

MATH 9850, Advanced Homological Algebra
 Spring 2020
 Project on polarization (conclusion)

Let k be a field, and set $R = k[X_1, \dots, X_d]$ and $\mathfrak{m} = \langle X_1, \dots, X_d \rangle \leq R$. For each positive integer n , set $[n] = \{1, \dots, n\}$. For each sequence of positive integers $\mathbf{n} = (n_1, \dots, n_d)$ set $R_{\mathbf{n}} = k[X_{i,j} \mid i \in [d] \text{ and } j \in [n_i]]$.

Let I be a monomial ideal of R , and let $\tilde{I} \leq R_{\mathbf{n}}$ be the polarization of I where \mathbf{n} is determined by the generators of I . The first part of this project concludes that the sequence consisting of polynomials $X_{i,j-1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\tilde{I}$. The goal of this part of the project is to show that the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\tilde{I}$.

Exercise 1. Read and understand the statement and proof of [1, Theorem 16.5(ii)].

Exercise 2. Let A be a noetherian ring, let M be a finitely generated R -module, and let $x_1, \dots, x_n, y_1, \dots, y_n \in A$ be such that $(x_1, \dots, x_n)A = (y_1, \dots, y_n)A$ and there exists an invertible n -by- n matrix $Q := (q_{i,j})$ such that $[y_1 \cdots y_n]^T = Q[x_1 \cdots x_n]^T$. Assume that one of the following holds:

- (α) (A, \mathfrak{m}) is a local ring and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathfrak{m}$, or
- (β) A is an \mathbb{N} -graded ring, M is an \mathbb{N} -graded A -module, and $x_1, \dots, x_n, y_1, \dots, y_n$ are homogeneous of positive degree.

- (a) Prove that $K^A(\underline{x}; M) \cong K^A(\underline{y}; M)$.

Proof. Since $K^A(\underline{x}; M) \cong K^A(\underline{x}) \otimes M$ and $K^A(\underline{y}; M) \cong K^A(\underline{y}) \otimes M$, it is equivalent to show that $K^A(\underline{x}) \cong K^A(\underline{y})$. Define $\psi_i : K^A(\underline{x})_i \rightarrow K^A(\underline{y})_i$ by $\psi_i = \bigwedge^i(Q^T)$ for $1 \leq i \leq n$ and $\psi_i = \text{id}$, otherwise. Since $K^A(\underline{x})_j = 0 = K^A(\underline{y})_j$ for $j \geq n+1$ or for $j \leq -1$ and $\bigwedge^i(Q^T)$ is invertible for $1 \leq i \leq n$, to show $K^A(\underline{x}) \cong K^A(\underline{y})$, it suffices to show that the following diagram with $1 \leq i \leq n$ commutes.

$$\begin{array}{ccc} K^A(\underline{y})_i & \xrightarrow{\partial_i^{K^A(\underline{y})}} & K^A(\underline{y})_{i-1} \\ \downarrow \bigwedge^i(Q^T) & & \downarrow \bigwedge^{i-1}(Q^T) \\ K^A(\underline{x})_i & \xrightarrow{\partial_i^{K^A(\underline{x})}} & K^A(\underline{x})_{i-1} \end{array}$$

Let $\{e_1, \dots, e_n\}$ be the standard A -basis of A^n . Then

$$\{e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i} \mid 1 \leq \lambda_1 < \cdots < \lambda_i \leq n\}$$

is an A -basis of $A^{\binom{n}{i}}$ for $2 \leq i \leq n$. Note that

$$\begin{aligned} \partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) &= \sum_{j=1}^i (-1)^{j-1} \partial_1^{K^A(\underline{y})}(e_{\lambda_j}) \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i} \\ &= \sum_{j=1}^i (-1)^{j-1} y_{\lambda_j} \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i}, \end{aligned}$$

and

$$\partial_i^{K^A(\underline{x})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) = \sum_{j=1}^i (-1)^{j-1} x_{\lambda_j} \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i}.$$

To show the above diagram commutes, it is enough to show that

$$\bigwedge^{i-1}(Q^T) \circ \partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) = \partial_i^{K^A(\underline{x})} \circ \bigwedge^i(Q^T)(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}). \quad (2.1)$$

Claim. For $v_1 \wedge \cdots \wedge v_i \in K^A(\underline{y})_i$, we have

$$\partial_i^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^i (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i.$$

Proof by induction. Base case $i = 1$ follows from the definition. Inductive step. By [2, Exercise 4.33] and inductive hypothesis, we have

$$\begin{aligned} & \partial_i^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_i) \\ &= \partial_i^{K^A(\underline{y})}((v_1 \wedge \cdots \wedge v_{i-1}) \wedge v_i) \\ &= \partial_{i-1}^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_{i-1}) \wedge v_i + (-1)^{i-1} \cdot v_1 \wedge \cdots \wedge v_{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \\ &= \left(\sum_{j=1}^{i-1} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1} \right) \wedge v_i \\ &\quad + (-1)^{i-1} \cdot v_1 \wedge \cdots \wedge v_{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \\ &= \sum_{j=1}^{i-1} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1} \wedge v_i \\ &\quad + (-1)^{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge \hat{v}_i \\ &= \sum_{j=1}^i (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i. \end{aligned}$$

On the one hand, the left hand side of (2.1) is

$$\begin{aligned} & \bigwedge^{i-1}(Q^T) \circ \partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) \\ &= \bigwedge^{i-1}(Q^T) \left(\sum_{j=1}^i (-1)^{j-1} y_{\lambda_j} e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i} \right) \\ &= \sum_{j=1}^i (-1)^{j-1} y_{\lambda_j} \bigwedge^{i-1}(Q^T)(e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i}) \\ &= \sum_{j=1}^i (-1)^{j-1} y_{\lambda_j} Q^T(e_{\lambda_1}) \wedge \cdots \wedge \hat{Q}^T(e_{\lambda_j}) \wedge \cdots \wedge Q^T(e_{\lambda_i}). \end{aligned}$$

On the other hand, the right hand side of (2.1) is

$$\begin{aligned} & \partial_i^{K^A(\underline{x})} \circ \bigwedge^i(Q^T)(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) \\ &= \partial_i^{K^A(\underline{x})}(Q^T(e_{\lambda_1}) \wedge \cdots \wedge Q^T(e_{\lambda_i})) \\ &= \sum_{j=1}^i (-1)^{j-1} \left(\partial_1^{K^A(\underline{x})}(Q^T(e_{\lambda_j})) \right) Q^T(e_{\lambda_1}) \wedge \cdots \wedge \hat{Q}^T(e_{\lambda_j}) \cdots \wedge Q^T(e_{\lambda_i}), \end{aligned}$$

where

$$\begin{aligned}
 \partial_1^{K^A}(\underline{x})(Q^T(e_{\lambda_j})) &= \partial_1^{K^A}(\underline{x})(q_{\lambda_j,1}e_1 + \cdots + q_{\lambda_j,n}e_n) \\
 &= \partial_1^{K^A}(\underline{x})(q_{\lambda_j,1}e_1) + \cdots + \partial_1^{K^A}(\underline{x})(q_{\lambda_j,n}e_n) \\
 &= q_{\lambda_j,1}x_1 + \cdots + q_{\lambda_j,n}x_n \\
 &= y_{\lambda_j}.
 \end{aligned}$$

Thus, (2.1) holds. \square

- (b) Use [1, Theorem 16.5(ii)] with part (a) to prove that x_1, \dots, x_n is M -regular if and only if y_1, \dots, y_n is M -regular.

Proof. “ \Rightarrow ”. Assume x_1, \dots, x_n is M -regular. Then $H_1(K^A(\underline{x}; M)) = H_1(\underline{x}; M) = 0$ by [3, Theorem VIII.6.15]. Since $K^A(\underline{x}; M) \cong K^A(\underline{y}; M)$ by (a), we have

$$H_1(\underline{y}; M) = H_1(K^A(\underline{y}; M)) \cong H_1(K^A(\underline{x}; M)) = 0.$$

So y_1, \dots, y_n is M -regular by [1, Theorem 16.5(ii)].

“ \Leftarrow ”. It is similar to the proof of “ \Rightarrow ”. \square

Exercise 3. Use Exercise 2 to prove that the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\tilde{I}$.

Proof. For $i = 1, \dots, d$, define Q_{n_i} to be a lower triangular n_i -by- n_i matrix such that all the non-zero entries are 1. Then

$$\begin{bmatrix} X_{1,1} - X_{1,2} \\ \vdots \\ X_{1,1} - X_{1,i} \\ \vdots \\ X_{1,1} - X_{1,n_1} \\ \vdots \\ X_{d,1} - X_{d,2} \\ \vdots \\ X_{d,1} - X_{d,i} \\ \vdots \\ X_{d,1} - X_{d,n_d} \end{bmatrix} = \begin{bmatrix} Q_{n_1} & & \\ & \ddots & \\ & & Q_{n_d} \end{bmatrix} \begin{bmatrix} X_{1,1} - X_{1,2} \\ \vdots \\ X_{1,i-1} - X_{1,i} \\ \vdots \\ X_{1,n_1-1} - X_{1,n_1} \\ \vdots \\ X_{d,1} - X_{d,2} \\ \vdots \\ X_{d,i-1} - X_{d,i} \\ \vdots \\ X_{d,n_d-1} - X_{d,n_d} \end{bmatrix}$$

Since $Q := \begin{bmatrix} Q_{n_1} & & \\ & \ddots & \\ & & Q_{n_d} \end{bmatrix}$ is a lower triangular matrix whose diagonal entries are 1,

we have Q is invertible. Since the first part of this project concludes that the sequence consisting of polynomials $X_{i,j-1} - X_{i,j}$ for all $i \in [d]$ and $1 < j < [n_i]$ is regular for $R_{\mathbf{n}}/\hat{I}$, we have the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is also regular for $R_{\mathbf{n}}/\tilde{I}$ by Exercise 2. \square

REFERENCES

1. H. Matsumura, *Commutative ring theory*, second ed., Studies in Advanced Mathematics, vol. 8, University Press, Cambridge, 1989. MR 90i:13001
2. Kristen A Beck and Sean Sather-Wagstaff. A somewhat gentle introduction to differential graded commutative algebra. In *Connections Between Algebra, Combinatorics, and Geometry*, pages 3–99. Springer, 2014.
3. Sean Sather-Wagstaff. Homological algebra notes. *Unpublished lecture notes. Available at <http://ssather.people.clemson.edu/notes.html>*, 2009.