, there is a $\not p$-minimal r-path vertex cover W'' of $\Sigma_{r-1}G$ such that $W'' \leq_{\not p} W'$.

Proof. If W' is itself a $\mathfrak{p}$-minimal r-path vertex cover for $\Sigma_{r-1}G$, then we are done. If W' is not $\mathfrak{p}$-minimal, then by Lemma 3.70, the size of $q(W')$ cannot be decreased, so for some $v_i \in q(W')$, the function $\gamma_{W'}(v_i)$ can be increased, which is done by increasing the second index of vertices of the form $v_{i,j}$ in W'. We increase $\gamma_{W'}(v_i)$ for each $v_i \in q(W')$ such that any further increase would cause the set not to be an r-path vertex cover. This process terminates in finitely many steps because $\gamma_{W'}(v_i) \leq r+1$ for each $v_i \in q(W')$. Denote the new set W''. Then W'' is minimal since the size of W'' cannot be decreased by Lemma 3.70. Thus, by construction, W'' is a $\mathfrak{p}$-minimal r-path vertex cover for $\Sigma_{r-1}G$ such that $W'' \leq_{\mathfrak{p}} W'$. $\square$

Lemma 3.72. Let V_1', V_2' be two minimal r-path vertex covers of $\Sigma_{r-1}G$. Then $V_1' \leq_{\mathfrak{p}} V_2'$ if and only if $P\big(q(V_1'), \gamma_{V_1'}\big) \subseteq P\big(q(V_2'), \gamma_{V_2'}\big)$.

Proof. $V_1' \leq_{\mathfrak{p}} V_2'$ if and only if $q(V_1') \subseteq q(V_2')$ and $\gamma_{V_1'}|_{q(V_1')} \geq \gamma_{V_2'}|_{q(V_1')}$ if and only if $P\big(q(V_1'), \gamma_{V_1'}\big) \subseteq P\big(q(V_2'), \gamma_{V_2'}\big)$. $\square$

Next, we present our third and final decomposition result which will yield the type computation in Theorem 3.74.

Theorem 3.73. *One has an irredundant parametric decomposition*

$$I_r(\Sigma_r G)R = \bigcap_{V'' \ \mathfrak{p}\text{-min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) + \mathfrak{m}^{[r+1]}.$$

Proof. By Fact 3.56 and [10, Theorem 7.5.3], to verify this result, it is enough to show that we have an irredundant decomposition

$$I_r(\Sigma_{r-1}G)R = \bigcap_{V'' \ \mathfrak{p}\text{-min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big).$$

Lemma 3.72 shows that this intersection is irredundant. So by Corollary 3.68, it is enough to show that

$$\bigcap_{V'' \text{ min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big) = \bigcap_{V'' \ \mathfrak{p}\text{-min. } r\text{-path v. cover of } \Sigma_{r-1}G} P\big(q(V''), \gamma_{V''}\big).$$

"$\subseteq$" follows as every $\mathfrak{p}$-minimal r-path vertex cover is a minimal r-path vertex cover.

"$\supseteq$" follows from Proposition 3.71 and Lemma 3.72. $\square$

The next theorem is the third main result of this thesis. It is Formula (***) from the abstract.

Theorem 3.74.

$$\operatorname{type}\left(\frac{R'}{I_r(\Sigma_r G)}\right) = \sharp\{\not p\text{-}\textit{minimal } r\textit{-path vertex covers of } \Sigma_{r-1}G\}.$$

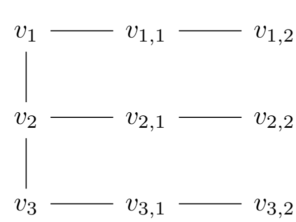
Proof. We compute

$$\begin{aligned}
\operatorname{type}\left(\frac{R'}{I_r(\Sigma_r G)}\right) &= \operatorname{type}\left(\frac{R'}{I_r(\Sigma_r G) + (X_i - X_{i,k} \mid 1 \leq i \leq d, 1 \leq k \leq r)R'}\right) \\
&= \operatorname{type}\left(\frac{R}{I_r(\Sigma_r G)R}\right) \\
&= \sharp\{\text{ideals in an irredundant parametric decomposition of } I_r(\Sigma_r G)R\} \\
&= \sharp\{\not p\text{-minimal } r\text{-path vertex covers of } \Sigma_{r-1}G\},
\end{aligned}$$

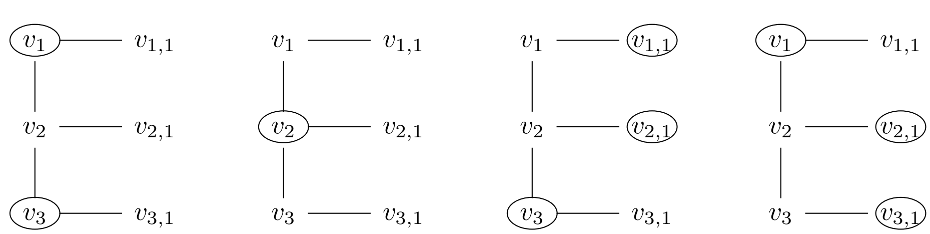
where the first equality is from Facts 2.95(a), 3.60(a) and 3.56, the second equality is from Fact 3.56, the third equality is from Fact 2.95(b) since $\dim\left(\frac{R}{I_r(\Sigma_r G)R}\right) = 0$, and the last equality is from Fact 3.73. ☐

Remark. Because of Fact 3.60, we can use Theorem 3.74 to compute $\operatorname{type}\big(R/I_r(G)\big)$ for all trees G such that $I_r(G)$ is Cohen-Macaulay.

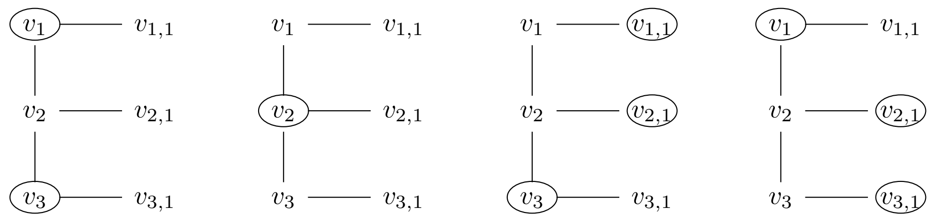
Example 3.75. Consider the following graph $\Sigma_2 P_2$ as in Example 3.46.



We depict the minimal 2-path vertex covers of $\Sigma_1 P_2 = \Sigma P_2$ in the following sketches.



It is straightforward to show that these are all $\mathcal{P}$-minimal, i.e., the $\mathcal{P}$-minimal 2-path vertex covers of ΣP_2 are the following.



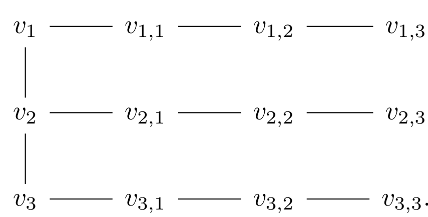
So by Theorem 3.74,

$$\operatorname{type}\big(R'/I_2(\Sigma_2 P_2)\big) = 4.$$

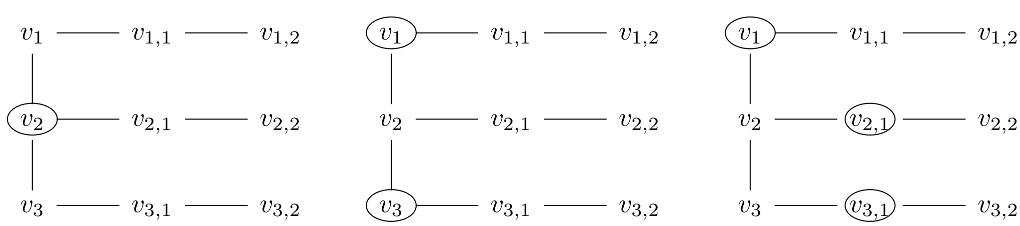
We observe that the smallest number of vertices for one of the 2-path vertex covers of $\Sigma_2 P_2$ is 3. Then by Facts 3.51 and 2.70, $\dim\big(R'/I_2(\Sigma_2 P_2)\big) = 9 - 3 = 6$. Since $R'/I_2(\Sigma_2 P_2)$ is Cohen-Macaulay by Fact 3.60(a), $\operatorname{depth}\big(R'/I_2(\Sigma_2 P_2)\big) = \dim\big(R'/I_2(\Sigma_2 P_2)\big) = 6$. Hence

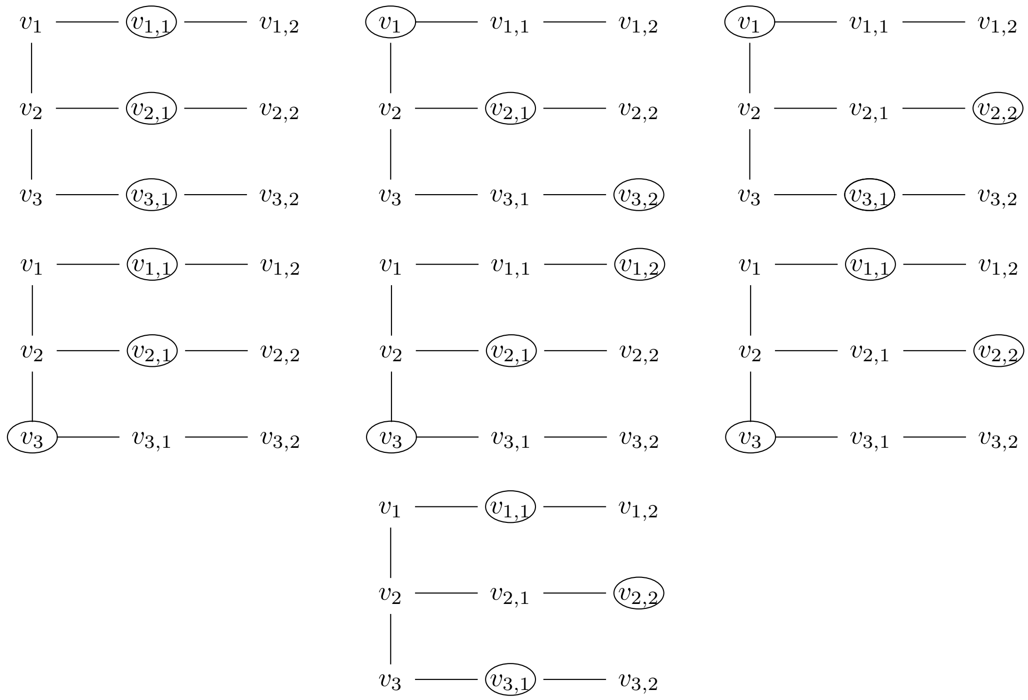
$$\operatorname{Ext}^6_{R'}\big(A, R'/I_2(\Sigma_2 P_2)\big) \cong A^4.$$

Example 3.76. Consider the following graph $\Sigma_3 P_2$ with $P_2 = \big(\, v_1 \text{ ——— } v_2 \text{ ——— } v_3 \,\big)$ is



We depict the minimal 3-path vertex covers of $\Sigma_2 P_2$ in the following sketches.

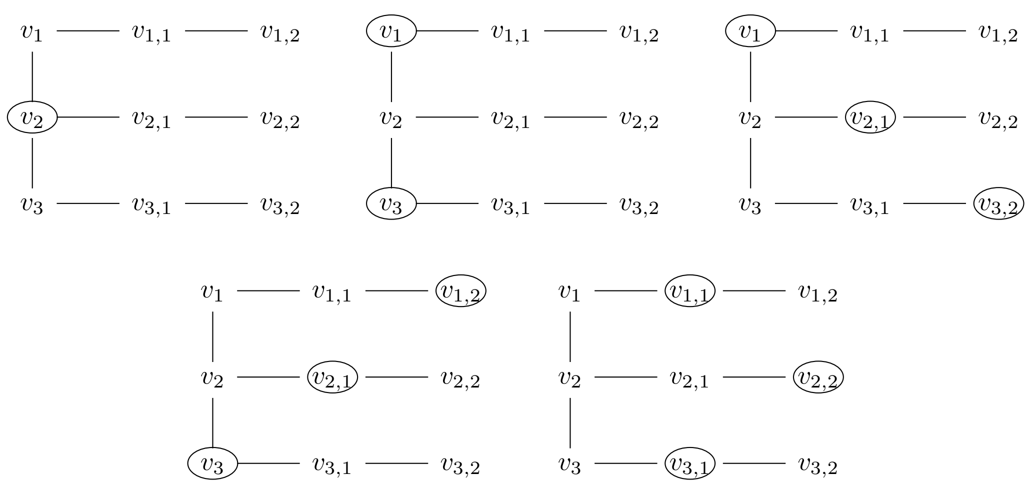




So by Corollary 3.68, we have a parametric decomposition

$$\begin{aligned}I_3(\Sigma_3 P_2)R = (X_2)R &\cap (X_1, X_3)R \cap (X_1, X_2^2, X_3^2)R \cap (X_1^2, X_2^2, X_3^2)R \cap (X_1, X_2^2, X_3^3)R \\ &\cap (X_1, X_2^3, X_3^2)R \cap (X_1^2, X_2^2, X_3)R \cap (X_1^3, X_2^2, X_3)R \cap (X_1^2, X_2^3, X_3)R \\ &\cap (X_1^2, X_2^3, X_3^2)R + \mathfrak{m}^3,\end{aligned}$$

which is a redundant decomposition since e.g., the last ideal $(X_1^2, X_2^3, X_3^2)R$ is contained in the second to last ideal $(X_1^2, X_2^3, X_3)R$. Note the $\wp$-minimal 3-path vertex covers of $\Sigma_2 P_2$ are the following.



So by Theorem 3.73 and we have an irredundant parametric decomposition

$$I_3(\Sigma_3 P_2)R = \big[(X_2)R \cap (X_1, X_3)R \cap (X_1, X_2^2, X_3^3)R \cap (X_1^3, X_2^2, X_3)R \cap (X_1^2, X_2^3, X_3^2)R\big] + \mathfrak{m}^3,$$

and by Theorem 3.74, we have

$$\operatorname{type}\big(R'/I_3(\Sigma_3 P_2)\big) = 5.$$

We observe that the smallest number of vertices for one of the 3-path vertex covers of $\Sigma_3 P_2$ is 3. Then by Facts 3.51 and 2.70, $\dim\big(R'/I_3(\Sigma_3 P_2)\big) = 12-3 = 9$. Since $R'/I_3(\Sigma_3 P_2)$ is Cohen-Macaulay by Fact 3.60(a), $\operatorname{depth}(R'/I_3(\Sigma_3 P_2)) = \dim\big(R'/I_3(\Sigma_3 P_2)\big) = 9$. Hence

$$\operatorname{Ext}^9_{R'}\big(A, R'/I_3(\Sigma_3 P_2)\big) \cong A^5.$$

3.5 f-weighted Path Ideals and the type of $I_{r,f}\big((\Sigma_r G)_\lambda\big)$

Let $r \geq 2$ be a positive integer and $R' = A\big[\{X_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\}\big]$. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For example, f may be max, min, gcd, or lcm, etc.

Definition 3.77. A *weighted r-path suspension* of G_ω is a weighted graph $(\Sigma_r G)_\lambda$ with weight function $\lambda : \Sigma_r G \to \mathbb{N}$ such that the underlying graph $\Sigma_r G$ is a r-path suspension of G and $\lambda(v_i v_j) = \omega(v_i v_j)$ for all $v_i v_j \in E(G)$, i.e., $\lambda|_{E(G)} = \omega$.

Example 3.78. A weighted 2-path suspension $(\Sigma_2 P_2)_\lambda$ of $G_\omega := (P_2)_\omega = \Big(\begin{tikzcd} v_1 \arrow[r, dash, "1"] & v_2 \arrow[r, dash, "2"] & v_3 \end{tikzcd} \Big)$ is

$$\begin{tikzcd}
v_1 \arrow[r, dash, "4"] \arrow[d, dash, "1"'] & v_{1,1} \arrow[r, dash, "3"] & v_{1,2} \\
v_2 \arrow[r, dash, "3"] \arrow[d, dash, "2"'] & v_{2,1} \arrow[r, dash, "3"] & v_{2,2} \\
v_3 \arrow[r, dash, "2"] & v_{3,1} \arrow[r, dash, "5"] & v_{3,2}.
\end{tikzcd}$$

Definition 3.79. Let Ω consist of the pairs (V', δ') with $V' \subseteq V$ and $\delta' : V' \to \mathbb{N}$.

Definition 3.80. We have the following definitions:

(a) The *f-weighted r-path ideal* associated to G_ω is the ideal $I_{r,f}(G_\omega) \subseteq R$ that is "generated by the f-weighted paths in G of length r":

$$I_{r,f}(G_\omega) = \left(X_{i_1}^{e_{i_1}} \dots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \dots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1} v_{i_2}), \\ e_{i_j} = f\{\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})\} \text{ for } 1 < j \leq r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r} v_{i_{r+1}}) \end{array} \right) R.$$

(b) Let $P(V', \delta') \subseteq R$ be the ideal "generated by the elements of $(V', \delta') \in \Omega$":

$$P(V', \delta') = \left(X_i^{\delta'(v_i)} \mid v_i \in V' \right) R.$$

Remark. When $f = \max$, we write that $I_r(G_\omega) := I_{r,f}(G_\omega)$ is the *weighted r-path ideal* associated to G_ω.

Definition 3.81. An *f-weighted r-path vertex cover* of G_ω is an ordered pair $(V', \delta') \in \Omega$ such that V' is an r-path vertex cover of G and such that for any r-path $v_{i_1} \dots v_{i_{r+1}}$ in G at least one of the following holds:

(a) $\delta'(v_{i_1}) \leq \omega(v_{i_1} v_{i_2})$;

(b) $\delta'(v_{i_{r+1}}) \leq \omega(v_{i_r} v_{i_{r+1}})$; or

(c) $\delta'(v_{i_j}) \leq f\big\{\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j} v_{i_{j+1}})\big\}$ for some $j \in \{2, \dots, r\}$.

The number $\delta'(v_{i_j})$ is the *weight* of v_{i_j}.

Remark. When $f = \max$, we write that (V', δ') is a *weighted r-path vertex cover* of G_ω.

Remark. For each f-weighted r-path vertex cover (V', δ') of G_ω, we also use $\big\{v_i^{\delta'(v_i)} \mid v_i \in V'\big\}$ to denote it, especially when we depict f-weighted r-path vertex covers of G_ω in sketches.

Definition 3.82. Given two f-weighted r-path vertex covers (V_1', δ_1') and (V_2', δ_2') of G_ω, we write $(V_2', \delta_2') \leq (V_1', \delta_1')$ if $V_2' \subseteq V_1'$ and $\delta_2'(v_i) \geq \delta_1'(v_i)$ for all $v_i \in V_2'$. An f-weighted r-path vertex cover (V', δ') is *minimal* if there does not exist another f-weighted r-path vertex cover (V'', δ'') such that $(V'', \delta'') < (V', \delta')$. An f-weighted r-path vertex cover (V', δ') of G_ω is *size-minimal* if $(V' \smallsetminus \{v_i\}, \delta'|_{V' \smallsetminus \{v_i\}})$ is not an f-weighted r-path vertex cover for any $v_i \in V'$. We define $|(V', \delta')| = |V'|$.

Fact 3.83. [9, Lemma 1.11] For every f-weighted r-path vertex cover (V', δ') of G_ω, there is a minimal f-weighted r-path vertex cover (V'', δ'') of G_ω such that $(V'', \delta'') \leq (V', \delta')$.

Fact 3.84. [9, Theorem 2.7]

$$I_{r,f}(G_\omega) = \bigcap_{(V',\delta')\ f\text{-w. } r\text{-path v. cover}} P(V', \delta') = \bigcap_{(V',\delta')\ \text{min. } f\text{-w. } r\text{-path v. cover}} P(V', \delta'),$$

where the first intersection is taken over all f-weighted r-path vertex covers of G_ω, and the second intersection is taken over all minimal f-weighted r-path vertex covers of G_ω. The second intersection is irredundant.

Theorem 3.85.

$$I_{r,f}(G_\omega) = \bigcap_{(V',\delta')\ \textit{size-min. } f\textit{-w. } r\textit{-path v. cover}} P(V', \delta'),$$

where the intersection is taken over all size-minimal f-weighted r-path vertex covers of G_ω.

Proof. It is similar to the proof of [9, Theorem 2.7]. □

Fact 3.86. [9, Lemma 2.11] If $I_{r,f}(G_\omega)$ is m-unmixed, then $I_r(G)$ is also m-unmixed.

Definition 3.87. Let v_i be a vertex of degree 1 in G that is not a part of any r-path in G. We write that v_i is an *r-pathless leaf* of G_ω. Let H_λ be the subgraph of G_ω induced by the vertex subset $V \setminus \{v_i\}$. We write that H_λ is obtained by *pruning* an r-pathless leaf from G_ω. A subgraph $\Gamma_{\lambda'}$ of G_ω is obtained by *pruning a sequence of r-pathless leaves* from G_ω if there exists a sequence of graphs $G_\omega = G^{(0)}_{\omega^{(0)}}, G^{(1)}_{\omega^{(1)}}, \dots, G^{(l)}_{\omega^{(l)}} = \Gamma_{\lambda'}$ such that each $G^{(i+1)}_{\omega^{(i+1)}}$ is obtained by pruning an r-pathless leaf from $G^{(i)}_{\omega^{(i)}}$.

Fact 3.88. [9, Lemma 3.3] Let H_λ be a weighted graph obtained by pruning a single r-pathless leaf v_i from G_ω.

(a) The set of r-paths in G is the same as the set of r-paths in H.

(b) The minimal f-weighted r-path vertex covers of G_ω are the same as the minimal f-weighted r-path vertex covers of H_λ.

Lemma 3.89. Let H_λ be a weighted graph obtained by pruning a sequence of r-pathless leaves from G_ω.

(a) The ideals $I_{r,f}(G_\omega)$ and $I_{r,f}(H_\lambda)$ have the same generators.

(b) The ideal $I_{r,f}(G_\omega)$ is m-unmixed if and only if $I_{r,f}(H_\lambda)$ is so.

(c) The ideal $I_{r,f}(G_\omega)$ is Cohen-Macaulay if and only if $I_{r,f}(H_\lambda)$ is so.

Proof. (a) By Fact 3.88(a), the set of r-paths in G is the same as the set of r-paths in H, and $\lambda(e) = \omega(e)$ for each edge $e \in E(H) \subseteq E(G)$. Then the claim about the generators now follows directly.

(b) It follows from Theorem 3.84 and Lemma 3.88(b).

(c) Part (a) implies that $(S'/I_{r,f}(H_\lambda))[X] \cong R/I_{r,f}(G_\omega)$, where $S' = A[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d]$. It follows that $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay if and only if $S'/I_{r,f}(H_\lambda)$ is so. $\square$

3.5.1 Weighted Path Ideals and the Type of $I_r((\Sigma_r G)_\lambda)$

In this subsection, we prove a weighted version of Theorem 3.74 or a path-version of Theorem 3.43. See Formula (****) from the abstract and Theorem 3.122.

Definition 3.90. The *weighted r-path ideal* associated to G_ω is the ideal $I_r(G_\omega) := I_{r,\max}(G_\omega) \subseteq R$ that is "generated by the max-weighted paths in G of length r":

$$I_r(G_\omega) = \left(X_{i_1}^{e_{i_1}} \ldots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \ldots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1} v_{i_2}), \\ e_{i_j} = \max\{\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})\} \text{ for } 1 < j \leq r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r} v_{i_{r+1}}) \end{array} \right) R.$$

Remark. (a) $I_r(G_1) = I_r(G)$, where $1 : E \to \mathbb{N}$ is the constant function $1(e) = 1$.

(b) $I_1(G_\omega) = I(G_\omega)$.

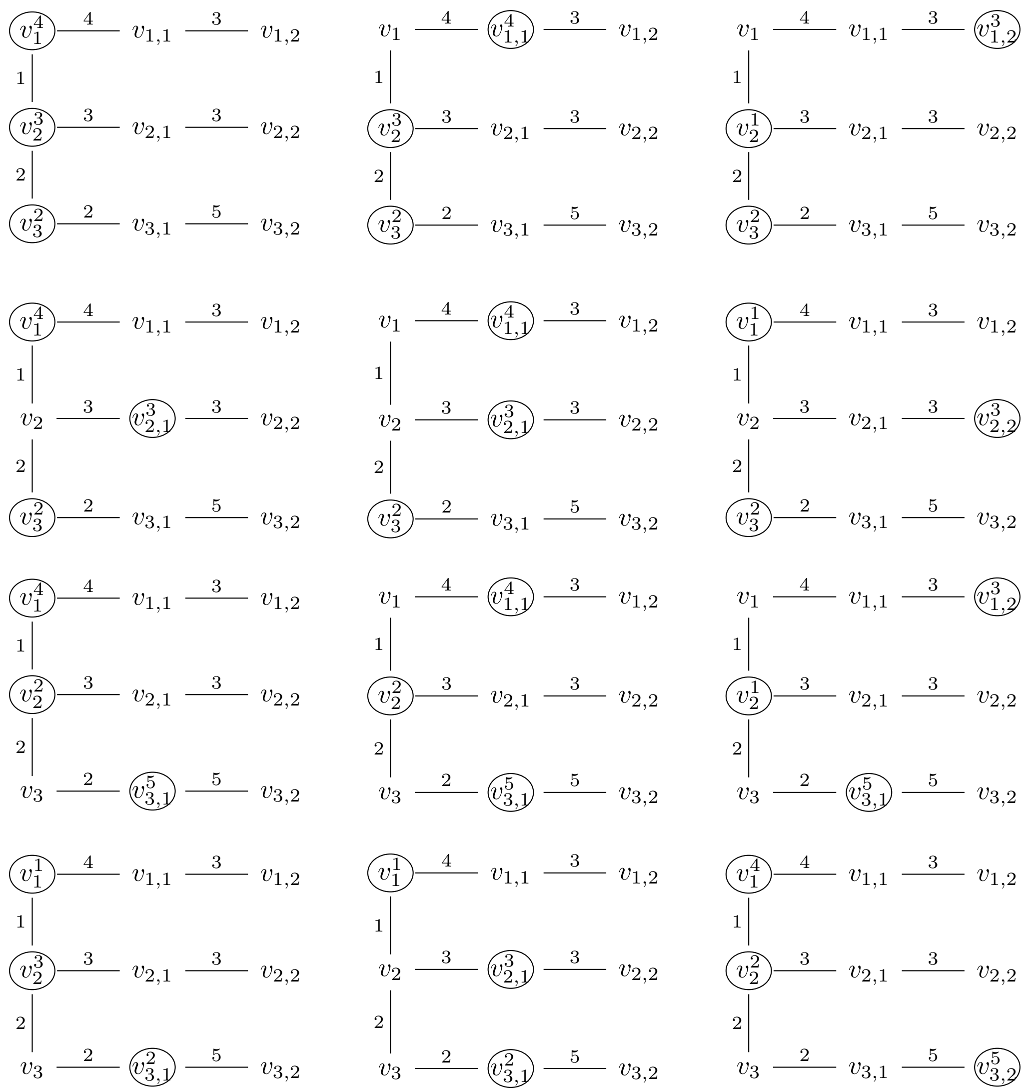
Example 3.91. Consider the following weighted graph $(\Sigma_2 P_2)_\lambda$ as in Example 3.78.

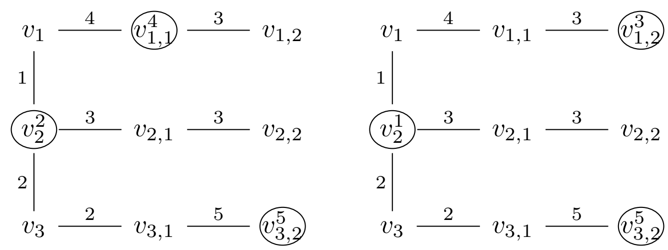
$$\begin{tikzcd}[arrows=-]
v_1 \arrow[r, "4"] \arrow[d, "1"'] & v_{1,1} \arrow[r, "3"] & v_{1,2} \\
v_2 \arrow[r, "3"] \arrow[d, "2"'] & v_{2,1} \arrow[r, "3"] & v_{2,2} \\
v_3 \arrow[r, "2"] & v_{3,1} \arrow[r, "5"] & v_{3,2}
\end{tikzcd}$$

Then the weighted 2-path ideal of $(\Sigma_2 P_2)_\lambda$ is

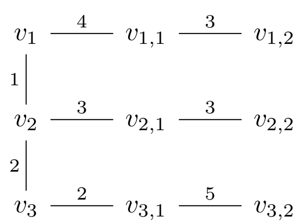
$$I_2\big((\Sigma_2 P_2)_\lambda\big) = \big(X_{1,2}^3 X_{1,1}^4 X_1^4, X_{1,1}^4 X_1^4 X_2, X_1 X_2^3 X_{2,1}^3, X_1 X_2^2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3,$$
$$X_{2,1}^3 X_2^3 X_3^2, X_2^2 X_3^2 X_{3,1}^2, X_{3,2}^5 X_{3,1}^5 X_3^2\big) R'.$$

Example 3.92. The minimal weighted 2-path vertex covers of $(\Sigma_2 P_2)_\lambda$ as in Example 3.78 are displayed in the following sketches.





Example 3.93. Consider the following graph $(\Sigma_2 P_2)_\lambda$ as in Example 3.78.



Then by Fact 3.84 and Example 3.92, the irredundant m-irreducible decomposition of $I_2\big((\Sigma_2 P_2)_\lambda\big)$ is given by

$$\begin{aligned} I_2\big((\Sigma_2 P_2)_\lambda\big) = {} & \big(X_1^4, X_2^3, X_3^2\big)R' \cap \big(X_{1,1}^4, X_2^3, X_3^2\big)R' \cap \big(X_{1,2}^3, X_2, X_3^2\big)R' \cap \big(X_1^4, X_{2,1}^3, X_3^2\big)R' \\ & \cap \big(X_{1,1}^4, X_{2,1}^3, X_3^2\big)R' \cap \big(X_1, X_{2,2}^3, X_3^2\big)R' \cap \big(X_1^4, X_2^2, X_{3,1}^5\big)R' \cap \big(X_{1,1}^4, X_2^2, X_{3,1}^5\big)R' \\ & \cap \big(X_{1,2}^3, X_2, X_{3,1}^5\big)R' \cap \big(X_1, X_2^3, X_{3,1}^2\big)R' \cap \big(X_1, X_{2,1}^3, X_{3,1}^2\big)R' \cap \big(X_1^4, X_2^2, X_{3,2}^5\big)R' \\ & \cap \big(X_{1,1}^4, X_2^2, X_{3,2}^5\big)R' \cap \big(X_{1,2}^3, X_2, X_{3,2}^5\big)R'. \end{aligned}$$

Definition 3.94. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. Define

$$\mathfrak{m}^{[\underline{a}(\lambda)]} = \big(X_1^{a_1}, \ldots, X_d^{a_d}\big)R,$$

where for $i = 1, \ldots, d$, $a_i = \sum_{k=0}^{r} e_{i,k}$ with

$$e_{i,k} = \begin{cases} \lambda(v_i v_{i,1}) & \text{if } k = 0, \\ \max\{\lambda(v_{i,k-1}v_{i,k}), \lambda(v_{i,k}v_{i,k+1})\} & \text{if } k = 1, \ldots, r-1, \\ \lambda(v_{i,r-1}v_{i,r}) & \text{if } k = r. \end{cases}$$

In words, $\mathfrak{m}^{\underline{a}(\lambda)}$ is the monomial ideal of R obtained from the monomial ideal $(g_1, \ldots, g_d)R'$ by setting $\mathfrak{m}^{\underline{a}(\lambda)} = \big(p(g_1), \ldots, p(g_d)\big)R$, where g_i is the corresponding generator in $I_r\big((\Sigma_r G)_\lambda\big)$ of the

r-whisker $v_iv_{i,1}\dots v_{i,r}$ from $(\Sigma_r G)_\lambda$ for $i = 1,\dots,d$.

Example 3.95. In Example 3.93, $\mathfrak{m}^{[\underline{a}(\lambda)]} = \big(X_1^{a_1}, X_2^{a_2}, X_3^{a_3}\big)R$ with

$$a_1 = \sum_{k=0}^{2} e_{1,k} = \lambda(v_1v_{1,1}) + \max\{\lambda(v_1v_{1,1}), \lambda(v_{1,1}v_{1,2})\} + \lambda(v_{1,1}v_{1,2}) = 4+4+3 = 11,$$
$$a_2 = \sum_{k=0}^{2} e_{2,k} = \lambda(v_2v_{2,1}) + \max\{\lambda(v_2v_{2,1}), \lambda(v_{2,1}v_{2,2})\} + \lambda(v_{2,1}v_{2,2}) = 3+3+3 = 9,$$
$$a_3 = \sum_{k=0}^{2} e_{3,k} = \lambda(v_3v_{3,1}) + \max\{\lambda(v_3v_{3,1}), \lambda(v_{3,1}v_{3,2})\} + \lambda(v_{3,1}v_{3,2}) = 2+5+5 = 12.$$

Fact 3.96. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \le \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \le \lambda(v_j, v_{j,1})$ for all edges $v_iv_j \in E$. Then $I_r\big((\Sigma_r G)_\lambda\big)$ is the polarization of $I_r\big((\Sigma_r G)_\lambda\big)R$ by e.g., [9, Proposition 3.7]. So by Fact 2.89, the list $X_i - X_{i,k}, 1 \le i \le d, 1 \le k \le r$ is a maximal homogeneous regular sequence for $\frac{R'}{I_r((\Sigma_r G)_\lambda)}$ and

$$\frac{R}{I_r((\Sigma_r G)_\lambda)R} \cong \frac{R'}{I_r((\Sigma_r G)_\lambda) + (X_i - X_{i,k} \mid 1 \le i \le d, 1 \le k \le r)R'}.$$

Furthermore, it is straightforward to show that

$$I_r\big((\Sigma_r G)_\lambda\big)R = I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]}, \text{ where } \lambda' = \lambda|_{\Sigma_{r-1}G}.$$

Because of the following fact, the main result of this subsection gives a formula to compute $\operatorname{type}\big(R/I_r(G_\omega)\big)$ for all trees such that $R/I_r(G_\omega)$ is Cohen-Macaulay.

Fact 3.97. [9, Proposition 3.7 and Theorem 3.11] Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \le \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_iv_j \in E$.

(a) $R'/I_r\big((\Sigma_r G)_\lambda\big)$ is Cohen-Macaulay.

(b) If $\Gamma_{\lambda'}$ is a weighted tree and $R/I_r(\Gamma_{\lambda'})$ is Cohen-Macaulay, then there exists a weighted tree $H_{\omega'}$ such that $(\Sigma_r H)_{\lambda''}$ is obtained by pruning a sequence of r-pathless leaves from $\Gamma_{\lambda'}$ with $\lambda'' = \lambda'|_{\Sigma_r H}$ and the weight function λ' satisfies the above condition.

Definition 3.98. Let $(\Sigma_{r-1} G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω. We define $q : V\big((\Sigma_{r-1} G)_\lambda\big) \to V(G)$ as $q(v_{i,j}) = v_i$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_{r-1} G)_\lambda\big)$ and

$\delta'' : V'' \to \mathbb{N}$. Then

$$q(V'') = \{v_i \mid \exists\ v_{i,j} \in V''\}.$$

Set

$$\mathrm{WCA}_i(\mathfrak{P}) = \{v_{i,j} \in V'' \mid \delta''(v_{i,j}) \le \lambda(v_{i,j}v) \text{ for some edge } v_{i,j}v \text{ in } (\Sigma_{r-1}G)_\lambda\},\ \forall\ i = 1, \dots, d,$$

and

$$h_{i,k} = \max\big\{\lambda(v_{i,k}v) \mid v_{i,k}v \in E\big((\Sigma_{r-1}G)_\lambda\big)\big\},\ \forall\ i = 1, \dots, d,\ k = 0, \dots, r-1.$$

Define

$$\begin{aligned} \gamma_{(V'',\delta'')} : q(V'') &\longrightarrow \mathbb{N} \sqcup \{\infty\} \\ v_i &\longmapsto \begin{cases} \min\Big\{\delta''(v_{i,j}) + \sum_{k=0}^{j-1} h_{i,k} \;\Big|\; v_{i,j} \in \mathrm{WCA}_i(\mathfrak{P})\Big\} & \text{if } \mathrm{WCA}_i(\mathfrak{P}) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Proposition 3.99. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_{r-1}G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Assume $\mathrm{WCA}_i(\mathfrak{P}) \neq \emptyset$ for some $i \in \{1, \dots, d\}$. Let $v_{i,j_1}, v_{i,j_2} \in \mathrm{WCA}_i(\mathfrak{P})$ with $j_1 < j_2$, then

$$\delta''(v_{i,j_1}) + \sum_{k=0}^{j_1-1} h_{i,k} < \delta''(v_{i,j_2}) + \sum_{k=0}^{j_2-1} h_{i,k}.$$

So

$$\gamma_{(V'',\delta'')}(v_i) = \delta''(v_{i,j_0}) + \sum_{k=0}^{j_0-1} h_{i,k},\ \text{ where } j_0 := \min\{j \mid v_{i,j} \in \mathrm{WCA}_i(\mathfrak{P})\}.$$

Proof. Suppose $\delta''(v_{i,j_1}) + \sum_{k=0}^{j_1-1} h_{i,k} \ge \delta''(v_{i,j_2}) + \sum_{k=0}^{j_2-1} h_{i,k}$. Then $\delta''(v_{i,j_1}) \ge \delta''(v_{i,j_2}) + \sum_{k=j_1}^{j_2-1} h_{i,k}$. So

$$h_{i,j_1} < \delta''(v_{i,j_2}) + h_{i,j_1} \le \delta''(v_{i,j_2}) + \sum_{k=j_1}^{j_2-1} h_{i,k} \le \delta''(v_{i,j_1}),\ \text{ i.e., } h_{i,j_1} < \delta''(v_{i,j_1}),$$

contradicted by that $v_{i,j_1} \in \mathrm{WCA}(\mathfrak{P})$. $\square$

The following theorem is a key for decomposing $I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ with $\lambda' = \lambda|_{\Sigma_{r-1}G}$ and hence $I_r\big((\Sigma_r G)_\lambda\big)R$. The proof is somewhat technical. The reader may wish to follow the argument with the succeeding example.

Theorem 3.100. *Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_{r-1}G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Then $I_r\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$ if and only if (V'', δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$.*

Proof. "$\Rightarrow$". Assume $I_r\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Let $P_r := v_{p_1,q_1} \dots v_{p_{r+1},q_{r+1}}$ be an r-path in $(\Sigma_{r-1}G)_\lambda$. Set

$$e_{p_k,q_k} = \begin{cases} \lambda(v_{p_1,q_1}v_{p_2,q_2}) & \text{if } k = 1, \\ \max\{\lambda(v_{p_{k-1},q_{k-1}}v_{p_k,q_k}), \lambda(v_{p_k,q_k}v_{p_{k+1},q_{k+1}})\} & \text{if } k = 2, \dots, r, \\ \lambda(v_{p_r,q_r}v_{p_{r+1},q_{r+1}}) & \text{if } k = r+1. \end{cases}$$

Then $X_{p_1}^{e_{p_1,q_1}} \dots X_{p_{r+1}}^{e_{p_{r+1},q_{r+1}}} \in [\![I_r((\Sigma_{r-1}G)_\lambda)R]\!] \subseteq [\![P(q(V''), \gamma_{(V'',\delta'')})]\!]$. So

$$X_{i_0}^{\gamma_{(V'',\delta'')}(v_{i_0})} \mid X_{p_1}^{e_{p_1,q_1}} \dots X_{p_{r+1}}^{e_{p_{r+1},q_{r+1}}} \text{ for some } v_{i_0} \in q(V'').$$

Hence $v_{i_0} = v_{p_l}$ for some $l \in \{1, \dots, r+1\}$ and

$$\min_{v_{i_0,j} \in \mathrm{WCA}_{i_0}(\mathfrak{P})} \left\{ \delta''(v_{i_0,j}) + \sum_{k=0}^{j-1} h_{i_0,k} \right\} = \gamma_{(V'',\delta'')}(v_{i_0}) \leq \sum_{k=0}^{r+1} \mathbb{1}_k \cdot e_{p_k,q_k}, \text{ where } \mathbb{1}_k = \begin{cases} 1 & \text{if } p_k = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

So $v_{p_l} = v_{i_0} \in q(V'')$. Since P_r is an r-path in $\Sigma_{r-1}G$, P_r is of the following form.

$$\begin{tikzcd}[arrows=dash]
v_{p_{1+q_1},0} \arrow[d, equal] & & & \\
v_{p_1,0} \arrow[r] \arrow[d] & v_{p_1,1} \arrow[r] & \cdots \arrow[r] & v_{p_1,q_1} \\
\vdots \arrow[d] & & & \\
v_{p_{r+1},0} \arrow[r] \arrow[d, equal] & v_{p_{r+1},1} \arrow[r] & \cdots \arrow[r] & v_{p_{r+1},q_{r+1}} \\
v_{p_{1+r-q_{r+1}},0} & & &
\end{tikzcd}$$

where q_1 or q_{r+1} may be 0. Let $M_0 := \max_{1\le k\le r+1}\{q_k \mid i_0 = p_k\}$. Then

$$M_0 = \begin{cases} q_1 & \text{if } i_0 = p_1, \\ q_{r+1} & \text{if } i_0 = p_{r+1}. \end{cases}$$

Since $\gamma_{(V'',\delta'')}(v_i) < \infty$, we have $\mathrm{WCA}_{i_0}(\mathfrak{P}) \neq \emptyset$. Set $j_0 := \min\{j \mid v_{i_0,j} \in \mathrm{WCA}_{i_0}(\mathfrak{P})\}$. Then by Proposition 3.99,

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} = \min_{v_{i_0,j}\in V''}\left\{\delta''(v_{i_0,j}) + \sum_{k=0}^{j-1} h_{i_0,k}\right\} \le \sum_{k=0}^{r+1} \mathbb{1}_k \cdot e_{p_k,q_k} = \sum_{k=0}^{M_0} e_{i_0,k}. \qquad (3.100.1)$$

Suppose $j_0 > M_0$. Then since $e_{i_0,k} \le h_{i_0,k}$ for $k = 0,\dots,M_0$, by (3.100.1), we have

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{M_0} e_{i_0,k} \le \delta''(v_{i_0,j_0}) + \sum_{k=0}^{M_0} h_{i_0,k} \le \delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} \le \sum_{k=0}^{M_0} e_{i_0,k}, \text{ i.e., } \delta''(v_{i_0,j_0}) \le 0,$$

contradicted by that $\delta''(v_{i_0,j_0}) \ge 1$. So $j_0 \le M_0$ and there must exist a sub-path of P_r of the form

$$v_{i_0,0} \;\text{——}\; v_{i_0,1} \;\text{——}\; \cdots \;\text{——}\; v_{i_0,M_0}.$$

Since $0 \le j_0 \le M_0$, there exists a vertex in this path of the form $v_{i_0,j_0} = v_{p_k,q_k}$ for some k in $\{1,\dots,r+1\}$. So $v_{p_k,q_k} = v_{i_0,j_0} \in \mathrm{WCA}_i(\mathfrak{P}) \subseteq V''$.

(a) Assume $0 = j_0 < M_0$. Since $\lambda(v_iv_j) \le \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_iv_j \in E$ and $M_0 \ge 1$, we have $e_{i_0,0} = \lambda(v_{i_0,0}v_{i_0,1}) = h_{i_0,0}$. Since $v_{i_0,j_0} \in \mathrm{WCA}_{i_0}(\mathfrak{P})$, we have $\delta''(v_{i_0,0}) \le h_{i_0,0} = e_{i_0,0}$.

(b) Assume $0 < j_0 < M_0$. Since $v_{i_0,j_0} \in \mathrm{WCA}_{i_0}(\mathfrak{P})$, v_{i_0,j_0} weighted-covers the edge $v_{i_0,j_0-1}v_{i_0,j_0}$ or $v_{i_0,j_0}v_{i_0,j_0+1}$, i.e., $\delta''(v_{i_0,j_0}) \le \max\{\lambda(v_{i_0,j_0-1}v_{i_0,j_0}), \lambda(v_{i_0,j_0}v_{i_0,j_0+1})\} = e_{i_0,j_0}$.

(c) Assume $j_0 = M_0$. Since $e_{i_0,k} \le h_{i_0,k}$ for $k = 0,\dots,j_0-1$, by (3.100.1), we have

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} e_{i_0,k} \le \delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} \le \sum_{k=0}^{M_0} e_{i_0,k} = \sum_{k=0}^{j_0} e_{i_0,k}, \text{ i.e., } \delta''(v_{i_0,j_0}) \le e_{i_0,j_0}.$$

So v_{i_0,j_0} weighted-covers P_r. Thus, V'' is a weighted r-path vertex cover of $\Sigma_{r-1}G$.

"$\Leftarrow$". Assume (V'',δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. We need to show every monomial generator of $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$ is in $P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Let $\underline{X}^{\underline{b}} := X_{i_1}^{e_{i_1,j_1}} \dots X_{i_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ be such a generator corresponding to an r-path $P_r := v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $(\Sigma_{r-1}G)_\lambda$. We need to show $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Note that $X_{i_1,j_1}^{e_{i_1,j_1}} \cdots X_{i_{r+1},j_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ is of the following form.

$$\begin{tikzcd}
X^{e_{i_{1+j_1},0}}_{i_{1+j_1},0} \arrow[d, equal] & & & \\
X^{e_{i_1,0}}_{i_1,0} \arrow[r, dash] \arrow[d, dash] & X^{e_{i_1,1}}_{i_1,1} \arrow[r, dash] & \cdots \arrow[r, dash] & X^{e_{i_1,j_1}}_{i_1,j_1} \\
\vdots \arrow[d, dash] & & & \\
X^{e_{i_{r+1},0}}_{i_{r+1},0} \arrow[r, dash] \arrow[d, equal] & X^{e_{i_{r+1},1}}_{i_{r+1},1} \arrow[r, dash] & \cdots \arrow[r, dash] & X^{e_{i_{r+1},j_{r+1}}}_{i_{r+1},j_{r+1}} \\
X^{e_{i_{1+r-j_{r+1}},0}}_{i_{1+r-j_{r+1}},0} & & &
\end{tikzcd}$$

where j_1 or j_{r+1} may be 0. Since P_r is an r-path in $(\Sigma_{r-1}G)_\lambda$ and (V'', δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, we have v_{i_l,j_l} weighted-covers the r-path P_r for some $l \in \{1, \dots, r+1\}$. So $v_{i_l,j_l} \in \mathrm{WCA}_{i_l}(\mathfrak{P})$ and then

$$\gamma_{(V'',\delta'')}(v_{i_l}) = \min_{v_{i_l,t} \in \mathrm{WCA}_{i_l}(\mathfrak{P})} \left\{ \delta''(v_{i_l,t}) + \sum_{k=0}^{t-1} h_{i_l,k} \right\} \le \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k}.$$

Let $M_0 := \max_{1\le k\le r+1}\{j_k \mid i_l = i_k\}$. Then $j_l \le M_0$. Since v_{i_l,j_l} weighted-covers the r-path P_r, $\delta''(v_{i_l,j_l}) \le e_{i_l,j_l}$. So

$$\delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \le e_{i_l,j_l} + \sum_{k=0}^{j_l-1} e_{i_l,k} = \sum_{k=0}^{j_l} e_{i_l,k} \le \sum_{k=0}^{M_0} e_{i_l.k}, \text{ i.e., } \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \le \sum_{k=0}^{M_0} e_{i_l,k}.$$

(a) Assume $j_l = 0$. Then

$$\gamma_{(V'',\delta'')}(v_{i_l}) \le \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k} = \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \le \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} \cdot e_{i_k,j_k} = b_{i_l},$$

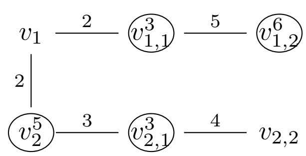
where $\mathbb{1}_{l,k} = \begin{cases} 1 & \text{if } i_k = i_l \\ 0 & \text{otherwise} \end{cases}$, $\forall\ k = 1, \dots, r+1$.

(b) Assume $j_l > 0$. Then $M_0 \ge 1$. Since $\lambda(v_iv_j) \le \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_iv_j \in E$, we have $e_{i_0,0} = \lambda(v_{i_0,0}v_{i_0,1}) = h_{i_0,0}$. Also, since $e_{i_0,k} = h_{i_0,k}$ for $k = 1, \ldots, j_l - 1$, we have

$$\gamma_{(V'',\delta'')}(v_{i_l}) \leq \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k} = \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} \cdot e_{i_k,j_k} = b_{i_l}.$$

So $X_{i_l}^{\gamma_{(V'',\delta'')}(v_{i_l})} \mid \underline{X}^{\underline{b}}$. Thus, $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. $\square$

Example 3.101. A weighted 2-path suspension $(\Sigma_2 P_1)_\lambda$ of $G_\omega := (P_1)_\omega = \big(\ v_1 \xrightarrow{\quad 2 \quad} v_2\ \big)$ with a weighted 3-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_2 P_1)_\lambda$ is given in the following sketch.



Since $I_3\big((\Sigma_2 P_1)_\lambda\big) = \big(X_{1,2}^5 X_{1,1}^5 X_1^2 X_2^2, X_{1,1}^2 X_1^2 X_2^3 X_{2,1}^3, X_1^2 X_2^3 X_{2,1}^4 X_{2,2}^4\big)R'$, we have

$$I_3\big((\Sigma_2 P_1)_\lambda\big)R = \big(X_1^{12} X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R.$$

Note $q(V'') = \{v_1, v_2\}$. Since $\delta''(v_{1,1}) = 3 < 5 = \lambda(v_{1,1}v_{1,2})$ and $\delta''(v_{1,2}) = 6 > 5 = \lambda(v_{1,1}v_{1,2})$, we have $\mathrm{WCA}_1(\mathfrak{P}) = \{v_{1,1}\}$. Similarly, $\mathrm{WCA}_2(\mathfrak{P}) = \{v_{2,1}\}$. So

$$\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,1}) + \sum_{k=0}^{1-1} h_{1,k} = \delta''(v_{1,1}) + \max\{\lambda(v_1v_2), \lambda(v_1v_{1,1})\} = 3 + \max\{2, 2\} = 5,$$

$$\gamma_{(V'',\delta'')}(v_2) = \delta''(v_{2,1}) + \sum_{k=0}^{1-1} h_{2,k} = \delta''(v_{2,1}) + \max\{\lambda(v_1v_2), \lambda(v_2v_{2,1})\} = 3 + \max\{2, 3\} = 6.$$

Then

$$P\big(V'', \gamma_{(V'',\delta'')}\big) = \big(X_1^5, X_2^6\big)R \supseteq \big(X_1^{12} X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R = I_3\big((\Sigma_2 P_1)_\lambda\big)R.$$

Proposition 3.102. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_j, v_{j,1})$ for all $v_iv_j \in E$. The monomial ideal $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with $V'' \subseteq V(\Sigma_{r-1}G)$ and $\delta'' : V'' \to \mathbb{N}$.

Proof. Fact 2.85 gives a decomposition of $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$. Let $J := \big(X_{b_1}^{\beta_{b_1}}, \ldots, X_{b_s}^{\beta_{b_s}}\big)R$ occur in the decomposition. Without loss of generality, assume $b_1, \ldots, b_s \in \mathbb{N}$ are distinct. Let $k \in \{1, \ldots, s\}$.

Then by Fact 2.85, there exists a generator $p(X_{i_1,j_1}\dots X_{i_{r+1},j_{r+1}})$ with $v_{i_1,j_1}\dots v_{i_{r+1},j_{r+1}}$ an r-path in $(\Sigma_{r-1}G)_\lambda$ such that for some $c(k)\in\{1,\dots,r+1\}$, we have $i_{c(k)}=b_k$ and

$$\beta_{b_k}=\begin{cases} e_{b_k,0} & \text{if } M_k=0,\\ \lambda(v_{b_k,M_k}v_{b_k,M_k-1})+\sum_{l=0}^{M_k-1}h_{b_k,l} & \text{if } M_k\geq 1,\end{cases}$$

where $M_k:=\max_{1\leq n\leq r+1}\{j_n\mid b_k=i_n\}\leq r-1$ and

$$e_{i_m,j_m}=\begin{cases} \lambda(v_{i_1,j_1}v_{i_2,j_2}) & \text{if } m=1,\\ \max\{\lambda(v_{i_{m-1},j_{m-1}}v_{i_m,j_m}),\lambda(v_{i_m,j_m}v_{i_{m+1},j_{m+1}})\} & \text{if } m=2,\dots,r,\\ \lambda(v_{i_r,j_r}v_{i_{r+1},j_{r+1}}) & \text{if } m=r+1.\end{cases}$$

Repeat the process for each $k\in\{1,\dots,s\}$ and set $V''=\{v_{b_1,M_1},\dots,v_{b_s,M_s}\}$. Then $q(V'')=\{v_{b_1},\dots,v_{b_s}\}$. Define

$$\begin{aligned}\delta'':V''&\longrightarrow\mathbb{N}\\ v_{b_k,M_k}&\longmapsto\begin{cases}\lambda(v_{b_k,M_k}v_{b_k,M_k-1}) & \text{if } M_k\geq 1\\ \beta_{b_k}(=e_{b_k,0}) & \text{if } M_k=0\end{cases},\ \forall\ k=1,\dots,s.\end{aligned}$$

Claim. $J=P\big(q(V''),\gamma_{(V'',\delta'')}\big)$. It is enough to show that $\gamma_{(V'',\delta'')}(v_{b_k})=\beta_{b_k}$ for $k=1,\dots,s$. Let $\mathfrak{P}:=(V'',\delta'')$. Since $|V''|=|q(V'')|$, we have $|\mathrm{WCA}_k(\mathfrak{P})|\leq 1$ for $k=1,\dots,s$.

(a) Assume $M_k\geq 1$. Since $v_{b_k,M_k}\in V''$ and $\delta''(v_{b_k,M_k})=\lambda(v_{b_k,M_k}v_{b_k,M_k-1})$, $v_{b_k,M_k}\in\mathrm{WCA}_{b_k}(\mathfrak{P})$. So $\mathrm{WCA}_{b_k}(\mathfrak{P})=\{v_{b_k,M_k}\}$. Hence

$$\gamma_{(V'',\delta'')}(v_{b_k})=\delta''(v_{b_k,M_k})+\sum_{l=0}^{M_k-1}h_{b_k,l}=\lambda(v_{b_k,M_k}v_{b_k,M_k-1})+\sum_{l=0}^{M_k-1}h_{b_k,l}=\beta_{b_k}.$$

(b) Assume $M_k=0$. Then $j_{c(k)}\in\{0,\dots,M_k\}=\{0\}$ and so $j_{c(k)}=0$. Then $v_{i_{c(k)},M_k}=v_{b_k,0}=$

$v_{b_k,M_k} \in V''$ and

$$\begin{aligned}\delta''(v_{i_{c(k)},j_{c(k)}}) &= \delta''(v_{b_k,0}) = \beta_{b_k} = e_{b_k,0} = e_{i_{c(k)},j_{c(k)}} \\ &= \begin{cases} \lambda(v_{i_1,j_1}v_{i_2,j_2}) & \text{if } c(k)=1, \\ \lambda(v_{i_r,j_r}v_{i_{r+1},j_{r+1}}) & \text{if } c(k)=r+1, \\ \max\bigl\{\lambda(v_{i_{c(k)-1},j_{c(k)-1}}v_{i_{c(k)},j_{c(k)}}), \lambda(v_{i_{c(k)},j_{c(k)}}v_{i_{c(k)+1},j_{c(k)+1}})\bigr\} & \text{if } 2 \le c(k) \le r-1. \end{cases}\end{aligned}$$

So $v_{b_k,0} \in \mathrm{WCA}_{b_k}(\mathfrak{P})$ and then $\mathrm{WCA}_{b_k}(\mathfrak{P}) = \{v_{b_k,0}\}$. Hence

$$\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,0}) + \sum_{l=0}^{0-1} h_{b_k,l} = \delta''(v_{b_k,0}) = \beta_{b_k}. \qquad \square$$

Example 3.103. Consider the following graph $(\Sigma_2 P_1)_\lambda$ as in Example 3.101.

$$\begin{tikzcd} v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2} \\ v_2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2} \end{tikzcd}$$

By Example 3.101, $I_3\bigl((\Sigma_2 P_1)\bigr)R = (X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11})R$. By Fact 2.85,

$$\begin{aligned} I_3\bigl((\Sigma_2P_1)\bigr)R &= \bigl(X_1^{12}, X_1^4, X_1^2\bigr)R \cap \bigl(X_1^{12}, X_1^4, X_2^{11}\bigr)R \cap \bigl(X_1^{12}, X_2^6, X_1^2\bigr)R \cap \bigl(X_1^{12}, X_2^6, X_2^{11}\bigr)R \\ &\quad \cap \bigl(X_2^2, X_1^4, X_1^2\bigr)R \cap \bigl(X_2^2, X_1^4, X_2^{11}\bigr)R \cap \bigl(X_2^2, X_2^6, X_1^2\bigr)R \cap \bigl(X_2^2, X_2^6, X_2^{11}\bigr)R \\ &= \bigl(X_1^2\bigr)R \cap \bigl(X_1^4, X_2^{11}\bigr)R \cap \bigl(X_1^2, X_2^6\bigr)R \cap \bigl(X_1^{12}, X_2^6\bigr)R \\ &\quad \cap \bigl(X_1^2, X_2^2\bigr)R \cap \bigl(X_1^4, X_2^2\bigr)R \cap \bigl(X_1^2, X_2^2\bigr)R \cap \bigl(X_2^2\bigr)R. \end{aligned}$$

Let $J_1 = (X_1^2)R$. Then $b_1 = 1$ and $\beta_{b_1} = \beta_1 = 2$. Consider the generator $X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}}X_{2,2}^{e_{2,2}} := X_1^2X_2^3X_{2,1}^4X_{2,2}^4$ of $I_3\bigl((\Sigma_2P_1)_\lambda\bigr)$. Then $M_1 := 0$ and $\beta_1 = e_{1,0} = 2$. Let $V'' = \{v_{1,0}\}$ and $\delta'' : V'' \longrightarrow \mathbb{N}$ given by $v_1 \longmapsto e_{1,0} = 2$. Since $q(V'') = \{v_1\}$, $\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,0}) = 2$. So

$$P\bigl(q(V''), \gamma_{(V'',\delta'')}\bigr) = P\bigl(\{v_1^2\}\bigr) = \bigl(X_1^2\bigr)R = J_1.$$

Let $J_2 = (X_1^4, X_2^{11})R$. Then $b_1 = 1, b_2 = 2$, and $\beta_{b_1} = \beta_1 = 4$ and $\beta_{b_2} = \beta_2 = 11$. Consider the generator $X_{1,1}^{e_{1,1}}X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}} := X_{11}^2X_1^2X_2^3X_{2,1}^3$ of $I_3\bigl((\Sigma_2P_1)_\lambda\bigr)$. Then $M_1 := 1$ and $\beta_1 = \lambda(v_{1,1}v_{1,0}) + h_{1,0} = 2 + 2 = 4$. Consider the generator $X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}}X_{2,2}^{e_{2,2}} := X_1^2X_2^3X_{2,1}^4X_{2,2}^4$ of

$I_3\big((\Sigma_2 P_1)_\lambda\big)$. Then $M_2 := 2$ and $\beta_2 = \lambda(v_{2,2}v_{2,1}) + h_{2,0} + h_{2,1} = 4+3+4 = 11$. Let $V'' = \{v_{1,1}, v_{2,2}\}$ and $\delta'' : V'' \longrightarrow \mathbb{N}$ given by $v_1 \longmapsto \lambda(v_{1,1}v_{1,0}) = 2$ and $v_{2,2} \longmapsto \lambda(v_{2,2}v_{2,1}) = 4$. Then $q(V'') = \{v_1, v_2\}$ and so $\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,1}) + h_{1,0} = 2+2$ and $\gamma_{(V'',\delta'')}(v_2) = \delta''(v_{2,2}) + h_{2,0} + h_{2,1} = 4+3+4 = 11$. So

$$P\big(q(V''), \gamma(V'', \delta'')\big) = P\big(\{v_1^4, v_2^{11}\}\big) = \big(X_1^4, X_2^{11}\big)R = J_2.$$

The next result gives our first decomposition needed for computing $\operatorname{type}\big(R'/I_r((\Sigma_r G)_\lambda)\big)$.

Theorem 3.104. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E$. One has*

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big), \text{ where } \lambda' = \lambda|_{\Sigma_{r-1}G},$$

and

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]}.$$

Proof. Since $I_r\big((\Sigma_r G)_\lambda\big)R = I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]}$ by Fact 3.96, we have by [10, Theorem 7.5.3], it is enough to show that

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

By Proposition 3.102, the monomial ideal $I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with $V'' \subseteq V(\Sigma_{r-1}G)$ and $\delta'' : V'' \to \mathbb{N}$. Then by Theorem 3.100,

$$\begin{aligned}
I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R &\subseteq \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\
&\subseteq \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'} \text{ in the decomp. of } I_r((\Sigma_{r-1}G)_{\lambda'})R} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\
&= I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R.
\end{aligned}$$

So

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big). \qquad \square$$

Example 3.105. Consider the following weighted 3-path suspension $(\Sigma_3 P_1)_\lambda$ of $G_\omega := (P_1)_\omega = \big(v_1 \overset{2}{\text{——}} v_2 \big)$.

$$\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2} \arrow[r, dash, "2"] & v_{1,3} \\
v_2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2} \arrow[r, dash, "2"] & v_{2,3}
\end{tikzcd}$$

Let $\lambda' = \lambda|_{\Sigma_2 P_1}$. Since $I_3\big((\Sigma_2 P_1)_{\lambda'}\big) = (X_{1,2}^5 X_{1,1}^5 X_1^2 X_2^2, X_{1,1}^2 X_1^2 X_2^3 X_{2,1}^3, X_1^2 X_2^3 X_{2,1}^4 X_{2,2}^4)$, by Theorem 3.104, we have two infinite intersection:

$$I_3\big((\Sigma_2 P_1)_{\lambda'}\big)R = \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R = \bigcap_{(V'',\delta'')\text{ w. } r\text{-path v. cover of } (\Sigma_2 P_1)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big),$$

and

$$\begin{aligned}
I_3\big((\Sigma_3 P_1)_{\lambda'}\big)R &= \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R + (X_1^{14}, X_2^{13})R \\
&= \bigcap_{(V'',\delta'')\text{ w. } r\text{-path v. cover of } (\Sigma_2 P_1)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).
\end{aligned}$$

The next result is key for our second decomposition result, Corollary 3.108.

Lemma 3.106. Let $\mathfrak{p} := (V_1'', \delta_1''), \mathfrak{P} := (V_2'', \delta_2'')$ be such that $V_1'', V_2'' \subseteq V\big((\Sigma_{r-1}G)_\lambda\big)$ and $\delta_1'', \delta_2'' : V'' \to \mathbb{N}$. If $(V_1'', \delta_1'') \leq (V_2'', \delta_2'')$, then $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big) \subseteq P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$.

Proof. Let $X_i^{\gamma_{(V_1'',\delta_1'')}(v_i)}$ be a generator of $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big)$. Then V_1'' "$\subseteq V_2''$" implies we have $X_i^{\gamma_{(V_2'',\delta_2'')}(v_i)} \in P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$ and

$$\begin{aligned}
\gamma_{(V_1'',\delta_1'')}(v_i) &= \min\Big\{\delta_1''(v_{i_j,t}) + \sum_{k=0}^{t-1} h_{i_j,k} \;\Big|\; v_{i_j,t} \in \mathrm{WCA}_{i_j}(\mathfrak{p})\Big\} \\
&\geq \min\Big\{\delta_2''(v_{i_j,t}) + \sum_{k=0}^{t-1} h_{i_j,k} \;\Big|\; v_{i_j,t} \in \mathrm{WCA}_{i_j}(\mathfrak{P})\Big\} = \gamma_{(V_2'',\delta_2'')}(v_i).
\end{aligned}$$

So $X_i^{\gamma_{(V_2'',\delta_2'')}(v_i)} \mid X_i^{\gamma_{(V_1'',\delta_1'')}(v_i)}$. Hence $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big) \subseteq P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$. $\square$

Example 3.107. Consider the following two pairs of sets $\mathfrak{p} := (V_1'', \delta_1'') := \{v_{1,1}^4, v_2^5, v_{2,1}^6\}$ and

$\mathfrak{P} := (V_2'', \delta_2'') := \{v_{1,1}^3, v_{1,2}^6, v_2^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.101.

$$\begin{tikzcd} v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^4 \arrow[r, dash, "5"] & v_{1,2} \\ v_2^5 \arrow[r, dash, "3"] & v_{2,1}^6 \arrow[r, dash, "4"] & v_{2,2} \end{tikzcd} \qquad \begin{tikzcd} v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^3 \arrow[r, dash, "5"] & v_{1,2}^6 \\ v_2^5 \arrow[r, dash, "3"] & v_{2,1}^3 \arrow[r, dash, "4"] & v_{2,2} \end{tikzcd}$$

Since $V_1'' \subseteq V_2''$ and $\delta_1'' \geq \delta_2''|_{V_1''}$, we have $(V_1'', \delta_1'') \leq (V_2'', \delta_2'')$. Similar to Example 3.101, we have $\mathrm{WCA}_1(\mathfrak{p}) = \{v_{1,1}\}$ and $\mathrm{WCA}_2(\mathfrak{p}) = \emptyset$. So $\gamma_{(V_1'', \delta_1'')}(v_2) = \infty$ and

$$\gamma_{(V_1'', \delta_1'')}(v_1) = \delta_1''(v_{1,1}) + \sum_{k=0}^{1-1} h_{1,k} = \delta_1''(v_{1,1}) + \max\{\lambda(v_1 v_2), \lambda(v_1 v_{1,1})\} = 4 + \max\{2, 2\} = 5.$$

Also, since $q(V_1'') = \{v_1, v_2\}$, we have $P\big(q(V_1''), \gamma_{(V_1'', \delta_1'')}\big) = (X_1^5, X_2^\infty)R = (X_1^5)R$. Then from Example 3.101 we have

$$P\big(q(V_2''), \gamma_{(V_2'', \delta_2'')}\big) = (X_1^5, X_2^6)R \supseteq (X_1^5)R = P\big(q(V_1''), \gamma_{(V_1'', \delta_1'')}\big).$$

Here is our second decomposition result for computing $\mathrm{type}\big(R'/I_r((\Sigma_r G)_\lambda)\big)$.

Corollary 3.108. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_i v_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_i v_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_i v_j \in E$. One has

$$I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1} G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big), \text{ where } \lambda' = \lambda|_{\Sigma_{r-1} G},$$

and

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1} G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]}.$$

Proof. By Fact 3.96 and [10, Theorem 7.5.3], it is enough to prove that

$$I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1} G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big).$$

By Theorem 3.104, it is enough to show that

$$\begin{aligned}
&\bigcap_{(V'',\delta'')\text{ weighted. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big)\\
=&\bigcap_{(V'',\delta'')\text{ min. weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).
\end{aligned}$$

"$\subseteq$" follows because every minimal weighted r-path vertex cover is a weighted r-path vertex cover.

"$\supseteq$" follows from Fact 3.83 and Lemma 3.106. $\square$

Example 3.109. Consider the following weighted 3-path suspension $(\Sigma_3P_1)_\lambda$ of $G_\omega := (P_1)_\omega = \big(v_1 \overset{2}{\text{——}} v_2 \big)$.

$$\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2} \arrow[r, dash, "2"] & v_{1,3} \\
v_2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2} \arrow[r, dash, "2"] & v_{2,3}
\end{tikzcd}$$

We depict the minimal weighted 3-path vertex covers of $(\Sigma_2P_1)_{\lambda'}$ with $\lambda' = \lambda|_{\Sigma_2P_1}$ in the following sketches.

$$\begin{tikzcd}
v_1^2 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2} \\
v_2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2} \\
v_2^2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^5 \arrow[r, dash, "5"] & v_{1,2} \\
v_2^3 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}$$

$$\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2}^5 \\
v_2^3 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^2 \arrow[r, dash, "5"] & v_{1,2} \\
v_2 \arrow[r, dash, "3"] & v_{2,1}^4 \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^5 \arrow[r, dash, "5"] & v_{1,2} \\
v_2 \arrow[r, dash, "3"] & v_{2,1}^3 \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}$$

$$\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1} \arrow[r, dash, "5"] & v_{1,2}^5 \\
v_2 \arrow[r, dash, "3"] & v_{2,1}^3 \arrow[r, dash, "4"] & v_{2,2}
\end{tikzcd}
\qquad
\begin{tikzcd}
v_1 \arrow[r, dash, "2"] \arrow[d, dash, "2"'] & v_{1,1}^2 \arrow[r, dash, "5"] & v_{1,2} \\
v_2^2 \arrow[r, dash, "3"] & v_{2,1} \arrow[r, dash, "4"] & v_{2,2}^4
\end{tikzcd}$$

Since $I_3\big((\Sigma_2P_1)_{\lambda'}\big) = (X_{1,2}^5X_{1,1}^5X_1^2X_2^2, X_{1,1}^2X_1^2X_2^3X_{2,1}^3, X_1^2X_2^3X_{2,1}^4X_{2,2}^4)$, by Corollary 3.108, we have

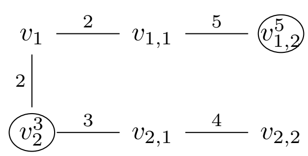
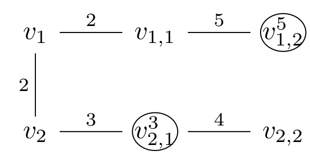
$$\begin{aligned}
I_3\big((\Sigma_2P_1)_{\lambda'}\big)R = \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R &= \big(X_1^2\big)R \cap \big(X_2^2\big)R \cap \big(X_1^7, X_2^3\big)R \cap \big(X_1^{12}, X_2^3\big)R\\
&\cap \big(X_1^4, X_2^7\big) \cap \big(X_1^7, X_2^6\big) \cap \big(X_1^{12}, X_2^6\big)R \cap \big(X_1^4, X_2^{11}\big)R.
\end{aligned}$$

Thus, the decomposition in Corollary 3.108 may be redundant.

In light of the preceding example, we define another order from which we can produce an irredundant decomposition. Lemma 3.118 is the key for understanding how this ordering helps with irredundancy.

Definition 3.110. Given two minimal weighted r-path vertex covers $(V_1'', \delta_1''), (V_2'', \delta_2'')$ of $(\Sigma_{r-1}G)_\lambda$, we write $(V_1'', \delta_1'') \leq_{\wp} (V_2'', \delta_2'')$ if $q(V_1'') \subseteq q(V_2'')$ and $\gamma_{(V_1'', \delta_1'')} \geq \gamma_{(V_2'', \delta_2'')}|_{q(V_1'')}$. A minimal weighted r-path vertex cover (V'', δ'') is *$\wp$-minimal* if there is not another minimal weighted r-path vertex cover (V''', δ''') such that $(V'', \delta'') <_{\wp} (V''', \delta''')$.

Example 3.111. Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,1}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.101.

Then $q(V_1'') = \{v_1, v_2\} = q(V_2'')$. Since

$$\gamma_{(V_1'', \delta_1'')}(v_1) = \delta_1''(v_{1,2}) + h_{1,1} + h_{1,0} = 5 + 5 + 2 = \delta_2''(v_{1,2}) + h_{1,1} + h_{1,0} = \gamma_{(V_2'', \delta_2'')}(v_1),$$

and $\gamma_{(V_1'', \delta_1'')}(v_2) = \delta_1''(v_2) = 3 < 3 + 3 = \delta_2''(v_{2,1}) + h_{2,0} = \gamma_{(V_2'', \delta_2'')}(v_2)$, we have $\gamma_{(V_1'', \delta_1'')} < \gamma_{(V_2'', \delta_2'')}$. So $(V_1'', \delta_1'') >_{\wp} (V_2'', \delta_2'')$. Hence (V_1'', δ_1'') is not $\wp$-minimal.

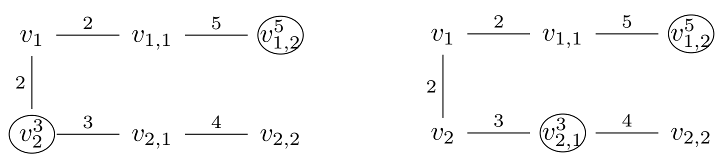
Lemma 3.112. Let $\mathfrak{p} := (W', \delta'), \mathfrak{P} := (W'', \delta'')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_{\wp} (W', \delta')$, then $|(W'', \delta'')| = |(W', \delta')|$ and $q(W'') = q(W')$.

Proof. Since (W', δ') is a minimal weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, for distinct pair $v_{i_1,j_1}, v_{i_2,j_2} \in W'$, we have $i_1 \neq i_2$. Also, since $q(W'') \subseteq q(W')$, $|W''| = |q(W'')| \leq |q(W')| = |W'|$. Suppose $|W''| < |W'|$. Then there exists $v_{i,j} \in W'$ such that $v_i \notin q(W'')$. Since (W', δ') is a minimal weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, there is an r-path P_r in $(\Sigma_{r-1}G)_\lambda$ that can only be weighted-covered by $v_{i,j}$. By assumption, P_r can be weighted-covered by some $v_{k,l} \in W''$, so $v_k \in q(W'')$. Also, since $v_i \notin q(W'')$, we have $k \neq i$. Let $\alpha = \min\{b \mid v_{k,b} \in \mathrm{WCA}_k(\mathfrak{p})\}$ and $\beta = \min\{b \mid v_{k,b} \in \mathrm{WCA}_k(\mathfrak{P})\}$. So $\alpha, \beta \leq l$. Since $\gamma_{(W'', \delta'')} \geq \gamma_{(W', \delta')}$, we have $\alpha \leq \beta$ similar to the proof of Proposition 3.99. If $\alpha < l$, then it is straightforward to show that P_r can also be weighted-covered by $v_{k,\alpha} \in W'$, a contradiction. Assume $\alpha = l$. Then $\alpha = \beta = l$ and so $v_{k,\beta} \in W''$ weighted-cover P_r. Since

$$\delta''(v_{k,\alpha}) + \sum_{b=0}^{\alpha-1} h_{k,b} = \gamma_{(W'',\delta'')}(v_k) \geq \gamma_{(W',\delta')}(v_k) = \delta'(v_{k,\alpha}) + \sum_{b=0}^{\alpha-1} h_{k,b},$$

we have $\delta'(v_{k,\alpha}) \leq \delta''(v_{k,\alpha})$. So P_r can also be weighted-covered by $v_{k,\alpha} \in W'$, a contradiction. Hence $|W''| = |W'|$ and thus $|q(W'')| = |q(W')|$. Since $q(W'') \subseteq q(W')$, we have $q(W'') = q(W')$. $\square$

Example 3.113. Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,1}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.115(a).



By Example 3.115(a), $(V_1'', \delta_1'') <_{\wp} (V_2'', \delta_2'')$. Then $|(V_1'', \delta_1'')| = |\{v_{1,2}, v_2\}| = 2 = |\{v_{1,2}, v_{2,1}\}| = |(V_2'', \delta_2'')|$ and $q(V_1'') = \{v_1, v_2\} = q(V_2'')$.

The following theorem can be used as an algorithm to find the set of $\wp$-minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ from the set of minimal weighted r-path vertex covers.

Theorem 3.114. *Let $\mathfrak{p} := (V_1'', \delta_1''), \mathfrak{P} := (V_2'', \delta_2'')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$. Then $(V_1'', \delta_1'') \leq_{\wp} (V_2'', \delta_2'')$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$: $j_{1,l} > j_{2,l}$ or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) \geq \delta_2''(v_{i_l,j_{2,l}})$ with $j_{1,l} := \{j \mid v_{i_l,j} \in V_1''\}$ and $j_{2,l} = \{j \mid v_{i_l,j} \in V_2''\}$.*

Proof. By Lemma 3.112, $(V_1'', \delta_1'') \leq_{\wp} (V_2'', \delta_2'')$ if and only if $q(V_1'') = q(V_2'')$ and $\gamma_{(V_1'',\delta_1'')}|_{q(V_1'')} \geq \gamma_{(V_2'',\delta_2'')}|_{q(V_1'')}$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$, $\gamma_{(V_1'',\delta_1'')}(v_{i_l}) \geq \gamma_{(V_2'',\delta_2'')}(v_{i_l})$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$, $\delta_1''(v_{i_l,j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k} \geq \delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}$ by Proposition 3.99. Claim. For $v_{i_l} \in q(V_1'')$, $\delta_1''(v_{i_l,j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k} \geq \delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}$ if and only if $j_{1,l} > j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) \geq \delta_2''(v_{i_l,j_{2,l}})$. Then we are done.

"$\Leftarrow$". Assume $j_{1,l} > j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) \geq \delta_2''(v_{i_l,j_{2,l}})$. Then

$$\alpha := \left(\delta_1''(v_{i_l,j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k}\right) - \left(\delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}\right) = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=j_{2,l}}^{j_{1,l}-1} h_{i_l,k}.$$

To prove our statement, it is equivalent to show $\alpha \geq 0$.

(a) If $j_{1,l} > j_{2,l}$, then $\alpha \geq \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) + h_{i_l,j_{2,l}} > h_{i_l,j_{2,l}} - \delta_2''(v_{i_l,j_{2,l}}) \geq 0$.

(b) If $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) \geq \delta_2''(v_{i_l,j_{2,l}})$, then $\alpha = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) \geq 0$.

"$\Rightarrow$". Suppose $j_{1,l} < j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) < \delta_2''(v_{i_l,j_{2,l}})$. Then

$$\alpha := \left(\delta_1''(v_{i_l,j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k}\right) - \left(\delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}\right) = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) - \sum_{k=j_{1,l}}^{j_{2,l}-1} h_{i_l,k}.$$

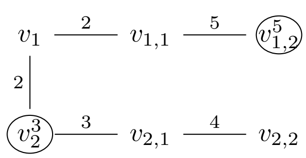
To prove our statement, it is equivalent to show $\alpha < 0$.

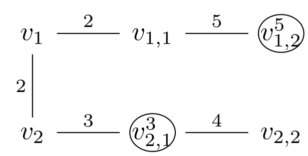
(a) If $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) < \delta_2''(v_{i_l,j_{2,l}})$, then $\alpha = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) < 0$.

(b) Assume $j_{1,l} < j_{2,l}$. Since $v_{i_l,j_{1,l}} \in V_1''$ and V_1'' is a minimal weighted r-path vertex cover, $\delta_1''(v_{i_l}, j_{1,l}) \leq h_{i_l,j_{1,l}}$. So $\alpha = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) - \sum_{k=j_{1,l}}^{j_{2,l}-1} h_{i_l,k} < \delta_1''(v_{i_l,j_{1,l}}) - h_{i_l,j_{1,l}} \leq 0$. $\square$

Example 3.115. We have the following examples.

(a) Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,1}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.111.





Then $q(V_1'') = \{v_{i_1} := v_1, v_{i_2} := v_2\} = q(V_2'')$. Note

$$j_{1,1} = \min\{j \mid v_{i_1,j} \in V_1''\} = \min\{j \mid v_{1,j} \in V_1''\} = 2,$$

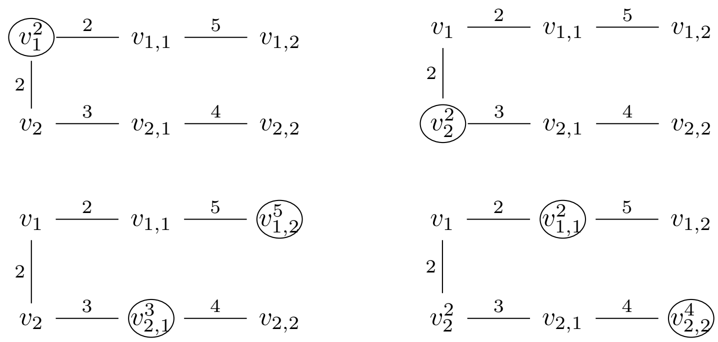
$$j_{1,2} = \min\{j \mid v_{i_2,j} \in V_1''\} = \min\{j \mid v_{2,j} \in V_1''\} = 0,$$

$$j_{2,1} = \min\{j \mid v_{i_1,j} \in V_2''\} = \min\{j \mid v_{2,j} \in V_2''\} = 2,$$

$$j_{2,2} = \min\{j \mid v_{i_2,j} \in V_2''\} = \min\{j \mid v_{2,j} \in V_2''\} = 1.$$

Since $j_{1,1} = 2 = j_{2,1}$ and $\delta_1''(v_{1,j_{1,1}}) = \delta_1''(v_{1,2}) = 5 = \delta_2''(v_{1,2}) = \delta_2''(v_{1,j_{2,1}})$, and $j_{1,2} = 0 < 1 = j_{2,2}$, we have $(V_1'', \delta_1'') <_{\wp} (V_2'', \delta_2'')$ by Theorem 3.114.

(b) Consider all the minimal weighted 3-path vertex covers of $(\Sigma_2 P_1)_{\lambda'}$ as in Example 3.109. Apply Theorem 3.114 repeatedly, we get all the $\wp$-minimal weighted 3-path vertex covers in the following.

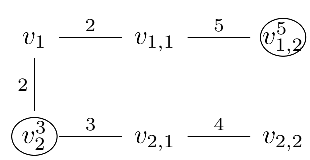


The next two results are key for our third and final decomposition result.

Proposition 3.116. For every minimal weighted r-path vertex cover $\mathfrak{p} := (W', \delta')$ of $(\Sigma_{r-1}G)_\lambda$, there is a $\wp$-minimal weighted r-path vertex cover (W'', δ'') of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$.

Proof. If (W', δ') is itself a $\wp$-minimal weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$, then we are done. If (W', δ') is not $\wp$-minimal, then by Lemma 3.112, the size of $q(W')$ cannot be decreased, so for some $v_i \in q(W')$ the function $\gamma_{(W',\delta')}(v_i) = \delta'(v_{i,j_0}) + \sum_{k=0}^{j_0-1} h_{i,k}$ with $j_0 := \{j \mid v_{i,j} \in \mathrm{WCA}_i(\mathfrak{p})\}$ from Proposition 3.99 can be increased, which is done by increasing j_0 and assigning an appropriate value to $\delta'(v_{i,j_0})$ since (W', δ') is minimal. We increase $\gamma_{(W',\delta')}(v_i)$ for each $v_i \in q(W')$ such that any further increase would cause the set not to be a weighted r-path vertex cover. This process terminates in finitely many steps because $j_0 \leq r$. Denote the new set (W'', δ''). Then (W'', δ'') is minimal since the size of W'' cannot be decreased by Lemma 3.112 and δ'' cannot be increased. Thus, by construction, (W'', δ'') is a $\wp$-minimal weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$. $\square$

Example 3.117. Consider the following minimal weighted 3-path vertex cover $\mathfrak{p} := (V_1'', \delta_1'') := \{v_{1,1}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.115(a).

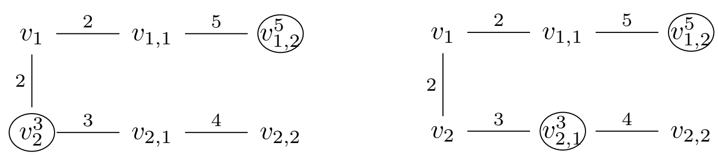


Note that $\gamma_{(V_1'',\delta_1'')}(v_1)$ cannot be increased. Assume $v_{2,1} \in V''$. Then set $\delta''(v_{2,1}) = 3$, we have $\mathfrak{p}' := (V_1''', \delta_1''') = \{v_{1,2}^5, v_{2,1}^3\}$ is a minimal weighted 3-path vertex cover by Example 3.115(a). However, since $v_{1,2} \in V''$, we have $v_{2,2}$ cannot be in V''', otherwise the 3-path $v_{1,1}v_1v_2v_{2,1}$ will be left uncovered. Thus, (V_1''', δ_1''') is $\mathscr{p}$-minimal and $(V_1''', \delta_1''') <_{\mathscr{p}} (V_1'', \delta_1'')$.

Lemma 3.118. Let $(V_1', \delta_1'), (V_2', \delta_2')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$. Then $(V_1', \delta_1') \leq_{\mathscr{p}} (V_2', \delta_2')$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$.

Proof. $(V_1', \delta_1') \leq_{\mathscr{p}} (V_2', \delta_2')$ if and only if $q(V_1') \subseteq q(V_2')$ and $\gamma_{(V_1',\delta_1')}|_{q(V_1')} \geq \gamma_{(V_2',\delta_2')}|_{q(V_1')}$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$. □

Example 3.119. Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,1}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ as in Example 3.115(a).



Then $(V_2'', \delta_2'') <_{\mathscr{p}} (V_1'', \delta_1'')$ by Example 3.115(a). Note also

$$P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big) = \big(X_1^{12}, X_2^6\big)R \subseteq \big(X_1^{12}, X_2^3\big)R = P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big).$$

Next, we present our third and final decomposition result which will yield the type computation in Theorem 3.122.

Theorem 3.120. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_iv_j \in E$. One has an irredundant parametric decomposition*

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'')\ \mathscr{p}\text{-min. w. } r\text{-path v. c. of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]},\ \ \lambda' = \lambda|_{\Sigma_{r-1}G}.$$

Proof. By Fact 3.96 and [10, Theorem 7.5.3], to verify this result, it is enough to show that we have an irredundant decomposition

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\ \mathscr{p}\text{-min. w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

Lemma 3.118 shows that this intersection is irredundant. So by Corollary 3.108, it is enough to show that

$$\bigcap_{(V'',\delta'')\text{ min. weighted }r\text{-path v. cover of }(\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big)$$
$$= \bigcap_{(V'',\delta'')\ \not{p}\text{-min. weighted }r\text{-path v. cover of }(\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).$$

"$\subseteq$" follows as every $\not{p}$-minimal weighted r-path vertex cover is a minimal weighted r-path vertex cover.

"$\supseteq$" follows from Proposition 3.116 and Lemma 3.118. $\square$

Example 3.121. Consider the graph $(\Sigma_3 P_1)_\lambda$ as in Example 3.109. Then by Theorem 3.120 and Example 3.115(b), we have an irredundant parametric decomposition

$$\begin{aligned} I_3\big((\Sigma_3 P_1)_\lambda\big) &= \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]} \\ &= \big[\big(X_1^2\big)R \cap \big(X_2^2\big)R \cap \big(X_1^{12}, X_2^6\big)R \cap \big(X_1^4, X_2^{11}\big)R\big] + \big(X_1^{14}, X_2^{13}\big)R. \end{aligned}$$

The next theorem is the fourth main result of this thesis. It is Formula (****) from the abstract.

Theorem 3.122. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_iv_j \in E$.*

$$\operatorname{type}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda)}\right) = \sharp\,\{\not{p}\text{-minimal weighted } r\text{-path vertex covers of } (\Sigma_{r-1}G)_{\lambda'}\},\ \lambda' = \lambda|_{\Sigma_{r-1}G}.$$

Proof. We compute

$$\begin{aligned} \operatorname{type}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda)}\right) &= \operatorname{type}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda) + (X_i - X_{i,k} \mid 1 \leq i \leq d, 1 \leq k \leq r)R'}\right) \\ &= \operatorname{type}\left(\frac{R}{I_r((\Sigma_r G)_\lambda)R}\right) \\ &= \sharp\,\{\text{ideals in an irredundant parametric decomposition of } I_r\big((\Sigma_r G)_\lambda\big)R\} \\ &= \sharp\,\{\not{p}\text{-minimal weighted } r\text{-path vertex covers of } (\Sigma_{r-1}G)_{\lambda'}\}, \end{aligned}$$

where the first equality is from Facts 2.95(a), 3.97(a) and 3.96, the second equality is from Fact 3.96, the third equality is from Fact 2.95(b) since $\dim\bigl(\frac{R}{I_r((\Sigma_r G)_\lambda)R}\bigr) = 0$, and the last equality is from Fact 3.120. $\square$

Remark. Because of Fact 3.97, we can use Theorem 3.122 to compute $\operatorname{type}\bigl(R/I_r(G_\omega)\bigr)$ for all weighted trees G_ω such that $I_r(G_\omega)$ is Cohen-Macaulay.

Example 3.123. Consider Example 3.121. Then by Theorem 3.122, we have

$$\operatorname{type}\bigl(R'/I_3(\Sigma_3 P_1)_\lambda\bigr) = 4.$$

We observe that the smallest number of vertices for one of the 3-path vertex covers of $(\Sigma_3 P_1)_\lambda$ is 2. Then by Facts 3.84 and 2.70, $\dim\bigl(R'/I_3((\Sigma_3 P_1)_\lambda)\bigr) = 8 - 2 = 6$. Since $R'/I_3((\Sigma_3 P_1)_\lambda)$ is Cohen-Macaulay by Fact 3.97(a), $\operatorname{depth}\bigl(R'/I_3((\Sigma_3 P_1)_\lambda)\bigr) = \dim\bigl(R'/I_3((\Sigma_3 P_1)_\lambda)\bigr) = 6$. Hence

$$\operatorname{Ext}^6_{R'}\bigl(A, R'/I_3((\Sigma_3 P_1)_\lambda)\bigr) \cong A^4.$$

3.5.2 f-weighted Path Ideals and the Type of $I_{r,f}((\Sigma_r G)_\lambda)$

In this subsection, we prove an f-weighted version of Theorem 3.74. See Formula (*******) from the abstract and Theorem 3.122(*******).

Definition 3.124. Let $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Define a ring homomorphism $p_{\underline{n}}$ by

$$\begin{aligned} p_{\underline{n}} : R' &\longrightarrow A[X_{1,0}, \ldots, X_{1,\min\{n_1-1,r\}}, \ldots, X_{d,0}, \ldots, X_{d,\min\{n_d-1,r\}}] =: S \\ a &\longrightarrow a, \ \forall\ a \in A, \\ X_{i,j} &\longmapsto X_{i,n_i-1}, \ \forall\ i = 1, \ldots, d,\ j = n_i, \ldots, r. \end{aligned}$$

Let $I \subseteq R'$ be a monomial ideal. Then $p_{\underline{n}}(I)S$ is the monomial ideal of S obtained from I by setting $X_{i,j} = X_{i,n_i-1}$ for any $X_{i,j} \in I$ such that $n_i \leq j \leq r$. It is straightforward to show that if $f_1, \ldots, f_m$ is a monomial generating sequence for I, then $p_{\underline{n}}(f_1), \ldots, p_{\underline{n}}(f_m)$ is a monomial generating sequence for $p_{\underline{n}}(I)S$.

Fact 3.125. (a) If $\underline{n} = (1, \ldots, 1) \in \mathbb{N}^d$, then $p_{\underline{n}} = p$, where p is from Definition 3.53.

(b) If $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ such that $n_1, \ldots, n_d > r$, then $S = R'$ and $p_{\underline{n}}(I)S = I$ for any monomial ideal $I \subseteq R'$.

Example 3.126. Consider the following graph $(\Sigma_2 P_2)_\lambda$ with $G_\omega := (P_2)_\omega = (\ v_1 \xrightarrow{\ 1\ } v_2 \xrightarrow{\ 2\ } v_3\ )$.

$$\begin{array}{ccccc}
v_1 & \overset{4}{\text{———}} & v_{1,1} & \overset{3}{\text{———}} & v_{1,2} \\
1\Big| & & & & \\
v_2 & \overset{3}{\text{———}} & v_{2,1} & \overset{3}{\text{———}} & v_{2,2} \\
2\Big| & & & & \\
v_3 & \overset{2}{\text{———}} & v_{3,1} & \overset{5}{\text{———}} & v_{3,2}
\end{array}$$

Let

$$\begin{aligned}
I := I_{2,\min}\big((\Sigma_2 P_2)_\lambda\big) = \big(&X_{1,2}^3 X_{1,1}^3 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\
&X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^2 X_{3,1}^2, X_{3,2}^5 X_{3,1}^2 X_3^2\big) R'.
\end{aligned}$$

Let $\underline{n} = (2,3,1) \in \mathbb{N}^3$. Then $S = R[X_{1,0}, X_{1,1}, X_{2,0}, X_{2,1}, X_{2,2}, X_{3,0}]$ and setting $X_{1,2} = X_{1,1}$, $X_{3,1} = X_3$ and $X_{3,2} = X_3$ in I we have

$$\begin{aligned}
p_{\underline{n}}(I)S &= \big(X_{1,1}^3 X_{1,1}^3 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\
&\qquad X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^2 X_3^2, X_3^5 X_3^2 X_3^2\big) R' \\
&= \big(X_{1,1}^6 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\
&\qquad X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^4, X_3^9\big) R'.
\end{aligned}$$

Definition 3.127. Let $\iota = r$ or $r-1$, $(\Sigma_\iota G)_\lambda$ a weighted ι-suspension of G_ω, $I := I_{r,f}\big((\Sigma_\iota G)_\lambda\big)$, $\underline{n} \in \mathbb{N}^d$ and P_r an r-path in $(\Sigma_\iota G)_\lambda$ with the corresponding generator $\underline{X}^{\underline{\alpha}}$ in I. We write

$$P_r \overset{\underline{n}}{\rightsquigarrow} v_{i_1,j_1} \cdots v_{i_m,j_m} =: \wp$$

if the reduction $\operatorname{red}\big(p_{\underline{n}}(\underline{X}^{\underline{\alpha}})\big) = X_{i_1,j_1} \dots X_{i_m,j_m}$. We call that $\wp$ is a *path* in $p_{\underline{n}}(I)$.

Remark. If $\underline{n}$ is known from the context, we usually write $P_r \rightsquigarrow \wp$ instead of $P_r \overset{\underline{n}}{\rightsquigarrow} \wp$.

Example 3.128. In Example 3.126, $P_2 := v_{1,2} v_{1,1} v_{1,0} \overset{\underline{n}}{\rightsquigarrow} v_{1,1} v_{1,0} =: \wp$ since $X_{1,2}^3 X_{1,1}^3 X_{1,0}^4$ is the corresponding generator of P_r in I and $\operatorname{red}\big(p_{\underline{n}}(X_{1,2}^3 X_{1,1}^3 X_{1,0}^4)\big) = \operatorname{red}(X_{1,1}^3 X_{1,1}^3 X_{1,0}^4) = X_{1,1} X_{1,0}$.

Definition 3.129. Let $\iota = r$ or $r-1$, $(\Sigma_\iota G)_\lambda$ a weighted ι-suspension of G_ω, $I := I_{r,f}\big((\Sigma_\iota G)_\lambda\big)$, $\underline{n} \in \mathbb{N}^d$, P_r an r-path in $(\Sigma_\iota G)_\lambda$ with the corresponding generator $\underline{X}^{\underline{\alpha}}$ in I and $P_r \overset{\underline{n}}{\rightsquigarrow} \wp$. Let

$\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_\iota G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Denote $v_{i,j} \smile (P_r \overset{\underline{n}}{\rightsquigarrow} \wp, \mathfrak{P})$ if $v_{i,j} \in V'' \cap V(\wp)$ and $X_{i,j}^{\delta''(v_{i,j})} \mid p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$, and denote $v_{i,j} \not\smile (P_r \overset{\underline{n}}{\rightsquigarrow} \wp, \mathfrak{P})$ otherwise. In particular, if $P_r = \wp$, then denote $v_{i,j} \smile (\wp, \mathfrak{P})$ or $v_{i,j} \not\smile (\wp, \mathfrak{P})$ if $v_{i,j} \in V'' \cap V(\wp)$ and $X_{i,j}^{\delta''(v_{i,j})} \mid p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$.

Remark. If $\underline{n}$ and $\mathfrak{P}$ is known from the context, we write $v_{i,j} \smile (P_r \rightsquigarrow \wp)$ or $v_{i,j} \not\smile (P_r \rightsquigarrow \wp)$. In particular, if $P_r = \wp$, then write $v_{i,j} \smile \wp$ or $v_{i,j} \not\smile \wp$.

Example 3.130. A weighted suspension $(\Sigma G)_\lambda$ of $G_\omega := (P_1)_\omega = \big(\; v_1 \overset{1}{\text{———}} v_2 \; \big)$ is

$$\begin{tikzcd}[arrows=dash]
v_{1,1} \arrow[d, "2"] & v_{2,1} \arrow[d, "3"] \\
v_1 \arrow[r, "1"] & v_2.
\end{tikzcd}$$

Let $I := I_{2,\min}\big((\Sigma P_1)_\lambda\big)$ and $\underline{n} := (1,1,1)$. Then $p_{\underline{n}}(I)$ is obtained from I by setting $X_{1,1} = X_{1,0}$ and $X_{2,1} = X_{2,0}$ in I. We have $P_2 := v_{1,1}v_1v_2 \overset{\underline{n}}{\rightsquigarrow} v_1v_2$. Let $\underline{X}^{\underline{\alpha}} := X_{1,1}^2 X_{1,0} X_{2,0}$ be the corresponding generator of $v_{1,1}v_1v_2$ in I. Then $p_{\underline{n}}(\underline{X}^{\underline{\alpha}}) = X_{1,0}^3 X_{2,0}$. Let $\mathfrak{p} = \{v_{1,0}^2, v_{2,0}^2, v_{1,1}\}$. Then $v_{1,0} \smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{p})$ since $X_{1,0}^2 \mid X_{1,0}^3 X_{2,0}$, $v_{2,0} \not\smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{p})$ since $X_{2,0}^2 \nmid X_{1,0}^3 X_{2,0}$, $v_{1,1} \not\smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{p})$ since $v_{1,1} \notin \{v_{1,0}, v_{2,0}\}$.

Lemma 3.131. Let $\iota = r$ or $r-1$, $(\Sigma_\iota G)_\lambda$ a weighted ι-suspension of G_ω, $I := I_{r,f}\big((\Sigma_\iota G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$. Then for any path $\wp$ in $p_{\underline{n}}(I)$ such that $P_r \rightsquigarrow \wp$, we have $v_{k,l} \smile (P_r \rightsquigarrow \wp)$ for some $v_{k,l} \in V''$.

Proof. Assume $\wp := v_{i_1,j_1} \dots v_{i_m,j_m}$ and $\underline{X}^{\underline{\alpha}}$ is the corresponding generator of the r-path P_r in I. Then $\operatorname{red}\big(p_{\underline{n}}(\underline{X}^{\underline{\alpha}})\big) = X_{i_1,j_1} \dots X_{i_m,j_m}$ and $p_{\underline{n}}\big(\underline{X}^{\underline{\alpha}}\big) \in p_{\underline{n}}(I) \subseteq P(V'', \delta'')$. So there exists some $v_{k,l} \in V''$ such that $v_{k,l} \in V(\wp)$ and $X_{k,l}^{\delta''(v_{k,l})} \mid p_{\underline{n}}\big(\underline{X}^{\underline{\alpha}}\big)$. Hence $v_{k,l} \smile (P_r \rightsquigarrow \wp)$. $\square$

Remark. One can think of $p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$ as the corresponding generator of $(P_r \rightsquigarrow \wp)$.

Definition 3.132. Let $\iota = r$ or $r-1$, $(\Sigma_\iota G)_\lambda$ a weighted ι-suspension of G_ω, $I := I_{r,f}\big((\Sigma_\iota G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. For $v_{i,j} \in V''$, set

$$\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big) := \big\{ P_r \overset{\underline{n}}{\rightsquigarrow} \wp \mid v_{i,j} \smile (P_r \rightsquigarrow \wp) \text{ but } v_{k,l} \not\smile (P_r \rightsquigarrow \wp) \; \forall \; v_{k,l} \in V'' \smallsetminus \{v_{i,j}\} \big\}.$$

If $(P_r \rightsquigarrow \wp) \in \mathfrak{P}\big(p_{\underline{n}}(I)\big)$ such that $P_r = \wp$, then we write $P_r \in \mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big)$. So in particular, we have

$$\mathfrak{P}_{i,j}(I) = \big\{P_r \mid P_r \text{ an } r\text{-path in } (\Sigma_\iota G)_\lambda \text{ such that } v_{i,j} \smile P_r \text{ but } v_{k,l} \not\smile P_r\ \forall\ v_{k,l} \in V'' \smallsetminus \{v_{i,j}\}\big\}.$$

Remark. If $p_{\underline{n}}(I)$ is known from the context, we usually write $\mathfrak{P}_{i,j}$ instead of $\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big)$.

Lemma 3.133. Let $\iota = r$ or $r-1$, $(\Sigma_\iota G)_\lambda$ a weighted ι-suspension of G_ω, $I := I_{r,f}\big((\Sigma_\iota G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$. Then $\mathfrak{P}_{i,j}(p_{\underline{n}}(I)) \neq \emptyset$ for any $v_{i,j} \in V''$. In particular, if $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$, then we have $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$.

Proof. Suppose $\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big) = \emptyset$ for some $v_{i,j} \in V''$. Then since $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$, we have $p_{\underline{n}}(I) \subseteq P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big)$, a contradiction.

Let $I_k := P(V'', \delta'')$ occur in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I) = \bigcap_{i=1}^n I_i$ with $k \in \{1, \ldots, n\}$ such that $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ occur in an m-irreducible decomposition of $p_{\underline{n}}(I)$ for some $v_{i,j} \in V''$. Then since $p_{\underline{n}}(I) \subseteq I_k = P(V'', \delta'')$, we have

$$p_{\underline{n}}(I) \subseteq P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \subsetneq P(V'', \delta'') = I_k.$$

So

$$\bigcap_{i=1}^n I_i = p_{\underline{n}}(I) = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \cap \bigcap_{i=1}^n I_i = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \cap \bigcap_{i=1, i\neq k}^n I_i.$$

By Fact 2.67, the number of ideals in any irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$ is n, so the above decomposition on the right is also an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then Fact 2.67 implies $I_k = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big)$, a contradiction. $\square$

Proposition 3.134. Let $\iota = 2$ or 1. Let $(\Sigma_\iota G)_\lambda$ be a weighted ι-path suspension of G_ω such that

(a) $\lambda(v_i v_j) \leq f\{\lambda(v_i v_j), \lambda(v_i v_{i,1})\} \leq \lambda(v_i v_{i,1})$ and $\lambda(v_i v_j) \leq f\{\lambda(v_i v_j), \lambda(v_j v_{j,1})\} \leq \lambda(v_j v_{j,1})$ for all edge $v_i v_j \in E\big((\Sigma_\iota G)_\lambda\big)$,

(b) $\lambda(v_i v_{i,1}) \leq f\{\lambda(v_i v_{i,1}), \lambda(v_{i,1} v_{i,2})\}$ for $i = 1, \ldots, d$,

(c) $f\{\lambda(v_iv_j), \lambda(v_jv_k)\} \leq \min\{f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\}, f\{\lambda(v_kv_j), \lambda(v_jv_{j,1})\}\}$ for all 2-paths $v_iv_jv_k$ in G_ω,

(d) for all 3-paths $v_iv_jv_kv_l$ in G_ω: if $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} < \lambda(v_jv_k)$, then $f\{\lambda(v_jv_k), \lambda(v_kv_l)\} \geq \lambda(v_jv_k)$,

(e) for all 3-cycles $v_iv_jv_kv_i$ in G_ω: if $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < \lambda(v_iv_k)$, then either $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} \geq \lambda(v_kv_i)$, or $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} \geq \lambda(v_kv_j)$ and $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} \leq \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.

Let $I := I_{2,f}\big((\Sigma_\iota G)_\lambda\big)$ and $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_\iota G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then there exists at most one $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$. In particular, if $\iota = 2$, then there exists a unique $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$, so $p_{\underline{n}}(I)$ is m-unmixed.

Proof. Suppose there exist $v_{i,\alpha}, v_{i,\beta} \in V''$ with $0 \leq \alpha < \beta \leq 2$ for some $i \in \{1, \ldots, d\}$.

(a) Suppose $\alpha = 1$ and $\beta = 2$. Then $\iota = 2$. By Lemma 3.133, we have $\mathfrak{P}_{i,2}\big(p_{\underline{n}}(I)\big) = \{v_{i,2}v_{i,1}v_{i,0}\}$ and $v_{i,1}v_{i,0}v_{j,0} \in \mathfrak{P}_{i,1}$ for some edge $v_{i,0}v_{j,0} \in E(G_\omega)$. So $v_{i,1} \smile v_{i,1}v_{i,0}v_{j,0}$ and then $\delta''(v_{i,1}) \leq \lambda(v_{i,0}v_{i,1}) \leq f\{\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,1}v_{i,2})\}$ by the condition (b), implying $v_{i,1} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(b) Suppose $\alpha = 0$ and $\beta = 2$. Then $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$ and $\mathfrak{P}_{i,0} \neq \emptyset$ by Lemma 3.133. So we have the following 3 cases.

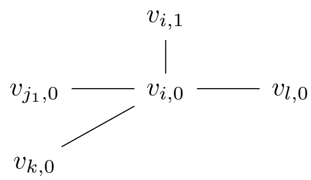
(1) Assume $v_{i,1}v_{i,0}v_{j,0} \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq f\{\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})\} \leq \lambda(v_{i,0}v_{i,1})$ by the condition (a), implying $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(2) Assume $v_{i,0}v_{j,0}v_{k,0} \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq \lambda(v_{i,0}v_{j,0}) \leq f\{\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})\} \leq \lambda(v_{i,0}v_{i,1})$ by the condition (a), implying $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

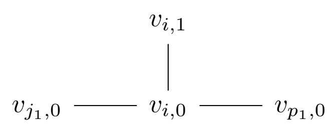
(3) Assume $v_{j,0}v_{i,0}v_{k,0} \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq f\{\lambda(v_{j,0}v_{i,0}), \lambda(v_{i,0}v_{k,0})\} \leq \lambda(v_iv_{i,1})$ by the condition (c) and (a), implying $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(c) Suppose $\alpha = 0$ and $\beta = 1$. Suppose $(v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0}) \in \mathfrak{P}_{i,1}$, then $v_{i,2}v_{i,1}v_{i,0}$ is not a path in $p_{\underline{n}}(I)$ and similar to (b), we have $v_{i,0} \smile (v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0})$, a contradiction. Similarly we have $v_{i,2}v_{i,1}v_{i,0} \notin \mathfrak{P}_{i,1}$. So there exists $v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$. Then $(v_{j_1,1}v_{j_1,0}v_{i,0} \rightsquigarrow v_{j_1,0}v_{i,0}) \notin \mathfrak{P}_{i,0}$, $v_{k,0}v_{j_1,0}v_{i,0} \notin \mathfrak{P}_{i,0}$ for any $v_{k,0}v_{j_1,0} \in E(G_\omega)$, and $v_{j_1,0}v_{i,0}v_{l,0} \notin \mathfrak{P}_{i,0}$ for any $v_{i,0}v_{l,0} \in E(G_\omega)$.

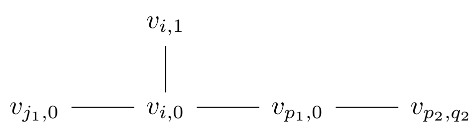
(1) Assume $v_{k,0}v_{i,0}v_{l,0} \in \mathfrak{P}_{i,0}$ with $k \neq j_1$. Since $v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$, we have $v_{j_1,0}, v_{i,0} \not\smile v_{j_1,0}v_{i,0}v_{l,0}$ through the condition (a) and (c). Then since $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$, we have $v_{l,0} \smile v_{j_1,0}v_{i,0}v_{l,0}$ by Lemma 3.131. So $v_{l,0} \smile v_{k,0}v_{i,0}v_{l,0}$, contradicted by $v_{k,0}v_{i,0}v_{l,0} \in \mathfrak{P}_{i,0}$.



(2) Assume $(v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$. Similar to the case (c).(1), we have $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$ and then $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0})$, contradicted by $(v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$.



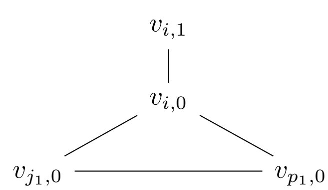
(3) Assume $v_{i,0}v_{p_1,0}v_{p_2,p_2} \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$.



Similar to the case (c).(1), we have $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$, we have $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$. So $f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} < \delta''(v_{p_1,0}) \leq \lambda(v_{i,0}v_{p_1,0})$ and hence $q_2 = 0$. Since $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{i,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}$, we have $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$.

i. Assume $j_1 \neq p_2$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 3-path in G_ω. Since $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \lambda(v_{i,0}v_{p_1,0})$ by the condition (d), contradicted by that $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$.

ii. Assume $j_1 = p_2$. Then $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$. So by the condition (e), we have the following 2 cases.



A. Assume $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$. Similar to the case (c).(1), we have $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$. So $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{j_1,0}$, contradictied by that $v_{i,0}v_{p_1,0}v_{j_1,0} \in \mathfrak{P}_{i,0}$.

B. Assume now $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})\} \geq \lambda(v_{p_1,0}v_{j_1,0})$ and $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\}$. Then since $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{i,1}$ and $v_{j_1,0} \not\smile v_{j_1,0}v_{p_1,0}v_{i,0}$, we have $v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}$; and since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{j_1,0}$, we have $v_{p_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicted by Lemma 3.131.

In particular, if $\iota = 2$, then for $i = 1, \ldots, d$, by definition of $p_{\underline{n}}(I)$ we have there exists a generator where all variables are of the form X_{i,i_l} with $i_l \in \{0, 1, 2\}$, so there exists a $v_{i,i_j} \in V''$. □

Remark. If $\iota = 1$, $f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_iv_{i,1})$ and $f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_\iota G)_\lambda\big)$ from condition (a) is redundant.

Proposition 3.135. Let $\iota = 3$ or 2. Let $(\Sigma_\iota G)_\lambda$ be a weighted ι-path suspension of G_ω such that

(a) $\lambda(v_iv_j) \leq f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\} \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_\iota G)_\lambda\big)$,

(b) $\lambda(v_{i,k}v_{i,k+1}) \leq f\{\lambda(v_{i,k}v_{i,k+1}), \lambda(v_{i,k+1}v_{i,k+2})\}$ for $i = 1, \ldots, d$ and $k = 0, 1$,

(c) $f\{\lambda(v_iv_j), \lambda(v_jv_k)\} \leq \min\big\{f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\}, f\{\lambda(v_kv_j), \lambda(v_jv_{j,1})\}\big\}$ for all 2-paths $v_iv_jv_k$ in G_ω,

(d) for all 4-paths $v_iv_jv_kv_lv_m$ in G_ω: if $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} < \lambda(v_jv_k)$ or $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < \lambda(v_kv_l)$, then $f\{\lambda(v_iv_j), \lambda(v_jv_k)\} \geq \lambda(v_jv_k)$ or $f\{\lambda(v_kv_l), \lambda(v_lv_m)\} \geq \lambda(v_kv_l)$,

(e) for all 3-cycles $v_iv_jv_kv_i$ in G_ω:

(1) if $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < f\{\lambda(v_iv_{i,1}), \lambda(v_iv_k)\}$ or there exists $v_iv_l \in E(G_\omega)$ with $j \neq l \neq k$ such that $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < f\{\lambda(v_iv_l), \lambda(v_iv_k)\}$, then $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} \leq \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$ and $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} \geq \lambda(v_kv_j)$, and

(2) if $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < \lambda(v_iv_k)$, then

i. $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} \geq \lambda(v_kv_i)$, and ii.

$$\begin{aligned}&\forall\ v_jv_{l_1,l_2}\in E\big((\Sigma_\iota G)_\lambda\big) \text{ with } v_i\neq v_{l_1,l_2}\neq v_k:\\ &\qquad\begin{cases} f\{\lambda(v_jv_i),\lambda(v_jv_{l_1,l_2})\}\leq \max\big\{\lambda(v_jv_i), f\{\lambda(v_jv_k),\lambda(v_jv_{l_1,l_2})\}\big\}, \text{ and}\\ f\{\lambda(v_jv_i),\lambda(v_jv_{l_1,l_2})\}\leq \max\big\{f\{\lambda(v_jv_i),\lambda(v_jv_k)\}, f\{\lambda(v_jv_k),\lambda(v_jv_{l_1,l_2})\}\big\},\end{cases}\end{aligned}$$

and

iii.

$$\begin{cases} f\{\lambda(v_kv_j),\lambda(v_kv_{l_1,l_2})\}\leq \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_i),\lambda(v_kv_{l_1,l_2})\}\big\}\\ \qquad\qquad\qquad\qquad \forall\ v_kv_{l_1,l_2}\in E\big((\Sigma_\iota G)_\lambda\big) \text{ with } v_i\neq v_{l_1,l_2}\neq v_k,\\ \text{or } f\{\lambda(v_jv_i),\lambda(v_jv_k)\}\geq\lambda(v_jv_i),\end{cases}$$

(f) for all 4-cycles $v_iv_jv_kv_lv_i$: if $f\{\lambda(v_{i,1}v_i),\lambda(v_iv_j)\}<\lambda(v_iv_l)$, then either

(1)

$$f\{\lambda(v_kv_j),\lambda(v_kv_l)\}\geq\lambda(v_kv_l) \text{ and } f\{\lambda(v_jv_i),\lambda(v_jv_k)\}\geq\lambda(v_jv_i),$$

or

(2) i. $f\{\lambda(v_lv_i),\lambda(v_lv_k)\}\geq\min\{\lambda(v_lv_i),\lambda(v_lv_k)\}$, and

ii. $f\{\lambda(v_jv_i),\lambda(v_jv_k)\}\leq\max\{\lambda(v_jv_i),\lambda(v_jv_k)\}$, and

iii.

$$\begin{cases} \text{either } f\{\lambda(v_kv_j),\lambda(v_kv_l)\}\geq\lambda(v_kv_l),\\ \text{or } f\{\lambda(v_lv_i),\lambda(v_lv_k)\}\geq\lambda(v_lv_k) \text{ and}\\ \quad \text{if } v_jv_l\in E(G_\omega), \text{ then } f\{\lambda(v_jv_i),\lambda(v_jv_k)\}\leq\max\big\{\lambda(v_jv_k), f\{\lambda(v_jv_i),\lambda(v_jv_l)\}\big\},\\ \text{or } f\{\lambda(v_kv_j),\lambda(v_kv_l)\}\geq\lambda(v_kv_j) \text{ and } f\{\lambda(v_lv_i),\lambda(v_lv_k)\}\geq\lambda(v_lv_i) \text{ and}\\ \quad \text{if } v_jv_l\in E(G_\omega), \text{ then } f\{\lambda(v_jv_i),\lambda(v_jv_k)\}\leq\max\big\{\lambda(v_jv_k), f\{\lambda(v_jv_i),\lambda(v_jv_l)\}\big\},\end{cases}$$

and

iv.

$$\begin{cases} \text{either } f\{\lambda(v_jv_i),\lambda(v_jv_k)\} \geq \lambda(v_jv_i), \\ \text{or } f\{\lambda(v_lv_i),\lambda(v_lv_k)\} \geq \lambda(v_lv_i), \\ \text{or } f\{\lambda(v_lv_i),\lambda(v_lv_k)\} \geq \min\{\lambda(v_lv_i),\lambda(v_lv_k)\} \text{ and } f\{\lambda(v_kv_j),\lambda(v_kv_l)\} \leq \lambda(v_kv_l), \\ \text{or } \forall\ v_kv_{l_1,l_2} \in E\big((\Sigma_\iota G)_\lambda\big) \text{ with } v_j \neq v_{l_1,l_2} \neq v_l : \\ \quad \text{either } f\{\lambda(v_kv_j),\lambda(v_kv_l)\} \leq \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_l),\lambda(v_kv_{l_1,l_2})\}\big\}, \\ \quad \text{or } f\{\lambda(v_kv_j),\lambda(v_kv_{l_1,l_2})\} \leq \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_l),\lambda(v_kv_{l_1,l_2})\}\big\}, \end{cases}$$

and

v.

$$\begin{cases} \text{either } f\{\lambda(v_kv_j),\lambda(v_kv_l)\} \geq \lambda(v_kv_l), \\ \text{or } f\{\lambda(v_lv_i),\lambda(v_lv_k)\} \geq \lambda(v_lv_i), \\ \text{or } \forall\ v_jv_{l_1,l_2} \in E\big((\Sigma_\iota G)_\lambda\big) \text{ with } v_i \neq v_{l_1,l_2} \neq v_k : \\ \quad \text{either } f\{\lambda(v_jv_i),\lambda(v_jv_k)\} \leq \max\big\{\lambda(v_jv_k), f\{\lambda(v_jv_i),\lambda(v_jv_{l_1,l_2})\}\big\}, \\ \quad \text{or } f\{\lambda(v_jv_k),\lambda(v_jv_{l_1,l_2})\} \leq \max\big\{\lambda(v_jv_k), f\{\lambda(v_jv_i),\lambda(v_jv_{l_1,l_2})\}\big\}. \end{cases}$$

Let $I := I_{3,f}\big((\Sigma_\iota G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_\iota G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then there exists at most one $v_{i,i_j} \in V''$ for $i = 1, \dots, d$. In particular, if $\iota = 3$ and $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$, then there exists a unique $v_{i,i_j} \in V''$ for $i = 1, \dots, d$, so $p_{\underline{n}}(I)$ is m-unmixed.

Proof. Suppose there exist $v_{i,\alpha}, v_{i,\beta} \in V''$ with $0 \leq \alpha < \beta \leq 3$ for some $i \in \{1, \dots, d\}$.

(a) Suppose $\alpha = 1$ or 2. Since $\mathfrak{P}_{i,\alpha} \neq \emptyset$ by Lemma 3.133, we have

$$\delta''(v_{i,\alpha}) \leq \max\big\{\lambda(v_{i,\alpha}v_{i,\alpha-1}), f\{\lambda(v_{i,\alpha+1}v_{i,\alpha}, \lambda(v_{i,\alpha}v_{i,\alpha-1}\}\big\} = f\{\lambda(v_{i,\alpha+1}v_{i,\alpha}, \lambda(v_{i,\alpha}v_{i,\alpha-1}\},$$

by the condition (b). Then for any $(P_r \rightsquigarrow \wp) \in \mathfrak{P}_{i,\beta}$ we have $v_{i,\alpha} \smile (P_r \rightsquigarrow \wp)$, a contradiction.

(b) Suppose $\alpha = 0$ and $\beta = 3$. Then $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$ and $\mathfrak{P}_{i,0} \neq \emptyset$ by Lemma 3.133. So we have the following 3 cases.

(1) Assume $v_{i,2}v_{i,1}v_{i,0}v_{j,0} \in \mathfrak{P}_{i,0}$ or $v_{i,1}v_{i,0}v_{j,0}v_{k,l} \in \mathfrak{P}_{i,0}$ or $(v_{i,1}v_{i,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j,0}) \in \mathfrak{P}_{i,0}$. Then we have $\delta''(v_{i,0}) \leq f\{\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})\} \leq \lambda(v_{i,0}v_{i,1})$ by the condition (a), implying $v_{i,0} \smile$

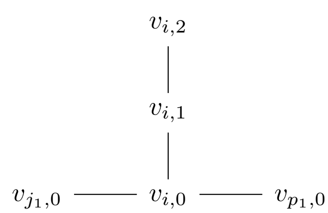
$v_{i,3}v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$.

(2) Assume $v_{i,0}v_{j,0}v_{k,0}v_{l,m} \in \mathfrak{P}_{i,0}$ or $v_{i,0}v_{j,0}v_{j,1}v_{j,2} \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i,0}v_{j,0}v_{j,1}) \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq \lambda(v_{i,0}v_{j,0}) \leq f\{\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})\} \leq \lambda(v_{i,0}v_{i,1})$ by the condition (a), implying $v_{i,0} \smile v_{i,3}v_{i,2}v_{i,1}v_{i,0}$, contradicted by $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$.

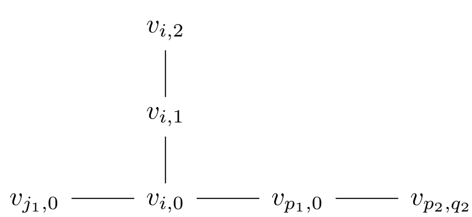
(3) Assume $v_{j,0}v_{i,0}v_{k,0}v_{l,m} \in \mathfrak{P}_{i,0}$ or $(v_{j,0}v_{i,0}v_{k,1}v_{k,0} \rightsquigarrow v_{j,0}v_{i,0}v_{k,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq f\{\lambda(v_{j,0}v_{i,0}), \lambda(v_{i,0}v_{k,0})\} \leq \lambda(v_i v_{i,1})$ by the condition (c) and (a), implying $v_{i,0} \smile v_{i,3}v_{i,2}v_{i,1}v_{i,0}$, contradicted by that $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$.

(c) Suppose $\alpha = 0$ and $\beta = 2$. Then similar to the case (b), $v_{i,3}v_{i,2}v_{i,1}v_{i,0}, (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,2}v_{i,1}v_{i,0}) \notin \mathfrak{P}_{i,2}$. So $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G_\omega)$. Then one can check that $(P_r \rightsquigarrow \wp) \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,2} \in V(\wp)$. So we have the following 11 cases.

(1) Assume $(v_{i,1}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$. Then we have $v_{p_1,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, through the condition (c) we have $v_{j_1,0}, v_{i,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$, contradicted by Lemma 3.131.

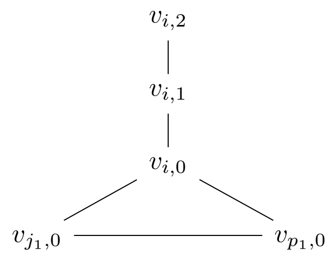


(2) Assume $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$. Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have $v_{j_1,0}, v_{i,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ through the condition (c). Note that $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$, so $v_{p_1,0}, v_{p_2,q_2} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, contradicted by that $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$ and Lemma 3.131.



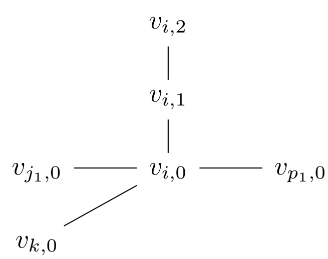
(3) Assume $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{j_1,0} = v_{p_2,q_2}$. Then $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a

3-cycle in G_ω.

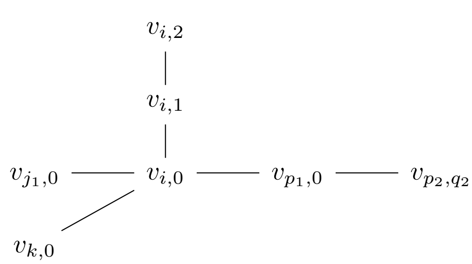


Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have $v_{i,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ and $f\{\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,0}v_{j_1,0})\} < f\{\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,0}v_{p_1,0})\}$. So $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\} \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\}$ and $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})\} \geq \lambda(v_{p_1,0}v_{j_1,0})$ by the condition (e).(1). Hence we have $v_{j_1,0}, v_{p_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ by way of contradiction. By the case (a) we have $v_{i,1} \notin V''$, so $v_{i,1} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicted by Lemma 3.131.

(4) Assume $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $k \neq j_1 \neq p_1$. Then we have $v_{p_1,0}, v_{p_2,q_2} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. So $v_{j_1,0} \smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ or $v_{i,0} \smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by Lemma 3.131. Hence $v_{j_1,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ or $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ by the condition (c), contradicted by $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$.



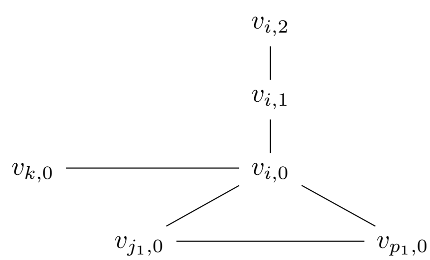
(5) Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $k \neq j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$. Then we have $v_{p_1,0}, v_{p_2,q_2} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. So $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ or $v_{i,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Lemma 3.131. Hence $v_{j_1,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ or $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ by the condition (c), contradicted by $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$.



(6) Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $k \neq j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$. Then $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G_ω. Since $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$, we have $v_{i,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$ by the condition (c). Since $v_{k,0} \not\smile v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have $v_{k,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$. Since $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ and $v_{i,0} \smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$, we have

$$f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \delta''(v_{i,0}) \leq f\{\lambda(v_{k,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\}.$$

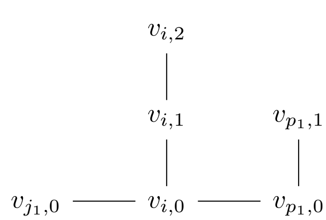
So $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\} \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\}$ and $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})\} \geq \lambda(v_{p_1,0}v_{j_1,0})$ by the condition (e).(1). Hence similar to the case (c).(3), $v_{j_1,0}, v_{p_1,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicted by Lemma 3.131. (Note also $v_{i,1}, v_{i,0}, v_{j_1,0}, v_{p_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$.)



(7) Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,1}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$. By the condition (b) we have

$$\lambda(v_{p_1,0}v_{p_1,1}) \leq f\{\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})\} < f\{\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})\} + \lambda(v_{p_1,1}v_{p_1,2}).$$

So $v_{p_1,1} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Also, we have $v_{i,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$ by the condition (c), and $v_{j_1,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$, contradicted by Lemma 3.131.



(8) Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$. By the condition (b) we have

$$\begin{aligned}
& f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\} + \lambda(v_{p_1,0}v_{p_1,1}) \\
\leq\ & f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\} + f\{\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})\} \\
<\ & f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\} + f\{\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})\} + \lambda(v_{p_1,1}v_{p_1,2}),
\end{aligned}$$

we have $v_{p_1,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Also, we have $v_{i,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by the condition (c), and $v_{j_1,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{i,2} \arrow[d, dash] & \\
 & v_{i,1} \arrow[d, dash] & \\
v_{j_1,0} \arrow[r, dash] & v_{i,0} \arrow[r, dash] & v_{p_1,0}
\end{tikzcd}$$

(9) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1 \neq p_2$. By the condition (a),

$$\lambda(v_{p_1,0}v_{p_2,0}) \leq f\{\lambda(v_{p_1,0}v_{p_2,0}), \lambda(v_{p_2,0}v_{p_2,1})\} < f\{\lambda(v_{p_1,0}v_{p_2,0}), \lambda(v_{p_2,0}v_{p_2,1})\} + \lambda(v_{p_2,0}v_{p_2,1}).$$

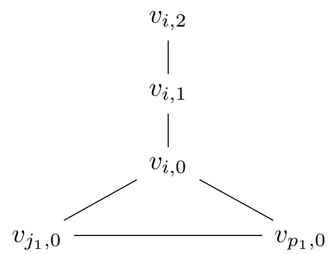
So $v_{p_2,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Also, we have $v_{i,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by the condition (c), and $v_{j_1,0}, v_{p_1,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{i,2} \arrow[d, dash] & & \\
 & v_{i,1} \arrow[d, dash] & & \\
v_{j_1,0} \arrow[r, dash] & v_{i,0} \arrow[r, dash] & v_{p_1,0} \arrow[r, dash] & v_{p_2,0}
\end{tikzcd}$$

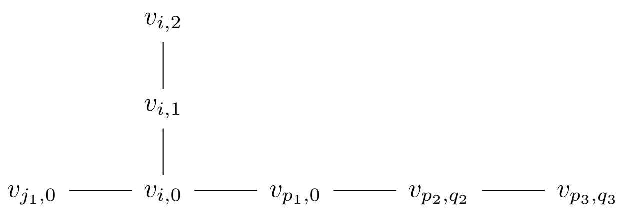
(10) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1 = p_2$. By the condition (a),

$$\lambda(v_{i,0}v_{j_1,0}) \leq f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} < f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} + \lambda(v_{j_1,1}v_{j_1,2}).$$

So $v_{j_1,0} \not\succ (v_{j_1,1}v_{j_1,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{i,1})$. Also, we have $v_{i,1}, v_{i,0} \not\succ (v_{j_1,1}v_{j_1,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{i,1})$, contradicted by Lemma 3.131.



(11) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$.



Then $v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$. Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have $v_{j_1,0}, v_{i,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by the condition (c). So $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Lemma 3.131. Hence

$$f\{\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\} < \delta''(v_{p_2,q_2}) \leq \lambda(v_{p_1,0}v_{p_2,q_2}).$$

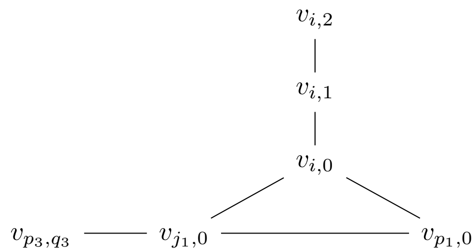
Thus, by the condition (b) we have $q_2 = 0$ and then $q_3 = 0$ by the condition (a). Since $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ and $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$, we have $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$.

i. Assume $p_2 \neq j_1 \neq p_3$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G_ω. So by the condition (d) and (c) we have

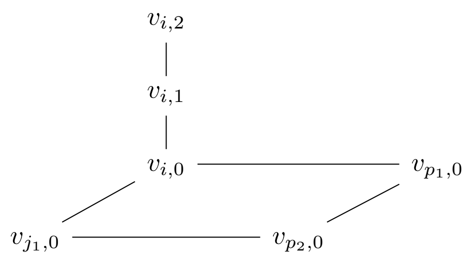
$$\lambda(v_{i,0}v_{p_1,0}) \leq f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} \leq f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\},$$

a contradiction.

ii. Assume $j_1 = p_2$. Then by the condition (e).(2).i, we have $v_{p_1,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$. Observe that $v_{p_3,q_3}, v_{i,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{i,1}, v_{i,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$. By the condition (e).(2).ii, we have $v_{j_1,0} \not\leftarrow v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.



iii. Assume $j_1 = p_3$. Then $v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}v_{i,0}$ is a 4-cycle in G_ω. So we have the following 3 cases.



A. Assume $f\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} \geq \lambda(v_{p_2,0}v_{p_1,0})$ by the condition (f).(1) or (f).(2).iii. Then we have $v_{p_2,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. But $v_{j_1,0}, v_{i,0}, v_{p_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicted by Lemma 3.131.

Note that $\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} \geq \min\{\lambda(v_{p_2,0}v_{p_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\}$ also works in this case but not wok in the corresponding case in the following case (d).(1).

B. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\} \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\}$ and

$$f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \lambda(v_{p_1,0}v_{p_2,0}),$$

by the condition (f).(2).ii and (f).(2).iii. Then we have $v_{j_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$ and $v_{p_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$. But $v_{i,0}, v_{p_2,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$, contradicted by Lemma 3.131.

C. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\} \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\}$ and

$f\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} \geq \lambda(v_{p_2,0}v_{j_1,0})$ and $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$,

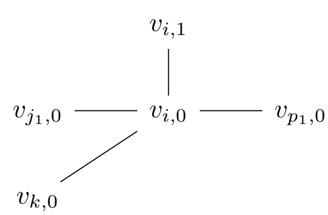
by the condition (f).(2).ii and (f).(2).iii. Then we have $v_{j_1,0} \not\leftarrow v_{p_2,0}v_{j_1}v_{i,0}v_{p_1,0}$, $v_{p_2,0} \not\leftarrow v_{p_2,0}v_{j_1}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\leftarrow v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. But $v_{i,0} \not\leftarrow v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.

(d) Suppose $\alpha = 0$ and $\beta = 1$. Then similar to the case (c), we have $v_{i,3}v_{i,2}v_{i,1}v_{i,0}, (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,2}v_{i,1}v_{i,0}), (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0}) \notin \mathfrak{P}_{i,1}$.

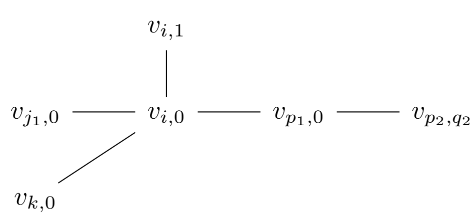
(1) Assume $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$ for some $v_{i,0}v_{j_1,0} \in E(G_\omega)$. Then one can check that $(P_r \rightsquigarrow \wp) \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,2} \in V(\wp)$. So one can also check that the remaining 11 cases are totally similar to the ones in the case (c).

(2) Assume $(v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0}) \in \mathfrak{P}_{i,1}$. Then $\wp \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,1} \in V(\wp)$.

i. Assume $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $k \neq j_1 \neq p_1$. Then $v_{j_1,0}, v_{k,0} \not\leftarrow (v_{j_1,1}v_{j_1,0}v_{i,0}v_{k,0} \rightsquigarrow v_{j_1,0}v_{i,0}v_{k,0})$. Also, we have $v_{i,0} \not\leftarrow (v_{j_1,1}v_{j_1,0}v_{i,0}v_{k,0} \rightsquigarrow v_{j_1,0}v_{i,0}v_{k,0})$ by the condition (c), contradicted by Lemma 3.131.



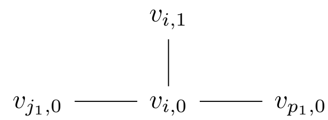
ii. Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $k \neq j_1 \neq p_1$. Then it is similar to the case (d).(2).i.



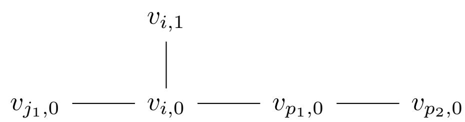
Note that in this case, we may have $v_{j_2,k_2} = v_{p_2,q_2}$.

iii. Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$. Then $v_{p_1,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by the condition (a). Also, we have $v_{i,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by the condition (c), and $v_{j_1,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow$

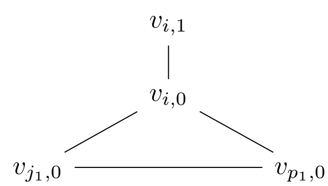
$v_{p_1,0}v_{i,0}v_{j_1,0}$), contradicted by Lemma 3.131.



iv. Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1 \neq p_2$. Then we have $v_{p_2,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by the condition (a). Also, we have $v_{i,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by the condition (c) and $v_{p_1,0}, v_{j_1,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicted by Lemma 3.131.



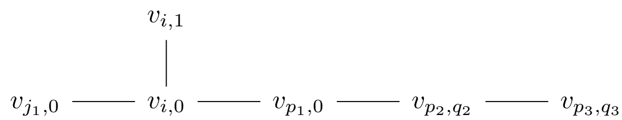
v. Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1 = p_2$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$. So $v_{p_1,0} \not\succ (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Since $v_{j_1,0} \not\succ (v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0})$, we have that $v_{j_1,0} \not\succ (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Also, we have $v_{i,0} \not\succ (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by the condition (c).



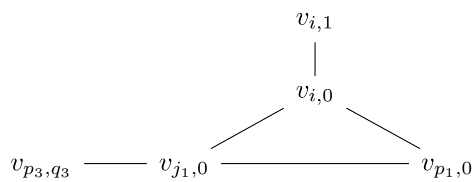
vi. Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $j_1 \neq p_1$ and $v_{p_2,q_2} \neq v_{j_1,0} \neq v_{p_3,q_3}$. By way of contradiction, we have $v_{p_2,q_2} \smile v_{j,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\succ v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $f\{\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\} < \lambda(v_{p_1,0}v_{p_2,q_2})$. So $q_3 = 0$ and then $q_2 = 0$. So we have $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G_ω. Hence by the condition (d)

$$f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} \geq \lambda(v_{i,0}v_{p_1,0}) > f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\},$$

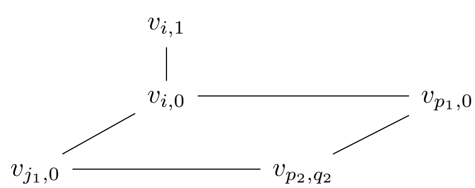
contradiction by the condition (c).



vii. Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_1,0} = v_{p_2,q_2}$. Then similar to (d).(2).v, we have $v_{p_1,0}, v_{i,0}, v_{j_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$, contradicted by Lemma 3.131.



viii. Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_1,0} = v_{p_3,q_3}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}v_{i,0}$ is a 4-cycle in G_ω, we have the following 2 cases by the condition (f).



A. Assume $f\{\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})\} \geq \lambda(v_{p_2,q_2}v_{p_1,0})$ by the condition (f).(1). Then $v_{p_2,q_2} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Since

$$\lambda(v_{j_1,1}v_{j_1,0}) + f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} > f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} \geq \lambda(v_{i,0}v_{j_1,0}),$$

by the condition (a), we have $v_{j_1,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Also, $v_{i,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$ by condition (c) and $v_{p_1,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$, contradicted by Lemma 3.131.

B. $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\}$ by the condition (f).(2).i. Since $v_{j_1,0} \not\smile (v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0})$, we have $v_{j_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Also, since $v_{i,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by the condition (c), we have $v_{p_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. So $v_{p_1,0} \smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow$

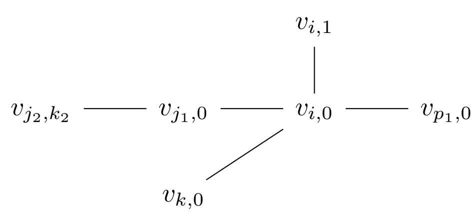
$v_{p_1,0}v_{i,0}v_{j_1,0}$) by Lemma 3.131. So $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since

$$
\begin{aligned}
f\{\lambda(v_{i,0}v_{j_1,0}), \lambda(v_{j_1,0}v_{p_2,q_2})\} &\leq f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} \\
&< \lambda(v_{j_1,1}v_{j_1,0}) + f\{\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\}
\end{aligned}
$$

by the condition (c), we have $v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Also, $v_{i,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$ by the condition (a) and $v_{p_2,q_2} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$, contradicted by Lemma 3.131.

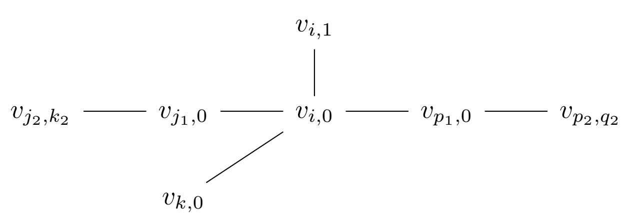
(3) Assume $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$. Then $\wp \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,1} \in V(\wp)$.

i. Assume $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$. Then $v_{j_2,k_2}, v_{j_1,0}, v_{k,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{k,0}$. Also, we have $v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{k,0}$ by the condition (c), contradicted by Lemma 3.131.



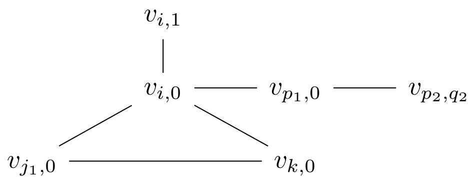
Note that in this case, we may have $v_{j_2,k_2} = v_{p_1,0}$.

ii. Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$. Then it is similar to the case (d).(3).i.

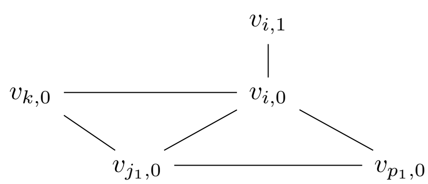


Note that in this case, we may have $v_{j_2,k_2} = v_{p_1,0}$, etc.

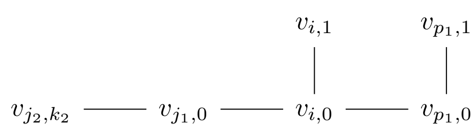
iii. Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$. Then by way of contradiction, we have $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{k,0}$. Since $v_{p_1,0} \not\smile v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have $f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} < \lambda(v_{i,0}v_{p_1,0})$ and then $q_2 = 0$. So $v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 4-path in G_ω. Hence $f\{\lambda(v_{k,0}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$, contradicted by that $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{k,0}$.



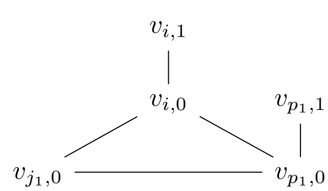
iv. Assume $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$. Then $v_{k,0}, v_{j_1,0} \not\smile v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $f\{\lambda(v_{k,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} > f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\}$ and $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$ by the condition (e).(1). So $v_{p_1,0} \not\smile v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have $v_{i,0} \not\smile v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$ by the condition (c), contradicted by Lemma 3.131.



v. Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{p_1,0} \neq v_{j_2,k_2}$. By the condition (a), we have $v_{p_1,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have $v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ by the condition (c), and $v_{j_2,k_2}, v_{j_1,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.



vi. Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.



By the condition (b), we have

$$\lambda(v_{p_1,1}v_{p_1,0}) \leq f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} < f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,2}v_{p_1,1}).$$

So $v_{p_1,1} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$ and $v_{p_1,1} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Since $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by the condition (e).(2).iii.

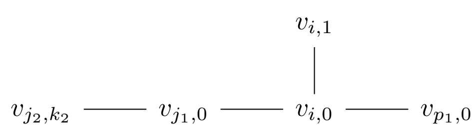
A. Assume $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\} \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0})\}, f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})\}\big\}$. Then $v_{p_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$. Also, we have $v_{i,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$ by the condition (a) and $v_{j_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$, contradicted by Lemma 3.131.

B. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Also, we have $v_{i,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$ by the condition (c) and $v_{p_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$, contradicted by Lemma 3.131.

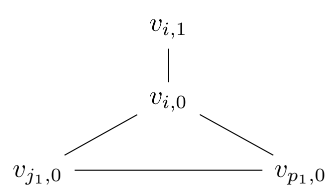
vii. Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{p_1,0} \neq v_{j_2,k_2}$. Since

$$\begin{aligned}\lambda(v_{i,0}v_{p_1,0}) &\leq f\{\lambda(v_{p_1,1}v_{p_1,0}), \lambda(v_{i,0}v_{p_1,0})\} \\ &< f\{\lambda(v_{p_1,1}v_{p_1,0}), \lambda(v_{i,0}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,2}v_{p_1,1}),\end{aligned}$$

we have $v_{p_1,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have $v_{i,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ by the condition (c), and $v_{j_2,k_2}, v_{j_1,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.

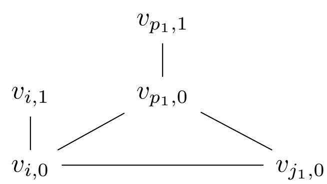


viii. Assume $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.



Since $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G_ω and $\lambda(v_{i,0}v_{p_1,0}) > f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\}$, we have the following 2 cases by the condition (e).(2).iii.

A. Assume $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\} \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})\}\big\}$. Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})\} < f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\}$. Since $v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_1,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j_1,0}v_{p_1,0}), \lambda(v_{j_1,0}v_{i,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$ by the condition (e).(1). Then it comes to the following case B.



Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})\} \geq f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})\}$. Then by the condition (b),

$$\begin{aligned}
& f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,1}v_{p_1,0}) \\
\leq\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} \\
<\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,2}v_{p_1,1}).
\end{aligned}$$

So $v_{p_1,0} \not\leftarrow (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$. Also, we have $v_{i,0} \not\leftarrow (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$ by the condition (a) and $v_{j_1,0} \not\leftarrow (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$, contradicted by Lemma 3.131.

B. Assume $f\{\lambda(v_{j_1,0}v_{i_1,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. By the condition (b) we have

$$\begin{aligned}
& f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,1}v_{p_1,0}) \\
\leq\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} \\
<\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,2}v_{p_1,1}).
\end{aligned}$$

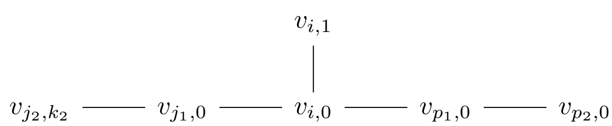
So $v_{p_1,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Also, we have $v_{i,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by the condition (c), contradicted by Lemma 3.131.

ix. Assume $f\{\lambda(v_{j_1,0}v_{i_1,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$. Then we have $v_{j_1,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. By the condition (b) we have

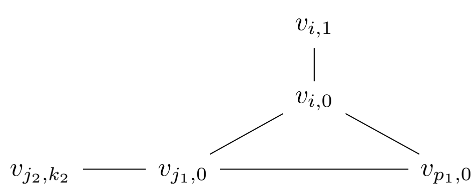
$$\begin{aligned}
& f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,1}v_{p_1,0}) \\
\leq\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} \\
<\ & f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})\} + f\{\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})\} + \lambda(v_{p_1,2}v_{p_1,1}).
\end{aligned}$$

So $v_{p_1,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Also, we have $v_{i,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by the condition (c), contradicted by Lemma 3.131.

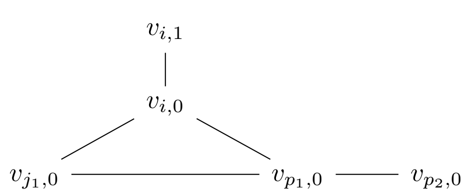
(4) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$ and $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$. By way of contradiction, $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have $f\{\lambda(v_{j_2,k_2}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} < \lambda(v_{j_1,0}v_{i,0})$. So $k_2 = 0$ and then $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ is a 4-path in G_ω. Since $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \lambda(v_{i,0}v_{p_1,0})$ by the condition (d). So $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$, a contradiction.



(5) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $j_1 = p_2$. Then by way of contradiction, we have $v_{j_2,k_2}, v_{j_1,0}, v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω and $\lambda(v_{i,0}v_{p_1,0}) > f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\}$, we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$ by the condition (e).(2).i. So $v_{p_1,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.



(6) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_1,0}$. Then by way of contradiction, we have $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ and $v_{i,0}, v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by the condition (e).(2).iii.



i. Assume $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}\big\}$. So we have $v_{p_1,0} \smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ or $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$ since $v_{p_1,0} \smile$

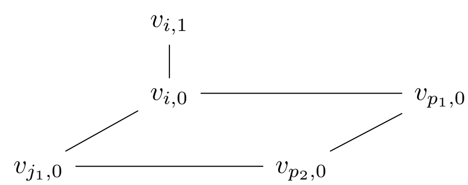
$v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$, a contradiction.

ii. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$, a contradiction.

(7) Assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_2,0}$ and $j_1 \neq p_1$. Then

$$\begin{aligned} f\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} &\leq f\{\lambda(v_{p_2,0}v_{p_1,0}), \lambda(v_{p_2,0}v_{p_2,1})\} \\ &< f\{\lambda(v_{p_2,0}v_{p_1,0}), \lambda(v_{p_2,0}v_{p_2,1})\} + \lambda(v_{p_2,0}v_{p_2,1}). \end{aligned}$$

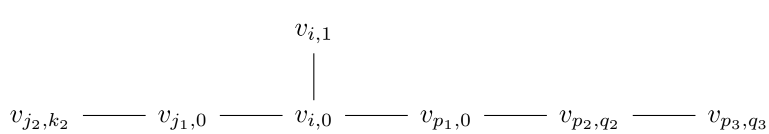
So $v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. By way of contradiction, we have $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. So $v_{p_1,0} \smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{j_1,0}v_{i,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by the condition (f).



i. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$ by the condition (f).(1). Then we have $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. By way of contradiction, $v_{i,0}, v_{p_1,0}, v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicted by Lemma 3.131.

ii. Assume we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ by the condition (f).(2).i. Then since $v_{p_1,0} \smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$, we have $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,0}$. But by way of contradiction, we have $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,0}$, contradicted by Lemma 3.131.

(8) Assume $v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$, $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$ and $v_{j_2,k_2} \neq v_{p_3,q_3} \neq v_{j_1,0}$.



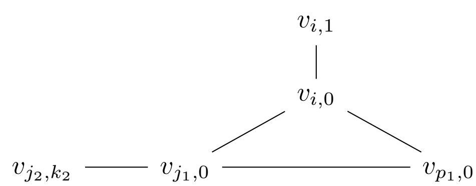
By way of contradiction, we have $v_{p_1,0} \smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $q_2 = 0$. By way of contradiction, we have $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ or $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$.

i. Suppose $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have $k_2 = 0$ and

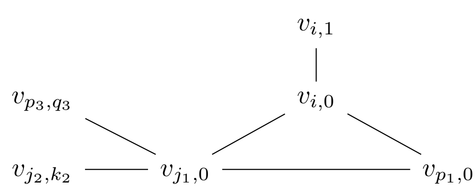
$f\{\lambda(v_{j_2,k_2}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})\} < \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 4-path in G_ω. Also, since $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \lambda(v_{i,0}v_{p_1,0})$ by the condition (d), contradicted by that $v_{p_1,0} \smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$.

ii. Suppose $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $q_3 = 0$ and $f\{\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\} < \lambda(v_{p_1,0}v_{p_2,q_2})$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path. Also, since $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} \geq \lambda(v_{i,0}v_{p_1,0})$ by the condition (d), contradicted by $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ and $v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$.

(9) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_2,k_2}$. Then by way of contradiction, we have $v_{j_2,k_2}, v_{j_1,0}, v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$ by the condition (e).(2).i. So $v_{p_1,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.



(10) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ with $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_2,k_2}$. Then it is similar to the case (3).xiv.

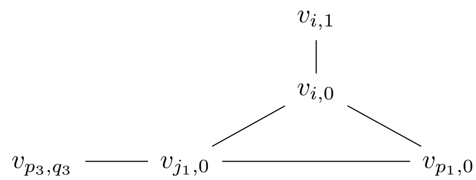


(11) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_1,0} = v_{p_2,q_2}$ and $v_{j_2,k_2} = v_{p_1,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω, $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, by the condition (e).(2).ii we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$ and

$$f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_3,q_3})\} \leq \max\big\{f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\}, f\{\lambda(v_{j_1,0}v_{p_3,q_3}), \lambda(v_{j_1,0}v_{p_1,0})\}\big\}.$$

So $v_{p_1,0}, v_{j_1,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have $v_{p_3,q_3}, v_{i,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by

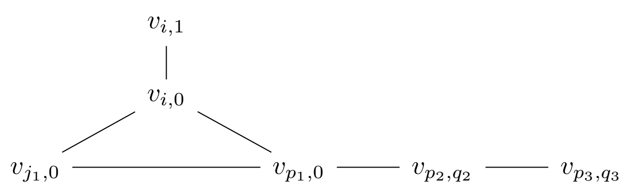
Lemma 3.131.



(12) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} \neq v_{j_1,0}$. Then by way of contradiction, we have $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$ and $v_{i,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. Suppose $v_{p_2,q_2} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. Then since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $q_2 = 0$ by the condition (b) and then $q_3 = 0$ by the condition (a). So $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_3,0}$ is a 4-path in G_ω. Since $\lambda(v_{p_1,0}v_{p_2,q_2}) > f\{\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\}$, by the condition (d) we have

$$\lambda(v_{i,0}v_{p_1,0}) \leq f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} \leq f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\},$$

a contradiction. Hence $v_{p_2,q_2} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$ and so $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. By the condition (e).(2).iii, we have the following 2 cases.

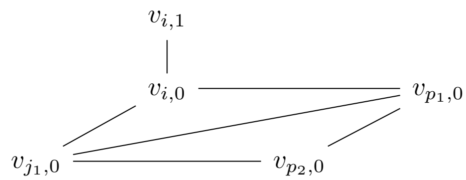


i. Assume $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\}\big\}$. Then since $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$, we have $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction.

ii. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Also, since $v_{i,0}, v_{p_2,q_2} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. So we have $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction.

(13) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} = v_{j_1,0}$. So $q_2 = 0$. Since $v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0} \in \mathfrak{P}_{i,1}$, we have $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$ and $v_{i,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}\big\}$ or $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\} \geq \lambda(v_{j_1,0}v_{i,0})$ by the condition (e).(2).iii. So $v_{p_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ or $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Hence by way of contradiction, we have $v_{p_2,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ or $v_{p_2,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So $\delta''(v_{p_2,0}) \leq \lambda(v_{p_2,0}v_{p_1,0})$. Since $v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}v_{i,0}$ is a 4-cycle in G_ω and $v_{j,0}v_{l,0} \in E(G_\omega)$, we have the follow-

ing 3 cases by the condition (f).



i. Assume $f\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} \geq \lambda(v_{p_2,0}v_{p_1,0})$ by the condition (f).(1) or (f).(2).iii. Since $\delta''(v_{p_2,0}) \leq \lambda(v_{p_2,0}v_{p_1,0})$, we have $v_{p_2,0} \smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$, a contradiction.

ii. Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \lambda(v_{p_1,0}v_{p_2,0})$ and

$$f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\} \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\}\big\},$$

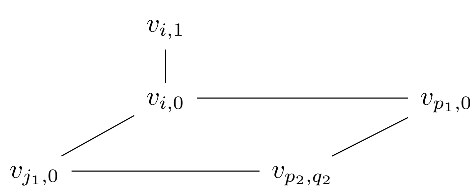
by the condition (f).(2).iii. Then $v_{p_1,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$. But we have $v_{i,0}, v_{p_2,0} \not\smile v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})\} \geq \lambda(v_{p_2,0}v_{j_1,0})$, $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \lambda(v_{p_1,0}v_{i,0})$ and

$$f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\} \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\}\big\},$$

by the condition (f).(2).iii. Then $v_{p_2,0}, v_{p_1,0}, v_{j_1,0}, v_{i,0} \not\smile v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. But we have $v_{i,0} \not\smile v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicted by Lemma 3.131.

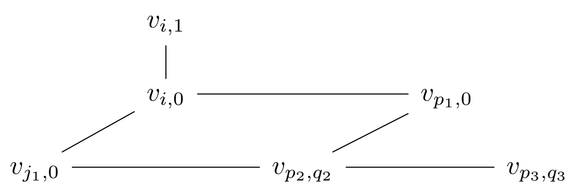
(14) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_1,0}$. Then by way of contradiction, we have $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$ and $v_{p_1,0} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by the condition (f).



i. Assume that $f\{\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})\} \geq \lambda(v_{p_2,q_2}v_{p_1,0})$ and $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,q_2})\} \geq \lambda(v_{j_1,0}v_{i,0})$ by the condition (f).(1). Then $v_{p_2,q_2}, v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, contradicted by Lemma 3.131.

ii. Assume we have $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ by the condition (f).(2).i. Then $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction.

(15) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_1,0} \neq v_{p_1,0}$. Then $q_2 = 0$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 5 cases by the condition (f).



i. Assume $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,q_2})\} \geq \lambda(v_{j_1,0}v_{i,0})$ by the condition (f).(1) or (f).(2).iv. Then we have $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $q_3 = 0$ and $\lambda(v_{p_2,q_2}v_{p_1,0}) > f\{\lambda(v_{p_2,q_2}v_{p_1,0}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\}$. So $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G_ω. Hence by the condition (d),

$$\lambda(v_{i,0}v_{p_1,0}) \leq f\{\lambda(v_{i,0}v_{j_1,0}), \lambda(v_{i,0}v_{p_1,0})\} \leq f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})\},$$

a contradiction.

ii. Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \lambda(v_{p_1,0}v_{i,0})$ by the condition (f).(2).iv. Then $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{j_1,0}, v_{p_2,q_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\}$ and

$$f\{\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})\} \leq \lambda(v_{p_2,q_2}v_{p_1,0}),$$

by the condition (f).(2).iv. By way of contradiction, we have $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. So $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$. Since $v_{i,0}, v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$, we have $v_{p_2,q_2} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$. So $v_{p_2,q_2} \smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $q_3 = 0$. So $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G_ω. Since $f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have $\lambda(v_{p_1,0}v_{p_2,q_2}) \leq f\{\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})\}$, contradicted by $v_{p_2,q_2} \smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$ and $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$.

iv. Assume $f\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,q_2})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,q_2})\}$ and

$$f\{\lambda(v_{p_2,q_2}v_{j_1,0}),\lambda(v_{p_2,q_2}v_{p_1,0})\} \leq \max\big\{\lambda(v_{p_2,q_2}v_{j_1,0}), f\{\lambda(v_{p_2,q_2}v_{p_1,0}),\lambda(v_{p_2,q_2}v_{p_3,q_3})\}\big\},$$

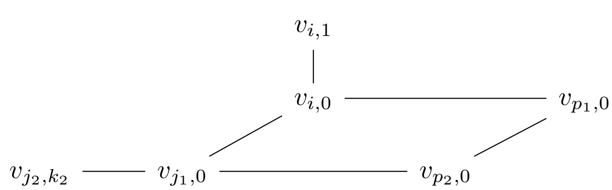
by the condition (f).(2).i and (f).(2).iv. Then $v_{p_2,q_2} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since $v_{j_1,0}, v_{i,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$, we have $v_{p_1,0} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. So $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{j_1,0}, v_{p_2,q_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$, contradicted by Lemma 3.131.

v. Assume

$$f\{\lambda(v_{p_2,q_2}v_{j_1,0}),\lambda(v_{p_2,q_2}v_{p_3,q_3})\} \leq \max\big\{\lambda(v_{p_2,q_2}v_{j_1,0}), f\{\lambda(v_{p_2,q_2}v_{p_1,0}),\lambda(v_{p_2,q_2}v_{p_3,q_3})\}\big\},$$

by the condition (f).(2).iv. Then $v_{p_2,q_2} \not\smile v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_3,q_3}$. But we have $v_{i,0}, v_{j_1,0}, v_{p_3,q_3} \not\smile v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_3,q_3}$, contradicted by Lemma 3.131.

(16) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_1,0} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_2,k_2} \neq v_{p_2,0}$. Then by way of contradiction, we have $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G_ω and $f\{\lambda(v_{i,1}v_{i,0}),\lambda(v_{i,0}v_{j_1,0})\} < \lambda(v_{i,0}v_{p_1,0})$, we have the following 4 cases by the condition (f).



i. Assume $f\{\lambda(v_{p_2,0}v_{j_1,0}),\lambda(v_{p_2,0}v_{p_1,0})\} \geq \lambda(v_{p_2,0}v_{p_1,0})$ by the condition (f).(1) or (f).(2).v. Then $v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Since $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, we have $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have $k = 0$ and $f\{\lambda(v_{j_2,k_2}v_{j_1,0}),\lambda(v_{j_1,0}v_{i,0})\} < \lambda(v_{j_1,0}v_{i,0})$. So $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ is a 4-path in G_ω. Hence $\lambda(v_{i,0}v_{p_1,0}) \geq f\{\lambda(v_{i,0}v_{p_1,0}),\lambda(v_{p_1,0}v_{p_2,0})\}$, contradicted by $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$.

ii. Assume $f\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,0})\}$ by the condition (f).(2).v. Then $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. But $v_{i,0}, v_{j_1,0}, v_{j_2,k_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,0})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,0})\}$ and

$$f\{\lambda(v_{j_1,0}v_{i,0}),\lambda(v_{j_1,0}v_{p_2,0})\} \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f\{\lambda(v_{j_1,0}v_{i,0}),\lambda(v_{j_1,0}v_{j_2,k_2})\}\big\},$$

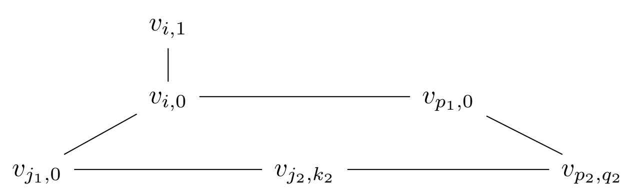
by the condition (f).(2).i and (f).(2).v. So we have $v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. By way of contradiction we have $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. Since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$ we have $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. But $v_{i,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$, contradicted by Lemma 3.131.

iv. Assume $f\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\} \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ and

$$f\{\lambda(v_{j_1,0}v_{p_2,0}), \lambda(v_{j_1,0}v_{j_2,k_2})\} \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_2,k_2})\}\big\},$$

by the condition (f).(2).i and (f).(2).v. Then $v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$. Similar to (C) right before, we have $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$. But $v_{j_2,k_2}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$, contradicted by Lemma 3.131.

(17) Assume $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ with $v_{j_2,k_2} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_1,0} \neq v_{p_2,q_2}$. Then $k_2 = 0 = q_2$ and it is similar to the case xiii.



In particular, if $\iota = 3$, then for $i = 1, \ldots, d$, by definition of $p_{\underline{n}}(I)$ we have there exists a generator where all variables are of the form X_{i,i_l} with $i_l \in \{0, 1, 2, 3\}$, so there exists a $v_{i,i_j} \in V''$. □

Remark. If $\iota = 2$, $f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_iv_{i,1})$ and $f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_\iota G)_\lambda\big)$ from condition (a) is redundant.

Proposition 3.136. Assume $r \geq 4$. Let $\iota = r$ or $r-1$. Let $(\Sigma_\iota G)_\lambda$ be a weighted ι-path suspension of G_ω such that $\lambda(v_iv_j) \leq f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\} \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_\iota G)_\lambda\big)$ and $\lambda(v_{i,k}v_{i,k+1}) \leq f\{\lambda(v_{i,k}v_{i,k+1}), \lambda(v_{i,k+1}v_{i,k+2})\}$ for $i = 1, \ldots, d$ and $k = 0, \ldots, r-2$, $f\{\lambda(v_iv_j), \lambda(v_jv_k)\} \leq \lambda(v_jv_{j,1})$ and for all 2-paths $v_iv_jv_k$ in G_ω:

$$f\{\lambda(v_iv_j), \lambda(v_jv_k)\} \leq f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\}\} = f\{\lambda(v_kv_j), \lambda(v_jv_{j,1})\}.$$

Let $I := I_{r,f}\big((\Sigma_rG)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_rG)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then there exists a unique $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$. In particular, if $\iota = r$, then there exists a unique $v_{i,i_j} \in V''$

for $i = 1, \dots, d$, so $p_{\underline{n}}(I)$ is m-unmixed.

Proof. Suppose there exist $v_{i,\alpha}, v_{i,\beta} \in V''$ with $0 \leq \alpha < \beta \leq r$. Suppose $v_{i,\beta} \smile (P_r \rightsquigarrow \wp)$ for some r-path P_r and some path $\wp \in p_{\underline{n}}(I)$, then we must have $v_{i,\alpha} \in V(\wp)$. But since $\mathfrak{P}_{i,\alpha} \neq \emptyset$ by Lemma 3.133, it is straightforward to show that $v_{i,\alpha} \smile (P_r \rightsquigarrow \wp)$. So $\mathfrak{P}_{i,\beta} = \emptyset$, contradicted by Lemma 3.133. In particular, if $\iota = r$, then for $i = 1, \dots, d$, by definition of $p_{\underline{n}}(I)$ we have there exists a generator where all variables are of the form X_{i,i_l} with $i_l \in \{0, \dots, r\}$, so there exists a $v_{i,i_j} \in V''$. $\square$

Remark. If $\iota = r - 1$, $f\{\lambda(v_i v_j), \lambda(v_i v_{i,1})\} \leq \lambda(v_i v_{i,1})$ and $f\{\lambda(v_i v_j), \lambda(v_i v_{i,1})\} \leq \lambda(v_j v_{j,1})$ for all edges $v_i v_j \in E\big((\Sigma_\iota G)_\lambda\big)$ from condition (a) is redundant.

Theorem 3.137. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. Then $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is Cohen-Macaulay, $\{X_{i,j} - X_{i,j-1} \mid i = 1, \dots, d, j = 1, \dots, r\}$ is a homogeneous regular sequence for R'/I and*

$$\frac{R'}{I + (X_{i,j} - X_{i,j-1} \mid i = 1, \dots, d, j = 1, \dots, r)R'} \cong \frac{R}{IR}.$$

Proof. Let $k \in \{1, \dots, (r-1)d\}$, $i_k = \lfloor \frac{k+d-1}{d} \rfloor$ and $j_k = k + (1 - i_k)d$. Let

$$\underline{n}_k = (\underbrace{r - i_k + 1, \dots, r - i_k + 1}_{j_k \text{ times}}, r - i_k + 2, \dots, r - i_k + 2) \in \mathbb{N}^d.$$

For $k = 1, \dots, (r-1)d$, define a polynomial ring R_k by

$$R_k = A \begin{bmatrix} 0, & \cdots & 0, & 0, & \cdots & 0, \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0, & \cdots & 0, & 0, & \cdots & 0, \\ 0, & \cdots & 0, & X_{j_k+1, r-i_k+1}, & \cdots & X_{d, r-i_k+1} \\ X_{1, r-i_k}, & \cdots & X_{j_k, r-i_k}, & X_{j_k+1, r-i_k}, & \cdots & X_{d, r-i_k}, \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{1,0}, & \cdots & X_{j_k,0}, & X_{j_k+1,0}, & \cdots & X_{d,0} \end{bmatrix}.$$

Then for $k = 1, \dots, (r-1)d$, $p_{\underline{n}_k}(I)R_k$ is the monomial ideal of R_k obtained from I by setting $X_{a,b} = X_{a,r-i_k}$ for $a = 1, \dots, j_k$ and $b = r - i_k + 1, \dots, r$ and setting $X_{a,b} = X_{a,r-i_k+1}$ for

$a = j_k + 1, \ldots, d$ and $b = r - i_k + 2, \ldots, r$. Note that

$$\frac{R_1}{p_{\underline{n}_1}(I)} \cong \frac{R'}{I + (X_{1,r} - X_{1,r-1})},$$

and for $k = 2, \ldots, (r-1)d$ we have inductively

$$\begin{aligned}
\frac{R_k}{p_{\underline{n}_k}(I)} &\cong \frac{R_{k-1}}{p_{\underline{n}_{k-1}(I)} + (X_{j_k,r-i_k+1} - X_{j_k,r-i_k})} \cong \frac{R_{k-1}/p_{\underline{n}_{k-1}}(I)}{(X_{j_k,r-i_k+1} - X_{j_k,r-i_k})} \\
&\cong \frac{R_{k-2}/(p_{\underline{n}_{k-2}}(I) + (X_{j_{k-1},r-i_{k-1}+1} - X_{j_{k-1},r-i_{k-1}}))}{(X_{j_k,r-i_k+1} - X_{j_k,r-i_k})} \\
&\cong \frac{R_{k-2}}{p_{\underline{n}_{k-2}}(I) + (X_{j_{k-1},r-i_{k-1}+1} - X_{j_{k-1},r-i_{k-1}}, X_{j_k,r-i_k+1} - X_{j_k,r-i_k})} \\
&\cong \cdots \\
&\cong \frac{R_1}{p_{\underline{n}_1}(I) + (X_{j_l,r-i_l+1} - X_{j_l,r-i_l} \mid l = 2, \ldots, k)} \\
&\cong \frac{R'}{I + (X_{j_l,r-i_l+1} - X_{j_l,r-i_l} \mid l = 1, \ldots, k)},
\end{aligned}$$

since $j_1 = 1$, $r - i_1 + 1 = r - 1 + 1 = r$ and $r - i_1 = r - 1$. Hence

$$\begin{aligned}
\frac{R}{IR} = \frac{R_{(r-1)d}}{p_{\underline{n}_{(r-1)d}}(I)} &\cong \frac{R'}{I + (X_{j_l,r-j_l+1} - X_{j_l,r-j_l} \mid l = 1, \ldots, (r-1)d)} \\
&= \frac{R'}{I + (X_{i,j} - X_{i,j-1} \mid i = 1, \ldots, d,\ j = 2, \ldots, r)}.
\end{aligned}$$

Let $k \in \{1, \ldots, (r-1)d\}$. Then by Proposition 3.134, 3.135 and 3.136, for any (V'', δ'') such that $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}_k}(I)$, we have there exists a unique $v_{i,i_\alpha} \in V''$ for $i = 1, \ldots, d$. So in $R_k/p_{\underline{n}_k}(I)$, the associated primes of $0 = p_{\underline{n}_k}(I)/p_{\underline{n}_k}(I)$ are of the form $\overline{(X_{1,\beta_1}, \ldots, X_{d,\beta_d})R_k}$. So $X_{j_k,r-i_k+1} - X_{j_k,r-i_k} \in \mathrm{NZD}\big(R_{k-1}/p_{\underline{n}_{k-1}}(I)\big)$, where we set $R_0 := R'$ and $\underline{n}_0 := (r+1, \ldots, r+1)$. Since $(X_{j_k,r-i_k+1} - X_{j_k,r-i_k})R_{k-1} + p_{\underline{n}_{k-1}}(I) \subsetneq R_{k-1}$, we have $R_{k-1}/p_{\underline{n}_{k-1}}(I) \xrightarrow{\cdot(X_{j_k,r-i_k+1} - X_{j_k,r-i_k})} R_{k-1}/p_{\underline{n}_{k-1}}(I)$ is not surjective. Hence $X_{j_k,r-i_k+1} - X_{j_k,r-i_k}$ is $R_{k-1}/p_{\underline{n}_{k-1}}(I)$-regular. So by definition

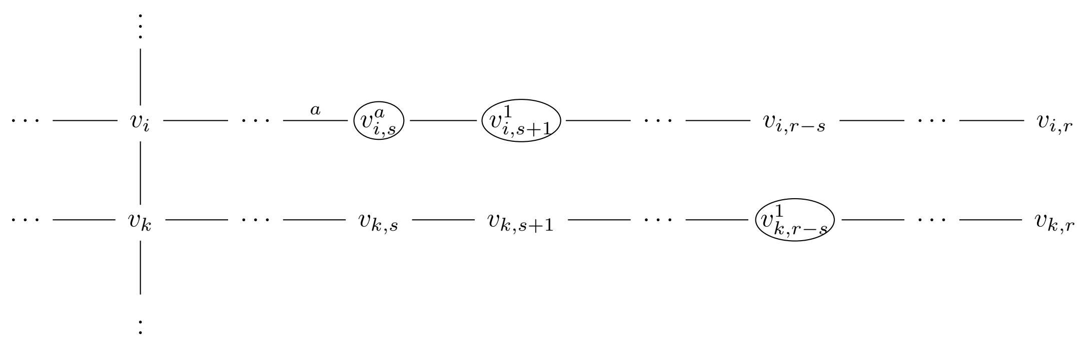
$$\{X_{i,j} - X_{i,j-1} \mid i = 1, \ldots, d, j = 2, \ldots, r\} = \{X_{j_l,r-j_l+1} - X_{j_l,r-j_l} \mid l = 1, \ldots, (r-1)d\}$$

is a homogeneous regular sequence for R'/I. Since R/IR is Artinian, it is Cohen-Macaulay. So by Fact 2.93, we have R'/I is Cohen-Macaulay. $\square$

Lemma 3.138. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. If $I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is m-unmixed, then $\lambda(v_iv_j) \le f\{\lambda(v_iv_j), \lambda(v_iv_{i,1})\} \le \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \le f\{\lambda(v_iv_j), \lambda(v_jv_{j,1})\} \le \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E(G_\omega)$ and $\lambda(v_{i,k}v_{i,k+1}) \le f\{\lambda(v_{i,k}v_{i,k+1}), \lambda(v_{i,k+1}v_{i,k+2})\}$ for $i = 1, \ldots, d$ and $k = 0, \ldots, r-2$, $f\{\lambda(v_{i_1}v_{i_2}), \lambda(v_{i_2}v_{i_3})\} \le \lambda(v_{i_2}v_{i_2,1})$, $f\{\lambda(v_{i_1}v_{i_2}), \lambda(v_{i_2}v_{i_3})\} \le f\{\lambda(v_{i_1}v_{i_2}), \lambda(v_{i_2}v_{i_2,1})\}$ and $f\{\lambda(v_{i_1}v_{i_2}), \lambda(v_{i_2}v_{i_3})\} \le f\{\lambda(v_{i_3}v_{i_2}), \lambda(v_{i_2}v_{i_2,1})\}$ for all 2-paths $v_{i_1}v_{i_2}v_{i_3}$ in G_ω.

Proof. Since $\{v_1^1, \ldots, v_d^1\}$ is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$, by Fact 3.83, there exists a minimal f-weighted r-path vertex cover (V''', δ''') of $(\Sigma_r G)_\lambda$ such that $(V''', \delta''') \le \{v_1^1, \ldots, v_d^1\}$. By the minimality of V''', we have $V''' = \{v_1, \ldots, v_d\}$ and so $|V'''| = d$. Hence by [9, Theorem 2.7], it suffices to show that if the conditions on weights are not satisfied, then there exists an f-weighted r-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$ and $\mathfrak{P}_{i,j} \ne \emptyset$ for each $v_{i,j} \in V''$.

(a) Suppose $a := \lambda(v_{i,s-1}v_{i,s}) > f\{\lambda(v_{i,s-1}v_{i,s}), \lambda(v_{i,s}v_{i,s+1})\}$ for some $i \in \{1, \ldots, d\}$ and some $s \in \{1, \ldots, r-1\}$.

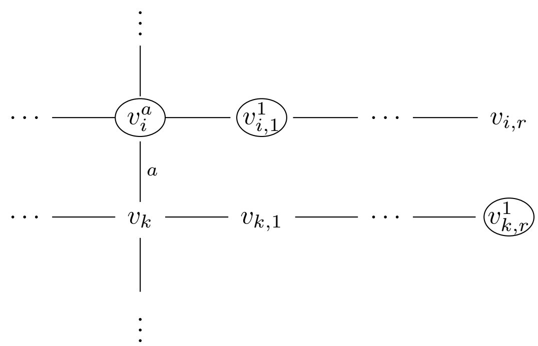


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,s+1}^1, v_{i,s}^a, v_{k,r-s}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i,k\}\right\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$, $v_{i,r} \cdots v_{i,1}v_i \in \mathfrak{P}_{i,s+1}$, $v_{i,s} \cdots v_{i,1}v_iv_kv_{k,1} \cdots v_{k,r-s-1} \in \mathfrak{P}_{i,s}$, $v_{k,r} \cdots v_{k,1}v_k \in \mathfrak{P}_{k,r-s}$ and $v_{t,r} \cdots v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any t in $\{1, \ldots, d\} \smallsetminus \{i,k\}$.

(b) Suppose $a := \lambda(v_i v_k) > f\{\lambda(v_{i,1} v_i), \lambda(v_i v_k)\}$ for some $i \in \{1, \ldots, d\}$ and some $v_i v_k \in E(G_\omega)$.

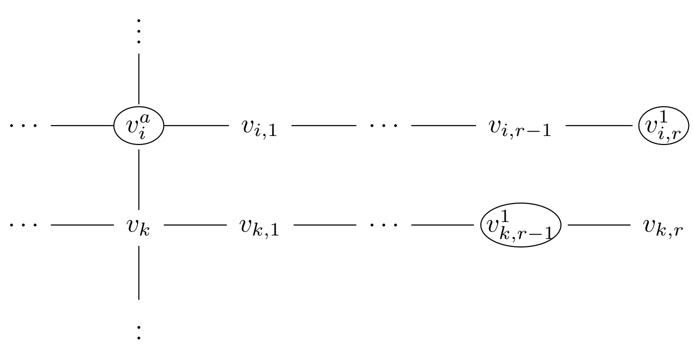


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{k,r}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, k\}\right\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$, $v_{i,1} v_i v_k v_{k,1} \cdots v_{k,r-2} \in \mathfrak{P}_{i,1}$, $v_i v_k v_{k,1} \cdots v_{k,r-1} \in \mathfrak{P}_{i,0}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,r}$ and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, k\}$.

(c) Suppose $a := f\{\lambda(v_{i,1} v_i), \lambda(v_i v_k)\} > \lambda(v_i v_{i,1})$ for some $i \in \{1, \ldots, d\}$ and some $v_i v_k \in E(G_\omega)$.

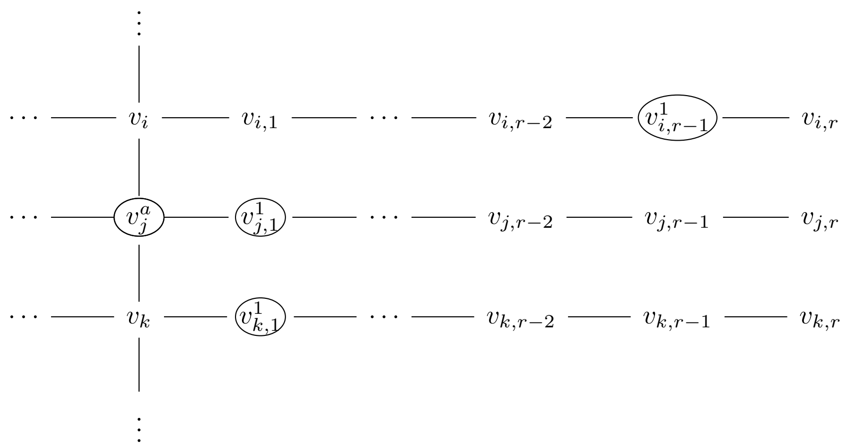


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,r}^1, v_i^a, v_{k,r-1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, k\}\right\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$, $v_{i,1} v_i v_k v_{k,1} \cdots v_{k,r-2} \in \mathfrak{P}_{i,0}$, $v_i \cdots v_{i,r} \in \mathfrak{P}_{i,r}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,r-1}$ and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, k\}$.

(d) Suppose $a := f\{\lambda(v_j v_i), \lambda(v_j v_k)\} > f\{\lambda(v_i v_j), \lambda(v_j v_{j,1})\}$ for some 2-path $v_i v_j v_k$ in G_ω.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v^1_{i,r-1}, v^a_j, v^1_{j,1}, v^1_{k,1}\} \sqcup \{v^1_m \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\}$$

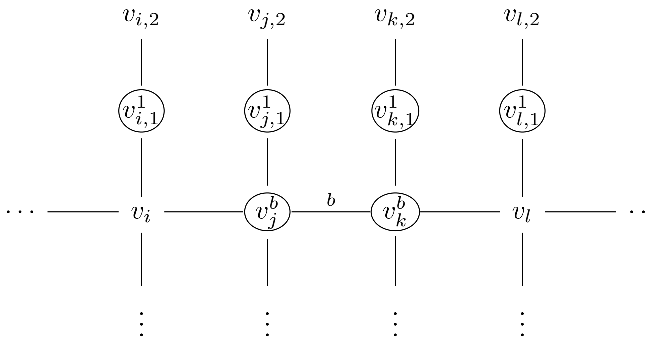
is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d + 1$, $v_{i,r} \cdots v_{i,1} v_i \in \mathfrak{P}_{i,r-1}$, $v_{i,r-2} \cdots v_i v_j v_k \in \mathfrak{P}_{j,0}$, $v_{i,r-2} \cdots v_i v_j v_{j,1} \in \mathfrak{P}_{j,1}$, $v_{k,r} \ldots v_{k,1} v_k \in \mathfrak{P}_{k,1}$ and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$. $\square$

Theorem 3.139. *Let $(\Sigma_2 G)_\lambda$ be a weighted 2-path suspension of G_ω. If $I_{2,f}\big((\Sigma_2 G)_\lambda\big)$ is m-unmixed, then the weight function λ satisfies the constraints in Proposition 3.134.*

Proof. By Lemma 3.138 and its proof, it is enough to show that if the constraints on 3-paths or 3-cycles are not satisfied, then there exists an f-weighted 2-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_2 G)_\lambda$ such that $|V''| = d + 1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. Without loss of generality, we assume the weight function λ satisfies constraints in Lemma 3.138.

(a) Let $v_i v_j v_k v_l$ be a 3-path in G_ω such that $f\{\lambda(v_{j,1} v_j), \lambda(v_j v_i)\} < \lambda(v_j v_k) =: b$. Suppose we have $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_j v_k) = b$.

(1) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_l)\} < b$.

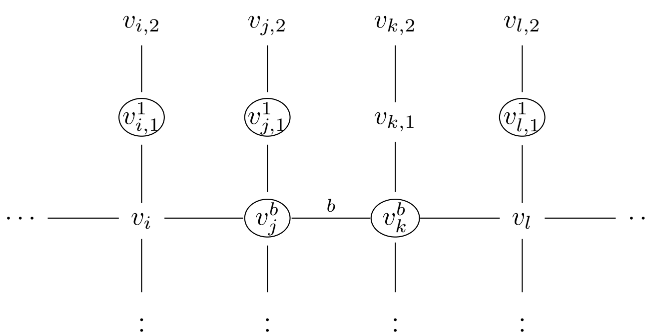


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^b, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, $v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_{k,1}v_kv_l \in \mathfrak{P}_{k,1}$, $v_iv_jv_k \in \mathfrak{P}_{k,0}$, $v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k, l\}$.

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_l)\} \geq b$.



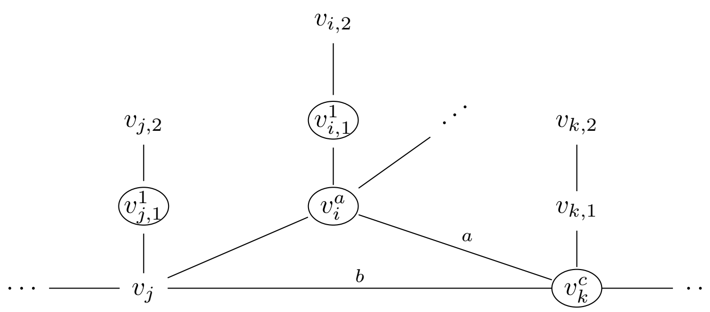
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^b, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, $v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_iv_jv_k \in \mathfrak{P}_{k,0}$, $v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k, l\}$.

(b) Let $v_iv_jv_kv_i$ be a 3-cycle in G_ω with $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_k) =: a$. Suppose we have

$f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \lambda(v_kv_i) = a$ and $f\{\lambda(v_kv_i), \lambda(v_kv_j) < \lambda(v_kv_j) =: b$. So $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \min\{a, b\} =: c$.



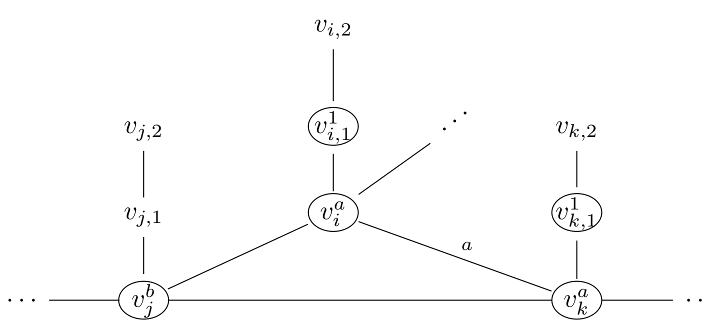
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,1}^1, v_i^a, v_k^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, $v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1}$, $v_{i,1}v_iv_j \in \mathfrak{P}_{i,1}$, $v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,0}$ and $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(c) Let $v_iv_jv_kv_i$ be a 3-cycle in G_ω with $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_k) =: a$. Suppose we have $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \lambda(v_kv_i) = a$ and $b := f\{\lambda(v_jv_i), \lambda(v_jv_k) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.

(1) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < a$.

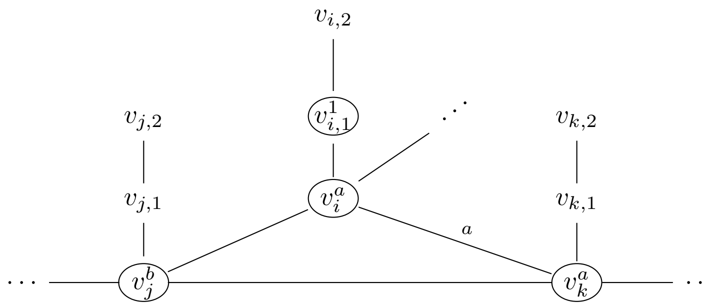


It is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,1}^1, v_i^a, v_k^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, $v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_j \in \mathfrak{P}_{i,1}$, $v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_jv_iv_k \in \mathfrak{P}_{k,0}$, $v_{k,1}v_kv_j \in \mathfrak{P}_{k,1}$ and $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} \geq a$.

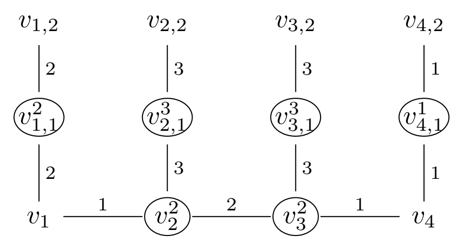


It is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,1}^1, v_i^a, v_k^a\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, $v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_j \in \mathfrak{P}_{i,1}$, $v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_jv_iv_k \in \mathfrak{P}_{k,0}$, and $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$. $\square$

Example 3.140. Consider a minimal min-weighted 2-path vertex cover of the following weighted 2-path suspension $(\Sigma_2 P_3)_\lambda$ of $G_\omega := (P_3)_\omega = \left(v_1 \xrightarrow{\;1\;} v_2 \xrightarrow{\;2\;} v_3 \xrightarrow{\;1\;} v_4 \right)$.

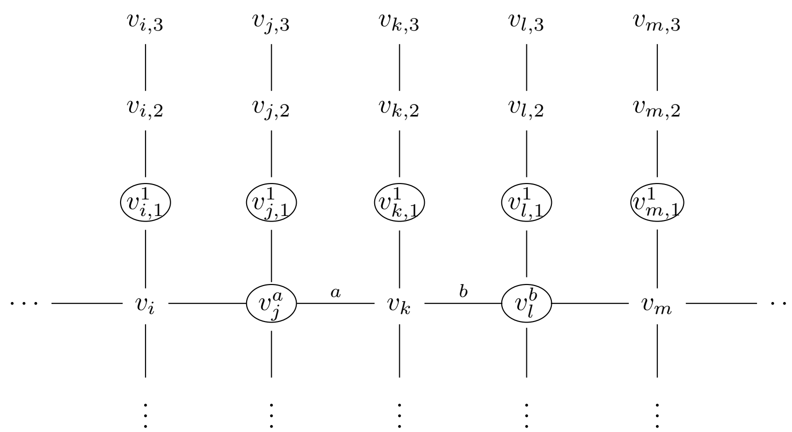


Theorem 3.141. *Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω. If $I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ is m-unmixed, then the weight function λ satisfies the constraints in Proposition 3.135.*

Proof. By Lemma 3.138 and its proof, it is enough to show that if the constraints on 4-paths or 3-cycles or 4-cycles are not satisfied, then there exists an f-weighted 3-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_3 G)_\lambda$ such that $|V''| = d + 1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. Without loss of generality, we assume the weight function λ satisfies constraints in Lemma 3.138.

(a) Let $v_iv_jv_kv_lv_m$ be a 4-path in G_ω such that $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} < \lambda(v_jv_k) =: a$. Suppose $f\{\lambda(v_iv_j), \lambda(v_jv_k)\} < \lambda(v_jv_k) = a$ and $f\{\lambda(v_kv_l), \lambda(v_lv_m)\} < \lambda(v_kv_l) =: b$.

(1) Assume $f\{\lambda(v_{l,1}v_l), \lambda(v_l v_m)\} < b$.

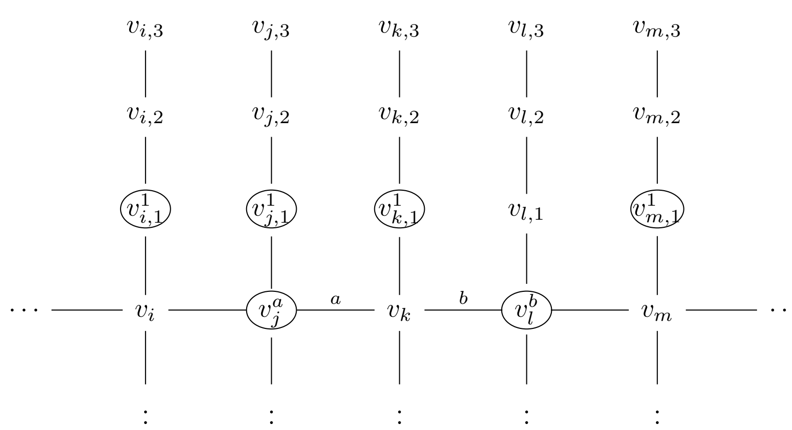


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_{l,1}^1, v_l^b, v_{m,1}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and we have $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,2}v_{j,1}v_j v_i \in \mathfrak{P}_{j,1}$, $v_j v_k v_l v_m \in \mathfrak{P}_{j,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, $v_{l,2}v_{l,1}v_l v_m \in \mathfrak{P}_{l,1}$, $v_i v_j v_k v_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,1}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$.

(2) Assume $f\{\lambda(v_{l,1}v_l), \lambda(v_l v_m)\} \geq b$.



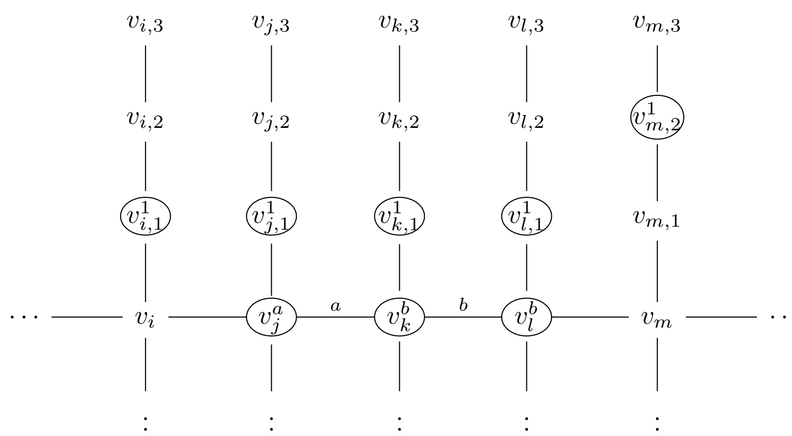
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_l^b, v_{m,1}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_lv_m \in \mathfrak{P}_{j,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l,m\}$.

(b) Let $v_iv_jv_kv_lv_m$ be a 4-path in G_ω such that $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < \lambda(v_kv_l) =: b$. Suppose $f\{\lambda(v_jv_k), \lambda(v_jv_i)\} < \lambda(v_jv_k) =: a$ and $f\{\lambda(v_kv_l), \lambda(v_lv_m)\} < \lambda(v_kv_l) = b$.

(1) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} < a$ and $f\{\lambda(v_{l,1}v_l), \lambda(v_lv_m)\} < b$.

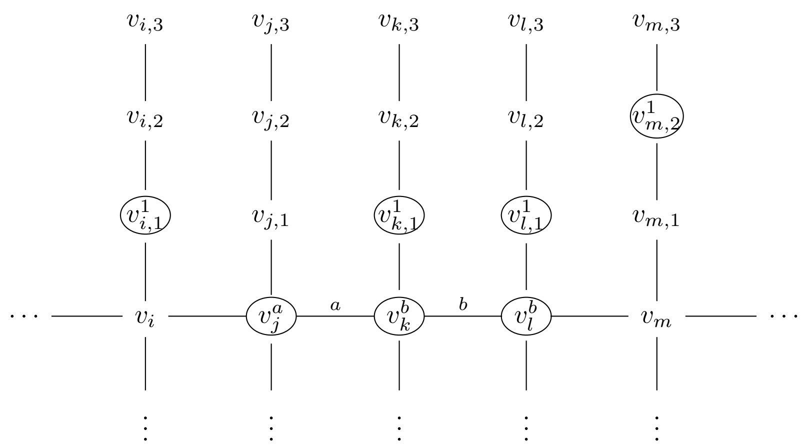


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{i,1}, v^1_{j,1}, v^a_j, v^1_{k,1}, v^b_k, v^1_{l,1}, v^b_l, v^1_{m,2}\right\} \sqcup \left\{v^1_n \mid n \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l,m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, we have $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_lv_m \in \mathfrak{P}_{j,0}$, $v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1}$, $v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0}$, $v_{l,2}v_{l,1}v_lv_m \in \mathfrak{P}_{l,1}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l,m\}$.

(2) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} \geq a$ and $f\{\lambda(v_{l,1}v_l), \lambda(v_lv_m)\} < b$.

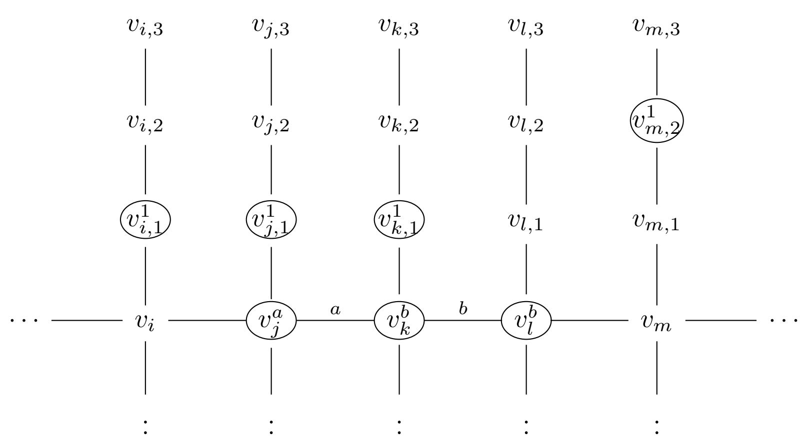


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_{l,1}^1, v_l^b, v_{m,2}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and we have $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_jv_kv_lv_m \in \mathfrak{P}_{j,0}$, $v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1}$, $v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0}$, $v_{l,2}v_{l,1}v_lv_m \in \mathfrak{P}_{l,1}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, m\}$.

(3) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} < a$ and $f\{\lambda(v_{l,1}v_l), \lambda(v_lv_m)\} \geq b$.

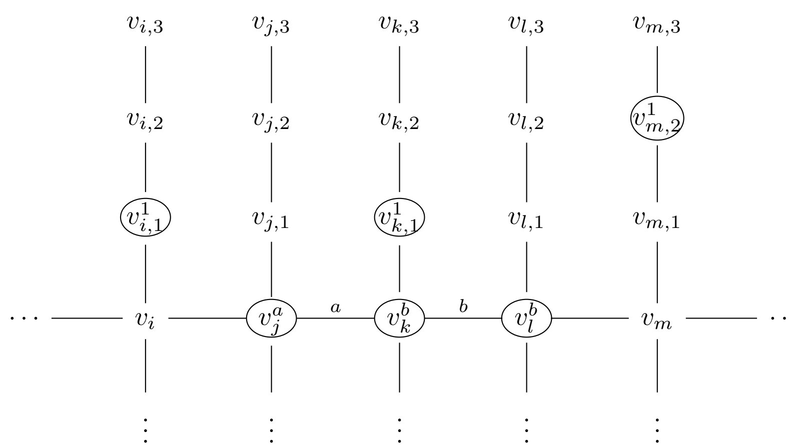


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_l^b, v_{m,2}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and we have $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_lv_m \in \mathfrak{P}_{j,0}$, $v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1}$, $v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \setminus \{i,j,k,l,m\}$.

(4) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} \geq a$ and $f\{\lambda(v_{l,1}v_l), \lambda(v_lv_m)\} \geq b$.

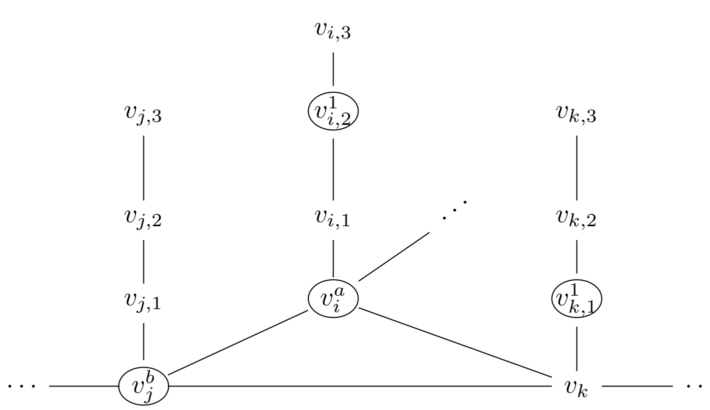


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{i,1}, v^a_j, v^1_{k,1}, v^b_k, v^b_l, v^1_{m,2}\right\} \sqcup \left\{v^1_n \mid n \in \{1,\dots,d\} \setminus \{i,j,k,l,m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, we have $v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_jv_kv_lv_m \in \mathfrak{P}_{j,0}$, $v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1}$, $v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$, $v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \setminus \{i,j,k,l,m\}$.

(c) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < f\{\lambda(v_iv_{i,1}), \lambda(v_iv_k)\} =: a$. Suppose $b := f\{\lambda(v_jv_i), \lambda(v_jv_k)\} > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$



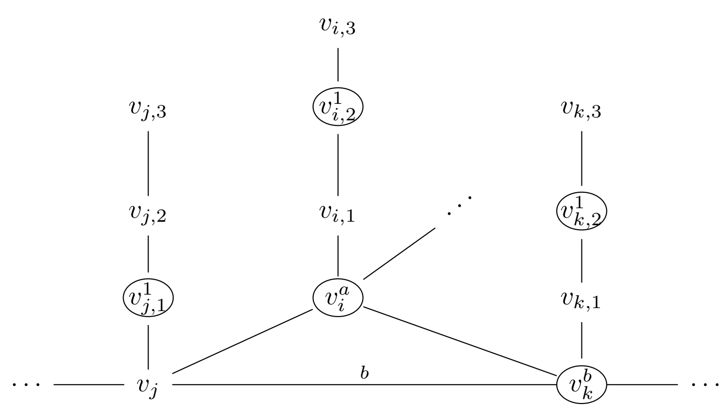
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(d) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < f\{\lambda(v_iv_{i,1}), \lambda(v_iv_k)\} =: a$. Suppose $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \lambda(v_kv_j) =: b$.

(1) Assume $\lambda(v_iv_k) < a$ and $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_i)\} < b$.

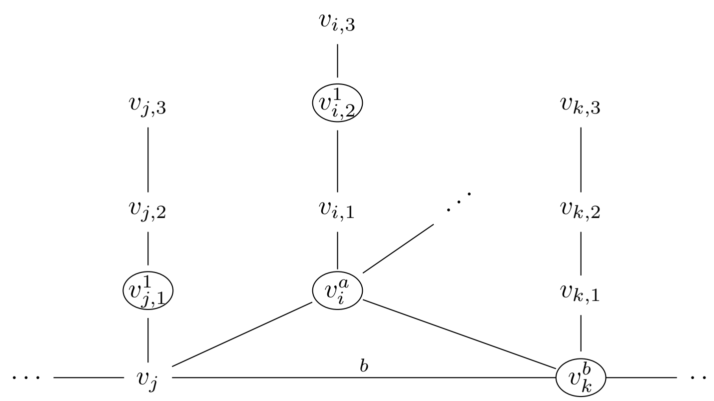


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_{k,2}^1, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{k,2}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume $\lambda(v_iv_k) \geq a$ or $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_i)\} \geq b$.

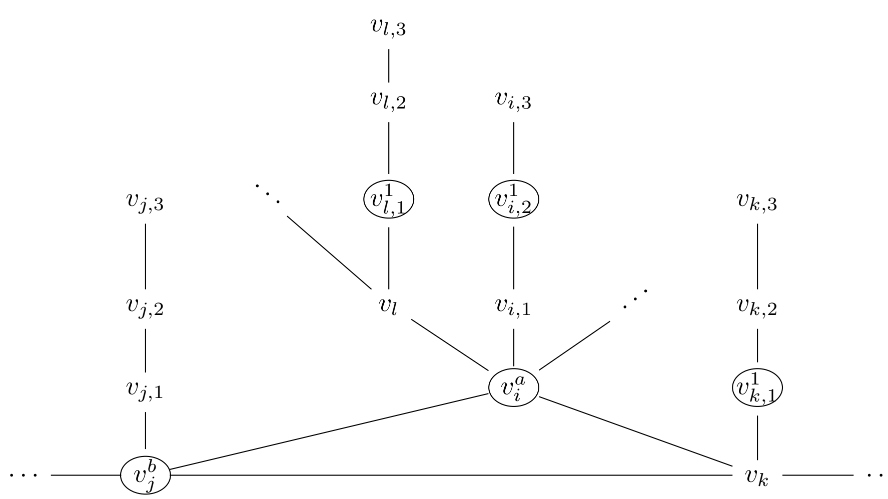


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(e) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < f\{\lambda(v_iv_l), \lambda(v_iv_k)\} =: a$ for some $v_iv_l \in E(G_\omega)$ with $j \neq l \neq k$. Suppose $b := f\{\lambda(v_jv_i), \lambda(v_jv_k)\} > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.



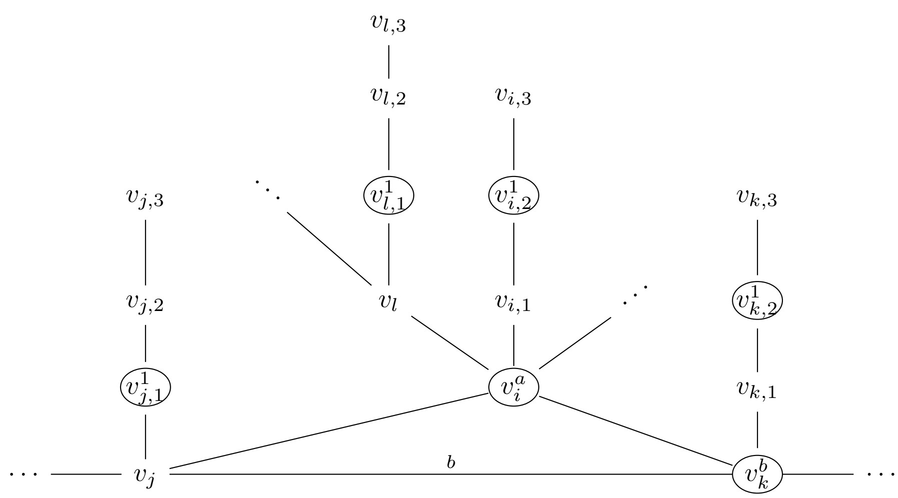
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_lv_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(f) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < f\{\lambda(v_iv_l), \lambda(v_iv_k)\} =: a$ for some $v_iv_l \in E(G_\omega)$ with $j \neq l \neq k$. Suppose $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \lambda(v_kv_j) =: b$.

(1) Assume $\lambda(v_iv_k) < a$ and $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_i)\} < b$.

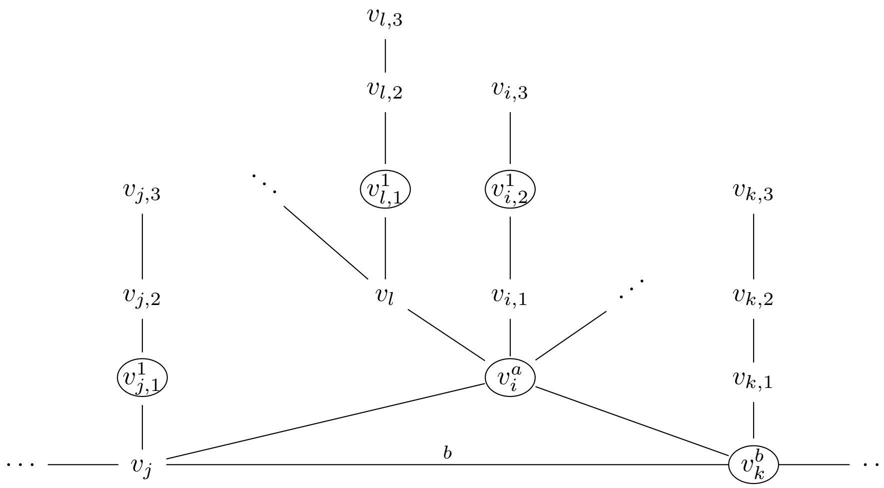


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_lv_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{k,2}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(2) Assume $\lambda(v_iv_k) \geq a$ or $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_i)\} \geq b$.



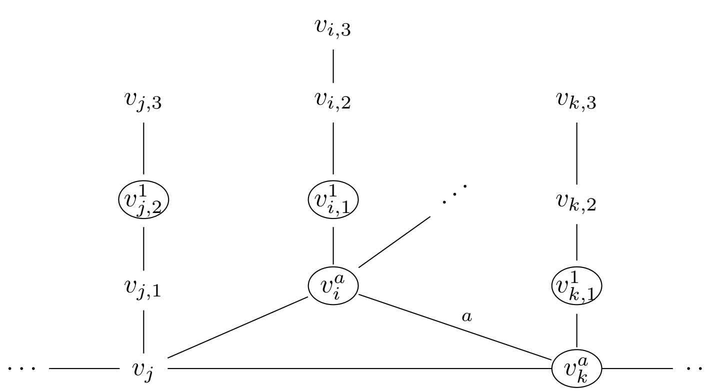
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{j,1}, v^1_{i,2}, v^a_i, v^b_k, v^1_{l,1}\right\} \sqcup \left\{v^1_m \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_lv_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(g) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < \lambda(v_iv_k) =: a$. Suppose we have $f\{\lambda(v_kv_i), \lambda(v_kv_j)\} < \lambda(v_kv_i) = a$.

(1) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < a$.

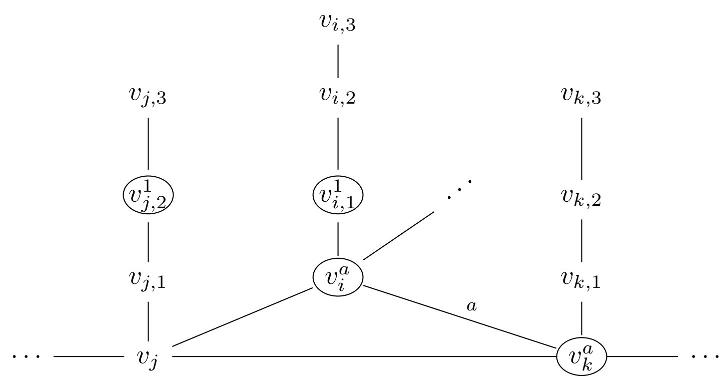


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_{i,1}^1, v_i^a, v_{k,1}^1, v_k^a\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,2}$, $v_{i,1}v_iv_jv_{j,1} \in \mathfrak{P}_{i,1}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{j,1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1}$, $v_{j,1}v_jv_iv_k \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} \geq a$.



Then it is straightforward to show that

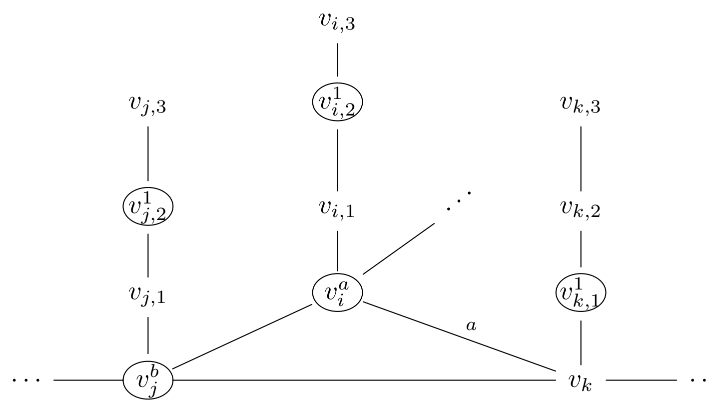
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_{i,1}^1, v_i^a, v_k^a\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,2}$, $v_{i,1}v_iv_jv_{j,1} \in \mathfrak{P}_{i,1}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{j,1}v_jv_iv_k \in \mathfrak{P}_{k,0}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(h) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < \lambda(v_iv_k) =: a$ and there exists $v_jv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_k$. Suppose

$$b := f\{\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})\} > \max\{\lambda(v_jv_i), f\{\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2})\}\}.$$

(1) Assume $l_1 = j$. Then $l_2 = 1$.



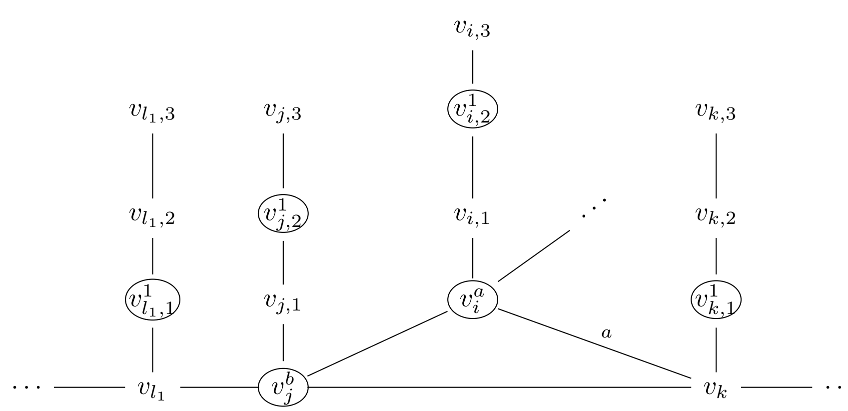
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2}$, $v_{j,1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \dots, d\} \setminus \{i, j, k\}$.

(2) Assume $l_1 \neq j$. Then $l_2 = 0$.

i. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} < b$.

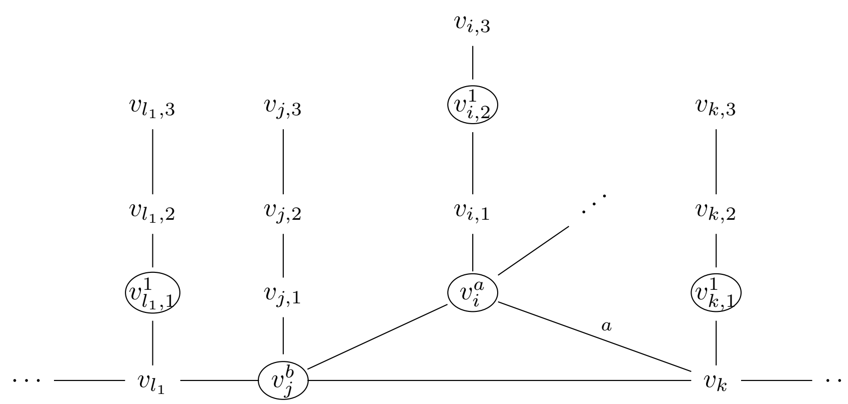


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{l_1,1}^1, v_{j,2}^1, v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \setminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,0}$, $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2}$, $v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l\}$.

ii. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} \geq b$.



Then it is straightforward to show that

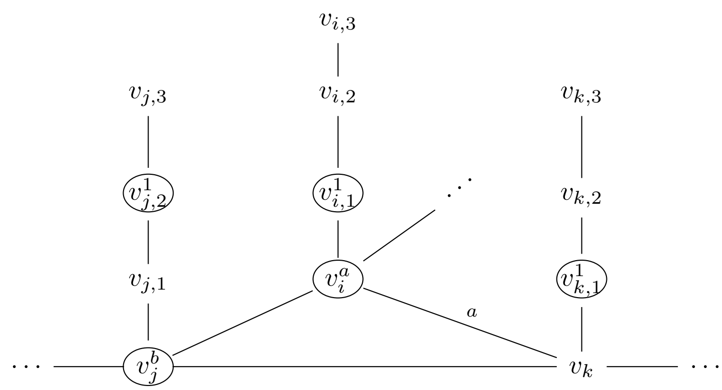
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{l_1,1}^1, v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,0}$, $v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l\}$.

(i) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < \lambda(v_iv_k) =: a$ and there exists $v_jv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_k$. Suppose

$$b := f\{\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})\} > \max\{f\{\lambda(v_jv_i), \lambda(v_jv_k)\}, f\{\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2})\}.$$

(1) Assume $l_1 = j$. Then $l_2 = 1$.



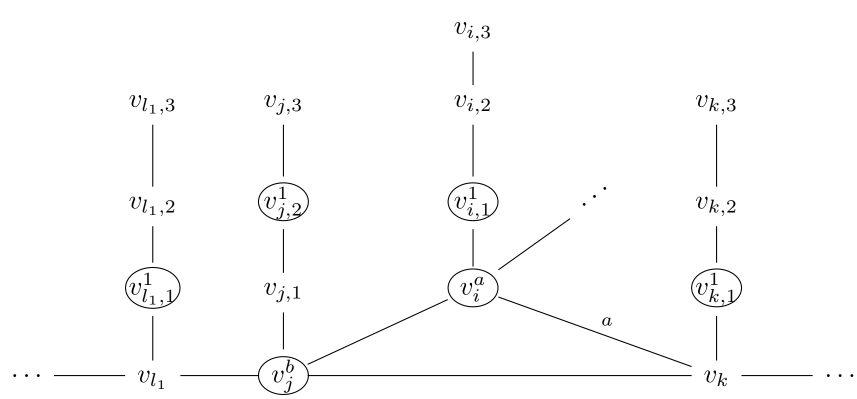
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_j^b, v_{i,1}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2}$, $v_{j,1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{j,1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume $l_1 \neq j$. Then $l_2 = 0$.

i. Assume $f\{\lambda(v_jv_{j,1}), \lambda(v_jv_k)\} < b$.

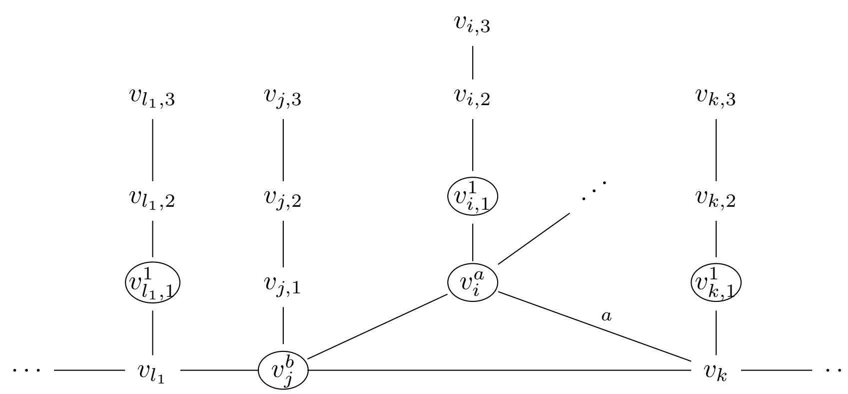


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{l_1,1}^1, v_{j,2}^1, v_j^b, v_{i,1}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1}$, $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2}$, $v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_kv_jv_{l_1} \in \mathfrak{P}_{i,0}$, $v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l_1\}$.

ii. Assume $f\{\lambda(v_jv_{j,1}), \lambda(v_jv_k)\} \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{l_1,1}^1, v_j^b, v_{i,1}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1}$, $v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_kv_jv_{l_1} \in \mathfrak{P}_{i,0}$, $v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l_1\}$.

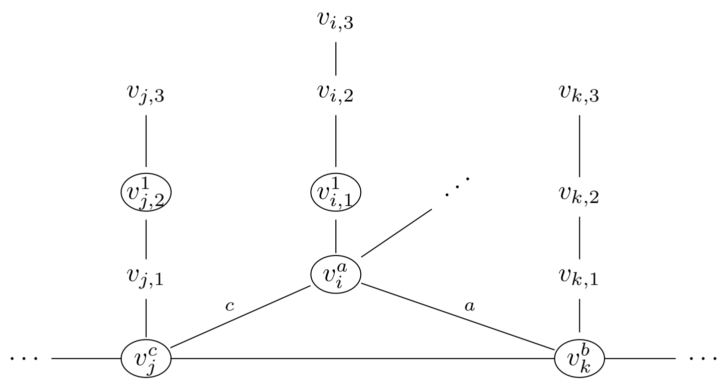
(j) Let $v_iv_jv_k$ be a 3-cycle in G_ω such that $f\{\lambda(v_iv_{i,1}, \lambda(v_iv_j))\} < \lambda(v_iv_k) =: a$ and there exists $v_kv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_j$. Suppose

$$b := f\{\lambda(v_kv_j), \lambda(v_kv_{l_1,l_2})\} > \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_i), \lambda(v_kv_{l_1,l_2})\}\big\}$$

and $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i) =: c$.

(1) Assume $l_1 = k$. Then $l_2 = 1$.

i. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} < c$.

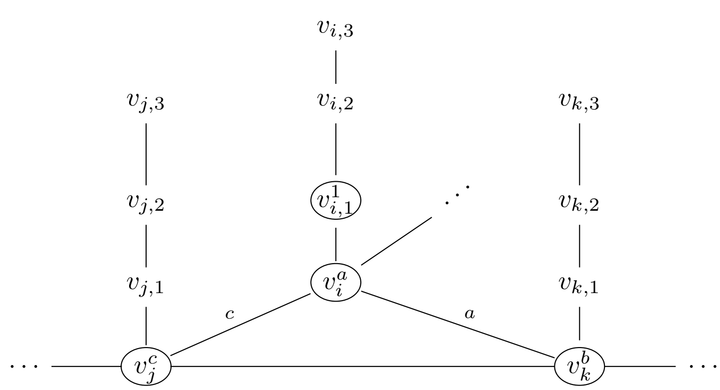


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_j^c, v_{i,1}^1, v_i^a, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2}$, $v_jv_iv_kv_{k,1} \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_{k,1} \in \mathfrak{P}_{k,0}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

ii. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} \geq c$.



Then it is straightforward to show that

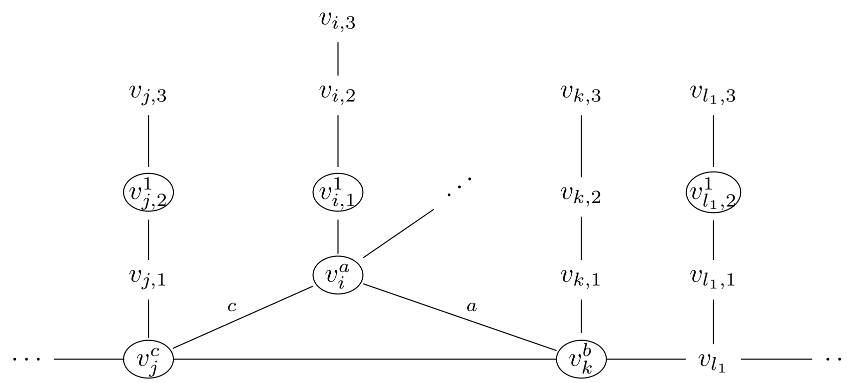
$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^c, v_{i,1}^1, v_i^a, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_jv_iv_kv_{k,1} \in \mathfrak{P}_{j,0}$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$,

$v_i v_k v_{k,1} v_{k,2} \in \mathfrak{P}_{i,0}$, $v_i v_j v_k v_{k,1} \in \mathfrak{P}_{k,0}$, and $v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

(2) Assume $l_1 \neq k$. Then $l_2 = 0$.

i. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_j v_k)\} < c$.

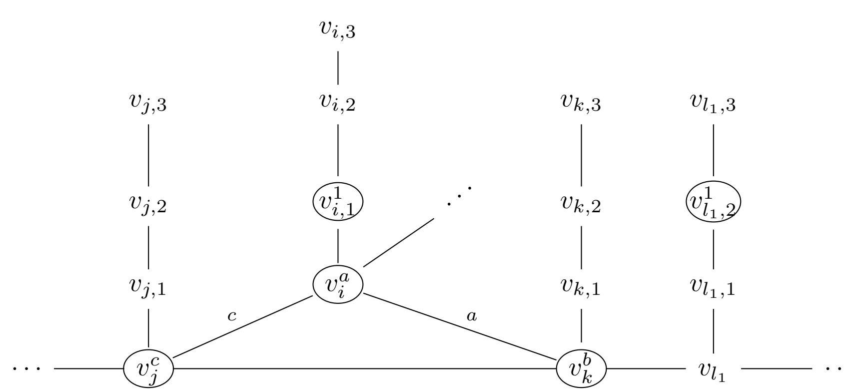


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_j^c, v_{i,1}^1, v_i^a, v_k^b, v_{l_1,2}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{j,2} v_{j,1} v_j v_k \in \mathfrak{P}_{j,2}$, $v_j v_i v_k v_{l_1} \in \mathfrak{P}_{j,0}$, $v_{i,1} v_i v_j v_k \in \mathfrak{P}_{i,1}$, $v_i v_k v_{l_1,0} v_{l_1,1} \in \mathfrak{P}_{i,0}$, $v_i v_j v_k v_{l_1} \in \mathfrak{P}_{k,0}$, $v_{l_1,3} v_{l_1,2} v_{l_1,1} v_{l_1} \in \mathfrak{P}_{l_1,2}$ and we have $v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l_1\}$.

ii. Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_j v_k)\} \geq c$.



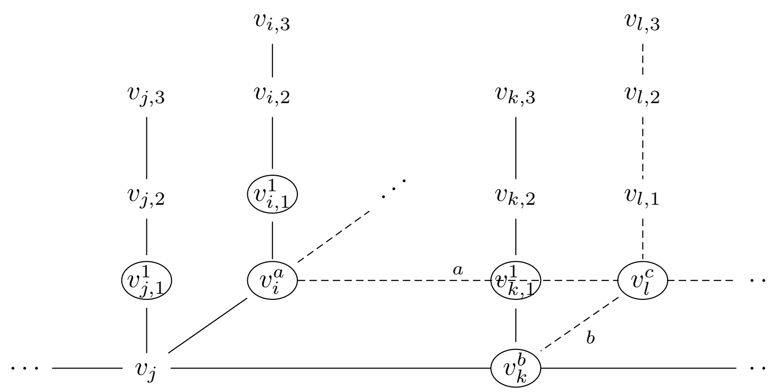
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^c, v_{i,1}^1, v_i^a, v_k^b, v_{l_1,2}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_j v_i v_k v_{l_1} \in \mathfrak{P}_{j,0}$, $v_{i,1} v_i v_j v_k \in \mathfrak{P}_{i,1}$, $v_i v_k v_{l_1,0} v_{l_1,1} \in \mathfrak{P}_{i,0}$, $v_i v_j v_k v_{l_1} \in \mathfrak{P}_{k,0}$, $v_{l_1,3} v_{l_1,2} v_{l_1,1} v_{l_1} \in \mathfrak{P}_{l_1,2}$ and we have $v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l_1\}$.

(k) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_k v_l) =: b$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \min\{\lambda(v_l v_i), \lambda(v_l v_k)\} =: c$.

(1) Assume $f\{\lambda(v_{k,1} v_k), \lambda(v_k v_j)\} < b$.

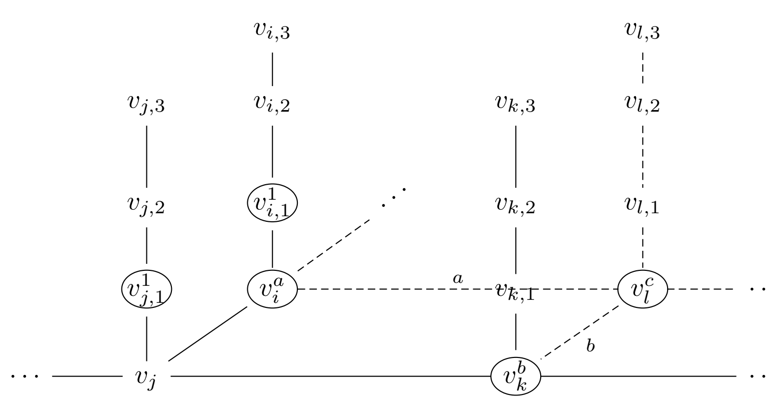


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and we have $v_{i,1} v_i v_j v_k \in \mathfrak{P}_{i,1}$, $v_i v_l v_k v_j \in \mathfrak{P}_{i,0}$, $v_{j,3} v_{j,2} v_{j,1} v_j \in \mathfrak{P}_{j,1}$, $v_i v_j v_k v_{k,1} \in \mathfrak{P}_{k,1}$, $v_j v_i v_l v_k \in \mathfrak{P}_{k,0}$, $v_i v_j v_k v_l \in \mathfrak{P}_{l,0}$ and $v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(2) Assume $f\{\lambda(v_{k,1} v_k), \lambda(v_k v_j)\} \geq b$.



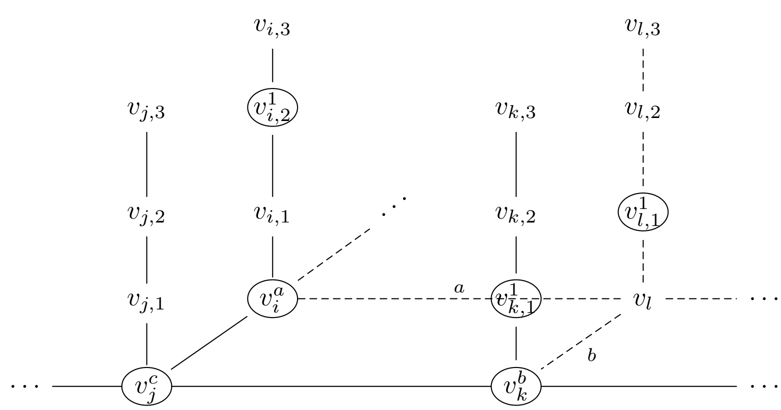
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(l) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $c := f\{\lambda(v_jv_i), \lambda(v_jv_k)\} > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.

(1) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < b$.

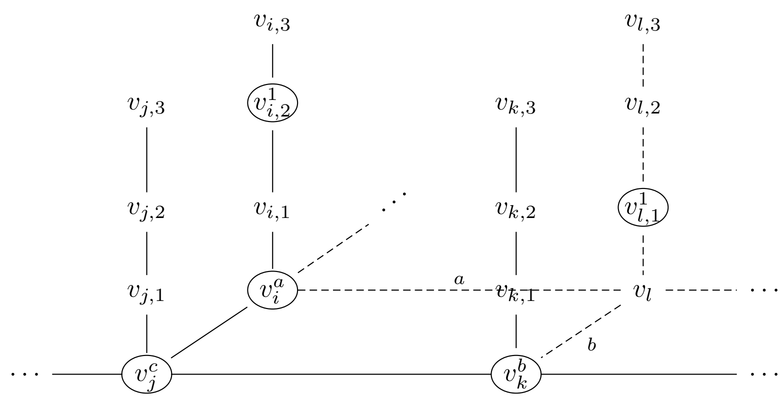


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_j^c, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_{k,2}v_{k,1}v_kv_j \in \mathfrak{P}_{k,1}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} \geq b$.

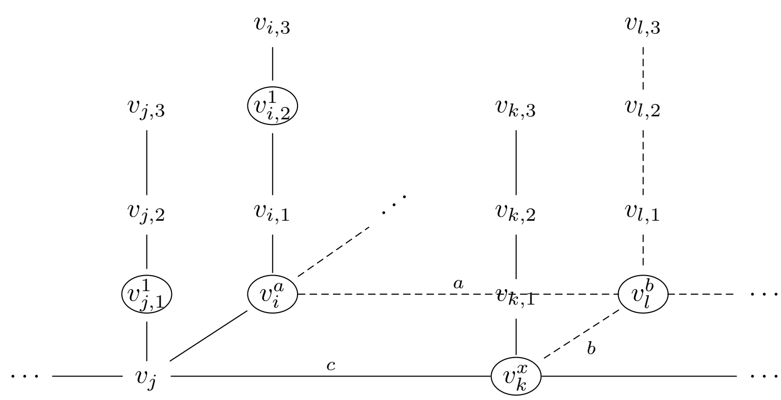


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_j^c, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(m) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_k) = b$ and $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_j) =: c$. Let $x := \min\{b, c\}$.

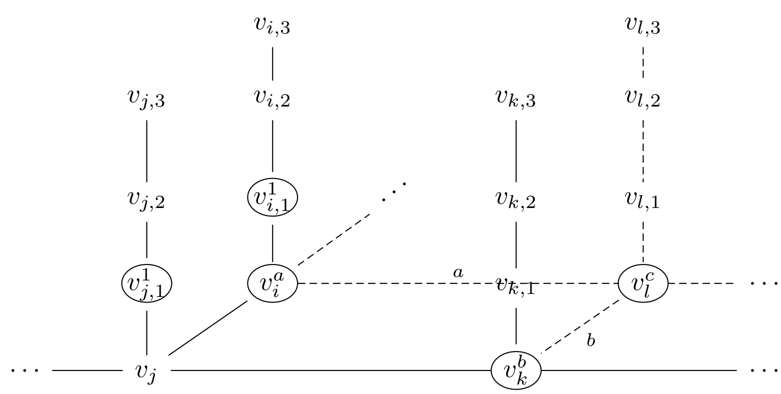


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_{j,1}, v_{k,1}^1, v_k^x, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(n) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_k) = b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_i) = a$. Let $c := \min\{a, b\}$.



Then it is straightforward to show that

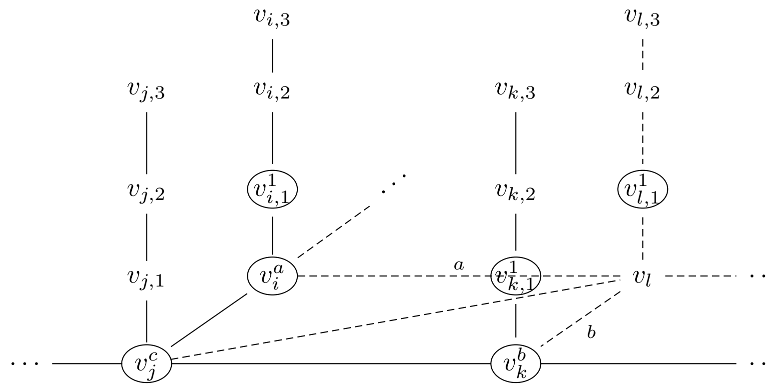
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}, v_{k,1}^1, v_k^x, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(o) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $v_jv_l \in E(G_\omega)$ and

$$c := f\{\lambda(v_jv_i), \lambda(v_jv_k)\} > \max\left\{\lambda(v_jv_k), f\{\lambda(v_jv_i), \lambda(v_jv_l)\}\right\}.$$

(1) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} < b$.

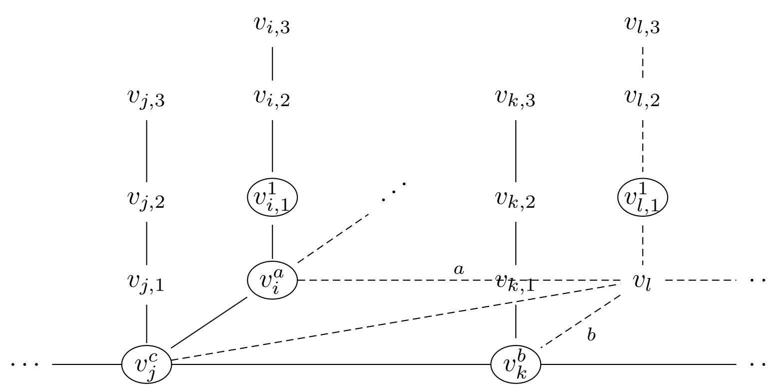


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{i,1}, v^a_i, v^c_j, v^1_{k,1}, v^b_k, v^1_{l,1}\right\} \sqcup \left\{v^1_m \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, $v_{i,1}v_iv_jv_l \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_{k,2}v_{k,1}v_kv_j \in \mathfrak{P}_{k,1}$, $v_iv_jv_lv_k \in \mathfrak{P}_{k,0}$, $v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l\}$.

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_kv_j)\} \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{i,1}, v^a_i, v^c_j, v^b_k, v^1_{l,1}\right\} \sqcup \left\{v^1_m \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, $v_{i,1}v_iv_jv_l \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{j,0}$,

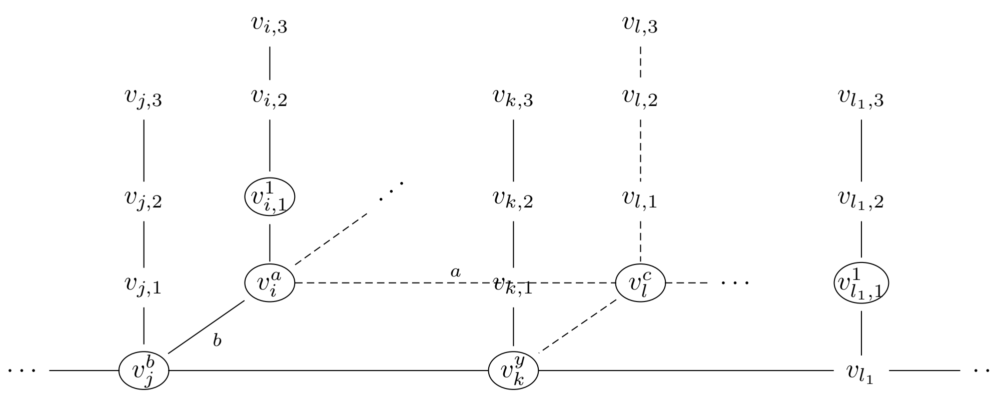
$v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l\}$.

(p) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l)$ and $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_i) = a$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \min\{\lambda(v_lv_i), \lambda(v_lv_k)\} =: c$ and there exists $v_kv_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f\{\lambda(v_kv_j), \lambda(v_kv_l)\} > \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2})\}\big\},$$

$$y := f\{\lambda(v_kv_j), \lambda(v_kv_{l_1,l_2})\} > \max\big\{\lambda(v_kv_j), f\{\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2})\}\big\}.$$

Then $l_1 \neq k$ and so $l_2 = 0$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \big\{v_{i,1}^1, v_i^a, v_j^b, v_k^y, v_{l,1}^1, v_l^a\big\} \sqcup \big\{v_m^1 \mid m \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l,l_1\}\big\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and that $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_{l_1} \in \mathfrak{P}_{i,0}$, $v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0}$, $v_iv_jv_kv_{l_1} \in \mathfrak{P}_{k,0}$, $v_lv_iv_jv_k \in \mathfrak{P}_{l,0}$, $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l,l_1\}$. (Suppose the weight of v_k is z with $z := \min\{x,y\} \leq y = f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l)$, then we get $v_k \smile v_kv_lv_iv_j$ and so $v_k \smile v_iv_lv_kv_{l_1}$ since $v_jv_iv_lv_kv_{l_1}$ is a 4-path in G_ω, a contradiction. Similarly, we have $v_jv_iv_lv_k \in \mathfrak{P}_{j,0}$.)

(q) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l)$ and $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_i) = a$ and $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} > \lambda(v_kv_l)$ and there exists $v_kv_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_j \neq$

$v_{l_1,l_2} \neq v_l$ such that

$$x := f\{\lambda(v_k v_j), \lambda(v_k v_l)\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\},$$

$$y := f\{\lambda(v_k v_j), \lambda(v_k v_{l_1,l_2})\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\}.$$

Then we have $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_k v_l) < f\{\lambda(v_k v_j), \lambda(v_k v_l)\}$, a contradiction.

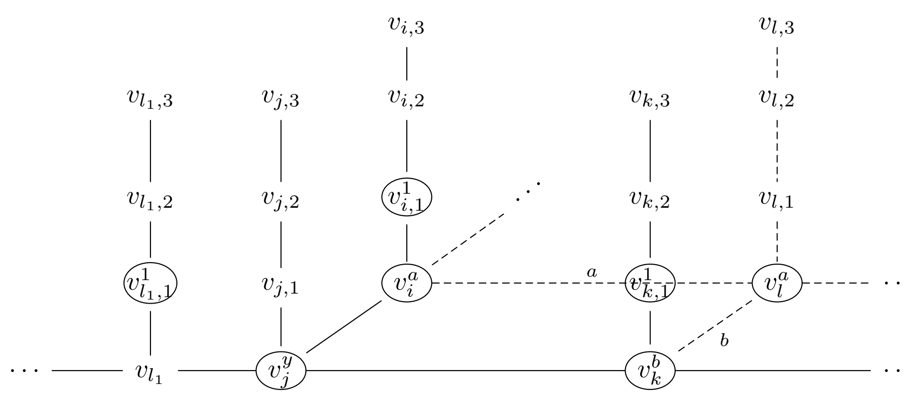
(r) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_k v_l) =: b$, $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \lambda(v_l v_i) = a$, and we have there exists $v_j v_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_i \neq v_{l_1,l_2} \neq v_k$ such that

$$c := f\{\lambda(v_j v_i), \lambda(v_j v_k)\} > \max\big\{\lambda(v_j v_k), f\{\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2})\}\big\},$$

$$x := f\{\lambda(v_j v_k), \lambda(v_j v_{l_1,l_2})\} > \max\big\{\lambda(v_j v_k), f\{\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2})\}\big\}.$$

Then $l_1 \neq k$ and so $l_2 = 0$. Let $y := \min\{x, c\}$.

(1) Assume $f\{\lambda(v_{k,1} v_k), \lambda(v_k v_j)\} < b$.



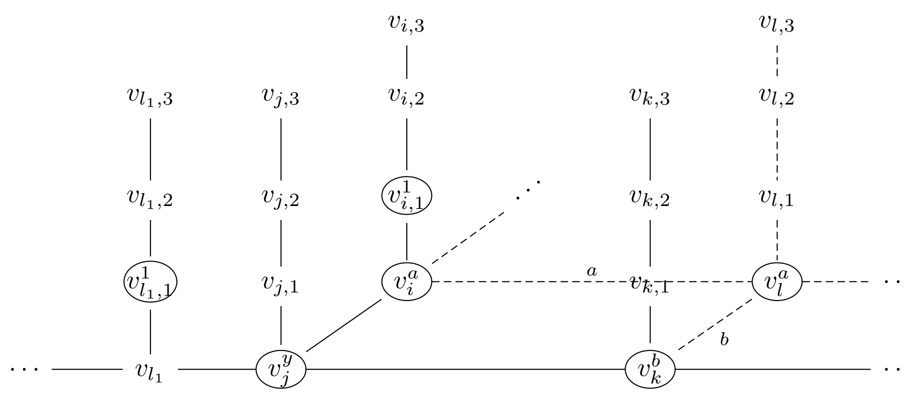
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \big\{v_{i,1}^1, v_i^a, v_j^y, v_{k,1}^1, v_k^b, v_l^a, v_{l_1,1}^1\big\} \sqcup \big\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, l_1\}\big\}$$

is an f-weighted 3-path vertex cover, and that $v_{i,1} v_i v_j v_{l_1} \in \mathfrak{P}_{i,1}$, $v_i v_l v_k v_j \in \mathfrak{P}_{i,0}$, $v_i v_j v_k v_l \in \mathfrak{P}_{j,0}$, $v_{k,2} v_{k,1} v_k v_j \in \mathfrak{P}_{k,1}$, $v_j v_i v_l v_k \in \mathfrak{P}_{k,0}$, $v_l v_i v_j v_{l_1} \in \mathfrak{P}_{l,0}$, $v_{l_1,3} v_{l_1,2} v_{l_1,1} v_{l_1} \in \mathfrak{P}_{l_1,1}$, $v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$. (We have $v_j \not\succ v_j v_i v_l v_k$ is because that $v_{l_1} v_j v_i v_l v_k$ is a 4-path in

G_ω and $\lambda(v_l v_i) > f\{\lambda(v_k v_l), \lambda(v_l v_i)\}$.)

(2) Assume $f\{\lambda(v_{k,1}v_k), \lambda(v_k v_j)\} \geq b$.



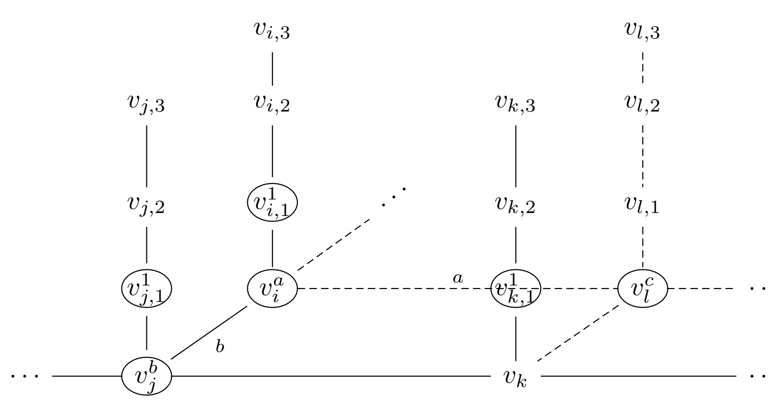
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^y, v_k^b, v_l^a, v_{l_1,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, l_1\}\right\}$$

is an f-weighted 3-path vertex cover, and that $v_{i,1}v_iv_jv_{l_1} \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_iv_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_jv_iv_lv_k \in \mathfrak{P}_{k,0}$, $v_lv_iv_jv_{l_1} \in \mathfrak{P}_{l,0}$, $v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1}$, and we have $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(s) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \min\{\lambda(v_lv_i), \lambda(v_lv_k)\} =: c$.

(1) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} < b$.

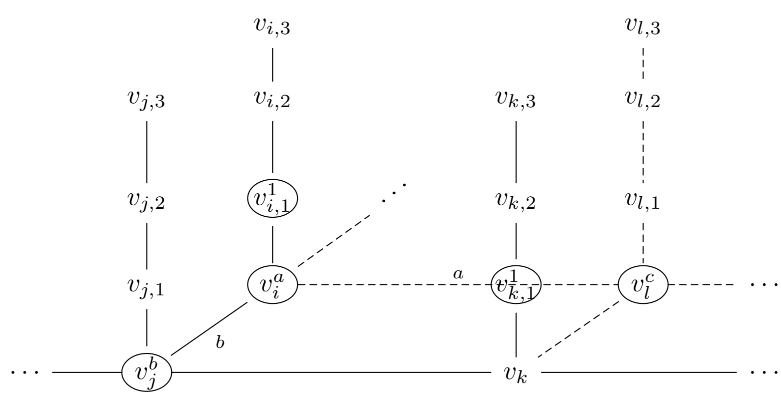


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_j^b, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and we have $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,1}$, $v_jv_iv_lv_k \in \mathfrak{P}_{j,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(2) Assume $f\{\lambda(v_{j,1}v_j), \lambda(v_jv_k)\} \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^b, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1}$, $v_iv_lv_kv_j \in \mathfrak{P}_{i,0}$, $v_jv_iv_lv_k \in \mathfrak{P}_{j,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, $v_iv_jv_kv_l \in \mathfrak{P}_{l,0}$ and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(t) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i) =: b$ and $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\} \geq b$, a contradiction.

(u) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i)$ and $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} < \lambda(v_lv_k) = b$ and $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_j) =: c$. Then it is similar to the case (m).

(v) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} < \lambda(v_iv_l) =: a$. Suppose we have $f\{\lambda(v_jv_i), \lambda(v_jv_k)\} < \lambda(v_jv_i)$ and $f\{\lambda(v_kv_j), \lambda(v_kv_l)\} < \lambda(v_kv_l) =: b$ and $f\{\lambda(v_lv_i), \lambda(v_lv_k)\} <$

$\lambda(v_l v_k) = b$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \lambda(v_l v_i) = a$. Then it is similar to the case (n).

(w) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_j v_i), \lambda(v_j v_k)\} < \lambda(v_j v_i)$ and $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_k v_l) =: b$ and $v_j v_l \in E(G_\omega)$ and

$$c := f\{\lambda(v_j v_i), \lambda(v_j v_k)\} > \max\big\{\lambda(v_j v_k), f\{\lambda(v_j v_i), \lambda(v_j v_l)\}\big\}.$$

Then it is similar to the case (o).

(x) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_j v_i), \lambda(v_j v_k)\} < \lambda(v_j v_i) =: b$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \lambda(v_l v_i) =: c$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \min\{\lambda(v_l v_i)\}$ and there exists $v_k v_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f\{\lambda(v_k v_j), \lambda(v_k v_l)\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\},$$

$$y := f\{\lambda(v_k v_j), \lambda(v_k v_{l_1,l_2})\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\}.$$

Then it is similar to the case (p).

(y) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_j v_i), \lambda(v_j v_k)\} < \lambda(v_j v_i) =: b$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \lambda(v_l v_i) =: c$ and $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} > \lambda(v_k v_l)$ and there exists $v_k v_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f\{\lambda(v_k v_j), \lambda(v_k v_l)\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\},$$

$$y := f\{\lambda(v_k v_j), \lambda(v_k v_{l_1,l_2})\} > \max\big\{\lambda(v_k v_j), f\{\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2})\}\big\}.$$

Then it is also similar to the case (p).

(z) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f\{\lambda(v_{i,1} v_i), \lambda(v_i v_j)\} < \lambda(v_i v_l) =: a$. Suppose we have $f\{\lambda(v_j v_i), \lambda(v_j v_k)\} < \lambda(v_j v_i)$ and $f\{\lambda(v_k v_j), \lambda(v_k v_l)\} < \lambda(v_k v_l) =: b$ and $f\{\lambda(v_l v_i), \lambda(v_l v_k)\} < \lambda(v_l v_i) = a$ and there exists $v_j v_{l_1,l_2} \in E((\Sigma_\iota G)_\lambda)$ with $v_i \neq v_{l_1,l_2} \neq v_k$ such that

$$c := f\{\lambda(v_j v_i), \lambda(v_j v_k)\} > \max\big\{\lambda(v_j v_k), f\{\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2})\}\big\},$$

$$x := f\{\lambda(v_j v_k), \lambda(v_j v_{l_1,l_2})\} > \max\big\{\lambda(v_j v_k), f\{\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2})\}\big\}.$$

Then it is similar to the case (r). $\square$

Theorem 3.142. *Assume $r \geq 4$. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. If $I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is m-unmixed, then the weight function λ satisfies the constraints in Proposition 3.136.*

Proof. By Lemma 3.138 and its proof, it is enough to show that if $a := f\{\lambda(v_{i,1}v_i), \lambda(v_iv_j)\} > f\{\lambda(v_{i,1}v_i), \lambda(v_iv_k)\} =: b$ for a 2-path $v_jv_iv_k$ in G_ω, then there exists an f-weighted r-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. It is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \big\{v^1_{j,r-2}, v^a_i, v^1_{i,r-1}, v^1_{k,1}\big\} \sqcup \big\{v^1_m \mid m \in \{1, \ldots, d\} \smallsetminus \{i,j,k\}\big\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$, $v_{j,r} \cdots v_{j,1}v_j \in \mathfrak{P}_{j,r-2}$, $v_{i,r-1} \cdots v_{i,1}v_iv_k \in \mathfrak{P}_{i,r-1}$, $v_{i,r-2} \cdots v_{i,1}v_iv_jv_{j,1} \in \mathfrak{P}_{i,0}$, $v_{k,r} \cdots v_{k,1}v_k \in \mathfrak{P}_{k,1}$, and to show that $v_{t,r} \cdots v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i,j,k\}$.

$$\begin{array}{ccccccccccccc}
 & & \vdots & & & & & & & & & & \\
 & & | & & & & & & & & & & \\
\cdots & \text{---} & v_j & \text{---} & v_{j,1} & \text{---} & \cdots & \text{---} & v^1_{j,r-2} & \text{---} & v_{i,r-1} & \text{---} & v_{j,r} \\
 & & | & & & & & & & & & & \\
\cdots & \text{---} & v^a_i & \text{---} & v_{i,1} & \text{---} & \cdots & \text{---} & v_{i,r-2} & \text{---} & v^1_{i,r-1} & \text{---} & v_{i,r} \\
 & & | & & & & & & & & & & \\
\cdots & \text{---} & v_k & \text{---} & v^1_{k,1} & \text{---} & \cdots & \text{---} & v_{i,r-2} & \text{---} & v_{i,r-1} & \text{---} & v_{k,r} \\
 & & | & & & & & & & & & & \\
 & & \vdots & & & & & & & & & &
\end{array}$$

where just two edge weights are showed. $\square$

Corollary 3.143. Let $f = \max$. Then the constraints for λ in Proposition 3.134, 3.135 and 3.136 becomes

$$\lambda(v_iv_j) \leq \min\{\lambda(v_iv_{i,1}), \lambda(v_jv_{j,1})\}, \ \forall \ v_iv_j \in E(G_\omega).$$

Corollary 3.144. Let $f = \operatorname{lcm}$. Then the constraints for λ in Proposition 3.134, 3.135 and 3.136 becomes

$$\lambda(v_iv_j) \mid \lambda(v_iv_{i,1}) \text{ and } \lambda(v_iv_j) \mid \lambda(v_jv_{j,1}), \ \forall \ v_iv_j \in E(G_\omega).$$

Corollary 3.145. Let $f = \min$. Then the constraints for λ in Proposition 3.134, 3.135 and 3.136 becomes

$$\lambda(v_iv_j) \le \min\{\lambda(v_iv_{i,1}), \lambda(v_jv_{j,1})\},\ \forall\ v_iv_j \in E(G_\omega),$$

$$\lambda(v_{i,k}, v_{i,k+1}) \le \lambda(v_{i,k+1}v_{i,k+2}),\ \forall\ i = 1, \ldots, d \text{ and } k = 0, \ldots, r-2,$$

and

$$\begin{cases} \lambda(v_iv_j) \ge \lambda(v_jv_k) \text{ or } \lambda(v_kv_l) \ge \lambda(v_jv_k) \text{ for all 3-paths } v_iv_jv_kv_l \text{ in } G_\omega, & \text{if } r = 2, \\ \lambda(v_iv_j) \ge \lambda(v_jv_k) \text{ or } \lambda(v_lv_m) \ge \lambda(v_kv_l) \text{ for all 4-paths } v_iv_jv_kv_lv_m \text{ in } G_\omega, & \\ \text{the weights on edges satisfies } a = b \ge c \text{ for all 3-cycles in } G_\omega, & \text{if } r = 3 \\ \text{all edges in } G_\omega \text{ have the same weight,} & \text{if } r \ge 4. \end{cases}$$

Proof. We first show the equivalence for weight constraints on 4-paths in G_ω when $r = 3$. Let $v_iv_jv_kv_lv_m$ be a 4-path in G_ω. On one hand, let $\lambda(v_iv_j) \ge \lambda(v_jv_k)$ or $\lambda(v_lv_m) \ge \lambda(v_kv_l)$, then by the condition (a) in Proposition 3.135 we have $\min\{\lambda(v_iv_j), \lambda(v_jv_k)\} = \lambda(v_jv_k) \ge \lambda(v_jv_k)$ or $\min\{\lambda(v_kv_l), \lambda(v_lv_m)\} = \lambda(v_kv_l) \ge \lambda(v_kv_l)$, so the condition (d) in Proposition 3.135 holds. On the other hand, without loss of generality, assume $\lambda(v_iv_j) < \lambda(v_jv_k)$, then $\min\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} = \lambda(v_jv_i) < \lambda(v_jv_k)$ by the condition (a) in Proposition 3.135 and $\min\{\lambda(v_iv_j), \lambda(v_jv_k)\} < \lambda(v_jv_k)$, so $\min\{\lambda(v_kv_l), \lambda(v_lv_m)\} \ge \lambda(v_kv_l)$ by the condition (d) in Proposition 3.135, hence $\lambda(v_lv_m) \ge \lambda(v_kv_l)$.

We then show the equivalence for weight constraints on 3-cycles in G_ω when $r = 3$. Let $v_iv_jv_kv_i$ be a 3-cycle in G_ω. On one hand, let the weights on edges of $v_iv_jv_kv_i$ satisfies $a = b \ge c$, without loss of generality, assume $\lambda(v_iv_j) \le \lambda(v_jv_k) = \lambda(v_kv_i)$, then we have

$$\min\{\lambda(v_jv_i), \lambda(v_jv_k)\} = \lambda(v_jv_i) < \lambda(v_jv_k) = \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$$

and $\min\{\lambda(v_kv_i), \lambda(v_kv_j)\} = \lambda(v_kv_j) \ge \lambda(v_kv_j)$, and $\min\{\lambda(v_kv_i), \lambda(v_kv_j)\} = \lambda(v_kv_j) \ge \lambda(v_kv_i)$, so we proved the condition (e).(1) and (e).(2).i in Proposition 3.135, it is straightforward to show that the condition (e).(2).ii and (e).(2).iii in Proposition 3.135 holds. On the other hand, if $\lambda(v_iv_j) = \lambda(v_jv_k) = \lambda(v_kv_i)$, then we are done, so without loss of generality, assume $\lambda(v_iv_j) < \lambda(v_iv_k)$, then $\min\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < \min\{\lambda(v_iv_{i,1}), \lambda(v_iv_k)\}$ by the condition (a).(1) in Proposition 3.135, so $\min\{\lambda(v_kv_i), \lambda(v_kv_j)\} \ge \lambda(v_kv_j)$ by the condition (e).(1) in Proposition 3.135, hence $\lambda(v_kv_i) \ge \lambda(v_kv_j)$, since $\min\{\lambda(v_iv_{i,1}), \lambda(v_iv_j)\} < \lambda(v_iv_k)$ similarly, we have $\min\{\lambda(v_kv_i), \lambda(v_kv_j)\} \ge \lambda(v_kv_i)$

by the condition (e).(2) in Proposition 3.135, so $\lambda(v_kv_j) \geq \lambda(v_kv_i)$, hence $\lambda(v_kv_i) \geq \lambda(v_kv_j) \geq \lambda(v_kv_i)$ and so $\lambda(v_kv_i) = \lambda(v_kv_j) > \lambda(v_iv_j)$.

We show there is no weight constraint on any 4-cycles in G_ω. It suffices to show that the condition f.(2) in Proposition 3.135 holds automatically provided that $f = \min$. It is straightforward to show that the condition (2).i, (2).ii, (2).iv and (2).v in Proposition 3.135 holds automatically. But the condition (2).iii in Proposition 3.134 is equivalent to that either $\lambda(v_kv_j) \geq \lambda(v_kv_l)$, or $\lambda(v_lv_i) \geq \lambda(v_lv_k)$, or $\lambda(v_kv_l) \geq \lambda(v_kv_j)$ and $\lambda(v_lv_k) \geq \lambda(v_lv_i)$, which is equivalent to $\lambda(v_kv_l) \leq \max\{\lambda(v_kv_j), \lambda(v_lv_i)\}$ or $\lambda(v_kv_l) \geq \max\{\lambda(v_kv_j), \lambda(v_lv_i)\}$, but this holds automatically.

It is straightforward to show the equivalence for $r = 2$ and $r = 4$. □

Corollary 3.146. Let $f = \gcd$. Then the constraints for λ in Proposition 3.134, 3.135 and 3.136 becomes

$$\lambda(v_iv_j) \mid \lambda(v_iv_{i,1}) \text{ and } \lambda(v_iv_j) \mid \lambda(v_jv_{j,1}),\ \forall\ v_iv_j \in E(G_\omega),$$

$$\lambda(v_{i,k}, v_{i,k+1}) \mid \lambda(v_{i,k+1}v_{i,k+2}),\ \forall\ i = 1, \ldots, d \text{ and } k = 0, \ldots, r-2,$$

and

(a) if $r = 2$, then $\lambda(v_jv_k) \mid \lambda(v_iv_j)$ or $\lambda(v_jv_k) \mid \lambda(v_kv_l)$ for all 3-paths $v_iv_jv_kv_l$ in G_ω,

(b) if $r = 3$, then

(1) for all 4-paths $v_iv_jv_kv_lv_m$ in G_ω: if $\lambda(v_jv_i) < \lambda(v_jv_k)$, then $\lambda(v_kv_l) \mid \lambda(v_lv_m)$,

(2) for all 3-cycles $v_iv_jv_kv_i$ in G_ω: if $\lambda(v_iv_j) < \lambda(v_iv_k)$, then $\lambda(v_kv_i) = \lambda(v_kv_j)$ and

$$\begin{aligned}&\gcd\{\lambda(v_jv_i), \lambda(v_jv_{n,0})\} \leq \max\big\{\gcd\{\lambda(v_jv_i), \lambda(v_jv_k)\}, \gcd\{\lambda(v_jv_k), \lambda(v_jv_{n,0})\}\big\}\\ &\qquad\qquad \forall\ v_jv_{n,0} \in E(G_\omega) \text{ with } i \neq n \neq k,\end{aligned}$$

(3) for all 4-cycles $v_iv_jv_kv_lv_i$: if $\lambda(v_iv_j) < \lambda(v_iv_l)$, then

$$\begin{cases} \text{either } \lambda(v_kv_l) \mid \lambda(v_kv_j) \text{ and } \lambda(v_jv_i) \mid \lambda(v_jv_k), \\ \text{or } \lambda(v_lv_i) \mid \lambda(v_lv_k), \\ \text{or } \lambda(v_lv_k) \mid \lambda(v_lv_i), \end{cases}$$

and

$$\begin{cases} \text{either } \lambda(v_kv_l) \mid \lambda(v_kv_j), \\ \text{or } \lambda(v_lv_k) \mid \lambda(v_lv_i), \\ \text{or } \lambda(v_kv_j) \mid \lambda(v_kv_l) \text{ and } \lambda(v_lv_i) \mid \lambda(v_lv_k). \end{cases}$$

(c) if $r = 4$, then all edges in G_ω have the same weight.

Proof. We first show the equivalence for weight constraints on 4-paths in G_ω when $r = 3$. Let $v_iv_jv_kv_lv_m$ be a 4-path in G_ω. On one hand, let $\lambda(v_iv_j) < \lambda(v_jv_k)$ and assume $\lambda(v_kv_l) \mid \lambda(v_lv_m)$, then $\gcd\{\lambda(v_kv_l), \lambda(v_lv_m)\} = \lambda(v_kv_l) \geq \lambda(v_kv_l)$, so the condition (d) in Proposition 3.135 holds. On the other hand, assume $\lambda(v_iv_j) < \lambda(v_jv_k)$, then $\gcd\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} = \lambda(v_jv_i) < \lambda(v_jv_k)$ by the condition (a) in Proposition 3.135 and $\gcd\{\lambda(v_iv_j), \lambda(v_jv_k)\} < \lambda(v_jv_k)$, so $\gcd\{\lambda(v_kv_l), \lambda(v_lv_m)\} \geq \lambda(v_kv_l)$ by the condition (d) in Proposition 3.135, hence $\lambda(v_kv_l) \mid \lambda(v_lv_m)$.

We then show the equivalence for weight constraints on 3-cycles in G_ω when $r = 3$. Let $v_iv_jv_kv_i$ be a 3-cycle in G_ω. On one hand, let $\lambda(v_iv_j) < \lambda(v_iv_k)$, assume $\lambda(v_kv_i) = \lambda(v_kv_j)$ and

$$\gcd\{\lambda(v_jv_i), \lambda(v_jv_{n,0})\} \leq \max\big\{\gcd\{\lambda(v_jv_i), \lambda(v_jv_k)\}, \gcd\{\lambda(v_jv_k), \lambda(v_jv_{n,0})\}\big\}$$
$$\forall\ v_jv_{n,0} \in E(G_\omega) \text{ with } i \neq n \neq k,$$

then $\gcd\{\lambda(v_jv_i), \lambda(v_jv_k)\} \leq \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$ and $\gcd\{\lambda(v_kv_i), \lambda(v_kv_j)\} = \lambda(v_kv_j) \geq \lambda(v_kv_j)$, and $\gcd\{\lambda(v_kv_i), \lambda(v_kv_j)\} = \lambda(v_kv_i) \geq \lambda(v_kv_i)$, so the condition (e).(1) and (e).(2).i in Proposition 3.135 was proved, it is straightforward to show that the condition (e).(2).ii in Proposition 3.135 holds and the condition (e).(2).iii in Proposition 3.135 holds automatically. On the other hand, then it is straightforward to show that we can deduce the condition (b).(2) in the corollary from the condition (e) in Proposition 3.135. □

Proposition 3.147. Assume H_λ is obtained by pruning a sequence of r-pathless leaves from G_ω and that H_λ is an r-path suspension of a weighted graph Γ_μ. Then the following conditions are equivalent.

(i) $I_{r,f}(G_\omega)$ is Cohen-Macaulay;

(ii) $I_{r,f}(G_\omega)$ is m-unmixed; and

(iii) the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136, where we rename the vertices of H_λ such that $V(\Gamma_\mu) = \{v_i \mid i = 1, \ldots, d\}$,

$$V(H_\lambda) = \{v_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\} \text{ with } v_{i,0} = v_i, \ \forall\ i = 1, \ldots, d,$$

and $\{v_{i,0}v_{i,1} \cdots v_{i,r}\}_{i=1}^{d}$ are all the d r-whiskers.

Proof. "(i)⇒(ii)". It follows from Fact 2.96.

"(ii)⇒(iii)". Assume $I_{r,f}(G_\omega)$ is m-unmixed. By Lemma 3.89(b) $I_{r,f}(H_\lambda)$ is also m-unmixed. Then (iii) follows from Theorem 3.139, 3.141 and 3.142.

"(iii)⇒(i)". Assume condition (iii) holds. Then Theorem 3.137 implies that $I_{r,f}(H_\lambda)$ is Cohen-Macaulay. So Lemma 3.89(c) implies that $I_{r,f}(G_\omega)$ is as well. □

Because of the following fact and Theorem 3.137, the main result of this subsection gives a formula to compute $\operatorname{type}\big(R/I_{r,f}(G_\omega)\big)$ for all trees such that $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay.

Theorem 3.148. *Assume that G_ω is a weighted tree. Then the following conditions are equivalent.*

(i) $I_{r,f}(G_\omega)$ is Cohen-Macaulay;

(ii) $I_{r,f}(G_\omega)$ is m-unmixed; and

(iii) there exists a weighted tree Γ_μ and an r-path suspension H_λ of Γ_μ such that H_λ is obtained by pruning a sequence of r-pathless leaves from G_ω, the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136, where we rename the vertices of H_λ such that we have $V(\Gamma_\mu) = \{v_i \mid i = 1, \ldots, d\}$,

$$V(H_\lambda) = \{v_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\} \textit{ with } v_{i,0} = v_i, \ \forall\ i = 1, \ldots, d,$$

and $\{v_{i,0}v_{i,1} \cdots v_{i,r}\}_{i=1}^{d}$ are all the d r-whiskers.

Proof. "(iii)⇒(i)⇒(ii)" follows from Proposition 3.147.

"(ii)⇒(iii)". Assume $I_{r,f}$ is m-unmixed. Since G is finite, prune a sequence of r-pathless leaves from G_ω to obtain a weighted graph H_λ that has no r-pathless leaves. Lemma 3.89(b) implies that $I_{r,f}(H_\lambda)$ is m-unmixed. So $I_r(H_\lambda)$ is m-unmixed by Lemma 3.86. Hence H is an r-path suspension of a tree Γ by [3, Theorem 3.8 and Remark 3.9]. Finally, Proposition 3.147 implies the weight conditions on $E(H_\lambda)$. □

Definition 3.149. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. Let $\mathfrak{P} := (V'', \delta'')$ be an size-minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. Define

$$\begin{aligned} \gamma_{(V'',\delta'')} : q(V'') &\longrightarrow \mathbb{N} \\ v_i &\longmapsto \delta''(v_{i,j_i}) + \sum_{k=0}^{j_i-1} h_{i,k}(\mathfrak{P}) \qquad \text{if } v_{i,j_i} \in V'' \text{ with } j_i \geq 1, \end{aligned}$$

where

$$h_{i,k}(\mathfrak{P}) = \begin{cases} f\{\lambda(v_{i,k+1}v_{i,k}), \lambda(v_{i,k}v_{i,k-1})\} & \text{if } 1 \leq k \leq j_i - 1, \\ \min\big\{f\{\lambda(v_{i,1}v_i), \lambda(v_i v_\alpha) \mid v_{i,j_i} \smile P_r, \text{ where } P_r \text{ is an } r\text{-path} & \\ \text{in } \mathfrak{P}_{i,k}(I), \text{ such that } v_\alpha \in V(P_r) \text{ and } v_i v_\alpha \in E(G_\omega)\big\} & \text{if } k = 0. \end{cases}$$

Remark. $\gamma_{(V'',\delta'')}$ is well-defined by Theorem 3.85, Proposition 3.134, 3.135 and 3.136.

The following theorem is a key for decomposing $I_{r,f}\big((\Sigma G)_{\lambda'}\big)R$ with $\lambda' = \lambda|_{\Sigma G}$ and hence $I_{r,f}\big((\Sigma_r G)_\lambda\big)R$. The proof is somewhat technical. The reader may wish to follow the argument with the succeeding example.

Theorem 3.150. *Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. Let $\mathfrak{P} := (V'', \delta'')$ be an size-minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. Then we have $I_{r,f}\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$.*

Proof. To show that $I_{r,f}\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$, it is enough to show that every monomial generator of $I_{r,f}\big((\Sigma_{r-1}G)_\lambda\big)R$ is in $P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Let $\underline{X}^{\underline{b}} := X_{i_1}^{e_{i_1,j_1}} \dots X_{i_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ be such a generator corresponding to an r-path $P_r := v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $(\Sigma_{r-1}G)_\lambda$. We need to show that $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Note that $X_{i_1,j_1}^{e_{i_1,j_1}} \cdots X_{i_{r+1},j_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ is of the following form.

$$\begin{array}{ccccccc}
X_{i_{1+j_1},0}^{e_{i_{1+j_1},0}} & & & & & & \\
\| & & & & & & \\
X_{i_1,0}^{e_{i_1,0}} & \text{———} & X_{i_1,1}^{e_{i_1,1}} & \text{———} & \cdots & \text{———} & X_{i_1,j_1}^{e_{i_1,j_1}} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
X_{i_{r+1},0}^{e_{i_{r+1},0}} & \text{———} & X_{i_{r+1},1}^{e_{i_{r+1},1}} & \text{———} & \cdots & \text{———} & X_{i_{r+1},j_{r+1}}^{e_{i_{r+1},j_{r+1}}} \\
\| & & & & & & \\
X_{i_{1+r-j_{r+1}},0}^{e_{i_{1+r-j_{r+1}},0}} & & & & & &
\end{array}$$

where j_1 or j_{r+1} may be 0. Since P_r is an r-path in $(\Sigma_{r-1}G)_\lambda$ and (V'', δ'') is an f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, we have there exists $l \in \{1, \ldots, r+1\}$ such that $v_{i_l,j_l} \smile P_r$. So $v_{i_l,j_l} \in V''$ and $\delta''(v_{i_l,j_l}) \leq e_{i_l,j_l}$. Let $M_0 := \max_{1 \leq k \leq r+1}\{j_k \mid i_l = i_k\}$. Then $j_l \leq M_0$. Hence

$$\delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq e_{i_l,j_l} + \sum_{k=0}^{j_l-1} e_{i_l,k} = \sum_{k=0}^{j_l} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l.k}, \text{ i.e., } \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k}.$$

(a) Assume $j_l = 0$. Then

$$\gamma_{(V'',\delta'')}(v_{i_l}) = \delta''(v_{i_l,0}) = \delta''(v_{i_l,0}) + \sum_{k=0}^{0-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} \cdot e_{i_k,j_k} = b_{i_l},$$

where $\mathbb{1}_{l,k} = \begin{cases} 1 & \text{if } i_k = i_l \\ 0 & \text{otherwise} \end{cases}$, $\forall\ k = 1, \ldots, r+1$.

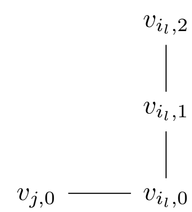
(b) Assume $j_l \geq 1$. Then $M_0 \geq 1$. Without loss of generality, assume if $v_{k,l} \smile P_r$ for some $v_{k,l} \in V''$, then $l > 1$.

(1) Assume $r = 2$. Then $j_l = 1$ and P_2 is of the following form. Since $v_{j,0}, v_{i_l,0} \not\smile P_2$ by assumption, we have $P_2 \in \mathfrak{P}_{i_l,1}$. So $h_{i_l,0}(\mathfrak{P}) \leq e_{i_l,0}$.

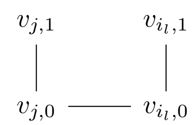
$$\begin{array}{ccc}
 & & v_{i_l,1} \\
 & & | \\
v_{j,0} & \text{———} & v_{i_l,0}
\end{array}$$

(2) Assume $r = 3$.

i. Assume P_3 is of the following form. Then similar to (1), we have $P_3 \in \mathfrak{P}_{i_l,2}$ or $P_3 \in \mathfrak{P}_{i_l,1}$ and so $h_{i_l,0}(\mathfrak{P}) \leq e_{i_l,0}$.

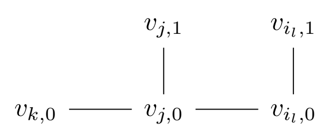


ii. Assume P_3 is of the following form. Then $j_l = 1$ and $v_{i_l,1} \smile P_2$.

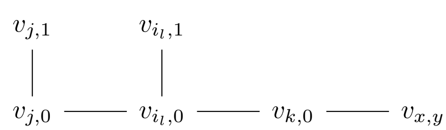


If $v_{j,1} \not\smile P_3$, then similar to (1), we have $P_3 \in \mathfrak{P}_{i_l,1}$ and so $h_{i_l,0}(\mathfrak{P}) \leq e_{i_l,0}$. Assume $v_{j,1} \smile P_3$. Then $P_r \notin \mathfrak{P}_{i_1,1}$ and $P_r \notin \mathfrak{P}_{j,1}$. So we have the following 3 cases.

A. Assume $v_{i_l,1}v_{i_l,0}v_{j,0}v_{k,0} \in \mathfrak{P}_{i_l,1}$. Then $h_{i_l,0}(\mathfrak{P}) \leq e_{i_l,0}$.



B. Assume $v_{i_l,1}v_{i_l,0}v_{k,0}v_{x,y} \in \mathfrak{P}_{i_l,1}$ with $v_{x,y} \neq v_{j,0}$.



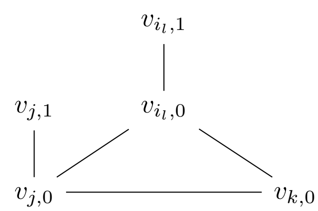
By way of contradiction, we have $v_{k,0}, v_{x,y} \not\smile v_{j,0}v_{i_l,0}v_{k,0}v_{x,y}$. By assumption $v_{i_l,0} \not\smile P_3$, so $v_{i_l,0} \frown v_{j,0}v_{i_l,0}v_{k,0}v_{x,y}$. Since $\mathfrak{P}$ is an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$, we have $v_{j,0} \smile v_{j,0}v_{i_l,0}v_{k,0}v_{x,y}$ by Lemma 3.131. Hence $v_{j,0}, v_{j,1} \in V''$, contradicted by Proposition 3.135 and Theorem 3.85 since $\mathfrak{P} = (V'', \delta'')$ is an size-minimal f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.

C. Assume $v_{i_l,1}v_{i_l,0}v_{k,0}v_{x,y} \in \mathfrak{P}_{i_l,1}$ with $v_{x,y} = v_{j,0}$. If $f\{\lambda(v_{i_l,1}v_{i_l,0}), \lambda(v_{i_l,0}v_{k,0})\} \leq e_{i_l,0}$, then

$h_{i_l,0}(\mathfrak{P}) \leq f\{\lambda(v_{i_l,1}v_{i_l,0}), \lambda(v_{i_l,0}v_{k,0})\} \leq e_{i_l,0}$. Assume

$$f\{\lambda(v_{i_l,1}v_{i_l,0}), \lambda(v_{i_l,0}v_{k,0})\} > e_{i_l,0} = f\{\lambda(v_{i_l,1}v_{i_l,0}), \lambda(v_{i_l,0}v_{j,0})\}.$$

Since $v_{i_l,0}v_{j,0}v_{k,0}v_{i_l,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{k,0}v_{j,0}), \lambda(v_{k,0}v_{i_l,0})\} \geq \lambda(v_{k,0}v_{j,0})$ by the condition (e).(1) in Proposition 3.135. Then since $v_{k,0} \not\succ v_{i_l,1}v_{i_l,0}v_{k,0}v_{j,0}$, we have $v_{k,0} \not\succ v_{i_l,1}v_{i_l,0}v_{j,0}v_{k,0}$. Since $v_{i_l,1}, v_{j,1} \in V''$, we have $v_{i_l,0}, v_{j,0} \notin V''$ by Proposition 3.135 and Fact 3.85. So $v_{i_l,0}, v_{j,0} \not\succ v_{i_l,1}v_{i_l,0}v_{j,0}v_{k,0}$. Hence $v_{i_l,1}v_{i_l,0}v_{j,0}v_{k,0} \in \mathfrak{P}_{i_l,1}$ by Lemma 3.131. So $h_{i_l,0}(\mathfrak{P}) \leq f\{\lambda(v_{i_l,1}v_{i_l,0}), \lambda(v_{i_l,0}v_{j,0})\} = e_{i_l,0}$.



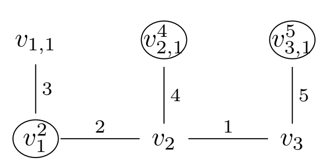
(3) Assume $r = 4$. Then it is straightforward to show that $h_{i_l,0}(\mathfrak{P}) = e_{i_l,0}$.

Thus, by (1), (2) and (3) we have $h_{i_l,0}(\mathfrak{P}) \leq e_{i_l,0}$. Since $h_{i_l,k}(\mathfrak{P}) = f\{\lambda(v_{i_l,k+1}v_{i_l,k}), \lambda(v_{i_l,k}v_{i_l,k-1})\} = e_{i_l,k}$ for $k = 1, \ldots, j_l - 1$, we have

$$\begin{aligned}\gamma_{(V'',\delta'')}(v_{i_l}) &= \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k}(\mathfrak{P}) \leq \delta''(v_{i_l,j_l}) + e_{i_l,0} + \sum_{k=1}^{j_l-1} e_{i_l,k} \\ &= \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} \cdot e_{i_k,j_k} = b_{i_l}.\end{aligned}$$

Therefore, by (a),(b) and (c) we have $X_{i_l}^{\gamma_{(V'',\delta'')}(v_{i_l})} \mid \underline{X}^{\underline{b}}$. So $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. $\square$

Example 3.151. A weighted suspension $(\Sigma P_2)_\lambda$ of $G_\omega := (P_2)_\omega = \big(\ v_1 \xrightarrow{\ 2\ } v_2 \xrightarrow{\ 1\ } v_3\ \big)$ with a minimal min-weighted 2-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma P_2)_\lambda$ is given in the following sketch



Since $I_{2,\min}\big((\Sigma P_2)_\lambda\big) = \big(X_{1,1}^3X_{1,0}^2X_{2,0}^2, X_{1,0}^2X_{2,0}^2X_{2,1}^4, X_{1,0}^2X_{2,0}X_{3,0}, X_{2,1}^4X_{2,0}X_{3,0}, X_{2,0}X_{3,0}X_{3,1}^5\big)R'$, we have

$$I_{2,\min}\big((\Sigma P_2)_\lambda\big)R = \big(X_1^5X_2^2, X_1^2X_2^6, X_1^2X_2X_3, X_2^5X_3, X_2X_3^6\big)R.$$

Note that $\gamma_{(V'',\delta'')}(v_1) = \delta''(v_1) = 2$,

$$\gamma_{(V'',\delta'')}(v_2) = h_{2,1}(\mathfrak{P}) + h_{2,0}(\mathfrak{P}) = \lambda(v_{2,1}v_2) + \min\{\lambda(v_{2,1}v_2), \lambda(v_2v_3)\} = 4+1 = 5,$$

$$\gamma_{(V'',\delta'')}(v_3) = h_{3,1}(\mathfrak{P}) + h_{3,0}(\mathfrak{P}) = \lambda(v_{3,1}v_3) + \min\{\lambda(v_{3,1}v_3), \lambda(v_3v_2)\} = 5+1 = 6.$$

Thus,

$$P\big(V'', \gamma_{(V'',\delta'')}\big) = \big(X_1^2, X_2^5, X_3^6\big)R \supseteq \big(X_1^5X_2^2, X_1^2X_2^6, X_1^2X_2X_3, X_2^5X_3, X_2X_3^6\big)R = I_{2,\min}\big((\Sigma P_2)_\lambda\big)R.$$

Proposition 3.152. Let $(\Sigma G)_\lambda$ be a weighted suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134. Let $I = I_{2,f}\big((\Sigma G)_\lambda\big)$. Then the monomial ideal $IR = I_{2,f}\big((\Sigma G)_\lambda\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with (V'', δ'') a minimal f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$.

Proof. Let $P(V', \delta') := \big(X_{b_1}^{\beta_{b_1}}, \ldots, X_{b_s}^{\beta_{b_s}}\big)R$ occur in an irredundant m-irreducible decomposition of the ideal IR with $V' = \{v_{b_1}, \ldots, v_{b_s}\}$ and $\delta'(v_{b_k}) = \beta_{b_k}$ for $k = 1, \ldots, s$. Let $\mathfrak{p} := (V', \delta')$. Define

$$V''_{b_k} := \begin{cases} \{v_{b_k,1}\} & \text{if } \beta_{b_k} > \lambda(v_{b_k,1}v_{b_k,0}), \\ \{v_{b_k,0}\} & \text{if } \beta_{b_k} \le \lambda(v_{b_k,1}v_{b_k,0}), \end{cases} \quad \forall\ k = 1, \ldots, s,$$

Set $V'' := \bigsqcup_{k=1}^s V''_{b_k}$ and define

$$\begin{aligned} \delta'' : V'' &\longrightarrow \mathbb{N} \\ v_{b_k,j} &\longmapsto \begin{cases} \lambda(v_{b_k,1}v_{b_k,0}) & \text{if } j = 1, \\ \beta_{b_k} & \text{if } j = 0, \end{cases} \quad \forall\ k = 1, \ldots, s. \end{aligned}$$

Claim. $\mathfrak{P} := (V'', \delta'')$ is an f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$. Let P_2 be a 2-path in $(\Sigma G)_\lambda$ and $\underline{X}^{\underline{\alpha}}$ the corresponding generator in I. Then $p(\underline{X}^{\underline{\alpha}}) \in IR \subseteq P(V', \delta')$. So $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}})$ for some $k \in \{1, \ldots, s\}$.

(a) Assume P_2 is of the form

$$\begin{tikzcd} v_{i_1,0} \arrow[r, dash] & v_{i_2,0} \arrow[r, dash] & v_{i_3,0}. \end{tikzcd}$$

Then $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}}) = \underline{X}^{\underline{\alpha}}$ and $b_k \in \{i_1, i_2, i_3\}$. So $\beta_{b_k} \leq \lambda(v_{b_k,1}v_{b_k,0})$. Then $v_{b_k,0} \in V''_{b_k} \subseteq V''$ and so $\delta''(v_{b_k,0}) = \beta_{b_k}$. Hence $X_{b_k,0}^{\delta''(v_{b_k,0})} \mid \underline{X}^{\underline{\alpha}}$. Thus, $v_{b_k,0} \smile (P_2, \mathfrak{P})$.

(b) Assume P_2 is of the form

$$\begin{tikzcd} & v_{i_1,1} \arrow[d, dash] \\ v_{i_3,0} \arrow[r, dash] & v_{i_1,0}. \end{tikzcd}$$

If $b_k = i_3$, then similar to (a) we have $v_{b_k,0} \smile (P_2, \mathfrak{P})$. So we assume $b_k = i_1$. If $\beta_{i_1} > \lambda(v_{i_1,1}v_{i_1,0})$, then $v_{i_1,1} \in V''_{i_1} \subseteq V''$ and $\delta''(v_{i_1,1}) = \lambda(v_{i_1,1}v_{i_1,0})$, so $v_{i_1,1} \smile (P_2, \mathfrak{P})$. So we assume $\beta_{i_1} \leq \lambda(v_{i_1,1}v_{i_1,0})$. Then $v_{i_1,0} \in V''$ and $\delta''(v_{i_1,0}) = \beta_{i_1}$. If $\beta_{i_1} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$, then $\delta''(v_{i_1,0}) \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$ and so $v_{i_1,0} \smile (P_2, \mathfrak{P})$ since $v_{i_1,0} \in V(P_2) \cap V''$. So we assume $\beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$. Then $v_{i_1,0} \not\smile (P_2, \mathfrak{P})$. So we need to show that $v_{i_3,0} \smile (P_2, \mathfrak{P})$. Claim. $\mathfrak{p}_{i_1,0}(IR) \neq \{v_{i_3,0}v_{i_1,0}v_{i_1,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}\}$. Suppose not. Define $\sigma' : V' \to \mathbb{N}$ by $\sigma'(v_{b_m}) := \delta'(v_{b_m})$ for $m \in \{1, \ldots, s\} \smallsetminus \{k\}$ and

$$\sigma'(v_{b_k,0}) = \sigma'(v_{i_1,0}) := \lambda(v_{i_1,1}v_{i_1,0}) + f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} > \lambda(v_{i_1,1}v_{i_1,0}) \geq \beta_{i_1} = \delta'(v_{i_1,0}),$$

we have $v_{i_1,0} \smile \big(v_{i_1,1}v_{i_1,0}v_{i_3,0} \rightsquigarrow v_{i_1,0}v_{i_3,0}, (V', \sigma')\big)$ and so $IR \subseteq P(V', \sigma') \subsetneq P(V', \delta')$, a contradiction. Thus, $\mathfrak{p}_{i_1,0}(IR) \neq \{v_{i_1,0}v_{i_3,0}\}$. Since $\delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq \max\big\{\lambda(v_{i_1,0}v_{i_3,0}), f\{\lambda(v_{i_3,0}v_{i_3,0}), \lambda(v_{i,0}v_{j,0})\}\big\}$ for any $v_{i_1,0}v_{j,0} \in E(G_\omega)$ with $j \neq i_3$, we have

$$(v_{i_3,1}v_{i_3,0}v_{i_1,0} \rightsquigarrow v_{i_3,0}v_{i_1,0}), v_{k,0}v_{i_3,0}v_{i_1,0}, v_{i_1,0}v_{i_3,0}v_{j,0} \notin \mathfrak{p}_{i_1,0}(IR).$$

Since $P(V', \delta')$ occurs in an irredundant m-irreducible decomposition of I and $v_{i_1,0} \in V'$, we have $\mathfrak{p}_{i_1,0}(IR) \neq \emptyset$ by Lemma 3.133. So it can only be one of the following 4 cases.

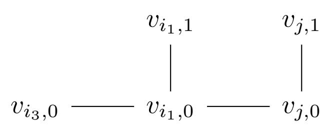
(1) There exists $(v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3$. Since

$$\delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\},$$

we have $v_{i_1,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since

$$\lambda(v_{j,1}v_{j,0}) + f\{\lambda(v_{j,1}v_{j,0}), \lambda(v_{j,0}v_{i_1,0})\} > f\{\lambda(v_{j,1}v_{j,0}), \lambda(v_{j,0}v_{i_1,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$$

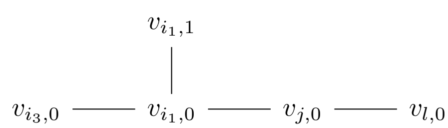
by the condition (a), we have $v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Hence similar to (a) we have $v_{i_3,0} \smile (P_2, \mathfrak{P})$.



(2) There exists $v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq l$. Then $\lambda(v_{i_1,0}v_{j,0}) \geq \delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$ and $v_{i_3}v_{i_1}v_jv_l$ is a 3-path in G_ω. So $f\{\lambda(v_{i_1,0}v_{j,0}), \lambda(v_{j,0}v_{l,0}) \geq \lambda(v_{i_1,0}v_{j,0})$ by the condition (d) in Proposition 3.134. Since $v_{j,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, we have $v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since

$$\delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\},$$

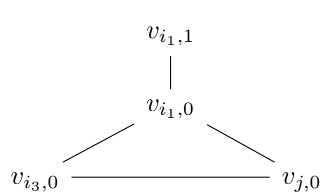
we have $v_{i_1,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Hence similar to (a) we have $v_{i_3,0} \smile (P_2, \mathfrak{P})$.



(3) There exists $v_{i_1,0}v_{j,0}v_{i_3,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 = l$. Then

$$\lambda(v_{i_1,0}v_{j,0}) \geq \delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\}.$$

Similarly we have $\delta'(v_{i_1,0}) > \lambda(v_{i_1,0}v_{i_3,0})$. So $v_{i_1,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$ and $v_{i_1,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have the following 2 cases by the condition (e) in Proposition 3.134.



i. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$. Then since $v_{j,0} \not\smile (v_{i_1,0}v_{j,0}v_{i_3,0}, \mathfrak{p})$, we have

$v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Hence similar to (a) we have $v_{i_3,0} \smile (P_2, \mathfrak{P})$.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_3,0})$ and

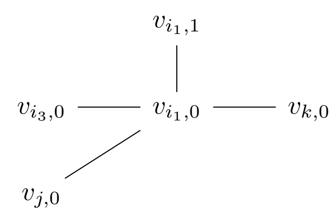
$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} \leq \max\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\}.$$

Since $v_j \not\smile (v_{i_1,0}v_{j,0}v_{i_3,0}, \mathfrak{p})$, we have $v_{j,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}, \mathfrak{p})$. So $v_{i_3,0} \smile (v_{i_1,0}v_{i_3,0}v_{j,0}, \mathfrak{p})$. Since $v_{i_3,0} \not\smile (v_{i_1,0}v_{j,0}v_{i_3,0}, \mathfrak{p})$, we have $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} > \lambda(v_{i_3,0}v_{j,0})$. Then we have $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} \leq \lambda(v_{i_3,0}v_{i_1,0})$ by assumption. So $v_{i_3,0} \smile (P_2 \rightsquigarrow v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Hence similar to (a) we have $v_{i_3,0} \smile (P_2, \mathfrak{P})$.

(4) There exists $v_{j,0}v_{i_1,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq k$. Then

$$\delta'(v_{i_1,0}) = \beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} > f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\}.$$

So $v_{i_1,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $v_{j,0} \not\smile (v_{j,0}v_{i_1,0}v_{k,0}, \mathfrak{p})$, we have $v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Hence similar to (a) we have $v_{i_3,0} \smile (P_2, \mathfrak{P})$.



Thus, by (a) and (b), (V'', δ'') is an f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$. Claim. (V'', δ'') is a minimal f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$. Let $k \in \{1, \ldots, s\}$.

(a) Assume $v_{b_k,1} \in V''$. Then $\beta_{b_k} > \lambda(v_{b_k,1}v_{b_k,0})$ and $\delta''(v_{b_k,1}) = \lambda(v_{b_k,1}v_{b_k,0})$. Since $P(V', \delta')$ occurs in an irredundant m-irreducible decomposition of the ideal IR, by Lemma 3.131 we have $(v_{b_k,1}v_{b_k,0}v_{j,0} \rightsquigarrow v_{b_k,0}v_{j,0}) \in \mathfrak{p}_{b_k,0}(I)$ for some $v_{b_k,0}v_{j,0} \in E(G_\omega)$ such that

$$\beta_{b_k} = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}.$$

So $v_{j,0} \not\smile (v_{b_k,1}v_{b_k,0}v_{j,0}, \mathfrak{P})$ by way of contradiction. Also, since $v_{b_k,0} \notin V''$, we have $v_{b_k,1}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,1}(I)$. Hence neither $\mathfrak{P} \smallsetminus \{v_{b_k,1}^{\delta''(v_{b_k,1})}\}$ nor $\mathfrak{P} \smallsetminus \{v_{b_k,1}^{\delta''(v_{b_k,1})}\} \sqcup \{v_{b_k,1}^{\delta''(v_{b_k,1})+1}\}$ is an f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$.

$$\begin{tikzcd} v_{b_k,1} \arrow[d, dash] & \\ v_{b_k,0} \arrow[r, dash] & v_{j,0} \end{tikzcd}$$

(b) Assume $v_{b_k,0} \in V''$. Then $\beta_{b_k} \le \lambda(v_{b_k,1}v_{b_k,0})$ and $\delta''(v_{b_k,0}) = \beta_{b_k}$. Since $P(V',\delta')$ occurs in an irredundant m-irreducible decomposition of the ideal IR, by Lemma 3.131 we have the following 3 cases.

(1) There exists $v_{i,0}v_{b_k,0}v_{j,0} \in \mathfrak{p}_{b_k,0}(IR)$ such that $\beta_{b_k} = f\{\lambda(v_{i,0}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Then by way of contradiction we have $v_{i,0}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,0}(I)$. Hence neither $\mathfrak{P} \smallsetminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ nor $\mathfrak{P} \smallsetminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\} \sqcup \{v_{b_k,0}^{\delta''(v_{b_k,0})+1}\}$ is an f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$.

$$\begin{tikzcd} v_{i,0} \arrow[r, dash] & v_{b_k,0} \arrow[r, dash] & v_{j,0} \end{tikzcd}$$

(2) There exists $v_{b_k,0}v_{i,0}v_{j,0} \in \mathfrak{p}_{b_k,0}(IR)$ such that $\beta_{b_k} = \lambda(v_{b_k,0}v_{i,0})$. Then it is similar to the case (1) right before.

$$\begin{tikzcd} v_{b_k,0} \arrow[r, dash] & v_{i,0} \arrow[r, dash] & v_{j,0} \end{tikzcd}$$

(3) There exists $(v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{b_k,0}v_{i,0}) \in \mathfrak{p}_{b_k,0}(IR)$ such that $\beta_{b_k} = \lambda(v_{b_k,0}v_{i,0})$.

$$\begin{tikzcd} & v_{i,1} \arrow[d, dash] & \\ v_{b_k,0} \arrow[r, dash] & v_{i,0} \arrow[r, dash] & v_{j,0} \end{tikzcd}$$

i. Assume $v_{i,1} \not\smile (v_{b_k,0}, v_{i,0}v_{i,1}, \mathfrak{P})$, then we have $v_{b_k,0}v_{i,0}v_{i,1} \in \mathfrak{P}_{b_k,0}(I)$.

ii. Assume $v_{i,1} \smile (v_{b_k,0}v_{i,0}v_{i,1}, \mathfrak{P})$. Then $v_{i,1} \in V''$ and so $\beta_i > \lambda(v_{i,1}v_{i,0})$. Also, we have $v_{i,0} \in V'$. So there exists $(v_{i,1}v_{i,0}v_{j,0} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{p}_{i,0}(IR)$ with $j \neq b_k$. Then $v_{j,0} \not\smile (v_{b_k,0}v_{i,0}v_{j,0}, \mathfrak{P})$ by way of contradiction. Also, since $v_{i,0} \notin V''$ by definition, we have $v_{b_k,0}v_{i,0}v_{j,0} \in \mathfrak{P}_{b_k,0}(I)$.

Hence neither $\mathfrak{P} \smallsetminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ nor $\mathfrak{P} \smallsetminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\} \sqcup \{v_{b_k,0}^{\delta''(v_{b_k,0})+1}\}$ is an f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$.

Thus, by (a) and (b), (V'',δ'') is a minimal f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$. Claim. $P(V',\delta') = P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. It is enough to show that $\gamma_{(V'',\delta'')}(v_{b_k}) = \beta_{b_k}$ for $k = 1, \dots, s$ since $V' = q(V'')$.

(a) Assume $v_{b_k,0} \in V''$, then $\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,0}) = \beta_{b_k}$.

(b) Assume $v_{b_k,1} \in V''$. Then $v_{b_k,0} \notin V''$, $\beta_{b_k} > \lambda(v_{b_k,1}v_{b_k,0})$ and $\delta''(v_{b_k,1}) = \lambda(v_{b_k,1}v_{b_k,0})$. Since (V'',δ'') is a minimal f-weighted 2-path vertex cover of $(\Sigma G)_\lambda$, we have $\mathfrak{P}_{b_k,1}(I) \neq \emptyset$ by Fact 3.84 and Lemma 3.133. So there exists $v_{b_k,1}v_{b_k,0}v_{i,0} \in \mathfrak{P}_{b_k,1}(I)$ such that $\beta_{b_k} = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{i,0})\}$ by the proof of that (V'',δ'') is a minimal f-weighted vertex cover. Claim. $\mathfrak{P}_{b_k,1}(I) = \{v_{b_k,1}v_{b_k,0}v_{i,0}\}$. Suppose not. Then there exists $v_{b_k,1}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,1}(I)$ with $j \neq i$. Then by way of contradiction, we have $v_{j,0}, v_{b_k,0}, v_{i,0} \not\prec (v_{j,0}v_{b_k,0}v_{i,0}, \mathfrak{p})$, a contradiction.

$$\begin{tikzcd}[arrows=dash]
 & v_{b_k,1} \arrow[d] & \\
v_{j,0} \arrow[r] & v_{b_k,0} \arrow[r] & v_{i,0}
\end{tikzcd}$$

So $h_{b_k,0} = f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{i,0})\} = \beta_{b_k} - \lambda(v_{b_k,1}v_{b_k,0})$. Therefore,

$$\gamma_{(V'',\delta'')}(v_{b_k}) = h_{b_k,1} + h_{b_k,0} = \lambda(v_{b_k,1}v_{b_k,0}) + \beta_{b_k} - \lambda(v_{b_k,1}v_{b_k,0}) = \beta_{b_k}. \qquad \square$$

Proposition 3.153. Let $(\Sigma_2 G)_\lambda$ be a weighted 2-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.135. Let $I = I_{3,f}\big((\Sigma_2 G)_\lambda\big)$. Then the monomial ideal $IR = I_{3,f}\big((\Sigma G)_\lambda\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with (V'',δ'') an size-minimal f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.

Proof. Let $P(V',\delta') := \big(X_{b_1}^{\beta_{b_1}}, \ldots, X_{b_s}^{\beta_{b_s}}\big)R$ occur in an irredundant m-irreducible decomposition of the ideal IR with $V' = \{v_{b_1}, \ldots, v_{b_s}\}$ and $\delta'(v_{b_k}) = \beta_{b_k}$ for $k = 1, \ldots, s$. Let $\mathfrak{p} := (V',\delta')$. For $k = 1, \ldots, s$, define

$$V''_{b_k} := \begin{cases} \{v_{b_k,2}\} & \text{if } \beta_{b_k} > \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\}, \\ \{v_{b_k,1}\} & \text{if } \lambda(v_{b_k,1}v_{b_k,0}) < \beta_{b_k} \leq \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\}, \\ \{v_{b_k,0}\} & \text{if } \beta_{b_k} \leq \lambda(v_{b_k,1}v_{b_k,0}), \end{cases},$$

Set $V'' := \bigcup_{k=1}^{s} V''_{b_k}$ and define

$$\delta'' : V'' \longrightarrow \mathbb{N}$$
$$v_{b_k,j} \longmapsto \begin{cases} \lambda(v_{b_k,2}v_{b_k,1}) & \text{if } j=2, \\ \lambda(v_{b_k,1}v_{b_k,0}) & \text{if } j=1, \\ \beta_{b_k} & \text{if } j=0. \end{cases}$$

Claim. $\mathfrak{P} := (V'', \delta'')$ is an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$. Let P_3 be a 3-path in $(\Sigma_2 G)_\lambda$ and $\underline{X}^{\underline{\alpha}}$ the corresponding generator in I. Then $p(\underline{X}^{\underline{\alpha}}) \in IR \subseteq P(V', \delta')$. So $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}})$ for some $k \in \{1, \ldots, s\}$.

(a) Assume P_3 is of the form

$$v_{i_1,0} \;\text{———}\; v_{i_2,0} \;\text{———}\; v_{i_3,0} \;\text{———}\; v_{i_4,0}.$$

Then $b_k \in \{i_1, i_2, i_3, i_4\}$ and $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}}) = \underline{X}^{\underline{\alpha}}$. So $\beta_{b_k} \leq \lambda(v_{b_k,1}v_{b_k,0})$. Then $v_{b_k,0} \in V''_k \subseteq V''$ and so $\delta''(v_{b_k,0}) = \beta_{b_k}$. Hence $X_{b_k,0}^{\delta''(v_{b_k,0})} \mid \underline{X}^{\underline{\alpha}}$. Thus, $v_{b_k,0} \smile (P_3, \mathfrak{P})$.

(b) Assume P_3 is of the form

$$\begin{array}{ccccc} & & & & v_{i_1,1} \\ & & & & | \\ v_{i_4,0} & \text{———} & v_{i_3,0} & \text{———} & v_{i_1,0}. \end{array}$$

If $b_k = i_3$ or i_4, then similar to (a) we have $v_{b_k,0} \smile (P_3, \mathfrak{P})$. So we assume $b_k = i_1$. Since $X_{i_1}^{\beta_{i_1}} \mid p(\underline{X}^{\underline{\alpha}})$, we have $\beta_{i_1} \leq \lambda(v_{i_1,2}v_{i_1,1}) + f\{\lambda(v_{i_1,2}v_{i_1,1}), \lambda(v_{i_1,1}v_{i_1,0})\}$. If $\lambda(v_{i_1,1}v_{i_1,0}) < \beta_{i_1} \leq \lambda(v_{i_1,2}v_{i_1,1}) + f\{\lambda(v_{i_1,2}v_{i_1,1}), \lambda(v_{i_1,1}v_{i_1,0})\}$, then $v_{i_1,1} \in V''_{i_1} \subseteq V''$ and $\delta''(v_{i_1,1}) = \lambda(v_{i_1,1}v_{i_1,0})$, so $v_{i_1,1} \smile (P_3, \mathfrak{P})$. So we assume $\beta_{i_1} \leq \lambda(v_{i_1,1}v_{i_1,0})$. Then $v_{i_1,0} \in V''$ and $\delta''(v_{i_1,0}) = \beta_{i_1}$. If $\beta_{i_1} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$, then $\delta''(v_{i_1,0}) = \beta_{i_1} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$, so $v_{i_1,0} \smile (P_3, \mathfrak{P})$. So we assume $\beta_{i_1} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$. Then $v_{i_1,0} \not\smile (P_3, \mathfrak{P})$. So we need to show that $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Similar to the proof of Proposition 3.152, we have

$$\mathfrak{p}_{i_1,0}(IR) \smallsetminus \{P_r \rightsquigarrow \wp \mid \wp \text{ is any path in } IR \text{ such that } v_{i_1,0}, v_{i_3,0} \in V(\wp)\} \neq \emptyset.$$

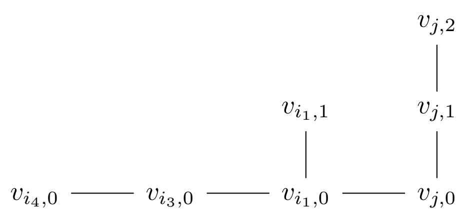
So it can only be the following 21 cases.

(1) There exists $(v_{i_1,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j \neq i_4$. Then $v_{j,0} \not\smile$

$(v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$ and

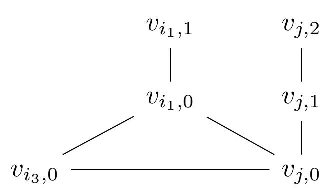
$$f\{\lambda(v_{i_1,0}v_{i_3,0}),\lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}),\lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}).$$

So $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$. Claim. $v_{i_3,0} \smile (P_3,\mathfrak{P})$ or $v_{i_4,0} \smile (P_3,\mathfrak{P})$. Suppose not. Then $v_{i_3,0}, v_{i_4,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.



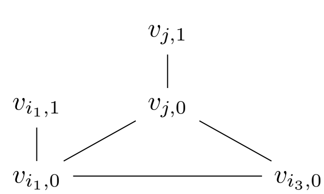
(2) There exists $(v_{i_1,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j = i_4$. Then

$$f\{\lambda(v_{i_1,0}v_{i_3,0}),\lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}),\lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}).$$



Claim. $v_{i_3,0} \smile (P_3,\mathfrak{P})$ or $v_{i_4,0} \smile (P_3,\mathfrak{P})$. Suppose not. Since $v_{i_1,0}v_{j,0}v_{i_3,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have the following 2 cases by the condition (e).(2).iii in Proposition 3.135.

i. Assume $f\{\lambda(v_{j,0}v_{i_3,0}),\lambda(v_{j,0}v_{j,1})\} < \max\big\{\lambda(v_{j,0}v_{i_3,0}), f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{j,1})\}\big\}$. Assume we have $f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{j,1})\} \leq f\{\lambda(v_{j,0}v_{i_3,0}),\lambda(v_{j,0}v_{j,1})\}$. Since $v_{j,0}v_{i_1,0}v_{i_3,0}v_{j,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{i_3,0}v_{j,0}),\lambda(v_{i_3,0}v_{i_1,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (e).(1) in Proposition 3.135. Then it comes to the following case ii.



Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{j,1})\} > f\{\lambda(v_{j,0}v_{i_3,0}), \lambda(v_{j,0}v_{j,1})\}$. Then

$$\begin{aligned}
&f\{\lambda(v_{j,0}v_{i_3,0}), \lambda(v_{j,0}v_{j,1})\} + \lambda(v_{j,1}v_{j,0}) \\
\leq\ &f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{j,1})\} + f\{\lambda(v_{j,2}v_{j,1}), \lambda(v_{j,1}v_{j,0})\} \\
<\ &f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{j,1})\} + f\{\lambda(v_{j,2}v_{j,1}), \lambda(v_{j,1}v_{j,0})\} + \lambda(v_{j,2}v_{j,1}).
\end{aligned}$$

So $v_{j,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_1,0}v_{i_3,0}v_{j,0}, \mathfrak{p})$. Since

$$\lambda(v_{i_1,0}v_{i_3,0}) \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}),$$

we have $v_{i_1,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $v_{i_3,0} \not\smile (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

ii. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$. Then we have $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. By way of contradiction, $v_{i_1,0}, v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.135.

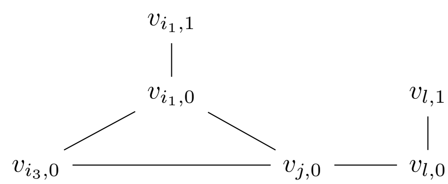
(3) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j \neq i_4$ and $i_3 \neq l \neq i_4$. Then $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0})$. So $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ and $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{i_3,0}, v_{i_4,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So we have $v_{j,0} \smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Since $v_{j,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, we have $f\{\lambda(v_{i_1,0}v_{j,0}), \lambda(v_{j,0}v_{l,0})\} < \lambda(v_{i_1,0}v_{j,0})$. Also, since $v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}$ is a 4-path in G_ω, we have $f\{\lambda(v_{i_4,0}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (d) in Proposition 3.135. Then since $v_{i_3,0} \not\smile (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by way of contradiction. But $v_{i_1,0}, v_{j,0}, v_{l,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by way of contradiction, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & & v_{i_1,1} \arrow[d, dash] & & v_{l,1} \arrow[d, dash] \\
v_{i_4,0} \arrow[r, dash] & v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] & v_{j,0} \arrow[r, dash] & v_{l,0}
\end{tikzcd}$$

(4) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j = i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{i_3,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since

$$\lambda(v_{i_1,0}v_{i_3,0}) \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$ and $v_{i_1,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, we have $v_{l,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$ and $v_{l,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. So $v_{j,0} \smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have the following 2 cases by the condition (e).(2).iii in Proposition 3.135.



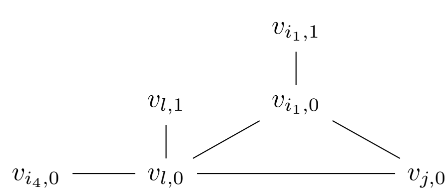
i. Assume $f\{\lambda(v_{j,0}v_{i_3,0}), \lambda(v_{j,0}v_{l,0})\} \leq f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\}$. Since $v_{j,0} \smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$, we have $v_{j,0} \smile (v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, a contradiction.

ii. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$. Then since $v_{i_3,0} \not\smile (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Also, we have $v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

(5) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 = l$. Claim. $v_{l,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{l,0} \not\smile (v_{i_4,0}v_{l,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ and $v_{i_4,0} \not\smile (v_{i_4,0}v_{l,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since

$$f\{\lambda(v_{i_1,0}v_{l,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{l,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

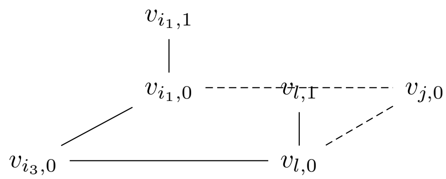
we have $v_{i_1,0} \not\smile (v_{i_4,0}v_{l,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{j,0} \smile (v_{i_4,0}v_{l,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (e).(2).i in Proposition 3.135. Hence $v_{j,0} \smile (v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, a contradiction.



(6) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3$ and $l = i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then we have $v_{l,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. Since

$$f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0},\mathfrak{p})$ and $v_{i_1,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$. So $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0},\mathfrak{p})$

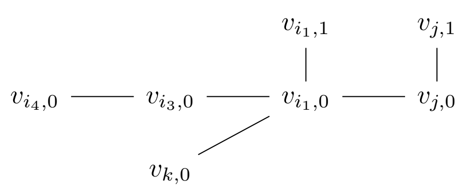


Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 2 cases by the condition (f) in Proposition 3.135.

i. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}),\lambda(v_{i_3,0}v_{l,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (f).(1). Since $v_{i_3,0} \not\smile (P_3,\mathfrak{P})$, we have $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$. Since $v_{i_1,1}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}(IR)$, we have $v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$. Also, since $\lambda(v_{l,1}v_{l,0}) + f\{\lambda(v_{l,1}v_{l,0}),\lambda(v_{l,0}v_{j,0})\} > \lambda(v_{l,0}v_{j,0})$ and $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$, we have $v_{l,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$, contradicted by Lemma 3.131.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0}) \geq \min\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})$ by the condition (f).(2).i. Since $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0},\mathfrak{p})$, we have $v_{j,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0},\mathfrak{p})$ by way of contradiction. Also, since $v_{i_1,0},v_{i_3,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0},\mathfrak{p})$, we have $v_{l,0} \smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0},\mathfrak{p})$. So we have $v_{l,0} \smile (v_{i,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i,0}v_{j,0}v_{l,0},\mathfrak{p})$, a contradiction.

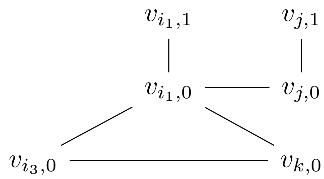
(7) There exists $(v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq k \neq i_4$. Claim. $v_{i_3,0} \smile (P_3,\mathfrak{P})$ or $v_{i_4,0} \smile (P_3,\mathfrak{P})$. Suppose not. Since $v_{i_1,0} \not\smile (P_3,\mathfrak{P})$, we have $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{k,0},\mathfrak{p})$. Also, by way of contradiction, we have $v_{i_4,0},v_{i_3,0},v_{k,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{k,0},\mathfrak{p})$, contradicted by Lemma 3.131.



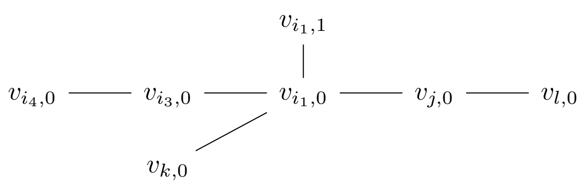
Note that in this case, we may have $v_{i_4,0} = v_{j,0}$.

(8) There exists $(v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3$ and $i_4 = k$. Claim. $v_{i_3,0} \smile (P_3,\mathfrak{P})$ or $v_{i_4,0} \smile (P_3,\mathfrak{P})$. Suppose not. Then $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{k,0},\mathfrak{p})$ by way of

contradiction. So $v_{j,0} \smile (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, a contradiction.

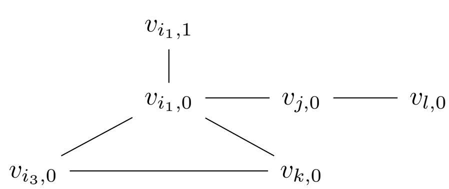


(9) There exists $v_{k,0}v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3$ and $i_3 \neq k \neq i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Since $v_{i_1,0} \not\smile (P_3, \mathfrak{P})$, we have $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{k,0}, \mathfrak{p})$. Also, by way of contradiction, we have $v_{i_4,0}, v_{i_3,0}, v_{k,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{k,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



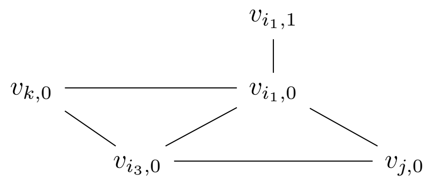
Note that in this case, we may have $v_{i_4,0} = v_{j,0}$, etc.

(10) There exists $v_{k,0}v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq l$ and $i_4 = k$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then by way of contradiction, we have $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ and $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{k,0}, \mathfrak{p})$. Since $v_{j,0} \not\smile (v_{k,0}v_{i_1,0}v_{j,0}v_{l,0}.\mathfrak{p})$, we have $f\{\lambda(v_{i_1,0}v_{j,0}), \lambda(v_{j,0}v_{l,0})\} < \lambda(v_{i_1,0}v_{j,0})$. Since $v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}$ is a 4-path in G_ω, we have $f\{\lambda(v_{k,0}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (d) in Proposition 3.135, contradicted by $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ and $v_{i_3,0} \not\smile (P_3, \mathfrak{P})$.



(11) There exists $v_{k,0}v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{P}_{i_1,0}$ with $j \neq i_3 = l$ and $i_4 = k$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then by way of contradiction, we have $v_{k,0}, v_{i_3,0}, v_{i_1,0} \not\smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $f\{\lambda(v_{i_1,0}v_{k,0}), \lambda(v_{i_1,0}v_{j,0})\} > f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,})\}$, by the condition (e).(1) in Proposition 3.135 we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$. So $v_{j,0} \not\smile$

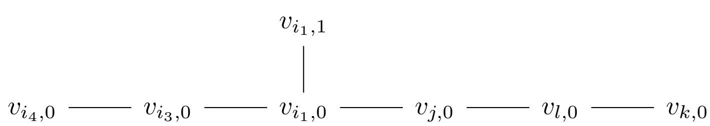
$(v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



(12) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j$, $i_3 \neq l$, $i_3 \neq k$ and $i_4 \neq j$, $i_4 \neq l$, $i_4 \neq k$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{i_3,0}, v_{i_4,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. So $v_{j,0} \smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by Lemma 3.131. By way of contradiction, we have $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ or $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$.



i. Suppose $v_{i_3,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{i_3,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, we have

$$f\{\lambda(v_{i_4,0}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} < \delta''(v_{i_3,0}) \leq \lambda(v_{i_3,0}v_{i_1,0}).$$

Since $v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}$ is a 4-path in G_ω, we have $f\{\lambda(v_{i_1,0}v_{j,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{i_1,0}v_{j,0})$, contradicted by that $v_{j,0} \smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ and $v_{j,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$.

ii. Suppose $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$, we have

$$f\{\lambda(v_{j,0}v_{l,0}), \lambda(v_{l,0}v_{k,0})\} < \delta''(v_{l,0}) \leq \lambda(v_{j,0}v_{l,0}).$$

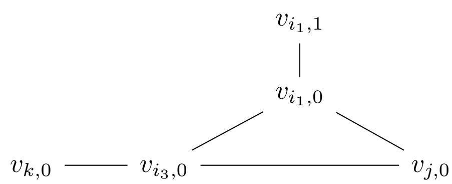
Since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}$ is a 4-path in G_ω, we have $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \geq \lambda(v_{i_1,0}v_{j,0})$, contradicted by that $v_{i_1,0} \smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$ and $v_{i_1,0} \not\smile (v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$.

(13) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $l = i_3$ and $k = i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or

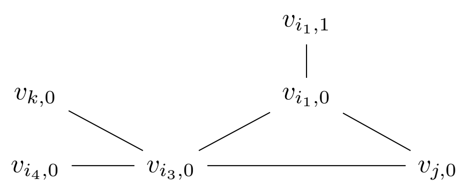
$v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{i_3,0}, v_{i_4,0} \not\smile (v_{k,0}v_{l,0}v_{i_1,0}v_{j,0})$. Since

$$f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0})$. So $v_{j,0} \smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0})$ by Lemma 3.131. Note that $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, so we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (e).(2).i in Proposition 3.135. So $v_{j,0} \smile (v_{i_1,0}v_{j,0}v_{i_3,0}v_{k,0}, \mathfrak{p})$, a contradiction.



(14) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $l = i_3$ and $k \neq i_4$. Then it is similar to (11).



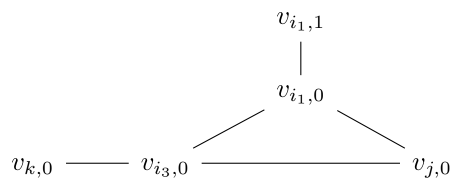
(15) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{P}_{i_1,0}(IR)$ with $l = i_3$ and $j = i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$. Suppose not. Since

$$f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, by the condition (e).(2).i and (e).(2).ii in Proposition 3.135 we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ and

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{k,0})\} \leq \max\big\{f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\}, f\{\lambda(v_{i_3,0}v_{k,0}), \lambda(v_{i_3,0}v_{j,0})\}\big\}.$$

So $v_{j,0}, v_{i_3,0} \not\smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$ by way of contradiction. But $v_{k,0} \not\smile (v_{k,0}v_{i_3,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



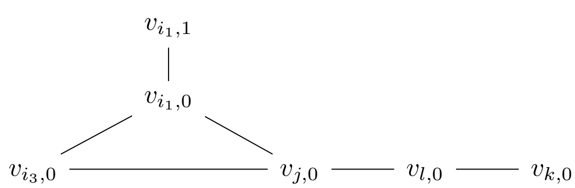
(16) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $i_3 \neq j = i_4$ and $k \neq i_3$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$. Suppose not. Then $v_{i_3,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since

$$\lambda(v_{i_1,0}v_{i_3,0}) \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}) \leq \lambda(v_{i_1,0}v_{j,0}),$$

we have $v_{i_1,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}$ is a 4-path in G_ω and

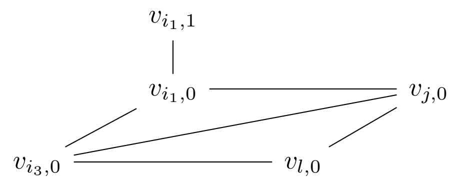
$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \lambda(v_{i_1,0}v_{j,0}),$$

we have $f\{\lambda(v_{j,0}v_{l,0}), \lambda(v_{l,0}v_{k,0})\} \geq \lambda(v_{j,0}v_{l,0})$ by the condition (d) in Proposition 3.135. Then since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$, we have $v_{l,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$. So $v_{j,0} \smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by Lemma 3.131. The remaining proof is totally similar to the corresponding part in (2).



(17) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j = i_4$ and $k = i_3$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$. Suppose not. Similarly, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{l,0})\} < \lambda(v_{i_1,0}v_{j,0})$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j,0}v_{i_3,0}), \lambda(v_{j,0}v_{l,0})\} \leq \max\big\{\lambda(v_{j,0}v_{i_3,0}), f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\}\big\}$ or $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (e).(2).iii in Proposition 3.135. So $v_{j,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$ or $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Hence by way of contradiction, we have $v_{l,0} \smile (v_{i_1,0}v_{i_3,0}v_{j,0}v_{l,0}, \mathfrak{p})$ or $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. So $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} < \lambda(v_{l,0}v_{j,0})$. Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 3 cases by the condition (f) in

Proposition 3.135.



i. Assume $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{j,0})$ by the condition (f).(1) or (f).(2).iii in Proposition 3.135, a contradiction.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{l,0})$ and

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \leq \max\big\{\lambda(v_{i_3,0}v_{l,0}), f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\}\big\},$$

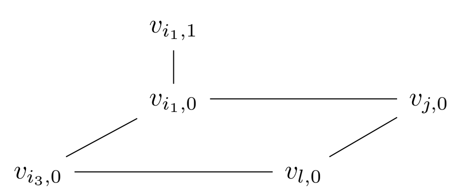
by the condition (f).(2).iii in Proposition 3.135. Then $v_{j,0}, v_{i_3,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}, \mathfrak{p})$. But we have $v_{i_1,0}, v_{l,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{i_3,0})$ and $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ and

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \leq \max\big\{\lambda(v_{i_3,0}v_{l,0}), f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{j,0})\}\big\},$$

by the condition (f).(2).iii in Proposition 3.135. Then $v_{l,0}, v_{j,0}, v_{i_3,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. But we have $v_{i_1,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

(18) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $l = i_4$ and $k = i_3$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Similarly, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \lambda(v_{i_1,0}v_{j,0})$. By way of contradiction we have $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$ and $v_{j,0} \smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 2 cases by the condition (f) in Proposition 3.135.

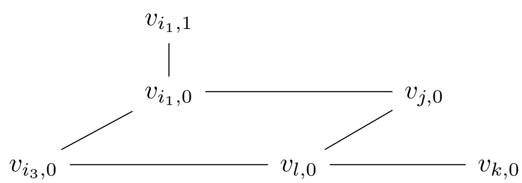


i. Assume $f\{\lambda(v_{l,0}v_{j,0}), \lambda(v_{l,0}v_{i_3,0})\} \geq \lambda(v_{l,0}v_{j,0})$ and $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (f).(1) in Proposition 3.135. Then $v_{l,0}, v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. But $v_{i,0}, v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \min\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\}$ by the condition (f).(2).i in

Proposition 3.135. Then $v_{j,0} \smile (v_{i_1,0}v_{j,0}v_{l,0}v_{i_3,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

(19) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq k$ and $l = i_4$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_4,0} \smile (P_3, \mathfrak{P})$. Suppose not. Similarly, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \lambda(v_{i_1,0}v_{j,0})$. Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 5 cases by the condition (f) in Proposition 3.135.



i. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ by the condition (f).(1) or (f).(2).iv in Proposition 3.135. Then $v_{i_3,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$. Since $v_{i_1,0}, v_{j,0} \not\smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$, we have $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by Lemma 3.131. Since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$, we have $f\{\lambda(v_{l,0}v_{j,0}), \lambda(v_{l,0}v_{k,0})\} < \lambda(v_{l,0}v_{j,0})$. Also, since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}$ is a 4-path in G_ω, we have $\lambda(v_{i_1,0}v_{j,0}) \leq f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\}$, a contradiction.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (f).(2).iv in Proposition 3.135. Then $v_{j,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. But $v_{i_1,0}, v_{i_3,0}, v_{l,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \min\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\}$ and

$$f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \leq \lambda(v_{l,0}v_{j,0}),$$

by the condition (f).(2).iv in Proposition 3.135. By way of contradiction we have $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. So $v_{j,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Since $v_{i_3,0}, v_{i_1,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$, we have $v_{l,0} \smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. So $v_{l,0} \smile (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}$ is a 4-path in G_ω and

$$f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,0}v_{i_1,1}), \lambda(v_{i_1,0}v_{i_3,0})\} < \lambda(v_{i_1,0}v_{j,0}),$$

we have $f\{\lambda(v_{j,0}v_{l,0}), \lambda(v_{l,0}v_{k,0})\} \geq \lambda(v_{j,0}v_{l,0})$, contradicted by $v_{l,0} \smile (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$ and $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}, \mathfrak{p})$.

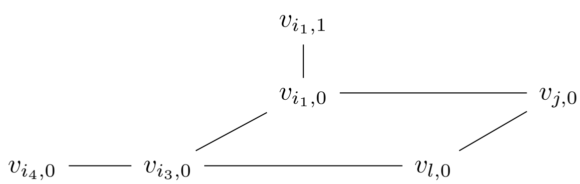
iv. Assume $f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})\}\geq\min\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})\}$ and

$$f\{\lambda(v_{l,0}v_{i_3,0}),\lambda(v_{l,0}v_{j,0})\}\leq\max\big\{\lambda(v_{l,0}v_{i_3,0}),f\{\lambda(v_{l,0}v_{j,0}),\lambda(v_{l,0}v_{k,0})\}\big\},$$

by the condition (f).(2).i and (f).(2).iv in Proposition 3.135. Then $v_{l,0}\not\smile(v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0},\mathfrak{p})$. Since $v_{i_1,0},v_{i_3,0}\not\smile(v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0},\mathfrak{p})$, we have $v_{j,0}\smile(v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0},\mathfrak{p})$ by Lemma 3.131. So $v_{j,0}\not\smile(v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0},\mathfrak{p})$. But $v_{i_1,0},v_{i_3,0},v_{l,0}\not\smile(v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0},\mathfrak{p})$, contradicted by Lemma 3.131.

v. Assume $f\{\lambda(v_{l,0}v_{i_3,0}),\lambda(v_{l,0}v_{k,0})\}\leq\max\big\{\lambda(v_{l,0}v_{i_3,0}),f\{\lambda(v_{l,0}v_{j,0}),\lambda(v_{l,0}v_{k,0})\}\big\}$ by the condition (f).(2).iv in Proposition 3.135. Then $v_{l,0}\not\smile(v_{i_1,0}v_{i_3,0}v_{l,0}v_{k,0},\mathfrak{p})$. But $v_{i_1,0},v_{i_3,0},v_{k,0}\not\smile(v_{i_1,0}v_{i_3,0}v_{l,0}v_{k,0},\mathfrak{p})$, contradicted by Lemma 3.131.

(20) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}\in\mathfrak{p}_{i_1,0}(IR)$ with $j\neq i_4$ and $k=i_3$. Claim. $v_{i_3,0}\smile(P_3,\mathfrak{P})$ or $v_{i_4,0}\smile(P_3,\mathfrak{P})$. Suppose not. Similarly, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}),\lambda(v_{i_1,0}v_{i_3,0})\}<\lambda(v_{i_1,0}v_{j,0})$. Since $v_{i_1,0}v_{j,0}v_{l,0}v_{k,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 4 cases by the condition (f) in Proposition 3.135.



i. Assume $f\{\lambda(v_{l,0}v_{k,0}),\lambda(v_{l,0}v_{j,0})\}\geq\lambda(v_{l,0}v_{j,0})$ by the condition (f).(1) or (f).(2).v in Proposition 3.135. Then $v_{l,0}\not\smile(v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$. Since $v_{i_1,0},v_{j,0}\not\smile(v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$, we have $v_{i_3,0}\smile(v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$. Since $v_{i_3,0}\not\smile(P_3,\mathfrak{P})$, we have $f\{\lambda(v_{i_4,0}v_{i_3,0}),\lambda(v_{i_3,0}v_{i_1,0})\}<\lambda(v_{i_3,0}v_{i_1,0})$. Since $v_{i_4,0}v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}$ is a 4-path in G_ω, we have $f\{\lambda(v_{i_1,0}v_{j,0}),\lambda(v_{j,0}v_{l,0})\}\geq\lambda(v_{i_1,0}v_{j,0})$. By way of contradiction we have $v_{j,0}\smile(v_{j,0}v_{i_1,0}v_{k,0}v_{i_4,0},\mathfrak{p})$. So we have $v_{j,0}\smile(v_{i_1,0}v_{j,0}v_{l,0}v_{i_3,0},\mathfrak{p})$, a contradiction.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})\}\geq\lambda(v_{j,0}v_{i_1,0})$ by the condition (f).(2).v in Proposition 3.135. Then $v_{j,0}\not\smile(v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_4,0},\mathfrak{p})$. But we have $v_{i_1,0},v_{i_3,0},v_{i_4,0}\not\smile(v_{i_1,0}v_{i_3,0}v_{i_4,0},\mathfrak{p})$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})\}\geq\min\{\lambda(v_{j,0}v_{i_1,0}),\lambda(v_{j,0}v_{l,0})\}$ and

$$f\{\lambda(v_{i_3,0}v_{i_1,0}),\lambda(v_{i_3,0}v_{l,0})\}\leq\max\big\{\lambda(v_{i_3,0}v_{l,0}),f\{\lambda(v_{i_3,0}v_{i_1,0}),\lambda(v_{i_3,0}v_{i_4,0})\}\big\},$$

by the condition (f).(2).i and (f).(2).v in Proposition 3.135. Then $v_{i_3,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}, \mathfrak{p})$. By way of contradiction we have $v_{j,0} \smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_4,0}, \mathfrak{p})$. Since $v_{j,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{i_3,0}, \mathfrak{p})$, we have $v_{j,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}, \mathfrak{p})$. But we have $v_{i_1,0}, v_{l,0} \not\smile (v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

iv. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \min\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\}$ and

$$f\{\lambda(v_{i_3,0}v_{l,0}), \lambda(v_{i_3,0}v_{i_4,0})\} \leq \max\big\{\lambda(v_{i_3,0}v_{l,0}), f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{i_4,0})\}\big\},$$

by the condition (f).(2).i and (f).(2).v in Proposition 3.135. Then $v_{i_3,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_4,0}, \mathfrak{p})$. Similar to iii right before, we have $v_{j,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_4,0}, \mathfrak{p})$. But we have $v_{i_4,0}, v_{l,0} \not\smile (v_{j,0}v_{l,0}v_{i_3,0}v_{i_4,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

(21) There exists $v_{k,0}v_{l,0}v_{j,0}v_{i_1,0} \in \mathfrak{p}_{i_1,0}(IR)$ with $j \neq i_3 \neq l$ and $k = i_4$. Then it is similar to (10).

$$\begin{tikzcd}[arrows=dash]
 & v_{i_1,1} \arrow[d] & & & \\
 & v_{i_1,0} \arrow[rr] \arrow[dl] & & v_{j,0} \arrow[dr] & \\
v_{i_3,0} \arrow[rr] & & v_{k,0} \arrow[rr] & & v_{l,0}
\end{tikzcd}$$

(c) Assume P_3 is of the form

$$\begin{tikzcd}[arrows=dash]
v_{i_3,1} \arrow[d] & v_{i_1,1} \arrow[d] \\
v_{i_3,0} \arrow[r] & v_{i_1,0}.
\end{tikzcd}$$

Without loss of generality, assume $b_k = i_1$. Then similar to (b), we can assume $v_{i_1,0} \in V''$ and $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} \leq \lambda(v_{i_1,1}v_{i_1,0})$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_3,1} \smile (P_3, \mathfrak{P})$. Suppose not. Similar to (b) we have

$$\mathfrak{p}_{i_1,0}(IR) \setminus \{P_r \rightsquigarrow \wp \mid \wp \text{ is any path in } IR \text{ such that } v_{i_1,0}, v_{i_3,0} \in V(\wp)\} \neq \emptyset.$$

So it can only be the following 8 cases.

(1) There exists $(v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}$. Then we have $v_{j,0} \not\smile (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $f\{\lambda(v_{k,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0})$, we have $v_{i_1,0} \not\smile (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Also, we have $v_{k,0} \not\smile (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow$

$v_{k,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
v_{i_3,1} \arrow[d, dash] & v_{i_1,1} \arrow[d, dash] & v_{j,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] \arrow[dl, dash] & v_{j,0} \\
v_{k,0} & &
\end{tikzcd}$$

(2) There exists $v_{k,0}v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}$. Then $v_{l,0}, v_{j,0}, v_{k,0} \not\succ (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0},\mathfrak{p})$. Since $f\{\lambda(v_{k,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0})$, we have $v_{i_1,0} \not\succ (v_{k,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
v_{i_3,1} \arrow[d, dash] & v_{i_1,1} \arrow[d, dash] & & \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] \arrow[dl, dash] & v_{j,0} \arrow[r, dash] & v_{l,0} \\
v_{k,0} & & &
\end{tikzcd}$$

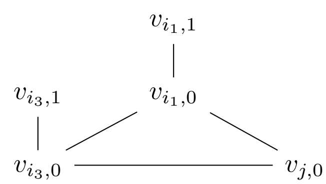
(3) There exists $(v_{i_1,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3$. Since $v_{i_3,0} \not\succ (P_3,\mathfrak{P})$ and $f\{\lambda(v_{i_3,0}v_{i_3,1}), \lambda(v_{i_3,0}v_{i_1,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$, we have $v_{i_3,0} \not\succ (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$. Since $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0})$, we have that $v_{i_1,0} \not\succ (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$. Also, we have $v_{j,0} \not\succ (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
& & v_{j,2} \arrow[d, dash] \\
v_{i_3,1} \arrow[d, dash] & v_{i_1,1} \arrow[d, dash] & v_{j,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] & v_{j,0}
\end{tikzcd}$$

(4) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3 \neq l$. Then by way of contradiction we have $v_{i_3,0}, v_{i_1,0}, v_{j,0}, v_{l,0} \not\succ (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.

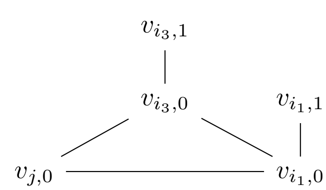
$$\begin{tikzcd}
v_{i_3,1} \arrow[d, dash] & v_{i_1,1} \arrow[d, dash] & & v_{l,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] & v_{j,0} \arrow[r, dash] & v_{l,0}
\end{tikzcd}$$

(5) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3 = l$.



i. Assume $f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \leq f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{j,0})\}$. Then since we have $v_{i_3,0} \not\smile (v_{i_1,0}v_{j,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{i_3,0}, \mathfrak{p})$, we have $v_{i_3,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{j,0}v_{i_3,0}v_{i_1,0}$ is a 3-cycle in G_ω, by the condition (e).(2).ii in Proposition 3.135 we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$. So $v_{j,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. But $v_{i_1,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

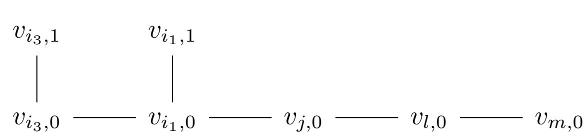
ii. Assume $f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} > f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{j,0})\}$. Since $v_{i_3,0}v_{j,0}v_{i_1,0}v_{i_3,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \geq \lambda(v_{i_1,0}v_{j,0})$ by the condition (e).(1) in Proposition 3.135, a contradiction.



(6) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3$ and $l \neq i_3 \neq m$. Then $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by way of contradiction. Since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{m,0}, \mathfrak{p})$, we have $f\{\lambda(v_{j,0}v_{l,0}), \lambda(v_{l,0}v_{m,0})\} < \lambda(v_{j,0}v_{l,0})$. Since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{m,0}$ is a 4-path in G_ω, by the condition (d) in Proposition 3.135 we have

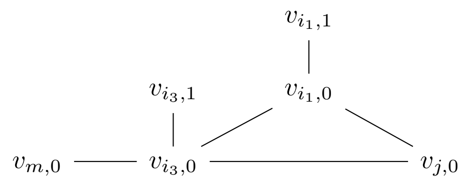
$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \geq \lambda(v_{i_1,0}v_{j,0}) > f\{\lambda(v_{i_1,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\},$$

a contradiction.

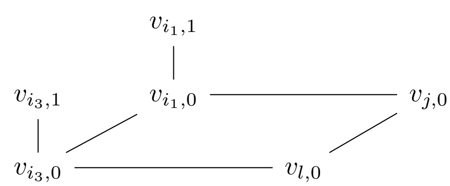


(7) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $i_3 = l$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (e).(2).i in Proposition 3.135. So $v_{j,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$. Since $f\{\lambda(v_{m,0}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} < f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\}$, we

have $v_{i_3,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$. Also, we have $v_{i_1,0}, v_{m,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



(8) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $i_3 = m$. Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 3 cases by the condition (f) in Proposition 3.135.



i. Assume $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{j,0})$ by the condition (f).(1) or (f).(2).iii in Proposition 3.135. Then $v_{l,0} \not\leftarrow (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Since $f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \geq \lambda(v_{i_3,0}v_{i_1,0})$ and $v_{i_3,0} \not\leftarrow (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\leftarrow (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Also, $v_{j,0}, v_{i_1,0} \not\leftarrow (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

ii. Assume $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{l,0})$ by the condition (f).(2).iii in Proposition 3.135. Then $v_{j,0} \not\leftarrow (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Since

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \leq f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\}$$

and $v_{i_3,0} \not\leftarrow (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\leftarrow (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Also, $v_{l,0}, v_{i_1,0} \not\leftarrow (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

iii. $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{i_3,0})$ and $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (f).(2).iii in Proposition 3.135. Then $v_{l,0}, v_{j,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. Since

$$f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \geq f\{\lambda(v_{i_3,0}v_{l,0}), \lambda(v_{i_3,0}v_{i_1,0})\}$$

and $v_{i_3,0} \not\leftarrow (P_3, \mathfrak{P})$, we have $v_{i_3,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. Also, $v_{i_1,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

(d) Assume P_3 is of the form

$$\begin{tikzcd}
 & v_{i_1,2} \arrow[d, dash] \\
 & v_{i_1,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0}.
\end{tikzcd}$$

Without loss of generality, assume $b_k = i_1$. Then similar to (c), we can assume $v_{i_1,0} \in V''$ and $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} \leq \lambda(v_{i_1,1}v_{i_1,0})$. Claim. $v_{i_3,0} \smile (P_3, \mathfrak{P})$ or $v_{i_3,1} \smile (P_3, \mathfrak{P})$. Suppose not. Similar to (b) we have

$$\mathfrak{p}_{i_1,0}(IR) \setminus \{P_r \rightsquigarrow \wp \mid \wp \text{ is any path in } IR \text{ such that } v_{i_1,0}, v_{i_3,0} \in V(\wp)\} \neq \emptyset.$$

So it can only be the following 8 cases.

(1) There exists $(v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}$. Then we have $v_{j,0} \nsucc (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $f\{\lambda(v_{k,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0})$, we have $v_{i_1,0} \nsucc (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Also, we have $v_{k,0} \nsucc (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{i_1,2} \arrow[d, dash] & \\
 & v_{i_1,1} \arrow[d, dash] & v_{j,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] \arrow[dl, dash] & v_{j,0} \\
v_{k,0} & &
\end{tikzcd}$$

(2) There exists $v_{k,0}v_{i_1,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{i_1,0}$. Then $v_{l,0}, v_{j,0}, v_{k,0} \nsucc (v_{k,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{k,0}v_{i_1,0}v_{j,0}, \mathfrak{p})$. Since $f\{\lambda(v_{k,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0})$, we have $v_{i_1,0} \nsucc$

$(v_{k,0}v_{i_1,0}v_{j,0}v_{l,0},\mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{i_1,2} \arrow[d, dash] & & \\
 & v_{i_1,1} \arrow[d, dash] & & \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] \arrow[dl, dash] & v_{j,0} \arrow[r, dash] & v_{l,0} \\
v_{k,0} & & &
\end{tikzcd}$$

(3) There exists $(v_{i_1,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i_1,0}v_{j,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3$. Since

$$f\{\lambda(v_{i_3,0}v_{i_1,0}),\lambda(v_{i_1,0}v_{j,0})\} \leq f\{\lambda(v_{i_1,1}v_{i_1,0}),\lambda(v_{i_1,0}v_{i_3,0})\} < \beta_{i_1} = \delta'(v_{i_1,0}),$$

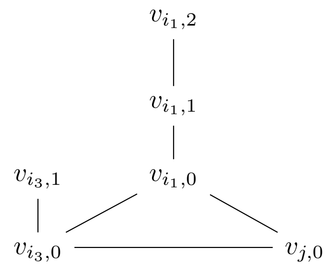
we have $v_{i_1,0} \not\leftarrow (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$. Also, we have $v_{j,0}, v_{i_3,0} \not\leftarrow (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{i_1,2} \arrow[d, dash] & v_{j,2} \arrow[d, dash] \\
 & v_{i_1,1} \arrow[d, dash] & v_{j,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] & v_{j,0}
\end{tikzcd}$$

(4) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3 \neq l$. Then by way of contradiction we have $v_{i_3,0}, v_{i_1,0}, v_{j,0}, v_{l,0} \not\leftarrow (v_{i_3,0}v_{i_1,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i_3,0}v_{i_1,0}v_{j,0},\mathfrak{p})$, contradicted by Lemma 3.131.

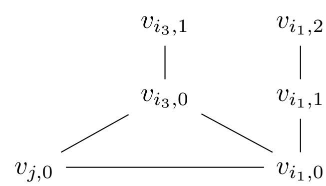
$$\begin{tikzcd}
 & v_{i_1,2} \arrow[d, dash] & & \\
 & v_{i_1,1} \arrow[d, dash] & & v_{l,1} \arrow[d, dash] \\
v_{i_3,0} \arrow[r, dash] & v_{i_1,0} \arrow[r, dash] & v_{j,0} \arrow[r, dash] & v_{l,0}
\end{tikzcd}$$

(5) There exists $(v_{i_1,0}v_{j,0}v_{l,0}v_{l,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3 = l$.



i. Assume $f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} \leq f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{j,0})\}$. Then since we have $v_{i_3,0} \not\smile (v_{i_1,0}v_{j,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{i_1,0}v_{j,0}v_{i_3,0}, \mathfrak{p})$, we have $v_{i_3,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Since $v_{i_1,0}v_{j,0}v_{i_3,0}v_{i_1,0}$ is a 3-cycle in G_ω, by the condition (e).(2).ii in Proposition 3.135 we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$. So $v_{j,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. But $v_{i_1,0} \not\smile (v_{j,0}v_{i_1,0}v_{i_3,0}v_{i_3,1} \rightsquigarrow v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

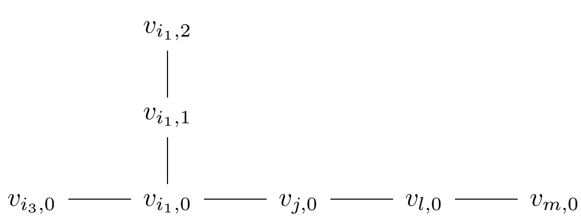
ii. Assume $f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{i_1,0})\} > f\{\lambda(v_{i_3,1}v_{i_3,0}), \lambda(v_{i_3,0}v_{j,0})\}$. Since $v_{i_3,0}v_{j,0}v_{i_1,0}v_{i_3,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{i_1,1}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\} \geq f\{\lambda(v_{i_1,0}v_{i_3,0}), \lambda(v_{i_1,0}v_{j,0})\} \geq \lambda(v_{i_1,0}v_{j,0})$ by the condition (e).(1) in Proposition 3.135, a contradiction.



(6) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $j \neq i_3$ and $l \neq i_3 \neq m$. Then $v_{l,0} \smile (v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}, \mathfrak{p})$ by way of contradiction. Since $v_{l,0} \not\smile (v_{i_1,0}v_{j,0}v_{l,0}v_{m,0}, \mathfrak{p})$, we have $f\{\lambda(v_{j,0}v_{l,0}), \lambda(v_{l,0}v_{m,0})\} < \lambda(v_{j,0}v_{l,0})$. Since $v_{i_3,0}v_{i_1,0}v_{j,0}v_{l,0}v_{m,0}$ is a 4-path in G_ω, by the condition (d) in Proposition 3.135 we have

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{j,0})\} \geq \lambda(v_{i_1,0}v_{j,0}) > f\{\lambda(v_{i_1,0}v_{i_1,0}), \lambda(v_{i_1,0}v_{i_3,0})\},$$

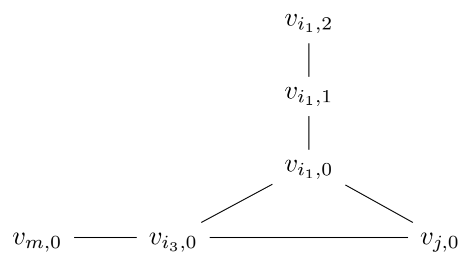
a contradiction.



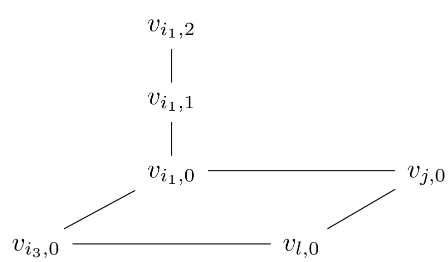
(7) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $i_3 = l$. Since $v_{i_1,0}v_{i_3,0}v_{j,0}v_{i_1,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{i_3,0})\} \geq \lambda(v_{j,0}v_{i_1,0})$ by the condition (e).(2).i in Proposition 3.135. So $v_{j,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$. Note that $v_{i_3,0} \not\leftarrow (P_3, \mathfrak{P})$, $v_{i_3,0} \not\leftarrow (v_{i_1,0}v_{j,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$ and

$$f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{m,0})\} \leq \max\big\{\lambda(v_{i_3,0}v_{i_1,0}), f\{\lambda(v_{i_3,0}v_{m,0}), \lambda(v_{i_3,0}v_{j,0})\}\big\}$$

by the condition (e).(2).ii in Proposition 3.135. So $v_{i_3,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$. Also, we have $v_{i_1,0}, v_{m,0} \not\leftarrow (v_{j,0}v_{i_1,0}v_{i_3,0}v_{m,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



(8) There exists $v_{i_1,0}v_{j,0}v_{l,0}v_{m,0} \in \mathfrak{p}_{i_1,0}$ with $i_3 = m$. Since $v_{i_1,0}v_{i_3,0}v_{l,0}v_{j,0}v_{i_1,0}$ is a 4-cycle in G_ω, we have the following 3 cases by the condition (f) in Proposition 3.135.



i. Assume $f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{j,0})$ by the condition (f).(1) in Proposition 3.135. Then $v_{l,0} \not\leftarrow (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$. Also, we have $v_{j,0}, v_{i_1,0}, v_{i_3,0} \not\leftarrow (v_{l,0}v_{j,0}v_{i_1,0}v_{i_3,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

ii. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \leq \max\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\}$ and

$$f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{l,0}),$$

by the condition (f).(2).ii and (f).(2).iii in Proposition 3.135. Then we have $v_{i_3,0}, v_{j,0} \not\leftarrow (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$. Also, $v_{l,0}, v_{i_1,0} \not\leftarrow (v_{j,0}v_{l,0}v_{i_3,0}v_{i_1,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

iii. Assume $f\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\} \leq \max\{\lambda(v_{i_3,0}v_{i_1,0}), \lambda(v_{i_3,0}v_{l,0})\}$ and

$$f\{\lambda(v_{l,0}v_{i_3,0}), \lambda(v_{l,0}v_{j,0})\} \geq \lambda(v_{l,0}v_{i_3,0}) \text{ and } f\{\lambda(v_{j,0}v_{i_1,0}), \lambda(v_{j,0}v_{l,0})\} \geq \lambda(v_{j,0}v_{i_1,0}),$$

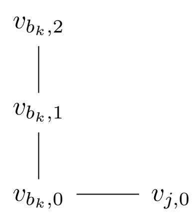
by the condition (f).(2).ii and (f).(2).iii in Proposition 3.135. Then we have $v_{i_3,0}, v_{l,0}, v_{j,0} \not\succ (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$. Also, $v_{i_1,0} \not\succ (v_{j,0}v_{i_1,0}v_{i_3,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

Thus, by (a), (b), (c) and (d), (V'', δ'') is an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$. Claim. (V'', δ'') is an size-minimal f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$. Let $k \in \{1, \ldots, s\}$.

(a) Assume $v_{b_k,2} \in V''$. Then there exists $(v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{j,0} \rightsquigarrow v_{b_k,0}v_{j,0}) \in \mathfrak{p}_{b_k,0}(IR)$ such that

$$\beta_{b_k} = \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\} + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}.$$

So $v_{j,0} \not\succ (v_{b_k,2}v_{b_k,2}v_{b_k,0}v_{j,0}, \mathfrak{P})$. Also, since $v_{b_k,0}, v_{b_k,1} \notin V''$, we have $v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,2}(I)$. Hence $\mathfrak{P} \setminus \{v_{b_k,2}^{\delta''(v_{b_k,2})}\}$ is not an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.



(b) Assume $v_{b_k,1} \in V''$. Then $v_{b_k,0} \notin V''$ and

$$\lambda(v_{b_k,1}v_{b_k,0}) < \beta_{b_k} \leq f\{\lambda(v_{b_k,2}v_{b_k,1})\} + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\}.$$

So we have the following 2 cases.

(1) There exists $(v_{b_k,1}v_{b_k,0}v_{i,0}v_{j,0} \rightsquigarrow v_{b_k,0}v_{i,0}v_{j,0}) \in \mathfrak{p}_{b_k,0}(IR)$ such that

$$\beta_{b_k} = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{i,0})\}.$$

Then we have $v_{b_k,0}, v_{i,0}, v_{j,0} \not\succ (v_{b_k,1}v_{b_k,0}v_{i,0}v_{j,0}, \mathfrak{P})$. So we have $v_{b_k,1}v_{b_k,0}v_{i,0}v_{j,0} \in \mathfrak{P}_{b_k,1}(I)$ with

$\delta''(v_{b_k,1}) = \lambda(v_{b_k,1}v_{b_k,0})$. Hence $\mathfrak{P} \setminus \{v_{b_k,1}^{\delta''(v_{b_k,1})}\}$ is not an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.

$$\begin{array}{ccccc} v_{b_k,1} & & & & \\ \vert & & & & \\ v_{b_k,0} & \text{———} & v_{i,0} & \text{———} & v_{j,0} \end{array}$$

(2) There exists $(v_{b_k,1}v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{b_k,0}v_{i,0}) \in \mathfrak{p}_{b_k,0}(IR)$ such that

$$\beta_{b_k} = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{i,0})\}.$$

Claim. $v_{i,1} \nleftarrow (v_{b_k,1}v_{b_k,0}v_{i,0}v_{i,1}, \mathfrak{P})$, then $v_{b_k,1}v_{b_k,0}v_{i,0}v_{i,1} \in \mathfrak{P}_{b_k,0}(I)$ since we have $v_{b_k,0}, v_{i,0} \nleftarrow (v_{b_k,1}v_{b_k,0}v_{i,0}v_{i,1}, \mathfrak{P})$, hence $\mathfrak{P} \setminus \{v_{b_k,1}^{\delta''(v_{b_k,1})}\}$ is not an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.

$$\begin{array}{ccc} v_{b_k,1} & & v_{i,1} \\ \vert & & \vert \\ v_{b_k,0} & \text{———} & v_{i,0} \end{array}$$

Proof of the Claim. Suppose not. Then $\delta'(v_{i,0}) = \beta_i > \lambda(v_{i,1}v_{i,0})$. Note that

$$\mathfrak{p}_{i,0}(IR) \setminus \{P_r \rightsquigarrow \wp \mid \wp \text{ is any path in } IR \text{ such that } v_{i,0}, v_{b_k,0} \in V(\wp)\} \neq \emptyset.$$

So we have the following 3 cases.

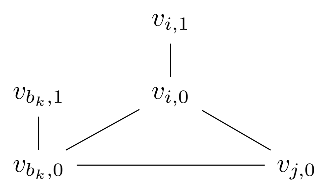
i. There exists $(v_{i,1}v_{i,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{p}_{i,0}(IR)$ with $j \neq b_k$. Then by way of contradiction we have $v_{b_k,0}, v_{i,0}, v_{j,0} \nleftarrow (v_{b_k,0}v_{i,0}v_{j,1}v_{j,0} \rightsquigarrow v_{b_k,0}v_{i,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{array}{ccccc} v_{b_k,1} & & v_{i,1} & & v_{j,1} \\ \vert & & \vert & & \vert \\ v_{b_k,0} & \text{———} & v_{i,0} & \text{———} & v_{j,0} \end{array}$$

ii. There exists $(v_{i,1}v_{i,0}v_{j,0}v_{l,1} \rightsquigarrow v_{i,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i,0}(IR)$ with $j \neq b_k \neq l$. Then by way of contradiction we have $v_{b_k,0}, v_{i,0}, v_{j,0}, v_{l,0} \nleftarrow (v_{b_k,0}v_{i,0}v_{j,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

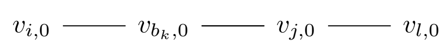
$$\begin{array}{ccccccc} v_{b_k,1} & & v_{i,1} & & & & \\ \vert & & \vert & & & & \\ v_{b_k,0} & \text{———} & v_{i,0} & \text{———} & v_{j,0} & \text{———} & v_{l,0} \end{array}$$

iii. There exists $(v_{i,1}v_{i,0}v_{j,0}v_{l,1} \rightsquigarrow v_{i,0}v_{j,0}v_{l,0}) \in \mathfrak{p}_{i,0}(IR)$ with $j \neq b_k = l$. Since we have $v_{i,0} \smile (v_{i,1}v_{i,0}v_{j,0}v_{b_k,0} \rightsquigarrow v_{i,0}v_{j,0}v_{b_k,0}, \mathfrak{p})$ and $v_{i,0} \not\smile (v_{b_k,1}v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{b_k,0}v_{i,0})$, we have that $f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j,0})\} > f\{\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{b_k,0})\}$. Since $v_{i,0}v_{b_k,0}v_{j,0}v_{i,0}$ is a 3-cycle in G_ω, we have $f\{\lambda(v_{j,0}v_{i,0}), \lambda(v_{j,0}v_{b_k,0})\} \geq \lambda(v_{j,0}v_{b_k,0})$. So $v_{j,0} \not\smile (v_{j,0}v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j,0}v_{b_k,0}v_{i,0}, \mathfrak{p})$. Since $\delta'(v_{b_k}) = \beta_{b_k} > \lambda(v_{b_k,1}v_{b_k,0})$, we have $v_{b_k,0} \not\smile (v_{j,0}v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j,0}v_{b_k,0}v_{i,0}, \mathfrak{p})$. Also, we have $v_{i,0} \not\smile (v_{j,0}v_{b_k,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j,0}v_{b_k,0}v_{i,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

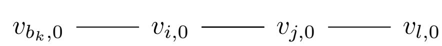


(c) Assume $v_{b_k,0} \in V''$. Then $\beta_{b_k} \leq \lambda(v_{b_k,1}v_{b_k,0})$. So we have the following 5 cases.

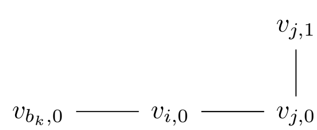
(1) There exists $v_{i,0}v_{b_k,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{b_k,0}(IR)$. Then $v_{i,0}v_{b_k,0}v_{j,0}v_{l,0} \in \mathfrak{P}_{b_k,0}(I)$. Hence $\mathfrak{P} \setminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ is not an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$.



(2) There exists $v_{b_k,0}v_{i,0}v_{j,0}v_{l,0} \in \mathfrak{p}_{b_k,0}(IR)$. Then it is similar to (1).



(3) There exists $(v_{b_k,0}v_{i,0}v_{j,0}v_{j,1} \rightsquigarrow v_{b_k,0}v_{i,0}v_{j,0}) \in \mathfrak{p}_{b_k,0}(IR)$. Then $v_{j,0} \not\smile (v_{b_k,1}v_{b_k,0}v_{i,0}v_{j,0}, \mathfrak{P})$. So $v_{b_k,1}v_{b_k,0}v_{i,0}v_{j,0} \in \mathfrak{P}_{b_k,0}(I)$. Hence $\mathfrak{P} \setminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ is not an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.



(4) There exists $(v_{i,0}v_{b_k,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i,0}v_{b_k,0}v_{j,0}) \in \mathfrak{p}_{b_k,0}(IR)$. Then $v_{j,0} \not\smile (v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{j,0}, \mathfrak{P})$. So $v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,0}$. Hence $\mathfrak{P} \setminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ is not an f-weighted 3-path vertex cover of

$(\Sigma_2 G)_\lambda$.

$$\begin{tikzcd}
 & & v_{j,1} \arrow[d, dash] \\
v_{i,0} \arrow[r, dash] & v_{b_k,0} \arrow[r, dash] & v_{j,0}
\end{tikzcd}$$

(5) There exists $(v_{b_k,0}v_{i,0}v_{i,1}v_{i,2} \rightsquigarrow v_{b_k,0}v_{i,0}) \in \mathfrak{p}_{b_k,0}(IR)$. Then $v_{i,0} \not\smile (v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{i,0}, \mathfrak{P})$. So $v_{b_k,2}v_{b_k,1}v_{b_k,0} \in \mathfrak{P}_{b_k,0}(I)$. Hence $\mathfrak{P} \setminus \{v_{b_k,0}^{\delta''(v_{b_k,0})}\}$ is not an f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$.

$$\begin{tikzcd}
 & v_{i,2} \arrow[d, dash] \\
 & v_{i,1} \arrow[d, dash] \\
v_{b_k,0} \arrow[r, dash] & v_{i,0}
\end{tikzcd}$$

Thus, by (a), (b) and (c), (V'', δ'') is an size-minimal f-weighted 3-path vertex cover of $(\Sigma_2 G)_\lambda$. Claim. $P(V', \delta') = P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. It is enough to show that $\gamma_{(V'',\delta'')}(v_{b_k}) = \beta_{b_k}$ for $k = 1, \ldots, s$ since $V' = q(V'')$.

(a) If $v_{b_k,0} \in V''$, then $\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,0}) = \beta_{b_k}$.

(b) Assume $v_{b_k,1} \in V''$. Then $\delta''(v_{b_k,1}) = \lambda(v_{b_k,1}v_{b_k,0})$ and

$$\lambda(v_{b_k,1}v_{b_k,0}) < \beta_{b_k} \leq \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\}.$$

Then there is $v_{b_k,1}v_{b_k,0}v_{j,0}v_{l,m} \in \mathfrak{P}_{b_k,1}(I)$ such that $\beta_{b_k} = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$ by the proof of that (V'', δ'') is an size-minimal f-weighted vertex cover. So we have $h_{b_k,0}(\mathfrak{P}) \leq f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Suppose $h_{b_k,1}(\mathfrak{P}) < f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Then there exists $v_{b_k,1}v_{b_k,0}v_{x,0}v_{y,z} \in \mathfrak{P}_{b_k,1}(I)$ with $x \neq j$ such that $h_{b_k,1}(\mathfrak{P}) = f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{x,0})\}$. So $f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{x,0})\} < f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$.

$$\begin{tikzcd}
 & & v_{b_k,1} \arrow[d, dash] & & \\
v_{y,z} \arrow[r, dash] & v_{x,0} \arrow[r, dash] & v_{b_k,0} \arrow[r, dash] & v_{j,0} \arrow[r, dash] & v_{l,m}
\end{tikzcd}$$

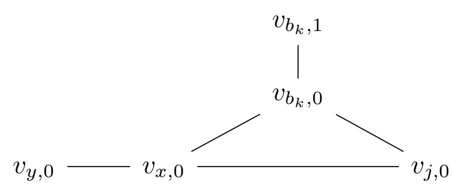
(1) Assume $v_{j,0} \neq v_{y,z} \neq v_{l,m}$ and $v_{j,0} \neq v_{x,0} \neq v_{l,m}$. By way of contradiction, we have $v_{j,0} \smile (v_{y,z}v_{x,0}v_{b_k,0}v_{j,0}, \mathfrak{P})$ and $v_{x,0} \smile (v_{l,m}v_{j,0}v_{b_k,0}v_{x,0}, \mathfrak{P})$. So $\lambda(v_{j,0}v_{b_k,0}) > f\{\lambda(v_{b_k,0}v_{j,0}), \lambda(v_{j,0}v_{l,m})\}$ and $\lambda(v_{x,0}v_{b_k,0}) > f\{\lambda(v_{y,z}v_{x,0}), \lambda(v_{x,0}v_{b_k,0})\}$. So $m = 0 = z$. Hence $v_{y,z}v_{x,0}v_{b_k,0}v_{j,0}v_{l,m}$ is a 4-path

in G_ω, a contradiction.

(2) Assume $v_{x,0} = v_{l,m}$ and $z = 0$. Since

$$\begin{aligned}\delta'(v_{b_k,0}) = \beta_k &= \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}\\ &> \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{x,0})\},\end{aligned}$$

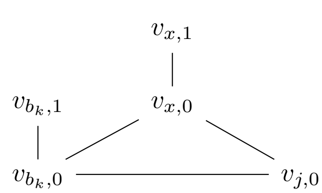
we have $v_{b_k,0} \not\succ (v_{b_k,1}v_{b_k,0}v_{x,0}v_{y,0} \rightsquigarrow v_{b_k,0}v_{x,0}v_{y,0}, \mathfrak{p})$. Since $v_{y,0}, v_{x,0} \not\succ (v_{b_k,1}v_{b_k,0}v_{x,0}v_{y,0}, \mathfrak{P})$, we have $v_{y,0}, v_{x,0} \not\succ (v_{b_k,1}v_{b_k,0}v_{x,0}v_{y,0} \rightsquigarrow v_{b_k,0}v_{x,0}v_{y,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



(3) Assume $v_{x,0} = v_{l,m}$ and $z = 1$. Then $v_{y,z} = v_{x,1}$. Since $v_{x,1} \not\succ (v_{b_k,1}v_{b_k,0}v_{x,0}v_{x,1}, \mathfrak{P})$, we have $v_{x,1} \not\succ (v_{x,1}v_{x,0}v_{j,0}v_{b_k,0}, \mathfrak{P})$. Since $v_{b_k,0} \notin V''_{b_k} \subseteq V''$, we have $v_{b_k,0} \not\succ (v_{x,1}v_{x,0}v_{j,0}v_{b_k,0}, \mathfrak{P})$. Since $v_{j,0} \not\succ (v_{b_k,1}v_{b_k,0}v_{j,0}v_{x,0})$, we have $v_{j,0} \not\succ (v_{x,1}v_{x,0}v_{j,0}v_{b_k,0}, \mathfrak{P})$. So $v_{x,0} \smile (v_{x,1}v_{x,0}v_{j,0}v_{b_k,0}, \mathfrak{P})$. Also, since $v_{x,0} \not\succ (v_{x,1}v_{x,0}v_{b_k,0}v_{b_k,1})$, we have

$$f\{\lambda(v_{x,1}v_{x,0}), \lambda(v_{x,0}v_{b_k,0})\} < f\{\lambda(v_{x,1}v_{x,0}), \lambda(v_{x,0}v_{j,0})\}.$$

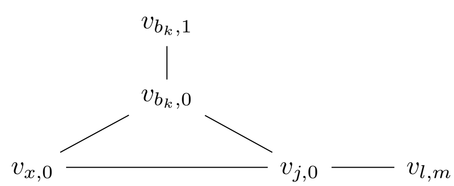
But $v_{x,0}v_{j,0}v_{b_k,0}v_{x,0}$ is a 3-cycle in G_ω, so $f\{\lambda(v_{j,0}v_{x,0}), \lambda(v_{j,0}v_{b_k,0})\} \geq \lambda(v_{j,0}v_{b_k,0})$ by the condition (e).(1) in Proposition 3.135. Since $v_{x,1}, v_{x,0}, v_{b_k,0} \not\succ (v_{x,1}v_{x,0}v_{b_k,0}v_{j,0}, \mathfrak{P})$, we have $v_{j,0} \smile (v_{x,1}v_{x,0}v_{b_k,0}v_{j,0}, \mathfrak{P})$ by Lemma 3.131. Hence $v_{j,0} \smile (v_{b_k,1}v_{b_k,0}v_{j,0}v_{x,0}, \mathfrak{P})$, a contradiction.



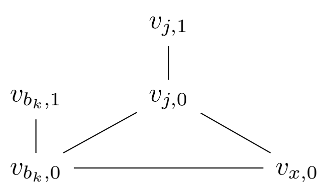
(4) Assume $v_{y,z} = v_{j,0}$ and $m = 0$. Since

$$\begin{aligned}\delta'(v_{b_k,0}) = \beta_k &= \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}\\ &> \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{x,0})\},\end{aligned}$$

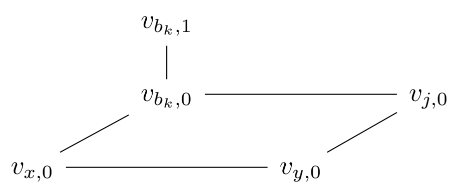
we have $v_{b_k,0} \not\swarrow (v_{b_k,1}v_{b_k,0}v_{x,0}v_{j,0} \rightsquigarrow v_{b_k,0}v_{x,0}v_{j,0}, \mathfrak{p})$. Since $v_{j,0}, v_{x,0} \not\swarrow (v_{b_k,1}v_{b_k,0}v_{x,0}v_{j,0}, \mathfrak{P})$, we have $v_{j,0}, v_{x,0} \not\swarrow (v_{b_k,1}v_{b_k,0}v_{x,0}v_{j,0} \rightsquigarrow v_{b_k,0}v_{x,0}v_{j,0}, \mathfrak{p})$, contradicted by Lemma 3.131.



(5) Assume $v_{y,z} = v_{j,0}$ and $m = 1$. Then $v_{l,m} = v_{j,1}$. So it is similar to (3) right before.



(6) Assume $v_{y,z} = v_{l,m}$. Then $z = 0 = m$. Then it is similar to (2) and (4) right before.



Hence by (1)-(6), we have $h_{b_k,0}(\mathfrak{P}) = f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Therefore,

$$\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,1}) + h_{b_k,0}(\mathfrak{P}) = \lambda(v_{b_k,1}v_{b_k,0}) + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\} = \beta_{b_k}.$$

(c) Assume $v_{b_k,2} \in V''$. Then $\beta_{b_k} > \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\}$ and $\delta''(v_{b_k,2}) = \lambda(v_{b_k,2}v_{b_k,1})$. Also, there exists $v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{j,0} \in \mathfrak{P}_{b_k,2}(I)$ such that

$$\beta_{b_k} = \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\} + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}.$$

So $h_{b_k,0}(\mathfrak{P}) \leq f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Suppose $h_{b_k,0}(\mathfrak{P}) < f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. Then there exists $v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{l,0} \in \mathfrak{P}_{b_k,2}(I)$ with $l \neq j$ such that $h_{b_k,0}(\mathfrak{P}) = f\{\lambda(v_{b_k,1}v_{b_k,0}), (v_{b_k,0}v_{l,0})$. So

$$\delta'(v_{b_k,0}) = \beta_{b_k} > \lambda(v_{b_k,2}v_{b_k,1}) + f\{\lambda(v_{b_k,2}v_{b_k,1}), \lambda(v_{b_k,1}v_{b_k,0})\} + f\{\lambda(v_{b_k,1}v_{b_k,0}), \lambda(v_{b_k,0}v_{l,0})\}.$$

Then $v_{b_k,0} \not\prec (v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{l,0} \rightsquigarrow v_{b_k,0}v_{l,0}, \mathfrak{p})$. Since $v_{l,0} \not\prec (v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{l,0}, \mathfrak{P})$, we have $v_{l,0} \not\prec (v_{b_k,2}v_{b_k,1}v_{b_k,0}v_{l,0} \rightsquigarrow v_{b_k,0}v_{l,0}, \mathfrak{p})$, contradicted by Lemma 3.131.

$$\begin{tikzcd}
 & v_{b_k,2} \arrow[d, dash] & \\
 & v_{b_k,1} \arrow[d, dash] & \\
v_{l,0} \arrow[r, dash] & v_{b_k,0} \arrow[r, dash] & v_{j,0}
\end{tikzcd}$$

$\square$

Proposition 3.154. Let $r \geq 4$ and $(\Sigma_{r-1}G)_\lambda$ a weighted $(r-1)$-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.136. Let $I = I_{r,f}\big((\Sigma_{r-1}G)_\lambda\big)$. Then the monomial ideal $IR = I_{r,f}\big((\Sigma G)_\lambda\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with (V'', δ'') a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$.

Proof. Let $P(V', \delta') := \big(X_{b_1}^{\beta_{b_1}}, \ldots, X_{b_s}^{\beta_{b_s}}\big)R$ occur in an irredundant m-irreducible decomposition of the ideal IR with $V' = \{v_{b_1}, \ldots, v_{b_s}\}$ and $\delta'(v_{b_k}) = \beta_{b_k}$ for $k = 1, \ldots, s$. Let $\mathfrak{p} := (V', \delta')$. Let $C := f\{\lambda(v_{1,1}v_{1,0}), \lambda(v_{1,0}v_{2,0})\}$. For $k = 1, \ldots, s$, define

$$V''_{b_k} := \begin{cases} \{v_{b_k,j}\} & \text{if } \beta_{b_k} = C + \lambda(v_{b_k,j}v_{b_k,j-1}) + \sum_{i=1}^{j-1} f\{\lambda(v_{b_k,i+1}v_{b_k,i}), \lambda(v_{b_k,i}v_{b_k,i-1})\}, \\ \{v_{b_k,0}\} & \text{if } \beta_{b_k} \leq \lambda(v_{b_k,1}v_{b_k,0}). \end{cases}$$

Set $V'' = \bigcup_{k=1}^{s} V''_{b_k}$. Define

$$\begin{aligned} \delta'' : V'' &\longrightarrow \mathbb{N} \\ v_{b_k,j} &\longmapsto \begin{cases} \lambda(v_{b_k,j}v_{b_k,j-1}) & \text{if } j \in \{1, \ldots, r-1\}, \\ \beta_{b_k} & \text{if } j = 0. \end{cases} \end{aligned}$$

Claim. (V'', δ'') is an f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. Let P_r be an r-path in $(\Sigma_{r-1}G)_\lambda$ and $\underline{X}^{\underline{\alpha}}$ the corresponding generator of I. Then $p(\underline{X}^{\underline{\alpha}})$ is a generator of IR. Also, since $IR \subseteq P(V', \delta')$, we have $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}})$ for some $k \in \{1, \ldots, s\}$.

(a) Assume $\beta_{b_k} \leq \lambda(v_{b_k,0}v_{b_k,1})$. Then $v_{b_k,0} \in V(P_r)$, $V''_{b_k} = \{v_{b_k,0}\}$ and $\delta''(v_{b_k,0}) = \beta_{b_k} \leq \lambda(v_{b_k,0}v_{b_k,1})$. If $v_{b_k,1} \notin V(P_r)$, then the exponent of X_{b_k} in $\underline{X}^{\underline{\alpha}}$ is equal to the exponent of X_{b_k} in $p(\underline{X}^{\underline{\alpha}})$. So $X_{b_k}^{\delta''(v_{b_k,0})} = X_{b_k}^{\beta_{b_k}} \mid \underline{X}^{\underline{\alpha}}$. So $v_{b_k,0} \smile (P_r, \mathfrak{P})$. Assume $v_{b_k,1} \in V(P_r)$. Then the exponent of $X_{b_k,0}$ in $\underline{X}^{\underline{\alpha}}$ is C. Since $\beta_{b_k} \leq \lambda(v_{b_k,0}v_{b_k,1})$, we have there exists an edge $v_{b_k,0}v_{j,0} \in E(G_\omega)$ such that $\beta_{b_k} = \lambda(v_{b_k,0}v_{j,0})$ or a 2-path $v_{i,0}v_{b_k,0}v_{j,0}$ in G_ω such that $\beta_{b_k} = f\{\lambda(v_{i,0}v_{b_k,0}), \lambda(v_{b_k,0}v_{j,0})\}$. So $\delta''(v_{b_k,0}) = \beta_{b_k} \leq C$ and hence $v_{b_k,0} \smile (P_r, \mathfrak{P})$.

(b) Assume we have $\beta_{b_k} = C + \lambda(v_{b_k,j}v_{b_k,j-1}) + \sum_{i=1}^{j-1} f\{\lambda(v_{b_k,i+1}v_{b_k,i}), \lambda(v_{b_k,i}v_{b_k,i-1})$ for some $j \in \{1, \ldots, r-1\}$. Then $\delta''(v_{b_k,j}) = \lambda(v_{b_k,j}v_{b_k,j-1})$. Since $X_{b_k}^{\beta_{b_k}} \mid p(\underline{X}^{\underline{\alpha}})$, we have P_r passes $v_{b_k,j}$. So $v_{b_k,j} \smile (P_r, \mathfrak{P})$.

Hence by (a) and (b), (V'', δ'') is an f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. Claim. (V'', δ'') is a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$.

(a) Assume $v_{b_k,j_1} \in V''$ for some $j_1 \in \{1, \ldots, r-1\}$. Then $\delta''(v_{b_k,j_1}) = \lambda(v_{b_k,j_1}v_{b_k,j_1-1})$ and $\beta_{b_k} = C + \lambda(v_{b_k,j_1}v_{b_k,j_1-1}) + \sum_{l=1}^{j_1-1} f\{\lambda(v_{b_k,l+1}v_{b_k,l}), \lambda(v_{b_k,l}v_{b_k,l-1})$. So there exists an r-path $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $(\Sigma_{r-1}G)_\lambda$ with $i_1 = b_k$ and a path $\wp$ in IR such that $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}} \rightsquigarrow \wp \in \mathfrak{p}_{b_k,0}(IR)$. Since $v_{i_1,0}, \ldots, v_{i_1,j_1-1} \notin V''$, we have $v_{i_2,j_2}, \ldots, v_{i_{1+j_1},0} \not\smile (v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}, \mathfrak{P})$. By way of contradiction, $v_{i_{2+j_1},0}, \ldots, v_{i_{r+1},0} \not\smile (v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}, \mathfrak{P})$.

$$\begin{array}{ccccccc}
v_{i_{1+j_1},0} & & & & & & \\
\| & & & & & & \\
v_{i_1,0} & \text{——} & v_{i_1,1} & \text{——} & \cdots & \text{——} & v_{i_1,j_1} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
v_{i_{r+1},0} & \text{——} & v_{i_{r+1},1} & \text{——} & \cdots & \text{——} & v_{i_{r+1},j_{r+1}} \\
\| & & & & & & \\
v_{i_{1+r-j_{r+1}},0} & & & & & &
\end{array}$$

Suppose $v_{i_{r+1},\alpha} \smile (v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}, \mathfrak{P})$ for some $1 \leq \alpha \leq j_{r+1}$. Then $v_{i_{r+1},\alpha} \in V''$ and so

$$\begin{aligned}
\delta'(v_{b_k,0}) = \beta_{i_{r+1}} &= C + \lambda(v_{i_{r+1},\alpha} v_{i_{r+1},\alpha-1}) + \sum_{l=1}^{\alpha-1} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\} \\
&\leq C + \lambda(v_{i_{r+1},j_{r+1}} v_{i_{r+1},j_{r+1}-1}) + \sum_{l=1}^{j_{r+1}-1} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\},
\end{aligned}$$

where the inequality follows when $\alpha = j_{r+1}$, and it follows when $\alpha < j_{r+1}$ because

$$\begin{aligned}
& C + \lambda(v_{i_{r+1},\alpha} v_{i_{r+1},\alpha-1}) + \sum_{l=1}^{\alpha-1} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\} \\
\leq\ & C + f\{\lambda(v_{i_{r+1},\alpha+1} v_{i_{r+1},\alpha}), \lambda(v_{i_{r+1},\alpha} v_{i_{r+1},\alpha-1})\} + \sum_{l=1}^{\alpha-1} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\} \\
=\ & C + \sum_{l=1}^{\alpha} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\} \\
<\ & C + \lambda(v_{i_{r+1},j_{r+1}} v_{i_{r+1},j_{r+1}-1}) + \sum_{l=1}^{j_{r+1}-1} f\{\lambda(v_{i_{r+1},l+1} v_{i_{r+1},l}), \lambda(v_{i_{r+1},l} v_{i_{r+1},l-1})\}.
\end{aligned}$$

So $v_{i_{r+1},0} \smile (v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}} \rightsquigarrow \wp, \mathfrak{p})$, contradicted by that $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}} \rightsquigarrow \wp \in \mathfrak{p}_{b_k,0}(IR)$. So $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}} \in \mathfrak{P}_{b_k,j_1}(I)$ with $\delta''(v_{b_k,j_1}) = \lambda(v_{b_k,j_1} v_{b_k,j_1-1})$. Hence neither $\mathfrak{P} \smallsetminus \{v_{b_k,j_1}^{\delta''(v_{b_k,j_1})}\}$ nor $\mathfrak{P} \smallsetminus \{v_{b_k,j_1}^{\delta''(v_{b_k,j_1})}\} \sqcup \{v_{b_k,j_1}^{\delta''(v_{b_k,j_1})+1}\}$ is an f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$.

(b) Assume $v_{b_k,0} \in V''$. Then $\beta_{b_k} \leq \lambda(v_{b_k,1} v_{b_k,0})$. So there exists an r-path $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $(\Sigma_{r-1}G)_\lambda$ with $i_1 = b_k$ and $j_1 = 0$ and a path $\wp$ in IR such that $v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}} \rightsquigarrow \wp \in \mathfrak{p}_{b_k,0}(IR)$. Then it similar to (a).

$$\begin{array}{ccccccc}
v_{i_1,0} & & & & & & \\
\| & & & & & & \\
v_{i_1,j_1} & & & & & & \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
v_{i_{r+1},0} & \text{———} & v_{i_{r+1},1} & \text{——} & \cdots & \text{——} & v_{i_{r+1},j_{r+1}} \\
\| & & & & & & \\
v_{i_{1+r-j_{r+1}},0} & & & & & &
\end{array}$$

Hence by (a) and (b), (V'', δ'') is a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. Claim.

$P(V',\delta') = P\big(q(V''),\gamma_{(V'',\delta'')}\big)$. It is enough to show that $\gamma_{(V'',\delta'')}(v_{b_k}) = \beta_{b_k}$ for $k = 1,\ldots,s$ since $V' = q(V'')$.

(a) If $v_{b_k,0} \in V''$, then $\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,0}) = \beta_{b_k}$.

(b) Assume $v_{b_k,j} \in V''$ for some $j \in \{1,\ldots,r-1\}$. Then $h_{b_k,0}(\mathfrak{P}) = C$. So

$$\begin{aligned}
\gamma_{(V'',\delta'')}(v_{b_k}) &= \delta''(v_{b_k,j}) + \sum_{i=0}^{j-1} h_{b_k,i}(\mathfrak{P}) \\
&= \delta''(v_{b_k,j}) + h_{b_k,0}(\mathfrak{P}) + \sum_{i=1}^{j-1} f\{\lambda(v_{b_k,i+1}v_{b_k,i}), \lambda(v_{b_k,i}v_{b_k,i-1})\} \\
&= C + \lambda(v_{b_k,j}v_{b_k,j-1}) + \sum_{i=1}^{j-1} f\{\lambda(v_{b_k,i+1}v_{b_k,i}), \lambda(v_{b_k,i}v_{b_k,i-1})\} \\
&= \beta_{b_k}. \qquad \square
\end{aligned}$$

Definition 3.155. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. Define

$$\mathfrak{m}^{[\underline{b}(\lambda)]} = \big(X_1^{b_1},\ldots,X_d^{b_d}\big)R,$$

where for $i = 1,\ldots,d$, $b_i = \sum_{k=0}^{r} e_{i,k}$ with

$$e_{i,k} = \begin{cases} \lambda(v_i v_{i,1}) & \text{if } k = 0, \\ f\{\lambda(v_{i,k-1}v_{i,k}), \lambda(v_{i,k}v_{i,k+1})\} & \text{if } k = 1,\ldots,r-1, \\ \lambda(v_{i,r-1}v_{i,r}) & \text{if } k = r. \end{cases}$$

In words, $\mathfrak{m}^{\underline{a}(\lambda)}$ is the monomial ideal of R obtained from the monomial ideal $(g_1,\ldots,g_d)R'$ by setting $\mathfrak{m}^{\underline{a}(\lambda)} = \big(p(g_1),\ldots,p(g_d)\big)R$, where g_i is the corresponding generator in $I_{r,f}\big((\Sigma_r G)_\lambda\big)$ of the r-whisker $v_i v_{i,1}\ldots v_{i,r}$ from $(\Sigma_r G)_\lambda$ for $i = 1,\ldots,d$.

Example 3.156. In Example 3.126, $\mathfrak{m}^{[\underline{b}(\lambda)]} = \big(X_1^{b_1}, X_2^{b_2}, X_3^{b_3}\big)R$ with

$$b_1 = \sum_{k=0}^{2} e_{1,k} = \lambda(v_1v_{1,1}) + \min\{\lambda(v_1v_{1,1}), \lambda(v_{1,1}v_{1,2})\} + \lambda(v_{1,1}v_{1,2}) = 4+3+3 = 10,$$
$$b_2 = \sum_{k=0}^{2} e_{2,k} = \lambda(v_2v_{2,1}) + \min\{\lambda(v_2v_{2,1}), \lambda(v_{2,1}v_{2,2})\} + \lambda(v_{2,1}v_{2,2}) = 3+3+3 = 9,$$
$$b_3 = \sum_{k=0}^{2} e_{3,k} = \lambda(v_3v_{3,1}) + \min\{\lambda(v_3v_{3,1}), \lambda(v_{3,1}v_{3,2})\} + \lambda(v_{3,1}v_{3,2}) = 2+2+5 = 9.$$

Theorem 3.157. *Let $r = 2$ or $r \geq 4$, and $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.136. One has*

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big),\ \ \lambda' = \lambda|_{\Sigma_{r-1}G},$$

and

$$I_{r,f}\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'')\text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{b}(\lambda)]}.$$

Proof. It is straightforward to see that $I_{r,f}\big((\Sigma_r G)_\lambda\big)R = I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{b}(\lambda)]}$. Then by [10, Theorem 7.5.3], it is enough to show that

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

By Proposition 3.152 and 3.154, the monomial ideal $I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with (V'', δ'') a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_{\lambda'}$. Then by Theorem 3.150,

$$\begin{aligned}
&I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R \\
\subseteq &\bigcap_{(V'',\delta'')\text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\
\subseteq &\bigcap_{(V'',\delta'')\text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'} \text{ in the decomp. of } I_{r,f}((\Sigma_{r-1}G)_{\lambda'})R} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\
= &I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R.
\end{aligned}$$

So

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big). \qquad \square$$

Theorem 3.158. *Let $r=3$, and $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.135. One has*

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \textit{ size-minimal } f\textit{-w. } r\textit{-path v. cover of } (\Sigma G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big), \;\; \lambda' = \lambda|_{\Sigma_{r-1}G},$$

and

$$I_{r,f}\big((\Sigma_r G)_{\lambda}\big)R = \bigcap_{(V'',\delta'') \textit{ size-minimal } f\textit{-w. } r\textit{-path v. cover of } (\Sigma G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{b}(\lambda)]}.$$

Proof. It is straightforward to see that $I_{r,f}\big((\Sigma_r G)_\lambda\big)R = I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{b}(\lambda)]}$. Then by [10, Theorem 7.5.3], it is enough to show that

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ size-minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).$$

By Proposition 3.153, the monomial ideal $I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1},\ldots,v_{i_t}\},\gamma_{(V'',\delta'')}\big)$ with (V'',δ'') an size-minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_{\lambda'}$. Then by Theorem 3.150,

$$\begin{aligned}
&I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R \\
&\subseteq \bigcap_{(V'',\delta'') \text{ size-minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big) \\
&\subseteq \bigcap_{(V'',\delta'') \text{ size-minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'} \text{ in the decomp. of } I_{r,f}((\Sigma_{r-1}G)_{\lambda'})R} P\big(q(V''),\gamma_{(V'',\delta'')}\big) \\
&= I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R.
\end{aligned}$$

So

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ size-minimal } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big). \qquad \square$$

In light of the preceding example, we define another order from which we can produce an irredundant decomposition. Lemma 3.162 is the key for understanding how this ordering helps with irredundancy.

Definition 3.159. Given two size-minimal f-weighted r-path vertex covers $(V_1'', \delta_1''), (V_2'', \delta_2'')$ of $(\Sigma_{r-1}G)_\lambda$, we write $(V_1'', \delta_1'') \leq_{\wp} (V_2'', \delta_2'')$ if $q(V_1'') \subseteq q(V_2'')$ and $\gamma_{(V_1'', \delta_1'')} \geq \gamma_{(V_2'', \delta_2'')}|_{q(V_1'')}$. An size-minimal f-weighted r-path vertex cover (V'', δ'') is *$\wp$-minimal* if there is not another size-minimal f-weighted r-path vertex cover (V''', δ''') such that $(V'', \delta'') <_{\wp} (V''', \delta''')$.

Lemma 3.160. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. Let $\mathfrak{p} := (W', \delta'), \mathfrak{P} := (W'', \delta'')$ be two size-minimal f-weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_{\wp} (W', \delta')$, then $|(W'', \delta'')| = |(W', \delta')|$ and $q(W'') = q(W')$. Moreover, if $\mathfrak{p} := (W', \delta'), \mathfrak{P} := (W'', \delta'')$ are two minimal f-weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_{\wp} (W', \delta')$, then $|(W'', \delta'')| = |(W', \delta')|$ and $q(W'') = q(W')$

Proof. The first part follows from definition. Let $I := I_{r,f}\big((\Sigma_{r-1}G)_\lambda\big)$. Since (W', δ') is a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, for distinct pair $v_{i_1,j_1}, v_{i_2,j_2} \in W'$, we have $i_1 \neq i_2$ by Fact 3.84, Proposition 3.134, 3.135 and 3.136. Also, since $q(W'') \subseteq q(W')$,we have $|W''| = |q(W'')| \leq |q(W')| = |W'|$. Suppose $|W''| < |W'|$. Then there exists $v_{i,j} \in W'$ such that $v_i \notin q(W'')$. Since (W', δ') is a minimal f-weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, we have there exists an r-path $P_r \in \mathfrak{p}_{i,j}(I)$. Since (W'', δ'') is an f-weighted r-path vertex cover, we have $v_{k,\alpha} \smile (P_r, \mathfrak{P})$ for some $v_{k,\alpha} \in W''$, so $v_k \in q(W'')$. Also, since $v_i \notin q(W'')$, we have $k \neq i$.

(a) Assume $\alpha = 0$. Since $(W'', \delta'') \leq_{\wp} (W', \delta')$, we have $v_k \in q(W')$ and

$$\gamma_{(W', \delta')}(v_k) \leq \gamma_{(W'', \delta'')}(v_k) = \delta''(v_{k,0}) \leq \lambda(v_{k,0}v_{k,1}).$$

So $v_{k,0} \in W'$ and then $\delta'(v_{k,0}) = \gamma_{(W', \delta')}(v_k) \leq \gamma_{(W'', \delta'')}(v_k) = \delta''(v_{k,0})$. But $v_{k,0} \smile (P_r, \mathfrak{P})$, so $v_{k,0} \smile (P_r, \mathfrak{p})$, contradicted by that $P_r \in \mathfrak{p}_{i,0}(I)$ and $k \neq i$.

(b) Assume $\alpha \geq 1$. Then $j < r-1$. Assume P_r is of the following form.

$$
\begin{array}{ccccccccc}
v_{i,0} & \text{———} & v_{i,1} & \text{———} & \cdots & \text{———} & v_{i,j} & \text{———} & \cdots \\
| & & & & & & & & \\
\vdots & & & & & & & & \\
| & & & & & & & & \\
v_{k,0} & \text{———} & v_{k,1} & \text{———} & \cdots & \text{———} & v_{k,\alpha} & \text{———} & \cdots
\end{array}
$$

Consider the r-path P_r' starting at $v_{k,0}$, passing along the r-path P_r and ending with $v_{i,M}$ for some $M \in \{j+1, \dots, r-1\}$. Since $v_i \notin q(W'')$, we have $v_{i,l} \not\prec (P_r', \mathfrak{P})$ for $l = 0, \dots, M$. Also, similar to (a), we have $v_{a,0} \not\prec (P_r', \mathfrak{P})$ for any other $v_{a,0} \in V(P_r')$, a contradiction.

Hence $|W''| = |W'|$ and thus $|q(W'')| = |q(W')|$. Since $q(W'') \subseteq q(W')$, we have $q(W'') = q(W')$. $\square$

The next two results are key for our second decomposition result.

Proposition 3.161. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. For every size-minimal f-weighted r-path vertex cover $\mathfrak{p} := (W', \delta')$ of $(\Sigma_{r-1}G)_\lambda$, there is a $\wp$-minimal f-weighted r-path vertex cover (W'', δ'') of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$.

Proof. If (W', δ') is itself a $\wp$-minimal f-weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$, then we are done. If (W', δ') is not $\wp$-minimal, then by Lemma 3.160, the size of $q(W')$ cannot be decreased, so for some $v_i \in q(W')$, the function $\gamma_{(W',\delta')}(v_i)$ can be increased. We increase $\gamma_{(W',\delta')}(v_i)$ for each $v_i \in q(W')$ such that any further increase would cause the set not to be a weighted r-path vertex cover. This process terminates in finitely many steps because if $v_{i,0} \in W'$, then $\gamma_{(W'',\delta')}(v_i) = \delta'(v_{i,0}) \leq \lambda(v_{i,1}v_{i,0})$, if $v_{i,j} \in W'$ for some $j \in \{1, \dots, r-2\}$, then

$$
\begin{aligned}
\gamma_{(W'',\delta')}(v_i) = \sum_{l=0}^{j} h_{i,l} &= h_{i,0} + \lambda(v_{i,j}v_{i,j-1}) + \sum_{l=1}^{j-1} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\} \\
&\leq \lambda(v_{i,1}v_{i,0}) + f\{\lambda(v_{i,j+1}v_{i,j}), \lambda(v_{i,j}v_{i,j-1})\} + \sum_{l=1}^{j-1} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\} \\
&= \lambda(v_{i,1}v_{i,0}) + \sum_{l=1}^{j} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\} \\
&< \lambda(v_{i,1}v_{i,0}) + \lambda(v_{i,r-1}v_{i,r-2}) + \sum_{l=1}^{r-2} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\},
\end{aligned}
$$

and if $v_{i,r-1} \in W'$, then

$$\gamma_{(W'',\delta')}(v_i) = \sum_{l=0}^{r-1} h_{i,l} = h_{i,0} + \lambda(v_{i,r-1}v_{i,r-2}) + \sum_{l=1}^{r-2} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\}$$
$$\leq \lambda(v_{i,1}v_{i,0}) + \lambda(v_{i,r-1}v_{i,r-2}) + \sum_{l=1}^{r-2} f\{\lambda(v_{i,l+1}v_{i,l}), \lambda(v_{i,l}v_{i,l-1})\}.$$

Denote the new set (W'', δ''). Then (W'', δ'') is $\wp$-minimal since the size of W'' cannot be decreased by Lemma 3.160 and $\gamma_{(W'',\delta'')}$ cannot be increased. Thus, by construction, (W'', δ'') is a $\wp$-minimal f-weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$. □

Lemma 3.162. Let $(V_1', \delta_1'), (V_2', \delta_2')$ be two size-minimal min-weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$. Then $(V_1', \delta_1') \leq_\wp (V_2', \delta_2')$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$.

Proof. $(V_1', \delta_1') \leq_\wp (V_2', \delta_2')$ if and only if $q(V_1') \subseteq q(V_2')$ and $\gamma_{(V_1',\delta_1')}|_{q(V_1')} \geq \gamma_{(V_2',\delta_2')}|_{q(V_1')}$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$. □

Next, we present our second decomposition result which will yield the type computation in Theorem 3.122.

Theorem 3.163. *Let $r = 2$ or $r \geq 4$, and $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.136. One has an irredundant parametric decomposition*

$$I_{r,f}\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'')\ \wp\text{-min. } f\text{-w. } r\text{-path v. c. of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]},$$

where $\lambda' = \lambda|_{\Sigma_{r-1}G}$.

Proof. By Fact 3.96 and [10, Theorem 7.5.3], to verify this result, it is enough to show that we have an irredundant decomposition

$$I_{r,f}\big((\Sigma_{r-1} G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\ \wp\text{-min. } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

Lemma 3.162 shows that this intersection is irredundant. So by Theorem 3.157, it is enough to show that

$$\bigcap_{(V'',\delta'')\text{ min. } f\text{-weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big)$$
$$= \bigcap_{(V'',\delta'')\ \mathpzc{p}\text{-min. } f\text{-weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).$$

"$\subseteq$" follows as every $\mathpzc{p}$-minimal f-weighted r-path vertex cover is a minimal f-weighted r-path vertex cover.

"$\supseteq$" follows from Proposition 3.161 and Lemma 3.162. $\square$

Theorem 3.164. *Let $r = 3$ and $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.135. One has an irredundant parametric decomposition*

$$I_{r,f}\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'')\ \mathpzc{p}\text{-min. } f\text{-w. } r\text{-path v. c. of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]},$$

where $\lambda' = \lambda|_{\Sigma_{r-1}G}$.

Proof. By Fact 3.96 and [10, Theorem 7.5.3], to verify this result, it is enough to show that we have an irredundant decomposition

$$I_{r,f}\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\ \mathpzc{p}\text{-min. } f\text{-w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).$$

Lemma 3.162 shows that this intersection is irredundant. So by Theorem 3.158, it is enough to show that

$$\bigcap_{(V'',\delta'')\text{ size-min. } f\text{-weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big)$$
$$= \bigcap_{(V'',\delta'')\ \mathpzc{p}\text{-min. } f\text{-weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''),\gamma_{(V'',\delta'')}\big).$$

"$\subseteq$" follows as every $\mathpzc{p}$-minimal f-weighted r-path vertex cover is an size-minimal f-weighted r-path vertex cover.

"$\supseteq$" follows from Proposition 3.161 and Lemma 3.162. $\square$

The next theorem is part of the fifth main result of this thesis. It is Formula (****) from the abstract.

Theorem 3.165. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Proposition 3.134 or 3.135 or 3.136. Then*

$$\operatorname{type}\left(\frac{R'}{I_{r,f}((\Sigma_r G)_\lambda)}\right) = \sharp\{\wp\text{-minimal } f\text{-weighted } r\text{-path vertex covers of } (\Sigma_{r-1}G)_{\lambda'}\},$$

where $\lambda' = \lambda|_{\Sigma_{r-1}G}$.

Proof. We compute

$$\begin{aligned}\operatorname{type}\left(\frac{R'}{I_{r,f}((\Sigma_r G)_\lambda)}\right) &= \operatorname{type}\left(\frac{R'}{I_{r,f}((\Sigma_r G)_\lambda) + (X_{i,j} - X_{i,j-1} \mid i = 1,\ldots,d, j = 1,\ldots,r)R'}\right)\\ &= \operatorname{type}\left(\frac{R}{I_{r,f}((\Sigma_r G)_\lambda)R}\right)\\ &= \sharp\{\text{ideals in an irredund. parametric decomposition of } I_{r,f}\big((\Sigma_r G)_\lambda\big)R\}\\ &= \sharp\{\wp\text{-minimal } f\text{-weighted } r\text{-path vertex covers of } (\Sigma_r G)_{\lambda'}\},\end{aligned}$$

where the first equality is from Facts 2.95(a) and Theorem 3.137, the second equality is from Theorem 3.137, the third equality is from Fact 2.95(b) since $\dim\big(\frac{R}{I_{r,f}((\Sigma_r G)_\lambda)R}\big) = 0$, and the last equality is from Theorem 3.163. □

Remark. Because of Theorem 3.148, we can use Theorem 3.165 to compute $\operatorname{type}\big(R/I_{r,f}(G_\omega)\big)$ for all weighted trees G_ω such that $I_{r,f}(G_\omega)$ is Cohen-Macaulay.

3.6 Edge Ideals $I(D)$ of Oriented Graphs D and the Type of $R'/I(\Sigma D)$

Let $D = (V(D), E(D))$ be an oriented graph with the underlying undirected graph G, i.e., D arises from the undirected G by directing every edge from one of its ends to the other. Let $v_i v_j$ represent an edge in D directed from v_i to v_j. Let $R' = A[X_1,\ldots,X_d,Y_1,\ldots,Y_d]$. Let $\mathbb{N} = \{1,2,3,\ldots\}$ be the set of positive integers.

Definition 3.166. A *vertex-weight function* on the oriented graph D is a function $\omega : V \to \mathbb{N}$ that assigns a *weight* to each vertex. A *weighted oriented graph* D_ω is an oriented graph D equipped with a vertex-weight function ω. For simplicity, let $\omega_i = \omega(v_i)$ for $i = 1, \dots, d$.

Assumption. For the remainder of this section, we assume D is a weighted oriented graph with vertex-weight function ω such that the underlying graph is G.

Definition 3.167. The *edge ideal* associated to D is the ideal $I(D) \subseteq R$ that is "generated by the edges of D":

$$I(D) = \Big(X_i X_j^{\omega_j} \mid v_i v_j \in E(D)\Big)R.$$

Example 3.168. The edge ideal of $D = (\ v_1^2 \longleftarrow v_2^3 \longrightarrow v_3^5\ )$ is $I(D) = (X_1^2 X_2, X_2 X_3^5)R$.

Convention 3.169. The Cohen-Macaulay property of $I(D)$ is independent of the weight we assign to a source or a sink. So when we study this property, we shall always assume $\omega(v_i) = 1$ if v_i is a source or sink.

Definition 3.170. A *vertex cover* of D is a subset of $V' \subseteq V$ such that for each edge $v_i v_j \in E(D)$ we have $v_i \in V'$ or $v_j \in V'$. A vertex cover V' is *minimal* if it does not properly contain another vertex cover of D.

Definition 3.171. Let $v_i \in V$. Define the *out-neighbourhood* of v_i by

$$N_D^+(v_i) = \{v_j \in V \mid v_i v_j \in E(D)\},$$

and define the *in-neighbourhood* of v_i by

$$N_D^-(v_i) = \{v_j \in V \mid v_j v_i \in E(D)\}.$$

Furthermore, the *neighbourhood* of v_i is the set

$$N_D(v_i) = N_D^+(v_i) \sqcup N_D^-(v_i).$$

Definition 3.172. [12, Definition 2.3] Let V' be a vertex cover of D. Define

$$L_1(V') = \big\{v_i \in V' \;\big|\; N_D^+(v_i) \not\subseteq V'\big\},$$

$$L_2(V') = \big\{v_i \in V' \;\big|\; N_D^-(v_i) \not\subseteq V'\big\} \smallsetminus L_1(V'),$$

and

$$\begin{aligned}L_3(V') = V' \smallsetminus \{L_1(V') \sqcup L_2(V')\} &= \left\{v_i \in V' \;\middle|\; N_D^+(v_i) \subseteq V' \text{ and } N_D^-(v_i) \subseteq V'\right\} \\ &= \{v_i \in V' \mid N_D(v_i) \subseteq V'\}.\end{aligned}$$

Fact 3.173. [12, Proposition 2.5] Let V' be a vertex cover of D. Then $L_3(V') = \emptyset$ if and only if V' is a minimal vertex cover of D.

Definition 3.174. A vertex cover V' of D is strong if for each $v_i \in L_3(V')$ there is $v_j v_i \in E(D)$ such that $v_j \in L_2(V') \cup L_3(V')$ with $\omega_j \geq 2$.

Fact 3.175. If V' is a minimal vertex cover, then V' is strong.

Definition 3.176. [12, Definition 3.4] Let V' be a vertex cover of D. Define

$$I_{V'} = \Big(L_1(V') \sqcup \left\{X_j^{\omega_j} \;\middle|\; v_j \in L_2(V') \sqcup L_3(V')\right\}\Big)R.$$

Fact 3.177. [12, Theorem 3.11]

$$I(D) = \bigcap_{V' \text{ s. v. cover}} I_{V'},$$

where the intersection is taken over all strong vertex covers of D. The intersection is irredundant.

Example 3.178. Let $D = (\; v_1^2 \longleftarrow v_2^3 \longrightarrow v_3^5 \;)$ as in Example 3.168. Then the strong vertex covers are $\{v_1, v_3\}, \{v_2\}$. Since $I_{\{v_1,v_3\}} = (X_1^2, X_3^5)R$ and $I_{\{v_2\}} = (X_2)R$, by Fact 3.177, we have

$$I(D) = I_{\{v_1,v_3\}} \cap I_{\{v_2\}} = \big(X_1^2, X_3^5\big)R \cap \big(X_2\big)R.$$

Definition 3.179. A *suspension* of D is a weighted oriented graph ΣD with vertex set $V \sqcup \{w_1, \ldots, w_d\}$ such that the underlying graph ΣG is a suspension of G.

Example 3.180. A vertex suspension ΣD of $D = (\; v_1^2 \longleftarrow v_2^3 \longrightarrow v_3^5 \;)$ is

$$\begin{tikzcd} w_1^1 & w_2^6 \arrow[d] & w_3^3 \\ v_1^2 \arrow[u] & v_2^3 \arrow[l] \arrow[r] & v_3^5. \arrow[u] \end{tikzcd}$$

Fact 3.181. Let ΣD be a suspension of D such that $\omega_i = \omega(v_i) = 1$ for any edge $v_iw_i \in \Sigma D$. Then by the proof of [7, Theorem 3.1], $I(\Sigma D)$ is the polarization of $I(D) + \left(X_1^{1+\omega_1}, \ldots, X_d^{1+\omega_d}\right)R$. So by Fact 2.89, the list $X_1 - Y_1, \ldots, X_d - Y_d$ is a maximal regular sequence for $\frac{R'}{I(\Sigma D)}$ and

$$\frac{R}{I(D) + \left(X_1^{1+\omega_1}, \ldots, X_d^{1+\omega_d}\right)R} \cong \frac{R'}{I(\Sigma D) + (X_1 - Y_1, \ldots, X_d - Y_d)R'}.$$

Because of the following fact, the main result of this section gives a formula to compute the type$(R/I(D))$ for all trees such that $R/I(D)$ is Cohen-Macaulay.

Fact 3.182. [7, Theorem 3.1] Let ΣD be a suspension of D such that $\omega(v_i) = 1$ for any edge $v_iw_i \in D$.

(a) Then $S/I(\Sigma D)$ is Cohen-Macaulay.

(b) Let Z be a weighted oriented graph such that the underlying simple graph is Γ. If Γ is a tree and $R/I(\Gamma)$ is Cohen-Macaulay, then $\Gamma = \Sigma H$ for some subtree H.

The following theorem is the fourth main result of this thesis. It is Formula (*****) from the abstract.

Theorem 3.183. *Let ΣD be a suspension of D such that $\omega(v_i) = 1$ for any edge $v_iw_i \in D$. Then*

$$\operatorname{type}\left(\frac{R'}{I(\Sigma D)}\right) = \sharp\,\{\textit{strong vertex covers of } D\}.$$

Proof. We compute

$$\begin{aligned}
\operatorname{type}\left(\frac{R'}{I(\Sigma D)}\right) &= \operatorname{type}\left(\frac{R'}{I(\Sigma D) + (X_1 - Y_1, \ldots, X_d - Y_d)R'}\right) \\
&= \operatorname{type}\left(\frac{R}{I(G_\omega) + (X_1^{1+\omega_1}, \ldots, X_d^{1+\omega_d})R}\right) \\
&= \sharp\,\{\text{ideals in an irredundant parametric decomposition of} \\
&\qquad I(D) + \left(X_1^{1+\omega_1}, \ldots, X_d^{1+\omega_d}\right)\} \\
&= \sharp\,\{\text{ideals in an irredundant m-irreducible decomposition of } I(D)\} \\
&= \sharp\,\{\text{strong vertex covers of } D\},
\end{aligned}$$

where the first equality is from Facts 2.95(a), 3.182 and 3.181, the second equality is from Fact 3.181, the third equality is from Fact 2.95(b) since $\dim\left(\frac{R}{I(G_\omega)+\left(X_1^{1+\omega_1},\ldots,X_d^{1+\omega_d}\right)R}\right)=0$, the fourth equality is from [10, Exercise 7.5.10], and the last equality is from Fact 3.177. $\square$

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