Modules.Know

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problem 5.1

function $\texttt{opposite} :: \texttt{Ring} \to \texttt{Ring}$

$\texttt{opposite}\, R = R^{\circ} := \{R, \bullet := \Lambda a, b \in R \,.\, b \cdot_R a\}$

function $\texttt{toOpposite} :: \prod R : \texttt{Ring} \,.\, R \to R^{\circ}$

$\texttt{toOpposite}\, r = \pi r := r$

thm $\forall R : \texttt{Ring}$ **iff** $\pi_R : \texttt{Iso} \,.\, R : \text{Commutative}$

proof $= R : \texttt{Ring} \vdash \pi_R : \text{Isomorphism} \vdash$

$a, b \in R \vdash$

$ab = \pi_R\, ab = \pi_R a \pi_R b = a \bullet b = ba \dashv$

$\to R : \text{Commutative} \dashv$

$\to$ **if** $\pi_R : \text{Isomorphism} \,.\, R : \text{Commutative} \multimap (\Rightarrow)$

$R : \text{Commutative} \vdash$

$a, b \in R \vdash$

$\pi_R a \pi_R b = a \bullet b = ba = ab = \pi_R ab \dashv$

$\to \pi_R : \texttt{Iso} \vdash$

$\to$ **if** $R : \text{Commutative} \,.\, \pi_R : \text{Isomorphism} \;- (\Rightarrow) \dashv$

$\forall R : \texttt{Ring}$ **iff** $\pi_R : \texttt{Iso} \,.\, R : \text{Commutative} \quad \square$

$$A \mapsto A^\top : \text{Isomorphism } \mathcal{M}^n(\mathbb{R})\, \mathcal{M}^n(\mathbb{R})^\circ$$

you can define Left R-Module as R°-Module

problem 5.4

$$\begin{aligned} &\texttt{predicate}\quad \texttt{Simple} :: ?R\text{-Module} \\ &\qquad M : \texttt{Simple} \Leftrightarrow \forall S : \text{Submodule } M \ .\ S = \{0\} \vee S = M \end{aligned}$$

$\texttt{thm}$ Schur's Lemma :: $\forall M, N : \texttt{Simple}\ .\ \forall \phi : \texttt{Homo}\ M\ N\ .\ \phi : \text{Isomorphism } \vee \phi = 0$

If ϕ is not Isomorphism or 0 then either $R \neq \ker\phi \neq \{0\}$, or $R \neq \operatorname{Im}\phi \neq \{0\}$. As $\ker\phi$ is also a submodule, first situation contradicts with a simplicity of M. And the second situation contradicts the simplicity of N as $\operatorname{Im}\phi$ is also a submodule. $\square$

problem 5.5

$\texttt{thm}$ $\texttt{MorphIsomorph}$ :: $\forall R : \text{Commutative}\ .\ \forall M : R\text{-Module}\ .\ \operatorname{Hom}_{R\text{-Mod}}(R, M) \cong M$

we will define isomorphism map explicitly:

$$\begin{aligned} &\texttt{function}\quad f :: \operatorname{Hom}_{R\text{-Mod}}(R, M) \to M \\ &\qquad f\,\phi = \phi(1_R) \end{aligned}$$

$$\begin{aligned} &\texttt{function}\quad g :: M \to \operatorname{Hom}_{R\text{-Mod}}(R, M) \\ &\qquad g\,m\,r = rm \end{aligned}$$

It is easy to see that both maps are homorphisms:

$$f\ r\phi + s\psi = r\phi(1_R) + s\psi(1_R) = r f\,\phi + s f\,\psi$$

$$g\,(xm + yn)\,r = r(xm + yn) = rxm + ryn = xrm + yrn = (xg\,m + yg\,n)\,r$$

Its also easy to prove that maps are inverses

$$f\,(g\,m) = (g\,m)\,1_R = 1_R m = m$$

$$(g\,(f\,\phi))\,r = g\,\phi(1_R)\,r = r\phi(1_R) = \phi(r)$$

So this modules are indeed isomorphic. $\square$

problem 5.6
Firstly, we will show that a group with $\mathbb{Q}$-vector space structure must have all elements all infinite order . Assume that a non-zero element a of finite order $k > 1$ exists. then we know that $ka = 0$, however we also know that $(1/k)(k)a = a$. Which means that
$(1/k)0 \neq 0$ and brings us to a contradiction.

Now assume that $\cdot$ and $*$ are both $\mathbb{Q}$-vector space structures over abelian group G . Let's arbitrary select elements $n \in \mathbb{Z}$ and $g \in G$. Then we can show b multiples of this elements will be equal :

$$n\,((1/n) \cdot g) = n \cdot (1/n) \cdot g = g = n * (1/n) * g = b\,((1/n) * g)$$

As our group has infinite order we can factor b out and get an equality $(1/n) \cdot g = (1/n) * g$ from which we can easily derive that
$\forall q \in \mathbb{Q}\ .\ \forall g \in G\ .\ q \cdot g = q * g$ and hence $\cdot = *$. $\square$

problem 5.9
$\forall R : \text{Commutative}\ .\ \forall M : R\text{-Module}\ .\ \text{End}_{R\text{-}\mathsf{Mod}}\,(M) : R\text{-Algebra}$
Lets repeat definition of R-Algebra.

$$\texttt{data}\quad \text{Algebra} :: \prod R : \text{Commutative}\ .\ \sum S : \texttt{Ring}\ .\ \texttt{Homo}\ R\quad \text{center}\, S$$

So we will define this homomorphism in following way:

$$\begin{aligned}\texttt{function}\quad & f :: R \to \text{center}\,\text{End}_{R\text{-}\mathsf{Mod}}\,(M)\\ & f(r)\,m = rm\end{aligned}$$

It's easy to sea tht this function i indeed a function into center as
$\forall r \in R\ .\ \forall \phi \in \text{End}_{R\text{-}\mathsf{Mod}}\, M\ .\ \forall m \in M$:

$$mf(r)\phi = \phi(rm) = r\phi(m) = m\phi f(r)$$

It's also easy to check that this function is a homomorphism $\forall a, b, c \in R\ .\ \forall m \in M$:

$$f(ab+c)m = (ab+c)m = abm+cm = f(a)f(b)m+f(c)m = (f(a)f(b)+f(c))m$$

So we can claim $(\text{Aut}_{R\text{-}\mathsf{Mod}}\,(M), f) : R$-Algebra.

$\mathcal{M}^n(R)$ is an R-Algebra in natural order as $\mathcal{M}^n(R) \cong \mathrm{End}_{R\text{-}\mathsf{Mod}}\, R^n$. $\square$

problem 5.10

$\forall R : \mathrm{Commutative}\ .\ M : \mathtt{Simple}\, R\ .\ \mathrm{End}_{R\text{-}\mathsf{Mod}}\, M : \mathtt{Division}$ As M is Simple we deduce that every its endomorphism is either a zero map or automorphism. And each automorphism is a bijection, hence has an inverse. So, every non-zero element of $\mathrm{End}_{R\text{-}\mathsf{Mod}}\, M$ has an inverse which makes it into a division algebra. $\square$

problem 5.11

$$\forall R : \mathrm{Commutative}\ .\ M : R\text{-Module}\ .\\ .\ \{f : \mathtt{Homo}\ R[x]\ \mathrm{End}_{\mathsf{Abb}} M\} \cong \mathrm{End}_{R\text{-}\mathsf{Mod}}\, M \quad (1)$$

We will begin with constructing bijection math explicitly.

$$\begin{aligned}
&\texttt{function} \quad f :: \mathtt{Homo}\ R[x]\ \mathrm{End}_{\mathsf{Abb}}\, M \to \mathrm{End}_{R\text{-}\mathsf{Mod}}\, M\\
&\qquad\qquad\quad f\,\phi\,m = \phi(x)m\\
&\texttt{function} \quad g :: \mathrm{End}_{R\text{-}\mathsf{Mod}}\, M \to \mathtt{Homo}\ R[x]\ \mathrm{End}_{\mathsf{Abb}}\, M\\
&\qquad\qquad\quad g\,\phi\,p\,m = \sum_{n=0}^{\infty} p_n \phi^n(m)\\
&m\phi g f = m x \phi g = \phi(m) = m\phi\\
&m p \phi f g = m p \phi(x) g = m = \sum_{n=0}^{\infty} p_n (\phi(x))^n(m) = m p \phi\\
&g = f^{-1} \quad \square
\end{aligned}$$

problem 5.13 ::

$$\forall R : \text{IntegralDomain} . \forall I : \text{Principle } R . \textbf{ if } I \neq (0) . I \cong_{\textsf{R-Mod}} R$$

$\vdash R : \text{IntegralDomain} \multimap (ID)$

$\vdash I : \text{Principle } R \to \exists a \in R . I = (a) \twoheadrightarrow a \in R; I = (a)$

$\vdash I \neq 0 \to a \neq 0$

`function` $f : R \to I$

$f(r) = ra$

$\vdash x, y, z \in R$

$f(xy + z) = (xy + z)a = xya + za = xf(y) + f(z) \dashv$

$$\left.\begin{array}{l} \dashv f : \text{Linear } R\, I \\ \left.\begin{array}{l} I = (a) = f[R] \to f : \text{Surjection} \\ (ID) \wedge a \neq 0 \to f : \text{Injection} \end{array}\right\} \to f : \text{Bijection} \end{array}\right\} \to f : \text{Isomorphism} \to$$

$\to I \cong_{\textsf{R-Mod}} R \dashv_3$

$\dashv_3 \forall R : \text{IntegralDomain} . \forall I : \text{Principle } R . \textbf{ if } I \neq (0) . I \cong_{\textsf{R-Mod}} R \quad \square$

problem 5.14 ::

$$\forall M : R\text{-Module} \; . \; \forall N, P : \text{Submodule } M \; . \; N + P : \text{Submodule } M$$

This is true by distributivity $r(a+b) = ra + rb$

$$\forall M : R\text{-Module} \; . \; \forall N, P : \text{Submodule } M \; . \; N \cap P : \text{Submodule } M$$

$$\begin{array}{l} \vdash r \in R \vdash a \in N \cap P \\ \left.\begin{array}{l} a \in N \to ra \in N \\ a \in P \to ra \in P \end{array}\right\} \to ra \in N \cap P \prec \end{array}$$

$$\frac{N+P}{N} \cong' \frac{N}{N} + \frac{P}{P \cap N} = 0 + \frac{P}{P \cap N} = \frac{P}{P \cap N}$$

? $\square$

problem 5.15

$$I(\frac{R}{J}) \cong \frac{I}{J \cap I} \cong \frac{I+J}{J} \quad \square$$

problem 5.16 ::

$\forall R : \text{Commutative} \; . \; \forall M : R\text{-Module} \; . \; \forall a : \text{Nilpotent } R \; . \; \textbf{iff } M = 0 \; . \; aM = M$

($\Leftarrow$) Simply, $a0 = 0$, hence $aM = M$.
($\Rightarrow$) Assume that $aM = M$.
As a is nilpotent $\exists n \in \mathbb{Z}_+ \; . \; a^n = 0$.
This means that with application of simple induction
$0 = a^n M = a^{n-1} M = \ldots = aM = M$
$\square$

problem 5.17

$$\texttt{function}\quad \text{Rees} :: \prod R : \text{Commutative} \;.\; \text{Ideal}\, R \to R\text{-Algebra}$$

$$\text{Rees}\, I = \Big(\bigoplus_{i=0}^{\infty} I^i, \Lambda\, r \;.\; [r] \oplus \bigoplus_{i=0}^{\infty} 0\Big)$$

$$\texttt{thm}\quad \forall R : \text{Commutative} \;.\; \forall a : \text{NZD} \;.\; \text{Rees}\, R\, (a) \cong_{R\text{-}\mathsf{Alg}} R[x]$$

After problem 5.13 we know that $(a) \cong R$, furthermore applying the same result we can show that $(a)^n \cong (a)^{n-1} \cong \ldots (a) \cong R$ which provides us with the sequence of isomorphisms functions

$$f : \prod n \in \mathbb{Z}_+ \;.\; (a)^n \leftrightarrow R$$

Then we construct a map $\phi : v \mapsto \sum_{i=0}^{\infty} f_n(v_i)x^i$. Inherently, this map is a bijection.

$$\phi(vw) = \phi\left(\bigoplus_{n=0}^{\infty} \sum_{i+j=n} v_i w_j\right) = \sum_{n=0}^{\infty} f_n(\sum_{i+j=n} v_i w_j) = \sum_{n=0}^{\infty} \sum_{i+j=n} f_n(v_i w_j)x^n =$$

$$= \sum_{n=0}^{\infty} \sum_{i+j=n} f_n(bca^n)x^n = \sum_{n=0}^{\infty} \sum_{i+j=n} bcx^n = \sum_{n=0}^{\infty} \sum_{i+j=n} f_i(v_i)f_j(w_j)x^n = \phi(v)\phi(w)$$

$$\phi(v+w) = \phi\left(\bigoplus_{n=0}^{\infty} v_n + w_n\right) = \sum_{n=0}^{\infty} f_n(v_n+w_n)x^n = \sum_{n=0}^{\infty} f_n(v_n)x^n + \sum_{n=0}^{\infty} f_n(w_n)x^n = \phi(v)+\phi(w)$$

$$\phi(rw) = \phi\left(\left([r] \oplus \bigoplus_{i=0}^{\infty} 0\right) w\right) = \phi\left([r] \oplus \bigoplus_{i=0}^{\infty}\right) \phi(w) = r\phi(w)$$

So we can say that ϕ is R-Algebra. As ϕ is also a bijection we can claim it to be an isomorphism. $\square$