Complex Numbers

Uncultured Tramp

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Contents

1 Complex Plane

1.1 Algebraic Definition

$\texttt{complexNumbers} :: \texttt{Field}$

$\texttt{complexNumbers}\,() = \mathbb{C} := \dfrac{\mathbb{R}}{(x^2+1)}$

$\texttt{complexNumbersDimension} :: \dim_{\mathbb{R}} \mathbb{C} = 2$

$\texttt{Proof} \ =$

$[1] := \texttt{DegreeOfSimpleExtension}\,\mathcal{I}\mathbb{C} : \deg \mathbb{C} = 2,$

$[*] := \mathcal{I} \deg[1] : \dim_{\mathbb{R}} \mathbb{C} = 2;$

$\square$

$\texttt{imaginaryUnit} :: \mathbb{C}$

$\texttt{imaginaryUnit}\,() = \mathbf{i} := \pi_{\mathbb{C}}(x)$

$\texttt{ImagenaryUnitSquare} :: \mathbf{i}^2 = -1$

$\texttt{Proof} \ =$

$[1] := \mathcal{I}\mathbb{R}\text{-ALGE}\Big(, \mathbb{C}, \pi_{\mathbb{C}}\Big)\mathcal{I}\mathbb{C} : \mathbf{i}^2 + 1 = \pi_{\mathbb{C}}^2(x) + 1 = \pi_{\mathbb{C}}(x^2+1) = 0,$

$[*] := [1] - 1 : \mathbf{i}^2 = -1;$

$\square$

$\texttt{ImagenraryUnitInverse} :: \mathbf{i}^{-1} = -\mathbf{i}$

$\texttt{Proof} \ =$

$[1] := \mathcal{I}\text{ABEL}(\mathbb{C}^\times)\texttt{ImaginaryUnitSquare}()\texttt{DoubleNegation}(\mathbb{C}) : \mathbf{i}(-\mathbf{i}) = -\mathbf{i}^2 = -(-1) = 1,$

$[*] := \mathcal{I}^{-1}\texttt{Inverse}[1] : \mathbf{i}^{-1} = -\mathbf{i};$

$\square$

$\texttt{ComplexBasis} :: \texttt{Basis}\Big(\mathbb{R}, \mathbb{C}, (1, \mathbf{i})\Big)$

$\texttt{Proof} \ =$

$[1] := \texttt{PositiveSquare}(\mathbb{R}) : \forall a \in \mathbb{R} \,.\, a^2 \geq 0,$

$[2] := \texttt{ImaginaryUnitSquare}()[1] : \forall a \in \mathbb{R} \,.\, a \neq \mathbf{i},$

$[3] := \mathcal{I}^{-1}\texttt{LinearlyIndependent}(\mathbb{R}, \mathbb{C})[2] : \texttt{LinearlyIndependent}\Big(\mathbb{R}, \mathbb{C}, (1, \mathbf{i})\Big),$

$[4] := \mathcal{I}\texttt{Basis}[3]\texttt{ComplexNumberDimension}() : \texttt{Basis}\Big(\mathbb{R}, \mathbb{C}, (1, \mathbf{i})\Big);$

$\square$

$\texttt{ComplexAlgebraicPresentation} :: \forall z \in \mathbb{C} \,.\, \exists! a, b \in \mathbb{R} \,.\, z = a + b\mathbf{i}$

$\texttt{Proof} \ =$

$[*] := \texttt{ComplexBasis}()\mathcal{I}\texttt{Basis}(\mathbb{R}, \mathbb{C}) : \forall z \in \mathbb{C} \,.\, \exists! a, b \in \mathbb{R} \,.\, z = a + b\mathbf{i};$

$\square$

$\texttt{realPart} :: \mathbb{C} \to \mathbb{R}$

$\texttt{realPart}\,(a + b\mathbf{i}) = \Re(a + b\mathbf{i}) := a$

$\texttt{imaginablePart} :: \mathbb{C} \to \mathbb{R}$

$\texttt{imaginablePart}\,(a + b\mathbf{i}) = \Im(a + b\mathbf{i}) := b$

$\texttt{ComplexGaloisGroup} :: G(\mathbb{R};\mathbb{C}) = \{\mathrm{id}, \gamma\} \quad \texttt{where} \quad \gamma = \Lambda a + b\mathbf{i} \,.\, a - b\mathbf{i}$

$\texttt{Proof} \;=$

$[1] := \daleth\mathbb{C} : \texttt{minimal}(\mathbb{R};\mathbb{C}) = x^2 + 1,$

$[2] := \daleth\mathbf{i} : \rho(x^2 + 1) = \{+\mathbf{i}, -\mathbf{i}\},$

$[3] := \texttt{GaloisTHM}\daleth\mathbb{C} : \Big|G(\mathbb{R};\mathbb{C})\Big| = \dim_{\mathbb{R}} \mathbb{C} = 2,$

$\Big(\gamma, [4]\Big) := \daleth\,\mathsf{GRP}\Big(G(\mathbb{R}\ \mathbb{C})\Big)[3] : \sum \gamma \in G(\mathbb{R};\mathbb{C}) \,.\, \gamma \neq \mathrm{id} \;\&\; G(\mathbb{R};\mathbb{C}) = \{\mathrm{id}, \gamma\},$

$[*] := \daleth G(\mathbb{R};\mathbb{C})[2][4] : \gamma(1) = 1 \;\&\; \gamma(\mathbf{i}) = -\mathbf{i},$

$\square$

$\texttt{conjugation} :: \mathbb{C} \xrightarrow{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{C}$

$\texttt{conjugation}\,(a + \mathbf{i}b) = \overline{a + \mathbf{i}b} := a - \mathbf{i}b$

$\texttt{ConjugataProductIsRealNonNeg} :: \forall z \in \mathbb{C} \,.\, z\bar{z} \in \mathbb{R}_+$

$\texttt{Proof} \;=$

$\Big(a, b, [1]\Big) := \texttt{ComplexAlgebraicPresentation}[2] : \sum a, b \in \mathbb{R} \,.\, z = a + b\mathbf{i},$

$[*] := [1]\daleth\,\texttt{conjugation}\daleth\,\texttt{iSquareSumNonNeg}(\mathbb{R}) : z\bar{z} = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2 \geq 0;$

$\square$

$\texttt{ComplexIsConjugationField} :: \texttt{ConjugationField}(\mathbb{R}.\mathbb{C})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RealPartByConjugation} :: \forall z \in \mathbb{C} \,.\, \Re(z) = \dfrac{z + \bar{z}}{2}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ImaginablePartByConjugation} :: \forall z \in \mathbb{C} \,.\, \Im(z) = \dfrac{z - \bar{z}}{2\mathbf{i}}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

1.2 Geometric Representation

$\texttt{SOAlgebraStructure} :: \forall A \in \Big\langle \mathbf{SO}(\mathbb{R},2) \Big\rangle_{\mathbb{R}\text{-ALGE}} \;.\; \exists r \in \mathbb{R}_+ : \exists T \in \mathbf{SO}(\mathbb{R},2) : A = rT$

$\texttt{Proof} \;=$

$\Big(n, S, a, [1]\Big) := \mathcal{C}\!\!\!/\,\mathbb{R}\text{-ALGE}\,\mathcal{C}\!\!\!/\,\text{GRP} : \sum n \in \mathbb{N} \;.\; \sum S : n \to \mathbf{SO}(\mathbb{R},n) \;.\; \sum a : n \to \mathbb{R}\; A = \sum_{i=1}^{n} a_i S_i,$

$[2] := \texttt{TrigonometricRepresentation}(\mathbb{R}^2, S) : \forall i \in n \;.\; S_i = \begin{bmatrix} \cos S_i & \sin S_i \\ -\sin S_i & \cos S_i \end{bmatrix},$

$\Big(A, B, [3]\Big) := [1][2] : A = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$

$\texttt{Assume}\ [4] : (A, B) \neq 0,$

$T := \dfrac{1}{A^2+B^2} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} : \mathbf{SO}(\mathbb{R},2),$

$[4.*] := \mathcal{D}T[3] : A = (A^2+B^2)T;$

$\rightsquigarrow [4] := I(\Rightarrow) : (A,B) \neq 0 \Rightarrow \exists r \in \mathbb{R}_+ : \exists T \in \mathbf{SO}(\mathbb{R},2) : A = rT,$

$\texttt{Assume}\ [5] : (A, B) = 0,$

$[5.*] := [5][4] : A = 0 = 0I;$

$\rightsquigarrow [4] := I(\Rightarrow) : (A,B) = 0 \Rightarrow \exists r \in \mathbb{R}_+ : \exists T \in \mathbf{SO}(\mathbb{R},2) : A = rT,$

$[*] := E(|)\texttt{LEM}((A,B) = 0)[5][4] : \exists r \in \mathbb{R}_+ : \exists T \in \mathbf{SO}(\mathbb{R},2) : A = rT;$

$\square$

$\texttt{matrixRepresentation} :: \mathbb{C} \xleftrightarrow{\mathbb{R}\text{-VS}} \Big\langle \mathbf{SO}(\mathbb{R},2) \Big\rangle_{\mathbb{R}\text{-ALGE}}$

$\texttt{matrixRepresentation}\,(a + b\mathbf{i}) = \mathrm{mat}(a + b\mathbf{i}) := \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

$\texttt{SOAlgebraIsComplexNumbers} :: \mathbb{C} \cong_{\mathbb{R}\text{-ALGE}} \Big\langle \mathbf{SO}(\mathbb{R},2) \Big\rangle_{\mathbb{R}\text{-ALGE}}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a + b\mathbf{i}, c + d\mathbf{i} : \mathbb{C},$

$[\ldots *] := \mathcal{C}\!\!\!/\,\mathbf{i}\,\mathcal{C}\!\!\!/\,\mathrm{mat}\,\mathcal{C}\!\!\!/\,\texttt{matrixMult}(\mathbb{R}^2)\mathcal{C}\!\!\!/^{-1}\mathrm{mat} : \mathrm{mat}\Big((a+b\mathbf{i})(c+d\mathbf{i})\Big) = \mathrm{mat}\Big((ac-bd) + (ad+bc)\mathbf{i}\Big) =$

$= \begin{bmatrix} ac-bd & ad+bc \\ ad+bc & ac-bd \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \mathrm{mat}(a+b\mathbf{i})\mathrm{mat}(c+d\mathbf{i});$

$\rightsquigarrow [1] := \mathcal{C}\!\!\!/^{-1}\mathbb{R}\text{-ALGE} : \texttt{Isomorphism}\Big(\mathbb{R}\text{-ALGE}, \mathbb{C}, \Big\langle \mathbf{SO}(\mathbb{R},2) \Big\rangle_{\mathbb{R}\text{-ALGE}}, \mathrm{mat}\Big),$

$[*] := \mathcal{C}\!\!\!/^{-1}\texttt{Isomorphic}[1] : \mathbb{C} \cong_{\mathbb{R}\text{-ALGE}} \Big\langle \mathbf{SO}(\mathbb{R},2) \Big\rangle_{\mathbb{R}\text{-ALGE}};$

$\square$

$\texttt{ComplexPolarPresentation} :: \forall z \in \mathbb{C} \;.\; \exists! T \in \mathbf{SO}(\mathbb{R},2) : z = |z|\cos T + \mathbf{i}|z|\sin T$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{argument} :: \mathbb{C}^\times \xrightarrow{\text{GRP}} \mathbf{SO}(\mathbb{R},2)$

$\texttt{argument}\,(|z|\cos T + \mathbf{i}|z|\sin T) = \mathrm{Arg}\Big(|z|\cos T + \mathbf{i}|z|\sin T\Big) := T$

DeMuavreFormula :: $\forall T \in \mathbf{SO}(\mathbb{R}, 2) \; . \; \forall n \in \mathbb{N} \; . \; \Big(| \cos T + \mathbf{i} \sin T \Big)^n = \cos T^n + \mathbf{i} \sin T^n$

Proof $=$

$\ldots$

$\square$

1.3 Roots

$$
\begin{aligned}
&\texttt{ComplexHasSquareRoots} :: \forall z \in \mathbb{C} \,.\, \exists\sqrt{z}\\
&\texttt{Proof} \;=\\
&\Big(x, y, [1]\Big) := \texttt{ComplexAlgebraicPresentation}(z) : \sum x, y \in \mathbb{C} \,.\, z = x + \mathbf{i}y,\\
&\texttt{Assume}\ [1] : z \notin \mathbb{R}_-,\\
&\texttt{Assume}\ a + \mathbf{i}b : \sqrt{z},\\
&[2] := [1]\mathcal{I}(a + \mathbf{i}b) : a^2 - b^2 = x \;\&\; 2ab = y,\\
&[3] := \frac{[2.2]}{2a} : b = \frac{y}{2a},\\
&[4] := [2.1][3] : a^2 - \frac{y^2}{4a^2} = x,\\
&[5] := a^2([4] - x) : a^4 - xa^2 - \frac{y^2}{4} = 0,\\
&[*.1] := \texttt{RootsOfParabola} : a = \pm\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} = \pm\sqrt{\frac{x + |z|}{2}} \in \mathbb{R},\\
&[*.2] := [6][3] : b = \pm\sqrt{\frac{|z| - x}{2}};\\
&\rightsquigarrow [2] := \mathcal{I}\,\texttt{TwoElementSet} :\\
&\quad : \sqrt{z} = \pm\left(\sqrt{\frac{x + |z|}{2}} + \sqrt{\frac{|z| - x}{2}}\mathbf{i}\right),\\
&[3] := \mathcal{I}\,\texttt{absoluteValue}(\mathbb{C})\texttt{MonotonicSquareRoot}(\mathbb{R}) : x + |z|, |z| - x \geq 0,\\
&[1.*] := [3][2] : \exists\sqrt{z};\\
&\rightsquigarrow [1] := I(\Rightarrow) : z \notin \mathbb{R}_- \Rightarrow \exists\sqrt{z},\\
&\texttt{Assume}\ [2] : z \in \mathbb{R}_{--},\\
&[2.*] := \mathcal{I}\mathbf{i}[2] : \sqrt{z} = |z|\mathbf{i};\\
&\rightsquigarrow [2] := I(\Rightarrow) : z \in \mathbb{R}_- \Rightarrow \exists\sqrt{z},\\
&[*] := E(|)\texttt{LEM}[1][2] : \exists\sqrt{z};\\
&\square
\end{aligned}
$$

$\texttt{ComplexHasAllRoots} :: \forall z \in \mathbb{C}^\times . \forall n \in \mathbb{N} . \left|\sqrt[n]{z}\right| = n$

$\texttt{Proof} =$

$T := \operatorname{Arg} z : \mathbf{SO}(\mathbb{R}, 2),$

$t := \operatorname{arc} T(1) : \dfrac{\mathbb{R}}{2\pi\mathbb{Z}},$

$S := \Lambda k \in n . \operatorname{rot}\left(\dfrac{1}{n}t + \dfrac{2(k-1)\pi}{n}\right) : n \to \mathbf{SO}(\mathbb{R}, 2),$

$u := \Lambda k \in n . \sqrt[n]{|z|}\Big(\cos S_k + \mathbf{i}\sin S_k\Big) : n \to \mathbb{C}^\times,$

$\texttt{Assume}\ k : n,$

$[1] := \mathcal{A} S_k^n \mathcal{A}\,\mathsf{GRP}\left(\operatorname{rot}, \dfrac{\mathbb{R}}{2\pi\mathbb{Z}}, \mathbf{SO}(\mathbb{R}, 2)\right)\mathcal{A} t :$

$\quad : S_k^n = \operatorname{rot}^n\left(\dfrac{1}{n}t + \dfrac{2(k-1)\pi}{n}\right) = \operatorname{rot}\left(t + 2(k-1)\pi\right) = \operatorname{rot}(t) = T,$

$[2] := \mathcal{A} u_k \texttt{DeMuavreFormula}(u_k, n)[1]\texttt{ComplexPolarPresentation} :$

$\quad : u_k^n = |z|\Big(\cos S_k^n + \mathbf{i}\sin S_k^n\Big) = |z|\Big(\cos T + \mathbf{i}\sin T\Big) = z,$

$[1.*] := \mathcal{A}\texttt{NRoot}[2] : u_k = \sqrt[n]{u};$

$\rightsquigarrow [1] := I(\forall) : \forall k \in n . u_k = \sqrt[n]{u},$

$\texttt{Assume}\ k, l : n,$

$\texttt{Assume}\ [2] : k \neq l,$

$[3] := \mathcal{A}\dfrac{\mathbb{R}}{2\pi\mathbb{Z}}\mathcal{A}(k, l)[2] : \dfrac{1}{n}t + \dfrac{2(k-1)\pi}{n} - \dfrac{1}{n}t - \dfrac{2(l-1)\pi}{n} = \dfrac{2(k-l)\pi}{n} \neq 0,$

$[4] := \mathcal{A}\texttt{Isomorphism}(\operatorname{rot})\mathcal{A}^{-1} S : S_l \neq S_k,$

$[*] := \mathcal{A} u : u_l \neq u_k;$

$\rightsquigarrow [2] := I(\forall) : \forall k, l \in n . k \neq l \Rightarrow u_l \neq u_k,$

$[*] := \texttt{RootNumber}[1][2] : \left|\sqrt[n]{z}\right| = n;$

$\square$

$\texttt{circleGroup} :: \texttt{Subgroup}(\mathbb{C}^\times)$

$\texttt{circleGroup}\,() = \mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$

$\texttt{rootsOfUnity} :: \prod_{n=1}^{\infty} n \to \mathbb{S}$

$\texttt{rootsOfUnity}\,(k) = \xi_{n,k} := \operatorname{mat}^{-1}\operatorname{rot}\left(\dfrac{2\pi k}{n}\right)$

$\texttt{PrimitiveRootsOfUnity} :: \prod n \in \mathbb{N} . ?\sqrt[n]{1}$

$z : \texttt{PrimitiveRootsOfUnity} \iff z \in \mathrm{P}_n(\mathbb{C}) \iff \forall k \in (n-1)_{\mathbb{N}} . z^k \neq 1$

$\texttt{RootsOfUnityTruePower} :: \forall n \in \mathbb{N} . \forall k \in n . \min\{t \in n : \xi_{n,k}^t = 1\} = \dfrac{n}{\gcd(n,k)}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{totientFunctionOfEuler} :: \mathbb{N} \to \mathbb{N}$

$\texttt{totientFunctionOfEuler}\,(n) = \varphi(n) := \Big|\{k : \texttt{Coprime}(n) : k < n\}\Big|$

$\texttt{PrimitiveRootsCardinality} :: \forall n \in \mathbb{N} \,.\, \Big|\mathrm{P}(n)\Big| = \varphi(n)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; k : n,$

$[k.*] := \text{Ɑ}\xi_{n,k}\text{Ɑ}\mathrm{P}(n)\text{Ɑ}\texttt{CoprimeRootsOfUnity} :$

$\quad : \texttt{Coprime}(n,k) \iff \gcd(n,k) = 1 \iff \dfrac{n}{\gcd(n,k)} = n \iff \xi_{n,k} \in \mathrm{P}(n);$

$\leadsto [*] := \text{Ɑ}^{-1}\texttt{SetEq}\text{Ɑ}^{-1} : |\mathrm{P}(n)| = \varphi(n);$

$\square$

$\texttt{PrimitiveRootsDontIntersect} :: \forall n, m \in \mathbb{N} \,.\, n \neq m \Rightarrow \mathrm{P}(n) \cap \mathrm{P}(m) = \emptyset$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a : \mathrm{P}(n),$

$\texttt{Assume}\; b : \mathrm{P}(m),$

$[1] := \text{Ɑ}\mathrm{P}(n,a) : \min\{k \in \mathbb{N} : a^k = 1\} = n,$

$[2] := \text{Ɑ}\mathrm{P}(n,a) : \min\{k \in \mathbb{N} : b^k = 1\} = m,$

$[a.*] := I(\rightarrow, \#)[1][2] : a \neq b;$

$\leadsto [*] := \text{Ɑ}\texttt{Intersection} : \mathrm{P}(n) \cap \mathrm{P}(m) = \emptyset;$

$\square$

$\texttt{RootsOfUnityDecomposition} :: \forall n \in \mathbb{N} \,.\, \sqrt[n]{1} = \bigsqcup_{k:n} \mathrm{P}(n)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a : \sqrt[n]{1},$

$k := \min\{k \in \mathbb{N} : a^k = 1\} : n,$

$[1] := \text{Ɑ}\sqrt[n]{1}\text{Ɔ}k : k|n,$

$[a.*] := \text{Ɑ}\mathrm{P}(k)\text{Ɔ}k : a \in \mathrm{P}(k);$

$\leadsto [*] := \texttt{PrimitiveRootsDontIntersect}(\ldots)\texttt{RootsOfUnityTruePower}(n) : \sqrt[n]{1} = \bigsqcup_{k:n} \mathrm{P}(n);$

$\square$

$\texttt{EulerTotientSum} :: \sum_{k:n} \varphi(k) = n$

$\texttt{Proof} \;=$

$[*] := \texttt{ComplexHasAllRoots}(n)\texttt{RootsOfUnityDecomposition}(n)\texttt{CardinalityOfDisjoinUnion}(\ldots)$

$\quad \texttt{PrimitiveRootsCardinality}(k) : n = \Big|\sqrt[n]{z}\Big| = \sum_{k:n}\Big|\mathrm{P}(k)\Big| = \sum_{k:n}\varphi(k);$

$\square$

$\texttt{ComplexQuadraticSplits} :: \forall P(x) : \texttt{Monic}(\mathbb{C})\forall[0] : \deg P = 2 \;.\; \exists a, b \in \mathbb{C} : P(x) = (x-a)(x-b)$

$\texttt{Proof} \;=$

$\Big(\alpha, \beta, [1]\Big) := [0]\mathit{\Pi} \deg P \mathit{\Pi}\texttt{Monic}(\mathbb{C}) : \sum \alpha, \beta \in \mathbb{C} \;.\; P(x) = x^2 + \alpha x + \beta,$

$a := \dfrac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} : \mathbb{C},$

$b := \dfrac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} : \mathbb{C},$

$[2] := \mathit{\Omega} a : P(a) = 0,$

$[3] := \mathit{\Omega} b : P(b) = 0,$

$[4] := \texttt{RootNumber}[2][3] : P(x) = (x-a)(x-b);$

$\square$

$\texttt{RealIrreducibleHasConjugateRoots} :: \forall P(x) : \texttt{Monic} \,\&\, \texttt{Irreducible}(\mathbb{R}) \;.\; \forall[0] \deg P(x) = 2 \;.$
$\;.\; \exists z \in \mathbb{C} \;.\; P(x) = (x-z)(x-\overline{z})$

$\texttt{Proof} \;=$

$\Big(a, b, [1]\Big) := \texttt{ComplexQuadraticSplits}\Big(P, [0]\Big)] : \sum a, b \in \mathbb{C} \;.\; P(x) = (x-a)(x-b),$

$[2] := \mathit{\Pi}\mathbb{C}[x](P(x))[1] : P(x) = (x-a)(x-b) = x^2 - (a+b)x + ab,$

$[3] := \mathit{\Pi}\mathbb{R}[x](P(x))[2] : a+b, ab \in \mathbb{R},$

$[4] := \mathit{\Pi}\texttt{Irreducible}(\mathbb{R}, P(x))[1] : \Im a \neq 0 \neq \Im b,$

$[5] := [2.1]\mathit{\Pi}(\Im a, \Im b) : \Im a = -\Im b,$

$[6] := [2.2]\texttt{ConjugationProductIsRealNoneg}\mathit{\Pi}\texttt{Field}(\mathbb{C}) : b \in \mathbb{R}\bar{a},$

$[7] := \mathit{\Pi}^{-1}\texttt{ComplexConjugation}[5][6] : b = \bar{a},$

$[*] := [7][1] : P(x) = (x-a)(x-\bar{a});$

$\square$

1.4 Circles

$\texttt{circle} :: \mathbb{R}_{++} \times \mathbb{C} \to ?\mathbb{C}$
$\texttt{circle}\,(c, r) = \mathbb{S}(c, r) := \{z \in \mathbb{C} : |z - c| = r\}$

$\texttt{circles} := \mathcal{S} = \mathbb{S}(\mathbb{C}, \mathbb{R}_{++}) :??\mathbb{C};$

$\texttt{CircleDiameter} :: \forall c \in \mathbb{C} \,.\, \forall r \in \mathbb{R}_{++} \,.\, \sup_{x,y \in \mathbb{S}(c,r)} |x - y| = 2r$

$\texttt{Proof} \;=$
$\texttt{Assume}\; x, y : \mathbb{S}(c, r),$
$[\ldots *] := \texttt{TriangleIneq}(\mathbb{C}) \Game \mathbb{S}(c, r, xy) : |x - y| \le |x - c| + |c - y| = 2r;$
$\leadsto [1] := \texttt{SupBound}(\mathbb{R}) : \sup_{x,y \in \mathbb{S}(c,r)} |x - y| \le 2r;$
$\Big(a, b, [2]\Big) := \texttt{ComplexAlgebraicPresentation} : \sum a, b \in \mathbb{R} \,.\, z = a + \mathbf{i}b,$
$x := z = (a + r) + \mathbf{i}b : \mathbb{C},$
$y := z = (a - r) + \mathbf{i}b : \mathbb{C},$
$[3] := [2] \Im x : |c - x| = r,$
$[4] := [2] \Im y : |c - y| = r,$
$[5] := \Game \mathbb{S}(c, r) : x, y \in \mathbb{S}(c, r),$
$[6] := \Im x \Im y : |x - y| = 2r,$
$[*] := [1][2] \Game\, \texttt{supremum} : \sup_{x,y \in \mathbb{S}(c,r)} |x - y| = 2r;$

$\square$

$\texttt{CircleIsUniquelyDefineded} :: \texttt{Injective}(\mathbb{C} \times \mathbb{R}_+, ?\mathbb{C}, \mathbb{S})$
$\texttt{Proof} \;=$
$\texttt{Assume}\; x, y : \mathbb{C},$
$\texttt{Assume}\; r, s : \mathbb{R}_{++},$
$\texttt{Assume}\; [1] : \mathbb{S}(x, r) = \mathbb{S}(y, s),$
$[2] := \texttt{circleDiameter}(\ldots)[1] I(\to, \#) : r = s,$
$t := |x - y| : \mathbb{R}_+,$
$\texttt{Assume}\; [3] : t \ne 0,$
$A := x - \frac{r}{t}(y - x) : \mathbb{C},$
$[4] := \Im A \Im t : |A - x| = r,$
$[5] := \Game \mathbb{S}(c, r)[4] : A \in \mathbb{S}(x, r),$
$[6] := E(=)[1][5][2] : A \in \mathbb{S}(y, r),$
$[7] := \Game \mathbb{S}(y, r) \Im A \Im t : r = |A - y| = \left|(1 + \frac{r}{t})(y - x)\right| = r + t,$
$[8] := [7] - r : t = 0,$
$[3.*] := [3][8] : \bot;$
$\leadsto [3] := E(\bot) : t = 0,$
$[\ldots *] := \texttt{AbsValueIsMetric}[3] \Im t : x = y;$
$\leadsto [*] := \Game^{-1} \texttt{Injective} : \texttt{Injective}(\mathbb{C} \times \mathbb{R}_+, ?\mathbb{C}, \mathbb{S});$
$\square$

$\texttt{center} :: \mathcal{S} \to \mathbb{C}$

$\texttt{center}\,(\mathbb{S}(c,r)) := c$

$\texttt{radius} :: \mathcal{S} \to \mathbb{R}_{++}$

$\texttt{radius}\,(\mathbb{S}(c,r)) := r$

$$\texttt{HermitianMatrix} :: \prod_{n=1}^{\infty} ?\mathbb{C}^{n\times n}$$

$$H : \texttt{HermitianMatrix} \iff H \in \mathbf{H}(n) \iff \overline{H}^{\top} = H$$

$\texttt{HermitianMatrixDeterminesSelfAdjointOperator} :: \forall n \in \mathbb{N} \,.\, \forall H \in \mathbb{C}^{n\times n} \,.$

$$.\, H \in \mathbf{H}(n) \iff \forall e : \texttt{Orthonormal}(\mathbb{C}^n) \,.\, \texttt{SelfAdjoint}\Big(\mathbb{C}^n, H_{e,e}\Big)$$

Proof $=$

$\dots$

$\square$

$\texttt{HermitianMatrixHasRealDiagonal} :: \forall n \in \mathbb{N} \,.\, \forall H \in \mathbf{H}(n) \,.\, \operatorname{diag} H \in \mathbb{R}^n$

Proof $=$

$\dots$

$\square$

$\texttt{HermitianHasRealEigenvlues} :: \forall n \in \mathbb{N} \,.\, \forall H \in \mathbf{H}(n) \,.\, \forall \lambda : \texttt{Eigenvalue}(H) \,.\, \lambda \in \mathbb{R}$

Proof $=$

$\Big(v, [1]\Big) := \mathcal{I}\texttt{Eigenvalue}(H, \lambda) : \sum v \in \mathbb{C}^n \,.\, vH = \lambda v \;\&\; v \neq 0,$

$[2] := \mathcal{I}\texttt{HermitianProduct}[1]\texttt{HermitianMatrixDeterminesSelfAdjointOperator}(n, H)[1]$

$\quad \mathcal{I}\texttt{HermitianProduct} : \lambda\langle v, v\rangle = \langle vH, v\rangle = \langle v, vH\rangle = \overline{\lambda}\langle v, v\rangle,$

$[3] := \dfrac{[2]}{\langle v, v\rangle} : \lambda = \bar{\lambda},$

$[*] := \mathcal{I}\texttt{complexConjugation}[3] : \lambda \in \mathbb{R};$

$\square$

$\texttt{HermitianMatrixDeterminant} :: \forall n \in \mathbb{N} \,.\, \forall H \in \mathbf{H}(n) \,.\, \det H \in \mathbb{R}$

Proof $=$

$$[*] := \texttt{DetBySpectre}(\mathbb{C}^n, H)\texttt{HermitianHasRealEigenvalues}(n, H) : \det H = \prod_{\lambda \in \mathbb{C}} \lambda^{\sigma_T(H)} \in \mathbb{R};$$

$\texttt{realHermitianCircle} :: \mathbf{H}(2) \to ?\mathcal{C}$

$$\texttt{realHermitianCircle}\,(H) = \mathbb{S}_{\mathbb{R}}(H) := \Big\{z \in \mathbb{C} : \langle vH_{e,e}, v\rangle = 0 \quad \texttt{where} \quad v = (z, 1)\Big\}$$

$\texttt{EveryCircleIsHermitian} :: \forall S \in \mathcal{S} . \exists H \in \mathbf{H}(2) : S = \mathbb{S}_{\mathbb{R}}(H)$

$\texttt{Proof} \; =$

$c := \texttt{center}(S) : \mathbb{C},$

$r := \texttt{radius}(S) : \mathbb{R}_{++},$

$H := \begin{bmatrix} 1 & -\overline{c} \\ -c & |c|^2 - r^2 \end{bmatrix} : \mathbb{C}^{2\times 2},$

$[2] := \mathcal{I}H : H \in \mathbf{H}(2),$

$\texttt{Assume } z : \mathbb{C},$

$v := (z, 1) : \mathbb{C}^2,$

$[3] := \mathcal{I}H\mathcal{I}\texttt{hermitianProduct}(\mathbb{C}^2)\mathcal{I}^{-1}\texttt{absValue} : \langle vH, v\rangle = \Big\langle (z - \overline{c}, -zc + |c|^2 - r^2), (z, 1) \Big\rangle =$

$\quad = z\overline{z} - \overline{cz} - zc + |c|^2 - r^2 = |z - c|^2 - r^2,$

$[z.*] := \sqrt{[3]} : \langle vH, v\rangle = 0 \iff |z - c| = r;$

$\rightsquigarrow [*] := \mathcal{I}r, \mathcal{I}c\mathcal{I}\mathbb{S}(H) : S = \mathbb{S}(c, r) = \mathbb{S}_{\mathbb{R}}(H);$

$\square$

$\texttt{GeneralizedCircels} = \mathcal{S}' := \dfrac{\mathbf{H}(2) \setminus \{0\}}{\mathbb{R}^\times} : \texttt{Type};$

$\texttt{body} :: \mathcal{S}' \to ?\mathbb{C}$

$\texttt{body}\,([H]) = [H] := \mathcal{S}(H)$

$\texttt{discriminant} :: \mathcal{S}' \to ?\mathbb{R}$

$\texttt{discriminant}\,([H]) = \Delta(H) := \mathbb{R}^2 \det H$

$\texttt{orientability} :: \mathcal{S}' \to ?\mathbb{R}$

$\texttt{orientability}\,([H]) = o(H) := \mathbb{R}^\times H_{1,1}$

$\texttt{RealCircle} :: ?\mathcal{S}'$

$S : \texttt{RealCircle} \iff S \in \Re\mathcal{S}' \iff \exists c \in \mathbb{C} : r \in \mathbb{R} : S = \mathbb{S}(c, r)$

$\texttt{ImaginablelCircle} :: ?\mathcal{S}'$

$S : \texttt{ImaginableCircle} \iff S \in \Im\mathcal{S}' \iff S =_{\mathsf{SET}} \emptyset \;\&\; o(S) \neq 0$

$\texttt{PointCircle} :: ?\mathcal{S}'$

$S : \texttt{PoinCircle} \iff \exists z \in \mathbb{C} : S = \{z\}$

$\texttt{LineCircle} :: ?\mathcal{S}'$

$S : \texttt{LineCircle} \iff \exists a, b \in \mathbb{C} : S = a \vee_{\mathbb{R}} b$

$\texttt{InfinityCircle} :: ?\mathcal{S}'$

$S : \texttt{InfinityCircle} \iff S =_{\mathsf{SET}} \emptyset \;\&\; o(S) = 0$

$\texttt{RealCircleCharacterization} :: \forall S \in \mathcal{S}' \,.\, S \in \Re\mathcal{S}' \iff \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) \neq \{0\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : s \in \Re\mathcal{S}',$

$\Big(c, r, [2]\Big) := \text{ᗡ}\Re\mathcal{S}'[1] : \sum c \in \mathbb{C} \,.\, r \in \mathbb{R}_{++} \,.\, S = \mathbb{S}(c, r),$

$[3] := \texttt{EveryCircleIsHermitian}[2] : S = \left[\!\left[\begin{array}{cc} 1 & \overline{c} \\ c & |c|^2 - r^2 \end{array} \right]\!\right],$

$[1.*.1] := \text{ᗡ}\Delta(S)\text{ᗡ}\det S[3]\texttt{NoZeroSquarePositive}(\mathbb{R})\texttt{InversePositiveIsNegative}(\mathbb{R}) :$

$\quad : \Delta(S) = \mathbb{R}^2(\det S) = \mathbb{R}^2(|c|^2 - r^2 - |c|^2| = -\mathbb{R}^2 r^2 < 0,$

$[1.*.2] := \text{ᗡ}o(S)[3]\text{ᗡ}\mathbb{R}^\times : o(S) = \mathbb{R}^\times \neq \{0\};$

$\leadsto [1] := I(\Rightarrow) : S \in \Re S \Rightarrow \Delta(S) = -\mathbb{R}^2 o(S) \neq \{0\},$

$\texttt{Assume}\ [2] : \Delta(S) = -\mathbb{R}^2 o(S) \neq \{0\},$

$\Big(a, b, z, [3]\Big) := \texttt{HermitianHasRealDiagonal}(2, S) : \exists a, b \in \mathbb{R} \,.\, \exists z \in \mathbb{C} \,.\, S = \left[\!\left[\begin{array}{cc} a & \overline{z} \\ z & b \end{array} \right]\!\right],$

$[4] := [2.2]\text{ᗡ}o(S)[3] : a \neq 0,$

$c := \dfrac{z}{a} : \mathbb{C},$

$[5] := \text{ᗡ}\Delta(S)[2.1][3] : 0 < \Delta(S) = \dfrac{b}{a} - |z|^2,$

$r := \sqrt{|z|^2 - \dfrac{b}{a}} : \mathbb{R}_{++},$

$[2.*] := \mathcal{O}z\mathcal{O}[3] : S = \mathbb{S}(c, r);$

$\leadsto [*] := I(\iff)[1] : S \in \Re\mathcal{S}' \iff \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) \neq \{0\};$

$\square$

$\texttt{ImaginableCircleCharacterization} :: \forall S \in \mathcal{S}' \,.\, S \in \Im\mathcal{S}' \iff \Delta(S) = \mathbb{R}_{++}$

$\texttt{Proof} \ =$

$\Big(a, b, z, [2]\Big) := \texttt{HermitianHasRealDiagonal}(2, S) : \exists a, b \in \mathbb{R} \,.\, \exists z \in \mathbb{C} \,.\, S = \left[\!\left[\begin{array}{cc} a & \overline{z} \\ z & b \end{array} \right]\!\right],$

$\texttt{Assume}\ [2] : S \in \Im\mathcal{S}',$

$[3] := \text{ᗡ}\Im\mathcal{S}'(S) : S = \emptyset,$

$[4] := [1][3] : \forall u \in \mathbb{C} \,.\, a|u|^2 + uz + \overline{uz} + b \neq 0,$

$[5] := \text{ᗡ}\Im\mathcal{S}'[1] : a \neq 0,$

$c := \dfrac{\overline{z}}{a} : \mathbb{C},$

$[6] := [4][5] : \forall u \in \mathbb{C} \,.\, |u + \overline{c}|^2 \neq |c|^2 - \dfrac{b}{a},$

$[7] := [6]\text{ᗡ}\texttt{absVsl} : |c|^2 - \dfrac{b}{a} < 0,$

$[2.*] := \text{ᗡ}\Delta(S)[1] : \Delta(S) = \mathbb{R}_{++}\left(\dfrac{b}{a} - |c|^2\right) = \mathbb{R}_{++};$

$\leadsto [2] := I(\Rightarrow) : S \in \Im\mathcal{S}' \Rightarrow \Delta(S) = \mathbb{R}_{++},$

$\texttt{Assume}\ [3] : \Delta(S) = \mathbb{R}_{++},$

$[4] := [3]\text{ᗡ}\Delta(S)[2] : \mathbb{R}_{++} = \Delta(S) = \mathbb{R}_{++}\Big(ab - |z|^2\Big),$

$[5] := \text{ᗡ}\mathbb{R}_{+=}[4] : ab - |z|^2 > 0,$

$[6] := \text{ᗡ}\texttt{absValue}(\mathbb{C})[5] : a \neq 0 \neq b,$

$[7] := \text{ᗡ}^{-1}o(S)[6] : o(S) \neq \{0\},$

$c := \dfrac{\overline{z}}{a} : \mathbb{C},$

$\texttt{Assume}\ u : \mathbb{C}$,

$[8] := \mathcal{I}^{-1}\texttt{absVal}(\mathbb{C})[5] : |u|^2 + u\bar{c} + c\bar{u} + \frac{b}{a} = |u+c|^2 + \frac{b}{a} - |c|^2 > 0$,

$[9] := \texttt{TrichtomyRule}[8] : |u|^2 + u\bar{c} + c\bar{u} + \frac{b}{a} \neq 0$;

$\rightsquigarrow [8] := \mathcal{I}^{-1}\mathbb{S}(S) : S = \emptyset$,

$[3.*] := \mathcal{I}^{-1}\Im\mathcal{S}' : S \in \Im\mathcal{S}'$;

$\rightsquigarrow [*] := I(\iff)[2] : S \in \Im\mathcal{S}' \iff \Delta(S) = \mathbb{R}_{++}$;

$\square$

$\texttt{PointCircleCharacterization} :: \forall S \in \mathcal{S}' \,.\, \texttt{PointCircle}(S) \iff \Delta(S) = 0 \,\&\, o(S) = \mathbb{R}^\times$

$\texttt{Proof}\ =$

$\Big(a, b, z, [2]\Big) := \texttt{HermitianHasRealDiagonal}(2, S) : \exists a, b \in \mathbb{R} \,.\, \exists z \in \mathbb{C} \,.\, S = \left[\left[\begin{array}{cc} a & \bar{z} \\ z & b \end{array}\right]\right]$,

$\texttt{Assume}\ [2] : \texttt{PointCircle}(S)$,

$\Big(v, [3] := \mathcal{I}\texttt{PointCircle}(S) : \sum v \in \mathbb{C} \,.\, S = \{v\}$,

$[4] := \mathcal{I}\texttt{hermitianSphere}[3][1] : \forall u \in \mathbb{C} \,.\, a|u|^2 + uz + \overline{uz} + b = 0 \Rightarrow u = v$,

$[5] := [4] \ldots : a \neq 0$,

$c := -\frac{\bar{z}}{a} : \mathbb{C}$,

$[6] := [4]\mathcal{D}c : \forall u \in \mathbb{C} \,.\, |u|^2 - u\bar{c} - \bar{u}c + \frac{b}{a} = |u-c|^2 + \frac{b}{a} - |c|^2 = 0 \Rightarrow u = v$,

$[7] := [6] \ldots : c = v \,\&\, \frac{b}{a} - |c|^2 = 0$;

$[2.*] := \mathcal{I}^{-1}o(S)[4]\mathcal{I}^{-1}\Delta(S)[7] : \Delta(S) = 0 \,\&\, o(S) = \mathbb{R}^\times$;

$\rightsquigarrow [2] := I(\Rightarrow) : \texttt{PointCircle}(S) \Rightarrow \Delta(S) = 0 \,\&\, o(S) = \mathbb{R}^\times$,

$\texttt{Assume}\ [3] : \Delta(S) = 0 \,\&\, o(S) = \mathbb{R}^\times$,

$[4] := \mathcal{I}o(S)[1][3] : a \neq 0$,

$v := -\frac{\bar{z}}{a} : \mathbb{C}$,

$[5] := \mathcal{I}\Delta(S)[1][3]\mathcal{D}v : \frac{b}{a} - |v|^2 = 0$,

$\texttt{Assume}\ u : S$,

$[5] := \mathcal{I}S(u)\mathcal{I}\texttt{absVal}[5] : 0 = |u|^2 - u\bar{c} - \bar{u}c + \frac{b}{a} = |u-c|^2 + \frac{b}{a} - |c|^2 = |u-c|^2$,

$[u.*] := \texttt{AbsValueIsMetric}[6] : u = v$;

$\rightsquigarrow [3.*] := \mathcal{I}^{-1}\texttt{Singleton} : S = \{v\}$;

$\rightsquigarrow [*] := I(\iff) : \texttt{PointCircle}(S) \iff \Delta(S) = 0 \,\&\, o(S) = \mathbb{R}^\times$;

$\square$

$\texttt{LineCircleCharacterization} :: \forall S \in \mathcal{S}' \,.\, \texttt{LineCircle}(S) \iff \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) = 0$

$\texttt{Proof} \;=$

$\Big(a, b, z, [2]\Big) := \texttt{HermitianHasRealDiagonal}(2, S) : \exists a, b \in \mathbb{R} \,.\, \exists z \in \mathbb{C} \,.\, S = \left[\left[\begin{array}{cc} a & \overline{z} \\ z & b \end{array}\right]\right],$

$\texttt{Assume}\ [2] : \texttt{LineCircle}(S),$

$\Big(u, v, [3]\Big) := ⅁\texttt{LineCircle}(S) : \sum u, v \in \mathbb{C} \,.\, u \vee v = S,$

$f := \Lambda w \in \mathbb{C} \,.\, \Big\langle S(w, 1), (w, 1) \Big\rangle : \mathbb{C} \to \mathbb{R},$

$[4] := [3]\texttt{AnalyticLineEquation}(\mathbb{R}, \mathbb{C})Ↄf⅁S : \texttt{Affine}(\mathbb{C}, \mathbb{R}, f),$

$[5] := [1][4]Ↄf : a = 0,$

$[2.*.1] := ⅁^{-1} o(S)[1][5] : o(S) = 0,$

$[2.*.2] := ⅁^{-1}\Delta(S)⅁ \det S[1][2.*.1] : \Delta(S) = -\mathbb{R}_{++};$

$\rightsquigarrow [2] := I(\Rightarrow) : \texttt{LineCircle}(S) \Rightarrow \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) = 0,$

$\texttt{Assume}\ [3] : \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) = 0,$

$[4] := ⅁o(S)[1][3] : a = 0,$

$\texttt{Assume}\ u : S,$

$[5.*] := ⅁S(u)⅁\texttt{absVal}[5] : 0 = a|u|^2 + uz + \overline{uz} + b = uz + \overline{uz} + b = 2\Re uz + b;$

$\rightsquigarrow [3.*] := ⅁\texttt{AnalyticLineEquation} : \texttt{Line}(S, \mathbb{C});$

$\rightsquigarrow [*] := I(\iff)[2] : \texttt{LineCircle}(S) \iff \Delta(S) = -\mathbb{R}_{++} \,\&\, o(S) = 0;$

$\dots$

$\square$

$\texttt{GeneralizedCirclesClassification} :: \mathcal{S}' = \Re\mathcal{S}' \sqcup \Im\mathcal{S}' \sqcup \texttt{PointCircle} \sqcup \texttt{LineCircle} \sqcup \texttt{InfinityCircle}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OriantableGeneralizedCircle} = \mathcal{S}'' := \dfrac{\mathcal{S}'}{\mathbb{R}_{++}} : \texttt{Type};$

$\texttt{forgetOrientation} :: \mathcal{S}'' \to ?\mathbb{R}$

$\texttt{forgetOrientation}\,([H]) = [H] := \pm[H]$

$\texttt{orientation} :: \mathcal{S}'' \to S''$

$\texttt{orientation}\,([H]) = O[H] := \texttt{if}\ H_{1,1} \neq 0 \,.\, \mathbb{R}_{++}H_{1,1}$

$\texttt{pencil} :: \Big(\mathcal{S}' \times \mathcal{S}'\Big) \setminus \texttt{diagonal}(\mathcal{S}') \to ?\mathcal{S}'$

$\texttt{pencil}\,([A], [B]) = \mathbf{p}\Big([A], [B]\Big) := \Big\{[\alpha A + \beta B] \Big| (\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}\Big\}$

$\texttt{crossDeterminant} :: \prod_{R \in \mathsf{RNG}} R^{2\times 2} \times R^{2\times 2} \to R$

$\texttt{crossDeterminant}\,(A, B) = \det(A, B) := A_{1,1}B_{2,2} + A_{2,2}B_{1,1} - A_{1,2}B_{2,1} - A_{1,2}B_{2,1}$

$\texttt{LinearCombinationDeterminant} :: \forall R \in \mathsf{RNG} . \forall \alpha, \beta \in R . \forall A, B \in R^{2\times 2} .$

$. \det(\alpha A + \beta B) = \alpha^2 \det A + \beta^2 \det B + \alpha\beta \det(A, B)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{RealCircleCrossDeterminant} :: \forall A, B \in \mathbf{H}(2) . \forall [0] : [A], [B] \in \Re\mathcal{S}' . \det(A, B) = A_{1,1}A_{2,2}\left(\delta^2 - r^2 - s^2\right)$

$\texttt{where} a = \texttt{center}(A), b = \texttt{center}(B), \delta = |a - b|, r = \texttt{radius}(A), s = \texttt{radius}(B)$

$\texttt{Proof} =$

$[1] := \mathcal{I}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : A = A_{1,1}\left[\begin{array}{cc} 1 & a \\ \bar{a} & |a|^2 - r^2 \end{array}\right],$

$[2] := \mathcal{I}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : B = B_{1,1}\left[\begin{array}{cc} 1 & b \\ \bar{b} & |b|^2 - s^2 \end{array}\right],$

$[*] := \mathcal{I} \det(A, B)[1][2]\mathcal{I}\texttt{conjugation}(\mathbb{C})\mathcal{O}^{-1} : \det(A, B) = A_{1,1}B_{1,1}\left(|a|^2 - r^2 + |b|^2 - s^2 - a\bar{b} - b\bar{a}\right) =$

$. A_{1,1}B_{1,1}\left(|a - b|^2 - r^2 - s^2\right) = A_{1,1}B_{1,1}\left(\delta^2 - r^2 - s^2\right);$

$\square$

$\texttt{affineWindingFunction} :: \mathbb{R}\mathcal{S}' \to \pm 1 \to C^\infty(\mathbb{R}, \mathbb{C})$

$\texttt{affineWindingFunction}(S, s) = w_{S,s} := \Lambda t \in \mathbb{R} . w(st)T \quad \texttt{where} \quad T \in \mathrm{Di}_{\mathbb{R}}(\mathbb{C}) \,\&\, T\mathbb{S}^1 = S$

$\texttt{IntersectingRealCircles} :: ?(\Re\mathcal{S}'' \times \Re\mathcal{S}'')$

$A, B : \texttt{IntersectingRealCircle} \iff A \cap B \neq \emptyset$

$\texttt{intersectionAngle} :: \texttt{IntersectingCircles} \to \texttt{Angle}(\mathbb{R}, \mathbb{C})$

$\texttt{intersectionAngle}(A, B) = \omega(A, B) := \angle \dot{w}_{A,a}|_t \dot{w}_{B,b}|_s$

$\texttt{where} \quad a = \mathrm{sign}\, O(A), b = \mathrm{sign}\, O(B), t, s \in \mathbb{R} : w_{A,a}(t) = w_{B,b}(s)$

$\texttt{IntersectionAngleAnalyticExpression} :: \forall A, B : \texttt{IntersectingCircles} .$

$. \cos\omega(A, B) = \mp\frac{\det(A, B)}{2\sqrt{\det A \det B}}$

$\texttt{Proof} =$

$\left[t, s, [1]\right] := \mathcal{I}\texttt{IntersectingCircle}(A, B)\mathcal{I}^{-1} w_A, w_B : \sum t, s \in \mathbb{R} . w_A(t) = w_B(s),$

$p := w_A(t) : A \cap B,$

$a := \texttt{center}(A) : \mathbb{C},$

$b := \texttt{center}(B) : \mathbb{C},$

$\rho := \texttt{radius}(A) : \mathbb{R}_{++},$

$\sigma := \texttt{radius}(B) : \mathbb{R}_{++},$

$\delta := |a - b| : \mathbb{R}_+,$

$[2] := \texttt{CircleTangentIsOrthogonalToRadian} : \overrightarrow{pa} \perp \dot{w}_A|_t \,\&\, \overrightarrow{pb} \perp \dot{w}_B|_s,$

$\left(\xi, \zeta, [3]\right) := \texttt{ComplexMatrixReprezentation}[2] : \sum \xi, \zeta \in \{+1, -1\} . \overrightarrow{pa} = \rho\xi\mathbf{i}\dot{w}_A|_t \,\&\, \rho\overrightarrow{pb} = \sigma\zeta\mathbf{i}\dot{w}_B|_s,$

$[*] := \mathcal{I}\omega(A, B)\texttt{RotationPreservesCosine}[3]\texttt{LawOfCosines}(\mathbb{R}, \mathbb{C})\mathcal{I}^{-1}$

$\texttt{RealCircleCrossDeterminant}(A, B)\mathcal{I}^{-1} \det A \det B :$

$: \cos\omega(A, B) = \cos\angle\dot{w}_{A,a}|_t \dot{w}_{B,b}|_s = \mp\cos\overrightarrow{pa}\overrightarrow{pb} = \mp\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = \mp\frac{\det(A, B)}{2\sqrt{\det A \det B}};$

$\square$

$\texttt{NonsingularCircles} = \mathcal{S}_* := \Re\mathcal{S}' \sqcup \Im\mathcal{S}' \sqcup \texttt{LineCircle} :?\mathcal{S}';$

$\texttt{commonInvariant} :: \mathcal{S}_* \times \mathcal{S}_* \to \mathbb{C}$

$$\texttt{commonInvariant}\,(A,B) = \Omega(A,B) := \frac{\det(A,B)}{2\sqrt{\det A \det B}}$$

$\texttt{Orthogonal} :: ?(\mathcal{S}' \times \mathcal{S}')$

$A, B : \texttt{Orthogonal} \iff A \perp B \iff \det(A,B) = 0$

$\texttt{KissingCircles} :: ?\Big(\Re\mathcal{S}'' \times \Re\mathcal{S}''\Big)$

$A, B : \texttt{KissingCircles} \iff |\text{conv } A \cap \text{conv } B| = 1$

$\texttt{KissingCirclesRadiCharacterization} :: \forall A, B \in \Re\mathcal{S}' \,.\, \texttt{KissingCircles}(A,B) \iff \rho + \sigma = \delta$

$\quad \texttt{where} \rho = \texttt{radius}(A), \sigma = \texttt{radius}(B), a = \texttt{center}(A), b = \texttt{center}(b), \delta = |a - b|$

$\texttt{Proof} =$

$\texttt{KissingCirclesCommonInvariant} :: \forall A, B \in \Re\mathcal{S}'' \,.\, \texttt{KissingCircles}(A,B) \iff \Omega(A,B) = \mp 1$

$\texttt{Proof} =$

$a := \texttt{center}(A) : \mathbb{C},$

$b := \texttt{center}(B) : \mathbb{C},$

$\rho := \texttt{radius}(A) : \mathbb{R}_{++},$

$\sigma := \texttt{radius}(B) : \mathbb{R}_{++},$

$\delta := |a - b| : \mathbb{R}_{+},$

$[1] := \text{Ɑ}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : A = A_{1,1}\begin{bmatrix} 1 & a \\ \bar{a} & |a|^2 - \rho^2 \end{bmatrix},$

$[2] := \text{Ɑ}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : B = B_{1,1}\begin{bmatrix} 1 & b \\ \bar{b} & |b|^2 - \sigma^2 \end{bmatrix},$

$\texttt{Assume } [3] : \texttt{KissingCircles}(A,B),$

$[3.*] := \text{Ɑ}\Omega(A,B)\texttt{RealCircleCrossDeterminant}(A,B)\texttt{BinomialExpansion}(\rho,\sigma)$

$\quad \texttt{KissingCirclesRadiCharacterization}(A,B)\text{Ɑ}\texttt{Inverse}(A,B) :$

$$: \Omega(A,B) = \mp\frac{\Delta(A,B)}{\sqrt{\det A \det B}} = \mp\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = \mp\frac{\delta^2 - (\rho+\sigma)^2 + 2\rho\sigma}{\rho\sigma} = \mp\frac{\delta^2 - \delta^2 + 2\rho\sigma}{2\rho\sigma} = \mp 1;$$

$\leadsto [3] := I(\Rightarrow) : \texttt{KissingCircles}(A,B) \Rightarrow \Omega(A,B) = \mp 1,$

$\texttt{Assume } [4] : \Omega(A,B) = \mp 1,$

$$[5] := [4]\text{Ɑ}\Omega(A,B)\texttt{RealCircleCrossDeterminant}(A,B) : \mp 1 = \Omega(A,B) = \mp\frac{\Delta(A,B)}{\sqrt{\det A \det B}} = \mp\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma};$$

$[6] := 2\rho\sigma[5] : 2\rho\sigma = \delta^2 - \rho^2 - \sigma^2,$

$[7] := \Big([6] + \rho^2 + \sigma^2\Big)\texttt{BinomialExpansion}(\rho,\sigma) : \delta^2 = (\rho+\sigma)^2,$

$[4.*] := \sqrt{[7]}\texttt{KissingCirclesRadiCharacterization}(A,B) : \texttt{KissingCircles}(A,B);$

$\leadsto [*] := I(\iff)[3] : \texttt{KissingCircles}(A,B) \iff \Omega(A,B) = \mp 1;$

$\square$

$\texttt{CirclesTouchingInside} :: ?\left(\Re\mathcal{S}'' \times \Re\mathcal{S}''\right)$

$A, B : \texttt{Circles} \iff |A \cap B| = 1 \ \& \ !\,\texttt{KissingCircles}(A, B)$

$\texttt{CirclesTouchingInsideRadiCharacterization} :: \forall A, B \in \Re\mathcal{S}' \,.\, \texttt{KissingCircles}(A, B) \iff |\rho - \sigma| = \delta$

$\quad \texttt{where} \rho = \texttt{radius}(A), \sigma = \texttt{radius}(B), a = \texttt{center}(A), b = \texttt{center}(b), \delta = |a - b|$

$\texttt{Proof} \ =$

$\texttt{TouchingInsideCommonInvariant} :: \forall A, B \in \Re\mathcal{S}'' \,.\, \texttt{CirclesTouchingInside}(A, B) \iff \Omega(A, B) = \pm 1$

$\texttt{Proof} \ =$

$a := \texttt{center}(A) : \mathbb{C},$

$b := \texttt{center}(B) : \mathbb{C},$

$\rho := \texttt{radius}(A) : \mathbb{R}_{++},$

$\sigma := \texttt{radius}(B) : \mathbb{R}_{++},$

$\delta := |a - b| : \mathbb{R}_{+},$

$[1] := Cl\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : A = A_{1,1} \begin{bmatrix} 1 & a \\ \bar{a} & |a|^2 - \rho^2 \end{bmatrix},$

$[2] := Cl\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : B = B_{1,1} \begin{bmatrix} 1 & b \\ \bar{b} & |b|^2 - \sigma^2 \end{bmatrix},$

$\texttt{Assume}\ [3] : A =_{\mathcal{S}'} B,$

$[3.*] := Cl\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)\texttt{BinomialExpansion}(\rho, \sigma)$

$\quad \texttt{IdenticalCirclesRadiCharacterization}(A, B)Cl\texttt{Inverse}(A, B) :$

$$: \Omega(A, B) = \mp \frac{\Delta(A, B)}{\sqrt{\det A \det B}} = \mp \frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = \mp \frac{(\rho + \sigma)^2 - 2\rho\sigma}{\rho\sigma} = \mp \frac{-2\rho\sigma}{2\rho\sigma} = \pm 1;$$

$\rightsquigarrow [3] := I(\Rightarrow) : A =_{\mathcal{S}'} B \Rightarrow \Omega(A, B) = \pm 1,$

$\texttt{Assume}\ [3] : \texttt{KissingCircles}(A, B),$

$[3.*] := Cl\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)\texttt{BinomialExpansion}(\rho, \sigma)$

$\quad \texttt{KissingCirclesRadiCharacterization}(A, B)Cl\texttt{Inverse}(A, B) :$

$$: \Omega(A, B) = \mp \frac{\Delta(A, B)}{\sqrt{\det A \det B}} = \mp \frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = \mp \frac{\delta^2 - (\rho + \sigma)^2 + 2\rho\sigma}{\rho\sigma} = \mp \frac{\delta^2 - \delta^2 + 2\rho\sigma}{2\rho\sigma} = \mp 1;$$

$\rightsquigarrow [3] := I(\Rightarrow) : \texttt{KissingCircles}(A, B) \Rightarrow \Omega(A, B) = \mp 1,$

$\texttt{Assume}\ [4] : \Omega(A, B) = \mathbf{p}, 1,$

$[5] := [4]Cl\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B) : \pm 1 = \Omega(A, B) = \mp \frac{\Delta(A, B)}{\sqrt{\det A \det B}} = \mp \frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma};$

$[6] := 2\rho\sigma[5] : -2\rho\sigma = \delta^2 - \rho^2 - \sigma^2,$

$[7] := \left([6] + \rho^2 + \sigma^2\right)\texttt{BinomialExpansion}(\rho, \sigma) : \delta^2 = (\rho - \sigma)^2,$

$[4.*] := \sqrt{[7]}\texttt{TouchingInsideRadiCharacterization}(A, B) : \texttt{CirclesTouchingInside}(A, B);$

$\rightsquigarrow [*] := I(\iff)[3] : \texttt{KissingCircles}(A, B) \iff \Omega(A, B) = \mp 1;$

$\square$

$\texttt{IdenticalCommonInvariant} :: \forall A, B \in \Re\mathcal{S}'' \,.\, A =_{\mathcal{S}'} B \Rightarrow \Omega(A, B) = \pm 1$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{IntersectingCircleCommonInvariant} :: \forall A, B \in \Re\mathcal{S}'' \,.\, \texttt{IntersectingCircles}(A, B) \iff \Big|\Omega(A, B)\Big| \le 1$

$\texttt{Proof} =$

$a := \texttt{center}(A) : \mathbb{C},$

$b := \texttt{center}(B) : \mathbb{C},$

$\rho := \texttt{radius}(A) : \mathbb{R}_{++},$

$\sigma := \texttt{radius}(B) : \mathbb{R}_{++},$

$\delta := |a - b| : \mathbb{R}_{+},$

$[1] := \mathcal{A}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : A = A_{1,1}\begin{bmatrix} 1 & a \\ \bar{a} & |a|^2 - \rho^2 \end{bmatrix},$

$[2] := \mathcal{A}\Re\mathcal{S}'[0]\texttt{EveryCircleIsHermitian}(A) : B = B_{1,1}\begin{bmatrix} 1 & b \\ \bar{b} & |b|^2 - \sigma^2 \end{bmatrix},$

$\texttt{Assume } [3] : \texttt{IntersectingCircles}(A, B),$

$\Big(p, [4]\Big) := \mathcal{A}\texttt{IntersectingCircles}(A, B) : \sum p \in \mathbb{C} \,.\, p \in A \cap B,$

$[5] := \mathcal{D}\delta\texttt{TriangleIneq}(\mathbb{C})(a, b, p)\mathcal{A}\texttt{RealsCircle}(A \,\&\, B)\mathcal{D}\rho\mathcal{D}\sigma : \delta = |a - b| \le |a - p| + |b - p| = \rho + \sigma,$

$[6] := \mathcal{D}\rho\mathcal{D}\sigma\texttt{InverseTriangleIneq}(\mathbb{C}, a, b, p)\mathcal{D}^{-1}\delta : |\rho - \sigma| = \Big||a - p| - |b - p|\Big| \le |a - b| = \delta,$

$[7] := [6]^2 : \rho^2 - 2\rho\sigma + \sigma^2 \le \delta^2,$

$\texttt{Assume } [8] : \rho^2 + \sigma^2 - \delta^2 \ge 0,$

$[8.*] := \mathcal{A}\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)[8][7]\mathcal{A}\texttt{Inverse}(\mathbb{R}, 2\rho\sigma) :$

$$= |\Omega(A, B)| = \left|\frac{\Delta(A, B)}{\sqrt{\det A \det B}}\right| = \left|\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma}\right| = \frac{\rho^2 + \sigma^2 - \delta^2}{2\rho\sigma} \le \frac{2\rho\sigma}{2\rho\sigma} = 1;$$

$\rightsquigarrow [8] := I(\Rightarrow) : \rho^2 + \sigma^2 - \delta^2 \ge 0 \Rightarrow \Big|\Omega(A, B)\Big| \le 1,$

$\texttt{Assume } [9] : \rho^2 + \sigma^2 - \delta^2 < 0,$

$[9.*] := \mathcal{A}\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)[9][5]\mathcal{A}\texttt{Inverse}(\mathbb{R}, 2\rho\sigma) :$

$$= |\Omega(A, B)| = \left|\frac{\Delta(A, B)}{\sqrt{\det A \det B}}\right| = \left|\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma}\right| = \frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = \frac{\delta^2 -)(\rho - \sigma)^2}{2\rho\sigma} + \frac{2\rho\sigma}{2\rho\sigma} \le 1;$$

$\rightsquigarrow [9] := I(\Rightarrow) : \rho^2 + \sigma^2 - \delta^2 < 0 \Rightarrow \Big|\Omega(A, B)\Big| \le 1,$

$[3.*] := E(|)[8][9]\texttt{TrichtomyTHM}(\mathbb{R}) : \Big|\Omega(A, B)\Big| \le 1;$

$\rightsquigarrow [3] := I(\Rightarrow) : \texttt{IntersectingCircles}(A, B) \Rightarrow \Big|\Omega(A, B)\Big| \le 1,$

$\texttt{Assume } [-1] : |\Omega(A, B)| < 1,$

$\texttt{Assume } [0] : \delta^2 - \rho^2 - \sigma^2 \ge 0,$

$[4] := [-1]\mathcal{A}\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)[0]\mathcal{A}\texttt{Inverse}(\mathbb{R}, 2\rho\sigma) :$

$$: 1 \ge |\Omega(A, B)| = \left|\frac{\Delta(A, B)}{\sqrt{\det A \det B}}\right| = \left|\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma}\right| = \frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma} = 1 + \frac{\delta^2 - (\rho + \sigma)^2}{2\rho\sigma},$$

$[5] := \texttt{PositiveSumIsGreater}[4] : \delta \le \rho + \sigma,$

$[6] := [4]^2 : -2\rho\sigma \le \rho^2 + \sigma^2 - \delta^2,$

$[7] := \texttt{BinomialExpansion}(\mathbb{R}, \rho, \sigma)[6][0] : (\rho - \sigma)^2 = \rho^2 - 2\rho\sigma + \sigma^2 \le 2\rho^2 + 2\sigma^2 - \delta^2 \le \delta^2,$

$[0.*] := \texttt{IntersectingCirclesRadiCharacterization}[5][7] : \texttt{IntersectingCircles}(A, B);$

$\rightsquigarrow [0] := I(\Rightarrow) : \delta^2 - \rho^2 - \sigma^2 \ge 0 \Rightarrow \texttt{IntersectingCircles}(A, B),$

$\texttt{Assume } [00] : \delta^2 - \rho^2 - \sigma^2 < 0,$

$[4] := [-1]\mathcal{A}\Omega(A, B)\texttt{RealCircleCrossDeterminant}(A, B)[00]\mathcal{A}\texttt{Inverse}(\mathbb{R}, 2\rho\sigma) :$

$$: 1 \ge |\Omega(A, B)| = \left|\frac{\Delta(A, B)}{\sqrt{\det A \det B}}\right| = \left|\frac{\delta^2 - \rho^2 - \sigma^2}{2\rho\sigma}\right| = \frac{\rho^2 + \sigma^2 - \delta^2}{2\rho\sigma} = \frac{(\rho + \sigma)^2 - \delta^2}{2\rho\sigma} - 1,$$

$[5] := [4] 2\rho\sigma - 4\rho\sigma : (\rho-\sigma)^2 \le \delta^2,$
$[6] := [00]\texttt{NonNegSumGreater}(2\rho\sigma)\texttt{BinomialExapansion}(\mathbb{R},\rho,\sigma) : \delta^2 \le \rho^2 + \delta^2 \le \rho^2 + 2\rho\sigma + \sigma^2 = (\rho+\sigma)^2,$
$[00.*] := \texttt{IntersectingCirclesRadiCharacterization}[5][7] : \texttt{IntersectingCircles}(A,B);$
$\rightsquigarrow [00] := I(\Rightarrow) : \delta^2 - \rho^2 - \sigma^2 < 0 \Rightarrow \texttt{IntersectingCircles}(A,B),$
$[-1.*] := I(|)[0][00]\texttt{TrichomyTHM} : \texttt{IntersectingCircles}(A,B);$
$\rightsquigarrow [*] := I(\iff)[3][-1] : \texttt{IntersectingCircles}(A,B) \iff \Big|\Omega(A,B)\Big| \le 1;$
$\square$

$\texttt{Pencel} :: ??\mathcal{S}'$
$P : \texttt{Pencel} \iff \exists A,B \in \mathcal{S}' : P = \mathbf{p}(A,B)$

$\texttt{PencelInvariant} :: \forall P : \texttt{Pencel} . \forall A,B,C,D \in P . \forall[0] : [A] \ne [B] \;\&\; [C] \ne [D] .$
$. \mathbb{R}_{++}(\det A \det B + \frac{1}{4}\det^2(A,B)) = \mathbb{R}_{++}(\det C \det D + \frac{1}{4}\det^2(A,C))$
$\texttt{Proof} =$
$\Big(\alpha,\beta,[1]\Big) := \mathcal{O}\texttt{Pencel}(P,A,B,C) : \sum(\alpha,\beta) \in \mathbb{R}^2 \setminus \{0\} . C = \alpha A + \beta B,$
$\Big(\gamma,\delta,[2]\Big) := \mathcal{O}\texttt{Pencel}(P,A,B,D) : \sum(\gamma,\delta) \in \mathbb{R}^2 \setminus \{0\} . D = \gamma A + \delta B,$
$$q := \begin{bmatrix} \det A & \frac{\det(A,B)}{2} \\ \frac{\det(A,B)}{2} & \det B \end{bmatrix} : \mathbb{R}^{2\times 2},$$
$$q' := \begin{bmatrix} \det C & \frac{\det(C,D)}{2} \\ \frac{\det(C,D)}{2} & \det C \end{bmatrix} : \mathbb{R}^{2\times 2},$$
$$T := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \mathbf{GL}(\mathbb{R},2),$$
$[3] := \texttt{QuadraticFormChangeOfBasis}[1][2] : q' = T^\top q T,$
$[*] := \texttt{DetHomo}(\mathbb{R},2)\texttt{DetOfTranspose}(\mathbb{R},2)[3] : \det q' = (\det T)^2 \det q;$
$\square$

$\texttt{pencelDiscriminant} :: \texttt{Pencel} \to \frac{\mathbb{R}}{\mathbb{R}_{++}}$
$\texttt{pencelDiscriminant}(P) = \Delta(P) := \mathbb{R}_{++}\det(A+B) \quad \texttt{where} \quad [A],[B] \in P \;\&\; [A] \ne [B]$
$\square$

$\texttt{EllipticPencel} :: ?\texttt{Pencel}$
$P : \texttt{EllipticPencel} \iff \Delta(P) > 0$

$\texttt{ParabolicPencel} :: ?\texttt{Pencel}$
$P : \texttt{ParabolicPencel} \iff \Delta(P) = 0$

$\texttt{HyperbolicPencel} :: ?\texttt{Pencel}$
$P : \texttt{HyperbolicPencel} \iff \Delta(P) < 0$

$\texttt{GeneralEllipticPencel} :: ??\mathcal{S}'$

$P : \texttt{GeneralEllipticPencel} \iff \exists a, b \in \mathbb{C} : a \neq b \;\&\; \forall A, B \in \mathcal{S}' \,.\, A \cap B = \{a, b\} \iff A, B \in P$

$\texttt{GeneralEllipticPencelIsEllipticPencel} :: \forall P : \texttt{GeneralEllipticPencel} \,.\, \texttt{EllipticPencel}(P)$

$\texttt{Proof} \;=$

$\Big(a, b, [1]\Big) := \mathcal{O}\!\!\!\!\mathcal{I}\,\texttt{GeneralEllipticPencel}(P) : \sum a, b \in \mathbb{C} \,.\, a \neq b \;\&\; \forall A, B \in \mathcal{S}' \,.\, A \cap B = \{a, b\} \iff A, B \in P$

$T := \Lambda A \in \mathbb{C}^{2\times 2} \,.\, \begin{pmatrix} |a|^2 A_{1,1} + a A_{1,2} + \bar{a} A_{2,1} + A_{2,2} \\ |b|^2 A_{1,1} + b A_{1,2} + \bar{b} A_{2,1} + A_{2,2} \end{pmatrix} : \mathbb{C}^{2\times 2} \xrightarrow{\mathbb{C}\textbf{-AFF}} \mathbb{C}^2,$

$[2] := \mathcal{D}T[1.1]\texttt{VandermontDeterminantTHM}(\mathbb{C}, 2) : \operatorname{rank} T = 2,$

$[4] := \texttt{KerRankTHM}[2] : \dim \ker T = 2,$

$[5] := \mathcal{D}T[1.2]\mathcal{O}\!\!\!\!\mathcal{I}\mathcal{S}' : P = \dfrac{T^{-1}(0) \cap \mathbf{H}(2)}{\mathbb{R}^\times},$

$[6] := [4][5]\texttt{HermitianRealStucture}(2)\mathcal{O}\!\!\!\!\mathcal{I}^{-1}\texttt{Pencel} : \texttt{Pencel}(P),$

$\texttt{Assume } A, B : P \cap \Re \mathcal{S}',$

$\texttt{Assume } [7] : A \neq B,$

$[8] := [1.2](A, B) : A \cap B = \{a, b\},$

$[9] := \texttt{IntersectingCircleCommonInvariant}[8] : \left| \dfrac{\det(A, B)}{2\sqrt{\det A \det B}} \right| < 1,$

$[10] := [9]^2 : \dfrac{\det^2(A, B)}{4 \det A \det B} < 1,$

$\Big[(A, B).*\Big] := \det A \det B - [10] \det A \det B : \det A \det B - \det^2(A, B) > 0;$

$\rightsquigarrow [7] := \texttt{PencelInvariant}(P) : \Delta(P) > 0,$

$[*] := \mathcal{O}\!\!\!\!\mathcal{I}^{-1}\texttt{EllipticPencel}[7] : \texttt{EllipticPencel}(P);$

$\square$

$\texttt{IntersectingCirclesGenerateGeneralEllipticPencel} ::$

$:: \forall A, B \in \mathcal{S}_* \,.\, |A \cap B| = 2 \Rightarrow \texttt{GeneralEllipticPencel}\Big(\mathbf{p}(A, B)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SpecialEllipticPencel} :: ??\texttt{LineCircle}$

$P : \texttt{SpecialEllipticPencel} \iff \exists z \in \mathbb{C} \,.\, \forall l, m : \texttt{LineCircle} \,.\, l \cap m = \{z\} \iff l, m \in P$

$\texttt{SpecialEllipticPencelIsEllipticPencel} :: \forall P : \texttt{SpecialEllipticPencel} \,.\, \texttt{EllipticPencel}(P)$

$\texttt{Proof} \;=$

$\Big(z, [1]\Big) := \mathcal{I}\texttt{SpecialEllipticPencel}(P) : \sum z \in \mathbb{C} \,.\, \forall l, m : \texttt{LineCircle} \,.\, l \cap m = \{z\} \iff l, m \in P,$

$T := \Lambda A \in \mathbb{C}^{2\times 2} \,.\, \begin{pmatrix} |z|^2 A_{1,1} + z A_{1,2} + \bar{z} A_{2,1} + A_{2,2} \\ A_{1,1} \end{pmatrix} : \mathbb{C}^{2\times 2} \xrightarrow{\mathbb{C}\text{-}\mathbf{AFF}} \mathbb{C}^2,$

$[2] := \mathcal{D}T : \operatorname{rank} T = 2,$

$[4] := \texttt{KerRankTHM}[2] : \dim\ker T = 2,$

$[5] := \mathcal{D}T[1]\mathcal{I}\mathcal{S}' : P = \dfrac{T^{-1}(0) \cap \mathbf{H}(2)}{\mathbb{R}^\times},$

$[6] := [4][5]\texttt{HermitianRealStucture}(2)\mathcal{I}^{-1}\texttt{Pencel} : \texttt{Pencel}(P),$

$\texttt{Assume}\ l, m : P,$

$\texttt{Assume}\ [7] : l \neq m,$

$[8] := [1.2](l, m) : l \cap m = \{z\},$

$\Big(v, \alpha, [9]\Big) := \mathcal{I}\texttt{LineCircle}(l) : \sum v \in \mathbb{C}^\times \,.\, \sum \alpha \in \mathbb{R} \,.\, l = \left[\left[\begin{array}{cc} 1 & v \\ \bar{v} & \alpha \end{array}\right]\right],$

$\Big(u, \beta, [10]\Big) := \mathcal{I}\texttt{LineCircle}(m) : \sum u \in \mathbb{C}^\times \,.\, \sum \beta \in \mathbb{R} \,.\, m = \left[\left[\begin{array}{cc} 1 & u \\ \bar{u} & \beta \end{array}\right]\right],$

$[11] := \texttt{EuclidsFithPostulate}(\mathbb{R}, \mathbb{C})[8] : l \nparallel m,$

$[12] := \mathcal{I}\mathcal{S}'\mathcal{I}\texttt{Parallel}[11][9][8] : \langle v \rangle \neq \langle u \rangle,$

$[13] := \texttt{StrictCauchySchwarzIneqCondition}[12] : \|u\|\|v\| > \Big|\langle u, v\rangle\Big|,$

$\Big[(l, m).*\Big] := \mathcal{I}\det l[9]\mathcal{I}\det m[10]\mathcal{I}\det(A, B)[9][10]\mathcal{I}^{-1}\texttt{EucleadenProduct}(\mathbb{C})[13] :$

$\quad : \det(l)\det(m) - \dfrac{1}{4}\det^2(l, m) = |u|^2|v|^2 - \dfrac{1}{4}(-u\bar{v} - \bar{u}v)^2 = \Big(\|u\|\|v\|\Big)^2 - \langle u, v\rangle^2 > 0;$

$\leadsto [7] := \texttt{PencelInvariant}(P) : \Delta(P) > 0,$

$[*] := \mathcal{I}^{-1}\texttt{EllipticPencel}[7] : \texttt{EllipticPencel}(P);$

$\square$

$\texttt{IntersectingLinesGenerateGeneralEllipticPencel} ::$

$\quad :: \forall l, m : \texttt{LineCircle} \,.\, |l \cap m| = 1 \Rightarrow \texttt{SpecialEllipticPencel}\Big(\mathbf{p}(l, m)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{pointCircle} :: \mathbb{C} \leftrightarrow \texttt{PointCircle}$

$$\texttt{pointCircle}(z) = \mathrm{pt}(z) := \left[\left[\begin{array}{cc} 1 & -z \\ -\bar{z} & |z|^2 \end{array}\right]\right]$$

$\texttt{GeneralParabolicPencel} :: ?\texttt{Pencel}$

$P : \texttt{GeneralParabolicPencel} \iff \exists z \in \mathbb{C} \,.\, \mathrm{pt}(z) \in P \;\&\; \forall A, B \in P \,.\, A \cap B = \{z\}$

$\texttt{GeneralParabolicPencelIsParabolicPencel} :: \forall P : \texttt{GeneralParabolicPencel} \,.\, \texttt{ParabolicPencel}(P)$

$\texttt{Proof} \;=$

$\Big(z, [1]\Big) := \mathit{Л}\texttt{GeneralEllipticPencel}(P) : \sum z \in \mathbb{C} \,.\, \mathrm{pt}(z) \in P \;\&\; \forall A, B \in P \,.\, A \cap B = \{z\},$

$\Big(A, B, [2]\Big) := \texttt{SpecialEllipticPencelIsElliptic}[1] : \sum A, B : \Re\mathcal{S}\text{''} \cap P \,.\, A \neq B,$

$[3] := [1.2](A, B) : A \cap B = \{z\},$

$[4] := \texttt{KissingCircleCommonInvariant}[3] : \left|\dfrac{\det(A, B)}{2\sqrt{\det A \det B}}\right| = 1,$

$[5] := [4]^2 : \dfrac{\det^2(A, B)}{4 \det A \det B} = 1,$

$[6] := \det A \det B - [10] \det A \det B : \det A \det B - \det^2(A, B) = 0;$

$\rightsquigarrow [7] := \texttt{PencelInvariant}(P)[6] : \Delta(P) = 0,$

$[*] := \mathit{Л}^{-1}\texttt{ParabolicPencel}[7] : \texttt{ParabolicPencel}(P);$

$\square$

$\texttt{KissingCirclesGenerateGeneralParabolicPencel} ::$

$$:: \forall A, B \in \Re\mathcal{S}' \,.\, |A \cap B| = 1 \Rightarrow \texttt{GeneralParabolicPencel}\Big(\mathbf{p}(A, B)\Big)$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CirclesAndTangentLineGenerateGeneralParabolicPencel} ::$

$$:: \forall A \in \Re\mathcal{S}' \,.\, \forall B : \texttt{LineCircle} \,.\, |A \cap B| = 1 \Rightarrow \texttt{GeneralParabolicPencel}\Big(\mathbf{p}(A, B)\Big)$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CirclesAndPointOnGeneralGenerateParabolicPencel} ::$

$$:: \forall A \in \Re\mathcal{S}_* \,.\, \forall z \in A \,.\, \texttt{GeneralParabolicPencel}\Big(\mathbf{p}(A, \mathrm{pt}(z))\Big)$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{infinityCircle} :: \texttt{InfinityCircle}$

$$\texttt{infinityCirlcle}\,() = \mathrm{pt}(\infty) := \left[\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\right]$$

$\texttt{SpecialParabolicPencel} :: ?\texttt{Pencel}$

$P : \texttt{SpecialParabolicPencel} \iff \exists l : \texttt{LineCircle} \cap P \,.\, \forall m : \texttt{LineCircle} \,.\, m \parallel l \Rightarrow m \in P$

$\texttt{SpecialParabolicPencelIsParabolicPencel} :: \forall P : \texttt{SpecialParabolicPencel} \,.\, \texttt{ParabolicPencel}(P)$

$\texttt{Proof} =$

$\Big(l, [1]\Big) := \mathcal{I}\texttt{SpecialParabolicPencel}(P) : \sum l : \texttt{LineCircle} \cap P \,.\, \forall m : \texttt{LineCircle} \,.\, m \parallel l \Rightarrow m \in l,$

$\Big(v, \alpha, [2]\Big) := \mathcal{I}\texttt{LineCircle}(l) : \sum v \in \mathbb{C}^\times \,.\, \sum \alpha \in \mathbb{R} \,.\, l = \left[\left[\begin{array}{cc} 1 & v \\ \bar{v} & \alpha \end{array}\right]\right],$

$m := \tau_v(l) : \texttt{LineCircle},$

$[3] := \texttt{TranslationIsDilation}\mathcal{O}m : m \parallel l,$

$[4] := [1][3] : m \in P,$

$[5] := \mathcal{I}v\mathcal{I}\mathcal{S}'[1]\mathcal{O}m : m \neq l,$

$\Big(u, \beta, [6]\Big) := \mathcal{I}\texttt{LineCircle}(m) : \sum u \in \mathbb{C}^\times \,.\, \sum \beta \in \mathbb{R} \,.\, m = \left[\left[\begin{array}{cc} 1 & u \\ \bar{u} & \beta \end{array}\right]\right],$

$[7] := \mathcal{I}\mathcal{S}'\mathcal{I}\texttt{Parallel}[2][3][6] : \langle v \rangle = \langle u \rangle,$

$[8] := \texttt{StrictCauchySchwarzIneqCondition}[7] : \|u\|\|v\| = \Big|\langle u, v \rangle\Big|,$

$\Big[[10]\Big] := \mathcal{I} \det l[9]\mathcal{I} \det m[10]\mathcal{I} \det(A, B)[9][10]\mathcal{I}^{-1}\texttt{EucleadenProduct}(\mathbb{C})[13] :$

$$: \det(l)\det(m) - \frac{1}{4}\det^2(l, m) = |u|^2|v|^2 - \frac{1}{4}(-u\bar{v} - \bar{u}v)^2 = \Big(\|u\|\|v\|\Big)^2 - \langle u, v \rangle^2 = 0,$$

$\leadsto [11] := \texttt{PencelInvariant}(P) : \Delta(P) = 0,$

$[*] := \mathcal{I}^{-1}\texttt{ParabolicPencel}[7] : \texttt{ParabolicPencel}(P);$

$\square$

$\texttt{ParallelLinesGenerateGeneralParabolicPencel} ::$

$$:: \forall l, m : \texttt{LineCircle} \,.\, l \parallel m \texttt{SpecialEllipticPencel}\Big(\mathbf{p}(l, m)\Big)$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LineAndInfinityGenerateGeneralParabolicPencel} ::$

$$:: \forall l, m : \texttt{LineCircle} \,.\, \texttt{SpecialEllipticPencel}\Big(\mathbf{p}\big(l, \mathrm{pt}(\infty)\big)\Big)$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ImaginableCirclePencelInvariant} :: \forall A, B \in \Im\mathcal{S}' \,.\, \forall[0] : A \neq B \,.\, \det A \det B - \frac{1}{4}\det^2(A,B) < 0$

$\texttt{Proof} \ =$

$\Big(a, \rho, [1]\Big) := \Game\mathcal{S}' : \sum a \in \mathbb{C} \,.\, \sum \rho \in \mathbb{R}_{++} \,.\, A = \left[\left[\begin{array}{cc} 1 & a \\ \bar{a} & |a|^2 + \rho^2 \end{array}\right]\right],$

$\Big(b, \sigma, [2]\Big) := \Game\mathcal{S}' : \sum b \in \mathbb{C} \,.\, \sum \sigma \in \mathbb{R}_{++} \,.\, B = \left[\left[\begin{array}{cc} 1 & b \\ \bar{b} & |b|^2 + \sigma^2 \end{array}\right]\right],$

$[*] := \Game \det A[1] \Game \det B[2] \Game \det(A,B)[1][2] \Game^{-1} \texttt{absValue}(\mathbb{C}) \texttt{BinomialExpansion}(\mathbb{C})[0] :$

$: \det A \det B - \frac{1}{4}\det^2(A,B) = \rho^2\sigma^2 - \frac{1}{4}\Big(|b|^2 + \sigma^2 + |a|^2 + \rho^2 - a\bar{b} - \bar{a}b\Big)^2 =$

$= \rho^2\sigma^2 - \frac{1}{4}\Big(\rho^2 + \sigma^2 + |a-b|^2\Big)^2 \leq \frac{1}{2}\rho^2\sigma^2 - \frac{1}{4}\rho^4 - \frac{1}{4}\sigma^4 + \frac{1}{2}|a-b|^4 = -\frac{1}{4}(\rho^2 - \sigma^2)^2 - \frac{1}{4}|a-b|^4 < 0;$

$\square$

$\texttt{ImaginableCirclesExistInHyperbolicPencelsOnly} ::$

$:: \forall P : \texttt{Pencel} \,.\, \forall A \in P \cap \Im\mathcal{S}' \,.\, \texttt{HyperbolicPencel}(P)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DifferentPointsGenerateHyperbolicPencel} ::$

$:: \forall a, b \in \mathbb{C} \,.\, a \neq b \Rightarrow \texttt{HyperbolicPencel}\Big(\mathbf{p}\big(\mathrm{pt}(a), \mathrm{pt}(b)\big)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DisjointCirclesGenerateHyperbolicPencel} ::$

$:: \forall A, B \in \Re\mathcal{S} \,.\, A \cap B = \emptyset \Rightarrow \texttt{HyperbolicPencel}\Big(\mathbf{p}(A, B)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DisjointCircleAndALineGenerateHyperbolicPencel} ::$

$:: \forall A \in \Re\mathcal{S} \,.\, \forall B : \texttt{LineCircle} \,.\, A \cap B = \emptyset \Rightarrow \texttt{HyperbolicPencel}\Big(\mathbf{p}(A, B)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CentredCircle} := \Re\mathcal{S}' | \Im\mathcal{S}' | \texttt{PointCircle} : \texttt{Type};$

$\texttt{center} :: \texttt{CentredCircle} \to \mathbb{C}$

$\texttt{center}\,([A]) := -\frac{A_{1,2}}{A_{1,1}}$

$\texttt{Cocentric} :: \mathbb{C} \to ?\texttt{CentredCircle}$

$A : \texttt{Cocentric} \iff \Lambda z \in \mathbb{C} \;.\; \texttt{center}(A) = z$

$\texttt{SpecialHyperbolicPencel} :: ?\texttt{Pencel}$

$P : \texttt{SpecialHyperbolicPencel} \iff \exists z \in \mathbb{C} : \forall A : \texttt{Cocentric}(z) \;.\; A \in P$

$\texttt{SpecialHyperbolicPencelIsHyperbolicPencel} :: \forall P : \texttt{SpecialHyperbolicBundle} \;.\; \texttt{HyperbolicPencel}(P)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CocenticGenerateSpecialHyperbolicPencel} ::$

$:: \forall z \in \mathbb{C} \;.\; \forall A, B : \texttt{Cocentric}(z) \;.\; \texttt{SpecialHyperbolicPencel}\Big(\mathbf{p}(A, B)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CentredAndInfinityGenerateSpecialHyperbolicPencel} ::$

$:: \forall A : \texttt{CentredCircles} \;.\; \texttt{SpecialHyperbolicPencel}\Big(\mathbf{p}\big(A, \mathrm{pt}(\infty)\big)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{GeneralHyperbolicPencel} := \texttt{HyperbolicPencel} \setminus \texttt{SpecialHyperbolicPencel} :?\texttt{HyperbolicPencel};$

$\texttt{EllipticPencelClassification} :: \texttt{EllipticPencel} = \texttt{GeneralEllipticPencel}|\texttt{SpecialEllipticPencel}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ParabolicPencelClassification} ::$

$:: \texttt{ParabolicPencel} = \texttt{GeneralParabolicPencel}|\texttt{SpecialParabolicPencel}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{HyperbolicPencelClassification} ::$

$:: \texttt{HyperbolicPencel} = \texttt{GeneralHyperbolicPencel}|\texttt{SpecialHyperbolicPencel}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{PencelClassification} :: \texttt{Pencel} = \texttt{EllipticPencel}|\texttt{ParabolicPencel}|\texttt{HyperbolicPencel}$

$\texttt{Proof} =$

$\ldots$

$\square$

1.5 Inversion

$\texttt{GeneralPositionWRTTheCircle} :: \mathcal{S}' \to ?\mathbb{C}$

$z : \texttt{GeneralPositionWRTTheCircle} \iff \Lambda A \in \mathcal{S}' \,.\, z \notin A \,\&$

$\quad \& \texttt{ if } \texttt{CentredCircle}(A) \texttt{ then } z \neq \texttt{center}(A) \texttt{ else } \top$

$\texttt{InversionExists} :: \forall A \in \mathcal{S}_* \,.\, \forall z : \texttt{GeneralPositionWRTTheCircle}(A) \,.\, \exists! z' \in \mathbb{C} : z' \neq z \,\&$

$\quad \& \bigcap \{B \in \mathcal{S}' : z \in B \,\&\, A \perp B\} = \{z, z'\}$

$\texttt{Proof} =$

$P := \{B \in \mathcal{S}' : z \in B \,\&\, A \perp B\} : ?\mathcal{S}',$

$$T := \Lambda B \in \mathbb{C}^{2\times 2} \,.\, \begin{pmatrix} \det(A, B) \\ B((z,1)) \end{pmatrix} : \mathbb{C}^{2\times 2} \xrightarrow{\mathbb{C}\text{-AFF}} \mathbb{C}^2,$$

$[1] := \mathcal{C}\mathcal{S}_*(A) : A \,!\, \texttt{PointCircle},$

$[2] := \mathcal{D}(T)[1] : \operatorname{rank} T = 2,$

$[3] := \texttt{RankKerTHM}[2] : \dim_{\mathbb{C}} \ker T = 2,$

$$[4] := \mathcal{D}T\mathcal{D}P : P = \frac{\ker T \cap \mathbf{H}(2)}{\mathbb{R}^\times},$$

$[5] := \mathcal{D}T\texttt{HermitianRealStructure}[3] : \dim_{\mathbb{R}} \ker T \cap \mathbf{H}(2) = 2,$

$[6] := \mathcal{C}^{-1}\texttt{Pencel}[5] : \texttt{Pencel},$

$[7] := \mathcal{C}\texttt{GeneralPositionWRTTheCircle}(A, z)\texttt{PencelClassification}(P)\mathcal{D}P : \texttt{GeneralParabolicPencel}(P),$

$[*] := \mathcal{C}\texttt{GeneralParabolicPencel} : z' \neq z \,\&\, \bigcap\{B \in \mathcal{S}' : z \in B \,\&\, A \perp B\} = \{z, z'\};$

$\square$

$\texttt{inversion} :: \mathcal{S}_* \to \hat{\mathbb{C}} \leftrightarrow \hat{\mathbb{C}}$

$\texttt{inversion}\,(l, \infty) = \operatorname{Inv}_l(\infty) := \infty \quad \texttt{if} \quad \texttt{LineCircle}(l)$

$\texttt{inversion}\,(A, \infty) = \operatorname{Inv}_A(\infty) := \texttt{center}(A) \quad \texttt{if} \quad \texttt{CentredCircle}(A)$

$\texttt{inversion}\,(A, \texttt{center}(A)) = \operatorname{Inv}_A\Big(\texttt{center}(A)\Big) := \infty \quad \texttt{if} \quad \texttt{CentredCircle}(A)$

$\texttt{inversion}\,(A, z) = \operatorname{Inv}_A(z) := z \quad \texttt{if} \quad z \in A$

$\texttt{inversion}\,(A, z) = \operatorname{Inv}_A(z) := \texttt{InversionExists}(A, z) \quad \texttt{if} \quad \texttt{GeneralPositionWRTTheCircle}(A, z)$

$\texttt{InversionIsInvolution} :: \forall A \in \mathcal{S}_* \,.\, \operatorname{Inv}_A^2 = \operatorname{id}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{InversionAnalyticExpression} :: \forall A \in \mathcal{S}_* \,.\, \forall z : \texttt{GeneralPositionWRTTheCircle}(A) \,.$

$$.\, \operatorname{Inv}_A(z) = -\frac{A_{2,1}\bar{z} + A_{2,2}}{A_{1,1}\bar{z} + A_{1,2}}$$

$\texttt{Proof} =$

$P := \{B \in \mathcal{S}' : z \in B \,\&\, A \perp B\} : ?\mathcal{S}',$

$z' := \operatorname{Inv}_A(z) : \mathbb{C},$

$$T := \Lambda B \in \mathbb{C}^{2\times 2} \,.\, \begin{pmatrix} \det(A, B) \\ B((z,1)) \\ B((z',1)) \end{pmatrix} : \mathbb{C}^{2\times 2} \xrightarrow{\mathbb{C}\text{-AFF}} \mathbb{C}^3,$$

$[1] := \mathcal{D}z'\mathcal{C}\operatorname{Inv}_A\texttt{InversionExists}(A, z) : \operatorname{rank} T = 2,$

$B := \begin{bmatrix} 1 & -\bar{z} \\ -z' & z'\bar{z} \end{bmatrix} : \mathbb{C}^{2\times 2},$

$[2] := \supset B : B(z,1) = 0 = B(z',1),$

$[3] := \subset \operatorname{rank}[1][2] \subset \det(A,B) \supset B : 0 = \det(A,B) = A_{1,1}z'\bar{z} + A_{2,2} + A_{2,1}\bar{z} + A_{1,2}z' =$
$\quad = (A_{1,1}\bar{z} + A_{1,2})z' + A_{2,1}\bar{z} + A_{2,2},$

$[*] := \dfrac{[3] - A_{2,1}\bar{z} - A_{2,2}}{A_{1,1}\bar{z} + A_{1,2}} : \operatorname{Inv}_A(z) = z' = -\dfrac{A_{2,1}\bar{z} + A_{2,2}}{A_{1,1}\bar{z} + A_{1,2}};$

$\square$

`Inversion` $:: ?$`Bijection`$(\hat{\mathbb{C}})$
$f :$ `Invesion` $\iff \exists A \in \mathcal{S}_* \,.\, f = \operatorname{Inv}_A$

`EllipticInversion` $:: ?$`Inversion`$(\hat{\mathbb{C}})$
$f :$ `EllipticInvesion` $\iff \exists A \in \Re\mathcal{S}' \,.\, f = \operatorname{Inv}_A$

`HyperbolicInversion` $:: ?$`Inversion`$(\hat{\mathbb{C}})$
$f :$ `HyperbolicInvesion` $\iff \exists A \in \Im\mathcal{S}' \,.\, f = \operatorname{Inv}_A$

`RealGeneralizedCircles` $= \Re\mathcal{S}_* :=$ `LineCircle`$|\Re\mathcal{S}' :?\mathcal{S}_*;$

`circlesDilation` $:: \operatorname{Di}_{\mathbb{R}}(\mathbb{C}) \to \mathcal{S}' \to \mathcal{S}'$

`circlesDilation` $\left(\phi, \left[\left[\begin{matrix} 1 & -z \\ -\bar{z} & |z|^2 + \rho \end{matrix}\right]\right]\right) = \phi^* \left[\left[\begin{matrix} 1 & -z \\ -\bar{z} & |z|^2 + \rho \end{matrix}\right]\right] := \left[\left[\begin{matrix} 1 & -\phi(z) \\ -\overline{\phi(z)} & |\phi(z)|^2 + \left(\operatorname{rat}(\phi)\right)^2\rho \end{matrix}\right]\right]$

`circlesDilation` $\left(\phi, \left[\left[\begin{matrix} 0 & n \\ \bar{n} & \alpha \end{matrix}\right]\right]\right) = \phi^* \left[\left[\begin{matrix} 0 & n \\ \bar{n}z & \alpha \end{matrix}\right]\right] := \left[\left[\begin{matrix} 0 & \operatorname{rat}(\phi)n \\ \operatorname{rat}(\phi)\bar{n} & \operatorname{rat}^2(\phi)\alpha + 2\operatorname{rat}(\phi)\langle n, v_\phi\rangle \end{matrix}\right]\right]$

  `where` $\quad n \in \mathbb{S}^1$

`DilationMappingOfTheCirclesConsistency` $:: \forall\phi \in \operatorname{Di}_{\mathbb{R}}(\mathbb{C}) \,.\, \forall S \in \mathcal{S}' \,.\, \phi(S) =_{?\mathbb{C}} \phi^*(S)$

`Proof` $=$

$\ldots$

$\square$

`UnitCircleDilationIsBijective` $:: \forall S \in \Re\mathcal{S}' \,.\, \exists!\phi \in \operatorname{Di}^+_{\mathbb{R}}(\mathbb{C}) \,.\, S = \phi^*\mathbb{S}^1$

`Proof` $=$

$\ldots$

$\square$

`unitCircleCircleInversion` $:: \mathcal{S}' \to \mathcal{S}'$

`unitCircleCircleInversion` $\left(\left[\left[\begin{matrix} \alpha & z \\ \bar{z} & \beta \end{matrix}\right]\right]\right) = \operatorname{Inv}^*_{\mathbb{S}^1} \left[\left[\begin{matrix} \alpha & z \\ \bar{z} & \beta \end{matrix}\right]\right] := \left[\left[\begin{matrix} \beta & z \\ \bar{z} & \alpha \end{matrix}\right]\right]$

$\texttt{UnitCircleCircleInversionConsistency} :: \forall S = \left[\left[\begin{array}{cc} \alpha & z \\ \bar{z} & \beta \end{array}\right]\right] \in S' \,.\, \mathrm{Inv}_{\mathbb{S}^1}(S) =_{?\mathbb{C}} \mathrm{Inv}^*_{\mathbb{S}^1}(S)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ p : \texttt{In}(S \setminus \{0\}),$

$[1] := \mathit{ɑ}\mathcal{S}'(S)\mathit{ɑ}p : \beta|p|^2 + z\bar{p} + \bar{z}p + \alpha = 0,$

$[2] := \mathit{ɑ}\texttt{Field}(\mathbb{C})[1] : \dfrac{\alpha}{|p|^2} + \dfrac{z}{p} + \dfrac{\bar{z}}{\bar{p}} + \beta = \dfrac{\beta|p|^2 + z\bar{p} + \bar{z}p + \beta}{|p|^2} = 0,$

$[p.*] := \mathit{ɑ}\mathrm{Inv}^*_{\mathbb{S}^1}(S)[2] : \mathrm{Inv}_{\mathbb{S}^1}(z) \in \mathrm{Inv}^*_{\mathbb{S}^1}(S);$

$\rightsquigarrow [*] := \texttt{InversionConvolution} : \mathrm{Inv}_{\mathbb{S}^1}(S) =_{?\mathbb{C}} \mathrm{Inv}^*_{\mathbb{S}^1}(S);$

$\square$

$\texttt{RealCircleCircleInversion} :: \Re\mathcal{S}' \to \mathcal{S}' \to \mathcal{S}'$

$\texttt{RealCircleCircleInversion}\,(A, S) = \mathrm{Inv}^*_A(S) := \phi^*\mathrm{Inv}^*_{\mathbb{S}^1}\phi^{-1*}(S) \quad \texttt{where} \quad \phi \in \mathrm{Di}^+_{\mathbb{R}}(\mathbb{C}) : A = \phi^*\mathbb{S}^1$

$\texttt{RealCircleCircleInversionConsistency} :: \forall A \in \Re\mathcal{S}' \,.\, \forall S \in \mathcal{S}' \,.\, \mathrm{Inv}_A(S) =_{?\mathbb{C}} \mathrm{Inv}^*_A(S)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{anticircle} :: \Re\mathcal{S}' \leftrightarrow \Im\mathcal{S}'$

$$\texttt{anticircle}\left(S = \left[\left[\begin{array}{cc} 1 & z \\ \bar{z} & |z|^2 - \rho \end{array}\right]\right]\right) = \hat{S} := \left[\left[\begin{array}{cc} 1 & z \\ \bar{z} & |z|^2 + \rho \end{array}\right]\right]$$

$\texttt{unitAnticircleCircleInversion} :: \mathcal{S}' \to \mathcal{S}'$

$$\texttt{unitAnticircleCircleInversion}\left(\left[\left[\begin{array}{cc} \alpha & z \\ \bar{z} & \beta \end{array}\right]\right]\right) = \mathrm{Inv}^*_{\hat{\mathbb{S}}^1}\left[\left[\begin{array}{cc} \alpha & z \\ \bar{z} & \beta \end{array}\right]\right] := \left[\left[\begin{array}{cc} \beta & -z \\ -\bar{z} & \alpha \end{array}\right]\right]$$

$\texttt{UnitAnticircleCircleInversionConsistency} :: \forall S = \left[\left[\begin{array}{cc} \alpha & z \\ \bar{z} & \beta \end{array}\right]\right] \in S' \,.\, \mathrm{Inv}_{\mathbb{S}^1}(S) =_{?\mathbb{C}} \mathrm{Inv}^*_{\mathbb{S}^1}(S)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ p : \texttt{In}(S \setminus \{0\}),$

$[1] := \mathit{ɑ}\mathcal{S}'(S)\mathit{ɑ}p : \beta|p|^2 + z\bar{p} + \bar{z}p + \alpha = 0,$

$[2] := \mathit{ɑ}\texttt{Field}(\mathbb{C})[1] : \dfrac{\alpha}{|p|^2} + \dfrac{z}{p} + \dfrac{\bar{z}}{\bar{p}} + \beta = \dfrac{\beta|p|^2 + z\bar{p} + \bar{z}p + \beta}{|p|^2} = 0,$

$[p.*] := \mathit{ɑ}\mathrm{Inv}^*_{\hat{\mathbb{S}}^1}(S)[2] : \mathrm{Inv}_{\hat{\mathbb{S}}^1}(z) \in \mathrm{Inv}^*_{\hat{\mathbb{S}}^1}(S);$

$\rightsquigarrow [*] := \texttt{InversionConvolution} : \mathrm{Inv}_{\hat{\mathbb{S}}^1}(S) =_{?\mathbb{C}} \mathrm{Inv}^*_{\hat{\mathbb{S}}^1}(S);$

$\square$

$\texttt{imaginableCircleCircleInversion} :: \Im\mathcal{S}' \to \mathcal{S}' \to \mathcal{S}'$

$\texttt{imaginableCircleCircleInversion}\,(A, S) = \mathrm{Inv}^*_A(S) := \phi^*\mathrm{Inv}^*_{\hat{\mathbb{S}}^1}\phi^{-1*}(S) \quad \texttt{where} \quad \phi \in \mathrm{Di}_{\mathbb{R}}(\mathbb{C}) : S = \phi^*\mathbb{S}^1$

$\texttt{ImaginableCircleCircleInversionConsistency} :: \forall A \in \Re\mathcal{S}' \,.\, \forall S \in \mathcal{S}' \,.\, \mathrm{Inv}_A(S) =_{?\mathbb{C}} \mathrm{Inv}^*_A(S)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LineToCircle} :: \forall l : \texttt{LineCircle} \,.\, \exists \phi \in \mathrm{Di}_{\mathbb{R}}(\mathbb{C}) : \exists S \in \Re\mathcal{S}' : l = \phi^* \mathrm{Inv}^*_S(\mathbb{S}^1)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{LineCircleCircleInversion} :: \texttt{LineCircle} \to \mathcal{S}' \to \mathcal{S}'$

$\texttt{LineCircleCircleInversion}\,(A, S) = \mathrm{Inv}^*_A(S) := \phi^* \mathrm{Inv}^*_S \mathrm{Inv}^*_{\mathbb{S}^1} \mathrm{Inv}_S^{-1*} \phi^{-1*}(l) \quad \texttt{where}$

$\quad \texttt{where} \quad \phi \in \mathrm{Di}_{\mathbb{R}}(\mathbb{C}), S \in \Re\mathcal{S} : l = \phi^* \mathrm{Inv}^*_S \mathbb{S}^1$

$\texttt{LineCircleCircleInversionConsistency} :: \forall l : \texttt{LineCircle} \,.\, \forall S \in \mathcal{S}' \,.\, \mathrm{Inv}_l(S) =_{?\mathbb{C}} \mathrm{Inv}^*_l(S)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CircleInversion} :: \mathcal{S}_* \to \mathcal{S}' \to \mathcal{S}'$

$\texttt{CircleInversion}\,(A, S) = \mathrm{Inv}^*_A(S) := \mathrm{Inv}^*_A(S)$

$\texttt{CircleInversionConsistency} :: \forall A \in \mathcal{S}_* \,.\, \forall S \in \mathcal{S}' \,.\, \mathrm{Inv}_A(S) =_{?\mathbb{C}} \mathrm{Inv}^*_A(S)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CircleDilationPreservesDiscr} :: \forall \phi \in \mathrm{Di}_{\mathbb{R}}(\mathbb{C}) \,.\, \forall A, B \in \mathcal{S}' \,.\, \det\Big(\phi^* A, \phi^* B\Big) = \det(A, B)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{CentredCircle}(A \,\&\, B),$

$\Big(a, \rho, [2]\Big) := \text{ɗ}\texttt{CentredCircle}(A) : \sum a \in \mathbb{C} \,.\, \sum \rho \in \mathbb{R} \,.\, A = \left[\left[\begin{array}{cc} 1 & a \\ \bar{a} & |a|^2 + \rho \end{array}\right]\right],$

$\Big(b, \sigma, [3]\Big) := \text{ɗ}\texttt{CentredCircle}(B) : \sum b \in \mathbb{C} \,.\, \sum \sigma \in \mathbb{R} \,.\, B = \left[\left[\begin{array}{cc} 1 & b \\ \bar{b} & |b|^2 + \sigma \end{array}\right]\right],$

$[1.*] := \text{ɗ} \det(\phi^* A, \phi^* B) \text{ɗ} \phi^* A \text{ɗ} \phi^* B [2][3] \text{ɗ}^{-1} \texttt{absValue}(\mathbb{C})$

$\quad : \texttt{ComplexNorm}\Big(\phi(a) - \phi(b)\Big) \texttt{DilationLipschitzConstant}(\phi, a, b) \text{ɗ}^{-1} \det(A, B) \text{ɗ} \phi^* A \text{ɗ} \phi^* :$

$\quad : \det(\phi^* A, \phi^* B) = \Big|\phi(b)\Big|^2 + \mathrm{rat}^2(\phi)\sigma\Big|\phi(a)\Big|^2 + \mathrm{rat}^2(\phi)\rho - \phi(a)\overline{\phi(b)} - \overline{\phi(a)}\phi(b) =$

$\quad = \Big|\phi(a) - \phi(b)\Big|^2 + \mathrm{rat}^2(\phi)(\sigma + \rho) = \Big\|\phi(a) - \phi(b)\Big\|^2 + \mathrm{rat}^2(\phi)(\sigma + \rho) = \Big(\mathrm{rat}(\phi)\Big)^2 \Big(\|a - b\|^2 + \rho^2 + \sigma^2\Big) =$

$\quad = \mathrm{rat}^2(\phi) \det(A, B);$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{CentredCircle}(A \,\&\, B) \Rightarrow \det(\phi^* A, \phi^* B) = \det(A, B),$

$\texttt{Assume}\ [2] : A, B \,!\, \texttt{CentredCircle},$

$\Big(v, \alpha, [3]\Big) := \text{ɗ}\texttt{CentredCircle}(A) : \sum v \in \mathbb{C} \,.\, \sum \alpha \in \mathbb{R} \,.\, A = \left[\left[\begin{array}{cc} 0 & v \\ \bar{v} & \alpha \end{array}\right]\right],$

$\Big(u, \beta, [4]\Big) := \text{ɗ}\texttt{CentredCircle}(B) : \sum b \in \mathbb{C} \,.\, \sum \beta \in \mathbb{R} \,.\, B = \left[\left[\begin{array}{cc} 0 & u \\ \bar{u} & \beta \end{array}\right]\right],$

$[2.*] := \text{ɗ} \det(\phi^* A, \phi^* B) \text{ɗ} \phi^* A \text{ɗ} \phi^* B [2][3] \text{ɗ}^{-1} \det(A, B) \text{ɗ} \phi^* A \text{ɗ} \phi^* :$

$\quad : \det(\phi^* A, \phi^* B) = \mathrm{rat}^2(\phi) u\bar{v} + \mathrm{rat}^2(\phi) v\bar{u} = \mathrm{rat}^2(\phi) \det(A, B);$

$\rightsquigarrow [2] := I(\Rightarrow) : A, B \,!\, \texttt{CentredCircle} \Rightarrow \det(\phi^* A, \phi^* B) = \det(A, B),$

$\texttt{Assume}\ [3] : \texttt{CentredCircle}(A)\ \&\ B\ !\ \texttt{CentredCircle},$

$\Big(a, \rho, [4]\Big) := \mathit{\Pi}\texttt{CentredCircle}(A) : \sum a \in \mathbb{C}\ .\ \sum \rho \in \mathbb{R}\ .\ A = \left[\left[\begin{matrix} 1 & a \\ \bar{a} & |a|^2 + \rho \end{matrix}\right]\right],$

$\Big(u, \beta, [5]\Big) := \mathit{\Pi}\texttt{CentredCircle}(B) : \sum u \in \mathbb{C}\ .\ \sum \beta \in \mathbb{R}\ .\ B = \left[\left[\begin{matrix} 0 & u \\ \bar{u} & \beta \end{matrix}\right]\right],$

$[3.*] := \mathit{\Pi}\det(\phi^* A, \phi^* B)\mathit{\Pi}\phi^* A\mathit{\Pi}\phi^* B[4][5]\texttt{TangentSpaceDilation}(0, \phi)\mathit{\Pi}\texttt{Bilinear}(\mathbb{C})\mathit{\Pi}\det(A, B)[4][5] :$

$: \det(\phi^* A, \phi^* B) = \mathrm{rat}^2(\phi)\alpha - \mathrm{rat}(\phi)\phi(z)\bar{u} - \mathrm{rat}(\phi)\overline{\phi(z)}u + 2\mathrm{rat}(\phi)\langle u, v_\phi\rangle =$

$= \mathrm{rat}^2(\phi)\alpha + 2\mathrm{rat}(\phi)\langle u, v_\phi\rangle + -2\mathrm{rat}(\phi)\Big\langle u, \mathrm{rat}(\phi)z + v_\phi\Big\rangle = \mathrm{rat}^2(\phi)\Big(\alpha - 2\langle u, z\rangle\Big) = \mathrm{rat}^2(\phi)\det(A, B);$

$\rightsquigarrow [3] := I(\Rightarrow) : \texttt{CentredCircle}(A)\ \&\ \texttt{CentredCircle}(B) \Rightarrow \det(\phi^* A, \phi^* B) = \det(A, B),$

$[*] := E(|)\ldots[1][2][3] : \det(\phi^* A, \phi^* B) = \det(A, B);$

□

$\texttt{UnitCircleInversionPreservesDiscr} :: \forall A, B \in \mathcal{S}'\ .\ \det\Big(\mathrm{Inv}^*_{\mathbb{S}^1}A, \mathrm{Inv}^*_{\mathbb{S}^1}B\Big) = \det(A, B)$

$\texttt{Proof}\ =$

...

□

$\texttt{CircleInversionPreservesDiscr} :: \forall S \in \mathcal{S}_*\ .\ \forall A, B \in \mathcal{S}'\ .\ \det\Big(\mathrm{Inv}^*_S A, \mathrm{Inv}^*_S B\Big) = \det(A, B)$

$\texttt{Proof}\ =$

...

□

$\texttt{InversionMapsLinesAndCirclesToLinesAndCircles} :: \forall S \in \mathcal{S}_*\ .\ \mathrm{Inv}_S \Re\mathcal{S}_* = \Re\mathcal{S}_*$

$\texttt{Proof}\ =$

...

□

1.6 Stereographic Projection

$\texttt{stereographicProjection} :: \mathbb{S}^2 \overset{\mathsf{TOP}}{\longleftrightarrow} \hat{\mathbb{C}}$

$\texttt{stereographicProjection}\,(0,0,1) = \mathrm{Stg}(0,0,1) := \infty$

$\texttt{stereographicProjection}\,(p) = \mathrm{Stg}(p) := \pi_{1,2} \mathcal{A}\, \texttt{Singleton}\Big(p \vee (0,0,1) \cap 0 \vee (1,0,0) \vee (0,1,0)\Big)$

$\texttt{StereographicProjojectionAnalyticExpression} :: \forall (x,y,z) \in \mathbb{S}^2 \ .\ z \neq 1 \Rightarrow \mathrm{Stg}(x,y,z) = \dfrac{x}{1-z} + \dfrac{\mathbf{i}y}{1-z}$

$\texttt{Proof} \ =$

$\Big(t,[1]\Big) := \texttt{LineParametrization}\ \mathrm{Stg}(x,y,z) : \prod t \in \mathbb{R} \ .\ \mathrm{Stg}(x,y,z) = t(0,0,1) + (1-t)(x,y,z),$

$[2] := \mathcal{A}\mathrm{Stg}(x,y,z)[1] : tz + (1-t) = 0,$

$[3] := [2]\mathcal{A}\texttt{Field}(\mathbb{R}) : t = \dfrac{1}{1-z},$

$[*] := [1][3] : \mathrm{Stg}(x,y,z) = \dfrac{x}{1-z} + \dfrac{\mathbf{i}y}{1-z};$

$\square$

$\texttt{StereographicProjectionInversion} :: \forall a + b\mathbf{i} \ .\ \mathrm{Stg}^{-1}(a+b\mathbf{i}) = \left(\dfrac{2a}{a^2+b^2+1}, \dfrac{2b}{a^2+b^2+1}, \dfrac{a^2+b^2-1}{a^2+b^2+1}\right)$

$\texttt{Proof} \ =$

$\Big(t,[1]\Big) := \texttt{LineParametrization}\ \mathrm{Stg}(x,y,z) : \prod t \in \mathbb{R} \ .\ \mathrm{Stg}^{-1}(a+b\mathbf{i}) = t(0,0,1) + (1-t)(a,b,0),$

$[2] := \mathcal{A}\mathrm{Stg}\mathcal{A}\texttt{EuclideanNorm}(\mathbb{R}^3) : 1 = \Big\|\mathrm{Stg}^{-1}(a+b\mathbf{i})\Big\| = t^2 + (1-t)^2a^2 + (1-t)^2b^2,$

$[3] := \texttt{BinomialExpansion}[2] - 1 : 0 = t^2(1+a^2+b^2) - 2(a^2+b^2)t + (a^2+b^2-1),$

$[4] := \dfrac{[3]}{1+a^2+b^2} : 0 = t^2 - \dfrac{2(a^2+b^2)}{1+a^2+b^2}t + \dfrac{a^2+b^2-1}{a^2+b^2+1},$

$[5] := [1][4] : 0 = (t-1)\left(t - \dfrac{a^2+b^2-1}{a^2+b^2+1}\right),$

$[6] := \mathcal{A}\mathrm{Stg} : \mathrm{Stg}^{-1}(a+b\mathbf{i}) \neq (0,0,1),$

$[7] := [5][6] : t = \dfrac{a^2+b^2-1}{a^2+b^2+1},$

$[*] := [7][1] : \mathrm{Stg}^{-1}(a+b\mathbf{i}) = \left(\dfrac{2a}{a^2+b^2+1}, \dfrac{2b}{a^2+b^2+1}, \dfrac{a^2+b^2-1}{a^2+b^2+1}\right);$

$\square$

$\texttt{ExtendedComplexPlaneIsHomeomorphicToSphere} :: \hat{\mathbb{C}} \cong_{\mathsf{TOP}} \mathbb{S}^2$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SphereCircle} = \mathcal{SS} := \texttt{Plane}(\mathbb{R}^3) :?(\mathbb{S}^{2*} \times \mathbb{R});$

$\texttt{circleStereographicProjection} :: \mathcal{SS} \leftrightarrow \mathcal{S}'$

$\texttt{circleStereographicProjection}\,(f,\alpha) = \mathrm{Stg}^*(f,\alpha) := \dfrac{1}{2}\left[\left[\begin{array}{cc} \alpha - f_3 & f_1 + \mathbf{i}f_2 \\ f_1 - \mathbf{i}f_2 & \alpha + f_3 \end{array}\right]\right]$

$\texttt{CircleStereographicProjectionConsistance} :: \forall S \in \mathcal{SS} \;.\; \mathrm{Stg}^* S =_{\mathsf{SET}} \mathrm{Stg}(S \cap \mathbb{S}^2)$

$\texttt{Proof} \;=$

$\Big(f, \alpha, [1]\Big) := \mathcal{I}\mathcal{SS}(S) : \sum f \in \mathbb{S}^{2*} \;.\; \sum \alpha \in \mathbb{R} \;.\; S = (f, \alpha),$

$\texttt{Assume}\ (x, y, z) : S \cap \mathbb{S}^2,$

$[2] := [1](x, y, z) : f(x, y, z) = -\alpha,$

$[3] := \texttt{StereographicPojectionAnalyticExpression}(x, y, z) : \mathrm{Stg}(x, y, z) = \dfrac{x}{1-z} + \dfrac{y\mathbf{i}}{1-z},$

$[4] := \mathcal{I}\mathbb{C}\mathcal{I}\mathbb{S}^2\mathcal{I}\mathbb{S}^{2*}[2] :$

$$: \frac{(\alpha - f_3)(x^2+y^2)}{2(1-z)^2} + \frac{(f_1 + \mathbf{i}f_2)(x - y\mathbf{i})}{2(1-z)} + \frac{(f_1 - \mathbf{i}f_2)(x + y\mathbf{i})}{2(1-z)} + + \frac{\alpha + f_3}{2} =$$

$$= \frac{\alpha\Big(x^2 + y^2 + (1-z)^2\Big) + 2f_1x(1-z) + 2f_2y(1-z) + f_3\Big((1-z)^2 - x^2 - y^2\Big)}{2(1-z)^2} =$$

$$= \frac{2\alpha\Big(1-z\Big) + 2f_1x(1-z) + 2f_2y(1-z) + f_3z\Big(1-z\Big)}{2(1-z)^2} =$$

$$= \frac{\Big(\alpha + f(x,y,z)\Big)}{1-z} = 0,$$

$\Big[(x, y, z).*\Big] := \mathcal{I}\mathcal{S}; [4] : \mathrm{Stg}(x, y, z) \in \mathrm{Stg}^*(S);$

$\rightsquigarrow [2] := \mathcal{I}^{-1}\texttt{Subset} : \mathrm{Stg}(S \cap \mathbb{S}^2) \subset \mathrm{Stg}^*(S),$

$\texttt{Assume}\ u : \mathrm{Stg}^*(S),$

$\Big((x, y, z), [3]\Big) := \mathcal{I}\,\texttt{Inverible}\mathrm{Stg}\texttt{StereographicProjectionAnalyticExpresion} :$

$$: \sum x, y, z \in \mathbb{S}^2 \;.\; u = \frac{x}{1-z} + \frac{y\mathbf{i}}{1-z},$$

$[4] := \mathcal{I}\mathcal{S}'\mathcal{I}\mathrm{Stg}^*(S)[3]\mathcal{I}\mathbb{C}\mathcal{I}\mathbb{S}^2\mathcal{I}\mathbb{S}^{2*} :$

$$= 0 = (\alpha - f_3)|u|^2 - (f_1 + \mathbf{i}f_2)\bar{u} - (f_1 - \mathbf{i}f_2)u + \alpha + f_3 =$$

$$= \frac{(\alpha - f_3)(x^2+y^2)}{2(1-z)^2} + \frac{(f_1 + \mathbf{i}f_2)(x - y\mathbf{i})}{2(1-z)} + \frac{(f_1 - \mathbf{i}f_2)(x + y\mathbf{i})}{2(1-z)} + + \frac{\alpha + f_3}{2} =$$

$$= \frac{\Big(\alpha + f(x,y,z)\Big)}{1-z},$$

$[5] := (1-z)[4] : f(x, y, z) = -\alpha,$

$[u.*] := \mathcal{I}\mathcal{SS}(S) : u \in \mathrm{Stg}(S \cap \mathbb{S}^2);$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Subset}\mathcal{I}^{-1}\texttt{SetEq} : \mathrm{Stg}^*(S) = \mathrm{Stg}(S \cap \mathbb{S}^2);$

$\square$

$\texttt{polarPlane} :: \mathbb{R}^3 \setminus \{0\} \to SS$

$\texttt{polarPlane}\,(v) = \mathrm{pp}(v) := [v^*; -1]$

$\texttt{PolarPlane} :: ?SS$

$S : \texttt{PolarPlane} \iff \exists v \in \mathbb{R}^3 \setminus \{0\} : S = \mathrm{pp}(v)$

$\texttt{pole} :: \texttt{PolarPlane} \to \mathbb{R}^3 \setminus \{0\}$

$\texttt{pole}\,(S) := \mathcal{I}\texttt{PolarPlane}$

$\texttt{LawOfReciprocity} :: \forall S : \texttt{PolarPlane} \,.\, \forall q \in S \,.\, \texttt{pole}(S) \in \mathrm{pp}(q)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{NonSingularSphereCircle} :: ?\mathcal{SS}$

$S : \texttt{NonSingularSphereCircle} \iff S \in \mathcal{SS}_* \iff \mathrm{Stg}^*(S) \in \mathcal{S}_*$

$\texttt{sphericleCircleInversion} :: \mathcal{SS}_* \to \mathbb{S}^2 \to \mathbb{S}^2$

$\texttt{sphericalCircleInversion}\,(S, s) = \mathrm{Inv}_S(s) := s\mathrm{Stg}\;\mathrm{Inv}_{\mathrm{Stg}^* S}\;\mathrm{Stg}^{-1}$

$\texttt{SphericalInversionTHM} :: \forall S \in \mathcal{SS}_* \,\&\, \texttt{PolarPlane} \,.\, \forall s \in \mathbb{S}^2 \setminus S \,.\, \mathbb{S}^2 \cap \Big(s \vee \texttt{polar}(S)\Big) = \{\mathrm{Inv}_S(s), s\}$

$\texttt{Proof} \;=$

$p := \texttt{polar}(S) : \mathbb{R}^{3*},$

$[1] := \text{Д} p \text{Д} \texttt{polar}(S) : S = [p; -1],$

$[2] := \text{Д}\mathrm{Stg}^* S[1] : \mathrm{Stg}^* S = \dfrac{1}{2} \left[\!\!\left[\begin{array}{cc} -p_1 - 1 & p_2 + \mathbf{i}p_3 \\ p_2 - \mathbf{i}p_3 & p_1 - 1 \end{array} \right]\!\!\right],$

$z := \mathrm{Stg}s : \hat{\mathbb{C}},$

$[3] := \texttt{StereographicProjectionAnalyticExpression}\text{Д} z : z = \dfrac{s_1 + s_2\mathbf{i}}{1 - s_3},$

$\texttt{Assume}\; s' : \mathbb{S}^2 \cap s \vee p,$

$\texttt{Assume}\; [4] : s' \neq s,$

$\Big(t, [5]\Big) := \texttt{ParametricLineEquation}\text{Д} s' : \sum t \in \mathbb{R} \,.\, s' = tp + (1 - t)s,$

$[6] := \text{Д}\mathbb{S}^2(s')[5]\text{Д}\texttt{productOfEuclid} :$

$\quad : 1 = \|s'\|^2 = \Big\|tp + (1-t)s\Big\|^2 = t^2\Big(\|p\|^2 - 2\langle s, p\rangle + \|s\|s^2\Big) + 2t\Big(\langle s, p\rangle + \|s\|^2\Big) + \|s\|^2,$

$[7] := \text{Д}\mathbb{S}^2(s)\Big([6] - 1\Big) : 0 = t^2 + \dfrac{2\langle s, p\rangle - 2}{\|p - s\|^2} t,$

$[8] := [7][5][4] : t = 2\dfrac{1 - \langle s, p\rangle}{\|p - s\|^2},$

$[s'.*] := [5][8] : s' = 2\dfrac{1 - \langle s, p\rangle}{\|p - s\|^2} p + \dfrac{\|p\|^2 - 1}{\|p - s\|^2} s;$

$\rightsquigarrow [4] := \texttt{AnalyticSolution} : \mathbb{S}^2 \cap (s \vee p) = \left\{ s, 2\dfrac{1 - \langle s, p\rangle}{\|p - s\|^2} p + \dfrac{\|p\|^2 - 1}{\|p - s\|^2} s \right\},$

$t := 2\dfrac{1 - \langle s, p\rangle}{\|p - s\|^2} : \mathbb{R},$

$z' := \mathrm{Stg}\Big(tp + (1 - t)s\Big) : \hat{\mathbb{C}},$

$[5.1] := \ldots : 1 + |z|^2 = 1 + \dfrac{s_1^2 + s_2^2}{1 - 2s_3 + s_3^2} = \dfrac{2}{1 - s_3},$

$[5.2] := \ldots : 1 - |z|^2 = 1 - \dfrac{s_1^2 + s_2^2}{1 - 2s_3 + s_3^2} = \dfrac{1 - 2s_3 + s_3^2 - 1 + s_3^2}{(1 - s^3)^2} = \dfrac{2s_3}{1 - s^3},$

$[5] := \texttt{StereographiProjectionAnalyticExpression} \mathcal{I} z' :$

$$: z_1 = \frac{(1-t)(s_1 + s_2\mathbf{i}) + t(p_1 + \mathbf{i}p_2)}{1-(1-t)s_3 - tp_3} = \frac{2(1-t)z - 2t(\mathrm{Stg}^*S)_{1,2}(1+|z|^2)}{\left(1 + t\frac{(\mathrm{Stg}^*S)_{2,2}-(\mathrm{Stg}^*S)_{1,1}}{\operatorname{tr}\mathrm{Stg}^*S}\right)(1+|z|^2) - (1-t)(1-|z|^2)},$$

$[6] := \mathcal{I} t \mathcal{I} \texttt{productOfEuclid} \mathcal{I}^{-1} \mathrm{Stg}^*S(z) :$

$$: t = 2\frac{1-\langle s,p\rangle}{\|s-p^2\|^2} = \frac{\frac{2}{1-s_3} + (\mathrm{Stg}^*S)_{1,2}z + (\mathrm{Stg}^*S)_{1,2}\bar{z} - \frac{2p_3s_3}{1-s_3}}{(1+|z|^2)(2-2\langle s,p\rangle) + (\|p\|-1)(1+|z|^2)} = \frac{\mathrm{Stg}^*S(z)}{\mathrm{Stg}^*S(z) - \det\mathrm{Stg}^*S(1+|z|^2)},$$

$$[7] := 1 - [7] : 1 - t = \frac{-\det\mathrm{Stg}^*S(1+|z|^2)}{\mathrm{Stg}^*S(z) - \det\mathrm{Stg}^*(1+|z^2)},$$

$[8] := [5][6][7] :$

$$: z' = \frac{-2z\det\mathrm{Stg}^*S(1+\|z|^2) - 2(\mathrm{Stg}^*S)_{1,2}\mathrm{Stg}^*S(z)(1+|z|^2)}{(1+\mathrm{Stg}^*S(z)((\mathrm{Stg}^*S)_{2,2} - (\mathrm{Stg}^*S)_{1,1}))(1+|z|^2) + \det\mathrm{Stg}^*S(1-|z|^4)} =$$

$$= \frac{-2z\det\mathrm{Stg}^*S - 2(\mathrm{Stg}^*S)_{1,2}\mathrm{Stg}^*S(z)}{(1+\mathrm{Stg}^*S(z)((\mathrm{Stg}^*S)_{2,2} - (\mathrm{Stg}^*S)_{1,1})) + \det\mathrm{Stg}^*S(1-|z|^2)} =$$

$$= -\frac{(\mathrm{Stg}^*S)_{2,1}\bar{z} + (\mathrm{Stg}^*S)_{2,2}}{(\mathrm{Stg}^*S)_{1,1}\bar{z} + (\mathrm{Stg}^*S)_{1,2}} = \mathrm{Inv}_{\mathrm{Stg}^*S}(z),$$

$\texttt{OrthogonalityByPolarity} :: \forall S, S' : \texttt{PolarPlane} \,.\, \mathrm{Stg}^*\ S \perp \mathrm{Stg}^*\ S' \iff \texttt{pole}(S) \in S'$

$\texttt{Proof} =$

$\ldots$

$\square$

1.7 Circles on A Sphere

$\texttt{UniqueOrthogonalPencel} :: \forall P : \texttt{Pencel} \,.\, \exists! Q : \texttt{Pencel} : P \perp Q$

$\texttt{Proof} =$

$\texttt{bundleOfCircles} :: \texttt{LinearlyIndependent}\Big(3, \mathbf{H}(2)\Big) \to ?\mathcal{S}'$

$\texttt{bundleOfCircles}\,(A) = \mathbf{b}(A) := \dfrac{\operatorname{span}(A)}{\mathbb{R}^\times}$

$\texttt{Bundle} :: ?\mathcal{S}'$

$B : \texttt{Bundle} \iff \exists A : \texttt{LinearlyIndependent}(3, \mathbf{H}(2)) : B = \mathbf{b}(A)$

$\texttt{BundleOfPlanesTHM} :: \forall B : \texttt{Bundle} \,.\, \left|\bigcap_{S \in B} \operatorname{Stg}^* S\right| = 1$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{centerOfBundle} :: \texttt{Bundle} \to \hat{\mathbb{R}}^3$

$\texttt{centerOfBundle}\,(B) = O_B := \text{Ɑ}\texttt{Singleton} \bigcap_{S \in B} \operatorname{Stg}^* S$

$\texttt{OrthogonalCircleOfBundle} :: \forall B : \texttt{Bundle} \,.\, \exists! S \in \mathcal{S}' \,.\, S \perp B$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{orhogonalCircle} :: \texttt{Bundle} \to \mathcal{S}'$

$\texttt{orthogonalCircle}\,(B) = B^\perp := \texttt{OrthogonalCircleOfBundle}$

$\texttt{CenterAndOrthogonalRelation} :: \forall B : \texttt{Bundle} \,.\, O_B = \texttt{pole} \operatorname{Stg}^* B^\perp$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{EllipticBundle} :: ?\texttt{Bundle}$

$B : \texttt{EllipticBundle} \iff O_B \in \mathbb{B}^2$

$\texttt{ParabolicBundle} :: ?\texttt{Bundle}$

$B : \texttt{ParaboliccBundle} \iff O_B \in \mathbb{S}^2$

$\texttt{HyperbolicBundle} :: ?\texttt{Bundle}$

$B : \texttt{HyperbolicBundle} \iff O_B \in \mathbb{D}^{2\complement}$

1.8 Cross Ratio

$\texttt{simpleRatio} :: \hat{\mathbb{C}}^3 \to \hat{\mathbb{C}}$

$$\texttt{simpleRatio}\,(a, b, c) = \mathrm{sr}(a; b, c) := \frac{a - b}{a - c}$$

$\texttt{crossRatio} :: \hat{\mathbb{C}}^4 \to \hat{\mathbb{C}}$

$$\texttt{crossRatio}\,(a, b, c, d) = \mathrm{cr}(a, b; c, d) := \frac{\mathrm{sr}(a; c, d)}{\mathrm{sr}(b; c, d)}$$

$\texttt{CrossRatioCircleTheorem} :: \forall a, b, c, d \in \hat{\mathbb{C}}^4 \,.\, \exists S \in \mathcal{S}' : a, b, c, d \in S \iff \mathrm{cr}(a, b; c, d) \in \hat{\mathbb{R}}$

$\texttt{Proof} =$

...

$\square$

$\texttt{CrossRatioInversion} :: \forall a, b, c, d \in \hat{\mathbb{C}}^4 \,.\, \mathrm{cr}\left(\mathrm{Inv}_{\mathbb{S}^1} a, \mathrm{Inv}_{\mathbb{S}^1} b, \mathrm{Inv}_{\mathbb{S}^1} c, \mathrm{Inv}_{\mathbb{S}^1} d\right) = \overline{\mathrm{cr}(a, b, c, d)}$

$\texttt{Proof} =$

...

$\square$

1.9 Möbius Transform

$\texttt{transformOfMöbius} :: \mathbf{GL}(\mathbb{C},2) \to \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$\texttt{transformOfMöbius}\,(A,z) = \mathbf{M}_A(z) := \dfrac{A_{1,1}z + A_{1,2}}{A_{2,1}z + A_{2,2}}$

$\texttt{MöbiusTransformComposition} :: \forall A,B \in \mathbf{GL}(\mathbb{C},2) \;.\; \mathbf{M}_A\mathbf{M}_B = \mathbf{M}_{BA}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; z : \hat{\mathbb{C}},$

$[z.*] := \mho\mathbf{M}_A(z)\mho\mathbf{M}_B(z)\mho\texttt{Field}(\mathbb{C})\texttt{MatrixMultInCoordinates}(\mathbb{C}^2,B,A)\mho^{-1}\mathbf{M}_{BA}(z) :$

$$: z\mathbf{M}_A\mathbf{M}_B = \frac{A_{1,1}z + A_{1,2}}{A_{2,1}z + A_{2,2}}\mathbf{M}_B = \frac{B_{1,1}\frac{A_{1,1}z+A_{1,2}}{A_{2,1}z+A_{2,2}} + B_{1,2}}{B_{2,1}\frac{A_{1,1}z+A_{1,2}}{A_{2,1}z+A_{2,2}} + B_{2,2}} = \frac{(A_{1,1}B_{1,1} + A_{2,1}B_{1,2})z + A_{1,2}B_{1,1} + A_{2,2}B_{1,2}}{(A_{1,1}B_{2,1} + A_{2,1}B_{2,2})z + A_{1,2}B_{2,1} + A_{2,2}B_{2,2}} =$$

$$= \frac{(BA)_{1,1}z + (BA)_{1,2}}{(BA)_{2,1}z + (BA)_{2,2}} = \mathbf{M}_{BA}(z);$$

$\rightsquigarrow [z.*] := I(=,\to) : \mathbf{M}_A\mathbf{M}_B = \mathbf{M}_{BA};$

$\square$

$\texttt{groupOfMöbius} :: \mathsf{GRP}$

$\texttt{groupOfMöbius}\,() = \mathcal{M} := \mathbf{M}_{\mathbf{GL}(\mathbb{C},2)}$

$\texttt{MöbiusTronformFactorization} :: \dfrac{\mathbf{GL}(2,\mathbb{C})}{\mathbf{M}} \cong_{\mathsf{TOP}} \dfrac{\mathbf{GL}(2,\mathbb{C})}{\mathbb{C}^\times}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{circleRotation} :: \mathbf{SO}(\mathbb{R},2) \to \mathcal{S}' \to \mathcal{S}'$

$$\texttt{circleRotation}\left(T, \left[\!\!\left[\begin{array}{cc} 1 & -z \\ -\bar{z} & |z|^2+\rho \end{array}\right]\!\!\right]\right) = T^*\left[\!\!\left[\begin{array}{cc} 1 & -z \\ -\bar{z} & |z|^2+\rho \end{array}\right]\!\!\right] := \left[\!\!\left[\begin{array}{cc} 1 & -Tz \\ -\overline{Tz} & |Tz|^2+\rho \end{array}\right]\!\!\right]$$

$$\texttt{circleRotation}\left(T, \left[\!\!\left[\begin{array}{cc} 0 & v \\ \bar{v} & \alpha \end{array}\right]\!\!\right]\right) = T^*\left[\!\!\left[\begin{array}{cc} 0 & v \\ \bar{v} & \alpha \end{array}\right]\!\!\right] := \left[\!\!\left[\begin{array}{cc} 0 & Tv \\ \overline{Tv} & \alpha \end{array}\right]\!\!\right]$$

$\texttt{RotationsPreservesDisctiminant} :: \forall A, B \in \mathcal{S}' \,.\, \forall T \in \mathbf{SO}(\mathbb{R}, 2) \,.\, \det\big(T^*A, T^*B\big) = \det(A, B)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{CentredCircle}(A \,\&\, B),$

$\Big(a, \rho, [2]\Big) := \mathit{\Pi}\texttt{CentredCircle}(A) : \sum a \in \mathbb{C} \,.\, \sum \rho \in \mathbb{R} \,.\, A = \left[\left[\begin{array}{cc} 1 & a \\ \bar{a} & |a|^2 + \rho \end{array}\right]\right],$

$\Big(b, \sigma, [3]\Big) := \mathit{\Pi}\texttt{CentredCircle}(B) : \sum b \in \mathbb{C} \,.\, \sum \sigma \in \mathbb{R} \,.\, B = \left[\left[\begin{array}{cc} 1 & b \\ \bar{b} & |b|^2 + \sigma \end{array}\right]\right],$

$[1.*] := \mathit{\Pi} \det(\mathcal{T}^*A, \mathcal{T}^*B)\mathit{\Pi}T^*A\mathit{\Pi}T^*B[2][3]\mathit{\Pi}^{-1}\texttt{absValue}(\mathbb{C})$

$: \texttt{ComplexNorm}\Big(T(a) - T(b)\Big)\mathit{\Pi}\mathbf{SO}(\mathbb{R}, 2)\mathit{\Pi}^{-1} \det(A, B)\mathit{\Pi} :$

$: \det(T^*A, T^*B) = \Big|T(b)\Big|^2 + \sigma\Big|T(a)\Big|^2 + \rho - T(a)\overline{T(b)} - \overline{T(a)}T(b) =$

$= \Big|T(a) - T(b)\Big|^2 + \sigma + \rho = \Big\|T(a) - T(b)\Big\|^2 + (\sigma + \rho) = \|a - b\|^2 + \rho + \sigma =$

$= \det(A, B);$

$\leadsto [1] := I(\Rightarrow) : \texttt{CentredCircle}(A \,\&\, B) \Rightarrow \det(T^*A, T^*B) = \det(A, B),$

$\texttt{Assume}\ [2] : A, B \,!\, \texttt{CentredCircle},$

$\Big(v, \alpha, [3]\Big) := \mathit{\Pi}\texttt{CentredCircle}(A) : \sum v \in \mathbb{C} \,.\, \sum \alpha \in \mathbb{R} \,.\, A = \left[\!\left[\begin{array}{cc} 0 & v \\ \bar{v} & \alpha \end{array}\right]\!\right],$

$\Big(u, \beta, [4]\Big) := \mathit{\Pi}\texttt{CentredCircle}(B) : \sum b \in \mathbb{C} \,.\, \sum \beta \in \mathbb{R} \,.\, B = \left[\!\left[\begin{array}{cc} 0 & u \\ \bar{u} & \beta \end{array}\right]\!\right],$

$[2.*] := \mathit{\Pi} \det(T^*A, T^*B)\mathit{\Pi}T^*A\mathit{\Pi}T^*B[2][3]\texttt{InnerProductByConjugation}\mathit{\Pi}\mathbf{SO}(\mathbb{R}, 2)\mathit{\Pi}^{-1} \det(A, B)\mathit{\Pi}T^*A\mathit{\Pi}T^* :$

$: \det(T^*A, T^*B) = T(u)\overline{T(v)} + T(v)\overline{T(u)} = 2\langle T(u), T(v)\rangle = 2\langle u, v\rangle = \det(A, B);$

$\leadsto [2] := I(\Rightarrow) : A, B \,!\, \texttt{CentredCircle} \Rightarrow \det(T^*A, T^*B) = \det(A, B),$

$\texttt{Assume}\ [3] : \texttt{CentredCircle}(A) \,\&\, B \,!\, \texttt{CentredCircle},$

$\Big(a, \rho, [4]\Big) := \mathit{\Pi}\texttt{CentredCircle}(A) : \sum a \in \mathbb{C} \,.\, \sum \rho \in \mathbb{R} \,.\, A = \left[\left[\begin{array}{cc} 1 & a \\ \bar{a} & |a|^2 + \rho \end{array}\right]\right],$

$\Big(u, \beta, [5]\Big) := \mathit{\Pi}\texttt{CentredCircle}(B) : \sum u \in \mathbb{C} \,.\, \sum \beta \in \mathbb{R} \,.\, B = \left[\left[\begin{array}{cc} 0 & u \\ \bar{u} & \beta \end{array}\right]\right],$

$[3.*] := \mathit{\Pi} \det(T^*A, T^*B)\mathit{\Pi}T^*A\mathit{\Pi}T^*B[4][5]\texttt{ConjugationInnerProduct}\mathit{\Pi}\mathbf{SO}(\mathbb{R}, 2)\mathit{\Pi} \det(A, B)[4][5] :$

$: \det(T^*A, T^*B) = \alpha - T(z)\overline{T(u)} - \overline{T(z)}T(u) =$

$= \alpha - \Big\langle T(z), T(u)\Big\rangle = \alpha - \langle z, u\rangle = \det(A, B);$

$\leadsto [3] := I(\Rightarrow) : \texttt{CentredCircle}(A) \,\&\, \texttt{CentredCircle}(B) \Rightarrow \det(T^*A, T^*B) = \det(A, B),$

$[*] := E(|) \ldots [1][2][3] : \det(T^*A, T^*B) = \det(A, B);$

$\square$

$\texttt{basicInversion} :: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

$\texttt{basicInversion}\,(\infty) = \mathrm{inv}(\infty) := 0$

$\texttt{basicInversion}\,(0) = \mathrm{inv}(0) := \infty$

$\texttt{basicInversion}\,(z) = \mathrm{inv}(z) := z^{-1}$

$\texttt{basicCircleInversion} :: \mathcal{S}' \to \mathcal{S}'$

$\texttt{basicCircleInversion}\,(S) = \mathrm{inv}^*S := \mathrm{Inv}^*_{\mathbb{R}}\mathrm{Inv}^*_{\mathbb{S}^1}(S)$

$\texttt{MöbiusTransformElementaryDecomposition} :: \forall M \in \mathcal{M} \,.\, \exists a, b \in \mathbb{C} : \exists r \in \mathbb{R}^\times : \exists R \in \mathbf{SO}(2) :$

$: M = R\ \sigma_a\ \tau_a\ \text{inv}\ \tau_b$

$\texttt{Proof}\ =$

$\Big(A, [1]\Big) := \mathcal{U}(M) : \sum A \in \mathbf{GL}(\mathbb{C}, 2) \,.\, M = \mathbf{M}_A,$

$\texttt{Assume}\ [0] : A_{2,1} \neq 0,$

$\texttt{Assume}\ z : \hat{\mathbb{C}},$

$$[2] := [1]\mathcal{U}\mathbf{M}_A\mathcal{U}\texttt{Field}(\mathbb{C})\mathcal{U}^{-1} \det A : M(z) = \mathbf{M}_A(z) = \frac{A_{1,1}z + A_{1,2}}{A_{2,1}z + A_{2,2}} = \frac{A_{1,1}z + \frac{A_{1,1}}{A_{2,1}}A_{2,2}}{A_{2,1}z + A_{2,2}} + \frac{A_{1,2} - \frac{A_{1,1}}{A_{2,1}}A_{2,2}}{A_{2,1}z + A_{2,2}} =$$

$$= \frac{A_{1,1}}{A_{2,1}} + \frac{1}{\frac{A_{2,1}}{A_{1,2} - \frac{A_{1,1}}{A_{2,1}}A_{2,2}}z + \frac{A_{2,2}}{A_{1,2} - \frac{A_{1,1}}{A_{2,1}}A_{2,2}}} = \frac{A_{1,1}}{A_{2,1}} - \frac{1}{\frac{A_{2,1}^2}{\det A}z + \frac{A_{2,2}A_{2,1}}{\det A}},$$

$a := -\dfrac{A_{2,2}A_{2,1}}{\det A} : \mathbb{C},$

$b := \dfrac{A_{1,1}}{A_{2,1}} : \mathbb{C},$

$r := \left|\dfrac{A_{2,1}^2}{\det A}\right| : \mathbb{R}^\times,$

$R := \operatorname{Arg}\left(\dfrac{A_{2,1}^2}{\det A}\right) : \mathbf{SO}(2),$

$[0.*] := [2] \ldots : M(z) = z\ R\ \sigma_r\ \tau_a\ \text{inv}\ \tau_b;$

$\rightsquigarrow [*] := \ldots : M = R\ \sigma_r\ \tau_a \text{inv}\ \tau_b;$

$\square$

$\texttt{circleMöbiusTransform} :: \mathcal{M} \to \mathcal{S}' \to \mathcal{S}'$

$\texttt{circleMöbiusTransform}\,(M, S) = M^*(S) := S\ R^*\ \sigma_r^*\ \tau_a^*\ \text{inv}^*\ \tau_b^*$

$\quad \texttt{where}\quad (a, b, r, R) = \texttt{MöbiusTransformElementaryDecomposition}(M)$

$\texttt{MöbiusTransformPreservesDiscriminant} :: \forall A, B \in \mathcal{S}' \,.\, \forall M \in \mathcal{M} \,.\, \det(M^*A, M^*B) = \det(A, B)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{MöbiusTransformMapsLinesAndCirclesToLinesAndCircles} :: \forall M \in \mathcal{M} \,.\, M^*\Re\mathcal{S}_* = \Re\mathcal{S}_*$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{CrossRatioInvariant} :: ?\Big(\hat{\mathbb{C}} \to \hat{\mathbb{C}}\Big)$

$f : \texttt{CrossRatioInvariant} \iff \forall a, b, c, d \in \mathbb{C} \,.\, \operatorname{cr}\Big(f(a), f(b), f(c), f(d)\Big) = \operatorname{cr}(a, b, c, d)$

$\texttt{MöbiusTransformIsCrossRatioInvariant} :: \forall M \in \mathcal{M} \,.\, \texttt{CrossRatioInvariant}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a, b, c, d : \hat{\mathbb{C}},$

$\texttt{Assume}\ v : \mathbb{C},$

$[v.*] := \reflectbox{$D$}\mathrm{cr}(a+v, b+v; c+v, d+v)\texttt{InverseCancelation}(\mathbb{C}, v)\reflectbox{D}\mathrm{cr}(a, b; c, d) :$

$$: \mathrm{cr}(a+v, b+v; c+v, d+v) = \frac{a+v-c-v}{a+v-d-v}\frac{b+v-d-v}{b+v-c-v} = \frac{a-c}{a-d}\frac{b-d}{b-c} = \mathrm{cr}(a, b; c, d);$$

$\leadsto [1] := I(\forall) : \forall v \in \mathbb{C} \,.\, \mathrm{cr}(a+v, b+v; c+v, d+v) = \mathrm{cr}(a, b; c, d),$

$\texttt{Assume}\ z : \mathbb{C},$

$[z.*] := \reflectbox{$D$}\mathrm{cr}(za, zb; zc, zd)\texttt{InverseCancelation}(\mathbb{C}, z)\reflectbox{D}\mathrm{cr}(a, b; c, d) :$

$$: \mathrm{cr}(za, zb; zc, zd) = \frac{za-zc}{za-zd}\frac{zb-zd}{zb-zc} = \frac{a-c}{a-d}\frac{b-d}{b-c} = \mathrm{cr}(a, b; c, d);$$

$\leadsto [2] := I(\forall) : \forall z \in \mathbb{C} \,.\, \mathrm{cr}(za, zb; zc, zd) = \mathrm{cr}(a, b; c, d),$

$[3] := \reflectbox{$D$}\mathrm{inv}\texttt{CrossRatioInversion}(\ldots)\texttt{ConugationInvolution}(\ldots) :$

$$: \mathrm{cr}(\mathrm{inv}\ a, \mathrm{inv}\ b, \mathrm{inv}\ c, \mathrm{inv}\ d) = \overline{\overline{\mathrm{cr}(a, b, c, d)}} = \mathrm{cr}(a, b, c, d),$$

$\Big[(a, b, c, d). *\Big] := \texttt{MöbiusTransformElementaryDecomposition}(M)[1][2][3] :$

$$: \mathrm{cr}(Ma, Mb, Mc, Md) = \mathrm{cr}(a, b, c, d);$$

$\leadsto [*] := \reflectbox{D}^{-1}\texttt{CrossRatioInvariant} : \texttt{CrossRatioInvariant}(M);$

$\square$

$\texttt{MöbiusTransformIsDeterminedByThreePoints} :: \forall x, y : 3 \hookrightarrow \hat{\mathbb{C}} \,.\, \exists! M \in \mathcal{M} : M(x) = y$

$\texttt{Proof} \ =$

$(A, [1]) := \reflectbox{$D$}\mathcal{M} : \sum A \in \mathbf{GL}(\mathbb{R}, 2) \,.\, M = \mathbf{M}_A,$

$\texttt{Assume}\ [2] : M(x) = x,$

$[3] := [2]\reflectbox{$D$}M(x)[1]\reflectbox{$D$}\mathbf{M}_A : x = M(x) = \mathbf{M}_A(x) = \dfrac{A_{1,1}x + A_{1,2}}{A_{2,1}x + A_{2,2}},$

$[4] := [3]\Big(A_{2,1}x + A_{2,2}\Big) : A_{2,1}x^2 + A_{2,2}x = A_{1,1}x + A_{1,2},$

$[5] := \reflectbox{$D$}x[4] : A_{2,1} = 0 \ \&\ A_{2,2} = A_{1,1} \ \&\ A_{1,2} = 0,$

$[2.*] := [5][1] : M = \mathrm{id};$

$\leadsto [2] := I(\Rightarrow) : M(x) = x \Rightarrow M = \mathrm{id},$

$\texttt{Assume}\ [3] : M(x) = (0, 1, \infty),$

$[4] := [3][1]\reflectbox{D}0 : A_{1,1}x_1 + A_{1,2} = 0,$

$[5] := \dfrac{[4]}{x_1} : A_{1,1} = -\dfrac{A_{1,2}}{x_1},$

$[6] := [3][1]\reflectbox{D}1 : A_{1,1}x_2 + A_{1,2} = A_{2,1}x_2 + A_{2,2},$

$[7] := \dfrac{[6]}{x_2} : A_{1,1} - A_{2,1} = \dfrac{A_{2,2} - A_{1,2}}{x_2},$

$[8] := [3][1]\reflectbox{D}\infty : A_{2,1}x_3 + A_{2,2} = 0,$

$[9] := \dfrac{[8]}{x_3} : A_{2,1} = -\dfrac{A_{2,2}}{x_3},$

$[10] := [5][7][9] : -\dfrac{A_{1,2}}{x_1} + \dfrac{A_{2,2}}{x_3} = \dfrac{A_{2,2}}{x_2} - \dfrac{A_{1,2}}{x_2},$

$[11] := \reflectbox{$D$}\texttt{Field}(\mathbb{C}) : A_{1,2} = \dfrac{x_1(x_2 - x_3)}{x_3(x_2 - x_1)}A_{2,2},$

$[3.*] := [11][5][7][1][3] : A \cong_{\mathcal{M}} \left[\!\!\left[\begin{matrix} \frac{x_3 - x_2}{x_3(x_2 - x_1)} & \frac{x_1(x_2 - x_3)}{x_3(x_2 - x_1)} \\ -\frac{1}{x_3} & 1 \end{matrix}\right]\!\!\right];$

$\rightsquigarrow [3] := I(\Rightarrow) : M(x) = (0, 1, \infty) \Rightarrow A \cong_{\mathcal{M}} \left[\!\!\left[\begin{matrix} \frac{x_3 - x_2}{x_3(x_2 - x_1)} & \frac{x_1(x_2 - x_3)}{x_3(x_2 - x_1)} \\ -\frac{1}{x_3} & 1 \end{matrix} \right]\!\!\right],$

Assume $[4] : M(0, 1, \infty) = y,$

$[5] := [1][4]\mathcal{I}0 : \dfrac{A_{1,2}}{A_{2,2}} = y_1,$

$[6] := A_{2,2}[5] : A_{1,2} = y_1 A_{2,2},$

$[7] := [1][4]\mathcal{I}1 : \dfrac{A_{1,1} + A_{1,2}}{A_{2,1} + A_{2,2}} = y_2,$

$[8] := [1][4]\mathcal{I}\infty : \dfrac{A_{1,1}}{A_{2,1}} = y_3,$

$[9] := A_{1,2}[8] : A_{1,1} = y_3 A_{2,1},$

$[10] := [9][7][6] : y_3 A_{2,1} + y_1 A_{2,2} = A_{2,1} y_2 + A_{2,2} y_2,$

$[11] := \mathcal{I}\texttt{Field}\mathbb{C} : A_{2,1} = \dfrac{y_2 - y_1}{y_3 - y_2} A_{2,2},$

$[4.*] := [11][6][9][4][1] : A \cong_{\mathcal{M}} \left[\!\!\left[\begin{matrix} \frac{y_2 - y_1}{y_3 - y_2} y_3 & y_1 \\ \frac{y_2 - y_1}{y_3 - y_2} & 1 \end{matrix} \right]\!\!\right];$

$\rightsquigarrow [4] := I(\Rightarrow) : M(0, 1, \infty) = y \Rightarrow A \cong_{\mathcal{M}} \left[\!\!\left[\begin{matrix} \frac{y_2 - y_1}{y_3 - y_2} y_3 & y_1 \\ \frac{y_2 - y_1}{y_3 - y_2} & 1 \end{matrix} \right]\!\!\right],$

$[6] := [3][4] : \left[\!\!\left[\begin{matrix} \frac{y_2 - y_1}{y_3 - y_2} y_3 & y_1 \\ \frac{y_2 - y_1}{y_3 - y_2} & 1 \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} \frac{x_3 - x_2}{x_3(x_2 - x_1)} & \frac{x_1(x_2 - x_3)}{x_3(x_2 - x_1)} \\ -\frac{1}{x_3} & 1 \end{matrix} \right]\!\!\right] x = y,$

$[7] := [6][2]\mathcal{I}^{-1}\texttt{Unique} : \exists! M \in \mathcal{M} : M(x) = y;$

$\square$

$\texttt{CrossRatioInvariantIsMöbiusTransform} :: \forall f : \texttt{Bijection} \,\&\, \texttt{CrossRatioInvariant}\left(\hat{\mathbb{C}}\right) . f \in \mathcal{M}$

Proof $=$

$M := \texttt{MöbiusTransformDeterminedByThreePoints}\Big((0, 1, \infty), \big(f(0), f(1), f(\infty)\big)\Big) : \mathcal{M},$

Assume $z : \hat{\mathbb{C}},$

$[1] := \mathcal{I}\texttt{CrossRatioInvariant}(f) : \mathrm{cr}\Big(f(0), f(1); f(\infty), f(z)\Big) = \mathrm{cr}(0, 1; \infty, z),$

$[2] := \mathcal{I}M : M =_{\mathcal{M}} \left[\!\!\left[\begin{matrix} \frac{f(1) - f(0)}{f(\infty) - f(1)} f(\infty) & f(0) \\ \frac{f(1) - f(0)}{f(\infty) - f(1)} & 1 \end{matrix} \right]\!\!\right],$

$[3] := [1]\mathcal{I}\mathrm{cr} : \dfrac{z - 1}{z} = \dfrac{f(0) - f(\infty)}{f(1) - f(\infty)} \dfrac{f(1) - f(z)}{f(0) - f(z)},$

$[4] := z[3] + 1 : z = z \dfrac{f(0) - f(\infty)}{f(1) - f(\infty)} \dfrac{f(1) - f(z)}{f(0) - f(z)} + 1,$

$[5] := [4](f(0) - f(z)) : z\Big(f(0) - f(z)\Big) = \dfrac{z(f(0) - f(\infty))(f(1) - f(z)) + (f(0) - f(z))(f(1) - f(\infty))}{f(1) - f(\infty)},$

$[6] := \mathcal{I}\texttt{Field}\mathbb{C}[5] : f(z)\left(\dfrac{f(1) - f(0)}{f(1) - f(\infty)} z + 1\right) = \dfrac{z(f(0) - f(1))f(\infty) + f(0)(f(1) - f(\infty))}{f(1) - f(\infty)},$

$[z.*] := [6][2] : f(z) = M(z);$

$\rightsquigarrow [*] := I(=, \rightarrow) : f = M;$

$\square$

$\texttt{MöbiusTransformInvolutionCriterion} :: \forall M \in \mathcal{M} \,.\, M \neq \mathrm{id} \Rightarrow \texttt{Involution}(M) \iff \operatorname{tr} M = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \operatorname{tr} M = 0,$

$\Big(a, [2]\Big) := \texttt{SpectralTrace}[1] : \sum a \in \mathbb{C}^\times : M \sim_{\mathcal{M}} \left[\!\!\left[\begin{matrix} a & \\ & -a \end{matrix} \right]\!\!\right],$

$[4] := [2]\texttt{SimilarMatrixMult}\,\mathcal{O}\mathcal{M} : M^2 \sim_{\mathcal{M}} \left[\!\!\left[\begin{matrix} a^2 & \\ & a^2 \end{matrix} \right]\!\!\right] =_{\mathcal{M}} \left[\!\!\left[\begin{matrix} 1 & \\ & 1 \end{matrix} \right]\!\!\right],$

$[1.*] := \mathcal{O}\mathcal{M}[4] : M^2 = \mathrm{id};$

$\rightsquigarrow [1] := I(\Rightarrow) : \operatorname{tr} M = 0 \Rightarrow \texttt{Involution}(M),$

$\texttt{Assume}\ [2] : \texttt{Involution}(M),$

$[3] := \mathcal{O}\texttt{Involution}(M) : M^2 = \mathrm{id},$

$[4] := \mathcal{O}\mathcal{M}[3] : M_{1,1}^2 + M_{1,2}M_{2,1} = 1 \ \&\ M_{1,1}M_{1,2} + M_{1,2}M_{2,2} = 0\ \&$

$\quad \&\ M_{2,2}M_{2,1} + M_{2,1}M_{1,1} = 0 \ \&\ M_{2,2}^2 + M_{1,2}M_{2,1} = 1,$

$\texttt{Assume}\ [5] : \operatorname{tr} \mathcal{M} \neq 0,$

$[6] := [5][4] : M_{1,2} = 0 = M_{2,1},$

$[7] := [6][4] : M_{1,1} = M_{2,2},$

$[5.*] := [7](M \neq \mathrm{id}) : \bot;$

$\rightsquigarrow [2.*] := E(\bot) : \operatorname{tr} M = 0;$

$\rightsquigarrow [*] := I(\iff) : \operatorname{tr} M = 0 \iff \texttt{Involution}(M);$

$\square$

$\texttt{MöbiusConjugate} :: \mathcal{M} \to \hat{\mathbb{C}}^2$

$(z, z') : \texttt{MöbiusConjugate} \iff \Lambda M \in \mathcal{M} \,.\, z \sim_M z' \iff \Lambda M \in \mathcal{M} \,.\, M(z) = z' \ \&\ z' = M(z)$

$\texttt{InvolutionByConjugates} :: \forall M \in \mathcal{M} \,.\, \texttt{Involution}(M) \iff \exists \texttt{ConjugatePair}(M)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.10 Applications to Projective Geometry

2 Hypercomplex Numbers

2.1 Dual and Double Numbers

2.2 Dual Numbers as orientated Lines

3 Gaussian Numbers