Problem 4.9 :: $\forall R : \texttt{Commutative} \,.\, \forall f : \texttt{ZD}\ R[x] \,.\, \exists c \in R \,.\, fc = 0 \wedge c \neq 0$
Proof =

$\vdash R : \texttt{Commutative}$

$\quad \vdash f : \texttt{ZD}\ R[x] \rightarrow$

$\quad\quad \rightarrow \exists g \in R[x] \,.\, fg = 0 \wedge g \neq 0 \multimap (*)$

$\quad\quad G := \left\{ g \in R[x] | gf = 0 \bullet g \neq 0 \right\} \quad | \quad G : \texttt{Subset}\ R[x]$

$\quad\quad (*) \rightarrow G \neq \emptyset \rightarrow \exists g \in G \,.\, \forall h \in G \,.\ \deg g \leq \deg h \twoheadrightarrow g$

$\quad\quad n := \deg f \quad d := \deg g \quad | \quad n, d \in \mathbb{Z}_+$

$\quad\quad ☿ :: \mathbb{I}_{0,n} \rightarrow \mathbb{T}$

$\quad\quad\quad ☿(k) = \forall i \in \mathbb{I}_{0,k} \,.\, f_{n-i} g = 0$

$\quad\quad \vdash f_n g \neq 0$

$$\left.\begin{array}{l} (:= g) \rightarrow fg = 0 \rightarrow f_n g_d = 0 \rightarrow \deg f_n g < \deg g \\ (:= g) \rightarrow fg = 0 \rightarrow f f_n g = 0 \rightarrow f_n g \in G \end{array}\right\} \rightarrow \bot$$

$\quad\quad \bot \rightarrow ☿(0) \multimap (0)$

$\quad\quad \vdash k \in \mathbb{I}_{0,n-1}$

$\quad\quad\quad \vdash ☿(k) \multimap (☿)$

$$F := \sum_{i=0}^{n-k-1} f_i x^i \quad | \quad F \in R[x]$$

$\quad\quad\quad (☿) - (:= g) \rightarrow 0 = fg = Fg \rightarrow f_{n-k-1} g_d = 0 \rightarrow \deg f_{n-k-1} g < \deg f \rightarrow$

$$\left.\begin{array}{l} \rightarrow f_{n-k-1} g \notin G \\ (:= g) \rightarrow f f_{n-k-1} g = 0 \end{array}\right\} \rightarrow f_{n-k-1} g = 0 - (☿) \rightarrow ☿(k+1)$$

$\quad\quad \dashv\dashv : \forall k \in \mathbb{I}_{0,n-1} \,.\, ☿(k) \Rightarrow ☿(k+1) - (0) \rightarrow ☿(n) \multimap (☿)$

$\quad\quad \vdash i \in \mathbb{I}_{0,n}$

$\quad\quad\quad (☿) \rightarrow ☿(i) \rightarrow f_i g_d = 0$

$$\left.\begin{array}{l} \dashv : \forall i \in \mathbb{I}_{0,n} \,.\, f_i g_d = 0 \rightarrow f g_d = 0 \\ (:= d) \rightarrow g_d \neq 0 \end{array}\right\} \rightarrow \exists \,.\, c \in R \,.\, fc = 0 \wedge c \neq 0 \quad [g_d]$$

$\dashv\dashv : \forall R : \texttt{Commutative} \,.\, \forall f : \texttt{ZD}\ R[x] \,.\, \exists c \in R \,.\, fc = 0 \wedge c \neq 0 \quad \square$

Problem 4.10

$$d \in \{n \in \mathbb{Z} \mid \forall m \in \mathbb{Z} \,.\, n \neq m^2\}$$
$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$$

$(a) :: \mathbb{Q}(\sqrt{d}) : \texttt{Subring}\ \mathbb{C}$

Proof $=$

Obviously, $\mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$

Assume that $a + b\sqrt{d}, x + y\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

then $(a + b\sqrt{d}) + (x + y\sqrt{d}) = (a + x) + (b + y)\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

and $0 = 0 + 0\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

and $\exists - a - b\sqrt{d} \in \mathbb{Q}(\sqrt{d}) \,.\, - a - b\sqrt{d} = -(a + b\sqrt{d})$

So, $\mathbb{Q}(\sqrt{d}) : \texttt{Abelian}(+_{\mathbb{C}})$

Assume that $a + b\sqrt{d}, x + y\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

then $(a + b\sqrt{d})(x + y\sqrt{d}) = (ax + byd) + (ay + bx)\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

and $1 = 1 + 0\sqrt{d} \in \mathbb{Q}(\sqrt{d})$

So, $\mathbb{Q}(\sqrt{d}) : \texttt{Monoid}(\cdot_{\mathbb{C}})$

Hence, $\mathbb{Q}(\sqrt{d}) : \texttt{Subring}\ \mathbb{C} \quad \square$

`def` $N :: \mathbb{Q}(\sqrt{d}) \to \mathbb{Q}$

$N(a + b\sqrt{d}) = a^2 - b^2 d$

`def` $\texttt{Norm} :: \prod R : \texttt{Ring} \,.\, \prod M : R\text{–}\texttt{Module} \,.\, ?M \to R$

$N : \texttt{Norm} \Leftrightarrow \forall r \in R \,.\, \forall v \in M \,.\, N(rv) = N(r)N(v) \wedge (N(v) = 0 \Rightarrow v = 0)$

$$(b) :: N : \texttt{Norm}\ \mathbb{Q}(\sqrt{d})\ \mathbb{Q}(\sqrt{d})$$

Proof $= \vdash v, w \in \mathbb{Q}(\sqrt{d})$

$$(:= \mathbb{Q}(\sqrt{d})) \multimap v = a + b\sqrt{d}$$

$$(:= \mathbb{Q}(\sqrt{d})) \multimap w = x + y\sqrt{d}$$

$$N(vw) = N\big((ax+byd)+(ay+bx)\sqrt{d}\big) = (ax+byd)^2 - (ay+bx)^2 d =$$
$$= a^2x^2 + 2axbyd + b^2y^2d^2 - a^2y^2d - 2axbyd - x^2b^2d =$$
$$= a^2x^2 - a^2y^2d - x^2b^2d + b^2y^2d^2 = (a^2-b^2d)(x^2-y^2d) = N(v)N(w)$$

$$\left.\begin{array}{r} \vdash N(v) = 0 \to a^2 - b^2 d = 0 \\ \vdash v \neq 0 \to a \neq 0 \vee b \neq 0 \end{array}\right\} \to a \neq 0 \wedge b \neq 0 \to$$

$$\left.\begin{array}{l} \to d = \frac{a^2}{b^2} \\ d \in \mathbb{Z} \end{array}\right\} \to \sqrt{d} = \frac{a}{b} \in \mathbb{Z} \to \perp :$$

$$: v = 0 \dashv: N(v) = 0 \Rightarrow v = 0 \dashv:$$

$: N : \texttt{Norm}\ \mathbb{Q}(\sqrt{d})\ \mathbb{Q}(\sqrt{d})$ $\square$

$$(c) :: \mathbb{Q}(\sqrt{d}) : \texttt{Field} \wedge \forall K : \texttt{Subfield}(\mathbb{C})\ .\ \texttt{if}\ \mathbb{Q} \subset K \wedge \sqrt{d} \in K\ .\ \mathbb{Q}(\sqrt{d}) \subset K$$

Proof $= \vdash v \in \mathbb{Q}(\sqrt{d})\ \vdash v \neq 0 \to N(v) \neq 0$

$$(:= \mathbb{Q}(\sqrt{d})) \multimap v = a + b\sqrt{d}$$

$$(a+b\sqrt{d})(a-b\sqrt{d})/N(v) = (a^2 - b^2 d)/N(v) = N(v)/N(v) = 1$$

$$(a-b\sqrt{d})/N(v) \in \mathbb{Q}(\sqrt{d}) \to \exists w \in \mathbb{Q}(\sqrt{d})\ .\ vw = 1 \dashv\dashv:$$

$$\left.\begin{array}{l} : \mathbb{Q}(\sqrt{d}) : \texttt{Division} \\ \mathbb{C} : \texttt{Field} \to \mathbb{Q}(\sqrt{d}) : \texttt{Commutative} \end{array}\right\} \to \mathbb{Q}(\sqrt{d}) : \texttt{Field}$$

$\vdash\ K : \texttt{Subfield}(\mathbb{C})$

$\vdash\ \mathbb{Q} \subset K \wedge \sqrt{d} \in K$

$\vdash v \in \mathbb{Q}(\sqrt{d})$

$$(:= \mathbb{Q}(\sqrt{d})) \multimap v = a + b\sqrt{d} \wedge a, b \in \mathbb{Q} \to$$

$$\to v = a + b\sqrt{d} \in K \dashv: \mathbb{Q}(\sqrt{d})) \subset K \dashv\dashv: \quad \square$$

$$(b) :: \mathbb{Q}(\sqrt{d}) \cong \frac{\mathbb{Q}[x]}{(x^2 - d)}$$

Proof =

$\texttt{def} \quad \phi :: \mathbb{Q}[x] \to \mathbb{Q}(\sqrt{d})$

$$\phi\, p = \sum_{i=0}^{\deg p} p_i (\sqrt{d})^i$$

$\phi : \texttt{Homo}\ \mathbb{Q}[x]\ \mathbb{Q}(\sqrt{d})$

$\vdash\ a + b\sqrt{(}d) \in \mathbb{Q}(\sqrt{d})$

$\quad p := a + bx \to \phi p = a + b\sqrt{d} \to \exists p \in \mathbb{Q}[x]\ .\ \phi p = a + b\sqrt{d} \dashv:$

$\phi : \texttt{Surjictive} \multimap (0)$

$$\forall p \in \mathbb{Q}[x]\ .\ p \in \ker\phi \Leftrightarrow \sum_{i=0}^{\deg p} p_i(\sqrt{d})^i = 0 \Leftrightarrow p \in (x^2 - d) \to \quad (\sqrt{(}d) \notin \mathbb{Q}) \to$$

$$\to \ker\phi = (x^2 - d) -(0)- \texttt{RingIsoThm1} \to \mathbb{Q}(\sqrt{d}) \cong \frac{\mathbb{Q}[x]}{(x^2 - d)} \quad \square$$

Problem 4.11 :: $\forall R : \texttt{Commutative} \,.\, \forall n \in \mathbb{N} \,.\, \forall f : \mathbb{I}_n \to R[x] \,.\, \forall a \in R \,.$
$(a) :: \left(\mathbf{L}_{i=1}^n f_i \frown [x-a]\right) = \left(\mathbf{L}_{i=1}^n f_i(a) \frown [x-a]\right)$
$\forall i \in \mathbb{I}_n \,.$
by division with reminder $f_i = p(x-a) + r$ where $p \in R[x]$ and $r \in R$.
We set $g_i = p$ and note that $f_i(a) = g_i(a-a) + r = r$.
So we acquired $g : \mathbb{I}_n \to R[x]$ with mentioned properties.
$\forall p \in R[x].p \in \left(\mathbf{L}_{i=1}^n f_i \frown [x-a]\right) \Leftrightarrow p = \sum_{i=1}^n q_i f_i + (q_{n+1})(x-a) =$
$= \sum_{i=1}^n q_i(g_i(x-a) + f_i(a)) + (q_{n+1})(x-a) =$
$= \sum_{i=1}^n q_i f_i(a) + (\sum_{i=1}^n q_i g_i + q_{n+1})(x-a) =$
$= \sum_{i=1}^n q_i f_i(a) + (q'_{n+1})(x-a) \Leftrightarrow p \in \left(\mathbf{L}_{i=1}^n f_i(a) \frown [x-a]\right)$
Hence, $\left(\mathbf{L}_{i=1}^n f_i \frown [x-a]\right) = \left(\mathbf{L}_{i=1}^n f_i(a) \frown [x-a]\right) \quad \square$

$$(b) :: \frac{R[x]}{\left(\mathbf{L}_{i=1}^n f_i \frown [x-a]\right)} \cong \frac{R}{\left(\mathbf{L}_{i=1}^n f_i(a)\right)}$$

$$\frac{R[x]}{\left(\mathbf{L}_{i=1}^n f_i \frown [x-a]\right)} = \frac{R[x]}{\left(\mathbf{L}_{i=1}^n f_i(a) \frown [x-a]\right)} \cong \frac{R[x]/(x-a)}{(\mathbf{L}_{i=1}^n f_i(a))} \cong \frac{R}{\left(\mathbf{L}_{i=1}^n f_i(a)\right)} \square$$

Problem 4.12

$$\frac{R[x_1, \ldots, x_n]}{(x_1 - a_1, \ldots, x_n - a_n)} \cong \frac{R[x_1, \ldots, x_{n-1}][y]}{(x_1 - a_1, \ldots, x_{n-1} - a_{n-1}, y - a_n)} \cong$$

$$\cong \frac{R[x_1, \ldots, x_{n-1}][y]/(y - a_n)}{(x_1 - a_1, \ldots, x_{n-1} - a_{n-1})} \cong \frac{R[x_1, \cdots, x_{n-1}]}{(x_1 - a_1, \ldots, x_{n-1} - a_{n-1})} \cong \cdots \cong$$

$$\cong \frac{R[x]}{(x - a_1)} \cong R \quad \square$$

Problem 4.13 ::
$:: \forall R : \texttt{IntegralDomain} \;.\; \forall n \in \mathbb{N} \;.\; \forall k \in \mathbb{I}_n \;.\; (\mathbf{L}_{i=1}^{k} x_i) : \texttt{Prime}\ R[\mathbf{L}_{i=1}^{n} x_i]$

$$
\begin{array}{l}
\forall R : \texttt{IntegralDomain} \;. \\
\quad \forall n \in \mathbb{N} \;. \\
\quad\quad \forall k \in \mathbb{I}_n \;. \\
\quad\quad\quad \dfrac{R[x_1, \ldots, x_n]}{(x_1, \ldots, x_k)} \cong R[x_1, \ldots x_{n-k}] : \texttt{IntegralDomain} \to (x_1, \ldots, x_k) : \texttt{Prime}\ R[x_1, \ldots, x_n] \\
\forall R : \texttt{IntegralDomain} \;.\; \forall n \in \mathbb{N} \;.\; \forall k \in \mathbb{I}_n \;.\; (\mathbf{L}_{i=1}^{k} x_i) : \texttt{Prime}\ R[\mathbf{L}_{i=1}^{n} x_i] \quad \square
\end{array}
$$

Problem 4.14 :: $\forall R : \texttt{Ring} \;.\; \forall I : \texttt{Maximal}\ R \;.\; I : \texttt{Prime}\ R$
(Quotients are banned)

$$
\begin{array}{l}
\forall R : \texttt{Ring} \;. \\
\quad \forall I : \texttt{Maximal}\ R \;. \\
\quad\quad \texttt{if}\ I : !\texttt{Prime}\ R \;. \\
\quad\quad\quad \exists a, b \in I^{\complement} \;.\; ab \in I \twoheadrightarrow a, b \;. \\
\quad\quad\quad \texttt{if}\ a : \texttt{Unit}\ R \;. \\
\quad\quad\quad\quad a^{-1}ab = b \to b \in I \to \bot \to \\
\quad\quad\quad \to a : !\texttt{Unit}\ R \to 1 \notin (a) \\
\quad\quad\quad \texttt{if}\ (a) + I = (1) \;. \\
\quad\quad\quad\quad \exists i \in I \;.\; \exists j \in R \;.\; i + ja = 1 \\
\quad\quad\quad\quad b = ib + jab \\
\quad\quad\quad\quad \left.\begin{array}{l} i \in I \to ib \in I \\ ab \in I \to jab \in I \end{array}\right\} \to b \in I \to \bot \to \\
\quad\quad\quad \left.\begin{array}{l} \to (a) + I \subsetneq R \\ a \in (a) \to a \in I + (a) \\ a \in I^{\complement} \to a \notin I \end{array}\right\} \to I \subsetneq I + (a) \subsetneq R \to \\
\quad\quad\quad \to I : !\texttt{Maximal}\ R \to \bot \to \\
\quad\quad \to I : \texttt{Prime}\ R \to \\
\forall R : \texttt{Ring} \;.\; \forall I : \texttt{Maximal}\ R \;.\; I : \texttt{Prime}\ R \quad \square
\end{array}
$$

Problem 4.16 :: $\forall R : \texttt{Commutative} \,.\, \forall P : \texttt{Prime}\ R \,.\, \texttt{if}$ $\forall p \in P \,.\, \texttt{if}\ p : \texttt{ZD}\ R \,.\, p = 0 \,.\, R : \texttt{IntegralDomain}$

$\forall R : \texttt{Commutative}\ .$

$\quad \forall P : \texttt{Prime}\ R\ .$

$\quad\quad \texttt{if} \quad \forall p \in P \,.\, \texttt{if}\ p : \texttt{ZD}\ R \,.\, p = 0\ .$

$\quad\quad\quad \texttt{LawOfExcludedMiddle} \to P = (0) \vee P \neq (0)$

$\quad\quad\quad \texttt{if} \quad P = (0)\ .$

$\quad\quad\quad\quad \forall a, b \in R\ .$

$\quad\quad\quad\quad\quad \texttt{if} \quad ab = 0 \to ab \in P \to a \in P \vee b \in P \to a = 0 \vee b = 0 \to$

$\quad\quad\quad P = (0) \Rightarrow R : \texttt{IntegralDomain}$

$\quad\quad\quad \texttt{if} \quad P \neq (0) \to \exists p \in P \,.\, p \neq 0 \twoheadrightarrow p \to p : \texttt{!ZD}\ R$

$\quad\quad\quad\quad \forall a, b \in R$

$\quad\quad\quad\quad\quad \texttt{if} \quad ab = 0 \to pab = 0 \to a = 0 \vee pa \neq 0 \wedge pa \in P \to$

$\quad\quad\quad\quad\quad\quad \to a = 0 \vee pa : \texttt{!ZD} \ \to a = 0 \vee b = 0 \to$

$\quad\quad\quad P \neq (0) \Rightarrow R : \texttt{IntegralDomain} \to$

$\quad\quad\quad R : \texttt{IntegralDomain} \to$

$\forall R : \texttt{Commutative} \,.\, \forall P : \texttt{Prime}\ R\ .$

$\texttt{if}\ \forall p \in P \,.\, \texttt{if}\ p : \texttt{ZD}\ R \,.\, p = 0 \,.\, R : \texttt{IntegralDomain} \quad \square$

Problem 4.17

K : `Compact`

$R = (C^0(K), +_{\mathbb{R}}, \cdot_{\mathbb{R}})$

$$\texttt{def } M :: K \to \texttt{Ideal } R$$
$$M_p = \{f \in R | f(p) = 0\}$$

(a) $:: \forall p \in K \ . \ M_p : \texttt{Maximal } R$

$\forall p \in K$.

$P := \bigcap_{U \in \mathcal{U}(p)} U |\ P \subset K$

$\Big(\forall f \in C^0(K) \ . \ \forall p' \in P . f(p') = f(p)\Big)$

$\dfrac{R}{M_p} = \dfrac{C^0(K)}{M_p} \cong \big\{ f(p') \ | f \in C^0(K) \ , p' \in P \big\} = \big\{ f(p) \ | f \in M_p \big\} = \mathbb{R}$

$\mathbb{R} : \texttt{Field } \to M_p : \texttt{Maximal } R \to$

$\to \forall p \in K \ . \ M_p : \texttt{Maximal } R \quad \square$

(b) $:: \forall n \in \mathbb{N} \ . \ \forall f : \mathbb{I}_n \to C^0(K)$.

`if` $\ \forall p \in K \ . \ \exists i \in \mathbb{I}_n \ . \ f_i(p) \neq 0 \ . \ (f) = (1)$

$\forall n \in \mathbb{N}$.

$f : \mathbb{I}_n \to C^0(K)$.

`if` $\ \forall p \in K \ . \ \exists i \in \mathbb{I}_n \ . \ f_i(p) \neq 0 \multimap (1)$.

$F := \sum_{i=1}^{n} f_i^2 \ | \ F \in C^0(K)$

$(1) \to 0 \notin \texttt{Im}\, F \to 1/F \in C^0(K) \to$

$(1) \to 1 = \sum_{i=1}^{n} \dfrac{f_i}{F} f_i \in (f) \to (f) = (1) \to$

$\forall n \in \mathbb{N} \ . \ \forall f : \mathbb{I}_n \to C^0(K) \ . \ \texttt{if} \ \ \forall p \in K \ . \ \exists i \in \mathbb{I}_n \ . \ f_i(p) \neq 0 \ . \ (f) = (1) \quad \square$

(c) :: $\forall I : \texttt{Maximal}\ R\ .\ \exists p \in K\ .\ I = M_p$

$\forall I : \texttt{Maximal}\ R\ .$

$\quad\texttt{if}\quad \forall p \in K\ .\ \exists f \in I\ .\ f(p) \neq 0 \multimap (0)\ .$

$\quad\quad\forall p \in K\ .$

$\quad\quad\quad f_p := (0)(p) \quad | \quad f : K \to I$

$$U_p := f_p(p) \neq 0 \to \exists U \in \mathcal{U}(p)\ .\ 0 \notin f_p[U] \twoheadrightarrow \quad | \quad U : \prod_{p \in K} \mathcal{U}(p)$$

$$K : \texttt{Compact} \to \exists n \in \mathbb{N}\ .\ \exists p : \mathbb{I}_n \to K\ .\ \bigcup_{i=1}^{n} U_{p_i} = K \twoheadrightarrow n, p$$

$\quad\quad(b) \to (f_p) = (1) = R$

$\quad\quad(f_p) \subset I \neq R \to \bot \to$

$\quad\to \exists p \in K\ .\ I(p) = \{0\} \twoheadrightarrow p$

$\quad I \subset M_p \to I = M_p \to$

$\forall I : \texttt{Maximal}\ R\ .\ \exists p \in K\ .\ I = M_p \quad \square$

problem 4.18 :: $\forall R : \texttt{Commutative}\ .\ \forall P : \texttt{Prime}\ R\ .\ \mathfrak{nil}(R) \subset P$

$\forall R : \texttt{Commutative}\ .$

$$\forall P : \texttt{Prime}\ R \to \frac{R}{P} : \texttt{IntegralDomain}\ .\ \multimap (p)\ .$$

$\quad\forall n \in \mathfrak{nil}(R) \to \exists k \in \mathbb{N} \forall i \in \mathbb{I}_{k-1}\ .\ n^i \neq 0 \wedge n^k = 0 \twoheadrightarrow k\ .$

$\quad\quad\texttt{if}\quad n \notin P \to n + P \neq P$

$$\left.\begin{array}{l} (p) \to n^k + P \neq P \\ n^k + P = 0 + P = P \end{array}\right\} \to \bot$$

$\quad\quad n \in P \to$

$\quad\to \forall n \in \mathfrak{nil}(R)\ .\ n \in P \to \mathfrak{nil}(R) \subset P \to$

$\forall R : \texttt{Commutative}\ .\ \forall P : \texttt{Prime}\ R\ .\ \mathfrak{nil}(R) \subset P \quad \square$

problem 4.19 :: $\forall R : \texttt{Commutative} \;.\; \forall P : \texttt{Prime}\ R \;.\; \forall n \in \mathbb{N} \;.$
$.\; \forall I : \mathbb{I}_n \to \texttt{Ideal}\ R$
(a) :: $\texttt{if} \quad \prod_{i=1}^{n} I_i \subset P \;.\; \exists i \in \mathbb{I}_n \;.\; I_i \subset P$

$$\forall a : \prod_{i=1}^{n} I_i \;.$$

$$\texttt{if} \quad \prod_{i=1}^{n} a_i \in P \;.$$

as $(P : \texttt{Prime}\ R)$ by Induction $\exists i \in \mathbb{I}_n \;.\; a_i \in P$

$\multimap (\alpha)$

$$\texttt{if} \quad \prod_{i=1}^{n} I_i \subset P \multimap (*) \;.$$

$$\texttt{if} \quad \forall i \in \mathbb{I}^n \;.\; I_i \not\subset P$$

$$\exists a : \prod_{i=1}^{n} (I_i \setminus P) \twoheadrightarrow a$$

$$(*) \to \prod_{i=1}^{n} a_i \in P -\!(\alpha)\!\to \exists i \in \mathbb{I}_n \;.\; a_i \in P \to \bot$$

$$\texttt{if} \quad \prod_{i=1}^{n} I_i \subset P \;.\; \exists i \in \mathbb{I}_n \;.\; I_i \subset P \quad \square$$

(b) ? $\forall I : \mathbb{N} \to \texttt{Ideal}\ R \;.\; \texttt{if}\ \bigcap_{i=1}^{\infty} I_i \subset P \;.\; \exists i \in \mathbb{N} \;.\; I_i \subset P$
This is false. We give counterexample: take $R = \mathbb{Z}$,$P = 3\mathbb{Z}$,$I_n = 2^n\mathbb{Z}$.
Then, $\bigcap_{i=1}^{\infty} I_i = \bigcap_{i=1}^{\infty} 2^n\mathbb{Z} = (0) \subset P$.
However, consequent does not hold: $\forall n \in \mathbb{N} \;.\; 2^n \in I_n \wedge 2^n \notin P$.
$\square$

problem 4.20 :: $\forall R : \texttt{Ring} \,.\, \forall M : \texttt{Maximal}\ R \,.\, R/M : \texttt{Simple}$

$$
\begin{aligned}
&\forall R : \texttt{Ring}\ . \\
&\quad \forall M : \texttt{Maximal}\ R \to \\
&\qquad \to \forall I : \texttt{Ideal}\ R \,.\, \texttt{if}\ M \subset I \land I \neq R \,.\, M \mathrel{=\!\!\multimap} (\alpha) \\
&\qquad \to \exists f : R \to \frac{R}{M} \,.\, fa \mapsto a + M \rightarrowtriangle f\ . \\
&\qquad \ker f = M \\
&\qquad \texttt{if}\ \frac{R}{M} : !\texttt{Simple} \to \exists I : \texttt{Ideal}\frac{R}{M} \,.\, I \neq (0) \land I \neq \frac{R}{M} \rightarrowtriangle I \\
&\qquad\quad I : \texttt{Ideal}\frac{R}{M} \to \exists f : \frac{R}{M} \to \frac{R/M}{I} \,.\, f(a+M) \mapsto a + I \rightarrowtriangle g \\
&\qquad\quad I = \ker g \\
&\qquad\quad \left.\begin{aligned}
&I \neq \tfrac{R}{M} \to \ker g \neq \tfrac{R}{M} = \operatorname{Im} f \to \ker fg \neq R \\
&I \neq (0) \to \ker fg \neq f^{-1}(0) = M \\
&\ker f \subset \ker fg \to M \subset \ker fg
\end{aligned}\right\} \to \\
&\qquad\quad \to M \subsetneq \ker fg \subsetneq R -\!(\alpha)\!\to \bot \\
&\forall R : \texttt{Ring} \,.\, \forall M : \texttt{Maximal}\ R \,.\, R/M : \texttt{Simple} \quad \square
\end{aligned}
$$

problem 4.21 :: $\forall K : \texttt{AlgebraiclyClosedField} \,.\, \forall I : \texttt{Ideal}\ K[x]\ .$
$.\ \texttt{iff}\ I : \texttt{Maximal}\ K[x] \,.\, \exists c \in K \,.\, I = (x - c)$
($\Leftarrow$) It easily can be seen that $\forall c \in K \,.\, (x - c) : \texttt{Maximal}\ K[x]$.
Indeed, $K[x]/(x - c) \cong K : \texttt{Field}$.
($\Rightarrow$) Assume that $M : \texttt{Maximal}\ K[x]$ and that it contains polynomials which have no common root. Then, by application of Euclidean algorithm we can show that where is $r \in M$ such that $\deg r = 0$. This means that $1 \in M$, which contradicts maximality of M.
So all polynomials in any maximal ideal M of $K[x]$ must have a common root, say c. This means that $M \subset (x - c)$ and by maximality $M = (x - c)$.
$\square$

problem 4.22 :: $(x^2 + 1) : \texttt{Maximal}\ \mathbb{R}[x]$
Indeed, $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C} : \texttt{Field}$, so $(x^2 + 1)$ is Maximal.
$\square$

problem 4.23 :: Fields and Boolean algebras have Krull dimension 0.
Case of Fields is obvious as the only prime ideal of any field K is (0), which means that (0) is also the only maximal ideal of a field. So $\dim K = 0$.
Assume that B is a boolean algebra with a prime ideal P. We know that B/P is integral domain, but this means that $B/P \cong \mathbb{Z}/2\mathbb{Z} : \texttt{Field}$, so P is maximal ideal. So, all prime ideals of B are maximal, which implies that $\dim B = 0$.
$\square$

problem 4.24 :: $\dim \mathbb{Z}[x] \geq 2$
Idea: $(0) \subset (2x-2) \subset (2) \subset \mathbb{Z}[x]$
We inspect (2), that is space of polynomials with even coefficients. Note that $\mathbb{Z}[x]/(2) = \cong \mathbb{Z}/2\mathbb{Z}[x]$ which is an integral domain, hence (2) is prime.
Moreover, as $\mathbb{Z}/2\mathbb{Z}$ is a field $\mathbb{Z}/2\mathbb{Z}[x]$ has $(x-1)$ as it's maximal ideal, so (2) is not maximal in $\mathbb{Z}[x]$, which implies that $\dim \mathbb{Z}[x] \geq 2$.
$\square$