Curves And Surfaces

Uncultured Tramp

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Contents

1 Smooth Curves

1.1 Natural Parametrization

$$\texttt{RegularCurve} :: \prod [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \;.\; \prod n \in \mathbb{N} \;.\; ?C^\infty([a,b],\mathbb{R}^n)$$

$$r : \texttt{RegularCurve} \iff r \in \mathcal{R}(a,b,n) \iff \forall t \in (a,b) \;.\; \|\mathrm{D}r|_t\| > 0$$

$$\texttt{lengthFunc} :: \prod [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \;.\; \prod n \in \mathbb{N} \;.\; C^1([a,b],\mathbb{R}^n) \to [a,b] \to \mathbb{R}_+$$

$$\texttt{lengthFunc}\,(r,t) = L_r(t) := \int_a^t \left\|\mathrm{D}r|_s\right\| \,\mathrm{d}s$$

$$\texttt{RegularArclengthIsMonotontonic} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \;.\; \forall n \in \mathbb{N} \;.\; \forall r \in \mathcal{R}(a,b,n) \;.$$

$$.\; L_r : \texttt{Increasing}\Big([a,b],\mathbb{R}_+\Big)$$

$\texttt{Proof}$ =

$\texttt{Assume}\ t,t' : [a,b],$

$\texttt{Assume}\ [t.1] : t < t',$

$$[t.*] := \eth L_r \texttt{AdditiveInteegral}(a,t,t',\|\mathrm{D}r\|)(t.1)\texttt{PositiveIntegral}\Big(\eth\mathcal{R}\Big)(t.1) :$$

$$: L_r(t') - L_r(t) = \int_a^{t'} \left|\mathrm{D}r|_s\right| \,\mathrm{d}s - \int_a^t \left\|\mathrm{D}r|_s\right\| \,\mathrm{d}s = \int_t^{t'} \left\|\mathrm{D}r|_s\right\| \,\mathrm{d}s > 0;$$

$$\rightsquigarrow [*] := \eth^{-1}\texttt{Increasing} : \Big(L_r : \texttt{Increasing}\big([a,b],\mathbb{R}_+\big)\Big),$$

$\square$

$$\texttt{NaturallyParametrized} :: ?\mathcal{R}(a,b,n)$$

$$r : \texttt{NaturallyParametrized} \iff \|\mathrm{D}r\| = 1$$

$$\texttt{NaturalParametrizationExists} :: \forall r \in \mathcal{R}(a,b,n) \;.\; \exists s : C^\infty\Big([0,L_r],[a,b]\Big) \;.\; .$$

$$.\; r \circ s : \texttt{NaturallyParametrized}\Big(0, L_r(b), n\Big)$$

$\texttt{Proof}$ =

$$s := L_r^{-1} : \texttt{Increasing}\Big([0,L_r(b)],[a,b]\Big),$$

$$[1] := \texttt{InverseDifferentiation}(s) : \mathrm{D}s = \frac{1}{\|\mathrm{D}r_s\|},$$

$\texttt{Assume}\ t : [0,L_r(b)],$

$$[t.1] := \texttt{DerivativeComposition}(r,s) : Dr \circ s = \frac{Dr|_s}{\|Dr|_s\|},$$

$$[t.*] := \texttt{NormHomogen}(\mathbb{R}^n)(t.1)\eth\texttt{Inverse} : \|\mathrm{D}r \circ s\| = \frac{\|\mathrm{D}r|_s\|}{\|\mathrm{D}r|_s\|} = 1;$$

$$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturallyParametrized} : \Big(r \circ s : \texttt{NaturallyParametrized}(a,b,n)\Big);$$

$\square$

$\texttt{ReparametrizationClassOfACurve} :: \mathcal{R}(a,b,n) \to ?\sum [c,d] : \texttt{ClosedInterval}(\mathbb{R}) \,.\, \mathcal{R}(c,d,n)$

$\Big([c,d],\gamma\Big) : \texttt{ReparametrizationClassOfACurve} \iff \Lambda r \in R(a,b,n) \,.\, \Big([c,d],\gamma\Big) \in [r] \iff$

$\iff \exists \tau : (\mathbb{R},[a,b]) \xleftrightarrow{\mathsf{DIFF}(\infty)} (\mathbb{R},[c,d]) \;\&\; \texttt{increasing} : \gamma = r \circ \tau$

$\texttt{arclength} :: C^1\Big([a,b],\mathbb{R}^n\Big) \to \mathbb{R}_+$

$\texttt{arclength}\,(r) = L(r) := L_r(b)$

$\texttt{ReparametrizationClass} :: ?\sum [a,b] : \texttt{ClosedInterval}(R) \,.\, \mathcal{R}(a,b,n)$

$X : \texttt{ReparametrizationClass} \iff X \in [\mathcal{R}(n)] \iff \exists [a,b] : \texttt{ClosedInterval}(R) : \exists r \in \mathcal{R}(a,b,n) : X = [r]$

$\texttt{ReparametrizationPreservesArclength} :: \forall [r] \in [\mathcal{R}(n)] \,.\, \forall \Big([a,b],\alpha\Big), \Big([c,d],\beta\Big) \in [r] \,.\, L(\alpha) = L(\beta)$

$\texttt{Proof} =$

$\Big(\tau,[1]\Big) := \eth[\mathcal{R}(n)]([r])(\alpha,\beta) : \sum \tau : C^\infty \;\&\; \texttt{increasing}\Big([c,d],[a,b]\Big) \,.\, \beta = \alpha \circ \tau,$

$[2] := \texttt{DerivativeOfIncreasing}(\tau) : \mathrm{D}\tau > 0,$

$[*] := \eth L(\beta)[1]\texttt{ChainRule}(\alpha,\tau)[2]\texttt{ChangeOfVariable}(\tau)\eth^{-1}(L(\alpha)) :$

$: L(\beta) = \int_c^d \Big\|\mathrm{D}\beta|_s\Big\| \,\mathrm{d}s = \int_c^d \mathrm{D}\tau|_s \Big\|\mathrm{D}\alpha|_{\tau(s)}\Big\| \,\mathrm{d}s = \int_a^b \Big\|\mathrm{D}\alpha|_s\Big\| \,\mathrm{d}s = L(\alpha);$

$\square$

$\texttt{classLength} :: [\mathcal{R}(n)] \to \mathbb{R}_+$

$\texttt{classLength}\,([r]) = L\big([r]\big) := L(r)$

$\texttt{NaturalParametrizationIsUnique} :: \forall X \in [\mathcal{R}(n)] \,.\, \exists! \Big([0,L(X)],r\Big) \in X :$

$: \Big(r : \texttt{NaturallyParametrized}(0,L(X),n)\Big)$

$\texttt{Proof} =$

$\texttt{Assume}\ \Big([0,L(X)],\alpha\Big), \Big([0,l(X)],\beta : X,$

$\texttt{Assume}\ [1] : \Big(r : \texttt{NaturallyParametrized}(0,L(X),n)\Big),$

$\Big(\tau,[1.1]\Big) := \eth[\mathcal{R}(n)](X)(\alpha,\beta) : \sum \tau : C^\infty \;\&\; \texttt{increasing}\Big([0,L(X)],[0,L(X)]\Big) \,.\, \beta = \alpha \circ \tau,$

$[1.2] := \texttt{DerivativeOfIncreasing}(\tau) : \mathrm{D}\tau > 0,$

$[1.3] := \eth\texttt{NaturallyParametrized}(0,L(X),n)(\beta)(s)[1.1]\texttt{ChainRule}(\alpha,\tau)[1.2]$

$\eth\texttt{NaturallyParametrized}(0,L(X),n) : 1 = \Big\|\mathrm{D}\beta\Big\| = \mathrm{D}\tau\Big\|\mathrm{D}\alpha|_\tau\Big\| = \mathrm{D}\tau,$

$[1.4] := \texttt{AntiderivativeOfUnity}[3] : \tau = \mathrm{id},$

$[1.*] := [1.4][1.1] : \alpha = \beta;$

$\square$

$\texttt{naturalParametrization} :: \prod X \in [\mathcal{R}(n)] \,.\, \texttt{NaturallyParametrized}(0,L(X),n)$

$\texttt{naturalParametrization}\,(X) = X := \texttt{NaturalParametrizationIsUnique}(X)$

1.2 Frenet Theory

$\texttt{FrenetCurve} :: ?[\mathcal{R}(n)]$
$r : \texttt{FrenetCurve} \iff (D^i r)_{i=1}^n : [0, L] \to \texttt{LinearlyIndependent}(\mathbb{R}^n)$

$\texttt{FrenetPropertyInClass} :: \forall r : \texttt{FrenetCurve}(n) \,.\, \forall\Big([a,b],\gamma\Big) \in r \,.$
$\quad .\, (\mathrm{D}^i\gamma)_{i=1}^n : [a,b] \to \texttt{LinearlyIndependent}(\mathbb{R}^n)$
$\texttt{Proof} \;=$
$\Big(\tau,[1]\Big) := \eth\mathcal{R}[n](r)\Big([a,b],\gamma\Big) : \sum \tau \in C^\infty \;\&\; \texttt{Increasing}([a,b],[0,L]) \,.\, \gamma = r\circ\tau,$
$[2] := \texttt{DerivativeOfIncreasing}(\tau)\eth\texttt{StriclyGreater} : \forall t \in (a,b) \,.\, \mathrm{D}\tau|_t \neq 0,$
$\Big(D,[3]\Big) := \eth^{-1}\texttt{LowerTriangularmatrixArange}\ \texttt{HigherOrderChainRule}([1]) :$
$\quad : \exists D : [a,b] \to \texttt{LowerTriangualar}(\mathbb{R},n) \,.\, D(\mathrm{D}^i r|_\tau)_{i=1}^n = (\mathrm{D}\gamma)_{i=1}^n \;\&\; \forall i \in n \,.\, D_{i,i} = \mathrm{D}\tau,$
$[4] := \texttt{NonDegenerateByDeterminantDeterminantOfTheTriangular}[2][3] : D \in \mathrm{GL}(n,\mathbb{R}),$
$[*] := \texttt{LindMap}(D)\Big([4],[3]\Big) : \forall t \in (a,b) \,.\, (\mathrm{D}^i\gamma|_t)_{i=1}^n : \texttt{LinearlyIndependent}(\mathbb{R}^n);$
$\square$

$\texttt{curvature} :: \texttt{FrenetCurve}(n) \to C^\infty([0,L],\mathbb{R}_+)$
$\texttt{curvature}\,(r,s) = k_r(s) := \Big\|\mathrm{D}^2 r|_s\Big\|$

$\texttt{velocity} :: \texttt{FrenetCurve}(n) \to C^\infty([0,L],\mathbb{S}^{n-1}(0,1))$
$\texttt{velocity}\,(r,s) = v_r(s) := \mathrm{D}r$

$\texttt{normal} :: \texttt{FrenetCurve}(n) \to (n-1) \to C^\infty([0,L],\mathbb{S}^{n-1}(0,1))$
$\texttt{normal}\,(i,r,s) = n_{r,i}(s) := \texttt{GrammSmidt}(\mathrm{D}^j r|_s)_{i=1}^n(i+1)$

$\texttt{torsion} :: \texttt{FrenetCurve}(n) \to (n-2) \to C^\infty([0,L],\mathbb{R})$
$\texttt{torsion}\,(i,r,s) = \tau_{r,i}(s) := \langle \mathrm{D}n_{r,i+1}|_s, n_{r,i}(s)\rangle$

$\texttt{frenetFrame} :: \texttt{FrenetCurve}(n) \to n \to C^\infty([0,L],\texttt{Orhtonormal}(\mathbb{R}^n))$
$\texttt{frenetFrame}\,(r) = f_r := v_r \oplus n_r$