Convex Analysis

Uncultured Tramp

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Contents

1 Convex Functions

1.1 Subject

epigraph $:: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \left(D \to \overset{\infty}{\mathbb{R}} \right) \to ?(V \oplus \mathbb{R})$

epigraph $(f) = \operatorname{epi} f := \{(x, \phi) | x \in D, \phi \in \mathbb{R}, \phi \geq f(x)\}$

Convex $:: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . ?\left(D \to \overset{\infty}{\mathbb{R}} \right)$

$f : \text{Convex} \iff \text{Convex}(V \oplus \mathbb{R}, \operatorname{epi} f)$

effectiveDomain $:: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \text{Convex}(V, D) \to ?D$

effectiveDomain $(f) = \operatorname{dom} f := \pi_1 \operatorname{epi} f$

DomainIsConvex $:: \forall V \in \mathbb{R}\text{-VS} . \forall D \subset V . \forall f : \text{Convex}(V, D) . \text{Convex}(V, \operatorname{dom} f)$

Proof $=$

As a linear image of convex set.

□

ProperConvexFunction $:: \prod V : \mathbb{R}\text{-VS} . ?\text{Convex}(V, V) .$

$f : \text{ProperConvexFunction} \iff \forall x \in V . f(x) > -\infty \ \& \ \exists x \in V . f(x) < +\infty$

InterpolationProperty $::$

$\quad :: \forall V : \mathbb{R}\text{-VS} . \forall C : \text{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$

$\quad . \text{Convex}(V, C, f) \iff \forall x, y \in C . \forall \lambda \in [0, 1] .$

$\quad . f\left(\lambda x + (1 - \lambda)y \right) \leq \lambda f(x) + (1 - \lambda)f(y)$

Proof $=$

$(\Rightarrow)$: assume that f is convex.

Then f has convex epigraph.

Take arbitrary $x, y \in C$ and $\lambda \in [0, 1]$.

If f takes value $+\infty$ either in x or y, then the inequality follows, so assume the contrary.

Then $\left(x, f(x) \right), \left(y, f(y) \right)$ trivially belong to the epigraph,

so by convexity $\left(\lambda x + (1 - \lambda)y, \lambda f(x) + (1 - \lambda)f(y) \right)$ is also in epigraph.

The definition of epigraph produces the inequality $f\left(\lambda x + (1 - \lambda)y \right) \leq \lambda f(x) + (1 - \lambda)f(y)$.

$(\Leftarrow)$: now assume that inequality always hold.

Assume $(x, \phi), (y, \psi)$ belong to the epigraph and $\lambda \in [0, 1]$.

Then $\lambda \phi + (1 - \lambda)\psi \geq \lambda f(x) + (1 - \lambda)f(y) \geq f\left(\lambda x + (1 - \lambda)y \right)$.

So $\lambda(x, \phi) + (1 - \lambda)(y, \psi)$ belong to the epigraph.

Thus, epigraph is convex and f is convex.

`JensensIneq` ::

    :: $\forall V : \mathbb{R}\text{-}\mathsf{VS}$. $\forall C : \mathtt{Convex}(V)$. $\forall f : C \to (-\infty, +\infty]$.

    . $\forall n \in \mathbb{N}$. $\forall \lambda \in \mathbb{R}_+^n$. $\forall \aleph : \sum_{k=1}^n \lambda_k = 1$. $\forall v \in V^n$. $f\left(\sum_{k=1}^n \lambda_k v_k\right) \leq \sum_{k=1}^n \lambda_k f(v_k)$

`Proof` $=$

Iterate the interpolation property.

□

`SecondDerivativeConvexityTest` :: $\forall I : \mathtt{OpenInterval}(\mathbb{R})$. $\forall f \in C^2(I)$.

    . $\mathtt{Convex}(\mathbb{R}, I, f) \iff f'' \geq 0$

`Proof` $=$

$(\Rightarrow)$: assume there is a $t \in I$ such that $f''(t) < 0$.

As f'' must be continous there is whole open interval (a, b) such that $f''(j) < 0$ for all $j \in (a, b)$.

Take some $x, y \in (a, b)$ with $x < y$ and define $z = \lambda x + (1 - \lambda)y$ for siome $\lambda \in (0, 1)$.

Then $f(z) - f(x) = \int_x^z f'(t)\, dt > f'(z)(z - x)$ and $f(y) - f(z) = \int_z^y f'(t)\, dt < f'(z)(y - z)$.

Then from definiton of z we get $f(z) > f(x) - (1 - \lambda)f'(z)(y - x)$ and $f(z) > f(y) + \lambda f'(z)(y - x)$.

By adding two inequalities with multipliers λ and $(1 - \lambda)$ one gets $f\left(\lambda x + (1 - \lambda)y\right) > \lambda f(x) + (1 - \lambda)f(y)$.

But this contradicts a convexity. .

$(\Rightarrow)$: use same inequalities but with different sign to prove the convexity.

□

`ExponentIsConvexity` :: $\forall \alpha \in \mathbb{R}$. $\mathtt{Convex}\left(\mathbb{R}, \mathbb{R}, \Lambda t \in \mathbb{R} . e^{\alpha t}\right)$

`Proof` $=$

write $f(t) = e^{\alpha t}$.

Then $f''(t) = \alpha^2 e^{\alpha t} \geq 0$.

□

`MonomialConvexity1` :: $\forall p \in [1, +\infty)$. $\mathtt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . t^p\right)$

`Proof` $=$

Write $f(t) = t^p$.

Then $f''(t) = p(p - 1)t^{p-2} \geq 0$ for $t > 0$.

□

`MonomialConvexity2` :: $\forall p \in [0, 1)$. $\mathtt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . - t^p\right)$

`Proof` $=$

Write $f(t) = t^p$.

Then $f''(t) = p(1 - p)t^{p-2} \geq 0$ for $t > 0$.

□

`MonomialConvexity3` :: $\forall p \in (-\infty, 0]$. `Convex`$\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . t^p\right)$

`Proof` $=$
write $f(t) = t^p$.
Then $f''(t) = p(p-1)t^{p-2} \geq 0$ for $t > 0$.
$\square$

`GeneralizedArcsinDerivativeIsConvex` :: $\forall \alpha \in \mathbb{R}_{++}$. `Convex`$\left(\mathbb{R}, (-\alpha, \alpha), \Lambda t \in \mathbb{R} . \dfrac{1}{\sqrt{\alpha^2 - t^2}}\right)$

`Proof` $=$

Write $f(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}}$.

Then $f'(t) = \dfrac{t}{\sqrt{\alpha^2 - t^2}^3}$.

And $f''(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}^3} + \dfrac{3t^2}{\sqrt{\alpha^2 - t^2}^5} > 0$ for $t \in (-\alpha, \alpha)$.

$\square$

`NegativeLogIsConvex` :: `Convex`$\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . -\ln(t)\right)$

`Proof` $=$
Write $f(t) = -\ln(t)$.

Then $f''(t) = \dfrac{1}{t^2} > 0$ for $t > 0$.

$\square$

`NegativeEntropyIsConvex` :: `Convex`$\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . t\ln(t)\right)$

`Proof` $=$
Write $f(t) = t\ln(t)$.
Then $f'(t) = \ln(t) + 1$.

And $f''(t) = \dfrac{1}{t} > 0$ for $t > 0$.

$\square$

`Concave` :: $\prod V : \mathbb{R}\text{-VS}$. $\prod D \subset V$. $?\left(D \xrightarrow{\infty} \mathbb{R}\right)$

$f :$ `Concave` $\iff$ `Convex`$(V, D, -f)$

`SecondDerivativeConvexityTest2` :: $\forall V :$ `EucledeanSpace` . $\forall U :$ `Open` $\&$ `Convex`(V) . $\forall f \in C^2(U)$.
 . `Convex`$(\mathbb{R}, U, f) \iff \mathbf{D}^2 f \geq 0$

`Proof` $=$
For $x \in U$ and $v \in V \setminus \{0\}$ define $\phi_{x,v}(t) = f(x + tv)$ with a domain $I_{x,v} = \{t \in \mathbb{R} | x + tv \in C\}$.
Then f is convex iff every $\phi_{x,v}$ does.
But $\phi''_{x,v}(t) = \langle v, \mathbf{D}^2 f|_y v \rangle$, where $y = x + tv$.
So f is convex iff $\mathbf{D}^2 f$ is positive-semidefinite.
$\square$

`GeometricMeanIsConcave` ::

$$:: \forall V : \text{EucledeanSpace} . \text{Concave}\left(V, V_{++}, \Lambda x \in V . \prod_{k=1}^{n} \sqrt[n]{x_k}\right) \quad \text{where} \quad n = \dim V$$

`Proof` =

write $f(x) = \prod_{k=1}^{n} \sqrt[n]{x_k}$.

Then $\nabla f|_x = \left(\dfrac{1}{n \sqrt[n]{x_i}^{n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}\right)_{i=1}^{n}$.

And $\mathbf{D}_{i,j}^2 f|_x = \dfrac{1}{n^2 \sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k}$ when $i \neq j$, and $\mathbf{D}_{i,i}^2 f|_x = -\dfrac{n-1}{n^2 \sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}$.

So, $\mathbf{D}^2 f|_x(v, v) = -\dfrac{n-1}{n^2} \sum_{i=1}^{n} \dfrac{v_i^2}{\sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j} + \dfrac{1}{n^2} \sum_{i \neq j}^{n} \dfrac{v_i v_j}{\sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k} =$

$$= f(x) \left(-\dfrac{n-1}{n^2} \sum_{i=1}^{n} \dfrac{v_i^2}{x_i^2} + \dfrac{1}{n^2} \sum_{i \neq j}^{n} \dfrac{v_i v_j}{x_i x_j}\right) = -\dfrac{f(x)}{n^2} \left(n \sum_{i=1}^{n} \dfrac{v_i^2}{x_i^2} - \left(\sum_{i=1}^{n} \dfrac{v_i}{x_i}\right)^2\right) \leq 0 .$$

This follows from obvious matching schema.

□

`NormsAreConvex` :: $\forall V : \mathbb{R}\text{-VS} . \forall \eta : \text{Norm}(V)\text{Convex}(V, V, \eta)$

`Proof` =

Write $\eta(v) = \|v\|$.

Just use triangle inequality $\left\|\lambda x + (1 - \lambda)y\right\| \leq \|\lambda x\| + \left\|(1 - \lambda)y\right\| = \lambda\|x\| + (1 - \lambda)\|y\|$.

□

`convexIndicator` :: $\forall V : \mathbb{R}\text{-VS} . \text{Convex}(V) \to \text{Convex}(V, V)$

`convexIndicator` $(C) = \Lambda x \in V . \chi(x|C) := \Lambda x \in V . \infty\big[x \in C^{\complement}\big]$

`supportFunction` :: $\forall V : \mathbb{R}\text{-HIL} . \text{Convex}(V) \to \text{Convex}(V, V)$

`supportFunction` $(C) = \Lambda x \in V . \chi^*(x|C) := \sup_{y \in C}\langle x, y\rangle$

`gauge` :: $\forall V : \mathbb{R}\text{-VS} . \text{Convex}(V) \to \text{Convex}(V, V)$

`gauge` $(C) = \Lambda x \in V . \gamma(x|C) := \Lambda x \in V . \inf\left\{\lambda \in \mathbb{R}_{++}\Big| x \in \lambda C\right\}$

`ConvexFunctionHasConvexLevelSets` ::

$$:: \forall V \in \mathbb{R}\text{-VS} . \forall f : \text{Convex}(V, V) . \forall \alpha \in \overset{\infty}{\mathbb{R}} . \text{Convex}\left(V, \{v \in V : f(v) \geq \alpha\}\right)$$

`Proof` =

. . .

□

`ConvexFunctionHasConvexStrictLevelSets` ::

$$:: \forall V \in \mathbb{R}\text{-VS} . \forall f : \text{Convex}(V, V) . \forall \alpha \in \overset{\infty}{\mathbb{R}} . \text{Convex}\left(V, \{v \in V : f(v) > \alpha\}\right)$$

`Proof` =

. . .

`ConvexlyBoundedRegionIsConvex` ::

:: $\forall V \in \mathbb{R}\text{-VS} . \forall I \in \mathsf{SET} . \forall \alpha : I \to \overset{\infty}{\mathbb{R}} . \forall f : I \to \mathtt{Convex}(V,V) . \mathtt{Convex}\Big(V, \{v \in V : \forall i \in I . f_i(v) > \alpha_i\}\Big)$

`Proof =`

`GeneralizedAMGMIneq` :: $\forall n \in \mathbb{N} . \forall \lambda : \mathbb{R}^n_+ . \forall x : \mathbb{R}^n_{++} . \forall \aleph : \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}$

`Proof =`

By Jensen inequality for natural logarithm $\ln\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i \ln(x_i).$

Then by exponentiating both parts $\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$

□

`PositivelyHomogeneous` :: $\prod V : \mathbb{R}\text{-VS} . ?\Big(V \to (-\infty, +\infty]\Big)$

$f : \mathtt{PositivelyHomogeneous} \iff \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . f(\alpha v) = \alpha f(v)$

`PositiveHomogeneousZeroPositivity` :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V) . f(0) \geq 0$

`Proof =`

Note that $f(0) = f(t0) = t f(0)$ for all $t \in \mathbb{R}_{++}$.

This means that $f(0)$ is either 0 or $+\infty$.

□

`PositiveHomogeneousConvexity` :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V) .$

. $\mathtt{Convex}(V,V,f) \iff \forall x, y \in V . f(x+y) \leq f(x) + f(y)$

`Proof =`

$(\Rightarrow)$: assume f is convex.

Then $f(x+y) = f\left(\frac{2}{2}x + \frac{2}{2}y\right) \leq \frac{1}{2}f(2x) + \frac{1}{2}f(2y) = f(x) + f(y)$ for any $x, y \in V$.

$(\Leftarrow)$: assume the inequality holds .

Then $f\Big(\lambda x + (1 + \lambda)y\Big) \leq f(\lambda x) + f\Big((1-\lambda)y\Big) = \lambda f(x) + (1-\lambda)f(y)$ when $\lambda \in (0,1)$ and $x, y \in V$.

Otherwise, when $\lambda = 0, 1$, convexity condition holds trivially.

□

`Conic` := $\lambda V \in \mathbb{R}\text{-VS} . \mathtt{Convex}(V,V) \times \mathtt{PositivelyHomogeneous}(V) : \mathbb{R}\text{-VS} \to \mathtt{Type};$

`ConicIneq` :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{Convex}(V,V) \,\&\, \mathtt{PositivelyHomogeneous}(V) . \forall n \in \mathbb{N} . \forall x \in V^n .$

. $\forall \lambda \in \mathbb{R}^n_{++} . f\left(\sum_{i=1}^{n} \lambda_i x\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$

`Proof =`

Iterate previous theorem.

□

`ConicEpigraph` $:: \forall V \in \mathbb{R}\text{-VS} . \forall f : V \to (-\infty, +\infty) . \texttt{Conic}(V, f) \iff \texttt{ConvexCone}(V, \operatorname{epi} f)$

`Proof =`

$\ldots$

$\square$

`ConicIsSupersymmetric` $:: \forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . \forall v \in V . f(v) \geq -f(-v)$

`Proof =`

Write $f(x) + f(-x) \geq f(x - x) = f(x) \geq 0$.

So $f(x) \geq -f(-x)$.

$\square$

`ConicIsLinearIffSymmetric` $:: \forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . f \in V^* \iff \forall v \in V . f(-v) = -f(v)$

`Proof =`

$(\Rightarrow)$: this is trival.

$(\Leftarrow)$: assume that the property holds .

Let $x, y \in V$.

Then $f(x) + f(y) \geq f(x + y) \geq -f(-x - y) \geq -f(-x) - f(-y) = f(x) + f(y)$.

This mean $f(x) + f(y) = f(x + y)$.

But as x and y were arbitrary f must be additive and hence linear.

$\square$

`StrictlyConvexFunction` $:: \prod V : \mathbb{R}\text{-VS} . \texttt{?ProperConvexFunction}(V)$

$f : \texttt{StrictlyConvexFunction} \iff$

$\qquad \iff \forall \lambda \in [0, 1] . \forall x, y \in V . \forall \aleph : x \neq y . f\left(\lambda x + (1 - \lambda)y\right) < \lambda f(x) + (1 - \lambda)f(y)$

`SquareNoremIsStrictlyConvex` $:: \forall V : \texttt{NormedSpace}(\mathbb{R}) . \texttt{StrictConvexFunction}\left(V, \| \bullet \|^2\right)$

`Proof =`

$\forall x, y \in V, \lambda \in (0, 1) . \left\|\lambda x + (1 - \lambda)y\right\|^2 \leq \left(\lambda\|x\| + (1 - \lambda)\|y\|\right)^2 < \lambda^2\|x\|^2 + (1 - \lambda)^2\|y\|^2 <$

$< \lambda\|x\|^2 + (1 - \lambda)\|y\|^2.$

$\square$

1.2 Convexity Preserving Operations

ConvexComposition :: $\forall V \in \mathbb{R}\text{-VS} . \forall D \subset V \forall f : \text{Convex}(V, D) . \forall \phi : \text{Convex} \& \text{Increasing}(\mathbb{R}, \mathbb{R}) . \text{Convex}(V,$

Proof $=$

Assume $x, y \in \text{dom } f, \lambda \in [0, 1]$.

Then $\phi\Big(f\big(\lambda x + (1 - \lambda)y\big)\Big) \leq \phi\Big(\lambda f(x) + (1 - \lambda)f(y)\Big) \leq \lambda \phi \circ f(x) + (1 - \lambda)\phi \circ f(y)$.

$\square$

ConvexFunctionFromSet :: $\forall V \in \mathbb{R}\text{-VS} . \forall C : \text{Convex}(V \oplus \mathbb{R}) . \text{Convex}\Big(V, V, \Lambda v \in V . \inf \big\{t | (v, t) \in C\big\}\Big)$

Proof $=$

This is function has cinvex epigraph.

$\square$

InfimalConvolutionIsConvex :: $\forall V \in \mathbb{R}\text{-VS} . \forall n \in \mathbb{N} . \forall f : \{1, \ldots, n\} \to \text{ProperConvexFunction}(V) .$

$. \text{Convex}\left(V, V, \Lambda x \in V . \inf \left\{\sum_{k=1}^{n} f_k(v_k)\Big| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}\right)$

Proof $=$

Let $g = \inf \left\{\sum_{k=1}^{n} f_k(v_k)\Big| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$.

$C = \sum_{k=1}^{n} \text{epi } f_k$ is convex.

A tuple $(x, \phi) \in C$ if there is a sequence $(v, \psi) \in (V \oplus \mathbb{R})^n$ such that $x = \sum_{k=1}^{n} v_k, \phi = \sum_{k=1}^{n} \psi_k$

and $f(v_k) \leq \psi_k$ for every $k \in \{1, \ldots, n\}$.

Thus $\phi = \sum_{k=1}^{n} \psi_k \geq \sum_{k=1}^{n} f(v_k) \geq g(x)$, so $(x, \phi) \in \text{epi } g$.

Then g can be constructed from set C.

$\square$

infimalConvolution :: $\prod_{V \in \mathbb{R}\text{-VS}} \prod_{n=1}^{\infty} \Big(\{1, \ldots, n\} \to \text{ProperConvexFunction}(V)\Big) \to \text{Convex}(V, V)$

infimalConvolution $(f) = \mathop{\square}\limits_{k=1}^{n} f_i := \inf \left\{\sum_{k=1}^{n} f_k(v_k)\Big| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$

ConvexDelta :: $\prod_{V \in \mathbb{R}\text{-VS}} V \to \text{ProperConvexFunction}(V)$

convexDelta $(a) = \delta_a := \Lambda x \in V . \text{ if } x = a \text{ then } 0 \text{ else } +\infty$

`GraphTranslationByInfimalConvolution` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \text{ProperConvexFunction}(V) . \forall a, v \in V . (\delta_a \Box f$

`Proof` =

Clearly $(\delta_a \Box f)(v) = \min \left\{ f(v - a), +\infty \right\}$.

$\Box$

`InfimalConvolutionDomain` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \text{ProperConvexFunction}(V) . \forall a, v \in V . \text{dom}(f \Box g) = \text{dom } f$

`Proof` =

Obvious .

$\Box$

`DistanceExpression` :: $\forall V : \text{NormedSpace}(\mathbb{R}) . \forall C : \text{Convex}(V) . d_V(C, \bullet) = \| \bullet \| \Box \chi(\bullet | C)$

`Proof` =

$\Box$

`InfimalConvolutionDefinesCommutativeMonoid` ::

 :: $\forall V \in \mathbb{R}\text{-VS}$.

 . $\text{CommutativeMonoid}\left(\text{Convex}(V, V), \Lambda f, g \in \text{Convex}(V, V) . \Lambda x \in V . \inf \left\{\phi \Big| (v, \phi) \in (\text{epi } f + \text{epi } g)\right\}\right)$

`Proof` =

δ_0 is a neutral element, comutativity and associativity is almost obvious.

$\Box$

`rightScalarMultiplication` :: $\displaystyle\prod_{V \in \mathbb{R}\text{-VS}}$. $\text{Convex}(V, V) \to \mathbb{R}_+ \to \text{Convex}(V, V)$

`rightScalarMultiplication` $(f, \lambda) = f\lambda := \text{ConvexFunctionFromSet}(V, \lambda \text{ epi } f)$

`RightScalarMultiplicationExpression` ::

 :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \text{Convex}(V, V) . \forall \lambda \in \mathbb{R}_{++} . \forall x \in X . f\lambda(x) = \lambda f(\lambda^{-1} x)$

`Proof` =

Obvious.

$\Box$

`RightScalarMultiplicationByZero` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \text{Convex}(V, V) . f0(x) = \delta(0)$

`Proof` =

Obvious.

$\Box$

`ConicityByRightMultiplication` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \text{Convex}(V, V) . \text{Conic}(V, f) \iff \forall \lambda \in \mathbb{R}_{++} . f\lambda = f$

`Proof` =

Follows from the expression for right multiplication.

$\Box$

$\texttt{conicClosure} :: \prod\limits_{V \in \mathbb{R}\text{-VS}} \texttt{ConvexFunction}(V) \to \texttt{Conic}(V)$

$\texttt{conicClosure} \, (\text{cone} \, f) := \texttt{ConvexFunctionFromSet}\Big(V.\, \text{cone} \, \text{epi} \, f\Big)$

$\texttt{GaugeExpression} :: \forall V \in \mathbb{R}\text{-VS} \, . \, \forall C : \texttt{Convex} \, \& \, \texttt{NonEmpy}(V) \, . \, \gamma(\bullet|C) = \text{cone}\Big(\chi(\bullet|C) + 1\Big)$

$\texttt{Proof} =$

$(x, \phi) \in \text{epi} \, \gamma(\bullet|C)$ iff $x \in \lambda C$ and $0 < \lambda \leq \phi$.

This means that $(x, \lambda) \in \text{cone} \, C \times \{1\} \subset \text{cone} \, \text{epi}\Big(\chi(\bullet|C) + 1\Big)$.

So $(x, \phi) \in \text{cone} \, C \times \{\phi/\lambda\} \subset \text{cone} \, \text{epi}\Big(\chi(\bullet|C) + 1\Big) = \text{epi} \, \text{cone}\Big(\chi(\bullet|C) + 1\Big)$.

On the other hand id $(x, \psi) \in \text{epi} \, \text{cone}\Big(\chi(\bullet|C) + 1$ then there exists $\lambda \in \mathbb{R}_{++}$ such that $\lambda x \in C$ and $\lambda\psi \geq 1$.

But this means that $\psi \geq \lambda^{-1} \geq \gamma(x|C)$.

Thus $(x, \psi) \in \gamma(\bullet|C)$.

And both functions are equal by equality of epigraphs.

$\square$

$\texttt{SupremumIsConvex} :: \forall V \in \mathbb{R}\text{-VS} \, . \, \forall I \in \mathsf{SET} \, . \, \forall f : I \to \texttt{ConvexFunction}(V) \, . \, \texttt{ConvexFunction}(V, \sup\limits_{i \in I} f_i)$

$\texttt{Proof} =$

$\text{epi} \sup\limits_{i \in I} f_i = \bigcap\limits_{i \in I} \text{epi} \, f_i$ is convex.

$\texttt{convexHull} :: \prod\limits_{V \in \mathbb{R}\text{-VS}} \prod\limits_{I \in \mathsf{SET}} \Big(I \to V \overset{\infty}{\to} \mathbb{R}\Big) \to \texttt{ConvexFunction}(V)$

$\texttt{convexHull} \, (f) = \text{conv}_{i \in I} \, f_i := \texttt{ConvexFunctionFromSet}\Bigg(V, \text{conv} \bigcup\limits_{i \in \mathcal{I}} \text{epi} \, f_i\Bigg)$

$\texttt{ConvexHullExpression} :: \forall V \in \mathbb{R}\text{-VS} \, . \, \forall I \in \mathsf{SET} \, . \, \forall f : I \to V \to (-\infty, +\infty] \, . \, \forall v \in V \, .$

$. \, \text{conv}_{i \in I} \, f_i(x) = \inf \left\{ \sum\limits_{i \in I} \lambda_i f_i(v_i) \, \middle| \, \lambda \in \mathbb{R}_+^{\oplus I}, v : I \to V, \sum\limits_{i \in I} \lambda_i = 1, \sum\limits_{i \in I} \lambda_i v_i = x \right\}$

$\texttt{Proof} =$

This follows from the thorough examination of the definition.

$\texttt{convexPullback} :: \prod V, W \in \mathbb{R}\text{-VS} \, . \, \mathbb{R}\text{-VS}(V, W) \to \texttt{ConvexFunction}(W) \to \texttt{ConvexFunction}(V)$

$\texttt{convexPullback} \, (f, T) = fT := f \circ T$

$\texttt{convexPushforward} :: \prod V, W \in \mathbb{R}\text{-VS} \, . \, \mathbb{R}\text{-VS}(V, W) \to \texttt{ConvexFunction}(V) \to \texttt{ConvexFunction}(W)$

$\texttt{convexPullback} \, (f, T) = T_* f := \Lambda w \in W \, . \, \inf\{f(v)|w = Tv\}$

1.3 Metric and Topological Properties

SphericalBound ::

    :: $\forall V \in \mathsf{BAN}(\mathbb{R}) . \forall f : \texttt{ProperConvexFunction}(V) . \forall c \in \operatorname{dom} f . \forall \rho \in \mathbb{R}_{++} . \forall \aleph : \eta < +\infty . \forall \alpha \in (0,1) .$

    $. \forall x \in \mathbb{B}_V(c, \alpha\rho) . \left| f(x) - f(c) \right| \leq \alpha \left(\eta - f(c) \right)$

    where $\eta = \sup f\left(\mathbb{B}_V(c, \rho) \right)$

Proof =

By convexity $f(x) - f(c) = f\left((1-\alpha)c + \alpha \left(\dfrac{x - (1-\alpha)c}{\alpha} \right) \right) - f(c) \leq \alpha \left(f\left(c + \dfrac{x-c}{\alpha} \right) - f(c) \right).$

If $x \in \mathbb{B}_V(c, \alpha\rho)$ then $c + \dfrac{x-c}{\alpha} \in \mathbb{B}_V(c, \rho)$.

So $f(x) - f(c) \leq \alpha \left(\eta - f(c) \right).$

On the other hand $f(c) - f(x) = f\left(\dfrac{x}{1+\alpha} + \dfrac{\alpha}{1-\alpha} \dfrac{(1+\alpha)c - x}{\alpha} \right) - f(x) \leq$

$\leq \dfrac{\alpha}{1+\alpha} \left(f\left(c + \dfrac{c-x}{\alpha} \right) - f(x) \right) \leq \dfrac{\alpha}{1+\alpha} (\eta - f(x)) = \dfrac{\alpha}{1+\alpha} \left(\eta - f(c) \right) + \dfrac{\alpha}{1+\alpha} \left(f(c) - f(x) \right).$

So by rearanging inequalities one gets $f(c) - f(x) \leq \alpha \left(\eta - f(x) \right).$

LocalLipschitzContinuity ::

    :: $\forall V \in \mathsf{BAN}(\mathbb{R}) . \forall f : \texttt{ProperConvexFunction}(V) . \forall c \in \operatorname{dom} f . \forall \rho \in \mathbb{R}_{++} . \forall \aleph : \delta < +\infty .$

    $. \left(\dfrac{\delta}{\rho} \right)\text{-Lip}\left(\mathbb{B}(c, \rho), \mathbb{R}, f_{|\mathbb{B}(c,\rho)} \right)$

    where $\eta = \operatorname{diam} f\left(\mathbb{B}_V(c, \rho) \right)$

Proof =

Assume $x, y \in \mathbb{B}_V(c, \rho)$ sych that $x \neq y$.

Let $\alpha = \dfrac{\|x-y\|}{\|x-y\| + \rho} < \dfrac{\|x-y\|}{\rho}$ and $z = x + \dfrac{1-\alpha}{\alpha}(x-y)).$

Then, $\|z - c\| \leq \|z - x\| + \|x - c\| \leq \dfrac{1-\alpha}{\alpha}\|x-y\| + \rho \leq 2\rho$, so $z \in \mathbb{B}_V(c, 2\rho).$

Thus, by convexity $f(x) = f\left(\alpha z + (1-\alpha)y \right) \leq f(y) + \alpha(f(z) - f(y) \leq f(y) + \alpha\delta \leq f(y) + \dfrac{\delta}{\rho}\|x-y\|$.

From symmetry $|f(x) - f(y)| \leq \dfrac{\delta}{\rho}\|x-y\|$.

$\square$

`ContinuityByBound` ::

:: $\forall V \in \mathsf{BAN}(\mathbb{R})$. $\forall f : \mathtt{ProperConvexFunction}(V)$. $\forall x \in V$. $\forall U \in \mathcal{U}(x)$.

. $\forall \aleph : \mathtt{Bounded}(V, U, f_{|U})$. $\operatorname{int} \operatorname{dom} f \xrightarrow{f_{|\operatorname{int} \operatorname{dom} f}} \mathbb{R} : \mathsf{TOP}$

`Proof` =

Assume $v \in U \cap \operatorname{rel} \operatorname{int} \operatorname{dom} f$.

As v belongs to relative intrior there are $w \in \operatorname{dom} f, \rho \in Reals_+, \lambda \in (0,1)$

such that $v \in \lambda \mathbb{B}(x, \rho) + (1 - \lambda)w$ and $\mathbb{B}(x, \rho) \subset U$.

Then $f(x) \le \lambda f(y) + (1 - \lambda)f(w) \le \lambda\beta + (1 - \lambda)f(w)$,

where $x \in \lambda \mathbb{B}(x, \rho) + (1 - \lambda)w, y \in \mathbb{B}(y, \rho)$ and β is the bound for U.

So by the previous theorem f is locally Lipshitz and continuous on $\operatorname{rel} \operatorname{int} \operatorname{dom} f$

and so f is actually continuous on $\operatorname{rel} \operatorname{int} \operatorname{dom} f$.

$\square$

`ContinuityByFiniteDimension` ::

:: $\forall V \in \mathtt{EucledeanSpace}$. $\forall f : \mathtt{ProperConvexFunction}(V)$. $\operatorname{int} \operatorname{dom} f \xrightarrow{f_{|\operatorname{int} \operatorname{dom} f}} \mathbb{R} : \mathsf{TOP}$

`Proof` =

Let $n = \dim V$.

Assume $x \in V$ and $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}(x, \rho) \subset \operatorname{dom} f$.

Then there exists a simplicital set $\{v_1, \ldots, v_{n+1}\}$ such that $\mathbb{B}(x, \rho) \subset \operatorname{conv}(v_1, \ldots, v_{n+1}) \subset \operatorname{dom} f$.

But this means that $\sup f\left(\mathbb{B}(x, \rho)\right) \le \max_{i=1,\ldots,n+1} f(v_i)$.

So by the previous theorem f is locally Lipshitz and continuous on $\operatorname{rel} \operatorname{int} \operatorname{dom} f$

and so f is actually continuous on $\operatorname{rel} \operatorname{int} \operatorname{dom} f$.

$\square$

`NonEmptyInteriorCondition` :: $\forall V \in \mathsf{BAN}(\mathbb{R})$. $\forall f : \mathtt{ProperConvexFunction}(V)$. $\forall x \in V$. $\forall U \in \mathcal{U}(x)$.

. $\forall \aleph : \mathtt{Bounded}(V, U, f_{|U})$. $\operatorname{int} \operatorname{epi} f \ne \emptyset$

`Proof` =

We can find radius ρ Lipschitz constant β such that $\left| f(v) - f(w) \right| \le \beta \|v - w\| \le \beta\rho$ for every $v, w \in \mathbb{B}(x, \rho)$.

Select $\delta \in (2\beta\rho, +\infty)$ and set $\gamma = \min(\rho, \delta/2) > 0$.

And let $(y, \phi) \in V \oplus \mathbb{R}$ such that $\left\| (y, \phi) - \left(x, f(x) + \rho\right) \right\|^2 \le \gamma^2$.

Then $\|y - x\| \le \gamma \le \rho$ and $\left| \phi - \left(f(x) + \delta\right) \right| \le \gamma \le \delta/2$.

So, $f(y) < f(x) + \delta/2 = f(x) + \delta - \delta/2 \le f(x) + \delta - \gamma \le \phi$.

Thus $(y, \phi) \in \operatorname{epi} f$.

As (y, ϕ) was arbitrary, the whole sphere is a subset of $\operatorname{epi} f$.

So $\operatorname{int} \operatorname{epi} f \ne \emptyset$.

$\square$

InteriorLevelSet ::

    $:: \forall V : \mathbb{R}\text{-BAN} . \forall f : \texttt{LowerSemicontinuous}\left(V, (-\infty, +\infty]\right) . \forall x \in V . \forall \aleph : f(x) < 0 . x \in \text{int } f^{-1}(-\infty, 0]$

Proof =

...

□

ConvexLevelSetInterior $:: \forall V : \mathbb{R}\text{-BAN} . \forall f : \texttt{ConvexFunction}(V) . \forall x \in V . \forall \aleph : f(x) < 0 .$

    $. \text{int } f^{-1}(-\infty, 0] \subset f^{-1}(-\infty, 0)$

Proof =

...

□

ConvexUsCLevelSetInteriorEq ::

    $. \forall V : \mathbb{R}\text{-BAN} . \forall f : \texttt{ConvexFunction}(V) . \forall \aleph : \texttt{LowerSemicontinuous}\left(f_{|f^{-1}(-\infty,0)}, (-\infty, 0)\right) .$

    $. \text{int } f^{-1}(-\infty, 0] = \text{int } f^{-1}(-\infty, 0)$

Proof =

...

□

ConvexEucLevelSetInteriorEq ::

    $. \forall V \in \texttt{EucledeanSpace} . \forall f : \texttt{ConvexFunction}(V) . \forall \aleph : \text{dom } f \in \mathcal{T}(V) . \text{int } f^{-1}(-\infty, 0] = \text{int } f^{-1}(-\infty, 0)$

Proof =

...

□

1.4 Closures

LowerSemicontinuityByEpigraph :: $\forall V \in$ EucledeanSpace . $\forall f : V \to \mathbb{R}$-TOPVS .

$\quad$ LowerSemicontinuous$(V, \overset{\infty}{\mathbb{R}}, f) \iff$ Closed$(V \oplus \mathbb{R}, \text{epi} f)$

Proof $=$

If f is lower semicontinuous then $\lim\inf\limits_{x \to v} f(x) = f(v)$.

So the epigraph must be closed.

Thus, result is basically obvious.

closure :: $\prod\limits_{V \in \mathbb{R}\text{-TOPVS}}$ ConvexFunction$(V) \to \Big($ ConvexFunction(V) & LowerSemicontinuous$(V, \mathbb{R})\Big)$

closure$(f) = \text{cl} f :=$ if $f > -\infty$ then FunctionFromSet$(V, \text{cl epi} f)$ else $-\infty$

ClosedFunction :: $\prod V \in \mathbb{R}$-TOPVS . ?ConvexFunction(V)

$f :$ ClosedFunction $\iff \text{cl} f = f$

ImproperDomain :: $\forall V \in \mathbb{R}$-TOPVS . $\forall f :$ ConvexFunction(V) .

$\quad$. $\forall \aleph : -\infty \in \text{Im} f$. $\forall x \in \text{rel int dom} f$. $f(x) = -\infty$

Proof $=$

If there is a point $p \in V$ such that $f(p) = -\infty$ then $p \in \text{dom} f$.

Also assume that $u \in \text{rel int dom} f$.

By propertirs of relative interiot there exists $x \in \text{dom} f$ suxh that $u \in (p, x)$.

So there is $\lambda \in (0, 1)$ suxh that $u = \lambda u + (1 - \lambda)x$.

But $(p, \alpha) \in \text{epi} f$ for any arbitrary $\alpha \in \mathbb{R}$.

Thus $\Big(u, \lambda\alpha + (1 - \lambda)f(x)\Big) \in \text{epi} f$ fo any α .

And by taking the limit $\alpha \to -\infty$ we see that it must be the case that $f(u) = -\infty$.

$\square$

ContinuityByClosedness ::

$\quad$ :: $\forall V \in$ BAN$(\mathbb{R})$. $\forall f :$ ClosedFunction(V) . $\text{int dom} f \xrightarrow{f_{|\text{int dom} f}} \mathbb{R}$: TOP

Proof $=$

. . .

$\square$

ConvexLsCLevelSetInteriorEq ::

$\quad$. $\forall V \in$ EucledeanSpace . $\forall f :$ ClosedFunction(V) . $\forall \aleph : \text{dom} f \in \mathcal{T}(V)$.

$\quad$. $\text{int} f^{-1}(-\infty, 0] = \text{int} f^{-1}(-\infty, 0)$

Proof $=$

. . .

$\square$

ClosedFunctionSupremum $:: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to \mathtt{ClosedFunction}(V) \,.$
   $.\, \sup f \in \mathtt{ClosedFunction}(V)$

Proof $=$

 The epigraph of the supremum is the intersection of epigraphs.

 Then use the fact that intersection of closed sets is closed.

 $\square$

ClosedFunctionSum $::$
   $.\, \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f, g \in \mathtt{ClosedFunction} \,\&\, \mathtt{ProperConvexFunction}(V) \,.$
   $.\, f + g \in \mathtt{ClosedFunction} \,\&\, \mathtt{ProperConvexFunction}(V)$

Proof $=$

 $\ldots$

 $\square$

ClosedFunctionSum2 $::$
   $.\, \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : \mathbb{N} \to \mathtt{ClosedFunction}(V) \,.$
   $.\, \forall \aleph : \inf_{i \in I} f_i \geq 0 \,.\, \sum_{i \in I} f_i \in \mathtt{ClosedFunction}(V)$

Proof $=$

 $\ldots$

 $\square$

DegenerateClosedForm $::$
   $.\, \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \mathtt{ClosedFunction}(V) \,.\, \forall \aleph : -\infty \in \operatorname{Im} f \,.\, \operatorname{Im} f = \{-\infty, +\infty\}$

Proof $=$

 $\ldots$

 $\square$

ClosureLevelSets $::$
   $: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \mathtt{ConvexFunction}(V) \,.\, \operatorname{cl}\left(f^{-1}(-\infty, 0)\right) = \operatorname{cl}\left(f^{-1}(-\infty, 0]\right) = \left(\operatorname{cl} f\right)^{-1}(-\infty, 0)$

Proof $=$

 $\ldots$

 $\square$

ProperClosedFunction $:= \mathtt{ProperConvexFunction} \,\&\, \mathtt{ClosedFunction} : \mathbb{R}\text{-}\mathsf{TOPVS} \to \mathsf{Type};$

ConvexLimit $:: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \mathtt{ProperClosedFunction}(V) \,.\, \forall x \in \operatorname{dom} f \,.\, \forall y \in V \,.\, \lim_{\lambda \downarrow 0} g(\lambda) = f(y)$

   where $g = \Lambda \lambda \in [0, 1] \,.\, f\left(\lambda x + (1 - \lambda)y\right)$

Proof $=$

 In $f\left(\lambda x + (1 - \lambda)y\right) \leq \lambda f(x) + (1 - \lambda)f(y)$ the majorant is continuous in $lambda$.

 So in $f(x) \leq \lim_{\lambda \to \infty} f\left(\lambda x + (1 - \lambda)y\right) \leq f(x)$ holds.

 $\square$

`ProperConvexContinuity` $::$ $f \in$ `ProperClosedFunction`$(\mathbb{R})$ $.$ $f_{|\operatorname{cl}\operatorname{dom} f} \in \mathsf{TOP}\left(\operatorname{cl}\operatorname{dom} f, \overset{\infty}{\mathbb{R}}\right)$

`Proof` $=$

Use convex limits as above.

$\square$

`ConvexExtension` $::$

$\quad \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} . \forall f, g \in$ `ProperClosedFunction`$(V \oplus \mathbb{R})$ $.$

$\quad . \forall \aleph : \operatorname{dom} f \cup \operatorname{dom} g \subset V \oplus \mathbb{R}_+ . \forall \beth : f_{|V \times \mathbb{R}_{++}} = g_{|V \times \mathbb{R}_{++}} . f = g$

`Proof` $=$

$\ldots$

$\square$

Sources

1. Convex Analysis — R. T. Rockaffeler 1972

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