Multilinear Algebra

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1 Basis of a Module: Proofs

$$\underline{\texttt{thm}}\quad \texttt{maxLInd} :: \forall M : R\text{-}\texttt{Module} \,.\, \forall v : \texttt{LinearlyIndependent}(M) \,.\, \exists w : \texttt{MaximalLI}(M) : v \subset w$$

$$\texttt{maxLInd}(M)(v) =$$

$$\underline{\texttt{def}}\quad L = \{w : \texttt{LinearlyIndependent}(M) : v \subset w\}$$

$$(v : \texttt{LinearlyIndependent}(M), \texttt{ExtThm}(v) : v \subset v) \to (*) : v \in L \to (1) : L \neq \emptyset$$

$$|C : \texttt{Chain}\ L\ (\subset) \vdash$$

$$\left((*) : v \in L, \triangleq \bigcup\right) \to (2) : v \subset \bigcup C$$

$$|A : \bigcup C \;!\; \texttt{LinearlyIndependent}(M) \vdash$$

$$A \to \exists \alpha \in F^R\left(\bigcup C\right) : \sum_{x \in \bigcup C} \alpha_x x = 0 : \alpha \neq 0 \multimap \alpha$$

$$\alpha \in F^R\left(\bigcup C\right) \to \left\{x \in \bigcup C : \alpha_x \neq 0\right\} : \texttt{Finite} - (C : \texttt{Chain}) \to$$

$$\to \exists w \in C : \sum_{x \in w} \alpha_x x = 0 \multimap w$$

$$(\alpha \neq 0, \triangleq w : \sum_{x \in w} \alpha_x x = 0) \to (3) : w \;!\; \texttt{LinearlyIndependent}(M)$$

$$w \in C \subset L \to w : \texttt{LinearlyIndependent}(M) - (3) \to \bot| : \bigcup C :$$

$$: \texttt{LinearlyIndependent}(M) - (2) \to (3) : \bigcup C \in L$$

$$(\triangleq \texttt{Chain}\ L\ (\subset), \triangleq \bigcup) \to \bigcup C : \texttt{UB}(C)| : \forall C : \texttt{Chain}\, L\, (\subset) \,.\, \exists B : \texttt{UB}(C) : B \in L -$$

$$- (1) - \texttt{ZornLemma} \to \exists \texttt{Maximal}(L) \square$$

$\underline{\texttt{thm}}$ $\texttt{freeBasis} :: \forall M : R\text{-}\texttt{Module} . (\exists S : \texttt{Set} : M \cong_{\mathsf{SET}} F^R(S)) \iff \texttt{Basis}(M)$

$\texttt{freeBasis}(M) =$

$|S : \texttt{Set} : M \cong_{\mathsf{SET}} F^R(S) \vdash$

$\triangleq S : M \cong_{\mathsf{SET}} F^R(S) \to \exists \phi : \texttt{Bijective}(F^R(S), M) \multimap \phi$

$\underline{\texttt{def}} \quad b = \Lambda s \in S . \Lambda x \in S . \delta_{x,s}$

$b : \texttt{Basis}(F^R(S)) \to \phi(b) : \texttt{Basis}(F^R(S))$

$| : (\exists S : \texttt{Set} : M \cong_{\mathsf{SET}} F^R(S)) \Rightarrow \texttt{Basis}(M) \multimap (\Rightarrow)$

$|(I, b) : \texttt{Basis}(M) \vdash$

$(I, b) : \texttt{Basis}(M) \to \exists \texttt{Bijective}(M, F^R(I)) \to F^R(I) \cong_{\mathsf{SET}} M$

$| : \texttt{Basis}(M) \Rightarrow (\exists S : \texttt{Set} : M \cong_{\mathsf{SET}} F^R(S)) \multimap (\Leftarrow)$

$(\Leftarrow, \Rightarrow) \to (\exists S : \texttt{Set} : M \cong_{\mathsf{SET}} F^R(S)) \iff \texttt{Basis}(M) \square$

$\underline{\texttt{thm}}$ $\texttt{basisFree} :: \forall M : R\text{-}\texttt{Module} . \forall B : \texttt{Subset}(M) .$

$(B, \mathrm{id}) : \texttt{Basis}(M) \iff L_B : \texttt{Isomorphism}_{R\text{-}\mathsf{MOD}}$

$\texttt{basisFree}(M, B) =$

$\triangleq \texttt{Basis}(M)$

$$\underline{\texttt{thm}} \quad \texttt{maxIsBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \mathrm{id} : \texttt{MaximalLI}(V) \to \texttt{Basis}(V)$$

$$\texttt{maxIsBasis}\, V\, (I, v) =$$

$$(I, v) : \texttt{MaximalLI}(V) \to (I, v) : (\texttt{LinearlyIndependent} \to L_v : \texttt{Injective}(F^K(I), V)$$

$$(L_v, |A : L_v \,!\, \texttt{Surjective}(F^K(I), V) \vdash$$

$$A \to \exists \texttt{In}(V) \wedge ! \texttt{In}(\operatorname{Im} L) \multimap x$$

$$\underline{\texttt{def}} \quad w : I|\{I\} \to V$$

$$w(\texttt{right}, I) = x$$

$$w(\texttt{left}, i) = v_i$$

$$(*) = |\alpha \in F^K(I|\{I\}) : \alpha \neq 0 \vdash$$

$$\texttt{ExMid}(\alpha_I, 0) : (\alpha_I = 0|\alpha_I \neq 0) \to (\exists i \in I : \alpha_i \neq 0|\alpha_I \neq 0) \wedge \sum_{i \in I} w_i = \sum_{i \in I} v_i \to$$

$$\to (L_v(\alpha) \neq 0|L_v(\alpha) \neq -\alpha_I x) \wedge L_w(\alpha) = L_v(\alpha) + \alpha_I x \to$$

$$(L_w(\alpha) \neq 0|L_w(\alpha) \neq 0) \to L_w(\alpha) \neq 0| : L_w : \texttt{Injective}(F^K(I|\{I\}), V) \to$$

$$(I|\{I\}, w) : \texttt{LinearlyIndependent}(V) - (\triangleq w : v \subset w) \to (I, v)\,!\,\texttt{MaximalLI}(V) \to$$

$$\to \perp|) : \texttt{Surjective}(F^K(I), V) \to L_v : \texttt{Bijective}(F^K(I), V) \to$$

$$\to (I, v) : \texttt{Basis}(V) \square$$

$\underline{\texttt{thm}}$ $\texttt{completeBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \forall v : \texttt{LinearlyIndependent}(V) \,.$
$\exists b : \texttt{Basis}(V) : (v \subset b)$
$\texttt{completeBasis}(V, v) = \texttt{maxIsBasis}\, V\ \texttt{maxLInd}(V, v)$

$\underline{\texttt{thm}}$ $\texttt{minIsBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \mathrm{id} : \texttt{MinimalGenerator}(V) \to \texttt{Basis}(V)$

$\texttt{minIsBasis}\, V\, (I, v) =$

$(I, v) : \texttt{MinimalGenerator}(V) \to (I, v) : \texttt{Generates}(V) \to L_v : \texttt{Surjective}(F^K(I), V)$

$(L_v, |A : L_v\ !\ \texttt{Injective}(F^K(I), V) \vdash$

$\quad A \to \exists \alpha \in F^K(I) : \alpha \neq 0 : L_v(\alpha) = 0 \multimap (1)$

$\quad (1)_2 : \alpha \neq 0 \to \exists k \in I : \alpha_k \neq 0 \multimap (2)$

$\quad \underline{\texttt{def}} \quad J = I \setminus \{k\}$

$\quad \underline{\texttt{def}} \quad w = v_{|J}$ (1

$\quad (*) = |x \in V \vdash$

$\quad\quad L_v : \texttt{Surjective}(F^K(I), V) \to \exists \beta \in F^K(I) : L_v(\beta) = x \multimap (4)$

$$(4)_2 : x = \sum_{i \in I} \beta_i v_i =_{(3)} \sum_{j \in J} (\beta_j + \beta_k \alpha_k^{-1} \alpha_j) v_j \to \exists \beta' \in F^K(J) : x = L_w(\beta')$$

$\quad | : \forall x \in V \,.\, \exists \beta \in F^K(J) : x = L_w(\beta) \to L_w : \texttt{Surjective}(F^K(J), V) \to$

$\quad \to (J, w) : \texttt{Generates}(V) - (J \subset I) \to (I, v)\ !\ \texttt{MinimalGenerator}(V) \to$

$\quad \to \bot|) : \texttt{Injective}(F^K(I), V) \to L_v : \texttt{Bijective}(F^K(I), V) \to$

$(I, v) : \texttt{Basis}(V)$

$\underline{\texttt{lemma}}$ $\texttt{fractionalLInd} :: \forall R : \texttt{IntegralDomain} \; . \; \forall S : \texttt{Set} \; . \; \forall v : \texttt{IndexedSet}(F^R(S)) \; .$

$v : \texttt{LinearlyIndependent}(F^R(S)) \iff v : \texttt{LinearlyIndependent}(F^{\text{fractions}\, R}(S))$

$\texttt{fractionalLInd}\, R\; S\; (I, v) =$

$\underline{\texttt{def}} \quad K = \text{fractions}\, R$

$(\Rightarrow) = |(I, v) : \texttt{LinearlyIndependent}(F^R(S)) \vdash$

$((I, v), \quad |A : (I, v) \; ! \; \texttt{LinearlyIndependent}(F^K(S)) \vdash$

$$A \to \exists \frac{a}{b} \in F^K(I) : L_v\left(\frac{a}{b}\right) = 0 : \frac{a}{b} \neq 0 \multimap (1)$$

$$(1)_1 \to \exists E : \texttt{Finite}(I) : E = \{i \in I : a_i \neq 0\} \multimap (2)$$

$$\underline{\texttt{def}} \quad p = \prod_{i \in E} b_i \in R$$

$$\triangleq p \to (3) : p\frac{a}{b} \in F^R(I)$$

$$(1)_2 \to (4) : L_v\left(p\frac{a}{b}\right) = pL_v\left(\frac{a}{b}\right) = 0$$

$$((1)_3, 3, 4) \to (I, v) \; ! \; \texttt{LinearlyIndependent}(F^R(S)) \to \bot|$$

$|) : \texttt{LinearlyIndependent}(F^R(S))| :$

$| : \big((I, v) : \texttt{LinearlyIndependent}(F^R(S)\big) \Rightarrow \big((I, v) : \texttt{LinearlyIndependent}(F^K(S))\big)$

$(\Leftarrow) = |(I, v) : \texttt{LinearlyIndependent}(F^K(S)) \vdash$

$|A : \alpha \in F^R(I) : L_v(\alpha) = 0 \vdash$

$$\triangleq K \to \alpha \in F^K(I)$$

$$(I, v) : \texttt{LinearlyIndependent}(F^K(S)) - (A) \to \alpha = 0| :$$

$| : \forall \alpha \in F^R(I) : L_v(\alpha) = 0 \; . \; \alpha = 0 \to (I, v) : \texttt{LinearlyIndependent}(F^R(S))| :$

$| : \big((I, v) : \texttt{LinearlyIndependent}(F^K(S)\big) \Rightarrow \big((I, v) : \texttt{LinearlyIndependent}(F^R(S))\big) \to$

$\to (I, v) : \texttt{LinearlyIndependent}(F^R(S)) \iff (I, v) : \texttt{LinearlyIndependent}(F^K(S)) \square$

$\underline{\texttt{thm}}$ $\texttt{LIndBound} :: \forall R : \texttt{IntegralDomain} \ . \ \forall S : \texttt{Set} \ .$

$\forall (I,b) : \texttt{MaximalLI}\, F^R(S) \ . \ \forall (J,v) : \texttt{LinearlyIndependent}\, F^R(S) \ . \ |J| \le |I|$

$\texttt{LIndBound}(R,S,b,v) =$

$\underline{\texttt{def}} \quad K = \text{fractions}\, R; M = F^R(S); V = F^K(S)$

$(\texttt{fractionalLInd}) \to (I,b),(J,v) : \texttt{LinearlyIndependent}(V)$

$(\texttt{fractionalLInd}^*) \to (I,b) : \texttt{MaximalLI}(V) - (\texttt{maxIsBasis}) \to (I,b) : \texttt{Basis}(V)$

$\texttt{wellOrdering}(J) : \exists (<) : \texttt{TotalOrder}(J) \multimap$

$\underline{\texttt{def}} \quad m = \min(J,<); \underline{\texttt{def}} \quad B_m = b$

$\texttt{foreach}\ j \in (J,<)|$

$(I,B_j) : \texttt{Basis}(V) \to \exists \alpha \in F^K(I) : v_j = \sum_{i \in I} \alpha_i B_j(i) \multimap (1)$

$(J,v) : \texttt{LinearlyIndependent}(V) - (2) \to \exists \psi \in I : \alpha_\psi \neq 0 : B_j(I) \notin \operatorname{Im} v \multimap (2)$

$\underline{\texttt{def}} \quad f(j) = \psi$

$(*)_j : \forall k < j \ . \ B_j(f(k)) \in \operatorname{Im} v \to f : \texttt{Injective}(\mathbb{I}_j(<), I)$

$\underline{\texttt{def}} \quad B_{j++}(k) = \texttt{if}\ k == \psi\ \texttt{then}\ v_j\ \texttt{else}\ B_j(k)$

$(*)_{j++} = (*)_j \wedge B_{j++}(f(j)) = B_{j++}(\psi) = v_j \in \operatorname{Im} v$

$(1,2) \to \exists c \in F^K(I) : B_j(\psi) = L_{B_{j++}}(c) \multimap (3)$

$|X : \beta \in F^k(I) : L_{B_{j++}}(\beta) = 0 \vdash$

$|Y : \beta_\psi \neq 0 \vdash$

$Y - (1,2,X) \to L_{B_j}(\alpha) = \beta_\psi^{-1} L_{B_j}(\beta_{\psi \to 0}) \to$

$\to (4) : B_j(\psi) = \alpha_\psi^{-1} \beta_\psi^{-1} L_{B_j}(\beta_{\psi \to 0}) - \alpha_\psi^{-1} L_{B_j}(\alpha_{\psi \to 0}) \to$

$(I,B_j) : \texttt{LinearlyIndependent}(V) - (4) \to \bot| : \beta_\psi = 0 \to$

$\to (4) : 0 = L_{B_{j++}}(\beta) = L_{B_j}(\beta)$

$(I,B_j) : \texttt{LinearlyIndependent}(V) - (4) \to \beta = 0$

$| : \forall \beta \in F^K(I) : L_{B_{j++}}(\beta) = 0 \ . \ \beta = 0 \to (I,B_{j++}) : \texttt{LinearlyIndependent}(V)$

$|x \in V \vdash$

$(I,B_k) : \texttt{Generates}(V) \to \exists \beta \in F^K(I) : x = L_{B_j}(\beta) \multimap (4)$

$(3,4) \to x = L_{B_j}(\beta) = L_{B_{j++}}(\beta_{\psi \to 0}) + \beta_\psi L_{B_{j++}}(c) = L_{B_{j++}}(\beta_{\psi \to 0} + \beta_\psi c)$

$| : \forall x \in V \ . \ \exists \beta \in F^K(I) \ . \ x = L_{B_{j++}}(\beta) \to (I,B_{j++}) : \texttt{Generates}(V) \to$

$\to (I,B_{j++}) : \texttt{Basis}(V) \models f : \texttt{Injective}(J,I) \to |J| \le |I| \square$