Topological Vector Spaces 2

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Contents

1 Abstract Topological Vector Spaces

1.1 Locally Convex Spaces

1.1.1 Intro and Definition

$$\texttt{TopologicalVectorSpace} :: \prod k : \texttt{TopologicalField} \,.\, ?\sum_{V \in k\text{-}\mathsf{VS}} \texttt{Topology}(V)$$

$$(V,\tau) : \texttt{TopologicalVectorSpace} \iff \cdot_V \in \mathsf{TOP}\Big(k \times (V,\tau), (V,\tau)\Big) \ \&\ +_V \in \mathsf{TOP}\Big((V,\tau) \times (V,\tau), (V,\tau)\Big)$$

$k :: \texttt{TopologicalField};$

$$\texttt{VectorTopology} := \Lambda V \in k\text{-}\mathsf{VS} \,.\, \texttt{TopologicalVectorSpace}(V) : \prod_{V \in k\text{-}\mathsf{VS}} V \,.\, ?\texttt{Topology}(V);$$

$$\texttt{categoryOfTopologicalVectorSpaces} :: \texttt{TopologicalField} \to \mathsf{CAT}$$
$$\texttt{categoryOfTopologicalVectorSpaces}\,(k) = k\text{-}\mathsf{TVS} :=$$
$$:= (\texttt{TopologicalVectorSpace}(k), k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$$

$$\texttt{categoryOfTopologicalVectorSpaces} :: \texttt{TopologicalField} \to \mathsf{CAT}$$
$$\texttt{categoryOfHausdorffTopologicalVectorSpaces}\,(k) = k\text{-}\mathsf{HTVS} :=$$
$$:= (\texttt{TopologicalVectorSpace}(k)\ \&\ \texttt{T2}, k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$$

$$\texttt{asTopologicalGroup} :: k\text{-}\mathsf{TVS} \to \mathsf{TGRP}$$
$$\texttt{asTopologicalGroup}\,(V) = V := V$$

$$\texttt{asVectorSpace} :: k\text{-}\mathsf{TVS} \to k\text{-}\mathsf{VS}$$
$$\texttt{asVectorSpace}\,(V) = V := V$$

1.1.2 Absorbent and Balanced Sets

$k :: \texttt{AbsoluteValueField}(\mathbb{R});$

$$\texttt{Balanced} :: \prod_{V:k\text{-TVS}} ??V$$

$$B : \texttt{Balanced} \iff \mathbb{D}_k(0,1)B \subset B$$

$$\texttt{Absorbent} :: \prod k : \texttt{AbsoluteValueField}(\mathbb{R}) \;.\; \prod_{V:k\text{-TVS}} ??V$$

$$A : \texttt{Absorbent} \iff \forall v \in V \;.\; \exists \rho \in \mathbb{R}_{++} \;.\; \forall \alpha \in \mathbb{D}_k(0,\rho) \;.\; \alpha v \in A$$

$\texttt{VectorSubspaceIsBalanced} :: \forall V \in k\text{-TVS} \;.\; \forall U \subset_{k\text{-VS}} V \;.\; \texttt{Balanced}(V,U)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{AbsorbentVectorSubspaceIswhole} :: \forall V \in k\text{-TVS} \;.\; \forall U \subset_{k\text{-VS}} V \;.\; \texttt{Absorbent}(V,U) \Rightarrow V$

$\texttt{Proof} =$

Take $v \in V$.

By definition of absorbent there is $\alpha \in k_*$ such that $\alpha v \in U$.

But then $v = \alpha^{-1}\alpha v \in U$.

So, $U = V$.

□

$\texttt{BalancedSetsAreDedikindComplete} :: \forall V \in k\text{-TVS} \;.\; \texttt{OrderDedekindComplete}\Big(\texttt{Balanced}(V)\Big)$

$\texttt{Proof} =$

Assume β is a set of balanced sets in V.

If $v \in \bigcup \beta$, then there is a $B \in \beta$ such that $v \in B$.

And by definition of balanced $\alpha v \in B \subset \bigcup \beta$ for any $\alpha \in \mathbb{B}_k(0,1)$.

So $\bigcup \beta$ is Balanced.

if $v \in \bigcap \beta$, then $v \in B$ for any $B \in \beta$.

And by definition of balanced $\alpha v \in B \subset \bigcup \beta$ for any $\alpha \in \mathbb{B}_k(0,1)$ and for all $B \in \beta$.

So $\bigcap \beta$ is Balanced.

□

$\texttt{AbsorbentAreClosedUnderUnions} :: \forall V \in k\text{-TVS} \;.\; \forall \alpha :?\texttt{Absorbent}(V) \;.\; \texttt{Absorbent}\left(V, \bigcup \alpha\right)$

$\texttt{Proof} =$

This is obvious.

□

$\texttt{AbsorbentAreClosedUnderFiniteIntersections}$ ::

$$:: \forall V \in k\text{-TVS} \,.\, \forall \alpha : \texttt{Finite}\Big(\texttt{Absorbent}(V)\Big) \,.\, \texttt{Absorbent}\left(V, \bigcap \alpha\right)$$

$\texttt{Proof}$ =

Say $n = |\alpha|$.

if $n = 0$, then $\bigcap \alpha = V$ which is always absorbent.

otherwisr represent $\alpha = \{A_1, \dots, A_n\}$ and assume $v \in V$.

Select a finite sequence $\rho : \{1, \dots, n\} \to \mathbb{R}_{++}$, with ρ_i absorbing v for A_i.

Let $\sigma = \min\{\rho_1, \dots, \rho_n\}$.

Then σ is absorbing for every A_i, so it is absorbing for $\bigcap \alpha$.

$\square$

In case of infinite intersiction the minimum may not exit.

$$\texttt{balancedHull} :: \prod_{V:k\text{-TVS}} 2^V \to \texttt{Balanced}(V)$$

$$\texttt{balancedHull}\,(A) = \operatorname{bal} A := \bigcap \Big\{B : \texttt{Balanced}(V), A \subset B\Big\}$$

$\texttt{BalancedHullProductExpression}$ :: $\forall_{V \in k\text{-TVS}} \forall A \subset V \,.\, \operatorname{bal} A = \mathbb{B}_k(0,1)A$

$\texttt{Proof}$ =

Clearly $\mathbb{B}_k(0,1)A$ is balanced.

Assume that B is a balanced set such that $A \subset B$.

Then $\mathbb{B}_k(0,1)A \subset \mathbb{B}_k(0,1)B \subset B$ as B as balanced.

This proves the result as balanced hull of A may beviewed as the smallest balanced set containing A.

$\square$

$$\texttt{balancedCore} :: \prod_{V:k\text{-TVS}} 2^V \to \texttt{Balanced}(V)$$

$$\texttt{balancedCore}\,(A) = A^{\mathrm{bal}} := \bigcup \Big\{B : \texttt{Balanced}(V), B \subset A\Big\}$$

$\texttt{BalancedCoreAsInterscti on}$ :: $\forall_{V \in k\text{-TVS}} \forall A \subset V \,.\, \operatorname{bal} A = \bigcap_{\alpha \in \mathbb{B}^{\complement}_k(0,1)} \alpha A$

$\texttt{Proof}$ =

Firstly, I show that $B = \bigcap_{\alpha \in \mathbb{B}^{\complement}_k(0,1)} \alpha A$ is balanced.

Assume $v \in B$.

Then, $v \in \alpha A$ for all $\alpha \in \mathbb{B}^{\complement}_k(0,1)$.

Thus $\mathbb{B}_k(0,1)v \subset A$.

By definition A^{bal} as a union this means, that $v \in A^{\mathrm{bal}}$, so $B \subset A^{\mathrm{bal}}$.

Assume now that $v \in A^{\mathrm{bal}}$.

Then $\mathbb{B}_k(0,1)v \subset \mathbb{B}_k(0,1)A^{\mathrm{bal}} \subset A^{\mathrm{bal}} \subset A$ As A^{bal} is a union of subsets.

But this mean that $v \in B$, so $A = B$.

$\square$

$\texttt{ClosedBalancedCoreIsOpen} :: \forall V : k\text{-}\mathsf{TVS} \,.\, \forall F : \texttt{Closed}(V) \,.\, \texttt{Closed}(V, F^{\mathrm{bal}})$

$\texttt{Proof} \;=$

Multiplication by non-zero scalar is a homeomorphism.

So result follows from intersection representation as αF will be closed.

$\square$

$\texttt{LinearMapsBalancedToBalanced} ::$

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall B : \texttt{Balanced}(V) \,.\, \texttt{Balanced}\Big(W, T(B)\Big)$$

$\texttt{Proof} \;=$

Assume $w \in T(B)$ and $\alpha \in \mathbb{D}_k(0,1)$.

Then there is $v \in B$ such that $T(v) = w$.

as B is balanced $\alpha v \in B$.

Thus $\alpha w = \alpha T(v) = T(\alpha v) \in T(B)$.

This proves that $T(B)$ is balanced.

$\square$

$\texttt{LinearSurjectiveMapsAbsorbentToAbsorbent} ::$

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS} \;\&\; \texttt{Surjective}(V, W) \,.\, \forall A : \texttt{Absorbent}(V) \,.\, \texttt{Absorbent}\Big(W, T(A)\Big)$$

$\texttt{Proof} \;=$

Assume $w \in W$.

Then there is $v \in V$ such that $T(v) = w$ as T is surjective.

Then there exists $\rho \in \mathbb{R}_{++}$ such that $\mathbb{D}(0, \rho)v \subset A$ as A is absorbent.

Take $\alpha \in \mathbb{D}(0, \rho)$.

Then $\alpha w = \alpha T(v) = T(\alpha v) \in T(A)$.

This proves that $T(A)$ is absorbent.

$\square$

$\texttt{BalancedPreimageIsBalanced} ::$

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall B : \texttt{Balanced}(W) \,.\, \texttt{Balanced}\Big(V, T^{-1}(B)\Big)$$

$\texttt{Proof} \;=$

Take $v \in T^{-1}(B)$ and $\alpha \in \mathbb{D}_k(0,1)$.

Then $T(v) \in B$, but also $T(\alpha v) = \alpha T(v) \in B$ as B is balanced .

But this means that $\alpha v \in T^{-1}(B)$.

$\square$

$\texttt{BalancedPreimageIsBalanced} ::$

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall A : \texttt{Absorbent}(W) \,.\, \texttt{Absorbent}\Big(V, T^{-1}(A)\Big)$$

$\texttt{Proof} \;=$

Take $v \in V$.

Then there is $\rho \in \mathbb{R}_{++}$ such that $T(\alpha v) = \alpha T(v) \in A$ for any $\alpha \in \mathbb{D}_k(0, \rho)$ as A is absorbent.

But this means that $\alpha v \in T^{-1}(A)$.

$\square$

1.1.3 Topology and Convexity

$$\texttt{Disc} := \Lambda V \in k\text{-}\mathsf{TVS} \,.\, \texttt{Convex} \,\&\, \texttt{Balanced}(V) : \prod_{V \in k\text{-}\mathsf{TVS}} ??V;$$

$\texttt{DiscCharacterization}$ ::

:: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall D \subset V \,.\, \texttt{Disc}(V, D) \iff \forall v, w \in D \,.\, \forall \alpha, \beta \in k \,.\, |\alpha| + |\beta| \le 1 \Rightarrow \alpha v + \beta w \in D$

$\texttt{Proof}$ =

Firstly, assume that D is a Disc.

Take $v, w \in D$ and $\alpha, \beta \in k$ such that $|\alpha| + |\beta| \le 1$.

$\alpha v, \beta w \in D$ as D is balanced.

So if $\alpha = 0$ or $\beta = 0$ then $\alpha v + \beta w = \alpha v \in V$ or $\alpha v + \beta w = \beta w \in V$.

Otherwise, $|\alpha| + |\beta| \ne 0$ and $\dfrac{|\alpha|}{|\alpha| + |\beta|} + \dfrac{|\beta|}{|\alpha| + |\beta|} = 1$.

Also, $\dfrac{|\alpha| + |\beta|}{|\alpha|}\alpha v, \dfrac{|\alpha| + |\beta|}{|\beta|}\beta w \in D$ as $|\alpha| + |\beta| \le 1$ and D is absorbent.

Then $\alpha v + \beta w = \dfrac{|\alpha|}{|\alpha| + |\beta|}\dfrac{|\alpha| + |\beta|}{|\alpha|}\alpha v + \dfrac{|\beta|}{|\alpha| + |\beta|}\dfrac{|\alpha| + |\beta|}{|\beta|}\beta w \in D$ as D is convex.

Now assume that the condition holds.

Then convexity and being balanced is obvious.

□

$\texttt{DiskedHull}$ :: $\forall V \in K\text{-}\mathsf{TVS} \,.\, \forall A \subset V \,.\, \bigcap \Big\{ D : \texttt{Disc}(V), A \subset D \Big\} = \operatorname{conv} \operatorname{bal} A$

$\texttt{Proof}$ =

Firstly we need to show that $\operatorname{conv} \operatorname{bal} A$ is balanced.

Assume $v \in \operatorname{conv} \operatorname{bal} A$ and $\alpha \in \mathbb{D}_k(0, 1)$.

If $\alpha = 0$ then $\alpha v = 0 \in \operatorname{bal} A \subset \operatorname{conv} \operatorname{bal} A$.

Otherwise, if C is convex in V, then $\dfrac{\alpha}{|\alpha|}C$ is also convex.

Also if $\operatorname{bal} A \subset C$ then $\operatorname{bal} A = \dfrac{\alpha}{|\alpha|} \operatorname{bal} A \subset \dfrac{\alpha}{|\alpha|}C$ as $\operatorname{bal} A$ is balanced.

Thus, $\dfrac{\alpha}{|\alpha|}v \in \operatorname{conv} \operatorname{bal} A$.

Also, as it was said $0 \in \operatorname{bal} A \subset \operatorname{conv} \operatorname{bal} A$.

So $\alpha v = \dfrac{|\alpha|}{|\alpha|}\alpha v + \Big(1 - |\alpha|\Big)0 \in \operatorname{conv} \operatorname{bal} A$ as $\operatorname{conv} \operatorname{bal} A$ is convex.

So $\operatorname{conv} \operatorname{bal} A$ is a disk and $B = \bigcap \Big\{ D : \texttt{Disc}(V), A \subset D \Big\} \subset \operatorname{conv} \operatorname{bal} A$.

Now assume that D is a disk such that $A \subset D$.

Then $\operatorname{bal} A \subset D$ as D is balanced.

Furthermore, $\operatorname{conv} \operatorname{bal} A \subset D$ as D is convex.

Thus $\operatorname{conv} \operatorname{bal} A = B$.

□

$\texttt{TVSIsConnected} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \texttt{Connected}(k) \Rightarrow \texttt{Connected}(V)$

$\texttt{Proof} \; =$

Note that $V = \bigcup_{v \in V} kv$.

Each kv is connected as continuous image of connected k.

Then all lines kv intersect at 0, so V is connected.

$\square$

$\texttt{AbsorbentNeighborhoodsOfZero} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall U \in \mathcal{U}_V(0) \,.\, \texttt{Absorbent}(V, U)$

$\texttt{Proof} \; =$

Assume $v \in V$.

Then $\lim_{\alpha \to 0} \alpha v = 0$.

So, there exists $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}_k(0, \rho)v \subset U$.

Then $\mathbb{D}_k\left(0, \frac{\rho}{2}\right) v \subset \mathbb{B}_k(0, \rho)v \subset U$.

Thus, U is absorbent.

$\square$

$\texttt{NeighborhoodsOfZeroScaling} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall U \in \mathcal{U}_V(0) \,.\, \forall \alpha \in k_* \,.\, \alpha U \in \mathcal{U}_V(0)$

$\texttt{Proof} \; =$

$\alpha \cdot \bullet$ is a homeomorphism, so αU i s open .

Obviously, $0 = \alpha 0 \in \alpha U$ as $0 \in U$.

Thus, $U \in \mathcal{U}_V(0)$.

$\square$

$\texttt{EachNeighborhoodsOfZeroContainsBalancedNeighborhoods} ::$

$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall U \in \mathcal{U}_V(0) \,.\, \exists W \in \mathcal{U}_V(0) \,.\, W \subset U \,\&\, \texttt{Balanced}(V, W)$

$\texttt{Proof} \; =$

$(\cdot)^{-1}(U)$ is open in $k \times V$.

So there exist $W \in \mathcal{U}_V(0)$ and $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}_k(0, \rho) \times W \subset (\cdot)^{-1}(U)$ as $0 \in (\cdot)^{-1}(U)$.

This means that $\mathbb{B}_k(0, \rho)W \subset U$.

Also, note that $\mathbb{B}_k(0, \rho)W = \bigcup_{|\alpha| < \rho} \alpha W \in \mathcal{U}_V(0)$.

Assume $v \in \mathbb{B}_k(0, \rho)W$ and $\alpha \in \mathbb{D}_k(0, 1)$.

Then there is $w \in W$ and $\beta \in \mathbb{B}_k(0, \rho)$ such that $v = w\beta$.

But $\alpha\beta$ is also in $\mathbb{B}_k(0, \rho)$ and so $\alpha v = \alpha\beta w \in \mathbb{B}_k(0, \rho)W$.

Thus, $\mathbb{B}_k(0, \rho)W$ is balanced.

$\square$

$\texttt{ClosedAndBlancedNeighborhoodBase} ::$

$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \exists \mathcal{F} : \texttt{Filterbase}(V, \mathcal{U}_V(0)) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{Closed} \,\&\, \texttt{Balanced}(V, F)$

$\texttt{Proof} \; =$

Pretty obvious.

$\square$

$\mathtt{LocallyConvexSpace} :: ?k\text{-TVS}$

$V : \mathtt{LocallyConvexSpace} \iff \exists \mathcal{F} : \mathtt{Filterbase}\Big(V, \mathcal{N}_V(0)\Big) \,.\, \forall F \in \mathcal{F} \,.\, \mathtt{Convex}(F, \mathcal{F})$

$\mathtt{categoryOfLocallyConvexSpaces} :: \mathtt{AbsoluteValueField}(\mathbb{R}) \to \mathsf{CAT}$

$\mathtt{categoryOfLocallyConvexSpaces}\,(k) = k\text{-LCS} :=$

$:= (\mathtt{LocallyConvexSpace}(k), k\text{-VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$

$\mathtt{categoryOfTopologicalVectorSpaces} :: \mathtt{AbsoluteValueField}(\mathbb{R}) \to \mathsf{CAT}$

$\mathtt{categoryOfHausdorffTopologicalVectorSpaces}\,(k) = k\text{-LCHS} :=$

$:= (\mathtt{LocallyConvexSpace}(k)\ \&\ \mathtt{T2}, k\text{-VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$

$\mathtt{NormedSpaceIsLocallyConvex} :: \mathsf{NORM}(k) \subset k\text{-LCHS}$

$\mathtt{Proof}\ =$

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

□

$\mathtt{NormedSpaceIsLocallyConvex} :: \mathsf{NORM}(k) \subset k\text{-LCHS}$

$\mathtt{Proof}\ =$

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

□

$\mathtt{LCSHasDiscBase} :: \forall V \in k\text{-LCS} \,.\, \exists \mathcal{F} : \mathtt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \mathtt{Disc}(V, F)$

$\mathtt{Proof}\ =$

Take $U \in \mathcal{N}_V(0)$.

Then there exists a convex neighborhood $C \in \mathcal{N}_V(0)$ with $C \subset U$ as V is locally convex.

Then there is $B \subset C$ which is a balanced neiborhood which was proved for all topological vector spaces.

Then $\operatorname{conv} B \subset C$ is convex and still an neighborhood of zero.

But convex hull of the balanced set is balanced,hence $\operatorname{conv} B$ is a disc .

□

$\mathtt{LCSHasOpenDiscBase} :: \forall V \in k\text{-LCS} \,.\, \exists \mathcal{F} : \mathtt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \mathtt{Disc}\ \&\ \mathtt{Open}(V, F)$

$\mathtt{Proof}\ =$

. . .

□

$\mathtt{LCSHasClosedDiscBase} :: \forall V \in k\text{-LCS} \,.\, \exists \mathcal{F} : \mathtt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \mathtt{Disc}\ \&\ \mathtt{Closed}(V, F)$

$\mathtt{Proof}\ =$

. . .

□

VectorTopologyByAbsorbentAndBalancedSets ::

$$:: \forall V \in k\text{-}\mathsf{VS} \;.\; \forall \mathcal{F} : \mathtt{GroupFilterbase}(V) \;.\; \forall \aleph : \mathcal{F} \subset \mathtt{Balanced} \;\&\; \mathtt{Absorbent}(V) \;.\; \Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big) \in k\text{-}\mathsf{TVS}$$

Proof =

As $F \in \mathcal{F}$ is balanced, then $F = -F$, so $\langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ is a group topology for $(V, +)$.

Now assume $F \in \mathcal{F}$ and $\alpha \in k_*$.

Then there exists balanced $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ such that $0 \in U$ and $2U \subset U + U \subset F$.

Then there exists balanced $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ such that $0 \in U$ and $2U \subset U + U \subset F$.

This can be generalized to th case when $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ and $2^n U \subset F$.

So, we can take such U that $|\alpha| \le 2^n$ and $\alpha U \subset 2^n U \subset F$ for any $\alpha \in k_*$ and $F \in \mathcal{F}$.

Now consider $\alpha \in k_*$, $v \in V$ and $F \in \mathcal{F}$.

There exists $U \in \mathcal{F}(0)$ such that $U + U + U \subset F$.

As U is absorbent there is $\rho \in (0, 1)$ such that $\mathbb{B}(0, \rho)v \subset U \subset F$.

Thus, $Cell(0, \rho)(v + U) = \mathbb{B}(0, \rho)v + \mathbb{B}(0, \rho)U = U + U \subset F$.

Now, assume $\alpha \neq 0$.

There is $U' \in \mathcal{F}$ such that $\alpha U' \subset U$.

Then there is also a $W \in \mathcal{F}$ such that $W \subset U' \cap U$.

Thus, $\mathbb{B}(\alpha, \rho)(v + W) = \alpha v + \alpha W + \mathbb{B}(0, \rho)(v + W) \subset \alpha v + U + U + U \subset \alpha v + F$.

This proves that scalar multiplication is continuous.

□

LocallyConvexTopologyByDiscFilterbase ::

$$:: \forall V \in k\text{-}\mathsf{VS} \;.\; \forall \mathcal{F} : \mathtt{Filterbase}(V) \;.\; \forall \aleph : \mathcal{F} \subset \mathtt{Disc} \;\&\; \mathtt{Absorbent}(V) \;.$$

$$. \forall \beth : \forall F \in \mathcal{F} \;.\; \exists \alpha \in (0, 1/2) \;.\; \alpha F \in \mathcal{F} \;.\; \Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big) \in k\text{-}\mathsf{LCS}$$

Proof =

We need to show that $\mathcal{F}$ is a group filterbase.

Assume $F \in \mathcal{F}$.

By assumption there are $\alpha \in (0, 1/2)$ such that $\alpha F \in \mathcal{F}$.

Then, as αF is convex and F is absorbent $\alpha F + \alpha F = 2\alpha F \subset F$.

Thus, by previous theorem $\Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big)$ is a topolofical vector space .

And it is locally convex as there is a filterbase consising of disks by construction.

□

1.1.4 Semimetrization

$\texttt{FSeminorm} :: \prod V \in k\text{-VS} \, . \, ?(V \to \mathbb{R}_+)$

$\sigma : \texttt{FSeminorm} \iff \Big(\forall \alpha \in \mathbb{D}_k(0,1) \, . \, \forall v \in V \, . \, \sigma(\alpha v) \le \sigma(v)\Big) \, \&$

$\quad \& \left(\forall v \in V \, . \, \lim_{n\to\infty} \sigma\left(\frac{v}{n}\right)\right) \, \& \, (\forall v, w \in V \, . \, \sigma(v+w) \le \sigma(v) + \sigma(w))$

$\texttt{FNorm} :: \prod V \in k\text{-VS} \, . \, ?\texttt{FSeminorm}(V)$

$\sigma : \texttt{FNorm} \iff \forall v \in V \, . \, \sigma(v) = 0 \iff v = 0$

$\texttt{FSeminormSemimetrization} :: \forall V \in k\text{-VS} \, . \, \forall \sigma : \texttt{FSeminorm} \, . \, \exists \tau : \texttt{VectorTopology}(V) \, . \, \sigma \in C(V, \tau)$

$\texttt{Proof} \; =$

I will show that σ is a value.

Firstly, note that $\sigma(-v) \le \sigma(v)$ and $\sigma(v) \le \sigma(-v)$, so $\sigma(v) = \sigma(-v$.

Also $\sigma(0) = \sigma\left(\frac{0}{n}\right) \to 0$, so $\sigma(0)$.

Other properties of value follows trivially by commutativity of $+_V$.

Now I show that scalar multiplication is continuous in topology defined by semimetric $\rho(v,w) = \sigma(v-w)$.

There are neighborgoods of zero defined by relation $\sigma(v) < \varepsilon$.

By first property of F-seminorm these balls are ballanced .

And by second property of F-seminorm these balls are absorbent.

So produced topology of ρ is a vector space topplogy.

$\square$

$\texttt{FNormSemimetrization} :: \forall V \in k\text{-VS} \, . \, \forall \sigma : \texttt{FNorm} \, . \, \exists \tau : \texttt{VectorTopology}(V) \, . \, \sigma \in C(V, \tau) \, \& \, \texttt{T2}(V, \tau)$

$\texttt{Proof} \; =$

In this case ρ is a metric, so resulting topology musy be Hausdorff.

$\square$

$\texttt{subspaceSeminorm} :: \prod V \in k\text{-VS} \, . \, \prod U \subset_{k\text{-VS}} V \, . \, \texttt{FSeminorm}(V) \to \texttt{FSeminorm}\left(\frac{V}{U}\right)$

$\texttt{subspaceSeminorm}\,(\sigma) = [\sigma]_U := \Lambda[v] \in \frac{V}{U} \, . \, \inf_{u \in U} \sigma(v+u)$

$\texttt{SubspaceSemimetrization} :: \forall V \in k\text{-TVS} \, \& \, \texttt{Semimetrizable} \, . \, \forall U \subset_{k\text{-VS}} V \, . \, \texttt{Semimetrizable}\left(\frac{V}{U}\right)$

$\texttt{Proof} \; =$

$\dots$

$\square$

1.1.5 Completion

$$\texttt{Completion} :: \prod_{V \in k\text{-TVS}} ? \sum_{W \in k\text{-TVS}} \texttt{TopologicalEmbedding}(V, W)$$

$$(W, \iota) : \texttt{Completion} \iff \texttt{Complete}(V) \,\&\, \texttt{Dense}\Big(W, \iota(V)\Big)$$

$\texttt{EveryTVSHasACompletion} :: \forall V \in k\text{-TVS} \,.\, \exists\texttt{Completion}(V)$

$\texttt{Proof} =$

As with topological Groups.

□

$$\texttt{TopologicalVectorSpaceSubset} :: \prod_{V \in k\text{-TVS}} ??V$$

$$U : \texttt{TopologicalVectorSpaceSubset} \iff U \subset_{k\text{-TVS}} V \iff U \subset_{k\text{-VS}} V \,\&\, \texttt{Closed}(V, U)$$

$$\texttt{CompleteteQuotient} :: \forall V \in k\text{-TVS} \,.\, \forall U \subset k\text{-TVS}V \,.\, \texttt{Complete}(V) \Rightarrow \texttt{Complete}\left(\frac{V}{U}\right)$$

$\texttt{Proof} =$

As with topological groups.

□

$\texttt{BalancedHullOfTotallyBoundedIsTotallyBounded} ::$

$:: \forall V \in k\text{-TVS} \,.\, \forall B : \texttt{TotallyBounded}(V) \,.\, \texttt{TotallyBounded}(V, \operatorname{bal} B)$

$\texttt{Proof} =$

Embed B in a completion of $\hat{V}$ of V.

Then $\operatorname{cl} B$ is a compact in $\hat{V}$.

As $\mathbb{D}_k(0,1)$ is comapct in k, then $\mathbb{D}_k(0,1)\operatorname{cl}_{\hat{V}} B$ is compact is continuous image of compact $\mathbb{D}_k(0,1) \times \operatorname{cl}_{\hat{V}} B$.

Then $\operatorname{bal} B = \mathbb{D}_k(0,1)B$ is totally bounded as a subset of compact $\mathbb{D}_k(0,1)\operatorname{cl}_{\hat{V}} B$.

□

$\texttt{BalancedHullOfCompactIsCompacts} ::$

$:: \forall V \in k\text{-TVS} \,.\, \forall K : \texttt{CompactSubset}(V) \,.\, \texttt{CompactSubset}(V, \operatorname{bal} K)$

$\texttt{Proof} =$

$\mathbb{D}_k(0,1)K$ is compact as am image of compact $\mathbb{D}_k(0,1) \times K$.

□

`ConvexHullofTotallyBoundedAsTotallyBounded` ::

:: $\forall V \in k\text{-}\mathsf{LCS} \,.\, \forall B : \texttt{TotallyBounded}(V) \,.\, \texttt{TotallyBounded}(V, \operatorname{conv} B)$

`Proof` =

In order to show that $\operatorname{conv} B$ is totally bounded we need to show that $convB$ can be covered by finite number of translates $(U + v_i)_{i=1}^n$ for any $U \in \mathcal{U}_V(0)$.
Select disc $D \in \mathcal{U}_V(0)$ such that $D + D \subset U$.
This is possible as V is loclly convex.
As K totally bounded there are a finite set of translates such that $K \subset (D + v_i)_{i=1}^n \subset \operatorname{conv}\{v_1, \ldots, v_n\} + D$.
As sum of convex sets is convex $\operatorname{conv} K \subset \operatorname{conv}\{v_1, \ldots, v_n\} + D$.
As $\operatorname{conv}\{v_1, \ldots, v_n\}$ is compact it is possible to select a finite set of m translates u_i of D such that
$\operatorname{conv} K \subset \bigcup_{i=1}^{m}(D + u_i)$.
So $\operatorname{conv} K$ is totally bounded.
□

`ConvexHullofTotallyBoundedAsTotallyBounded` ::

:: $\forall V \in k\text{-}\mathsf{LCS}\texttt{Complete} \,.\, \forall K : \texttt{CompactSubset}(V) \,.\, \texttt{CompactSubset}(V, \operatorname{conv} K)$

`Proof` =

$\operatorname{conv} K$ is closed.
And as it was shown in the previous theorem $\operatorname{conv} K$ is also totally bounded, hence compact.
□

1.1.6 Continuous Decompositions

$\texttt{TopologicalComplement} :: \prod V : k\text{-TVS} \, . \, ?\texttt{LinearComplement}(V)$

$(U, W) : \texttt{TopologicalComplement} \iff V =_{k\text{-TVS}} U \oplus W \iff$

$\iff \texttt{Homeomorphism}\Big(U \oplus W, V, \Lambda(u, w) \in U \oplus W \, . \, u + w\Big)$

$\texttt{TopologicalComplementsByContinuousProjection} ::$

$:: \forall V \in k\text{-TVS} \, . \, \forall U, W : \texttt{LinearComplement}(V) \, . \, U \oplus W =_{k\text{-TVS}} V \iff P_{U,W} \in \mathrm{End}_{\mathsf{TOP}}(V)$

Proof $=$

Define $T : U \oplus W \to V$ by $T(u, w) = u + w$.

$(\Rightarrow)$: Assume that T is a homeomorphism.

There is an expression $P_{U,W} = T^{-1} P_1 I_U$, where $P_1 : U \oplus W \to U$ is a projection,

and $I_U : U \to V$ is a natural embedding.

Thus, $P_{U,W}$ is continuous as composition of continuous functions.

$(\Leftarrow)$:Assume $(\Delta, u_\delta + w_\delta)$ is a net in V converging to 0 .

Then by continuity $0 = P_{U,W}(0) = P_{U,W}(\lim_{\delta \in \Delta} u_\delta + w_\delta) = \lim_{\delta \in \Delta} P_{U,W}(u_\delta + w_\delta) = \lim_{\delta \in \Delta} u_\delta$.

Also $E - P_{U,W} = P_{W,U}$ is continuous.

So by the argument similar to one above $\lim_{\delta \in \Delta} w_\delta = 0$.

Thus, $\lim_{\delta \in \Delta}(u_\delta, w_\delta) = 0$ and T^{-1} is continuous meaning that T is homeomorphism.

$\square$

$\texttt{TopologicalComplementsByIsomorphicQuotient} ::$

$:: \forall V \in k\text{-TVS} \, . \, \forall U, W : \texttt{LinearComplement}(V) \, . \, U \oplus W =_{k\text{-TVS}} V \iff \texttt{Homeomorphism}\left(W, \frac{V}{U}, \pi_{U|W}\right)$

Proof $=$

π_U is a quotient map, and hence continuous.

$(\Rightarrow)$:Assume $(\Delta, [U + w_\delta])$ is a net in $\frac{V}{U}$ converging to zero.

But this means that $\lim_\delta w_\delta = 0$ and $\lim_\delta \pi^{-1}_{U|W}[U + \mathbf{w}_\delta] = \lim_\delta w_\delta = 0$.

So $\pi_{U|W}$ is homeomorphism.

$(\Leftarrow)$: write $P_{U,W} = \pi_U \pi^{-1}_{U|W} I_W$.

This is continuous a as composition of continuous functions.

So by the previous theorem $V = U \oplus_{k\text{-TVS}} W$.

$\square$

$\texttt{ComplementedImpliesClosed} :: \forall V \in k\text{-TVS} \forall (U, W) : \texttt{TopologicalComplement}(V) \, . \, \texttt{Closed}(V, U)$

Proof $=$

By previous theorem $P_{W,U}$ is continuous.

Thus, $U = \ker P_{W,U}$ is closed.

$\square$

$\texttt{MaximalSubspace} :: \prod_{V \in k\text{-VS}} ?\texttt{VectorSubspace}(V)$

$U : \texttt{MaximalSubspace} \iff \forall W \subset_{k\text{-VS}} V \, . \, U \subsetneq W \Rightarrow W = V$

$\texttt{MaximalClosedSubspace} ::$

$:: \forall V \in k\text{-TVS} \, . \, \forall U \subset_{k\text{-VS}} V \, .$

$. \, \texttt{MaximalSubspace} \; \& \; \texttt{Closed}(V, U) \iff \exists f \in \mathsf{TOP}(V, k) \, . \, U = \ker f \; \& \; f \neq 0$

$\texttt{Proof} \; =$

$(\Rightarrow)$: Assume U is closed and maximal subspace in V.

As U is maximal it should have a codimension 1.

So where exists $v \in U^{\complement}$ such that $V = U \oplus \langle v \rangle$.

As U is closed, where exists a balanced open subset $O \in \mathcal{U}_V(0)$ such that $(O + v) \cap U = \emptyset$.

assume $u + \alpha v \in O$ is such that $|\alpha| > 1$ and $u \in U$.

Then, as O is balanced, $\alpha^{-1} u + v \in O$.

But, then $(\alpha^{-1} u + v) - v = \alpha^{-1} u \in (O + v) \cap U$, which is a contradiction.

Thus, $u + \alpha t \in \sigma O$ implies that $|\alpha| < |\sigma|$.

Define $f(u + \alpha v) = \alpha : V \to k$.

Consider a net $v_\delta = u_\delta + \alpha_\delta v$ converging to zero with u_δ in U.

But the previous remark shows that $f(v_\delta) = \alpha_\delta$ converges to zero.

$\texttt{SchroederBernsteinTHM} ::$

$:: \forall V, V' \in k\text{-TVS} \, . \, \forall \aleph : V \cong_{k\text{-TVS}} V \oplus V \, . \, \forall \beth : V' \cong_{k\text{-TVS}} V' \oplus V' \, .$

$. \, \forall \beth : \texttt{TopologicalComplement}(V, V') \, . \, \forall \daleth : \texttt{TopologicalComplement}(V', V') \, . \, V \cong_{k\text{-TVS}} V'$

$\texttt{Proof} \; =$

Write $V \cong V' \oplus U = (V' \oplus V') \oplus U \cong V' \oplus (V' \oplus U) \cong V' \oplus V$.

Symmetricaly, $V' \cong V' \oplus V$.

Thus, $V \cong V \oplus V' \cong V'$.

$\square$

1.1.7 Finite Dimension Conditions

OneDimTVS :: $\forall V \in k\text{-HTVS} \,.\, \dim V = 1 \iff V \cong_{k\text{-TVS}} k$

Proof =

As dimension is invarint for linear isomorphism $(\Leftarrow)$ is obvious .

$(\Rightarrow)$: As $\dim V = 1$ there is a $v \in V$ such that $v \neq 0$ and $V = kv$.

Then the map $T(\alpha v) = \alpha$ is a linear isomorphism .

fix some $\rho \in \mathbb{R}_{++}$.

As V is Hausdorff there must exist an open set $U \in \mathcal{U}_V(0)$ such that $\rho v \notin U$.

Furthermore, U must have a balanced subset $W \in \mathcal{U}_V(0)$.

As W is balanced, $W \subset \mathbb{B}(0, \rho)v$.

So, $\alpha_\delta v \to 0 \iff \alpha_\delta \to 0$.

Thus, T must be a homeomorphism.

$\square$

FinDimIsomorphism ::

$\forall V \in k\text{-HTVS} \,.\, \forall n \in \mathbb{N} \,.\, \dim V = n \iff V \cong_{k\text{-TVS}} (k^n, \| \bullet \|_\infty)$

Proof =

I modify the proof of the previous theorem.

By algebraic there must exist a base $\mathbf{e} = (e_1, \ldots, e_n)$ of V .

fix ρ in $\mathbb{R}_{++}$.

As V is Hausdorff and each $e_i \neq 0$ there $U \subset \mathcal{U}_V(0)$ such $\rho e_i \notin U$ for any $i \in \{1, \ldots, n\}$.

So there exists a blanced subset W of U such that $W \subset \mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \rho) \cdot \mathbf{e}$.

Thus, the mapping $\alpha \cdot \mathbf{e} \mapsto \alpha$ is continuous.

Also, if $U \in \mathcal{U}_V(0)$ the set U must be absorbent,

so there is a sequence $\rho_1, \ldots, \rho_n \in \mathbb{R}_{++}$ such that $\mathbb{D}_k(0, \rho_i)e_i \subset U$.

Let $\sigma = \min(e_1, \ldots, e_n) \in \mathbb{R}_{++}$.

Then $\mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \sigma) \cdot \mathbf{e} \subset U$.

So, the inverse $\alpha \mapsto \alpha \cdot \mathbf{e}$ is also continuous.

$\square$

FDimdSubspaceIsClosed :: $\forall V \in k\text{-HTVS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \dim U < \infty \Rightarrow$ Closed(V, U)

Proof =

U is Hausdorff as a subset of Hausdorff space.

Then U is isomorphic to $\ell^\infty_{k, \dim U}$ which is complete.

So, U can be viewed as an uniform embedding of complete space into V, and hence must be closed.

$\square$

$\texttt{ClosedFDimSum} :: \forall V \in k\text{-TVS} . \forall U \subset_{k\text{-TVS}} V . \forall W \subset_{k\text{-VS}} V . \dim W < \infty \Rightarrow \texttt{Closed}(V, U + W)$

$\texttt{Proof} =$

As U is closed in V the quotient $\frac{V}{U}$ must be Hausdorff.

As $\dim P_U(W) \leq \dim \dim W$ the image $P_U(W)$ is still finite dimensional.

So by previous theorem $P_U(W)$ is closed in $\frac{V}{U}$.

But then the preimage $U + W = P_U^{-1} P_U(W)$ is closed as quotient map P_U is continuous.

□

$\texttt{FiniteDimensionalDomain} :: \forall V, U \in k\text{-HTVS} . \forall T \in k\text{-VS}(V, U) .$

$. \dim V < \infty \Rightarrow T \in k\text{-TVS}(V, U)$

$\texttt{Proof} =$

$\dim T(V) \leq \dim V$, thus $T(V)$ must be finite dimensional.

Thus both V and $T(V)$ are isomorphic to copies of l_k^∞ with coresponding finite dimensions.

And T must be continuous as any mapping between such spaces does.

$\texttt{FiniteDimensionalCodomain} :: \forall V, U \in k\text{-HTVS} . \forall T \in k\text{-TVS} \,\&\, \texttt{Surjective}(V, U) .$

$. \dim U < \infty \Rightarrow \texttt{Open}(V, U, T)$

$\texttt{Proof} =$

By isomorphism theorem $\frac{V}{\ker T} \cong_{k\text{-VS}} T(V) = U$.

So $\dim \frac{V}{\ker T} < \infty$.

Also $\frac{V}{\ker T}$ is Haussdorf as T is continuous .

So by prvious theorem the isomorphism must $\frac{V}{\ker T} \cong_{k\text{-VS}} U$ must be continuous.

So U is also finite dimensional Hausdorff this bijection is homeomorphism and so $\frac{V}{\ker T} \cong_{k\text{-TVS}} U$.

Denote this homeomorphism by S.

Then T factors as $P_{\ker T} S$ and both these maps are open.

□

$\texttt{FDimIffLocallyCompact} :: \forall V \in k\text{-}\mathsf{HTVS} \, . \, \dim V < \infty \iff \texttt{LocallyCompact}(V)$

$\texttt{Proof} \; =$

$(\Rightarrow)$: V is homeomorphic to $l^\infty_{k,\dim V}$ and this space is locally compact..

This can be easily shown by considering a base of closed cubes.

So V is locally compact.

$(\Leftarrow)$: now consider the case when V is locally compact.

Then there exists a compact balansed neighborhood of zero, say K.

Take K to be any another open neighborhood and choose $W \in \mathcal{U}_V(0)$ such balanced set that $W + W \subset U$.

As K is compact, it is totally bounded and hence can be covered by a finite set of translates $K \subset \bigcup_{i=1}^n W + v_i$.

As W is absorbent and balanced there is $\rho \in (1, +\infty)$ such that each $v_i \in \rho U$.

Then $K \subset \bigcup_{i=1}^n W + v_i \subset W + \rho W \subset \rho W + \rho W = \rho(W + W) \subset \rho U$.

Thus, sets of form $2^{-n}K$ form base at zero.

As K is totally bounded it can can be covered by a finite set of translates $K \subset \bigcup_{i=1}^n \frac{1}{2}K + e_i$.

$F = \operatorname{span} e$ is finite-dimensional and hence closed.

$K \subset \bigcup n_{i=1} \frac{1}{2}K + e_i \subset \frac{1}{2}K + F$.

But also $\alpha F = F$ or any non-zero scalar α.

So $\frac{1}{2}K \subset \frac{1}{4}K + F$.

Iterating this relation and substituting we get the result that $K \subset \frac{1}{2^n}K + F$ for any $n \in \mathbb{N}$.

This can be rewriten as $K \subset \bigcap_{n=1}^\infty \frac{1}{2^n}K + F = F$.

But K spans whole V, and so $V = F$ which is finite dimensional.

$\square$

$\texttt{FDimCompactConvexHullIsCompact} ::$

$:: \forall V \in k\text{-}\mathsf{TVS} \, . \, \forall K : \texttt{CompactSubset}(V) \, . \, \dim V < \infty \Rightarrow \texttt{CompactSubset}(V, \operatorname{conv} K)$.

$\texttt{Proof} \; =$

Let $n = \dim V$.

$\operatorname{conv} K$ consists of convex combination of form $\sum_{i=1}^{2n+1} \lambda_i x_i$ where $\lambda \geq 0$ and $\sum_{i=1}^{2n+1} \lambda_i = 1$ and $x_i \in K$.

This condition can be express as $\lambda \in \triangle_{2n+1} \subset k^{2n+1}$.

But $\triangle_{2n+1}$ is also compact, ans so is $\triangle_{2n+1} \times K^{2n+1}$ by Tychonoff's theorem.

So $\operatorname{conv} K = (\cdot)(\triangle_{2n+1} \times K^{2n+1})$ is compact as a continuous image of a compact.

$\square$

1.1.8 Case of Ultravalued Field

k : `UltravaluedField`;

`AbsolutelyKConvex` :: $\prod_{V:k\text{-TVS}} ??V$

A : `AbsolutelyKConvex` $\iff \mathbb{D}_k(0,1)A + \mathbb{D}_k(0,1)A = A$

`KConvex` :: $\prod_{V:k\text{-TVS}} ??V$

V : `KConvex` $\iff \exists v \in V \,.\, \exists A :$ `AbsolutelyKConvex`$(V) \,.\, C = A + v$

`AbsolutelyKConvexByZeroContaintment` :: $\forall V \in k\text{-TVS} \,.\, \forall C :$ `KConvex`$(V) \,.\, 0 \in C \Rightarrow$ `AbsolutelyKConvex`$(V$

`Proof` $=$

C must be a translate of absolutely K-Convex set, so write $C = A + v$.

As A is absolutely K-Convex, then $\alpha(x+v) + \beta(y+v) - v \in C$ for any $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

Take $\alpha = \beta = 1, y = 0$.

Then the expression above reduces to $x + v \in C$.

But this means that $A \subset C$.

On the other hand, $\alpha(x+v) + \beta(y+v) \in A$ for any $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

Taking $\alpha = 1, \beta = -1, y = 0$, produces $x \in A$.

Thus $C \subset A$ and $C = A$ is absolutely K-convex.

$\square$

`TripleCombinationKConvexityCondition` ::

:: $\forall V \in k\text{-TVS} \,.\, \forall C \subset V \,.$

. `KConvex`$(V, C) \iff \forall x, y, z \in C \,.\, \forall \alpha, \beta, \gamma \in \mathbb{D}_k(0,1) \,.\, \alpha + \beta + \gamma = 1 \Rightarrow \alpha x + \beta y + \gamma z \in C$

`Proof` $=$

1 $(\Rightarrow)$: assume that C is K-convex.

1.1 C must be a translate of absolutely K-Convex set, so write $C = A + v$.

1.2 Then $\alpha x + \beta y + \gamma z = \alpha(x - v) + \beta(y - v) + \gamma(z - v) + v \in C$.

2 $(\Leftarrow)$.

2.1 If $C = \emptyset$ then it is trivially K-convex, so assume the contrary.

2.2 Take $v \in V$ and let $A = C - v$.

2.3 A is absolutely K-convex.

2.3.1 Assume $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

2.3.2 $1 - \alpha - \beta \in \mathbb{D}_k(0,1)$.

2.3.2.1 $|1 - \alpha - \beta| \leq \max\Big(1, |\alpha|, |\beta|\Big) = 1$.

2.3.3 Then by the hypothesis $\alpha x + \beta y + (1 - \alpha - \beta)v \in C$.

2.3.4 Translating by $-v$ gives $\alpha(x - v) + \beta(y - v) = \alpha x + \beta y + (1 - \alpha - \beta)v - v \in A$.

$\square$

$\texttt{convexCombinationKConvexityCondition}$ ::

:: $\forall V \in k\text{-}\mathsf{TVS}$. $\forall \aleph$: res char $k \neq 2$. $\forall C \subset V$.

. $\texttt{KConvex}(V, C) \iff \forall x, y \in C$. $\forall \alpha \in \mathbb{D}_k(0,1)$. $\alpha x + (1-\alpha)y + \gamma z \in C$

$\texttt{Proof}$ =

1 ($\Rightarrow$) This direction is obvious.

1.1 The convex combination is a weaker form of triple combination in the previous result.

2 ($\Leftarrow$) .

2.1 If $C = \emptyset$ then it is trivially K-convex, so assume the contrary.

2.2 Take $v \in V$ and let $A = C - v$.

2.3 A is absolutely K-convex.

2.3.1 Assume $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

2.3.2 Rewrite $\alpha(x-v) + \beta(y-v) + v = \frac{1}{2}\big(2\alpha x + (1-2\alpha)v\big) + \frac{1}{2}\big(2\beta y + (1-2\beta)v\big)$.

2.3.3 Both $\frac{1}{2}\big(2\alpha x + (1-2\alpha)v\big)$ and $\frac{1}{2}\big(2betay + (1-2\beta)v\big)$ in C.

2.3.3.1 for ultravalue $|2\alpha| = |\alpha + \alpha| \leq |\alpha| = 1$.

2.3.3.2 Same holds for β.

2.3.3.3 So the convex combination hypothesis can be applied.

2.3.4 clearlly $\frac{1}{2} + \frac{1}{2} = 1$, so $\alpha(x-v) + \beta(y-v) \in A$.

2.3.4.1 $\left|\frac{1}{2}\right| = 1$ as residual characteristic of the field is not 2.

□

$\texttt{AbsolutelyKConvexIntersection}$ :: $\forall V : k\text{-}\mathsf{TVS}$. $\forall I \in \mathsf{SET}$.

. $\forall A : I \to \texttt{AbsolutelyKConvex}(V)$. $\texttt{AbsolutelyKConvex}\left(V, \bigcap_{i \in I} A_i\right)$

$\texttt{Proof}$ =

Obvious.

□

$\texttt{KConvexIntersection} :: \forall V : k\text{-}\mathsf{TVS} \,.\, \forall I \in \mathsf{SET} \,.$

$$. \forall C : I \to \texttt{KConvex}(V) \,.\, \texttt{KConvex}\left(V, \bigcap_{i \in I} C_i\right)$$

Proof $=$

1 Assume that $\bigcap_{i \in I} C_i \neq \emptyset$.

1.1 Otherwise the condition is trivial.

2 Take any $v \in \bigcap_{i \in I} C_i$.

3 Then $\left(\bigcap_{i \in I} C_i\right) - v$ is absolutely K-convex and $\bigcap_{i \in I} C_i$ is K-convex.

3.1 $\left(\bigcap_{i \in I} C_i\right) - v = \bigcap_{i \in I} (C_i - v)$ as translation by v is bijective.

3.2 Then every $C_i - v$ are K-convex sets, which contain zero, so they are absolutely K-Convex.

3.3 So, the intersection $\bigcap_{i \in I} (C_i - v)$ is also absoluterly K-Convex.

□

$\texttt{kConvexHull} :: \prod_{V : k\text{-}\mathsf{TVS}} (?V) \to \texttt{KConvex}(V)$

$\texttt{kConvexHull}\,(X) = K\text{-conv } X := \bigcap \Big\{ C : \texttt{KConvex}(V), X \subset C \Big\}$

$\texttt{KConvexHullByLinearCombinations} ::$

$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall X \subset V \,.$

$$. K\text{-conv } X = \left\{ x_{n+1} + \sum_{i=1}^{n} \alpha_i (x_i - x_{n+1}) \middle| n \in \mathbb{Z}_+, \alpha : \{1, \dots, n\} \to \mathbb{D}_k(0,1), x : \{1, \dots, n+1\} \to X \right\}$$

Proof $=$

1 Let B denote the set defined above.

2 B is K-Convex.

2.1 Note, that x_{n+1} in definition can be fixed.

2.2 Then $B - x_{n+1}$ is obviously absolutely K-convex.

3 $X \subset B$.

3.1 Just take $n = 1, \alpha_1 = 1$.

4 So K-conv $X \subset B$.

5 If C is K-convex, then $B \subset C$.

5.1 Some $x_{n+1} \in X$ must also be contained in C .

5.2 So $C - x_{n+1}$ is absolutely K-convex. .

5.3 So by induction $\sum_{i=1}^{n} \alpha_i (x_i - x_{n+1}) \in C - x_{n+1}$.

6 Thus, $B \subset K$-conv X, and so $B = K$-conv X .

□

$\texttt{kDiskHull} :: \prod_{V:k\text{-TVS}} (?V) \to \texttt{AbsolutelyKConvex}(V)$

$\texttt{kDiscHull}\,(X) = K\text{-disc}\ X := \bigcap \Big\{ C : \texttt{AbsolutelyKConvex}(V), X \subset C \Big\}$

$\texttt{AbsolutelyKConvexInterior} :: \forall V : k\text{-TVS} \,.\, \forall A : \texttt{AbsolutelyKConvex}(V) \,.\, \operatorname{int} A = \emptyset | \operatorname{int} A = A$

$\texttt{Proof} =$

1 assume $\operatorname{int} A \neq \emptyset$.

2 Take $v \in \operatorname{int} A$.

3 Without loss of generality assume $v = 0$.

3.1 Then $A - v$ is an isomorphic absolutely convex set with $0 \in \operatorname{int} A$.

4 Take any $U \in \mathcal{U}_V(0)$ such that $U \subset \operatorname{int} A \subset A$.

5 Now take arbitrary $v \in A$.

6 Then $U + v \subset A$.

6.1 $U + v$ consists of elements $u + v$ with $u \in U \subset A$.

6.2 As $v \in A$ also and A is absolutely K-convex it must be the case that $u + v \in A$.

7 As translation is a homeomorphism $U + v$ is open and so $v \in \operatorname{int} A$.

□

$\texttt{OpenKDiscHull} :: \forall V : k\text{-TVS} \,.\, \forall U : \texttt{Open}(V) \,.\, \texttt{Open}(V, K\text{-disc}\ U)$

$\texttt{Proof} =$

1 K-disc U is absolutely K-convex.

2 $U \subset K$-disc U, so int K-disc $U \neq 0$.

3 But this means that K-disc U is open.

□

$\texttt{LocallyKConvexSpace} :: ?k\text{-TVS}$

$V : \texttt{LocallyKConvexSpace} \iff \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{U}_V(0)\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{KConvex}(V, F)$

$\texttt{NonarchimedeanVSHasZeroTopDim} :: \forall V : \texttt{LocallyKConvexSpace}(k) \ \& \ \texttt{T2} \ . \ \dim_{\mathsf{TOP}} V = 0$

$\texttt{Proof} \ =$

1 V has a base of closed K-discs.

1.1 Consider $U \in \mathcal{U}_V(0)$.

1.2 Then there exists an open K-disic D such that $0 \in D \subset \overline{D} \subset U$.

1.3 Then $\overline{D}$ is a K-disk.

1.3.1 If $u, v \in \overline{D}$ it means that every their open neighborhood meet D .

1.3.2 Assume $\alpha, \beta \in \mathbb{D}_k(0,1)$.

1.3.3 Consider an open neighborhood W of $\alpha u + \beta v$.

1.3.4 Then there is an open neighborhood of zero $O + O \subset W - \alpha u - \beta v$.

1.3.5 Consider the case $\alpha \neq 0 \neq \beta$.

1.3.6 Then there must be some $u' \in D \cap \dfrac{1}{\alpha}(O + \alpha u)$.

1.3.7 Then there is also $v' \in D \cap \dfrac{1}{\beta}(O + \beta v)$.

1.3.8 Then $\alpha u' + \beta v' \in D$ as D is absoluterly K-convex.

1.3.9 Also $\alpha u' + \beta v' \in O + O + \alpha u + \beta v \subset W$.

1.3.10 As W was arbitrary this means that $\alpha u + \beta v \in \overline{D}$.

1.4 $\overline{D} \subset U$.

1.4.1 This is true as V is Hausdorff, and Hence regular.

2 But then every K-disc in this base is clopen.

2.1 To be in base every K-disc D should contain an element of $U_V(0)$.

2.2 Hence D has non-empty interior.

2.3 But This means that D is open.

3 Thus $\dim_{\mathsf{TOP}} V = 0$.

$\square$

$\texttt{RelativelyKConvex} :: \prod_{V_k\text{-}\mathsf{TVS}} \prod_{A \subset V} ??A$

$R : \texttt{RelativelyKConvex} \iff \exists C : \texttt{KConvex}(K) \ . \ R = C \cap A$

$\texttt{KConvexFilterbase} :: \prod V : k\text{-}\mathsf{TVS} \ . \prod_{A \subset V} ?\texttt{Filterbase}(A)$

$\mathcal{F} : \texttt{KConvexFilterbase} \iff \forall F \in \mathcal{F} \ . \ \texttt{RelativelyKConvex}(V, A, F)$

$\texttt{CCompact} :: \prod_{V_k\text{-}\mathsf{TVS}} ??V$

$K : \texttt{CCompact} \iff \forall \mathcal{F} : \texttt{KConvexFilterbase}(V, K) \ . \ \exists \texttt{AdherencePoint}\Big(V, \mathcal{F}\Big)$

$|\cdot| \neq \Lambda \alpha \in k \ . \ [\alpha \neq 0]$

$\texttt{EveryCompactIsCCompact} :: \forall V : k\text{-TVS} . \forall K : \texttt{Compact}(V, K) . \texttt{CCompact}(V, K)$

$\texttt{Proof} =$

1 Assume $\mathcal{F}$ is a K-Convex filterbase on K.

2 Then associated ultrafilter must have a limit.

3 This limit is an adherence point of $\mathcal{F}$.

$\square$

$\texttt{ClosedSubsetOfCCompact} :: \forall V : k\text{-HTVS} . \forall K : \texttt{CCompact}(V) . \forall L : \texttt{Closed}(K) \,\&\, \texttt{KConvex}(V) . \texttt{CCompact}(V,$

$\texttt{Proof} =$

1 Assume $\mathcal{F}$ is a K-Convex filterbase on L.

2 Then the $\mathcal{F}$ is also a K-Convex filterbase for K.

3 Then, there is an adherence poiint $p \in K$ fo $\mathcal{F}'$.

4 p is also an adherence point for $\mathcal{F}$.

4.1 Take any $U \in \mathcal{U}_V(p)$.

4.2 Then $F \cap K \cap U \neq \emptyset$ for any $F \in \mathcal{F}$.

4.3 Bat all these $F \subset L$.

4.4 Thus $p \in \operatorname*{cl}_K L = L$.

$\square$

$\texttt{MaximalConvexFilterbase} ::$

$:: \forall V : \texttt{LocallyKConvexSpace}(k) . \forall C : \texttt{KConvex}(V) . \forall \mathcal{F} \in \max \texttt{KConvexFilterbase}(V, C) .$

$. \forall p \in \mathcal{C} . \texttt{AherencePoint}(C, \mathcal{F}, p) \iff \lim \mathcal{F} = p$

$\texttt{Proof} =$

1 $(\Rightarrow)$: Assume p is an adherence point for $\mathcal{F}$ in $\mathcal{C}$.

1.1 Then $\forall F \in \ . \forall U \in \mathcal{U}_V(p) . U \cap F \neq \emptyset$.

1.2 Assume that $U \in \mathcal{U}_C(p)$.

1.3 Then there exist a K-convex D and open $W \in \mathcal{U}_C(p)$ such that $W \subset D \subset V$.

1.4 Then $\forall F \in \mathcal{F} . D \cap F \neq \emptyset$.

1.4.1 $\forall F \in \mathcal{F} . W \cap F \neq \emptyset$.

1.4.2 $W \subset D$.

1.5 As $\mathcal{F}$ is maximal $D \in \mathcal{F}$.

1.6 Thus, $p = \lim \mathcal{F}$.

2 $(\Leftarrow)$: Now Assume $p = \lim \mathcal{F}$.

2.1 Then $\forall U \in \mathcal{U}_C(p) . \exists F \in \mathcal{F} . F \subset U$.

2.2 Take arbitrary $U \in \mathcal{U}_C(p)$ and $F \in \mathcal{F}$.

2.3 Then by (2.1) there exits $G \in \mathcal{F}$ such that $G \subset Y$.

2.4 As $\mathcal{F}$ is a filterbase $G \cap F \neq \emptyset$.

2.5 Thus $F \cap U \neq \emptyset$.

2.6 This proves that p is and adherence point for $\mathcal{F}$.

$\square$

$\texttt{KConvexAndCcompactIsClosed} ::$

$:: \forall V : \texttt{LocallyKConvexSpace}(k) \,.\, \forall K : \texttt{CCompact} \,\&\, \texttt{KConvex}(V) \,.\, \texttt{Closed}(V, K)$

$\texttt{Proof} \ =$

1 Assume p is a Limit point for K.

2 Then there exists an filter $\mathcal{F}$ in K such that $p = \lim \mathcal{F}$.

2.1 Take $\mathcal{N}_V(p) \cap K$ for example.

3 Then p is an adherence point of $\mathcal{F}$.

4 construct a K-convex filterbase $\mathcal{C}$ from $\mathcal{F}$.

4.1 For example, use the fact that V is locally K-convex.

4.2 Let C be the intersections of K and K-convex neighborhoods of p.

5 Then p is still a limit point of $\mathcal{C}$ in V.

6 There also must exist an adherence point of $\mathcal{C}$ in K, say q.

7 But as V is Hausdorff and $\mathcal{C}$ has a limit it must be the case $q = p$.

8 Thus K has all its limit points and must be closed.

$\square$

$\texttt{CCompactProduct} :: \forall I \in \texttt{Set} \,.\, \forall V : I \to k\text{-}\mathsf{TVS} \,.\, \forall C : \prod_{i\in I} \texttt{CCompact}(V_i) \,.\, \texttt{CCompact}\left(\prod_{i\in I} V_i, \prod_{i\in I} C_i\right)$

$\texttt{Proof} \ =$

Same proof as Tychonoff's theorem's proof with filters, but with k-convex sets.

$\square$

$\texttt{CCompactCombination} :: \forall V : \texttt{LocallyKConvexSpace}k \,.\, \forall n \in \mathbb{Z}_+ \,.\, \forall D : \{1, \ldots, n\} \to \texttt{AbsolutelyKConvex} \,\&\, \texttt{C}$

$\texttt{Proof} \ =$

1 I will give a proof by induction.

2 K-conv $\bigcup_{i=1}^{n} D_i = \emptyset$ in case $n = 0$ and is trivially c-compact.

3 K-conv $\bigcup_{i=1}^{n+1} D_i = K$-conv $\left(D_{n+1} + \bigcup_{i=1}^{n} D_i\right)$ by the result expressing K-convex hulls by linear combinations.

4 So for the induction step we need to prove case of two c-compacts D_1 and D_2.

5 assume $\mathcal{F}$ is a closed k-convex filterbase on K-conv $D_1 \cup D_2$.

6 Let $\mathcal{F}' = \left\{\{(x, y) \in D_1 \times D_2 : \exists \alpha, \beta \in \mathbb{D}_k(0, 1) \,.\, \alpha x + \beta y \in F\} \middle| F \in \mathcal{F}\right\}$.

7 Then $\mathcal{F}'$ is a k-convex fiterbase on $D_1 \times D_2$.

8 $D_1 \times D_2$ is c-compact.

9 So there is an adherence point (x, y) of $\mathcal{F}'$.

10 Let $C = K$-disc$\{x, y\}$.

11 Then C is c-compact K-disc.

12 Then $\overline{F} \cap C \neq \emptyset$ fo all $F \in \mathcal{F}$.

13 So $\mathcal{F}'' = \{\overline{F} \cap C | F \in \mathcal{F}\}$ is a filterbas on C.

14 So there exists and adherence point P of $\mathcal{F}''$.

15 But p is als an adherence point of $\mathcal{F}$ then.

$\square$

$\texttt{CCompactIffSphericallyComplete} :: \texttt{CCompact}(k) \iff \texttt{SphericallyComplete}(k)$

$\texttt{Proof} =$

1 $(\Rightarrow)$: Assume that k is c-compact.

1.1 Let $B : \mathbb{N} \to 2^k$ be a deacrising sequence of closed balls.

1.2 Then $\mathcal{B} = \{B_i | i \in \mathbb{N}\}$ is a k-convex filter.

1.3 So there must exist and adherence point β of $\mathcal{B}$.

1.4 Then $\beta \in B_n$ for every $n \in \mathbb{N}$.

1.4.1 $B_n \cap U \neq \emptyset$for every $U \in \mathcal{U}_k(\beta)$.

1.4.2 This means that $\beta \in \overline{B}_n$.

1.4.3 But $B_n = \overline{B}_n$ as B_n is closed.

1.5 Which can be rendered as $\beta \in \bigcap_{n=1}^{\infty} B$.

2 $(\Rightarrow)$: Assume that k is sphercally complete.

2.1 we claim that every k-convex set in k is either $\emptyset$ or a ball.

2.1.1 Assume A is an absolutely k-convex set such that $\emptyset \neq A \neq k$.

2.1.2 Take $\omega \in A^\complement$.

2.1.3 Then $\omega \neq 0$.

2.1.4 Then every ω' such that $|\omega| \leq |\omega'|$ is not in A.

2.1.4.1 Assume there is some $\omega' \in A$ such that $|\omega| \leq |\omega'|$.

2.1.4.2 Then $\left|\frac{\omega}{\omega'}\right| \leq 1$.

2.1.4.3 Thus, as A is a k-disc, $\omega = \frac{\omega}{\omega'}\omega' \in A$.

2.1.5 So the set $R = \left\{|\omega| \middle| \omega \in A^\complement\right\}$ is bounded from above.

2.1.6 Let $r = \sup R$.

2.1.7 Take $\alpha \in A$ and $\beta \in k$ with $|\beta| \leq |\alpha|$.

2.1.8 Then $\beta \in A$.

2.1.9 so A is a ball of radius r open or closed depending on iclusion of r to R.

2.2 Also note, that in non-archimedian space any balls are either disjoin or contained in one or another.

2.3 So any k-convex filterbase $\mathcal{F}$ in k can be represented as a decreasing sequence of balls, closed or open.

2.4 Construct sequence of closed balls $\mathcal{B}$ by taking closures.

2.4.1 radii of balls will form a set R bounded from below by 0.

2.4.2 let $\delta = \inf R$.

2.4.3 Then there exists a decreasing sequence of balls B with respective radi r such that $\lim_{n\to\infty} r_n = \delta$.

2.4.3.1 This is true as all elements in the filterbase $\mathcal{F}$ must have non-empty intersecion.

2.5 Then there exists $\beta \in \bigcap \mathcal{B}$.

2.4.4 Take $\mathcal{B} = \{B_n | n \in \mathbb{N}\}$.

2.6 β is an adherence point of $\mathcal{F}$.

2.6.1 There is some $B \in \mathcal{B}$ such $\beta \in B \subset \overline{F}$ for very element $F \in \mathcal{F}$.

2.6.2 Then $F \cap U \neq \emptyset$ for every $U \in \mathcal{U}_k(\beta)$.

$\square$

1.1.9 Some Interesting Examples

k :: AbsoluteValueField

NonLocallyConvexSpace :: $\exists V : k\text{-TVS} . \neg$LocallyConvexSpace$(V)$

Proof =

1 Let $V = L^p(\mathbb{R}, \lambda)$ for $p \in (0, 1)$.

2 Its topology can be metrized by the metroc $\rho(f, g) = \displaystyle\int |f - g|^p$.

2.1 we use inequelity of form $\left(\sum_{i=1}^n \alpha_i\right)^p \le \sum_{i=1}^n \alpha_i$ for $\alpha_i > 0$.

3 on the other hand $\operatorname{conv} \mathbb{B}_V(0, \sigma) \subset \mathbb{B}_V(0, 2^{p-1}\sigma)$.

3.1 Assume $f \in \mathbb{B}_V(0, \sigma)$.

3.2 Define $F(t) = \displaystyle\int_{-\infty}^t |f|^p$.

3.3 Then F is a continuos function on $[-\infty, +\infty]$ such that $F(-\infty) = 0$ and $F(+\infty) = \rho(0, f)$.

3.4 By intermidieat value theorem there exists $t \in \mathbb{R}$ such that $F(t) = \dfrac{\rho(0, f)}{2}$.

3.5 Let $g(x) = f(x)\delta_x(-\infty, t), h(x) = f(x)\delta_x(t, +\infty)$.

3.6 Then $\rho(g, 0) \le \dfrac{\sigma}{2}$ and $\rho(h, 0) \le \dfrac{\sigma}{2}$ and $f = h + g = \dfrac{2}{2}f + \dfrac{2}{2}g$.

3.7 But $2g, 2h \in \mathbb{B}_V(0, 2^{2p-1}\sigma)$, so $f \in \operatorname{conv} \mathbb{B}_V(0, 2^{2p-1}\sigma)$.

4 By iterating one gets $\operatorname{conv} \mathbb{B}_V(0, \sigma) = V$.

5 So there are no non-trivial convex neighborhoods of 0.

□

NonCompactConvexHullOfTheCompact :: $\exists V : k\text{-TVS} . \exists K :$ CompactSubset$(V) . \neg$CompactSubset$(V, \operatorname{conv} K)$

Proof =

1 Let $V = \ell^1$.

2 Let $K = \left\{0, \delta_1^\bullet, \ldots, \dfrac{1}{n}\delta_n^\bullet, \ldots\right\}$.

3 Define $\xi_n = \dfrac{1}{\sum_{i=1}^n 2^{-i}} \displaystyle\sum_{t=1}^n \frac{2^{-t}}{t}\delta_t^\bullet \in \operatorname{conv} K$.

4 Then $\zeta = \displaystyle\lim_{n\to\infty} \xi_n = \sum_{t=1}^\infty \frac{2^{-t}}{t}\delta_t^\bullet$.

5 But then $\zeta_i \ne 0$ for all $i \in \mathbb{N}$, but this means that $\zeta \notin \operatorname{conv} K$, so K is not compact.

□

NoncomplimentedClosedSubpaceExist :: $\exists V : k\text{-}\mathsf{TVS}\ .\ \exists U \subset_{k\text{-}\mathsf{TVS}} V\ .\ \neg$TopologicalComplement$(V, U)$

Proof =

1 Let $V = \ell^\infty$.

2 Let $U = c_0$.

...

$\square$

k :: UltravaluedField

PathologicalConvexSet ::

:: res char$(k) = 2 \Rightarrow \exists V : k\text{-}\mathsf{TVS}\ .\ \exists A : \neg$KConvex$(V)\ .\ \forall a, b \in A\ .\ \forall \lambda \in \mathbb{D}_k(0,1)\ .\ \lambda a + (1-\lambda)b \in A$

Proof =

1 Let $V = k^3$ and let $A = \Big\{a \in \mathbb{D}_k(0,1) : \exists i \in \{1,2,3\}\ .\ a_i \in \mathbb{B}_k(0,1)\Big\}$.

2 A has desired property for convex combinations of two elements.

2.1 Assume $\lambda \in \mathbb{D}_k(0,1)$ and $a, b \in A$.

2.2 Note, either $|\lambda| = 1$ or $|1 - \lambda| = 1$.

2.2.1 $1 = [1] = [1 - \lambda + \lambda] = [1 - \lambda] + [\lambda]$ in a residue1 field $\mathbb{F}_2$.

2.3 There exists some $i, j \in \{1,2,3\}$ such that $|a_i| < 1$ and $|b_j| < 1$.

2.4 So $|\lambda a_i| = |\lambda||a_i| < 1$ and $|(1-\lambda)b_j| = |1-\lambda||b_j| < 1$.

2.5 so either $|\lambda a_i + (1-\lambda)b_i| < 1$ or $|\lambda a_j + (1-\lambda)b_j| < 1$.

3 A is not K-convex.

3.1 $(-1,1,1) \notin A$.

3.1.1 $|-1| = |1| = 1$.

3.2 on the othe hand $(-1,1,1) = -1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3 \in K\text{-conv}\ A$.

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