Order Induced Topology

Uncultured Tramp

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Contents

1 Topology of Partial Order

1.1 Types of Intervals

$$\texttt{openInterval} :: \prod X : \texttt{Poset} \,.\, X \times X \to ?X$$
$$\texttt{openInterval}\,(a,b) = (a,b) := \{x \in X : a < x < b\}$$

$$\texttt{closedInterval} :: \prod X : \texttt{Poset} \,.\, X \times X \to ?X$$
$$\texttt{closedInterval}\,(a,b) = [a,b] := \{x \in X : a \le x \le b\}$$

$$\texttt{rightHalfInterval} :: \prod X : \texttt{Poset} \,.\, X \times X \to ?X$$
$$\texttt{rightHalfInterval}\,(a,b) = (a,b] := \{x \in X : a < x \le b\}$$

$$\texttt{leftHalfInterval} :: \prod X : \texttt{Poset} \,.\, X \times X \to ?X$$
$$\texttt{leftHalfInterval}\,(a,b) = [a,b) := \{x \in X : a \le x < b\}$$

$$\texttt{leftOpenRay} :: \prod X : \texttt{Poset} \,.\, X \to ?X$$
$$\texttt{leftOpenRay}\,(a) = (-\infty,a) := \{x \in X : x < a\}$$

$$\texttt{leftClosedRay} :: \prod X : \texttt{Poset} \,.\, X \to ?X$$
$$\texttt{leftClosedRay}\,(a) = (-\infty,a] := \{x \in X : x \le a\}$$

$$\texttt{rightOpenRay} :: \prod X : \texttt{Poset} \,.\, X \to ?X$$
$$\texttt{rightOpenRay}\,(a) = (a,\infty) := \{x \in X : x > a\}$$

$$\texttt{rightClosedRay} :: \prod X : \texttt{Poset} \,.\, X \to ?X$$
$$\texttt{rightClosedRay}\,(a) = [a,\infty) := \{x \in X : x \ge a\}$$

$$\texttt{OpenInterval} :: \prod X : \texttt{Poset} \,.\, ??X$$
$$A : \texttt{OpenInterval} \iff \exists a,b \in X : a \le b \;\&\; \Big(A = (a,b) \Big| A = (a,\infty) \Big| A = (-\infty,b)\Big) \Big| A = \mathbb{R}$$

$$\texttt{ClosedInterval} :: \prod X : \texttt{Poset} \,.\, ??X$$
$$A : \texttt{ClosedInterval} \iff \exists a,b \in X : a \le b \;\&\; \Big(A = [a,b] \Big| A = [a,\infty) \Big| A = (-\infty,b]\Big) \Big| A = \mathbb{R}$$

$\texttt{OpenIntersecton} :: \forall X : \texttt{Toset} \,.\, \forall (a,b),(c,d) : \texttt{OpenInterval}(X) \,.\, (a,b) \cap (c,d) \neq \emptyset \Rightarrow$
$\Rightarrow (a,b) \cap (c,d) : \texttt{OpenInterval}(X)$

Proof =

...

□

$\texttt{ClosedIntersection} :: \forall X : \texttt{Toset} \,.\, \forall [a,b],[c,d] : \texttt{ClosedInterval}(X) \,.\, [a,b] \cap [c,d] \neq \emptyset \Rightarrow$
$\Rightarrow [a,b] \cap [c,d] : \texttt{ClosedInterval}(X)$

Proof =

...

□

$\texttt{OpenUnion} :: \forall X : \texttt{Toset} \,.\, \forall (a,b),(c,d) : \texttt{OpenInterval}(X) \,.\, (a,b) \cap (c,d) \neq \emptyset \Rightarrow$
$\Rightarrow (a,b) \cup (c,d) : \texttt{OpenInterval}(X)$

Proof =

...

□

$\texttt{ClosedUnion} :: \forall X : \texttt{Toset} \,.\, \forall [a,b],[c,d] : \texttt{ClosedInterval}(X) \,.\, [a,b] \cup [c,d] \neq \emptyset \Rightarrow$
$\Rightarrow [a,b] \cup [c,d] : \texttt{ClosedInteval}(X)$

Proof =

...

□

1.2 Left And Right Topology

$\texttt{leftTopology} :: \texttt{Poset} \to \mathsf{TOP}$

$\texttt{leftTopology}\,(X) := \Big\langle \{(-\infty, x] | x \in X\} \Big\rangle_{\mathsf{TOP}}$

$\texttt{rightTopology} :: \texttt{Poset} \to \mathsf{TOP}$

$\texttt{rightTopology}\,(X) := \Big\langle \{[x, +\infty) | x \in X\} \Big\rangle_{\mathsf{TOP}}$

$\texttt{LeftGrounded} :: \prod X : \texttt{POSET} \,.\, ?X$

$A : \texttt{LeftGrounded} \iff \forall a \in A \,.\, \forall x \in X \,.\, x \leq a \Rightarrow a \in X$

$\texttt{LeftGroundedIntersect} :: \forall X \in \texttt{Poset} \,.\, \forall I : \mathsf{SET} \,.\, \forall A : I \to \texttt{LeftGrounded}(X) \,.\, \bigcap_{i \in I} A_i : \texttt{LeftGrounded(X)}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LeftGroundedUnion} :: \forall X \in \texttt{Poset} \,.\, \forall I : \mathsf{SET} \,.\, \forall A : I \to \texttt{LeftGrounded}(X) \,.\, \bigcup_{i \in I} A_i : \texttt{LeftGrounded(X)}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LeftTopologyOpenSet} :: \forall X \in \texttt{Poset} \,.\, \forall U \subset X \,.\, U \in \mathcal{T}\Big(\texttt{leftTopology}(X)\Big) \iff U : \texttt{LeftGounded}(X)$

$\texttt{Proof} \;=$

$[1] := \texttt{LeftGroundedUnion}\eth\texttt{leftTopology} : \mathcal{T}\Big(\texttt{leftTopology}(X)\Big) \subset \texttt{LeftGrounded}(X),$

$\texttt{Assume}\ A : \texttt{LeftGrounded}(X),$

$U := \bigcup_{a \in A} (-\infty, a] : \texttt{Open}\ \texttt{leftTopology}\ X,$

$[2] := \jmath(-\infty, a]\jmath U \eth^{-1}\texttt{Subset} : A \subset U,$

$[3] := \jmath(-\infty, a]\jmath U \eth^{-1}\eth\texttt{LeftGrounded}(A)\texttt{Subset} : U \subset A,$

$[A.*] := \eth^{-1}\texttt{SetEq}[2][3] : U = A;$

$\rightsquigarrow [2] := I\texttt{Subset} : \texttt{LeftGrounded}(X) \subset \texttt{Open}\ \texttt{leftTopology}\ X,$

$[*] := I\texttt{SetEq}[1][2] : \texttt{LeftGrounded}(X) = \texttt{Open}\ \texttt{leftTopology}\ X;$

$\square$

$\texttt{LowerIntersection} :: \forall X : \texttt{Poset} \,.\, \forall I : \texttt{Set} \,.\, \forall U : I \to \mathcal{T}\Big(\texttt{leftTopology}(X)\Big) \,.$

$\quad .\, \bigcap_{i \in I} U_i \in \mathcal{T}\Big(\texttt{leftTopology}(X)\Big)$

$\texttt{Proof} \;=$

$[*] := \texttt{LeftGroundedIntersection}(X)\texttt{LeftTopologyOpenSet}(X) : \texttt{This};$

$\square$

$\texttt{leftOpenSets} :: \prod X : \texttt{Poset} \,.\, \texttt{Topology}(X)$

$\texttt{leftOpenSets}\,() = L(X) := \texttt{topology leftToplogy}(X)$

$\texttt{LeftTopologyIsT0} :: \forall X : \texttt{Poset} \,.\, \big(X, L(X)\big) : \texttt{T0}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x, y : X,$

$\texttt{Assume}\ [1] : x \neq y,$

$[2] := \texttt{PosetQuadrohtomy}[1] : x < y | y < x | y \# x,$

$\Big[(x, y).*\Big] := \eth\texttt{leftOpenRay}[1][2] : y \notin (-\infty, x] | x \notin (-\infty, y];$

$\rightsquigarrow [*] := \eth^{-1}\texttt{T0} : \Big(\big(X, L(X)\big) : \texttt{T0}\Big);$

$\square$

$\texttt{ClosedPointsAreMaximal} :: \forall X : \texttt{Poset} \,.\, \forall x \in X \,.\, \{x\} : \texttt{Closed}\big(X, L(X)\big) \iff x \in \max X$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ClosedPointsAreMinimal} :: \forall X : \texttt{Poset} \,.\, \forall x \in X \,.\, \{x\} : \texttt{Open}\big(X, L(X)\big) \iff x \in \min X$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{PointClosure} :: \forall X : \texttt{Poset} \,.\, \forall x \in X \,.\, \overline{\{x\}} = [x, \infty)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{IntersectionClosed} :: ?\mathsf{TOP}$

$X : \texttt{IntersectionClosed} \iff \forall I \in \mathsf{SET} \,.\, \forall U : I \to \mathcal{T}(X) \,.\, \bigcap_{i \in I} U_i \in \mathcal{T}(X)$

$\texttt{Leftable} :: ?\mathsf{TOP}$

$X : \texttt{Leftable} \iff \exists o : \texttt{Order}(X) \,.\, X \cong_{\mathsf{TOP}} \big(X, L(X, o)\big)$

$\texttt{LeftableIffIC} :: \forall X \in \mathsf{TOP} \;.\; X : \texttt{IntersectionClosed} \iff X : \texttt{Leftable}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : (X : \texttt{IntersectionClosed}),$

$$U := \Lambda x \in X \;.\; \bigcap_{U \in \mathcal{U}(x)} U : X \to \mathcal{T}(X),$$

$$o := \Big\{(x, y) \in X^2 : U_x \subset U_y\Big\} : \texttt{Order}(X),$$

$$[2] := \eth^{-1}\texttt{Base}\jmath U : \Big(\operatorname{Im} U : \texttt{Base}(X)\Big),$$

$$[3] := \jmath U \eth \texttt{leftClosedRay}(X, o) : \forall x \in X \;.\; U_x = (-\infty, x]_o,$$

$$[*] := \eth^{-1}\texttt{leftTopology}[3]\eth^{-1}\texttt{Base}[2] : X = \Big(X, L(X, o)\Big);$$

$\square$

2 Topology of Total Order

2.1 Topology Induced by Intervals

$\texttt{OpenIntervalsAreBase} :: \forall X : \texttt{Toset} \,.\, \texttt{OpenIntervals}(X) : \texttt{Base}(X)$

$\texttt{Proof} =$

$(1) := \eth\texttt{OpenInterval}(X) : X \in \texttt{OpenInterval}(X),$

$\texttt{Assume}\ A, B : \texttt{OpenInterval}(X),$

$\texttt{Assume}\ x : \texttt{In}(A \cap B),$

$(2) := \eth\emptyset(\eth x) : A \cap B \neq \emptyset,$

$(3) := \texttt{OpenIntersection}(A, B) : A \cap B : \texttt{OpenInterval}(x),$

$(3) := \eth x\texttt{SetEq}^{-1} : x \in A \cap B \subset A \cap B;$

$\rightsquigarrow (2) := I(\forall)I(\forall)I(\exists)(A \cap B) : \forall A, B : \texttt{OpenIntervals}(X) \,.\, \forall x \in A \cap B \,.\, \exists Z : \texttt{OpenInterval} : x \in Z \subset A \cap$

$(*) := \eth^{-1}\texttt{Base}(1)(2) : \Big(\texttt{OpenIntervals}(X) : \texttt{Base}(X)\Big);$

$\square$

$\texttt{orderTopology} :: \prod X : \texttt{Toset} \,.\, \texttt{Topology}(X)$

$\texttt{orderTopology}\,() := \texttt{genTop}(\texttt{OpenInterval}(X))$

$\texttt{synecdoche} :: \texttt{Toset} \to \mathsf{TOP}$

$\texttt{synecdoche}\,(X) := (X, \texttt{orderTopology})$

$\texttt{OrderableTopologicalSpace} :: ?\mathsf{TOP}$

$X : \texttt{OrderableTopologicalSpace} \iff \exists R : \texttt{TotalOrder}(X) \,.\, \mathcal{T}(X, R) = \mathcal{T}(X)$

$\texttt{TotallyOrderedSeparation} :: \forall X : \texttt{Toset} \,.\, X : \texttt{T4}$

$\texttt{Proof} =$

$\ldots$

$\square$