Metric Topology

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Contents

1 Metric Spaces

1.1 Distance Metric

$$\texttt{Semimetric} :: \prod X \in \mathsf{SET} \;.\; \texttt{Symmetric}(X, X; \mathbb{R}_+)$$
$$d : \texttt{Semimetric} \iff \forall x, y, z \in X \;.\; d(x,x) = 0 \;\&\; d(x,y) \le d(x,z) + d(z,y)$$

$$\texttt{SemimetricSpace} := \sum_{X \in \mathsf{SET}} \texttt{Semimetric}(X) : \texttt{Type};$$

$$\texttt{Metric} :: \prod x \in \mathsf{SET} \;.\; \texttt{Semimetric}(X, X; \mathbb{R}_+)$$
$$d : \texttt{Metric} \iff \forall x, y \in X \;.\; d(x,y) = 0 \iff x = y$$

$$\texttt{MetricSpace} := \sum_{X \in \mathsf{SET}} \texttt{Metric}(X) : \texttt{Type};$$

$$\texttt{semimetric} :: \prod (X, \rho) : \texttt{SemimetricSpace} \;.\; \texttt{Semimetrc}(X)$$
$$\texttt{semimetric}\,() = d_X := \rho$$

$$\texttt{synecdoche} :: \prod (X, \rho) : \texttt{SemimetricSpace} \;.\; \mathsf{SET}$$
$$\texttt{synecdoche}\,() = (X, \rho) := X$$

$$\texttt{TriangleInequality} :: \forall (X, d) : \texttt{SemimetricSpace} \;.\; \forall x, y, z \in X \;.\; d(x,y) \le d(x,z) + d(z,y)$$

$\texttt{Proof}$ $=$

...

□

$$\texttt{ReversedTriangleInequality} :: \forall (X, d) : \texttt{SemmimetricSpace} \;.\; \forall x, y, z \in X . \Big| d(x,y) - d(y,z) \Big| \le d(x,z)$$

$\texttt{Proof}$ $=$

$[1] := \texttt{TriangleInequality}(x, y, z) : d(x,y) \le d(x,z) + d(z,y),$

$[2] := \eth\texttt{Semimetric}(d)(z, y)[1] : d(x,y) \le d(x,z) + d(y,z),$

$[3] := [2] - d(y,z) : d(x,y) - d(y,z) \le d(x,z),$

$[4] := \texttt{TriangleInequality}(y, z, x) : d(y,z) \le d(y,x) + d(x,z),$

$[5] := \eth\texttt{Semimetric}(d)(x, y)[4] : d(y,z) \le d(y,x) + d(x,z),$

$[6] := [5] - d(y,z) : d(y,z) - d(x,y) \le d(x,z),$

$[*] := \eth^{-1}\texttt{absValue}[3][6] : \Big| d(x,y) - d(y,z) \Big| \le d(x,z);$

□

$\texttt{Lipschitz} :: \prod X, Y : \texttt{SemimetricSpace} \,.\, \mathbb{R}_+ \to ?(X \to Y)$

$f : \texttt{Lipschitz} \iff f \in L\text{-Lip} \iff \Lambda L \in \mathbb{R}_+ \,.\, \forall x, y \in X \,.\, d\Big(f(x), f(y)\Big) \le L d(x, y)$

$\texttt{SemiisometryCategory} :: \mathsf{CAT}$

$\texttt{SemiisometryCategory}\,() = \mathsf{SMS}_{\circ\to.} := \Big(\texttt{SemimetricSpace}, 1\text{-Lip}, \circ, \mathrm{id}\Big)$

$\texttt{IsometryCategory} :: \mathsf{CAT}$

$\texttt{IsometryCategory}\,() = \mathsf{MS}_{\circ\to.} := \Big(\texttt{MetricSpace}, 1\text{-Lip}, \circ, \mathrm{id}\Big)$

$\texttt{Semiisometry} :: \prod X, Y : \texttt{SemimetricSpace} \,.\, ?(X \to Y)$

$f : \texttt{Semiisometry} \iff \forall a, b \in X \,.\, d\Big(f(a), f(b)\Big) = d(a, b)$

$\texttt{Isometry} :: \prod X, Y : \texttt{MetricSpace} \,.\, ?(X \to Y)$

$f : \texttt{Isometry} \iff \forall a, b \in X \,.\, d\Big(f(a), f(b)\Big) = d(a, b)$

$\texttt{distance} :: \prod X : \texttt{SemimetricSpace} \,.\, ?X \to X \to \mathbb{R}_+$

$\texttt{distance}\,(A, x) = d(A, x) := \inf_{a \in A} d(a, x)$

$\texttt{distanceBetweenSets} :: \prod X : \texttt{SemimetricSpace} \,.\, ?X \to ?X \to \widehat{\mathbb{R}}_+$

$\texttt{distanceBetweenSets}\,(A, B) = d(A, B) := \sup_{b \in B} d(A, b)$

$\texttt{diameter} :: \prod X : \texttt{SemimetricSpace} \,.\, ?X \to \widehat{\mathbb{R}}_+$

$\texttt{diameter}\,(A) = \mathrm{diam}\, A := \sup_{x, y \in A} d(x, y)$

$\texttt{Bounded} :: \prod X \in \mathsf{SMS}_{\circ\to.} \,.\, ?_{\texttt{Semiiso}} ?X$

$A : \texttt{Bounded} \iff \mathrm{diam}\, A < \infty$

$\texttt{BoundedSubset} :: \forall X \in \mathsf{SMS}_{\circ\to.} \,.\, \forall A : \texttt{Bounded}(X) \,.\, \forall B \subset A \,.\, B : \texttt{Bounded}(X)$

$\texttt{Proof} =$

$[1] := \eth\texttt{Bounded}(X)(A)\eth\, \mathrm{diam}\, \texttt{SubsetSupremum}(A, B)\eth^{-1}\, \mathrm{diam} :$

$: \infty > \mathrm{diam}\, A = \sup_{x, y \in A} d(x, y) \ge \sup_{x, y \in B} d(x, y) = \mathrm{diam}\, B,$

$[2] := \eth^{-1}\texttt{Bounded}[1] : \Big(B : \texttt{Bounded}(X)\Big);$

$\square$

$\texttt{disk} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} \,.\, X \to \mathbb{R}_+ \to ?X$

$\texttt{disk}\,(x,r) = \mathbb{D}(r,x) := \Big\{ y \in X : d(x,y) \le r \Big\}$

$\texttt{cell} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} \,.\, X \to \mathbb{R}_+ \to ?X$

$\texttt{cell}\,(x,r) = \mathbb{B}(r,x) := \Big\{ y \in X : d(x,y) < r \Big\}$

$\texttt{sphere} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} \,.\, X \to \mathbb{R}_+ \to ?X$

$\texttt{sphere}\,(x,r) = \mathbb{S}(r,x) := \Big\{ y \in X : d(x,y) = r \Big\}$

$\texttt{Cellbound} :: \forall X \in \mathsf{SMS}_{\circ\to\cdot} \,.\, \forall A \subset X \,.\, A : \texttt{Bounded}(X) \iff \exists x \in X : \exists r \in \mathbb{R}_+ : A \subset \mathbb{B}(x,r)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{BoundedMetric1} :: \forall X \in \mathsf{SET} \,.\, \forall d : \texttt{Semimetric}(X) \,.\, \dfrac{d}{1+d} : \texttt{Semimetric}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y, z : X,$

$[\dots *] := \texttt{PlusUnityRecipricol}(d(x,z))\texttt{TrinagleIneq}(X,d)\texttt{PlusUnityRecipricol}(d(x,y)+d(y,z))$

$$\texttt{PositiveSumIsGreater} : \frac{d(x,z)}{1+d(x,z)} = 1 - \frac{1}{1+d(x,z)} \le 1 - \frac{1}{1+d(x,y)+d(y,z)} =$$

$$= \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} = \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)};$$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Semimetric} : \left(\dfrac{d}{1+d} : \texttt{Semimetric}(X) \right),$

$\square$

$\texttt{BoundedMetric2} :: \forall X \in \mathsf{SET} \,.\, \forall d : \texttt{Semimetric}(X) \,.\, \min(1,d) : \texttt{Semimetric}$

$\texttt{Proof} \;=$

$\dots$

$\square$

1.2 Topology

$\texttt{metricTopology} :: \mathsf{SMS}_{\circ\rightarrow.} \rightarrow \mathsf{TOP}$

$\texttt{metricTopology}\,(X) = X := \Big\langle \big\{\mathbb{B}(x,r)\big| r \in \mathbb{R}_{++}, x \in X\big\} \Big\rangle_{\mathsf{TOP}}$

$\texttt{semimetricSpaceCategory} :: \texttt{Category}$

$\texttt{semimetricSpaceCategory}\,() = \mathsf{SMS} := \Big(\texttt{SemimetricSpace}, C, \circ, \mathrm{id}\Big)$

$\texttt{metricSpaceCategory} :: \texttt{Category}$

$\texttt{metricSpaceCategory}\,() = \mathsf{MS} := \Big(\texttt{metricSpace}, C, \circ, \mathrm{id}\Big)$

$\texttt{OpenCells} :: \forall X \in \mathsf{SMS} \,.\, \forall x \in X \,.\, \forall r \in \mathbb{R}_{++} \,.\, \mathbb{B}(x,r) \in \mathcal{T}(X)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{MetricOpennesCriterion} :: \forall X \in \mathsf{SMS} \,.\, \forall U \subset X \,.\, U \in \mathcal{T}(X) \iff \forall x \in X \,.\, \exists r \in \mathbb{R}_{++} : \mathbb{B}(x,r) \subset U$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ClosureByDistance} :: \forall X \in \mathsf{SMS} \,.\, \forall A \subset X \,.\, \overline{A} = \{x \in X : d(x,A) = 0\}$

$\texttt{Proof} \ =$

$B := \{x \in X : d(x,A) = 0 : \texttt{Subset}(X),$

$\texttt{Assume}\ x : B,$

$[1] := \jmath B(x) : d(x,A) = 0,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$\Big(r, [2]\Big)) := \texttt{MetricOpennessCriterion}(U,x) : \sum r \in \mathbb{R}_{++} \,.\, \mathbb{B}(x,r) \subset U,$

$[U.*] := [1]\eth\texttt{distanceToSet}[2] : U \cap A \neq \texttt{Emptyset};$

$\rightsquigarrow [x.*] := \texttt{AltClosure}(A) : x \in \overline{A};$

$\rightsquigarrow [1] := bd^{-1}\texttt{Subset} : B \subset \overline{A},$

$\texttt{Assume}\ x : \overline{A},$

$\texttt{Assume}\ r : \mathbb{R}_{++},$

$[2] := \texttt{OpenCell}(x,r)\texttt{AltClosure}(A) : \mathbb{B}(x,r) \cap A = \emptyset,$

$[r.*] := \eth^{-1}\texttt{Functor}\,(distanceToSet, ()\, x, A)[2] : d(x,A) < r;$

$\rightsquigarrow [2] := \texttt{InfimumInduction} : d(x,A) = 0,$

$[x.*] := \jmath B[2] : x \in B;$

$\rightsquigarrow [*] := \texttt{SetEq}[1] : A = B;$

$\square$

$\texttt{DiscIsClosed} :: \forall X \in \mathsf{SMS} \,.\, \forall x \in X \, \forall r \in \mathbb{R}_{++} \, \mathbb{D}(x,r) : \texttt{Closed}(X)$

$\texttt{Proof} \;=$

$D := \mathbb{D}(x,r) : \texttt{Subset}(X),$

$\texttt{Assume}\ y : X,$

$\texttt{Assume}\ [1] : d(y,D) = 0,$

$\Big(a,[2]\Big) := \eth\texttt{distanceToSet}(y,D)[1]\eth\texttt{infimum} : \sum a : \mathbb{N} \to D \,.\, \lim_{n \to infty} d(y,a_n) = 0,$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(n,[3]\Big) := \eth\texttt{Limit}[2](\varepsilon) : \sum n \in \mathbb{N}\, d(y,a_n) < \varepsilon,$

$[\varepsilon.*] := \texttt{TriangleIneq}\eth\texttt{disc}(D)[2]\texttt{IneqSum} : d(x,y) \le d(x,a_n) + d(a_n,y) < r + \varepsilon;$

$\leadsto [3] := \texttt{IneqLim} : d(x,y) \le r,$

$[y.*] := \eth\texttt{disc}(D) : y \in D;$

$\leadsto [1] := \texttt{ClosureByDistance} : D = \overline{D},$

$[*] := \eth\texttt{closure} : \Big(D : \texttt{Closed}(X)\Big);$

$\square$

$\texttt{SphereIsClosed} :: \forall X \in \mathsf{SMS} \,.\, \forall x \in X \, \forall r \in \mathbb{R}_{++} \, \mathbb{S}(x,r) : \texttt{Closed}(X)$

$\texttt{Proof} \;=$

$S := \mathbb{S}(x,r) : \texttt{Subset}(X),$

$[1] := \texttt{TrichtomyPrinciple}\eth\texttt{sphere}(S) : S^{\complement} = \mathbb{B}(x,r) \cup \mathbb{D}^{\complement}(x,r),$

$[2] := \eth\texttt{ClosedDiscIsClosed}(x,r) : \mathbb{D}^{\complement}(x,r) \in \mathcal{T}(X),$

$[3] := \texttt{OpenCell}(x,r) : \mathbb{B}(x,r) \in \mathcal{T}(X),$

$[4] := [1]\texttt{OpenUnion}[3][2] : S^{\complement} \in \mathcal{T}(X),$

$[*] := \eth^{-1}\texttt{Closed}(X)[4] : \Big(S : \texttt{Closed}(X)\Big);$

$\square$

$\texttt{SemimetricBalance} :: \forall X \in \mathsf{SMS} \,.\, \forall x,y \in X \,.\, \forall U \in \mathcal{U}(x) \,.\, y \in U \iff \forall U \in \mathcal{U}(y) \,.\, x \in U$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{BoundedMetricTopology1} :: \forall (X,d) \in \mathsf{SMS} \,.\, (X,d) \cong_{\mathsf{SMS}} \left(X, \frac{d}{1+d}\right)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{BoundedMetricTopology2} :: \forall (X,d) \in \mathsf{SMS} \,.\, (X,d) \cong_{\mathsf{SMS}} \Big(X, \min(d,1)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MetricSpaceRegularity} :: \forall X \in \mathsf{MS} \,.\, X : \texttt{T4}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A, B : \texttt{Closed}(X),$

$\texttt{Assume}\ [1] : A \cap B = \emptyset,$

$\texttt{Assume}\ a : A,$

$\Big(r, [2]\Big) := \texttt{ClosureByDistance}[1] : \sum r \in \mathbb{R}_{++} \,.\, d(a, B) > r,$

$O(a) := \mathbb{B}\left(a, \frac{r}{4}\right) : \mathcal{T}(X);$

$\rightsquigarrow O := I(\rightarrow) : A \to \mathcal{T}(X),$

$U := \bigcap_{a \in A} O(a) : \mathcal{T}(X),$

$[1] := \jmath U : A \subset U,$

$[2] := \jmath U : B \cap U = \emptyset,$

$\texttt{Assume}\ b : B \cap \overline{U},$

$\Big(u, [3]\Big) := \texttt{ClosureByDistance}\eth\texttt{distanceToSet} : \sum u : \mathbb{N} \to U \,.\, \lim_{n\to\infty} d(b, u_n) = 0,$

$\texttt{Assume}\ n : \mathbb{N},$

$\Big(a_n, [4]\Big) := \jmath U[3] : \sum a_n \in A \,.\, u_n \in O(a_n),$

$r := d(a_n, B) : \mathbb{R}_{++},$

$[5] := \jmath O(a_n) \jmath r : d(a_n, b) \geq r,$

$[6] := \eth\texttt{cell} : d(a_n, u_n) < \frac{r}{2},$

$\texttt{Assume}\ [7] : \frac{r}{2} \geq d(u_n, b),$

$[8] := \texttt{TriangleIneq}(a_n, b)[7][6] : d(a_n, b) \leq d(a_n, u_n) + d(u_n, b) < r,$

$[7.*] := \texttt{TrichtomyPrinciple}[5][8] : \bot;$

$\rightsquigarrow [7] := E(\bot) : \frac{r}{2} < d(u_n, b),$

$[n.*] := \texttt{ReverseTriangularIneq}(a_n, u_n, b)[6][7] : \Big|d(a_n, b) - d(u_n, b)\Big| \leq d(a_n, u_n) < \frac{r}{2} < d(u_n, b);$

$\rightsquigarrow \Big(a, [4]\Big) :=: \sum a : \mathbb{N} \to A \,.\, \lim_{n\to\infty} d(a_n, b) = 0,$

$[5] := \texttt{ClodureByDistancr} : b \in A,$

$[b.*] := \texttt{EmptyNonempy}[1][5] : \bot;$

$\rightsquigarrow [3]] := E(\bot) : B \cap \overline{U} = \emptyset,$

$V := \left(\overline{U}\right)^{\complement} : \texttt{Open}(X),$

$[*] := \eth^{-1}\texttt{T4} : \Big(X : \texttt{T4}\Big);$

$\square$

$\texttt{SemimetricSeparability} :: \forall X \in \mathsf{SMS} \,.\, X : \texttt{T0} \Rightarrow X \in \mathsf{MS}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SemimetricSeparability} :: \forall X \in \mathsf{SMS} \,.\, X : \texttt{T0} \Rightarrow X : \texttt{T4}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SemimetricCountability} :: \forall X \in \mathsf{SMS} \,.\, X : \texttt{FirstCountable}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{SemimetricCountability} :: \forall X \in \mathsf{SMS} \,.\, X : \texttt{Separable} \Rightarrow X : \texttt{SecondCountable}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{DeltaEpsilonFormalism} :: \forall X, Y \in \mathsf{SMS} \,.\, \forall f : X \to Y \,.\, f \in C(X,Y) \iff \forall \epsilon \in \mathbb{R}_+ \,.\, \forall x \in X \,.$
$\quad .\, \exists \delta \in \mathbb{R}_+ : \forall a, b \in \mathbb{B}(x, \delta) \,.\, d(f(a), f(b)) < \epsilon$

$\texttt{Proof} \; =$

$\texttt{Assume}\ [1] : f \in C(X,Y),$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ \epsilon : \mathbb{R}_+,$

$y := f(x) : \texttt{Element}(Y),$

$B := \mathbb{B}(y, \epsilon) : \texttt{Open}(Y),$

$U := f^{-1}(B) : \texttt{Open}(X),$

$[1] := \jmath U : x \in U,$

$\Big(\delta, [2]\Big) := \texttt{MetricOpennesCriterion}[1] : \sum \delta \in \mathbb{R}_+ \,.\, \mathbb{B}(x, \delta) \subset U,$

$[1.*] := \jmath U[2] : f\mathbb{B}(x, \delta) \subset \mathbb{B}(y, \epsilon);$

$\rightsquigarrow [1] := \eth\texttt{Cell}I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \texttt{Right},$

$\texttt{Assume}\ U : \texttt{Open}(Y),$

$[U.*] := \texttt{MetricOpennesCriterion}\eth\texttt{Right}[2](U) : f^{-1}(U) \in \mathcal{T}(X);$

$\rightsquigarrow [2.*] := \eth^{-1} C(X,Y) : f \in C(X,Y);$

$\rightsquigarrow [*] := I(\iff) : \texttt{THIS};$

$\square$

$\texttt{LipschitzIsContinuous} :: \forall X, Y \in \mathsf{SMS} \,.\, \forall L \in \mathbb{R}_{++} \,.\, \forall f \in \texttt{L}\text{-Lip}(X,Y) \,.\, f \in C(X,Y)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{IsometricBijectionIsHomeo} :: \forall X, Y \in \mathsf{SMS} \,.\, \forall f : \texttt{IsometryBijection}(X,Y) \,.\, f : \texttt{Home}(X,Y)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{SemiisometryFullEmbedding} :: \mathsf{SMS}_{\circ\to .} : \texttt{FullEmbedding}(\texttt{SMS})$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{IsometryFullEmbedding} :: \mathsf{MS}_{\circ\to\cdot} : \texttt{FullEmbedding}(\texttt{MS})$

$\texttt{Proof} =$

$\ldots$

$\square$

1.3 Calculus of Convergence

$\texttt{NEpsilonFormalism} :: \forall X \in \mathsf{SMS} \,.\, \forall D : \texttt{DirectedSet} \,.\, \forall x : \texttt{Net}(D, x) \,.\, \forall L \in X \,.\, \lim_{n \in D} x_n = L \iff$

$\iff \forall \epsilon \in \mathbb{R}_{++} \,.\, \exists N \in D : \forall n \in D \,.\, n \geq N \Rightarrow x_n \in \mathbb{B}(L, \epsilon)$

$\texttt{Proof} =$

...

□

$\texttt{SemimetricConvergence} :: \forall X \in \mathsf{SMS} \,.\, \forall D : \texttt{DirectedSet} \,.\, \forall x : \texttt{Net}(D, x) \,.\, \forall L \in X \,.\, \lim_{n \in D} x_n = L \iff$

$\iff \lim_{n \in D} d(x_n, L) = 0$

$\texttt{Proof} =$

...

□

$\texttt{FrechetCombination} :: \forall D : \texttt{DirectedSet} \,.\, \forall x : \mathbb{N} \to \texttt{Net}(D, \mathbb{R}_+) \,.\, \lim_{\delta \in D} \sum_{n=1}^{\infty} \frac{x_{n.\delta}}{2^n(1 + x_{n,\delta})} = 0 \iff$

$\iff \forall n \in \mathbb{N} \,.\, \lim_{\delta \in D} \,.\, x_{n,\delta} = 0$

$\texttt{Proof} =$

...

□

$\texttt{FrechetWeakSemimetric} :: \forall X \in \texttt{Set} \,.\, \forall d : \mathbb{N} \to \texttt{Semimetric}(X) \,.\, \forall D : \texttt{DirectedSet} \,.$

$. \forall x : \texttt{Net}(X, D) \,.\, x : \texttt{Convergent}\left(D, \left(X, \sum_{i=1}^{n} \frac{d}{2^n(1 + d)}\right)\right) \iff \forall n \in \mathbb{N} \,.\, x : \texttt{Convergent}(D, (X, d_n))$

$\texttt{Proof} =$

...

□

$\texttt{ConvergenceByDenceSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall A : \texttt{Dense}(X) \,.\, \forall L \in X \,.\, \forall D : \texttt{DirectedSet} \,.\, \forall x : \texttt{Net}(D, X) \,.$

$. \lim_{n \in D} x_n = L \iff \forall a \in A \,.\, \lim_{n \in D} d(x_n, a) = d(L, a)$

$\texttt{Proof} =$

...

□

1.4 Compacta

$\texttt{TotallyBounded} :: ?\texttt{SMS}$

$$X : \texttt{TotallyBounded} \iff \forall \epsilon \in \mathbb{R} \,.\, \exists n \in \mathbb{N} : \exists x : n \to X : X \subset \bigcup_{i=1}^{n} \mathbb{B}(x_n, \epsilon)$$

$\texttt{EmptySpaceIsTotallyBounded} :: \emptyset : \texttt{TotallyBounded}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CountablyCompactIsTotallyBounded} :: \forall X \in \texttt{SMS} \,.\, X : \texttt{CountablyCompact} \Rightarrow X : \texttt{TotallyBounded}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : X\ !\ \texttt{TotallyBounded},$

$\Big(\epsilon, [2]\Big) := \eth\texttt{TotallyBounded} : \sum \epsilon \in \mathbb{R}_{++} \,.\, \forall n \in \mathbb{N} \,.\, \forall x : n \to X \,.\, \bigcup_{i=1}^{n} \mathbb{B}(x_i; \epsilon) \neq X,$

$(x_0, [3]) := \eth\texttt{NonEmptyTotallyBounded} : \sum x_0 : \texttt{type}(X) \,.\, x_0 \in X,$

$\texttt{Assume}\ n : \mathbb{N},$

$(x_n, [4_n]) :=: \sum x_n \in X : \forall i \in_{\mathbb{Z}_+} n - 1 \,.\, d(x_i, x_n) > \epsilon;$

$\rightsquigarrow (x, [4]) := I\Big(\sum\Big) : \sum x : \mathbb{N} \to X \,.\, \forall m, n \in \mathbb{Z}_+ \,.\, n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon,$

$c := \texttt{CountablyCompactHasClusters}(X, x) : \texttt{Cluster}(X, x),$

$(n, m, [6, 7, 8]) := \eth\texttt{Cluster}(X, x, c) : \sum n, m \in \mathbb{Z}_+ \,.\, n \neq m \,\&\, d(x_n, c) < \frac{\epsilon}{2} \,\&\, d(x_m, c) < \frac{\epsilon}{2},$

$[9] := \texttt{TriangleIneq}(x_n, x_m, c)\texttt{IneqSum}[7][9] : d(x_n, x_m) \leq d(x_n, c) + d(c, x_m) < \epsilon,$

$[1.*] := [4](n, m)[6][9]\texttt{TrichtomyPrinciple}(\mathbb{R}_+) : \perp;$

$\rightsquigarrow [*] := E(\perp) : \Big(X : \texttt{TotallyBounded}\Big);$

$\square$

$\texttt{TotallyBoundedIsSecondCountable} :: \forall X \in \texttt{SMS} \,.\, X : \texttt{TotallyBounded} \Rightarrow X : \texttt{SecondCountable}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ q : \mathbb{Q} \cap (0,1),$

$$(n, x, [1]) := \eth\texttt{TotallyBounded}(q) : \sum n \in \mathbb{N} \,.\, \sum x : n \to X \,.\, X = \bigcup_{i=1}^{n} \mathbb{B}(x_i, q),$$

$\beta_q := \left\{ \mathbb{B}(x_i, q) | i \in n \right\} : \texttt{Finite}\ \mathcal{T}(X);$

$\rightsquigarrow \beta := I(\to) : \mathbb{Q} \cap (0,1) \to \texttt{Finite}\mathcal{T}(X),$

$$\mathcal{B} := \bigcup_{q \in \mathbb{Q} \cap (0,1)} \beta_q :?\mathcal{T}(X),$$

$[1] := \texttt{CountableFiniteUnion}(\mathcal{B}, \mathbb{Q} \cap (0,1), \beta, \jmath\mathcal{B}) : |\mathcal{B}| \leq \aleph_0,$

$\texttt{Assume}\ U : \texttt{Open}(X),$

$\texttt{Assume}\ u : U,$

$(r, [2]) := \texttt{MetricOpennesCriterion}(U, x) : \sum r \in \mathbb{R}_{++} \,.\, \mathbb{B}(u, r) \subset U,$

$(q, [3]) := \texttt{ArchimedianLimit}(r) : \sum q \in (0,1) \cap \mathbb{Q} \,.\, q < \dfrac{r}{2},$

$(x, [4]) := \jmath beta_q(u) : \sum x \in X \,.\, u \in \mathbb{B}(x, q),$

$\texttt{Assume}\ y : \mathbb{B}(x, q),$

$[5] := \texttt{TriangleInequality}(u, y, x)[4]\eth\texttt{cell}(X, x, q)[3] : d(u, y) \leq d(u, x) + d(x, y) < 2q < r,$

$[y.*] := \eth\texttt{cell}(X, u, r)[2] : y \in U;$

$\rightsquigarrow [5] := \eth^{-1}\texttt{Subset} : \mathbb{B}(x, q) \subset U,$

$[u.*] := I(\exists)(\mathcal{B})[5][4] : \exists B \in \mathcal{B} : u \in B \subset U;$

$\rightsquigarrow [2] := I(\forall) : \forall u \in U \,.\, \exists B \in \mathcal{B} : u \in B \subset U,$

$[U.*] := \texttt{InnerCover}[2] : \exists \mathcal{A} \subset \mathcal{B} \,.\, \bigcup \mathcal{A} = U;$

$\rightsquigarrow [2] := \eth^{-1}\texttt{Base} : \Big(\mathcal{B} : \texttt{Base}(X)\Big),$

$[*] := \eth^{-1}\texttt{SecondCountable}(\mathcal{B})[1][2] : \Big(X : \texttt{SecondCountable}\Big);$

$\square$

$\texttt{CountablyCompactIsCompact} :: \forall X : \texttt{SMS} \,.\, X : \texttt{CountablyCompact} \Rightarrow X : \texttt{Compact}$

$\texttt{Proof} \ =$

$[1] := \texttt{CountablyCompactIsTotallyBounded}(X) : \Big(X : \texttt{TotallyBounded}\Big),$

$[2] := \texttt{TotallyBoundedIsSecondCountable}(X) : \Big(X : \texttt{SecondCountable}\Big),$

$[3] := \texttt{SecondCountableIsLindelof}(X) : \Big(X : \texttt{Lindelof}\Big),$

$[*] := \texttt{LindelofAndCountablyCompactIsCompact} : \Big(X : \texttt{Compact}\Big);$

$\square$

$\texttt{SequantiallyCompactIffCompact} :: \forall X : \texttt{SMS} \,.\, X : \texttt{SequentiallyCompact} \iff X : \texttt{Compact}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{TotallyBoundedByDenseSubset} :: \forall X \in \mathsf{SMS} \, . \, \forall A : \texttt{Dense}(X) \, . \, A : \texttt{TotallyBounded} \Rightarrow X : \texttt{TotallyBounded}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : X \,!\, \texttt{TotallyBounded},$

$\Big(\epsilon, [2]\Big) := \eth\texttt{TotallyBounded} : \sum \epsilon \in \mathbb{R}_{++} \, . \, \forall n \in \mathbb{N} \, . \, \forall x : n \to X \, . \, \bigcup_{i=1}^{n} \mathbb{B}(x_i; \epsilon) \neq X,$

$(x_0, [3]) := \eth\texttt{NonEmptyTotallyBounded} : \sum x_0 : \texttt{type}(X) \, . \, x_0 \in X,$

$\texttt{Assume}\ n : \mathbb{N},$

$(x_n, [4_n]) :=: \sum x_n \in X : \forall i \in_{\mathbb{Z}_+} n-1 \, . \, d(x_i, x_n) > \epsilon;$

$\rightsquigarrow (x, [4]) := I\Big(\sum\Big) : \sum x : \mathbb{N} \to X \, . \, \forall m, n \in \mathbb{Z}_+ \, . \, n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon,$

$(a, [5]) := \eth\texttt{Dense}(A)\left(x, \frac{\epsilon}{3}\right) : \sum a : \mathbb{Z}_+ \to A \, . \, \forall n \in \mathbb{Z}_+ \, . \, d(a_n, x_n) < \frac{\epsilon}{3},$

$\Big(n, [6,7]\Big) := \eth\texttt{TotallyBounded}(A)\left(q, \frac{\epsilon}{3}\right) : \sum n \in \mathbb{N} \, . \, d(a_n, a_0) < \frac{\epsilon}{3} \ \& \ n \neq 0,$

$[8] := \texttt{TriangleIneq}[6][5]\texttt{SumIneq} : d(x_n, x_0) \leq d(x_n, a_n) + d(a_n, a_0) + d(a_0, x_0) < \epsilon,$

$[1.*] := [8][4](0, n)[7]\texttt{TrichtomyPrinciple}(\mathbb{R}_{++}) : \bot;$

$\rightsquigarrow [*] := E(\bot)) : \Big(X : \texttt{TotallyBounded}\Big);$

$\square$

$\texttt{Equicontinuous} :: \prod X, Y \in \mathsf{SMS} \, . \, ??C(X, Y)$

$\Phi : \texttt{Equicontinuous} \iff \Phi \in \mathcal{EC}(X, Y) \iff \forall x \in X \, . \, \forall \varepsilon \in \mathbb{R}_{++} \, . \, \exists U \in \mathcal{U}(x) : \forall f \in \Phi \, . \, \operatorname{diam} f(U) < \varepsilon$

$\texttt{EquicontinuousAtAPoint} :: \prod X, Y \in \mathsf{SMS} \, . \, X \to ??C(X, Y)$

$\Phi : \texttt{Equicontinuous} \iff \Phi \in \mathcal{EC}_x(X, Y) \iff \Lambda x \in X \, . \, \forall \varepsilon \in \mathbb{R}_{++} \, . \, \exists U \in \mathcal{U}(x) : \forall f \in \Phi \, . \, \operatorname{diam} f(U) < \varepsilon$

$\texttt{FiniteContinuousIsEquicontinuous} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall \Phi : \texttt{Finite}\ C(X, Y) \, . \, \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FiniteEquicontinuousUnion} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall n \in \mathbb{N} \, . \, \forall \Phi : n \to \mathcal{EC}(X, Y) \, . \, \bigcup \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{EquicontinuityIsLocal} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall \Phi : ??C(X, Y) \, . \, \Big(\forall x \in X \, . \, \Phi \in \mathcal{EC}_x(X, Y)\Big) \Rightarrow \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{EquicontinuityByMonotonicConvergence} :: \forall X \in \mathsf{SMS} \,.\, f : \mathbb{N} \to C(X) \,.\, \forall \varphi \in C(X) \,.$
$f \downarrow \varphi \Rightarrow \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \texttt{In}(X),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(N, [1]) := \eth\texttt{Limit}(f(x), \varphi(x), \varepsilon/10) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \frac{\varepsilon}{10},$

$\Big(\delta_1, [2]\Big) := \eth C(X)(\varphi, \varepsilon/10, x) : \sum \delta_1 \in \mathbb{R}_{++} \,.\, \varphi\Big(\mathbb{B}(x, \delta_1)\Big) \subset \left(\varphi(x) - \frac{\varepsilon}{10}, \varphi(x) + \frac{\varepsilon}{10}\right),$

$\Big(\delta_2, [3]\Big) := \eth C(X)(f_N, \varepsilon/10, x) : \sum \delta_2 \in \mathbb{R}_{++} \,.\, f_N\Big(\mathbb{B}(x, \delta_2)\Big) \subset \left(f_N(x) - \frac{\varepsilon}{10}, f_N(x) + \frac{\varepsilon}{10}\right),$

$\delta := \min(\delta_1, \delta_2) : \mathbb{R}_{++},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [4] : n \geq N,$

$\texttt{Assume}\ y : \mathbb{B}(x, \delta),$

$[5] := \eth\texttt{MonotonicConvergence}(f, \varphi, y)[4] : |f_N(y) - \varphi(y)| > |f_n(y) - \varphi(x)|,$

$[n.*] := \texttt{TriangleIneq}^4[1,2,3,4,5] : |f_n(y) - f_n(x)| \leq |f_n(y) - \varphi(y)| + |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| <$
$|f_N(y) - \varphi(y)| + \frac{2\varepsilon}{10} \leq |f_N(y) - f_N(x)| + |f_N(x) - \varphi(x)| + |\varphi(x) - \varphi(y)| + \frac{2\varepsilon}{10} < \frac{\varepsilon}{2};$

$\leadsto [4] := \eth^{-1}\mathcal{EC}_x : \{f_n | n \geq N\} \in \mathcal{EC}_x(X, \mathbb{R}),$

$[x.*] := \texttt{FiniteContinuousIsEquiContinuous}\{f_n | n < N\}\texttt{FiniteEqyiContinuousUnion}[4] :$
$: \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}_x(X, \mathbb{R});$

$\leadsto [*] := \eth^{-1}\mathcal{EC}_x : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, \mathbb{R});$

$\square$

$\texttt{maricFunction} :: \prod X \in \mathsf{SMS} \,.\, \texttt{Pointwise}(X) \to \mathbb{R}_{++} \to X \to \mathbb{R}$

$\texttt{maricFunction}\,(f, \varepsilon) = M_{f,\varepsilon} := \Lambda x \in X \,.\, \sum_{n=1}^{\infty} \min\Big(0, |f_m(x) - \lim_{m \to \infty} f_n(x)| - \varepsilon\Big)$

$\texttt{MaricFunctionContinuity} :: \forall X \in \mathsf{SMS} \,.\, \forall \varphi \in C(X) \,.\, f : \texttt{Pointwise}(X, \varphi) \,.\, \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$

$\texttt{Proof} \ =$

$\varphi := \lim_{n \to \mathbb{N}} f_n : X \to \mathbb{R},$

$\texttt{Assume}\ x : X,$

$\Big(N, [1]\Big) := \eth\texttt{Pointwise}(X) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \frac{\varepsilon}{3} \ \&\ n < N \Rightarrow |f_n(x) - \varphi(y)| >$

$\varepsilon' := \min(|f_{N-1}(x) - \varphi(x)| - \varepsilon, \varepsilon) : \mathbb{R}_{++},$

$\Big(\delta_1, [2]\Big) := \eth\mathcal{EC} : \prod \delta \in \mathbb{R}_{++} \,.\, \forall n \in \mathbb{N} \,.\, f_n\mathbb{B}(x, \delta) \subset \mathbb{B}\left(f(x), \frac{\varepsilon'}{3}\right),$

$\Big(\delta_2, [3]\Big) := \eth C(X) : \prod \delta \in \mathbb{R}_{++} \,.\, \varphi\mathbb{B}(x, \delta) \subset \mathbb{B}\left(\varphi(x), \frac{\varepsilon'}{3}\right),$

$\delta := \min(\delta_1, \delta_2, \delta_3) : \mathbb{R}_+,$

$\texttt{Assume}\ y : \mathbb{B}(x, \delta),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [4] : n \geq N,$

$[y.*] := \texttt{TriangleIneq}[1][2][3] : |\varphi(y) - f_n(y)| \leq |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon;$

$\leadsto [4] := I(\forall) : \forall n \geq N \,.\, \forall y \in \mathbb{B}(x, \delta) \,.\, |\varphi(y) - f_n(x)| < \varepsilon,$

$\texttt{Assume}\ y : \mathbb{B}(x, \delta),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [5] : n < N,$

$[y.*] := [1][2][3][5] : |\varphi(y) - f_n(y)| > |\varphi(x) - f_n(x)| - \varepsilon' > \varepsilon;$

$\rightsquigarrow [5] := I(\forall) : \forall y \in \mathbb{B}(x,\delta) \ .\ \forall n \in \mathbb{N} \ .\ n < N \Rightarrow |\varphi(x) - f_n(x)| < \varepsilon,$

$$[6] := [4][5]\eth^{-1}\texttt{maricFunction} : \forall y \in \mathbb{B}(x,\delta) \ .\ M_{f,\varepsilon}(y) = \sum_{n=1}^{N-1} |f_n(y) - \varphi(y)| - \varepsilon,$$

$[y.*] := \texttt{ContinuousSum} : M_{f,\varepsilon|\mathbb{B}(x,\delta)} \in C\Big(\mathbb{B}(x,\delta)\Big);$

$\rightsquigarrow [*] := \texttt{ContinuityIsLocal} : M_{f,\varepsilon} \in C(X);$

$\square$

$\texttt{DiniLemma} :: \forall X : \texttt{Pseudocompact}\ \&\ \mathsf{SMS} \ .\ \forall f : \mathbb{N} \to C(X) \ .\ \forall \varphi \in C(X) \ .$

$\quad .\ f \downarrow \varphi \Rightarrow \varphi : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), \varphi\Big)$

$\texttt{Proof}\ =$

$[1] := \texttt{EquicontinuityByMonotonicConvergence} : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X,\mathbb{R}),$

$[2] := \texttt{MaricFunctionContinuity} : \forall \varepsilon \in \mathbb{R}_{++} \ .\ M_{f,\varepsilon} \in C(X),$

$[a,3] := \eth\texttt{Paracompact}(M_f) : \sum \mathbb{R}_{++} \to \mathbb{R}_+ \ .\ \forall \varepsilon \in \mathbb{R}_{++} \ .\ \max_{x \in X} M_{f,\varepsilon}(x) = a_\varepsilon,$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [4] : n > \left\lceil \frac{a_\varepsilon}{\varepsilon} \right\rceil,$

$\texttt{Assume}\ [5] : \|f_n - \varphi\|_\infty \geq \varepsilon,$

$\Big(x, [6]\Big) := \eth\texttt{uniformNorm}[5]\eth\texttt{Paracompact}(X) : \sum x \in X : |f_n(x) - \varphi(x)| \geq \varepsilon,$

$[7] := \eth\texttt{MonotonicConvergence}[5] : \forall m \in \mathbb{N} \ .\ m \leq n \Rightarrow |f_m(x) - \varphi(x)| \geq \varepsilon,$

$$[8] := \eth\texttt{maricFunction}[4][7] : M_{f,\varepsilon}(x) \geq \sum_{i=1}^{n} \|f(x) - \varphi(x)\| > a_\varepsilon,$$

$[n.*] := \eth\texttt{Maximum}(a_n, M_{\varepsilon,x})[8] : \bot;$

$\rightsquigarrow [\varepsilon.*] := E(\bot) : \forall n \in \mathbb{N} \ .\ n > \left\lceil \frac{a_\varepsilon}{\varepsilon} \right\rceil \Rightarrow \|f_n - \varphi\|_\infty < \varepsilon;$

$\rightsquigarrow [*] := \texttt{MetricLimit} : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), f, \varphi\Big);$

$\square$

$\texttt{DiniSpace} :: ?\mathsf{TOP}$

$X : \texttt{DiniSpace} \iff \forall f : \mathbb{N} \to C(X) \ .\ \forall \varphi \in C(X) \ .$

$\quad .\ f \downarrow \varphi \Rightarrow \varphi : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), \varphi\Big)$

$\texttt{InverseDiniLemma} :: \forall X : \texttt{DiniSpace} \ .\ X : \texttt{Paracompact}$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{EquicontinuousNet} :: \prod X, Y \in \textsf{SMS} \,.\, ?\texttt{Net Pointwise}(X, Y)$

$(D, f) : \texttt{EquicontinuousNet} \iff \{f_\delta | \delta \in D\} \in \mathcal{EC}(X, Y)$

$\texttt{EquicontinuousLimitIsContinuous} :: \forall X, Y \in \textsf{SMS} \,.\, \forall (D, f) : \texttt{EquicontinuousNet}(X) \,.\, \forall F : \texttt{Limit}(D, f) \,.$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : X,$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_+,$

$(\delta, [1]) := \eth\texttt{EquicontinuousNet}(D, f)\left(x, \frac{\varepsilon}{y}\right) : \sum \delta' \in \mathbb{R}_{++} \,.\, \forall n \in D \,.\, \operatorname{diam} f_n \mathbb{B}(x, \delta) < \frac{\varepsilon}{3},$

$\texttt{Assume}\; y : Y,$

$(N, [2]) := \eth\texttt{Limit}(f, d)(F) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow d(f_n(x), F(x)) < \frac{\varepsilon}{3},$

$(M, [3]) := \eth\texttt{Limit}(f, d)(F) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow d(f_n(y), F(y)) < \frac{\varepsilon}{3},$

$K := \max(N, M) : D,$

$[y.*] := \texttt{TriangleIneq}[1][2][3] : d(F(x), F(y)) \leq d(F(x), f_K(x)) + d(f_K(x), f_K(y)) + f(f_K(y), F(y)) < \varepsilon;$

$\rightsquigarrow [x.*] := I(\forall) : \forall y \in \mathbb{B}(x, \delta) \,.\, d(F(x), F(y)) < \varepsilon;$

$\rightsquigarrow [*] := \eth^{-1} C(X, Y) : F \in C(X, Y);$

$\square$

$\texttt{ConvergentContinuously} :: \prod X, Y \in \textsf{TOP} \,.\, ?\texttt{PointwiseNet}(X, Y)$

$(D, f) : \texttt{ConvergentContinuously} \iff \forall x : \texttt{ConvergentNet}(D, X) \,.\, \lim_{n \in D} f_n(x_n) = \left(\lim_{n \in D} f\right)\left(\lim_{n \in D} x_n\right)$

$\texttt{UniformIsConvergentContinuously} :: \forall X, Y \in \textsf{SMS} \,.\, \forall (D, f) : \texttt{UniformNet}(X, Y) \,.\, (D, f) : \texttt{ConvergentCont}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UniformIsConvergentContinuously} :: \forall X, Y \in \textsf{SMS} \,.\, \forall (D, f) : \texttt{EquicontinuousNet}(X, Y) \,.\, (D, f) : \texttt{Converg}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UniformConvergenceOnCompact} :: \forall Y \in \mathsf{SMS} . \forall X : \texttt{Compact} \ \& \ \mathsf{SMS} . \forall (D, f) : \texttt{Net}\ C(X, Y) .$

$\quad . \forall F \in C(X, Y) . \forall [1] : \texttt{PointwiseLimit}(D, f, F) . \texttt{UniformLimit}(D, f, F)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ x : X,$

$\Big(r', [1]\Big) := \eth\texttt{EquicontinuosNet}(D, f) \left(x, \frac{\varepsilon}{3}\right) : \sum r' \in \mathbb{R}_{++} . \forall n \in D . \ \operatorname{diam} f_n \mathbb{B}(x, r') < \frac{\varepsilon}{3},$

$\Big(r'', [2]\Big) := \eth C(X, Y)(F) \left(x, \frac{\varepsilon}{3}\right) : \sum r'' \in \mathbb{R}_{++} . \ \operatorname{diam} F\mathbb{B}(x, r') < \frac{\varepsilon}{3},$

$r(x) := \min(r', r'') : \texttt{In}\ \mathbb{R}_{++},$

$\Big(N(x), [3]\Big) := \eth\texttt{PointwisLimit}(D, f, F) \left(x, \frac{\varepsilon}{3}\right) : \sum N(x) \in D : \forall n \in D . n \geq N \Rightarrow d(f_n(x), F(x)) < \frac{\varepsilon}{3},$

$\texttt{Assume}\ y : \texttt{In}\ \mathbb{B}\Big(x, r(x)\Big),$

$\texttt{Assume}\ n : \texttt{In}(D),$

$\texttt{Assume}\ [4] : n \geq N(x),$

$[y.*] := \texttt{TriangleIneq}[1, 2, 3] : d\Big(f_n(y), F(y)\Big) \leq d(f_n(y), f_n(x)) + d(f_n(x), F(x)) + d(F(x), F(y)) < \varepsilon;$

$\rightsquigarrow [x.*] := I(\forall) : \forall y \in \mathbb{B}\Big(x, r(x)\Big) . d\Big(f_n(y), F(y)\Big) < \varepsilon;$

$\rightsquigarrow \Big(r, N, [1]\Big)] := I\left(\prod\right) : \prod_{x \in X} \Big(r(x), N(x)\Big) : \mathbb{R}_{++} \times D . \forall y \in \mathbb{B}\Big(x, r(x)\Big) . \forall n \in D .$

$\quad . n \geq N(x) \Rightarrow d\Big(f_n(y), F(y)\Big) < \varepsilon,$

$[2] := \texttt{FullCover}\mathbb{B}(\cdot, r) : X = \bigcup_{x \in X} \mathbb{B}\Big(x, r(x)\Big),$

$\Big(n, x, [3]\Big) := \eth\texttt{Compact}(X) : \sum n \in \mathbb{N} . \sum x : n \to X . X = \bigcup_{i=1}^{n} \mathbb{B}\Big(x_i, r(x_i)\Big),$

$\Delta := \max_{i \in n} N(x_i) : D;$

$\texttt{Assume}\ y : X,$

$\texttt{Assume}\ \delta : \texttt{In}\ D,$

$\texttt{Assume}\ [4] : \delta \geq \Delta,$

$\Big(i, [5]\Big) := [2](y) : \sum i \in n . y \in \mathbb{B}\Big(x_i, r(x_i)\Big),$

$[6] := [4]\jmath\Delta(i) : \delta \geq N(x_i),$

$[\varepsilon.*] := [1][5][6] : d\Big(f_\delta(y), F(y)\Big) < \varepsilon;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{UniformLimit} : \texttt{UniformLImit}(D, f, F);$

$\square$

1.5 Completeness and Completion

$$\texttt{CauchyFilterBase} :: \prod X \in \mathsf{SMS} \,.\, ?\texttt{FilterBase}(X)$$

$$\mathcal{F} : \texttt{CauchyFilterBase} \iff \inf_{F \in \mathcal{F}} \operatorname{diam} F = 0$$

$$\texttt{CauchyFilter} := \prod X \in \mathsf{SMS} \,.\, \texttt{Filter} \,\&\, \texttt{CauchyFilterBase}(X) : \mathsf{SMS} \to \texttt{Type};$$

$\texttt{ConvergentFilterBaseIsCauchy} :: \forall X \in \mathsf{SMS} \,.\, \forall \mathcal{F} : \texttt{ConvergentFilterBase}(X) \,.$

$\quad . \,\mathcal{F} : \texttt{CauchyFilterBase}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$$\texttt{CauchyNet} :: \prod X \in \mathsf{SMS} \,.\, \prod D : \texttt{DirectedSet} \,.\, ?\texttt{Net}(D, x)$$

$$x : \texttt{CauchyNet} \iff \forall \varepsilon \in \mathbb{R} \,.\, \exists \Delta \in D : \forall \delta, \delta' \in D \,.\, d(x_\delta, x_{\delta'}) < \varepsilon$$

$$\texttt{CauchySequence} := \prod X \in \mathsf{SMS} \,.\, \texttt{CauchyNet}(\mathbb{N}, X) : \prod_{X \in \mathsf{SMS}} ?(\mathbb{N} \to X);$$

$\texttt{CauchyFilterAndNetEquivalence} :: \forall X \in \mathsf{SMS} \,.\, \forall (D, x) : \texttt{Net}(X) \,.$

$\quad . \, x : \texttt{CauchyNet}(D, X) \iff \mathcal{F}(D, x) : \texttt{CauchyFilter}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$$\texttt{Complete} :: ?\mathsf{SMS}$$

$$X : \texttt{Complete} \iff \forall \mathcal{F} : \texttt{CauchyFilterBase}(X) \,.\, \mathcal{F} : \texttt{ConvergentFilterBase}(X)$$

$\texttt{CompleteAltDef} :: \forall X \in \mathsf{SMS} \,.\, X : \texttt{Complete} \iff \forall x : \texttt{CauchySequence}(X) \,.\, x : \texttt{Convergent}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ R : \forall x : \texttt{CauchySequence}(X) \,.\, x : \texttt{Convergent},$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase},$

$$\Big(F, [1]\Big) := \Lambda n \in \mathbb{N} \,.\, \eth\texttt{CauchyFilter}(X, \mathcal{F}) \left(\frac{1}{n}\right) : \prod_{n \in \mathbb{N}} \sum_{F_n \in \mathcal{F}} \operatorname{diam} F_n < \frac{1}{n},$$

$$x := \eth\texttt{CauchyFilterbase}(\mathcal{F})(F) : \prod_{n \in \mathbb{N}} \bigcap_{i=1}^{n} F_i,$$

$[2] := \eth^{-1}\texttt{CauchySequnce}(X)[1] : \texttt{CauchySequence}(X, x),$

$[3] := R[2] : \texttt{Convergent}(X, x),$

$L := \lim_{n \to \infty} x_n : X,$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(N,[4]\Big) := \eth\texttt{Limit}(X,x,L)\Big(\frac{\varepsilon}{2}\Big) : \sum_{N\in\mathbb{N}} : \forall n \in \mathbb{N} \,.\, d(L,x_n) < \frac{\varepsilon}{2},$

$\Big(M,[5]\Big) := \texttt{InverseArchimedeanLimit} : \sum M \in \mathbb{N} \,.\, \frac{1}{M} < \frac{\varepsilon}{2},$

$k := \max(M,N) : \mathbb{N},$

$\texttt{Assume}\ y : F_k,$

$[\varepsilon.*] := \texttt{TriangleIneq}(X,L,y,x_k)[1,4,5] : d(L,y) \le d(L,x_k) + d(x_k,y) < \varepsilon;$

$\rightsquigarrow [4] := \eth^{-1}\texttt{FilterbaseLimit} : L = \lim \mathcal{F},$

$[\mathcal{F}.*] := \eth^{-1}\texttt{ConvergentFilterBase}[4] : \texttt{ConvergentFilterBase}(X,\mathcal{F});$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Complete} : \texttt{Complete}(X);$

$\square$

$\texttt{CauchyFilterbaseClustersAreLimits} :: \forall X : \mathsf{SMS} \,.\, \forall \mathcal{F} : \texttt{CauchyFilterBase}(X) \,.$

$\quad .\, \forall c : \texttt{Cluster}(\mathcal{F}) \,.\, c : \texttt{Limit}(\mathcal{F})$

$\texttt{Proof}\ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(F,[1]) := \eth^{-1}\texttt{CauchyFilterBase}(\mathcal{F})\left(\frac{\varepsilon}{2}\right) : \sum_{F\in\mathcal{F}} \operatorname{diam} F < \frac{\varepsilon}{2},$

$(x,[2]) := \eth\texttt{Cluster}(\mathcal{F},c)\left(F, \mathbb{B}\left(\frac{\varepsilon}{2}\right)\right) : \sum_{x\in F} d(x,c) < \frac{\varepsilon}{2},$

$\texttt{Assume}\ y : F,$

$[3] := \texttt{TriangleIneq}(X)(c,y,x) : d(c,y) \le d(c,x) + d(x,y) < \varepsilon,$

$[y.*] := \eth^{-1}\texttt{cell}[3] : y \in \mathbb{B}(c,\varepsilon);$

$\rightsquigarrow [F.*] := \eth^{-1}\texttt{Subset} : F \subset \mathbb{B}(c,\varepsilon);$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Limit} : \texttt{Limit}(X,\mathcal{F},c);$

$\square$

$\texttt{CauchyPartialLimitTHM} :: \forall X \in \mathsf{SMS} \,.\, \forall x : \texttt{CauchySequance}(X) \,.\, \forall L : \texttt{PartialLimit}(X,x) \,.\, \lim_{n\to\infty} x_n = L$

$\texttt{Proof}\ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(N,[1]\Big) := \eth\texttt{CauchySequance}(X,x)\left(\frac{\varepsilon}{2}\right) : \sum_{N\in\mathbb{N}} \forall n,m \in \mathbb{N} \,.\, n,m \ge N \Rightarrow d(x_n,x_m) < \frac{\varepsilon}{2},$

$\Big(M,[2]\Big) := \eth\texttt{PartialLimit}(X,x,p)\left(p,\frac{\varepsilon}{2}\right) : \sum_{M\in\mathbb{N}} M \ge N \,\&\, d(x_M,L) < \frac{\varepsilon}{2},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [3] : n \ge N,$

$[4] := \texttt{TriangleIneq}[1][2] : d(x_n,L) < d(x_n,x_M) + d(x_M,L) < \varepsilon;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Limit} : \lim_{n\to\infty} x_n = L;$

$\square$

$\texttt{CompactIffCompleteAndTotallyBounded} :: \forall X \in \mathsf{SMS} \,.\, \texttt{Compact}(X) \iff \texttt{Complete} \,\&\, \texttt{TotallyBounded}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ L : \texttt{Compact}(X),$

$[1] := \texttt{CountablyCompactIsTottalyBounded}(X, L) : \texttt{TotallyBounded}(X),$

$[2] := \texttt{SequantiallyCompactIffCompact}(X, L) : \texttt{SequantiallyCompact}(X),$

$\texttt{Assume}\ x : \texttt{CauchySequence}(X),$

$p := \eth\texttt{SequantiallyCompact}(X)(x) : \texttt{PartialLimit}(X, x),$

$[x.*] := \texttt{CauchyPartialLimitTHM}(X, x, p) : \lim_{n\to\infty} x_n = p;$

$\leadsto [x.*] := \texttt{CompleteAltDef} : \texttt{Complete}(X);$

$\leadsto [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : \texttt{Complete} \,\&\, \texttt{TotallyBounded}(X),$

$\texttt{Assume}\ x : \mathbb{N} \to X,$

$u^1 := x : \mathbb{N} \to X,$

$\texttt{Assume}\ n : \mathbb{N},$

$$\Big[m, c, [2]\Big] := \eth\texttt{TotallyBounded}(X)\left(\frac{1}{n}\right) : \sum_{m\in\mathbb{N}} \sum_{c:m\to X} X = \bigcup_{i=1}^{m} \mathbb{B}\left(c_i, \frac{1}{n}\right),$$

$$\Big[i, [3]\Big] := \texttt{PigionholePrinciple}[2] : \sum_{i\in m} \left|\left\{k \in \mathbb{N} : u^n_k \in \mathbb{B}\left(c_i, \frac{1}{n}\right)\right\}\right| = \aleph_0,$$

$$\Big(u^{n+1}, [n.*]\Big) := \texttt{subsequance}[3] : \sum u^{n+1} : \texttt{Subsequance}(X, u^{n+1}) \,.\, \operatorname{Im} u^{n+1} \subset \mathbb{B}\left(c_i, \frac{1}{n}\right);$$

$$\leadsto \Big(u, [2]\Big) := I\left(\prod\right) : \prod_{u:\mathbb{N}\to\mathbb{N}\to X} \,.\, \forall n \in \mathbb{N} \,.\, u^{n+1} : \texttt{Subsequance}(X, u^n) \,\&\, \exists c \in X : \operatorname{Im} u^{n+1}\mathbb{B}\left(c, \frac{1}{n}\right),$$

$\Delta := \Lambda n \in \mathbb{N} \,.\, u^n_n : \texttt{Subsequance}(X, x),$

$[3] := \jmath\Delta[2] : \forall N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n, m > N \to d(x_n, x_m) < \dfrac{2}{N},$

$[4] := \eth^{-1}\texttt{CauchySequance}[3] : \texttt{CauchySequance}(X, \Delta),$

$[5] := \texttt{CompleteAltDef}(X)[4] : \texttt{Convergent}(X, \Delta),$

$\delta := \lim_{n\to\infty} \Delta_n : X,$

$[x.*] := \eth^{-1}\texttt{PartialLimit}(x) : \texttt{PartialLimit}(X, x, \delta);$

$\leadsto [*.R] := \texttt{SequentiallyCompactIffCompact} : \texttt{Compact}(X);$

$\leadsto [*] := I(\iff) : \texttt{Left} \iff \texttt{Right};$

$\square$

$$\texttt{ConvergentByCompleteSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall \mathcal{F} : \texttt{CauchyFilter}(X) \,.\, \Big(\exists F \in \mathcal{F} : F : \texttt{Complete}\Big) \Rightarrow$$

$\quad \Rightarrow \texttt{Convergent}(X, \mathcal{F})$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConvergentByCompactSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall \mathcal{F} : \texttt{CauchyFilter}(X) \,.\, \Big(\exists F \in \mathcal{F} : F : \texttt{Compact}\Big) \Rightarrow$

$\quad \Rightarrow \texttt{Convergent}(X, \mathcal{F})$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CompletionCategory} :: \mathsf{SMS} \to \mathsf{CAT}$

$$\texttt{CompletionCategory}\,(X) = \mathsf{COMP}(X) := \Bigg(\sum Y : \texttt{Complete} \,.\, f : \texttt{Isometry}(X, Y),$$

$$, \Big((Y, f), (Z, g)\Big) \mapsto \Big\{\varphi : \texttt{Isometry}(Y, Z)\Big| f\varphi = g\Big\}, \circ, \mathrm{id}\Bigg)$$

$\texttt{Completion} := \Lambda X \in \mathsf{SMS} \,.\, \texttt{Initial}\; \mathsf{COMP}(X) : \mathsf{SMS} \to \texttt{Type};$

$\texttt{CompleteByDenseSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall A : \texttt{Dense}(X) \,.\, \Big(\forall a : \texttt{CauchySequance}(A) \,.\, a : \texttt{Convergent}(X)\Big) \Rightarrow$

$\quad \Rightarrow X : \texttt{Complete}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : \texttt{CauchySequance}(X),$

$\Big(a, [1]\Big) := \Lambda n \in \mathbb{N} \,.\, \eth\texttt{Dense}(X)(x_n) : \sum a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, d(a_n, x_n) < \dfrac{1}{n},$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$\Big(K, [2]\Big) := \eth\texttt{CauchySequance}(X, x)\left(\dfrac{\varepsilon}{3}\right) : \sum K \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n, m \geq K \Rightarrow d(x_n, x_m) < \dfrac{\varepsilon}{3},$

$\Big(L, [3]\Big) := \texttt{ReducioInfinuma}\left(\dfrac{\varepsilon}{3}\right) : \sum L \in \mathbb{N} \,.\, \dfrac{1}{L} < \dfrac{\varepsilon}{3},$

$N := \max(K, L) : \mathbb{N},$

$\texttt{Assume}\; n, m : \mathbb{N},$

$\texttt{Assume}\; [4] : n, m \geq N,$

$\Big[\varepsilon.*\Big] := \texttt{TriangleIneq}^2(\ldots)[1,2,3,4] : d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(a_m, x_m) < \varepsilon;$

$\leadsto [4] := \eth^{-1}\texttt{CauchySequance} : \texttt{CauchySequance}(A, a),$

$L := \lim\limits_{n\to\infty} a_n : \texttt{In}\; X,$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$\Big(K, [5]\Big) := \eth\texttt{Limit}(X, a, L)\left(\dfrac{\varepsilon}{2}\right) : \sum K \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n \geq K \Rightarrow d(a_n, L) < \dfrac{\varepsilon}{2},$

$\Big(M, [6]\Big) := \texttt{ReducioInfinuma}\left(\dfrac{\varepsilon}{2}\right) : \sum M \in \mathbb{N} \,.\, \dfrac{1}{M} < \dfrac{\varepsilon}{2},$

$N := \max(K, M) : \mathbb{N},$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; [7] : n > N,$

$[\varepsilon.*] := \texttt{TrianglIneq}[1,5,6,7] : d(x_n, L) \leq d(x_n, a_n) + d(a_n, L) < \varepsilon;$

$\leadsto [x.*] := \eth^{-1}\texttt{Limit} : \lim\limits_{n\to\infty} x_n = L;$

$\leadsto [*] := \texttt{CompleteAltDef} : \texttt{Complete}(X);$

$\square$

$\texttt{CompletionIsUnique} :: \forall X \in \mathsf{SMS} \,.\, \forall (A,a),(B,b) : \texttt{Completion}(X) \,.\, (A,B) : \texttt{Semiisometric}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RealsAreComplete} :: \mathbb{R} : \texttt{Complete}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CauchySequanceDistanceExists} :: \forall X \in \mathsf{SMS} \,.\, \forall x,y : \texttt{CauchySequance}(X) \,.\, d(x,y) : \texttt{Convergent}(\mathbb{R}_+)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(L,[1]\Big) := \eth\texttt{CauchySequance}(X,x)\left(\frac{\varepsilon}{2}\right) : \sum L \in \mathbb{N} \,.\, \forall n,m \in \mathbb{N} \,.\, n,m > L \Rightarrow d(x_n,x_n) < \frac{\varepsilon}{2},$

$\Big(M,[2]\Big) := \eth\texttt{CauchySequance}(X,y)\left(\frac{\varepsilon}{2}\right) : \sum M \in \mathbb{N} \,.\, \forall n,m \in \mathbb{N} \,.\, n,m > M \Rightarrow d(y_n,y_m) < \frac{\varepsilon}{2},$

$N := \max(L,M) : \mathbb{N},$

$\texttt{Assume}\ n,m : \mathbb{N},$

$\texttt{Assume}\ [3] : n,m \geq N,$

$[\varepsilon.*] := \texttt{TriangleIneq}\Big(d(x_n,y_n), d(x_m,y_m), d(x_n,y_m)\Big)$

$\quad \texttt{ReversedTriangleIneq}(x_n,y_n,y_m)\texttt{ReversedTriangleIneq}(x_n,x_m,y_m)[1,2,3] :$

$\quad : \Big|d(x_n,y_n) - d(x_m,y_m)\Big| \leq \Big|d(x_n,y_n) - d(x_n,y_m)\Big| + \Big|d(x_n,y_m) - d(x_m,y_m)\Big| \leq d(x_n,x_m) + d(y_n,y_m) < \varepsilon;$

$\rightsquigarrow [4] := \eth^{-1}\texttt{CauchySequance} : \texttt{CauchySequance}(\mathbb{R}, d(x,y)),$

$[*] := \texttt{RealsAreComplete}\ \texttt{CompleteAltDef}(\mathbb{R}) : \texttt{Convergent}(\mathbb{R}, d(x,y));$

$\square$

$\texttt{CauchyMetric} :: \forall X \in \mathsf{SMS} \,.\, \lim d : \texttt{Semimetric}\ \texttt{CauchySequance}(X)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{completion} :: \mathsf{SMS} \to \mathsf{MS}$

$$\texttt{completion}\,(X,d) = (\hat{X},\hat{d}) := \frac{\Big(\texttt{CauchySequance}(X), \lim d\Big)}{\lim d}$$

$\texttt{CompletionTHM} :: \forall X \in \mathsf{SMS} \ . \ \left(\hat{X}, \Lambda x \in X \ . \ \Lambda n \in \mathbb{N} \ . \ x\right) : \texttt{Completion}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ p : \hat{X},$

$\left(x, [1]\right) := \eth \hat{X}(p) : \sum x : \texttt{CauchySequance}(X) \ . \ p = [x],$

$A := \Lambda n \in \mathbb{N} \ . \ \Lambda m \in \mathbb{N} \ . \ \texttt{if} \ m < n \ \texttt{then} \ x_m \ \texttt{else} \ x_n : \mathbb{N} \to \texttt{Convergent}(X),$

$\texttt{Assume} \ \varepsilon : \mathbb{R}_{++},$

$\left(N, [2]\right) := \eth \texttt{CauchySequance}(X, x)(\varepsilon) : \sum N \in \mathbb{N} \ . \ \forall n, m \in \mathbb{N} \, n, m \geq \mathbb{N} \Rightarrow d(x_n, x_m) < \varepsilon,$

$\texttt{Assume} \ n : \mathbb{N},$

$\texttt{Assume} \ [3] : n \geq N,$

$[p.*] := \eth \texttt{completion}[1] \jmath A[2,3] : \hat{d}(p, [A_n]) = \lim_{m \to \infty} d(x_m, A_{n,m}) = \lim_{m \to \infty} d(x_m, x_n) < \varepsilon;$

$\leadsto [1] := \eth^{-1} \texttt{Dense} : \texttt{Dense}(\hat{X}, X),$

$[2] := \texttt{CompleteByDenseSubset}(\hat{X}, X) : \texttt{Complete}(\hat{X}),$

$\phi := \Lambda x \in X \ . \ \left[\Lambda n \in \mathbb{N} \ . \ x\right] : \texttt{Isometry}(X, \hat{X}),$

$\texttt{Assume} \ (Y, f) : \texttt{In} \ \mathsf{COMP}(X),$

$\psi := \Lambda [x] \in \hat{X} \ . \ \lim_{n \to \infty} f(x_n) : \texttt{Isometry}(\hat{X}, Y),$

$\texttt{Assume} \ x : X,$

$[x.*] := \jmath \phi \jmath \psi \texttt{ConstantSeq} : x \phi \psi = [n \mapsto x] \psi = \lim_{n \to \infty} f(x) = f(x);$

$\leadsto [3] := \eth^{-1} \mathsf{COMP}(X) : \left(\psi : \hat{X} \xrightarrow{\mathsf{COMP}(X)} Y\right),$

$\texttt{Assume} \ \psi' : \hat{X} \xrightarrow{\mathsf{COMP}(X)} Y,$

$\texttt{Assume} \ p : \hat{X},$

$\left(x, [4]\right) := \eth \hat{X}(p) : \sum x : \texttt{CauchySequance}(X) \ . \ p = [x],$

$A := \Lambda n \in \mathbb{N} \ . \ \Lambda m \in \mathbb{N} \ . \ \texttt{if} \ m < n \ \texttt{then} \ x_m \ \texttt{else} \ x_n : \mathbb{N} \to \texttt{Convergent}(X),$

$\left[(Y, f)\right] := [4] \jmath^{-1} A \texttt{ContinuousLimit} \jmath^{-1} \phi \eth \mathsf{COMP}(X) : \psi'(p) = \psi'[x] = \psi' \lim_{n \to \infty} [A_n] = \lim_{n \to \infty} \psi'[A_n] =$

$= \lim_{n \to \infty} \psi' \phi(x_n) = \lim_{n \to \infty} f(x_n) \lim_{n \to \infty} \psi \phi(x_n) = \psi(p);$

$\leadsto [*] := \eth^{-1} \texttt{Completion} : \texttt{Completion}\left(X, (\hat{X}, \varphi)\right);$

$\square$

$\texttt{Contraction} := \Lambda X \in \mathsf{MS} \ . \ \sum \alpha \in (0,1) \ . \ \alpha\text{-Lip}(X) : \mathsf{MS} \to \texttt{Type};$

$\mathtt{BanachFixedPointTHM} :: \forall X \in \mathsf{MS} \,\&\, \mathtt{Complete} \,\&\, \mathtt{NonEmpty} \,.\, \forall f : \mathtt{Contraction}(X) \,.\, \exists! x \in X : f(x) = x$

$\mathtt{Proof} \ =$

$$\Big(\alpha, [1]\Big) := \eth\mathtt{Contraction}(f) : \sum_{\alpha \in (0,1)} \forall x, y \in X \,.\, d(f(x), f(y)) \leq alpha * d(f(x), f(y)),$$

$x := \eth\mathtt{NonEmpty}(X) : \mathtt{In}(X),$

$\delta := d(x, f(x)) : \mathbb{R}_+,$

$\mathtt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$$\Big(N, [2]\Big) := \mathtt{PowerReduction} : \sum_{N \in \mathbb{N}} \alpha^N < \frac{(1-\alpha)\varepsilon}{\delta},$$

$\mathtt{Assume}\ n, m : \mathbb{N},$

$\mathtt{Assume}\ [3] : n, m \geq N,$

$\mathtt{Assume}\ [4] : n \geq m,$

$[\varepsilon.*] := \mathtt{TriangleInequality}[4]\mathtt{NonNegSum}[3][1]\mathtt{PowerSum}[2][3] :$

$$: d\Big(f^n(x), f^m(x)\Big) \leq \sum_{i=0}^{n-m} d\Big(f^{m+i}(x), f^{m+i+1}(x)\Big) \leq \sum_{i=0}^{\infty} d\Big(f^{N+i}(x), f^{N+i+1}(x)\Big) \leq \sum_{i=0}^{\infty} \delta\alpha^{N+i} = \frac{\delta\alpha^N}{1-\alpha} < \varepsilon;$$

$\rightsquigarrow [2] := \eth^{-1}\mathtt{CauchySequance} : \mathtt{CauchySequance}\Big(X, \Lambda n \in \mathbb{N} \,.\, f^n(x)\Big),$

$[3] := \mathtt{AltCompleteDef}[2] : \mathtt{Convergent}\Big(X, \Lambda n \in \mathbb{N} \,.\, f^n(x)\Big),$

$p := \lim_{n \to \infty} f^n(x) : \mathtt{In}X,$

$\mathtt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$$\Big(M, [4]\Big) := \eth\mathtt{Limit}(X, \Lambda n \in \mathbb{N} \,.\, f^n(x), L) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow d(f^n(x), p) < \frac{\varepsilon}{3},$$

$$\Big(K, [5]\Big) := \eth\mathtt{CauchySequance}(X, \Lambda n \in \mathbb{N} \,.\, f^n(x)) : \sum K \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n, m \geq N \Rightarrow d(f^n(x), f^m(x) < \frac{\varepsilon}{3},$$

$N := \max(M, K) : \mathbb{N},$

$\mathtt{Assume}\ n : \mathbb{N},$

$\mathtt{Assume}\ [6] : n \geq N,$

$[\varepsilon.*] := \mathtt{TriangleIneq}[1](f^n\ x, p)[4,5,6] : d(p, f\ x) \leq d(p, f^n\ x) + d(f^n\ x, f^{n+1}\ x) + d(f^{n+1}(x), f\ p) <$

$< 2d(p, f^n\ x) + d(f^n\ x, f^{n+1}\ x) < \varepsilon;$

$\rightsquigarrow [4] := \mathtt{ZeroBound} : d(L, f\ p) = 0,$

$[5] := \eth\mathsf{MS}[4] : p = f\ p;$

$\mathtt{Assume}\ q : \mathtt{Type}InX,$

$\mathtt{Assume}\ [6] : q = f\ q,$

$[7] := [1](p, q)[5][6] : d(p, q) \leq \alpha d(f\ , f\ q) = \alpha d(p, q),$

$[8] := \mathtt{FixedMultiolicationIsZero}[7] : d(p, q) = 0,$

$[q.*] := \eth^{-1}\mathsf{MS}[8] : p = q;$

$\rightsquigarrow [*] := \eth^{-1}\mathtt{Unique} : \exists! x \in X : f(x) = x;$

$\square$

1.6 Baire Category

$\texttt{CantorIntersectionTheorem} :: \forall X \in \mathsf{SMS} \,\&\, \texttt{Complete} \,.\, \forall A : \texttt{Decreasing}\left(\texttt{Closed}(X) \,\&\, \texttt{NonEmpty}\right) .$

$$. \forall [0] : \lim_{n\to\infty} \operatorname{diam} A_n = 0 \,.\, \bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

$\texttt{Proof} \;=$

$a := \eth\texttt{NonEmpty}(A) : \prod_{n\in\mathbb{N}} A_n,$

$[1] := \eth^{-1}\texttt{CauchySequance}[0]\jmath a : \texttt{CauchySequance}(X, a),$

$[2] := \texttt{AltCompleteDef} : \texttt{Convergent}(X, a),$

$K := \lim_{n\to\infty} a_n : \texttt{NonEmpty}(X),$

$\texttt{Assume}\ x : K,$

$[3] := \Lambda n \in \mathbb{N} \,.\, \texttt{ClosedLimit}(X, A_n, a_{+n}, x) : \forall n \in \mathbb{N} \,.\, x \in A_n,$

$[x.*] := \eth\texttt{Intersection} : x \in \bigcap_{n=1}^{\infty} A_n;$

$\rightsquigarrow [3] := \eth^{-1}\texttt{Subset} : K \subset \bigcap_{n=1}^{\infty} A_n,$

$[*] := \eth\texttt{NonEmpty}[3] : \bigcap_{n=1}^{\infty} A_n \neq \emptyset;$

$\square$

$\texttt{MetricBaireCategoryTheorem} :: \forall X \in \mathsf{SMS} \,\&\, \texttt{Complete} \,.\, X : \texttt{Baire}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ U : \mathbb{N} \to \texttt{Dense} \,\&\, \texttt{Open}(X),$

$\texttt{Assume}\ x : \texttt{In}\ X,$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$u_1 := x : \texttt{In}\ X,$

$r_1 := \varepsilon : \mathbb{R}_{++},$

$\delta_1 := 1 : \mathbb{R}_{++},$

$\texttt{Assume}\ n : \mathbb{N},$

$[1] := \texttt{OpenDenseIntersection} : \bigcap_{i=1}^{n} U_i : \texttt{Open} \,\&\, \texttt{Dense}(X),$

$V := \mathbb{B}(x_n, r_n) \cap \bigcap_{i=1}^{n} U_i : \texttt{Open} \,\&\, \texttt{NonEmpty}(X),$

$v := \eth\texttt{NonEmpty}(X, V : \texttt{In}(V),$

$\Delta := d(v, \partial V) : \mathbb{R}_{++},$

$\delta_{n+1} := \dfrac{\min(\Delta, \delta_n)}{2} : \mathbb{R}_{++},$

$K_n := \{v \in V : d(v, \partial V) \le \delta_{n+1}\} : \texttt{NonEmpty}(V),$

$[2] := \texttt{ContinuousDistance}(\overline{V}, \partial V)\texttt{ClosedPreimage}\jmath K_n : \texttt{Closed}(X, K_n),$

$u_{n+1} := \eth\texttt{NonEmpty} : K_n,$

$r_{n+1} := \min\left(\dfrac{r_n}{2}\,\delta_{n+1}\right) : \mathbb{R}_{++};$

$$\rightsquigarrow \Big(K,[1]\Big) := I\left(\sum\right) : \sum K : \texttt{Decreasing CLosed}(X) \;.\; \lim_{n\to\infty} \operatorname{diam} K_n = 0 \;\&\; \forall n \in \mathbb{N} \;.\; K_n \subset \bigcap_{i=1}^{n} U_n \cap \mathbb{B}(x,\varepsilon),$$

$$[2] := \texttt{CantorIntersectionTheorem}(X,K,[1]) : \bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

$$[x.*] := \texttt{IntersectionSubset}(X,K,[1],[2]) : \bigcap_{n=1}^{\infty} K_n \subset \mathbb{B}(x,\varepsilon) \cap \bigcap_{n=1}^{\infty} U_n;$$

$$\rightsquigarrow [U.*] := \eth^{-1}\texttt{Dense} : \texttt{Dense}\left(X, \bigcup_{n=1}^{\infty} U_n\right);$$

$$\rightsquigarrow [*] := \eth^{-1}\texttt{Baire} : \texttt{Baire}(X);$$

$\square$

$$\texttt{FirstCategory} :: \prod X \in \mathsf{TOP} \;.\; ??X$$

$$A : \texttt{FirstCategory} \iff A \neq \emptyset \;\&\; \exists P : \mathbb{N} \to \texttt{NowhereDense}(X) \;.\; \bigcup_{n=1}^{\infty} P_n = A$$

$\texttt{FirstCategoryTheorem1} :: \forall X : \texttt{Baire} \;.\; \forall U \in \mathcal{T}(X) \;.\; U \,!\, \texttt{FirstCategory}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{FirstCategory}(X,U),$

$[2] := \eth_1\texttt{FirstCategory}(X,U)[1] : U \neq \emptyset,$

$$\Big(A,[3]\Big) := \eth_2\texttt{FirstCategory}(X,U)[1] : \sum A : \mathbb{N} \to \texttt{NowhereDense}(X) \;.\; U = \bigcap_{n=1}^{\infty} A_n,$$

$[3] := \texttt{DualBaireProperty}[3] : \texttt{Codense}(U),$

$[4] := \eth\texttt{Codense}(X,U) : \texttt{Dense}(X, U^{\complement}),$

$[5] := \texttt{DenseOpenIntersection}[4][2] : U \cap U^{\complement} \neq \emptyset,$

$[1.*] := \texttt{ComplementIntersection}(U)\texttt{EmptyIsNonEmpty} : \bot;$

$\rightsquigarrow [*] := E(\bot) :!\texttt{FirstCategory}(X,U);$

$\square$

$\texttt{FirstCategoryTheorem2} :: \forall X \in \mathsf{TOP} \;.\; \forall [0] : \forall U \in \mathcal{T}(X) \;.\; U \,!\, \texttt{FirstCategory} \;.\; X : \texttt{Baire}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ U : \mathbb{N} \to \texttt{Open} \;\&\; \texttt{Dense}(X),$

$$\texttt{Assume}\ [1] : \bigcap_{n=1}^{\infty} U_n \,!\, \texttt{Dense}(X),$$

$$\Big(V,[2]\Big) := \eth\texttt{Dense}[1] : \sum V \in T(X) \;.\; V \cap \bigcap_{n=1}^{\infty} U_n = \emptyset,$$

$A := V \cap U^{\complement} : \mathbb{N} \to \texttt{Codense}(X),$

$$[3] := \jmath A[2] : V = \bigcup_{i=1}^{\infty} A_i,$$

$[1.*] := [0](V)\eth^{-1}\texttt{FirstCategory}[3] : \bot;$

$$\rightsquigarrow [U.*] := E(\bot) : \bigcap_{n=1}^{\infty} U_n = \texttt{Dense}(X);$$

$$\rightsquigarrow [*] := \eth^{-1}\texttt{Baire} : \texttt{Baire}(X);$$

$\square$

1.7 Hausdorff Metric

$\texttt{metricOfHausdorff} :: \mathsf{MS} \to \mathsf{MS}$

$\texttt{metricOfHausdorff}\,(X) = \mathcal{H}(X) = \Big(\mathcal{H}(X), d_H\Big) := \Big(\texttt{Closed} \,\&\, \texttt{Bounded} \,\&\, \texttt{Nonempty}(X), A, B \mapsto \max(\sup_{x\in A} d(x$

$\texttt{Assume}\ A, B, C : \texttt{NonEmpty} \,\&\, \texttt{Compact}(X),$

$\texttt{Assume}\ [1] : d(A, B) = \sup_{x\in A} d(x, B),$

$\Big(a, [2]\Big) := \eth\texttt{supremum} : \sum a : \mathbb{N} \to A\ .\ d(A, B) = \lim_{n\to\infty} d(a_n, B),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ c : C,$

$[c.*] := [1]\eth\texttt{setDistance}(B)\forall x \in A\ .\ \forall y \in B\texttt{TriangleIneq}(x, y, c)\texttt{InfReduction}(B)\texttt{SupReduction}(B)$

$\quad \texttt{SumIneq}(\mathbb{R}, \ldots)\texttt{SupremumIntroduction}(C)\eth^{-1}\texttt{metricOfHausforff} :$

$\quad : d(a_n, B) = \inf_{b\in B} d(a_n, b) =\leq d(a_n, c) + \inf_{b\in B} d(c, b) \leq d(a_n, c) + \sup_z \inf_b d(z, b) \leq d(a_n, c) + d(C, B);$

$\rightsquigarrow [1.*] := \texttt{UniversalIneq} : d(a_n, B) \leq \inf_{z\in C} d(a_n, z) + d(C, B) \leq d(A, C) + d(C, B),$

$[1] := I(\Rightarrow) : d(A, B) = \sup_{x\in A} d(x, B) \Rightarrow d(A, B) \leq \sup_{x\in A} d(x, B) + d(C, B);$

$\square$

$\texttt{blowUp} :: \prod X \in \mathsf{SMS}\ .\ \mathbb{R}_{++} \to (?X) \to (?X)$

$\texttt{blowUp}\,(\varepsilon, A) = A_\varepsilon := \bigcup_{a\in A} \mathbb{B}(a, \varepsilon)$

$\texttt{HausdorffMetricAltDef} :: \forall X \in \mathsf{MS}\ .\ \forall A, B : \texttt{Compact}(X)\ .\ d(A, B) = \inf \Big\{\varepsilon \in \mathbb{R}_{++} \Big| B \subset A_\varepsilon \,\&\, A \subset B_\varepsilon\Big\}$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{HausdorffFunctor} :: \mathsf{MS}_{\circ\to\cdot} \xrightarrow{\mathsf{CAT}} \mathsf{MS}_{\circ\to\cdot}$

$\texttt{HausdorffFunctor}\,(X) = \mathcal{H}(X) := \mathcal{H}(X)$

$\texttt{HausdorffFunctor}\,(X, Y, f) = \mathcal{H}(f) := \Lambda A \in \mathcal{H}(X)\ .\ \operatorname{cl} f(A)$

$\texttt{CauchyDiamConverge} :: \forall X \in \mathsf{MS}\ .\ \forall A : \texttt{Cauchy}(X)\ .\ \operatorname{diam} A : \texttt{Convergent}(\mathbb{R})$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{HausdorffMetrizationANdCompletionCommute} :: \forall X \in \mathsf{MS}.\mathcal{H}(\widehat{X}) \cong_{\mathsf{MS}_{\circ\to}.} \widehat{\mathcal{H}(X)}$

$\texttt{Proof} =$

$\texttt{Assume } A : \mathcal{H}(\widehat{X}),$

$\texttt{Assume } a : A,$

$\Big(x^a, [1]\Big) := \eth\widehat{(X)}(a) : \sum x : \texttt{Cauchy}(X) \;.\; a = [x^a] \;\&\; \forall N \in \mathbb{N} \;.\; \forall n, m \in \mathbb{N} \;.\; n, m \geq N \Rightarrow d(x^a_m, x^a_m) \leq N^{-1};$

$\leadsto (x, [1]) := I\,(A) : \prod_{a \in A} \sum x : \texttt{Cauchy}(X) \;.\; a = [x^a] \;\&\; \forall N \in \mathbb{N} \;.\; \forall n, m \in \mathbb{N} \;.\; n, m \geq N \Rightarrow d(x^a_m, x^a_m) \leq N^{-1},$

$B := \Lambda n \in \mathbb{N} \;.\; \operatorname*{cl}_X \{x^a_n | a \in A\} : \mathbb{N} \to \texttt{Closed}(X),$

$\texttt{Assume } n : \mathbb{N},$

$\texttt{Assume } b, b' : \texttt{In}(B_n),$

$\texttt{Assume } \varepsilon : \mathbb{R}_{++},$

$\Big(a', a, [2]\Big) := \eth\texttt{closure}\,\jmath B_n(b, b') : \sum a', a :\in A \;.\; d(b, x^a_n) < \varepsilon \;\&\; d(b', x^{a'}_n) < \varepsilon,$

$[n.*] := \texttt{TriangleIneq}^4[2][1] : d(b', b) \leq d(b, x^a_n) + d(x^a_n, a) + d(a, a') + d(a', x^{a'}_n) + d(x^{a'}_n, b') < \operatorname{diam} A + \frac{2}{n} + 2\varepsilon;$

$\leadsto [2] := I(\forall) : \forall n \in \mathbb{N} \;.\; \operatorname{diam} B_n \leq \operatorname{diam} A + \frac{2}{n},$

$\texttt{Assume } \varepsilon : \mathbb{R},$

$\Big(N, [3]\Big) := \texttt{ReductionInfinuma}(\varepsilon) : \prod N \in \mathbb{N} \;.\; \varepsilon < \frac{1}{\mathbb{N}},$

$\texttt{Assume } n, m : \mathbb{N},$

$\texttt{Assume } [4] : n, m \geq N,$

$[\varepsilon.*] := \eth\mathcal{H}(X)\eth\texttt{supremum}[1][3][4] : d(B_n, B_m) = \max\left(\sup_{a \in A} d(x^a_n, B_m), \sup_{a \in A} d(x^a_m, B_n)\right) \leq \sup_{a \in A} d(x^a_n, x^a_m) \leq \frac{1}{N} < \varepsilon;$

$\leadsto [3] := \eth^{-1}\texttt{CauchySequance} : \texttt{CauchySequance}\Big(\mathcal{H}(X), B\Big),$

$\varphi(X) := \lim_{n \to \infty} X_n : \widehat{\mathcal{H}(X)};$

$\leadsto \varphi := I(\to) : \mathcal{H}(\widehat{X}) \to \widehat{\mathcal{H}(X)},$

$\texttt{Assume } A, B : \mathcal{H}(\widehat{X}),$

$\Big(\alpha, [4]\Big) := \jmath\varphi(A) : \sum \alpha : \texttt{Cauchy}\Big(\mathcal{H}(X)\Big)\varphi(A) = [\alpha] \;\&\; \lim_{n \to \infty} \operatorname*{cl}_{\widehat{X}} \alpha_n = A,$

$\Big(\beta, [5]\Big) := \jmath\varphi(B) : \sum \beta : \texttt{Cauchy}\Big(\mathcal{H}(X)\Big)\phi(B) = [\beta] \;\&\; \lim_{n \to \infty} \operatorname*{cl}_{\widehat{X}} \beta_n = B,$

$\Big[A, B, [*]\Big] := [\eth\texttt{Complement}4][5] : d\Big(\varphi(A), \varphi(B)\Big) = \lim_{n \to \infty} d\Big(\alpha_n, \beta_n\Big) = d(A, B)\Big);$

$\leadsto [4] := \eth^{-1}\texttt{Isometry} : \texttt{Isometry}\Big(\mathcal{H}(\widehat{X}), \widehat{\mathcal{H}(X)}\Big),$

$\texttt{Assume } A : \widehat{\mathcal{H}(X)},$

$\Big(K, [1]) := \eth\texttt{Completion}(\mathcal{H}(X)) : \prod K : \texttt{CachySequance}\Big(\mathcal{H}(X)\Big) \;.\; A = [K],$

$C := \Big\{x : \texttt{CauchySequence}(X) | \forall n, m \in \mathbb{N} \;.\; x_n \in K_n \;\&\; d(x_n, x_m) \leq d(K_n, K_m)\Big\} :?\texttt{Cauchy}(X),$

$\psi(A) := \operatorname{cl}\{\lim_{n \to \infty} x_n | x \in C\} : \texttt{Closed}(\widehat{X}),$

$[A.*] := \texttt{CauchyDiamConverge} : \texttt{Bounded}(\widehat{X}.\psi(A));$

$\leadsto \psi := I(\to) : \widehat{\mathcal{H}(X)} \to \mathcal{H}(\widehat{X}),$

$\texttt{Assume}\ A, B : \widehat{\mathcal{H}(X)},$

$\Big(\alpha, [4]\Big) := \jmath\varphi(A) : \sum \alpha : \texttt{Cauchy}\Big(\mathcal{H}(X)\Big) \ . \ A = [\alpha],$

$\Big(\beta, [5]\Big) := \jmath\varphi(B) : \sum \beta : \texttt{Cauchy}\Big(\mathcal{H}(X)\Big) \ . \ B = [\beta],$

$C_A := \Big\{x : \texttt{CauchySequence}(X) | \forall n, m \in \mathbb{N} \ . \ x_n \in \alpha_n \ \& \ d(x_n, x_m) \le d(\alpha_n, \alpha_m)\Big\} :?\texttt{Cauchy}(X),$

$C_B := \Big\{x : \texttt{CauchySequence}(X) | \forall n, m \in \mathbb{N} \ . \ x_n \in \beta_n \ \& \ d(x_n, x_m) \le d(\beta_n, \beta_m)\Big\} :?\texttt{Cauchy}(X),$

$[(A,B).*] := \eth\texttt{hausdorffMetric}\eth\texttt{completionUniformCauchyExchange}[4][5]\eth^{-1}\texttt{completion} ::$

$$d(\psi(A), \psi(B)) = \max\left(\sup_{a \in \psi(A)} \inf_{b \in \psi(B)} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b)\right) =$$

$$= \max\left(\lim_{n \to \mathbb{N}} \lim_{m \to \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m), \lim_{n \to \mathbb{N}} \lim_{m \to \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m)\right) = \lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} d(\alpha_n, \beta_m) = d(A,B);$$

$\leadsto [5] := \eth^{-1}\texttt{Isometry} : \texttt{Isometry}\Big(\widehat{\mathcal{H}(X)}, \mathcal{H}(\widehat{X})\Big),$

$\texttt{Assume}\ A : \mathcal{H}(\widehat{X}),$

$\Big(\alpha, [6]\Big) := \jmath\varphi(A) : \sum \alpha : \texttt{Cauchy}\Big(\mathcal{H}(X)\Big)\varphi(A) = [\alpha] \ \& \ \lim_{n \to \infty} \operatorname{cl}_{\widehat{X}} \alpha_n = A,$

$[7] := [6]\texttt{ContinuousLimitCommute}\jmath\psi : d(\psi\varphi A, A) = d(\psi \lim_{n \to \infty} \alpha_n, A) = \lim_{n \to \infty} d(\psi\alpha_n, A) = \lim_{n \to \infty} d(\operatorname{cl}_{\widehat{X}} \alpha_n, A) = d(A$

$[A.*] := \eth^{-1}\mathsf{MS}[7][8] : \psi\varphi A = A;$

$\leadsto [6] := I(\rightarrow, =) : \psi\varphi = \mathrm{id},$

$[*] := \texttt{IsometricSurjectionIsBijection}(\psi)\texttt{RightInverseIsBijective}[6] : \psi = \varphi^{-1};$

$\square$

$\texttt{HausdorffCompleteIffComplete} :: \forall X \in \mathsf{MS} \ . \ \texttt{Complete}(X) \iff \texttt{Complete}\ \mathcal{H}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CompactHausdorffConvergence} :: \forall X : \texttt{Complete} \ . \ \forall K : \texttt{CauchySequance}(\mathcal{H}(X)) \ .$

$\quad . \ \forall [0] : \forall n \in \mathbb{N} \ . \ \texttt{Compact}(K_n) \ . \ \lim_{n \to \infty} \Big(\texttt{Compact}\Big)$

$\texttt{Proof} \ =$

$A := \operatorname{cl} \bigcup_{n=1}^{\infty} K_n : \texttt{Closed}(X),$

$[1] := \texttt{ClosedSubsetOfComplete}(X, A) : \texttt{Complete}(A),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(N, [2]\Big) := \eth\texttt{CauchySequance}(\mathcal{H}(X), K)\left(\frac{\varepsilon}{8}\right) : \sum_{N \in \mathbb{N}} \forall n, m \in \mathbb{N} \ . \ n, m \ge N \Rightarrow d(K_n, K_m) < \frac{\varepsilon}{8},$

$\Big(k, E, [3]\Big) := \texttt{CompactIsTotallyBounded}\Big((K_n)_{n=1}^{N}\Big)\eth\texttt{TotallyBounded} :$

$$: \prod_{n=1}^{N} \sum_{k \in \mathbb{N}} \sum_{E : k \to ?K_n} \forall i \in k_n \ . \ \operatorname{diam} E < \frac{\varepsilon}{2} \ \& \ \bigcup_{i=1}^{k_n} E_i = K_n,$$

$F := \Lambda n \in N \ . \ \Lambda i \in k_n \ . \ \Big\{x \in A \ . \ d(x, E_i) \le \frac{\varepsilon}{4}\Big\} : \prod_{n=1}^{N} k_n \to ?A,$

$[\varepsilon.1.*] := \jmath F \eth^{-1} \operatorname{diam} : \forall n \in N \ . \ \forall i \in k_n \ . \ \operatorname{diam} F_{n,i} \le \varepsilon,$

$\texttt{Assume}\ a : A,$

$\Big(x, [5]\Big) := \texttt{MetricCLosure}\jmath A : \sum x \in \bigcup_{n=1}^{\infty} K_n \ . \ d(x,a) < \frac{\varepsilon}{8},$

$\Big(n, [6]\Big) := \eth\texttt{Union}(K, x) : \sum n \in \mathbb{N} \ . \ x \in K_n,$

$\Big(m, [7]\Big) := \texttt{DirechletPrinciple}[2] : \sum m \in N \ . \ d(K_m, K_n) < \frac{\varepsilon}{8},$

$\Big(y, [8]\Big) := \eth^{-1}\texttt{distanceOfHausdorff}[7] : \sum y \in K_m \ . \ d(y, x) < \frac{\varepsilon}{8},$

$[9] := \texttt{TriangleIneq}[5][8] : d(a,y) \leq d(a,x) + d(x,y) < \frac{\varepsilon}{4},$

$\Big(i, [10]\Big) := [3](y) : \sum i \in k_i \ . \ y \in E_i,$

$[a.*] := \jmath[10][9] : a \in F_{n,i};$

$\leadsto [\varepsilon.2.*] := \eth^{-1}\texttt{Union} : \bigcup_{n=1}^{N} \bigcup_{i=1}^{k_n} F_i = A;$

$\leadsto [2] := \eth^{-1}\texttt{TotallyBounded} : \texttt{TotallyBounded}(A),$

$[3] := \texttt{CompleteAndTotallyBoundedIsCompact}[2] : \texttt{Comcpact}(A),$

$[4] := \eth\mathcal{H}(X) : \lim_{n\to\infty} K_n \subset A,$

$[*] := \texttt{ClosedSubsetCompact}[3,4] : \texttt{Compact}(\lim_{n\to\infty} K_n);$

$\square$

$\texttt{SingletonHausdorffLimit} :: \forall X \in \mathsf{MS} \ . \ \forall p : \texttt{Converging}\Big(\mathcal{X}\Big) \ . \ \forall n \in \mathbb{N} \ . \ |p_n| = 1 \Rightarrow \Big|\lim_{n\to\infty} x_n\Big|$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CompactHausdorffMetrIffCompact} :: \forall X \in \mathsf{MS} \ . \ \texttt{Compact}(X) \iff \texttt{Compact}(\mathcal{H}(X))$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [0] : \texttt{Compact(X)},$

$[1] := \texttt{HausdorffMetrizationAndCompletionCommute}(X) : \texttt{Complete}\Big(\mathcal{X}\Big),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(n, A, [2]\Big) := \texttt{CompactTotallyBounded}(X)\eth\texttt{TotallyBounded} : \sum_{n=1}^{\infty} \sum_{A:n\to?X} \bigcup_{i=1}^{n} A_i = X \ \& \ \forall i \in n \ . \ \operatorname{diam} A \leq \varepsilon,$

$B := \Lambda b \in 2^n \ . \ \{K \in \mathcal{H}(X) : \forall i \in \ . \ b_i = 1 \iff A_i \cap K \neq \emptyset\} : 2^n \to ?\mathcal{H}(X),$

$\texttt{Assume}\ b : 2^n,$

$\texttt{Assume}\ K, K' : B_b,$

$[b.*] := \jmath B\eth\texttt{distanceOfHaussdorff}[2] : d(K, K') \leq \max_{i\in b} d(K \cap A_i, K \cap A_i) \leq \varepsilon;$

$\leadsto [\varepsilon. * .1] := I(\forall) : \forall b \in 2^n \ . \ \operatorname{diam} B_b \leq \varepsilon,$

$[\varepsilon. * .2] := \jmath B[2] : \bigcup_{b\in 2^n} = \mathcal{H}(X);$

$\leadsto [2] := \eth^{-1}\texttt{TotallyBounded} : \texttt{TotallyBounded}\Big(\mathcal{H}(X)\Big),$

$[0.*] := \texttt{CompactIfCompleteAndTotallyBounded}(\mathcal{H}(X)) : \texttt{Compact}(\mathcal{H}(X));$

$\leadsto LR := I(\Rightarrow) : \texttt{Compact}(X) \Rightarrow \texttt{Compact}(\mathcal{H}(X)),$

$\texttt{Assume}\ R : \texttt{Compact}(\mathcal{H}(X)),$

$\texttt{Assume}\ x : \mathbb{N} \to X,$

$[P',] := \texttt{SequanceCompactIffCompact}(\mathcal{H}(X))\eth\texttt{SequanceCompact} : \sum P' : \texttt{Subsequence}\{x\} \ .\ P' : \texttt{Converging}$

$P := \lim_{n\to\infty} p' : \texttt{In}\mathcal{H}(X),$

$(p, [1]) := \texttt{SingletonHausdorffLimit}(P) : \sum p \in X \ .\ P = \{p\},$

$(x', [x.*]]) := [1]\jmath p\eth\texttt{distanceOfHaussdorff} : \sum x' : \texttt{Subsequence}(X) \ .\ p = \lim_{n\to\infty} x_n;$

$\rightsquigarrow [R.*] := \texttt{SequanceCompactIffCompact}(X) : \texttt{Compact}(X);$

$\rightsquigarrow [R.*] := I(\iff) : \texttt{Compact}(X) \iff \texttt{Compact}(\mathcal{H}(X));$

$\square$

1.8 Geodesic Paths and Hopf-Rinow Theorem

$$\texttt{RectifiablePath} :: \prod X \in \mathsf{MS} \,.\, \prod x, y \in X \,.\, ?\Omega(x, y)$$

$$\gamma : \texttt{RectifiablePath} \iff \gamma \in R(x, y) \iff$$

$$\iff \texttt{ConvergingNet}\left(\mathbb{R}_+, M(\gamma), \Lambda(n, \omega) \in M(\gamma) \,.\, \sum_{i=1}^{n} d_X\big(\omega_i(0), \omega_i(1)\big)\right)$$

$$\texttt{LengthSpace} :: ?\mathsf{MS}$$

$$X : \texttt{LengthSpace} \iff \forall x, y \in X \,.\, R(x, y) \neq \emptyset \iff$$

$$\texttt{arclength} :: \prod X \in \mathsf{MS} \,.\, \prod x, y \in X \,.\, R(x, y) \to \mathbb{R}_+$$

$$\texttt{arclength}\,(\gamma) = |\gamma| := \lim_{(n,\omega) \in M(\gamma)} \sum_{i=1}^{n} d_X\big(\omega_i(0), \omega_i(1)\big)$$

$$\texttt{ArclengthSum} :: \forall X \in \mathsf{MS} \,.\, \forall x, y, z \in X \,.\, \forall \alpha \in R(x, y) \,.\, \forall \beta \in R(y, z) \,.\, |\alpha\beta| = |\alpha| + |\beta|)$$

Proof $=$

. . .

□

$$\texttt{ArclengthInversion} :: \forall X \in \mathsf{MS} \,.\, \forall x, y \in X \,.\, \forall \gamma \in R(x, y) \,.\, \left|\gamma^{\frown}\right| = \gamma$$

Proof $=$

. . .

□

$$\texttt{ConstantArclength} :: \forall X \in \mathsf{MS} \,.\, \forall x \in X \,.\, \Big|\texttt{constant}\big([0, 1], X, x\big)\Big| = 0$$

Proof $=$

$$\texttt{IntrinsicMetric} :: \prod X : \texttt{LengthSpace} \,.\, \texttt{Metric}(X)$$

$$\texttt{IntrinsicMetric}\,(x, y) = d_I(x, y) := \inf_{\gamma \in R(x,y)} |\gamma|$$

$$\texttt{Geodesic} :: \prod X \in \mathsf{MS} \,.\, ?R(X, Y)$$

$$\gamma : \texttt{Geodesic} \iff |\gamma| = \inf_{\gamma \in R(x,y)} |\gamma|$$

$$\texttt{GeodesicSpace} :: ?\texttt{LengthSpace}(X)$$

$$X : \texttt{GeodesicSpace} \iff \forall x, y \in X \,.\, \exists \gamma : \texttt{Geodesic}(x, y)$$

$\texttt{GeodesicSpaceIsPathConnected} :: \forall X : \texttt{Geodesic} \ . \ X : \texttt{PathConnected}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{HopfRinow} :: ?\mathsf{MS}$

$X : \texttt{HopfRinow} \iff \forall x, y \in X \ . \ \forall a, b \in \mathbb{R}_+ \ . \ a + b < d(x,y) \Rightarrow d\Big(\mathbb{B}(x,a), \mathbb{B}(y,b)\Big) = d(x,y) - a - b$

$\texttt{HopfRinowLemma} :: \forall X : \texttt{Complete} \ \& \ \texttt{HopRinow} \ . \ \forall x \in X \ . \ \forall r \in \mathbb{R}_{++} \Bigg(\forall r' \in \mathbb{R}_{++} \ . \ \texttt{Compact}\Big(\mathbb{D}(x, r'), \varepsilon\Big) \Rightarrow \texttt{Com}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ y : X,$

$\texttt{Assume} \ [1] : d(x,y) = r,$

$\texttt{Assume} \ t : (0, r),$

$\texttt{Assume} \ s : (0, r - t),$

$\Big(u, v[2]\Big) := \eth\texttt{HopfRinov}(X, x, y, t, s) : \forall t \in (0,r) \ . \ \forall s \in (0, r-t) \ . \ \exists u, v \in X : d(x,u) = t \ \& \ d(y,v) = s \ \& \ d(u,v$

$[y.3] := \texttt{TriangleIneq} : d(y,u) \le d(y,v) + d(u,v) = r - t;$

$\rightsquigarrow [1] := \eth^{-1}\texttt{metricSpace} : \forall t \in (0,r) \ . \ d\Big(\mathbb{D}(x,t), \mathbb{D}(x,r) \le r - t,$

$[2] := \texttt{CompactHausdorffConvergence}(X)[1] : \texttt{Compact}\Big(X, \mathbb{D}(x,r)\Big);$

$\square$

$\texttt{CompactBlowUp} :: \forall X \in \mathsf{MS} \ \& \ \texttt{LocallyComapct} \ . \ \forall K : \texttt{Compact}(X) \ . \ \exists U \in \mathcal{U}(K) : \texttt{Compact}(\overline{U})$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{HopfRinowTHM1} :: \forall X : \texttt{LocallyCompact} \ \& \ \texttt{HopfRinow} \ \& \ \texttt{Complete} \ . \ \forall x \in X \ . \ \forall r \in \mathbb{R}_{++} \ . \ \texttt{Compact}\Big(\mathbb{D}(x,r)\Big)$

$\texttt{Proof} \ =$

$R := \Big\{ r \in \mathbb{R}_{++} \Big| \texttt{Compact}\big(X, \mathbb{D}(x,r)\big) \Big\} :?\mathbb{R}_{++},$

$[1] := \eth\texttt{LocallyCompact}(X)\jmath R : R \ne \emptyset,$

$\texttt{Assume} \ t : \sup R,$

$[2] := \texttt{HopfRinowLemma}(X, x, t)\eth\texttt{Supremum}(R)\jmath R : t \in R,$

$K := \mathbb{D}(x,t) : \texttt{Compact},$

$\Big(U, [3]\Big)] := \texttt{CompactBlowUp}\jmath R[2]\eth\texttt{Supremum}(R) : \sum U \in \mathcal{U}(K) \ . \ \overline{U} : \texttt{Compact},$

$A := X \setminus U : \texttt{Closed}(X),$

$\Big(s, [4]\Big) := \texttt{DishointClosedDistance}\jmath A[3] : \sum s \in \mathbb{R}_{++} \ . \ d(K, A) > s,$

$[5] := [4]\jmath K[3] : \mathbb{D}(x, t+s) \subset \overline{U},$

$[6] := \texttt{CompactClosedSubset}[5] : \texttt{Compact}\Big(X, \mathbb{D}(x, t+s)\Big),$

$[4] := \eth t \jmath R[6] : \bot;$

$\rightsquigarrow [2] := \texttt{CompactSubset}E(\bot) : R = \mathbb{R}_{++};$

$\square$

$\texttt{CompleteByCompactDiscs} :: \forall X \in \mathsf{MS} . \forall [0] : \left(\forall x \in X . \forall r \in \mathbb{R}_{++} . \forall \texttt{Compact}\Big(X, \mathbb{D}(x,r)\Big) . \right) . \texttt{Complete}(X)$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterBase}(X),$

$\texttt{Assume}\ F : \mathcal{F},$

$\texttt{Assume}\ [1] : \operatorname{diam} F < \infty,$

$\delta := \operatorname{diam} F : \mathbb{R}_{++},$

$\mathcal{G} := \{G \in \mathcal{F} | G \subset\} : \texttt{CauchyFilterBase}(X),$

$x := \eth\texttt{NonEmpty}(F) : F,$

$[2] := [0](x, \delta) : \texttt{Compact}\Big(X, \mathbb{D}(x,r)\Big),$

$[3] := \texttt{CompactIsComplete}[2] : \texttt{Complete}\Big(X, \mathbb{D}(x,r)\Big),$

$[F.*] := \eth\texttt{Complete}[3](\mathcal{G}) : \texttt{ConvergingFilterBase}\Big(X, \mathcal{G}\Big);$

$\rightsquigarrow [\mathcal{F}.*] := \eth\texttt{CauchyFilterBase}\eth^{-1}\texttt{ConcvergingFilterBase} : \texttt{ConvergingFilterBase}(X, \mathcal{F});$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Complete} : \texttt{Complete}(X);$

$\square$

$\texttt{MidpointTHM} :: \forall X : \texttt{Complete} \,\&\, \texttt{LocallyCompat} \,\&\, \texttt{HopfRinow} . \forall x, y \in X . \exists z \in X :$

$$d(x,z) = d(z,y) = \frac{1}{2} d(x,y)$$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : x \neq y,$

$\delta := d(x,y) : \mathbb{R}_{++},$

$\texttt{Assume}\ t : (0, \delta),$

$\Big(a, b, [2]\Big) := \eth\texttt{HopfRinow}\eth\texttt{HausdorffDistance} :$

$$\sum a : \mathbb{N} \to \mathbb{D}\left(x, \frac{\delta - t}{2}\right) . \sum b : \mathbb{N} \to \mathbb{D}\left(x, \frac{\delta - t}{2}\right) .$$

$$. \lim_{n\to\infty} d(a_n, b_n) = t \,\&\, \lim_{n\to\infty} d(a_n, x) = \frac{\delta - t}{2} \,\&\, \lim_{n\to\infty} d(b_n, y) = \frac{\delta - t}{2},$$

$[3] := \texttt{HopfRinowTHM1}(X)\texttt{CompactIsSequanceCompact} : \texttt{PartiallyConverging}(X, (a,b)),$

$\alpha_t := \texttt{partial} \lim_{n\to\infty} a_n : \mathbb{D}\left(x, \frac{\delta - t}{2}\right),$

$\beta_t := \texttt{partial} \lim_{n\to\infty} b_n : \mathbb{D}\left(y, \frac{\delta - t}{2}\right),$

$[t.*] := [2]\jmath(\alpha, \beta) : d(x, \alpha_t) = \frac{\delta - t}{2} \,\&\, d(\alpha_t, \beta_t) = t \,\&\, d(y, \beta_t) = \frac{\delta - t}{2};$

$\rightsquigarrow (\alpha, \beta, [2]) := I\left(\prod\right) I\left(\sum\right) : \prod t \in (0, \delta) . \sum \alpha_t \in \mathbb{D}(x, \delta) . \sum \beta_t \in \mathbb{D}(y, \delta) . d(x, \alpha_t) = \frac{\delta - t}{2} \,\&\, d(\alpha_t, \beta$

$a := \Lambda n \in \mathbb{N} . \alpha_{\frac{1}{n}} : \mathbb{N} \to \mathbb{D}(x, \delta),$

$b := \Lambda n \in \mathbb{N} . \beta_{\frac{1}{n}} : \mathbb{N} \to \mathbb{D}(y, \delta),$

$[3] := \texttt{HopfRinowTHM1}(X)\texttt{CompactIsSequanceCompact} : \texttt{PartialyConverging}(X, (a,b)),$

$\alpha_t := \texttt{partial} \lim_{n\to\infty} a_n : X,$

$\beta_t := \texttt{partial} \lim_{n\to\infty} b_n : X,$

$[4] := [2]\jmath(\alpha, \beta) : d(x, \alpha) = \frac{\delta}{2} \,\&\, d(\alpha, \beta) = 0 \,\&\, d(y, \beta) = \frac{\delta}{2},$

$[*] := \eth^{-1}\mathsf{MS}[4] : \alpha = \beta;$

$\square$

$\texttt{GeneralizedMidpointTHM} :: \forall X : \texttt{Complete} \,\&\, \texttt{LocallyCompat} \,\&\, \texttt{HopfRinow} \,.$

$\quad . \, \forall x, y \in X \,.\, \exists \xi : \mathbb{Q}_2 \cap [0,1] \to X :$

$\quad : \forall a, b \in \mathbb{Q}_2 \cap [0,1] \,.\, d\Big(\xi(a), \xi(b)\Big) = |a-b| d(x,y) \,\&\, \xi(0) = x \,\&\, \xi(1) = y$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{HopfRinowTHM2} :: \forall X : \texttt{Complete} \,\&\, \texttt{LocallyCompat} \,\&\, \texttt{HopfRinow} \,.$

$\quad . \, \forall x, y \in X \,.\, \exists \gamma : \Omega(x,y\Big) :$

$\quad : \forall a, b \in [0,1] \,.\, d\Big(\gamma(a), \gamma(b)\Big) = |a-b| d(x,y)$

$\texttt{Proof} \; =$

$\delta := d(x,y) : \mathbb{R}_{++},$

$\Big(\xi, [1]\Big) := \texttt{GeneralizedMidpointTHM}(X, x, y) : \sum \xi : \mathbb{Q}_2 \cap [0,1] \to X \,.\, \forall a, b \in \mathbb{Q}_2 \cap [0,1] \,.$

$\quad . \, d\Big(\xi(a), \xi(b)\Big) = |a-b| d(x,y) \,\&\, \xi(0) = x \,\&\, \xi(1) = y,$

$[2] := \texttt{LipschitzIsUC} : \texttt{UniformltContinuous}(\mathbb{Q}_2, X, \xi),$

$\Big(\gamma, [3]\Big) := \texttt{UniformlyContinuousExtension}[1,2] : \sum \gamma : \Omega(x,y\Big) :$

$\quad : \forall a, b \in [0,1] \,.\, d\Big(\gamma(a), \gamma(b)\Big) = |a-b| d(x,y);$

$\square$

$\texttt{HopfRinowGeodesic} :: \forall X : \texttt{Complete} \,\&\, \texttt{LocallyCompat} \,\&\, \texttt{HopfRinow} \,.\, X : \texttt{GeodesicSpace}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.9 Lipschitz Connected Spaces

$\texttt{LipschitzConnected} :: \mathbb{R}_{++} \to ?\mathsf{MS}$

$X : \texttt{LipschitzConnected} \iff \Lambda C \in \mathbb{R}_{++} \,.\, 1\text{-Lip}(X) \iff$

$\iff \forall x, y \in X \,.\, \forall \varepsilon \in \mathbb{R}_{++} \,.\, \exists n \in \mathbb{N} : \exists q : n \to X \,.$

$$. \, q_1 = x \,\&\, q_n = y \,\&\, \,\&\, \forall i \in (n-1) \,.\, d(q_i, q_j) < \varepsilon \,\&\, \sum_{i=1}^{n} \leq C d(x, y)$$

$\texttt{OneLipschitzConnectedIsGeodesic} :: \forall X : 1\text{-Lip}(X) \,.\, \texttt{GeodesicSpace}(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{GeodesicIsOneLipschitz} :: \forall X : \texttt{GeodesicSpace}(X) \,.\, 1\text{-Lip}(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

2 Metrization

$\texttt{Metrizable} :: ?\mathsf{TOP}$

$X : \texttt{Metrizable} \iff \exists d : \texttt{Metric}(X) \,.\, (X,d) \cong_{\mathsf{TOP}} X$

$\texttt{UrysohnMetrization} :: \forall X : \texttt{SecondCountabe} \,\&\, \texttt{T3} \,.\, \texttt{Metrizable}(X)$

$\texttt{Proof} \;=$

$\mathcal{B} := \eth\texttt{SecondCountable}(X) : \texttt{Countable} \,\&\, \texttt{Base}(X),$

$\texttt{Assume}\ V, W : \mathcal{B},$

$\texttt{Assume}\ [1] : \overline{V} \subset W,$

$\Big(f_{V,W}, [2]\Big) := \texttt{UrysohnLemma}(\overline{V}, W^{\complement}, [1]) : \sum f_{V,W} : C\Big(X, [0,1]\Big) \,.\, W^{\complement} = f^{-1}\{0\} \,\&\, \overline{V} = f^{-1}\{1\};$

$$\rightsquigarrow \Big(f, [1]\Big) := I\left(\prod\right) : \prod_{V,W \in \mathcal{B} : \overline{V} \subset W} \;\sum_{f : X \xrightarrow{\mathsf{TOP}} [0,1]} W^{\complement} = f^{-1}\{0\} \,\&\, \overline{V} = f^{-1}\{1\},$$

$\Phi := \operatorname{Im} f :?C\Big(X, [0,1]\Big),$

$[2] := \jmath\Phi\eth\mathcal{B} : |\Phi| \le \aleph_0,$

$\phi := \texttt{enumerate}(\Phi) : \mathbb{N} \leftrightarrow \Phi,$

$$d := \Lambda x, y \in X \,.\, \sum_{n=1}^{\infty} \frac{|\phi_n(x) - \phi_n(y)|}{n!} : \texttt{Semimetric}(X),$$

$[3] := \jmath d[1]\eth\texttt{T1}(X) : \texttt{Metric}(X, d),$

$[4] := \texttt{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$[*] := \eth\texttt{T3} : \mathcal{T}(X) = \mathcal{T}(X, d);$

$\square$

$\texttt{SeparableMetrization} :: \forall X : \texttt{Separable} \,.\, \texttt{Metrizable}(X) \iff \texttt{SecondCountable} \,\&\, \texttt{T3}(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ContinuousByLocallyFinite} :: \forall X, Y \in \mathsf{TOP} . \forall f : X \to Y . \forall \mathcal{A} : \texttt{LocallyFinite} \ \& \ \texttt{Cover}(X) .$

$. \left(\forall A \in \mathcal{A} . f_{|\overline{A}} \in C(\overline{A}, X)\right) \Rightarrow f \in C(X, Y)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ x : X,$

$\mathcal{A}' := \left\{A \in \mathcal{A} | x \in A\right\} :?\mathcal{A},$

$[2] := \jmath\mathcal{A}'\eth\texttt{LocallyFinite}(X, \mathcal{A}) : |\mathcal{A}'| < \infty,$

$\left(U, [3]\right) := \eth\texttt{LocallyFinite}(X) : \sum U \in \mathcal{U}(x) . \forall A \in \mathcal{A} . A \cap U \neq \emptyset \Rightarrow A \in \mathcal{A}',$

$[4] := \eth^{-1}\texttt{Cover}(X)(\mathcal{A}') : \mathcal{A}' \neq \emptyset,$

$\texttt{Assume} \ V : \mathcal{U}\left(f(x)\right),$

$\texttt{Assume} \ A : \mathcal{A}',$

$W := f^{-1}_{|A}(V) :?A,$

$[5] := \eth\mathsf{TOP} : W_A \in \mathcal{T}(A),$

$\left(O_A, [A.*]\right) := \eth\texttt{subsetTopology}[5] : \sum O_A \in \mathcal{U}_X(x) . W = O_A \cap A;$

$\leadsto \left(O, [5]\right) := I\left(\sum\right) I(\to) : \sum O : \mathcal{A} \to \mathcal{U}_X(x) . \forall A \in \mathcal{A}' . O_A \cap A = f^{-1}_{|A}(V),$

$\theta := \bigcap_{A \in \mathcal{A}'} O_A \cap U : \mathcal{U}(x),$

$[*.1] := \jmath\theta[3][5] : f(\theta) \subset U;$

$\leadsto [*] := \texttt{ContinuityLocalProof}(X, Y, f) : f \in C(X, Y);$

$\square$

$\texttt{CountableSupport} :: \prod X \in \mathsf{SET} . ?(X \to \mathbb{R})$

$f : \texttt{CountableSupport} \iff \left|\{x \in X : f(x) \neq 0\}\right| \leq \aleph_0$

$\texttt{CountableSupport} :: \prod X \in \mathsf{SET} . ?\texttt{CountableSupport}(X)$

$f : \texttt{CountableSupport} \iff f \in L_1(X, \#) \iff \sum_{x \in x} |f(x)| < \infty$

$\texttt{ContinuousIndicators} :: \prod X \in \mathsf{TOP} . \prod S :??X . ?\left(S \to C(X)\right)$

$f : \texttt{ContinuousIndicators} \iff \forall A \in S . f_A(A^\complement) = \{0\}$

$\texttt{NagataSmirnovFunc} :: \prod X \in \mathsf{TOP} . \prod S :??X . \texttt{ContinuousIndicators}(X) \to S \to X \to \mathbb{R}$

$\texttt{NagataSmirnovFunc}\,(f, A) = \mathrm{NS}_{f,x}(A) := f_A(x)$

$\texttt{SemimetricBaseTheorem} :: \forall X \in \mathsf{SMS} \;.\; \exists \sigma\text{-}\texttt{LocallyFinite} \,\&\, \texttt{Base}(X)$

$\texttt{Proof} \;=$

$(\leq) := \texttt{WellOrderingTHM} : \texttt{WellOrder}(X),$

$$U := \Lambda x \in X \;.\; \Lambda k \in \mathbb{N} \;.\; \Lambda n \in \texttt{after}(k) \;.\; \mathbb{B}\left(x, \frac{1}{k} - \frac{1}{n}\right) \setminus \overline{\bigcap_{y<x} \mathbb{D}\left(y, \frac{1}{k} - \frac{1}{n+1}\right)} :$$

$$: X \to \prod_{k=1}^{\infty} \Big(\texttt{after}(k) \to \mathcal{T}(X)\Big),$$

$$\mathcal{A} := \Lambda k \in \mathbb{N} \;.\; \Big\{U(x,k,n) | x \in X, n \in \texttt{after}(k)\Big\} : \mathbb{N} \to ?\mathcal{T}(X),$$

$\texttt{Assume } k : \mathbb{N},$

$\texttt{Assume } z : X,$

$$R := \left\{x \in X : z \in \mathbb{B}\left(x, \frac{1}{k}\right)\right\} : ?X,$$

$[1] := \eth\texttt{cell}\jmath R : z \in R,$

$x := \min R : R,$

$$\Big(n, [2]\Big) := \jmath R \jmath x : \sum_{n=k+1}^{\infty} \frac{1}{n} \leq \frac{1}{k} - d(x,z),$$

$[k.*] := \jmath x \jmath R \jmath U[2] : z \in U(z,k,n);$

$\rightsquigarrow [1] := I(1) : \forall k \in \mathbb{N} \;.\; \texttt{Cover}\Big(X, \mathcal{A}_k\Big),$

$$[2] := \eth^{-1}\texttt{Base}\jmath\mathcal{A} : \texttt{Base}\left(X, \bigcup_{1=k}^{\infty} \mathcal{A}_k\right),$$

$\texttt{Assume } k : \mathbb{N},$

$\texttt{Assume } n : \texttt{after}(k),$

$\texttt{Assume } x, y : X,$

$\texttt{Assume } [3] : x \neq y,$

$$[*] := \jmath U : d\Big(U(x,k,n), U(z,k,n)\Big) \geq \frac{1}{n(n+1)};$$

$$\rightsquigarrow [3] := I(\forall) : \forall x \in X \;.\; \forall k \in \mathbb{N} \;.\; \forall n : \texttt{after}(k) \;.\; d\Big(U(x,k,n), U(z,k,n)\Big) \geq \frac{1}{n(n+1)},$$

$[4] := \eth^{-1}\texttt{LocallyFinite}[3] : \forall k \in \mathbb{N} \;.\; \texttt{LocallyFinite}(X, \|),$

$$[*] := \eth^{-1}\sigma\text{-}\texttt{LocallyFinite}(X)[4] : \sigma\text{-}\texttt{LocallyFinite}(X)\left(X, \bigcap_{k=1}^{\infty} \mathcal{A}_k\right);$$

$\square$

$\texttt{NagataSmirnovLemma} :: \forall X \in \mathsf{TOP} . \forall S : \texttt{LocallyFinite}(X) . \forall f : \texttt{ContinuousIndicators}(X, S) .$

$. \mathrm{NS}_f \in C\Big(X, L_1(X, \#)\Big)$

$\texttt{Proof} =$

$\texttt{Assume } (D, x) : \texttt{ConvergingNet}(X),$

$L := \lim_{n \to \infty} x_n : X,$

$\Big(U, [1]\Big) := \eth\texttt{LocallyFinite}(X, S, L) : \sum U \in \mathcal{U}(L) . \Big|\{s \in S : s \cap U \neq \emptyset\}\Big| < \infty,$

$\Big(\delta, [2]\Big) := \eth\texttt{ConvergingNet}(X, D, x, U) : \sum \delta' \in D . \forall \delta' \in D . \delta'' \geq \delta \Rightarrow x_{\delta'} \in U,$

$S' := \big\{s \in S : s \cap U \neq \emptyset\big\} : \texttt{Finite}(S),$

$[3] := \eth L_1[1][2]\eth\texttt{ContinuousIndicators}(X, S, f)\texttt{ContinuousSum}(X, S', \ldots)$

$\texttt{ContinuousConvergence}(X, f_s, D, x) :$

$$: \lim_{n \in D} \Big\|\mathrm{NS}_{f, x_n} - \mathrm{NS}_{f,L}\Big\|_1 = \lim_{n \in D} \sum_{s \in S} \Big|f_s(x_n) - f_s(L)\Big| = \lim_{n \in D} \sum_{s \in S'} \Big|f_s(x_n) - f_s(L)\Big| =$$

$$= \sum_{s \in S'} \lim_{n \in D} \Big|f_s(x_n) - f_s(L)\Big| = 0,$$

$* \ldots] := \texttt{NormConvergence} : \lim_{n \in D} \mathrm{NS}_{f, x_n} = \mathrm{NS}_{f,L};$

$\rightsquigarrow [*] := \texttt{ContinuousConvergence} : \mathrm{NS}_f \in C\Big(X, L_1(X, \#)\Big);$

$\square$

$\texttt{SmirnoffNagataMetrizationTHM} :: \forall X : \texttt{T4} . \forall \mathcal{B} : \texttt{Base} \,\&\, \sigma\text{-}\texttt{LocallyFinite}(X) . X : \texttt{Metrizable}(X)$

$\texttt{Proof} =$

$$\Big(\mathcal{B}', [1]\Big) := \eth\sigma\text{-}\texttt{LocallyCompact}(X) : \sum \mathcal{B}' : \mathbb{N} \to \texttt{LocallyFinite}(X) . \mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}',$$

$\texttt{Assume } m, n : \mathbb{N},$

$\texttt{Assume } V : \mathcal{B}_n,$

$A := \bigcup\{W \in \mathcal{B}_m : \overline{W} \subset V\} : \mathcal{T}(X),$

$[2] := \texttt{LocallyFiniteClosure} : \overline{A} \subset V,$

$\Big(f_V, [3]\Big) := \texttt{UrysohnLemma}(X, \overline{A}, V^{\complement}) : \sum f \in C(X) . f^{-1}\{1\} = \overline{A} \,\&\, f^{-1}\{0\} = V^{\complement};$

$\rightsquigarrow \Big(f_{n,m}, [4]\Big) := I\Big(\sum\Big) : \sum f_{n,m} : \mathcal{B}_n \to C(X) . \forall V \in \mathcal{B}_n . f^{-1}_{n,m,V}\{1\} = \overline{A} \,\&\, f^{-1}_{n,m,V}\{0\} = V^{\complement},$

$[2] := \eth^{-1}\texttt{ContinuousIndicators} : \texttt{ContinuousIndicators}(X, \mathcal{B}_n, f_{n,m}),$

$h_{n,m} := \mathrm{NS}_{f_{n,m}} : X \xrightarrow{\mathsf{TOP}} L_1(\mathcal{B}_n, \#);$

$\rightsquigarrow h := I(\to) : \mathbb{N} \times \mathbb{N} \to X \xrightarrow{\mathsf{TOP}} L_1(\mathcal{B}_n, \#),$

$H := \operatorname{Im} h :?C\Big(X, L_1(\mathcal{B}_n, \#)\Big),$

$[2] := \texttt{ImageCardinality} : |H| \leq \aleph_0,$

$f := \texttt{enumerate}(H) : \mathbb{N} \to C\Big(X, L_1(\mathcal{B}_n, \#)\Big);$

$[4] := \texttt{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$$d := \Lambda x, y \in X . \sum_{n=1}^{\infty} \frac{\|f_n(x) - f_n(y)\|_1}{n!} : \texttt{Metric}(X),$$

$\ldots$

$\square$

3 Uniform Spaces

3.1 Uniform Topology

$$\texttt{Connector} :: \prod_{X \in \mathsf{SET}} ?(X \to 2^X)$$
$$U : \texttt{Connector} \iff \forall x \in X \,.\, x \in U_x$$

$$\texttt{connectorSet} :: \prod_{X \in \mathsf{SET}} \texttt{Connector}(X) \to ?(X \times X)$$
$$\texttt{connectorSet}\,(U) = U := \{(x,y) | x \in X, y \in U(x)\}$$

$$\texttt{swap} :: \prod_{X \in \mathsf{SET}} ?(X \times X) \to ?(X \times X)$$
$$\texttt{swap}\,(U) = U^{-1} := \{(y,x) | (x,y) \in U\}$$

$$\texttt{connectorComposition} :: \prod X \in \mathsf{SET} \,.\, \texttt{Connector}(X) \times \texttt{Connector}(X)$$
$$\texttt{connectorComposition}\,(A,B) = AB = B \circ A := \Lambda x \in X \,.\, \bigcup_{a \in A(x)} B(a)$$

$$\texttt{Uniformity} :: \prod X \in \mathsf{SET} \,.\, \texttt{Filter}(X \times X)$$
$$\mathcal{U} : \texttt{Uniformity} \iff \forall U \in \mathcal{U} \,.\, \Delta\, X \subset U \;\&\; U^{-1} \in \mathcal{U} \;\&\; \exists V \in \mathcal{U} : VV \subset U$$

$$\texttt{UniformSpace} := \sum_{X \in \mathsf{SET}} \texttt{Unifomity}(X) : \texttt{Type};$$

$$\texttt{BaseOfUniformity} :: \prod X \in \mathsf{SET} \,.\, \texttt{FilterBase}(X \times X)$$
$$\mathcal{B} : \texttt{BaseOfUniformity} \iff \forall U \in \mathcal{B} \,.\, \Delta\, X \subset U \;\&\; \exists V \in \mathcal{U} : U^{-1} \subset V \;\&\; \exists V \in \mathcal{U} : VV \subset U$$

$$\texttt{UniformityGeneration} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{B} : \texttt{BaseOfUniformity}(X) \,.\, \texttt{Uniformity}(X, \mathcal{F}\,\mathcal{B})$$

Proof $=$

...

$\square$

$$\texttt{ConnectorOfSpace} :: \prod (X,\mathcal{U}) : \texttt{UniformSpace} \,.\, ?(X \to 2^X)$$
$$U : \texttt{ConnectorOfSpace} \iff \texttt{Connector}\Big((X,\mathcal{U}), U\Big) \iff U \in \mathcal{U}$$

$$\texttt{metricUniformity} :: \prod_{X \in \mathsf{SET}} (\texttt{Metric} \to \texttt{Uniformity})(X)$$
$$\texttt{metricUniformity}\,(d) = \mathcal{U}_d := \Big\{\{(x,y) \in X \times X : d(x,y) < r\}\Big| r \in \mathbb{R}_{++}\Big\}$$

$$\texttt{metricAsUniform} :: \mathsf{SMS} \to \texttt{UniformSpace}$$
$$\texttt{metricAsUniform}\,(X,d) = (X,d) := (X,\mathcal{U}_d)$$

$\texttt{uniformTopology} :: \prod_{X \in \mathsf{SET}} (\texttt{Uniformity} \to \texttt{Topology})(X)$

$\texttt{uniformTopology}\,(\mathcal{U}) = \mathcal{T}_{\mathcal{U}} := \Big\langle \{U(x) | U : \texttt{SpaceConnector}(X,\mathcal{U})\} \Big\rangle_{\mathsf{TOP}}$

$\texttt{uniformAsTopology} :: \mathsf{SMS} \to \texttt{UniformSpace}$

$\texttt{uniformAsTopology}\,(X,\mathcal{U}) = (X,\mathcal{U}) := (X, \mathcal{T}_{\mathcal{U}})$

$\texttt{uniformity} :: \prod (X,\mathcal{U}) : \texttt{UniformSpace} \,.\, \texttt{Uniformity}(X)$

$\texttt{uniformity}\,() = \mathcal{U}_{(X,\mathcal{U})} := \mathcal{U}$

$\texttt{Symmetric} :: \prod_{X \in \mathsf{SET}} ??(X \times X)$

$A : \texttt{Symmetric} \iff A = A^{-1} \iff$

$\texttt{UniformityGeneratingBase} :: \prod_{X \in \mathsf{SET}} \texttt{Uniformity}(X) \to ?\texttt{BaseOfUniformity}(X)$

$\mathcal{B} : \texttt{UniformityGeneratingBase} \iff \Lambda\mathcal{U} : \texttt{Uniformity}(X) \,.\, \mathcal{F}\mathcal{B} = \mathcal{U} \iff$

$\texttt{SymmetricConnectorsBase} :: \forall X : \texttt{UniformSpace} \,.\, \exists \mathcal{B} : \texttt{UniformityGeneratingBase}(X) :$

$\quad : \forall U \in \mathcal{B} \,.\, \texttt{Symmetric}(X, U)$

Proof $=$

...

□

$\texttt{uniformityElementAsConnector} :: \prod X : \texttt{UniformSpace} \,.\, \mathcal{U}_X \to \texttt{Connector}(X)$

$\texttt{uniformityElementAsConnector}\,(U) = U := \Lambda x \in X \,.\, \{y \in X : (x,y) \in U\}$

$\texttt{UniformSpaceClosure} :: \forall X : \texttt{UniformSpace} \,.\, \forall A \subset X \,.\, \overline{A} = \bigcap_{U \in \mathcal{U}_X} \bigcup_{a \in A} U(a)$

Proof $=$

...

□

$\texttt{UniformSpaceIsRegular} :: \forall X : \texttt{UniformSpace} \,.\, \texttt{Regular}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ [1] : x \notin X,$

$$\Big(U, [2]\Big) := \eth\texttt{closure}(X)\texttt{UniformSpaceClosure}(X, A)\eth\texttt{intersection}(X) : \sum U \in \mathcal{U} \,.\, x \notin \bigcup_{a\in A} U(a),$$

$$\Big(V, [3]\Big) := \eth\texttt{Uniformity}(\mathcal{U})(U) : \sum V \in \mathcal{U} \,.\, V \circ V \subset U \;\&\; ,$$

$$[5] := \texttt{UniformSpaceClosure}\left(X, \bigcup_{a\in A} V(a)\right) : \overline{\bigcup_{a\in A} V(a)} = \bigcap_{U\in\mathcal{U}} \bigcup_{v\in\bigcup_{a\in A} V(a)} U(v),$$

$$\texttt{Assume}\ [6] : x \in \overline{\bigcup_{a\in A} V(a)},$$

$$[7] := \eth\texttt{Intersection}[6][5] : x \in \bigcup_{v\in\bigcup_{a\in A} V(a)} V(v),$$

$$\Big(v, [8]\Big) := \eth\texttt{Connector}[7] : \sum v \in \bigcup_{a\in A} V(a) \,.\, (v, x) \in V,$$

$$\Big(a, [9]) := \eth\texttt{Connector}[8] : \sum a \in A \,.\, (a, v) \in V,$$

$[10] := [3, 8, 9] : (a, x) \in U,$

$$[11] := \eth\texttt{Union}(U)\eth\texttt{Connector}[10] : x \in \bigcup_{a\in A} U(a),$$

$[6.*] := [11][2] : \bot;$

$$\rightsquigarrow [6] := \eth\mathsf{TOP}(X)\eth\texttt{uniformAsTopological}[2] : x \notin \overline{\bigcup_{a\in A} V(a)},$$

$$[A.*] := \eth\texttt{uniformAsTopological}(X) : \bigcup_{a\in A} V(a) \in \mathcal{T}(X);$$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Regular} :;$

$\square$

$\texttt{UniformSpaceT3Criterion} :: \forall X : \texttt{UniformSpace} \,.\, \bigcap \mathcal{U}_X = \Delta\ X \iff \texttt{T3}(X)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ClosedConnectorTHM} :: \forall X : \texttt{UniformSpace} \,.\, \forall U \in \mathcal{U}_X \,.\, VUV = \bigcap \left\{ VUV \middle| V : \texttt{Symmetric}(\mathcal{U}) \right\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (x,y) : \texttt{In}\ \overline{U}^{\complement},$

$\Big(O,[1]\Big) := \texttt{ClosureAltDef}(U)(x,y) : O \in \mathcal{U}(x,y) \,.\, O \cap U = \emptyset,$

$\Big(V,[2]\Big) := \texttt{UniformSymmetricBase}(X)\texttt{ProductTopologyByBase} : \sum V : \texttt{Symmetric}(\mathcal{U}_X) \,.\, V(x) \times V(y) \subset O,$

$\texttt{Assume}\ [3] : (x,y) \in VUV,$

$\Big(a,b,[4]\Big) := \eth\texttt{connectorComposition}(V,U,V)[3] : \sum (a,b) \in U \,.\, (x,a),(b,y) \in V,$

$[5] := \eth\texttt{Symmetric}(V)[4][2] : (a,b) \in U \cap O,$

$[3.*] := [6][1] : \bot;$

$\leadsto [1.*] := E(\bot) : (x,y) \in (VUV)^{\complement};$

$\leadsto [1] := \eth^{-1}\texttt{Subset} : \overline{U}^{\complement} \subset \bigcap_V (VUV)^{\complement},$

$[2] := \texttt{ComplementSubset} : \bigcap_V VUV \subset \overline{U},$

$\texttt{Assume}\ (x,y) : \texttt{In}(\overline{U}),$

$[3] := \texttt{ClosureAltDef}\Big(U,(x,y)\Big) : \forall V \in \mathcal{U}(x,y) \,.\, V \cap U \neq \emptyset,$

$\texttt{Assume}\ V : \texttt{Symmetric}(\mathcal{U}_X),$

$[4] := \eth\texttt{productTopology}(X,X)[1](V(x) \times V(y)) : V(x) \times V(y) \cap U \neq \emptyset,$

$\Big((a,b),[5]\Big) := \eth\texttt{Connector}[4] : \sum (a,b) \in U \,.\, (x,a) \in V(x) \,\&\, (y,b) \in V(y),$

$[V.*] := [1]\eth\texttt{connectorComposition}\eth\texttt{Symmetric}(\mathcal{U}_X) : (x,y) \in VUV;$

$\leadsto [(x,y).*] := \eth^{-1}\texttt{Intersection} : (x,y) \in \bigcap_V VUV;$

$\leadsto [*] := \eth^{-1}\texttt{SetEq} : \overline{U} = \bigcap_V VUV;$

□

$\texttt{ClosedConnectorBase} :: \forall X : \texttt{UniformSpace} \,.\, \exists \mathcal{B} : \texttt{BaseOfUniformity}(X) : \forall V \in \mathcal{B} \,.$

$\quad . \Big(\texttt{Symmetric}(X) \,\&\, \texttt{Closed}(X \times X)\Big)(V)$

$\texttt{Proof} \ =$

$\dots$

□

3.2 Uniform Category

$\texttt{UniformContinuity} :: \prod X, Y \in \mathsf{TOP} \ . \ ?(X \xrightarrow{\mathsf{TOP}} Y)$

$f : \texttt{UniformContinuity} \iff f \in \mathrm{UC}(X) \iff \forall V \in \mathcal{U}_Y \ . \ \exists U \in \mathcal{U}_X : (f \times f)(U) \subset V$

$\texttt{uniformCategory} :: \mathsf{CAT}$

$\texttt{uniformCategory}\,() = \mathsf{UNI} := \Big(\texttt{UniformSpace}, \mathrm{UC}, \circ, \mathrm{id}\Big)$

$\texttt{uniformSeparatedCategory} :: \mathsf{CAT}$

$\texttt{uniformCategory}\,() = \mathsf{UNIS} := \Big(\texttt{UniformSpace} \ \& \ \texttt{T3}, \mathrm{UC}, \circ, \mathrm{id}\Big)$

$\texttt{UniformCover} :: \prod_{X \in \mathsf{UNI}} ?\texttt{Cover}(X)$

$\mathcal{A} : \texttt{UniformCover} \iff \exists U \in \mathcal{U}_X : \forall x \in X \ . \ \exists A \in \mathcal{A} : U(x) \subset A$

$\texttt{UniformContinuityByUniformCover} :: \forall X, Y \in \mathsf{UNI} \ . \ \forall f : X \xrightarrow{\mathsf{TOP}} Y \ . \ f : X \xrightarrow{\mathsf{UNI}} Y \iff$

$\iff \forall V \in \mathcal{U}_Y \ . \ \texttt{UniformCover}\Big(X, \big\{f^{-1}\, V(x) \big| x \in X\big\}\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{EveryOpenCoverIsUniformForACompactSpace} :: \forall X \in \mathsf{UNI} \ . \ \texttt{Compact}(X) \Rightarrow$

$\Rightarrow \forall \mathcal{O} : \texttt{OpenCover}(X) \ . \ \texttt{UniformCover}(X, \mathcal{O})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \texttt{In}\ X,$

$\Big(O_x, [1]\Big) := \eth\texttt{Cover}(O, X)(x) : \sum O_x \in \mathcal{O} \ . \ x \in O_x,$

$\Big(U_x, [2]\Big) := \eth\texttt{connectorTopology}(X)(O_x) : U_x \in \mathcal{U}_X \ . \ U_x(x) \subset O_x,$

$\Big(V_x, [x.*]\Big) := \texttt{UniformSymmetricBase} : \sum V : \texttt{Symmetric}(\mathcal{U}_X) \ . \ V_x \circ V_x \subset U_x;$

$\leadsto \Big(O, U, V, [1]\Big) := I\left(\prod\right) : \sum (O, U, V) : X \to \mathcal{O} \times \mathcal{U}_X \times \texttt{Symmetric}(\mathcal{U}_X) \ . \ \forall x \in X \ . \ U_x(x) \subset O_x \ \& \ VV \subset U$

$\Big(n, x, [2]\Big) := \eth\texttt{Compact}(X)\big(V_\bullet(\bullet)\big) : \sum_{n=1}^{\infty} \ . \ \sum_{x : n \to X} \texttt{OpenCover}(X, V_x(x)),$

$V' := \bigcup_{i=1}^{n} V_{x_i} : \mathcal{U}_X,$

$\texttt{Assume}\ y : X,$

$\Big(i, [3]\Big) := \eth\texttt{OpenCover}(X, U_x(x)) : \sum_{i=1}^{n} y \in V_{x_i}(x_i),$

$[y.*] := [3]\eth^{-1}\texttt{connectorComposition}(V, V')\texttt{IntersectionSubset}(V) : V'(y) \subset V_{x_i}V'(x_i) \subset V_{x_i}V_{x_i}(x_i) \subset U_{x_i}(x$

$\leadsto [*] := \eth^{-1}\texttt{UniformCover} : \texttt{UniformCover}(X, O);$

$\square$

$\texttt{UniforlmyContinuousByCompact} :: \forall X, Y \in \mathsf{UNI} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} Y \;.\; \texttt{Comcpact}(X) \Rightarrow X \xrightarrow{\mathsf{UNI}} Y$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CompactsUniformlyIsomorphic} :: \forall X \in \mathsf{SET} \;.\; \forall \mathcal{U}, \mathcal{V} : \texttt{Uniformity}(X) \;.\; \texttt{Compact}(X,\mathcal{U}) \;\&\; \texttt{Compact}(X,\mathcal{V}) \Rightarrow$
$\Rightarrow (X,\mathcal{U}) \cong_{\mathsf{UNI}} (X,\mathcal{V})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.3 Complete Uniform Spaces

$$\texttt{CauchyFilterbase} :: \prod_{X \in \mathsf{UNI}} . ?\texttt{Filterbase}$$

$$\mathcal{F} : \texttt{CauchyFilterbase} \iff \forall U \in \mathcal{U}_X . \exists A \in \mathcal{F} : \exists x \in X : A \times A \subset U$$

$$\texttt{ConvergingFilterbaseIsCaouchy} :: \forall X \in \mathsf{UNI} . \forall \mathcal{F} : \texttt{ConvergingFilterbase}(X) .$$
$$. \texttt{CauchyFilterbase}(X, \mathcal{F})$$

Proof $=$

...

□

$$\texttt{UCPreservesCauchyFilterbase} :: \forall X, Y \in \mathsf{UNI} . \forall \mathcal{F} : \texttt{CauchyFilterbase}(X) . \forall f : X \xrightarrow{\mathsf{UNI}} Y .$$
$$f\ \mathcal{F} : \texttt{CauchyFilterbas}(Y)$$

Proof $=$

...

□

$$\texttt{Complete} :: ?\mathsf{UNI}$$

$$X : \texttt{Complete} \iff \forall \mathcal{F} : \texttt{CauchyFilterbase}(X) . \texttt{ConvergingFilterbase}(X, \mathcal{F})$$

$$\texttt{CauchySequance} :: \prod_{X \in \mathsf{UNI}} ?(\mathbb{N} \to X)$$

$$x : \texttt{CauchySequance} \iff \texttt{CauchyFilterbase}\left(X, \left\{\{x_{n+m} | n \in \mathbb{N}\} \middle| m \in \mathbb{N}\right\}\right)$$

$$\texttt{SequentiallyComplete} :: ?\mathsf{UNI}$$

$$X : \texttt{SequentiallyComplete} \iff \forall x : \texttt{CauchySequence}(X) . \texttt{Convergent}(X, x)$$

$$\texttt{UniformEmbedding} :: \prod_{X, Y \in \mathsf{UNI}} ?(X \xrightarrow{\mathsf{UNI}} Y)$$

$$f : \texttt{UnifrormEmbedding} \iff f_{|f(X):}X \xleftrightarrow{\mathsf{UNI}} Y$$

$$\texttt{UniformImagePreservesCompletenes} :: \forall X, Y \in \mathsf{UNI} . \forall f : X \xrightarrow{\mathsf{UNI}} Y .$$
$$. \texttt{Complete}(X)\ \&\ \texttt{UniformEmbedding}(X, Y, f)$$

Proof $=$

...

□

$\texttt{WeakerIsComplete} :: \forall X \in \mathsf{SET} . \forall \mathcal{U}, \mathcal{V} : \texttt{Uniformity}(X) . \mathcal{U} \subset \mathcal{V} \ \& \ (X,\mathcal{U}) \cong_{\mathsf{TOP}} (X,\mathcal{V}) \Rightarrow$

$\Rightarrow \Big(\texttt{Complete}(X,\mathcal{U}) \Rightarrow \texttt{Complete}(X,\mathcal{V})\Big)$

Proof $=$

$\dots$

$\square$

$\texttt{WeakerIsSequentiallyComplete} :: \forall X \in \mathsf{SET} . \forall \mathcal{U}, \mathcal{V} : \texttt{Uniformity}(X) . \mathcal{U} \subset \mathcal{V} \ \& \ (X,\mathcal{U}) \cong_{\mathsf{TOP}} (X,\mathcal{V}) \Rightarrow$

$\Rightarrow \Big(\texttt{SequantiallyComplete}(X,\mathcal{U}) \Rightarrow \texttt{SequentiallyComplete}(X,\mathcal{V})\Big)$

Proof $=$

$\dots$

$\square$

$\texttt{CauchyCluster} :: \forall X \in \mathsf{UNI} . \forall \mathcal{F} : \texttt{CauchyClasterbase} . \forall x : \texttt{Cluster}(X,\mathcal{F}) . x = \lim \mathcal{F}$

Proof $=$

$\dots$

$\square$

$\texttt{UniformExtension} :: \forall X, Y \in \mathsf{UNIS} . \forall A : \texttt{Dense}(X) . \forall f : A \xrightarrow{\mathsf{UNI}} Y . \texttt{Complete}(Y) \Rightarrow$

$\Rightarrow \exists F : X \xrightarrow{\mathsf{UNI}} Y . F_{|A} = f$

Proof $=$

Assume $x : X$,

$\Big((D,u),[1]\Big) := \texttt{DenseLimit}(X,A,x) : \sum (D,u) : \texttt{Net}(A) . \lim_{n \in D} u_n = X,$

$[2] := \texttt{ConvengNetIsConvergingFilterbase} : \texttt{ConvergingFilterBase}\Big(A, \mathcal{F}(D,u)\Big),$

$[3] := \texttt{ConvergingIsCauchy}[2] : \texttt{CauchyFilterBase}\Big(A, \mathcal{F}(D,u)\Big),$

$[4] := \texttt{UCPreservesCauchy}(A,Y,f)[3] : \texttt{CauchyFilterBase}\Big(Y, f\ \mathcal{F}(D,U)\Big),$

$[5] := \eth^{-1}\texttt{Complete}(Y)[4] : \texttt{ConvergingFilterBase}\Big(Y, f\ \mathcal{F}(D,U)\Big),$

$F(x) := \lim f\ \mathcal{F}(D,U) : Y;$

$\rightsquigarrow F := I(\to) : X \to Y,$

$[*.1] := \texttt{ContinuousPreserveLimits}(A,Y,f)\jmath F : F_{|A} = f,$

Assume $V : \mathcal{U}_Y$,

$\Big(V', [2]\big) := \texttt{ClosedSymmetricUniformityBase} : \sum V' \in \mathcal{U}_Y . \texttt{Closed}(V', Y \times T) \ \& \ V' \subset V,$

$\Big(U', [3]\Big) := \eth \mathrm{UC}(A,Y)(f) : \sum U' \in \mathcal{U}_A . (f \times f)(U') \subset V',$

$\Big(U, [4]\Big) := \eth\texttt{UniformSubset}(X,A)(U') : \sum U \in \mathcal{U}_X . U' = (A \times A) \cap U,$

$[V.*] := \texttt{ClosedContainsLimits}(Y,V')F[4][3][2] : (F \times F)(U) \subset \overline{(f \times f)(U')} \subset V' \subset V;$

$\rightsquigarrow [*.2] := \eth^{-1}\mathrm{UC} : F \in \mathrm{UC}(X,Y);$

$\square$

$\texttt{TotallyBounded} :: \prod_{X \in \mathsf{UNI}} ??X$

$A : \texttt{TotallyBounded} \iff \forall U \in \mathcal{U} \,.\, \exists F : \texttt{FiniteCover}(X, A) : \forall S \in F \,.\, \exists x \in X : S \subset U(x)$

$\texttt{TotallyBoundedAltDef1} :: \forall X \in \mathsf{UNI} \,.\, \forall A \subset X \,.\, \texttt{TotallyBounded}(X) \iff$

$$\iff \forall U \in \mathcal{U} \,.\, \exists n \in \mathbb{N} : \exists x : n \to X : A \subset \bigcup_{i=1}^{n} U(x_i)$$

...

$\square$

$\texttt{TotallyBoundedAltDef2} :: \forall X \in \mathsf{UNI} \,.\, \forall A \subset X \,.\, \texttt{TotallyBounded}(X) \iff$

$$\iff \forall U \in \mathcal{U} \,.\, \exists n \in \mathbb{N} : \exists a : n \to A : A \subset \bigcup_{i=1}^{n} U(a_i)$$

...

$\square$

$\texttt{TotallyBoundedByUltrafilters} :: \forall X \in \mathsf{UNI} \,.\, \texttt{TotallyBounded}(X) \iff$

$\iff \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{CauchyFilter}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactIffCompleteAndTotallyBounded} :: \forall X \in \mathsf{UNI} \,.\, \texttt{Compact}(X) \iff \texttt{Complete} \,\&\, \texttt{TotallyBounded}(X)$

$\texttt{Proof} =$

...

$\square$

3.4 Uniformization

3.5 Metrization of a Uniform Space

3.6 Completion of a Uniform Space

4 Metric Dimension

4.1 Covering Dimension

$$\texttt{CoveringDimensionLE} :: \prod_{X \in \mathsf{MS}} ?\Big(\texttt{Compact}(X) \times \mathbb{Z}_+\Big)$$

$$(A, n) : \texttt{CoveringDimensionLE} \iff \dim A \leq n \iff \forall r \in \mathbb{R}_{++} \,.\, \exists k \in \mathbb{N} : \exists x : k \to X :$$
$$: A \subset \bigcup_{i=1}^{k} \mathbb{B}(x_i, r) \;\&\; \forall a \in A \,.\, \Big|\{i \in K : a \in \mathbb{B}(x_i, r)\}\Big| \leq n + 1$$

$$\texttt{CoveringDimensionGreater} :: \prod_{X \in \mathsf{MS}} ?\Big(\texttt{Compact}(X) \times \mathbb{Z}_+\Big)$$

$$(A, n) : \texttt{CoveringDimensionGreater} \iff \dim A > n \iff !(\dim A \leq n)$$

$$\texttt{FiniteDimensionalCompact} :: \prod_{X \in \mathsf{MS}} ?\texttt{Compact}(X)$$

$$K : \texttt{FiniteDimensionalCompact} \iff \exists n \in \mathbb{Z}_+ \,.\, \dim X \leq n$$

$$\texttt{coveringDimension} :: \prod_{X \in \mathsf{MS}} \texttt{FiniteDimensionalCompact} \to \mathbb{Z}_+$$

$$\texttt{coveringDimension}\,(K) = \dim K := \min\{n \in \mathbb{Z}_+ : \dim X \leq n\}$$

$\texttt{IntervalDimension} :: \dim[0,1] = 1$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \dim[0,1] \leq 1,$

$$\Big(k, x, [2], [3]\Big) := \eth\texttt{CoveringDimensionLE}[1]\left(\frac{1}{3}\right) : \sum k \in \mathbb{N} \ . \ \sum x : k \to [0,1] \ .$$

$$. \ [0,1] \subset \bigcup_{i=1}^{k} \mathbb{B}\left(x_i, \frac{1}{3}\right) \ \& \ \forall a \in [0,1] \ . \ \Big|\Big\{i \in k : a \in \mathbb{B}(x_i, r)\Big\}\Big| \leq 1,$$

$$[4] := \eth^{-1}\texttt{DisjointUnion} : [0,1] = \bigsqcup_{i=1}^{k} \mathbb{B}\left(x_i, \frac{1}{3}\right),$$

$\texttt{Assume}\ [5] : k = 1,$

$$[6] := \eth \operatorname{diam}[0,1]\texttt{BallDiam}\left(x_i, \frac{1}{3}\right) : 1 = \operatorname{diam}[0,1] = \operatorname{diam}\mathbb{B}\left(x_i, \frac{1}{3}\right) < \frac{2}{3},$$

$[5.*] := \texttt{OneIsNotZero}[6] : \perp;$

$\rightsquigarrow [5] := E(\perp)\texttt{UnitIsMinimal}(\mathbb{N}) : k > 1,$

$[6] := \texttt{DisjointOpenUnionDisconnected} : \texttt{Disconnected}[0,1],$

$[1.*] := \texttt{IntervalIsConnected}[6] : \perp;$

$\rightsquigarrow [1] := E(\perp) : \dim[0,1] > 0,$

$\texttt{Assume}\ r : \mathbb{R}_{++},$

$[2] := \eth\texttt{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > 1\} \neq \emptyset,$

$k := \min\{N \in \mathbb{N} : Nr > 1\} : \mathbb{N},$

$x := \Lambda i \in k \ . \ (i-1)r : k \to [0,1],$

$\texttt{Assume}\ t : [0,1],$

$[2] := \eth\texttt{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > t\} \neq \emptyset,$

$i := \min\{N \in \mathbb{N} : Nr > t\} \neq \emptyset : \mathbb{N},$

$[3] := \jmath i \eth t : i \leq k,$

$[t.*.1] := \jmath i \jmath x : t \in [(i-1)r, ir) \subset \mathbb{B}(x_i, r),$

$\texttt{Assume}\ j, l : k,$

$\texttt{Assume}\ [4] : t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r),$

$$[5] := \jmath^{-1}x : r|l-j|\texttt{TriangleIneq}\Big([0,1], x_j, x_l, t\Big) \leq |x_j - x_l| \leq |x_l - t| + |t - x_i| < 2r,$$

$$[(j,l).*] := \frac{[5]}{r} : |l-j| \leq 1;$$

$\rightsquigarrow [4] := I(\forall) : \forall j, l \in k \ . \ t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r) \Rightarrow |l-j| \leq 1,$

$$[t.*.2] := \eth\mathbb{N}[2] : \Big|\Big\{t \in k : a \in \mathbb{B}(x_i, r)\Big\}\Big| \leq 2;$$

$\rightsquigarrow [2] := \eth^{-1}\texttt{CoveringDimensionLE} : \dim[0,1] \leq 1,$

$[*] := \eth\mathbb{N}[1][2] : \dim[0,1] = 1;$

$\square$

4.2 Embedding Theorem

$\texttt{MengerMap} :: \prod X, Y \in \mathsf{MS} \, . \, \mathbb{R} \to ?C(X,Y)$

$f : \texttt{MengerMap} \iff \Lambda \varepsilon \in \mathbb{R}_{++} \, . \, \forall a, b \in X \, . \, f(a) = f(b) \Rightarrow d(a,b) < \varepsilon$

$\texttt{MengerMapsIsOpen} :: \forall K : \mathsf{MS} \,\&\, \texttt{Compact} \, . \, \forall n \in \mathbb{N} \, . \, \forall \varepsilon \in \mathbb{R}_{++} \, . \, \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon) \in \mathcal{T}\Big(C(X, \mathbb{R}^n)\Big)$

$\texttt{Proof} \ =$

$[1] := \texttt{CompactProduct}(K,K) : \texttt{Compact}(K \times K),$

$\texttt{Assume}\ f : \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon),$

$A := \Big(d\Big)^{-1}[\varepsilon, +\infty) : \texttt{Closed}(K \times K),$

$[2] := \texttt{ClosedCompactSubset} : \texttt{Compact}(K \times K, A),$

$\delta := \inf_{(a,b) \in A} \|f(a) - f(b)\| : \mathbb{R}_{++},$

$\texttt{Assume}\ g : \mathbb{B}\left(f, \frac{\delta}{2}\right),$

$\texttt{Assume}\ x, y : K,$

$\texttt{Assume}\ [4] : g(x) = g(y),$

$[5] := \texttt{TriangleIneq}(f(x), g(x), g(y), g(y))[4]\eth g :$

$\quad : |f(x) - f(y)| \le |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$

$[(x,y).*] := \jmath\delta\jmath A[5] : d(x,y) < \varepsilon;$

$\rightsquigarrow [g.*] := \eth^{-1}\texttt{MengerMap} : \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon, g);$

$\rightsquigarrow [f.*] := I(\exists)\eth^{-1}\texttt{Subset} : \exists U \in \mathcal{U}(f) \, . \, U \subset \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon);$

$\rightsquigarrow [*] := \texttt{OpenByCover} : \texttt{Open}\ C(X, \mathbb{R}^n)\ \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon);$

$\square$

$\texttt{CoveringSequance} :: \prod K : \texttt{FiniteDimensionalCompact}(X) \, . \, \mathbb{R}_{++} \to ? \sum_{n=1}^{\infty} n \to X$

$x : \texttt{CoveringSequance} \iff \Lambda r \in \mathbb{R}_{++} \, . \, X \subset \bigcup_{i=1}^{k} \mathbb{B}(x_i, r) \,\&\, \forall X \in A \, . \, \Big|\{i \in K : a \in \mathbb{B}(x_i, r)\}\Big| \le n + 1$

$\texttt{PontryaginTwinSequance} :: \prod X : \texttt{Compact} \,\&\, \mathsf{MS} \, . \, \prod r \in \mathbb{R}_{++} \, . \, \prod (n, x) : \texttt{CoveringSequance}(X, r) \, .$

$\quad . \, (X \xrightarrow{\mathsf{TOP}} \mathbb{R}^{2*\dim K+1}) \to \mathbb{R} \to ?\Big(n \to \mathbb{R}^{2\dim X+1}\Big)$

$y : \texttt{PontryaginTwinSequance} \iff \Lambda f : (X \xrightarrow{\mathsf{TOP}} \mathbb{R}^{2*\dim K+1}) \, . \, \Lambda s \in \mathbb{R}_{++} \, . \, \forall i \in n \, . \, d\Big(f\ \mathbb{B}(x_i, r), y_i\Big) < s \,\&$

$\quad \& \, \forall V : \texttt{ProperAffineSubspace}(\mathbb{R}^{2\dim K}) \, . \, |\operatorname{Im} y \cap V| \le \dim V + 2$

$\texttt{PontryaginTwinSequanceExists} :: \forall X : \texttt{FiniteDimensionalCompact} \;.\; \forall f \in C(X, \mathbb{R}^{1+2\dim X}) \;.\; \forall r, s \in \mathbb{R}_{++} \;.$
$\;.\; \forall (n, x) : \texttt{CoveringSequance}(X, r) \;.\; \exists \texttt{PontryaginTwinSequance}(X, f, x, s)$

$\texttt{Proof} \;=$

$\left(y^0, [1]\right) := \texttt{RealOpenIsInfinite}(\ldots) : \sum y^0 : n \hookrightarrow X \;.\; \forall i \in \|y^0_i - f(x_i)\| < \dfrac{s}{1+2^n},$

$\mathcal{A}^0 := \left\{A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : ||A \cap \Im y^0| > \dim A + 1\right\} :$
$\;: \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}),$

$w := \Lambda \mathcal{A} : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \;.\; \sum_{A \in \mathcal{A}} |A \cap \Im y^0| - \dim A - 1 :$
$\;: \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \to \mathbb{Z}_+,$

$M := w(\mathcal{A}^0) : \mathbb{Z}_+,$

$[0] := \jmath M \jmath \mathcal{A}_0 : M \le 2^n,$

$\texttt{Assume } m : (M-1)_+,$

$\texttt{Assume } [2] : \mathcal{A}^m \ne \emptyset,$

$(k, A) := \texttt{enumerate}(\mathcal{A}^n) : \sum k \in \mathbb{N} \;.\; A : k \leftrightarrow \mathcal{A}^m,$

$i := \max\{i \in n : x^m_i \in A_1\} : n,$

$\delta := \min\left\{d(y^n_i, A)\middle| A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \;\&\; \operatorname{Im} y^n \cap A \ne \emptyset \;\&\; y^n_i \notin A\right\} : \mathbb{R}_{++},$

$\left(u, [3]\right) := \texttt{RealOpenIsInfinite}(\ldots) : \sum u \in \mathbb{R}^{1+2\dim X} \;.\; d(y^n_i, u) = \min\left(\delta, \dfrac{s}{1+2^n}\right),$

$y^{m+1} := \Lambda j \in n \;.\; \texttt{if } i == j \texttt{ then } u \texttt{ else } y^n_i : n \to \mathbb{R}^{1+2\dim X},$

$\mathcal{A}^{m+1} := \left\{A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : ||A \cap \Im y^{m+1}| > \dim A + 1\right\} :$
$\;: \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}),$

$[2.*] := \jmath W \jmath \delta \jmath u \jmath y^{m+1} \jmath \mathcal{A}^0 : w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m);$

$\rightsquigarrow [2] := I(\Rightarrow) : \mathcal{A}^m \ne \emptyset \Rightarrow w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m),$

$\texttt{Assume } [2] : \mathcal{A}^m = \emptyset,$

$y^{m+1} := y^m : n \to \mathbb{R}^{1+2*\dim X},$

$\mathcal{A}^{m+1} := \mathcal{A}^m : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X});$

$\rightsquigarrow y, \mathcal{A}, [2] := I\left(\prod\right) : \prod_{m=0}^{M} \sum y^m : n \to \mathbb{R}^{1+2\dim X} \;.\; \sum \mathcal{A}^m : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \;.$
$\;.\; \mathcal{A}^m = \left\{A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : ||A \cap \Im y^m| > \dim A + 1\right\} \;\&$
$\;\forall m \in (M-1) \;.\; \mathcal{A}^m \ne \emptyset w(\mathcal{A}^m) < w(\mathcal{A}^{m+1}),$

$[3] := [2] \jmath w \texttt{InverseFiniteInduction} : \mathcal{A}^M = \emptyset,$

$[4] := \jmath y^M : \forall i \in n \;.\; d(f(x_i), y^M_i) \le \|f(x_i) - y^1_i\| + \sum_{j=1}^{M} \|y^j_i - y^{j+1}_i\| < s,$

$[*] := \eth^{-1}\texttt{PontryaginTwinSequance} : \texttt{PontryaginTwinSequance}(X, f, x, s, y^M);$

$\square$

$\texttt{functionOfNebeling} :: \prod X : \texttt{FiniteDimensionalCompact} . \prod f \in C(X, \mathbb{R}^{1+2\dim X}) . \prod r, s \in \mathbb{R}_{++} .$
$. \prod (n, x) : \texttt{CoveringSequance}(X, r) . \prod p : \texttt{PontryaginTwinSequance}(X, f, x, s) . X \xrightarrow{\text{TOP}} \mathbb{R}^{1+2\dim X}$

$$\texttt{functionOfNebeling}\,(u) = N_p(u) := \frac{\sum_{i=1}^n d\Big(u, \mathbb{B}^{\complement}(x_i, r), rBig)p_i}{\sum_{i=1}^n d\Big(u, \mathbb{B}^{\complement}(x_i), r\Big)}$$

$\texttt{FunctionOfNebelingIsClose} :: \forall X : \texttt{FiniteDimensionalCompact} . \forall f \in C(X, \mathbb{R}^{1+2\dim X}) . \forall r, s \in \mathbb{R}_{++} .$
$. \forall (n, x) : \texttt{CoveringSequance}(X, r) . \forall p : \texttt{PontryaginTwinSequance}(X, f, x, s) .$
$[0] . \forall [0] : \forall i \in \ . \omega_f(x_i, 2r) < s . \|N_p - f\| < 2s$

$\texttt{Proof} \ =$

$\texttt{Assume}\ u : X,$

$\Big(n', k, [1]\Big) := \eth\texttt{CoveringSequance}(x) : \sum n' \in 1 + \dim X . \sum k : n' \to n . \forall i \in n .$
$. u \in \mathbb{B}(x_i, r) \iff \exists j \in n' : i = k_i,$

$y := N_p(u) : \texttt{In}\ \mathbb{R}^{1+2*\dim X},$

$[2] := \jmath y \jmath N_p[1] : y \in \operatorname{conv} \operatorname{Im} p_k,$

$\Big(v, [3]\Big) := \eth\texttt{PontryaginTwinSequance}(p) : \prod_{i=1}^{n'} \mathbb{B}(x_{k_i}, r) . \forall i \in n' . d(f(v_i), p_{k_i}) < s,$

$$y' := \frac{\sum_{i=1}^{n'} d\Big(u, \mathbb{B}^{\complement}(x_i, r), r\Big) f(v_i)}{\sum_{i=1}^n \sum_{i=1}^{n'} d\Big(u, \mathbb{B}^{\complement}(x_i), r\Big)} : \texttt{In}\ \mathbb{R}^{1+2\dim X},$$

$[4] := [3][2]\jmath y \jmath N_p \jmath y' : \|y - y'\| < s,$

$[5] := [0]\jmath y' : \|f(u) - y'\| < s,$

$[u.*] := \texttt{TriangleIneq}[4][5] : d\Big(f(u), N_p(u)\Big) < 2s;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{uniformNorm} : \|f - N_P\| < 2s;$

$\square$

$\texttt{NebelingsAreMenger} :: \forall X : \texttt{FiniteDimensionalCompact} \,.\, \forall f \in C(X, \mathbb{R}^{1+2\dim X}) \,.\, \forall r, s \in \mathbb{R}_{++} \,.$

$\quad .\, \forall (n, x) : \texttt{CoveringSequance}(X, r) \,.\, \forall p : \texttt{PontryaginTwinSequence}(X, f, x, s) \,.$

$\quad [0] \,.\, N_p : \texttt{MengerMap}(X, \mathbb{R}^{2*dim(X)+1}, 2r)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ u, v : X,$

$\texttt{Assume}\ [1] : N_p(u) = N_p(v),$

$y := N_p(u) : \texttt{In}\ \mathbb{R}^{1+2*\dim X},$

$\Big(m, k, [1]\Big) := \eth\texttt{CoveringSequance}(X, n, x, u) :$

$\quad \sum m \in 1 + \dim X \,.\, \sum k : m \hookrightarrow n \,.\, \forall i \in n \,.\, u \in \mathbb{B}(x_i, r) \iff \exists j \in m : i = k_i,$

$\Big(m', k', [1']\Big) := \eth\texttt{CoveringSequance}(X, n, x, v) :$

$\quad \sum m' \in 1 + \dim X \,.\, \sum k' : m' \hookrightarrow n \,.\, \forall i \in n \,.\, v \in \mathbb{B}(x_i, r) \iff \exists j \in m' : i = k'_i,$

$[2] := \jmath y \jmath N_p[1] : y \in \operatorname{conv} \operatorname{Im} p_k,$

$[2'] := \jmath y \jmath N_p[1'] : y \in \operatorname{conv} \operatorname{Im} p_{k'},$

$\Big(\alpha, [3], [4]\Big) := \eth\operatorname{conv}[2] : \sum \alpha : m \to \mathbb{R} \,.\, \sum_{i=1}^{m} \alpha_i = 1 \;\&\; \sum_{i=1}^{m} \alpha_i p_{k_i} = y,$

$\Big(\alpha', [3'], [4']\Big) := \eth\operatorname{conv}[2'] : \sum \alpha : m' \to \mathbb{R} \,.\, \sum_{i=1}^{m'} \alpha'_i = 1 \;\&\; \sum_{i=1}^{m'} \alpha'_i p_{k'_i} = y,$

$[5] := [4][4'] : \sum_{i=1}^{m} \alpha_i p_{k_i} = y = \sum_{i=1}^{m'} \alpha_i p_{k'_i},$

$\Big(i, [6]\Big) := [3]\texttt{ZeroSumIsZero} : \sum i \in m \,.\, \alpha_i \neq 0,$

$m'' := m - 1 : \mathbb{Z}_+,$

$k'' := \hat{k}_i : m'' \to n,$

$\alpha'' := \hat{\alpha}_i : m'' \to \mathbb{R},$

$[7] := [6][5] \ldots : p_{k_i} = \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} p_{k'_j} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} p_{k''_j},$

$[8] := [3][3'][6] \jmath \alpha'' \eth\texttt{Field}(\mathbb{R}) : \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} = \frac{1}{\alpha_i} + \frac{\alpha_i - 1}{\alpha_i} = \frac{\alpha_i}{\alpha_i} = 1,$

$[9] := \eth^{-1}\operatorname{conv}[7][8] : p_{k_i} \in \operatorname{conv} p_{k'' \oplus k'},$

$[10] := \texttt{AffineCombinationDim}(p_{k'' \oplus k'}) \eth m' \eth m \jmath m'' :$

$\quad : \dim \operatorname{conv} p_{k \oplus k'} \leq m' + m'' - 1 \leq 1 + \dim X + \dim X - 1 = 2 \dim X,$

$\Big(j, [11]\Big) := \eth\texttt{PontryaginTwinSequance}(p)[9][10] : \sum j \in m' + m'' \,.\, (k' \otimes k'')_j = k_i,$

$[12] := [11] \eth j \jmath k'' \texttt{injective}(k) : k'_j = k_i,$

$l := k'_j : n,$

$[13] := \jmath[12][1][1'] : u, v \in \mathbb{B}(x_l, r),$

$\Big[(u, v). *\Big] := [13]\eth\texttt{cell}(X)\texttt{TriangleIneq}(X) : d(u, v) < 2r;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{MengerMap} : \texttt{MengerMap}(X, \mathbb{R}^{2*dim(X)+1}, 2r),$

$\square$

$\texttt{AllMengerIsInjective} :: \forall X, Y \in \mathsf{MS} \,.\, \forall f : X \xrightarrow{\mathsf{MS}} Y \,.\, \Big(\forall q \in \mathbb{Q}_{++} \,.\, f : \texttt{MengerMap}(X, Y, q)\Big) \,.$

$\quad . \, \texttt{Injective}(X, Y, f)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MengerIsDense} :: \forall X : \texttt{FinteDimensionalCompact} \,.\, \forall \varepsilon \in \mathbb{R}_{++} \,.$

$\quad . \, \texttt{Dense}\Big(C(X, \mathbb{R}^{1+2\dim X}) \; \texttt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, \varepsilon)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : C(X, \mathbb{R}^n),$

$\texttt{Assume}\ s : \mathbb{R}_{++},$

$[1] := \texttt{UCByCompact}(X, f) : f \in \mathrm{UC}\Big(X, \mathbb{R}^{1+2\dim X}\Big),$

$\Big(\delta, [2]\Big) := \eth\mathrm{UC}\Big(X, \mathbb{R}^{1+2\dim X}\Big)(f)\left(\frac{s}{2}\right) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall x \in X \,.\, \operatorname{diam} f\mathbb{B}(x, \delta) < \frac{s}{2},$

$r := \min\left(\delta, \frac{\varepsilon}{2}\right) : \mathbb{R}_{++},$

$(n, x) := \eth\texttt{FiniteDimensionalCompact}(X)\,(r) : \texttt{CoveringSequance}\,(X, r)\,,$

$[2] := \jmath r \eth\texttt{CoveringSequance}\,(X, r, n, x) : \forall i \in n \,.\, \omega_f(x_i, r) < \frac{s}{2},$

$p := \texttt{PontryaginDualSequanceExists}\left(X, f, x, \frac{s}{2}\right) : \texttt{PontryaginDualSequance}\left(X, f, x, \frac{s}{2}\right),$

$[f.*] := \texttt{FunctionOfNebelingIsClose}(X, p, [2]) : d(N_p, f) < s;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Dense} : \texttt{Dense}\Big(C(X, \mathbb{R}^{1+2\dim X}) \; \texttt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, \varepsilon);$

$\dots$

$\square$

$\texttt{MengerNebelingPontryaginEmbeddingTHM} :: \forall X : \texttt{FinteDimensionalCompact} \,.$

$\quad \exists\texttt{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2\dim X})$

$\texttt{Proof} \;=$

$[1] := \texttt{DualBaireProperty}\Big(C(X, \mathbb{R}^{1+2\dim X}), \Lambda q \in \mathbb{Q}_{++} \,.\, \texttt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, q)\Big)$

$\quad \texttt{AllMengerIsInjective}(X, \mathbb{R}^{1+2\dim X}) : \texttt{Dense}\Big(C(X, \mathbb{R}^{1+2\dim X}), \texttt{Injective}(X, \mathbb{R}^{1+2\dim X})\Big),$

$[2] := \texttt{ConstantIsContinuous}(X, \mathbb{R}^{1+2\dim X}, 0) : 0 \in C(X, \mathbb{R}^{1+2\dim X}),$

$[3] := \eth^{-1}\texttt{NonEmpty} : C(X, \mathbb{R}^{1+2\dim X}) \neq \emptyset,$

$f := [3][1] : \texttt{Injective}(X, \mathbb{R}^{1+2\dim X}),$

$[*] := \texttt{CompactHausdorffIsClosed}\eth^{-1}\texttt{HomeomorphicEmbedding} : \texttt{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2\dim X}, f);$

$\square$