Metric Topology

Uncultured Tramp

July 28, 2022

Contents

1 Metric Spaces

1.1 Distance Metric

$\texttt{Semimetric} :: \prod X \in \mathsf{SET} \;.\; \texttt{Symmetric}(X, X; \mathbb{R}_+)$

$d : \texttt{Semimetric} \iff \forall x, y, z \in X \;.\; d(x, x) = 0 \;\&\; d(x, y) \leq d(x, z) + d(z, y)$

$\texttt{SemimetricSpace} := \sum\limits_{X \in \mathsf{SET}} \texttt{Semimetric}(X) : \texttt{Type};$

$\texttt{Metric} :: \prod x \in \mathsf{SET} \;.\; \texttt{Semimetric}(X, X; \mathbb{R}_+)$

$d : \texttt{Metric} \iff \forall x, y \in X \;.\; d(x, y) = 0 \iff x = y$

$\texttt{MetricSpace} := \sum\limits_{X \in \mathsf{SET}} \texttt{Metric}(X) : \texttt{Type};$

$\texttt{semimetric} :: \prod (X, \rho) : \texttt{SemimetricSpace} \;.\; \texttt{Semimetrc}(X)$

$\texttt{semimetric}\,() = d_X := \rho$

$\texttt{synecdoche} :: \prod (X, \rho) : \texttt{SemimetricSpace} \;.\; \mathsf{SET}$

$\texttt{synecdoche}\,() = (X, \rho) := X$

$\texttt{TriangleInequality} :: \forall (X, d) : \texttt{SemimetricSpace} \;.\; \forall x, y, z \in X \;.\; d(x, y) \leq d(x, z) + d(z, y)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ReversedTriangleInequality} :: \forall (X, d) : \texttt{SemmimetricSpace} \;.\; \forall x, y, z \in X. \left| d(x, y) - d(y, z) \right| \leq d(x, z)$

$\texttt{Proof} =$

$[1] := \texttt{TriangleInequality}(x, y, z) : d(x, y) \leq d(x, z) + d(z, y),$

$[2] := \eth\texttt{Semimetric}(d)(z, y)[1] : d(x, y) \leq d(x, z) + d(y, z),$

$[3] := [2] - d(y, z) : d(x, y) - d(y, z) \leq d(x, z),$

$[4] := \texttt{TriangleInequality}(y, z, x) : d(y, z) \leq d(y, x) + d(x, z),$

$[5] := \eth\texttt{Semimetric}(d)(x, y)[4] : d(y, z) \leq d(y, x) + d(x, z),$

$[6] := [5] - d(y, z) : d(y, z) - d(x, y) \leq d(x, z),$

$[*] := \eth^{-1}\texttt{absValue}[3][6] : \left| d(x, y) - d(y, z) \right| \leq d(x, z);$

$\square$

$\texttt{Lipschitz} :: \prod X, Y : \texttt{SemimetricSpace} . \mathbb{R}_+ \to ?(X \to Y)$

$f : \texttt{Lipschitz} \iff f \in L\text{-Lip} \iff \Lambda L \in \mathbb{R}_+ . \forall x, y \in X . d\Big(f(x), f(y)\Big) \leq L d(x, y)$

$\texttt{SemiisometryCategory} :: \mathsf{CAT}$

$\texttt{SemiisometryCategory}\,() = \mathsf{SMS}_{\circ \to \cdot} := \Big(\texttt{SemimetricSpace}, 1\text{-Lip}, \circ, \mathrm{id}\Big)$

$\texttt{IsometryCategory} :: \mathsf{CAT}$

$\texttt{IsometryCategory}\,() = \mathsf{MS}_{\circ \to \cdot} := \Big(\texttt{MetricSpace}, 1\text{-Lip}, \circ, \mathrm{id}\Big)$

$\texttt{Semiisometry} :: \prod X, Y : \texttt{SemimetricSpace} . ?(X \to Y)$

$f : \texttt{Semiisometry} \iff \forall a, b \in X . d\Big(f(a), f(b)\Big) = d(a, b)$

$\texttt{Isometry} :: \prod X, Y : \texttt{MetricSpace} . ?(X \to Y)$

$f : \texttt{Isometry} \iff \forall a, b \in X . d\Big(f(a), f(b)\Big) = d(a, b)$

$\texttt{distance} :: \prod X : \texttt{SemimetricSpace} . ?X \to X \to \mathbb{R}_+$

$\texttt{distance}\,(A, x) = d(A, x) := \inf_{a \in A} d(a, x)$

$\texttt{distanceBetweenSets} :: \prod X : \texttt{SemimetricSpace} . ?X \to ?X \to \widehat{\mathbb{R}}_+$

$\texttt{distanceBetweenSets}\,(A, B) = d(A, B) := \sup_{b \in B} d(A, b)$

$\texttt{diameter} :: \prod X : \texttt{SemimetricSpace} . ?X \to \widehat{\mathbb{R}}_+$

$\texttt{diameter}\,(A) = \operatorname{diam} A := \sup_{x, y \in A} d(x, y)$

$\texttt{Bounded} :: \prod X \in \mathsf{SMS}_{\circ \to \cdot} . ?_{\texttt{Semiiso}} ?X$

$A : \texttt{Bounded} \iff \operatorname{diam} A < \infty$

$\texttt{BoundedSubset} :: \forall X \in \mathsf{SMS}_{\circ \to \cdot} . \forall A : \texttt{Bounded}(X) . \forall B \subset A . B : \texttt{Bounded}(X)$

$\texttt{Proof} =$

$[1] := \eth \texttt{Bounded}(X)(A) \eth \operatorname{diam} \texttt{SubsetSupremum}(A, B) \eth^{-1} \operatorname{diam} :$

$\quad : \infty > \operatorname{diam} A = \sup_{x, y \in A} d(x, y) \geq \sup_{x, y \in B} d(x, y) = \operatorname{diam} B,$

$[2] := \eth^{-1} \texttt{Bounded}[1] : \Big(B : \texttt{Bounded}(X)\Big);$

$\square$

$\texttt{disk} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} .\ .\ X \to \mathbb{R}_+ \to ?X$

$\texttt{disk}\,(x,r) = \mathbb{D}(r,x) := \left\{ y \in X : d(x,y) \leq r \right\}$

$\texttt{cell} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} .\ .\ X \to \mathbb{R}_+ \to ?X$

$\texttt{cell}\,(x,r) = \mathbb{B}(r,x) := \left\{ y \in X : d(x,y) < r \right\}$

$\texttt{sphere} :: \prod X \in \mathsf{SMS}_{\circ\to\cdot} .\ .\ X \to \mathbb{R}_+ \to ?X$

$\texttt{sphere}\,(x,r) = \mathbb{S}(r,x) := \left\{ y \in X : d(x,y) = r \right\}$

$\texttt{Cellbound} :: \forall X \in \mathsf{SMS}_{\circ\to\cdot} .\ .\ \forall A \subset X\ .\ A : \texttt{Bounded}(X) \iff \exists x \in X : \exists r \in \mathbb{R}_+ : A \subset \mathbb{B}(x,r)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{BoundedMetric1} :: \forall X \in \mathsf{SET}\ .\ \forall d : \texttt{Semimetric}(X)\ .\ \dfrac{d}{1+d} : \texttt{Semimetric}$

$\texttt{Proof} =$

$\text{Assume } x,y,z : X,$

$[\ldots *] := \texttt{PlusUnityRecipricol}(d(x,z))\texttt{TrinagleIneq}(X,d)\texttt{PlusUnityRecipricol}(d(x,y)+d(y,z))$

$\qquad \texttt{PositiveSumIsGreater} : \dfrac{d(x,z)}{1+d(x,z)} = 1 - \dfrac{1}{1+d(x,z)} \leq 1 - \dfrac{1}{1+d(x,y)+d(y,z)} =$

$\qquad = \dfrac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} = \dfrac{d(x,y)}{1+d(x,y)} + \dfrac{d(y,z)}{1+d(y,z)};$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Semimetric} : \left(\dfrac{d}{1+d} : \texttt{Semimetric}(X) \right),$

$\square$

$\texttt{BoundedMetric2} :: \forall X \in \mathsf{SET}\ .\ \forall d : \texttt{Semimetric}(X)\ .\ \min(1,d) : \texttt{Semimetric}$

$\texttt{Proof} =$

$\ldots$

$\square$

1.2 Topology

`metricTopology` $::$ $\mathsf{SMS}_{\circ\to\cdot} \to \mathsf{TOP}$

`metricTopology` $(X) = X := \left\langle \left\{ \mathbb{B}(x,r) \middle| r \in \mathbb{R}_{++}, x \in X \right\} \right\rangle_{\mathsf{TOP}}$

`semimetricSpaceCategory` $::$ `Category`

`semimetricSpaceCategory` $() = \mathsf{SMS} := \left(\mathrm{SemimetricSpace}, C, \circ, \mathrm{id} \right)$

`metricSpaceCategory` $::$ `Category`

`metricSpaceCategory` $() = \mathsf{MS} := \left(\texttt{metricSpace}, C, \circ, \mathrm{id} \right)$

`OpenCells` $::$ $\forall X \in \mathsf{SMS} . \forall x \in X . \forall r \in \mathbb{R}_{++} . \mathbb{B}(x,r) \in \mathcal{T}(X)$
`Proof` $=$
$\dots$
$\square$

`MetricOpennesCriterion` $::$ $\forall X \in \mathsf{SMS} . \forall U \subset X . U \in \mathcal{T}(X) \iff \forall x \in X . \exists r \in \mathbb{R}_{++} : \mathbb{B}(x,r) \subset U$
`Proof` $=$
$\dots$
$\square$

`ClosureByDistance` $::$ $\forall X \in \mathsf{SMS} . \forall A \subset X . \overline{A} = \{ x \in X : d(x,A) = 0 \}$
`Proof` $=$
$B := \{ x \in X : d(x,A) = 0 : \texttt{Subset}(X),$
`Assume` $x : B,$
$[1] := {}_\jmath B(x) : d(x,A) = 0,$
`Assume` $U : \mathcal{U}(x),$
$\left(r, [2] \right)) := \texttt{MetricOpennessCriterion}(U,x) : \sum r \in \mathbb{R}_{++} . \mathbb{B}(x,r) \subset U,$
$[U.*] := [1]\eth\texttt{distanceToSet}[2] : U \cap A \neq \texttt{Emptyset};$
$\rightsquigarrow [x.*] := \texttt{AltClosure}(A) : x \in \overline{A};$
$\rightsquigarrow [1] := bd^{-1}\texttt{Subset} : B \subset \overline{A},$
`Assume` $x : \overline{A},$
`Assume` $r : \mathbb{R}_{++},$
$[2] := \texttt{OpenCell}(x,r)\texttt{AltClosure}(A) : \mathbb{B}(x,r) \cap A = \emptyset,$
$[r.*] := \eth^{-1}\texttt{Functor}\,(distanceToSet, ()\,x, A)[2] : d(x,A) < r;$
$\rightsquigarrow [2] := \texttt{InfimumInduction} : d(x,A) = 0,$
$[x.*] := {}_\jmath B[2] : x \in B;$
$\rightsquigarrow [*] := \texttt{SetEq}[1] : A = B;$
$\square$

`DiscIsClosed` $:: \forall X \in \mathsf{SMS} . \forall x \in X \, \forall r \in \mathbb{R}_{++} \, \mathbb{D}(x,r) : \mathtt{Closed}(X)$

`Proof` $=$

$D := \mathbb{D}(x,r) : \mathtt{Subset}(X),$

`Assume` $y : X,$

`Assume` $[1] : d(y, D) = 0,$

$\left(a, [2]\right) := \eth\,\texttt{distanceToSet}(y, D)[1]\eth\texttt{infimum} : \sum a : \mathbb{N} \to D . \lim_{n \to infty} d(y, a_n) = 0,$

`Assume` $\varepsilon : \mathbb{R}_{++},$

$\left(n, [3]\right) := \eth\,\texttt{Limit}[2](\varepsilon) : \sum n \in \mathbb{N} \, d(y, a_n) < \varepsilon,$

$[\varepsilon.*] := \texttt{TriangleIneq}\,\eth\texttt{disc}(D)[2]\texttt{IneqSum} : d(x,y) \leq d(x, a_n) + d(a_n, y) < r + \varepsilon;$

$\rightsquigarrow [3] := \texttt{IneqLim} : d(x,y) \leq r,$

$[y.*] := \eth\texttt{disc}(D) : y \in D;$

$\rightsquigarrow [1] := \texttt{ClosureByDistance} : D = \overline{D},$

$[*] := \eth\texttt{closure} : \left(D : \texttt{Closed}(X)\right);$

$\square$

`SphereIsClosed` $:: \forall X \in \mathsf{SMS} . \forall x \in X \, \forall r \in \mathbb{R}_{++} \, \mathbb{S}(x,r) : \texttt{Closed}(X)$

`Proof` $=$

$S := \mathbb{S}(x,r) : \texttt{Subset}(X),$

$[1] := \texttt{TrichtomyPrinciple}\,\eth\texttt{sphere}(S) : S^{\complement} = \mathbb{B}(x,r) \cup \mathbb{D}^{\complement}(x,r),$

$[2] := \eth\texttt{ClosedDiscIsClosed}(x,r) : \mathbb{D}^{\complement}(x,r) \in \mathcal{T}(X),$

$[3] := \texttt{OpenCell}(x,r) : \mathbb{B}(x,r) \in \mathcal{T}(X),$

$[4] := [1]\texttt{OpenUnion}[3][2] : S^{\complement} \in \mathcal{T}(X),$

$[*] := \eth^{-1}\texttt{Closed}(X)[4] : \left(S : \texttt{Closed}(X)\right);$

$\square$

`SemimetricBalance` $:: \forall X \in \mathsf{SMS} . \forall x, y \in X . \forall U \in \mathcal{U}(x) . y \in U \iff \forall U \in \mathcal{U}(y) . x \in U$

`Proof` $=$

$\ldots$

$\square$

`BoundedMetricTopology1` $:: \forall (X, d) \in \mathsf{SMS} . (X, d) \cong_{\mathsf{SMS}} \left(X, \dfrac{d}{1+d}\right)$

`Proof` $=$

$\ldots$

$\square$

`BoundedMetricTopology2` $:: \forall (X, d) \in \mathsf{SMS} . (X, d) \cong_{\mathsf{SMS}} \left(X, \min(d, 1)\right)$

`Proof` $=$

$\ldots$

$\square$

`MetricSpaceRegularity` :: $\forall X \in \mathsf{MS} \, . \, X : \mathsf{T4}$

`Proof` $=$

`Assume` $A, B : \mathtt{Closed}(X),$

`Assume` $[1] : A \cap B = \emptyset,$

`Assume` $a : A,$

$\left(r, [2]\right) := \mathtt{ClosureByDistance}[1] : \sum r \in \mathbb{R}_{++} \, . \, d(a, B) > r,$

$O(a) := \mathbb{B}\left(a, \dfrac{r}{4}\right) : \mathcal{T}(X);$

$\leadsto O := I(\rightarrow) : A \to \mathcal{T}(X),$

$U := \bigcap\limits_{a \in A} O(a) : \mathcal{T}(X),$

$[1] := \jmath U : A \subset U,$

$[2] := \jmath U : B \cap U = \emptyset,$

`Assume` $b : B \cap \overline{U},$

$\left(u, [3]\right) := \mathtt{ClosureByDistance} \eth \mathtt{distanceToSet} : \sum u : \mathbb{N} \to U \, . \, \lim\limits_{n \to \infty} d(b, u_n) = 0,$

`Assume` $n : \mathbb{N},$

$\left(a_n, [4]\right) := \jmath U[3] : \sum a_n \in A \, . \, u_n \in O(a_n),$

$r := d(a_n, B) : \mathbb{R}_{++},$

$[5] := \jmath O(a_n) \jmath r : d(a_n, b) \geq r,$

$[6] := \eth \mathtt{cell} : d(a_n, u_n) < \dfrac{r}{2},$

`Assume` $[7] : \dfrac{r}{2} \geq d(u_n, b),$

$[8] := \mathtt{TriangleIneq}(a_n, b)[7][6] : d(a_n, b) \leq d(a_n, u_n) + d(u_n, b) < r,$

$[7.*] := \mathtt{TrichtomyPrinciple}[5][8] : \bot;$

$\leadsto [7] := E(\bot) : \dfrac{r}{2} < d(u_n, b),$

$[n.*] := \mathtt{ReverseTriangularIneq}(a_n, u_n, b)[6][7] : \left| d(a_n, b) - d(u_n, b) \right| \leq d(a_n, u_n) < \dfrac{r}{2} < d(u_n, b);$

$\leadsto \left(a, [4]\right) :=: \sum a : \mathbb{N} \to A \, . \, \lim\limits_{n \to \infty} d(a_n, b) = 0,$

$[5] := \mathtt{ClodureByDistancr} : b \in A,$

$[b.*] := \mathtt{EmptyNonempy}[1][5] : \bot;$

$\leadsto [3]] := E(\bot) : B \cap \overline{U} = \emptyset,$

$V := \left(\overline{U}\right)^{\complement} : \mathtt{Open}(X),$

$[*] := \eth^{-1}\mathtt{T4} : \left(X : \mathtt{T4}\right);$

$\square$

`SemimetricSeparability` :: $\forall X \in \mathsf{SMS} \, . \, X : \mathsf{T0} \Rightarrow X \in \mathsf{MS}$

`Proof` $=$

$\ldots$

$\square$

`SemimetricSeparability` :: $\forall X \in \mathsf{SMS} \, . \, X : \mathsf{T0} \Rightarrow X : \mathsf{T4}$

`Proof` $=$

$\ldots$

$\square$

SemimetricCountability :: $\forall X \in$ SMS . X : FirstCountable

Proof $=$

. . .

□

SemimetricCountability :: $\forall X \in$ SMS . X : Separable $\Rightarrow X$: SecondCountable

Proof $=$

. . .

□

DeltaEpsilonFormalism :: $\forall X, Y \in$ SMS . $\forall f : X \to Y$. $f \in C(X,Y) \iff \forall \epsilon \in \mathbb{R}_+$. $\forall x \in X$.
. $\exists \delta \in \mathbb{R}_+ : \forall a, b \in \mathbb{B}(x, \delta)$. $d(f(a), f(b)) < \epsilon$

Proof $=$

Assume $[1] : f \in C(X,Y)$,

Assume $x : X$,

Assume $\epsilon : \mathbb{R}_+$,

$y := f(x) :$ Element(Y),

$B := \mathbb{B}(y, \epsilon) :$ Open(Y),

$U := f^{-1}(B) :$ Open(X),

$[1] := {}_{J}U : x \in U$,

$\left(\delta, [2]\right) :=$ MetricOpennesCriterion$[1] : \sum \delta \in \mathbb{R}_+$. $\mathbb{B}(x, \delta) \subset U$,

$[1.*] := {}_{J}U[2] : f\mathbb{B}(x, \delta) \subset \mathbb{B}(y, \epsilon)$;

$\leadsto [1] := \eth \mathrm{Cell} I(\Rightarrow) :$ Left $\Rightarrow$ Right,

Assume $[2] :$ Right,

Assume $U :$ Open(Y),

$[U.*] :=$ MetricOpennesCriterion$\eth$Right$[2](U) : f^{-1}(U) \in \mathcal{T}(X)$;

$\leadsto [2.*] := \eth^{-1}C(X,Y) : f \in C(X,Y)$;

$\leadsto [*] := I(\iff) :$ THIS;

□

LipschitzIsContinuous :: $\forall X, Y \in$ SMS . $\forall L \in \mathbb{R}_{++}$. $\forall f \in$ L-Lip(X,Y) . $f \in C(X,Y)$

Proof $=$

. . .

□

IsometricBijectionIsHomeo :: $\forall X, Y \in$ SMS . $\forall f :$ IsometryBijection(X,Y) . $f :$ Home(X,Y)

Proof $=$

. . .

□

SemiisometryFullEmbedding :: SMS$_{\circ\to}$. : FullEmbedding(SMS)

Proof $=$

. . .

□

```
IsometryFullEmbedding :: MS∘→. : FullEmbedding(MS)
Proof =
...
□
```

1.3 Calculus of Convergence

$\texttt{NEpsilonFormalism} :: \forall X \in \mathsf{SMS} . \forall D : \texttt{DirectedSet} . \forall x : \texttt{Net}(D, x) . \forall L \in X . \lim\limits_{n \in D} x_n = L \iff$

$\qquad \iff \forall \epsilon \in \mathbb{R}_{++} . \exists N \in D : \forall n \in D . n \geq N \Rightarrow x_n \in \mathbb{B}(L, \epsilon)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SemimetricConvergence} :: \forall X \in \mathsf{SMS} . \forall D : \texttt{DirectedSet} . \forall x : \texttt{Net}(D, x) . \forall L \in X . \lim\limits_{n \in D} x_n = L \iff$

$\qquad \iff \lim\limits_{n \in D} d(x_n, L) = 0$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FrechetCombination} :: \forall D : \texttt{DirectedSet} . \forall x : \mathbb{N} \to \texttt{Net}(D, \mathbb{R}_+) . \lim\limits_{\delta \in D} \sum\limits_{n=1}^{\infty} \dfrac{x_{n.\delta}}{2^n(1 + x_{n,\delta})} = 0 \iff$

$\qquad \iff \forall n \in \mathbb{N} . \lim\limits_{\delta \in D} . x_{n,\delta} = 0$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FrechetWeakSemimetric} :: \forall X \in \texttt{Set} . \forall d : \mathbb{N} \to \texttt{Semimetric}(X) . \forall D : \texttt{DirectedSet} .$

$\quad . \forall x : \texttt{Net}(X, D) . x : \texttt{Convergent}\left(D, \left(X, \sum\limits_{i=1}^{n} \dfrac{d}{2^n(1+d)}\right)\right) \iff \forall n \in \mathbb{N} . x : \texttt{Convergent}(D, (X, d_n))$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ConvergenceByDenseSubset} :: \forall X \in \mathsf{SMS} . \forall A : \texttt{Dense}(X) . \forall L \in X . \forall D : \texttt{DirectedSet} . \forall x : \texttt{Net}(D, X) .$

$\quad . \lim\limits_{n \in D} x_n = L \iff \forall a \in A . \lim\limits_{n \in D} d(x_n, a) = d(L, a)$

$\texttt{Proof} =$

$\dots$

$\square$

1.4 Compacta

$\texttt{TotallyBounded} :: ?\texttt{SMS}$

$X : \texttt{TotallyBounded} \iff \forall \epsilon \in \mathbb{R} . \exists n \in \mathbb{N} : \exists x : n \to X : X \subset \bigcup_{i=1}^{n} \mathbb{B}(x_n, \epsilon)$

$\texttt{EmptySpaceIsTotallyBounded} :: \emptyset : \texttt{TotallyBounded}$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{CountablyCompactIsTotallyBounded} :: \forall X \in \texttt{SMS} . X : \texttt{CountablyCompact} \Rightarrow X : \texttt{TotallyBounded}$
$\texttt{Proof} =$
$\texttt{Assume } [1] : X \,!\, \texttt{TotallyBounded},$

$\left(\epsilon, [2]\right) := \eth\texttt{TotallyBounded} : \sum \epsilon \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . \forall x : n \to X . \bigcup_{i=1}^{n} \mathbb{B}(x_i; \epsilon) \neq X,$

$(x_0, [3]) := \eth\texttt{NonEmptyTotallyBounded} : \sum x_0 : \texttt{type}(X) . x_0 \in X,$

$\texttt{Assume } n : \mathbb{N},$

$(x_n, [4_n]) :=: \sum x_n \in X : \forall i \in_{\mathbb{Z}_+} n - 1 . d(x_i, x_n) > \epsilon;$

$\rightsquigarrow (x, [4]) := I\left(\sum\right) : \sum x : \mathbb{N} \to X . \forall m, n \in \mathbb{Z}_+ . n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon,$

$c := \texttt{CountablyCompactHasClusters}(X, x) : \texttt{Cluster}(X, x),$

$(n, m, [6, 7, 8]) := \eth\texttt{Cluster}(X, x, c) : \sum n, m \in \mathbb{Z}_+ . n \neq m \,\&\, d(x_n, c) < \frac{\epsilon}{2} \,\&\, d(x_m, c) < \frac{\epsilon}{2},$

$[9] := \texttt{TriangleIneq}(x_n, x_m, c)\texttt{IneqSum}[7][9] : d(x_n, x_m) \leq d(x_n, c) + d(c, x_m) < \epsilon,$

$[1.*] := [4](n, m)[6][9]\texttt{TrichtomyPrinciple}(\mathbb{R}_+) : \bot;$

$\rightsquigarrow [*] := E(\bot) : \left(X : \texttt{TotallyBounded}\right);$

$\square$

$\texttt{TotallyBoundedIsSecondCountable} :: \forall X \in \texttt{SMS} \,.\, X : \texttt{TotallyBounded} \Rightarrow X : \texttt{SecondCountable}$

$\texttt{Proof} =$

$\texttt{Assume } q : \mathbb{Q} \cap (0,1),$

$(n, x, [1]) := \eth\texttt{TotallyBounded}(q) : \displaystyle\sum n \in \mathbb{N} \,.\, \sum x : n \to X \,.\, X = \bigcup_{i=1}^{n} \mathbb{B}(x_i, q),$

$\beta_q := \left\{ \mathbb{B}(x_i, q) \mid i \in n \right\} : \texttt{Finite } \mathcal{T}(X);$

$\leadsto \beta := I(\to) : \mathbb{Q} \cap (0,1) \to \texttt{Finite}\mathcal{T}(X),$

$\mathcal{B} := \displaystyle\bigcup_{q \in \mathbb{Q} \cap (0,1)} \beta_q \,:? \mathcal{T}(X),$

$[1] := \texttt{CountableFiniteUnion}(\mathcal{B}, \mathbb{Q} \cap (0,1), \beta, \jmath\mathcal{B}) : |\mathcal{B}| \leq \aleph_0,$

$\texttt{Assume } U : \texttt{Open}(X),$

$\texttt{Assume } u : U,$

$(r, [2]) := \texttt{MetricOpennesCriterion}(U, x) : \displaystyle\sum r \in \mathbb{R}_{++} \,.\, \mathbb{B}(u, r) \subset U,$

$(q, [3]) := \texttt{ArchimedianLimit}(r) : \displaystyle\sum q \in (0,1) \cap \mathbb{Q} \,.\, q < \frac{r}{2},$

$(x, [4]) := \jmath beta_q(u) : \displaystyle\sum x \in X \,.\, u \in \mathbb{B}(x, q),$

$\texttt{Assume } y : \mathbb{B}(x, q),$

$[5] := \texttt{TriangleInequality}(u, y, x)[4]\eth\texttt{cell}(X, x, q)[3] : d(u, y) \leq d(u, x) + d(x, y) < 2q < r,$

$[y.*] := \eth\texttt{cell}(X, u, r)[2] : y \in U;$

$\leadsto [5] := \eth^{-1}\texttt{Subset} : \mathbb{B}(x, q) \subset U,$

$[u.*] := I(\exists)(\mathcal{B})[5][4] : \exists B \in \mathcal{B} : u \in B \subset U;$

$\leadsto [2] := I(\forall) : \forall u \in U \,.\, \exists B \in \mathcal{B} : u \in B \subset U,$

$[U.*] := \texttt{InnerCover}[2] : \exists \mathcal{A} \subset \mathcal{B} \,.\, \bigcup \mathcal{A} = U;$

$\leadsto [2] := \eth^{-1}\texttt{Base} : \left(\mathcal{B} : \texttt{Base}(X) \right),$

$[*] := \eth^{-1}\texttt{SecondCountable}(\mathcal{B})[1][2] : \left(X : \texttt{SecondCountable} \right);$

$\square$

$\texttt{CountablyCompactIsCompact} :: \forall X : \texttt{SMS} \,.\, X : \texttt{CountablyCompact} \Rightarrow X : \texttt{Compact}$

$\texttt{Proof} =$

$[1] := \texttt{CountablyCompactIsTotallyBounded}(X) : \left(X : \texttt{TotallyBounded} \right),$

$[2] := \texttt{TotallyBoundedIsSecondCountable}(X) : \left(X : \texttt{SecondCountable} \right),$

$[3] := \texttt{SecondCountableIsLindelof}(X) : \left(X : \texttt{Lindelof} \right),$

$[*] := \texttt{LindelofAndCountablyCompactIsCompact} : \left(X : \texttt{Compact} \right);$

$\square$

$\texttt{SequantiallyCompactIffCompact} :: \forall X : \texttt{SMS} \,.\, X : \texttt{SequentiallyCompact} \iff X : \texttt{Compact}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{TotallyBoundedByDenseSubset} :: \forall X \in \mathsf{SMS} . \forall A : \texttt{Dense}(X) . A : \texttt{TotallyBounded} \Rightarrow X : \texttt{TotallyBounded}$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : X\ !\ \texttt{TotallyBounded},$

$\left(\epsilon, [2]\right) := \eth\texttt{TotallyBounded} : \sum \epsilon \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . \forall x : n \to X . \bigcup_{i=1}^{n} \mathbb{B}(x_i; \epsilon) \neq X,$

$(x_0, [3]) := \eth\texttt{NonEmptyTotallyBounded} : \sum x_0 : \texttt{type}(X) . x_0 \in X,$

$\texttt{Assume}\ n : \mathbb{N},$

$(x_n, [4_n]) :=: \sum x_n \in X : \forall i \in_{\mathbb{Z}_+} n - 1 . d(x_i, x_n) > \epsilon;$

$\rightsquigarrow (x, [4]) := I\left(\sum\right) : \sum x : \mathbb{N} \to X . \forall m, n \in \mathbb{Z}_+ . n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon,$

$(a, [5]) := \eth\texttt{Dense}(A)\left(x, \dfrac{\epsilon}{3}\right) : \sum a : \mathbb{Z}_+ \to A . \forall n \in \mathbb{Z}_+ . d(a_n, x_n) < \dfrac{\epsilon}{3},$

$\left(n, [6, 7]\right) := \eth\texttt{TotallyBounded}(A)\left(q, \dfrac{\epsilon}{3}\right) : \sum n \in \mathbb{N} . d(a_n, a_0) < \dfrac{\epsilon}{3}\ \&\ n \neq 0,$

$[8] := \texttt{TriangleIneq}[6][5]\texttt{SumIneq} : d(x_n, x_0) \leq d(x_n, a_n) + d(a_n, a_0) + d(a_0, x_0) < \epsilon,$

$[1.*] := [8][4](0, n)[7]\texttt{TrichtomyPrinciple}(\mathbb{R}_{++}) : \bot;$

$\rightsquigarrow [*] := E(\bot)) : \left(X : \texttt{TotallyBounded}\right);$

$\square$

$\texttt{Equicontinuous} :: \prod X, Y \in \mathsf{SMS} . ??C(X, Y)$

$\Phi : \texttt{Equicontinuous} \iff \Phi \in \mathcal{EC}(X, Y) \iff \forall x \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists U \in \mathcal{U}(x) : \forall f \in \Phi . \operatorname{diam} f(U) < \varepsilon$

$\texttt{EquicontinuousAtAPoint} :: \prod X, Y \in \mathsf{SMS} . X \to ??C(X, Y)$

$\Phi : \texttt{Equicontinuous} \iff \Phi \in \mathcal{EC}_x(X, Y) \iff \Lambda x \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists U \in \mathcal{U}(x) : \forall f \in \Phi . \operatorname{diam} f(U) < \varepsilon$

$\texttt{FiniteContinuousIsEquicontinuous} :: \forall X, Y \in \mathsf{SMS} . \forall \Phi : \texttt{Finite}\ C(X, Y) . \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{FiniteEquicontinuousUnion} :: \forall X, Y \in \mathsf{SMS} . \forall n \in \mathbb{N} . \forall \Phi : n \to \mathcal{EC}(X, Y) . \bigcup \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{EquicontinuityIsLocal} :: \forall X, Y \in \mathsf{SMS} . \forall \Phi : ??C(X, Y) . \left(\forall x \in X . \Phi \in \mathcal{EC}_x(X, Y)\right) \Rightarrow \Phi \in \mathcal{EC}(X, Y)$

$\texttt{Proof} =$

$\ldots$

$\square$

`EquicontinuityByMonotonicConvergence` $:: \forall X \in \mathsf{SMS} . f : \mathbb{N} \to C(X) . \forall \varphi \in C(X) .$
$\quad f \downarrow \varphi \Rightarrow \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$
`Proof` $=$
`Assume` $x : \mathrm{In}(X),$
`Assume` $\varepsilon : \mathbb{R}_{++},$
$(N, [1]) := \eth\texttt{Limit}(f(x), \varphi(x), \varepsilon/10) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \dfrac{\varepsilon}{10},$
$\left(\delta_1, [2]\right) := \eth C(X)(\varphi, \varepsilon/10, x) : \sum \delta_1 \in \mathbb{R}_{++} . \varphi\left(\mathbb{B}(x, \delta_1)\right) \subset \left(\varphi(x) - \dfrac{\varepsilon}{10}, \varphi(x) + \dfrac{\varepsilon}{10}\right),$
$\left(\delta_2, [3]\right) := \eth C(X)(f_N, \varepsilon/10, x) : \sum \delta_2 \in \mathbb{R}_{++} . f_N\left(\mathbb{B}(x, \delta_2)\right) \subset \left(f_N(x) - \dfrac{\varepsilon}{10}, f_N(x) + \dfrac{\varepsilon}{10}\right),$
$\delta := \min(\delta_1, \delta_2) : \mathbb{R}_{++},$
`Assume` $n : \mathbb{N},$
`Assume` $[4] : n \geq N,$
`Assume` $y : \mathbb{B}(x, \delta),$
$[5] := \eth\texttt{MonotonicConvergence}(f, \varphi, y)[4] : |f_N(y) - \varphi(y)| > |f_n(y) - \varphi(x)|,$
$[n.*] := \texttt{TriangleIneq}^4[1, 2, 3, 4, 5] : |f_n(y) - f_n(x)| \leq |f_n(y) - \varphi(y)| + |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| <$
$\quad |f_N(y) - \varphi(y)| + \dfrac{2\varepsilon}{10} \leq |f_N(y) - f_N(x)| + |f_N(x) - \varphi(x)| + |\varphi(x) - \varphi(y)| + \dfrac{2\varepsilon}{10} < \dfrac{\varepsilon}{2};$
$\leadsto [4] := \eth^{-1}\mathcal{EC}_x : \{f_n | n \geq N\} \in \mathcal{EC}_x(X, \mathbb{R}),$
$[x.*] := \texttt{FiniteContinuousIsEquiContinuous}\{f_n | n < N\}\texttt{FiniteEqyiContinuousUnion}[4] :$
$\quad : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}_x(X, \mathbb{R});$
$\leadsto [*] := \eth^{-1}\mathcal{EC}_x : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, \mathbb{R});$
$\square$

$\;$

`maricFunction` $:: \prod X \in \mathsf{SMS} . \texttt{Pointwise}(X) \to \mathbb{R}_{++} \to X \to \mathbb{R}$

`maricFunction` $(f, \varepsilon) = M_{f, \varepsilon} := \Lambda x \in X . \displaystyle\sum_{n=1}^{\infty} \min\left(0, |f_m(x) - \lim_{m \to \infty} f_n(x)| - \varepsilon\right)$

$\;$

`MaricFunctionContinuity` $:: \forall X \in \mathsf{SMS} . \forall \varphi \in C(X) . f : \texttt{Pointwise}(X, \varphi) . \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$
`Proof` $=$
$\varphi := \displaystyle\lim_{n \to \mathbb{N}} f_n : X \to \mathbb{R},$
`Assume` $x : X,$
$\left(N, [1]\right) := \eth\texttt{Pointwise}(X) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \dfrac{\varepsilon}{3} \;\&\; n < N \Rightarrow |f_n(x) - \varphi(y)| >$
$\varepsilon' := \min(|f_{N-1}(x) - \varphi(x)| - \varepsilon, \varepsilon) : \mathbb{R}_{++},$
$\left(\delta_1, [2]\right) := \eth\mathcal{EC} : \prod \delta \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . f_n\mathbb{B}(x, \delta) \subset \mathbb{B}\left(f(x), \dfrac{\varepsilon'}{3}\right),$
$\left(\delta_2, [3]\right) := \eth C(X) : \prod \delta \in \mathbb{R}_{++} . \varphi\mathbb{B}(x, \delta) \subset \mathbb{B}\left(\varphi(x), \dfrac{\varepsilon'}{3}\right),$
$\delta := \min(\delta_1, \delta_2, \delta_3) : \mathbb{R}_+,$
`Assume` $y : \mathbb{B}(x, \delta),$
`Assume` $n : \mathbb{N},$
`Assume` $[4] : n \geq N,$
$[y.*] := \texttt{TriangleIneq}[1][2][3] : |\varphi(y) - f_n(y)| \leq |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon;$
$\leadsto [4] := I(\forall) : \forall n \geq N . \forall y \in \mathbb{B}(x, \delta) . |\varphi(y) - f_n(x)| < \varepsilon,$
`Assume` $y : \mathbb{B}(x, \delta),$

Assume $n : \mathbb{N}$,

Assume $[5] : n < N$,

$[y.*] := [1][2][3][5] : |\varphi(y) - f_n(y)| > |\varphi(x) - f_n(x)| - \varepsilon' > \varepsilon$;

$\rightsquigarrow [5] := I(\forall) : \forall y \in \mathbb{B}(x, \delta) \cdot \forall n \in \mathbb{N} \cdot n < N \Rightarrow |\varphi(x) - f_n(x)| < \varepsilon$,

$[6] := [4][5]\eth^{-1}\texttt{maricFunction} : \forall y \in \mathbb{B}(x, \delta) \cdot M_{f,\varepsilon}(y) = \sum_{n=1}^{N-1} |f_n(y) - \varphi(y)| - \varepsilon$,

$[y.*] := \texttt{ContinuousSum} : M_{f,\varepsilon|\mathbb{B}(x,\delta)} \in C\Big(\mathbb{B}(x, \delta)\Big)$;

$\rightsquigarrow [*] := \texttt{ContinuityIsLocal} : M_{f,\varepsilon} \in C(X)$;

$\square$

$\texttt{DiniLemma} :: \forall X : \texttt{Pseudocompact} \ \& \ \textsf{SMS} \cdot \forall f : \mathbb{N} \to C(X) \cdot \forall \varphi \in C(X) \cdot$

$\quad \cdot f \downarrow \varphi \Rightarrow \varphi : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), \varphi\Big)$

$\texttt{Proof} =$

$[1] := \texttt{EquicontinuityByMonotonicConvergence} : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, \mathbb{R})$,

$[2] := \texttt{MaricFunctionContinuity} : \forall \varepsilon \in \mathbb{R}_{++} \cdot M_{f,\varepsilon} \in C(X)$,

$[a, 3] := \eth\texttt{Paracompact}(M_f) : \sum \mathbb{R}_{++} \to \mathbb{R}_+ \cdot \forall \varepsilon \in \mathbb{R}_{++} \cdot \max_{x \in X} M_{f,\varepsilon}(x) = a_\varepsilon$,

Assume $\varepsilon : \mathbb{R}_{++}$,

Assume $n : \mathbb{N}$,

Assume $[4] : n > \left\lceil \dfrac{a_\varepsilon}{\varepsilon} \right\rceil$,

Assume $[5] : \|f_n - \varphi\|_\infty \geq \varepsilon$,

$\Big(x, [6]\Big) := \eth\texttt{uniformNorm}[5]\eth\texttt{Paracompact}(X) : \sum x \in X : |f_n(x) - \varphi(x)| \geq \varepsilon$,

$[7] := \eth\texttt{MonotonicConvergence}[5] : \forall m \in \mathbb{N} \cdot m \leq n \Rightarrow |f_m(x) - \varphi(x)| \geq \varepsilon$,

$[8] := \eth\texttt{maricFunction}[4][7] : M_{f,\varepsilon}(x) \geq \sum_{i=1}^{n} \|f(x) - \varphi(x)\| > a_\varepsilon$,

$[n.*] := \eth\texttt{Maximum}(a_n, M_{\varepsilon,x})[8] : \bot$;

$\rightsquigarrow [\varepsilon.*] := E(\bot) : \forall n \in \mathbb{N} \cdot n > \left\lceil \dfrac{a_\varepsilon}{\varepsilon} \right\rceil \Rightarrow \|f_n - \varphi\|_\infty < \varepsilon$;

$\rightsquigarrow [*] := \texttt{MetricLimit} : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), f, \varphi\Big)$;

$\square$

$\texttt{DiniSpace} :: ?\textsf{TOP}$

$X : \texttt{DiniSpace} \iff \forall f : \mathbb{N} \to C(X) \cdot \forall \varphi \in C(X) \cdot$

$\quad \cdot f \downarrow \varphi \Rightarrow \varphi : \texttt{Limit}\Big(\Big(C(X), \|\cdot\|_\infty\Big), \varphi\Big)$

$\texttt{InverseDiniLemma} :: \forall X : \texttt{DiniSpace} \cdot X : \texttt{Paracompact}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{EquicontinuousNet} :: \prod X, Y \in \mathsf{SMS} \, . \, \texttt{?Net Pointwise}(X, Y)$

$(D, f) : \texttt{EquicontinuousNet} \iff \{f_\delta | \delta \in D\} \in \mathcal{EC}(X, Y)$

$\texttt{EquicontinuousLimitIsContinuous} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall (D, f) : \texttt{EquicontinuousNet}(X) \, . \, \forall F : \texttt{Limit}(D, f) \, .$

$\texttt{Proof} =$

$\texttt{Assume } x : X,$

$\texttt{Assume } \varepsilon : \mathbb{R}_+,$

$(\delta, [1]) := \eth \texttt{EquicontinuousNet}(D, f) \left(x, \dfrac{\varepsilon}{y} \right) : \sum \delta' \in \mathbb{R}_{++} \, . \, \forall n \in D \, . \, \operatorname{diam} f_n \mathbb{B}(x, \delta) < \dfrac{\varepsilon}{3},$

$\texttt{Assume } y : Y,$

$(N, [2]) := \eth \texttt{Limit}(f, d)(F) : \sum N \in \mathbb{N} \, . \, \forall n \in \mathbb{N} \, . \, n \geq N \Rightarrow d(f_n(x), F(x)) < \dfrac{\varepsilon}{3},$

$(M, [3]) := \eth \texttt{Limit}(f, d)(F) : \sum N \in \mathbb{N} \, . \, \forall n \in \mathbb{N} \, . \, n \geq N \Rightarrow d(f_n(y), F(y)) < \dfrac{\varepsilon}{3},$

$K := \max(N, M) : D,$

$[y.*] := \texttt{TriangleIneq}[1][2][3] : d(F(x), F(y)) \leq d(F(x), f_K(x)) + d(f_K(x), f_K(y)) + f(f_K(y), F(y)) < \varepsilon;$

$\rightsquigarrow [x.*] := I(\forall) : \forall y \in \mathbb{B}(x, \delta) \, . \, d(F(x), F(y)) < \varepsilon;$

$\rightsquigarrow [*] := \eth^{-1} C(X, Y) : F \in C(X, Y);$

$\square$

$\texttt{ConvergentContinuously} :: \prod X, Y \in \mathsf{TOP} \, . \, \texttt{?PointwiseNet}(X, Y)$

$(D, f) : \texttt{ConvergentContinuously} \iff \forall x : \texttt{ConvergentNet}(D, X) \, . \, \lim\limits_{n \in D} f_n(x_n) = \left(\lim\limits_{n \in D} f \right) \left(\lim\limits_{n \in D} x_n \right)$

$\texttt{UniformIsConvergentContinuously} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall (D, f) : \texttt{UniformNet}(X, Y) \, . \, (D, f) : \texttt{ConvergentCont}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{UniformIsConvergentContinuously} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall (D, f) : \texttt{EquicontinuousNet}(X, Y) \, . \, (D, f) : \texttt{Converg}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{UniformConvergenceOnCompact} :: \forall Y \in \mathsf{SMS} \,.\, \forall X : \texttt{Compact} \ \& \ \mathsf{SMS} \,.\, \forall (D, f) : \texttt{Net}\ C(X, Y) \,.$
$\quad .\ \forall F \in C(X, Y) \,.\, \forall [1] : \texttt{PointwiseLimit}(D, f, F) \,.\, \texttt{UniformLimit}(D, f, F)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ x : X,$

$\left(r', [1] \right) := \eth \texttt{EquicontinuosNet}(D, f) \left(x, \dfrac{\varepsilon}{3} \right) : \sum r' \in \mathbb{R}_{++} \,.\, \forall n \in D \,.\, \operatorname{diam} f_n \mathbb{B}(x, r') < \dfrac{\varepsilon}{3},$

$\left(r'', [2] \right) := \eth C(X, Y)(F) \left(x, \dfrac{\varepsilon}{3} \right) : \sum r'' \in \mathbb{R}_{++} \,.\, \operatorname{diam} F\mathbb{B}(x, r') < \dfrac{\varepsilon}{3},$

$r(x) := \min(r', r'') : \texttt{In}\ \mathbb{R}_{++},$

$\left(N(x), [3] \right) := \eth \texttt{PointwisLimit}(D, f, F) \left(x, \dfrac{\varepsilon}{3} \right) : \sum N(x) \in D : \forall n \in D \,.\, n \geq N \Rightarrow d(f_n(x), F(x)) < \dfrac{\varepsilon}{3},$

$\texttt{Assume}\ y : \texttt{In}\ \mathbb{B} \left(x, r(x) \right),$

$\texttt{Assume}\ n : \texttt{In}(D),$

$\texttt{Assume}\ [4] : n \geq N(x),$

$[y.*] := \texttt{TriangleIneq}[1, 2, 3] : d \left(f_n(y), F(y) \right) \leq d(f_n(y), f_n(x)) + d(f_n(x), F(x)) + d(F(x), F(y)) < \varepsilon;$

$\leadsto [x.*] := I(\forall) : \forall y \in \mathbb{B} \left(x, r(x) \right) \,.\, d \left(f_n(y), F(y) \right) < \varepsilon;$

$\leadsto \left(r, N, [1] \right)] := I \left(\prod \right) : \prod_{x \in X} \left(r(x), N(x) \right) : \mathbb{R}_{++} \times D \,.\, \forall y \in \mathbb{B} \left(x, r(x) \right) \,.\, \forall n \in D \,.$

$\quad .\ n \geq N(x) \Rightarrow d \left(f_n(y), F(y) \right) < \varepsilon,$

$[2] := \texttt{FullCover}\ \mathbb{B}(\cdot, r) : X = \bigcup_{x \in X} \mathbb{B} \left(x, r(x) \right),$

$\left(n, x, [3] \right) := \eth \texttt{Compact}(X) : \sum n \in \mathbb{N} \,.\, \sum x : n \to X \,.\, X = \bigcup_{i=1}^{n} \mathbb{B} \left(x_i, r(x_i) \right),$

$\Delta := \max_{i \in n} N(x_i) : D;$

$\texttt{Assume}\ y : X,$

$\texttt{Assume}\ \delta : \texttt{In}\ D,$

$\texttt{Assume}\ [4] : \delta \geq \Delta,$

$\left(i, [5] \right) := [2](y) : \sum i \in n \,.\, y \in \mathbb{B} \left(x_i, r(x_i) \right),$

$[6] := [4] \jmath \Delta(i) : \delta \geq N(x_i),$

$[\varepsilon.*] := [1][5][6] : d \left(f_\delta(y), F(y) \right) < \varepsilon;$

$\leadsto [*] := \eth^{-1} \texttt{UniformLimit} : \texttt{UniformLImit}(D, f, F);$

$\square$

1.5 Completeness and Completion

$\texttt{CauchyFilterBase} :: \prod X \in \mathsf{SMS} . \, ?\texttt{FilterBase}(X)$

$\mathcal{F} : \texttt{CauchyFilterBase} \iff \inf\limits_{F \in \mathcal{F}} \operatorname{diam} F = 0$

$\texttt{CauchyFilter} := \prod X \in \mathsf{SMS} . \, \texttt{Filter} \,\&\, \texttt{CouchyFilterBase}(X) : \mathsf{SMS} \to \texttt{Type};$

$\texttt{ConvergentFilterBaseIsCauchy} :: \forall X \in \mathsf{SMS} . \, \forall \mathcal{F} : \texttt{ConvergentFilterBase}(X) .$
$\quad . \, \mathcal{F} : \texttt{CauchyFilterBase}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CauchyNet} :: \prod X \in \mathsf{SMS} . \, \prod D : \texttt{DirectedSet} . \, ?\texttt{Net}(D, x)$

$x : \texttt{CauchyNet} \iff \forall \varepsilon \in \mathbb{R} . \, \exists \Delta \in D : \forall \delta, \delta' \in D . \, d(x_\delta, x_{\delta'}) < \varepsilon$

$\texttt{CauchySequence} := \prod X \in \mathsf{SMS} . \, \texttt{CauchyNet}(\mathbb{N}, X) : \prod\limits_{X \in \mathsf{SMS}} ?(\mathbb{N} \to X);$

$\texttt{CauchyFilterAndNetEquivalence} :: \forall X \in \mathsf{SMS} . \, \forall (D, x) : \texttt{Net}(X) .$
$\quad . \, x : \texttt{CauchyNet}(D, X) \iff \mathcal{F}(D, x) : \texttt{CauchyFilter}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Complete} :: ?\mathsf{SMS}$

$X : \texttt{Complete} \iff \forall \mathcal{F} : \texttt{CauchyFilterBase}(X) . \, \mathcal{F} : \texttt{ConvergentFilterBase}(X)$

$\texttt{CompleteAltDef} :: \forall X \in \mathsf{SMS} . \, X : \texttt{Complete} \iff \forall x : \texttt{CauchySequence}(X) . \, x : \texttt{Convergent}(X)$

$\texttt{Proof} =$

$\texttt{Assume } R : \forall x : \texttt{CauchySequence}(X) . \, x : \texttt{Convergent},$

$\texttt{Assume } \mathcal{F} : \texttt{CauchyFilterbase},$

$\left(F, [1] \right) := \Lambda n \in \mathbb{N} . \, \eth\texttt{CauchyFilter}(X, \mathcal{F}) \left(\dfrac{1}{n} \right) : \prod\limits \sum\limits_{n \in \mathbb{N} \, F_n \in \mathcal{F}} \operatorname{diam} F_n < \dfrac{1}{n},$

$x := \eth\texttt{CauchyFilterbase}(\mathcal{F})(F) : \prod\limits_{n \in \mathbb{N}} \bigcap\limits_{i=1}^{n} F_i,$

$[2] := \eth^{-1}\texttt{CauchySequnce}(X)[1] : \texttt{CauchySequence}(X, x),$

$[3] := R[2] : \texttt{Convergent}(X, x),$

$L := \lim\limits_{n \to \infty} x_n : X,$

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [4]\right) := \eth\texttt{Limit}(X, x, L)\left(\frac{\varepsilon}{2}\right) : \sum\limits_{N \in \mathbb{N}} : \forall n \in \mathbb{N} . d(L, x_n) < \frac{\varepsilon}{2}$,

$\left(M, [5]\right) := \texttt{InverseArchimedeanLimit} : \sum M \in \mathbb{N} . \frac{1}{M} < \frac{\varepsilon}{2}$,

$k := \max(M, N) : \mathbb{N}$,

Assume $y : F_k$,

$[\varepsilon.*] := \texttt{TriangleIneq}(X, L, y, x_k)[1, 4, 5] : d(L, y) \le d(L, x_k) + d(x_k, y) < \varepsilon$;

$\rightsquigarrow [4] := \eth^{-1}\texttt{FilterbaseLimit} : L = \lim \mathcal{F}$,

$[\mathcal{F}.*] := \eth^{-1}\texttt{ConvergentFilterBase}[4] : \texttt{ConvergentFilterBase}(X, \mathcal{F})$;

$\rightsquigarrow [*] := \eth^{-1}\texttt{Complete} : \texttt{Complete}(X)$;

$\square$

$\texttt{CauchyFilterbaseClustersAreLimits} :: \forall X : \mathsf{SMS} . \forall \mathcal{F} : \texttt{CauchyFilterBase}(X) .$
$\quad . \forall c : \texttt{Cluster}(\mathcal{F}) . c : \texttt{Limit}(\mathcal{F})$

$\texttt{Proof} =$

Assume $\varepsilon : \mathbb{R}_{++}$,

$(F, [1]) := \eth^{-1}\texttt{CauchyFilterBase}(\mathcal{F})\left(\frac{\varepsilon}{2}\right) : \sum\limits_{F \in \mathcal{F}} \operatorname{diam} F < \frac{\varepsilon}{2}$,

$(x, [2]) := \eth\texttt{Cluster}(\mathcal{F}, c)\left(F, \mathbb{B}\left(\frac{\varepsilon}{2}\right)\right) : \sum\limits_{x \in F} d(x, c) < \frac{\varepsilon}{2}$,

Assume $y : F$,

$[3] := \texttt{TriangleIneq}(X)(c, y, x) : d(c, y) \le d(c, x) + d(x, y) < \varepsilon$,

$[y.*] := \eth^{-1}\texttt{cell}[3] : y \in \mathbb{B}(c, \varepsilon)$;

$\rightsquigarrow [F.*] := \eth^{-1}\texttt{Subset} : F \subset \mathbb{B}(c, \varepsilon)$;

$\rightsquigarrow [*] := \eth^{-1}\texttt{Limit} : \texttt{Limit}(X, \mathcal{F}, c)$;

$\square$

$\texttt{CauchyPartialLimitTHM} :: \forall X \in \mathsf{SMS} . \forall x : \texttt{CauchySequance}(X) . \forall L : \texttt{PartialLimit}(X, x) . \lim\limits_{n \to \infty} x_n = L$

$\texttt{Proof} =$

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [1]\right) := \eth\texttt{CauchySequance}(X, x)\left(\frac{\varepsilon}{2}\right) : \sum\limits_{N \in \mathbb{N}} \forall n, m \in \mathbb{N} . n, m \ge N \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2}$,

$\left(M, [2]\right) := \eth\texttt{PartialLimit}(X, x, p)\left(p, \frac{\varepsilon}{2}\right) : \sum\limits_{M \in \mathbb{N}} M \ge N \,\&\, d(x_M, L) < \frac{\varepsilon}{2}$,

Assume $n : \mathbb{N}$,

Assume $[3] : n \ge N$,

$[4] := \texttt{TriangleIneq}[1][2] : d(x_n, L) < d(x_n, x_M) + d(x_M, L) < \varepsilon$;

$\rightsquigarrow [*] := \eth^{-1}\texttt{Limit} : \lim\limits_{n \to \infty} x_n = L$;

$\square$

CompactIffCompleteAndTotallyBounded $:: \forall X \in \mathsf{SMS}\,.\,\texttt{Compact}(X) \iff \texttt{Complete}\ \&\ \texttt{TotallyBounded}(X)$

Proof $=$

Assume $L : \texttt{Compact}(X)$,

$[1] := \texttt{CountablyCompactIsTottalyBounded}(X, L) : \texttt{TotallyBounded}(X)$,

$[2] := \texttt{SequantiallyCompactIffCompact}(X, L) : \texttt{SequantiallyCompact}(X)$,

Assume $x : \texttt{CauchySequence}(X)$,

$p := \eth\texttt{SequantiallyCompact}(X)(x) : \texttt{PartialLimit}(X, x)$,

$[x.*] := \texttt{CauchyPartialLimitTHM}(X, x, p) : \lim\limits_{n \to \infty} x_n = p;$

$\leadsto [x.*] := \texttt{CompleteAltDef} : \texttt{Complete}(X);$

$\leadsto [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right}$,

Assume $R : \texttt{Complete}\ \&\ \texttt{TotallyBounded}(X)$,

Assume $x : \mathbb{N} \to X$,

$u^1 := x : \mathbb{N} \to X$,

Assume $n : \mathbb{N}$,

$\left[m, c, [2]\right] := \eth\texttt{TotallyBounded}(X)\left(\dfrac{1}{n}\right) : \sum\limits_{m \in \mathbb{N}} \sum\limits_{c:m \to X} X = \bigcup\limits_{i=1}^{m} \mathbb{B}\left(c_i, \dfrac{1}{n}\right)$,

$\left[i, [3]\right] := \texttt{PigionholePrinciple}[2] : \sum\limits_{i \in m} \left| \left\{ k \in \mathbb{N} : u_k^n \in \mathbb{B}\left(c_i, \dfrac{1}{n}\right) \right\} \right| = \aleph_0$,

$\left(u^{n+1}, [n.*]\right) := \texttt{subsequance}[3] : \sum u^{n+1} : \texttt{Subsequence}(X, u^{n+1})\,.\, \operatorname{Im} u^{n+1} \subset \mathbb{B}\left(c_i, \dfrac{1}{n}\right);$

$\leadsto \left(u, [2]\right) := I\left(\prod\right) : \prod\limits_{u:\mathbb{N} \to \mathbb{N} \to X} .\ \forall n \in \mathbb{N}\,.\, u^{n+1} : \texttt{Subsequence}(X, u^n)\ \&\ \exists c \in X : \operatorname{Im} u^{n+1}\mathbb{B}\left(c, \dfrac{1}{n}\right)$,

$\Delta := \Lambda n \in \mathbb{N}\,.\, u_n^n : \texttt{Subsequence}(X, x)$,

$[3] := \jmath\Delta[2] : \forall N \in \mathbb{N}\,.\, \forall n, m \in \mathbb{N}\,.\, n, m > N \to d(x_n, x_m) < \dfrac{2}{N}$,

$[4] := \eth^{-1}\texttt{CauchySequance}[3] : \texttt{CauchySequance}(X, \Delta)$,

$[5] := \texttt{CompleteAltDef}(X)[4] : \texttt{Convergent}(X, \Delta)$,

$\delta := \lim\limits_{n \to \infty} \Delta_n : X$,

$[x.*] := \eth^{-1}\texttt{PartialLimit}(x) : \texttt{PartialLimit}(X, x, \delta);$

$\leadsto [*.R] := \texttt{SequantiallyCompactIffCompact} : \texttt{Compact}(X);$

$\leadsto [*] := I(\iff) : \texttt{Left} \iff \texttt{Right};$

$\square$

ConvergentByCompleteSubset $:: \forall X \in \mathsf{SMS}\,.\,\forall \mathcal{F} : \texttt{CauchyFilter}(X)\,.\, \left(\exists F \in \mathcal{F} : F : \texttt{Complete}\right) \Rightarrow$

$\qquad \Rightarrow \texttt{Convergent}(X, \mathcal{F})$

Proof $=$

$\dots$

$\square$

$\text{ConvergentByCompactSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall \mathcal{F} : \text{CauchyFilter}(X) \,.\, \Big(\exists F \in \mathcal{F} : F : \text{Compact} \Big) \Rightarrow$

$\qquad \Rightarrow \text{Convergent}(X, \mathcal{F})$

$\text{Proof} =$

$\ldots$

$\square$

$\text{CompletionCategory} :: \mathsf{SMS} \to \mathsf{CAT}$

$\text{CompletionCategory}(X) = \text{COMP}(X) := \Big(\sum Y : \text{Complete} \,.\, f : \text{Isometry}(X, Y),$

$\qquad , \Big((Y, f), (Z, g) \Big) \mapsto \Big\{ \varphi : \text{Isometry}(Y, Z) \Big| f \varphi = g \Big\}, \circ, \text{id} \Big)$

$\text{Completion} := \Lambda X \in \mathsf{SMS} \,.\, \text{Initial COMP}(X) : \mathsf{SMS} \to \text{Type};$

$\text{CompleteByDenseSubset} :: \forall X \in \mathsf{SMS} \,.\, \forall A : \text{Dense}(X) \,.\, \Big(\forall a : \text{CauchySequence}(A) \,.\, a : \text{Convergent}(X) \Big) \Rightarrow$

$\qquad \Rightarrow X : \text{Complete}(X)$

$\text{Proof} =$

$\text{Assume } x : \text{CauchySequence}(X),$

$\Big(a, [1] \Big) := \Lambda n \in \mathbb{N} \,.\, \eth \text{Dense}(X)(x_n) : \sum a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, d(a_n, x_n) < \dfrac{1}{n},$

$\text{Assume } \varepsilon : \mathbb{R}_{++},$

$\Big(K, [2] \Big) := \eth \text{CauchySequence}(X, x) \Big(\dfrac{\varepsilon}{3} \Big) : \sum K \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n, m \geq K \Rightarrow d(x_n, x_m) < \dfrac{\varepsilon}{3},$

$\Big(L, [3] \Big) := \text{ReducioInfinuma} \Big(\dfrac{\varepsilon}{3} \Big) : \sum L \in \mathbb{N} \,.\, \dfrac{1}{L} < \dfrac{\varepsilon}{3},$

$N := \max(K, L) : \mathbb{N},$

$\text{Assume } n, m : \mathbb{N},$

$\text{Assume } [4] : n, m \geq N,$

$\Big[\varepsilon . * \Big] := \text{TriangleIneq}^2(\ldots)[1, 2, 3, 4] : d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(a_m, x_m) < \varepsilon;$

$\leadsto [4] := \eth^{-1} \text{CauchySequence} : \text{CauchySequence}(A, a),$

$L := \lim\limits_{n \to \infty} a_n : \text{In } X,$

$\text{Assume } \varepsilon : \mathbb{R}_{++},$

$\Big(K, [5] \Big) := \eth \text{Limit}(X, a, L) \Big(\dfrac{\varepsilon}{2} \Big) : \sum K \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, n \geq K \Rightarrow d(a_n, L) < \dfrac{\varepsilon}{2},$

$\Big(M, [6] \Big) := \text{ReducioInfinuma} \Big(\dfrac{\varepsilon}{2} \Big) : \sum M \in \mathbb{N} \,.\, \dfrac{1}{M} < \dfrac{\varepsilon}{2},$

$N := \max(K, M) : \mathbb{N},$

$\text{Assume } n : \mathbb{N},$

$\text{Assume } [7] : n > N,$

$[\varepsilon . *] := \text{TrianglIneq}[1, 5, 6, 7] : d(x_n, L) \leq d(x_n, a_n) + d(a_n, L) < \varepsilon;$

$\leadsto [x . *] := \eth^{-1} \text{Limit} : \lim\limits_{n \to \infty} x_n = L;$

$\leadsto [*] := \text{CompleteAltDef} : \text{Complete}(X);$

$\square$

CompletionIsUnique :: $\forall X \in \mathsf{SMS} \, . \, \forall (A, a), (B, b) : \mathtt{Completion}(X) \, . \, (A, B) : \mathtt{Semiisometric}$

Proof $=$

...

□

RealsAreComplete :: $\mathbb{R} : \mathtt{Complete}$

Proof $=$

...

□

CauchySequanceDistanceExists :: $\forall X \in \mathsf{SMS} \, . \, \forall x, y : \mathtt{CauchySequence}(X) \, . \, d(x, y) : \mathtt{Convergent}(\mathbb{R}_+)$

Proof $=$

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(L, [1] \right) := \eth \mathtt{CauchySequence}(X, x) \left(\dfrac{\varepsilon}{2} \right) : \sum L \in \mathbb{N} \, . \, \forall n, m \in \mathbb{N} \, . \, n, m > L \Rightarrow d(x_n, x_n) < \dfrac{\varepsilon}{2}$,

$\left(M, [2] \right) := \eth \mathtt{CauchySequence}(X, y) \left(\dfrac{\varepsilon}{2} \right) : \sum M \in \mathbb{N} \, . \, \forall n, m \in \mathbb{N} \, . \, n, m > M \Rightarrow d(y_n, y_m) < \dfrac{\varepsilon}{2}$,

$N := \max(L, M) : \mathbb{N}$,

Assume $n, m : \mathbb{N}$,

Assume $[3] : n, m \geq N$,

$[\varepsilon.*] := \mathtt{TriangleIneq}\Big(d(x_n, y_n), d(x_m, y_m), d(x_n, y_m) \Big)$

$\quad \mathtt{ReversedTriangleIneq}(x_n, y_n, y_m) \mathtt{ReversedTriangleIneq}(x_n, x_m, y_m)[1, 2, 3] :$

$\quad : \Big| d(x_n, y_n) - d(x_m, y_m) \Big| \leq \Big| d(x_n, y_n) - d(x_n, y_m) \Big| + \Big| d(x_n, y_m) - d(x_m, y_m) \Big| \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon;$

$\leadsto [4] := \eth^{-1} \mathtt{CauchySequence} : \mathtt{CauchySequence}(\mathbb{R}, d(x, y))$,

$[*] := \mathtt{RealsAreComplete} \, \mathtt{CompleteAltDef}(\mathbb{R}) : \mathtt{Convergent}(\mathbb{R}, d(x, y));$

□

CauchyMetric :: $\forall X \in \mathsf{SMS} \, . \, \lim d : \mathtt{Semimetric} \, \mathtt{CauchySequence}(X)$

Proof $=$

...

□

completion :: $\mathsf{SMS} \to \mathsf{MS}$

completion $(X, d) = (\hat{X}, \hat{d}) := \dfrac{\Big(\mathtt{CauchySequence}(X), \lim d \Big)}{\lim d}$

CompletionTHM $:: \forall X \in \mathsf{SMS} . \left(\hat{X}, \Lambda x \in X . \Lambda n \in \mathbb{N} . x \right) : \mathtt{Completion}(X)$

Proof $=$

Assume $p : \hat{X}$,

$\left(x, [1] \right) := \eth\hat{X}(p) : \sum x : \mathtt{CauchySequance}(X) . p = [x]$,

$A := \Lambda n \in \mathbb{N} . \Lambda m \in \mathbb{N} . \text{if } m < n \text{ then } x_m \text{ else } x_n : \mathbb{N} \to \mathtt{Convergent}(X)$,

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [2] \right) := \eth\mathtt{CauchySequance}(X, x)(\varepsilon) : \sum N \in \mathbb{N} . \forall n, m \in \mathbb{N} \, n, m \geq \mathbb{N} \Rightarrow d(x_n, x_m) < \varepsilon$,

Assume $n : \mathbb{N}$,

Assume $[3] : n \geq N$,

$[p.*] := \eth\mathtt{completion}[1]\jmath A[2,3] : \hat{d}(p, [A_n]) = \lim_{m \to \infty} d(x_m, A_{n,m}) = \lim_{m \to \infty} d(x_m, x_n) < \varepsilon$;

$\rightsquigarrow [1] := \eth^{-1}\mathtt{Dense} : \mathtt{Dense}(\hat{X}, X)$,

$[2] := \mathtt{CompleteByDenseSubset}(\hat{X}, X) : \mathtt{Complete}(\hat{X})$,

$\phi := \Lambda x \in X . \left[\Lambda n \in \mathbb{N} . x \right] : \mathtt{Isometry}(X, \hat{X})$,

Assume $(Y, f) : \mathtt{In} \, \mathsf{COMP}(X)$,

$\psi := \Lambda [x] \in \hat{X} . \lim_{n \to \infty} f(x_n) : \mathtt{Isometry}(\hat{X}, Y)$,

Assume $x : X$,

$[x.*] := \jmath\phi\jmath\psi\mathtt{ConstantSeq} : x\phi\psi = [n \mapsto x]\psi = \lim_{n \to \infty} f(x) = f(x)$;

$\rightsquigarrow [3] := \eth^{-1}\mathsf{COMP}(X) : \left(\psi : \hat{X} \xrightarrow{\mathsf{COMP}(X)} Y \right)$,

Assume $\psi' : \hat{X} \xrightarrow{\mathsf{COMP}(X)} Y$,

Assume $p : \hat{X}$,

$\left(x, [4] \right) := \eth\hat{X}(p) : \sum x : \mathtt{CauchySequance}(X) . p = [x]$,

$A := \Lambda n \in \mathbb{N} . \Lambda m \in \mathbb{N} . \text{if } m < n \text{ then } x_m \text{ else } x_n : \mathbb{N} \to \mathtt{Convergent}(X)$,

$\left[(Y, f) \right] := [4]\jmath^{-1}\mathtt{AContinuousLimit}\jmath^{-1}\phi\eth\mathsf{COMP}(X) : \psi'(p) = \psi'[x] = \psi' \lim_{n \to \infty} [A_n] = \lim_{n \to \infty} \psi'[A_n] =$

$\quad = \lim_{n \to \infty} \psi'\phi(x_n) = \lim_{n \to \infty} f(x_n) \lim_{n \to \infty} \psi\phi(x_n) = \psi(p)$;

$\rightsquigarrow [*] := \eth^{-1}\mathtt{Completion} : \mathtt{Completion}\left(X, (\hat{X}, \varphi) \right)$;

$\square$

Contraction $:= \Lambda X \in \mathsf{MS} . \sum \alpha \in (0, 1) . \alpha\text{-}\mathrm{Lip}(X) : \mathsf{MS} \to \mathtt{Type}$;

`BanachFixedPointTHM` :: $\forall X \in \mathsf{MS}$ & `Complete` & `NonEmpty` . $\forall f :$ `Contraction`(X) . $\exists! x \in X : f(x) = x$

`Proof` $=$

$\left(\alpha, [1]\right) := \eth$`Contraction`$(f) : \sum\limits_{\alpha \in (0,1)} \forall x, y \in X . d(f(x), f(y)) \leq alpha * d(f(x), f(y))$,

$x := \eth$`NonEmpty`$(X) : \mathrm{In}(X)$,

$\delta := d(x, f(x)) : \mathbb{R}_+$,

`Assume` $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [2]\right) :=$ `PowerReduction` $: \sum\limits_{N \in \mathbb{N}} \alpha^N < \dfrac{(1 - \alpha)\varepsilon}{\delta}$,

`Assume` $n, m : \mathbb{N}$,

`Assume` $[3] : n, m \geq N$,

`Assume` $[4] : n \geq m$,

$[\varepsilon.*] :=$ `TriangleInequality`$[4]$`NonNegSum`$[3][1]$`PowerSum`$[2][3]$:

$$: d\left(f^n(x), f^m(x)\right) \leq \sum_{i=0}^{n-m} d\left(f^{m+i}(x), f^{m+i+1}(x)\right) \leq \sum_{i=0}^{\infty} d\left(f^{N+i}(x), f^{N+i+1}(x)\right) \leq \sum_{i=0}^{\infty} \delta\alpha^{N+i} = \frac{\delta\alpha^N}{1 - \alpha} < \varepsilon;$$

$\rightsquigarrow [2] := \eth^{-1}$`CauchySequence` : `CauchySequence`$\left(X, \Lambda n \in \mathbb{N} . f^n(x)\right)$,

$[3] :=$ `AltCompleteDef`$[2]$: `Convergent`$\left(X, \Lambda n \in \mathbb{N} . f^n(x)\right)$,

$p := \lim\limits_{n \to \infty} f^n(x) : \mathrm{In}X$,

`Assume` $\varepsilon : \mathbb{R}_{++}$,

$\left(M, [4]\right) := \eth$`Limit`$(X, \Lambda n \in \mathbb{N} . f^n(x), L) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow d(f^n(x), p) < \dfrac{\varepsilon}{3}$,

$\left(K, [5]\right) := \eth$`CauchySequence`$(X, \Lambda n \in \mathbb{N} . f^n(x)) : \sum K \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(f^n(x), f^m(x)) < \dfrac{\varepsilon}{3}$,

$N := \max(M, K) : \mathbb{N}$,

`Assume` $n : \mathbb{N}$,

`Assume` $[6] : n \geq N$,

$[\varepsilon.*] :=$ `TriangleIneq`$[1](f^n\ x, p)[4, 5, 6] : d(p, f\ x) \leq d(p, f^n\ x) + d(f^n\ x, f^{n+1}\ x) + d(f^{n+1}(x), f\ p) <$

$\quad < 2d(p, f^n\ x) + d(f^n\ x, f^{n+1}\ x) < \varepsilon;$

$\rightsquigarrow [4] :=$ `ZeroBound` $: d(L, f\ p) = 0$,

$[5] := \eth$`MS`$[4] : p = f\ p;$

`Assume` $q : \mathrm{Type}In X$,

`Assume` $[6] : q = f\ q$,

$[7] := [1](p, q)[5][6] : d(p, q) \leq \alpha d(f\ , f\ q) = \alpha d(p, q)$,

$[8] :=$ `FixedMultiolicationIsZero`$[7] : d(p, q) = 0$,

$[q.*] := \eth^{-1}$`MS`$[8] : p = q;$

$\rightsquigarrow [*] := \eth^{-1}$`Unique` $: \exists! x \in X : f(x) = x;$

$\square$

1.6 Baire Category

`CantorIntersectionTheorem` $:: \forall X \in \mathsf{SMS}\ \&\ \texttt{Complete}\ .\ \forall A : \texttt{Decreasing}\left(\texttt{Closed}(X)\ \&\ \texttt{NonEmpty}\right).$

$\qquad . \forall[0] : \lim\limits_{n\to\infty} \operatorname{diam} A_n = 0\ .\ \bigcap\limits_{n=1}^{\infty} A_n \neq \emptyset$

`Proof` $=$

$a := \eth\texttt{NonEmpty}(A) : \prod\limits_{n\in\mathbb{N}} A_n,$

$[1] := \eth^{-1}\texttt{CauchySequance}[0]\jmath a : \texttt{CauchySequance}(X, a),$

$[2] := \texttt{AltCompleteDef} : \texttt{Convergent}(X, a),$

$K := \lim\limits_{n\to\infty} a_n : \texttt{NonEmpty}(X),$

`Assume` $x : K,$

$[3] := \Lambda n \in \mathbb{N}\ .\ \texttt{ClosedLimit}(X, A_n, a_{+n}, x) : \forall n \in \mathbb{N}\ .\ x \in A_n,$

$[x.*] := \eth\texttt{Intersection} : x \in \bigcap\limits_{n=1}^{\infty} A_n;$

$\leadsto [3] := \eth^{-1}\texttt{Subset} : K \subset \bigcap\limits_{n=1}^{\infty} A_n,$

$[*] := \eth\texttt{NonEmpty}[3] : \bigcap\limits_{n=1}^{\infty} A_n \neq \emptyset;$

$\square$

`MetricBaireCategoryTheorem` $:: \forall X \in \mathsf{SMS}\ \&\ \texttt{Complete}\ .\ X : \texttt{Baire}$

`Proof` $=$

`Assume` $U : \mathbb{N} \to \texttt{Dense}\ \&\ \texttt{Open}(X),$

`Assume` $x : \texttt{In}\ X,$

`Assume` $\varepsilon : \mathbb{R}_{++},$

$u_1 := x : \texttt{In}\ X,$

$r_1 := \varepsilon : \mathbb{R}_{++},$

$\delta_1 := 1 : \mathbb{R}_{++},$

`Assume` $n : \mathbb{N},$

$[1] := \texttt{OpenDenseIntersection} : \bigcap\limits_{i=1}^{n} U_i : \texttt{Open}\ \&\ \texttt{Dense}(X),$

$V := \mathbb{B}(x_n, r_n) \cap \bigcap\limits_{i=1}^{n} U_i : \texttt{Open}\ \&\ \texttt{NonEmpty}(X),$

$v := \eth\texttt{NonEmpty}(X, V : \texttt{In}(V),$

$\Delta := d(v, \partial V) : \mathbb{R}_{++},$

$\delta_{n+1} := \dfrac{\min(\Delta, \delta_n)}{2} : \mathbb{R}_{++},$

$K_n := \{v \in V : d(v, \partial V) \leq \delta_{n+1}\} : \texttt{NonEmpty}(V),$

$[2] := \texttt{ContinuousDistance}(\overline{V}, \partial V)\texttt{ClosedPreimage}\jmath K_n : \texttt{Closed}(X, K_n),$

$u_{n+1} := \eth\texttt{NonEmpty} : K_n,$

$r_{n+1} := \min\left(\dfrac{r_n}{2}\,\delta_{n+1}\right) : \mathbb{R}_{++};$

$$\leadsto \left(K, [1]\right) := I\left(\sum\right) : \sum K : \texttt{Decreasing CLosed}(X) \,.\, \lim_{n\to\infty} \operatorname{diam} K_n = 0 \,\&\, \forall n \in \mathbb{N} \,.\, K_n \subset \bigcap_{i=1}^{n} U_n \cap \mathbb{B}(x, \varepsilon),$$

$$[2] := \texttt{CantorIntersectionTheorem}(X, K, [1]) : \bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

$$[x.*] := \texttt{IntersectionSubset}(X, K, [1], [2]) : \bigcap_{n=1}^{\infty} K_n \subset \mathbb{B}(x, \varepsilon) \cap \bigcap_{n=1}^{\infty} U_n;$$

$$\leadsto [U.*] := \eth^{-1}\texttt{Dense} : \texttt{Dense}\left(X, \bigcup_{n=1}^{\infty} U_n\right);$$

$$\leadsto [*] := \eth^{-1}\texttt{Baire} : \texttt{Baire}(X);$$

$\square$

$$\texttt{FirstCategory} :: \prod X \in \textsf{TOP} \,.\, ??X$$

$$A : \texttt{FirstCategory} \iff A \neq \emptyset \,\&\, \exists P : \mathbb{N} \to \texttt{NowhereDense}(X) \,.\, \bigcup_{n=1}^{\infty} P_n = A$$

$\texttt{FirstCategoryTheorem1} :: \forall X : \texttt{Baire} \,.\, \forall U \in \mathcal{T}(X) \,.\, U \,!\, \texttt{FirstCategory}$

$\texttt{Proof} =$

$\texttt{Assume } [1] : \texttt{FirstCategory}(X, U),$

$[2] := \eth_1\texttt{FirstCategory}(X, U)[1] : U \neq \emptyset,$

$$\left(A, [3]\right) := \eth_2\texttt{FirstCategory}(X, U)[1] : \sum A : \mathbb{N} \to \texttt{NowhereDense}(X) \,.\, U = \bigcap_{n=1}^{\infty} A_n,$$

$[3] := \texttt{DualBaireProperty}[3] : \texttt{Codense}(U),$

$[4] := \eth\texttt{Codense}(X, U) : \texttt{Dense}(X, U^{\complement}),$

$[5] := \texttt{DenseOpenIntersection}[4][2] : U \cap U^{\complement} \neq \emptyset,$

$[1.*] := \texttt{ComplementIntersection}(U)\texttt{EmptyIsNonEmpty} : \bot;$

$\leadsto [*] := E(\bot) :! \texttt{FirstCategory}(X, U);$

$\square$

$\texttt{FirstCategoryTheorem2} :: \forall X \in \textsf{TOP} \,.\, \forall [0] : \forall U \in \mathcal{T}(X) \,.\, U \,!\, \texttt{FirstCategory} \,.\, X : \texttt{Baire}$

$\texttt{Proof} =$

$\texttt{Assume } U : \mathbb{N} \to \texttt{Open} \,\&\, \texttt{Dense}(X),$

$$\texttt{Assume } [1] : \bigcap_{n=1}^{\infty} U_n \,!\, \texttt{Dense}(X),$$

$$\left(V, [2]\right) := \eth\texttt{Dense}[1] : \sum V \in T(X) \,.\, V \cap \bigcap_{n=1}^{\infty} U_n = \emptyset,$$

$A := V \cap U^{\complement} : \mathbb{N} \to \texttt{Codense}(X),$

$$[3] := {}_J A[2] : V = \bigcup_{i=1}^{\infty} A_i,$$

$[1.*] := [0](V)\eth^{-1}\texttt{FirstCategory}[3] : \bot;$

$$\leadsto [U.*] := E(\bot) : \bigcap_{n=1}^{\infty} U_n = \texttt{Dense}(X);$$

$\leadsto [*] := \eth^{-1}\texttt{Baire} : \texttt{Baire}(X);$

$\square$

1.7 Hausdorff Metric

`metricOfHausdorff` $::$ MS $\to$ MS

`metricOfHausdorff` $(X) = \mathcal{H}(X) = \left(\mathcal{H}(X), d_H\right) := \left(\texttt{Closed} \,\&\, \texttt{Bounded} \,\&\, \texttt{Nonempty}(X), A, B \mapsto \max(\sup_{x \in A} d(x$

Assume $A, B, C : \texttt{NonEmpty} \,\&\, \texttt{Compact}(X)$,

Assume $[1] : d(A, B) = \sup_{x \in A} d(x, B)$,

$\left(a, [2]\right) := \eth\texttt{supremum} : \sum a : \mathbb{N} \to A \,.\, d(A, B) = \lim_{n \to \infty} d(a_n, B)$,

Assume $n : \mathbb{N}$,

Assume $c : C$,

$[c.*] := [1]\eth\texttt{setDistance}(B)\forall x \in A \,.\, \forall y \in B\texttt{TriangleIneq}(x, y, c)\texttt{InfReduction}(B)\texttt{SupReduction}(B)$
$\quad \texttt{SumIneq}(\mathbb{R}, \ldots)\texttt{SupremumIntroduction}(C)\eth^{-1}\texttt{metricOfHausforff} :$
$\quad : d(a_n, B) = \inf_{b \in B} d(a_n, b) =\leq d(a_n, c) + \inf_{b \in B} d(c, b) \leq d(a_n, c) + \sup_z \inf_b d(z, b) \leq d(a_n, c) + d(C, B);$

$\leadsto [1.*] := \texttt{UniversalIneq} : d(a_n, B) \leq \inf_{z \in C} d(a_n, z) + d(C, B) \leq d(A, C) + d(C, B),$

$[1] := I(\Rightarrow) : d(A, B) = \sup_{x \in A} d(x, B) \Rightarrow d(A, B) \leq \sup_{x \in A} d(x, B) + d(C, B);$

$\square$

`blowUp` $:: \prod X \in \texttt{SMS} \,.\, \mathbb{R}_{++} \to (?X) \to (?X)$

`blowUp` $(\varepsilon, A) = A_\varepsilon := \bigcup_{a \in A} \mathbb{B}(a, \varepsilon)$

`HausdorffMetricAltDef` $:: \forall X \in \texttt{MS} \,.\, \forall A, B : \texttt{Compact}(X) \,.\, d(A, B) = \inf \left\{ \varepsilon \in \mathbb{R}_{++} \middle| B \subset A_\varepsilon \,\&\, A \subset B_\varepsilon \right\}$

`Proof` $=$

$\ldots$

$\square$

`HausdorffFunctor` $:: \texttt{MS}_{\circ \to \cdot} \xrightarrow{\text{CAT}} \texttt{MS}_{\circ \to \cdot}$

`HausdorffFunctor` $(X) = \mathcal{H}(X) := \mathcal{H}(X)$

`HausdorffFunctor` $(X, Y, f) = \mathcal{H}(f) := \Lambda A \in \mathcal{H}(X) \,.\, \operatorname{cl} f(A)$

`CauchyDiamConverge` $:: \forall X \in \texttt{MS} \,.\, \forall A : \texttt{Cauchy}(X) \,.\, \operatorname{diam} A : \texttt{Convergent}(\mathbb{R})$

`Proof` $=$

$\ldots$

$\square$

$\texttt{HausdorffMetrizationANdCompletionCommute} :: \forall X \in \mathsf{MS}.\mathcal{H}(\widehat{X}) \cong_{\mathsf{MS}_{\circ\to\cdot}} \widehat{\mathcal{H}(X)}$

$\texttt{Proof} =$

$\texttt{Assume } A : \mathcal{H}(\widehat{X}),$

$\texttt{Assume } a : A,$

$\left(x^a, [1]\right) := \eth\widehat{(X)}(a) : \sum x : \texttt{Cauchy}(X) \, . \, a = [x^a] \, \& \, \forall N \in \mathbb{N} \, . \, \forall n, m \in \mathbb{N} \, . \, n, m \geq N \Rightarrow d(x_m^a, x_m^a) \leq N^{-1};$

$\rightsquigarrow (x, [1]) := I(A) : \prod_{a \in A} \sum x : \texttt{Cauchy}(X) \, . \, a = [x^a] \, \& \, \forall N \in \mathbb{N} \, . \, \forall n, m \in \mathbb{N} \, . \, n, m \geq N \Rightarrow d(x_m^a, x_m^a) \leq N^{-1},$

$B := \Lambda n \in \mathbb{N} \, . \, \underset{X}{\mathrm{cl}}\{x_n^a | a \in A\} : \mathbb{N} \to \texttt{Closed}(X),$

$\texttt{Assume } n : \mathbb{N},$

$\texttt{Assume } b, b' : \texttt{In}(B_n),$

$\texttt{Assume } \varepsilon : \mathbb{R}_{++},$

$\left(a', a, [2]\right) := \eth\texttt{closure}_{J}B_n(b, b') : \sum a', a :\in A \, . \, d(b, x_n^a) < \varepsilon \, \& \, d(b', x_n^{a'}) < \varepsilon,$

$[n.*] := \texttt{TriangleIneq}^4[2][1] : d(b', b) \leq d(b, x_n^a) + d(x_n^a, a) + d(a, a') + d(a', x_n^{a'}) + d(x_n^{a'}, b') < \mathrm{diam}\, A + \dfrac{2}{n} + 2\varepsilon;$

$\rightsquigarrow [2] := I(\forall) : \forall n \in \mathbb{N} \, . \, \mathrm{diam}\, B_n \leq \mathrm{diam}\, A + \dfrac{2}{n},$

$\texttt{Assume } \varepsilon : \mathbb{R},$

$\left(N, [3]\right) := \texttt{ReductionInfinuma}(\varepsilon) : \prod N \in \mathbb{N} \, . \, \varepsilon < \dfrac{1}{N},$

$\texttt{Assume } n, m : \mathbb{N},$

$\texttt{Assume } [4] : n, m \geq N,$

$[\varepsilon.*] := \eth\mathcal{H}(X)\eth\texttt{supremum}[1][3][4] : d(B_n, B_m) = \max\left(\sup_{a \in A} d(x_n^a, B_m), \sup_{a \in A} d(x_m^a, B_n)\right) \leq \sup_{a \in A} d(x_n^a, x_m^a) \leq \dfrac{1}{N} < \varepsilon;$

$\rightsquigarrow [3] := \eth^{-1}\texttt{CauchySequance} : \texttt{CauchySequance}\left(\mathcal{H}(X), B\right),$

$\varphi(X) := \lim_{n \to \infty} X_n : \widehat{\mathcal{H}(X)};$

$\rightsquigarrow \varphi := I(\to) : \mathcal{H}(\widehat{X}) \to \widehat{\mathcal{H}(X)},$

$\texttt{Assume } A, B : \mathcal{H}(\widehat{X}),$

$\left(\alpha, [4]\right) := {}_{J}\varphi(A) : \sum \alpha : \texttt{Cauchy}\left(\mathcal{H}(X)\right) \varphi(A) = [\alpha] \, \& \, \lim_{n \to \infty} \underset{\widehat{X}}{\mathrm{cl}}\, \alpha_n = A,$

$\left(\beta, [5]\right) := {}_{J}\varphi(B) : \sum \beta : \texttt{Cauchy}\left(\mathcal{H}(X)\right) \phi(B) = [\beta] \, \& \, \lim_{n \to \infty} \underset{\widehat{X}}{\mathrm{cl}}\, \beta_n = B,$

$\left[A, B, [*]\right] := [\eth\texttt{Complement}4][5] : d\left(\varphi(A), \varphi(B)\right) = \lim_{n \to \infty} d\left(\alpha_n, \beta_n\right) = d(A, B));$

$\rightsquigarrow [4] := \eth^{-1}\texttt{Isometry} : \texttt{Isometry}\left(\mathcal{H}(\widehat{X}), \widehat{\mathcal{H}(X)}\right),$

$\texttt{Assume } A : \widehat{\mathcal{H}(X)},$

$\left(K, [1]\right) := \eth\texttt{Completion}(\mathcal{H}(X)) : \prod K : \texttt{CachySequance}\left(\mathcal{H}(X)\right) \, . \, A = [K],$

$C := \left\{x : \texttt{CauchySequence}(X) | \forall n, m \in \mathbb{N} \, . \, x_n \in K_n \, \& \, d(x_n, x_m) \leq d(K_n, K_m)\right\} :? \texttt{Cauchy}(X),$

$\psi(A) := \mathrm{cl}\{\lim_{n \to \infty} x_n | x \in C\} : \texttt{Closed}(\widehat{X}),$

$[A.*] := \texttt{CauchyDiamConverge} : \texttt{Bounded}(\widehat{X}.\psi(A));$

$\rightsquigarrow \psi := I(\to) : \widehat{\mathcal{H}(X)} \to \mathcal{H}(\widehat{X}),$

Assume $A, B : \widehat{\mathcal{H}(X)}$,

$\left(\alpha, [4]\right) := {}_{J}\varphi(A) : \sum \alpha : \mathtt{Cauchy}\left(\mathcal{H}(X)\right) . A = [\alpha],$

$\left(\beta, [5]\right) := {}_{J}\varphi(B) : \sum \beta : \mathtt{Cauchy}\left(\mathcal{H}(X)\right) . B = [\beta],$

$C_A := \left\{ x : \mathtt{CauchySequence}(X) | \forall n, m \in \mathbb{N} . x_n \in \alpha_n \,\&\, d(x_n, x_m) \leq d(\alpha_n, \alpha_m) \right\} :?\mathtt{Cauchy}(X),$

$C_B := \left\{ x : \mathtt{CauchySequence}(X) | \forall n, m \in \mathbb{N} . x_n \in \beta_n \,\&\, d(x_n, x_m) \leq d(\beta_n, \beta_m) \right\} :?\mathtt{Cauchy}(X),$

$[(A, B).*] := \eth\mathtt{hausdorffMetric}\eth\mathtt{completionUniformCauchyExchange}[4][5]\eth^{-1}\mathtt{completion} ::$

$$d(\psi(A), \psi(B)) = \max\left(\sup_{a \in \psi(A)} \inf_{b \in \psi(B)} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right) =$$

$$= \max\left(\lim_{n \to \mathbb{N}} \lim_{m \to \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m), \lim_{n \to \mathbb{N}} \lim_{m \to \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m) \right) = \lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} d(\alpha_n, \beta_m) = d(A, B);$$

$\rightsquigarrow [5] := \eth^{-1}\mathtt{Isometry} : \mathtt{Isometry}\left(\widehat{\mathcal{H}(X)}, \mathcal{H}(\widehat{X}) \right),$

Assume $A : \mathcal{H}(\widehat{X})$,

$\left(\alpha, [6]\right) := {}_{J}\varphi(A) : \sum \alpha : \mathtt{Cauchy}\left(\mathcal{H}(X)\right) \varphi(A) = [\alpha] \,\&\, \lim_{n \to \infty} \mathrm{cl}_{\widehat{X}} \alpha_n = A,$

$[7] := [6]\mathtt{ContinuousLimitCommute}{}_{J}\psi : d(\psi\varphi A, A) = d(\psi \lim_{n \to \infty} \alpha_n, A) = \lim_{n \to \infty} d(\psi \alpha_n, A) = \lim_{n \to \infty} d(\mathrm{cl}_{\widehat{X}} \alpha_n, A) = d(A$

$[A.*] := \eth^{-1}\mathtt{MS}[7][8] : \psi\varphi A = A;$

$\rightsquigarrow [6] := I(\to, =) : \psi\varphi = \mathrm{id},$

$[*] := \mathtt{IsometricSurjectionIsBijection}(\psi)\mathtt{RightInverseIsBijective}[6] : \psi = \varphi^{-1};$

$\square$

$\mathtt{HausdorffCompleteIffComplete} :: \forall X \in \mathsf{MS} . \mathtt{Complete}(X) \iff \mathtt{Complete}\,\mathcal{H}(X)$

$\mathtt{Proof} =$

$\ldots$

$\square$

$\mathtt{CompactHausdorffConvergence} :: \forall X : \mathtt{Complete} . \forall K : \mathtt{CauchySequence}(\mathcal{H}(X)) .$

$. \forall[0] : \forall n \in \mathbb{N} . \mathtt{Compact}(K_n) . \lim_{n \to \infty}\left(\mathtt{Compact} \right)$

$\mathtt{Proof} =$

$A := \mathrm{cl}\bigcup_{n=1}^{\infty} K_n : \mathtt{Closed}(X),$

$[1] := \mathtt{ClosedSubsetOfComplete}(X, A) : \mathtt{Complete}(A),$

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [2]\right) := \eth\mathtt{CauchySequance}(\mathcal{H}(X), K)\left(\frac{\varepsilon}{8}\right) : \sum_{N \in \mathbb{N}} \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(K_n, K_m) < \frac{\varepsilon}{8},$

$\left(k, E, [3]\right) := \mathtt{CompactIsTotallyBounded}\left((K_n)_{n=1}^{N} \right)\eth\mathtt{TotallyBounded} :$

$: \prod_{n=1}^{N} \sum_{k \in \mathbb{N}} \sum_{E:k \to ?K_n} \forall i \in k_n . \mathrm{diam}\, E < \frac{\varepsilon}{2} \,\&\, \bigcup_{i=1}^{k_n} E_i = K_n,$

$F := \Lambda n \in N . \Lambda i \in k_n . \left\{ x \in A . d(x, E_i) \leq \frac{\varepsilon}{4} \right\} : \prod_{n=1}^{N} k_n \to ?A,$

$[\varepsilon.1.*] := {}_{J}F\eth^{-1}\mathrm{diam} : \forall n \in N . \forall i \in k_n . \mathrm{diam}\, F_{n,i} \leq \varepsilon,$

Assume $a : A$,

$\left(x, [5]\right) := \mathtt{MetricCLosure}_{J}A : \sum x \in \bigcup_{n=1}^{\infty} K_n \ . \ d(x, a) < \frac{\varepsilon}{8}$,

$\left(n, [6]\right) := \eth\mathtt{Union}(K, x) : \sum n \in \mathbb{N} \ . \ x \in K_n$,

$\left(m, [7]\right) := \mathtt{DirechletPrinciple}[2] : \sum m \in N \ . \ d(K_m, K_n) < \frac{\varepsilon}{8}$,

$\left(y, [8]\right) := \eth^{-1}\mathtt{distanceOfHausdorff}[7] : \sum y \in K_m \ . \ d(y, x) < \frac{\varepsilon}{8}$,

$[9] := \mathtt{TriangleIneq}[5][8] : d(a, y) \leq d(a, x) + d(x, y) < \frac{\varepsilon}{4}$,

$\left(i, [10]\right) := [3](y) : \sum i \in k_i \ . \ y \in E_i$,

$[a.*] := {}_{J}[10][9] : a \in F_{n,i}$;

$\rightsquigarrow [\varepsilon.2.*] := \eth^{-1}\mathtt{Union} : \bigcup_{n=1}^{N} \bigcup_{i=1}^{k_n} F_i = A$;

$\rightsquigarrow [2] := \eth^{-1}\mathtt{TotallyBounded} : \mathtt{TotallyBounded}(A)$,

$[3] := \mathtt{CompleteAndTotallyBoundedIsCompact}[2] : \mathtt{Comcpact}(A)$,

$[4] := \eth\mathcal{H}(X) : \lim_{n\to\infty} K_n \subset A$,

$[*] := \mathtt{ClosedSubsetCompact}[3, 4] : \mathtt{Compact}(\lim_{n\to\infty} K_n)$;

$\square$

$\mathtt{SingletonHausdorffLimit} :: \forall X \in \mathsf{MS} \ . \ \forall p : \mathtt{Converging}\left(\mathcal{X}\right) \ . \ \forall n \in \mathbb{N} \ . \ |p_n| = 1 \Rightarrow \left| \lim_{n\to\infty} x_n \right|$

$\mathtt{Proof} \ =$

$\ldots$

$\square$

$\mathtt{CompactHausdorffMetrIffCompact} :: \forall X \in \mathsf{MS} \ . \ \mathtt{Compact}(X) \iff \mathtt{Compact}(\mathcal{H}(X))$

$\mathtt{Proof} \ =$

$\mathtt{Assume} \ [0] : \mathtt{Compact(X)}$,

$[1] := \mathtt{HausdorffMetrizationAndCompletionCommute}(X) : \mathtt{Complete}\left(\mathcal{X}\right)$,

$\mathtt{Assume} \ \varepsilon : \mathbb{R}_{++}$,

$\left(n, A, [2]\right) := \mathtt{CompactTotallyBounded}(X)\eth\mathtt{TotallyBounded} : \sum_{n=1}^{\infty} \sum_{A:n\to?X} \bigcup_{i=1}^{n} A_i = X \ \& \ \forall i \in n \ . \ \mathrm{diam}\, A \leq \varepsilon$,

$B := \Lambda b \in 2^n \ . \ \{K \in \mathcal{H}(X) : \forall i \in \ . \ b_i = 1 \iff A_i \cap K \neq \emptyset\} : 2^n \to ?\mathcal{H}(X)$,

$\mathtt{Assume} \ b : 2^n$,

$\mathtt{Assume} \ K, K' : B_b$,

$[b.*] := {}_{J}B\eth\mathtt{distanceOfHaussdorff}[2] : d(K, K') \leq \max_{i \in b} d(K \cap A_i, K \cap A_i) \leq \varepsilon$;

$\rightsquigarrow [\varepsilon.*.1] := I(\forall) : \forall b \in 2^n \ . \ \mathrm{diam}\, B_b \leq \varepsilon$,

$[\varepsilon.*.2] := {}_{J}B[2] : \bigcup_{b\in 2^n} = \mathcal{H}(X)$;

$\rightsquigarrow [2] := \eth^{-1}\mathtt{TotallyBounded} : \mathtt{TotallyBounded}\left(\mathcal{H}(X)\right)$,

$[0.*] := \mathtt{CompactIfCompleteAndTotallyBounded}(\mathcal{H}(X)) : \mathtt{Compact}(\mathcal{H}(X))$;

$\rightsquigarrow LR := I(\Rightarrow) : \mathtt{Compact}(X) \Rightarrow \mathtt{Compact}(\mathcal{H}(X))$,

Assume $R : \texttt{Compact}(\mathcal{H}(X))$,

Assume $x : \mathbb{N} \to X$,

$[P',] := \texttt{SequanceCompactIffCompact}(\mathcal{H}(X))\breve{\partial}\texttt{SequanceCompact} : \sum P' : \texttt{Subsequence}\{x\} \, . \, P' : \texttt{Converging}$

$P := \lim\limits_{n \to \infty} p' : \texttt{In}\mathcal{H}(X)$,

$(p, [1]) := \texttt{SingletonHausdorffLimit}(P) : \sum p \in X \, . \, P = \{p\}$,

$(x', [x.*]]) := [1]\jmath p \breve{\partial}\texttt{distanceOfHaussdorff} : \sum x' : \texttt{Subsequence}(X) \, . \, p = \lim\limits_{n \to \infty} x_n;$

$\leadsto [R.*] := \texttt{SequanceCompactIffCompact}(X) : \texttt{Compact}(X);$

$\leadsto [R.*] := I(\iff) : \texttt{Compact}(X) \iff \texttt{Compact}(\mathcal{H}(X));$

$\square$

1.8 Geodesic Paths and Hopf-Rinow Theorem

$\texttt{RectifiablePath} :: \prod X \in \mathsf{MS} . \prod x, y \in X . ?\Omega(x, y)$

$\gamma : \texttt{RectifiablePath} \iff \gamma \in R(x, y) \iff$

$$\iff \texttt{ConvergingNet}\left(\mathbb{R}_+, M(\gamma), \Lambda(n, \omega) \in M(\gamma) . \sum_{i=1}^{n} d_X\big(\omega_i(0), \omega_i(1)\big)\right)$$

$\texttt{LengthSpace} :: ?\mathsf{MS}$

$X : \texttt{LengthSpace} \iff \forall x, y \in X . R(x, y) \neq \emptyset \iff$

$\texttt{arclength} :: \prod X \in \mathsf{MS} . \prod x, y \in X . R(x, y) \to \mathbb{R}_+$

$$\texttt{arclength}(\gamma) = |\gamma| := \lim_{(n, \omega) \in M(\gamma)} \sum_{i=1}^{n} d_X\big(\omega_i(0), \omega_i(1)\big)$$

$\texttt{ArclengthSum} :: \forall X \in \mathsf{MS} . \forall x, y, z \in X . \forall \alpha \in R(x, y) . \forall \beta \in R(y, z) . |\alpha\beta| = |\alpha| + |\beta|)$

$\texttt{Proof} =$

...

□

$\texttt{ArclengthInversion} :: \forall X \in \mathsf{MS} . \forall x, y \in X . \forall \gamma \in R(x, y) . \left|\gamma^\frown\right| = \gamma$

$\texttt{Proof} =$

...

□

$\texttt{ConstantArclength} :: \forall X \in \mathsf{MS} . \forall x \in X . \left|\texttt{constant}\big([0, 1], X, x\big)\right| = 0$

$\texttt{Proof} =$

$\texttt{IntrinsicMetric} :: \prod X : \texttt{LengthSpace} . \texttt{Metric}(X)$

$\texttt{IntrinsicMetric}(x, y) = d_I(x, y) := \inf_{\gamma \in R(x, y)} |\gamma|$

$\texttt{Geodesic} :: \prod X \in \mathsf{MS} . ?R(X, Y)$

$\gamma : \texttt{Geodesic} \iff |\gamma| = \inf_{\gamma \in R(x, y)} |\gamma|$

$\texttt{GeodesicSpace} :: ?\texttt{LengthSpace}(X)$

$X : \texttt{GeodesicSpace} \iff \forall x, y \in X . \exists \gamma : \texttt{Geodesic}(x, y)$

```
GeodesicSpaceIsPathConnected :: ∀X : Geodesic . X : PathConnected
Proof =
...
□


HopfRinow :: ?MS
X : HopfRinow ⟺ ∀x, y ∈ X . ∀a, b ∈ ℝ₊ . a + b < d(x, y) ⇒ d(𝔹(x, a), 𝔹(y, b)) = d(x, y) − a − b


HopfRinowLemma :: ∀X : Complete & HopRinow . ∀x ∈ X . ∀r ∈ ℝ₊₊ (∀r′ ∈ ℝ₊₊ . Compact(𝔻(x, r′), ε) ⇒ Com

Proof =
Assume y : X,
Assume [1] : d(x, y) = r,
Assume t : (0, r),
Assume s : (0, r − t),
(u, v[2]) := ðHopfRinov(X, x, y, t, s) : ∀t ∈ (0, r) . ∀s ∈ (0, r − t) . ∃u, v ∈ X : d(x, u) = t & d(y, v) = s & d(u, 
[y.3] := TriangleIneq : d(y, u) ≤ d(y, v) + d(u, v) = r − t;
⤳ [1] := ð⁻¹metricSpace : ∀t ∈ (0, r) . d(𝔻(x, t), 𝔻(x, r) ≤ r − t,
[2] := CompactHausdorffConvergence(X)[1] : Compact(X, 𝔻(x, r));
□


CompactBlowUp :: ∀X ∈ MS & LocallyComapct . ∀K : Compact(X) . ∃U ∈ 𝒰(K) : Compact(Ū)
Proof =
...
□


HopfRinowTHM1 :: ∀X : LocallyCompact & HopfRinow & Complete . ∀x ∈ X . ∀r ∈ ℝ₊₊ . Compact(𝔻(x, r))
Proof =
R := {r ∈ ℝ₊₊ | Compact(X, 𝔻(x, r))} :?ℝ₊₊,
[1] := ðLocallyCompact(X)ⱼR : R ≠ ∅,
Assume t : sup R,
[2] := HopfRinowLemma(X, x, t)ðSupremum(R)ⱼR : t ∈ R,
K := 𝔻(x, t) : Compact,
(U, [3]) := CompactBlowUpⱼR[2]ðSupremum(R) : ∑ U ∈ 𝒰(K) . Ū : Compact,
A := X \ U : Closed(X),
(s, [4]) := DishointClosedDistanceⱼA[3] : ∑ s ∈ ℝ₊₊ . d(K, A) > s,
[5] := [4]ⱼK[3] : 𝔻(x, t + s) ⊂ Ū,
[6] := CompactClosedSubset[5] : Compact(X, 𝔻(x, t + s)),
[4] := ðtⱼR[6] : ⊥;
⤳ [2] := CompactSubsetE(⊥) : R = ℝ₊₊;
□
```

CompleteByCompactDiscs $:: \forall X \in \mathsf{MS} . \forall[0] : \Big(\forall x \in X . \forall r \in \mathbb{R}_{++} . \forall \texttt{Compact} \Big(X, \mathbb{D}(x,r) \Big) . \Big) . \texttt{Complete}(X)$

Proof $=$

Assume $\mathcal{F} : \texttt{CauchyFilterBase}(X),$

Assume $F : \mathcal{F},$

Assume $[1] : \operatorname{diam} F < \infty,$

$\delta := \operatorname{diam} F : \mathbb{R}_{++},$

$\mathcal{G} := \{ G \in \mathcal{F} | G \subset \} : \texttt{CauchyFilterBase}(X),$

$x := \eth\texttt{NonEmpty}(F) : F,$

$[2] := [0](x, \delta) : \texttt{Compact} \Big(X, \mathbb{D}(x,r) \Big),$

$[3] := \texttt{CompactIsComplete}[2] : \texttt{Complete} \Big(X, \mathbb{D}(x,r) \Big),$

$[F.*] := \eth\texttt{Complete}[3](\mathcal{G}) : \texttt{ConvergingFilterBase} \Big(X, \mathcal{G} \Big);$

$\leadsto [\mathcal{F}.*] := \eth\texttt{CauchyFilterBase}\eth^{-1}\texttt{ConcvergingFilterBase} : \texttt{ConvergingFilterBase}(X, \mathcal{F});$

$\leadsto [*] := \eth^{-1}\texttt{Complete} : \texttt{Complete}(X);$

$\square$

MidpointTHM $:: \forall X : \texttt{Complete} \ \& \ \texttt{LocallyCompat} \ \& \ \texttt{HopfRinow} . \forall x, y \in X . \exists z \in X :$

$$d(x,z) = d(z,y) = \frac{1}{2} d(x,y)$$

Proof $=$

Assume $[1] : x \neq y,$

$\delta := d(x,y) : \mathbb{R}_{++},$

Assume $t : (0, \delta),$

$\Big(a, b, [2] \Big) := \eth\texttt{HopfRinow}\eth\texttt{HausdorffDistance} :$

$$\sum a : \mathbb{N} \to \mathbb{D} \left(x, \frac{\delta - t}{2} \right) . \sum b : \mathbb{N} \to \mathbb{D} \left(x, \frac{\delta - t}{2} \right) .$$

$$. \lim_{n \to \infty} d(a_n, b_n) = t \ \& \ \lim_{n \to \infty} d(a_n, x) = \frac{\delta - t}{2} \ \& \ \lim_{n \to \infty} d(b_n, y) = \frac{\delta - t}{2},$$

$[3] := \texttt{HopfRinowTHM1}(X)\texttt{CompactIsSequanceCompact} : \texttt{PartiallyConverging}(X, (a,b)),$

$\alpha_t := \texttt{partial} \lim_{n \to \infty} a_n : \mathbb{D} \left(x, \frac{\delta - t}{2} \right),$

$\beta_t := \texttt{partial} \lim_{n \to \infty} b_n : \mathbb{D} \left(y, \frac{\delta - t}{2} \right),$

$[t.*] := [2]\jmath(\alpha, \beta) : d(x, \alpha_t) = \frac{\delta - t}{2} \ \& \ d(\alpha_t, \beta_t) = t \ \& \ d(y, \beta_t) = \frac{\delta - t}{2};$

$\leadsto (\alpha, \beta, [2]) := I \left(\prod \right) I \left(\sum \right) : \prod t \in (0, \delta) . \sum \alpha_t \in \mathbb{D} (x, \delta) . \sum \beta_t \in \mathbb{D} (y, \delta) . d(x, \alpha_t) = \frac{\delta - t}{2} \ \& \ d(\alpha_t, \beta$

$a := \Lambda n \in \mathbb{N} . \alpha_{\frac{1}{n}} : \mathbb{N} \to \mathbb{D} (x, \delta),$

$b := \Lambda n \in \mathbb{N} . \beta_{\frac{1}{n}} : \mathbb{N} \to \mathbb{D} (y, \delta),$

$[3] := \texttt{HopfRinowTHM1}(X)\texttt{CompactIsSequanceCompact} : \texttt{PartialyConverging}(X, (a,b)),$

$\alpha_t := \texttt{partial} \lim_{n \to \infty} a_n : X,$

$\beta_t := \texttt{partial} \lim_{n \to \infty} b_n : X,$

$[4] := [2]\jmath(\alpha, \beta) : d(x, \alpha) = \frac{\delta}{2} \ \& \ d(\alpha, \beta) = 0 \ \& \ d(y, \beta) = \frac{\delta}{2},$

$[*] := \eth^{-1}\mathsf{MS}[4] : \alpha = \beta;$

$\square$

GeneralizedMidpointTHM :: $\forall X$: Complete & LocallyCompat & HopfRinow .

 . $\forall x, y \in X$. $\exists \xi : \mathbb{Q}_2 \cap [0, 1] \to X$:

 : $\forall a, b \in \mathbb{Q}_2 \cap [0, 1]$. $d\Big(\xi(a), \xi(b)\Big) = |a - b| d(x, y)$ & $\xi(0) = x$ & $\xi(1) = y$

Proof $=$

. . .

□

HopfRinowTHM2 :: $\forall X$: Complete & LocallyCompat & HopfRinow .

 . $\forall x, y \in X$. $\exists \gamma : \Omega\Big(x, y\Big)$:

 : $\forall a, b \in [0, 1]$. $d\Big(\gamma(a), \gamma(b)\Big) = |a - b| d(x, y)$

Proof $=$

$\delta := d(x, y) : \mathbb{R}_{++}$,

$\Big(\xi, [1]\Big) :=$ GeneralizedMidpointTHM$(X, x, y) : \sum \xi : \mathbb{Q}_2 \cap [0, 1] \to X$. $\forall a, b \in \mathbb{Q}_2 \cap [0, 1]$.

 . $d\Big(\xi(a), \xi(b)\Big) = |a - b| d(x, y)$ & $\xi(0) = x$ & $\xi(1) = y$,

$[2] :=$ LipschitzIsUC : UniformltContinuous$(\mathbb{Q}_2, X, \xi)$,

$\Big(\gamma, [3]\Big) :=$ UniformlyContinuousExtension$[1, 2] : \sum \gamma : \Omega\Big(x, y\Big)$:

 : $\forall a, b \in [0, 1]$. $d\Big(\gamma(a), \gamma(b)\Big) = |a - b| d(x, y)$;

□

HopfRinowGeodesic :: $\forall X$: Complete & LocallyCompat & HopfRinow . X : GeodesicSpace

Proof $=$

. . .

□

1.9 Lipschitz Connected Spaces

$\texttt{LipschitzConnected} :: \mathbb{R}_{++} \to ?\text{MS}$

$X : \texttt{LipschitzConnected} \iff \Lambda C \in \mathbb{R}_{++} . \text{ 1-Lip}(X) \iff$

$\qquad \iff \forall x, y \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists n \in \mathbb{N} : \exists q : n \to X .$

$\qquad . \ q_1 = x \ \& \ q_n = y \ \& \ \& \ \forall i \in (n-1) . \ d(q_i, q_j) < \varepsilon \ \& \ \sum_{i=1}^{n} \leq Cd(x, y)$

$\texttt{OneLipschitzConnectedIsGeodesic} :: \forall X : \text{1-Lip}(X) . \texttt{GeodesicSpace}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{GeodesicIsOneLipschitz} :: \forall X : \texttt{GeodesicSpace}(X) . \text{1-Lip}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

2 Metrization

2.1 Metrization Theorems

$\texttt{Metrizable} :: ?\mathsf{TOP}$

$X : \texttt{Metrizable} \iff \exists d : \texttt{Metric}(X) . (X, d) \cong_{\mathsf{TOP}} X$

$\texttt{UrysohnMetrization} :: \forall X : \texttt{SecondCountabe} \ \& \ \texttt{T3} . \texttt{Metrizable}(X)$

$\texttt{Proof} =$

$\mathcal{B} := \eth\texttt{SecondCountable}(X) : \texttt{Countable} \ \& \ \texttt{Base}(X),$

$\texttt{Assume } V, W : \mathcal{B},$

$\texttt{Assume } [1] : \overline{V} \subset W,$

$\left(f_{V,W}, [2]\right) := \texttt{UrysohnLemma}\left(\overline{V}, W^{\complement}, [1]\right) : \sum f_{V,W} : C\left(X, [0,1]\right) . W^{\complement} = f^{-1}\{0\} \ \& \ \overline{V} = f^{-1}\{1\};$

$\rightsquigarrow \left(f, [1]\right) := I\left(\prod\right) : \prod_{V,W \in \mathcal{B} : \overline{V} \subset W} \ \sum_{f : X \xrightarrow{\ \mathsf{TOP}\ } [0,1]} W^{\complement} = f^{=1}\{0\} \ \& \ \overline{V} = f^{-1}\{1\},$

$\Phi := \operatorname{Im} f :? C\left(X, [0,1]\right),$

$[2] := {}_{J}\Phi\eth\mathcal{B} : |\Phi| \leq \aleph_0,$

$\phi := \texttt{enumerate}(\Phi) : \mathbb{N} \leftrightarrow \Phi,$

$d := \Lambda x, y \in X . \sum_{n=1}^{\infty} \frac{|\phi_n(x) - \phi_n(y)|}{n!} : \texttt{Semimetric}(X),$

$[3] := {}_{J}d[1]\eth\texttt{T1}(X) : \texttt{Metric}(X, d),$

$[4] := \texttt{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$[*] := \eth\texttt{T3} : \mathcal{T}(X) = \mathcal{T}(X, d);$

$\square$

$\texttt{SeparableMetrization} :: \forall X : \texttt{Separable} . \texttt{Metrizable}(X) \iff \texttt{SecondCountable} \ \& \ \texttt{T3}(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

ContinuousByLocallyFinite $:: \forall X, Y \in \mathsf{TOP} . \forall f : X \to Y . \forall \mathcal{A} : \texttt{LocallyFinite} \,\&\, \texttt{Cover}(X) .$

$\quad . \left(\forall A \in \mathcal{A} . f_{|\overline{A}} \in C(\overline{A}, X) \right) \Rightarrow f \in C(X, Y)$

Proof $=$

Assume $x : X,$

$\mathcal{A}' := \left\{ A \in \mathcal{A} \,|\, x \in A \right\} :?\mathcal{A},$

$[2] := {}_{\jmath}\mathcal{A}'\eth\texttt{LocallyFinite}(X, \mathcal{A}) : |\mathcal{A}'| < \infty,$

$\left(U, [3] \right) := \eth\texttt{LocallyFinite}(X) : \sum U \in \mathcal{U}(x). \forall A \in \mathcal{A} . A \cap U \neq \emptyset \Rightarrow A \in \mathcal{A}',$

$[4] := \eth^{-1}\texttt{Cover}(X)(\mathcal{A}') : \mathcal{A}' \neq \emptyset,$

Assume $V : \mathcal{U}\left(f(x) \right),$

Assume $A : \mathcal{A}',$

$W := f_{|A}^{-1}(V) :?A,$

$[5] := \eth\mathsf{TOP} : W_A \in \mathcal{T}(A),$

$\left(O_A, [A.*] \right) := \eth\texttt{subsetTopology}[5] : \sum O_A \in \mathcal{U}_X(x) . W = O_A \cap A;$

$\rightsquigarrow \left(O, [5] \right) := I \left(\sum \right) I (\to) : \sum O : \mathcal{A} \to \mathcal{U}_X(x) . \forall A \in \mathcal{A}' . O_A \cap A = f_{|A}^{-1}(V),$

$\theta := \bigcap_{A \in \mathcal{A}'} O_A \cap U : \mathcal{U}(x),$

$[*.1] := {}_{\jmath}\theta[3][5] : f(\theta) \subset U;$

$\rightsquigarrow [*] := \texttt{ContinuityLocalProof}(X, Y, f) : f \in C(X, Y);$

$\square$

CountableSupport $:: \prod X \in \mathsf{SET} . ?(X \to \mathbb{R})$

$f : \texttt{CountableSupport} \iff \left| \left\{ x \in X : f(x) \neq 0 \right\} \right| \leq \aleph_0$

CountableSupport $:: \prod X \in \mathsf{SET} . ?\texttt{CountableSupport}(X)$

$f : \texttt{CountableSupport} \iff f \in L_1(X, \#) \iff \sum_{x \in x} |f(x)| < \infty$

ContinuousIndicators $:: \prod X \in \mathsf{TOP} . \prod S :??X . ?\left(S \to C(X) \right)$

$f : \texttt{ContinuousIndicators} \iff \forall A \in S . f_A(A^{\complement}) = \{0\}$

NagataSmirnovFunc $:: \prod X \in \mathsf{TOP} . \prod S :??X . \texttt{ContinuousIndicators}(X) \to S \to X \to \mathbb{R}$

NagataSmirnovFunc $(f, A) = \mathrm{NS}_{f,x}(A) := f_A(x)$

$\texttt{SemimetricBaseTheorem} :: \forall X \in \mathsf{SMS} . \exists \sigma\text{-}\texttt{LocallyFinite} \,\&\, \texttt{Base}(X)$

$\texttt{Proof} =$

$(\leq) := \texttt{WellOrderingTHM} : \texttt{WellOrder}(X),$

$$U := \Lambda x \in X . \Lambda k \in \mathbb{N} . \Lambda n \in \texttt{after}(k) . \mathbb{B}\left(x, \frac{1}{k} - \frac{1}{n}\right) \setminus \overline{\bigcap_{y<x} \mathbb{D}\left(y, \frac{1}{k} - \frac{1}{n+1}\right)} :$$

$$: X \to \prod_{k=1}^{\infty}\left(\texttt{after}(k) \to \mathcal{T}(X)\right),$$

$$\mathcal{A} := \Lambda k \in \mathbb{N} . \left\{U(x,k,n) | x \in X, n \in \texttt{after}(k)\right\} : \mathbb{N} \to ?\mathcal{T}(X),$$

$\texttt{Assume } k : \mathbb{N},$

$\texttt{Assume } z : X,$

$$R := \left\{x \in X : z \in \mathbb{B}\left(x, \frac{1}{k}\right)\right\} : ?X,$$

$[1] := \eth\texttt{cell}\jmath R : z \in R,$

$x := \min R : R,$

$$\left(n, [2]\right) := \jmath R \jmath x : \sum_{n=k+1}^{\infty} \frac{1}{n} \leq \frac{1}{k} - d(x,z),$$

$[k.*] := \jmath x \jmath R \jmath U[2] : z \in U(z,k,n);$

$$\rightsquigarrow [1] := I(1) : \forall k \in \mathbb{N} . \texttt{Cover}\left(X, \mathcal{A}_k\right),$$

$$[2] := \eth^{-1}\texttt{Base}\jmath\mathcal{A} : \texttt{Base}\left(X, \bigcup_{1=k}^{\infty} \mathcal{A}_k\right),$$

$\texttt{Assume } k : \mathbb{N},$

$\texttt{Assume } n : \texttt{after}(k),$

$\texttt{Assume } x,y : X,$

$\texttt{Assume } [3] : x \neq y,$

$$[*] := \jmath U : d\left(U(x,k,n), U(z,k,n)\right) \geq \frac{1}{n(n+1)};$$

$$\rightsquigarrow [3] := I(\forall) : \forall x \in X . \forall k \in \mathbb{N} . \forall n : \texttt{after}(k) . d\left(U(x,k,n), U(z,k,n)\right) \geq \frac{1}{n(n+1)},$$

$[4] := \eth^{-1}\texttt{LocallyFinite}[3] : \forall k \in \mathbb{N} . \texttt{LocallyFinite}(X, \|),$

$$[*] := \eth^{-1}\sigma\text{-}\texttt{LocallyFinite}(X)[4] : \sigma\text{-}\texttt{LocallyFinite}(X)\left(X, \bigcap_{k=1}^{\infty} \mathcal{A}_k\right);$$

$\square$

$\texttt{NagataSmirnovLemma} :: \forall X \in \mathsf{TOP} . \forall S : \texttt{LocallyFinite}(X) . \forall f : \texttt{ContinuousIndicators}(X, S) .$

$\quad . \mathrm{NS}_f \in C\Big(X, L_1(X, \#)\Big)$

$\texttt{Proof} =$

$\texttt{Assume } (D, x) : \texttt{ConvergingNet}(X),$

$L := \lim\limits_{n \to \infty} x_n : X,$

$\Big(U, [1]\Big) := \eth\texttt{LocallyFinite}(X, S, L) : \sum U \in \mathcal{U}(L) . \Big|\{s \in S : s \cap U \neq \emptyset\}\Big| < \infty,$

$\Big(\delta, [2]\Big) := \eth\texttt{ConvergingNet}(X, D, x, U) : \sum \delta' \in D . \forall \delta' \in D . \delta'' \geq \delta \Rightarrow x_{\delta'} \in U,$

$S' := \{s \in S : s \cap U \neq \emptyset\} : \texttt{Finite}(S),$

$[3] := \eth L_1 [1][2] \eth \texttt{ContinuousIndicators}(X, S, f) \texttt{ContinuousSum}(X, S', \ldots)$

$\quad \texttt{ContinuousConvergence}(X, f_s, D, x) :$

$\quad\quad : \lim\limits_{n \in D} \Big\|\mathrm{NS}_{f, x_n} - \mathrm{NS}_{f, L}\Big\|_1 = \lim\limits_{n \in D} \sum\limits_{s \in S} \Big| f_s(x_n) - f_s(L) \Big| = \lim\limits_{n \in D} \sum\limits_{s \in S'} \Big| f_s(x_n) - f_s(L) \Big| =$

$\quad\quad = \sum\limits_{s \in S'} \lim\limits_{n \in D} \Big| f_s(x_n) - f_s(L) \Big| = 0,$

$* \ldots] := \texttt{NormConvergence} : \lim\limits_{n \in D} \mathrm{NS}_{f, x_n} = \mathrm{NS}_{f, L};$

$\rightsquigarrow [*] := \texttt{ContinuousConvergence} : \mathrm{NS}_f \in C\Big(X, L_1(X, \#)\Big);$

$\square$

$\texttt{SmirnoffNagataMetrizationTHM} :: \forall X : \mathsf{T4} . \forall \mathcal{B} : \texttt{Base} \, \& \, \sigma\text{-}\texttt{LocallyFinite}(X) . X : \texttt{Metrizable}(X)$

$\texttt{Proof} =$

$\Big(\mathcal{B}', [1]\Big) := \eth \sigma\text{-}\texttt{LocallyCompact}(X) : \sum \mathcal{B}' : \mathbb{N} \to \texttt{LocallyFinite}(X) . \mathcal{B} = \bigcup\limits_{n=1}^{\infty} \mathcal{B}',$

$\texttt{Assume } m, n : \mathbb{N},$

$\texttt{Assume } V : \mathcal{B}_n,$

$A := \bigcup \{W \in \mathcal{B}_m : \overline{W} \subset V\} : \mathcal{T}(X),$

$[2] := \texttt{LocallyFiniteClosure} : \overline{A} \subset V,$

$\Big(f_V, [3]\Big) := \texttt{UrysohnLemma}(X, \overline{A}, V^\complement) : \sum f \in C(X) . f^{-1}\{1\} = \overline{A} \, \& \, f^{-1}\{0\} = V^\complement;$

$\rightsquigarrow \Big(f_{n,m}, [4]\Big) := I\Big(\sum\Big) : \sum f_{n,m} : \mathcal{B}_n \to C(X) . \forall V \in \mathcal{B}_n . f_{n,m,V}^{-1}\{1\} = \overline{A} \, \& \, f_{n,m,V}^{-1}\{0\} = V^\complement,$

$[2] := \eth^{-1} \texttt{ContinuousIndicators} : \texttt{ContinuousIndicators}(X, \mathcal{B}_n, f_{n,m}),$

$h_{n,m} := \mathrm{NS}_{f_{n,m}} : X \xrightarrow{\ \mathsf{TOP}\ } L_1(\mathcal{B}_n, \#);$

$\rightsquigarrow h := I(\to) : \mathbb{N} \times \mathbb{N} \to X \xrightarrow{\ \mathsf{TOP}\ } L_1(\mathcal{B}_n, \#),$

$H := \mathrm{Im}\, h \, :? C\Big(X, L_1(\mathcal{B}_n, \#)\Big),$

$[2] := \texttt{ImageCardinality} : |H| \leq \aleph_0,$

$f := \texttt{enumerate}(H) : \mathbb{N} \to C\Big(X, L_1(\mathcal{B}_n, \#)\Big) ;$

$[4] := \texttt{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$d := \Lambda x, y \in X . \sum\limits_{n=1}^{\infty} \dfrac{\|f_n(x) - f_n(y)\|_1}{n!} : \texttt{Metric}(X),$

$\ldots$

$\square$

2.2 Metrization Preserving Operations

MetrizationIsPreservedBySupTopology ::

$$: \forall X \in \mathsf{SET} \, . \, \forall \tau : \mathbb{N} \to \mathtt{Metrizable}(X) \, . \, \mathtt{Metrizable}\left(X, \bigvee_{n \in \mathbb{N}} \tau_n\right)$$

Proof $=$

Let ρ_n be a metrizing distance metric for topology τ_n.

Then the metric $d = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n}{1 + \rho_n}$ metrizes $\bigvee_{n \in \mathbb{N}} \tau_n$ by considerations of convergence.

$\square$

3 Metric Dimension

3.1 Covering Dimension

$$\texttt{CoveringDimensionLE} :: \prod_{X \in \mathsf{MS}} ?\Big(\texttt{Compact}(X) \times \mathbb{Z}_+\Big)$$

$$(A, n) : \texttt{CoveringDimensionLE} \iff \dim A \leq n \iff \forall r \in \mathbb{R}_{++} \, . \, \exists k \in \mathbb{N} : \exists x : k \to X :$$

$$: A \subset \bigcup_{i=1}^{k} \mathbb{B}(x_i, r) \; \& \; \forall a \in A \, . \, \Big|\{i \in K : a \in \mathbb{B}(x_i, r)\}\Big| \leq n + 1$$

$$\texttt{CoveringDimensionGreater} :: \prod_{X \in \mathsf{MS}} ?\Big(\texttt{Compact}(X) \times \mathbb{Z}_+\Big)$$

$$(A, n) : \texttt{CoveringDimensionGreater} \iff \dim A > n \iff !(\dim A \leq n)$$

$$\texttt{FiniteDimensionalCompact} :: \prod_{X \in \mathsf{MS}} ?\texttt{Compact}(X)$$

$$K : \texttt{FiniteDimensionalCompact} \iff \exists n \in \mathbb{Z}_+ \, . \, \dim X \leq n$$

$$\texttt{coveringDimension} :: \prod_{X \in \mathsf{MS}} \texttt{FiniteDimensionalCompact} \to \mathbb{Z}_+$$

$$\texttt{coveringDimension}\,(K) = \dim K := \min\{n \in \mathbb{Z}_+ : \dim X \leq n\}$$

$\texttt{IntervalDimension} :: \dim[0,1] = 1$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \dim[0,1] \le 1,$

$\left(k, x, [2], [3]\right) := \eth\texttt{CoveringDimensionLE}[1]\left(\dfrac{1}{3}\right) : \sum k \in \mathbb{N} . \sum x : k \to [0,1] .$

$\qquad . \ [0,1] \subset \displaystyle\bigcup_{i=1}^{k} \mathbb{B}\left(x_i, \dfrac{1}{3}\right)\ \&\ \forall a \in [0,1] . \left|\{i \in k : a \in \mathbb{B}(x_i, r)\}\right| \le 1,$

$[4] := \eth^{-1}\texttt{DisjointUnion} : [0,1] = \displaystyle\bigsqcup_{i=1}^{k} \mathbb{B}\left(x_i, \dfrac{1}{3}\right),$

$\texttt{Assume}\ [5] : k = 1,$

$[6] := \eth\,\mathrm{diam}[0,1]\texttt{BallDiam}\left(x_i, \dfrac{1}{3}\right) : 1 = \mathrm{diam}[0,1] = \mathrm{diam}\,\mathbb{B}\left(x_i, \dfrac{1}{3}\right) < \dfrac{2}{3},$

$[5.*] := \texttt{OneIsNotZero}[6] : \bot;$

$\rightsquigarrow [5] := E(\bot)\texttt{UnitIsMinimal}(\mathbb{N}) : k > 1,$

$[6] := \texttt{DisjointOpenUnionDisconnected} : \texttt{Disconnected}[0,1],$

$[1.*] := \texttt{IntervalIsConnected}[6] : \bot;$

$\rightsquigarrow [1] := E(\bot) : \dim[0,1] > 0,$

$\texttt{Assume}\ r : \mathbb{R}_{++},$

$[2] := \eth\texttt{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > 1\} \ne \emptyset,$

$k := \min\{N \in \mathbb{N} : Nr > 1\} : \mathbb{N},$

$x := \Lambda i \in k . (i-1)r : k \to [0,1],$

$\texttt{Assume}\ t : [0,1],$

$[2] := \eth\texttt{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > t\} \ne \emptyset,$

$i := \min\{N \in \mathbb{N} : Nr > t\} \ne \emptyset : \mathbb{N},$

$[3] := \jmath i \eth t : i \le k,$

$[t.*.1] := \jmath i \jmath x : t \in [(i-1)r, ir) \subset \mathbb{B}(x_i, r),$

$\texttt{Assume}\ j, l : k,$

$\texttt{Assume}\ [4] : t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r),$

$[5] := \jmath^{-1} x : r|l-j|\texttt{TriangleIneq}\left([0,1], x_j, x_l, t\right) \le |x_j - x_l| \le |x_l - t| + |t - x_i| < 2r,$

$[(j,l).*] := \dfrac{[5]}{r} : |l - j| \le 1;$

$\rightsquigarrow [4] := I(\forall) : \forall j, l \in k . t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r) \Rightarrow |l - j| \le 1,$

$[t.*.2] := \eth\mathbb{N}[2] : \left|\{t \in k : a \in \mathbb{B}(x_i, r)\}\right| \le 2;$

$\rightsquigarrow [2] := \eth^{-1}\texttt{CoveringDimensionLE} : \dim[0,1] \le 1,$

$[*] := \eth\mathbb{N}[1][2] : \dim[0,1] = 1;$

$\square$

3.2 Embedding Theorem

$\texttt{MengerMap} \ :: \ \prod X, Y \in \mathsf{MS} \ . \ \mathbb{R} \to? C(X, Y)$

$f : \texttt{MengerMap} \iff \Lambda \varepsilon \in \mathbb{R}_{++} \ . \ \forall a, b \in X \ . \ f(a) = f(b) \Rightarrow d(a, b) < \varepsilon$

$\texttt{MengerMapsIsOpen} \ :: \ \forall K : \mathsf{MS} \ \& \ \texttt{Compact} \ . \ \forall n \in \mathbb{N} \ . \ \forall \varepsilon \in \mathbb{R}_{++} \ . \ \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon) \in \mathcal{T}\Big(C(X, \mathbb{R}^n)\Big)$

$\texttt{Proof} \ =$

$[1] := \texttt{CompactProduct}(K, K) : \texttt{Compact}(K \times K),$

$\text{Assume } f : \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon),$

$A := \Big(d\Big)^{-1}[\varepsilon, +\infty) : \texttt{Closed}(K \times K),$

$[2] := \texttt{ClosedCompactSubset} : \texttt{Compact}(K \times K, A),$

$\delta := \inf\limits_{(a,b) \in A} \|f(a) - f(b)\| : \mathbb{R}_{++},$

$\text{Assume } g : \mathbb{B}\left(f, \dfrac{\delta}{2}\right),$

$\text{Assume } x, y : K,$

$\text{Assume } [4] : g(x) = g(y),$

$[5] := \texttt{TriangleIneq}(f(x), g(x), g(y), g(y))[4]\eth g :$

$\quad : |f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| < \dfrac{\delta}{2} + \dfrac{\delta}{2} = \delta,$

$[(x, y).*] := \jmath \delta \jmath A[5] : d(x, y) < \varepsilon;$

$\leadsto [g.*] := \eth^{-1}\texttt{MengerMap} : \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon, g);$

$\leadsto [f.*] := I(\exists)\eth^{-1}\texttt{Subset} : \exists U \in \mathcal{U}(f) \ . \ U \subset \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon);$

$\leadsto [*] := \texttt{OpenByCover} : \texttt{Open} \ C(X, \mathbb{R}^n) \ \texttt{MengerMap}(X, \mathbb{R}^n, \varepsilon);$

$\square$

$\texttt{CoveringSequance} \ :: \ \prod K : \texttt{FiniteDimensionalCompact}(X) \ . \ \mathbb{R}_{++} \to? \sum\limits_{n=1}^{\infty} n \to X$

$x : \texttt{CoveringSequance} \iff \Lambda r \in \mathbb{R}_{++} \ . \ X \subset \bigcup\limits_{i=1}^{k} \mathbb{B}(x_i, r) \ \& \ \forall X \in A \ . \ \Big|\{i \in K : a \in \mathbb{B}(x_i, r)\}\Big| \leq n + 1$

$\texttt{PontryaginTwinSequence} \ :: \ \prod X : \texttt{Compact} \ \& \ \mathsf{MS} \ . \ \prod r \in \mathbb{R}_{++} \ . \ \prod (n, x) : \texttt{CoveringSequance}(X, r) \ .$

$\quad . \ (X \xrightarrow{\texttt{TOP}} \mathbb{R}^{2*\dim K + 1}) \to \mathbb{R} \to? \Big(n \to \mathbb{R}^{2\dim X + 1}\Big)$

$y : \texttt{PontryaginTwinSequence} \iff \Lambda f : (X \xrightarrow{\texttt{TOP}} \mathbb{R}^{2*\dim K + 1}) \ . \ \Lambda s \in \mathbb{R}_{++} \ . \ \forall i \in n \ . \ d\Big(f \ \mathbb{B}(x_i, r), y_i\Big) < s \ \&$

$\quad \& \ \forall V : \texttt{ProperAffineSubspace}(\mathbb{R}^{2\dim K}) \ . \ |\operatorname{Im} y \cap V| \leq \dim V + 2$

$\texttt{PontryaginTwinSequanceExists} :: \forall X : \texttt{FiniteDimensionalCompact} . \forall f \in C(X, \mathbb{R}^{1+2\dim X}) . \forall r, s \in \mathbb{R}_{++} .$
$\quad . \forall (n, x) : \texttt{CoveringSequence}(X, r) . \exists \texttt{PontryaginTwinSequence}(X, f, x, s)$

$\texttt{Proof} =$

$\left(y^0, [1] \right) := \texttt{RealOpenIsInfinite}(\ldots) : \sum y^0 : n \hookrightarrow X . \forall i \in \|y_i^0 - f(x_i)\| < \dfrac{s}{1 + 2^n},$

$\mathcal{A}^0 := \left\{ A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : \||A \cap \Im y^0| > \dim A + 1 \right\} :$

$\quad : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}),$

$w := \Lambda \mathcal{A} : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) . \displaystyle\sum_{A \in \mathcal{A}} |A \cap \Im y^0| - \dim A - 1 :$

$\quad : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \to \mathbb{Z}_+,$

$M := w(\mathcal{A}^0) : \mathbb{Z}_+,$

$[0] := \jmath M \jmath \mathcal{A}_0 : M \leq 2^n,$

$\texttt{Assume } m : (M - 1)_+,$

$\texttt{Assume } [2] : \mathcal{A}^m \neq \emptyset,$

$(k, A) := \texttt{enumerate}(\mathcal{A}^n) : \displaystyle\sum k \in \mathbb{N} . A : k \leftrightarrow \mathcal{A}^m,$

$i := \max\{i \in n : x_i^m \in A_1\} : n,$

$\delta := \min \left\{ d(y_i^n, A) \Big| A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) \ \& \ \operatorname{Im} y^n \cap A \neq \emptyset \ \& \ y_i^n \notin A \right\} : \mathbb{R}_{++},$

$\left(u, [3] \right) := \texttt{RealOpenIsInfinite}(\ldots) : \displaystyle\sum u \in \mathbb{R}^{1+2\dim X} . d(y_i^n, u) = \min \left(\delta, \dfrac{s}{1 + 2^n} \right),$

$y^{m+1} := \Lambda j \in n . \texttt{ if } i == j \texttt{ then } u \texttt{ else } y_i^n : n \to \mathbb{R}^{1+2\dim X},$

$\mathcal{A}^{m+1} := \left\{ A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : \||A \cap \Im y^{m+1}| > \dim A + 1 \right\} :$

$\quad : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}),$

$[2.*] := \jmath W \jmath \delta \jmath u \jmath y^{m+1} \jmath \mathcal{A}^0 : w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m);$

$\leadsto [2] := I(\Rightarrow) : \mathcal{A}^m \neq \emptyset \Rightarrow w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m),$

$\texttt{Assume } [2] : \mathcal{A}^m = \emptyset,$

$y^{m+1} := y^m : n \to \mathbb{R}^{1+2*\dim X},$

$\mathcal{A}^{m+1} := \mathcal{A}^m : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X});$

$\leadsto y, \mathcal{A}, [2] := I\left(\displaystyle\prod \right) : \prod_{m=0}^{M} \sum y^m : n \to \mathbb{R}^{1+2\dim X} . \sum \mathcal{A}^m : \texttt{Finite ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) .$

$\quad . \mathcal{A}^m = \left\{ A : \texttt{ProperAffineSubspace}(\mathbb{R}^{1+2\dim X}) : \||A \cap \Im y^m| > \dim A + 1 \right\} \ \&$

$\quad \forall m \in (M - 1) . \mathcal{A}^m \neq \emptyset w(\mathcal{A}^m) < w(\mathcal{A}^{m+1}),$

$[3] := [2] \jmath w \texttt{InverseFiniteInduction} : \mathcal{A}^M = \emptyset,$

$[4] := \jmath y^M : \forall i \in n . d(f(x_i), y_i^M) \leq \|f(x_i) - y_i^1\| + \displaystyle\sum_{j=1}^{M} \|y_i^j - y_i^{j+1}\| < s,$

$[*] := \eth^{-1} \texttt{PontryaginTwinSequence} : \texttt{PontryaginTwinSequence}(X, f, x, s, y^M);$

$\square$

$\texttt{functionOfNebeling} :: \prod X : \texttt{FiniteDimensionalCompact} \ . \ \prod f \in C(X, \mathbb{R}^{1+2\dim X}) \ . \ \prod r, s \in \mathbb{R}_{++} \ .$

$\quad . \ \prod (n, x) : \texttt{CoveringSequance}(X, r) \ . \ \prod p : \texttt{PontryaginTwinSequence}(X, f, x, s) \ . \ X \xrightarrow{\ \text{TOP}\ } \mathbb{R}^{1+2\dim X}$

$\texttt{functionOfNebeling}\,(u) = N_p(u) := \dfrac{\sum_{i=1}^{n} d\Big(u, \mathbb{B}^{\complement}(x_i, r), rBig\Big)p_i}{\sum_{i=1}^{n} d\Big(u, \mathbb{B}^{\complement}(x_i), r\Big)}$

$\texttt{FunctionOfNebelingIsClose} :: \forall X : \texttt{FiniteDimensionalCompact} \ . \ \forall f \in C(X, \mathbb{R}^{1+2\dim X}) \ . \ \forall r, s \in \mathbb{R}_{++} \ .$

$\quad . \ \forall (n, x) : \texttt{CoveringSequance}(X, r) \ . \ \forall p : \texttt{PontryaginTwinSequence}(X, f, x, s) \ .$

$\quad [0] \ . \ \forall [0] : \forall i \in \ . \ \omega_f(x_i, 2r) < s \ . \ \|N_p - f\| < 2s$

$\texttt{Proof} \ =$

$\texttt{Assume}\ u : X,$

$\Big(n', k, [1]\Big) := \eth\texttt{CoveringSequance}(x) : \sum n' \in 1 + \dim X \ . \ \sum k : n' \to n \ . \ \forall i \in n \ .$

$\quad . \ u \in \mathbb{B}(x_i, r) \iff \exists j \in n' : i = k_i,$

$y := N_p(u) : \texttt{In}\ \mathbb{R}^{1 + 2*\dim X},$

$[2] := \jmath y \jmath N_p[1] : y \in \operatorname{conv} \operatorname{Im} p_k,$

$\Big(v, [3]\Big) := \eth\texttt{PontryaginTwinSequence}(p) : \prod_{i=1}^{n'} \mathbb{B}(x_{k_i}, r) \ . \ \forall i \in n' \ . \ d(f(v_i), p_{k_i}) < s,$

$y' := \dfrac{\sum_{i=1}^{n'} d\Big(u, \mathbb{B}^{\complement}(x_i, r), r\Big) f(v_i)}{\sum_{i=1}^{n} \sum_{i=1}^{n'} d\Big(u, \mathbb{B}^{\complement}(x_i), r\Big)} : \texttt{In}\ \mathbb{R}^{1 + 2\dim X},$

$[4] := [3][2]\jmath y \jmath N_p \jmath y' : \|y - y'\| < s,$

$[5] := [0]\jmath y' : \|f(u) - y'\| < s,$

$[u.*] := \texttt{TriangleIneq}[4][5] : d\Big(f(u), N_p(u)\Big) < 2s;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{uniformNorm} : \|f - N_P\| < 2s;$

$\square$

$\texttt{NebelingsAreMenger} :: \forall X : \texttt{FiniteDimensionalCompact} \, . \, \forall f \in C(X, \mathbb{R}^{1+2\dim X}) \, . \, \forall r, s \in \mathbb{R}_{++} \, .$
$\quad . \, \forall (n, x) : \texttt{CoveringSequance}(X, r) \, . \, \forall p : \texttt{PontryaginTwinSequence}(X, f, x, s) \, .$
$\quad [0] \, . \, N_p : \texttt{MengerMap}(X, \mathbb{R}^{2*dim(X)+1}, 2r)$

$\texttt{Proof} \; = $

$\texttt{Assume } u, v : X,$

$\texttt{Assume } [1] : N_p(u) = N_p(v),$

$y := N_p(u) : \texttt{In } \mathbb{R}^{1+2*\dim X},$

$\left(m, k, [1] \right) := \eth \texttt{CoveringSequance}(X, n, x, u) :$

$\quad \sum m \in 1 + \dim X \, . \, \sum k : m \hookrightarrow n \, . \, \forall i \in n \, . \, u \in \mathbb{B}(x_i, r) \iff \exists j \in m : i = k_i,$

$\left(m', k', [1'] \right) := \eth \texttt{CoveringSequance}(X, n, x, v) :$

$\quad \sum m' \in 1 + \dim X \, . \, \sum k' : m' \hookrightarrow n \, . \, \forall i \in n \, . \, v \in \mathbb{B}(x_i, r) \iff \exists j \in m' : i = k'_i,$

$[2] := {}_Jy_JN_p[1] : y \in \operatorname{conv} \operatorname{Im} p_k,$

$[2'] := {}_Jy_JN_p[1'] : y \in \operatorname{conv} \operatorname{Im} p_{k'},$

$\left(\alpha, [3], [4] \right) := \eth \operatorname{conv}[2] : \sum \alpha : m \to \mathbb{R} \, . \, \sum_{i=1}^{m} \alpha_i = 1 \; \& \; \sum_{i=1}^{m} \alpha_i p_{k_i} = y,$

$\left(\alpha', [3'], [4'] \right) := \eth \operatorname{conv}[2'] : \sum \alpha : m' \to \mathbb{R} \, . \, \sum_{i=1}^{m'} \alpha'_i = 1 \; \& \; \sum_{i=1}^{m'} \alpha'_i p_{k'_i} = y,$

$[5] := [4][4'] : \sum_{i=1}^{m} \alpha_i p_{k_i} = y = \sum_{i=1}^{m'} \alpha_i p_{k'_i},$

$\left(i, [6] \right) := [3]\texttt{ZeroSumIsZero} : \sum i \in m \, . \, \alpha_i \neq 0,$

$m'' := m - 1 : \mathbb{Z}_+,$

$k'' := \hat{k}_i : m'' \to n,$

$\alpha'' := \hat{\alpha}_i : m'' \to \mathbb{R},$

$[7] := [6][5] \ldots : p_{k_i} = \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} p_{k'_j} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} p_{k''_j},$

$[8] := [3][3'][6]{}_J\alpha''\eth\texttt{Field}(\mathbb{R}) : \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} = \frac{1}{\alpha_i} + \frac{\alpha_i - 1}{\alpha_i} = \frac{\alpha_i}{\alpha_i} = 1,$

$[9] := \eth^{-1}\operatorname{conv}[7][8] : p_{k_i} \in \operatorname{conv} p_{k'' \oplus k'},$

$[10] := \texttt{AffineCombinationDim}(p_{k'' \oplus k'})\eth m'\eth m{}_J m'' :$

$\quad : \dim \operatorname{conv} p_{k \oplus k'} \leq m' + m'' - 1 \leq 1 + \dim X + \dim X - 1 = 2\dim X,$

$\left(j, [11] \right) := \eth\texttt{PontryaginTwinSequence}(p)[9][10] : \sum j \in m' + m'' \, . \, (k' \otimes k'')_j = k_i,$

$[12] := [11]\eth j_J k'' \texttt{injective}(k) : k'_j = k_i,$

$l := k'_j : n,$

$[13] := {}_J[12][1][1'] : u, v \in \mathbb{B}(x_l, r),$

$\left[(u, v). * \right] := [13]\eth\texttt{cell}(X)\texttt{TriangleIneq}(X) : d(u, v) < 2r;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{MengerMap} : \texttt{MengerMap}(X, \mathbb{R}^{2*dim(X)+1}, 2r),$

$\square$

`AllMengerIsInjective` :: $\forall X, Y \in \mathsf{MS}$. $\forall f : X \xrightarrow{\mathsf{MS}} Y$. $\left(\forall q \in \mathbb{Q}_{++} \ . \ f : \mathtt{MengerMap}(X, Y, q) \right)$.

   . $\mathtt{Injective}(X, Y, f)$

`Proof` $=$

$\dots$

$\square$

`MengerIsDense` :: $\forall X : \mathtt{FinteDimensionalCompact}$. $\forall \varepsilon \in \mathbb{R}_{++}$.

   . $\mathtt{Dense}\left(C(X, \mathbb{R}^{1+2\dim X}) \ \mathtt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, \varepsilon) \right)$

`Proof` $=$

`Assume` $f : C(X, \mathbb{R}^n)$,

`Assume` $s : \mathbb{R}_{++}$,

$[1] := \mathtt{UCByCompact}(X, f) : f \in \mathrm{UC}\left(X, \mathbb{R}^{1+2\dim X} \right)$,

$\left(\delta, [2] \right) := \eth\mathrm{UC}\left(X, \mathbb{R}^{1+2\dim X} \right)(f)\left(\dfrac{s}{2} \right) : \sum \delta \in \mathbb{R}_{++}$. $\forall x \in X$. $\ \mathrm{diam} \ f\mathbb{B}(x, \delta) < \dfrac{s}{2}$,

$r := \min\left(\delta, \dfrac{\varepsilon}{2} \right) : \mathbb{R}_{++}$,

$(n, x) := \eth\mathtt{FiniteDimensionalCompact}(X)(r) : \mathtt{CoveringSequence}(X, r)$,

$[2] := \mathit{jr}\eth\mathtt{CoveringSequence}(X, r, n, x) : \forall i \in n$. $\omega_f(x_i, r) < \dfrac{s}{2}$,

$p := \mathtt{PontryaginDualSequanceExists}\left(X, f, x, \dfrac{s}{2} \right) : \mathtt{PontryaginDualSequence}\left(X, f, x, \dfrac{s}{2} \right)$,

$[f.*] := \mathtt{FunctionOfNebelingIsClose}(X, p, [2]) : d(N_p, f) < s$;

$\rightsquigarrow [*] := \eth^{-1}\mathtt{Dense} : \mathtt{Dense}\left(C(X, \mathbb{R}^{1+2\dim X}) \ \mathtt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, \varepsilon) \right)$;

$\dots$

$\square$

`MengerNebelingPontryaginEmbeddingTHM` :: $\forall X : \mathtt{FinteDimensionalCompact}$.

  $\exists\mathtt{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2\dim X})$

`Proof` $=$

$[1] := \mathtt{DualBaireProperty}\left(C(X, \mathbb{R}^{1+2\dim X}), \Lambda q \in \mathbb{Q}_{++} \ . \ \mathtt{MengerMap}(X, \mathbb{R}^{1+2\dim X}, q) \right)$

  $\mathtt{AllMengerIsInjective}(X, \mathbb{R}^{1+2\dim X}) : \mathtt{Dense}\left(C(X, \mathbb{R}^{1+2\dim X}), \mathtt{Injective}(X, \mathbb{R}^{1+2\dim X}) \right)$,

$[2] := \mathtt{ConstantIsContinuous}(X, \mathbb{R}^{1+2\dim X}, 0) : 0 \in C(X, \mathbb{R}^{1+2\dim X})$,

$[3] := \eth^{-1}\mathtt{NonEmpty} : C(X, \mathbb{R}^{1+2\dim X}) \neq \emptyset$,

$f := [3][1] : \mathtt{Injective}(X, \mathbb{R}^{1+2\dim X})$,

$[*] := \mathtt{CompactHausdorffIsClosed}\eth^{-1}\mathtt{HomeomorphicEmbedding} : \mathtt{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2\dim X}, f)$;

$\square$

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