Topological Vector Spaces

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Contents

1 Normed Spaces

1.1 Norms and Seminorms

$\texttt{Seminorm} :: \prod K : \texttt{ValuationField} \,.\, \prod V : \texttt{VectorSpace}(K) \,.\, ?(V \to \mathbb{R}_+$
$N : \texttt{Seminorm} \iff \forall v, w \in V \,.\, \forall a \in K \,.\, N(av) = |a|N(v)\ \&\ N(a+v) \le N(w) + N(v)$

$\texttt{SeminormedSpace} = \sum V : \texttt{VectorSpace}(K) \,.\, \texttt{Seminorm}(V)$

$\texttt{seminorm} :: \prod (V, N) : \texttt{SeminormedSpace} \,.\, \texttt{Seminorm}(V)$
$\texttt{seminorm}((V, N)) = \|\cdot\|_{(V,N)} := N$

$\texttt{Norm} :: ?\texttt{Seminorm}(V)$
$N : \texttt{Norm} \iff \forall v \in V : v \ne 0 \,.\, N(v) \ne 0$

$\texttt{NormedSpace} = \sum V : \texttt{VectorSpace}(K) \,.\, \texttt{Norm}(K)$

$\texttt{norm} :: \prod (V, N) : \texttt{NormedSpace} \,.\, \texttt{Norm}(V)$
$\texttt{norm}((V, N)) = \|\cdot\|_{(V,N)} := N$

$\texttt{NormAsDistance} :: \forall V : \texttt{NormedSpace} \,.\, \Lambda(x, y) \in V \times V \,.\, \|x - y\| : \texttt{Distance}$
$\texttt{Proof} =$
$d := \Lambda(x, y) \in V \times V \,.\, \|x - y\| : V \times V \to \mathbb{R}_+,$
$\texttt{Assume}\ x : V,$
$(1) := \eth(d)(x, x)\eth{-}_V\eth\texttt{Seminorm}(\|\cdot\|) : d(x, x) = \|x - x\| = \|0\| = 0;$
$\leadsto (1) := \cdot : \forall x \in V \,.\, d(x, x) = 0,$
$(2) := \eth\texttt{Norm}(\|\cdot\|) : \forall (x, y) \in V \times V : d(x, y) = \|x - y\| = 0 \,.\, x = y,$
$(3) := \eth\texttt{Seminorm}(\|\cdot\|) : \forall x, y, z \in V \,.\, d(x, y) = \|x - y\| \le \|x - z\| + \|z - y\| \le d(x, z) + d(z, y),$
$(4) := \eth\texttt{Seminorm}(\|\cdot\|) : \forall x, y \in V \,.\, d(x, y) = \|x - y\| = \|(-1)(y - x)\| = |-1|\|y - x\| = \|y - x\| = d(y, x),$
$(*) := \eth^{-1}\texttt{Distance}(V)(d) : (d : \texttt{Distance}(V));$
$\square$

$\texttt{normedAsMetric} :: \texttt{NormedSpace} \to \texttt{MetricSpace}$
$\texttt{normedAsMetric}(X) = \texttt{implicit}\ X := (X, \Lambda(x, y) \in X \times X \,.\, \|x - y\|)$

$\texttt{Stronger} :: \prod V : \texttt{VectorSpace}(K) \,.\, ?(\texttt{Norm}(V) \times \texttt{Norm}(V))$
$(N, M) : \texttt{Stronger} \iff \exists c \in \mathbb{R}_{++} \,.\, \forall v \in V \,.\, cN(v) \ge M(v)$

$\texttt{Equavalent} :: \prod V : \texttt{VectorSpace}(K) \,.\, ?(\texttt{Norm}(V) \times \texttt{Norm}(V))$
$(N, M) : \texttt{Equavalent} \iff N \cong M \iff ((N, M) : \texttt{Stronger}(V))\ \&\ ((M, N) : \texttt{Stronger}(V))$

$\texttt{seminormTopology} :: \texttt{Seminorm}(V) \to \texttt{Topology}(V)$

$\texttt{seminormTopology}\,(\|\cdot\|) := \texttt{fromBase}\{\{y \in V | \|x - y\| < r\} | x \in V, r \in \mathbb{R}_{++}\}$

$\texttt{seminormedAsTopologic} :: \texttt{SeminormedSpace}(K) \to \mathsf{TOP}$

$\texttt{seminormedAsTopological}\,(V, \|\cdot\|) = \texttt{implicit}(V, \|\cdot\|) := (V, \texttt{seminormTopology}(\|\cdot\|))$

$\texttt{NormDistanceCharacteristic} :: \forall V : \texttt{VectorSpace}\,(K)\,.\,\forall d : \texttt{Distance}(V)\,.$

$.\,(\forall v, w, x \in V\,.\,d(v + x, w + x) = d(v, w)\ \&\ \forall a \in K\,.\,d(av, 0) = |a|d(v, 0)) \iff$

$\iff \exists \|\cdot\| : \texttt{Norm}(V)\,.\,d = \texttt{NormAsDistance}(\|\cdot\|)$

$\texttt{Proof} =$

$\texttt{Assume}\ L : \texttt{Left},$

$\|\cdot\| := \Lambda v \in V\,.\,d(v, 0) : V \to \mathbb{R}_+,$

$(1) := \texttt{eval}\|\cdot\|, 0 : \|0\| = 0,$

$\texttt{Assume}\ v : V : \|v\| = 0,$

$(2) := \eth)3\texttt{Distance}(d)(\eth\|\cdot\|\eth x) : x = 0;$

$\rightsquigarrow (2) := \cdot : \forall x \in V : \|x\| = 0\,.\,x = 0,$

$\texttt{Assume}\ x, y : V,$

$(3) := \eth\|\cdot\|(x + y)\eth_1 d(x + y, 0, -y)\eth_3\texttt{Distance}(d)(x, -y)\eth_2\texttt{Distance}(0, -y)\eth_2 d(y, -1)$

$\eth^{-1}\|\cdot\|(x)\eth^{-1}\|\cdot\|(y) : \|x + y\| = d(x + y, 0) = d(x, -y) \le d(x, 0) + d(0, -y) = d(x, 0) + d(y, 0) =$

$= \|x\| + \|y\|;$

$\rightsquigarrow (3) := \cdot : \forall x, y \in V\,.\,\|x + y\| \le \|x\| + \|y\|,$

$\texttt{Assume}\ (x, a) : V \times K,$

$(4) := \eth\|ax\|\eth_2 d(x, a)\eth^{-1}\|\cdot\|(x) : \|ax\| = d(ax, 0) = |a|d(x, 0) = |a|\|x\|;$

$\rightsquigarrow (4) := \cdot : \forall x \in V\,.\,\forall a \in K\,.\,\|ax\| = |a|\|x\|,$

$(5) := \eth^{-1}\texttt{Norm}(V)(1, 3, 4, 2) : (\|\cdot\| : \texttt{Norm}(V)),$

$\texttt{Assume}\ x, y : V,$

$(6) := \eth_1 d(x, y, -y)\eth^{-1}\|\cdot\|(x - y) : d(x, y) = d(x - y, 0) = \|x - y\|;$

$\rightsquigarrow (6) := \texttt{NormAsMetric}^{-1}(\cdot) : d = \texttt{NormAsMetric}(\|\cdot\|;$

$\rightsquigarrow (\Rightarrow) := \cdot : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : \texttt{Right},$

$\texttt{Assume}\ v, w, x : V,$

$(1) := R(v + x, w + x)\eth\texttt{Inverse}(x)R^{-1}(v, w) : d(v + x, w + x) = \|v + x - w - x\| = \|v - w\| = d(v, w);$

$\rightsquigarrow (1) := \texttt{UniversalIntroduction} : \forall x, y, z \in V\,.\,d(v + x, w + x) = d(v, w),$

$\texttt{Assume}\ v : V,$

$\texttt{Assume}\ a : K,$

$(2) := R(av, 0)\eth\texttt{Zero}(0)\eth_2\texttt{Norm}R^{-1}(v, 0) : d(av, 0) = \|av - 0\| = \|av\| = |a|\|v\| = |a|d(v, 0);$

$\rightsquigarrow (2) := \texttt{UniversalIntroduction} : \forall v \in V\,.\,\forall a \in A\,.\,d(av, 0) = |a|d(v, 0),$

$(3) := (1, 2) : \texttt{Left};$

$\rightsquigarrow (*) := \texttt{IffIntroduction} : \texttt{Left} \iff \texttt{Right},$

$\square$

1.2 Geometric Construction

$K = \mathbb{R}|\mathbb{C}$

$\texttt{interval} :: \prod V : \texttt{VectorSpace}(K) : \mathbb{R} \subset K \,.\, V \times V \to \texttt{Set}(V)$

$\texttt{interval}(a,b) = [a,b] := \{ta + (1-t)b | t \in [0,1]\}$

$\texttt{Convex} :: \prod V : \texttt{VectorSpace}(K) : \mathbb{R} \subset K \,.\, ?\texttt{Set}(V)$

$A : \texttt{Convex} \iff \forall a, b \in A \,.\, [a,b] \subset A$

$V = \texttt{VectorSpace}(K)$

$\texttt{TriangularEquelity} :: \forall a, b \in V \,.\, \forall z \in [a,b] \,.\, \|a - b\| = \|a - z\| + \|z - b\|$

$\texttt{Proof} =$

$t := \eth\texttt{interval}(a,b)(z) : [0,1] : z = ta + (1-t)b,$

$(*) := \eth(t) : \|a - z\| + \|z - b\| = \|(1-t)a - (1-t)b\| + \|ta - tb\| = (1-t)\|a - b\| + t\|a - b\| = \|a - b\|;$

$\square$

$\texttt{ConvexIntersection} :: \forall A : \mathbb{N} \to \texttt{Convex}(V) \,.\, \bigcap_{n=1}^{\infty} A_n : \texttt{Convex}(V)$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b : \bigcap_{n=1}^{\infty} A_n,$

$\texttt{Assume}\ t : [0,1],$

$\texttt{Assume}\ n : \mathbb{N},$

$(1) := \eth\texttt{intersection}(a,b) : a, b : A_n,$

$(2) := \eth\texttt{Convex} : ta + (1-t)b \in A_n;$

$\leadsto (3) := (\cdot) : ta + (1-t)b \in \bigcap_{n=1}^{\infty} A_n;$

$\leadsto (4) := \eth^{-1}\texttt{Convex}\left(\bigcup_{n=1}^{\infty} A_n\right) : \left(\bigcup_{n=1}^{\infty} A_n : \texttt{Convex}(V)\right);$

$\square$

$\texttt{BallIsConvex} :: \mathbb{B}_V(0,1) : \texttt{Convex}$

$\texttt{Proof} =$

$\|a\|, \|b\| \leq 1$

$\|ta + (1-t)b\| = t\|a\| + (1-t)\|b\| \leq t + (1-t) = 1$

$\square$

$\texttt{Balanced} :: ??V$

$A : \texttt{Balanced} \iff \forall u \in K : |u| = 1 \,.\, \forall v \in A \,.\, uv \in A$

$\texttt{Disclike} :: ?\texttt{Convex} \,\&\, \texttt{Balanced}(V)$

$A : \texttt{Disclike} \iff \forall v \in V : v \neq 0 \,.\, \exists a \in K : a \neq 0 \,.\, av \in A$

$\texttt{minkowskiFunctional} :: \texttt{Disclike} \to V \to \mathbb{R}_+$

$\texttt{minkowskyFunctional}\,(D, v) = M(D)(v) := \inf\{t \in \mathbb{R}_{++} \,.\, t^{-1}v \in D\}$

$\texttt{MinkowskiNorm} :: \forall D : \texttt{Disclike}(V) \,.\, M(D) : \texttt{Seminorm}(V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{MinkowskiCharacterisation} :: \forall V : \texttt{VectorSpace}(K) \,\&\, \texttt{TopologicalSpace} \,.$

$\quad .\, V : \texttt{NormedSpace} \iff \exists D : \texttt{Disclike}(V) : \mathcal{T}_V = \texttt{fromBase}\{aD + v : a \in K : a \neq 0, v \in V\}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LineOfCircleCriterion} :: \forall V : \texttt{NormedSpace} \,.\, \exists x, y \in V : x \neq y : [x, y] \subset \mathbb{S}_V(0, 1) \iff$

$\quad \iff \exists x, y \in V : \{x, y\} : \texttt{LinearlyIndependant}(V) : \|x + y\| = \|x\| + \|y\|$

$\texttt{Proof} =$

$\dots$

$\square$

characterize intersection of two circles in V centred in v and $-v$ of radius $\|v\|$.

$\texttt{NormedSpaceIsHopfRinov} :: \forall x, y \in V \,.\, \forall r, s \in \mathbb{R}_{++} \,.\, \mathbb{B}_V(x, r) \cap \mathbb{B}_V(y, s) = \emptyset \iff \|x - y\| > r + s$

$\texttt{Proof} =$

$\dots$

$\square$

1.3 Topological Properties

$$\texttt{normedQuetient} :: \prod X : \texttt{NormedSpace} \,.\, \texttt{Closed} \,\&\, \texttt{Subspace}(X) \to \texttt{NormedSpace}$$

$$\texttt{normedQuetient}\,(K) = \left(\frac{X}{K}\right)_{\mathsf{NVS}} := \left(\left(\frac{X}{K}\right)_{\mathsf{VS}(K)}, \Lambda[x] \in \left(\frac{X}{K}\right)_{\mathsf{VS}(K)} \inf_{y \in K} \|x - y\|\right)$$

$$\texttt{Isometry} :: ?\mathcal{M}_{\mathsf{VS}(K)}(V, W)$$

$$T : \texttt{Isometry} \iff \forall v \in V \,.\, \|Tv\| = \|v\|$$

$$\texttt{NormedCompletion} :: \forall V : \texttt{NormedSpace} \,.\, \exists \hat{V} : \texttt{NormedSpace} \,\&\, \texttt{Complete} :$$

$$: \exists T : \texttt{Isometry}(V, \hat{V}) : T(V) : \texttt{Dense}(\hat{V})$$

Proof $=$

...

□

$$\texttt{Banach} = \texttt{NormedSpace} \,\&\, \texttt{Complete}$$

$$\texttt{ContAddition} :: \forall V : \texttt{NormedSpace} \,.\, (+) : \texttt{UniformlyCont}(V \times V, V)$$

Proof $=$

...

□

$$\texttt{IntersecrionQuatientDimension} :: \forall V : \texttt{NormedSpace} \,.\, \forall n \in \mathbb{N} \,.$$

$$. \forall J : n \to \texttt{Subspace} \,\&\, \texttt{Closed}(V) : \forall i \in n \,.\, \dim\left(\frac{V}{J_i}\right) = 1 \,.\, \dim\left(\frac{V}{\bigcap_{i=1}^{n} J_i}\right) \le n$$

Proof $=$

...

□

$$\texttt{FiniteDimensionIsClosed} :: \forall M : \texttt{Subspace} \,\&\, \texttt{Closed}(V) \,.\, \forall N : \texttt{Subspace}(V) : \dim N < \infty \,.$$

$$. vM + N : \texttt{Closed}(V)$$

Proof $=$

...

□

$$\texttt{BanachQuatient} :: \forall V : \texttt{Banach} \,.\, \forall W : \texttt{Subspace} \,\&\, \texttt{Closed}(V) \,.\, \frac{V}{W} : \texttt{Banach}$$

Proof $=$

...

□

2 Inner Product spaces

2.1 Preinnner and Inner Product

$$K : \texttt{ConjugationField} : \mathbb{R} \subset K$$

$$V : \texttt{VectorSpace}\,(K)$$

$$\texttt{PreinnerProduct} :: ?V \otimes \overline{V} \to_{\mathsf{VS}(K)} K$$
$$I : \texttt{PreinnerProduct} \iff (\forall x \in V \,.\, I(v \otimes v) \in \mathbb{R}_+) \;\&\; \forall x, y \in V \,.\, I(x \otimes y) = \overline{I(y \otimes x)}$$

$$\texttt{InnerProduct} :: ?\texttt{PreinnerProduct}(V)$$
$$I : \texttt{InnerProduct} \iff \forall x \in V \,.\, I(x \otimes x) = 0 \iff x = 0$$

$$\texttt{PrehilbertSpace} := \sum H : \mathsf{VS}(K) \,.\, \texttt{PreinnerProduct}(H)$$

$$\texttt{preinnerProduct} :: \prod (H, I) : \texttt{PrehilbertSpace}(K) \,.\, H \times H \to K$$
$$\texttt{preinnerProduct}\,(v, w) = \langle v, w \rangle := I(v \otimes w)$$

$$\texttt{InnerProductSpace} := \sum H : \mathsf{VS}(K) \,.\, \texttt{InnerProduct}(H)$$

$$\texttt{quadraticForm} :: (V \otimes \overline{V} \to_{\mathsf{VS}(K)} K) \to V \to K$$
$$\texttt{quadraticForm}\,(I, v) = Q_I(v) := I(v \otimes v)$$

$$\texttt{PolarDecomposition} :: \forall I : V \otimes \overline{V} \to_{\mathsf{VS}(K)} K \,.\, \forall v, w \in V \,.\, I(v \otimes w) = \sum_{i=1}^{4} \frac{i^4}{4} Q_I(v + i^k w)$$

Proof $=$

$\dots$

$\square$

$$\texttt{PreinnerProductIsSeminorm} :: \forall H : \texttt{PrehilbertSpace}(K) \,.\, (\Lambda x \in H \,.\, \sqrt{\langle x, x \rangle}) : \texttt{Seminorm}(X)$$

Proof $=$

$\dots$

$\square$

$$\texttt{InnerProductIsNorm} :: \forall H : \texttt{InnerProductSpace}(K) \,.\, (\Lambda x \in H \,.\, \sqrt{\langle x, x \rangle}) : \texttt{Norm}(X)$$

Proof $=$

$\dots$

$\square$

$\texttt{asSeminormed} :: \texttt{PrehilbertSpace}(K) \to \texttt{SeminormedSpace}(K)$
$\texttt{asSeminormed}\,(H, I) = \texttt{implicit}(H, I) := (H, \texttt{PreinnerProductIsSeminorm}(H, I))$

$\texttt{asNormed} :: \texttt{InnerProductSpace}(K) \to \texttt{NormedSpace}(K)$
$\texttt{asNormed}\,(H, I) = \texttt{implicit}(H, I) := (H, \texttt{InnerProductIsNorm}(H, I))$

$\texttt{InnerProductIsCont} :: \forall H : \texttt{PrehilbertSpace}(K) \,.\, (\Lambda(x, y) \in H \times H \,.\, \langle x, y \rangle) : H \times H \to_{\mathsf{TOP}} K$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{PrehilbertCharacterisation} :: \forall V : \texttt{PrehilbertSpace}(K) \,.$
$\quad .\, \forall x, y \in V \,.\, \|x + y\| = \|x\| + \|y\| \Rightarrow \{x, y\} : \texttt{LinearlyIndependant}(V)$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{ParallelagramLaw} :: ?\texttt{SeminormedSpace}(K)$
$V : \texttt{ParallelagramLaw} \iff \forall x, y \in V \,.\, \|x + y\| + \|x - y\| = 2\|x\| + 2\|y\|$

$\texttt{ParallelagramTHM} :: \forall H : \texttt{SeminormedSpace}(K) \,.\, H : \texttt{ParallelagramLaw} \iff H : \texttt{PrehilbertSpace}$
$\texttt{Proof} =$
$\ldots$
$\square$

2.2 Geometric Properties: Orthogonality and Projections

$H : \texttt{InnerProductSpace}(\mathbb{C})$

$\texttt{Orthogonal} :: ?H \times H$

$(v, w) : \texttt{Orthogonal} \iff v \perp w \iff \langle v, w \rangle$

$\texttt{OrthogonalSystem} :: ??H$

$S : \texttt{OrthogonalSystem} \iff \perp(S) \iff \forall v, w \in S : v \neq w \;.\; v \perp w$

$\texttt{Orthonormal} :: ?\texttt{OrthogonalSystem}(H)$

$S : \texttt{Orthonormal} \iff \forall v \in S \;.\; \|v\| = 1$

$\texttt{OrthogonalIsLInd} :: \forall \perp(S) \;.\; S : \texttt{LinearlyIndependant}(H)$

$\texttt{Proof} =$

...

$\square$

$$\texttt{Pythagorus} :: \forall n \in \mathbb{N} \;.\; \forall e : n \to H : \perp(\operatorname{Im} e) \;.\; \left\| \sum_{i=1}^{n} e_i \right\|^2 = \sum_{i=1}^{n} \|e_i\|^2$$

$\texttt{Proof} =$

...

$\square$

$\texttt{OrthogonalBound} :: H : \texttt{Separable} \Rightarrow \forall \perp(S) \;.\; |S| \leq \aleph_0$

$\texttt{Proof} =$

...

$\square$

$\texttt{GramSchmidtProcess} :: \forall S : \texttt{LinearlyIndependant}(H) \;.\; \exists E : \texttt{Orthonormal}(H) \;.\; \operatorname{span} S = \operatorname{span} E$

$\texttt{Proof} =$

...

$\square$

$\texttt{furieCoifficients} :: \prod E : \texttt{Orthonormal}(H) \;.\; H \to E \to K$

$\texttt{furieCoifficients}\,(v, e) = c_e(v) := \langle v, e \rangle$

$\texttt{furieSerias} :: \texttt{Orthonormal}(H) \to H \to H$

$$\texttt{furieSerias}\,(E, v) := \sum_{e \in E} c_e(v)$$

$\texttt{ProjectionTHM} :: \forall n \in \mathbb{N} \,.\, \forall e : n \to H : \perp(\operatorname{Im} e) \,.\, \forall v \in H \,.\, d\,(v, \texttt{FurieSerias}(e, v)) = \inf_{w \in \texttt{span}(e)} d(v, w)$

$\texttt{Proof} \; =$

...

$\square$

$\texttt{BesselIneq} :: \forall x \in H \,.\, \forall E : \texttt{Orthonormal}(H) \,.\, \sum_{e \in E} |\langle v, e\rangle|^2 \leq \|v\|^2$

$\texttt{Proof} \; =$

...

$\square$

$\texttt{Total} :: \prod V : \texttt{SeminormedSpace} \,.\, ??V$

$E : \texttt{Total} \iff \texttt{span}(E) : \texttt{Dense}(V)$

$\texttt{TotalExists} :: \forall H : \texttt{InnerProductSpace}(K) \;\&\; \texttt{Separable} \,.\, \exists \texttt{Orthonormal} \;\&\; \texttt{Total}(V)$

$\texttt{Proof} \; =$

...

$\square$

$\texttt{FurieSpaceTheorem} :: \forall E : \texttt{Orthonormal} \;\&\; \texttt{Total}(V) \,.\, \forall v \in V \,.\, \texttt{furieSerias}(E, v) = v$

$\texttt{Proof} \; =$

...

$\square$

$\texttt{Shauder} :: \prod E : \texttt{SeminormedSpace}(K) \,.\, ?\mathbb{N} \to H$

$e : \texttt{Shauder} \iff \forall x \in E \,.\, \exists! a : \mathbb{N} \to K \,.\, \sum_{n=1}^{\infty} a_n e_n = v$

$\texttt{ShauderExists} :: \forall H : \texttt{InnerProductSpace} \;\&\; \texttt{Separable} \,.\, \exists \texttt{Shauder}(H)$

$\texttt{Proof} \; =$

...

$\square$

3 Banach and Hilbert Spaces

3.1 Definition

$\texttt{Banach} := \texttt{NormedSpace} \,\&\, \texttt{Complete}$

$\texttt{Hilbert} := \texttt{InnerProductSpace} \,\&\, \texttt{Complete}$

$\texttt{IsoBanach} :: \forall V : \texttt{Banach} \,.\, \forall W : \texttt{NormedSpace} : W \cong_{\mathsf{NORM}} V \,.\, W : \texttt{Banach}$

$\texttt{Proof} \;=$

$T := \eth W \cong_{\mathsf{NORM}} V : W \leftrightarrow_{\mathsf{NORM}} V,$

$C := \texttt{TopIsoChar}(T) : \mathbb{R}_+ : \forall x \in W \,.\, \|Tx\| \le Cx,$

$\texttt{Assume}\ x : \texttt{Cauchy}(W),$

$\texttt{Assume}\ k : \mathbb{N},$

$() := \eth_1 \mathcal{B}(W, V)(T)(\lim_{n\to\infty} \|T(x_n) - T(x_{n+k})\|)\eth C \eth \texttt{Cauchy}(W)(x) :$

$\quad : \lim_{n\to\infty} \|T(x_n) - T(x_{n+k})\| = \lim_{n\to\infty} \|T(x_n - x_{n+k})\| \le \lim_{n\to\infty} C\|x_n - x_{n+k}\| = 0;$

$\rightsquigarrow () := \texttt{UniIntro} : \forall k \in \mathbb{N} \,.\, \lim_{n\to\infty} \|T(x_n) - T(x_{n+k}) = 0\|),$

$() := \eth^{-1}\texttt{Cauchy}(V) : (T\,x : \texttt{Cauchy}(V)),$

$Y := \lim_{n\to\infty} T\,x_n : V,$

$X := T^{-1}\,Y : W,$

$(2) := \texttt{MultUnity}(\lim_{n\to\infty} x_n, \lim_{n\to infty} T^{-1}Tx_n)(\eth T)\texttt{ContLimit}(T^{-1})\eth Y \eth X :$

$\quad : \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^{-1}Tx_n = T^{-1} \lim_{n\to\infty} Tx_n = T^{-1}Y = X,$

$() := \eth^{-1}\texttt{Convergent}(W)(x)(2) : (x : \texttt{Convergrnt}(x));$

$\rightsquigarrow (*) := \eth\texttt{Complete}(W) : (W : \texttt{Banach}),$

$\square$

$\texttt{BanachSubspaceIsClosed} :: \forall V : \texttt{NormedSpace} \,.\, \forall S \subset_{\mathsf{NORM}} V : \texttt{Banach} \,.\, S : \texttt{Closed}(V)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{BanachSubspaceIsClosed} :: \forall V : \texttt{Banach} \,.\, \forall S \subset_{\mathsf{NORM}} V : \texttt{Closed}(V) \,.\, S : \texttt{Banach}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.2 Finite-Dimensional Spaces

$\texttt{FiniteDimensionalClassification} :: \forall K : \texttt{AVField} \,\&\, \texttt{Complete} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : \texttt{NormedSpace}(K) :$
$\quad : \dim V = n \,.\, V \cong_{\text{NORM}} K_1^n$

$\texttt{Proof} \ =$

$\texttt{Assume}\ n : \mathbb{N},$

$\ddagger(n) := \forall m \in \mathbb{N} : m \leq n \,.\, \forall V : \texttt{NormedSpace}(K) : \dim V = m \,.\, V \cong_{\text{NORM}} K_1^m : \texttt{Type};$

$\rightsquigarrow \ddagger := \texttt{FuncIntro} : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ V : \texttt{NormedSpace}(K) : \dim v = 1,$

$v := \eth \dim \eth V : \texttt{In}(V) : V = Kv,$

$\texttt{Assume}\ k : K,$

$Tk := kv : \texttt{In}(V);$

$() := \eth T \eth \texttt{Norm} : \|Tk\| = \|kT1\| = |k|\|1\|;$

$\rightsquigarrow T := \texttt{FuncIntro} : \mathcal{B}(K, V),$

$(1) := \eth v \eth T : (T : K \leftrightarrow_{\text{NORM}} V),$

$() := \eth \cong_{\text{NORM}} (1) : K_1 \cong_{\text{NORM}} V;$

$\rightsquigarrow (1) := \eth\ddagger : \ddagger(1),$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ H : \ddagger(m),$

$\texttt{Assume}\ V : \texttt{NormedSpace}(K) : \dim V = m + 1,$

$$v := \eth \dim \eth V : m + 1 \to V : V = \sum_{i=1}^{m+1} K v_i = V : \texttt{LinearlyIndependend}(V),$$

$\texttt{Assume}\ i : m + 1,$

$S := \operatorname{span}\{v_j | j \in m + 1 : j \neq i\} : \texttt{Subspace}(V),$

$T := H(S) : S \leftrightarrow K_1^m,$

$(2) := \texttt{IsoBanach}(S, K_1^m, T) : (S : \texttt{Banach}),$

$(3) := \texttt{BanachSubspaceIsClosed}(V, (S, 2)) : (S : \texttt{Closed}(V)),$

$$C_i := \texttt{ClosedSubspaceIsGeometric}(V, (S, 3)) : \mathbb{R}_+ : \forall \sum_{j=1}^{m+1} x_j v_j \in V \,.\, |x_i| \leq C_i \left\| \sum_{j=1}^{m+1} x_j v_j \right\|;$$

$$\rightsquigarrow C := \texttt{FuncIntro} : m + 1 \to \mathbb{R}_+ : \forall i \in m + 1 \,.\, \forall \sum_{j=1}^{m+1} x_j v_j \in V \,.\, |x_i| \leq C_i \left\| \sum_{j=1}^{m+1} x_j v_j \right\|,$$

$$c := \sum_{i=1}^{m+1} C_i : \mathbb{R}_+,$$

$$T := \Lambda \sum_{j=1}^{m+1} x_j v_j \in V \,.\, (x_i)_{i=1}^n : V \to K^{m+1},$$

$$\texttt{Assume} \sum_{j=1}^{n} x_j v_j : \texttt{In}(V),$$

$() := \eth T \eth c : \left\| T \sum_{j=1}^{m+1} x_j v_j \right\| = \sum_{i=1}^{m+1} |x_i| \le c \left\| \sum_{j=1}^{m+1} x_j v_j \right\| ;$

$\rightsquigarrow (2) := \eth^{-1} \mathcal{B}(V, K_1^{m+1}) : (T : \mathcal{B}(B, K^{m+1})),$

$b := \sum_{i=1}^{m+1} \|v_i\| : \mathbb{R}_+,$

$\texttt{Assume}\ x : K_1^{m+1},$

$() := \eth T \eth \texttt{Norm}\,\texttt{Majorize}(\|v_i\|, b) \eth \| \cdot \|_1 : \|T^{-1}x\| = \left\| \sum_{i=1}^{m+1} x_i v_i \right\| \le \sum_{i=1}^{m+1} |x_i| \|v_i\| \le \sum_{i=1}^{m+1} |x_i| \sum_{j=1}^{m+1} \|v_j\| = b\|x\|;$

$\rightsquigarrow (4) := \eth^{-1} V \leftrightarrow_{\mathsf{NORM}} K_1^{m+1}(2) : (T : V \leftrightarrow_{\mathsf{NORM}} K_1^{m+1}),$

$() := \eth^{-1} \cong_{\mathsf{NORM}} (4) : V \cong_{\mathsf{NORM}} K_1^{m+1};$

$\rightsquigarrow (2) := \texttt{UniIntro} : \forall V : \texttt{NormedSpace}(K) : \dim V = m+1 \,.\, V \cong_{\mathsf{NORM}} K_1^{m+1},$

$H^+ := \texttt{UniUpdate}(H, 2) : \ddagger(m+1);$

$\rightsquigarrow (*) := \texttt{Induction}(m+1) : \forall n \in \mathbb{N} \,.\, \ddagger(n),$

$\square$

$K :: \texttt{AVField} \ \& \ \texttt{Complete}$

$\texttt{FinDimIsBanach} :: \forall V : \texttt{NormedSpace}(K) : \dim V < \infty \,.\, V : \texttt{Banach}$

$\texttt{Proof} \ =$

$n := \dim V : \mathbb{N},$

$(1) := \texttt{FiniteDimensionalClassification}(V, n) : V \cong_{\mathsf{NORM}} K_1^n,$

$() := \texttt{IsoBanach}(1) : (V : \texttt{Banach});$

$\square$

$\texttt{FinDimMajorization} :: \forall V : \texttt{NormedSpace}(K) : \dim V < \infty \,.\, \forall s : \texttt{seminorm}(V) \,.\, s < \| \cdot \|_V$

$\texttt{Proof} \ =$

$n := \dim V : \mathbb{N},$

$(1) := \texttt{FiniteDimensionalClassification}(V, n) : V \cong_{\mathsf{NORM}} K_1^n,$

$(c, C) := \eth V \cong_{\mathsf{NORM}} K_1^n : \mathbb{R}_+ \times \mathbb{R}_+ : \forall x \in V \,.\, c\|x\|_1 \le \|x\| \le C\|x\|_1,$

$v := \eth \dim(V, n)(1) : n \to V : V = \operatorname{span}(v) : \forall i \in n \,.\, \|v_i\|_1 = 1,$

$H := \max_{1 \le i \le n} s(v_i) : \mathbb{R}_+,$

$\texttt{Assume}\ x : V,$

$a := \eth(v, x) : K^n : x = \sum_{i=1}^{n} a_i v,$

$() := \eth \texttt{Seminorm}(V)(s)(x, \eth a) \eth H \eth \| \cdot \|_1 \eth C : s(x) \le \sum_{i=1}^{n} |a_i| s(v_i) \le H\|x\|_1 \le CH\|x\|;$

$\rightsquigarrow () := \eth^{-1} s \le \| \cdot \|_V : (s \le \| \cdot \|_V);$

$\square$

$\texttt{FinDimOperator} :: \forall V : \texttt{NormedSpace}(K) : \dim V < \infty \ . \ \forall W : \texttt{NormedSpace}(K) \ .$
$\quad . \ \forall T : \mathcal{L}(V, W) \ . \ T : \mathcal{B}(V, W)$

$\texttt{Proof} \ =$

$n := \dim V : \mathbb{N},$

$(1) := \texttt{FiniteDimensionalClassification}(V, n) : V \cong_{\mathsf{NORM}} K^n_1,$

$(c, C) := \eth V \cong_{\mathsf{NORM}} K^n_1 : \mathbb{R}_+ \times \mathbb{R}_+ : \forall x \in V \ . \ c\|x\|_1 \leq \|x\| \leq C\|x\|_1,$

$v := \eth \dim(V, n)(1) : n \to V : V = \operatorname{span}(v) : \forall i \in n \ . \ \|v_i\|_1 = 1,$

$H := \max_{1 \leq i \leq n} \|Tv_i\| : \mathbb{R}_+,$

$\texttt{Assume} \ x : V,$

$a := \eth(v, x) : K^n : x = \sum_{i=1}^n a_i v,$

$() := \eth \mathcal{L}(T)(x, \eth a)\texttt{Seminorm}(W)\eth H \eth \| \cdot \|_1 \eth C : \|Tx\| \leq \sum_{i=1}^n |a_i| \|Tv_i\| \leq H\|v\|_1 \leq HC\|v\|;$

$\rightsquigarrow () := \eth^{-1} s \leq \| \cdot \|_V : (s \leq \| \cdot \|_V);$

$\square$

$\texttt{FinDimTopCompletable} :: \forall V : \texttt{NormedSpace}(K) \ .$
$\quad . \ \forall S : \texttt{Sub}(\mathsf{NORM}(K), V) : \dim S < \infty \ . \ S : \texttt{TopologicalyCompletable}(V)$

$\texttt{Proof} \ =$

$n := \dim S : \mathbb{N},$

$R := \texttt{FinDimComplement}(V, S) : \texttt{Sub}(\mathsf{NORM}(K), V) : V \cong_{\mathsf{VS}(K)} S \oplus R : (R : \texttt{Closed}(V)),$

$W := \left(\frac{V}{R}\right)_{\mathsf{NORM}} : \texttt{NormedSpace}(K),$

$(1) := \texttt{QuetientDim}(\eth W, \eth R) : \dim W = n,$

$T := \pi_{R|S} : \mathcal{B}(S, R),$

$(2) := \texttt{EqDimIso}(S, W, T, 1, \eth T) : (T : S \leftrightarrow_{\mathsf{VS}(K)} W),$

$(3) := \texttt{FinDimOperator}^2(S, W, T)(2)(W, S, T^{-1}) : (T : S \leftrightarrow_{\mathsf{NORM}(K)} W),$

$P := T^{-1} \circ \pi_R : V \to_{\mathsf{NORM}} S,$

$\texttt{Assume} \ x : S,$

$() := \eth P \texttt{AdHocContraction}(S, x)\eth T(x) : Tx = \pi^{-1}_{R|S}\pi_{R|S}\,x = x;$

$\rightsquigarrow (4) := \texttt{UniIntro} : \forall x \in S \ . \ Px = x,$

$\texttt{Assume} \ x : R,$

$() := \eth P \eth \texttt{quatProjection}(R, S) : Px = \pi^{-1}_{R|S}\pi_R\,x = \pi^{-1}_{R|S} 0 = 0,$

$\rightsquigarrow (5) := \texttt{UniIntro} : \forall x \in R \ . \ Px = 0,$

$() := \eth^{-1}\texttt{ProjectionOnAlong}(S, R)(4, 5) : (P : \texttt{ProjectionOnAlong}(S, R)),$

$(7) := \texttt{ProjectionTopComplement}(P) : V =_{\mathsf{NORM}(K)} S \oplus R,$

$() := \eth^{-1}\texttt{TopologicalyCompletable}(V)(7) : (S : \texttt{TopologicalyCompletable}(V));$

$\square$

3.3 Banach Space of Bounded Operators

$\texttt{BanachOperators} :: \forall V : \texttt{Banach}(K) \, . \, \forall W : \texttt{SeminormedSpace}(K) \, . \, \mathcal{B}(W,V) : \texttt{Banach}(K)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ T : \texttt{Cauchy}\, \mathcal{B}(W,V),$

$\texttt{Assume}\ x : W,$

$(1) := \eth^{-1}\texttt{Cauchy}\eth\texttt{OperatorNorm}\eth\texttt{Cauchy}(T) : (Tx : \texttt{Cauchy}\, V),$

$(2) := \eth\texttt{Banach}(V)(Tx) : Tx : \texttt{Convergent}(V),$

$Ax := \lim_{n\to\infty} T_n x : V;$

$\rightsquigarrow A := \texttt{FuncIntro} : \mathcal{L}(W,V),$

$N := \eth\texttt{Cauchy}(T)(1) : \mathbb{N} : \forall n,m \in \mathbb{N} : (n,m) \ge (N,N) \, . \, \|T_n - T_m\| \le 1,$

$C := \|T_N\| : \mathbb{R}^+,$

$\texttt{Assume}\ x : W : \|x\| = 1,$

$() := \eth\texttt{Norm}(V)((A - T_N)x, T_N x)\eth N \eth C : \|A\,x\| \le \|(A - T_N)x\| + \|T_N x\| \le 1 + C;$

$\rightsquigarrow (1) := \eth^{-1}\mathcal{B}(W,V) : (T : \mathcal{B}(W,V)),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_+,$

$N := \eth\texttt{Cauchy}(T)(\varepsilon/4) : \mathbb{N} : \forall n,m \in \mathbb{N} : (n,m) \ge (N,N) \, . \, \|T_n - T_m\| \le \frac{\varepsilon}{4},$

$\texttt{Assume}\ n : \mathbb{N} : n \ge N,$

$\texttt{Assume}\ x : W : \|x\| = 1,$

$m := \max(\eth T(x), N) : \mathbb{N} : \forall k \in \mathbb{N} : k \ge m \, . \, \|T_m x - Tx\| \le \frac{\varepsilon}{4},$

$() := \eth\texttt{Norm}(Tx - T_m x, T_m x - T_n x)\eth n \eth m \eth N \texttt{PositiveUpperBound}\left(\frac{\varepsilon}{4}, 2\right) :$

$: \|Tx - T_n x\| \le \|Tx - T_m x\| + \|T_m x - T_n x\| \le \frac{\varepsilon}{2} < \varepsilon;$

$\rightsquigarrow () := \eth\texttt{operatorNorm}\eth \sup(\Lambda x \in \mathbb{S}_W \, . \, \|Tx - T_n x\|) : \|T - T_n\| \le \varepsilon;$

$; \rightsquigarrow () := \eth^{-1} \lim(\mathcal{B}(W,B)) : \lim_{n\to\infty} T_n = T;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Banach} : \mathcal{B}(W,V) : \texttt{Banach};$

$\square$

$\texttt{ReflexiveIsBanach} :: \forall V : \texttt{Reflexive} \,\&\, \texttt{NormedSpace}(K) \, . \, V : \texttt{Banach}(K)$

$\texttt{Proof} \ =$

$(1) := \texttt{BanachOperators}(K, V^*) : V^{**} : \texttt{Banach},$

$(2) := \eth\texttt{Reflexive}(V) : V \cong_{\mathsf{NORM}} V^{**},$

$(*) := \texttt{IsoBanach}(1,2) : V : \texttt{Banach};$

$\square$

3.4 Absolutely Convergent Series

$\texttt{AbsolutelyConvergent} :: \prod V : \texttt{Banach}(K) \,.\, ?\mathbb{N} \to V$

$$v : \texttt{AbsolutelyConvergent} \iff \sum_{n=1}^{\infty} \|v_n\| < \infty$$

$\texttt{AbsoluteConvergenceTHM} :: \forall V : \texttt{Banach}(K) \,.\, \forall v : \texttt{AbsolutelyConvergent}(V) \,.\, v : \texttt{ConvergentSeria}(V)$

$\texttt{Proof} =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_+,$

$$N := \eth\texttt{ConvergentSeria}(\eth\texttt{AbsolutelyConvergent}(V)v)(\varepsilon) : \mathbb{N} : \forall n \in \mathbb{N} : n \geq N \,.\, \sum_{k=n}^{\infty} \|v_k\| \leq \varepsilon,$$

$\texttt{Assume}\ n : \mathbb{N} : n \geq N,$

$$() := \eth_1\texttt{Norm}(\{v_n\}_{k=n}^{\infty})\eth(N,n) : \left\|\sum_{k=n}^{\infty} v_k\right\| \leq \sum_{k=n}^{\infty} \|v_k\| \leq \varepsilon;$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{ConvergentSeria}(V) : (v : \texttt{ConvergentSeria}(V));$

$\square$

$\texttt{AbsCIsBanach} :: \forall V : \texttt{NormedSpace}(K) : \forall v : \texttt{AbsolutelyConvergent}(V) \,.\, v : \texttt{ConvergentSeria}(V) \,.$
$\quad .\, V : \texttt{Banach}(K)$

$\texttt{Proof} =$

$\texttt{Assume}\ v : \texttt{Cauchy}(V),$

$\texttt{Assume}\ n : \mathbb{N},$

$$N := \eth\texttt{Cauchy}(V)(v)\left(\frac{1}{2^n}\right) : \mathbb{N} : \forall k, l \in \mathbb{N} : (k,l) \geq (N,N) \,.\, \|v_k - v_l\| \leq \frac{1}{2_n},$$

$u_n := v_N : V;$

$$\rightsquigarrow u := \texttt{FuncIntro} : \mathbb{N} \to V : \forall n \in \mathbb{N} \,.\, \|u_n - u_{n-1}\| \leq \frac{1}{2^n},$$

$x := \Lambda n \in \mathbb{N} \,.\, \texttt{if}\ n == 1\ \texttt{then}\ u_1\ \texttt{else}\ u_n - u_{n-1} : \mathbb{N} \to V,$

$(1) := \eth V(\eth x \eth u) : (x : \texttt{ConvergentSeria}(V)),$

$$(2) := \eth x(1) : \sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} u_n,$$

$$() := \texttt{CauchySubseq}(2) : \lim_{n\to\infty} v_n = \sum_{n=1}^{\infty} x_n;$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Banach} : (V : \texttt{Banach}(K));$

$\square$

$\texttt{BanachQuetientBanach} :: \forall V : \texttt{Banach}(K) \,.\, \forall S : \texttt{Closed}(V) : S \subset_{\mathsf{NORM}} V \,.\, \left(\frac{V}{S}\right)_{\mathsf{NORM}} : \texttt{Banach}(K)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ [x] : \texttt{AbsolutelyConvergent} \left(\frac{V}{S}\right)_{\mathsf{NORM}},$

$x := \eth\texttt{normedQuetient}(V)(S)([x])(2) : \mathbb{N} \to V : \forall n \in \mathbb{N} \,.\, \|x_n\| \leq 2\big\|[x]\big\|,$

$(1) := \eth x \eth\texttt{AbsolutelyConvergent}([x]) : \sum_{n=1}^{\infty} \|x_i\| \leq 2 \sum_{n=1}^{\infty} \|[x_i]\| < \infty,$

$(2) := \eth^{-1}\texttt{AbsolutelyConvergent}(1) : (x : \texttt{AbsolutelyConvergent}(V)),$

$(3) := \texttt{AbsoluteConvergenceTHM}(V) : (x : \texttt{ConvergentSeria}(V)),$

$X := \sum_{n=1}^{\infty} x_n : V,$

$(4) := \eth\texttt{quetientNorm} : \left\| \sum_{n=1}^{\infty} [x] - [X] \right\| \leq \left\| \sum_{n=1}^{\infty} x - X \right\| = 0,$

$() := \eth^{-1}\texttt{ConvergentSeria}(4) : \left([x] : \texttt{ConvergentSeria} \left(\frac{V}{S}\right)_{\mathsf{NORM}} \right);$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall [x] : \texttt{AbsolutelyConvergent} \,.\, [x] : \texttt{ConvergentSeria},$

$() := \texttt{AbsIsBanach}(1) : \left(\frac{V}{S}\right)_{\mathsf{NORM}} : \texttt{Banach}(K);$

$\square$

3.5 Continuous Extension of Operators

$\texttt{ContinuousExtension} :: \forall V : \texttt{SeminormedSpace}(K) \,.\, \forall S : \texttt{Dense}(V) : S \subset_{\mathsf{NORM}} V \,.\, \forall W : \texttt{Banach}(K) \,.$
$\quad . \forall T : \mathcal{B}(S, W) \,.\, \exists! A : \mathcal{B}(V, W) : A_{|S} = T$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : v,$

$v := \eth\texttt{Dense}(V)(S) : \mathbb{N} \to S : \lim_{n\to\infty} v_n = x,$

$() := \texttt{ConvergentIsCauchy}(v) : (v : \texttt{Cauchy}(V)),$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_+,$

$N := \eth\texttt{Cauchy}(V)(v)(\varepsilon) : \mathbb{N} : \forall n, m \in \mathbb{N} : (n, m) \geq (N, N) \,.\, \|v_n - v_m\| \leq \varepsilon/\|T\|,$

$\texttt{Assume}\; n, m : \mathbb{N} : (n, m) \geq (N, N),$

$() := \eth\mathcal{B}(S, W)(T)(v_n - v_m)\eth N(\eth(n, m)) : \|Tv_n - Tv_m\| \leq \|T\|\|v_n - v_m\| \leq \varepsilon;$

$\rightsquigarrow () := \eth^{-1}\texttt{Cauchy} : (Tv : \texttt{Cauchy}(W)),$

$() := \eth\texttt{Banach}(W)(Tv) : (Tv : \texttt{Convergent}),$

$Ax := \lim_{n\to\infty} Tv_n : W,$

$\texttt{Assume}\; w : \mathbb{N} \to S : \lim_{n\to\infty} w_n = x,$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_+,$

$N := \eth \lim(v)\left(\frac{\varepsilon}{2\|T\|}\right) : \mathbb{N} : \forall n \in \mathbb{N} : n \geq N \,.\, \|v_n - x\| \leq \frac{\varepsilon}{2\|T\|},$

$N' := \eth \lim(w)\left(\frac{\varepsilon}{2\|T\|}\right) : \mathbb{N} : \forall n \in \mathbb{N} : n \geq N' \,.\, \|w_n - Ax\| \leq \frac{\varepsilon}{2\|T\|},$

$M := \max(N, N') : \mathbb{N},$

$\texttt{Assume}\; n : \mathbb{N} : n \geq M,$

$() := \ldots : \|Tv_n - Tw_n\| \leq \|T\|\|v_n - w_n\| \leq T\|v_n - x\| + T\|x - w_n\| \leq \varepsilon;$

$\rightsquigarrow () := \texttt{LimitsAgree} : \lim_{n\to\infty} Tw_n = Ax;;$

$\rightsquigarrow A := \texttt{FuncClassIntro} : \mathcal{B}(V, W) : A_{|S} = T,$

$\texttt{Assume}\; B : \mathcal{B}(V, W) : B_{|S} = T,$

$() := \texttt{DenceContEq}(A, B) : A = B;$

$\rightsquigarrow (*) := \texttt{UniqueIntro} : (\Lambda A : \mathcal{B}(V, W) \,.\, A_{|S} = T, A) : \texttt{Unique},$

$\square$

$\texttt{denseExtension} :: \prod V : \texttt{SeminormedSpace}(K) \,.\, \prod S : \texttt{Dense}(V) : S \subset_{\mathsf{NORM}} V \,.\, \prod W : \texttt{Banach}(K) \,.$
$\quad . \mathcal{B}(S, W) \to \mathcal{B}(V, W)$

$\texttt{denseExtension}\,(T) := \texttt{ContinuousExtension}(T)$

IsometryExtension :: $\forall V$: SeminormedSpace(K) . $\forall S$: Dense(V) : $S \subset_{\text{NORM}} V$. $\forall W$: Banach(K) .

. $\forall T$: NonExpanding(S, W) . denseExtension(T) : NonExpanding(V, W)

Proof =

$\|Ax\| = \lim_{n\to\infty} \|Tv_n\| \le \lim_{n\to\infty} \|T\|\|v_n\| = \|T\|\|x\| \le \|x\|$

$\square$

TopInjExtension :: $\forall V$: SeminormedSpace(K) . $\forall S$: Dense(V) : $S \subset_{\text{NORM}} V$. $\forall W$: Banach(K) .

. $\forall T$: TopologicalInjection(S, W) . denseExtension(T) : TopologicalInjection(V, W)

Proof =

$\|Ax\| = \lim_{n\to\infty} \|Tv_n\| \ge \lim_{n\to\infty} c\|v_n\| = c\|x\|$

$\square$

SepRealHahnBanach :: $\forall V$: Banach$(\mathbb{R})$ & Separable . $\forall A$: Subspace(V) .

. $\forall f \in A^*$. $\exists F \in V^*$: $F_{|A} = f$ & $\|F\| = \|f\|$

Proof =

Assume $(1) : f = 0$,

$F := 0$: In(V^*),

$(2) := \eth F_{|A}(1) : F_{|A} = 0 = f$;

$(3) := \eth F \eth f : \|F\| = \|0\| = \|f\|$;

$\rightsquigarrow (1) :=$ ImplicationInto : $f = 0 \Rightarrow$ RealHahnBanach,

Assume $(2) : f \ne 0$,

$g := \dfrac{f}{\|f\|} : A^* : \|g\| = 1$,

HahnBanachLemma :: $\operatorname{codim}_V A = 1 \Rightarrow$ RealHahnBanach

Proof =

$() := \eth \operatorname{codim}_V A = 1 : A^{\complement}$: NonEmpty,

$x := \eth$NonEmpty$(A^{\complement}) : A^{\complement}$,

Assume $a, b : A$,

$(3) := \eth$abs$(g(a-b))\eth$operatorNorm$(g)(a-b)$AddSubstract$(a-b, x)\eth_2\| \cdot \|((x+a), (x-a))$:

: $g(a-b) \le |g(a-b)| \le \|a-b\| \le \|(x+a)-(x-b)\| \le \|x+a\| + \|x+b\|$,

$(4) :=$ SumIneq$(3, g(a), -g(b), \|x+a\|, \|x+b\|) : -g(b) - \|x+b\| \le \|x+a\| - g(a)$,

$X_b := -g(b) - \|x+b\| : \mathbb{R}$;

$Y_a := \|x+a\| - g(a) : \mathbb{R}$;

$\rightsquigarrow (X, Y)) :=$ FuncIntro : $A \times A \to \mathbb{R} \times \mathbb{R} : \forall (a, b) \in A \times A . X_b \le Y_a$,

$C_x := \inf_{a\in A} Y_a : \mathbb{R}$,

$c_x := \sup_{a\in A} X_a : \mathbb{R}$,

$(3) := \eth(X, Y) : c_x \le C_x$,

$r :=$ IntermidiateReal$(c_x, C_x) : \mathbb{R} : c_x \le r \le C_x$,

$(4) := \eth(X, Y, r) : \forall a \in A . |r + g(a)| \le \|x+a\|$,

$\texttt{Assume}\ v : V,$

$(a,s) := \eth\text{codim}_V A = 1(v,x) : A \times \mathbb{R} : sx + av = sx + a,$

$G(v) := g(a) + sr : \mathbb{R};$

$\texttt{Assume}\ O : v \in A,$

$(5) := \eth(s,a)O : v = a,$

$(6) := \texttt{EqEl}(|G(v)|, \eth F, (5))\eth_2 g \eth a : |G(v)| = |g(a)| \le \|a\| = \|v\|;$

$\rightsquigarrow (5) := \texttt{ImplyIntro} : v \in A \Rightarrow |G(v)| \le \|v\|,$

$\texttt{Assume}\ O : v \notin A,$

$(5) := \eth(s,a)O : s \ne 0,$

$(6) := \texttt{EqEl}(|G(v)|, \eth F,)\eth\texttt{AbsVal}(\mathbb{R})(sc + g(a), s)\eth_2 \mathcal{L}(A,K)(g)(s^{-1}, a)(4)\left(\frac{a}{s}\right)$

$\quad \eth_2^{-1}\texttt{Norm}(V)(|s|, x + s^{-1}a)\eth^{-1}(a,s) :$

$\quad : |G(v)| = |sr + g(a)| = |s|\left|r + \frac{g(a)}{s}\right| = |s|\left|r + g\left(\frac{a}{s}\right)\right| \le |s|\left\|x + \frac{a}{s}\right\| = \|sx + a\| = \|v\|;$

$\rightsquigarrow (6) := \texttt{ImplyIntro} : v \notin A \Rightarrow |G(v)| \le \|v\|,$

$(7) := \texttt{OrEl}(v \in A | v \notin A)(5,6) : |G(v)| \le \|v\|;$

$\rightsquigarrow G := \texttt{FuncIntro} : V^* : \|G\| \le 1\ \&\ G_{|A} = g,$

$(5) := \eth_2 G \eth g : \|G\| \ge \|g\| = 1,$

$(6) := \texttt{TwofoldIneq}\eth_1 G(5) : \|g\| = 1,$

$F := \|f\| G : V^* : \|F\| = \|f\|\ \&\ F_{|A} = f,$

$(*) := \eth\texttt{RealHahnBanach}(F) : \texttt{RealHahnBanach};$

$D := \eth\texttt{Separable}(V) : \texttt{Dense}(V),$

$W := \text{span}(D) : \texttt{Sub}(\mathsf{NORM}, V),$

$(n,e) := \eth W \eth S : \big(\texttt{Cardinal} : n \le \aleph_0\big) \times \big(n \to W : \texttt{Basis}(W)\big),$

$h_0 := g : \mathcal{B}(S,K),$

$U_0 := S : \texttt{Sub}(\mathsf{NORM}, V),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ A : e_n \in U_{n-1},$

$U_n := U_{n-1} : \texttt{Sub}(\mathsf{NORM}, V),$

$h_n := h_{n-1} : \mathcal{B}(U_n, K),$

$\texttt{Assume}\ A : e_n \notin U_{n-1},$

$U_n := U_{n-1} + \text{span}\, e_n : \texttt{Sub}(\mathsf{NORM}, V),$

$h_n := \texttt{HahnBanachLemma}(U_n, U_{n-1}, h_{n-1}) : \mathcal{B}(U_n, K);$

$\rightsquigarrow (U,h) := \texttt{RecursiveFunc} : \forall n \in \mathbb{N}\ .\ \sum U_n : \texttt{Sub}(\mathsf{NORM}, V)\ .\ \mathcal{B}(U_n, K),$

$\texttt{Assume}\ v : W,$

$$(m,k) := \eth\texttt{Basis}(W)(e)(v) : n \times m \to n : \exists a : m \to K\ .\ v = \sum_{i=1}^{m} a_i e_{k_i},$$

$Hv) := h_m(v) : K;$

$\rightsquigarrow H := \texttt{FuncIntro} : \mathcal{B}(W,K),$

$G := \texttt{ContinuousExtension}(W, H) : \mathcal{B}(V, K),$
$F := \|f\| G : \mathcal{B}(V, K) : \|F\| = \|f\| : F_{|S} = f,$
$\square$

$\texttt{ContinuousIsoExtension} :: \forall V, W : \texttt{Banach} \;.\; \forall S : \texttt{Sub}(\mathsf{NORM}, V) \;.\; \forall R : \texttt{Sub}(\mathsf{NORM}, W) \;.$
$\;.\; \forall T : S \leftrightarrow_{\mathsf{NORM}} T \;.\; \exists! A : V \leftrightarrow_{\mathsf{NORM}} W : A_{|S} = T$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{denseIsoExtension} :: \prod V : \texttt{Banach}(K) \;.\; \prod S : \texttt{Dense}(V) : S \subset_{\mathsf{NORM}} V \;.\; \prod W : \texttt{Banach}(K) \;.$
$\prod R : \texttt{Dense}(V) : R \subset_{\mathsf{NORM}} V \;.\; \mathcal{B}(S, W) \to \mathcal{B}(V, W)$
$\texttt{denseIsoExtension}\,(T) := \texttt{ContinuousIsoExtension}(T)$

$\texttt{ContinuousIsometryExtension} :: \forall V, W : \texttt{Banach} \;.\; \forall S : \texttt{Sub}(\mathsf{NORM}, V) \;.\; \forall R : \texttt{Sub}(\mathsf{NORM}, W) \;.$
$\;.\; \forall T : S \leftrightarrow_{\mathsf{NORM}_{\circ\to\cdot}} T \;.\; \exists! A : V \leftrightarrow_{\mathsf{NORM}_{\circ\to\cdot}} W : A_{|S} = T$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{ContinuousUnitaryExtension} :: \forall V, W : \texttt{Banach} \;.\; \forall S : \texttt{Sub}(\mathsf{NORM}, V) \;.\; \forall R : \texttt{Sub}(\mathsf{NORM}, W) \;.$
$\;.\; \forall T : \texttt{Unitary}(S, T) \;.\; \exists! A : \texttt{Unitary}(V, W) : A_{|S} = T$
$\texttt{Proof} \;=$
$\dots$
$\square$

3.6 Orthogonal Complements

$$x = \left(\frac{1}{n}\right)_{n=1}^{\infty} : l_2$$
$$e = ((\delta_{i,n})_{i=1}^{\infty})_{n=1}^{\infty} : \texttt{Schauder}(l_2)$$
$$Y = \operatorname{span}(e_2)_{n=2}^{\infty} \subset_{\mathsf{NORM}} l_2$$
$$y = \arg\min_{y \in l_2} \|x - y\| = (0) \oplus \left(\frac{1}{n}\right)_{n=2}^{\infty} : l_2$$
$$y \notin Y$$
$$\exists v : \mathbb{N} \to Y : \lim_{n\to\infty} v_n = y \Rightarrow \nexists y \in Y : y = \arg\min_{y\in Y} \|x - y\|$$

$\texttt{NearestVector} :: \forall H : \texttt{Hilbert}(K) \,.\, \forall v \in H \,.\, \forall X \subset_{\mathsf{NORM}} H \,.\, \exists! x \in X : x = \arg\min_{x\in X} \|x - v\|$

$\texttt{Proof} \ =$

$d := \inf_{x\in X} \|x - v\| : \mathbb{R}_+,$

$x := \eth \inf(\eth d) : \mathbb{N} \to X : \lim_{n\to\infty} \|x_n - v\| = d,$

$\texttt{Assume}\ m, n : \mathbb{N},$

$() := \texttt{ParalellogramLaw}(v - x_n, v - x_m) :$

$\quad : \|(x - y_n) + (x - y_m)\|^2 + \|(x - y_n) - (x - y_m)\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2;$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall n, m \in \mathbb{N} \,.$

$\quad . \ \|(x - y_n) + (x - y_m)\|^2 + \|y_n - y_m\|^2 = 2\|x - y_m\|^2 + 2\|x - y_n\|^2,$

$(2) := \texttt{Lim}(\eth d, \texttt{RearangeI}(1, \|(x - y_n) + (x - y_m)\|^2)) : \lim_{n,m\to\infty} \|y_n - y_m\|^2 = 4d - 4d = 0,$

$() := \eth^{-1}\texttt{Cauchy} : (x : \texttt{Cauchy}(X)),$

$() := \texttt{ClosedSubspaceIsBanach}(H, X) : (X : \mathsf{HIL}(K)),$

$y := \eth\texttt{Hilbert}(X)(x) : (X : \lim_{n\to\infty} y = x),$

$(*) := \eth y \eth d : y = \arg\min_{x\in X} \|x - v\|;$

$\texttt{Assume}\ z : \texttt{In}(X) : z = \arg\min_{x\in X} \|x - v\|,$

$(3) := \texttt{ParalellogramLaw}(v - y, v - z)(*, \eth d)(\eth z, \eth d) :$

$\quad : 4\left\|v - \frac{y - z}{2}\right\|^2 + \|y - z\|^2 = 2\|v - y\|^2 + 2\|v - z\|^2 = 4d^2,$

$(4) := (*)\eth d\left(4\left\|v - \frac{y - z}{2}\right\|^2\right) : 4\left\|v - \frac{y - z}{2}\right\|^2 \le 4d^2,$

$() := \eth\texttt{Norm}(H)(3)(4) : y = z;$

$\rightsquigarrow (**) := \eth^{-1}\texttt{Unique} : (y : \texttt{Unique}(\Lambda x \in X \,.\, x = \arg\min_{x\in X} \|x - v\|));$

$\square$

$$f \in H^*$$
$$x \in H \setminus \ker f$$
$$\|f\| = \sup_{y\in H} \frac{\|fy\|}{\|y\|} = \sup_{y\in\ker f} \frac{\|f(x + y)\|}{\|x + y\|} = \sup_{y\in\ker f} \frac{\|fx\|}{\|x + y\|} = \frac{\|fx\|}{\inf_{y\in\ker f} \|x + y\|}$$

$\texttt{orthogonalComplement} :: \prod H : \texttt{InnerProductSpace}(K) \,.\, H \to ?H$

$\texttt{orthogonalComplement}\,(x) = x^\perp := \{v \in H : v \perp x\}$

$\texttt{setOrthogonalComplement} :: \prod H : \texttt{InnerProductSpace}(K) \,.\, ?H \to ?H$

$\texttt{setOrthogonalComplement}\,(A) = A^\perp := \bigcap_{a \in A} a^\perp$

$\texttt{OrthCIsCS} :: \forall H : \texttt{InnerProductSpace}(K) \,.\, \forall A \subset H \,.\, A^\perp \subset_{\mathsf{NORM}} H$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a : A,$

$f := \Lambda x \in H \,.\, \langle x, a\rangle : \texttt{In}(H^*),$

$(1) := \eth\texttt{orthogonalComplement}(a, f, \eth\texttt{Orthogonal}) : a^\perp = \ker f,$

$() := (1)\texttt{ClosedKernel} : a^\perp \subset_{\mathsf{NORM}} H;$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall a \in A \,.\, a^\perp \subset_{\mathsf{NORM}} H,$

$(*) := (1)\eth\texttt{Closed}(H)\texttt{SubspaceIntersection} : A^\perp \subset_{\mathsf{NORM}} H;$

$\square$

$\texttt{DoubleOrthCI} :: \forall H : \texttt{InnerProductSpace}(K) \,.\, \forall A \subset H \,.\, A \subset A^{\perp\perp}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OrthogonalDistanceLemma} :: \forall H : \texttt{InnerProductSpace}(K) \,.\, \forall A \subset_{\mathsf{NORM}} H \,.\, \forall x \in H \,.\, x \in A^\perp \iff$
$\iff d(x, A) = \|x\|$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : A^\perp,$

$\texttt{Assume}\ a : A,$

$() := \ldots : \|x - a\|^2 = \langle x - a, x - a\rangle = \|x\|^2 + \|a\|^2 \geq \|x\|^2;$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall a \in A \,.\, \|x - a\| \geq \|x\|,$

$(2) := \eth\texttt{Subspace}(H)(A) : 0 \in A,$

$() := (1)(2) : d(x, A) = \|x\|;$

$\rightsquigarrow (\Rightarrow) := \texttt{ImplicationIntro} : x \in A^\perp \Rightarrow d(x, A) = \|x\|,$

$\texttt{Assume}\ E : d(x, A) = \|x\|,$

$\texttt{Assume}\ a : \texttt{In}(A) : \langle x, a\rangle \neq 0,$

$\texttt{Assume}\ c : \mathbb{R},$

$\beta := c\overline{\langle x, a\rangle} : K,$

$() := \eth\beta : \|x - \beta a\|^2 = \|x\|^2 - 2c|\langle x, a\rangle|^2 + c^2|\langle x, a\rangle|^2\|a\|^2;$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall t \in \mathbb{R} \,.\, \exists v \in A : \|x - v\|^2 = \|x\|^2 - 2t|\langle x, a\rangle| + t^2\|a\|^2,$

$v := (1)\left(\frac{|\langle x, a\rangle|}{\|a\|^2}\right) : \texttt{In}(A) : \|x - v\| = \|x\|^2 - \frac{|\langle x, a\rangle|^2}{\|a\|^2},$

$(2) := \eth v \eth a E : \|x - v\| = \|x\|^2 - \dfrac{|\langle x, a\rangle|^2}{\|a\|^2} < \|x\|^2 = d(x, A),$

$() := \eth v (2) : \bot;$

$\rightsquigarrow (1) := \texttt{Contradiction} : \forall a \in A \ . \ a \bot x,$

$() := \eth \texttt{setOrthogonalComplement} : z \in A^{\bot};$

$\rightsquigarrow (\Leftarrow) := \texttt{ImplicationIntro} : x \in A^{\bot} \Leftarrow d(x, A) = \|x\|,$

$(*) := \texttt{IffIntro}(\Rightarrow, \Leftarrow) : x \in A^{\bot} \iff d(x, A) = \|x\|;$

$\square$

$\texttt{OrthCTHM} :: \forall H : \texttt{Hilbert}(K) \ . \ \forall A \subset_{\mathsf{NORM}} H \ . \ H =_{\mathsf{NORM}} A \oplus A^{\bot}$

$\texttt{Proof} \ =$

$\texttt{Assume } x : \texttt{In}(H),$

$y := \texttt{NearestVector}(H, A, x) : \texttt{In}(A) : d(x, A) = \|x - y\|,$

$z := x - y : \texttt{In}(H),$

$(1) := \eth \texttt{setDistance}(z, A) \texttt{AutoInf}(A, \Lambda a \in A \ . \ a - y) \eth^{-1} z \eth^{-1} \texttt{SetDistance}(z, A) \eth y \eth z :$
$: d(z, A) = \inf_{a \in A} \|z - a\| = \inf_{a \in A} \|z + y - a\| = \inf_{a \in A} \|x - a\| = d(x, A) = \|x - y\| = \|z\|,$

$(2) := \texttt{OrthogonalDistanceLemma}(1) : z \in H^{\bot},$

$(3) := \eth^{-1} z (2) : x \in A + A^{\bot},$

$\texttt{Assume } y' : \texttt{In}(A),$

$\texttt{Assume } z' : \texttt{In}(A^{\bot}) : x = y' + z',$

$(4) := \eth z \eth z' : y - y' = z - z',$

$(5) := \eth \texttt{setOrthogonalComplement}(H)(A)(y - y', z - z') : \langle y - y', z - z' \rangle = 0,$

$() := \texttt{SelfOrthogonal}(4, 5) : y = y' \ \& \ z = z';;$

$\rightsquigarrow () := \eth \texttt{Unique} : ((y, z) : \texttt{Unique}(A \times A^{\bot}, \Lambda(a, b) \in A \times A^{\bot} \ . \ x = a + b));$

$\rightsquigarrow (*) := \texttt{UniIntro}\,\texttt{Pythogorus} : H =_{\mathsf{NORM}} A \oplus A^{\bot};$

$\square$

$\texttt{DoubleHilbertOrthC} :: \forall H : \texttt{Hilbert}(K) \ . \ \forall A \subset_{\mathsf{VS}} H \ . \ A^{\bot\bot} = \overline{A}$

$\texttt{Proof} \ =$

$(1) := \texttt{OrthCIsCS}(H, A^{\bot}) : A^{\bot\bot} \subset_{\mathsf{NORM}} H,$

$(2) := \texttt{DoubleOrthC}(H, A) : A \subset A^{\bot\bot},$

$(3) := \eth \texttt{closure}(H, A, 1, 2) : \overline{A} \subset A^{\bot\bot},$

$(4) := \texttt{OrthCTHM}(H, \overline{A}) : H = \overline{A} \oplus \overline{A}^{\bot},$

$(5) := \eth \texttt{setOrthogonalComplement}(A) : \overline{A}^{\bot} \subset A^{\bot},$

$\texttt{Assume } x : A^{\bot\bot} : d(x, A) > 0,$

$(y, z) := (4)(x) : \overline{A} \times \overline{A}^{\bot} : x = y + z,$

$(6) := \eth (y, z)(3) : z \in A^{\bot\bot},$

$(7) := \texttt{Selforthogonal}(5, 6) : z = 0,$

$() := \eth (y, z)(7) : x \in \overline{A};$

$\rightsquigarrow (7) := \eth\mathtt{Subset} : A^{\perp\perp} \subset \overline{A},$

$(*) := \mathtt{SetEq}(3,7) : A^{\perp\perp} = \overline{A};$

$\square$

$\mathtt{TotalCriterion} :: \forall H : \mathtt{Hilbert}(K) \,.\, \forall A \subset H : A^{\perp} = \{0\} \,.\, A : \mathtt{Total}(H)$

$\mathtt{Proof}\ =$

$(1) := \eth A : A^{\top\top} = H,$

$(2) := (1)\mathtt{DoubleHilbertOrthC}(H,A) : \overline{A} = H,$

$(*) := \eth^{-1}\mathtt{Total}(H) : A : \mathtt{Total}(H);$

$\square$

$\mathtt{OrthogonalQuetient} :: \forall H : \mathtt{Hilbert}(K) \,.\, A \subset_{\mathsf{NORM}} H \,.\, \dfrac{H}{A} \cong_{\mathsf{NORM}_{\circ\to\cdot}} A^{\perp}$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [x] : \dfrac{H}{A},$

$(y,z) := \mathtt{OrthCTHM}(H,A)(x) : A \times A^{\perp} : x = y + z,$

$T[x] := z : A^{\perp},$

$() := \eth\mathtt{normedQuetient}(H,A)([x])\eth(y,z)\mathtt{Pythagorus}(y+a,z)\mathtt{NonNegativeInf}(\eth\mathtt{Norm})\eth^{-1}T[x] :$

$\quad : \big\|[x]\big\| = \min_{a\in A}\|x+a\| = \min_{a\in A}\|y+z+a\| = \min_{a\in A}\|y+a\| + \|z\| = \|z\| = \|T[x]\|;$

$\rightsquigarrow T := \mathtt{FuncIntro} : \mathtt{Isometry}\left(\dfrac{H}{A}, A^{\perp}\right),$

$\mathtt{Assume}\ x : A^{\perp},$

$() := \eth T : T[x]_A = x;$

$\rightsquigarrow (1) := \eth\mathtt{Iomorphism}(\mathsf{NORM}_{\circ\to\cdot}) : \left(T : \dfrac{H}{A} \leftrightarrow_{\mathsf{NORM}_{\circ\to\cdot}} A^{\perp}\right),$

$(*) := \eth\mathtt{Isomorphic}(\mathsf{NORM}_{\circ\to\cdot})(1) : \dfrac{H}{A} \cong_{\mathsf{NORM}_{\circ\to\cdot}} A^{\perp};$

$\mathtt{HilbertNormed} :: ?\mathsf{NORM}(K)$

$V : \mathtt{HilbertNormed} \iff \exists H : \mathtt{Hilbert}(K) : V \cong_{\mathsf{NORM}_{\circ\to\cdot}} H$

$\mathtt{HilbertQuetient} :: \forall H : \mathtt{Hilbert}(K) \,.\, \forall A \subset_{\mathsf{NORM}} H \,.\, \dfrac{H}{A} : \mathtt{HilbertNormed}(K)$

$\mathtt{Proof}\ =$

$\ldots$

$\square$

$\mathtt{Orthoprojector} :: \prod H : \mathtt{Hilbert}(K) \,.\, \prod A \subset_{\mathsf{NORM}} H \,.\, ?\mathtt{Projector}(H,A)$

$P : \mathtt{Orthoprojector} \iff \forall x \in A^{\perp} \,.\, Px = 0$

$\texttt{OrthoprojectorExists} :: \forall H : \mathsf{HIL}(K) \,.\, \forall A \subset_{\mathsf{NORM}} H \,.\, \exists! \texttt{Orthoprojector}(H, A)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{TopologicalyCompletableCriterion} :: \forall V : \texttt{SeminormedSpace}(K) \,.$

$\quad . \forall S : \texttt{Subspace}(V) \,.\, S : \texttt{TopologicalyCompletable} \iff \forall W : \texttt{SeminormedSpace}(K) \,.\, \forall T : \mathcal{B}(S, W) \,.$

$\quad . \exists A : \mathcal{B}(V, W) : A_{|S} = T$

$\texttt{Proof} \ =$

$\Rightarrow$

$P : \texttt{Projector}(H, S)$

$A := TP : \mathcal{B}(V, W),$

$\Leftarrow$

$A = \mathrm{Right}(I_S)$

$H = \mathrm{Im}\, A \oplus \ker A = S \oplus \ker A;$

$\square$

$\texttt{HilbertExtension} :: \forall H : \mathsf{HIL}(K) \,.\, \forall S : \texttt{Subspace}(H) \,.\, \forall W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}(S, W) \,.$

$\quad . \exists A : \mathcal{B}(H, V) : \|A\| = \|T\| : A_{|S} = T$

$\texttt{Proof} \ =$

$P := \texttt{OrthoprojectorExists}(H, \overline{S}) : P : \texttt{Orthoprojector}(H.\overline{S}),$

$B := \texttt{ContinousExtension}(T) : \mathcal{B}(\overline{S}, V) : \|B\| = \|T\| : B_{|S} = T,$

$A := BP : \mathcal{B}(H, V),$

$(1) := \eth A \texttt{OperatorProductNorm}(B, P) \eth \texttt{Orthoprojector}(P) \eth B :$

$\quad : \|A\| = \|BP\| \le \|B\|\|P\| = \|B\| = \|T\|,$

$\texttt{Assume}\ v : S,$

$() := \eth A \eth \texttt{Projector}(H, \overline{S})(P)(v) \eth B : Av = BPv = Bv = Tv;$

$\rightsquigarrow (*) := \eth \texttt{constrictUniIntro} : A_{|S} = T;$

$\square$

$\texttt{LindenschtrausCafrari} :: \forall V : \mathsf{BAN}(K) \,.\, \big(\forall S \subset_{\mathsf{BAN}} V \,.\, V : \texttt{TopologicalyCompletable}\big) \iff$

$\quad \iff V : \texttt{HilbertNormed}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

3.7 Category Structure

$$\mathsf{BAN} : \texttt{Category}$$
$$\mathsf{BAN} = (\texttt{Banach}, \Lambda A, B : \texttt{Banach} \;.\; \mathcal{B}(A, B), \circ)$$

$$\mathsf{BAN}_{\circ\to\cdot} : \texttt{Category}$$
$$\mathsf{BAN}_{\circ\to\cdot} = (\texttt{Banach}, \Lambda A, B : \;.\; \texttt{NonExpanding}(A, B), \circ)$$

$$\mathsf{HIL} : \texttt{Category}$$
$$\mathsf{HIL} = (\texttt{Hilbert}, \Lambda A, B : \;.\; \mathcal{B}(A, B), \circ)$$

$$\mathsf{HIL}_{\circ\to\cdot} : \texttt{Category}$$
$$\mathsf{HIL}_{\circ\to\cdot} = (\texttt{Hilbert}, \Lambda A, B : \;.\; \texttt{NonExpanding}(A, B), \circ)$$

$$\texttt{BanSum} :: \mathsf{BAN} \to \mathsf{BAN} \to \mathsf{BAN}$$
$$\texttt{BanSum}\,(A, B) = A \oplus B := (A \times B, \Lambda(a, b) \;.\; \|a\| + \|b\|)$$

$$\texttt{BanProduct} :: \mathsf{BAN} \to \mathsf{BAN} \to \mathsf{BAN}$$
$$\texttt{BanProduct}\,(A, B) = A \otimes B := \big(A \times B, \Lambda(a, b) \;.\; \max(\|a\|, \|b\|)\big)$$

$$\texttt{HilbertSum} :: \mathsf{HIL} \to \mathsf{HIL} \to \mathsf{HIL}$$
$$\texttt{HilbertSum}\,(A, B) = A \dot{\oplus} B := \Big(A \times B, \Lambda(a, b), (c, d) \;.\; \langle a, c\rangle + \langle b, d\rangle\Big)$$

3.8 Isomorphisms of Hilberts Spaces

$\texttt{FischerRiesz} :: \forall H, L : \mathsf{HIL} \,.\, \forall e : \mathbb{N} \to H : \texttt{Schauder}(H) \,.\, \forall f : \mathbb{N} \to L : \texttt{Schauder}(L) \,.$
$\quad \exists! U : H \leftrightarrow_{\mathsf{HIL}_{\circ\to\cdot}} L : \forall n \in \mathbb{N} \,.\, U(e_n) = f_n$

$\texttt{Proof} \ =$

$\texttt{Assume}\ v : H,$

$$a := \eth\texttt{Schauder}(H)(e)(v) : \mathbb{N} \to K : v = \sum_{n=1}^{\infty} a_i e_i,$$

$$U(v) := \sum_{n=1}^{\infty} a_i f_i : \texttt{FormalSeria}(L),$$

$(1) := \eth H \eth a : (a^2 : \texttt{ConvergingSeria})(K),$

$() := \eth^{-1} L(1) : (U(v) \in L);$

$\rightsquigarrow U := \eth\texttt{Unique}(\eth\texttt{SchauderFuncIntro} : H \hookrightarrow_{\mathsf{NORM}} L,$

$\texttt{Assume}\ v : L,$

$$a := \eth\texttt{Schauder}(L)(f)(v) : \mathbb{N} \to K : v = \sum_{n=1}^{\infty} a_i f_i,$$

$$w := \sum_{n=1}^{\infty} a_i e_i : \texttt{FormalSeria}(L),$$

$(1) := \eth L \eth a : (a^2 : \texttt{ConvergingSeria})(K),$

$(2) := \eth^{-1} H(1) : (w \in H),$

$() := \eth U(2)(w) : U(w) = u;$

$\rightsquigarrow U := \eth^{-1}\texttt{Bijection} : H \leftrightarrow_{\mathsf{NORM}} L,$

$\texttt{Assume}\ a, w : H,$

$$a := \eth\texttt{Schauder}(H)(e)(v) : \mathbb{N} \to K : v = \sum_{n=1}^{\infty} a_i e_i,$$

$$b := \eth\texttt{Schauder}(H)(e)(w) : \mathbb{N} \to K : w = \sum_{n=1}^{\infty} b_i e_i,$$

$$() := \ldots : \langle U(v), U(w) \rangle = \left\langle \sum_{n=1}^{\infty} a_i f_i, \sum_{n=1}^{\infty} b_i f_i \right\rangle = \sum_{n=1}^{\infty} a_n b_n = \left\langle \sum_{n=1}^{\infty} a_i e_i, \sum_{n=1}^{\infty} b_i e_i \right\rangle = \langle v, w \rangle;$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Unitary} : (U : H \leftrightarrow_{\mathsf{HIL}_{\circ\to\cdot}} L),$

$\square$

$\texttt{HilbertBasisExists} :: \forall H : \mathsf{HIL} \,.\, \exists e : \texttt{Total} \,\&\, \texttt{Orthonormal}(H)$

$\texttt{Proof} \ =$

$O := \{A \subset H : \texttt{Orthonormal}(H)\} :??H,$

$(0) := \eth O : \emptyset \in O,$

$(00) := \eth\texttt{NonEmpty} : O \neq \emptyset,$

$\texttt{Assume}\ C : \texttt{Chain}(O, \subset),$

$B := \bigcup_{A \in C} A :?H,$

$\texttt{Assume}\ x, y : B : x \neq y,$

$A := \eth\texttt{Chain}(H, \subset)\eth B(x) : \texttt{In}(C) : x, y \in A,$

$() := \eth\texttt{Orthonormal}(H)(A)(x, y) : x \perp y \,\&\, \|x\| = 1 \,\&\, \|y\| = 1;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Orthonormal}(H) : (B : \texttt{Orthonormal}(H)),$

$() := \texttt{UnionIsMaximal}(C, B) : (B : \texttt{Maximal}(C, \subset));$

$\rightsquigarrow A := \texttt{ZornLemma}(00) : \texttt{Maximal}(O, \subset),$

$(1) := \eth O \eth\texttt{Maximal}(O, \subset)(A) : A^\perp = \{0\},$

$(*) := \texttt{TotalCriterion}(1) : A : \texttt{Total};$

$\square$

$\texttt{HilbertBasisDim} :: \forall H : \mathsf{HIL} \,.\, \forall E, F : \texttt{Total} \,\&\, \texttt{Orthonormal}(H) \,.\, \#E = \#F$

$\texttt{Proof} \ =$

$\texttt{HilbertDimLemma} :: \forall E, F : \texttt{Total} \,\&\, \texttt{Orthonormal}(H) \,.\, \#E \leq \#F$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : F,$

$$\Big(N, e, a, (1)\Big) := \eth\texttt{furieSeria}\,\texttt{FurieSpaceTheorem}(H, E, f) :$$

$$: \sum N : \texttt{Countable} \,.\, \sum e : N \to E \,.\, \sum a : N \to K \setminus \{0\} \,.\, f = \sum_{n \in N} a_n e_n,$$

$\mathcal{E}(f) := \{e_n | n \in \mathbb{N}\} : \texttt{Subset}(H);$

$\rightsquigarrow \mathcal{E} := \texttt{FuncIntro} : F \to \texttt{Subset}(H),$

$\mathfrak{E} := \bigcup_{f \in F} \mathcal{E}(f) : \texttt{Subset}(H),$

$(1) := \eth\texttt{Total}(F)\eth\mathfrak{E} : (\mathfrak{E} : \texttt{Total}),$

$\texttt{Assume}\ (e, (2)) : \sum e \in E \,.\, e \notin \mathfrak{E},$

$$(N, e', (3)) := \eth\texttt{furieSeria}\,\texttt{FurieSpaceTheorem}(H, \mathfrak{E}, f) :$$

$$: \sum N : \texttt{Countable} \,.\, \sum e' : N \to \mathfrak{E} \,.\, e = \sum_{n \in \mathbb{N}} \langle e, e'_n \rangle e'_n,$$

$(4) := (3)\eth\texttt{Orthonormal}(E)\eth\mathfrak{E} : e = 0,$

$(5) := \eth\texttt{Orthonormal}(E)(e) : e \neq 0,$

$() := \texttt{Absurd} : \perp;$

$\rightsquigarrow (3) := \eth\mathfrak{E}\texttt{FromContradiction} : E = \mathfrak{E},$

$$() := \ldots : \#E = \#\mathfrak{E} = \#\bigcup_{f \in D} \mathcal{E}(f) \leq (\#F) \cdot \aleph_0 = \#F;$$

$\square$

$(1) := \texttt{HilbertDimLemma}(F, E) : \#F \leq \#E,$
$(2) := \texttt{HilbertDimLemma}(E, F) : \#E \leq \#F,$
$(*) := \texttt{EqChoice}(1, 2)) : \#E = \#F;$

$\texttt{HilbertDim} :: \mathsf{HIL} \to \texttt{Cardinal}$
$\texttt{HilbertDim}\,(H) = \dim_{\mathsf{HIL}} H := \#\texttt{HibertBasisExists}(H)$

$\texttt{FischerRieszII} :: \forall H, L : \mathsf{HIL} : \dim_{\mathsf{HIL}} H = \dim_{\mathsf{HIL}} L\ .$
$\quad . \forall E : \texttt{Total}\ \&\ \texttt{Orthonormal}(H)\ . \forall F : \texttt{Total}\ \&\ \texttt{Orthonormal}(L)\ .$
$\quad . \forall \varphi : E \leftrightarrow_{\mathsf{SET}} F\ . \exists! U : H \leftrightarrow_{\mathsf{HILI}} L : \forall e \in E\ . U(e) = \varphi(E)$
$\texttt{Proof} =$
$\texttt{Assume}\ v : H,$

$$(N, e, a) := \eth\texttt{Total}(H)(E)(v) : \sum N :?\mathbb{N}\ . \sum e : N \to E\ . \sum a : N \to K\ . v = \sum_{n=1}^{\infty} a_i e_i,$$

$$U(v) := \sum_{n=1}^{\infty} a_i \varphi(e_i) : \texttt{FormalSeria}(L),$$

$(1) := \eth H \eth a : (a^2 : \texttt{ConvergingSeria})(K),$
$() := \eth^{-1} L(1) : (U(v) \in L);$
$\rightsquigarrow U := \eth\texttt{Unique}(\eth\texttt{Orthonormal})\texttt{FuncIntro} : H \hookrightarrow_{\mathsf{NORM}} L,$
$\texttt{Assume}\ v : L,$

$$(N, f, a) := \eth\texttt{Total}(L)(f)(v) : \sum N :?\mathbb{N}\ . \sum f : N \to F\ . \sum a : N \to K\ . v = \sum_{n=1}^{\infty} a_i f_i,$$

$$w := \sum_{n=1}^{\infty} a_i \varphi^{-1}(f_i) : \texttt{FormalSeria}(L),$$

$(1) := \eth L \eth a : (a^2 : \texttt{ConvergingSeria})(K),$
$(2) := \eth^{-1} H(1) : (w \in H),$
$() := \eth U(2)(w) : U(w) = u;$
$\rightsquigarrow U := \eth^{-1}\texttt{Bijection} : H \leftrightarrow_{\mathsf{NORM}} L,$
$\texttt{Assume}\ a, w : H,$

$$[!]a := \eth\texttt{Schauder}(H)(e)(v) : \mathbb{N} \to K : v = \sum_{n=1}^{\infty} a_i e_i,$$

$$[!]b := \eth\texttt{Schauder}(H)(e)(w) : \mathbb{N} \to K : w = \sum_{n=1}^{\infty} b_i e_i,$$

$$[!]() := \dots : \langle U(v), U(w) \rangle = \left\langle \sum_{n=1}^{\infty} a_i f_i, \sum_{n=1}^{\infty} b_i f_i \right\rangle = \sum_{n=1}^{\infty} a_n b_n = \left\langle \sum_{n=1}^{\infty} a_i e_i, \sum_{n=1}^{\infty} b_i e_i \right\rangle = \langle v, w \rangle;$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Unitary} : (U : H \leftrightarrow_{\mathsf{HIL}_{\circ\to\cdot}} L),$
$\square$

$\texttt{hilbertMatrix} :: \prod H, L : \mathsf{HIL}(K) \; . \; \mathcal{B}(H, L) \to$

$\to \texttt{Total} \,\&\, \texttt{Orthonormal}(H) \to \texttt{Total} \,\&\, \texttt{Orthonormal}(L) \to \dim_{\mathsf{HIL}} H \to \dim_{\mathsf{HIL}} L \to K$

$\texttt{hilbertMatrix}\,(T, e, f, i, j) = \texttt{mat}(T, e, f)_{i,j} := \langle f_j, Te_i \rangle$

$\texttt{HilbertMatrix} :: \prod H, L : \mathsf{HIL}(K) \; . \; ?\dim_{\mathsf{HIL}} H \to \dim_{\mathsf{HIL}} L \to K$

$A : \texttt{HilberMatrix} \iff \exists e : \texttt{Total} \,\&\, \texttt{Orthonormal}(H) : \exists f : \texttt{Total} \,\&\, \texttt{Orthonormal}(L) :$

$: \exists T : \mathcal{B}(H, L) : A = \texttt{mat}(T, e, f)$

$\texttt{asOperator} :: \prod H, L : \mathsf{HIL}(K) \; . \; \texttt{HilbertMatrix}(H, L) \to$

$\to \texttt{Total} \,\&\, \texttt{Orthonormal}(H) \to \texttt{Total} \,\&\, \texttt{Orthonormal}(L) \to \mathcal{B}(L, K)$

$\texttt{hilbertMatrix}\,(A, e, f, v) = (e, f)\, Av := \sum_{j \in \dim_{\mathsf{HIL}} L} f_j \sum_{i \in \dim_{\mathsf{HIL}}} A_{i,j} \langle e_i, v \rangle$

$\texttt{HilbertMatrixBounded} :: \forall H, L : \mathsf{HIL}(K) \; . \; \forall A : \texttt{HilbertMatrix}(H, L) \; .$

$. \; \forall n \in \dim_{\mathsf{HIL}} H \; . \; \forall m \in \dim_{\mathsf{HIL}} L \; . \sum_{i \in \dim_{\mathsf{HIL}} L} |A_{n,i}|^2 + \sum_{i \in \dim_{\mathsf{HIL}} H} |A_{i,m}|^2 < \infty$

$\texttt{Proof} \;=$

$(T, f, e) := \eth\texttt{HilbertMatrixBounded} : \mathcal{B}(H, L) \times \texttt{Total} \,\&\, \texttt{Orthonormal}(H) \times \texttt{Total} \,\&\, \texttt{Orthonormal}(L),$

$\texttt{Assume}\; n : \dim_{\mathsf{HIL}} H,$

$\texttt{Assume}\; m : \dim_{\mathsf{HIL}} L,$

$(1) := \eth T : \sum_{i \in \dim_{\mathsf{HIL}} L} |A_{n,i}|^2 = \|T\, e_n\|^2 < \infty,$

$(2) := \eth T : \sum_{i \in \dim_{\mathsf{HIL}} H} |A_{i,m}|^2 = \|T^* f_m\|^2 < \infty,$

$() := (1) + (2) : \sum_{i \in \dim_{\mathsf{HIL}} L} |A_{n,i}|^2 + \sum_{i \in \dim_{\mathsf{HIL}} H} |A_{i,m}|^2 < \infty;$

$\leadsto (*) := \texttt{UniIntro} : \forall n \in \dim_{\mathsf{HIL}} H \; . \; \forall m \in \dim_{\mathsf{HIL}} L \; . \sum_{i \in \dim_{\mathsf{HIL}} L} |A_{n,i}|^2 + \sum_{i \in \dim_{\mathsf{HIL}} H} |A_{i,m}|^2 < \infty,$

$\square$

$\texttt{HilbertMatrixCriterion} :: \forall H, L : \mathsf{HIL}(K) \; . \; \forall A \in K^{\dim_{\mathsf{HIL}} H \times \dim_{\mathsf{HIL}} L} :$

$: \sum_{(i,j) \in \dim_{\mathsf{HIL}} H \times \dim_{\mathsf{HIL}} L} |A_{i,j}|^2 < \infty \; . \; A : \texttt{HilbertMatrix}(H, L)$

$\texttt{Proof} \;=$

$e := \texttt{HilbertBasisExists}(H) : \texttt{Total} \,\&\, \texttt{Orthonormal}(H),$

$f := \texttt{HilbertBasisExists}(L) : \texttt{Total} \,\&\, \texttt{Orthonormal}(L),$

$\texttt{Assume}\; V : H,$

$T\, v := \sum_{i \in \dim_{\mathsf{HIL}} H} \langle v, e_i \rangle \sum_{j \in \dim_{\mathsf{HIL}} L} A_{i,j} f_j : \texttt{FormalSeria}(L),$

$(1) := \eth\mathtt{Norm}(L)\ldots : \|Tv\|^2 = \sum\sum|\langle v, e_i\rangle A_{i,j}|^2 \leq \sum|\langle v, e_i\rangle|\sum\sum\left|A_{j,l}\right|^2 = \|v\|^2\sum\sum\left|A_{j,l}\right|^2,$

$() := \eth\mathtt{Hilert}(L) : Tv \in V;$

$\rightsquigarrow T := \eth^{-1}\mathtt{FuncIntro} : \mathcal{B}(H, L),$

$(1) := \eth\mathtt{mat} : \mathtt{mat}(e, f, T) = A,$

$(*) := \eth^{-1}\mathtt{HilbertMatrix}(H, L)(1) : (A : \mathtt{HilbertMatrix}(H, L)(1)),$

$\square$

$\mathtt{EquivalentMorphism} :: \prod V, W : \mathsf{BAN}(K) \;.\; ?\big(V \to_{\mathsf{BAN}} W \times V \to_{\mathsf{BAN}} W\big)$

$(A, B) : \mathtt{EquivalentMorphisms} \iff A \sim_{\mathsf{BAN}} B \iff$

$\iff \exists T : \mathtt{Automorphism}(\mathsf{BAN}(K), V) : \exists S : \mathtt{Automorphism}(\mathsf{BAN}(K), W) : SAT = B$

$\mathtt{IsometricMorphisms} :: \prod V, W : \mathsf{BAN}(K) \;.\; ?\big(V \to_{\mathsf{BAN}} W \times V \to_{\mathsf{BAN}} W\big)$

$(A, B) : \mathtt{IsometricMorphism} \iff A \sim_{\mathsf{BAN}_{\circ\to\cdot}} B \iff$

$\iff \exists T : \mathtt{Automorphism}(\mathsf{BAN}_{\circ\to\cdot}(K), V) : \exists S : \mathtt{Automorphism}(\mathsf{BAN}_{\circ\to\cdot}(K), W) : SAT = B$

$\mathtt{MatrixEquivalent} :: \forall H, L : \mathsf{HIL}(K)\,\forall A, B : H \to_{\mathsf{HIL}} L \;.\; A \sim_{\mathsf{BAN}_{\circ\to\cdot}} B \iff$

$\iff \exists e, e' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(H) : \exists f, f' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(L) : \mathtt{mat}(A, e, f) = \mathtt{mat}(B, e', f')$

$\mathtt{Proof} =$

$\mathtt{Assume}\ (\Rightarrow) : A \sim_{\mathsf{BAN}_{\circ\to\cdot}} B,$

$(T, S) := \eth\mathtt{IsometricMorphism}(\Rightarrow) :$

$: \mathtt{Automorphism}(\mathsf{HIL}_{\circ\to\cdot}(K), H) \times \mathtt{Automorphism}(\mathsf{HIL}_{\circ\to\cdot}(K), L) : TAS = B,$

$e := \mathtt{HilbertBasisExists}(H) : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(H),$

$f := \mathtt{HilbertBasisExists}(L) : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(L),$

$e' := Te : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(H),$

$f' := Sf : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(L),$

$() := \eth e', f'\eth\mathtt{mat}\eth B : \mathtt{mat}(A, e', f') = \mathtt{mat}(A, Te, Sf) = \mathtt{mat}(SAT, e, f) = \mathtt{mat}(B, e, f);$

$\rightsquigarrow (\Rightarrow) := \ldots : \ldots,$

$\mathtt{Assume}\ e, e' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(H),$

$\mathtt{Assume}\ f, f' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(L) : \mathtt{mat}(A, e, f) = \mathtt{mat}(B, e', f'),$

$T := \mathtt{FischerRieszII}(H)(e, e') : \mathtt{Unitary}(H) : e' = Te,$

$S := \mathtt{FischerRieszII}(L)(f, f') : \mathtt{Unitary}(L) : f' = Sf,$

$() := \ldots : B = (e', f')\,\mathtt{mat}(B, e', f') = (Te, Sf)\,\mathtt{mat}(A, e, f) = SAT;$

$\rightsquigarrow (\Leftarrow) := \ldots : \ldots,$

$(*) := \mathtt{IffIntro}(\Rightarrow, \Leftarrow) : A \sim_{\mathsf{BAN}_{\circ\to\cdot}} B \iff$

$\iff \exists e, e' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(H) : \exists f, f' : \mathtt{Total}\ \&\ \mathtt{Orthonormal}(L) : \mathtt{mat}(A, e, f) = \mathtt{mat}(B, e', f');$

$\square$

3.9 Hilbert Adjoints

$\texttt{adjointForm} :: \prod H : \mathsf{HIL}(K) \,.\, H \to H^*$
$\texttt{adjointForm}(v) = v^* := \Lambda w \in H \,.\, \langle w, v\rangle$

$\texttt{RieszTHM} :: \forall H : \mathsf{HIL}(K) \,.\, H^* \cong_{\mathsf{BAN}_{\circ\to\cdot}} H^{\mathrm{i}}$
$\texttt{Proof} =$
$E := \texttt{HilbertBasisExists}(H) : \texttt{Total} \,\&\, \texttt{Orthonormal}(H),$
$\texttt{Assume}\ (v, w, 1) : \sum v, w \in H \,.\, v^* = w^*,$
$\texttt{Assume}\ e : E,$
$() := \eth^{-1}\texttt{adjointForm}(v)(e)(1)\eth\texttt{adjointForm}(w)(e) : \langle e, v\rangle = v^*e = w^*e\langle e, w\rangle;$
$\rightsquigarrow () := \texttt{FurieSpaceTHM}(H, E, v)\texttt{UniIntroFurieSpaceTHM}(H, E, w) : v = w;$
$\rightsquigarrow (1) := \eth\texttt{Injective} : \texttt{adjointForm} : H \hookrightarrow H^*,$
$\texttt{Assume}\ f : H^*,$
$\texttt{Assume}\ v : (\ker f)^\perp : \|v\| = 1,$
$w := \overline{f(v)}v : (\ker f)^\perp,$
$(2) := \eth\texttt{Unity}(\eth v, f(v))\eth w\eth\texttt{adjointForm}(w)(v) : f(v) = f(v)\langle v, v\rangle = \langle v, w\rangle = w^*v,$
$\texttt{Assume}\ u : \ker f,$
$(3) := \eth\ker(f)(u) : f(u) = 0,$
$(4) := \eth\texttt{adjointForm}(w, u)\eth\texttt{orthogonalComplement}() : w^u = \langle v, u\rangle = 0;$
$\rightsquigarrow (3) := \eth\texttt{constric}\,\texttt{UniIntro} : f_{|\ker f} = w^*_{|\ker f},$
$() := \eth\texttt{Linear}(f, w^*m\texttt{OrthCTHM}(\ker f)) : f = w^*;$
$\rightsquigarrow (2) := \eth^{-1}\texttt{Bijection}(1, \eth^{-1}\texttt{Surjection}) : \texttt{adjointForm} : H \leftrightarrow_{\mathsf{SET}} H^*,$
$[!] \ldots$
$\square$

$\texttt{dualOfHilbertSpace} :: \mathsf{HIL}(K) \to \mathsf{HIL}(K)$
$\texttt{dualOfHilbertSpace}(H) = H^* := (\texttt{dual}(H), \Lambda(x*, y*) \in H^* \times H^* \,.\, \langle y, x\rangle)$

$\texttt{HilbertIsReflexive} :: \forall H : \mathsf{HIL}(K) \,.\, H : \texttt{Reflexive}$
$\texttt{Proof} =$
$\texttt{Assume}\ x : H^{**},$
$\big(g, (1)\big) := \texttt{ReiszTHM}(H^*, x) : \sum g \in H^* \,.\, x = g^*,$
$\big(v, (2)\big) := \texttt{ReiszTHM}(H, x) : \sum v \in H^* \,.\, g = v^*,$
$\texttt{Assume}\ f : H^*,$
$\big(w, (3)\big) := \texttt{ReiszTHM}(H, x) : \sum w \in H^* \,.\, f = w^*,$
$(4) := (1)(x(f))\eth\texttt{conjugateForm}(g, f)\eth\texttt{dualOfHilbertSpace}(2,3)(H)\eth^{-1}\texttt{cunjugateForm}(w, v)(3)^{-1}$
$\quad \eth\texttt{natural} : x(f) = g^*f = \langle f, g\rangle = \langle w^*, v^*\rangle = \langle v, w\rangle = w^*v = f(v) = \alpha_v f;;$
$\rightsquigarrow (*) := \eth^{-1}\texttt{Reflexive} : (H : \texttt{Reflexive});$

$\square$

$\texttt{BoundedConjugateBillinearForm} :: \prod H : \mathsf{HIL}(K) \,.\, \mathcal{L}(H, H^{\mathrm{i}}; K)$

$J : \texttt{BoundedConjugateBillinearForm} \iff J \in \mathcal{B}_2(H) \iff$

$\iff \sup\{|J(x,y)| | x \in H, y \in H^{\mathrm{i}} : \|x\| = \|y\| = 1\} < \infty$

$\texttt{bilinearConjugateNorm} :: \mathcal{B}_2(H) \to \mathbb{R}_+$

$\texttt{bilinearConjugateNorm}(J) = \|J\| := \sup\{|J(x,y)| | x \in H, y \in H^{\mathrm{i}} : \|x\| = \|y\| = 1\}$

$\texttt{associate} :: \mathcal{B}(H, H) \to \mathcal{B}_2(H)$

$\texttt{associate}(T) := \Lambda x \in H \,.\, \Lambda y \in H^{\mathrm{i}} \,.\, \langle Tx, y\rangle$

$|\texttt{associate}(T)(x,y)| = |\langle Tx, y\rangle| \le \|Tx\|\|y\| \le \|T\|\|x\|\|y\|$

$|\langle Tx, Tx\rangle| = \|Tx\| \Rightarrow \|\texttt{associate}(T)\| = \|T\|$

$\texttt{BijectiveAssociation} :: \forall H \in \mathsf{HIL}(K) \,.\, \texttt{associate} : \mathcal{B}(H,H) \leftrightarrow_{\mathsf{SET}} \mathcal{B}_2(H)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \big(A, B, (1)\big) : \sum (A, B) \in (\mathcal{B}(H,H))^2 \,.\, A \ne B,$

$(2) := \eth(-)(1) : A - B \ne 0,$

$() := \eth\texttt{associate}(2)(A, B) : \texttt{associate}(A) - \texttt{associate}(B) = \texttt{associate}(A - B) \ne 0;$

$\rightsquigarrow (0) := \eth\texttt{Injective} : \texttt{associate} : \mathcal{B}(H,H) \hookrightarrow_{\mathsf{SET}} \mathcal{B}_2(H),$

$\texttt{Assume}\ J : \mathcal{B}_2(H, H),$

$\texttt{Assume}\ x : H,$

$f := \Lambda y \in H \,.\, J(y, x) : H^*,$

$\big(v, (1)\big) := \texttt{ReiszTHM} : \sum v \in H \,.\, f = v^*,$

$T(x) := v : H,$

$\texttt{Assume}\ y : H,$

$(*) := \ldots : J(x,y) = \overline{J(y,x)} = \overline{f(y)} = \overline{v^* y} = \overline{\langle y, v\rangle} = \langle v, y\rangle = \langle Tx, y\rangle;$

$\rightsquigarrow (2) := \texttt{UniIntro} : \forall y \in H \,.\, J(x,y) = \langle Tx, y\rangle,$

$() := \ldots : \|Tx\| = \sup\{|\langle Tx, y\rangle| : y \in H : \|y\| = 1\} == \sup\{|J(x,y)| : y \in H : \|y\| = 1\} \le \|J\|\|x\|;$

$\rightsquigarrow T := \eth^{-1}\mathcal{B} : \mathcal{B}(H,H) : J = \texttt{associate}(T);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Bijective}((0), \eth^{-1}\texttt{Surjective}) : \texttt{associate} : \mathcal{B}(H,H) \leftrightarrow_{\mathsf{SET}} \mathcal{B}_2(H);$

$\square$

$\texttt{RealHilbertDual} :: \forall H \in \mathsf{HIL}(\mathbb{R}) \,.\, H^* = H$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.10 Inverse Operator Theorem

$\texttt{OpenMappingLemma} :: \forall V : \mathsf{BAN}(K) \,.\, \forall W : \mathsf{NORM}(K) \,.\, \forall T : \mathcal{B}(V,W) \,.\, \forall(\theta, Q) : \sum \theta \in [0,1] \,.$

$\quad . \, T\mathbb{B}_V(0,1) : \texttt{Dense}(\mathbb{B}_W(0,\theta)) \,.\, \mathbb{B}_W(0,\theta) \subset T\mathbb{B}_V(0,1)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; y : \mathbb{B}_W(0,\theta),$

$\big(t,(1)\big) := \texttt{OpenBallMultiplication}(W,\theta,y) : \sum t \in (1,\infty) \,.\, ty \in \mathbb{B}_W(0,\theta),$

$d := t^{-1} : (0,1),$

$Y_0 := ty : \mathbb{B}_W(0,\theta),$

$\texttt{Assume}\; n : \mathbb{N},$

$(x_n,(2)) := \eth\texttt{Dense}(\mathbb{B}_W(0,(1-d)^{n-1}\theta)(T\mathbb{B}_V(0,(1-d)^{n-1}), Y_{n-1}) : \sum x \in \mathbb{B}_V(0,1) \,.\, \|Y_{n-1} - Tx_n\| \le (1-d)^n\theta,$

$Y_n := Y_{n-1} - Tx_n : W,$

$() := \eth\texttt{Linear}(T)\eth\texttt{Dense}(\mathbb{B}_W(0,(1-d)^{n-1}\theta)(T\mathbb{B}_V(0,(1-d)^{n-1})) : T\mathbb{B}_V(0,(1-d)^n) : \texttt{Dense}(\mathbb{B}_W(0,(1-d)^n\theta)),$

$() := \eth Y_n(2) : (Y_n \in \mathbb{B}_W(0,\theta));$

$\rightsquigarrow \big(x,(2)\big) := \texttt{RecursiveFuncIntro} : \sum x : \prod n \in \mathbb{N} \,.\, \mathbb{B}_V(0,(1-d)^{n-1}) \,.$

$$. \forall n \in \mathbb{N} \,.\, \left\| ty - \sum_{i=1}^{n} Tx_n \right\| \le (1-d)^n\theta,$$

$$(3) := \texttt{IneqIntro}\left(\sum_{n=0}^{\infty} \|x_n\|, \eth\texttt{ball}(\eth x)\right) \texttt{SimplePowerSeria}(d)\eth\texttt{infinity} :$$

$$: \sum_{n=1}^{\infty} \|x_n\| \le \sum_{n=0}^{\infty} (1-d)^n = \frac{1}{d} < \infty,$$

$() := \eth^{-1}\texttt{AbsolutelyConvergent}(3) : (x : \texttt{AbsolutelyConvergent}),$

$$X := \sum_{n=1}^{\infty} x_n : W,$$

$(4) := \eth\texttt{ConvergentSeria}(2) : TX = ty,$

$(5) := d(4) : TdX = y,$

$(6) := \eth\texttt{ball}(3, dX) : dX \in \mathbb{B}_W(0,1),$

$() := (5)(6) : y \in T\mathbb{B}_W(0,1);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Subset} : \mathbb{B}_W(0,\theta) \subset T\mathbb{B}_V(0,1);$

$\square$

$\texttt{OpenMappingII} :: \prod V, W : \mathsf{BAN}(K) \,.\, ?\mathcal{B}(V,W)$

$T : \texttt{OpenMappingII} \iff \exists(\theta, Q) : \sum \theta \in [0,1] \,.\, T\mathbb{B}_V(0,1) : \texttt{Dense}(\mathbb{B}_W(0,\theta))$

$\texttt{OpenMappingInterpretion} :: \forall V, W : \mathcal{B}(V,W) \,.\, \forall T : \texttt{OpenMapping}(V,W) \,.$

$\quad . \, T : \texttt{TopologicalSurjection}(V,W)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{OpenMappingTHM} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B} \,\&\, \texttt{Surjective}(V, W) \,.\, T : \texttt{OpenMappingII}(V, W)$

$\texttt{Proof} \;=$

[!](Baire Category)

...

$\square$

$\texttt{BoundedInverseTHM} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B} \,\&\, \texttt{Bijective}(V, W) \,.\, T^{-1} : \mathcal{B}(V, W)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ClosedMappingTHM} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}(V, W) \,.$

$\quad T : \texttt{TopologicalyInjective}(V, W) \iff T : \texttt{Injective} \,\&\, \texttt{ClosedMapping}(V, W)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{OpenMappingByInverseTHM} :: \texttt{BoundedInverseTHM} \Rightarrow \texttt{OpenMappingTHM}$

$\texttt{Proof} \;=$

$V' := \dfrac{V}{\ker T} : \mathsf{BAN}(K),$

$T' := \Lambda[x] \in V' \,.\, Tx : \mathcal{B} \,\&\, \texttt{Bijective}(V', W),$

$(1) := \texttt{BoundedInverseTHM}(T') : (T' : V' \leftrightarrow_{\mathsf{BAN}} W),$

$(2) := \texttt{KnownTHM}(1) : (T : \texttt{TopologicalySurjective}(V, W)),$

$(*) := \texttt{OpenMappingInterpretion} : (T : \texttt{OpenMappingII}(V, W));$

$\square$

$\texttt{ClosedGraphTheorem} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{L}(V, W) \,.\, \forall \Gamma : \texttt{Graph}(T) : \texttt{Closed}(V \oplus W) \,.$

$\quad .\, T : \mathcal{B}(V, W)$

$\texttt{Proof} \;=$

$() := \texttt{ClosedSubspaceIsBanach}(\Gamma) : \texttt{Graph}(T) : \mathsf{BAN}(K),$

$P := \Lambda(x, T(X)) : \texttt{Graph}(T) \,.\, x : \mathcal{B}(\texttt{Graph}(T), V),$

$(1) := \texttt{BoundedInverseTHM}\eth\texttt{Graph}(T)\eth P : \Big(P^{-1} : \mathcal{B}\big(V, \texttt{Graph}(T)\big)\Big),$

$\texttt{Assume}\ x : V,$

$(2) := \eth^{-1}\texttt{Norm}(V \oplus W)(x, Tx)(1) : \|x\| + \big\|T(x)\big\| = \Big\|\big(x, T(x)\big)\Big\| \le \big\|P^{-1}\big\|\|x\|,$

$() := (2) - \|x\| : \|T(x)\| \le (\|P^{-1}\| - 1)\|x\|;$

$\rightsquigarrow (*) := \eth\mathcal{B} : T : \mathcal{B}(V, W);$

$\square$

$\texttt{MajorizedNormsAreSameIfBan} :: \forall V : \mathsf{VS}(K) \,.\, \forall (N, M, 1) :$

$: \sum N, M : \texttt{Norm}(V) \,.\, (V, N), (V, M) : \mathsf{BAN}(K) \,.\, \forall (2) : N \leq M \,.\, N \cong M$

$\texttt{Proof} \;=$

$(3) := \eth\texttt{Stronger}(2) : I_{M,N} : \mathcal{B}((V, M), (V, N)),$

$(4) := \texttt{BoundedInverseTHM}(I_{M,N}, 3) : I_{N,M} : \mathcal{B}((V, N), (V, M)),$

$(*) := \eth^{-1}\texttt{EqNorm}(\eth^{-1}\texttt{Stronger}(4), 2) : N \cong M;$

$\square$

$\texttt{BanachTopologicalCompliment} :: \forall V : \mathsf{BAN}(K) \,.\, \forall (A, B, 1) : \sum A, B \subset_{\mathsf{BAN}} V \,.$

$. \, V \cong_{\mathsf{VS}} A \oplus B \,.\, V \cong_{\mathsf{BAN}} A \oplus B$

$\texttt{Proof} \;=$

$T := \Lambda(a, b) \in A \oplus B \,.\, a + b : \mathcal{L}(A \oplus B, V),$

$(2) := \eth\texttt{DirectSum}(1) : T : \texttt{Bijection}(A \oplus B, V),$

$\texttt{Assume}\ (a, b) : A \oplus B,$

$() := \texttt{EqEl}(\|T(a, b)\|, \eth T(a, b))\texttt{TriangleIneq}(V, a, b)\eth\texttt{directProduct}(A, B) :$

$: \|T(a, b)\| = \|a + b\| \leq \|a\| + \|b\| = \|(a, b)\|;$

$\rightsquigarrow (3) := \eth^{-1}\mathcal{B} : (T : \mathcal{B}(A \oplus B, V)),$

$(4) := \eth\texttt{Isomorphism}(\mathsf{BAN})\texttt{BoundedInverseTHM}(2, 3) : (T : A \oplus B \leftrightarrow_{\mathsf{BAN}} B),$

$(5) := \eth\texttt{Isomorphic}(\mathsf{BAN}) : V \cong A \oplus B;$

$\square$

$\texttt{BanachProjector} :: \forall V : \mathsf{BAN}(K) \,.\, \forall A \subset_{\mathsf{BAN}} V \,.\, \forall P : \texttt{Projector}(V, A) \,.\, P : \mathcal{B}(V, V)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.11 Retractions of Banach Spaces

$\texttt{CoretractionInHIL} :: \forall V, W \in \mathsf{HIL}(K) . \forall T : V \to_{\mathsf{HIL}} W .$
$\quad . T : \texttt{Coretraction}(\mathsf{HIL}, V, W) \iff T : \texttt{Injective} \ \& \ \texttt{ClosedMapping}(V, W)$

$\texttt{Proof} =$

$\texttt{Assume Right} : (T\texttt{Coretraction}(\mathsf{HIL}, V, W)),$

$\big(A, (1)\big) := \eth\texttt{Coretraction}(\mathsf{HIL}, V, W)(T) : \sum A : W \to_{\mathsf{HIL}} V \ . TA = \mathrm{id}_V,$

$(2) := \texttt{ForgettingCoretraction}(T, \mathsf{SET}) : T : \texttt{Injective}(V, W),$

$(3) := \texttt{IdentityNorm}(V)(2)\texttt{OperatorProductNorm}(A, T) : 1 = \|I\| = \|AT\| \le \|A\|\|T\|,$

$\texttt{Assume } x : V,$

$(4) := \texttt{EqEl}(\|x\|, (1))\eth\texttt{operatorNorm}(A) : \|x\| = \|ATx\| \le \|A\|\|Tx\|,$

$() := \|A\|^{-1}(4) : \|Tx\| \ge \|A\|^{-1}\|x\|;$

$\leadsto (4) := \texttt{TopologicalInjectionCriterion} : (T : \texttt{TopolocalyInjectitive}(V, W)),$

$() := \texttt{ClosedMappingTheorem}(V, W, T, 4) : (T : \texttt{ClosedMapping}(V, W));$

$\leadsto (\Rightarrow) := \texttt{ImplicationIntro} : T : \texttt{Coretraction} \Rightarrow T : \texttt{Injective} \ \& \ \texttt{ClosedMapping}(V, W),$

$\texttt{Assume Left} : (T : \texttt{Injective} \ \& \ \texttt{ClosedMapping}(V, W)),$

$(1) := \eth\texttt{constrictImage}(\mathrm{Im}\, T)\texttt{Left} : (T^{|\,\mathrm{Im}\, T}) : \texttt{Bijective}(V, W),$

$(2) := \texttt{BoundedInverseTHM}(T^{|\,\mathrm{Im}\, T}) : \left(T^{|\,\mathrm{Im}\, T}\right)^{-1} : \mathcal{B}(\mathrm{Im}\, T, V),$

$(3) := \texttt{Left}\eth\texttt{ClosedMapping}(V, W)(T, V) : \mathrm{Im}\, T \subset_{\mathsf{HIL}} W,$

$P := \texttt{OrthoprojectorExists}(W, \mathrm{Im}\, T) : \texttt{Orthoprojector}(W, \mathrm{Im}\, T),$

$A := \left(T^{|\,\mathrm{Im}\, T}\right)^{-1} P : \mathcal{B}(W, V),$

$\texttt{Assume } x : V,$

$() := \eth A(Tx)\eth\texttt{Orthoprojector}(W, \mathrm{Im}\, T)(Tx)\eth\texttt{constrictImage}(T)\texttt{Inverse}(T)\eth :$
$\quad : ATx = \left(T^{|\,\mathrm{Im}\, T}\right)^{-1} PTx = \left(T^{|\,\mathrm{Im}\, T}\right)^{-1} Tx = x;$

$\leadsto (4) := \eth^{-1}\mathrm{id}_V : TA = \mathrm{id}_V,$

$() := \eth^{-1}\texttt{Retraction}(\mathsf{HIL}, V, W) : (T : \texttt{Retraction}(\mathsf{HIL}, V, W));$

$\leadsto (*) := \texttt{IffIntro}\big((\Rightarrow), \texttt{ImplicationIntro}\big) :$
$\quad : T : \texttt{Coretraction}(\mathsf{HIL}, V, W) \iff T : \texttt{Injective} \ \& \ \texttt{ClosedMapping}(V, W);$

$\square$

$\texttt{CoretractionInHILI} :: \forall V, W \in \mathsf{HIL}_{\circ\to\cdot}(K) . \forall T : V \to_{\mathsf{HIL}_{\circ\to\cdot}} W .$
$\quad . T : \texttt{Coretraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \iff T : \texttt{Isometry}(V, W)$

$\texttt{Proof} =$

$\texttt{Assume Right} : (T\texttt{Coretraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W)),$

$\big(A, (1)\big) := \eth\texttt{Coretraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W)(T) : \sum A : W \to_{\mathsf{HIL}_{\circ\to\cdot}} V \ . TA = \mathrm{id}_V,$

$\texttt{Assume } x : V,$

$(2) := \eth\mathsf{HIL}_{\circ\to\cdot}(x, Tx, ATx) : \|ATx\| \le \|Tx\| \le \|x\|,$

$(3) := \eth(1)(x) : \|ATx\| = \|x\|,$

$() := \texttt{DoubleIneq}(2,3) : \|Tx\| = \|x\|;$

$\rightsquigarrow (4) := \eth^{-1}\texttt{Isometry} : (T : \texttt{Isometry}(V,W));$

$\rightsquigarrow (\Rightarrow) := \texttt{ImplicationIntro} : T : \texttt{Coretraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \Rightarrow T : \texttt{Isometry}(V,W),$

$\texttt{Assume Left} : (T : \texttt{Isometry}(V,W)),$

$(1) := \eth\texttt{constrictImage}(\operatorname{Im} T)\texttt{Left} : (T^{|\operatorname{Im} T}) : \texttt{Bijective}(V,W),$

$(2) := \texttt{Left}\eth\texttt{ClosedMapping}(V,W)(T,V) : \operatorname{Im} T \subset_{\mathsf{BAN}} W,$

$P := \texttt{OrthoprojectoExists}(W, \operatorname{Im} T) : \texttt{Orthoprojector}(W, \operatorname{Im} T),$

$A := \left(T^{|\operatorname{Im} T}\right)^{-1} P : \mathcal{B}(W,V),$

$\texttt{Assume } y : W,$

$(x,v,3) := \texttt{OrthCTHM}(W, \Im T, v) : \sum (x,v) \in V \times (\operatorname{Im} T)^{\perp} \,.\, y = Tx + v),$

$() := \eth A \eth(x,v)\eth A(Tx)\eth\texttt{Orthoprojector}(W,\operatorname{Im} T)(Tx)\eth\texttt{constrictImage}(T)\eth\texttt{Inverse}(T)$
$\quad \texttt{Left}\eth^{-1}\texttt{Isometry}(V,W)(T,x)\texttt{PosetiveIneq}(\|v\|)\texttt{Pythagorus}(W, \Im T, Tx, v) :$
$$: \|Ay\| = \left\| \left(T^{|\operatorname{Im} T}\right)^{-1} P(Tx+v) \right\| = \|x\| = \|Tx\| \le \|Tx\| + \|v\| = \|y\|;$$

$\rightsquigarrow (4) := \eth^{-1}\texttt{Nonexpanding}(W,V) : \big(A : W \to_{\mathsf{HIL}_{\circ\to\cdot}} V\big),$

$\texttt{Assume } x : V,$

$() := \eth A(Tx)\eth\texttt{Orthoprojector}(W, \operatorname{Im} T)(Tx)\eth\texttt{constrictImage}(T)\eth\texttt{Inverse}(T) :$
$$: ATx = \left(T^{|\operatorname{Im} T}\right)^{-1} PTx = \left(T^{|\operatorname{Im} T}\right)^{-1} Tx = x;$$

$\rightsquigarrow (5) := \eth^{-1}\mathrm{id}_V : TA = \mathrm{id}_V,$

$() := \eth^{-1}\texttt{Retraction}(\mathsf{BAN}, V, W)(4,5) : (T : \texttt{Retraction}(\mathsf{BAN}, V, W));$

$\rightsquigarrow (*) := \texttt{IffIntro}\big((\Rightarrow), \texttt{ImplicationIntro}\big) :$
$$: T : \texttt{Coretraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \iff T : \texttt{Isometry}(V,W);$$

$\square$

$\texttt{RetractionInHIL} :: \forall V, W \in \mathsf{HIL}(K) \,.\, \forall T : V \to_{\mathsf{HIL}_{\circ\to\cdot}} W \,.$
$$. \, T : \texttt{Retraction}(\mathsf{HIL}, V, W) \iff T : \texttt{Surjective}(V,W)$$

$\texttt{Proof} =$

$\texttt{Assume Right} : (T\texttt{Retraction}(\mathsf{HIL}, V, W)),$

$\big(A, (1)\big) := \eth\texttt{Retraction}(\mathsf{HIL}, V, W)(T) : \sum A : W \to_{\mathsf{HIL}} V \,.\, AT = \mathrm{id}_W,$

$() := \texttt{RetractionInSET}\eth^{-1}\texttt{Retraction}(\mathsf{SET}, V, W)(1) : \big(T : \texttt{Surjective}(V,W)\big);$

$\rightsquigarrow (\Rightarrow) := \texttt{ImplicationIntro} : T : \texttt{Retraction}(\mathsf{HIL}, V, W) \Rightarrow T : \texttt{Surjective}(V,W),$

$\texttt{Assume Left} : (T : \texttt{Surjective}(V,W)),$

$S := (\ker T)^{\perp} : \texttt{Subobject}(\mathsf{BAN}, V),$

$() := \texttt{BoundedInverseTHM}(S, W, T^{|S}) : \left(\left(T^{|S}\right)^{-1} : \mathcal{B}(W,S)\right),$

$A := \left(T^{|S}\right)^{-1} : \mathcal{B}(W,S),$

$(2) := \eth A \eth\texttt{constrictImage}(T,S)\eth\texttt{Inverse}(T^{|S}) : AT = \mathrm{id}_W,$

$() := \eth^{-1}\texttt{Retraction}(\mathsf{HIL}, V, W)(2) : \big(T : \texttt{Retraction}(\mathsf{HIL}, V, W)\big);$

$\rightsquigarrow (*) := \texttt{IffIntro}\big((\Rightarrow), \texttt{ImplicationIntro}\big) :$

$: T : \texttt{Retraction}(\mathsf{HIL}, V, W) \iff T : \texttt{Bijection}(V, W);$

$\square$

$\texttt{RetractionInHILI} :: \forall V, W \in \mathsf{HIL}_{\circ\to\cdot}(K) \;.\; \forall T : V \to_{\mathsf{HIL}_{\circ\to\cdot}} W \;.$

$. \; T : \texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \iff T : \texttt{Coisometry}(V, W)$

$\texttt{Proof} \;=$

$\texttt{Assume Right} : (T : \texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W)),$

$\big(A, (1)\big) := \eth\texttt{Retraction}(\mathsf{HIL}, V, W)(T) : \sum A : W \to_{\mathsf{HIL}_{\circ\to\cdot}} V \;.\; AT = \mathrm{id}_W,$

$\texttt{Assume}\; x : \mathbb{B}_W,$

$(2) := \eth\mathsf{HIL}_{\circ\to\cdot}(A)(x)\eth\mathbb{B}_W : \|Ax\| \le \|x\| = 1,$

$(3) := (1)(x) : TAx = x,$

$(4) := \eth^{-1}\mathbb{B}_V(2) : Ax \in \mathbb{B}_V,$

$() := \eth\texttt{Image}(3, 4) : x \in T\mathbb{B}_V;$

$\rightsquigarrow (2) := \eth\texttt{Subset} : \mathbb{B}_W \subset T\mathbb{B}_V,$

$() := \eth^{-1}\texttt{Coretraction}(V, W)(2) : \big(T : \texttt{Corectraction}\big);$

$\rightsquigarrow (\Rightarrow) := \texttt{ImplicationIntro} : T : \texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \Rightarrow T : \texttt{Coisometry}(V, W),$

$\texttt{Assume Left} : (T : \texttt{Coisometry}(V, W)),$

$S := (\ker T)^{\perp} : \texttt{Subobject}(\mathsf{BAN}, V),$

$() := \eth\texttt{Coisometry}(S, W, T^{|S}) : \left(\left(T^{|S}\right)^{-1} : W \to_{\mathsf{HIL}_{\circ\to\cdot}} S\right),$

$A := \left(T^{|S}\right)^{-1} : W \to_{\mathsf{HIL}_{\circ\to\cdot}} S3,$

$(2) := \eth A\eth\texttt{constrictImage}(T, S)\eth\texttt{Inverse}(T^{|S}) : AT = \mathrm{id}_W,$

$() := \eth^{-1}\texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W)(2) : \big(T : \texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W)\big);$

$\rightsquigarrow (*) := \texttt{IffIntro}\big((\Rightarrow), \texttt{ImplicationIntro}\big) :$

$: T : \texttt{Retraction}(\mathsf{HIL}_{\circ\to\cdot}, V, W) \iff T : \texttt{Coisometry}(V, W);$

$\square$

$\texttt{CoretractionInBAN} :: \forall V, W \in \mathsf{BAN}(K) \;.\; \forall T : V \to_{\mathsf{BAN}} W \;.\; T : \texttt{Coretraction}(\mathsf{BAN}, V, W) \iff$

$\iff T : \texttt{ClosedMapping} \;\&\; \texttt{Injective}(V, W) : \exists S : \sum S \subset_{\mathsf{BAN}} W \;.\; W \cong_{\mathsf{BAN}} \operatorname{Im} T \oplus S$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RetractionInBANI} :: \forall V, W \in \mathsf{BAN}(K) \;.\; \forall T : V \to_{\mathsf{BAN}} W \;.\; T : \texttt{Retraction}(\mathsf{BAN}, V, W) \iff$

$\iff T : \texttt{Surjective}(V, W) \;\&\; \exists S : \sum S \subset_{\mathsf{BAN}} V \;.\; V \cong_{\mathsf{BAN}} \ker T \oplus S$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.12 Banach-Steinhaus Theorem

$\texttt{PointwiselyBoundedOperatorFamily} :: \prod X : \texttt{Set} \, . \, \prod V, W : \mathsf{NORM}(K) \, . \, ?(X \to \mathcal{B}(V, W))$

$T : \texttt{PointwiselyBoundedOperatorFamily} \iff \forall v \in V \, . \, \exists C \in \mathbb{R}_+ : \forall x \in X \, . \, \|T_x v\| \leq C$

$\texttt{UniformlyBoundedOperatorFamily} :: \prod X : \texttt{Set} \, . \, \prod V, W : \mathsf{NORM}(K) \, . \, ?(X \to \mathcal{B}(V, W))$

$T : \texttt{UniformlyBoundedOperatorFamily} \iff \exists C \in \mathbb{R}_+ : \forall x \in X \, . \, \|T_x\| \leq C$

$\texttt{BanachSteinhaus} :: \forall V : \mathsf{BAN}(K) \, . \, \forall X : \texttt{Set} \, . \, \forall W : \mathsf{NORM}(K) \, .$
$\quad . \, \forall T : \texttt{PointwiselyBoundedOperatorFamily}(X, V, W) \, .$
$\quad . \, T : \texttt{UniformlyBoundedOperatorFamily}(X, V, W)$

$\texttt{Proof} =$

$\texttt{BanachSteinhausLemma} :: \forall x \in V \, . \, \forall r \in \mathbb{R}_{++} \, . \, \forall T : \mathcal{B}(V, W) \, . \, r\|T\| \leq \sup \big\{ \|Tv\| \big| v \in \mathbb{B}_V(x, r) \big\}$

$\texttt{Proof} =$

$\texttt{Assume } z : V,$

$$(1) := \texttt{EqEl} \left(\|Tz\|, \eth_1^{-1} \mathcal{L}(T) \left(\frac{1}{2} T(x+z), \frac{1}{2} T(x-z) \right) \right)$$
$$\quad \texttt{TriangleIneq} \left(\frac{1}{2} T(x+z), \frac{1}{2} T(x-z) \right) \texttt{SumMaxIneq} \left(\|T(x+z)\|, \|T(x-z)\| \right) :$$
$$\quad : \|Tz\| = \left\| \frac{1}{2} T(x+z) - \frac{1}{2} T(x-z) \right\| \leq \frac{1}{2} \|T(x+z)\| + \frac{1}{2} \|T(x-z)\| \leq \max \left(\|T(x+z)\|, \|T(x-z)\| \right),$$

$\texttt{Assume } B : \|z\| \leq r,$

$(2) := \eth^{-1} \mathbb{B}_V(x, r) : x + z \in \mathbb{B}_V(x, r),$

$(3) := \eth^{-1} \mathbb{B}_V(x, r) : x - z \in \mathbb{B}_V(x, r),$

$() := (1) \eth^{-1} \eth^{-1} sup(2, 3) : \|Tz\| \leq \sup \big\{ \|Tv\| \big| v \in \mathbb{B}_V(x, r) \big\};;$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall z \in \mathcal{B}(0, r) \, . \, \|Tz\| \leq \sup \big\{ \|Tv\| \big| v \in \mathbb{B}_V(x, r) \big\},$

$(*) := \eth \texttt{operatorNorm}(1) : r\|T\| \leq \sup \big\{ \|Tv\| \big| v \in \mathbb{B}_V(x, r) \big\};$

$\square$

$\texttt{Assume } \Omega : \sup_{x \in X} \|T_x\| = \infty,$

$(x, 1) := \eth \sup(\Omega, \Lambda n \in \mathbb{N} \, . \, 4^n) : \sum x : \mathbb{N} \to W \, . \, \forall n \in \mathbb{N} \, . \, \left\| T_{x_n} \right\| \geq 4^n,$

$$(v_1, p_1) := \frac{1}{3} \eth \texttt{OperatorNorm} \left((1), 1, \frac{2}{3} \right) : \sum v_1 \in \mathbb{B}_V \left(0, \frac{1}{3} \right) \, . \, \|T_1 v_1\| \geq \frac{2}{9} \|T_1\|,$$

$\texttt{Assume } (n, 2) : \sum n \in \mathbb{N} \, . \, n > 1,$

$(v_n, p_n) := \texttt{BanachSteinhausLemma}(v_{n-1}, (1/3)^n, T_n) : \sum v_n \in W \, . \, \|v_n - v_{n-1}\| \leq 3^{-n} \; \&$
$$\quad \& \; \|T_n v_n\| \geq \frac{2}{3} \frac{1}{3^n} \|T_n\|;$$

$$\rightsquigarrow (v, p) := \texttt{RecursiveFuncIntro} : \sum \mathbb{N} \to V \, . \, \forall n \in \mathbb{N} \, . \, \|v_n - v_{n+1}\| \leq 3^{-n} \; \& \; \|T_n x_n\| \geq \frac{2}{3} \frac{1}{3^n} \|T_n\|,$$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\big(N,(2)\big) :=: \sum N \in \mathbb{N} \,.\, \frac{2}{3^N} \le \varepsilon,$

$\texttt{Assume}\ (n,m,3) : \sum n,m \in \mathbb{N} \,.\, n \ge N \ \&\ m \ge N,$

$() := \texttt{IteratedTriangleIneq}(v,n,m)p(3)\texttt{NonNegativeAddIneq}\,\texttt{SimplePowerSeria}(2) :$

$$: \|v_n - v_n\| \le \sum_{i=\min n,m}^{\max n,m-1} \|v_i - v_{i+1}\| \le \frac{1}{3^{N+1}} \sum_{i=0}^{|n-m|} \frac{1}{3^i} \le \frac{1}{3^N} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{2}{3^N} \le \varepsilon;;$$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Cauchy} : \big(v : \texttt{Cauchy}(V)\big),$

$w := \lim_{n\to\infty} v_n : V,$

$\texttt{Assume}\ n : \mathbb{N},$

$() := \texttt{EqEl}\big(\|T_{x_n} w\|, \eth_1^{-1}\mathcal{L}(T_{x_n}, v_n, w - v_n)\big)\texttt{InverseTriangleIneq}(v_n, w - v_n)(2, p_2)(1) :$

$$: \|T_{x_n} w\| = \|T_{x_n} v_n - T_{x_n}(w - v_n)\| \ge \|T_{x_n} v_n\| - \|T_{x_n}(w - v_n)\| \ge \frac{2}{3}\frac{1}{3^n}\|T_{x_n}\| - \frac{1}{3}\frac{1}{3^n}\|T_{x_n}\| \ge \frac{1}{6}\left(\frac{4}{3}\right)^n;$$

$\rightsquigarrow (3) := \eth\texttt{Limit} : \lim_{n\to\infty} \|T_{x_n} w\| = \infty,$

$() := (3)\eth\texttt{PointwiselyBoundedOperatorFamily} : \perp;$

$\rightsquigarrow (1) := \texttt{FromContradiction} : \sup_{x\in X} \|T_x\| < \infty,$

$(**) := \eth^{-1}\texttt{UniformlyBoundedOperatorFamily}(1) : (T : \texttt{UniformlyBoundedOperatorFamily}(X,V,W));$

$\square$

$\texttt{DualBoundednessTHM} :: \forall V : \mathsf{NORM}(K) \,.\, \forall A \subset V \,.\, \forall (0) : \forall f \in V^* \,.\, \{f(v) | v \in A\} : \texttt{Bounded}(K) \,.$

$A : \texttt{Bounded}(V)$

$\texttt{Proof}\ =$

$F := \Lambda v \in A \,.\, \Lambda f \in V^* \,.\, f(v) : A \to \mathcal{B}(V^*, K),$

$(1) := (0)(\eth F) : \big(F : \texttt{PointwiselyBoundedOperatorFamily}(A, V^*, K)\big),$

$(2) := \texttt{BanachSteinhaus}(F) : \big(F : \texttt{UniformlyBoundedOperatorFamily}(A, V^*, K)\big),$

$C := \eth\texttt{UniformlyBoundedOperatorFamily}(F) : \sum C \in \mathbb{R}_{++} \forall v \in A \,.\, \|F_v\| \le C,$

$(*) := \eth F(\eth C) : \big(A : \texttt{Bounded}(V)\big);$

$\square$

$\texttt{BilinearBanachBoundedness} :: \forall V : \mathsf{BAN}(K) \,.\, \forall W, U : \mathsf{NORM} K \,.\, \forall B : \texttt{DisjointlyBounded}(V,W,U) \,.$

$.\, B : \texttt{JointlyBounded}(V,W,U)$

$\texttt{Proof}\ =$

$\beta := \Lambda w \in \mathbb{B}_W \,.\, \Lambda v \in V \,.\, B(v,w) : W \to \mathcal{B}(V,U),$

$(1) := \eth\texttt{DisjointlyBounded}(B) : \big(\beta : \texttt{PointwiselyBoundedOperatorFamily}(\mathbb{B}_W, V, U)\big),$

$(2) := \texttt{BanachStenhaus}(\beta) : \big(\beta : \texttt{UniformlyBoundedOperatorFamily}(\mathbb{B}_W, V, U)\big),$

$:= \eth^{-1}\texttt{JointlyBounded}(\eth\texttt{UniformlyBoundedOperatorFamily}(\beta)) : \big(B : \texttt{DisjointlyBounded}(V,W,U)\big);$

$\square$

3.13 Functor of Banach Conjugacy

$\texttt{BanachMorphFunctor} :: \mathsf{BAN}(K) \to \texttt{ContravariantFunctor}(\mathsf{BAN}(K), \mathsf{BAN}(K))$
$\texttt{BanachMorphFunctor}\,(V) = \mathcal{B}(V, ?) := (\Lambda W \in \mathsf{BAN}(K)\ .\ \mathcal{B}(V, W),$
$\quad, \Lambda T : W \to_{\mathsf{BAN}} U\ .\ \Lambda A \in \mathcal{B}(V, W)\ .\ TA)$

$\texttt{BanachMorphContraFunctor} :: \mathsf{BAN}(K) \to \texttt{CovariantFunctorFunctor}\,((, \mathsf{BAN})\,(K), \mathsf{BAN}(K))$
$\texttt{BanachMorphContraFunctor}\,(V) = \mathcal{B}(?, V) := (\Lambda W \in \mathsf{BAN}(K)\ .\ \mathcal{B}(W, V),$
$\quad, \Lambda T : U \to_{\mathsf{BAN}} V\ .\ \Lambda A \in \mathcal{B}(W, V)\ .\ AT)$

$\texttt{BanachConjugacyFunctor} :: \texttt{ContravariantFunctor}(\mathsf{BAN}(K), \mathsf{BAN}(K))$
$\texttt{BanachConjugacyFunctor}\,(V, T) = (V^*, T^*) := \big(\mathcal{B}(V, K), \mathcal{B}(T, K)\big)$

$\texttt{DoubleBanachConjugacyFunctor} :: \texttt{CovariantFunctor}(\mathsf{BAN}(K), \mathsf{BAN}(K))$
$\texttt{DoubleBanachConjugacyFunctor}\,(V, T) = (V^{**}, T^{**}) := \big(\mathcal{B}(V^*, K), \mathcal{B}(T^*, K)\big)$

$\texttt{ScalarId} :: \mathcal{B}(K, ?) \cong \mathrm{id}_{\mathsf{BAN}(K)}$
$\texttt{Proof}\ =$
$N := \Lambda V \in \mathsf{BAN}(K)\ .\ \Lambda f \in \mathcal{B}(K, V)\ .\ f(1) : \prod V \in \mathsf{BAN}(K)\ .\ \mathcal{L}(\mathcal{B}(K, V), V),$
$\texttt{Assume}\ V : \mathsf{BAN}(K),$
$\texttt{Assume}\ (f, 1) : \sum f \in \mathcal{B}(K, V)\ .\ \|f\| = 1,$
$(2) := \eth\texttt{operatorNorm}(f)(1) : 1 = \|f(1)\|,$
$() := \eth N_V(2) : \|N_V f\| = \|f(1)\| = 1;$
$\leadsto (1) := \eth\texttt{BoundedOperator}\eth\texttt{operatorNorm} : (N_V \in \mathcal{B}(\mathcal{B}(K, V)))\ \&\ \|N_V\| = 1,$
$\texttt{Assume}\ v : V,$
$f := \Lambda c \in K\ .\ cv : \mathcal{B}(K, V),$
$() := \eth N_V : N_V f = f(1) = v;$
$\leadsto (2) := \eth^{-1}\texttt{Bijective} : (N_V : \mathcal{B}(K, V) \leftrightarrow V),$
$\texttt{Assume}\ (f, 3) : \sum f \in \mathcal{B}(K, V)\ .\ N_V f = 0,$
$(4) := \eth N_V(3) : f(1) = 0,$
$() := \eth_2\mathcal{L}(4) : f = 0;$
$\leadsto (3) := \texttt{LinearInjectivityProperty} : (N_V : \mathcal{B}(K, V) \hookrightarrow V),$
$(4) := \eth\texttt{Isomorphism}(\mathsf{BAN}(K)(1, 2, 3) : (N_V : \mathcal{B}(K, V) \leftrightarrow_{\mathsf{BAN}(K)} V),$
$\leadsto (1) := \texttt{UniIntro} : \forall V \in \mathsf{BAN}(K)\ .\ N_V : \mathcal{B}(K, V) \leftrightarrow_{\mathsf{BAN}(K)} V,$
$\texttt{Assume}\ V, W : \mathsf{BAN}(K),$
$\texttt{Assume}\ T : \mathcal{B}(V, W),$
$\texttt{Assume}\ v : V,$
$() := \eth N_V^{-1}\eth T^*\eth N_W : N_W T^* N_V^{-1} v = N_W T^*(\Lambda c \in K\ .\ cv) = N_W(\Lambda c \in K\ .\ cTv) = Tv;$
$\leadsto () := \eth^{-1}\texttt{SimmilarMorphism}(\mathsf{BAN}(K)) : (N_V, N_W)\,T \cong_{\mathsf{BAN}(K)} T^*;$
$\leadsto (*) := \eth^{-1}\texttt{IsoFunctor}(N_V, N_W)) : \mathcal{B}(K, ?) \cong \mathrm{id}_{\mathsf{BAN}(K)},$

ConjugacyAdditive :: $\forall V, W \in \mathsf{BAN} \,.\, \forall A, B : \mathcal{B}(V,W) \,.\, (A+B)^* = A^* + B^*$

Proof $=$

$\ldots$

$\square$

ConjugacyHomogen :: $\forall V, W \in \mathsf{BAN}(K) \,.\, \forall A : \mathcal{B}(V,W) \,.\, \forall c \in K \,.\, (cA)^* = cA^*$

Proof $=$

$\ldots$

$\square$

ConjugacyPreserveNorm :: $\forall V, W \in \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}(V,W) \,.\, \|T^*\| = \|T\|$

Proof $=$

Assume $(f, 1) : \sum f \in W^* \,.\, \|f\| = 1,$

$() := \eth^{-1}\texttt{operatorNorm}\big(T^* f\big)\eth T^* \eth\texttt{operatorNorm}(T)(1) :$

$$: \|T^* f\| = \sup\left\{\big\|(T^* f)v\big\|\Big| v \in \mathbb{B}_V\right\} = \sup\left\{\big\|f(Tv)\big\|\Big| v \in \mathbb{B}_V\right\} \le \sup\left\{\|fw\| : w \in \mathbb{B}_W\big(0, \|T\|\big)\right\} = \|T\|;$$

$\rightsquigarrow (1) := \texttt{UniIntro} : \forall f \in W^* \,.\, \big\|T^* f\big\| \le \|T\|,$

$\big(v, (2)\big) := \eth\texttt{operatorNorm}(T) : \sum v : \mathbb{N} \to V \,.\, \lim_{n\to\infty} \|Tv_n\| = \|T\| \;\&\; \forall n \in \mathbb{N} \,.\, \|v_n\| = 1,$

Assume $n : \mathbb{N},$

$f' := \Lambda c T v_n \in \mathrm{span}(Tv_n) \,.\, c\|Tv_n\| : \mathcal{B}\big(\mathrm{span}(v_n), K\big),$

$(3) := \eth f' : \|f'\| = 1,$

$f_n :=$ HahnBanach$(f, W) : \sum f_n \in W^* \,.\, \|f_n\| = 1 \;\&\; f_n T v_n = \|Tv_n\|;$

$\rightsquigarrow \big(f, (3)\big) := \texttt{FuncIntro} : \sum f : \mathbb{N} \to W^* \,.\, \forall n \in \mathbb{N} \,.\, \|f_n\| = 1 \;\&\; f_n T v_n = \|Tv_n\|,$

$(4) := \eth\texttt{operatorNorm}(T^* f_n)\eth T^* \texttt{SupRelax}(\mathbb{B}_V, v_n)((3)(n)_2(2) :$

$$\lim_{n\to\infty} \|T^* f_n\| = \lim_{n\to\infty} \sup\left\{\big\|(T^* f_n)\big\|v\big| v \in V : \|v\| = 1\right\} = \lim_{n\to\infty} \sup\left\{\big\|f_n(Tv)\big\|\big| v \in V : \|v\| = 1\right\} \ge$$

$$\ge \lim_{n\to\infty} \|f_n T v_n\| = \lim_{n\to\infty} \|Tv_n\| = \|T\|,$$

$(5) := \lim_{n\to\infty} (1)(f_n)\texttt{ConstantLimit}(\|T\|) : \lim_{n\to\infty} \big\|T^* f_n\big\| \le \lim_{n\to\infty} \|T\| = \|T\|,$

$(6) := \texttt{DoubleIneq}(4, 5) : \lim_{n\to\infty} \|T^* f_n\| = \|T\|,$

$(*) := \eth^{-1}\texttt{operatorNorm}(1, 6) : \|T^*\| = \|T\|;$

$\square$

ConjugacyReverseComposition :: $\forall V, W, U \in \mathsf{BAN}(K) \,.\, \forall A : \mathcal{B}(V,W) \,.\, \forall B : \mathcal{B}(W,U) \,.\, (AB)^* = B^* A^*$

Proof $=$

$\ldots$

$\square$

ConjugacyPreservesIdentity :: $\forall V \in \mathsf{BAN}(K) \,.\, \mathrm{id}_V^* = \mathrm{id}_{V^*}$

Proof $=$

$\ldots$

$\square$

$\texttt{DoubleConjugacyTHM} :: \alpha : \texttt{Natural}\Big(\mathsf{BAN}(K), \mathrm{id}, (\cdot)^{**}\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ V, W : \mathsf{BAN}(K),$

$\texttt{Assume}\ T : \mathcal{B}(V, W),$

$\texttt{Assume}\ v : V,$

$(1) := \eth\alpha_W(Tv) : \alpha_W Tv = \Lambda f \in W^* \ . \ fTv,$

$(2) := \eth\alpha_V(v)\eth T^{**} : T^{**}\alpha_V v = T^{**}\Lambda f \in V^* \ . \ f(v) = \Lambda f \in W^* \ . \ T^* fv = \Lambda f \in W^* \ . \ fTv,$

$() := (2)(1) : \alpha_W Tv = T^{**}\alpha_V v;$

$\rightsquigarrow () := \eth\texttt{FuncEq} : \alpha_W T = T^{**}\alpha_V;$

$(*) := \eth^{-1}\texttt{Natural} : \Big(\alpha : \texttt{Natural}\Big(\mathsf{BAN}(K), \mathrm{id}, (\cdot)^{**}\Big)\Big);$

$\square$

$\texttt{IsometryConjugation} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall T : \mathcal{B}(V, W) \ .$

$\quad . \ T : \texttt{Isometry}(V, W) \iff T^* : \texttt{Coisometry}(W, V)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{CoisometryConjugation} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall T : \mathcal{B}(V, W) \ .$

$\quad . \ T : \texttt{Coisometry}(V, W) \iff T^* : \texttt{Isometry}(W, V)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{TopologicalInjectionConjugation} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall T : \mathcal{B}(V, W) \ .$

$\quad . \ T : \texttt{TopologicalInjection}(V, W) \iff T^* : \texttt{TopologicalSurjection}(W, V)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{TopologicalSurjectionConjugation} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall T : \mathcal{B}(V, W) \ .$

$\quad . \ T : \texttt{TopologicalSurjection}(V, W) \iff T^* : \texttt{TopologicalInjection}(W, V)$

$\texttt{Proof} \ =$

...

$\square$

3.14 Homology of Banach Spaces

$\texttt{conjugateSeq} :: \prod N : \texttt{TOIndex} \,.\, \texttt{Sequance}\big(\mathsf{BAN}(K), N\big) \to \texttt{Sequance}\big(\mathsf{BAN}(K), \texttt{reverse}(N)\big)$

$\texttt{conjugatelSeq}\,(V_\bullet, T_\bullet) = (V_\bullet, T_\bullet)^* := (V_\bullet^*, T_\bullet^*)$

$\texttt{ConjugateExactSeq} :: \forall (V_\bullet, T_\bullet) : \texttt{Sequance}(\mathsf{BAN}(K), N)\,.$

$\quad .\,(V_\bullet, T_\bullet) : \texttt{Exact}(\mathsf{BAN}(K), N) \iff (V_\bullet, T_\bullet)^* : \texttt{Exact}(\mathsf{BAN}(K), \texttt{reverse}(N))$

$\texttt{Proof} \;=$

$\texttt{Assume Right} : \Big((V_\bullet, T_\bullet) : \texttt{Exact}(\mathsf{BAN}(K), N)\Big),$

$\texttt{Assume}\; n : \texttt{Inner}(N),$

$(1) := \eth \ker \eth T_n^* : \ker T_n^* = \{f \in V_n^* : \operatorname{Im} T_n \subset \ker f\},$

$(2) := \eth \operatorname{Im} \eth T_{n-}^* : \operatorname{Im} T_{n-}^* = \{g \circ T_{n-} | g \in V_{n-}^*\},$

$\texttt{Assume}\; f : \texttt{In}(\operatorname{Im} T_{n-}^*),$

$(g, 3) := (2)(f) : \sum g \in V_{n-}^* \,.\, f = g \circ T_{n-},$

$\texttt{Assume}\; v : \texttt{In}(\operatorname{Im} T_n),$

$(w, 4) := \eth \operatorname{Im}(T_n)(v) : \sum w \in V_{n+} \,.\, v = T_n(w),$

$() := (3)(4)(fv)\eth\texttt{Exact}(V, T)(n-, n) : fv = gT_{n-}T_n w = 0;$

$\rightsquigarrow := (2) : f \in \ker T_n^*;$

$\rightsquigarrow (3) := \eth\texttt{Subset} : \operatorname{Im} T_{n-}^* \subset \ker T_n^*,$

$\texttt{Assume}\; f : \texttt{In}(\ker T_n^*),$

$(4) := \eth f : \operatorname{Im} T_n \subset \ker f,$

$\texttt{Assume}\; A : f = 0,$

$() := AT_{n-}^* : f = T_{n-}^* 0;$

$\rightsquigarrow (5) := \texttt{ImplicationIntro} : f = 0 \Rightarrow f \in \operatorname{Im} T_{n-}^*,$

$\texttt{Assume}\; A : f \neq 0,$

$(v, 6) := \eth\texttt{NonZero}(f, A) : \sum v \in V_n \,.\, fv \neq 0,$

$(7) := \eth\texttt{Exact}(V, T)(n)(4) : \ker T_{n-} \subset \ker f,$

$(8) := \eth \ker(6, 7) : T_{n-} v \neq 0,$

$\texttt{Assume}\; R : T_{n-} v \in T_{n-} \ker f,$

$(x, 9) := \eth\texttt{Subset}(T_{n-}, \ker f) : \sum x \in \ker f \,.\, T_{n-} x = T_{n-} v,$

$(10 := \eth\texttt{Inverse}(V_n, x, v, 9)\eth_1\mathcal{L}(T_{n-1}(v, x)) : 0 = T_{n-} v - T_{n-} x = T_{n-}(v - x),$

$(11) := \eth \ker 10 : v - x \in \ker T_{n-},$

$(12 := \eth_1\mathcal{L}(f)(v, x)(9)(6) : f(v - x) = f(v) - f(x) = f(v) \neq 0,$

$(13) := \eth \ker(12) : v - x \notin \ker f,$

$() := \texttt{Contradiction}(7)(11, 13) : \perp;$

$\rightsquigarrow (0) := \texttt{Negation} : T_{n-} v \notin T_{n-} \ker f,$

$g' := \Lambda c T_{n-} v + x \in \operatorname{span}(T_{n-} v) \oplus T_{n-} \ker f \,.\, cf(v) : \mathcal{B}(\operatorname{span}(T_{n-} v) \oplus T_{n-} \ker f, K),$

$(g, 9) := \texttt{HahnBanach}(g', V_{n-}) : \sum g \in V_{n-}^* \,.\, gT_{n-} v = f(v) \;\&\; \forall x \in \ker f \,.\, gT_{n-} x = 0,$

$\texttt{Assume}\ x : V_n,$

$\big(a, b, y, (10)\big) := \texttt{FunctionalTopComplement}(f, v, 6) : \sum (a, b, y) \in K \times K \times \ker f \ .\ x = av + by,$

$() := (10)(9)\eth \ker(f)(y) : T^*_{n-} g(x) = T^*_{n-} g(av + by) = agT_{n-}v + bgT_{n-}x = af(v) = f(x);$

$\rightsquigarrow (10) := \texttt{FuncEqIntro} : f = T^*_{n-} g,$

$() := \eth^{-1} \operatorname{Im}(10) : f \in \operatorname{Im} T^*_{n-};$

$\rightsquigarrow (4) := \eth^{-1}\texttt{Subset} : \ker T^*_n \subset \operatorname{Im} T^*_{n-},$

$(5) := \eth^{-1}\texttt{SetEq} : \ker T^*_n = \operatorname{Im} T^*_{n-};$

$\rightsquigarrow () := \eth^{-1}\texttt{Exact} : (V_\bullet, T_\bullet)^* : \texttt{Exact}(\mathsf{BAN}(K), \texttt{reverse}(N));$

$\rightsquigarrow (\Rightarrow) := \texttt{ImplicationIntro} : (V_\bullet, T_\bullet) : \texttt{Exact}(\mathsf{BAN}(K), N) \Rightarrow (V_\bullet, T_\bullet)^* : \texttt{Exact}(\mathsf{BAN}(K), \texttt{reverse}(N)),$

$\texttt{Assume Left} : \Big((V_\bullet, T_\bullet)^* : \texttt{Exact}\big(\mathsf{BAN}(K), \texttt{reverse}(N)\big)\Big),$

$\texttt{Assume}\ n : \texttt{Inner}(N),$

$\texttt{Assume}\ A : \operatorname{Im} T_n \subsetneq \ker T_{n-},$

$(v, 1) := \eth \operatorname{Im} \eth \ker(A) : \sum v \in \operatorname{Im} T_n \ .\ T_{n-}(v) \neq 0,$

$(x, 2) := \eth \operatorname{Im}(v) : \sum x \in V_{n+} \ .\ T_n x = v,$

$f' := \Lambda c T_{n-} v \in \operatorname{span}(T_{n-} v) = 1 : \mathcal{B}(\operatorname{span} T_{n-} v, K),$

$(f, 3) := \texttt{HahnBanach}(f, V_{n-}) : \sum f \in V^*_{n-} \ .\ f T_{n-} v = 1,$

$(4) := \texttt{ConjugacyPreservesComposition}(T^*_n T^*_{n-})\eth(T_{n-} T_n)^*(2)(3)\texttt{FieldNontriviality}(K) :$

$\quad : \big(T^*_n T^*_{n-} f\big)(x) = f T_{n-} T_n x = f T_{n-} v = 1 \neq 0,$

$(5) := \eth^{-1}\texttt{NonZero}(V^*_{n+})(4) : T^*_n T^*_{n-} f \neq 0,$

$(6) := \eth^{-1}\texttt{NonZero}(\mathcal{B}(V^*_{n-}, V^*_{n+}) : T^*_n T^*_{n-} \neq 0,$

$() := \texttt{Contradiction}(\eth\texttt{Exact}(V^*_\bullet, T^*_\bullet)(N), 6) : \perp;$

$\rightsquigarrow (1) := \texttt{Negation} : \operatorname{Im} T_n \subset \ker T_{n-},$

$\texttt{Assume}\ A : \ker T_{n-} \subsetneq \operatorname{Im} T_n,$

$(v, 2) := \eth \ker \eth \operatorname{Im} A : \sum v \in V_n \ .\ T_{n-} v = 0\ \&\ \forall w \in V_{n+} \ .\ T_n w \neq v,$

$f' := \Lambda c v + x \in \operatorname{span}(v) \oplus T_n V_{n+} = c : \mathcal{B}(\operatorname{span}(v) \oplus T_n V_{n+}, K),$

$(f, 3) := \texttt{HahnBanach}(f', V_n) : \sum f \in V^*_n \ .\ f(v) = 1\ \&\ \forall w \in V_{n+} \ .\ f T_n w = 0,$

$(4) := (3)_2 : f \in \ker T^*_n,$

$(5) := (3)_1 : f \notin \operatorname{Im} T^*_{n-},$

$(6) := \eth\texttt{NotASubset}(4, 5) : \ker T^*_n \subsetneq \operatorname{Im} T^*_n,$

$() := \texttt{Contradiction}(\eth\texttt{Exact}(V^*_\bullet, T^*_\bullet)(N), 6) : \perp;$

$\rightsquigarrow () := \eth\texttt{SetEq}(1, \texttt{Negation}) : \operatorname{Im} T_n = \ker T_{n-};$

$\rightsquigarrow () := \eth^{-1}\texttt{Exact} : \Big((V_\bullet, T_\bullet) : \texttt{Exact}\big(\mathsf{BAN}(K), N\big)\Big);$

$\rightsquigarrow (*) := \ldots : (V_\bullet, T_\bullet) : \texttt{Exact}(\mathsf{BAN}(K), N) \iff (V_\bullet, T_\bullet)^* : \texttt{Exact}(\mathsf{BAN}(K), \texttt{reverse}(N));$

$\square$

$\texttt{ConjugacyPreservesIsomorphisms} :: \forall V, W \in \mathsf{BAN}(K) \;.\; \forall T : \mathcal{B}(V, W) \;.$
$\;.\; T : V \leftrightarrow_{\mathsf{BAN}} W \iff T^* : W^* \leftrightarrow_{\mathsf{BAN}} V^*$

$\texttt{Proof} \;=$

$\texttt{Assume}\;\texttt{Right} : \big(T : V \leftrightarrow_{\mathsf{BAN}} W\big),$

$(1) := \texttt{MinimalExactIso}(T, \texttt{Right}) : \Big(0 \to_0 V \to_T W \to_0 0 : \texttt{Exact}\big(\mathsf{BAN}(K)\big)\Big),$

$(2) := \texttt{ConjugateExactSeq}(1) : \Big(0 \to_0 W^* \to_{T^*} V^* \to_0 0 : \texttt{Exact}\big(\mathsf{BAN}(K)\big)\Big),$

$() := \texttt{BoundedInverseTHM}\,\texttt{MinimalExactIso}^{-1}(2) : \big(T^* : W^* \leftrightarrow_{\mathsf{BAN}} V^*\big);$

$\ldots$

$\texttt{Assume}\;\texttt{Left} : \big(T^* : W^* \leftrightarrow_{\mathsf{BAN}} V^*\big),$

$(1) := \texttt{MinimalExactIso}(T, \texttt{Left}) : \Big(0 \to_0 W^* \to_{T^*} V^* \to_0 0 : \texttt{Exact}\big(\mathsf{BAN}(K)\big)\Big),$

$(2) := \texttt{ConjugateExactSeq}^{-1}(1) : \Big(0 \to_0 V \to_T W \to_0 0 : \texttt{Exact}\big(\mathsf{BAN}(K)\big)\Big),$

$() := \texttt{BoundedInverseTHM}\,\texttt{MinimalExactIso}^{-1}(2) : \big(T : V \leftrightarrow_{\mathsf{BAN}} W\big);$

$\ldots$

$\square$

3.15 Products and Coproducts of Banach Spaces

3.16 Completion as a Universal Property

3.17 Tensor Products of Banach Spaces

3.18 Tensor Products of Hilbert Spaces

4 Weak Topology of a Banach Space

4.1 Definition of the Weak Topology

$$\texttt{weakTopologyBase} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{TopologyBase}(V)$$

$$\texttt{weakTopologyBase}\,() := \left\{ v + \left\{ x \in V : \forall i \in n \,.\, \left|f_i(x)\right| < a_i \right\} \middle| n \in \mathbb{N}, f \in n \to V^*, a \in n \to \mathbb{R}_{++}, v \in V \right\}$$

$$\texttt{weakTopology} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{Hausdorff}(V)$$

$$\texttt{weakTopology}\,() = \mathbf{w}(V) := \texttt{fromBase}\,\texttt{weakTopologyBase}(V)$$

$$\texttt{weakStarTopologyBase} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{TopologyBase}(V^*)$$

$$\texttt{weakTopologyBase}\,() := \left\{ g + \left\{ f \in V^* : \forall i \in n \,.\, \left|f(v_i)\right| < a_i \right\} \middle| n \in \mathbb{N}, v \in n \to V, a \in n \to \mathbb{R}_{++}, g \in V^* \right\}$$

$$\texttt{weakStarTopology} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{Hausdorff}(V^*)$$

$$\texttt{weakStarTopology}\,() = \mathbf{w}^*(V) := \texttt{fromBase}\,\texttt{weakStarTopologyBase}(V)$$

$\texttt{WeakIsWeak} :: \forall V : \mathsf{NORM}(K) \,.\, \mathbf{w}(V) \le \mathcal{T}(V)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U : \texttt{In}\big(\texttt{weakTopologyBase}(V)\big),$

$(N, f, a, 1) := \eth\texttt{weakTopologyBase}(V)(U) : \sum n \in \mathbb{N} \,.\, \sum f : n \to V^* \,.\, \sum a : n \to \mathbb{R} \,.$

$\quad .\ U = \bigcap_{i=1}^{n} f_i^{-1}(-a_i, a_i),$

$() := (1)\eth\texttt{Continuous}(V, K, (-a_i, a_i))\texttt{BoundedFunctionalIsContinuous}(f) : U \in \mathcal{T}(V);$

$\rightsquigarrow (*) := \eth\texttt{Coarser}\big(\eth\mathbf{w}(V)\texttt{BaseDefinesTopology}\big) : \mathbf{w}(V) \le \mathcal{T}(V);$

$\square$

$\texttt{WeakStarIsWeak} :: \forall V : \mathsf{NORM}(K) \,.\, \mathbf{w}^*(V) \le \mathcal{T}(V^*)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U : \texttt{In}\big(\texttt{weakStarTopologyBase}(V)\big),$

$(N, f, a, 1) := \eth\texttt{weakStarTopologyBase}(V)(U) : \sum n \in \mathbb{N} \,.\, \sum x : n \to V \,.\, \sum a : n \to \mathbb{R} \,.$

$\quad .\ U = \bigcap_{i=1}^{n} \alpha_{x_i}^{-1}(-a_i, a_i),$

$() := (1)\eth\texttt{Continuous}(V^*, K, (-a_i, a_i))\texttt{BoundedFunctionalIsContinuous}(\alpha_x) : U \in \mathcal{T}(V^*);$

$\rightsquigarrow (*) := \eth\texttt{Coarser}\big(\eth\mathbf{w}^*(V)\texttt{BaseDefinesTopology}\big) : \mathbf{w}^*(V) \le \mathcal{T}(V^*);$

$\square$

$\texttt{WeakStarIsWeaker} :: \forall V : \mathsf{NORM}(K) \,.\, \mathbf{w}^*(V) \leq \mathbf{w}(V^*)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ U : \texttt{In}\big(\texttt{weakStarTopologyBase}(V)\big),$

$(N, f, a, 1) := \eth\texttt{weakStarTopologyBase}(V)(U) : \sum n \in \mathbb{N} \,.\, \sum x : n \to V \,.\, \sum a : n \to \mathbb{R} \,.$

$$. \; U = \bigcap_{i=1}^{n} \alpha_{x_i}^{-1}(-a_i, a_i),$$

$() := \texttt{CanonicalIsometry}(x)\eth\mathbf{w}(V^*) : U \in \mathbf{w}(V^*);$

$\rightsquigarrow (*) := \eth\texttt{Coarser}\big(\eth\mathbf{w}^*(V)\texttt{BaseDefinesTopology}\big) : \mathbf{w}^*(V) \leq \mathbf{w}(V^*);$

$\square$

$\texttt{WeakConvergent} :: \prod V : \mathsf{NORM}(K) \,.\, ?(\mathbb{N} \to V)$

$v : \texttt{WeakConvergent} \iff \Big(v : \texttt{Convergent}\big(V, \mathbf{w}(V)\big)\Big)$

$\texttt{weakLimit} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{WeakConvergent}(V) \to V$

$\texttt{weakLimit}\,(v) = \mathbf{w} \lim_{n\to\infty} v_n := \texttt{limit}(V, \mathbf{w}(V), v)$

$\texttt{WeakStarConvergent} :: \prod V : \mathsf{NORM}(K) \,.\, ?(\mathbb{N} \to V^*)$

$v : \texttt{WeakStarConvergent} \iff \Big(v : \texttt{Convergent}\big(V^*, \mathbf{w}^*(V)\big)\Big)$

$\texttt{weakStarLimit} :: \prod V : \mathsf{NORM}(K) \,.\, \texttt{WeakStarConvergent}(V) \to V^*$

$\texttt{weakStarLimit}\,(f) = \mathbf{w}^* \lim_{n\to\infty} f_n := \texttt{limit}(V^*, \mathbf{w}^*(V), v)$

$\texttt{WeakLimitCharacteristic} :: \forall V : \mathsf{NORM}(K) \,.\, \forall x : \mathbb{N} \to V \,.\, \forall X \in V \,.$

$\quad .\, \mathbf{w} \lim_{n\to\infty} x_n = X \iff \forall f \in V^* \,.\, \lim_{n\to\infty} f(x_n) = f(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ L : \mathbf{w} \lim_{n\to\infty} x_n = X,$

$\texttt{Assume}\ f : \texttt{In}(V^*),$

$\texttt{Assume}\ \epsilon : \mathbb{R}_{++},$

$(N, 1) := \eth\texttt{WeakLimit}(L)\big(x, X, f^{-1}\mathbb{B}_K(f(X), \epsilon)\big) : \sum N \in \mathbb{N} \,.\, \forall (n, *) : \sum n \in \mathbb{N} \,.\, n \geq N \,.$

$\quad .\, |f(x_n) - f(X)| < \epsilon;$

$\rightsquigarrow () := \eth\texttt{Limit}(K) : \lim_{n\to\infty} f(x_n) = f(X);$

$\rightsquigarrow () := \texttt{UniIntro} : \forall f \in V^* \,.\, \lim_{n\to\infty} f(x_n) = f(X);$

$\rightsquigarrow \texttt{Left} := \texttt{ImplicationIntro} : \mathbf{w} \lim_{n\to\infty} x_n = X \implies \forall f \in V^* \,.\, \lim_{n\to\infty} f(x_n) = f(X),$

$\texttt{Assume}\ R : \forall f \in V^* \,.\, \lim_{n\to\infty} f(x_n) = f(X),$

$\texttt{Assume}\ U : \mathcal{U}_{\mathbf{w}(V)}(X),$

$(m, f, a, 1) := \texttt{NeighbourhoodInBase}\big(\eth\mathbf{w}(V)\big)(U) : \sum m \in \mathbb{N} \,.\, \sum (f, a) : m \to V^* \times \mathbb{R}_{++} \,.$

$\quad .\, \bigcap_{i=1}^{m} \{v \in V : |f_i(v) - f_i(X)| < a_i\} \subset U,$

$\texttt{Assume}\ (i, 2) : \sum i \in \mathbb{N} \,.\, i \leq n,$

$(3) := R(f_i) : \lim_{n\to\infty} f_i(x_n) = f_i(X),$

$(N_i, o_i) := \eth\texttt{Convergent}(f_i(x), a_i, 3) : \sum N_i \in \mathbb{N} \,.\, \forall (n, *) : \sum n \in \mathbb{N} \,.\, n \geq N_i \,.\, |f_i(x_n) - f_i(X)| \leq a_i;$

$\rightsquigarrow (N, o) := \texttt{FuncIntro} : \prod i \in n \,.\, \sum N_i \in \mathbb{N} \,.\, \forall n \in \sum (n, *) : \mathbb{N} \,.\, n \geq N_i \,.\, |f_i(x_n) - f_i(X)| \leq a_i,$

$M := \max_{i\in n} N_i : \mathbb{N},$

$() := \eth\texttt{Subset}\eth\texttt{Intersection}\texttt{IncreasingTipologicalSumPropogation}(M, o) :$

$\quad : \forall (n, *) \in \sum n \in \mathbb{N} \,.\, n \geq M \,.\, x_n \in U;$

$\rightsquigarrow () := \eth^{-1}\texttt{WeakLimit}\eth^{-1}\texttt{TopologicalLimit}(\mathbf{w}(V)) : \mathbf{w} \lim_{n\to\infty} x_n = X;$

$\rightsquigarrow (*) := \texttt{IffIntro}(\texttt{Left}, \texttt{ImplicationIntro}) : \mathbf{w} \lim_{n\to\infty} x_n = X \iff \forall f \in V^* \,.\, \lim_{n\to\infty} f(x_n) = f(X);$

$\square$

$\texttt{WeakStarLimitCharacteristic} :: \forall V : \mathsf{NORM}(K) \,.\, \forall f : \mathbb{N} \to V^* \,.\, \forall F \in V^*$

$\quad \mathbf{w}^* \lim_{n\to\infty} f_n = F \iff \forall v \in V \,.\, \lim_{n\to\infty} f_n(v) = F(v)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; L : \mathbf{w}^* \lim_{n\to\infty} f_n = F,$

$\texttt{Assume}\; v : \texttt{In}(V),$

$\texttt{Assume}\; \epsilon : \mathbb{R}_{++},$

$(N,1) := \eth\texttt{WeakStarLimit}(L)\big(f, F, \alpha_v^{-1}\mathbb{B}_K(F(v),\epsilon)\big) : \sum N \in \mathbb{N} \,.\, \forall (n,*) : \sum n \in \mathbb{N} \,.\, n \geq N \,.$

$\quad .\, |f_n(v) - F(v)| < \epsilon;$

$\rightsquigarrow () := \eth\texttt{Limit}(K) : \lim_{n\to\infty} f_n(v) = F(v);$

$\rightsquigarrow () := \texttt{UniIntro} : \forall v \in V \,.\, \lim_{n\to\infty} f_n(v) = F(v);$

$\rightsquigarrow \texttt{Left} := \texttt{ImplicationIntro} : \mathbf{w}^* \lim_{n\to\infty} f_n = F \Longrightarrow \forall v \in V \,.\, \lim_{n\to\infty} f_n(v) = F(v),$

$\texttt{Assume}\; R : \forall v \in V \,.\, \lim_{n\to\infty} f_n(v) = F(v),$

$\texttt{Assume}\; U : \mathcal{U}_{\mathbf{w}^*(V)}(F),$

$(m, v, a, 1) := \texttt{NeighbourhoodInBase}\big(\eth\mathbf{w}^*(V)\big)(U) : \sum m \in \mathbb{N} \,.\, \sum (f,a) : m \to V \times \mathbb{R}_{++} \,.$

$\quad .\, \bigcap_{i=1}^{m} \{g \in V^* : |g(v_i) - F(v_i)| < a_i\} \subset U,$

$\texttt{Assume}\; (i,2) : \sum i \in \mathbb{N} \,.\, i \leq n,$

$(3) := R(v_i) : \lim_{n\to\infty} f_n(v_i) = F(v_i),$

$(N_i, o_i) := \eth\texttt{Convergent}(f(v_i), a_i, 3) : \sum N_i \in \mathbb{N} \,.\, \forall (n,*) : \sum n \in \mathbb{N} \,.\, n \geq N_i \,.\, |f_n(v_i) - F(v_i)| \leq a_i;$

$\rightsquigarrow (N, o) := \texttt{FuncIntro} : \prod i \in n \,.\, \sum N_i \in \mathbb{N} \,.\, \forall n \in \sum (n,*) : \mathbb{N} \,.\, n \geq N_i \,.\, |f_n(v_i) - F(v_i)| \leq a_i,$

$M := \max_{i\in n} N_i : \mathbb{N},$

$() := \eth\texttt{Subset}\eth\texttt{Intersection}\texttt{IncreasingTipologicalSumPropogation}(M, o) :$

$\quad : \forall (n,*) \in \sum n \in \mathbb{N} \,.\, n \geq M \,.\, f_n \in U;$

$\rightsquigarrow () := \eth^{-1}\texttt{WeakStarLimit}\eth^{-1}\texttt{TopologicalLimit}(\mathbf{w}^*(V)) : \mathbf{w}^* \lim_{n\to\infty} f_n = F;$

$\rightsquigarrow (*) := \texttt{IffIntro}(\texttt{Left}, \texttt{ImplicationIntro}) : \mathbf{w}^* \lim_{n\to\infty} f_n = F \iff \forall v \in V \,.\, \lim_{n\to\infty} f_n(v) = F(v);$

$\square$

$\texttt{FiniteDimWeakIsNormal} :: \forall (V,d) : \sum V : \mathsf{NORM}(K) \ . \ \dim V < \infty \ . \ \mathbf{w}(V) = \mathcal{T}(V)$

$\texttt{Proof} \ =$

$(n,1) := \eth\texttt{Infinity}(d) : \sum n \in \mathbb{N} \ . \ \dim V = n,$

$(2) := \texttt{FiniteDimClasification}(1, V, K^n_\infty) : V \cong_{\mathsf{NORM}} K^n_\infty,$

$\texttt{Assume } v : K^n_\infty,$

$\texttt{Assume } r : \mathbb{R}_{++},$

$(3) := \eth\texttt{ball}(K^n_\infty)(v,r)\eth\texttt{maxnorm}\eth^{-1}\texttt{intersection}\eth^{-1}\texttt{orth}((K^n)^*) :$

$: \mathbb{B}^n_\infty(v,r) = \left\{x \in V : \|x-v\|_\infty < r\right\} = \{x \in V : \forall i \in n \ . \ |x_i - v_i| < r\} =$

$$= \bigcap_{i=1}^n \left\{x \in V : |x_i - v_i| < r\right\} = \bigcap_{i=1}^n \left\{x \in V : |e^*_i(x-v)| < r\right\},$$

$() := \eth\mathbf{w}\left(K^n_\infty\right)\eth\texttt{weakTopologyBase}(n, e^*, r)\mathbb{B}^n_\infty(v,r) : \mathbb{B}^n_\infty(v,r) \in \mathbf{w}(K^n_\infty);$

$\rightsquigarrow (3) := \eth^{-1}\texttt{SetEq}\Big(\eth\mathcal{T}\left(K^n_\infty\right)\texttt{BaseDefinesTopology}(\cdot), \texttt{WeakIsWeak}\left(K^n_\infty\right)\Big) : \mathcal{T}\left(K^n_\infty\right) = \mathbf{w}\left(K^n_\infty\right),$

$(*) := (3)(2) : \mathcal{T}(V) = \mathbf{w}(V);$

$\square$

4.2 Weak Boundedness

$\texttt{WeaklyOpenIsUnbounded} :: \forall (V,d) : \sum V : \mathsf{NORM}(K) \,.\, \dim V = \infty \,.\, \forall U \in \mathbf{w} \,\&\, \texttt{Nonempty}(V) \,.$

$\quad . \, U : \texttt{Unbounded}(V)$

$\texttt{Proof} \; =$

$(m,f,a,x,1) := \eth\mathbf{w}(V)(U) : \sum m \in \mathbb{N} \,.\, \sum (f,a) : m \to V^* \times \mathbb{R}_{++} \,.$

$$. \sum x \in V \,.\, \bigcap_{i=1}^{m} \{v \in V : |f_i(v) - f_i(x)| < a_i\} \subset U,$$

$$(2) := \texttt{CodimOfIntersction}\,\texttt{KerCodimIsDimOfIm}(f) : \operatorname{codim} \bigcap_{i=1}^{m} \ker f_i \le m,$$

$$(3) := \eth\texttt{codim}(d(2)) : \bigcap_{i=1}^{m} \ker f_i \ne \{0\},$$

$$(v,4) := \eth\texttt{NontrivialSubspace}(V)(3) : \sum v \in \bigcap_{i=1}^{m} \ker f_i \,.\, v \ne 0,$$

$(5) := \texttt{LineIsUnbounded}(V, 4, Kv + x) : \big(Kv + x : \texttt{Unbounded}(V)\big),$

$(6) := \eth\texttt{Subset}(1, \eth\ker(f)(v)) : Kv + x \subset U,$

$(*) := \texttt{UnboundednessPropogation}(5,6) : \big(U : \texttt{Unbounded}(V)\big);$

$\square$

$\texttt{BoundedSteinhausConvergence} :: \forall V : \mathsf{BAN}(K) \ . \ \forall W : \mathsf{NORM}(K) \ . \ \forall T : \mathbb{N} \to \mathcal{B}(V, W) \ .$

$\forall (A, c) : \sum A \in V \to W \ . \ \forall x \in V \ . \ \lim_{n\to\infty} T_n(x) = A(x) \ . \ A \in \mathcal{B}(V, W) \ \& \ \|A\| \le \lim_{n\to\infty} \inf \|T_n\|$

$\texttt{Proof} \ =$

$(1) := \texttt{ContAddition}(W)(\Lambda x, y \in V \ . \ T(x+y))\, \texttt{ContScalarMult}\big(\Lambda a in K \ . \ \Lambda x \in V \ . \ T(ax)\big) : (A : \mathcal{L}(V, W)),$

$(2) := \eth^{-1}\texttt{PoinwiselyBoundedOperatorFamily}(c) : \big(T : \texttt{PoinwiselyBoundedOperatorFamily}(\mathbb{N}, V, W)\big),$

$(3) := \texttt{BanachSteinhaus}(T, 2) : \big(T : \texttt{UniformlyoundedOperatorFamily}(\mathbb{N}, V, W)\big),$

$(C, 4) := \eth\texttt{UniformlyBoundedOperatorFamily}(T) : \sum C \in \mathbb{R}_{++} \ . \ \forall n \in \mathbb{N} \ . \ \|T_n\| \le C,$

$\texttt{Assume}\ x : V,$

$() := \texttt{EqEl}(\|Ax\|, c)\texttt{ContNorm}(T_n x)(4)\texttt{ConstantLimit} :$

$: \|Ax\| = \left\| \lim_{n\to\infty} T_n x \right\| = \lim_{n\to\infty} \|T_n x\| \le \lim_{n\to\infty} C\|x\| = C\|x\|;$

$\leadsto (5) := \eth^{-1}\mathcal{B}(V, W) : \big(A : \mathcal{B}(V, W)\big),$

$\texttt{Assume}\ x : V,$

$(6) := \eth\texttt{operatorNorm} : \forall n \in \mathbb{N} \ . \ \|T_n x\| \le \|T_n\|\|x\|,$

$(1) := \texttt{EqEl}(\|Ax\|, c)\texttt{ContNorm}(T_n x)\texttt{MajorizedConvergence}(6) :$

$: \|Ax\| = \left\| \lim_{n\to\infty} T_n x \right\| = \lim_{n\to\infty} \|T_n x\| \le \left(\liminf_{n\to\infty} \|T_n\| \right) \|x\|;$

$\leadsto (*) := \eth\texttt{operatorNorm}(A) : \|A\| \le \liminf_{n\to\infty} \|T_n\|;$

$\square$

$\texttt{WeaklyBounded} :: \prod V : \mathsf{NORM}(K) \ . \ ??V$

$A : \texttt{WeaklyBounded} \iff \forall f \in V^* \ . \ f(A) : \texttt{Bounded}(K)$

$\texttt{WeaklyStarBounded} :: \prod V : \mathsf{NORM}(K) \ . \ ??V^*$

$A : \texttt{WeaklyStarBounded} \iff \forall x \in V \ . \ \alpha_x(A) : \texttt{Bounded}(K)$

$\texttt{BanachSteinhausII} :: \forall A : \texttt{WeaklyBounded}(V) \ . \ A : \texttt{Bounded}(V)$

$\texttt{Proof} \ =$

Apply $\texttt{BoundedFunctionalsTHM}$

$\texttt{BanachSteinhausStar} :: \forall A : \texttt{WeaklyStarBounded}(V) \ . \ A : \texttt{Bounded}(V^*)$

$\texttt{Proof} \ =$

Apply $\texttt{BoundedFunctionalsTHM}$

$\texttt{WeakStarContinuousFunctionalsAreApplicants} :: \forall V \in \mathsf{NORM}(K) .$

$\quad . F \in V^{**} \ \& \ \texttt{Continuous}(V^*, \mathbf{w}^*(V)) . \exists x \in V : F = \alpha_x$

$\texttt{Proof} \ =$

$\texttt{Assume}\ d : \dim V < \infty,$

$() := \texttt{HilbertIsReflexive}\,\texttt{FiniteDimensionalClassification}(V, d) : \exists x \in V : F = \alpha_x;$

$\rightsquigarrow (1) := \texttt{ImplicationIntro} : \dim V < \infty \Rightarrow \exists x \in V : F = \alpha_x,$

$\texttt{Assume}\ d : \dim V = \infty,$

$(U, 2) := F^{-1}\mathbb{B}_K : \sum U \in \mathcal{U}_{\mathbf{w}^*(V)}(0) . \forall f \in U . |F(f)| < 1,$

$(m, v, a, 3) := \eth\mathbf{w}^*(V)(U) : \sum m \in \mathbb{N} . \sum (v, a) : m \to V \times \mathbb{R}_{++} . \bigcap_{i=1}^{m} \{f \in V^* : |f(v_i)| < a_i\} \subset U,$

$(\mathcal{F}, 4) := \texttt{KernelExtension}\big(d, \{v_i | i \in m\}\big) : \sum \mathcal{F} : \texttt{NontrivialSubspace}(V^*) .$

$\quad . \forall f \in \mathcal{F} . \{v_i | i \in m\} \subset \ker f,$

$\texttt{Assume}\ f : \texttt{In}(\mathcal{F}),$

$\texttt{Assume}\ c : \texttt{In}(K),$

$\texttt{Assume}\ i : \texttt{In}(m),$

$() := \texttt{EqEl}(|cf(v_i)|, \eth \ker(4))\texttt{ZeroAbsValuePositiveRealsLB}(a_i) : |cf(v_i)| = |0| = 0 < a_i;$

$\rightsquigarrow (5) := (3) : cf \in U,$

$() := \eth_2\mathcal{L}(F)(c, f)\eth_2\texttt{Norm}(c, F(f))(2)(5) : |c|\|F(f)\| = \|cF(f)\| < 1;$

$\rightsquigarrow () := \eth_3^{-1}\texttt{Norm}\,\texttt{MultiplicativelyNonIncreasingObject} : F(f) = 0;$

$\rightsquigarrow (5) := \eth^{-1} \ker : \mathcal{F} \subset \ker F,$

$(6) := \eth \ker \eth\alpha(v)(5) : \bigcap_{i=1}^{m} \ker \alpha_{v_i} \subset \mathcal{F} \subset \ker F,$

$(b, 7) := \texttt{KernelIntersectionLinearCombination}(6) : \sum b : m \to K . F = \sum_{i=1}^{m} b_i \alpha_{v_i},$

$x := \sum_{i=1}^{m} b_i v_i : V,$

$() := \eth\alpha\eth x(7) : F = \alpha_x;$

$\rightsquigarrow (*) := \texttt{OrElimination}(\dim V < \infty | \dim V = \infty, 1, \cdot) : \exists x \in V : F = \alpha_x;$

$\square$

4.3 Separation Theorems

$\texttt{HahnBanachSeparation} :: \forall V : \mathsf{BAN}(\mathbb{R}) \,.\, \forall (A, B, o) : \sum A, B : \texttt{Convex} \,\&\, \texttt{Closed}(V) \,.\, A \cap B = \emptyset \,.$
$\quad A : \texttt{Compact}(V, \mathbf{w}(V)) \implies \exists f \in V^* : \sup_{a \in A} f(a) < \inf_{b \in B} f(b)$

$\texttt{Proof} \;=$

$\texttt{Assume } x : A,$

$(f, c, 2) := \texttt{SeparatingHyperplaneExists}(x, B) : \sum (f, c) \in V^* \times \mathbb{R} \,.\, f(x) = 0 \,\&\, \forall b \in B \,.\, f(b) \geq c,$

$U(x) := \{x \in V : f(x) < c\} : \texttt{In}\Big(\mathcal{U}_{(V, \mathbf{w}(V))}(0)\Big),$

$(3) := \eth U(x)(2) : x + U(x) \cap B = \emptyset;$

$\leadsto U := \texttt{DepFuncIntro} : \prod x \in A \,.\, \sum U(x) \,.\, \mathcal{U}_{(V, \mathbf{w}(V))}(0) \,.\, x + U(x) \cap B = \emptyset,$

$\mathcal{O} := \Big\{x + U(x) : x \in X\Big\} :?\mathbf{w}(U),$

$(2) := \eth\texttt{union}\eth\mathcal{O}\eth U : A \subset \bigcup \mathcal{O},$

$(n, a, 3) := \eth\texttt{Compact}(V, \mathbf{w}(V)) : \sum n \in \mathbb{N} \,.\, \sum a : n \to A \,.\, A \subset \bigcup_{i=1}^{n} a_i + U(a_i),$

$Y := \bigcap_{i=1}^{n} U(a_i) : \mathcal{U}_{(V, \mathbf{w}(V))}(0),$

$(4) := \texttt{ConvexIntersection}\big(Y, U(a)\big) : \big(Y : \texttt{Convex}(V)\big),$

$O := A + Y : \texttt{Open} \,\&\, \texttt{Convex}(V),$

$(5) := \eth\texttt{NonIntersecting}\eth O\eth U : O \cap B = \emptyset,$

$X := O - B : \texttt{Open} \,\&\, \texttt{Convex}(V),$

$(6) := \eth(X)(5)(0) : 0 \notin X,$

$(f, c, 7) := \texttt{SeparatingHyperplaneExists}(0, X, 6) : \sum (f, c) \in V^* \times \mathbb{R}_{++} \,.\, f(0) = c \,\&\, \forall x \in X \,.\, f(x) < -c,$

$(8) := (7)\texttt{MorphismOfZero}(f) : c = f(0) = 0,$

$(y, 9) := \eth\mathcal{L}(f)\eth\mathcal{U}_{(V, \mathbf{w}(V))}(0) : \sum y \in Y \,.\, f(y) > 0,$

$\texttt{Assume } a : A,$

$\texttt{Assume } b : B,$

$(10) := \eth(X)(a - b) : a - b \in X,$

$() := (7)(a - b) : f(a) \leq f(b);;$

$\leadsto (11) := \texttt{UniIntro} : \forall a \in A \,.\, \forall b \in B \,.\, f(a) + f(y) < f(b),$

$(12) := \texttt{LimitIneq}(10) : \sup_{a \in A} f(a) + f(y) \leq \inf_{b \in B} f(b),$

$(*) := \texttt{NonNegSubstractIneq}(12, 9, f(y)) : \sup_{a \in A} f(a) < \inf_{b \in B} f(b);$

$\square$

$\texttt{WeakStarSeparation} :: \forall V : \mathsf{BAN}(\mathbb{R}) . \forall A : \texttt{Convex} \,\&\, \texttt{NonZero}(V^*) \,\&\, \texttt{Closed}(V^*, \mathbf{w}^*(V)) .$

$\quad . \forall f \in A^{\complement} . \exists x \in V . \sup_{a \in A} a(x) < f(x)$

$\texttt{Proof} \;=$

$\big(U, (1)\big) := A^{\complement} : \texttt{Open}\big(\mathbf{w}^*(V)\big),$

$(n, v, a, 1) := \eth\texttt{NeighbourhoodInBase}(\eth\mathbf{w}^*(V), U, f) : \sum n \in \mathbb{N} .$

$\quad . \sum (v, a) : n \to V \times \mathbb{R}_{++} . \bigcap_{i=1}^{n} \{g \in V^* : |g(v_i) - f(v_i)| < a_i\} \subset U,$

$O := \bigcap_{i=1}^{n} \{g \in V^* : |g(v_i) - f(v_i)| < a_i\} : \mathcal{U}_{\mathbf{w}^*(V)}(f),$

$(2) := \eth O : O : \texttt{Convex}(V^*),$

$(3) := \eth O(1) : O \cap A = \emptyset,$

$X := O - A : \mathbf{w}^*(V) \,\&\, \texttt{Convex}(V),$

$X := \eth X(3) : 0 \notin X,$

$(F, c, 4) := \texttt{SeparatingHyperplaneExists}(0, X, 6) : \sum (F, c) \in V^{**} \times \mathbb{R}_{++} .$

$\quad . F(0) = c \,\&\, \forall x \in X . F(x) > c,$

$(5) := (4)\texttt{MorphismOfZero}(F) : c = F(0) = 0,$

$a := \eth\texttt{NonEmpty}(A) : \texttt{In}(A),$

$\texttt{Assume}\ u : \texttt{In}(O - \{f\}),$

$() := (4)(f + u - a) : -F(u) < -F(a) + F(f);$

$\rightsquigarrow (6) := \texttt{UniIntro} : \forall u \in O - \{f\} . -F(u) < -F(a) + F(f),$

$(7) := \texttt{IneqLimit}(6) : \sup_{u \in O - \{f\}} -F(u) \leq -F(a) + F(f),$

$\texttt{Assume}\ g : \bigcap_{i=1}^{n} \ker \alpha_{v_i},$

$(8) := \eth O(g) : g \in O - \{f\},$

$\texttt{Assume}\ r : \mathbb{R},$

$() := \eth_2 \mathcal{L}(F)(r, g)\eth\texttt{supremum}(7)(8, rg : -rF(g) = -F(rg) \leq -F(a) + F(f);$

$\rightsquigarrow (9) := \texttt{UniIntro} : \forall r \in \mathbb{R} . -rF(g) \leq -F(a) + F(f),$

$(10) := \texttt{MulticativelyBoundedObject}(\mathbb{R})(9) : F(g) = 0,$

$() := \eth \ker(10) : g \in \ker F;$

$\rightsquigarrow (8) := \eth^{-1}\texttt{Subset} : \bigcap_{i=1}^{n} \ker \alpha_{v_i} \subset \ker F,$

$(b, 9) := \texttt{KernelIntersectionLinearCombination}(8) : \sum b : n \to \mathbb{R} . F = \sum_{i=1}^{n} b_i \alpha_{v_i},$

$x := \sum_{i=1}^{n} b_i v_i : V,$

$(10) := \eth\alpha\eth x(7) : F = \alpha_x,$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ u : O - \{f\},$

$() := (10)(7)(f + u - A) : f(x) + u(x) > a(x);$

$\rightsquigarrow (11) := \texttt{LimitIneq}\,\texttt{UniIntro}^2 : f(x) + \inf_{u \in O - \{f\}} u(x) \geq \sup_{a \in A} a(x),$

$(12) := \eth \mathcal{U}_{\mathbf{w}^*(V)}(o)(O - \{f\}) : \inf_{u \in O - \{f\}} u(x) < 0,$

$(*) := \texttt{PositiveAddIneq}\left(12, 11, \inf_{u \in O - \{f\}}\right) : f(x) > \sup_{a \in A} a(x);$

$\square$

$\texttt{Mazur} :: \forall V : \mathsf{BAN}(\mathbb{R}) \,.\, \forall A : \texttt{Convex} \,\&\, \texttt{Closed}(V) \,.\, A : \texttt{Closed}(V, \mathbf{w}(V))$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : A^{\complement},$

$(f, c, 1) := \texttt{SeparatingHyperplaneExists}(x, X, 6) : \sum (f, c) \in V^* \times \mathbb{R}_{++} \,.\, f(x) > c \,\&\, \forall a \in A \,.\, f(a) < c,$

$U(x) := \{v \in V : f(v) > c\} : \mathbf{w}(V),$

$() := \eth U(x)(1)(A) : A \cap U(x) = \emptyset,$

$() := \eth U(x) : x \in U(x);$

$\rightsquigarrow U := \texttt{DepFuncIntro} : \prod x \in A^{\complement} \,.\, \sum U(x) \in \mathbf{w}(U) \,.\, x \in U(x) \,\&\, U(x) \cap A = \emptyset,$

$(O) := \bigcup_{x \in A^{\complement}} U(x) : \mathbf{w}(V),$

$(2) := \eth O \eth_1 U : A^{\complement} \subset U(x),$

$(3) := \eth O \eth_2 U : U(x) \cap A = \emptyset,$

$(4) := \texttt{ComlempletionSubset}(2, 3) : O = A^{\complement},$

$(*) := \eth^{-1}\texttt{Closed}(O, 4) : \Big(A : \texttt{Closed}\big(\mathbf{w}(V)\big)\Big);$

$\square$

$\texttt{ConvexConvergence} :: \forall V : \mathsf{BAN}(K) \,.\, \forall x : \texttt{WeaklyConvergent}(V) \,.$
$\quad \exists y \in \mathbb{N} \to \operatorname{conv}(x(\mathbb{N})) : \texttt{Convergent}(V) : \lim_{n \to \infty} y_n = \mathbf{w} \lim_{n \to \infty} x_n$

$\texttt{Proof} \ =$

$A := \operatorname{cl} \operatorname{conv}\{x_n\}_{n=1}^{\infty} : \texttt{Convex} \,\&\, \texttt{Closed}(V),$

$(1) := \texttt{Mazur}(A) : \Big(A : \texttt{Closed}\big(\mathbf{w}(V)\big)\Big),$

$(2) := \texttt{ClosedConvergence}(1, x) : \mathbf{w} \lim_{n \to \infty} x_n \in A,$

$(y, 1) := \texttt{AllPointsAreLimits}\left(\operatorname{conv} A, \mathbf{w} \lim_{n \to \infty} x_n\right) : \sum y : \mathbb{N} \to \operatorname{conv}(A) \,.\, \lim_{n \to \infty} y_n = \mathbf{w} \lim_{n \to \infty} x_n;$

$\square$

4.4 Metrization and Separability

$\texttt{Alaoglu} :: \forall V : \mathsf{BAN}(K) \, . \, \overline{\mathbb{B}}_{V^*} : \texttt{Compact}(V^*, \mathbf{w}^*(V))$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{constrictTopology}(\eth\texttt{productTopology}(K), \eth\mathbf{w}^*(V)) : \mathbf{w}^*(V) = \texttt{constrictTopology}\Big(\mathcal{T}(K^V), V^*\Big),$

$\texttt{Assume}\ x : V,$

$A_x := \{a \in K : |a| \le \|x\|\} : \texttt{Set}(K),$

$() := \texttt{LocallyCompactField}(K, A_x) : (A_x : \texttt{Compact}(K));$

$\rightsquigarrow A := \texttt{FuncIntro} : V \to \texttt{Compact}(K),$

$B := \prod_{x \in V} A_x : \texttt{Set}\Big(K^V\Big),$

$(2) := \mathrm{GTOP}.\texttt{Tychonoff}\Big(K^V, B, \eth B\Big) : \Big(B : \texttt{Compact}\Big(K^V\Big)\Big),$

$(3) := \eth A \eth B \eth\texttt{ClosedBall}(V^*) : \overline{\mathbb{B}}_{V^*} \subset B,$

$(4) := \texttt{CompacConstriction}(1, B) : \Big(B \cap V^* : \texttt{Compact}\big(\mathbf{w}^*(V)\big)\Big),$

$\texttt{Assume}\ (\mathcal{A}, f) : \texttt{ConverginNet}(\overline{\mathbb{B}}_{V^*}, \mathbf{w}^*(V)),$

$F := \lim_{a \in \mathcal{A}} : V^*,$

$\texttt{Assume}\ x : V,$

$\texttt{Assume}\ a : \mathcal{A},$

$() := \eth\overline{\mathbb{B}}_{V^*} : |f_a(x)| \le \|x\|;$

$\rightsquigarrow (5) := \texttt{UniIntro} : \forall a \in \mathcal{A} \, . \, |f_a(x)| \le \|x\|,$

$(6) := \texttt{LimIneq}(5) : \lim_{a \in \mathcal{A}} f_a(x) \le \|x\|,$

$(7) := \eth F(x) : \lim_{a \in \mathcal{A}} f_a(x) = F(x),$

$() := (5)(6) : F(x) \le \|x\|;$

$\rightsquigarrow (5) := \eth^{-1}\overline{\mathbb{B}}_{V^*} : F \in \overline{\mathbb{B}}_{V^*};$

$\rightsquigarrow (5) := \mathrm{GTOP}.\texttt{ClosedByNets} : \Big(\overline{\mathbb{B}}_{V^*} : \texttt{Closed}\big(\mathbf{w}^*(V)\big)\Big),$

$(*) := \mathrm{GTOP}.\texttt{ClosedSubsetOfCompact}(4, 5) : \Big(\overline{\mathbb{B}}_{V^*} : \texttt{Compact}(V^*, \mathbf{w}^*(V))\Big);$

$\square$

$\texttt{WeakStarMetrization} :: \forall V : \mathsf{BAN}(K) \ . \ (\mathbb{B}^*_V, \mathbf{w}^*(V)) : \texttt{Metrizable} \iff V : \texttt{Separable}$

$\texttt{Proof} =$

$\texttt{Assume Right} : (V : \texttt{Separable}),$

$(Q, 1) := \eth\texttt{Separable}(V) : \sum Q : \texttt{Dense}(V) \ . \ \#Q = \aleph_0,$

$(q, 2) := \texttt{enumerate}(Q \cap \mathbb{S}_V, 1) : \sum q : \mathbb{N} \to Q \cap \mathbb{S}_V \ . \ Q \cap \mathbb{S}_V = \{q_n | n \in \mathbb{N}\},$

$$\rho := \Lambda f, g \in V^* \ . \ \sum_{n=1}^{\infty} \frac{|f(q_n) - g(q_n)|}{2^n} : V^* \times V^* \to \mathbb{R}_+,$$

$\texttt{Assume } f : V^*,$

$$() := \eth\rho(f, f)\eth^{-1}\texttt{Zero}(K)\texttt{ZeroSum} : \rho(f, f) = \sum_{n=1}^{\infty} \frac{|f(q_n) - f(q_n)|}{2^n} = \sum_{n=1}^{\infty} 0 = 0;$$

$\leadsto (3) := \texttt{UniIntro} : \forall f \in V^* \ . \ \rho(f, f) = 0,$

$\texttt{Assume } f, g : V^*,$

$() := \eth\rho(f, g)\texttt{AbsValSubtractCommute}(f(q), g(q))\eth^{-1}\rho(f, g) :$

$$: \rho(f, g) = \sum_{n=1}^{\infty} \frac{|f(q_n) - g(q_n)|}{2^n} = \sum_{n=1}^{\infty} \frac{|g(q_n) - f(q_n)|}{2^n} = \rho(g, f);$$

$\leadsto (4) := \texttt{UniIntro} : \forall f, g \in V^* \ . \ \rho(f, g) = \rho(g, f),$

$\texttt{Assume } f, g, h : V^*,$

$$() := \eth\rho(f, g)\texttt{TriangleIneq}(f(q), g(q), h(q))\texttt{SumIsLinear}\eth^{-1}\rho(f, g) : \rho(f, g) = \sum_{n=1}^{\infty} \frac{|f(q_n) - g(q_n)|}{2^n} \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{|f(q_n) - h(q_n)|}{2^n} + \frac{|h(q_n) - g(q_n)|}{2^n} = \sum_{n=1}^{\infty} \frac{|f(q_n) - h(q_n)|}{2^n} + \sum_{n=1}^{\infty} \frac{|h(q_n) - g(q_n)|}{2^n} = \rho(f, h) + \rho(h, g);$$

$\leadsto (5) := \texttt{UniIntro} : \forall f, g, h \in V^* \ . \ \rho(f, g) \leq \rho(f, h) + \rho(h, g),$

$(6) := \eth^{-1}\texttt{Distance}(3, 4, 5) : \big(\rho : \texttt{Distance}(V^*)\big),$

$\texttt{Assume } U : \mathcal{T}_{\mathbb{B}_{V^*}, \rho},$

$\texttt{Assume } f : \texttt{In}(U),$

$(\varepsilon, 7) := \texttt{MetricNeighbourhood}(\rho)(f, U) : \prod \varepsilon \in \mathbb{R}_+ \ . \ \mathbb{B}_{\mathbb{B}_{V^*}, \rho}(f, \varepsilon) \subset U,$

$B := \mathbb{B}_{\mathbb{B}_{V^*}}(f, \varepsilon) : \mathcal{T}(\rho),$

$(n, 8) := \texttt{ArchemedeanProperty Assymptot}(\Lambda x \in \mathbb{R}.2^{-x}, \frac{\varepsilon}{2}) :$

$$\prod n \in \mathbb{N} \ . \ 2^{-n} < \frac{\varepsilon}{4},$$

$$A_f := \bigcap_{i=1}^{n} \left\{ g \in \mathbb{B}_{V^*} : |g(q_i) - f(q_i)| < \frac{\varepsilon}{2n} \right\} : \mathbf{w}^*,$$

$\texttt{Assume } g : \texttt{In}(A_f),$

$(9) := \eth\rho(f, g)\eth A_f(g)\texttt{TriangleIneq}(f(q_i), g(q_i))\eth\texttt{OperarorNorm}(f)\eth\texttt{OperatorNorm}(g)\texttt{PowerSeria}(2^{-1})\eth n :$

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{|f(q_i) - g(q_i)|}{2^i} < \frac{\varepsilon}{2} + \sum_{i=n+1}^{\infty} \frac{|f(q_i) - g(q_i)|}{2^i} \leq \frac{\varepsilon}{2} + 2\sum_{i=n+1}^{\infty} 2^{-i} = \frac{\varepsilon}{2} + 2^{-n+1} < \varepsilon,$$

$() := \eth^{-1}B(g, 9) : g \in B;$

$\leadsto (9) := \eth^{-1}\texttt{Subset} : A_f \subset U,$

$10_f := (9)(7) : A_f \subset U,$

$() := \eth^{-1}A_f(f) : f \in A_f;$

$\rightsquigarrow (A,7) := I\left(\prod\right) : \prod f \in U \, . \, \sum A_f \in \mathbf{w}^*_{\mathbb{B}_{V^*}}(V) \, . \, f \in A_f \,\&\, A_f \subset U,$

$(8) := \eth\texttt{Union}(A)(7) : U = \bigcup_{f \in U} A_f,$

$() := (8)\eth A : U \in \mathbf{w}^*;$

$\rightsquigarrow (7) := \eth\texttt{Continuous} : \Big(\mathrm{id} : (\mathbf{B}_{V^*}, \mathbf{w}^*) \to_{\mathsf{TOP}} (\mathbf{B}_V, \rho)\Big),$

$8 := \mathrm{GTOP}.\texttt{HausdorfToCompactBijection}(\mathrm{id}, \mathrm{w}^*, \rho) : \Big(\mathrm{id} : (\mathbf{B}_{V^*}, \mathbf{w}^*) \leftrightarrow_{\mathsf{TOP}} (\mathbf{B}_V, \rho)\Big),$

$() := \eth^{-1}\texttt{Metrizable}(8) : (\mathbf{w}^* : \texttt{Metrizable});$

$\rightsquigarrow R := I(\Rightarrow) : \texttt{Rightarrow} \Rightarrow \texttt{Left},$

$\texttt{MetrizationWithCF} :: \forall X : \texttt{Hausdorff} \,\&\, \texttt{Compact} \, . \, C(X) : \texttt{Separable} \iff X : \texttt{Metrizable}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ L : \big(C(X) : \texttt{Separable}\big),$

$(1) := ../R\big(C(X), L\big) : \Bigg(\Big(\mathbb{B}_{C^*(X)}, \mathbf{w}^*\big(C(X)\big)\Big) : \texttt{Metrizable}\Bigg),$

$\big((X,d),\varphi\big) := \eth^{-1}\texttt{Metrizable}(1) : \sum (X,d) : \mathsf{MS} \, . \, (X,d) \leftrightarrow_{\mathsf{TOP}} \Big(\mathbb{B}_{C^*(X)}, \mathbf{w}^*\big(C(X)\big)\Big),$

$\texttt{Assume}\ x, y : \texttt{In}(X),$

$\alpha := \Lambda f \in C(X) \, . \, f(x) : \mathbb{B}_{C^*(X)},$

$\beta := \Lambda f \in C(X) \, . \, f(y) : \mathbb{B}_{C^*(X)},$

$\rho(x,y) := d\big(\varphi^{-1}(x), \varphi^{-1}(y)\big) : \mathbb{R}\eth\lessdot\sim_+;$

$\rightsquigarrow \rho := I(\to) : \rho : X \times X \to \mathbb{R}_+,$

$(2) := \eth\rho(\eth\varphi, \eth d) : \big(\rho : \texttt{Distance}(X)\big),$

$\texttt{Assume}\ U : \mathcal{T}_U,$

$K := U^{\complement} : \texttt{Closed}(X),$

$K := \eth X(\eth K) : (K : \texttt{Compact}),$

$\texttt{Assume}\ x : U,$

$\Delta := \Lambda y \in K \, . \, \rho(y,x) : \big((K \to \mathbb{R}_{++}\big),$

$(3) := \texttt{ContinuousComp}(\Delta) : \big(\Delta : K \to_{\mathsf{TOP}} \mathbb{R}_{++}\big),$

$(y,4) := \texttt{ExtremeValue}(K, \Delta, \min) : \sum y \in K \, . \, \Delta(y) = \min_{y \in K} \Delta(y),$

$A_x := \mathbb{B}_\rho(x, \Delta(y)) : \mathcal{T}_\rho,$

$5_x := \eth A_x : x \in A_x,$

$() := \eth y \eth\Delta\eth A_x : A_x \subset U;$

$\rightsquigarrow (A,3) := I\left(\prod\right) : \prod x \in U \, . \, \sum A_x \in \mathcal{T}_\rho \, . \, x \in A_x \,\&\, A_x \subset U,$

$(4) := \eth\texttt{Unioin}(A,3) : U = \bigcup_{x \in U} A_x,$

$() := \eth^{-1}\mathcal{T}_\rho : U \in \mathcal{T}_\rho;$

$\rightsquigarrow (3) := \eth\texttt{Continuous}(\mathrm{id}, \rho, \mathcal{T}_X) : \mathrm{id} : (X,\rho) \to_{\mathsf{TOP}} X,$

$(4) := \eth\rho : (X,\rho) \cong_{\mathsf{TOP}} (\delta(X), \mathbf{w}^*),$

$(5) := \texttt{Alaoglu}\mathrm{GTOP}.\texttt{SubspaceCompactness}\big(\delta(X)\big)(4) : \big((X,\rho) : \texttt{Compact}\big),$

$(6) := \mathrm{GTOP}.\texttt{HausdorffToCompactBijection}(\mathrm{id}, \rho, \mathcal{T}_X) : \mathrm{id} : (X,\rho) \leftrightarrow_{\mathsf{TOP}} X,$

$() := \eth^{-1}\texttt{Metrizable}(6) : X : \texttt{Metrizable};$

$\rightsquigarrow L := I(\Rightarrow) : C(X) : \texttt{Separable} \Rightarrow X : \texttt{Metrizable},$

$\texttt{Assume}\ R : X : \texttt{Metrizable},$

$(d,1) := \eth\texttt{Metrizable}(X) : \sum d\texttt{Distance}(X)\ .\ (X,d) \cong_{\mathsf{TOP}} X,$

$(2) := \texttt{MetricCompactIsSeparable} : \big((X,d) : \texttt{Separable}\big),$

$(x,3) := \texttt{denseSeq}(X,d) : \sum x : \mathbb{N} \to X\ .\ \{x_n : n \in \mathbb{N}\} : \texttt{Dense}(X),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ q : \mathbb{Q}_{++},$

$f_{n,q} := \texttt{CutOff}\big(x_n, \mathbb{B}(x_n,q)\big) : C(X);;$

$\rightsquigarrow f := I(\to) : \mathbb{N} \to \mathbb{Q}_{++} \to C(X),$

$A := \texttt{algebra}(\{f\} \cup \{1\}) : \texttt{Subalgebra}\big(C(X), \mathbb{Q}\big),$

$\texttt{Assume}\ (a,b,4) : \sum x,y \in X\ .\ x \neq y,$

$(n,5) := \eth\texttt{denseSeq}(x,y) : \sum n \in \mathbb{N}\ .\ d(x_n,x) < d(x_n,y),$

$(q,6) := \eth\texttt{Dense}(x,y) : \sum q \in \mathbb{Q}\ .\ d(x_n,x) < q < d(x_n,y),$

$(7) := \eth f_{n,q}(x)(6) : f_{n,q}(x) > 0,$

$(8) := \eth f_{n,q}(x)(6) : f_{n,q}(y) = 0,$

$() := (7)(8) : f_{n,q}(x) \neq f_{n,q}(y);$

$\rightsquigarrow (5) := \eth^{-1}\texttt{SeparatesPoints}(X) : A : \texttt{SeparatesPoints}(X),$

$(6) := \texttt{StoneWeierstrass}(C(X), A) : \Big(A : \texttt{Dense}\big(C(X)\big)\Big),$

$() := \eth^{-1}\texttt{Separable}(A,6) : \big(C(X) : \texttt{Separable}\big);$

$\rightsquigarrow (*) := I(\iff)(L) : C(X) : \texttt{Separable} \iff X : \texttt{Metrizable},$

$\square$

$\texttt{Assume}\ L : \Big(\big(\mathbb{B}_{V^*}, \mathbf{w}^*(V)\big) : \texttt{Metrizable}\Big),$

$(1) := \texttt{MetrizationsWithCF}(\mathbb{B}_{V^*}, \mathbf{w}^*(V)) : \Big(C(\mathbb{B}_{V^*}, \mathbf{w}^*(V)) : \texttt{Separable}\Big),$

$(2) := \eth\texttt{supNorm}\eth\texttt{unitBall}\eth\alpha : \alpha : V \to_{\mathsf{MS}} C(\mathbb{B}_{V^*}, \mathbf{w}^*(V)),$

$() := \texttt{TopologicalEmbeddingSep}(1,2) : \big(V : \texttt{Separable}\big);$

$\rightsquigarrow (*) := I(\iff)(R) : (\mathbb{B}^*_V, \mathbf{w}^*(V)) : \texttt{Metrizable} \iff V : \texttt{Separable};$

$\square$

$\texttt{WeakStarSeparability} :: \forall V : \mathsf{BAN}(K) \ \& \ \texttt{Separable} \ . \ (V^*, \mathbf{w}^*(V)) : \texttt{Separable}$

$\texttt{Proof} \ =$

$(1) := \texttt{WeakStarMetrization}(V) : \Big((\mathbb{B}_{V^*}, \mathbf{w}^*) : \texttt{Metrizable} \Big),$

$(2) := \text{GTOP}.\texttt{MetricCompactIsSeparable} : \Big((\mathbb{B}_{V^*}, \mathbf{w}^*) : \texttt{Separable} \Big),$

$(A, 3) := \eth\texttt{Separable}(\mathbb{B}_{V^*}, \mathbf{w}^*) : \sum A : \texttt{Countable}(\mathbf{B}_{Vj}) \ . \ \mathrm{cl}_{\mathbf{w}^*(V)}(A) = \mathbb{B}_{V^*},$

$(4) := \eth\texttt{zero}(\mathbb{Q})\eth A : 0 \in \mathbb{Q}A,$

$\texttt{Assume} \ (f, 5) : \sum f \in V^* \ . \ f \neq 0,$

$\texttt{Assume} \ U : \mathcal{U}_{\mathbf{w}^*(V)}(f),$

$(n, f, a, 6) := \eth\texttt{weakStarTopology}(U) : \sum n \in \mathbb{N} \ . \ \sum (x, a) : \mathbb{N} \to V \times \mathbb{R}_+ + \ . \ \bigcap_{i=1}^{n} \{g \in V^* : |f(x_i) - g(x_i)| < a_i\}$

$(h, 7) := (3) \left(\frac{f}{2\|f\|}, x, \frac{a}{2\|f\|}, \right) : \sum h \in A \ . \ h \in \bigcap_{i=1}^{n} \left\{ g \in V^* : \left| \frac{f(x_i)}{2\|f\|} - g(x_i) \right| < \frac{a_i}{2\|f\|} \right\},$

$(q, 8) := \texttt{RationalApproximation}(1/(2\|f\|), 6, 7) : \sum q \in \mathbb{Q} \ . \ qh \in U,$

$() := \eth^{-1}\texttt{SetProduct}(\mathbb{Q}, A)(q, h) : qh \in \mathbb{Q}A;$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Dense}(V^*) : (\mathbb{Q}A : \texttt{Dense}(V)),$

$(*) := \eth^{-1}\texttt{Separable} : (V : \texttt{Separable});$

$\square$

$\texttt{WeakSeparability} :: \forall V : \mathsf{BAN}(K) \ . \ (V, \mathbf{w}(V)) : \texttt{Separable} \Longrightarrow V : \texttt{Separable}$

$\texttt{Proof} \ =$

$S := \eth\texttt{Separable}(V, \mathbf{w}(V)) : \Big(S : \texttt{Dense} \ \& \ \texttt{Countable}\big(V, \mathbf{w}(V)\big) \Big),$

$D := \mathrm{span}(S) : \texttt{Subspace}(V),$

$C := \mathrm{span}(\mathbb{Q}, S) : \texttt{Countable}(V),$

$(1) := \eth D \eth C : \Big(C : \texttt{Dense}(D, \mathcal{T}_V) \Big),$

$(2) := \eth^{-1}\texttt{ConvexSet}\eth\,\mathrm{span}\,\eth D : (D : \texttt{ConvexSet}),$

$(3) := \texttt{Mazur}(D, 2) : \overline{D} = \overline{D}^{\mathbf{w}},$

$(4) := \eth S \eth D : \overline{D}^{\mathbf{w}} = V,$

$(5) := \eth\texttt{closure}(1)\eth D : \overline{C} = \overline{D},$

$(6) := (5)(3)(4) : \overline{C} = V,$

$(*) := \eth^{-1}\texttt{Separable}(6) : (V : \texttt{Separable});$

$\square$

$\texttt{Goldstine} :: \forall V : \mathsf{BAN}(K) \,.\, \texttt{closure}\Big(\big(V^{**}, \mathbf{w}^*(V^*)\big), \mathbb{B}_V\Big) = \mathbb{B}_{V^{**}}$

$\texttt{Proof} \ =$

$(1) := \texttt{Alaoglu}(V^*) : \big(\mathbb{B}_{V^{**}} : \texttt{Compact}(\mathbf{w}^*(V^*))\big),$

$(2) := \eth\texttt{unitBall}(V)\eth\texttt{unitBall}(V^{**}) : \mathbb{B}_V \subset \mathbb{B}_{V^{**}},$

$(3) := \eth\texttt{closure}\big(\mathbf{w}^*(V^*), \mathbb{B}_V\big)(1, 2 : \overline{\mathbb{B}}_V^{\mathbf{w}^*} \subset \mathbb{B}_{V^{**}},$

$D := \overline{\mathbb{B}}_V^{\mathbf{w}^*} : \texttt{Compact}(\mathbf{w}^*(v^*)),$

$\texttt{Assume}\ (x, 4) : \sum x \in \mathbb{B}_{V^{**}} \,.\, x \notin D,$

$(f, 5) := \texttt{WeakStarSeparation}(x, D, 4) : \sum f \in \mathbb{S}_{V^*} \,.\, x(f) > \sup_{y \in D} y(f),$

$(6) := \eth^{-1}\texttt{Subset}(V^**)\eth D(\mathbb{B}_V) : \mathbb{B}_V \subset D,$

$(7) := \texttt{GrowingSup}(6)\big(\sup_{y \in D} y(f)\big)\eth\texttt{naturalInjection}\eth^{-1}\texttt{operatorNorm}(f)\eth f\eth\texttt{unitSphere}(V^*) :$

$\quad \sup_{y \in D} y(f) \geq \sup_{y \in \mathbb{B}_V} y(f) = \sup_{y \in \mathbb{B}_V} f(y) = \|f\| = 1,$

$(8) := \eth\texttt{operatorNorm}(x)(f)\eth\texttt{unitSphere}(V^*)\eth\texttt{unitBall}(V^*) : x(f) \leq \|x\|\|f\| \leq 1,$

$() := I(\perp)\eth\texttt{Ineq}(7)(5)(8) : \perp;$

$\rightsquigarrow (4) := \eth\texttt{SetEq}(3) : \overline{\mathbb{B}}_V^{\mathbf{w}^*} = \mathbb{B}_{V^{**}},$

$\square$

$\texttt{WeakMetrization} :: \forall V : \mathsf{BAN}(K) \,.\, (\mathbb{B}_V, \mathbf{w}(V)) : \texttt{Metrizable} \iff V^* : \texttt{Separable}$

$\texttt{Proof} \ =$

$\dots$

$\square$

4.5 Extreme Constructs

$$\texttt{ExtremePoints} :: \prod V : \mathsf{VS}(K) \ . \ \prod A : \texttt{ConvexSet}(V) \ . \ ?A$$

$$v : \texttt{ExtremePoints} \iff v \in \mathrm{Ext}(A) \iff \forall x, y \in A : x \neq y \ . \ v \notin (x, y)_V$$

$$\texttt{SupportingManifold} :: \prod V : \mathsf{VS}(K) \ . \ \prod A : \texttt{ConvexSet}(V) \ . \ ?\texttt{Affine}(V)$$

$$H : \texttt{SupportingManifold} \iff H \cap A \neq \emptyset \ \& \ \forall x, y \in A : x \neq y : \exists a \in (x, y)_V : a \in A \ . \ [x, y] \subset H$$

5 Polynormed Spaces

5.1 Polynorms

$\texttt{Polynorm} :: \prod V : \mathsf{VS}(K) \, . \, \sum A : \texttt{NonEmpty} \, . \, A \to \texttt{Seminorm}(V)$

$\texttt{PolynormedSpace} :: \sum V : \mathsf{VS}(K) \, . \, \texttt{Polynorm}(V)$

$\texttt{RelatedSeminorm} :: \prod V : \, . \, \texttt{Polynorm}(V) \to ?\texttt{Seminorm}(V)$

$M : \texttt{RelatedSeminorm} \iff \exists \alpha : \texttt{Finite}(A) \, . \, \forall v \in V \, . \, M(v) = \max\{N_a(v) | a \in \alpha\}$

$\texttt{RelatedSeminormedSpace} :: \prod \big(V, (A, N)\big) : \texttt{PolynormedSpace}(K) \, . \, ?\texttt{SeminormedSpace}(K)$

$(V, M) : \texttt{RelatedSeminormedSpace} \iff M : \texttt{RelatedSeminorm}(A, N)$

$\texttt{CountablyNormedSpace} :: ?\texttt{PolynormedSpace}(K)$

$(V, (A, N)) : \texttt{CountablyNormedSpace} \iff A \cong_{\mathsf{SET}} \mathbb{N}$

$\texttt{PolynormedSubspace} :: \prod (V, (A, N)) : \texttt{PolynormedSpace}(K) \, . \, ?\texttt{PolynormedSpace}(K)$

$(S, (B, M)) : \texttt{PolymormedSubspace} \iff S : \texttt{Subspace}(V) \; \& \; A = B \; \& \; \forall a \in A \, . \, M_a = N_{a|S}$

$\texttt{strongestPolynorm} :: \prod V : \mathsf{VS}(K) \, . \, \texttt{Polynorm}(V)$

$\texttt{strongestPolynorm}\,(V) := (\texttt{Seminorm}(V), \mathrm{id})$

$\texttt{strongOperatorSpace} :: \texttt{SeminormedSpace}(K) \to \texttt{SeminormedSpace}(K) \to \texttt{PolynormedSpace}(K)$

$\texttt{strongOperatorSpace}\,(V, w) = so(V, W) := \Big(\mathcal{B}(V, W), \big(V, \Lambda v \in V \, . \, \Lambda T \in \mathcal{B}(V, W) \, . \, \|T(v)\|\big)\Big)$

$\texttt{weakOperatorSpace} :: \texttt{SeminormedSpace}(K) \to \texttt{SeminormedSpace}(K) \to \texttt{PolynormedSpace}(K)$

$\texttt{weakOperatorSpace}\,(V, w) = so(V, W) := \Big(\mathcal{B}(V, W), \big(V \times W^*, \Lambda(v, f) \in V \times W^* \, . \, \Lambda T \in \mathcal{B}(V, W) \, . \, |f\,T(v)|\big)\Big)$

$\texttt{polyball} :: \prod \big(V, (A, N)\big) \, . \, \texttt{Finite}(A) \to \mathbb{R}_{++} \to ?V$

$\texttt{polyball}\,(\alpha, r) := \{v \in V : \forall a \in \alpha \, . \, N_a(v) < r\}$

$\texttt{Polyinterior} :: \prod \big(V, (A, N)\big) \, . \, \prod U : ?V \, . \, ?U$

$V : \texttt{Polyinterior} \iff \exists \alpha : \texttt{Finitie}(A) \, . \, \exists r \in \mathbb{R}_{++} \, . \, \texttt{polyball}(\alpha, r) + v \subset U$

$\texttt{PolynormOpen} :: \prod \big(V, (A, N)\big) \, . \, ??V$

$U : \texttt{PolynormOpen} \iff \forall v \in U \, . \, v : \texttt{Polyinterior}\Big(\big(V, (A, N), \big), U\Big)$

$\texttt{PolynormTopology} :: \forall\big(V,(A,N)\big) : \texttt{PolynormedSpace}(K) \;.\; \texttt{PolynormOpen}\big(V,(A,N)\big) : \texttt{Topology}(V)$

$\texttt{Proof} \;=$

$(1) := \eth^{-1}\texttt{PolynormOpen}\big(V,(A,N)\big)(\emptyset) : \Big(\emptyset : \texttt{PolynormOpen}\big(V,(A,N)\big)\Big),$

$\texttt{Assume}\; v : \texttt{In}(V),$

$a := \eth\texttt{NonEmpty}(A) : \texttt{In}(a),$

$(2) := \eth\texttt{polyball}\big(\{a\},1\big)\eth_{+}\texttt{VectorSpace}(v)\eth\texttt{Subset}(V) : \texttt{polyball}\big(\{a\},1\big) + v \subset V,$

$() := \eth^{-1}\texttt{Polyinterior} : \Big(v : \texttt{Polyinterior}\Big(\big(V,(A,N)\big),V\Big)\Big);$

$\rightsquigarrow (2) := \eth^{-1}\texttt{PolynormOpen}\big(V,(A,N)\big) : \big(V : \texttt{PolynormOpen}\big(V,(A,N)\big)\big),$

$\texttt{Assume}\; X : \texttt{Set},$

$\texttt{Assume}\; U : X \to \texttt{PolynormOpen}\big(V,(A,N)\big),$

$\texttt{Assume}\; v : \texttt{In}\left(\bigcup_{x\in X} U_x\right),$

$(x,3) := \eth\texttt{union}(v) : \sum x \in X \;.\; v \in U_x,$

$(\alpha,r,4) := \eth\texttt{PolynormOpen}\big(V,(A,N)\big)(U_x)(v) : \sum(\alpha,r) : \texttt{Finite}(A)\times\mathbb{R}_{++} \;.\; \texttt{polyball}(\alpha,r) + v \subset U_x,$

$(5) := \texttt{UnionSubset}(4) : \texttt{polyball}(\alpha,r) + v \subset \bigcup_{x\in X} U_x,$

$() := \eth^{-1}\texttt{Polyinterior}(5) : \left(v : \texttt{Polyinterior}\left(\big(V,(A,N)\big),\bigcup_{x\in X} U_x\right)\right);$

$\rightsquigarrow () := \eth^{-1}\texttt{PolyormOpen}\big(V,(A,N)\big) : \left(\bigcup_{x\in X} U_x : \texttt{PolynormOpen}\big(V,(A,N)\big)\right);$

$\rightsquigarrow (3) := I(\forall) : \forall X : \texttt{Set} \;.\; \forall U : X \to \texttt{PolynormOpen}\big(V,(A,N)\big) \;.\; \bigcup_{x\in X} U_x : \texttt{PolynormOpen}\big(V,(A,N)\big),$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; U : n \to \texttt{PolynormOpen}\big(V,(A,N)\big),$

$\texttt{Assume}\; v : \texttt{In}\left(\bigcap_{i=1}^{n} U_i\right),$

$\texttt{Assume}\; i : \texttt{Range}(n),$

$(4) := \eth\texttt{intersection}\eth v(i) : v \in U_i,$

$(\alpha_i,r_i,5_i) := \eth\texttt{PolynormOpen}\big(V,(A,N)\big)(U_i) : \sum(\alpha_i,r_i) : \texttt{Finite}(A)\times\mathbb{R}_{++} \;.\; \texttt{polyball}(\alpha_i,r_i) + v \subset U_i;$

$\rightsquigarrow (\alpha,r,4) := I\left(\prod\right) : \prod i \in n \;.\; \sum(\alpha_i,r_i) : \texttt{Finite}(A)\times\mathbb{R}_{++} \;.\; \texttt{polyball}(\alpha_i,r_i) + v \subset U_i,$

$O := \texttt{polyball}\left(\bigcup_{i=1}^{n}\alpha_i, \min_{i\in n} r_i\right) + v :??V,$

$(5) := \eth\texttt{polyball}\eth O\eth\texttt{intersection} : O \subset \bigcap_{i=1}^{n}\texttt{polyball}(\alpha_i,r_i) + v,$

$(6) := \texttt{SubsetIntersection}(U,O,4,5) : O \subset \bigcap_{i=1}^{n} U_i,$

$() := \eth^{-}\texttt{Polyinterior}(6) : \left(v : \texttt{Polyinterior}\left(\big(V,(A,N)\big),\bigcap_{i=1}^{n} U_i\right)\right);$

$\rightsquigarrow () := \eth^{-1}\texttt{PolyormOpen}\big(V,(A,N)\big) : \left(\bigcap_{i=1}^{n} U_i : \texttt{PolynormOpen}\big(V,(A,N)\big)\right);$

$\rightsquigarrow (4) := I(\forall) : \forall n : \mathbb{N} \,.\, \forall U : n \to \texttt{PolynormOpen}\big(V,(A,N)\big) \,.\, \bigcap_{i=1}^{n} U_i : \texttt{PolynormOpen}\big(V,(A,N)\big),$

$(*) := \eth^{-1}\texttt{Topology}(V)(1,2,3,4) : \Big(\texttt{PolynormOpen}\big(V,(A,N)\big) : \texttt{Topology}(V)\Big);$

$\square$

$\texttt{implicit} :: \texttt{PolynormedSpace}(K) \to \mathsf{TOP}$

$\texttt{implicit}\,(X) := (X_1, \texttt{PolynormOpen}(X))$

$\texttt{polynorm} :: \prod \big(V,(A,N)\big) : \texttt{PolynormedSpace}(K) \,.\, \texttt{Polynorm}(V)$

$\texttt{polynorm}\,() = (A_V, \|\cdot\|_V) := (A,N)$

$\texttt{SumIsContnuousInPNS} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \Lambda x,y \in V \,.\, x+y \in C(V \times V, V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ScalarMultIsContInPNS} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \Lambda x \in V \,.\, \Lambda a \in K \,.\, ax \in C(V \times K, V$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SetAdditionInPNS} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \forall U \in \mathcal{T}_V \,.\, \forall X :?V \,.\, U + X \in \mathcal{T}_V$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SetMultInPNS} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \forall U \in \mathcal{T}_V \,.\, \forall a \in K \setminus \{0\} \,.\, aU \in \mathcal{T}_V$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{PolynormedConvergence} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \forall x : \mathbb{N} \to V \,.\, \forall X \in V \,.$

$\quad .\ \lim_{n\to\infty} x_n = X \iff \forall a \in A_V \,.\, \lim_{n\to\infty} \|x_n - X\|_{V,a} = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ R : \lim_{n\to\infty} x_n = X,$

$\texttt{Assume}\ a : A_V,$

$\texttt{Assume}\ r : \mathbb{R}_+,$

$() := \eth\texttt{Limit}(V,x,X)(\texttt{polyball}(\{a\},r)+X) : \exists N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \ge N \,.\, x_n \in \texttt{polyball}(\{a\},r)+X;$

$\rightsquigarrow () := \eth^{-1}\texttt{Limit}(\mathbb{R}, \|x_n - X\|_{V,a}, 0) : \lim_{n\to\infty} \|x_n - X\|_{V,a} = 0;$

$\rightsquigarrow (1) := T(\Rightarrow)I(\forall) : \lim_{n\to\infty} x_n = X \Rightarrow \forall a \in A_V \,.\, \lim_{n\to\infty} \|x_n - X\|_{V,a} = 0;$

$\texttt{Assume}\ L : \forall a \in A_V \,.\, \lim_{n\to\infty} \|x_n - X\|_{V,a} = 0,$

$\texttt{Assume}\ U : \mathcal{U}_V(X),$

$(\alpha, r, 2)) := \eth\texttt{PolynormOpen}(U) : \sum (\alpha, r) \in \texttt{Finite}(A) \times \mathbb{R}_{++} \,.\, X + \texttt{polyball}(\alpha, r) \subset U,$

$\texttt{Assume}\ a : \texttt{In}(\alpha),$

$(N_a, 3_a) := \eth\texttt{Limit}\big(L(a)\big)(\mathbb{B}(0,r)) : \sum N_a \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \ge N_a \,.\, \|x_n - X\|_{V,a} < r;$

$\rightsquigarrow (N,3) := I\left(\prod\right) : \forall a \in \alpha \,.\, \sum N_a \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \ge N_a \,.\, \|x_n - X\|_{V,a} < r,$

$(4) := \eth\texttt{polyball}(\alpha, r) : \texttt{polyball}(\alpha, r) = \bigcap_{a\in\alpha} \texttt{polyball}(\{a\}, r),$

$M := \max_{a\in\alpha}(N_a) : \mathbb{N},$

$() := (2)(3)(4)\eth M : \forall n \in \mathbb{N} \,.\, n \ge M \,.\, x_n \in U;$

$\rightsquigarrow () := \eth^{-1}\texttt{Limit}(V,x,X) : \lim_{n\to\infty} x_n = X;$

$(*) := I(\Rightarrow)I(\iff)(1) : \lim_{n\to\infty} x_n = X \iff \forall a \in A \,.\, \lim_{n\to\infty} \|x_n - X\|_{V,a} < r;$

$\square$

$\texttt{SeparatingPolynorm} :: ?\texttt{Polynorm}(V)$

$(A, N) : \texttt{SeparatingPolynorm} \iff \forall v \in V : v \ne 0 \,.\, \exists a \in A \,.\, N_a(v) > 0$

$\texttt{PolynormedSpaceIsHausdorff} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, V : \texttt{Hausdorff} \iff (A_V, \|\cdot\|) : \texttt{SeparatingPoly}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ L : (V : \texttt{Hausdorff}),$

$\texttt{Assume}\ (v, 1) : \sum v \in V \,.\, v \ne 0,$

$(U, 2) := \eth\texttt{Hausdorff}(V)(0, x, 1) : \sum U \in \mathcal{U}_V(0) \,.\, v \notin U,$

$(\alpha, r, 3) := \eth\texttt{PolynormOpen}(V)(U)(0) : \sum (\alpha, r) : \texttt{Finite}(A_V) \times \mathbb{R}_{++} \,.\, \texttt{polyball}(\alpha, r) \subset U,$

$(a, 4) := \eth\texttt{In}(2,3) : \sum a \in \alpha \,.\, \|v\|_a \ge r,$

$() := \text{RealLine}.\texttt{BoundedBelowByPositive}(4) : \|v\|_a \ne 0;;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{SeparatingPolynorm}\, I(\Rightarrow) : \Big(V : \texttt{Hausdorff} \Rightarrow (A_V, \|\cdot\|) : \texttt{SeparatingPolynorm}\Big),$

$\texttt{Assume}\ R : \Big((A_V, \|\cdot\|)\Big) : \texttt{SeparatingPolynorm},$

$\texttt{Assume}\ (x, y, 2) : \sum x, y \in V\ .\ x \neq y,$

$(a, 3) := \eth\texttt{SeparatingPolynorm}(A_V, \|\cdot\|)(x - y) : \sum a \in A_V\ .\ \|x - y\|_a \neq 0,$

$O := \texttt{polyball}(\{a\}, \|x - y\|_a/2) + x : \mathcal{U}_V(x),$

$O' := \texttt{polyball}(\{a\}, \|x - y\|_a/2) + y : \mathcal{U}_V(x),$

$\texttt{Assume}\ u : \texttt{In}(O \cap O'),$

$(4) := \text{Algebra}.\texttt{AddSubstract}(\|x - y\|_a, x - y, u)\eth_2\texttt{Seminorm}(x - u, u - y)\eth OQ' :$

$\quad : \|x - y\|_a = \|x - u + u - y\|_a \leq \|x - u\| + \|u - y\| < \|x - y\|,$

$() := I(\bot)\eth\texttt{Ineq}(4) : \bot;$

$\rightsquigarrow () := \eth\texttt{Disjoint}\eth\texttt{empty} : (O, O') : \texttt{Disjoint};$

$* := I(\iff)(1)I(\Rightarrow)\eth\texttt{Hausdorff} : \Big(V : \texttt{Hausdorff}) \iff (A_V, \|\cdot\|) : \texttt{SeparatingPolynorm}\Big);$

$\square$

$\texttt{ContinuousOperotorOfPNS} :: \forall V, W : \texttt{PolynormedSpace}(K)\ .\ \forall T : \mathcal{L}(V, W)\ .\ T : C(V, W) \iff$

$\quad \iff \forall a \in A_W\ .\ \exists N : \texttt{RelatedSeminorm}(V) : \exists c \in \mathbb{R}_{++}\ .\ \|T(\cdot)\|_a \leq cN$

$\texttt{Proof}\ =$

$\texttt{Assume}\ L : \big(T : C(V, W)\big),$

$\texttt{Assume}\ a : A_W,$

$(2) := \eth C(V, W)\ \texttt{polyball}(a, -1) : \big(T^{-1}\ \texttt{polyball}(a, 1) : \mathcal{U}_V(0)\big),$

$(\alpha, r, 3) := \eth\texttt{PolynormOpen}\Big(T^{-1}\ \texttt{polyball}(a, 1), 0\Big) : \sum(\alpha, r) : \texttt{Finite}(A) \times \mathbb{R}_{++}\ .\ \texttt{polyball}(\alpha, r) \subset T^{-1}\ \texttt{polyba}$

$\rightsquigarrow (4) := T(3)\eth\texttt{polyball} : \|T(\cdot)\|_a \leq r^{-1}\|\cdot\|_\alpha; ,$

$(1) := I(\Rightarrow)I(\forall)(\|\cdot\|_\alpha = N, c = r^{-1}) : \texttt{Left} \to \texttt{Right};$

$\texttt{Assume}\ R : \texttt{Right},$

$\texttt{Assume}\ U : \texttt{Open}(W),$

$\texttt{Assume}\ y : \texttt{In}(U),$

$(\alpha, r, 2) := \eth\texttt{PolynormOpen}\Big(U, x\Big) : \sum(\alpha, r) : \texttt{Finite}(A) \times \mathbb{R}_{++}\ .\ \texttt{polyball}(x, \alpha, r) \subset U,$

$\texttt{Assume}\ a : \texttt{In}(\alpha),$

$(c_a, N_a) := R(a) : \texttt{Right}(a);$

$\rightsquigarrow (c, N) := I\left(\prod\right) : \prod a \in \alpha\ .\ (c_a, N_a) : \texttt{Right}(a),$

$h := \max_{a \in \alpha} c_a : \mathbb{R}_+,$

$M := \Lambda v \in V\ .\ \max_{a \in \alpha} N_a(v) : \texttt{RelatedSeminorm}(V),$

$(3) := \eth\texttt{Right}\eth h\eth M : \forall v \in V\ .\ \|T\,v\|_\alpha \leq hM(v),$

$(4) := \text{Operator}.\texttt{BoundedIffContinuous}(3) : \Big(T : C\big((V, M), (W, \|\cdot\|_\alpha)\big)\Big),$

$(5) := \eth C\big((V, M), (W, \|\cdot\|_\alpha)\big)(\texttt{polyball}(x, \alpha, r)) : T^{-1}\texttt{polyball}(x, \alpha, r) \in \mathcal{T}(V, M),$

$(6) := \eth\texttt{PolynormOpen}(5) : T^{-1}\texttt{polyball}(x, \alpha, r) \in \mathcal{T}(V),$

$() := \text{SET}.\texttt{SubsetPreimage}(T, U.\texttt{polyball}(x, \alpha, r), 2) : T^{-1}\texttt{polyball}(x, \alpha, r) \subset T^{-1}U;$

$\rightsquigarrow () := \text{GTOP}.\texttt{OpenCharacteristic}(V) : T^{-1}U \in \mathcal{T}_V;$

$\rightsquigarrow (*) := I(\iff)(1)I(\Rightarrow)\eth^{-1}C(V, W) : \texttt{Left} \iff \texttt{Right};$

$\square$

$\texttt{SeminormDominatedByPolynorm} :: \prod V : \mathsf{VS}(K) \,.\, \texttt{Polynorm}(V) \to ?\texttt{Seminorm}(V)$

$M : \texttt{SeminormDominatedByPolynorm}(A, N) \iff M \leq (A, N) \iff$

$\iff \exists \alpha : \texttt{Finite}(A) \,.\, M \leq \Lambda v \in V \,.\, \max_{a \in \alpha} N_a(v)$

$\texttt{PolynormDominatedByPolynorm} :: \prod V : \mathsf{VS}(K) \,.\, \texttt{Polynorm}(V) \to ?\texttt{Plynorm}(V)$

$(B, M) : \texttt{PolynormDominatedByPolynorm}(A, N) \iff (B, M) \leq (A, N) \iff \forall b \in B \,.\, M_b \leq (A, N)$

$\texttt{EquevalentPolynorms} :: \prod V : \mathsf{VS}(K) \,.\, ?\Big(\texttt{Polynorm}(V) \times \texttt{Polynorm}\Big)$

$\Big((A, N), (B, M)\Big) : \texttt{EquevalentPolynorms} \iff (A, N) \cong (B, M) \iff (A, N) \leq (B, M) \;\&\; (B, M) \leq (A, N)$

$\texttt{StrictlyDominatedByPolynorm} :: \prod V : \mathsf{VS}(K) \,.\, \texttt{Polynorm}(V) \to ?\texttt{Plynorm}(V)$

$(B, M) : \texttt{StrictluDominatedByPolynorm}(A, N) \iff (B, M) < (A, N) \iff$

$\iff (B, M) \leq (A, N) \;\&\; (B, M) \ncong (A, N)$

$\texttt{PolynormTopologyRelation} :: \forall V : \mathsf{VS}(K) \,.\, \forall (A, N), (B, M) : \texttt{Polynorm}(V) \,.$

$.\, (A, N) \leq (B, M) \iff \mathcal{T}\Big(V, (A, N)\Big) \leq \mathcal{T}\Big(V, (A, N)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ L : (A, N) \leq (B, M),$

$\texttt{Assume}\ U : \texttt{Open}(V, (A, N)),$

$\texttt{Assume}\ x : \texttt{In}(x),$

$(\alpha, r, 1) := \eth\texttt{PolynormOpen}\Big(V, (A, N)\Big)(U)(x) : \sum (\alpha, r) : \texttt{Finite}(A) \times \mathbb{R}_{++} \,.\, \texttt{polyball}(\alpha, r) + a \subset U,$

$\texttt{Assume}\ a : \texttt{In}(\alpha),$

$(2) := \eth\texttt{PolynormDomintedByPolynorm}(L)(a) : N_a \leq (B, M),$

$(\beta_a, 3_a) := \eth\texttt{SeminormDominatedByPolynorm}(2) : \sum \beta : \texttt{Finite}(B) \,.\, N_a \leq \max_{b \in \beta_a} M_b;$

$\rightsquigarrow (\beta, 2) := I(\Pi) : \prod a \in \alpha \,.\, \sum \beta : \texttt{Finite}(B) \,.\, N_a \leq \max_{b \in \beta_a} M_b,$

$(3) := \max_{a \in \alpha}(2)_a : \max_{a \in \alpha} N_a \leq \max_{a \in \alpha} \max_{b \in \beta_a} M_b,$

$(4) := \eth\texttt{polyball}(3)(1) : \texttt{polyball}\left(\bigcup_{a \in \alpha} \beta_a, r\right) \subset \texttt{polyball}(\alpha, r) + x \subset U,$

$(5) := \mathrm{SET}.\texttt{FiniteIntersection}(\alpha, \beta) : \bigcup_{a \in \alpha} \beta_a : \texttt{Finite}(B),$

$() := \eth^{-1}\texttt{Polyinteror}(4, 5) : \Big(x : \texttt{Polyitnerior}\big(V, (B, M)\big)(U)\Big);$

$\rightsquigarrow () := \eth^{-1}\texttt{PolynormOpen} : U \in \mathcal{T}\big(V, (B, M)\big);$

$\rightsquigarrow (1) := I(\Rightarrow)\eth^{-1}\texttt{Coarser} : \big(A, N\big) \leq \big(B, M\big) \Rightarrow \mathcal{T}\big(V, (A, N)\big) \leq \mathcal{T}\big(V, (B, M)\big),$

$\texttt{Assume}\ R : \mathcal{T}\big(V,(A,N)\big) \le \mathcal{T}\big(V,(B,M)\big),$

$\texttt{Assume}\ a : \texttt{In}(A),$

$(1) := R\Big(\texttt{polyball}\big(\{a\},1\big)\Big) : \texttt{polyball}\big(\{a\},1\big) \in \mathcal{T}\big(V,(B,M)\big),$

$(\beta,r,2) := \eth\texttt{PolynormOpen}\Big(V,(B,M)\Big)\Big(\texttt{polyball}\big(\{a\},1\big)\Big)(0) :$

$: \sum(\beta,r) : \texttt{Finite}(B) \times \mathbb{R}_{++} \,.\, \texttt{polyball}(\beta,r) \subset \texttt{polyball}\big(\{a\},r\big),$

$(3) := \eth^{-1}\texttt{SeminormDominatedByPolynorm}\eth\texttt{polyball}(2) : N_a \le \max_{b\in\beta} M_b;$

$\rightsquigarrow () := I(\iff)(1)I(\Rightarrow)\eth\texttt{PolynormDominatedByPolynorm}I(\forall) :$

$.\ (A,N) \le (B,M) \iff \mathcal{T}\Big(V,(A,N)\Big) \le \mathcal{T}\Big(V,(A,N)\Big);$

$\square$

$\texttt{SeminormablePolynorm} :: \forall V : \mathsf{VS}(K)\,.\,\forall(A,N) : \texttt{Polynorm}(V)\,.\,\exists M : \texttt{Seminorm}(V) :$

$: (V,M) \cong_{\mathsf{TOP}} \big(V,(A,N)\big) \iff \exists\alpha : \texttt{Finite}(A) : (A,N) \cong (\alpha, N_{|\alpha})$

$\texttt{Proof} =$

$(\Rightarrow)$

$\forall a \in A\,.\, N_a \le M \le N_\alpha$

$(\Leftarrow)$

$M = \max_{a\in\alpha} N_a$

$\square$

$\texttt{MetrizablePolynorm} :: \forall V : \mathsf{VS}(K)\,.\,\forall(A,N) : \texttt{Polynorm}(V)\,.\,\big(V,(A,N)\big) : \texttt{Metrizable} \iff$

$\iff \exists\alpha : \texttt{Countable}(A)\,.\,(A,N) \cong (\alpha, N_{|\alpha}))$

$\texttt{Proof} =$

$\texttt{Assume}\ R : \Big((V,(A,N)) : \texttt{Metrizable}\Big),$

$(d,1) := \eth\texttt{Metrizable}\Big(V,(A,N)\Big) : \sum d : \texttt{Distance}(V)\,.\,(V,d) \cong_{\mathsf{TOP}} \Big(V,(A,N)\Big),$

$\texttt{Assume}\ n : \mathbb{N},$

$(\alpha_n, r_n, 2)n)) := \eth\texttt{PolynormOpen}\Big(V(A,N)\Big)\Big(\mathbb{B}_d(0,1/n),1\Big)(0) : \sum(\alpha_n,r_n) : \texttt{Finite}(A) \times \mathbb{R}_{++}\,.\,\texttt{polyball}(\alpha_n,r_n)$

$\rightsquigarrow (\alpha,r,2) := I(\Pi) : \prod n \in \mathbb{N}\,.\,\sum(\alpha_n,r_n) : \texttt{Finite}(A) \times \mathbb{R}_{++}\,.\,\texttt{polyball}(\alpha_n,r_n) \subset \mathbb{B}_d(0,1/n),$

$\mathcal{A} := \bigcup_{n\in\mathbb{N}} \alpha_n :?A,$

$(3) := textrmSET.\texttt{CountableUnionOfFinites}(\mathbb{N},\alpha) : \big(\mathcal{A} : \texttt{Countable}(A)\big),$

$\texttt{Assume}\ a : \texttt{In}(A),$

$(n,4) := \texttt{MetricLocalBase}\Big((V,d), \texttt{polyball}\big(\{a\},1\big),(1)\Big) : \sum n \in \mathbb{N}\,.\,\mathbb{B}_d(0,1/n) \subset \texttt{polyball}\big(\{a\},1\big),$

$(5) := (4)(2)_n : \texttt{polyball}\big(\alpha_n,r_n\big) \subset \texttt{polyball}\big(\{a\},1\big),$

$() := \eth\texttt{polyball}(5) : N_a \le r_n^{-1} \max_{x\in\alpha_n} N_x;$

$\rightsquigarrow (4) := \eth^{-1}\texttt{PolynormDominatedByPolynorm} : (A,N) \le (\mathcal{A}, N_{|\mathcal{A}}),$

$(5) := \eth^{-1}\texttt{PolynormDominatedByPlynorm}\eth\texttt{Subset}(\mathcal{A}) : (\mathcal{A}), N_{|\mathcal{A}} \le (A,N),$

$() := \eth^{-1}\texttt{EuevalentPolynorms}(4,5) : (A,N) \cong (\mathcal{A}, N_{|\mathcal{A}});$

$\rightsquigarrow (1) := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ (\alpha, R) : \sum \alpha : \texttt{Countable}(A)\ .\ (A, N) \cong (\alpha, N_{|\alpha}),$

$a := \texttt{enumerate}(\alpha) : \mathbb{N} \twoheadrightarrow_{\mathsf{SET}} A,$

$$d := \Lambda v, w \in V\ .\ \sum_{n=1}^{\infty} \frac{\|v - w\|_{a_n}}{2^n(1 + \|v - w\|)} : \texttt{Distance}(V),$$

$\ldots$

$\square$

5.2 Abstract Topological Vector Spaces

$\texttt{TopologicalVectorSpace} :: ? \sum V : \mathsf{VS}(K) \, . \, \texttt{Topology}(V)$

$(V, T) : \texttt{TopologicalVectorSpace} \iff (+)_V : C((V, T) \times (V, T), (V, T)) \; \& \; (\cdot)_V : C(K \times (V, T), (V, T))$

$\texttt{implicit} :: \texttt{TopologicalVectorSpace}(K) \to \mathsf{VS}(K)$

$\texttt{implicit}\,(V, T) := V$

$\texttt{continuousOperators} :: \texttt{TopologicalVectorSpace}(K) \times \texttt{TopologicalVectorSpace}(K) \to \mathsf{VS}(K)$

$\texttt{continuousOPerators}\,(V, W) = \mathcal{B}(V, W) := \mathcal{L}(V, W) \cap C(V, W)$

$\texttt{TopologyIsDefinedAtZero} :: \forall V : \texttt{VS}(K) \, . \, \forall T, S : \texttt{Topology}(V) :$

$: \Big((V, T), (V, S) : \texttt{TopologicalVectorSpace}(K)\Big) \, . \, T = S \iff \mathcal{U}_T(0) = \mathcal{U}_S(0)$

Proof =

$(\Rightarrow)$

Obvious.

$(\Leftarrow)$

Use that for open set U such that $x \in U$ maps $f(v) = v - x$ and $g(v) = v + x$ are continuous in both topologies.

□

$\texttt{TVSSeparatesPoints} :: \forall V : \texttt{TopologicalVectorSpace}(K) \, . \, \forall A : \texttt{Closed}(V) \, . \, \forall v \in V \, .$

$. \, \forall \Delta : v \notin A \, . \, \exists U \in \mathcal{U}(v) : \exists O \in \mathcal{U}(A) \, . \, U \cap O = \emptyset$

Proof =

every topological group is regular □

5.3 Locally Convex Vector Spaces

$\texttt{LocallyConvexSpace} :: ?\texttt{TopologicalVectorSpace}(K)$

$V : \texttt{LocallyConvexSpace} \iff \exists N : \texttt{NeighborhoodBase}(0) \,.\, \forall U \in N \,.\, N : \texttt{ConvexSet}(V)$

$\texttt{PolynormedAreLocallyConvex} :: \forall V : \texttt{TopologicalVectorSpace}(K) \,.\, V : \texttt{LocallyConvex}(K) \iff$

$\iff \exists (A, N) : \texttt{Polynorm}(V) \,.\, V \cong_{\mathsf{TOP}} \big(V, (A, n)\big)$

$\texttt{Proof} =$

Use Minkowski constructions.

$\square$

5.4 Abstract Duality

$\texttt{dualSpace} :: \texttt{TopologicalVectorSpace}(K) \to \mathsf{VS}(K)$
$\texttt{dualSpace}\,(V) = V^* := \mathcal{B}(V, K)$

$\texttt{HausdorffByFunctionals} :: \forall V : \texttt{PolynormedSpace}(K)\ .$
$\quad .\ \Big(V : \texttt{Hausdorff} \iff V^* : \texttt{SeparatesPoints}(V)\Big)$
$\texttt{Proof}\ =$
$\texttt{Assume}\ L : \big(V : \texttt{Hausdorff}\big),$
$\texttt{Assume}\ (v, 1) : \sum v \in V\ .\ v \neq 0,$
$(a, 2) := \texttt{PolynormedSpaceIsHausdorff}(V, L)(x) : \sum a \in A_V\ .\ \|v\|_a > 0,$
$f := \Lambda zv \in Kv\ .\ z\|v\|_a : \mathcal{B}\big((Kv, \|\cdot\|_a), K\big),$
$(F, 3) := \texttt{HahnBanach}(V, \|\cdot\|_a)(f) : \sum F : \mathcal{B}\big((V, \|\cdot\|_a), K\big)\ .\ \|F\|_a = 1,$
$(4)_v := \eth F(v)(2) : F(v) > 0,$
$() := \texttt{ContinuousOperatorsOfPNS}(F, 3) : F \in V^*;$
$\leadsto (1) := I(\Rightarrow)\eth^{-1}\texttt{SeparatesPoints} : \Big(V : \texttt{Hausdorff}V^* : \texttt{SeparatesPoints}(V)\Big),$
$\texttt{Assume}\ R : \big(V^* : \texttt{SeparatesPoints}(V)\big),$
$\texttt{Assume}\ (v, 2) : \sum v \in V\ .\ v \neq 0,$
$(f, 3) := \eth\texttt{SeparatesPoints}(V)(V^*)(v, 1) :\sim f \in V^*\ .\ f(v) \neq 0,$
$(a, 4) := \texttt{ContinuousOperatorsOfPNS}\Big(\eth V^*(f)\Big) : \sum a \in A_V\ .\ f \in \mathcal{B}\big((V, \|\cdot\|_a), K\big),$
$(5) := \eth\mathcal{B}\big((V, \|\cdot\|_a), K\big)(f) : \exists c \in \mathbb{R}_{++}\ .\ \forall v \in V\ .\ |f(v)| \leq c\|v\|_a,$
$(6) := \eth\texttt{PositiveIneqMult}\big((5), (3)(v)\big) : \|v\|a \neq 0;$
$\leadsto (*) := I(\iff)(1)I(\Rightarrow)\texttt{PolynormedSpaceIsHausforff} : \Big(V : \texttt{Hausdorff} \iff V^* : \texttt{SeparatesPoints}(V)\Big),$
$\square$

$\texttt{Enough} :: \prod V : \mathcal{VS}(K)\ .\ \texttt{Subspace}(V^{\#})$
$E : \texttt{Enough} \iff E : \texttt{SeparatesPoints}(V)$

$\texttt{specialWeakTopology} :: \prod V : \mathsf{VS}(K)\ .\ ?V^{\#} \to \texttt{Topology}(V)$
$\texttt{specialWeakTopology}\,(A) = \mathbf{w}(V, A) := \mathcal{T}\Big(V, (A, \Lambda f \in A\ .\ \Lambda v \in V\ .\ |f(v)|)\Big)$

$\texttt{SpecialWeakTopologyAsFromSpan} :: \forall V : \mathsf{VS}(K) \,.\, \forall A :?V^\# \,.\, \mathbf{w}(V,A) = \mathbf{w}\big(V, \mathrm{span}(A)\big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ g : \mathrm{span}\,\mathtt{A},$

$(n,f,a,1) := \eth\texttt{Span}(A) : \sum n \in \mathbb{N} \,.\, \sum (f,a) : n \to A \times K \,.\, g = \sum_{i=1}^{n} a_i f_i,$

$\texttt{Assume}\ v : \mathtt{V},$

$() := (1)\texttt{TriangleIneq}\eth_2\texttt{AbsaluteValueField}(a, f(v))\texttt{SumDominatedByMax}(n, \|a\|\|f(v)\|) :$

$\quad . \ |g(v)| \le \sum_{i=1}^{n} |a_i||f_i(v)| \le n\Big(\max_{i\in n}|a_i|\Big)\Big(\max_{i\in n}|f_i(v)|\Big);$

$\rightsquigarrow () := \eth^{-1}\texttt{SeminormDominatedByPolynorm} : |g(\cdot)| \le (A, \Lambda f \in A \,.\, |f(\cdot)|);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{PolynormDominatedByPolynorm} : (\mathrm{span}(A), \Lambda f \in \mathrm{span}(A) \,.\, |f(\cdot)|) \le (A, \Lambda f \in A \,.\, |f(\cdot)|),$

$(2) := \eth^{-1}\texttt{PolynormDominatedByPolynorm}\eth\texttt{Subset}(\mathrm{span}(A))(A) :$

$\quad : (A, \Lambda f \in A \,.\, |f(\cdot)|) \le (\mathrm{span}(A), \Lambda f \in \mathrm{span}(A) \,.\, |f(\cdot)|),$

$(3) := \texttt{PolynormTopologyRelation}(2,3) : \mathcal{T}\big(V, (A, \ldots)\big) = \mathcal{T}\big(V, (\mathrm{span}(A), \ldots)\big),$

$(*) := \eth\texttt{SpecialWeakTopology}(3) : \mathbf{w}(V,A) = \mathbf{w}\big(V, \mathrm{span}(A)\big);$

$\texttt{weakTopology} :: \prod V : \texttt{TopologicalVectorSpace}(K) \,.\, \texttt{Topology}(V)$

$\texttt{weakTopology}\,(V) = \mathbf{w}(V) := \mathbf{w}(V, V^*)$

$\texttt{WeakTopologyCoarser} :: \forall V : \texttt{PolynormedSpace}(K) \,.\, \mathbf{w}(V) \le \mathcal{T}(V)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{weakStarTopology} :: \prod V : \texttt{TopologicalVectorSpace}(K) \,.\, \texttt{Topology}(V^*)$

$\texttt{weakStarTopology}\,(V) = \mathbf{w}^*(V) := \mathbf{w}(V^*, \alpha_V)$

$\texttt{InfiniteDimWeakStructure} :: \forall V : \texttt{TopologicalVectorSpace}(K) \,.\, \forall d : \dim V = \infty \,.\, \forall A \subset V^* \,.$

$\quad . \ \forall U \in \mathcal{U}\big(\mathbf{w}(V,A)\big)(0) \,.\, \exists S : \texttt{Subspace}(\mathcal{VS}, V) : S \subset U$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SpecialWeakContinuity} :: \forall V : \texttt{TopologicalVectorSpace}(K) \,.\, \forall A \subset V^\# \,.\, \forall f \in V^\# \,.$

$\quad . \ f : C\Big(\big(V, \mathbf{w}(V,A)\big), K\Big) \iff f \in \mathrm{span}(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{EnoughIsDenseInWeakStar} :: \forall V : \texttt{TopologicalVectorSpace}(K) .$

$\quad . \forall A : \texttt{Enough}(V) \ \& \ C(V, K) \ . \ A : \texttt{Dense}\Big(V^*, \mathbf{w}^*(V)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ f : \texttt{In}(V^*),$

$\texttt{Assume} \ U : \texttt{In}\Big(\mathcal{U}\big(V^*, \mathbf{w}^*(V)\big)(f)\Big),$

$(X, r, 1) := \eth\texttt{Polyinterior}\eth\texttt{weakTopology}(V^*)(U, f) : \sum (X, r) : \texttt{Finite}(V) \times \mathbb{R}_{++} \ . \ \bigcap_{x \in X} \{g \in V^* : |f(x) - g(x)|$

$(Y, 2) := \texttt{biggestIndependant}(X) : \sum Y : \texttt{LinearlyIndependend}(V) \ . \ X \subset \mathrm{span}(Y),$

$(g, 3) := \texttt{GeneralFunctional}(V, A, (Y, 2), f) : \sum g \in A \ . \ \forall y \in Y \ . \ g(y) = f(y),$

$(4) := (2)(3) : \forall x \in X \ . \ g(x) = f(x),$

$() := (1)(4) : g \in U;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Dense}\Big(V^*, \mathbf{w}^*(V)\Big) : A : \texttt{Dense}\Big(V^*, \mathbf{w}^*(V)\Big),$

$\square$

5.5 Dual Operators

$$\texttt{dualOperator} :: \prod V, W : \texttt{TopologicalVectorSpace}(K) \,.\, \mathcal{B}(V, W) \to \mathcal{L}(W^*, V^*)$$
$$\texttt{dualOperator}\,(T) = T^* := \Lambda f \in W^* \,.\, \Lambda x \in V \,.\, f\,T\,x$$

$$\texttt{DualOperatorIsWeakStarContinuous} :: \forall V, W : \texttt{TopologicalVectorSpace}(K) \,.\, \forall T : \mathcal{B}(V, W) \,.$$
$$.\, T^* : C\Big((W^*, \mathbf{w}^*(W)), (V^*, \mathbf{w}^*(V))\Big)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : \texttt{In}(V),$

$\texttt{Assume}\; f : \texttt{In}(W^*),$

$(1) := \eth\texttt{dualOperator}(T)(|T\,f(v)|) : |T^*\,f(v)| = |f\,T(v)|,$

$(2) := \eth^{-1}\texttt{LessEq}(1) : |T^*\,f(v)| \le |f\,T(v)|;$

$\rightsquigarrow (3) := \texttt{ContinuousOperatorsOfPNS} : \bigg(T^* : C\Big(\big(W^*, \mathbf{w}^*(W)\big), \big(V^*, \mathbf{w}^*(V)\big)\Big)\bigg);$

$\square$

$$\texttt{StructureOfWeaklyContinuousOperators} :: \forall V, W : \mathsf{NORM}(K) \,.\, \forall T : \mathcal{B}\Big(\big(W^*, \mathbf{w}^*(W)\big), \big(V^*, \mathbf{w}^*(V)\big)\Big) \,.$$
$$\exists S \in \mathcal{B}(W, V) \,.\, T = S^*$$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : \texttt{In}(V),$

$(1) := \texttt{ContinuousComposition}\eth\texttt{dualOperator}(T)(\alpha v) : T^*\alpha_v = \alpha_v \circ T : \mathcal{B}\Big(\big(V^*, \mathbf{w}^*(W)\big), \big(W^*, \mathbf{w}^*(W)\big)\Big),$

$\big(Sv, 2\big) := \eth\,\mathrm{span}\,\texttt{SpecialWeakContinuity}\Big(T^*\alpha_v, \eth\texttt{weakStarTopology}(W)\Big) : \sum Sv \in W \,.\, \alpha_{Sv} = T^*\alpha_v;$

$\rightsquigarrow (S, 2) := I(\to)I(\forall) : \sum S : \mathcal{L}(V, W) \,.\, \forall v \in V \,.\, \alpha_{Sv} = T^*\alpha_v,$

$\texttt{Assume}\; v : \texttt{In}(\mathbb{B}_V),$

$\texttt{Assume}\; f : \texttt{In}(W^*),$

$() := \eth\alpha(Sv)\Big(\big|\alpha_{Sv}(f)\big|\Big)\eth S\eth^{-1}\texttt{operatorNorm}\eth v :$

$\quad : |\alpha_{Sv}(f)| = |f(Sv)| = |Tf(v)| \le \|Tf\|\|v\| \le \|Tf\|;$

$\rightsquigarrow (2) := I(\forall) : \forall v \in \mathcal{B}_V\, \forall f \in W^* \,.\, |\alpha_{Sv} f| \le \|Tf\|,$

$(3) := \texttt{BanachOperators}(W, K) : \Big(W^* : \mathsf{BAN}(K)\Big),$

$(c, 4) := \texttt{BanachSteinkhaus}(2, 3) : \sum c \in \mathbb{R}_{++} \,.\, \forall v \in \mathcal{B}_V \,.\, \forall f \in W^* \,.\, |\alpha_{Sv} f| \le c,$

$(5) := \eth\texttt{Norm}\eth\alpha(4) : \forall v \in V \,.\, \|Sv\| \le c\|v\|,$

$(*) := \eth^{-1}\mathcal{B}(5) : \Big(S : \mathcal{B}(V, W)\Big);$

$\square$

5.6 Bipolar Theorem

6 Ordered Vector Spaces