Order Theory

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Contents

1 Ordered Spaces

1.1 Objects

1.1.1 Ordered Sets

$$\texttt{Order} := \Lambda X \in \mathsf{SET} \,.\, \texttt{Reflexive}(X) \,\&\, \texttt{Antisymmetric}(X) \,\&\, \texttt{Transitive}(X) : \prod_{X \in \mathsf{SET}} \texttt{Relation}(X);$$

$$\texttt{Poset} := \sum_{X \in \mathsf{SET}} \texttt{Order} : \texttt{Type};$$

$$\texttt{asSet} :: \texttt{Poset} \to \mathsf{SET}$$
$$\texttt{asSet}\,((X, R)) = (X, R) := X$$

$$\texttt{order} :: \prod (X, R) : \texttt{Poset} \,.\, \texttt{Order}(X)$$
$$\texttt{order}\,() = \leq_{(X,R)} := R$$

$$\texttt{Comparable} :: \prod_{X \in \mathsf{SET}} \,.\, \texttt{Order}(X) \to ?X^2$$
$$(x, y) : \texttt{Comparable} \iff \Lambda(\leq) : \texttt{Order}(X) \,.\, x \leq y | y \leq x$$

$$\texttt{TotalOrder} :: \prod_{X \in \mathsf{SET}} ?\texttt{Order}(X)$$
$$R : \texttt{TotalOrder} \iff \forall x, y \in X^2 \,.\, \texttt{Comparable}\Big(X, R, (x, y)\Big)$$

$$\texttt{Toset} := \sum_{X \in \mathsf{SET}} \texttt{TotalOrder} : \texttt{Type};$$

$$\texttt{StrictlyLess} :: \prod X : \texttt{Poset} \,.\, ?X^2$$
$$(x, y) : \texttt{StrictlyLess} \iff x < y \iff x \leq y \,\&\, x \neq y$$

$$\texttt{orderedSubset} :: \prod X : \texttt{Poset} \,.\, ?X \to \texttt{Poset}$$
$$\texttt{orderedSubset}\,(A) = A := \Big(A, (\leq)_X \cap A \times A\Big)$$

$$\texttt{subsetPoset} :: \mathsf{SET} \to \texttt{Poset}$$
$$\texttt{subsetPoset}\,(X) = ?X := \Big(?X, (\subset)\Big)$$

1.1.2 Maximal and minimal elements

$\texttt{Maximal} :: \prod X : \texttt{Poset} \,.\, ?X$

$x : \texttt{Maximal} \iff x \in \max X \iff \forall y \in X \,.\, x \leq y \Rightarrow x = y$

$\texttt{Minimal} :: \prod X : \texttt{Poset} \,.\, ?X$

$x : \texttt{Minimal} \iff x \in \min X \iff \forall y \in X \,.\, x \geq y \Rightarrow x = y$

$\texttt{Top} :: \prod X : \texttt{Poset} \,.\, ?X$

$x : \texttt{Top} \iff x \in \top(X) \iff \forall y \in X \,.\, y \leq x$

$\texttt{Bottom} :: \prod X : \texttt{Poset} \,.\, ?X$

$x : \texttt{Bottom} \iff x \in \bot(X) \iff \forall y \in X \,.\, y \geq x$

$\texttt{TopIsMaximal} :: \forall X : \texttt{Poset} \,.\, \top(X) \subset \max(X)$

$\texttt{Proof} =$

$\texttt{Assume}\ x \in \top(X),$

$\texttt{Assume}\ y \in X,$

$\texttt{Assume}\ [1] : x \leq y,$

$[2] := \texttt{E}\top(X, x)(y) : y \leq x,$

$[y.*] := \texttt{EAntisymmetric}(X, \leq)[1, 2] : x = y;$

$\rightsquigarrow [x.*] := \texttt{I}(\Rightarrow)\texttt{I}(\forall)\texttt{I}(\max X) : x \in \max X;$

$\rightsquigarrow [*] := \texttt{ISubset} : \top(X) \subset \max X;$

$\square$

$\texttt{TopIsUnique} :: \forall X : \texttt{Poset} \,.\, \left|\top(X)\right| \leq 1$

$\texttt{Proof} =$

$\texttt{Assume}\ x, y \in\to (X),$

$[1] := \texttt{E}\top(X, x)(y) : y \leq x,$

$[2] := \texttt{E}\top(X, y)(x) : x \leq y,$

$\left[(x, y).*\right] := \texttt{EAntisymmetric}(X, \leq)[1, 2] : x = y;$

$\rightsquigarrow [*] := \texttt{I}\mathsf{CARD} : \left|\top(X)\right| \leq 1,$

$\square$

$\texttt{BottomIsMinimal} :: \forall X : \texttt{Poset} \,.\, \bot(X) \subset \min(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{BittomIsUnique} :: \forall X : \texttt{Poset} \,.\, \left|\bot(X)\right| \leq 1$

$\texttt{Proof} =$

$\ldots$

$\square$

1.1.3 Preorders

$$\texttt{Preorder} := \Lambda X \in \mathsf{SET}\ .\ \texttt{Reflexive}(X)\ \&\ \texttt{Transitive}(X) : \prod_{X \in \mathsf{SET}} \texttt{Relation}(X);$$

$$\texttt{PreorderedSet} := \sum_{X \in \mathsf{SET}} \texttt{Preorder} : \texttt{Type};$$

$$\texttt{asSet} :: \texttt{PreorderedSet} \to \mathsf{SET}$$
$$\texttt{asSet}\,((X, R)) =:= X$$

$$\texttt{preorder} :: \prod (X, R) : \texttt{PreorderedSet}\ .\ \texttt{Preorder}(X)$$
$$\texttt{preorder}\,() = \preceq_{(X,R)} := R$$

$$\texttt{orderQuotient} :: \texttt{PreorderedSet} \to \texttt{Poset}$$
$$\texttt{orderQuotient}\,(X) = \widehat{X} := \frac{X}{\{(x, y) \in X^2 : x \preceq y\ \&\ y \preceq x\}}$$

1.2 Categories

1.2.1 One and Infinity

$$\texttt{Monotonic} :: \prod X, Y : \texttt{Poset} \,.\, ?(X \to Y)$$
$$f : \texttt{Monotonic} \iff \forall a, b \in X \,.\, a \leq b \Rightarrow f(a) \leq f(b)$$

$$\texttt{posetCategory} :: \mathsf{CAT}$$
$$\texttt{posetCategory}\,() = \mathsf{POSET} := (\texttt{Poset}, \texttt{Monotonic}, \circ, \mathrm{id})$$

$$\texttt{imagePosetFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{POSET})$$
$$\texttt{imagePosetFunctor}\,(X) = \mathsf{P}(X) := ?X$$
$$\texttt{imagePosetFunctor}\,(X, Y, f) = \mathsf{P}_{X,Y}(f) := \texttt{image}(f)$$

$$\texttt{preimagePosetFunctor} :: \texttt{Contravariant}(\mathsf{SET}, \mathsf{POSET})$$
$$\texttt{preimagePosetFunctor}\,(X) = \mathsf{P}'(X) := ?X$$
$$\texttt{preimagePosetFunctor}\,(X, Y, f) = \mathsf{P}'_{X,Y}(f) := \texttt{preimage}(f)$$

$$\texttt{freePosetFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{POSET})$$
$$\texttt{freePosetFunctor}\,(X) = \mathsf{F}_{\mathsf{POSET}}(X) := \Big(X, \Delta(X)\Big)$$
$$\texttt{imagePosetFunctor}\,(X, Y, f) = \mathsf{F}_{\mathsf{POSET}}X, Y(f) := f$$

$$\texttt{forgetfulPosetFunctor} :: \texttt{Covariant}(\mathsf{POSET}, \mathsf{SET})$$
$$\texttt{forgetfulPosetFunctor}\,(X) = \mathsf{U}_{\mathsf{POSET}}(X) := X$$
$$\texttt{forgetfulPosetFunctor}\,(X, Y, f) = \mathsf{U}_{\mathsf{POSET}}X, Y(f) := f$$

$$\texttt{FreePosetAdjointness} :: \mathsf{F}_{\mathsf{POSET}} \dashv \mathsf{U}_{\mathsf{POSET}}$$
$$\texttt{Proof} =$$
$$\texttt{Assume}\ X : \mathsf{SET},$$
$$\texttt{Assume}\ Y : \mathsf{POSET},$$
$$[*] := \texttt{EFIU} : \mathsf{SET}\Big(\mathsf{F}(X), Y\Big) =_{\mathsf{SET}} \mathsf{SET}\Big(X, Y\Big) =_{\mathsf{SET}} \mathsf{POSET}\Big(X, \mathsf{U}(Y)\Big);$$
$$\rightsquigarrow [*] := \texttt{HomSetAdjunction} : \mathsf{F}_{\mathsf{POSET}} \dashv \mathsf{U}_{\mathsf{POSET}};$$
$$\square$$

$$\texttt{posetAsCategory} :: \mathsf{POSET} \to \mathsf{CAT}$$
$$\texttt{posetAsCategory}\,(X) =:= \Big(X, \Lambda x, y \in X \,.\, \texttt{if}\ x \leq y\ \texttt{then}\ \{1\}\ \texttt{else}\ \emptyset, (1,1) \mapsto 1, 1\Big)$$

$$\texttt{MonotonicAreFunctors} :: \forall X, Y \in \mathsf{POSET} \,.\, \forall f \in \mathsf{POSET}(X, Y) \,.\, \texttt{Covariant}(X, Y, f)$$
$$\texttt{Proof} =$$
$$\ldots$$
$$\square$$

1.2.2 products

$$\texttt{posetProduct} :: \prod \mathcal{I} \in \mathsf{SET} \,.\, (\mathcal{I} \to \mathsf{POSET}) \to \mathsf{POSET}$$

$$\texttt{posetProduct}\,(X) = \prod_{i\in\mathcal{I}} X_i := \left(\left(\prod_{i\in\mathcal{I}} X_i, \left\{(x,y) \in \left(\prod_{i\in\mathcal{I}} X_i\right)^2 : \forall i \in \mathcal{I} \,.\, x_i \le y_i\right\}\right), \pi\right)$$

$\texttt{posetProductIsProduct} :: \texttt{Product}\Big(\mathsf{POSET}, \texttt{posetProduct}\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mathcal{I} \in \mathsf{SET},$

$\texttt{Assume}\ X : \mathcal{I} \to \mathsf{POSET},$

$\texttt{Assume}\ P \in \mathsf{POSET},$

$\texttt{Assume}\ f : \prod_{i\in\mathcal{I}} \mathsf{POSET}(P, X_i),$

$h := \Lambda p \in P \,.\, \Lambda i \in \mathcal{I} \,.\, f_i(p) : P \to \prod_{i\in\mathcal{I}} X_i,$

$\texttt{Assume}\ a, b \in P,$

$\texttt{Assume}\ [1] : a \le b,$

$[2] := \forall i \in I \,.\, \mathtt{E}\mathsf{POSET}(P, X_i)(f_i)(a,b) : \forall i \in I \,.\, f_i(a) \le f_i(b),$

$\Big[(a,b).*\Big] := \mathtt{E}h\mathtt{E}\prod_{i\in\mathcal{I}} X_i[2]\mathtt{I}h : h(a) \le h(b);$

$\rightsquigarrow [\mathcal{I}.*] := \mathtt{I}\mathsf{POSET} : h \in \mathsf{POSET}\left(P, \prod_{i\in\mathcal{I}} X_i\right);$

$\rightsquigarrow [*] := \texttt{IProduct} : \texttt{Product}\Big(\mathsf{POSET}, \texttt{posetProduct}\Big);$

$\square$

$\texttt{PosetHasEqualisers} :: \texttt{WithEqualizers}(P,)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{PosetsAreComplete} :: \texttt{Complete}(\mathsf{POSET})$

$\texttt{Proof} \;=$

$\dots$

$\square$

1.2.3 Coproducts

$$\texttt{posetSum} :: \prod \mathcal{I} \in \mathsf{SET} \,.\, (\mathcal{I} \to \mathsf{POSET}) \to \mathsf{POSET}$$

$$\texttt{posetSum}\,(X) = \coprod_{i\in\mathcal{I}} X_i := \left(\left(\bigsqcup_{i\in\mathcal{I}} X_i, \bigcup_{i\in\mathcal{I}} \left\{ \Big((i,x),(i,y)\Big) \Big| (x,y) \in (\leq)_{X_i} \right\}\right), \iota\right)$$

$$\texttt{PosetSumIsCoproduct} :: \texttt{Coproduct}\Big(\mathsf{POSET}, \texttt{posetSum}\Big)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mathcal{I} \in \mathsf{SET}$,

$\texttt{Assume}\ X : \mathcal{I} \to \mathsf{POSET}$,

$\texttt{Assume}\ P \in \mathsf{POSET}$,

$\texttt{Assume}\ f : \prod_{i\in\mathcal{I}} \mathsf{POSET}(X_i, P)$,

$h := \Lambda(i,x) \in \coprod_{i\in\mathcal{I}} \,.\, f_i(x) : \coprod_{i\in\mathcal{I}} X_i \to P$,

$\texttt{Assume}\ (i,a),(j,b) \in \coprod_{i\in\mathcal{I}} X_i$,

$\texttt{Assume}\ [1] : (i,a) \leq (j,b)$,

$[2] := \mathtt{E}\coprod_{i\in\mathcal{I}} X_i[1] : i = j \;\&\; a \leq b$,

$[2] := \mathtt{E}\mathsf{POSET}(X_i, P)(f_i)(a,b) : f_i(a) \leq f_i(b)$,

$\left[\Big((i,a),(j,b)\Big).*\right] := \mathtt{I}h[1][2] : h(i,a) \leq h(j,b)$;

$\rightsquigarrow [\mathcal{I}.*] := \mathtt{I}\mathsf{POSET} : h \in \mathsf{POSET}\left(\coprod_{i\in\mathcal{I}} X_i, P\right)$;

$\rightsquigarrow [*] := \texttt{IProduct} : \texttt{Coproduct}\Big(\mathsf{POSET}, \texttt{posetSum}\Big)$;

$\square$

$$\texttt{Between} :: \forall X \in \mathsf{POSET} \,.\, X^2 \to ?X$$

$$a : \texttt{Between} \iff \Lambda x, y \in X \,.\, x \leq a \leq y | y \leq b \leq x$$

$$\texttt{PosetHasCoequalisers} :: \texttt{WithCoequalizers}(P, H)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ X, Y \in \mathsf{POSET}$,

$\texttt{Assume}\ f, g \in \mathsf{POSET}(X, Y)$,

$(\preceq) := \left\{ \Big([x],[y]\Big) \in \mathsf{coeq}_{\mathsf{SET}}(X,Y,f,g) : x \leq y \right\} : \texttt{Preorder}\Big(\mathsf{coeq}_{\mathsf{SET}}(X,Y,f,g)\Big)$,

$Z := \widehat{\mathsf{coeq}}_{\mathsf{SET}}(X,Y,f,g) : \mathsf{POSET}$,

$\texttt{Assume}\ a, b \in Y$,

$\texttt{Assume}\ [1] : a \leq b$,

$[2] := \mathtt{I}(\preceq) : [a] \preceq [b]$,

$[*.3] := \mathtt{I}\pi_Z[2] : \pi_Z(a) \leq \pi_Z(b)$;

$\rightsquigarrow [1] := \mathtt{I}\mathsf{POSET} : \pi_Z \in \mathsf{POSET}(Y, Z)$;

$[2] := \mathtt{E}Z\mathtt{E}\mathsf{coeq}\mathtt{I}\pi_Z : f\pi_Z = g\pi_Z$,

$\texttt{Assume}\ A \in \mathsf{POSET}$,

$\texttt{Assume}\ h \in \mathsf{POSET}(Y, A)$,

$\texttt{Assume}\ [3] : fh = gh$,

$\texttt{Assume}\ z \in Z$,

$\texttt{Assume}\ a, b \in z$,

$\Big(u, [4]\Big) := \texttt{E}Z\texttt{EEqClass}(z)(a, b) : \sum u \in X\ .\ f(u) \leq a, b \leq g(u) \Big| g(u) \leq a, b \leq f(u)$,

$[5] := \texttt{E}\mathsf{POSET}(h)[4] : fh(u) \leq h(a), h(b) \leq fh(u)$,

$[6] := [3](u) : fh(u) = gh(u)$,

$\Big[(a, b).\ *\Big] := \texttt{E}\mathsf{POSET}(A)\texttt{EAntisymmetric}[5][6] : h(a) = h(b)$;

$[4] := \texttt{I}(\exists)\texttt{I}(\forall) : \exists u \in X : \forall a, b \in z\ .\ h(a) = h(b) = fh(u)$,

$\hat{h}(z) := fh(u) : A$;

$\rightsquigarrow \hat{h} := \texttt{I}(\rightarrow) : \hat{h} : Z \rightarrow A$,

$[4] := \texttt{E}\hat{h} : \forall y \in Y\ .\ h(y) = \hat{h}[y]$,

$\texttt{Assume}\ [a], [b] \in Z$,

$\texttt{Assume}\ [5] : [a] \leq [b]$,

$[6] := \texttt{E}Z[4] : a \leq b$,

$[7] := \texttt{E}\mathsf{POSET}(Y, A)(h)[5] : h(a) \leq h(n)$,

$\Big[\big([a], [b]\big)i.\ *\Big] := [4][7] : \hat{h}[a] \leq \hat{h}[b]$;

$\rightsquigarrow [A.*] := \texttt{E}\mathsf{POSET} : \hat{h} \in \mathsf{POSET}(Z, A)$;

$\rightsquigarrow [*] := \texttt{ICoequalizer} : \texttt{Coequalizer}\Big(\mathsf{POSET}, X, Y, f, g\Big)$;

$\square$

$\texttt{PosetsAreBicomplete} :: \texttt{Bicomplete}(\mathsf{POSET})$

$\texttt{Proof}\ =$

$\ldots$

$\square$

1.2.4 min and max

$\texttt{maximum} :: \prod X \in \mathsf{POSET} \,.\, X^2 \to X$

$\texttt{maximum}\,(x, y) = \max(x, y) := \texttt{if}\ x \le y\ \texttt{then}\ y\ \texttt{else}\ x$

$\texttt{MaximumProperty} :: \forall X : \texttt{Toset} \,.\, \forall x, y \in X \,.\, x \le \max(x, y)\ \&\ y \le \max(x, y)$

$\texttt{Proof}\ =$

$[1] := \texttt{I}(\Rightarrow)\Lambda P : x \le y \,.\, \texttt{I}(\ \&\ )\Big(\texttt{E}(=)\Big(\texttt{E}\max(x, y)\texttt{EifElseThen}P, P\Big),$

$\texttt{E}(=, 2)\Big(\texttt{E}\max(x, y)\texttt{EifElseThen}P, \texttt{EReflexive}(\le_X)(y)\Big)\Big) : x \le y \Rightarrow x \le \max(x, y)\ \&\ y \le \max(x, y),$

$[2] := \texttt{I}(\Rightarrow)\Lambda P : \neg(x \le y) \,.\, \texttt{I}(\ \&\ )\Big(\texttt{E}(=, 2)\Big(\texttt{E}\max(x, y)\texttt{EifElseThen}P, \texttt{EReflexive}((\le_X), x)\Big),$

$\texttt{E}(=)\Big(\texttt{E}\max(x, y)\texttt{EifElseThen}P, \texttt{EToset}(X, P)\Big)\Big) : \neg(x \le y) \Rightarrow x \le \max(x, y)\ \&\ y \le \max(x, y),$

$[*] := \texttt{E}(|)\Big(\texttt{EBool}(x \le y), [1], [2]\Big) : x \le \max(x, y)\ \&\ y \le \max(x, y);$

$\square$

$\texttt{minimum} :: \prod X \in \mathsf{POSET} \,.\, X^2 \to X$

$\texttt{minimum}\,(x, y) = \min(x, y) := \texttt{if}\ x \le y\ \texttt{then}\ x\ \texttt{else}\ y$

$\texttt{MinimumProperty} :: \forall X : \texttt{Toset} \,.\, \forall x, y \in X \,.\, x \ge \min(x, y)\ \&\ y \ge \min(x, y)$

$\texttt{Proof}\ =$

$[*] := \texttt{dualize}(X, \texttt{MaximumProperty})(x, y) : \texttt{This};$

$\square$

1.2.5 Isomorphisms of finite Tosets

$\texttt{FiniteTosetHasTop} :: \forall X : \texttt{Toset} \,.\, 0 < |X| < \infty \Rightarrow \exists \top(X)$

$\texttt{Proof} \ =$

$\mathit{\Omega} := \Lambda n \in \mathbb{N} \,.\, \forall X \in \texttt{Toset} \,.\, |X| = n \Rightarrow \exists \top(X) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ X : \texttt{Toset},$

$\texttt{Assume}\ [1] : |X| = 1,$

$[2] := \texttt{SingltonByCardinality}[1] : \texttt{Singleton}(X),$

$\Big(x, [3]\Big) := \texttt{ESingleton}(x) : \sum x \in X \,.\, \{x\} = X,$

$[4] := \texttt{EReflexive}(X, x) : x \le x;$

$\texttt{Assume}\ y \in X,$

$[5] := [3](y) : y = x,$

$[y.*] := \texttt{E}(=, 1)[5][4] : y \le x;$

$\leadsto [5] := \texttt{I}\forall\texttt{I}\top : x \in \top X;$

$\leadsto [1] := \texttt{I}\mathit{\Omega} : \mathit{\Omega}(1),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [2] : \mathit{\Omega}(n),$

$\texttt{Assume}\ X : \texttt{Toset},$

$\texttt{Assume}\ [3] : |X| = n + 1,$

$[4] := \texttt{EmptyByCardinality}[3] : X \ne \emptyset,$

$x := \texttt{ENonEmpty}[4] \in X,$

$X' := X \setminus \{x\} : \texttt{Subset}(X),$

$[5] := \texttt{E}X'\texttt{CardinalDiff}(X)[3] : |X'| = n,$

$[6] := \texttt{E}\mathit{\Omega}(n)[2](X')[5] : \top(X') \ne \emptyset,$

$x' := \texttt{ENonEmpty}[6] \in \top(X'),$

$y := \max(x, x') \in X,$

$[7] := \texttt{E}X'\texttt{DifferenceStructure} : X = X' \sqcup \{x\},$

$\texttt{Assume}\ z \in X,$

$[8] := \texttt{EDisjointUnion}[7](z) : z \in X' | z \in \{x\},$

$[9] := \texttt{I}(\Rightarrow)\Lambda P : z \in X' \,.\, \texttt{ETransitive}(\le_X)\Big(\texttt{E}\top(X')(x')(z, P), \texttt{E}y\texttt{MaxProperty}(X, x, x')\pi_2\Big)$

$\quad \texttt{I}y : z \in X' \Rightarrow z \le y,$

$[10] := \texttt{I}(\Rightarrow)\Lambda P : z \in \{x\} \,.\, \texttt{ETransitive}(\le_X)\Big(\texttt{ESingleton}(P)\texttt{EReflexive}(\le_X)\texttt{E}y\texttt{MaxProperty}(X, x, x')\pi_1\Big)$

$\quad \texttt{I}y : z \in \{x\} \Rightarrow z \le y,$

$[z.*] := \texttt{E}(|)([8], [9], [10]) : z \le y,$

$\leadsto [n.*] := \texttt{I}(\forall)\texttt{I}(\top) : y \in \top(X);$

$\leadsto [2] := \texttt{I}(\exists)\texttt{E}(\Rightarrow)\texttt{E}(\forall)\texttt{I}(\mathit{\Omega})\texttt{E}\mathbb{N}[1]\texttt{E}\mathit{\Omega} : \forall n \in \mathbb{N} \,.\, \forall X : \texttt{Toset} \,.\, |X| = n \Rightarrow \exists \top(n);$

$[*] := \texttt{I}(\forall)\Lambda X : \texttt{Toset} \,.\, \texttt{I}(\Rightarrow)\Lambda P : 0 < |X| < \infty \,.\, [2]\big(|X|, P\big)(X)\Big(\texttt{I}(=)\Big(\mathbb{N}, \big(|X|, P\big)\Big)\Big) :$

$\quad .\ \forall X : \texttt{Toset} \,.\, |X| < \infty \Rightarrow \exists \top(X);$

$\square$

$\texttt{FiniteTosetHasBottom} :: \forall X : \texttt{Toset} \,.\, 0 < |X| < \infty \Rightarrow \exists\bot(X)$

$\texttt{Proof} \;=$

$[*] := \texttt{dualize}\Big(X, \texttt{FiniteTosetHasTop}\Big) : \exists\bot(X);$

$\square$

$\texttt{setMaximum} :: \prod X : \texttt{Toset} \,.\, ?X$

$\texttt{setMaximum}\,() = \max X := \texttt{ESingleton}\Big(\texttt{FiniteTosetHasTop}(X), \texttt{TopIsUnique}(X)\Big)$

$\texttt{setMinimum} :: \prod X : \texttt{Toset} \,.\, ?X$

$\texttt{setMunimum}\,() = \min X := \texttt{ESingleton}\Big(\texttt{FiniteTosetHasBottom}(X), \texttt{BottomIsUnique}(X)\Big)$

$\texttt{FiniteTosetsAreIsomorphic} :: \forall X, Y : \texttt{Toset} \,.\, |X| = |Y| < \infty \Rightarrow X \cong_{\mathsf{POSET}} Y$

$\texttt{Proof} \;=$

$\Omega := \Lambda n \in \mathbb{Z}_+ \,.\, \forall X, Y : \texttt{Toset} \,.\, |X| = |Y| = n \Rightarrow X \cong_{\mathsf{POSET}} Y : \mathbb{N} \to \texttt{Type},$

$[1] := \texttt{IIsomorphic}(\mathsf{POSET}, \mathrm{id}_\emptyset) : \emptyset \cong_{\mathsf{POSET}} \emptyset,$

$[2] := \texttt{I}\Omega\texttt{I}(\forall)\Lambda X, Y : \texttt{Toset} \,.\, \texttt{I}(\Rightarrow)\Lambda P : |X| = |Y| = 0 \,.\, \texttt{E}(=,1)\Big(\texttt{EmptyByCardinality}(P_1), \texttt{E}(=,2)(\texttt{EmptyByCar}$

$\texttt{Assume}\; n : \mathbb{Z}_+,$

$\texttt{Assume}\; [3] : \Omega(n),$

$\texttt{Assume}\; X, Y : \texttt{Toset},$

$\texttt{Assume}\; [4] : |X| = |Y| = n + 1,$

$x := \min X \in X,$

$y := \min Y \in Y,$

$X' := X \setminus \{x\} :?X,$

$Y' := Y \setminus \{y\} :?Y,$

$[5] := \texttt{E}X'\texttt{CardinalDiff}(X)[4] : |X'| = n,$

$[6] := \texttt{E}Y'\texttt{CardinalDiff}(Y)[4] : |Y'| = n,$

$[7] := \texttt{E}\Omega[3](X', Y')[5][6] : X' \cong_{\mathsf{POSET}} Y',$

$\Big(f', [8]\Big) := \texttt{EIsomorphic}[7] : \sum f' : X' \to Y' \,.\, X \overset{f'}{\leftrightarrow} Y : \mathsf{POSET},$

$f := \Lambda u \in X \,.\, \texttt{if}\; x == u \;\texttt{then}\; y \;\texttt{else}\; f(u) : X \to Y,$

$[8] := \texttt{E}X'\texttt{DifferenceStructure} : X = X' \sqcup \{x\},$

$\texttt{Assume}\; u, v \in X,$

$\texttt{Assume}\; [9] : u \le v,$

$[10] := \texttt{EDisjointUnion}[8](X, u) : u \in X' | u \in \{x\},$

$[11] := \Lambda P : u \in X' \,.\, \texttt{I}f\texttt{E}\mathsf{POSET}(X', Y', f')\Big((u, P), \Big(v, \texttt{I}X'\texttt{EStriclyLessTransitive}\Big(\texttt{E}x\texttt{E}\min\texttt{E}\bot(u, P), [9]\Big)\Big)$

$[12] := \Lambda P : u \in \{x\} \,.\, \texttt{E}(=)\Big(\texttt{ESingleton}(P)\texttt{I}f(x), \texttt{E}y\texttt{E}\min\texttt{E}\bot\big(f(v)\big)\Big) : u \in \{x\} \Rightarrow f(u) \le f(v),$

$\Big[(u, v). *\Big] := \texttt{E}(|)[10][11][12] : f(u) \le f(v);$

$\rightsquigarrow [9] := \texttt{E}\mathsf{POSET} : f \in \mathsf{POSET}(X, Y),$

$[10] := \texttt{E}Y'\texttt{DifferenceStructure} : Y = Y' \sqcup \{y\},$

$\texttt{Assume}\ u, v \in X,$

$\texttt{Assume}\ [11] : f(u) = f(v),$

$[12] := \texttt{EDisjointUnion}[10](Y, f(u)) : f(u) \in Y' | f(u) \in \{y\},$

$[13] := \Lambda P : f(u) \in Y' \,.\, \texttt{E}f\texttt{EIfThenElse}(P)\texttt{EInjective}(X', Y', f')[11] : f(u) \in Y' \Rightarrow u = v,$

$[14] := \Lambda P : f(u) \in \{y\} \,.\, \texttt{E}f\texttt{EIfThenElse}(P) : f(u) \in \{y\} \Rightarrow u = v,$

$\Big[(u, v). *\Big] := \texttt{E}(|)[12][13][14] : f(u) \leq f(v);$

$\rightsquigarrow [11] := \texttt{IInjective} : \texttt{Injective}(X, Y, f),$

$\texttt{Assume}\ z \in Y,$

$[13] := \texttt{EDisjointUnion}[10](Y, z) : z \in Y' | z \in \{y\},$

$[14] := \Lambda P : z \in Y' \,.\, \texttt{E}f\texttt{EIfThenElse}(P)\texttt{ESurjective}(X', Y', f')[12]\texttt{I}f : f(u) \in Y' \Rightarrow \exists f^{-1}(y),$

$[14] := \Lambda P : f(u) \in \{y\} \,.\, \texttt{E}f\texttt{EIfThenElse}(P) : f(u) \in \{y\} \Rightarrow \exists f^{-1}(y),$

$\Big[(u, v). *\Big] := \texttt{E}(|)[13][14][15] : f^{-1}(y);$

$\rightsquigarrow [12] := \texttt{ISurjectiv} : \texttt{Surjective}(X, Y, f),$

$[13] := \texttt{IIsomorphism}[9][11][12] : \texttt{Isomorphism}(\mathsf{POSET}, X, Y, f),$

$[n.*] := \texttt{EIsomorphic}[13] : X \cong_{\mathsf{POSET}} Y;$

$\rightsquigarrow [3] := \texttt{I} \Rightarrow \texttt{I}\forall\texttt{I}\Omega\texttt{EIE}\forall\texttt{E}\mathbb{N}[2]\texttt{E}\Omega : \forall n \in \mathbb{N} \,.\, \forall X, Y \in \texttt{Toset} \,.\, |X| = |Y| = n \Rightarrow X \cong_{\mathsf{POSET}} Y,$

$[*] := \texttt{I}(\forall)\Lambda X, Y : \texttt{Toset} \,.\, \texttt{I}(\Rightarrow)\Lambda P : |X| = |Y| < \infty \,.\, [2]\big(|X|, P\big)(X)\Big(\texttt{I}(=)\Big(\mathbb{N}, \big(|X|, P\big)\Big)\Big) :$

$\quad .\ \forall X, Y : \texttt{Toset} \,.\, |X| = |Y| < \infty \Rightarrow X \cong_{\mathsf{POSET}} Y;$

$\square$

1.3 Well Ordering

1.3.1 Well Founded Sets

$\texttt{WellFounded} :: ?\mathsf{POSET}$

$X : \texttt{WellFounded} \iff \forall T : X \to \texttt{Type} \,.\, \Big(\forall x \in X \,.\, (\forall y \in X \,.\, y < x \Rightarrow T(y)\Big) \Rightarrow T(X)\Big) \Rightarrow \forall x \in X \,.\, T(x)$

$\texttt{WellFoundedHasMin} :: \forall X : \texttt{WellFounded} \,.\, \forall A \subset X \,.\, A \neq \emptyset \Rightarrow \exists \min A$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \min A = \emptyset,$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ [2] : \forall y \in X \,.\, y < x \Rightarrow y \in A^{\complement},$

$\texttt{Assume}\ [3] : x \in A,$

$[4] := \texttt{I}\min[2][3] : x \in \min A,$

$[3.*] := \texttt{E}\emptyset[1][4] : \bot;$

$\rightsquigarrow [x.*] := \texttt{E}(\bot) : x \in A^{\complement};$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow)\texttt{I}(\forall)\texttt{E}\texttt{WellFounded} : \forall x \in X \,.\, x \in A^{\complement};$

$[3] := \texttt{ISubsetISetEq} : X = A^{\complement},$

$[4] := [3]^{\complement} : A = \emptyset,$

$[1.*] := \texttt{I}(\bot)[0][4] : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot : \exists \min A,$

$\square$

$\texttt{StrictlyDecreasing} :: \prod_{X,Y \in \mathsf{POSET}} ?\mathsf{POSET}(X,Y)$

$f : \texttt{StrictlyDeceasing} \iff \forall a, b \in X \,.\, a > b \Rightarrow f(a) < f(b) \iff$

$\texttt{StrictlyIncreasing} :: \prod_{X,Y \in \mathsf{POSET}} ?\mathsf{POSET}(X,Y)$

$f : \texttt{StrictlyIncreasing} \iff \forall a, b \in X \,.\, a > b \Rightarrow f(a) > f(b) \iff$

$\texttt{WellFoundedByAbscenceOfDecreasingSequences} ::$

$\quad : \forall X \in \mathsf{POSET} \,.\, \texttt{StrictlyDecreasing}(\mathbb{N}, X) = \emptyset \Rightarrow \texttt{WellFounded}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A :?X,$

$\texttt{Assume}\ [1] : A \neq \emptyset,$

$\texttt{Assume}\ [2] : \min A = \emptyset,$

$B_1 := A :?X,$

$[3_1] := [1] : A \neq \emptyset,$

$\texttt{Assume}\ n : \mathbb{N},$

$b_n := \texttt{ENonEmpty}[3_n] \in B_n,$

$B_{n+1} := \{a \in A : a < b_n\} :?A,$

$[3_{n+1}] := \texttt{E}\min[2]\texttt{E}B_{n+1} : B_{n+1} \neq \emptyset,$

$[n.*] := \texttt{E}b_n\texttt{E}B_{n+1} : B_{n+1} < b_n;$

$\leadsto b := \texttt{I}\left(\prod\right)\texttt{IStrictlyDecreasing} : \texttt{StrictlyDecreasing}(\mathbb{N}, A),$

$[A.*] := \texttt{E}\emptyset[0](b)\texttt{I}(\bot) : \bot;$

$\leadsto [1] := \texttt{E}(\bot)\texttt{I}(\Rightarrow)\texttt{I}(\forall) : \forall A \subset X \,.\, A \neq \emptyset \Rightarrow \exists \min A,$

$\texttt{Assume}\ T : X \to \texttt{Type},$

$\texttt{Assume}\ [2] : \forall x \in X \,.\, \Big(\forall y \in X \,.\, y < x \Rightarrow T(y)\Big) \Rightarrow T(X),$

$A := \{x \in X : \neg T(x)\} :?X,$

$\texttt{Assume}\ [3] : A \neq \emptyset,$

$a := [1](A)[3]\texttt{ENoneEmpty} \in \min A,$

$[4] := \texttt{E}\min A(a)\texttt{E}A : \forall x \in X \,.\, x < a \Rightarrow T(x),$

$[5] := [2](a)[4] : T(a),$

$[6] := \texttt{E}A(a) : \neg T(A),$

$[3.*] := [6][5] : \bot;$

$\leadsto [3] := \texttt{E}(\bot) : A = \emptyset,$

$[T.*] := \texttt{E}A[3] : \forall x \in X \,.\, T(x);$

$\leadsto [*] := \texttt{IWellFounded} : \texttt{WellFounded}(X);$

$\square$

$\texttt{WellFoundedSubset} :: \forall X : \texttt{WellFounded} \,.\, \forall A \subset X \,.\, \texttt{WellFounded}(A)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.3.2 Well Ordered Sets

$$\mathtt{WellOrdered} := \mathtt{Toset}\ \&\ \mathtt{WellFounded} :?\mathsf{POSET};$$

$$\mathtt{minimumWO} :: \prod X : \mathtt{WellOrdered}\ .\ ?X \to X$$
$$\mathtt{minimumWO}\,(A) = \min A := \mathtt{ETosetWellFoundedHasMin}(X, A)$$

$$\mathtt{Next} :: \prod_{X \in \mathsf{POSET}} X \to ?X$$
$$y : \mathtt{Next} \iff \Lambda x \in X\ .\ x < y\ \&\ \{z \in X : x < z < y\} = \emptyset$$

$$\mathtt{HasNext} :: \prod_{X \in \mathsf{POSET}} ?X$$
$$x : \mathtt{HasNext} \iff \exists \mathtt{Next}(X, x)$$

$$\mathtt{WellOrderedNextIsUnique} :: \forall X : \mathtt{WellOrdered}\ .\ \forall x : \mathtt{HasNext}(X)\ .\ \exists! \mathtt{Next}(X, x)$$
$\mathtt{Proof}\ =$
$\mathtt{Assume}\ y, z : \mathtt{Next}(X, x),$
$[1] := \mathtt{EToset}(X)(y, z) : y \le z | z \le y,$
$[2] := \mathtt{E}_2\mathtt{Next}(X, x, y) : \{u \in X : x < u < y\} = \emptyset,$
$[3] := \mathtt{E}_2\mathtt{Next}(X, x, z) : \{u \in X : x < u < z\} = \emptyset,$
$[4] := \mathtt{E}_1\mathtt{Next}(X, x, y) : x < y,$
$[5] := \mathtt{E}_1\mathtt{Next}(X, x, z) : x < z,$
$\Big[(y, z).*\Big] := \mathtt{E}(|)\Big(\Lambda P : y \le z\ .\ [4][3], \Lambda P : z \le y\ .\ [5][2]\Big) : y = z;$
$\rightsquigarrow [*] := \mathtt{I}\exists!\mathtt{EHasNext}(X, x) : \exists!\mathtt{Next}(X, x);$
$\square$

$$\mathtt{next} :: \prod X : \mathtt{WellOrdered}\ .\ \mathtt{HasNext}(X) \to X$$
$$\mathtt{next}\,(x) = \sigma(x) := \mathtt{WellOrderedNextIsUnique}(X)(x)$$

$$\mathtt{HasPredecessor} :: \prod_{X \in \mathsf{POSET}} ?X$$
$$y : \mathtt{HasPredecessor} \iff \exists x \in X : \mathtt{Next}(X, y, x)$$

$$\mathtt{pred} :: \prod X : \mathtt{WellOrdered}\ .\ \mathtt{HasPredecessor}(X) \to X$$
$$\mathtt{pred}\,(x) = p(x) := \mathtt{EHasPredecessor}(X, x)$$

$$\mathtt{Limit} := \neg\mathtt{HasPredecessor} : \prod_{X \in \mathsf{POSET}} ?X;$$

$\texttt{WellOrderedNextDecomposition} :: \forall X : \texttt{WellOrdered} \,.\, X = \texttt{HasNext}(X) \sqcup \max X$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \neg\texttt{HasNext}(X),$

$A := \{y \in X : x < y\} :?X,$

$\texttt{Assume}\ [0] : A \neq \emptyset,$

$a := \min A \in A,$

$\texttt{Assume}\ z \in X,$

$\texttt{Assume}\ [1] : x < z < a,$

$[2] := \texttt{E}A[1] : z \in A,$

$[3] := \texttt{E}a\texttt{E}\min A[2] : a \leq z,$

$[1.*] := \texttt{E}(z < a)[1][3] : \bot;$

$\rightsquigarrow [1] := \texttt{E}(\bot)\texttt{INext} : \texttt{Next}(X, x, a),$

$[0.*] := \texttt{E}x\texttt{IHasNext}[1] : \bot;$

$\rightsquigarrow [1] := \texttt{E}(\bot) : A = \emptyset,$

$[1.*] := \texttt{E}A\texttt{I}\max X : x \in \max X;$

$\rightsquigarrow [1] := \texttt{ISubset} : \neg\texttt{HasNext}(X) \subset \max X,$

$\texttt{Assume}\ x \in (\max X)^{\complement},$

$A := \{y \in X : x < y\} :?X,$

$[3] := \texttt{E}x\texttt{E}\max X\texttt{I}A : A \neq \emptyset,$

$a := \min A \in A,$

$\texttt{Assume}\ [4] : x < z < a,$

$[5] := \texttt{E}A[1] : z \in A,$

$[6] := \texttt{E}a\texttt{E}\min A[2] : a \leq z,$

$[4.*] := \texttt{E}(z < a)[1][3] : \bot;$

$\rightsquigarrow [4] := \texttt{E}(\bot)\texttt{INext} : \texttt{Next}(X, x, a),$

$[x.*] := \texttt{IHasNext} : \texttt{HasNext}(X, x);$

$\rightsquigarrow [2] := \texttt{ISubset} : (\max X)^{\complement} \Rightarrow \texttt{HasNext}(X, x);$

$[3] := \texttt{UnionCrossIntroduction}[1][2] : X = (\max X) \cup \texttt{HasNext}(X, x),$

$\texttt{Assume}\ x : \max X \sqcup \texttt{WithNext}(X),$

$y := \texttt{EHasNext}(X, x) : y,$

$4 := \texttt{E}_1\texttt{Next}(X, x, y) : x < y,$

$[5] := \texttt{E}\max X(x)(y) : y \leq x,$

$[6] := \texttt{TrichtomyPrincple}[4][5] : \bot;$

$\rightsquigarrow [4] := \texttt{I}\emptyset : \max X \cap \texttt{HasNext}(X) = \emptyset,$

$[*] := \texttt{IDisjoint}[3][4] : X = \texttt{HasNext}(X) \sqcup \max X;$

$\square$

$\texttt{LimitRepresentation} :: \forall X : \texttt{WellOrdered} \,.\, \forall x \in X \,.\, \exists n \in \mathbb{Z}_+ : \exists a : \texttt{Limit}(X) \,.\, x = \sigma^n(a)$

$\texttt{Proof} \ =$

$A := \left\{a \in X : \exists n \in \mathbb{Z}_+ : x = \sigma^n(a)\right\} :?X,$

$[1] := \texttt{E}A\texttt{E}\sigma^0\texttt{I}(=)(x) : x \in A,$

$[2] := \texttt{I}\emptyset[1] : A \neq \emptyset,$

$a := \min A \in a,$

$\Big(n, [3]\Big) := \texttt{E}A(a) : \sum n \in \mathbb{Z}_+ \,.\, x = \sigma^n(a),$

$\texttt{Assume}\ b \in X,$

$\texttt{Assume}\ [4] : \texttt{Next}(X, b, a),$

$[5] := \texttt{E}_1\texttt{Next}(X, b, a) : b < a,$

$[6] := \texttt{I}\sigma[4][3] : x = \sigma^{n+1}(b),$

$[7] := \texttt{E}A[6] : b \in A,$

$[8] := \texttt{E}\min A(a)(b) : a \leq a,$

$[b.*] := \texttt{TrichtomyPrinciple}[5][8] : \bot;$

$\rightsquigarrow [*] := \texttt{E}(\bot)\texttt{E}(\forall)\texttt{ILimit} : \texttt{Limit}(X, a);$

$\square$

$\texttt{zero} :: \prod X : \texttt{WellOrdered} \,\&\, \texttt{nonEmpty} \,.\, X$

$\texttt{zero}\,() = 0_X := \min X$

$\texttt{WellOrderedSubset} :: \forall X : \texttt{WellOrdered} \,.\, \forall A \subset X \,.\, \texttt{WellOrdered}(A)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.3.3 Initial Intervals

$$\texttt{InitialInterval} :: \prod_{X \in \mathsf{POSET}} ?X$$

$$I : \texttt{InitialInterval} \iff \forall a \in I \ . \ \forall x \in X \ . \ x \leq a \Rightarrow x \in I$$

$\texttt{InitialIntervalTransitivity} :: \forall X \in \mathsf{POSET} \ . \ \forall I : \texttt{InitialInterval}(X) \ . \ \forall J : \texttt{InitialInterval}(I) \ . \ \texttt{Ini}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ j \in J,$

$\texttt{Assume} \ x \in X,$

$\texttt{Assume} \ [1] : x \leq j,$

$[2] := \texttt{EInitialInterval}(X, I)[1] : x \in I,$

$[3] := \texttt{EInitialInterval}(I, J) : x \in J;$

$\rightsquigarrow [*] := \texttt{I}(\texttt{InitialInterval}) : \texttt{InitialInterval}(X, J);$

$\square$

$\texttt{InitialInervalIntersection} :: \forall X \in \mathsf{POSET} \ . \ \forall \mathcal{I} \in \mathsf{SET} \ . \ \forall I : \mathcal{I} \to \texttt{InitialInterval}(X) \ .$

$$. \ \texttt{InitialInterval}\left(X, \bigcap_{i \in \mathcal{I}} I_i\right)$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ a \in \bigcap_{i \in \mathcal{I}} I_i,$

$\texttt{Assume} \ x \in X,$

$\texttt{Assume} \ [1] : x \leq a,$

$[2] := \forall i \in \mathcal{I} \ . \ \texttt{EInitialInterval}(X, I_i)(a, x)[1] : \forall i \in \mathbf{i} \ . \ x \in I_i,$

$[a.*] := \texttt{Eintersection}[2] : x \in \bigcup_{i \in \mathcal{I}} I_i;$

$\rightsquigarrow [*] := \texttt{I}(\texttt{InitialInterval}) : \texttt{InitialInterval}\left(X, \bigcup_{i \in \mathcal{I}} I_i\right);$

$\square$

$\texttt{WellOrderedInitialIntervalRepresentation} ::$

$:: \forall X : \texttt{WellOrdered} . \forall I : \texttt{InitialInterval}(X) . I \neq X \Rightarrow \exists x \in X . I = \{i \in X : i < x\}$

$\texttt{Proof} =$

$[1] := [0]^{\complement} : I^{\complement} \neq \emptyset,$

$x := \min I^{\complement} \in I^{\complement},$

$\texttt{Assume } i \in I,$

$\texttt{Assume } [2] : x \leq i,$

$[3] := \texttt{EInitialInterval}[2] : x \in I,$

$[4] := \texttt{Ecomplement}(I, x) : x \notin I,$

$[2.*] := [3][4] : \perp;$

$\leadsto [i.*] := \texttt{TrichtomyPrinciple} : i < x;$

$\leadsto [2] := \texttt{ISubset} : I \subset \{i \in X . i < x\},$

$\texttt{Assume } i \in X,$

$\texttt{Assume } [3] : i < x,$

$[4] := \texttt{E}x\texttt{E} \min I^{\complement}(i)[3] : i \in I^{\complement\complement},$

$[i.*] := \texttt{IdempotentComplement}[4] : i \in I;$

$\leadsto [*] := \texttt{ISubsetISetEq}[2] : I = \{i \in X . i < x\};$

$\square$

$\texttt{InitialIntervalTotallity} :: \forall X : \texttt{WellOrdered} . \forall I, J : \texttt{InitialInterval}(X) . I \subset J | J \subset I$

$\texttt{Proof} =$

$\Big(x, [1] := \texttt{WellOrderedInitilIntervalRepresentation}(X, I) : \sum x \in X . I = \{i \in X : i < x\},$

$\Big(y, [2] := \texttt{WellOrderedInitilIntervalRepresentation}(X, J) : \sum y \in X . J = \{j \in X : j < y\},$

$[3] := \texttt{EToset}(X)(x, y) : x \leq y | y \leq x,$

$[*] := \texttt{ISubset}[1][2][3] : I \subset J | J \subset I;$

$\square$

$\texttt{WellOrderedInitialIntervals} :: \forall X : \texttt{WellOrdered} . \texttt{WellOrdered}\Big(\texttt{InitialInterval}(X), \subset\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{WellOrderedInitialIntervalsIsomorphisms} :: \forall X : \texttt{WellOrdered} . \texttt{InitialInterval}(X) \setminus \{X\} \cong_{\mathsf{POSET}} X$

$\texttt{Proof} =$

$\ldots$

$\square$

1.3.4 Isomorphisms Of Countable Tosets

$\texttt{NaturalNumbersAreWellOrdered} :: \texttt{WellOrdered}(\mathbb{N})$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Unbounded} :: ?\mathsf{POSET}$

$X : \texttt{Unbounded} \iff \max X = \min X = \emptyset$

$\texttt{Dense} :: ?\mathsf{POSET}$

$X : \texttt{Dense} \iff \forall x \in X \,.\, \texttt{Next}(x)$

$\texttt{CountableTosetIsoMorphism} :: \forall X, Y : \texttt{Toset} \,\&\, \texttt{Unbounded} \,\&\, \texttt{Dense} \,.\, |X| = |Y| = \aleph_0 \Rightarrow X \cong_{\mathsf{POSET}} Y$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{RationalNumbersClassifyCountableSubsets} :: \forall X : \texttt{Toset} \,.\, |X| \leq \aleph_0 \Rightarrow \exists A \subset \mathbb{Q} : A \cong_{\mathsf{POSET}} X$

$\texttt{Proof} =$

$\dots$

$\square$

1.4 The Choice

1.4.1 Transfinite Inducion

$\texttt{StriclyIncreasingWellOrdered} :: \forall X : \texttt{WellOrdered} : \forall f : \texttt{Strictlt Increasing}(X,X) \;.\; \forall x \in X \;.\; f(x) \geq x$

$\texttt{Proof} \;=$

$T := \Lambda x \in X \;.\; f(x) \geq x : X \to \texttt{Type},$

$\texttt{Assume}\; x \in X,$

$A := \{y \in X : y < x\} :?X,$

$\texttt{Assume}\; [0] : A \neq \emptyset,$

$[1] := \texttt{EStrictlyIncreasing}(X, X, f)\texttt{E}A : f(A) < f(x),$

$\texttt{Assume}\; [2] \in \forall a \in A \;.\; f(a) \geq a,$

$[3] := [1][2] : A < f(x),$

$\texttt{Assume}\; [4] : f(x) < x,$

$[5] := \texttt{E}A[4] : f(x) \in A,$

$[6] := [3][5] : f(x) < f(x),$

$[0.*] := \texttt{EStrictlyLess}(X, f(x))\texttt{I}(=)(X, f(x)) : \bot;$

$\leadsto [1] := \texttt{E}\bot\texttt{E}^2(\Rightarrow) : A \neq \emptyset \Rightarrow \forall a \in A \;.\; f(a) \geq a \Rightarrow x \leq f(x),$

$\texttt{Assume}\; [2] : A = \emptyset,$

$[3] := \texttt{EWellOrderedI} \min \texttt{I}0 : x = 0,$

$[2.*] := \texttt{E}0(f(x))\texttt{E}(=, 2)[3] : f(x) \geq 0 = x;$

$\leadsto [2] := \texttt{I}(\Rightarrow)\texttt{I}(\forall) : A = \emptyset \Rightarrow \forall a \in A \;.\; f(a) \geq a \Rightarrow x \leq f(x),$

$[x.*] := \texttt{E}(|)\texttt{LEM}(A = \emptyset)[1][2] : \forall a \in A \;.\; f(a) \geq a \Rightarrow x \leq f(x);$

$\leadsto [*] := \texttt{EWellFounded}(X) : \forall x \in X \;.\; x \leq f(x);$

$\square$

$\texttt{TransfiniteRecursion} :: \forall X : \texttt{WellOrdered} \,.\, \forall Y \in \mathsf{SET} \,.$

$$.\ \forall G : \left(\prod_{x\in X}[0,x) \to Y\right) \to Y \,.\, \exists! f : X \to Y : \forall x \in X \,.\, f(x) = G\Big(x, f_{|[0,x)}\Big)$$

$\texttt{Proof}\ =$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ [1] : \forall z \in X \,.\, z < x \Rightarrow \exists f'_y : [0,z] \to Y \,.\, \forall u \in [0,z] \,.\, f'(u) = G(u, f'_{|[0,u)}),$

$\texttt{Assume}\ u, v : [0,x),$

$\Big(f'_u, [2]\Big) := [1](u) : \sum f'_u : [0,u] \to Y \,.\, \forall a \in [0,u] \,.\, f(a) = G(a, f'_{u|[0,a)}),$

$\Big(f'_v, [3]\Big) := [1](v) : \sum f'_v : [0,v] \to Y \,.\, \forall a \in [0,v] \,.\, f(a) = G(a, f'_{v|[0,a)}),$

$m := \min(u,v) \in X,$

$\texttt{Assume}\ a \in [0,m),$

$\texttt{Assume}\ [5] : \forall b \in [0,m) \,.\, b < a \Rightarrow f'_v(b) = f'_u(b),$

$[6] := [5]\texttt{Iconstraint}[0,a) : f'_{v|[0,a)} = f'_{u|[0,a)},$

$[a.*] := [2][6][3] : f'_v(a) = G(a, f'_{v|[0,a)}) = G(a, f'_{u|[0,a)}) = f'_u(a);$

$\rightsquigarrow \Big[(u,v).*\Big] := \texttt{EWellFounded}\Big([0,m)\Big) : f'_{v|[0,m)} = f'_{U|[0,m)};$

$\rightsquigarrow [2] := \texttt{E}mI(\forall) : \forall u, v \in [0,x) \,.\, f'_{v|[0,\min(u,v)]} = f'_{u|[0,\min(u,v)]},$

$f'' := \Lambda a \in [0,x) \,.\, [1](a)(a) : [0,x) \to Y,$

$f'_x := \Lambda a \in [0,x] \,.\, \texttt{if}\ a < x\ \texttt{then}\ f''(a)\ \texttt{else}\ G\Big(x, f''\Big) : [0,x] \to Y,$

$[1.*] := \texttt{E}f'_x[2] : \forall a \in [0,x] \,.\, f'_x(a) = G\Big(x, f'_{|[0,a)}\Big);$

$\rightsquigarrow [1] := \texttt{EWellFounded}(X) : \forall a \in X \,.\, \exists f'_a(a) : [0,a] \to Y : \forall a \in [0,x] \,.\, f'_x(a) = G\Big(x, f'_{|[0,a)}\Big),$

$f := \Lambda a \in X \,.\, [1](a)(a) : X \to Y,$

$\texttt{Assume}\ u, v : [0,x),$

$\Big(f'_u, [2]\Big) := [1](u) : \sum f'_u : [0,u] \to Y \,.\, \forall a \in [0,u] \,.\, f(a) = G(a, f'_{u|[0,a)}),$

$\Big(f'_v, [3]\Big) := [1](v) : \sum f'_v : [0,v] \to Y \,.\, \forall a \in [0,v] \,.\, f(a) = G(a, f'_{v|[0,a)}),$

$m := \min(u,v) \in X,$

$\texttt{Assume}\ a \in [0,m),$

$\texttt{Assume}\ [5] : \forall b \in [0,m) \,.\, b < a \Rightarrow f'_v(b) = f'_u(b),$

$[6] := [5]\texttt{Iconstraint}[0,a) : f'_{v|[0,a)} = f'_{u|[0,a)},$

$[a.*] := [2][6][3] : f'_v(a) = G(a, f'_{v|[0,a)}) = G(a, f'_{u|[0,a)}) = f'_u(a);$

$\rightsquigarrow \Big[(u,v).*\Big] := \texttt{EWellFounded}\Big([0,m)\Big) : f'_{v|[0,m)} = f'_{U|[0,m)};$

$\rightsquigarrow [2] := \texttt{E}mI(\forall) : \forall u, v \in X \,.\, f'_{v|[0,\min(u,v)]} = f'_{u|[0,\min(u,v)]},$

$[3] := \texttt{E}f[2] : \forall a \in X \,.\, f'_x(a) = G\Big(x, f'_{|[0,a)}\Big);$

$\square$

$\texttt{WellOrderedTotality} :: \forall X, Y : \texttt{WellOrdered}\ .$

$\quad .\ \exists I : \texttt{InitialInterval}(X)\ .\ I \cong_{\mathsf{POSET}} X \Big| \exists I : \texttt{InitialInterval}(Y)\ .\ I \cong_{\mathsf{POSET}} Y$

$\texttt{Proof}\ =$

$G := \Lambda x \in X\ .\ \Lambda g : [0, x) \to Y \sqcup \{\infty\}\ .\ \texttt{if}\ (Y \sqcup \{\infty\}) \setminus \operatorname{Im} g \neq \emptyset\ \texttt{then}\ \min(Y \sqcup \{\infty\}) \setminus \operatorname{Im} g\ \texttt{else}\ \infty :$

$\quad : \prod_{x \in X} \Big([0, x) \to Y \sqcup \{\infty\}\Big) \to Y \sqcap \{\infty\},$

$\Big(f, [1]\Big) := \texttt{TranssfiniteRecursion}(X, Y \sqcup \{\infty\}, G) : \sum f : X \to Y \sqcup \{\infty\}\ .\ \forall x \in X\ .\ f(x) = G\Big(x, f_{|[0,x)}\Big),$

$[2] := [1]\texttt{E}G\texttt{E}\min \texttt{I}\mathsf{POSET} : \mathsf{POSET}(X, Y \sqcup \{\infty\}, f),$

$\texttt{Assume}\ [3] : \infty \notin \operatorname{Im} f,$

$[4] := \texttt{E}f\texttt{E}G[3] : \texttt{injective}(X, Y, f),$

$\texttt{Assume}\ y \in \operatorname{Im} f,$

$\texttt{Assume}\ a \in Y,$

$\texttt{Assume}\ [5] : a < y,$

$\Big(x, [6]\Big) := \texttt{E}\operatorname{Im} f(y) : \sum x \in X\ .\ y = f(x),$

$[7] := \texttt{E}f[6]\texttt{E}G : y = \min Y \setminus f([0, x)),$

$[y.*] := \texttt{EimageE}\min[7][5]\texttt{I}\operatorname{Im} f : a \in \operatorname{Im} f;$

$\rightsquigarrow [3.*] := \texttt{IInitialInterval} : \texttt{InitialInterval}(Y, \operatorname{Im} f);$

$[4] := \texttt{I}(\Rightarrow) : \infty \notin \operatorname{Im} f \Rightarrow \exists I : \texttt{InitialInterval}(Y)\ .\ I \cong_{\mathsf{POSET}} X,$

$\texttt{Assume}\ [4] : \infty \in \operatorname{Im} f,$

$I := f^{-1}(Y) :?X,$

$g := f_{|I} : I \to Y,$

$[5] := \texttt{E}f\texttt{E}G[3] : \texttt{Isomorphism}(\mathsf{POSET}, X, Y, g),$

$\texttt{Assume}\ i \in I,$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ [6] : x < i,$

$[7] := \texttt{E}I\texttt{E}f[6] : f(x) \neq \infty,$

$[i.*] := \texttt{E}I[7] : x \in I;$

$\rightsquigarrow [4.*] := \texttt{IInitialInterval} : \texttt{InitialInterval}(X, I);$

$\rightsquigarrow [4] := \texttt{I}(\Rightarrow) : \infty \in \operatorname{Im} f \Rightarrow \exists I : \texttt{InitialInterval}(X) I \cong_{\mathsf{POSET}} X,$

$[*] := \texttt{OrPushforwardLEM}(|)[3][4] :$

$\quad : \exists I : \texttt{InitialInterval}(X)\ .\ I \cong_{\mathsf{POSET}} Y \Big| \exists I : \texttt{InitialInterval}(Y)\ .\ I \cong_{\mathsf{POSET}} X;$

$\square$

$\texttt{InitialIntervalIsNotIsomorphic} :: \forall X : \texttt{WellOrdered} \,.\, \forall I : \texttt{InitialInterval}(X) \,.$
$\quad . \, I \neq X \Rightarrow \neg\Big(X \cong_{\mathsf{POSET}} I\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : X \cong_{\mathsf{POSET}} I,$

$f := \texttt{EIsomorphic}[1] : X \xleftrightarrow{\mathsf{POSET}} I,$

$x := \texttt{E}[0] : I^{\complement},$

$[2] := \texttt{EInitialInterval}(X)(x) : I < x,$

$[3] := \texttt{StrictlyIncreasingWellOrdered}(X, f, x) : x \leq f(x),$

$[4] := [2][3] : I < f(x),$

$[5] := \texttt{E}f : f(x) \in I,$

$[6] := \texttt{E}(I < f(x)) : f(x) \notin I,$

$[1.*] := [5][6] : \bot;$

$\rightsquigarrow [*] := \texttt{I}(\neg) : \neg\Big(X \cong_{\mathsf{POSET}} I\Big);$

$\square$

$\texttt{OrderType} :: ?(\texttt{WellOrdered} \times \texttt{WellOrdered})$
$X, Y : \texttt{OrderType} \iff X \leq_{\mathsf{ORD}} Y \iff \exists I : \texttt{InitialInterval}(Y) \,.\, I \cong_{\mathsf{POSET}} X$

$\texttt{OrderTypeIsWellOrdering} :: \forall \mathcal{X} : ?\texttt{WellOrdered} \,.\, \texttt{WellOrdered}(\mathcal{X}, \leq_{\mathsf{ORD}})$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.4.2 Zermelo's Theorem

$$\texttt{DiscriminationFunction} :: \prod_{X \in \mathsf{SET}} ?X \setminus \{X\} \to X$$

$$f : \texttt{DiscriminationFunction} \iff \forall A :?X \setminus \{X\} \; . \; f(A) \in A^{\complement}$$

$$\texttt{DiscriminationFunctionExists} :: \forall X \in \mathsf{SET} \; . \; X \neq \emptyset \Rightarrow \exists \texttt{DiscriminationFunction}(X)$$

$\texttt{Proof} =$

$$f := \texttt{Choice}\left\{A^{\complement} | A \subset X : A \neq \emptyset\right\} : \texttt{DiscriminationFunction}(X);$$

$\square$

$$\texttt{CorrectFragment} :: \prod_{X \in \mathsf{SET}} \prod f : \texttt{DiscriminationFunction}(X) \; . \; ? \sum_{A \subset X} \texttt{Order}(A)$$

$$(A, \leq) : \texttt{CorrectFragment} \iff \texttt{WellOrdered}(A, \leq) \;\&\; \forall a \in A \; . \; f[0, a) = a$$

$$\texttt{CorrectFragmentTotallity} :: \forall X \in \mathsf{SET} \,.\, \forall f : \texttt{DiscriminationFunction}(X) \,.$$
$$.\, \forall A, B : \texttt{CorrectFragment}(X, f) \,.\, A \leq_{\mathsf{ORD}} B | B \leq_{\mathsf{ORD}} A$$

$\texttt{Proof} =$

$[1] := \texttt{WellOrderedTotality}(A, B) :$

$: \exists I : \texttt{InitialInterval}(A) \,.\, I \cong_{\mathsf{POSET}} X \Big| \exists I : \texttt{InitialInterval}(B) \,.\, I \cong_{\mathsf{POSET}} Y,$

$\texttt{Assume}\ I : \texttt{InitialInterval}(A),$

$\texttt{Assume}\ [2] : I \cong_{\mathsf{POSET}} B,$

$g := \texttt{EIsomorphic}(\mathsf{POSET}) : \texttt{Isomorphism}(\mathsf{POSET}, I, B),$

$\texttt{Assume}\ i \in I,$

$\texttt{Assume}\ [3] : \forall j \in I \,.\, j < i \Rightarrow g(j) = j,$

$[4] := \texttt{ECorrectFragment}(A, i) : i = f[0, i),$

$[5] := \texttt{Eimage}[3]\texttt{ISubset} : [0, i) \subset B,$

$[6] := \texttt{E}\mathsf{POSET}(I, B)[5] : [0, i) \leq g(i),$

$\Big(j, [7]\Big) := \texttt{InitialIntervalStructure}\Big(B, [0, i)\Big) : \sum j \in B \,.\, [0, j)_B = [0, i)_A,$

$[8] := \texttt{E}\Big(\texttt{Poset}, I, B, g\Big)[7] : j = g(i),$

$[i.*] := [8]\texttt{ECorrectFragment}(B, j)[7] : g(i) = j = f[0, j)_B = f[0, i)_A = i;$

$\rightsquigarrow [3] := \texttt{EWellOrdered}(I) : g = \mathrm{id}_I,$

$[2.*] := \texttt{E}g[3]\texttt{ISubset} : B \subset_{\mathsf{POSET}} A;$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow: \Big(\exists I : \texttt{InitialInterval}(A) \,.\, I \cong_{\mathsf{POSET}} B\Big) \Rightarrow B \subset_{\mathsf{POSET}} A,$

$\texttt{Assume}\ I : \texttt{InitialInterval}(B),$

$\texttt{Assume}\ [3] : I \cong_{\mathsf{POSET}} A,$

$g := \texttt{EIsomorphic}(\mathsf{POSET}) : \texttt{Isomorphism}(\mathsf{POSET}, I, A),$

$\texttt{Assume}\ i \in I,$

$\texttt{Assume}\ [4] : \forall j \in I \,.\, j < i \Rightarrow g(j) = j,$

$[5] := \texttt{ECorrectFragment}(B, i) : i = f[0, i),$

$[6] := \texttt{Eimage}[4]\texttt{ISubset} : [0, i) \subset A,$

$[7] := \texttt{E}\mathsf{POSET}(I, A)[6] : [0, i) \leq g(i),$

$\Big(j, [8]\Big) := \texttt{InitialIntervalStructure}\Big(A, [0, i)\Big) : \sum j \in A \,.\, [0, j)_A = [0, i)_B,$

$[9] := \texttt{E}\Big(\texttt{Poset}, I, A, g\Big)[8] : j = g(i),$

$[i.*] := [9]\texttt{ECorrectFragment}(B, j)[8] : g(i) = j = f[0, j)_A = f[0, i)_B = i;$

$\rightsquigarrow [4] := \texttt{EWellOrdered}(I) : g = \mathrm{id}_I,$

$[3.*] := \texttt{E}g[3]\texttt{ISubset} : A \subset_{\mathsf{POSET}} B;$

$\rightsquigarrow [3] := \texttt{I} \Rightarrow: \Big(\exists I : \texttt{InitialInterval}(A) \,.\, I \cong_{\mathsf{POSET}} A\Big) \Rightarrow A \subset_{\mathsf{POSET}} B,$

$[*] := \texttt{OrPushforward}[1, 2, 3] :$

$\exists I : \texttt{InitialInterval}(A) \,.\, I \cong_{\mathsf{POSET}} X \Big| \exists I : \texttt{InitialInterval}(B) \,.\, I \cong_{\mathsf{POSET}} Y;$

$\square$

$\mathtt{correctFragmentUnion} :: \prod_{X \in \mathsf{SET}} . \prod f : \mathtt{DiscriminationFunction}(X) .$

$. \mathtt{CorrectFragment}^2(X, f) \to \mathtt{CorrectFragment}(X, f)$

$\mathtt{correctFragmentUnion}\,(A, B) = A \cup B := \mathtt{if}\ A \subset B\ \mathtt{then}\ B\ \mathtt{else}\ A$

$\mathtt{ZermeloTHM} :: \forall X \in \mathsf{SET} . \exists (\leq) : \mathtt{Order}(X) . \mathtt{WellOrdered}\Big(X, (\leq)\Big)$

$\mathtt{Proof}\ =$

$f := \mathtt{DiscriminationFunctionExists}(X) : \mathtt{DiscriminationFunctionExists}(X),$

$C := \mathtt{CorrectFragment}(X, f) \in \mathsf{SET},$

$[1] := \mathtt{ICorrectFragment}(X, f)(\emptyset) : \emptyset \in C,$

$a := \bigcap C :?X,$

$R := \{(x, y) \in a : \exists b \in A : x \leq_b y\} :?a^2,$

$[3] := \mathtt{CorrectFragmentTotality}(X, C) : \mathtt{Toset}(a, R),$

$\mathtt{Assume}\ A :?a,$

$\mathtt{Assume}\ [4] : A \neq \emptyset,$

$(c, [5]) := \mathtt{EaE}A[4] : \sum c \in C . c \cap A = \emptyset,$

$x := \min c \cap A \in c \cap A,$

$\mathtt{Assume}\ y \in A,$

$\mathtt{Assume}\ [6] : x < y,$

$(b, [7]) := \mathtt{EaE}A[6] : \sum b \in C . y \in b,$

$[8] := \mathtt{CorrectFragmentTotality}[5][6][7] : y \in c \cap A,$

$[y.*] := \mathtt{E}\min[6][8] : \perp;$

$\rightsquigarrow [A.*] := \mathtt{I}\min : x = \min A;$

$\rightsquigarrow [3] := \mathtt{IWellOrdered} : \mathtt{WellOrdered}(a),$

$\mathtt{Assume}\ [4] : a \neq X,$

$x := f(a) : a^{\complement},$

$b := a \cup \{x\} :?X,$

$R := \leq_a \cup \{(y, x) | y \in b\} : \mathtt{Order}(b),$

$[5] := \mathtt{E}R\mathtt{EWellOrdered}(a)\mathtt{IWellOrdered} : \mathtt{WellOrdered}(b, R),$

$[6] := \mathtt{E}b\mathtt{I}x : f[0, x)_b = f(a) = x,$

$[7] := [5][6]\mathtt{I}C : b \in c,$

$[8] := \mathtt{E}b\mathtt{E}x\mathtt{E}a\mathtt{I}b\mathtt{I}c : b \notin c,$

$[9] := [7][8] : \perp;$

$\rightsquigarrow [4] := \mathtt{E}\perp : X = a,$

$[2] := \mathtt{E}(=)[4][3] : \mathtt{WellOrdered}(a);$

$\square$

$\texttt{CardinalsAreComparable} :: \forall X, Y \in \mathsf{SET} \,.\, |X| \leq |Y| \Big| |Y| \leq |x|$

$\texttt{Proof} \ =$

$\Big(\leq_X, [1] \Big) := \texttt{ZermeloTHM}(X) : \sum \leq_X : \texttt{Order}(X) \,.\, \texttt{WellOrdered}(X, \leq_X),$

$\Big(\leq_Y, [2] \Big) := \texttt{ZermeloTHM}(X) : \sum \leq_Y : \texttt{Order}(Y) \,.\, \texttt{WellOrdered}(X, \leq_X),$

$[3] := \texttt{WellOrderedTotality} : \exists I : \texttt{InitialInterval}(X) \,.\, I \cong_{\mathsf{POSET}} Y \Big|$
$\Big| \exists I : \texttt{InitialInterval}(Y) \,.\, I \cong_{\mathsf{POSET}} X,$

$[*] := \texttt{OrPushforward}[3]\texttt{ICardinalityLess} : |X| \leq |Y| \Big| |Y| \leq |X|;$

$\square$

1.4.3 Zorn's Lemma

$$\texttt{Chain} :: \prod_{X \in \mathsf{POSET}} ??X$$

$$C : \texttt{Chain} \iff C \in \mathcal{C}(X) \iff \texttt{Toset}(X)$$

$$\texttt{UpperBound} :: \prod_{X \in \mathsf{POSET}} ?X \to ?X$$

$$x : \texttt{UpperBound} \iff \Lambda A \subset X \ . \ A \leq x$$

$\texttt{ZornsLemma} :: \forall X \in \mathsf{POSET} \ . \ \Big(\forall C \in \mathcal{C}(X) \ . \ \exists \texttt{UpperBound}(X, C)\Big) \Rightarrow \exists \max X$

$\texttt{Proof} \ =$

$\Big((\prec), [1]\Big) := \texttt{ZermeloTHM}(X) : \sum (\prec) : \texttt{Order}(X) \ . \ \texttt{WellOrdered}(X, \prec),$

$\texttt{Assume} \ x \in X,$

$\texttt{Assume} \ f : [0, x)_\prec \to X \sqcup \{\star\},$

$\texttt{Assume} \ [2] : f[0, x) \in \mathcal{C}(X, \leq),$

$[3] := [0]\Big(f[0, x), [1]\Big) : \texttt{UpperBound}\Big((X, \leq), f[0, x)\Big) \neq \emptyset,$

$G(x, f_{|[0,x)}) := \texttt{if} \ x \in \texttt{UpperBound}\Big((X, \leq), f[0, x)\Big) \ \texttt{then} \ x \ \texttt{else} \ \texttt{ENonEmptyUpperBound}\Big((X, \leq), f[0, x)\Big) :$

$\quad : \texttt{UpperBound}\Big((X, \leq), f[0, x)\Big),$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow)\texttt{I}\sqcup : f[0, x) \in \mathcal{C}(x, \leq) \Rightarrow X \sqcup \{\star\},$

$\texttt{Assume} \ [3] : f[0, x) \notin \mathcal{C}(x, \leq),$

$G(x, f) := \star : \texttt{UpperBound}\Big((X, \leq), f[0, x)\Big);$

$\rightsquigarrow [3] := \texttt{I}(\Rightarrow) Intro\sqcup : f[0, x) \notin \mathcal{C}(x, \leq) \Rightarrow X \sqcup \{\star\},$

$G(x, f) := \texttt{E}(|)\texttt{LEM}[2][3] : X \sqcup \{\star\};$

$\rightsquigarrow G := \texttt{I}\left(\prod\right) : \prod_{x \in X} f[0, x) \to X \sqcup \{\star\} \to X \sqcup \{\star\},$

$\Big(f, [2]\Big) := \texttt{TransfinitrRecursion}\Big(X, X \sqcup \{\star\}, G\Big) : \sum f : X \to X \sqcup \{\star\} \ . \ \forall x \in X \ . \ f(x) = G(x, f_{|[0,x)}),$

$\texttt{Assume} \ x \in X,$

$\texttt{Assume} \ [3] : \forall y \in Y \ . \ y \prec x \Rightarrow f(y) \neq \star,$

$[4] := [2]\texttt{E}G : \forall y \in [0, x) \ . \ f[0, y) \in \mathcal{C}(X),$

$[5] := \texttt{ImageUnion}\Big(f, [0, x)\Big)\texttt{E}\mathcal{C}(X)\texttt{EunionI}\mathcal{C}(X) : f[0, x) = \bigcup_{y \in [0, x)} f[0, y] \in \mathcal{C}(X),$

$[*] := [2](x)\texttt{E}G[5] : f(x) \neq \star;$

$\rightsquigarrow [3] := \texttt{EWellOrdered}(X) : \star \notin \operatorname{Im} f,$

$[4] := \texttt{E}f\texttt{E}G : \texttt{StrictlyIncreasing}\Big((X, \prec), (X, \leq)\Big),$

$[5] := \texttt{E}\mathsf{POSET}(X)\texttt{I}\mathcal{C} : f(X) \in \mathcal{C}(X),$

$x := [0]\Big(f(X)\Big) : \texttt{UpperBound}\Big((X, \leq), f(X)\Big),$

$\texttt{Assume} \ z \in X,$

$\texttt{Assume} \ [6] : f(z) < z,$

$[7] := [2]\texttt{E}f\texttt{E}G : f(z) \in \texttt{UpperBound}\Big((X, \prec), f[0, z)\Big),$

$[8] := [7]\texttt{EUpperBound}\Big((X, \prec), f[0, z)\Big)[7]\texttt{IUpperBound}\Big((X, \prec), f[0, z]\Big) : z \in \texttt{UpperBound}\Big((X, \prec), f[0, z)\Big),$

$[9] := [2]\texttt{E}f\texttt{E}G[9] : f(z) = z,$

$[[z.*]] := \texttt{EStrictlyLess}[6][9] : \bot;$

$\rightsquigarrow [6] := \texttt{E}(\bot)\texttt{I}(\forall) : \forall z \in X \ . \ z \not< f(z),$

$\texttt{Assume} \ y \in X,$

$\texttt{Assume} \ [7] : y > x,$

$[8] := \texttt{E}f\texttt{E}G[6] : f(y) < x < y,$

$[9] := [6][8] : \bot,$

$\rightsquigarrow [*] := \texttt{I} \max X : x \in \max X;$

$\square$

$\texttt{SpecialZornsLemma} :: \forall X : \mathsf{POSET} \; . \; \Big(\forall C \in \mathcal{C}(X) \; . \; \exists \texttt{UpperBound}(X, C)\Big) \forall x \in X \; . \; \exists m \in \max X \; . \; x \leq m$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{TotalExtensionExists} :: \forall X : \mathsf{POSET} \; . \; \exists R : \texttt{TotalOrder}(X) : (\leq_X) \subset R$

$\texttt{Proof} \; =$

$\texttt{Assume } C \in \mathcal{C}\Big(\texttt{Order}(X), \subset\Big),$

$R := \bigcup C :?(X^2),$

$[1] := \texttt{E}R\texttt{E}C\texttt{IOrderI}R : \texttt{Order}(X, R),$

$[C.*] := \texttt{E}R\texttt{EunionIUpperBound} : \texttt{UpperBound}\Big(\Big(X, \subset\Big), C, R\Big);$

$\leadsto [2] := \texttt{I}(\forall) : \forall C \in \mathcal{C}\Big(\texttt{Order}(X), \subset\Big) \; . \; \exists \texttt{UpperBound}\Big(\Big(\texttt{Order}(X), \subset\Big), C, R\Big),$

$\Big(R, [3]\Big) := \texttt{SpecialZornsLemma}\Big(\Big(\texttt{Order}(X), \subset\Big), [2], (\leq_X)\Big) : \sum R \in \max\Big(\texttt{Order}(X), \leq\Big) \; . \; (\leq)_X \subset R,$

$\texttt{Assume } x, y \in X,$

$\texttt{Assume } [4] \in \neg(xRy) \;\&\; \neg(yRx),$

$[5] := \texttt{ETransitive}(R)[4] : \forall z \in X \; . \; xRz \Rightarrow \neg(zRy) \;\&\; yRz \Rightarrow \neg(zRy) \;\&\; zRx \Rightarrow \neg(yRz) \;\&\; zRy \Rightarrow \neg(xRz),$

$[6] := \texttt{EReflexive}(R)[5] : x \neq y,$

$R' := \Big\{(a, b) \in X^2 \Big| n \in \mathbb{N}, z : [1, \ldots, n] \to X, a = z_1, b = z_n,$

$\quad , \forall i \in [1, \ldots, n-1] \; . \; (z_i, z_{i+1}) \in R | z_i = x \;\&\; z_{i+1} = y\Big\} :?X^2,$

$[7] := \texttt{E}R'\texttt{E}(2) : R \subset R',$

$[8] := \texttt{E}R'\texttt{E}(1) : \texttt{Reflexive}(X, R'),$

$[9] := \texttt{E}R' : \texttt{Transitive}(X, R'),$

$\texttt{Assume } x', y' \in X,$

$\texttt{Assume } [10] : x'R'y' \;\&\; y'Rx',$

$\texttt{Assume } [11] : x' \neq y',$

$\Big(n, u, [12]\Big) := \texttt{E}R[10.1] : \sum_{n=1}^{\infty} \sum y : [1, \ldots, n] \to X \; . \; x' = y_1 \;\&\; y' = y_n \;\&$

$\quad \& \; \forall i \in [1, \ldots, n-1] \; . \; \Big((u_i, u_{i+1}) \in R | (u_i = x \;\&\; u_{i+1} = y)\Big),$

$\Big(m, v, [13]\Big) := \texttt{E}R[10.2] : \sum_{m=1}^{\infty} \sum v : [1, \ldots, n] \to X \; . \; y' = v_1 \;\&\; x' = v_n \;\&$

$\quad \& \; \forall i \in [1, \ldots, m-1] \; . \; \Big((v_i, v_{i+1}) \in R | (v_i = x \;\&\; v_{i+1} = y)\Big),$

$[14] := \texttt{EAntisymmetric}(X, R)[4][11][12][13] : \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} v_i = x \;\&\; v_{i+1} = y \;\&\; u_j = x \;\&\; u_{j+1} = y,$

$[15] := \texttt{ETranstive}(X, R)[14] : x'Rx \;\&\; y'Rx \;\&\; yRx' \;\&\; yRy',$

$[11.*] := [5](x')'\Big[15.(1, 2)\Big] : \perp;$

$\leadsto \Big[(x', y').*\Big] := \texttt{E}(\perp) : x' \neq y';$

$\leadsto [10] := \texttt{IOrder}(X)[8, 9] : \texttt{Order}(X, R),$

$\Big[(x, y).*\Big] := \texttt{E}\max\Big(\texttt{Order}(X), \subset\Big)\Big(R', [10]\Big)[7] : \perp;$

$\leadsto [*] := \texttt{E}\perp\texttt{I}\forall\texttt{IToset} : \texttt{Toset}(X, R);$

$\square$

1.5 Ordinal

1.5.1 Numbers

$\texttt{ordinals} :: \mathsf{CAT}$

$\texttt{ordinals}\,() = \mathsf{ORD} := \Big(\texttt{WellOrdered}, \Lambda X, Y : \texttt{WellOrdered}\ \texttt{if}\ X \le_{\mathsf{ORD}} Y\ \texttt{then}\ 1\ \texttt{else}\ 0, (1,1) \mapsto 1, 1\Big)$

$\texttt{OrdinalsAreWellOrdered} :: \forall X \in \mathsf{SET} . \forall n : X \to \mathsf{ORD} . \texttt{WellOrdered}(\operatorname{Im} n)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ordinalOuterPredicatTransfer} :: \Big(\prod X : \texttt{WellOrdered} . ?X \to ?X\Big) \to ?\mathsf{ORD} \to ?\mathsf{ORD}$

$\texttt{ordinalOuterPredicatTransfer}\,(P) = P := \Lambda A \in \mathsf{ORD} . \Lambda a \in \mathsf{ORD} . \exists X \in \mathsf{SET} : \exists n : X \to \mathsf{ORD} .$
$. a \in \operatorname{Im} n \ \&\ P(\operatorname{Im} n \cap A, a) \ \&\ \forall Y \in \mathsf{SET} : \forall m : Y \to \mathsf{ORD} . \ \operatorname{Im} n \subset \operatorname{Im} m \Rightarrow P(\operatorname{Im} m \cap A, a)$

$\texttt{ordinalInnerPredicatTransfer} :: \Big(\prod X : \texttt{WellOrdered} . \prod_{A:?X} ?A\Big) \to \prod_{A:?\mathsf{ORD}} \mathsf{ORD}$

$\texttt{ordinalInnerPredicatTransfer}\,(P) = P := \Lambda A :?\mathsf{ORD} . \Lambda a \in A . \forall X \in \mathsf{SET} . \forall n : X \to \mathsf{ORD} . \ \operatorname{Im} n \subset A \Rightarrow$
$\Rightarrow \exists Y \in \mathsf{SET} . \exists m : Y \to \mathsf{ORD} : \operatorname{Im} n \subset \operatorname{Im} n \ \&\ a \in \operatorname{Im} n \ \&\ P(\operatorname{Im} n, a)$

$\texttt{OrdinalAsInterval} :: \forall n \in \mathsf{ORD} . n \cong_{\mathsf{POSET}} [0, n)_{\mathsf{ORD}}$

$\texttt{Proof} =$

...

$\square$

$\texttt{nextOrd} :: \mathsf{ORD} \xrightarrow{\mathsf{CAT}} \mathsf{ORD}$

$\texttt{nextOrd}\,(a) = \sigma(a) := \texttt{InitialInterval}(a)$

$\texttt{LimitOrdinal} :: ?\mathsf{ORD}$

$n : \texttt{LimiOrdinal} \iff \forall a \in \mathsf{ORD} . n \ncong_{\mathsf{ORD}} \sigma(a)$

$\texttt{Bounded} :: \prod_{X \in \mathsf{POSET}} ??X$

$A : \texttt{Bounded} \iff \exists \texttt{UpperBound}(X, A) \iff$

$\texttt{BoundedOrdinalsHaveLub} :: \forall A : \texttt{Bounded}(\mathsf{ORD}) . \exists \min \texttt{UpperBound}(\mathsf{ORD}, A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{TransitiveSet} :: ?\mathsf{SET}$

$A : \texttt{TransitiveSet} \iff \forall a \in A \,.\, \forall b \in a \,.\, b \in A$

$\texttt{ZFOrder} :: \prod_{X \in \mathsf{SET}} ?X^2$

$\texttt{ZFOrder}\,() = \leq_{\mathsf{ZF}} := \left\{(x, y) \in X^2 \middle| x = y | x \in y\right\}$

$\texttt{WellFoundnesAxiom} := \forall X \in \mathsf{SET} \,.\, X \neq \emptyset \Rightarrow \exists a \in X \,.\, a \cap X = \emptyset : \texttt{Type};$

$\texttt{OrdinalSet} :: ?\texttt{TransitiveSet}$

$A : \texttt{OrdinalSet} \iff \forall a \in A \,.\, \texttt{TransitiveSet}(a)$

$\texttt{OrdinalSetIsWellFounded} :: \texttt{WellFoundnesAxiom} \Rightarrow \forall A : \texttt{OrdinalSet} \,.\, \texttt{WellOrdered}(A, \leq_{\mathsf{ZF}})$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{OrdinalOrderCorrespondsToSubsetOrder} :: \forall X : \texttt{WellOrdered} \forall A \subset X \,.\, A \leq_{\mathsf{ORD}} X$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : X <_{\mathsf{ORD}} A,$

$\Big(I, [2]\Big) := \texttt{E}(<_{\mathsf{ORD}})[1] : \sum I : \texttt{InitialInterval}(A) \,.\, I \cong_{\mathsf{POSET}} X \ \&\ I \subsetneq A,$

$f := \texttt{EIsomorphism}[2.1] : \texttt{Isomorphism}(\mathsf{POSET}, X, I),$

$\Big(a, [3]\Big) := \texttt{EInitialInterval}(A, I)[2.2] : \sum a \in A \,.\, \forall i \in I \,.\, i < a,$

$[4] := [3]\Big(f(a)\Big) : f(a) < a,$

$[5] := \texttt{StrictlyIncreaingWellOrdered}(X, X, f)(a) : a \leq f(a),$

$[1.*] := \texttt{TrichtomyPrinciple}[4, 5] : \bot;$

$\rightsquigarrow [*] := \texttt{E}(\bot) : A \leq_{\mathsf{ORD}} X,$

$\square$

1.5.2 Arithmetics

$\texttt{ordinalSum} :: \mathsf{ORD} \times \mathsf{ORD} \to \mathsf{ORD}$

$\texttt{ordinalSum}\,(a, b) = a + b := \Big(a \sqcup b, (\leq)_a \sqcup (\leq)_b \sqcup \Big\{(x, y)\Big|x \in a, y \in b\Big\}\Big)$

$\texttt{ordinalProduct} :: \mathsf{ORD} \times \mathsf{ORD} \to \mathsf{ORD}$

$\texttt{ordinalProduct}\,(a, b) = ab := \left(a \times b, \left\{\Big((x, y), (x', y')\Big)\middle|x \leq x'|(x = x' \;\&\; y \leq y')\right\}\right)$

$\texttt{OrdinalSumIsAssoc} :: \forall a, b, c \in \mathsf{ORD} \,.\, (a + b) + c = a + (b + c)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrdinalSumNeutralElement} :: \forall a \in \mathsf{ORD} \,.\, 0 + a = a = 0 + a$

$\texttt{Proof} =$

$\texttt{OrdinalSumIncreasing} :: \forall a, b, b' \in \mathsf{ORD} \,.\, b < b' \Rightarrow a + b < a + b'$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrdinalSumNonDecreasing} :: \forall a, a', b \in \mathsf{ORD} \,.\, a \leq a' \Rightarrow a + b \leq a' + b$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrdinalEquationSolution} :: \forall a, b \in \mathsf{ORD} \,.\, a \leq b \Rightarrow \exists! c \in \mathsf{ORD} : a + c = b$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrdinalIndexedSum} :: \prod I : \texttt{WellOrdered} \,.\, (I \to \mathsf{ORD}) \to \mathsf{ORD}$

$\texttt{Proof} =$

$\texttt{ordinalIndexedSum}\,(a) = \sum_{i \in I} a_i := \left(\sum_{i \in I} a_i, \left\{\Big((x, i), (y, j)\Big)\middle|i < j|(i = j \;\&\; x \leq y)\right\}\right)$

OrdinalProductIsAssoc :: $\forall a, b, c \in \mathsf{ORD} \ . \ (ab)c = a(bc)$

Proof $=$

$\ldots$

$\square$

OrdinalProductNeutralElement :: $\forall a \in \mathsf{ORD} \ . \ 1a = a = a1$

Proof $=$

$\ldots$

$\square$

OrdinalProductZeroElement :: $\forall a \in \mathsf{ORD} \ . \ 0a = 0 = a0$

Proof $=$

$\ldots$

$\square$

OrdinalDistributivity :: $\forall a, b, c \in \mathsf{ORD} \ . \ a(b + c) = ab + ac$

Proof $=$

$\ldots$

$\square$

OrdinalProductIncreasing :: $\forall a, b, b' \in \mathsf{ORD} \ . \ b < b' \Rightarrow ab < ab'$

Proof $=$

$\ldots$

$\square$

OrdinalProductNonDecreasing :: $\forall a, a', b \in \mathsf{ORD} \ . \ a \leq a' \Rightarrow ab \leq a'b$

Proof $=$

$\ldots$

$\square$

OrdinalMultEquationSolution :: $\forall a, b, c \in \mathsf{ORD} \ . \ c \leq ab \Rightarrow \exists! d, e \in \mathsf{ORD} : c = ad + e$

Proof $=$

$\ldots$

$\square$

OrdinalReminder :: $\forall a, b \in \mathsf{ORD} \ . \ a > 0 \ \& \ b \geq a \Rightarrow \exists! c, r \in \mathsf{ORD} : c \leq b \ \& \ r \leq a \ \& \ b = ca + r$

Proof $=$

$\ldots$

$\square$

MultipleOrdinalReminder :: $\forall a, b \in \mathsf{ORD} \ . \ \forall n \in \mathbb{N} a > 0 \ \& \ a^{n+1} \geq b \geq a^n \Rightarrow \exists! r \in \mathsf{ORD} : \exists! c : n \to \mathsf{ORD} : \forall i$

Proof $=$

$\ldots$

$\square$

1.5.3 Powers

$\texttt{ordinalPower} :: \mathsf{ORD} \times \mathsf{ORD} \to \mathsf{ORD}$

$$\texttt{ordinalPower}\,(\alpha,\beta) = \alpha^\beta := \left(\left\{f : \beta \to \alpha : \Big|\{b \in \beta : f(b) \neq \emptyset\}\Big| < \infty\right\},\right.$$
$$\left.\left\{f, g \in \alpha : f = g \Big| \min\left\{b \in \beta : f(b) < g(a)\right\} < \min\left\{b \in \beta : g(a) < f(b)\right\}\right\}\right)$$

$\texttt{ZeroPower} :: \forall a \in \mathsf{ORD} \;.\; a^0 = 1$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{IncreasingPower} :: \forall a, b, b' \in \mathsf{ORD} \;.\; b \geq b' \Rightarrow a^b \geq a^{b'}$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CountablePower} :: \forall a, b \in \mathsf{ORD} \;.\; |a| \leq \aleph_0 \,\&\, |b| \leq \aleph_0 \Rightarrow \left|a^b\right| \leq \aleph_0$

$\texttt{Proof} \;=$

...

$\square$

$$\texttt{OrdinalPowerSeriesRepresentation} :: \forall a, b, z \in \mathsf{ORD} z < a^b \Rightarrow \exists! c : [0, b) \to [0, a) : z = \sum_{i \in [0,b)} a^i c_i$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{countablePower} :: \mathbb{Z}_+ \to \mathsf{ORD}$

$\texttt{countablePower}\,(0) = \omega_0 := \mathbb{N}$

$\texttt{countablePower}\,(n) = \omega_n := \mathbb{N}^{\omega_{n-1}}$

$\texttt{continualPower} :: \mathbb{Z}_+ \to \mathsf{ORD}$

$\texttt{continualPower}\,(0) = \epsilon_0 := \sup\{\omega_n | n \in \mathbb{Z}\}$

$\texttt{continualPower}\,(n) = \epsilon_n := (\epsilon_0)^{\epsilon_{n-1}}$