Analysis on Real Line

Uncultured Tramp

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Contents

1 The Real Type

1.1 Topology of an Archimedean Field

$\texttt{ReductioInfinima} :: \forall R : \texttt{Archimedean} \; . \; \lim_{n\to\infty} \frac{1}{n_R} = 0$

$\texttt{Proof} \;\; =$

$\texttt{Assume}\ U : \mathcal{U}_R(0),$

$\Big((-a,a),1\Big) := \eth\texttt{orderTopology}(R)\eth\texttt{Neighborhood}(R,0)(U) : \sum (-a,a) : \texttt{OpenInterval}(R) \; . \; (-a.a) \subset U,$

$(N,2) := \eth\texttt{Archemedean}(R)\big(a^{-1}\big) : \sum N \in \mathbb{N} \; . \; N > a^{-1},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N,$

$(4) := \eth\texttt{Transitive}\Big(\texttt{order}(\mathbb{N})\Big) : n > a^{-1},$

$(5) := (4)^{-1} : \frac{1}{n} < a,$

$(6) := \ldots : -a < 0 < \frac{1}{n},$

$(7) := \Big(\eth(-a,a)(5)(6)\Big)(1) : \frac{1}{n} \in U;$

$\rightsquigarrow (3) := I(\forall)I(\Rightarrow) : \forall n \in \mathbb{N} \; . \; n \geq N \Rightarrow \frac{1}{n} \in (-a,a);$

$\rightsquigarrow (2) := I(\forall)I(\exists)(N) : \forall U : \texttt{Open}(R) \; . \; \exists N \in \mathbb{N} : \forall n \in \mathbb{N} \; . \; n \geq N \Rightarrow \frac{1}{n} \in (-a,a),$

$(*) := \eth\texttt{Limit}(4) : \lim_{n\to\infty} \frac{1}{n} = 0;$

$\square$

$\texttt{ContinuousAddition} :: \forall R : \texttt{Archimedean} \; . \; \forall x,y : \texttt{Converging}(R) \; . \; \lim_{n\to\infty} x_n + y_n = \big(\lim_{n\to\infty} x_n\big) + \big(\lim_{n\to\infty} y_n\big)$

$\texttt{Proof} \;\; =$

$X := \lim_{n\to\infty} x_n : R,$

$Y := \lim_{n\to\infty} y_n : R,$

$\texttt{Assume}\ \varepsilon : R_{++},$

$(M',1) := \eth\texttt{Limit}(x,X)(\varepsilon/2) : \sum M' \in \mathbb{N} \; . \; \forall n \in \mathbb{N} : n \geq M' \; . \; |x_n - X| < \varepsilon/2,$

$(M,2) := \eth\texttt{Limit}(x,X)(\varepsilon/2) : \sum M \in \mathbb{N} \; . \; \forall n \in \mathbb{N} : n \geq M \; . \; |y_n - Y| < \varepsilon/2,$

$N := \max(M',M) : \mathbb{N},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N,$

$() := \texttt{TriangleIneq}(x_n - X, y_n - Y)(1,2)(\eth N(3)) : |x_n + y_n - X - Y| \leq |x_n - X| + |y_n - Y| < \epsilon;$

$\rightsquigarrow (*) := \eth\texttt{Limit}I(\forall)I(\exists)(N)I(\forall)I(\Rightarrow) : \lim_{n\to\infty} x_n + y_n = X + Y;$

$\square$

$\texttt{ContinuousMultiplication} :: \forall R : \texttt{Archimedean} . \forall x.y : \texttt{Converging}(R) . \lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right)$

$\texttt{Proof} \ =$

$X := \lim_{n\to\infty} x_n : R,$

$Y := \lim_{n\to\infty} y_n : R,$

$\Delta := x - X : \mathbb{N} \to R,$

$(1) := \texttt{ContinuousAddition}(x_n, -X) : \lim_{n\to\infty} \Delta_n = 0,$

$\Delta' := y\_Y : \mathbb{N} \to R,$

$(2) := \texttt{ContinuousAddition}(y_n, -Y) : \lim_{n\to\infty} \Delta'_n = 0,$

$\texttt{Assume}\ \varepsilon : R_{++},$

$\delta := \min\left(\frac{\varepsilon}{3|Y|}, \sqrt{\frac{\varepsilon}{3}}\right) : \hat{R},$

$\delta' := \min\left(\frac{\varepsilon}{3|X|}, \sqrt{\frac{\varepsilon}{3}}\right) : \hat{R},$

$(M, 3) := \eth\texttt{Limit}(1)(\delta) : \sum M \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq M . |\Delta_n| < \delta,$

$(M', 4) := \eth\texttt{Limit}(2)(\delta') : \sum M' \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq M' . |\Delta'_n| < \delta',$

$N := \max(M, M') : \mathbb{N},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (5) : n \geq N,$

$() := \eth^{-1}\Delta_n \eth^{-1}\Delta'_n \texttt{TriangleIneq}(X\Delta'_n, Y\Delta_n, \Delta_n\Delta'_n)\texttt{AbsoluteHomogenity}^2(X)(Y)(3,4)(5)\eth\delta\eth\delta' :$

$\quad : |x_n y_n - XY| \leq |X||\Delta'_n| + |Y||\Delta_n| + |\Delta_n\Delta'_n| < \varepsilon;$

$\rightsquigarrow (*) := \eth\texttt{Limit} I(\forall)I(\exists)(N)I(\forall)I(\Rightarrow) : \lim_{n\to\infty} x_n y_n = XY;$

$\square$

$\texttt{ContinuousInverse} :: \forall R : \texttt{Archimedean} . \forall x : \texttt{Converging}(R) . \forall(1) : \forall n \in \mathbb{N} . x_n \neq 0 .$

$\quad . \forall(2) : \lim_{n\to\infty} x_n \neq 0 . \lim_{n\to\infty} x_n^{-1} = \left(\lim_{n\to\infty} x_n\right)^{-1}$

$\texttt{Proof} \ =$

$X := \lim_{n\to\infty} x_n : R,$

$\Delta := \frac{x}{X} - 1 : \mathbb{N} \to R,$

$(1) := \texttt{ContinuousMultuplication}(x, X^{-1})\texttt{ContinuousAddition}\left(\frac{x}{X}, -1\right) : \lim_{n\to\infty} \Delta_n = 0,$

$\texttt{Assume}\ \varepsilon : R_{++},$

$\delta := \min\left(\frac{\varepsilon}{2}, \frac{1}{2}\right) : R_{++},$

$(N, 2) := \eth\texttt{Limit}(1) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} : n \geq N . |\Delta_n| < \delta,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N,$

$() := \eth^{-1}\Delta_n \texttt{AbsoluteHomogenity}\Delta_n((2)(3)\eth\delta)^2 : \left|\frac{X}{x_n} - 1\right| = \left|\frac{1}{1+\Delta_n} - 1\right| = \frac{|\Delta_n|}{|1+\Delta_n|} < 2|\Delta_n| < \varepsilon;$

$\rightsquigarrow (2) := \eth\texttt{Limit} I(\forall)I(\exists)(N)I(\forall)I(\Rightarrow) : \lim_{n\to\infty} \frac{X}{x_n} = 1,$

$:= \texttt{ContinuousMultiplication}\left(\frac{X}{x}, X^{-1}\right) : \lim_{n\to\infty} x_n^{-1} = X^{-1};$

$\square$

$\texttt{BernoulliIneq} :: \forall R : \texttt{OrderedField} \,.\, \forall x \in (-1, +\infty)_R \,.\, \forall n \in \mathbb{N} \,.\, (1+x)^n \geq 1 + nx$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{Reflexive}\Big(\texttt{order}(R)\Big)(1+x) : 1 + x \geq 1 + x,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (2) : \texttt{This}(n),$

$() := (2)\texttt{ReduceIneq}(nx^2) : (1+x)^{n+1} \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \geq 1 + (1+nx);$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall n \in \mathbb{N} \,.\, \texttt{This}(n+1) \Rightarrow (This)(n),$

$(*) := E(\mathbb{N})(1)(2) : \texttt{This};$

$\square$

$\texttt{PowerCompression} :: \forall R : \texttt{Archimedean} \,.\, \forall \gamma \in R \,.\, \forall (0) : 0 < |\gamma| < 1 \,.\, \lim_{n \to \infty} \gamma^n = 0$

$\texttt{Proof} \ =$

$\alpha := |\gamma|^{-1} : R,$

$(1) := (0)^{-1} : \alpha > 1,$

$\beta := \gamma - 1 : R_{++},$

$\texttt{Assume}\ \varepsilon : R_{++},$

$(N.2) := \texttt{ReductioInfima}\,(\beta\varepsilon) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow \dfrac{1}{n} \leq \beta\varepsilon,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N,$

$() := \texttt{AbsHomogen}^n(\gamma)\eth^{-1}(\beta)\texttt{BernoulliIneq}\ldots(2)\Big(n.(3)\Big) : \Big|\gamma^n\Big| = \dfrac{1}{(1+\beta)^n} \leq \dfrac{1}{1+n\beta} \leq \dfrac{1}{n\beta} < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Limit}I(\forall)I(\exists, N)I(\forall)I(\Rightarrow) : \lim_{n \to \infty} \gamma^n = 0;$

$\square$

1.2 Intermidiate Point Property

$\texttt{IntermidiatePointProperty} :: ?\texttt{Poset}$

$R : \texttt{IntermidiatePointProperty} \iff \forall A, B \subset R \,.\, A < B \Rightarrow \exists x \in R \,.\, A \leq x \leq B$

$\texttt{LowerUpperBound} :: \prod X : \texttt{Poset} \,.\, ?X \to ?X$

$x : \texttt{LowerUpperBound} \iff \Lambda A \subset X \,.\, x \geq A \;\&\; \forall y \in A \,.\, y \leq A \Rightarrow x \not> y$

$\texttt{UpperLowerBound} :: \prod X : \texttt{Poset} \,.\, ?X \to ?X$

$x : \texttt{UpperLowerBound} \iff \Lambda A \subset X \,.\, x \leq A \;\&\; \forall y \in A \,.\, y \geq A \Rightarrow x \not< y$

$\texttt{LUBExistsInIPP} :: \forall X : \texttt{IntermidiatePointProperty} \,.\, \forall A : \texttt{BoundedFromAbove}(X) \,.$
$\quad .\, A \neq \emptyset \Rightarrow \exists \texttt{LowerUpperBound}(A)$

$\texttt{Proof} =$

$B := \{x \in X : x > A\} : ?X,$

$(1) := \eth\texttt{BoundedFromAbove}(X)(\eth B) : B \neq \emptyset,$

$(2) := \eth\texttt{IntermidiatePointProperty}(X)(A, B) : \exists x \in X \,.\, A \leq x \leq B,$

$C := \{x \in X : A \leq x \leq B\} : ?X,$

$(3) := \eth\emptyset C : C \neq \emptyset,$

$(4) := \ldots : \min C \neq \emptyset,$

$(x) := \eth\emptyset(4) : \min C,$

$\Delta := x_n - X : \mathbb{N} \to R,$

$(1) := \texttt{ContinuousAddition}(x_n, -X) : \lim\limits_{n\to\infty} \Delta_n = 0,$

$\texttt{Assume}\ y : \texttt{In}(x),$

$\texttt{Assume}\ (5) : y \geq A,$

$\texttt{Assume}\ (6) : x > y,$

$(7) := (6)(2) : y \leq B,$

$(8) := \eth C(7) : y \in C,$

$(9) := \eth x \eth \min : x \not> y,$

$(10) := (6)(9) : \bot;$

$\rightsquigarrow (5) := I(\forall)I(\Rightarrow)E(\bot) : \forall y \in X \,.\, y \geq A \Rightarrow x \not> y,$

$(6) := \eth\texttt{LowerUpperBound}(X)(A)(5,2) : \Big(x : \texttt{LowerUpperBound}(A)\Big);$

$\square$

$\texttt{LUBsUniqueInToset} :: \forall X : \texttt{Toset} \,.\, \forall A \subset X \,.\, \forall x, y : \texttt{LowerUpperBound}(A) \,.\, x = y$

$\texttt{Proof} =$

$(1) := \eth_1\texttt{LowerUpperBound}(A)(x) : x \geq A,$

$(2) := \eth_2\texttt{LowerUpperBound}(A)(y)(x) : x \geq y,$

$(3) := \eth_1\texttt{LowerUpperBound}(A)(y) : y \geq A,$

$(4) := \eth_2\texttt{LowerUpperBound}(A)(x)(y) : y \geq x,$

$(*) := \eth\texttt{Antysimmetric}(\texttt{order}(X))(2,4) : x = y;$

$\square$

$\texttt{ULBExistsInIPP} :: \forall X : \texttt{IntermidiatePointProperty} \;.\; \forall A : \texttt{BoundedFromBelow}(X) \;.$
$\;.\; A \neq \emptyset \Rightarrow \exists \texttt{UpperLowerBound}(A)$

$\texttt{Proof} \;=$

$B := \{x \in X : x < A\} :?X,$

$(1) := \eth\texttt{BoundedFromBelow}(X)(\eth B) : B \neq \emptyset,$

$(2) := \eth\texttt{IntermidiatePointProperty}(X)(B, A) : \exists x \in X \;.\; B \leq x \leq A,$

$C := \{x \in X : B \leq x \leq A\} :?X,$

$(3) := \eth\emptyset C : C \neq \emptyset,$

$(4) := \ldots : \max C \neq \emptyset,$

$(x) := \eth\emptyset(4) : \max C,$

$\texttt{Assume}\; y : \texttt{In}(x),$

$\texttt{Assume}\; (5) : y \geq A,$

$\texttt{Assume}\; (6) : x > y,$

$(7) := (6)(2) : y \geq B,$

$(8) := \eth C(7) : y \in C,$

$(9) := \eth x \eth \max : x \not< y,$

$(10) := (6)(9) : \bot;$

$\rightsquigarrow (5) := I(\forall)I(\Rightarrow)E(\bot) : \forall y \in X \;.\; y \leq A \Rightarrow x \not< y,$

$(6) := \eth\texttt{UpperLowerBound}(X)(A)(5, 2) : \Big(x : \texttt{UpperLowerBound}(A)\Big);$

$\square$

$\texttt{ULBsUniqueInToset} :: \forall X : \texttt{Toset} \;.\; \forall A \subset X \;.\; \forall x, y : \texttt{UpperLowerBound}(A) \;.\; x = y$

$\texttt{Proof} \;=$

$(1) := \eth_1\texttt{UpperLowerBound}(A)(x) : x \leq A,$

$(2) := \eth_2\texttt{UpperLowerBound}(A)(y)(x) : x \leq y,$

$(3) := \eth_1\texttt{UpperLowerBound}(A)(y) : y \leq A,$

$(4) := \eth_2\texttt{UpperLowerBound}(A)(x)(y) : y \leq x,$

$(*) := \eth\texttt{Antysimmetric}(\texttt{order}(X))(2, 4) : x = y;$

$\square$

$\texttt{supremum} :: \prod X : \texttt{Toset} \,\&\, \texttt{IntermidiatePointProperty} \;.\; \texttt{BoundedFromAbove} \,\&\, \texttt{NonEmpty}(X) \to X$

$\texttt{supremum}\,(A) = \sup A := \texttt{LUBExistsInIPP} \,\&\, \texttt{LUBsUniqueToToset}(A)$

$\texttt{infimum} :: \prod X : \texttt{Toset} \,\&\, \texttt{IntermidiatePointProperty} \;.\; \texttt{BoundedFromBelow} \,\&\, \texttt{NonEmpty}(X) \to X$

$\texttt{infimum}\,(A) = \inf A := \texttt{ULBExistsInIPP} \,\&\, \texttt{ULBsUniqueToToset}(A)$

$\texttt{Real} := \texttt{Archimedean} \,\&\, \texttt{IntermidiatePointProperty}$

$\texttt{RealIsUncountable} :: \forall R : \texttt{Real} \,.\, \#R > \aleph_0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : \#R = \aleph_0,$

$r := \eth\texttt{Cardinal}(1) : \mathbb{N} \leftrightarrow_{\mathsf{SET}} R,$

$a_1 := r(0) : R,$

$(b_1, l_1, 2) := -\eth\texttt{Aechemedean}(R)(z) : \sum b_1 \,.\, b_1 < a_1,$

$I_0 := R :?R,$

$J_0 := R :?R,$

$\texttt{Assume}\ n : \mathbb{N},$

$I_n := \{r(m) : m > n \,\&\, b_n < r(m) < a_n\} :?R,$

$(3) := \texttt{SmallNumberLemma}(\eth I_n) : I_n \neq \emptyset,$

$a_{n+1} := \arg\min_{x \in I_n} r^{-1}(x) : \texttt{In}(I),$

$(2_n) := \eth I(bda_{n+1}) : b_n < a_{n+1} < a_n,$

$J_n := \{r(m) : m > n \,\&\, b_n < r(m) < a_{n+1}\} :?R,$

$(4) := \texttt{SmallNumberLemma}(\eth J_n) : J_n \neq \emptyset,$

$b_{n+1} := \arg\min_{x \in J_n} r^{-1}(x) : \texttt{In}(J_n),$

$(2'_n) := \eth I(bda_{n+1}) : b_n < b_{n+1} < a_{n+1};$

$\leadsto (I, J, a, b, 3) := I(\sum) : \sum(I, J) : \mathbb{N} \to ?R \,.\, \prod n \in \mathbb{N} \,.\, I_{n-1} \times J_{n-1} \,\&\, b < a \,\&\, b, a : \texttt{Monotonic},$

$(4) := \eth(I.j, a, b, 3) : r^{-1}(a), r^{-1}(b) : \texttt{Increasing},$

$A := a(\mathbb{N}) :?R,$

$B := b(\mathbb{N}) :?R,$

$(5) := \eth A \eth B(3) : B < A,$

$(z, 6) := \eth\texttt{IntermidiatePointProperty} : \sum z \in R \,.\, B \leq z \leq A,$

$(7) := \eth z \eth I \eth J(6) : \forall n \,.\, z \in I_n \,\&\, z \in J_n,$

$(8) := \eth a(7)(4) : a_{r^{-1}(z)} = z,$

$(*) := \eth A(3)(8)(6) : \bot;$

$\leadsto (1) := E(\bot) : \#R \neq \aleph_0,$

$(2) := \texttt{RationalsInCharZero}(R) : \mathbb{Q} \subset R,$

$(3) := \eth^{-1}\texttt{GeCardinality}(2)(1) : \#R > \aleph_0,$

$\square$

1.3 Construction By Cuts [!!]

$$\texttt{DedikindCuts} :: ?(?\mathbb{Q} \times ?\mathbb{Q})$$
$$(A, B) : \texttt{DedikindCuts} \iff A < B \ \& \ A \cup B = \mathbb{Q} \ \& \ \forall x \in A \ . \ x \, ! \, \texttt{LowerUpperBound}(A)$$

$$\texttt{DedikindAdd} :: \texttt{DedindCuts} \times \texttt{DedikindCuts} \to \texttt{DedikindCuts}$$
$$\texttt{DedikindAdd}\,((A, B), (C, D)) = (A, B) + (C, D) := (A + C, B + D)$$

1.4 Construction By Completion

$$\texttt{Cauchy} :: ?(\mathbb{N} \to \mathbb{Q})$$
$$x : \texttt{Cauchy} \iff \forall \varepsilon \in \mathbb{Q}_{++} \,.\, \exists N \in \mathbb{N} : \forall n, m \in \mathbb{N} \,.\, \forall(0) : n \geq N \;\&\; m \geq N \,.\, |x_n - x_m| \leq \varepsilon$$

$$\texttt{EqualCauchy} :: ?(\texttt{Cauchy} \times \texttt{Cauchy})$$
$$(x, u) : \texttt{EqualCauchy} \iff x = y \iff \lim_{n\to\infty} (x_n - y_n) = 0$$

$$\texttt{CauchyEquality} :: \Big(\texttt{EqualCauchy} : \texttt{Equality}(\texttt{Cauchy})\Big)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \texttt{Cauchy},$

$(1) := \eth\texttt{Inverse}(x) : x - x = 0,$

$(2) := \texttt{ConstantLimit}0 : \lim_{n\to\infty} 0 = 0,$

$() := \eth^{-1}\texttt{EqualCaucht}(2) : x = x;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Reflexive}I(\forall) : \Big(\texttt{EqualCauchy} : \texttt{Reflexive}\Big),$

$\texttt{Assume}\ x, y : \texttt{Cauchy},$

$\texttt{Assume}\ (2) : x = y,$

$(3) := \eth\texttt{Commutative}\big(\texttt{addition}(\mathbb{Q})\big)\eth\texttt{EqualCauchy}(2) : \lim_{n\to\infty} y_n - x_n = \lim_{n\to infty} x_n - y_n = 0,$

$() := \eth^{-1}\texttt{CauchyEqual}(3) : y = x;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Symmetric}I(\forall)I(\Rightarrow) : \Big(\texttt{EqualCauchy} : \texttt{Symmetric}\Big),$

$\texttt{Assume}\ x, y, z : \texttt{Cauchy},$

$\texttt{Assume}\ (3) : x = y \;\&\; y = z,$

$(4) := \texttt{AddZero}(-y_n)\texttt{ContinuousAddition}\eth\texttt{EqualCauchy}(3) :$

$\quad : \lim_{n\to\infty} x_n - z_n = \lim_{n\to\infty} x_n - y_n + y_n - z_n = \big(\lim_{n\to\infty} x_n - y_n\big) + \big(\lim_{n\to\infty} y_n - z_n\big) = 0,$

$() := \eth^{-1}\texttt{CauchyEqual}(4) : x = z;$

$\rightsquigarrow (3) := \eth^{-1}\texttt{Transitive}I(\forall)I(\Rightarrow) : \Big(\texttt{EqualCauchy} : \texttt{Transitive}\Big),$

$(*) := \eth\texttt{Transitive} : \Big(\texttt{EqualCauchy} : \texttt{Equality}\Big);$

$\square$

$$\mathbb{R} := \frac{\texttt{Cauchy}}{\texttt{EqualCauchy}} : ??\texttt{Cauchy},$$

$\texttt{CauchyAddition} :: \forall x, y : \texttt{Cauchy} \, . \, x + y : \texttt{Cauchy}$

$\texttt{Proof} \; =$

$\texttt{Assume}\; \varepsilon : \mathbb{Q}_{++},$

$(M, 1) := \eth\texttt{Cauchy}(x)(\varepsilon/2) : \sum M \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq N : m \geq M \, . \, |x_n - x_m| < \varepsilon/2,$

$(M', 2) := \eth\texttt{Cauchy}(y)(\varepsilon/2) : \sum M' \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq M' : m \geq N \, . \, |y_n - y_m| < \varepsilon/2,$

$N := \max(M, M') : \mathbb{N},$

$\texttt{Assume}\; n, m : \mathbb{N},$

$\texttt{Assume}\; (3) : n \geq N \;\&\; m \geq N,$

$() := \texttt{TriangleIneq}(x_n - x_m, y_n - y_m)(1, 2)(\eth N(3)) :$

$\quad : |x_n + y_n - x_m - y_m| \leq |x_n - x_m| + |y_n - y_m| < \varepsilon;$

$\leadsto (*) := \eth^{-1}\texttt{Cauchy}I(\forall)I(\exists)(N)I(\forall) : \Big(x + y : \texttt{Cauchy}\Big);$

$\square$

$\texttt{AddCauchyClass} :: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\texttt{AddCauchyClass}\,([a], [b]) = [a] + [b] := [a + b]$

$\texttt{CauchyClassAdditionIsWellDefined} :: \forall [a], [b] \in \mathbb{R} \, . \, \forall x \in [a] \, . \, \forall y \in [b] [x + y] = [a + b]$

$\texttt{Proof} \; =$

$(1) := \texttt{ContinuousAddition}(x_n - a_n, y_n - a_n)\eth\texttt{EqualCauchy}^2(x_n, a_n)(y_n, b_n) :$

$\quad : \lim_{n \to \infty} x_n + y_n - a_n - b_n = \left(\lim_{n \to \infty} x_n - a_n\right) + \left(\lim_{n \to \infty} y_n - b_n\right) = 0,$

$(*) := \eth\mathbb{R}\eth^{-1}\texttt{EqualCauchy}(1) : [x + y] = [a + b];$

$\square$

$\texttt{CauchyClassesAsGroup} :: (\mathbb{R}, +) : \texttt{Abelean}$

$\texttt{Proof} \; =$

$\texttt{Assume}\; [a], [b], [c] : \mathbb{R},$

$() := \eth(+, \mathbb{R})\eth(-, \mathbb{N} \to \mathbb{Q}) : [a] + [-a] = [a - a] = [0],$

$() := \eth(+, \mathbb{R})\eth(0, \mathbb{N} \to \mathbb{Q}) : [a] + [0] = [a + 0] = [a],$

$() := \ldots :$

$\quad : ([a] + [b]) + [c] = [a + b] + [c] = [(a + b) + c] = [a + (b + c)] = [a] + [b + c] = [a] + ([b] + [c]),$

$() := \ldots : [a] + [b] = [a + b] = [b + a] = [b] + [a];$

$\leadsto (*) := \eth^{-1}\texttt{Abelean}I(\forall) : \Big((\mathbb{R}, +) : \texttt{Abelean}\Big);$

$\square$

$\texttt{CauchyMult} :: \forall x, y : \texttt{Cauchy} \,.\, xy : \texttt{Cauchy}$

$\texttt{Proof} \;=$

$(K, 1) := \eth\texttt{Cauchy}(x)(1) : \sum K \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq K : m \geq K \,.\, |x_n - x_m| < 1,$

$(K', 2) := \eth\texttt{Cauchy}(y)(1) : \sum K' \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq K' : m \geq K \,.\, |y_n - y_m| < 1,$

$L := \max(K, K') : \mathbb{N},$

$\texttt{Assume}\ \varepsilon : \mathbb{Q}_{++},$

$\delta := \max\left(\frac{\varepsilon}{3(|y_L| + 1)}, \sqrt{\frac{\varepsilon}{3}}\right) : \hat{\mathbb{Q}},$

$\delta' := \max\left(\frac{\varepsilon}{3(|x_L| + 1))}, \sqrt{\frac{\varepsilon}{3}}\right) : \hat{\mathbb{Q}},$

$(M, 3) := \eth\texttt{Cauchy}(x)(\delta) : \sum M \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq M : m \geq M \,.\, |x_n - x_m| < \delta,$

$(M', 4) := \eth\texttt{Cauchy}(y)(\delta') : \sum M' \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq M' : m \geq M' \,.\, |y_n - y_m| < \delta',$

$N := \max(M, M', L) : \mathbb{N},$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (5) : n \geq N \,\&\, m \geq N,$

$\Delta := x_m - x_n : \mathbb{Q},$

$\Delta' := y_m - y_n : \mathbb{Q},$

$() := \eth^{-1}\Delta\eth^{-1}\Delta'\texttt{TriangleIneq}(\ldots)\texttt{AbsHomogen}^3(\ldots)\eth L\eth N(5, \eth\Delta, \eth\Delta')\eth\delta\eth\delta' :$

$\quad : |x_n y_n - x_m y_m| = |y_n\Delta + x_n\Delta' + \Delta\Delta'| \leq |y_n||\Delta| + |x_n||\Delta'| + |\Delta||\Delta'| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Cauchy}I(\forall)I(\exists)(N)I(\forall) : \Big(xy : \texttt{Cauchy}\Big);$

$\square$

$\texttt{multCauchyClass} :: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\texttt{multCauchyClass}\,([a], [b]) = [a][b] := [ab]$

$\texttt{CauchyClassMultiplicationIsWellDefined} :: \forall [a], [b] \in \mathbb{R} \,.\, \forall x \in [a] \,.\, \forall y \in [b] \,.\, [a][b] = [x][y]$

$\texttt{Proof} \;=$

$\Delta := a - x : \mathbb{N} \to \mathbb{Q},$

$\Delta' := b - y : \mathbb{N} \to \mathbb{Q},$

$(1) := \eth\texttt{EqualCauchy}(a, x) : \lim_{n\to\infty} \Delta_n = 0,$

$(2) := \eth\texttt{EqualCauchy}(b, y) : \lim_{n\to\infty} \Delta'_n = 0,$

$(3) := \eth^{-1}\Delta\eth^{-1}\Delta'\texttt{ContinuousMultiplication}^3(\Delta, a)(\Delta', b)(\Delta, \Delta') :$

$\quad : \lim_{n\to\infty} a_n b_n - x_n y_n = -\lim_{n\to\infty} \Delta_n a_n + \Delta'_n b_n + \Delta_n \Delta'_n = 0,$

$(*) := \eth^{-1}\texttt{EqualCauchy}(ab, xy) : ab = xy,$

$\square$

$\texttt{CauchyClassesAsRing} :: (\mathbb{R}, +, \cdot) : \texttt{CommutativeRing}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [a], [b], [c] : \mathbb{R},$

$() := \dots : [a][1] = [a],$

$() := \dots : [a][b] = [ab] = [ba] = [b][a],$

$() := \dots : ([a][b])[c] = [ab][c] = [(ab)c] = [a(bc)] = [a][bc] = [a]([b][c]),$

$() := \dots : ([a] + [b])[c] = [a + b][c] = [(a + b)c] = [ac + bc] = [ac] + [bc] = [a][c] + [b][c];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{CommutativeRing} : (\mathbb{R}, +, \cdot) : \texttt{CommutativeRing};$

$\square$

$\texttt{SeparatedFromZero} :: ?\texttt{Cauchy}$

$x : \texttt{SeparatedFromZero} \iff \exists s \in \mathbb{Q}_{++} : \forall n \in \mathbb{N} \,.\, |x_n| \geq s$

$\texttt{CauchyInverse} :: \forall x : \texttt{SeparatedFormZero} \,.\, x^{-1} : \texttt{Cauchy}$

$\texttt{Proof} \ =$

$(s, 0) := \eth\texttt{SeparatedFromZero}(x) : \sum s \in \mathbb{Q}_{++} \,.\, \forall n \in \mathbb{N} \,.\, |x_n| \geq s,$

$\texttt{Assume}\ \varepsilon : \mathbb{Q}_{++},$

$\delta := \min\left(\frac{s}{2}, \frac{s^2\varepsilon}{2}\right) : \mathbb{Q}_{++},$

$(N, 1) := \eth\texttt{Cauchy}(x)(\delta) : \sum N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} : n, m \geq N \,.\, |x_n - x_m| \leq \delta,$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (2) : \min(n, m) \geq N,$

$\Delta := x_m - x_n : \mathbb{Q},$

$() := \eth^{-1}\Delta\texttt{AbsHomogen}(\Delta)(1)(2)\eth\delta\eth\Delta(0)(1)(2)\eth\delta\eth\Delta :$

$$: \left|\frac{1}{x_n} - \frac{1}{x_m}\right| = \left|\frac{1}{x_n} - \frac{1}{x_n + \Delta}\right| = \frac{|\Delta|}{|x_n||(x_n + \Delta)|} \leq \frac{2|\Delta|}{s^2} \leq \varepsilon;$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Cauchy}I(\forall)I(\exists, N)I(\forall)I(\Rightarrow) : \left(x^{-1} : \texttt{Cauchy}\right);$

$\square$

$\texttt{CauchyClassInversion} :: \forall [x] \in \mathbb{R}.[x] \neq [0] \Rightarrow \exists [y] \in \mathbb{R} : [x][y] = [1]$

$\texttt{Proof} \ =$

$(s, 1) := \eth\texttt{EqualCauchy}([x], [0])\eth[x] : \sum s \in \mathbb{Q}_{++} \,.\, \forall N \in \mathbb{N} \,.\, \exists n \in \mathbb{N} : n \geq N \,.\, |x_n| \geq s,$

$n_1 := (1)(1) : \mathbb{N},$

$(z_1, 2_1) := x_{n_1} : \sum z_1 \in \mathbb{Q} : |z_1| \geq s,$

$\texttt{Assume}\ m : \mathbb{N},$

$(n_{m+1}3_m) := (1)(n_m + 1) : \sum n_{m+1} \in \mathbb{N} \,.\, n_{m+1} \geq n_m + 1 > n_m,$

$(z_{m+1}, 2_{m+1}) := x_{n_{m+1}} : \sum z_{m+1} \in \mathbb{Q} \,.\, |z_{m+1}| \geq s;$

$\rightsquigarrow (n, z, 2) := I\left(\sum\right) : \sum (n, z) : \texttt{Subsequencer} \times \mathbb{N} \to \mathbb{Q} \,.\, z = x(n) \,\&\, \forall m \in \mathbb{N} \,.\, |z_m| \geq s,$

$(3) := \eth^{-1}\texttt{Cauchy}(2) : \left(z : \texttt{Cauchy}\right),$

$(4) := \eth^{-1}\texttt{SeparetedFromZero}(2) : \left(z : \texttt{SeparatedFromZero}\right),$

Assume $\varepsilon : \mathbb{Q}_{++}$,

$(N, 5) := \eth\texttt{Cauchy}(x) : \sum N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} \,.\, \max(n, m) \geq N \,.\, |x_m - x_n| \leq \varepsilon$,

Assume $m : \mathbb{N}$,

Assume $(6) : m \geq N$,

$(7) := \eth\texttt{Subsequencer}(n)(m) : n_m \geq n$,

$() := (2)_1(m)(5)(6, 7) : |x_m - z_m| = |x_m - x_{n_m}| \leq \varepsilon$;

$\rightsquigarrow (5) := \eth^{-1}\texttt{Limit}I(\forall)I(\exists, N)I(\forall)I(\Rightarrow) : \lim\limits_{n\to\infty} x_n - z_n = 0$,

$(6) := \eth^{-1}\texttt{EqualCauchy}(5) : [z] = [x]$,

$(*) := E(=)(6)\Big([z^{-1}][x]\Big)\eth\texttt{multCauchyClass} : [z^{-1}][x] = [z^{-1}][z] = [1]$;

$\square$

CauchyClassesAreField :: $(\mathbb{R}, +, \cdot) : \texttt{Field}$

Proof $=$

...

$\square$

CauchyClassGE :: $?(\mathbb{R} \times \mathbb{R})$

$\big([a], [b]\big) : \texttt{CauchyClassGE} \iff [a] \geq [b] \iff \exists x \in [a] : \exists y \in [b] : \forall n \in \mathbb{N} \,.\, x_n \geq y_n$

CauchyClassOrder :: CauchyClassGE : $\texttt{Order}(\mathbb{R})$

Proof $=$

Assume $[x] : \mathbb{R}$,

$(1) := \eth\texttt{Reflexive}(\texttt{order}(\mathbb{N} \to \mathbb{Q})) : x \geq x$,

$() := \eth^{-1}\texttt{CauchyClassGe}(1) : [x] \geq [x]$;

$\rightsquigarrow (1) := \eth^{-1}\texttt{Reflexive}I(\forall) : \Big(\texttt{CauchyClassGe} : \texttt{Reflexive}(\mathbb{R})\Big)$,

Assume $[a], [b] : \mathbb{R}$,

Assume $(2) : [a] \geq [b]$ & $[b] \geq [a]$,

$(x, y, 3) := \eth(2)_1 : \sum (x, y) \in [a] \times [b] \,.\, x \geq y$,

$(x', y', 4) := \eth(2)_2 : \sum (x', y') \in [a] \times [b] \,.\, y' \geq x'$,

$(5) := \eth\mathbb{R}\Big([b]\Big)\eth\texttt{EqualCauchy}(y, y')(3)(4)\eth\mathbb{R}\Big([a]\Big)\eth\texttt{EqualCauchy}(x, x') :$

$\quad : 0 = \lim\limits_{n\to\infty} y_n - y'_n \leq \lim\limits_{n\to\infty} x_n - y'_n \leq \lim\limits_{n\to\infty} x_n - x'_n = 0$,

$(6) := \texttt{DoubleIneqLimit}(5) : \lim\limits_{n\to\infty} x_n - y'_n = 0$,

$() := \eth\mathbb{R}([a].[b])\eth^{-1}\texttt{EqualCauchy}\eth x\eth y'(6) : [a] = [b]$;

$\rightsquigarrow (2) := \eth^{-1}\texttt{Antysimmetric}I(\forall)I(\Rightarrow) : \Big(\texttt{CauchyClassGE} : \texttt{Antysimmetric}(\mathbb{R})\Big)$,

Assume $[a], [b], [c] : \mathbb{R}$,

Assume $(3) : [a] \geq [b]$ & $[b] \geq [c]$,

$(x, y, 4) := \eth(3)_1 : \sum (x, y) \in [a] \times [b] \,.\, x \geq y$,

$(x', z, 5) := \eth(3)_2 : \sum (y', z) \in [b] \times [c] \,.\, y' \geq z$,

$\Delta := y' - y : \mathbb{N} \to \mathbb{Q}$,

$(6) := \eth\texttt{EqualCauchy}(y, y')\eth^{-1}\Delta : \lim\limits_{n\to\infty} \Delta_n = 0$,

$(7) := (4)\Big(x + |\Delta|\Big)\eth^{-1}\Delta\texttt{AbsValueIsGreater}(5) :$

$: x + |\Delta| \geq y + |\Delta| = y' - \Delta + |\Delta| \geq y' \geq z,$

$(8) := \eth\texttt{inverse}(x_n)(6) : \lim_{n\to\infty} x_n + |\Delta_n| - x_n = \lim_{n\to\infty} |\Delta_n| = 0,$

$(9) := \eth\mathbb{R}\eth^{-1}\texttt{EqualCauchy}(*) : x + |\Delta| \in [a],$

$() := \eth\texttt{CauchyClassGe}(9)\eth z(8) : [a] \geq [c];$

$\rightsquigarrow (3) := \eth^{-1}\texttt{Transitive} : \Big(\texttt{CauchyClassGe} : \texttt{Transitive}(\mathbb{R})\Big),$

$(*) := \eth^{-1}\texttt{Order} : \Big(\texttt{CauchyClassGe} : \texttt{Order}(\mathbb{R})\Big);$

$\square$

$\texttt{CauchyClassOrderIsTotal} :: \forall [x],[y] \in \mathbb{R} \,.\, [x] \leq [y] \Big| [y] \leq [x]$

$\texttt{Proof} =$

$(1) := \texttt{LEM}([x] = [y]) : [x] = [y] \;\Big|\; [x] \neq [y],$

$\texttt{Assume}\ (2) : [x] = [y],$

$(3) := \eth\texttt{Reflexive}(\texttt{order}(\mathbb{R}))(2) : [x] \leq [y],$

$() := I(|)(3)([y] \leq [x]) : [x] \leq [y] \;\Big|\; [y] \leq [x];$

$\rightsquigarrow (2) := I(\Rightarrow) : [x] = [y] \Rightarrow \Big([x] \leq [y] \;\Big|\; [y] \leq [x]\Big),$

$\texttt{Assume}\ (3) : [x] \neq [y],$

$(s,4) := \eth\mathbb{Q}_{++}\eth\texttt{EqualCauchy}(3) : \sum s \in \mathbb{Q}_{++} \,.\, \forall N \in \mathbb{N} \,.\, \exists n \in \mathbb{N} : n \geq N \,.\, |x_n - y_n| \geq s,$

$(M,5) := \eth\texttt{Cauchy}(x)(s/4) : \sum M \in \mathbb{N} \,.\, \forall n,m \in \mathbb{N} : n \geq M \,\&\, m \geq M \,.\, |x_n - x_m| \leq \frac{s}{4},$

$(L,6) := \eth\texttt{Cauchy}(y)(s/4) : \sum L \in \mathbb{N} \,.\, \forall n,m \in \mathbb{N} : n \geq L \,\&\, m \geq L \,.\, |y_n - y_m| \leq \frac{s}{4},$

$N := \max(M,L) : \mathbb{N},$

$(7) := \eth\texttt{Total}(\texttt{order}(\mathbb{Q}))(x_N, y_N) : x_N \geq y_N \Big| y_N \geq x_N,$

$\texttt{Assume}\ (8) : x_N \geq y_N,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (9) : n \geq N,$

$(m,10) := (4)(N) : \sum m \in \mathbb{N} \,.\, m \geq N \,\&\, x_m - y_m > s,$

$() := \ldots : x_n - y_n \geq x_m - y_m - \frac{s}{2} \geq \frac{s}{2} > 0;$

$\rightsquigarrow (9) := \eth\texttt{order}(\mathbb{N} \to \mathbb{Q}) : x(+N) \geq y(+N),$

$(10) := \eth\texttt{CaichyClassGE}(9) : [x] \geq [y];$

$\rightsquigarrow (8) := I(\Rightarrow)I(I) : x_N \geq y_N \Rightarrow [x] \geq [y] \;\Big|\; [y] \geq [x],$

$\rightsquigarrow (9) := \texttt{RepeatInvert}(x,y) : y_N \geq x_N \Rightarrow [x] \geq [y] \;\Big|\; [y] \geq [x],$

$() := E(|)(7,8,9) : [x] \geq [y] \;\Big|\; [y] \geq [x];$

$\rightsquigarrow (3) := I(\Rightarrow) : [x] \neq [y] \Rightarrow \Big([x] \geq [y] \;\Big|\; [y] \geq [x]\Big),$

$(4) := E(|)(1,2,3) : [x] \geq [y] \;\Big|\; [y] \geq [x];$

$\square$

$\texttt{CauchyClassesAreOrderedField} :: (\mathbb{R}, \geq) : \texttt{OrderedField}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [a], [b], [c] : \mathbb{R},$

$\texttt{Assume}\ (1) : [a] \geq [b],$

$(x, y, 2) := \eth\texttt{CauchyClassGE}(x) : \sum (x, y) \in [a] \times [y] \;.\; x \geq y,$

$(3) := (2) + c : x + c \geq x + y,$

$() := \eth^{-1}\texttt{CauchyClassGE}(x) : [a] + [c] = [x + c] \geq [y + c] = [b] + [c];$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall [a], [b], [c] \in \mathbb{R} \;.\; [a] \geq [b] \Rightarrow [a] + [c] \geq [b] + [c],$

$\texttt{Assume}\ [a], [b] : \mathbb{R},$

$\texttt{Assume}\ (2) : [a] \geq [0]\ \&\ [b] \geq [0],$

$(x, 3) := \eth\texttt{CauchyClassGE}(x) : \sum x \in [a] \;.\; x \geq 0,$

$(y, 4) := \eth\texttt{CauchyClassGE}(y) : \sum y \in [b] \;.\; y \geq 0,$

$(5) := \eth\texttt{OrderedField}(\mathbb{Q})(x, y) : xy \geq 0,$

$(6) := \eth^{-1}\texttt{CauchyClassGE}(5) : [a][b] \geq [xy] \geq [0];$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall [a], [b] \in \mathbb{R} \;.\; [a] \geq 0\ \&\ [b] \geq 0 \Rightarrow [a][b] \geq [0],$

$(3) := \eth^{-1}\texttt{OrderedField}(1)(2) : \Big(\mathbb{R} : \texttt{OederedField}\Big);$

$\square$

$\texttt{CauchyClassesAreArchimedean} :: (\mathbb{R}, \geq) : \texttt{Archimedean}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [x] : \mathbb{R}_{++},$

$(N, 1) := \eth\texttt{Cauchy}(x)(1) : \sum N \in \mathbb{N} \;.\; \forall n, m \in \mathbb{N} \;.\; n \geq N\ \&\ m \geq N \;.\; |x_n - x_m| \geq 1,$

$n := \lceil x_N \rceil + 1 : \mathbb{N},$

$y := x(N+) : \texttt{Cauchy},$

$\texttt{Assume}\ m : \mathbb{N},$

$() := \eth(y_m)(1)(N + m)\eth\texttt{ceiling}(x_N)\eth^{-1}n : y_m = x_{N+m} \leq x_N + 1 \leq \lceil x_N \rceil + 1 = n;$

$\rightsquigarrow (1) := \eth\texttt{order}(\mathbb{N} \to \mathbb{Q}) : y \leq n,$

$(2) := \eth y(1)\eth\mathbb{R} : [x] \leq [n];$

$\rightsquigarrow (3) := I(\forall)I(\exists, n) : \forall [x] \in \mathbb{R}_{++} \;.\; \exists n \in \mathbb{N} : n \geq [x],$

$(*) := \eth^{-1}\texttt{Archimedean} : \Big(R : \texttt{Archimedean}\Big);$

$\square$

$\texttt{CauchyClassesAreReal} :: \mathbb{R} : \texttt{Real}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A, B :?\mathbb{R},$

$\texttt{Assume}\ (1) : A < B,$

$X := \{x \in \mathbb{R} : A < x < B\} :?\mathbb{R},$

$(2) := \texttt{LEM}\Big(X = \emptyset\Big) : X = \emptyset \Big| X \neq \emptyset,$

$\texttt{Assume}\ (3) : X \neq \emptyset,$

$(x, 4) := (3)(\eth X) : A \leq x \leq B;$

$\rightsquigarrow (3) := I(\rightarrow)I(\exists, x) : X \neq \emptyset \Rightarrow A \leq x \leq B,$

$\texttt{Assume}\ (4) : X = \emptyset,$

$(x_1, 1) := \eth\texttt{NonEmpty}(A) : \sum x_1 \in \mathbb{R}\ .\ x_1 \in A,$

$\texttt{Assume}\ n : \mathbb{N},$

$(x_{n+1}, 3) := (1)(4) : \sum x_{n+1} \in A\ .\ x_{n+1} + \dfrac{1}{2n} \geq A\ \&\ x_n \leq x_{n+1};$

$\rightsquigarrow (x, 3) := I\left(\prod\right) : \prod n \in \mathbb{N}\ .\ \sum x_n \in A\ .\ x_{n+1} + \dfrac{1}{2n} \geq A\ \&\ x_n \leq x_{n+1},$

$(4) := \eth^{-1}\texttt{Nondecreasing}(3) : \Big(x : \texttt{Nondecreasing}\Big),$

$\texttt{Assume}\ n : \mathbb{N},$

$(a^n, 5) := \eth\mathbb{R}(x_n)(4) : \sum a^n : \texttt{Cauchy}\ .\ a^n \in x_n\ \&\ a^n \leq a^{n+1}\ \&\ \forall k, l \in \mathbb{N}\ .\ |a_k^n - a_l^n| < \dfrac{1}{2n},$

$y_n := a_n : \mathbb{Q};$

$\rightsquigarrow y := I(\rightarrow) : \mathbb{N} \rightarrow \mathbb{Q},$

$\texttt{Assume}\ \varepsilon : \mathbb{Q}_{++},$

$(N, 5) := \texttt{ReductioInfima}(\varepsilon) : \dfrac{1}{N} \leq \varepsilon,$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (6) : n > N\ \&\ m > N,$

$() := \eth(a^n, a^m)(3)_1(5) : |x_n + x_m| < \varepsilon;$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Cauchy}\eth I(\forall)I(\exists, N+1)I(\forall)I(\Rightarrow) : \Big(y : \texttt{Cauchy}\Big),$

$\texttt{Assume}\ a : A,$

$(6) := \eth y : a \leq \Lambda n \in \mathbb{N}\ .\ y_n + \dfrac{1}{n},$

$(7) := \eth\mathbb{R}\eth\texttt{EqualCauchy}\left(y, \Lambda n \in \mathbb{N}\ .\ y_n + \dfrac{1}{n}\right) : \Lambda n \in \mathbb{N}\ .\ y_n + \dfrac{1}{n} \in [y],$

$(8) := (6)(7) : a \leq [y],$

$\rightsquigarrow (6)^* := \eth^{-1}\texttt{SetIneq} : A \leq [y],$

$\texttt{Assume}\ b : B,$

$(\beta, 7) := \eth\mathbb{R}(b) : \sum b \in \beta\ .\ \beta : \texttt{Decreasing},$

$\texttt{Assume}\ n : \mathbb{N}$,

$() := \eth y(1) : y_n = a_n^n \leq \beta_n;$

$\rightsquigarrow (8) := \eth\texttt{order}(\mathbb{N} \to \mathbb{Q}) : y \leq \beta,$

$(9) := \eth\mathbb{R}\eth\texttt{CauchyClassGe}\eth\beta : [y] \leq b;$

$\rightsquigarrow () := \eth^{-1}\texttt{SetIneq} : B \geq [y];$

$\rightsquigarrow (1) := \eth^{-1}\texttt{IntermidiatePointProperty} : \Big(\mathbb{R} : \texttt{IntermidiatePointProperty}\Big),$

$(2) := \eth^{-1}\mathbb{R}(1) : \Big(\mathbb{R} : \texttt{Reals}\Big);$

$\square$

1.5 Rationals in Reals

$\mathbb{R} : \texttt{Real}$

$\texttt{RationalApproximation} :: \forall r \in \mathbb{R}_+ + \ . \forall \varepsilon \in \mathbb{R} \ . \exists q \in \mathbb{Q} \ . |r - q| \leq \varepsilon$

$\texttt{Proof} \ =$

$u_0 := \lfloor r \rfloor : \mathbb{Z},$

$\texttt{Assume}\ n : \mathbb{N},$

$d_n := \lceil 10^n (r - u_{n-1}(-1)) \rceil : \mathbb{Z},$

$U_n := u_{n-1} + 10^{-n} d_n : \mathbb{R};$

$\rightsquigarrow (u, 1) := I\left(\sum\right) : \sum u : \mathbb{N} \to \mathbb{Q} \ . |u_n - r| < 10^{-n},$

$(*) := \texttt{PowerCompression}(1) : \exists n \in \mathbb{N} \ . |u_n - r| \leq \varepsilon;$

$\square$

$\texttt{RationalsDensity} :: \mathbb{Q} : \texttt{Dense}(\mathbb{R})$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DisjointIntervalsAreAtmostCountable} :: \forall U : \texttt{Disjoint}\Big(\texttt{OpenInterval}(\mathbb{R})\Big) \ . |U| \leq \aleph_0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ I : U,$

$(q_I, 1_I) := \eth\texttt{Dense}\Big(\texttt{RationalDensity}\Big)(I) : \sum q_I \in \mathbb{Q} \ . q_I \in I;$

$\rightsquigarrow q := I\left(\prod\right) : \prod I \in U \ . \mathbb{Q} \cup I,$

$\texttt{Assume}\ I, J : U,$

$\texttt{Assume}\ (1) : I \neq J,$

$(2) := \eth q_I : q_I \in I,$

$(3) := \eth q_J : q_j \in J,$

$(4) := \eth\texttt{Disjoint}(1, 2, 3) : q_I \neq q_J;$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Injection} : \Big(q : \texttt{Injection}(U, \mathbb{Q})\Big),$

$(6) := \texttt{InjectionCardinality}(5) : |U| \leq |\mathbb{Q}| = \aleph_0;$

$\square$

$\texttt{Period} :: \prod G : \texttt{Abelian} \ . \prod X : \texttt{Set} \ . G \to X \to ?G$

$p : \texttt{Period} \iff \Lambda f : G \to X \ . \forall g \in G \ . f(p + g) = f(p) \ \& \ p \neq 0$

$\texttt{Periodic} :: \prod G : \texttt{Abelian} \ . \prod X : \texttt{Set} \ . ?(G \to X)$

$f : \texttt{Periodic} \iff \exists p : \texttt{Period}(f)$

$\texttt{Coirrational} :: ?(\mathbb{R} \times \mathbb{R})$

$(a, b) : \texttt{Coirrational} \iff \forall q \in \mathbb{Q} \, . \, qa \neq b$

$\texttt{IrrationalGenDense} :: \forall r \in \mathbb{Q}^{\complement} \, . \, \{nr + m | n, m \in \mathbb{Z}\} : \texttt{Dense}$

$\texttt{Proof} =$

$\texttt{Assume}\ (0) : r > 0,$

$\Delta := \Lambda n \in \mathbb{Z} \, . \, \{nr\} : \mathbb{Z} \to [0, 1),$

$x := \inf \operatorname{Im} \Delta : [0, 1),$

$\texttt{Assume}\ (1) : x = \min \operatorname{Im} \Delta,$

$(2) := \eth\Delta\eth\mathbb{Q} : x \notin \mathbb{Q},$

$\mathcal{N} := \{n \in \mathbb{N} : nx \geq 1\} :?\mathbb{N},$

$(3) := \eth\texttt{Archemedean}(\mathbb{R})(1/x)\eth\mathcal{N} : \mathcal{N} \neq \emptyset,$

$n := \inf \mathcal{N} : \mathbb{N},$

$\texttt{Assume}\ (4) : nx - 1 \geq x,$

$(5) := (4) + 1 - x : (n - 1)x \geq 1,$

$(6) := \eth n \eth \min(5) : \bot;$

$\leadsto (4) := E(\bot) : nx - 1 < x,$

$(m, 5) := \eth x \eth r \eth \mathcal{N} \eth n : \sum m \in \mathbb{N} \, . \, nx - 1 = \Delta_m,$

$(6) := \eth x \eth \min(5) : \bot;$

$\leadsto (1) := E(\bot) : x \neq \min \operatorname{Im} \Delta,$

$(\delta, 2) := \eth \inf(1) : \sum \delta \in \operatorname{Im} \Delta \, . \, \lim_{n \to \infty} \delta_n = x,$

$\texttt{Assume}\ a : \mathbb{R},$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(N, 3) := \eth\texttt{Cauchy}(\delta) : \sum N \in \mathbb{N} \, . \, \forall n, m\mathbb{N} \, . \, |\delta_n - \delta_m| < \varepsilon,$

$(m, 4) := \eth\Delta(3)(2)(1) : \sum m \in \mathbb{N} \, . \, \Delta_m < \varepsilon,$

$\mathcal{N} := \{n \in \mathbb{Z} : n\Delta_m > r\} :?\mathbb{Z},$

$(5) := \eth\texttt{Archemedean}\left(\frac{r}{\Delta_m}\right)\eth\mathcal{N} : \mathcal{N} \neq \emptyset,$

$n := \arg \min_{n \in \mathcal{N}} |n| : \mathbb{Z},$

$() := \eth\mathcal{N}\eth n(4) : |n\Delta_m - r| < \varepsilon;$

$\leadsto (*) := \eth\texttt{Dense}\eth\forall\eth\forall : \texttt{This};$

$\square$

$\texttt{DensePeriodicImage} :: \forall f : \texttt{Periodic}(\mathbb{R},\mathbb{R}) \,\&\, C(\mathbb{R},\mathbb{R}) \,.\, \forall \Delta \in \mathbb{R}_{++} \,.$

$\quad .\, \forall (0) : \Big(\forall p : \texttt{Period}(f) \,.\, (p,\Delta) : \texttt{Coirrational}\Big) \,.\, \{f(n\Delta) \mid n \in \mathbb{N}\} : \texttt{Dense}\big(\operatorname{Im} f\big)$

$\texttt{Proof} \;=$

$p := \eth\texttt{Periodic}(f) : \texttt{Period}(p),$

$(1) := (0)(p) : \Big(p,\Delta\Big) : \texttt{Coirrational},$

$\texttt{Assume}\ y : \operatorname{Im} f,$

$(x,4) := \eth \operatorname{Im} f(y) : \sum x \in \mathbb{R} \,.\, f(y) = x,$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta,5) := \eth C(\mathbb{R},\mathbb{R})(x,\delta) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall z \in (x-\delta, x+\delta) \,.\, f(z) \in (y-\varepsilon, y+\varepsilon),$

$(m,z,6) := \texttt{DenseGenGense}(x/p, \delta/p) : \sum m,z \in \mathbb{Z} \,.\, \left|\dfrac{m\Delta}{p} + z - \dfrac{x}{p}\right| < \dfrac{\delta}{p},$

$(7) := p(6) : |m\Delta + pz - x| < \delta,$

$() := \eth\texttt{Period}(f)(p)(5)(7) : |f(m\Delta) - f(z)| = |f\,(m\Delta + pz) - y| \le \varepsilon;$

$\rightsquigarrow (*) := I(\forall)I(\forall)\eth^{-1}\texttt{Dense} : \texttt{This};$

$\square$

2 Real Sequences

2.1 Monotonic Sequences

$\texttt{NondecreasingAndBoundedConverge} :: \forall x : \texttt{Nondecreasing} \ \& \ \texttt{BoundedFromAbove}(\mathbb{N}, \mathbb{R}) \ . \ x : \texttt{Convergent}$

$\texttt{Proof} \ =$

$X := x(\mathbb{N}) :?\mathbb{R},$

$(1) := \eth X \eth \texttt{BoundedFomAbove} x : \Big(X : \texttt{BoundedFromAbove}\Big),$

$c := \sup X : \mathbb{R},$

$\texttt{Assume}\ (2) : \lim_{n\to\infty} x_n \neq c,$

$(\varepsilon, 3) := \eth\texttt{Limit}(2) : \sum \varepsilon \in \mathbb{R}_{++} \ . \ \forall n \in \mathbb{N} \ . \ \exists m \in \mathbb{N} : m \geq n \ \& \ |c - x_n| > \varepsilon,$

$(4) := (3)\eth c \eth \sup \eth X : \forall n \in \mathbb{N} \ . \ \exists m \in \mathbb{N} : m \geq n \ \& \ c - x_n > \varepsilon,$

$(5) := \eth\texttt{Nondecreasing}(4) : \forall n \in \mathbb{N} \ . \ c - x_n > \varepsilon,$

$(6) := \eth \sup \eth X(5) : c \neq \sup X,$

$() := (6)\eth c : \bot;$

$\rightsquigarrow (*) := E(\bot) : c = \lim_{n\to\infty} x_n;$

$\square$

$\texttt{NonincreasingAndBoundedConverge} :: \forall x : \texttt{Nonincreasing} \ \& \ \texttt{BoundedFromBelow}(\mathbb{N}, \mathbb{R}) \ . \ x : \texttt{Convergent}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{MonotonicAndBoundedConverges} :: \forall x : \texttt{Monotonic} \ \& \ \texttt{Bounded}(\mathbb{N}, \mathbb{R}) \ . \ x : \texttt{Convergent}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{limitSuperior} :: \mathbb{N} \to \mathbb{R} \to \overset{\infty}{\mathbb{R}}$

$\texttt{limitSupereior}\,(x) = \limsup x := \lim_{n\to\infty} \sup\{x_m | m \in \mathbb{N} : m \geq n\}$

$\texttt{limitInferior} :: \mathbb{N} \to \mathbb{R} \to \overset{\infty}{\mathbb{R}}$

$\texttt{limitInferior}\,(x) = \liminf x := \lim_{n\to\infty} \inf\{x_m | m \in \mathbb{N} : m \geq n\}$

$\texttt{LimitReverse} :: \forall x : \mathbb{N} \to \mathbb{R} \ . \ -\limsup x = \liminf -x$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LimSupStructure} :: \forall x : \mathbb{N} \to \mathbb{R} \;.\; \exists k : \texttt{Subsequencer} \;.\; \limsup x = \lim_{n\to\infty} x_{k_n}$

$\texttt{Proof} \;=$

$X := \Lambda n \in \mathbb{N} \;.\; \{x_m | m \in \mathbb{N} : m \geq n\} : \mathbb{N} \to ?\mathbb{R},$

$\texttt{Assume}\ (1) : \forall n \in \mathbb{N} \;.\; \exists y \in X_n : y = \max X_n,$

$(y_1, 2_1) := (1)(1) : \sum y_1 \in X_1 \;.\; y_1 = \max X_1,$

$(k_1, 3_1) := \eth X_1(y_1) : \sum k_1 \in \mathbb{N} \;.\; y_1 = x_{k_1},$

$\texttt{Assume}\ n : \mathbb{N},$

$(y_{n+1}, 2_{n+1}) := (1)(k_n + 1) : \sum y_{n+1} \in X_{k_n+1} \;.\; y_{n+1} = \max X_{k_n+1},$

$(k_{n+1}, 3_{n+1}) := \eth X_{k_n+1} \eth y_{n+1} : \sum k_{n+1} \;.\; y_{n+1} = x_{k_{n+1}},$

$() := \eth X_{k_n+1} \eth k_{n+1}(3_{n+1})(2_{n+1}) : k_{n+1} > k_n;$

$\rightsquigarrow (k, 2) := \eth \texttt{Subsequencer} I(\sum) : \sum k : \texttt{Subsequencer} \;.\; \forall n \in \mathbb{N} \;.\; x_{k_n} = \max X_{k_n},$

$() := \lim_{n\to\infty} (2)\eth^{-1} \sup \texttt{ConvergentSubseq}\left(\overset{\infty}{\mathbb{R}}\right)(\sup X_k)\eth^{-1} \limsup x :$

$\lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} \max X_{k_n} = \lim_{n\to\infty} \sup X_{k_n} = \lim_{n\to\infty} \sup X_n = \limsup x;$

$\rightsquigarrow (1) := I(\Rightarrow)I(\exists)(k) : \forall n \in \mathbb{N} \;.\; \exists y \in X_n : y = \max X_n \Rightarrow \texttt{This},$

$\texttt{Assume}\ (2) : \exists n \in \mathbb{N} \;.\; \forall y \in X_n \;.\; y \neq \max X_n,$

$(n, 3) := E(\exists)(2) : \sum n \in \mathbb{N} \;.\; \forall y \in X_n \;.\; y \neq X_n,$

$(k, 4) := \eth X_n \eth \sup : \exists k : \texttt{Subsequencer} \;.\; \lim_{m\to\infty} x_{k_m} = \sup X_n,$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ (5) : m \geq n,$

$(l, 6) := \eth X_n \eth X_m : \sum l \in \mathbb{N} \;.\; \forall d \in \mathbb{N} : d \geq l \;.\; x_{k_l} \in X_m,$

$() := \eth \sup(6) : \sup X_m = \lim_{m\to\infty} x_{k_m};$

$\rightsquigarrow () := \lim_{m\to\infty} \texttt{FinitelyReducedSequenve}\eth^{-1} \limsup : \lim_{m\to\infty} x_{k_m} = \lim_{m\to\infty} \sup X_m = \limsup x;$

$\rightsquigarrow (2) := I(\Rightarrow)I(\exists) : \exists n \in \mathbb{N} \;.\; \forall y \in X_n \;.\; y \neq \max X_n \Rightarrow \texttt{This},$

$(3) := \texttt{LEM}(\forall n \in \mathbb{N} \;.\; \exists x \in X_n : x = \max X_n) : \forall n \in \mathbb{N} \;.\; \exists x \in X_n : x = \max X_n \;\Big|$

$\Big|\; \exists n \in \mathbb{N} : \forall x \in X_n \;.\; x \neq \max X_n,$

$(*) := E(|)(1, 2, 3) : \texttt{This};$

$\square$

$\texttt{LimInfStructure} :: \forall x : \mathbb{N} \to \mathbb{R} \;.\; \exists k : \texttt{Subsequencee} \;.\; \lim_{n\to\inf} x_{k_n} = \liminf x$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ConvergenceByCoincidence} :: \forall x : \texttt{Bounded}(\mathbb{N}, \mathbb{R}) \,.\, x : \texttt{Convergent} \iff \liminf x = \limsup x$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : (x : \texttt{Convergent}),$

$X := \lim_{n\to\infty} x_n : \mathbb{R},$

$(k, 2) := \texttt{LimSupStructure}(x) : \sum k : \texttt{Subsequencer} \,.\, \lim_{n\to\infty} x_{k_n} = \limsup x,$

$(3) := \texttt{ConvergentSubseq}(2, \eth X) : \limsup x = X,$

$(l, 4) := \texttt{LimInfStructure}(x) : \sum l : \texttt{Subsequencer} \,.\, \lim_{n\to\infty} x_{l_n} = \liminf x,$

$(5) := \texttt{ConvergentSubseq}(4, \eth X) : \liminf x = X,$

$() := E(=)(3)(5) : \liminf x = \limsup x;$

$\rightsquigarrow (1) := E(\Rightarrow) : x : \texttt{Convergent} \Rightarrow \liminf x = \limsup x,$

$\texttt{Assume}\ (2) : \liminf x = \limsup x,$

$X := \limsup x : \mathbb{R},$

$\texttt{Assume}\ n : \mathbb{N},$

$() := \eth \sup \eth \inf : \inf\{x_m | m \in \mathbb{N} : m \geq n\} \leq x_n \leq \sup\{x_m | m \in \mathbb{N} : m \geq n\};$

$\rightsquigarrow (3) := \eth^{-1} X \texttt{DoubleIneq}(2) \eth^{-1} \limsup \eth^{-1} \liminf \lim_n \to \infty : \lim_{n\to\infty} = X,$

$() := \eth^{-1}\texttt{Convergent}(3) : \Big(x : \texttt{Convergent}\Big);$

$\rightsquigarrow (2) := I(\Rightarrow) : \liminf x = \limsup x \Rightarrow x : \texttt{Convergent},$

$(*) := I(\iff)(1, 2) : \liminf x = \limsup x \iff x : \texttt{Convergent};$

$\square$

2.2 Stolz-Cizaro Theorem

$\texttt{Stolz} :: ?(\mathbb{N} \to \mathbb{R})$

$x : \texttt{Stolz} \iff \exists y : \mathbb{N} \to \mathbb{R} : \exists z : \texttt{Increasing} \ . \ \lim_{n\to\infty} z_n = \infty \ \& \ x = \frac{y}{z}$

$\texttt{stolzOperator} :: \texttt{Stolz} \to \mathbb{N} \to \mathbb{R}$

$\texttt{stolzOperator}\left(\frac{x}{y}\right) = \frac{\Delta x}{\Delta y} := \Lambda n \in \mathbb{N} \ . \ \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$

$\texttt{StolzCizaro} :: \forall \frac{x}{y} : \texttt{Stolz} \ . \ \forall L \in \mathbb{R} \ . \ \lim_{n\to\infty} \frac{\Delta x_n}{\Delta y_n} = L \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = L$

$\texttt{Proof} =$

$\texttt{Assume} \ \varepsilon : \mathbb{R},$

$(N,1) := \eth\texttt{Limit}\left(\frac{\Delta x}{\Delta y}, L\right) : \sum N \in \mathbb{N} \ . \ \forall n \in \mathbb{N} : n \geq N \ . \ \left|\frac{\Delta x_n}{\Delta y_n} - L\right| < \varepsilon,$

$\texttt{Assume} \ n : \mathbb{N},$

$\texttt{Assume} \ (2) : n \geq N,$

$() := (1)(n,2)/(y_{n+1} - y_n) : (L-\varepsilon)(y_{n+1} - y_n) \leq x_{n+1} - x_n \leq (L+\varepsilon)(y_{n+1} + y_n);$

$\rightsquigarrow (2) := I(\forall) : \forall n \in \mathbb{N} : n \geq N \ . \ (L-\varepsilon)(y_{n+1} + y_n) \leq x_{n+1} - x_n \leq (L+\varepsilon)(y_{n+1} + y_n),$

$\texttt{Assume} \ k : \mathbb{N},$

$\texttt{Assume} \ (3) : k > N,$

$(4) := \sum_{n=N}^{k-1} (2)(n) : (L-\varepsilon)(y_k + y_N) \leq x_k + x_N \leq (L+\varepsilon)(y_k + y_N),$

$(5) := (2)/y_k : (L-\varepsilon)\left(1 + \frac{y_N}{y_k}\right) \leq \frac{x_k}{y_k} + \frac{x_N}{y_k} \leq (L+\varepsilon)\left(1 + \frac{y_N}{y_k}\right);$

$\rightsquigarrow () := \texttt{LimitIneq} : (L-\varepsilon) \leq \lim_{n\to\infty} \frac{x_n}{y_n} \leq (L+\varepsilon);$

$\rightsquigarrow (1) := \lim_{\varepsilon\to 0} I(\forall) : L \leq \lim_{n\to\infty} \frac{x_n}{y_n} \leq L,$

$(*) := \texttt{DoubleIneq}(1) : \lim_{n\to\infty} \frac{x_n}{y_n} = L;$

$\square$

2.3 Real Series

$$\texttt{partialSums} :: \prod G : \texttt{TopologicalGroup} \,.\, (\mathbb{N} \to G) \to \mathbb{N} \to G$$

$$\texttt{partialSums}\,(x) = S(x) := \Lambda n \in \mathbb{N} \,.\, \sum_{i=1}^{n} x_i$$

$$\texttt{ConvergentSeria} :: \prod G : \texttt{TopologicalGroup} \,.\, ?(\mathbb{N} \to G)$$

$$x : \texttt{ConvergentSums} \iff S(x) : \texttt{Convergent}$$

$$\texttt{infinitSum} :: \prod G : \texttt{TopologicalGroup} \,.\, \texttt{ConvergentSeria}(G) \to G$$

$$\texttt{infinitSum}\,(x) = \sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} S_n(x)$$

$$\texttt{SeriaSum} :: \forall a, b : \texttt{ConvergentSeria}(\mathbb{R}) \,.\, \sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{NthTermTest} :: \forall x : \texttt{ConvergentSeria}(\mathbb{R}) \,.\, \lim_{n\to\infty} x_n = 0$$

$\texttt{Proof} =$

$(1) := \texttt{ConvergentIsCauchy}\eth\texttt{ConvergentSeria}(x) : \Big(S(x) : \texttt{Cauchy}\Big),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(N, 2) := \eth\texttt{Cauchy}(x, 1)(\varepsilon) : \sum N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} : \min(n, m) \geq N \,.\, \Big|S_n(x) - S_m(x)\Big| < \varepsilon,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N + 1,$

$(4) := \eth^{-1}\texttt{partialSums}(2)(n, 3) : |x_n| = |S_{n+1}(x) - S_n(x)| < \varepsilon;$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Limit}I(\forall)I(\exists)(N + 1)I(\forall)I(\Rightarrow) : \lim_{n\to 0} x_n = 0,$

$\square$

$$\texttt{ComperissonTest} :: \forall a : \mathbb{N} \to \mathbb{R} \,.\, \forall b : \texttt{ConvergentSeria}(\mathbb{R}) \,.\, \forall(0) : |a| \leq b \,.\, a : \texttt{ConvergentSeria}(\mathbb{R})$$

$\texttt{Proof} =$

$(1) := \texttt{ConvergentIsCauchy}\eth\texttt{ConvergentSeria}(b) : \Big(S(b) : \texttt{Cauchy}\Big),$

$\texttt{Assume}\ \varepsilon : \mathbb{R},$

$(N, 2) := \eth\texttt{Cauchy}(b, 1)(\varepsilon) : \sum N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} : \max(n, m) \geq N \,.\, |S_n(b) - S_m(b)|,$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (3) : \max(n, m) \geq N,$

$() := ðS(a)\texttt{TriangeIneq}(a)(0)ð^{-1}S(b)(2)(n,m,3) :$

$$: |S_n(a) - S_m(a)| = \left|\sum_{i=n}^{m} a_i\right| \le \sum_{i=m}^{n} |a_i| \le \sum_{i=m}^{n} b_i = |S_n(b) - S_m(b)| < \varepsilon;$$

$\rightsquigarrow (5) := ð^{-1}\texttt{ConvergentSums}I(\forall)I(\exists)(N+1)I(\forall)I(\Rightarrow) : \Big(a : \texttt{ConvergentSeria}\Big),$

$\square$

$\texttt{alternatingSigns} :: \mathbb{N} \to \mathbb{Z}$
$\texttt{alternatingSigns}\,() = (-1)^n := \Lambda n \in \mathbb{N} \,.\, (-1)^n$

$\texttt{AlternatingTest} :: \forall x : \mathbb{N} \to \mathbb{R}_+ \,.\, \forall(0_1) : \lim_{n\to\infty} x_n = 0 \,.\, \forall(0_2) : \Big(|x| : \texttt{Decreasing}\Big) \,.$
$\quad .\, (-1)^n x : \texttt{ConvergingSeria}(\mathbb{R})$

$\texttt{Proof} =$

$$a := \Lambda n \in \mathbb{N} \,.\, \sum_{i=1}^{2n} (-1)^i x_i : \mathbb{N} \to \mathbb{R},$$

$$b := \Lambda n \in \mathbb{N} \,.\, \sum_{i=1}^{2n-1} (-1)^i x_i : \mathbb{N} \to \mathbb{R},$$

$\texttt{Assume}\ n : \mathbb{N},$
$(*_1) := ðað\texttt{Decreasing}(x) : a_{n+1} - a_n = x_{2n+2} - x_{2n+1} < 0,$
$(*_2) := ðbð\texttt{Decreasing}(x) : b_{n+1} - b_n = -x_{2n+1} + x_{2n} > 0,$
$() := ðbðaðxð\mathbb{R}_+ : a_n - b_n = x_{2n} \ge 0;$

$\rightsquigarrow (1) := ð^{-1}\texttt{Increasing}ð^{-1}\texttt{Decreasing}ð^{-1}\texttt{order}(\mathbb{N} \to \mathbb{R}) : \Big(a : \texttt{Decreasing}\Big) \,\&\, \Big(b : \texttt{Increasing}\Big) \,\&\, b \le a,$

$(2) := (1_1)(1_3) : b_1 \le a,$

$(3) := ð^{-1}\texttt{BoundedFromBelow}(2) : \Big(a : \texttt{BoundedFromBelow}\Big),$

$(4) := (1_2)(1_3) : b \le a_1,$

$(5) := ð^{-1}\texttt{BoundedFromAbove}(2) : \Big(b : \texttt{BoundedFromAbove}\Big),$

$6 := \texttt{NondecreasingAndBoundedConverge}(b, 1_2, 5) : \Big(b : \texttt{Convergent}\Big),$

$7 := \texttt{NonincreasingAndBoundedConverge}(a, 1_1, 3) : \Big(a : \texttt{Convergent}\Big),$

$A := \lim_{n\to\infty} a_n : \mathbb{R},$
$B := \lim_{n\to\infty} b_n : \mathbb{R},$
$(8) := ðAðB(A-B)\texttt{LimitSum}ðaðb(0_1) : A - B = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty} x_{2n} = 0,$
$(9) := ð\texttt{Inverse}(\mathbb{R})(8) : A = B,$

$$(*) := \texttt{SequenceCompositionLimits}(9, ða, ðb)ð^{-1}\texttt{infiniteSum} : \sum_{n=1}^{\infty} (-1)^n x_n = A;$$

$\square$

$\texttt{geometricSeria} :: \mathbb{R} \to \mathbb{N} \to \mathbb{R}$
$\texttt{geometricSeria}\,(a) = a^n := \Lambda n \in \mathbb{N} \,.\, a^{n-1}$

$\texttt{FiniteGeometricSum} :: \forall a \in \mathbb{R} : a \neq 1 \,.\, \forall n \in \mathbb{N} \,.\, \sum_{i=0}^{n} a^i = \frac{1 - a^{i+1}}{1 - a}$

$\texttt{Proof} \; =$

$(1) := I(=)(1) : \sum_{i=0}^{0} a^i = 1 = \frac{1 - a^1}{1 - a},$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; (2) : \texttt{This}(a, n),$

$(2) :=: \sum_{i=0}^{n+1} a^i = a^{n+1} + \frac{1 - a^{n+1}}{1 - a} = \frac{1 - a^{n+2}}{1 - a};$

$\rightsquigarrow (*) := I(\mathbb{N})(1) : \sum_{i=0}^{n} a^i = \frac{1 - a^{i+1}}{1 - a};$

$\square$

$\texttt{InfiniteGeometricSum} :: \forall a \in (-1, 1) \,.\, \sum_{n=0}^{\infty} a^n = \frac{1}{1 - a}$

$\texttt{Proof} \; =$

$(*) := \eth\texttt{infiniteSum}(a^n)\eth S_n(a^n)\texttt{FiniteGeometrisSum}(a)\texttt{PowerCompression}(a) :$

$$\sum_{n=0}^{\infty} a^n = \lim_{n\to\infty} S_n(a^n) = \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - a} = \frac{1}{1 - a};$$

$\square$

$\texttt{RatioTest} :: \forall x : \mathbb{N} \to \mathbb{R} \,.\, \forall r \in (0, 1) \,.\, \forall (0) : \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = r \,.\, x : \texttt{ConvergentSeria}$

$\texttt{Proof} \; =$

$\varepsilon := \frac{1 - r}{2} : (0, 1),$

$r' := r + \varepsilon : (0, 1),$

$(N, 1) := \eth\texttt{Limit}(0)(\varepsilon) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \geq N \,.\, \left| \frac{|x_{n+1}|}{|x_n|} - r \right| < \varepsilon,$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; (2) : n \geq N,$

$() := \Big(\eth\texttt{absValuse}(1)(n, 2) + r\Big)|x_n|\eth^{-1}r' : |x_{n+1}| < r'|x_n|;$

$\rightsquigarrow (2) := I(\forall) : \forall n \in \mathbb{N} : n \geq N \,.\, |x_{n+1}| < r'|x_n|,$

$(3) := \texttt{InductionIneq}(2) : \forall n \in \mathbb{N} \,.\, |x_{N+n}| < (r')^n |x_N|,$

$(*) := \texttt{ComparissonTest}(\texttt{InfiniteGeometricSum}(a), x_{+N}) + \sum_{i=1}^{N-1} x_i : \Big(x : \texttt{ConvergentSeria}\Big),$

$\square$

2.4 Absolute Convergence [!!]

$\texttt{AbsolutelyConvergent} :: ?\texttt{ConvergentSeria}(\mathbb{R})$

$x : \texttt{AbsolutelyConvergent} \iff |x| : \texttt{ConvergentSeria}(\mathbb{R})$

$\texttt{AbsConvStable} :: \forall x : \texttt{AbsolutelyConvergent}(\mathbb{R}) \,.\, \forall \sigma : \mathbb{N} \leftrightarrow_{\mathsf{SET}} \mathbb{N} \,.\, \sum_{i=1}^{\infty} x_{\sigma(i)} = \sum_{i=1}^{\infty} x_i$

$\texttt{Proof} =$

$\texttt{Assume}\ \varepsilon : \mathbb{R},$

$(N,1) := \eth\texttt{Cauchy}\eth\texttt{ConvergentSeria}\Big(\eth\texttt{AbsolutelyConvergent}\Big)(\varepsilon) :$

$\quad : \sum N \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} : \max(n,m) \geq N \,.\, \Big|S_n\big(|x|\big) - S_m\big(|x|\big)\Big|,$

$M := \max\{\sigma^{-1}(n) : 1 \leq n \leq N\} : \mathbb{N},$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (2) : n \geq M \ \&\ m \geq N,$

$(3) := \eth S(x)\texttt{TriangularIneq}\eth M(2)\eth^{-1}S(|x|)(1)(\eth N) :$

$\quad : \Big|S_{\sigma(n)}(x) - S_m(x)\Big| \leq \sum_{i=N}^{\sigma(n)} |x_i| + \sum_{i=N}^{m} |x_i| \leq 2\sum^{\sigma}(n)_{i=N}|x_i| = 2\Big(S_{\sigma(n)}(|x|) - S_N(|x|)\Big) < 2\varepsilon;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{infinitSumContinuousAddition}\eth^{-1}\texttt{Limit} : \sum_{i=1}^{\infty} x_{\sigma(i)} - \sum_{i=1}^{\infty} x_i = 0,$

$(*) := (1) + \sum_{i=1}^{\infty} x_i : \sum_{i=1}^{\infty} x_{\sigma(i)} = \sum_{i=1}^{\infty} x_i;$

$\square$

$\texttt{ConditionallyConvergent} := \texttt{ConvergentSeria}(\mathbb{R}) \ \&\ !\,\texttt{AbsolutelyConvergent} : \texttt{Type},$

$\texttt{support} :: \prod G : \texttt{Abelean} \,.\, (\mathbb{N} \to G) \to ?\mathbb{N}$

$\texttt{support}\,(x) = \operatorname{supp} x := \{n \in \mathbb{N} : x_n \neq 0\}$

$\texttt{CondConvStructure} :: \forall x : \texttt{ConditionallyConvergent} \,.\, \#\operatorname{supp} x^- = \infty = \#\operatorname{supp} x^+$

$\texttt{Proof} =$

$\texttt{Assume}\ (1) : \#\operatorname{supp} x^- < \infty,$

$(I,2) := \eth\texttt{Finite}(1) : \sum I : \texttt{Finite}(\mathbb{N}) \,.\, I = \operatorname{supp} x^-,$

$(3) := \eth_1\texttt{ConditionallyConvergent}(x)\eth^{-1}\texttt{absValue}\eth x^-(1)(2) : \sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} |x_i| + 2\sum_{i \in I} x_i,$

$() := \eth^{-1}\texttt{AsolutelyConvergent}\eth\texttt{ConditionallyConvergent}(x) : \bot;$

$\rightsquigarrow (3) := E(\bot) : \#\operatorname{supp} x^- = \infty,$

$\ldots$

$\square$

$\mathtt{RiemannRearangementTHM} :: \forall x : \mathtt{ConditionallyConvergent} \,.\, \forall r \in \mathbb{R} \,.\, \exists \sigma : \mathbb{N} \leftrightarrow_{\mathsf{SET}} \mathbb{N} \,.\, \sum_{n=1}^{\infty} x_{\sigma(n)} = r$

$\mathtt{Proof} \; =$

$I := \operatorname{supp} x^{+} : ?\mathbb{N},$

$J := \operatorname{supp} x^{-} : ?\mathbb{N},$

$n := \mathtt{CondConvStructure}(x)\text{þ}I\eth\mathtt{EqCard} : I \leftrightarrow_{\mathsf{SET}} \mathbb{N},$

$m := \mathtt{CondConvStructure}(x)\text{þ}J\eth\mathtt{EqCard} : J \leftrightarrow_{\mathsf{SET}} \mathbb{N},$

$(1) := \eth\mathtt{ConditionallyConvergent}(x)\eth\mathtt{absVal}\text{þ}^{-1}I\text{þ}^{-1}J\eth^{-1}n\eth^{-1}m :$

$$: \infty = \sum_{n=1}^{\infty} |x_n| = \sum_{i=1}^{\infty} x_{n_i} - \sum_{j=1}^{\infty} x_{m_j},$$

$$(2) := \text{þ}^{-1}I\text{þ}^{-1}J\eth^{-1}n\eth^{-1}m : \sum_{n=1}^{\infty} x_n = \sum_{i=1}^{\infty} x_{n_i} + \sum_{j=1}^{\infty} x_{m_j},$$

$$(3) := (1) + (2) : \sum_{i=1}^{\infty} x_{n_i} = \infty,$$

$$(4) := (1) - (2) : \sum_{j=1}^{\infty} x_{m_j} = -\infty,$$

$(N_0, M_0, K_0) := (0, 0) : \mathbb{Z},$

$\mathtt{Assume}\ a : \mathbb{Z}_{+},$

$\mathtt{Assume}\ i : \mathtt{In}\{0, 1\},$

$$R_{2a+i} := r - \sum_{j=1}^{K_{2a+i-1}} y_j : \mathbb{R},$$

$\mathtt{Assume}\ (5) : i = 0,$

$$\mathcal{K} := \left\{ k \in \mathbb{N} : \sum_{j=N_{a-1}+1}^{k} x_{n_j} \geq R_{2a+i} \right\} : ?\mathbb{N},$$

$(6) := (3)(\text{þ}\mathcal{K}) : \mathcal{K} \neq \emptyset,$

$N_a := \min \mathcal{K} : \mathbb{N},$

$K_{2a} := K_{2a-1} + N_a - N_{a-1} : \mathbb{Z}_{+},$

$\mathtt{Assume}\ K_{2a-1} + k : (K_{2a-1}, K_{2a}]_{\mathbb{N}},$

$y_{K_{2n}+k} := x_{N_a+k} : \mathbb{R};$

$\rightsquigarrow y := \mathtt{Extend}(\rightarrow) : (0, K_{2a}] \rightarrow \mathbb{R},$

$$(7_{n,i}) := \text{þ}y\text{þ}\mathcal{K} : r - \sum_{j=1}^{K_{2a}} \leq 0;$$

$\rightsquigarrow (5^{*}) := \mathtt{Alternative}\ (y) : \ldots,$

$\mathtt{Assume}\ (5) : i = 1,$

$$\mathcal{K} := \left\{ k \in \mathbb{N} : \sum_{j=M_{a-1}+1}^{k} x_{m_j} \leq R_{2a+i} \right\} : ?\mathbb{N},$$

$(6) := (3)(\text{þ}\mathcal{K}) : \mathcal{K} \neq \emptyset,$

$M_a := \min \mathcal{K} : \mathbb{N},$

$K_{2a+1} := K_{2a} + M_a - M_{a-1} : \mathbb{Z}_{+},$

$\texttt{Assume}\ K_{2a}+k:(K_{2a},K_{2a+1}]_{\mathbb{N}},$

$y_{K_{2n}+k}:=x_{N_a+k}:\mathbb{R};$

$\rightsquigarrow y:=\texttt{Extend}(\rightarrow):(0,K_{2a+1}]\to\mathbb{R},$

$$(7_{n,i}):=\text{þ}y\text{þ}\mathcal{K}:r-\sum_{j=1}^{K_{2a}}\leq 0;$$

$$\rightsquigarrow (y,K,5):=I\left(\sum\right):\sum y:\mathbb{N}\to\mathbb{R}\ .\ \sum K:\texttt{Increasing}(\mathbb{N},\mathbb{N})\ .$$

$$.\ \forall a\in(2,\infty)_{\mathbb{N}}\ .\ \forall b\in[K_a,K_{a+1})\ .\ \left|r-\sum_{i=1}^{b}y_i\right|\leq|y_{K_a}|,$$

$\texttt{Assume}\ \varepsilon:\mathbb{R}_{++},$

$(\sigma,6):=\text{þ}y:\sum\sigma:\mathbb{N}\leftrightarrow_{\mathsf{SET}}\mathbb{N}\ .\ y:\texttt{Subsequence}(x_\sigma),$

$$(7):=\texttt{SubseqLimit}\ \texttt{LimitRearangement}(x,x_\sigma)\Big(\texttt{NthTermTest}(\eth_1\texttt{ConditionallyConvergent}(x))\Big):$$

$$:\lim_{n\to\infty}y_n=0,$$

$(N,8):=\eth\texttt{Limit}(7)(\varepsilon):\sum N\in\mathbb{N}\ .\ \forall n\in\mathbb{N}:n\geq N\ .\ |y_n|<\varepsilon,$

$(M,9):=\eth\texttt{Increasing}(K)(N):\sum M\in\mathbb{N}:\forall m\in\mathbb{N}:m\geq M\ .\ K_m\geq N,$

$\texttt{Assume}\ n:\mathbb{N},$

$\texttt{Assume}\ (10):n\geq\max(K_M,K_2),$

$(m,11):=\text{þ}K(n):\sum m\in\mathbb{N}\ .\ n\in[K_m,K_{m+1})\ \&\ m\geq M,$

$(12):=\eth\texttt{increasing}(K)(11)(9):K_m\geq K_M\geq N,$

$$():=\eth\texttt{partialSum}(m,y)(5)(m)(8)(12):|S_m(y)-r|=\left|\sum_{i=1}^{m}y_i-r\right|\leq|y_{K_m}|<\varepsilon;$$

$$\rightsquigarrow (6):=\eth^{-1}\texttt{infiniteSum}\eth^{-1}\texttt{Limit}I(\forall)I(\exists)(K_M)I(\forall)I(\Rightarrow):\sum_{n=1}^{\infty}y_n=r,$$

$$(*):=\texttt{ZeroSeriaPart}(x_\sigma,y)(6):\sum_{n=1}^{\infty}x_{\sigma(n)}=r;$$

□

$$\texttt{RiemannRearangementDivergnece}::\forall x:\texttt{ConditionallyConvergent}\ .\ \exists\sigma:\mathbb{N}\leftrightarrow_{\mathsf{SET}}\mathbb{N}\ .\ \sum_{n=1}^{\infty}x_{\sigma(n)}=\infty$$

$\texttt{Proof}\ =$

...

□

$\texttt{productPartialSums} :: \prod R : \texttt{Ring} \,.\, (\mathbb{N} \to R^2) \to \mathbb{N} \to R$

$$\texttt{productPartialSums}\,(x, y) = S(x, y) := \Lambda n \in \mathbb{N} \,.\, \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$\texttt{CauchyProduct} :: \forall x, y : \texttt{AbsolutelyCovergent} \,.\, \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} x_m y_{n-m} = \left(\sum_{n=1}^{\infty} x_n\right)\left(\sum_{n=1}^{\infty} y_n\right)$$

$\texttt{Proof} \ =$

$\dots$

$\square$

2.5 Real Exponention [!!]

$$\texttt{realExponent} :: \mathbb{R} \to \mathbb{Z}_+ \to \mathbb{R}$$
$$\texttt{realExponent}\,(a, 0) = a^0 := 1$$
$$\texttt{realExponent}\,(a, n) = a^n := aa^{n-1}$$

$$\texttt{realNegativeExponent} :: \mathbb{R}^\times \to \mathbb{Z}_{--} \to \mathbb{R}^\times$$
$$\texttt{realNegativeExponent}\,(a, -n) = a^{-n} := \frac{1}{a^n}$$

$\texttt{PositiveRootExists} :: \forall a \in \mathbb{R}_+ \,.\, \forall n \in \mathbb{N} \,.\, \exists b \in \mathbb{R} \,.\, b = \sup\{x \in \mathbb{R} : x^n \leq a\}$

$\texttt{Proof} \;=$

$A := \{x \in \mathbb{R} : x^n \leq a\} :?\mathbb{R},$

$(m, 1) := \eth\texttt{Archimedean}(a) : \sum m \in \mathbb{N} \,.\, a < m,$

$(2) := \texttt{NaturalIneqExp}(1) : a < m \leq m^n,$

$(3) := \sqrt[n]{\flat A(2)} : A < m,$

$i(4) := \eth^{-1}\texttt{BoundedFomAbove}(3) : \Big(A : \texttt{BoundedFromAbove}(\mathbb{R})\Big),$

$b := \sup A : \mathbb{R};$

$\square$

$$\texttt{realRoot} :: \mathbb{N} \to \mathbb{R}_+ \to \mathbb{R}_+$$
$$\texttt{realRoot}\,(n, a) = \sqrt[n]{a} := \sup\{x \in \mathbb{R} : x^n \leq a\}$$

$$\texttt{realRationalExponent} :: \mathbb{R}_{++} \to \mathbb{Q} \to \mathbb{R}_{++}$$
$$\texttt{realsRationalExponent}\left(a, \frac{n}{m}\right) = a^{\frac{n}{m}} := \left(\sqrt[m]{a}\right)^n$$

$\texttt{RealRatExpIsConsistent} :: \forall x \in \mathbb{R}_{++} \,.\, \forall \frac{a}{b}, \frac{n}{m} \in \mathbb{Q}.\forall(0) : \frac{a}{b} =_{\mathbb{Q}} \frac{n}{m} \,.\, x^{\frac{a}{b}} = x^{\frac{n}{m}}$

$\texttt{Proof} \;=$

$(c, d, k, l, 1) := \eth\mathbb{Q}(1) : \sum cd, k \in \mathbb{N} \,.\, \sum l \in \mathbb{Z} \,.\, n = cl \;\&\; a = dl \;\&\; m = ck \;\&\; b = dk,$

$(2) := \eth x^{\frac{n}{m}}(1)\eth\texttt{RealRoot}(1)\eth^{-1}x^{\frac{a}{b}} : x^{\frac{n}{m}} = \left(\sqrt[m]{x}\right)^n = \left(\sqrt[ck]{x}\right)^{cl} = \left(\sqrt[b]{x}\right)^a = x^{\frac{a}{b}};$

$\square$

$\texttt{RootLimit} :: \forall a \in \mathbb{R}_{++} \,.\, \lim_{n\to\infty} a^{1/n} = 1$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : a \geq 1,$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (2) : n > m,$

$(3) := \eth\texttt{realRoot}(n, m)(a) : (\sqrt[n]{a})^n = a = (\sqrt{m}a)^m,$

$(4^*) := \texttt{IneqMult}(3_1)(1) : \sqrt[n]{a} \geq 1,$

$(5^*) := \texttt{IneqMult}(3_2)(1) : \sqrt[m]{a} \geq 1,$

$\texttt{Assume}\ (6) : \sqrt[n]{a} > \sqrt[m]{a},$

$(7) := \texttt{IneqMult}(4, 5)(2)(6) : (\sqrt[n]{a})^n > (\sqrt[m]{a})^m,$

$(8) := I(E)(3, 7) : a < a,$

$() := \eth\texttt{StricIneq}I(=)(a) : \perp;$

$\leadsto (6^*) := E(\perp) : \sqrt[n]{a} \leq \sqrt[m]{a};$

$\leadsto (2) := \eth^{-1}\texttt{Nonincreasing}\eth^{-1}\texttt{BoundedFromBelow} :$

$\quad : \Lambda n \in \mathbb{N} \,.\, \sqrt[n]{a} : \texttt{Nonincreasing}\ \&\ \texttt{BoundedFromBelow}(\mathbb{N}, \mathbb{R}_{++}),$

$(3) := \texttt{NonincreasingAndBoundedConverge}(2) : \Lambda n \in \mathbb{N} \,.\, \sqrt[n]{a} : \texttt{Converging}(\mathbb{R});$

$L := \lim_{n\to\infty} \sqrt[n]{a} = 1 : [1, a],$

$(4) := \eth^{-1}\texttt{Subseq} : \Lambda n \in \mathbb{N} \,.\, \sqrt[2^n]{a} : \texttt{Subseq}\Big(\Lambda b \in \mathbb{N} \,.\, \sqrt[n]{a}\Big),$

$(5) := \texttt{SubseqLim} : L = \lim_{n\to\infty} \sqrt[2^n]{a},$

$(6) := \texttt{LimitFixedPoint}(5) : \sqrt{L} = L,$

$() := \eth L(6) : L = 1;$

$\leadsto (1) := I(\Rightarrow) : a \geq 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1,$

$\texttt{Assume}\ (2) : a < 1,$

$(3) := (2)^{-1} : a^{-1} \geq 1,$

$(4) := \texttt{NegativeExponent}(1)(3) : \lim_{n\to\infty} \left(\sqrt[n]{a}\right)^{-1} = \lim_{n\to\infty} \sqrt[n]{a^{-1}} = 1,$

$() := \texttt{ContinuousDivision}(4) : \lim_{n\to\infty} \sqrt[n]{a} = 1;$

$\leadsto (2) := I(\Rightarrow) : a < 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1,$

$(*) := E(|)\texttt{IneqAlternative}(a, 1)(1)(2) : \lim_{n\to\infty} \sqrt[n]{a} = 1;$

$\square$

$\texttt{ContinuousExponentI} :: \forall a \in \mathbb{R}_{++} \,.\, \forall q : \texttt{Cauchy}(\mathbb{Q}) \,.\, a^q : \texttt{Converging}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{+}+,$

$(1) := \texttt{ConvergingIsBounded}\eth\texttt{Comlete}(\mathbb{R})(q) : \Big(q : \texttt{Bounded}\Big),$

$(m, 2) := \texttt{MonotonicExponent}(q, 1) : \sum m \in \mathbb{N} \,.\, m = \arg\max_n a^{q_n},$

$(N, 3) := \texttt{RootLimit}(a)\left(\frac{\varepsilon}{a^{q_m}}\right) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \geq N \,.\, \left|a^{1/n} - 1\right| < \frac{\varepsilon}{a^{q_m}},$

$(M, 4) := \eth\texttt{Cauchy}(q)(1/N) : \sum M \in \mathbb{N} \,.\, \forall n, m \in \mathbb{N} : \min(n, m) \geq M \,.\, |q_n - q_m| < \frac{1}{N},$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ (5) : \min(n, m) \geq M,$

$() := \texttt{AbsHomogen}(a^{q_n} - a^{q_m}, a^{\min q_n, q_m})(2)\texttt{MonotonicExponent}(a, (3)(4)(5)) :$

$\quad : |a^{q_n} - a^{q_m}| = a^{\min(q_n, q_m)}\left|a^{|q_n - q_m|} - 1\right| < a^{q_m}\frac{\varepsilon}{a^{q_m}} = \varepsilon;$

$\rightsquigarrow (*) := \eth\texttt{Complete}(\mathbb{R})\eth^{-1}\texttt{Cauchy}I(\forall)I(\exists)(M)I(\forall)I(\Rightarrow) : \Big(x^q : \texttt{Converging}\Big);$

$\square$

$\texttt{ContinuousExponentII} :: \forall a \in \mathbb{R}_{++} \,.\, \forall q, p : \texttt{Cauchy}(\mathbb{Q}) \,.\, \forall(0) : \lim_{n\to\infty} q_n = \lim_{m\to\infty} q_m \,.\, \lim_{n\to\infty} a^{q_n} = \lim_{n\to\infty} a^{p_n}$

$\texttt{Proof} \ =$

$(1) := \texttt{ContinuousAddition}(0) : \lim_{n\to\infty} q_n - p_n = 0,$

$c := \max\Big\{\max(a^{q_n}, a^{p_n}) : n \in \mathbb{N}\Big\} : \mathbb{R}_{++},$

$(N, 3) := \texttt{RootLimit}(a)\left(\frac{\varepsilon}{c}\right) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \geq N \,.\, \left|a^{1/n} - 1\right| < \frac{\varepsilon}{c},$

$(M, 4) := \eth\texttt{Cauchy}(q)(1/N) : \sum M \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \geq M \,.\ \,.\, |q_n - p_n| < \frac{1}{N},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (5) : n \geq M,$

$() := \texttt{AbsHomogen}(a^{q_n} - a^{p_n}, a^{\min q_n, p_n})\flat^{-1}c\texttt{MonotonicExponent}(a, (3)(4)(5)) :$

$\quad : |a^{q_n} - a^{p_n}| = a^{\min(q_n, p_n)}\left|a^{|q_n - p_m|} - 1\right| < c\frac{\varepsilon}{c} = \varepsilon;$

$\rightsquigarrow (*) := \texttt{ContinuousAddition}\eth^{-1}\texttt{Limit}I(\forall)I(\exists)(M)I(\forall)I(\Rightarrow) : \lim_{n\to\infty} a^{q_n} = \lim_{n\to\infty} a^{p_n};$

$\square$

$\texttt{realRealExponent} :: \mathbb{R}_{++} \to \mathbb{R} \to \mathbb{R}_{++}$

$\texttt{realRealExponent}\,(x, y) = x^y := \lim_{n\to\infty} x^{q_n}$

$\quad \texttt{where} \quad q = \texttt{RationalApproximation}(y)$

3 Topology of The Real Line

3.1 Open And Closed Sets

$\texttt{OpenRealStructure} :: \forall U : \texttt{Open}(\mathbb{R}) \;.\; \exists I : \texttt{Countable} \;.\; \exists (a,b) : \texttt{Disjoint}(I, \texttt{OpenInterval}(\mathbb{R})) \;.$

$$. \; U = \bigcup_{i \in I} (a_i, b_i)$$

$\texttt{Proof} \;=$

$(I, (a,b), 1) := \eth\texttt{topology}(\mathbb{R}) : \sum I : \texttt{Set} \;.\; \sum (a.b) : \texttt{Disjoint}(I, \texttt{openInterval}(\mathbb{R})) \;.\; U = \bigcup_{i \in I} (a_i, b_i),$

$(2) := \texttt{DisjointIntervalsAreAtmostCountable}(1) : \Big(I : \texttt{Countable} \Big);$

$\square$

$\texttt{ClosedRealStructure} :: \forall K : \texttt{Closed}(\mathbb{R}) \;.\; \exists I : \texttt{Countable} \;.\; \exists U : I \to \texttt{Open}(\mathbb{R}) \;.\; K = \bigcap_{i \in I} U_i$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.2 Nested Closed Intervals

$\texttt{length} :: \texttt{ClosedInterval}(\mathbb{R}) \to \overset{\infty}{\mathbb{R}}_+$

$\texttt{length}([a,b]) = \lambda[a,b] := b - a$

$\texttt{CantorIntersectionTheorem} :: \forall I : \texttt{Nested}(\mathbb{N}, \texttt{ClosedInterval}) \,.\, \forall(0) : \lim_{n\to\infty} \lambda(I_n) = 0 \,.$

$\quad .\ \exists x \in \mathbb{R} \,.\, \bigcap_{n=1}^{\infty} I_n = \{x\}$

$\texttt{Proof} \ =$

$a := \min I_n : \mathbb{N} \to \mathbb{R},$

$(1) := \text{þ}a\eth\texttt{Nested}(I) : \Big(a : \texttt{Nondecreasing} \,\&\, \texttt{BoundedFromAbove}(\mathbb{R})\Big),$

$(2) := \texttt{NondecreasingAndBoundedConverge} : \Big(a : \texttt{Converging}(\mathbb{R})\Big),$

$A := \lim_{n\to\infty} a_n : \mathbb{R},$

$\texttt{Assume}\ n : \mathbb{N},$

$(b, 3) := \eth\texttt{ClosedInterval} : \sum b \in \mathbb{R} \,.\, [a_n, b] = I_n,$

$(4) := \text{þ}A\eth b : a_n \le A \le b,$

$() := (4)(3) : A \in I_n;$

$\rightsquigarrow (3) := I(\forall) : \forall n \in \mathbb{N} \,.\, A \in I_n,$

$(4) := \eth\texttt{intersect} : A \in \bigcap_{i=1}^{n} I_n,$

$\texttt{Assume}\ B : \bigcap_{n=1}^{\infty} I_n,$

$\texttt{Assume}\ (5) : A \ne B,$

$\delta := |A - B| : \mathbb{R}_{++},$

$(n, 6) := \eth\texttt{Limit}(0)(\delta) : \sum n \in \mathbb{N} \,.\, \lambda(I_n) < \delta,$

$(7) := \eth\texttt{Intersect}(n)\eth A\eth B : A, B \in I_n,$

$(8) := \eth\texttt{ClosedInterval}\eth\texttt{length}(6) : |A - B| < \delta,$

$() := \eth\texttt{StrictIneq}(8)\text{þ}\delta : \bot;$

$\rightsquigarrow (6) := I(\forall)E(\bot) : \forall B \in \bigcap_{n=1}^{\infty} I_n \,.\, B = A,$

$(7) := \eth^{-1}\texttt{Singleton}(6) : \bigcap_{n=1}^{\infty} = \{A\};$

$\square$

$\texttt{BolzanoWeierstrass} :: \forall x : \texttt{Bounded}(\mathbb{N}, \mathbb{R}) \,.\, \exists n : \texttt{Subseqer} \,.\, x_n : \texttt{Converging}$

$\texttt{Proof} \;=$

$[a_1, b_1] := [\inf_n x_n, \sup_n y_n] : \texttt{ClosedInterval},$

$r := 2\lambda[a_1, b_1] : \mathbb{R}_+,$

$(1_1) := \text{þ} I_1 \eth \liminf x \eth \limsup y : |[a_1, b_1] \cap \operatorname{Im} x| = \infty,$

$\texttt{Assume}\ n : \mathbb{N},$

$I_1 := \left[a_n, \dfrac{a_n + b_n}{2}\right] : \texttt{ClosedInterval},$

$I_2 := \left[\dfrac{a_n + b_n}{2}, b_n\right] : \texttt{ClosedInterval},$

$(2)] := \text{þ} I_1 \text{þ} I_2 : I_1 \cup I_2 = [a_n, b_n],$

$(i, 3) := \texttt{PigionholePrinciple}(\aleph_0)(2)(1_n) : \sum i \in \{1, 2\} \,.\, |I_i \cap \operatorname{Im} x| = \infty,$

$[a_{n+1}, b_{n+1}] := I_i : \texttt{ClosedInterval},$

$(2_n) := \text{þ}[a_{n+1}, b_{n+1}]E(=)(3) : \Big|[a_i, b_i] \cap \operatorname{Im} x\Big| = \infty,$

$(4^*_n) := \eth^{-1}\texttt{Subset}\text{þ}[a_{n+1}, b_{n+1}]\text{þ}i\text{þ}I_1\text{þ}I_2 : [a_{n+1}, b_{n+1}] \subset [a_n, b_n],$

$(5^*_n) := \eth^{-1}\texttt{length}\text{þ}[a_{n+1}, b_{n+1}]\text{þ}i\text{þ}I_1\text{þ}I_2 : \lambda[a_{n+1}, b_{n+1}] = \dfrac{\lambda[a_n, b_n]}{2};$

$\rightsquigarrow ([a, b], 2]) := I\left(\sum\right) \eth^{-1}\texttt{Nested}(4)\texttt{RecursiveApplication}I(\forall) :$

$\quad : \sum [a, b] : \texttt{Nested}(\mathbb{N}, \texttt{ClosedInterval}) \,.\, \forall n \in \mathbb{N} \,.\, \lambda[a_n, b_n] = 2^{-n}r \;\&\; \Big|[a_n, b_n] \cap \operatorname{Im} x\Big| = \infty,$

$(3) := \texttt{PowerCompression}(2_1) : \lim_{n\to\infty} \lambda[a_n, b_n] = 0,$

$(X, 4) := \texttt{CantorIntersectionTheorem}([a, b])(3) : \sum X \in \mathbb{R} \,.\, \{X\} = \bigcap_{n=1}^{\infty} [a_n, b_n],$

$(n, 5) := \eth^{-1}\texttt{Subseq}\Lambda m \in \mathbb{N} \,.\, \texttt{InfSeq}(x)[a_m, b_m] : \sum n : \texttt{Subseqer} \,.\, \forall m \in \mathbb{N} \,.\, x_{n_m} \in [a_m, b_m],$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(N, 6) := \eth\texttt{Limit}(3)(\varepsilon) : \sum N \in \mathbb{N} \,.\, \forall m \in \mathbb{N} : m \geq N \,.\, \lambda[a_m, b_m] < \varepsilon,$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ (7) : m \geq N,$

$(8) := (5)(m) : x_{n_m} \in [a_m, b_m],$

$(9) := (4)\eth\texttt{intersect}(m) : X \in [a_m, b_m],$

$(10) := \eth\texttt{length}(8)(9)(6)(m, 7) : |X - x_{n_m}| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Limit}I(\forall)I(\exists)(N)I(\forall)I(\Rightarrow) : \lim_{m\to\infty} x_{n_m} = X;$

$\square$

3.3 Sets of Partial Limits[!]

3.4 Elementary Baire Category

$\texttt{NowhereDense} :: ??\mathbb{R}$

$A : \texttt{NowhereDense} \iff \forall U : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}) \,.\, \exists V : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}) : V \subset U \,\&\, V \cap A = \emptyset$

$\texttt{Meager} :: ??\mathbb{R}$

$$A : \texttt{Meager} \iff \exists Z : \mathbb{N} \to \texttt{NowhereDense} \,.\, A = \bigcup_{n=1}^{\infty} Z_n$$

$\texttt{Comeager} :: ??\mathbb{R}$

$$A : \texttt{Comeager} \iff \exists U : \mathbb{N} \to \texttt{Dense} \,\&\, \texttt{Open}(\mathbb{R}) \,.\, A = \bigcap_{n=1}^{\infty} U_n$$

$\texttt{RealBaireTheoremI} :: \forall U : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}) \,.\, U \,!\, \texttt{Meager}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (0) : U : \texttt{Meager},$

$(Z, 00) := \eth\texttt{Meager}(0) : \sum Z : \mathbb{N} \to \texttt{NowhereDense} \,.\, U = \bigcup_{n=1}^{\infty} Z_n,$

$(a, 1) := \eth\texttt{NonEmpty} : \sum a \in \mathbb{R} : a \in U,$

$(I_1, 2) := \eth\texttt{topology}(\mathbb{R})\eth\texttt{Open}(\mathbb{R})(U)(a) : \sum I_1 : \texttt{OpenInterval}(\mathbb{R}) \,.\, a \in I_1 \subset U,$

$(K_1, 3_1) := \eth^{-1}\texttt{ClosedInervalIntermidiateNumber}^2(I_1, 2) : \sum K_1 : \texttt{ClosedInterval}(\mathbb{R}) \,.\, K_1 \subset I_1,$

$\texttt{Assume}\ n : \mathbb{N},$

$(V, 4) := \eth^{-1}\texttt{OpenInterval}\eth\texttt{ClosedInerval}(K_n) : \sum V : \texttt{Open}(K_n) \,.\, V \subset K_n,$

$(I_{n+1}, 5) := \eth\texttt{NowhereDense}(Z_n) : \sum I_{n+1} : \texttt{Open}(\mathbb{R}) : I_{n+1} \subset V \,\&\, I_n \cap Z_n = \emptyset,$

$(K_{n+1}, 3_{n+1}) := \eth^{-1}\texttt{ClosedInervalIntermidiateNumber}^2(I_{n+1})\eth\texttt{topology}(\mathbb{R})\eth\texttt{Open}(\mathbb{R}) :$

$\quad : \sum K_{n+1} : \texttt{ClosedInterval} \,.\, K_{n+1} \subset I_{n+1},$

$6_n := (3_n)(5_1)(4) : K_{n+1} \subset K_n,$

$7_n := \texttt{IntersectSubset}(3_n)(5_2) : K_{n+1} \cap Z_n = \emptyset;$

$\rightsquigarrow (K, 4) := I\left(\sum\right) \ldots : \sum K : \texttt{Nested}\left(\mathbb{N}, \texttt{ClosedInterval}\right) \,.\, \forall n \in \mathbb{N} \,.\, K_n \cap Z_n = \emptyset,$

$$(5) := \texttt{CantorIntersectTheorem} : \bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

$$H := \bigcap_{n=1}^{\infty} K_n \neq \emptyset :??\mathbb{R},$$

$(6) := \text{þ}H\eth\texttt{Nested}(K)\eth\texttt{intersect}(3_1)(2) : H \subset U,$

$(7) := (00)(4)(H) : H \cap U = \emptyset,$

$() := (7)(6)(5) : \bot;$

$\rightsquigarrow (*) := E(\bot) : U \,!\, \texttt{Meager};$

$\square$

$\texttt{RealBaireTheoremII} :: \forall A : \texttt{Comeager} \,.\, A : \texttt{Dense}(\mathbb{R})$

$\texttt{Proof} \;=$

$(U,1) := \eth\texttt{Comeager}(A) : \sum U : \mathbb{N} \to \texttt{Open} \,\&\, \texttt{Dense}(\mathbb{R}) \,.\, A = \bigcap_{n=1}^{\infty} U_n,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ V : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}),$

$(x,2) := \eth\texttt{Dense}(\mathbb{R})(U)(V) : \sum x \in V \,.\, x \in U,$

$(W,3) := \texttt{OpenIntersect}(V)\eth\texttt{Open}(U,x) : \sum W : \texttt{Open}(\mathbb{R}) \,.\, x \in W \subset U \cap V,$

$(4) := \texttt{ComplementSubset}(3) : W \cap U^{\complement} = \emptyset;$

$\rightsquigarrow (2) := I(\forall)\eth^{-1}\texttt{NowhereDense}I(\forall) : \forall n \in \mathbb{N} \,.\, U_n^{\complement} : \texttt{NowhereDense},$

$(5) := \texttt{DeMorganLaw}(1)\eth^{-1}\texttt{Meager} : A^{\complement} = \bigcup_{n=1}^{\infty} U^{\complement} : \texttt{Meager},$

$\texttt{Assume}\ V : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}),$

$(6) := \texttt{RealBaireTheoremII}(A^{\complement}, V) : V \not\subset A^{\complement},$

$(a,7) := \eth\texttt{complement}(6) : \sum a \in A \,.\, a \in V;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Dense}I(\forall)I(\exists)(a) : (A : \texttt{Dense}(\mathbb{R}));$

$\square$

$\texttt{IrrationalsAreNotCountableUnionOfClosed} :: \forall C : \mathbb{N} \to \texttt{Closed}(\mathbb{R}) \,.\, \mathbb{Q}^{\complement} \neq \bigcup_{n=1}^{\infty} C_n$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : \mathbb{Q}^{\complement} = \bigcup_{n=1}^{\infty} C_n,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ U : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}),$

$\texttt{Assume}\ (2) : \forall V : \texttt{Open} \,\&\, \texttt{NonEmpty}(\mathbb{R}) \,.\, V \subset U \Rightarrow V \cap C_n \neq \emptyset,$

$(3) := \eth^{-1}\texttt{Dense}(2) : \Big(U \cap C_n : \texttt{Dense}(C_n)\Big),$

$(4) := \texttt{IntersectSubset}\eth\texttt{closeure}(U \cap C_n)\texttt{DenseClosure}(3) : U \subset C_n,$

$(q,5) := \eth\texttt{Subset}(4)\eth\texttt{Dense}(\mathbb{Q})(U) : \sum q \in \mathbb{Q} \,.\, q \in C_n,$

$() := \eth\texttt{complement}(1)\eth\texttt{union}(5) : \bot;$

$\rightsquigarrow (2) := I(\forall)\eth\texttt{NowhereDense}I(\forall)E(\bot) : \forall n \in \mathbb{N} \,.\, K_n : \texttt{NowhereDense},$

$q := \eth\texttt{EqCard} : \mathbb{N} \leftrightarrow_{\mathsf{SET}} \mathbb{Q},$

$C' := \Lambda n \in \mathbb{N} \,.\, C_n \cup \{q_n\} : \mathbb{N} \to \texttt{NowhereDense},$

$(3) := \flat C'\texttt{UnionCommute}(1)\eth q\texttt{ComplementUnion} : \bigcup_{n=1}^{\infty} C'_n = \bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n=1}^{\infty} \{q_n\} = \mathbb{Q}^{\complement} \cup \mathbb{Q} = \mathbb{R},$

$() := \texttt{RealBaireTheoremI}(\mathbb{R})(C')(3) : \bot;$

$\rightsquigarrow (*) := E(\bot) : \mathbb{Q}^{\complement} \neq \bigcup_{n=1}^{\infty} C_n,$

$\square$

3.5 Cantor Set[!]

3.6 Meshes on Reals Intervals

$$\texttt{Mesh} :: \sum [a,b] : \texttt{ClosedInterval} \,.\, \mathbb{R}_{++} \to ? \sum n \in \mathbb{N} \,.\, \texttt{Increasing}\Big(n, [a,b]\Big)$$

$$(n,t) : \varepsilon\text{-}\texttt{Mesh} \iff \bigcup_{i=1}^{n-1} [t_i, t_{i+1}] = [a,b] \;\&\; \forall i \in \mathbb{N} : i < n \,.\, t_{i+1} - t_i < \varepsilon$$

$$\texttt{LittleStepsTHM} :: \forall \delta \in \mathbb{R}_{++} \,.\, \forall n \in \mathbb{N} \,.\, \forall \Delta : n \to [-\delta, \delta] \,.\, \forall x : \texttt{Between}\left(\Delta_1, \sum_{i=1}^{n} \Delta_i\right) \,.$$

$$.\, \exists k \in n : \left| \sum_{i=1}^{k} \Delta_i - x \right| \le \delta$$

$\texttt{Proof} \;=$

$S := \sum_{i=1}^{n} \Delta_i : \mathbb{R},$

$\texttt{Assume}\ (1) : n = 1,$

$(2) := \text{þ}S(1) : \Delta_1 = S,$

$(3) := \eth\texttt{closedSet} : [\Delta_1, S] = \{S\},$

$(4) := \eth x(3) : x = \Delta_1,$

$(5) := \eth\texttt{absVal}(3)\eth\delta : |x - \Delta_1| = 0 < \delta;$

$\leadsto (1) := \eth^{-1}\texttt{This} : \texttt{This}(\delta, 1),$

$\texttt{Assume}\ (2) : \texttt{This}(\delta, n),$

$\texttt{Assume}\ \Delta' : (n+1) \to \mathbb{R} : \forall i \in n+1 \,.\, |\Delta_i| < \delta,$

$s_+ := \sum_{i=1}^{n+1} \Delta'_i : \mathbb{R},$

$s_- := \sum_{i=1}^{n} \Delta'_i : \mathbb{R},$

$(3) := \eth\texttt{Between}(\Delta_1, s_+)(s_1) : \Big(x : \texttt{Between}(\Delta, s_-) \;\Big|\; x : \texttt{Between}(s_-, s_+)\Big),$

$\texttt{Assume}\ (4) : x : \texttt{Between}(\Delta, s_-),$

$(5) := (2)(\Delta'_{|n}) : \texttt{This}(\delta, n+1, \Delta');$

$\leadsto (4) := I(\Rightarrow) : x : \texttt{Between}(\Delta, s_-) \Rightarrow \texttt{This}(\delta, n+1, \Delta'),$

$\texttt{Assume}\ (5) : x : \texttt{Between}(s_-, s_+),$

$(6) := \eth\Delta\text{þ}s_-\text{þ}s_+ : |s_- - s_+| < \delta,$

$(7) := (5)(6) : \texttt{This}(\delta, n+1, \Delta');$

$\leadsto (5) := I(\Rightarrow) : x : \texttt{Between}(s_-, s_+) \Rightarrow \texttt{This}(\delta, n+1, \Delta'),$

$() := E(|)(3)(4)(5) : \texttt{This}(\delta, n+1, \Delta');$

$\leadsto (*) := E(\mathbb{N}) : \texttt{This};$

$\square$

$\texttt{MeshExists} :: \forall [a,b] : \texttt{ClosedInterval} \;.\; \forall \varepsilon \in \mathbb{R}_{++} \;.\; \exists (n,t) : \varepsilon\text{-}\texttt{Mesh}[a,b]$

$\texttt{Proof} \;=$

$n := \left\lceil \dfrac{2(b-a)}{\varepsilon} \right\rceil : \mathbb{N},$

$t := \Lambda k \in n+1 \;.\; \min\left(a + \dfrac{(k-1)\varepsilon}{2}, b\right) : \texttt{Increassing}\Big(n, [a,b]\Big),$

$(2) := \text{þ} t \eth \varepsilon : \forall i \in n \;.\; t_{i+1} - t_i \le \dfrac{\varepsilon}{2} < \varepsilon,$

$\texttt{Assume}\; x : \texttt{In}[a,b],$

$(k,3) := \texttt{LittleStepsTHM}\left(\dfrac{\varepsilon}{2}, n+1, \Lambda k \in n+1 \;.\; [k>1](t_k - t_{k-1}), x-a\right) : k \in n \;.\; x \in [t_i, t_{i+1}],$

$() := \eth^{-1}\texttt{In}\eth\texttt{union}(3) : x \in \displaystyle\bigcup_{i=1}^{n} [t_i, t_{i+1}];$

$\rightsquigarrow (3) := \eth^{-1}\texttt{Subset} : [a,b] \subset \displaystyle\bigcup_{i=1}^{n} [t_i, t_{i+1}],$

$(4) := \text{þ} t_1 : t_1 = a,$

$(5) := \text{þ} t_{n+1} : t_{n+1},$

$(6) := \eth\texttt{Union}\eth\texttt{ClosedInterval}\eth\texttt{Increasing}(t)(4)(5) : \displaystyle\bigcup_{i=1}^{n} [t_i, t_{i+1}] \subset [a,b],$

$(7) := \eth\texttt{SetEq}(3)(6) : [a,b] = \displaystyle\bigcup_{i=1}^{n} [t_i, t_{i+1}],$

$(8) := \eth^{-1}\varepsilon\text{-}\texttt{Mesh}(2)(7) : \Big((n-1, t) : \varepsilon\text{-}\texttt{Mesh}[a,b]\Big);$

$\square$

$\texttt{mesh} :: \prod [a,b] : \texttt{ClosedInterval} \;.\; \prod \varepsilon \in \mathbb{R}_{++} \;.\; \varepsilon\text{-}\texttt{Mesh}[a,b]$

$\texttt{mesh}\,() := \texttt{MeshExists}\Big([a,b], \varepsilon\Big)$

$\texttt{partitionSystem} :: \prod [a,b] : \texttt{ClosedInterval} \;.\; ?? \sum n \in \mathbb{N} \;.\; \texttt{Increasing}(n, [a,b])$

$\texttt{partitionSystem}\,() = \mathfrak{P}[a,b] := \left\{ \{(n,t) : \varepsilon\text{-}\texttt{Mesh}\} \middle| \varepsilon \in \mathbb{R}_{++} \right\}$

$\texttt{PartitionSystemsDirectNet} :: \forall [a,b] : \texttt{ClosedInterval} \;.\; \Big(\mathfrak{P}[a,b], \subset\Big) : \texttt{NetIndex}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; P : \mathfrak{P}[a,b],$

$() := \texttt{MeshExists}\eth\mathfrak{P}[a,b](P) : P \neq \emptyset;$

$\rightsquigarrow (1) := I(\forall) : \forall P \in \mathfrak{P}a,b \;.\; P \neq \emptyset,$

$\texttt{Assume}\; P, Q : \mathfrak{P}[a,b],$

$(\varepsilon, 2) := \eth\mathfrak{P}[a,b](P) : \sum \varepsilon \in \mathbb{R}_{++} \;.\; P = \Big\{(n,t) : \varepsilon\text{-}\texttt{Mesh}[a,b]\Big\},$

$(\delta, 3) := \eth\mathfrak{P}[a,b](Q) : \sum \delta \in \mathbb{R}_{++} \;.\; Q = \Big\{(n,t) : \varepsilon\text{-}\texttt{Mesh}[a,b]\Big\},$

$\texttt{Assume}\; (4) : \delta \leq \varepsilon,$

$\texttt{Assume}\; (n,t) : \texttt{In}(Q),$

$(5) := \eth\delta\text{-}\texttt{Mesh}(n,t)(3)(4) : \forall i \in n-1 \;.\; t_{i+1} - t_i < \delta \leq \varepsilon,$

$(6) := \eth^{-1}\varepsilon\text{-}\texttt{Mesh}(5)(2) : (n,t) \in P;$

$\rightsquigarrow (7) := I(\Rightarrow)\eth^{-1}\texttt{Subset} : \delta \leq \varepsilon \Rightarrow Q \subset P,$

$(8) := \eth^{-1}\mathfrak{P}[a,b]\texttt{SubsetIntersection}(7) : P \cap Q = \Big\{(n,t) : \min(\delta,\varepsilon)\text{-}\texttt{Mesh}[a,b]\Big\} \in \mathfrak{P}[a,b];$

$\rightsquigarrow (2) := I(\forall) : \forall P, Q \in \mathfrak{P}[a,b] \;.\; P \cap Q \subset \mathfrak{P}[a,b],$

$(*) := \eth^{-1}\texttt{NetIndex}(2) : \Bigg(\Big(\mathfrak{P}[a,b], \subset\Big) : \texttt{NetIndex}\Bigg);$

$\square$

4 Continuous Functions

4.1 Limit of a function

$\texttt{UpperLimit} :: \prod U :?\mathbb{R} \,.\, ?\Big(U \cup \{\inf U\} \times (U \to \mathbb{R}) \times \mathbb{R}\Big)$

$(a, f, y) : \texttt{UpperLimit} \iff \lim\limits_{x \downarrow a} f(x) = y \iff \forall \epsilon \in \mathbb{R}_{++} \,.\, \exists \delta \in \mathbb{R}_{++} :$

$: \forall x \in (a, a + \delta) \cap U \,.\, f(x) \in (y - \varepsilon, y + \varepsilon)$

$\texttt{LowerLimit} :: \prod U :?\mathbb{R} \,.\, ?\Big(U \cup \{\sup U\} \times (U \to \mathbb{R}) \times \mathbb{R}\Big)$

$(a, f, y) : \texttt{LowerLimit} \iff \lim\limits_{x \uparrow a} f(x) = y \iff \forall \epsilon \in \mathbb{R}_{++} \,.\, \exists \delta \in \mathbb{R}_{++} :$

$: \forall x \in (a - \delta, a) \cap U \,.\, f(x) \in (y - \varepsilon, y + \varepsilon)$

$\texttt{TwoSidedFLimit} :: \prod U :?\mathbb{R} \,.\, \forall f : U \to \mathbb{R} \,.\, \forall a \in U \,.\, \forall (0) \lim\limits_{x \downarrow a} f(x) = f(a) = \lim\limits_{x \uparrow a} f(x) \,.\, f \in C(U, \mathbb{R}, a)$

$\texttt{Proof} =$

$\texttt{Assume}\ \varepsilon : \mathbb{R},$

$(\delta_+, 1) := (0_1)(\varepsilon) : \sum \delta_+ \in \mathbb{R}_{++} \,.\, \forall x \in (a, a + \delta_+) \,.\, |f(x) - f(a)| < \varepsilon,$

$(\delta_-, 2) := (0_1)(\varepsilon) : \sum \delta_- \in \mathbb{R}_{++} \,.\, \forall x \in (a - \delta_-, a) \,.\, |f(x) - f(a)| < \varepsilon,$

$\delta := \min(\delta_+, \delta_-) : \mathbb{R}_{++},$

$\texttt{Assume}\ x : (a - \delta, a + \delta),$

$() := (1)(2)\eth\delta\eth x : |f(x) - f(a)| < \varepsilon;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Limit} : \lim\limits_{x \to a} f(x) = f(a),$

$(*) := \texttt{SeqContAtAPoint}(1) : f \in C(C, \mathbb{R}, a);$

$\square$

$\texttt{ZeroAtInftyTest} :: \forall f : C\Big((0,\infty),\mathbb{R}\Big) . \forall (0) : \forall x \in (0,\infty) . \lim_{n\to\infty} f(nx) = 0 . \lim_{x\to\infty} f(x) = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : \lim_{x\to\infty} f(x) \neq 0,$

$(\varepsilon, 2) := \eth\texttt{Limit}(1) : \sum \varepsilon \in \mathbb{R}_{++} . \forall \delta \in \mathbb{R}_{++} . \exists x \in (\delta, +\infty) . |f(x)| \geq \varepsilon,$

$C := \Lambda n \in \mathbb{N} . \left\{x \in \mathbb{R}_{++} : |f(nx)| \leq \frac{\varepsilon}{2}\right\} : \texttt{Closed}(\mathbb{R}_{++}),$

$K := \Lambda n \in \mathbb{N} . \bigcap_{k=n}^{\infty} C_n : \texttt{Closed}(\mathbb{R}_{++}),$

$(3) := \text{þ}K(1) : \bigcup_{n=1}^{\infty} K_n = \mathbb{R},$

$(N, 4) := \texttt{RealBairCategoryI}(3) : \sum N \in \mathbb{N} . K_N \ ! \ \texttt{NowhereDense},$

$\Big((a,b), 5\Big) := \texttt{ClosedDense}\eth\texttt{NowhereDense}(4)\text{þ}K : \sum (a,b) \subset K_N : \forall x \in (a,b) . \ \& \forall n \in \mathbb{N} : n \geq N .$

$\quad . |f(nx)| \leq \frac{\varepsilon}{2},$

$M := \max\left(N, \left\lceil \frac{a}{b-a} \right\rceil\right) : \mathbb{N},$

$\texttt{Assume}\ k : \mathbb{N},$

$() := \eth M \eth(k,a,b) : (M+k)b - (M+k+1) = K(b-a) + M(b-a) - a \geq$

$\quad \geq K(b-a) + a - a = K(b-a) > 0;$

$\rightsquigarrow (6) := \eth^{-1}\texttt{union} : \bigcup_{n=M}^{\infty} (na, nb) = (Ma, +\infty),$

$(7) := (6)(5) : \forall x \in (Ma, +\infty) . |f(x)| < \varepsilon,$

$() := (7)(2) : \perp;$

$\rightsquigarrow (*) := E(\perp) : \lim_{x\to\infty} f(x) = 0;$

$\square$

$\ldots$

$\square$

4.2 Points of Discontinuity

$\texttt{RemovableDiscontinuity} :: \prod U :?\mathbb{R} \, . \, f : U \to \mathbb{R} \, . \, ?U$

$a : \texttt{RemovableDiscontinuity} \iff \exists b \in \mathbb{R} : \lim_{x \downarrow a} f(x) = b = \lim_{x \uparrow a} f(x) \ \& \ b \neq f(a)$

$\texttt{DiscontinuityI} :: \prod U :?\mathbb{R} \, . \, f : U \to \mathbb{R} \, . \, ?U$

$a : \texttt{DiscontinuityI} \iff \exists b, c \in \mathbb{R} : \lim_{x \downarrow a} f(x) = b \ \& \ \lim_{x \uparrow a} f(x) = c \ \& \ b \neq c$

$\texttt{DiscontinuityII} :: \prod U :?\mathbb{R} \, . \, f : U \to \mathbb{R} \, . \, ?U$

$a : \texttt{DiscontinuityII} \iff \left(\forall b \in \mathbb{R} \, . \, \lim_{x \downarrow a} f(x) \neq b \right) \Big| \left(\forall b \in \mathbb{R} \, . \, \lim_{x \uparrow a} f(x) \neq b \right)$

$\texttt{setOfDiscontinuities} :: \prod U :?\mathbb{R} \, . \, U \to \mathbb{R} \to ?U$

$\texttt{setOfDiscontinuities}(f) = \mathcal{D}(f) := \Big\{ x \in U : f \, ! \, C(U, \mathbb{R}, x) \Big\}$

$\texttt{oscilationInSet} :: \prod U :?\mathbb{R} \, . \, (U \to \mathbb{R}) \to ?U \to \overset{\infty}{\mathbb{R}}_+$

$\texttt{oscilationInSet}(f, X) = \omega(f, X) := \sup_{a, b \in X} |f(x) - f(y)|$

$\texttt{oscilationAtPoint} :: \prod U :?\mathbb{R} \, . \, (U \to \mathbb{R}) \to U \to \overset{\infty}{\mathbb{R}}_+$

$\texttt{oscilationAtPoint}(f, x) = \omega(f, x) := \lim_{t \to 0} \omega\Big(f, (x - t, x + t) \cap U \Big)$

$\texttt{OscilationZeroIffC} :: \forall U :?\mathbb{R} \, . \, \forall f : U \to \mathbb{R} \, . \, \forall x \in U \, . \, \omega(f, x) = 0 \iff f \in C(U, \mathbb{R}, x)$

Proof =

Straight from definitions.

□

$\texttt{DiscSetStructure} :: \forall f : \mathbb{R} \to \mathbb{R} \,.\, \exists C : \mathbb{N} \to \texttt{Closed}(\mathbb{R}) \,.\, \mathcal{D}(f) = \bigcup_{n=1}^{\infty} C_n$

$\texttt{Proof} \ =$

$C := \Lambda n \in \mathbb{N} \,.\, \left\{ x \in \mathbb{R} : \omega(f,x) \geq \frac{1}{n} \right\} : \mathbb{N} \to ?\mathbb{R},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ x : \texttt{In}\left(C_n^{\complement}\right),$

$(1) := \eth\texttt{complement}\eth x \text{þ} C_n : \omega(f,x) < \frac{1}{n},$

$(\Delta, 2) := \eth\texttt{StrictIneq}(1) : \sum \Delta \in \mathbb{R}_{++} \,.\, \omega(f,x) + \Delta < \frac{1}{n},$

$(t,3) := \eth\omega(f,x)(1) : \sum t \in \mathbb{R}_{++} \,.\, \omega\Big(f, (x-t, x+t)\Big) < \omega(f,x) + \Delta,$

$\texttt{Assume}\ y : \texttt{In}(x-t, x+t),$

$(4) := \eth\omega(f,y)(3)(2) : \omega(f,y) \leq \omega\Big(f, (x-t,x+t)\Big) < \omega(f,x) + \Delta < \frac{1}{n},$

$() := \eth^{-1}\texttt{complement}\text{þ} C_n : y \in C_n^{\complement};$

$\rightsquigarrow () := \eth^{-1}\texttt{Subset} : (x-t, x+t) \subset C_n^{\complement};$

$\rightsquigarrow (1) := I(\forall)\eth^{-1}\texttt{ClosedOpenByNeighbourhoods} : \forall n \in \mathbb{N} \,.\, C_n : \texttt{Closed},$

$(*) := \texttt{OscilationZeroIffC}(\text{þ} C) : \mathcal{D}(f) = \bigcup_{n=1}^{\infty} C_n;$

$\square$

$\texttt{DiscSetofMonotonicAtmostCountable} :: \forall f : \texttt{Monotonic}(\mathbb{R}, \mathbb{R}) \,.\, \#\mathcal{D}(f) \leq \aleph_0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : \Big(f : \texttt{NonDecreasing}(\mathbb{R})\Big),$

$\texttt{Assume}\ x : \mathcal{D}(f),$

$(a,b) := (\lim_{x \downarrow t} f(t), \lim_{x \uparrow t} f(t)) : \texttt{OpenInterval};$

$\rightsquigarrow (a,b) := I(\to) : \mathcal{D}(f) \to \texttt{OpenInterval},$

$(2) := \eth\texttt{Increasing}(f)\text{þ}(a,b) : \Big((a,b) : \texttt{Disjoint}(\mathcal{D}(f), \texttt{OpenInterval})\Big),$

$(*) := \texttt{DishointIntervalsAreAtmostCountable}(2) : \#\mathcal{D}(f) \leq \aleph_0;$

$\ldots$

$\square$

4.3 Uniformly Continuous Functions

$\texttt{GlobalUCCriterion} :: \forall f \in C(\mathbb{R},\mathbb{R}) \,.\, \forall a, b \in \mathbb{R} \,.\, \forall (1) : \lim_{x \to \infty} f(x) = a \;\&\; \lim_{x \to -\infty} f(x) = b \,.\, f \in UC(\mathbb{R},\mathbb{R})$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(t, 1) := \eth\texttt{LimToInfty}(0_1)\left(\frac{\varepsilon}{2}\right) : \sum t \in \mathbb{R}_{++} \,.\, \forall x \in (t, +\infty) \,.\, |f(x) - a| < \frac{\varepsilon}{2},$

$(s, 2) := \eth\texttt{LimToInfty}(0_2)\left(\frac{\varepsilon}{2}\right) : \sum s \in \mathbb{R}_{++} \,.\, \forall x \in (-\infty, s) \,.\, |f(x) - b| < \frac{\varepsilon}{2},$

$(\delta_+, 3) := \eth C(f)(t)(\varepsilon/2) : \sum \delta_+ \in \mathbb{R}_{++} : \forall x \in (t - \delta_+, t + \delta_+) \,.\, |f(x) - f(t)| < \frac{\varepsilon}{2},$

$(\delta_-, 4) := \eth C(f)(s)(\varepsilon/2) : \sum \delta_- \in \mathbb{R}_{++} : \forall x \in (s - \delta_-, s + \delta_-) \,.\, |f(x) - f(s)| < \frac{\varepsilon}{2},$

$I := [s - \delta_-, t + \delta_+] : \texttt{ClosedInterval},$

$(5) := \texttt{CompactUCCriterion}(f, I) : \Big(f_{|I} : UC(I, \mathbb{R})\Big),$

$(\delta_0, 6) := \eth UC(I, \mathbb{R})(5)(\varepsilon)\eth\texttt{constrict}(f, I) : \sum \delta_0 \in \mathbb{R}_{++} \,.\, \forall x, y \in \mathbb{R} \,.\, |f(x) - f(y)| < \varepsilon,$

$\delta := \min(\delta_-, \delta_0, \delta_+) : \mathbb{R}_{++},$

$\texttt{Assume}\ x, y : \mathbb{R},$

$\texttt{Assume}\ (7) : |x - y| < \delta,$

$(8) := \eth I(7) : x, y \in (-\infty, s) \;\Big|\; x, y \in I \;\Big|\; x, y \in (t, +\infty),$

$() := E(|)(1)(2)(6) : |f(x) - f(y)| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1} UC(\mathbb{R},\mathbb{R}) : f \in UC(\mathbb{R},\mathbb{R});$

$\square$

4.4 Intermidiate Value Theorem

$\texttt{IntermidiateValueTHM} :: \forall f \in C\Big([a,b],\mathbb{R}\Big) \,.\, \forall y \in [f(a), f(b)] \,.\, \exists x \in [a,b] : f(x) = y$

$\texttt{Proof} \;=$

$(1) := \texttt{CompactUCCritrion}(f,[a,b]) : \Big(f : UC\Big([a,b],\mathbb{R}\Big)\Big),$

$\texttt{Assume}\ n : \mathbb{N},$

$(\delta, 2) := \eth UC(1)(1/n) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall x, y \in [a,b] : |x - y| < \delta \,.\, |f(x) - f(y)| < \frac{1}{n},$

$(m, x, 3) := \texttt{mesh}\Big([a_n, b_n], \delta\Big) : \sum n \in \mathbb{N} \,.\, x : \texttt{Increasing}(n, [a_n, b_n]) \,.\, [a,b] = \bigcup_{i=1}^{n-1} [x_i, x_{i+1}]$

$\quad \& \ \forall i \in \mathbb{N} : i < m \,.\, x_{i+1} - x_i < \delta, ,$

$(4) := \flat(2)(r) : m \geq n,$

$(i, 5) := \arg\min_i |f(x_i) - y| : n,$

$u_n := x_i : \texttt{In}[a,b],$

$(6_n) := \texttt{LittleStepTHM}(3)(4)\flat(u_n)\flat(i) : |f(u_n) - y| < \frac{1}{n};$

$\rightsquigarrow (u, 2) := I\Big(\sum\Big) I(\forall) : \sum u : \mathbb{N} \to [a,b] \,.\, \forall n \in \mathbb{N} \,.\, |f(u_n) - y| < \frac{1}{n},$

$(3) := \texttt{TwoSideLimit}(0, \Lambda n \in \mathbb{N} \,.\, 1/n)\texttt{ReductioInfima}(2) : \lim_{n\to\infty} f(u_n) = y,$

$(m, 4) := \texttt{BolzanoWeierstrass}([a,b], u) : \sum m : \texttt{Subseqer} \,.\, u_m : \texttt{Converging},$

$x := \lim_{n\to\infty} u_{m_n} : \texttt{In}[a,b],$

$(*) := \texttt{SubseqLimit}(3, f(u))\texttt{SeqContinuous}(f, x)\flat x\Big(f(x)\Big) : f(x) = f\Big(\lim_{n\to\infty} u_{m_n}\Big) = \lim_{n\to\infty} f(u_{m_n}) = y;$

$\square$

$\texttt{FreshmensFixedPointTHM} :: \forall f : C\Big([0,1],[0,1]\Big) \,.\, \exists x \in [0,1] : f(x) = x$

$\texttt{Proof} \;=$

$g := \Lambda x \in [0,1] \,.\, f(x) - x : C\Big([0,1],\mathbb{R}\Big),$

$\texttt{Assume}\ (1) : f(0) \neq 0 \ \&\ f(1) \neq (1),$

$(2) := \eth g \eth [0,1](1) : 0 \in [g(1), g(0)],$

$(x, 3) := \texttt{IntermidiateValueTHM}(g)(2)(0) : \sum x \in [0,1] \,.\, g(x) = 0,$

$() := \eth g(3) : f(x) = x;$

$\rightsquigarrow (1) := I(\Rightarrow) I(\exists)(x) : \Big(f(0) \neq 0 \ \&\ f(1)\Big) \neq 1 \Rightarrow \exists x \in [0,1] \,.\, f(x) = x,$

$(*) := E(|)(\ldots)(1) : \exists x \in [0,1] \,.\, f(x) = x;$

$\square$

$\texttt{IncreasingHomeomorphism} :: \forall f : \texttt{Increasing} \,\&\, C\Big([a,b],\mathbb{R}\Big) \,.\, f : [a,b] \leftrightarrow_{\mathsf{TOP}} [f(a), f(b)]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : [a,b],$

$(1) := \eth x : a \le x \le b,$

$(2) := \eth\texttt{Increasing}(f)(1) : f(a) \le f(x) \le f(b),$

$() := \eth^{-1}[f(a), f(b)](2) : f(x) \in [f(a), f(b)];$

$\rightsquigarrow (0) := \eth^{-1}\texttt{Codomain} : \Big(f : [a,b] \to [f(a), f(b)]\Big),$

$\texttt{Assume}\ y : [f(a), f(b)],$

$() := \texttt{IntermidiateValueTHM}(f, y) : \exists x \in [a,b]\ f(x) = y;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Surjection} : \Big(f : [a,b] \twoheadrightarrow [f(a), f(b)]\Big),$

$\texttt{Assume}\ t, s : [a,b],$

$\texttt{Assume}\ (2) : f(t) = f(s),$

$() := \eth\texttt{Increasing}(2) : t = s;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Injection} : \Big(f : [a,b] \hookrightarrow [f(a), f(b)]\Big),$

$(3) := \eth^{-1}\texttt{Bijection}(1)(2) : \Big(f : [a,b] \leftrightarrow [f(a), f(b)]\Big),$

$\texttt{Assume}\ f(x) : \mathbb{N} \to [f(a), f(b)],$

$\texttt{Assume}\ f(X) : [f(a), f(b)],$

$\texttt{Assume}\ (4) : \lim_{n\to\infty} f(x_n) = f(X),$

$\texttt{Assume}\ (5) : \lim_{n\to\infty} x_n \ne X,$

$(\varepsilon, 6) := \eth\texttt{Limit}(5) : \sum \varepsilon \in \mathbb{R}_{++} \,.\, \forall N \in \mathbb{N} \,.\, \exists n \in \mathbb{N} : n \ge N \,.\, |x_n - X| \ge \varepsilon,$

$\delta := \min\Big(\big|f(X) - f(X-\varepsilon)\big|, \big|f(X) - f(X+\varepsilon)\big|\Big) : \mathbb{R}_{++},$

$\texttt{Assume}\ N : \mathbb{N},$

$(n, 7) := (6)(N) : \sum n \in \mathbb{N} : n \ge N \,.\, |x_n - X| \ge \varepsilon,$

$() := \eth\texttt{Increasing}(f)(7)\eth^{-1}\delta : |f(x_n) - f(X)| \ge \delta;$

$\rightsquigarrow (7) := \texttt{Negate}\eth\texttt{Limit} : \lim_{n\to\infty} f(x_n) \ne f(X),$

$() := (7)(4) : \bot;$

$\rightsquigarrow (4) := \texttt{SeqContinuous} I(\forall) I(\forall) I(\Rightarrow) E(\bot) : f^{-1} : [f(a), f(b)] \to_{\mathsf{TOP}} [a,b],$

$(*) := \eth^{-1}\texttt{Homeo}(3)(4) : f : [a,b] \leftrightarrow_{\mathsf{TOP}} [f(a), f(b)];$

$\square$

4.5 Continuous Wonders[!]

5 Convergence of Functions

5.1 Pointwise Topology

$\texttt{pointwisePolynorm} :: \prod A \subset \mathbb{R} \,.\, A \to (A \to \mathbb{R}) \to \mathbb{R}_+$

$\texttt{pointwisePolynorm}\,(x, f) = \mathbf{p}_x(f) := |f(x)|$

$\texttt{PointwiseIsPolynormed} :: \forall A \subset \mathbb{R} \,.\, \mathbf{p}(A) : \texttt{Polynorm}(\mathbb{R})$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \texttt{In}(A),$

$(1) := \eth \mathbf{p}_x(0) \eth \texttt{absValue} : \mathbf{p}_x(0) = |0(x)| = |0| = 0,$

$\texttt{Assume}\ f, g : A \to \mathbb{R},$

$() := \eth \mathbf{p}_x(f+g) \texttt{TriangleIneq}(\texttt{absValue}(\mathbb{R})) \eth^{-1} \mathbf{p}_x :$

$\quad : \mathbf{p}_x(f+g) = |f(x) + g(x)| \leq |f(x)| + |g(x)| = \mathbf{p}_x(f) + \mathbf{p}_x(g);$

$\rightsquigarrow (2) := I(\forall) : \forall f, g : A \to \mathbb{R} \,.\, \mathbf{p}_x(f+g) \leq \mathbf{p}_x(f) + \mathbf{p}_x(g),$

$\texttt{Assume}\ f : A \to \mathbb{R},$

$\texttt{Assume}\ \alpha : \mathbb{R},$

$() := \eth \mathbf{p}_x(\alpha f) \texttt{AbsHomogen}(\texttt{absValue}(\mathbb{R})) \eth^{-1} \mathbf{p}_x : \mathbf{p}_x(\alpha f) = |\alpha f(x)| = |\alpha||f(x)| = |\alpha| \mathbf{p}_x(f);$

$\rightsquigarrow (3) := I^2(\forall) : \forall f : A \to \mathbb{R} \,.\, \forall \alpha \in \mathbb{R} \,.\, \mathbf{p}_x(\alpha f) = |\alpha| \mathbf{p}_x(f),$

$() := \eth^{-1} \texttt{Seminorm}(1)(2)(3) : \Big(\mathbf{p}_x : \texttt{Seminorm}(\mathbb{R})\Big);$

$\rightsquigarrow (1) := I(\forall) : \forall x \in A \,.\, \mathbf{p}_x : \texttt{Seminorm}(\mathbb{R}),$

$(*) := \eth^{-1} \texttt{Polynorm}(x) : \Big(\mathbf{p} : \texttt{Polynorm}(\mathbb{R})\Big);$

$\square$

$\texttt{PointwiseContinuousLimit} :: \forall f : \mathbb{N} \to C(\mathbb{R},\mathbb{R}) \,.\, \forall \varphi : \mathbb{R} \to \mathbb{R} \,.\, \forall (0) : f \xrightarrow{\mathbf{P}} \varphi \,.\, \mathcal{D}^{\complement}(\varphi) : \texttt{Dense}(\mathbb{R})$

$\texttt{Proof} \; =$

$C := \Lambda i \in \mathbb{N} \,.\, \Lambda j \in \mathbb{N} \,.\, \Lambda r \in \mathbb{R} \,.\, \{x \in \mathbb{R} \,.\, |f_i(x) - f_j(x)| \le r\} : \mathbb{N} \to \mathbb{N} \to \mathbb{R} \to \texttt{Closed}(\mathbb{R}),$

$\texttt{Assume}\ U : \texttt{Open}\ \&\ \texttt{NonEmpty}(\mathbb{R}),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$$K := \Lambda n \in \mathbb{N} \,.\, \bigcap_{i=n}^{\infty} \bigcap_{j=i+1}^{\infty} C(i,j)(\varepsilon) : \mathbb{N} \to \texttt{Closed}(\mathbb{R}),$$

$$(1) := (0)\text{þ} : \bigcup_{n=1}^{\infty} K_n = \mathbb{R},$$

$(n,2) := \texttt{RealBairCategoryI}(U)(1) : \sum n \in \mathbb{N} \,.\, K_n \,!\, \texttt{NowhereDense}(U),$

$(V,3) := \eth\texttt{NowhereDense}(2) : \sum V : \texttt{Open}\ \&\ \texttt{NonEmpty}(U) \,.\, K_n \cap V : \texttt{Dense}(V),$

$(4) := \texttt{DenseClosed}(3) : V \subset K_n,$

$\texttt{Assume}\ x : \texttt{In}(V),$

$(\delta,5) := \eth C(f_{n|V})(x)(\varepsilon) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall y \in \mathbb{B}_V(x,\delta) \,.\, f(y) \in \mathbb{B}(f(x),\varepsilon),$

$\texttt{Assume}\ s,t : \texttt{In}\mathbb{B}_V(x,\delta),$

$() := \texttt{TriangelIneq}(f(s), -f_k(s), f_k(s), -f_k(x), f_k(x), -f_k(t), f_k(t), -f(t))\text{þ}^2 C\text{þ}K\eth V\eth s,t(5)(\eth s,t) :$

$\quad : |f(s) - f(t)| \le |f(s) - f_k(s)| + |f_k(s) - f_k(x)| + |f_k(x) - f_k(t)| + |f_k(t) - f(t)| < 4\varepsilon;$

$\rightsquigarrow () := I(\forall)\eth^{-1}\omega(f,x)I(\forall) : \forall x \in V \,.\, \omega(f,x) < 4\varepsilon;$

$\rightsquigarrow (1) := I(\forall)\eth^{-1}\texttt{NowhereDense}I(\forall)I(\exists)(V) : \forall \varepsilon \in \mathbb{R}_{++} \,.\, \{x \in \mathbb{R} : \omega(f,x) \ge 4\varepsilon\} : \texttt{NowhereDense},$

$$(2) := \eth^{-1}\mathcal{D}(f)\eth\omega(f,x) : \mathcal{D}(f) = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : \omega(f,x) \ge n^{-1}\},$$

$\texttt{Assume}\ U : \texttt{Open}\ \&\ \texttt{NonEmpty}(\mathbb{R}),$

$(3) := (2)\texttt{RealBairCategoryI}(U)(1) : U \not\subset \mathcal{D}(f),$

$(x,4) := \eth^{-1}\texttt{complement}\eth\texttt{Subset}(3) : \sum x \in U \,.\, x \in \mathcal{D}^{\complement}(f);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Dense} : \left(\mathcal{D}^{\complement}(f) : \texttt{Dense}(\mathbb{R})\right);$

$\square$

5.2 Relation between Pointwise and Uniform Convergence

$$\Big(\|\cdot\|_\infty\Big)\ \lim_{n\to\infty} f_n = \varphi \iff f \rightrightarrows \varphi$$

$\texttt{NonDecreasingConvergenceIsUniform} :: \forall f : \mathbb{N} \to \texttt{Nondecreasing}\ \&\ C\Big([a,b],\mathbb{R}\Big) . \forall \varphi \in C\Big([a,b],\mathbb{R}\Big) .$

$\quad . \forall (0) : f \xrightarrow{\mathbf{P}} \varphi . f \rightrightarrows \varphi$

$\texttt{Proof}\ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta, 00) := \eth UC\texttt{CompactUCCriterion}(\varphi)\left(\frac{\varepsilon}{2}\right) : \sum \delta \in \mathbb{R}_{++} . \forall x, y \in [a,b] : |x-y| < \delta . |\varphi(x) - \varphi(y)| < \varepsilon,$

$(n, t, 1) := \texttt{mesh}([a,b], \delta) : \sum n \in \mathbb{N} . \sum t : \texttt{NonDecreasing}\Big(n, [a,b]\Big) .$

$\quad . [a,b] = \bigcup_{i=1}^{n-1} [t_i, t_{i+1}]\ \&\ \forall i \in \mathbb{N} : i < n . t_{i+1} - t_i \le \delta,$

$(N, 2) := (0)(t)(\varepsilon/2) : \sum N \in \mathbb{N} . \forall i \in \mathbb{N} : i \le n\ \&\ \forall m \in \mathbb{N} : m \ge N . |f_n(t_i) - \varphi(t_i)| < \frac{\varepsilon}{2},$

$\texttt{Assume}\ s : [a,b],$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ (3) : m \ge N,$

$(i, 4) := (1_1)(t) : \sum i \in \mathbb{N} : i < n . s \in [t_i, t_{i+1}],$

$(5) := \eth\texttt{NonDecreasing}(f_m)(t_i, s, t_{i+1})(4) : f_m(t_i) \le f_m(s) \le f_m(t_{i+1}),$

$(6) := (5)(2)(3) : \varphi(t_i) - \frac{\varepsilon}{2} \le f_m(s) \le \varphi(t_{t+1}) + \frac{\varepsilon}{2},$

$(7) := (00)(1)(4)(6) : \varphi(s) - \varepsilon \le f_m(s) \le \varphi(s) + \varepsilon,$

$() := \eth^{-1}\texttt{absValue} : |f_m(s) - \varphi(s)| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1} f \rightrightarrows \varphi : f \rightrightarrows \varphi;$

$\square$

$\texttt{NonIncreasingConvergenceIsUniform} :: \forall f : \mathbb{N} \to \texttt{NonIncreasing}\ \&\ C\Big([a,b],\mathbb{R}\Big) . \forall \varphi \in C\Big([a,b],\mathbb{R}\Big) .$

$\quad . \forall (0) : f \xrightarrow{\mathbf{P}} \varphi . f \rightrightarrows \varphi$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{DiniCondition} :: \forall f : \mathbb{N} \to C\Big([a,b],\mathbb{R}\Big) \,.\, \forall \varphi \in C\Big([a,b],\mathbb{R}\Big) \,.\, \forall (0) : f \xrightarrow{\mathbf{p}} \varphi \,.$

$\quad . \,\forall Y : \forall x \in [a,b] \,.\, f(x) : \texttt{Monotonic}(\mathbb{N},\mathbb{R}) \,.\, f \rightrightarrows \varphi$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$U := \Lambda n \in \mathbb{N} \,.\, \Big\{x \in [a,b] \;\Big|\; |f_n(x) - \varphi(x)| < \varepsilon\Big\} : \mathbb{N} \to \texttt{Open}[a,b],$

$(1) := (0)\text{þ}(U) : \bigcup_{n=1}^{\infty} U_n = [a,b],$

$(n,k,2) := \eth\texttt{Compact}[a,b](U)(1) : \sum n \in \mathbb{N} \,.\, \sum k : n \to \mathbb{N} \,.\, \bigcup_{i=1}^{n} U_{k_i} = [a,b],$

$(3) := (0)(Y)\text{þ}(U) : \Big(U : \texttt{Increasing}(?[a,b])\Big),$

$() := (2)(3) : U_{n_k} = [a,b];$

$\rightsquigarrow (*) := \eth^{-1} f \rightrightarrows \varphi I(\forall)\eth U : f \rightrightarrows \varphi;$

$\square$

5.3 Pointwise Compactness

$\texttt{UniformlyBounded} :: ??(X \to \mathbb{R})$

$F : \texttt{UniformlyBounded} \iff \exists c \in \mathbb{R}_{++} \,.\, \forall f \in F \,.\, \forall x \in X \,.\, |f(x)| \le c$

$\texttt{SimpleHellySelection} :: \forall f : \mathbb{N} \to \texttt{Monotonic}\Big([a,b], \mathbb{R}\Big) \,.\, \forall (0) : \Big(\operatorname{Im} f : \texttt{UniformlyBounded}\Big) \,.$

$\quad . \exists n : \texttt{Subseqer} : \Big(f_n : \texttt{Converging}(\mathbf{p})\Big)$

$\texttt{Proof} \ =$

$(1, q) := \eth\texttt{EqCardRationalIntervalsAreCountable}[a,b] : \sum (1) : \top \,.\, q : \mathbb{N} \leftrightarrow_{\mathsf{Set}} \mathbb{Q} \cap [a,b],$

$n^1 := \Lambda k \in \mathbb{N} \,.\, k : \texttt{Subseqer},$

$\texttt{Assume}\ m : \mathbb{N},$

$(k, 2) := \texttt{CompactSubseq}(f_{n^m}(q_m))\eth\texttt{UniformlyBounded}(0) : \sum k : \texttt{Subseqer} \,.\, f_{n^m_k}(q_m) : \texttt{Converging}(\mathbb{R})),$

$n^{m+1} := n^m_k : \texttt{Subseqer};$

$\rightsquigarrow (n.2) := I\Big(\sum\Big)\ \texttt{SubseqLimit} : \sum n : \texttt{Decreasing}(\mathbb{N}, \texttt{Subseqer}) \,.\, \forall k \in \mathbb{N} \,.\, \forall i \in \mathbb{N} : i \le k \,.$

$\quad . f_{n^k}(q_i) : \texttt{Converging}[a,b],$

$n' := \Lambda k \in \mathbb{N} \,.\, n^k_k : \texttt{Subseqer},$

$(3) := \flat n'(2) : \forall k \in \mathbb{N} \,.\, f_{n'}(q_k) : \texttt{Converging}(\mathbb{R}),$

$\texttt{Assume}\ r : [a,b] \cap \mathbb{Q},$

$(l, 4) := \eth q(r) : \sum l \in \mathbb{N} \,.\, q_l = r,$

$\varphi(r) := \lim_{m \to \infty} f_{n'_m} : \mathbb{R};$

$\rightsquigarrow \varphi := I(\to) : [a,b] \cap \mathbb{Q} \to \mathbb{R},$

$\texttt{Assume}\ x : [a,b] \cap \mathbb{Q}^{\complement},$

$(q, 4) := \texttt{RationalApproximation}(x) : \sum q : \mathbb{N} \to [a,b] \cap \mathbb{Q} \,.\, q \uparrow x,$

$(5) := \eth\texttt{UniformlyBounded}(f)\eth\texttt{Monotonic}(f) : \Big(\varphi(q) : \texttt{Monotonic} \ \&\ \texttt{Bounde}\Big),$

$(6) := \texttt{MonotonicAndBoundedIsConverging}(5) : \Big(\varphi(q) : \texttt{Converging}\Big),$

$\varphi := I(\to) : [a,b] \to \mathbb{R};$

$(4) := \flat(\varphi)\eth\texttt{Monotonic}(f) : (\varphi : \texttt{Monotonic}),$

$(5) := \texttt{RepeatAndRepalce}(1)(\mathbb{Q} \cap [a,b], (\mathbb{Q} \cap [a,b]) \cup \mathcal{D}(\varphi)) : \forall x \in \mathcal{D}(f) \,.\, \lim_{k \to \infty} f_{n'_k}(x) = \varphi(x),$

$\texttt{Assume}\ x : \mathcal{D}^{\complement}(\varphi),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(6) := \eth x \eth \mathcal{D}(\varphi) : \varphi \in C\Big([a,b], \mathbb{R}, x\Big),$

$(\delta, 7) := \eth C\Big([a,b], \mathbb{R}, x\Big)(\varphi)(\varepsilon) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall t, s \in (x - \delta, x + \delta) \,.\, |\varphi(t) - \varphi(s)| < \varepsilon,$

$(p, q, 8) := \eth\texttt{Monotonic}(f, \varphi)\texttt{RationalApproximation} : \sum p, q \in \mathbb{Q} \cap (x - \delta, x + \delta) \,.\, \forall k \in \mathbb{N} \,.$

$\quad . \varphi(p) - f_{n'_k}(q) \le \varphi(p) - f_{n'_k}(q) \le \varphi(q) - f_{n'_k}(p),$

$() := \lim_{k\to\infty}(8)(k)\text{þ}(\varphi)(7)(\eth p, q)$:

$: \lim_{k\to\infty}\left|\varphi(x) - f_{n'_k}(x)\right| \le \lim_{k\to\infty}\max\left(\left|\varphi(p) - f_{n'_k}(q)\right|, \left|\varphi(q) - f_{n'_k}(p)\right|\right) = |\varphi(p) - \varphi(q)| < \varepsilon;$

$\rightsquigarrow (6) := I(\forall) : \forall\varepsilon \in \mathbb{R}_{++} \,.\, \lim_{k\to\infty}|f_{n'_k}(x) - \varphi(x)| < \varepsilon,$

$(7) := \lim_{\varepsilon\to 0}(6)(\varepsilon) : \lim_{k\to\infty}|f_{n'_k}(x) - \varphi(x)| = 0,$

$() := \eth^{-1}\texttt{Limit}(7) : \lim_{k\to\infty} f_{n'_k}(x) = \varphi(x);$

$\rightsquigarrow (6) := I(\forall) : \forall x \in \mathcal{D}(\varphi) \,.\, \lim_{k\to\infty} f_{n'_k}(x) = \varphi(x),$

$(*) := \eth^{-1} f_{n'} \xrightarrow{\mathbf{p}} \varphi(5)(6) : f_{n'} \xrightarrow{\mathbf{p}} \varphi;$

□

5.4 Approximation Theorems [!!]

$\texttt{partialBernsteinPolynomial} :: \prod n \in \mathbb{N} \,.\, n \to \mathbb{R}[\mathbb{Z}_+]$

$\texttt{partialBernsteinPolynomial}\,(k) = b^n_k := \Lambda x \in \mathbb{R} \,.\, \binom{n}{k} x^k (1-x)^{n-k}$

$\texttt{funcBernsteinPolynomial} :: \Big([0,1] \to \mathbb{R}\Big) \to \mathbb{N} \to \mathbb{R}[\mathbb{Z}_+]$

$\texttt{funcBernsteinPolynomial}\,(f, n) = B^n_f := \Lambda x \in \mathbb{R} \,.\, \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b^n_k(x)$

$\texttt{BernsteinLemmaI} :: \forall n \in \mathbb{N} \,.\, B^n_1 = 1$

$\texttt{Proof} =$

$\texttt{Assume}\ x : \mathbb{R},$

$(*) := \eth B^n_1(x)\texttt{BinomialExpansion}\eth^{-1}\texttt{Unity}(1) : B^n_1(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + 1 - x)^n = 1;$

$\square$

$\texttt{BernseinLemmaII} :: \forall n \in \mathbb{N} \,.\, \forall x \in \mathbb{R} \,.\, \sum_{k=0}^{n} k b^n_k(x) = nx$

$\texttt{Proof} =$

$f := \Lambda(x, y) \in \mathbb{R} \times \mathbb{R} \,.\, (x+y)^n : \mathbb{R}^2 \to \mathbb{R},$

$(1) := \texttt{BinomialExpansion}\,\texttt{LinearDifferentiation}\eth\frac{\partial f}{\partial x} :$

$: \forall x, y \in \mathbb{R} \,.\, \sum_{k=0}^{n} k x^{k-1} y^{n-k} = \sum_{k=0}^{n} \frac{\partial}{\partial x} x^k y^{n-k} = \frac{\partial f}{\partial x}(x, y) = n(x+y)^{n-1},$

$(2) := \Lambda x, y \in \mathbb{R} \,.\, x(1) : \forall x, y \in \mathbb{R} \,.\, \sum_{k=0}^{n} k x^k y^{n-k} = nx(x+y)^{n-1},$

$(*) := \eth^{-1} b^n_k \Lambda x \in \mathbb{R} \,.\, (2)(x, 1-x) : \forall x \in \mathbb{R} \,.\, \sum_{k=0}^{n} k b^n_k(x) = nx;$

$\square$

$\texttt{PositiveBernstein} :: \forall n \in \mathbb{N} \,.\, \forall k \in n \,.\, \forall x \in [0,1] \,.\, b^k_(x) \geq 0$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{BerensteinLemmaIII} :: \forall n \in \mathbb{N} . \forall x \in \mathbb{R} . \sum_{k=0}^{n} k^2 b_k^n(x) = n(n-1)x^2 + nx$

$\texttt{Proof} \ =$

$f := \Lambda(x, y) \in \mathbb{R} \times \mathbb{R} . (x+y)^n : \mathbb{R}^2 \to \mathbb{R},$

$(1) := \texttt{BinomialExpansion}\,\texttt{LinearDifferentiation}\eth\dfrac{\partial^2 f}{\partial x^2} :$

$$: \forall x, y \in \mathbb{R} . \sum_{k=0}^{n} k(k-1)x^{k-2}y^{n-k} = \sum_{k=0}^{n} \frac{\partial^2}{\partial x^2} x^k y^{n-k} = \frac{\partial^2 f}{\partial x^2}(x, y) = n(n-1)(x+y)^{n-2},$$

$$(2) := \Lambda x, y \in \mathbb{R} . x^2(1) : \forall x, y \in \mathbb{R} . \sum_{k=0}^{n} (k-1)kx^k y^{n-k} = n(n-1)x^2(x+y)^{n-2},$$

$$(3) := (2) + \frac{\partial f}{\partial x} : \forall x, y \in \mathbb{R} . \sum_{k=0}^{n} k^2 x^k y^{(n-k)} = n(n-1)x^2(x+y)^{n-2} + nx(x+y)^{n-1},$$

$$(*) := \eth^{-1}(3)(x, 1-x) : \forall x \in \mathbb{R} . \sum_{k=0}^{\infty} k^2 b_k^n(x) = n(n-1)x^2 + nx;$$

□

$\texttt{BernsteinLemmaIV} :: \forall n \in \mathbb{N} . \forall x \in \mathbb{R} . \sum_{k=0}^{n} (k - nx)^2 b_k^n(x) = nx(1-x)$

$\texttt{Proof} \ =$

$(*) := x^2 n^2 \texttt{BernsteinLemmaI} - 2xn\texttt{BernsteibLemmaII} + \texttt{BernsteinLemmaIII} :$

$$: \forall x \in \mathbb{R} . \sum_{k=0}^{n} (k - nx)^2 b_k^n(x) = n(n-1)x^2 + nx - 2n^2x^2 + n^2x^2 = nx(1-x),$$

□

$\texttt{BernsteinPolynomialApproximation} :: \forall f \in C\Big([0,1],\mathbb{R}\Big) \,.\, B_f \rightrightarrows f$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta, 1) := \texttt{CompactUCCriterion}\eth UC(f)(\varepsilon/2) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall x, y \in [0,1] : |x-y| < \delta \,.\, |f(x) - f(y)| < \frac{\varepsilon}{2},$

$N := \left\lceil \frac{\|f\|_\infty}{\delta^2} \right\rceil : \mathbb{N},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (3) : n \geq N,$

$\texttt{Assume}\ x : \texttt{In}[0,1],$

$J_1 := \left\{ k \in n : \left| \frac{k}{n} - x \right| < \delta \right\} :?n,$

$J_2 := J_1^{\complement} :?n,$

$(4) := \texttt{TriangleIneq}\text{þ}J_1\texttt{PositiveBernstein}(x)\texttt{BernsteinLemmaI}(n,x) :$

$$: \left| \sum_{k \in J_1} \left(f\left(\frac{k}{n}\right) - f(x) \right) b_k^n(x) \right| \leq \sum_{k \in J_1} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_k^n(x) < \frac{\varepsilon}{2} \sum_{k=0}^{n} b_k^n(x) = \frac{\varepsilon}{2},$$

$(5) := \texttt{BernsteinLemaIV}(x)\texttt{PositiveBernstein}(J_2)\text{þ}J_2 :$

$$: nx(1-x) = \sum_{k=0}^{n} (k - nx)^2 b_k^n(x) \geq \sum_{k \in J_2} (k - nx)^2 b_k^n(x) \geq \sum_{k \in J_2} \delta^2 n^2 b_k^n(x),$$

$(6) := (5)\texttt{MaxIneq}(\Lambda x \,.\, x(1-x)) : \sum_{k \in J_2} B_k^n(x) \leq \frac{nx(1-x)}{\delta^2 n^2} \leq \frac{1}{4\delta n},$

$(7) := \texttt{TriangleIneq}\eth^{-1}\|f\|_\infty(6)(3)\text{þ}(N) :$

$$: \left| \sum_{k \in J_2} \left(f\left(\frac{k}{n}\right) - f(x) \right) b_k^n(x) \right| \leq \sum_{k \in J_2} \left| f\left(\frac{k}{n}\right) - f(x) \right| b_k^n(x) \leq 2\|f\|_\infty \sum_{k \in J_2} b_k^n(x) \leq \frac{\|f\|_\infty}{2n\delta^2} < \frac{\varepsilon}{2},$$

$() := (4)(7) : |B_f(x) - f(x)| < \varepsilon;$

$\rightsquigarrow (4) := I(\forall) : \forall x \in [0,1] \,.\, |B_f^n(x) - f(x)| < \varepsilon,$

$(5) := \eth^{-1}\|\cdot\|_\infty\texttt{CompactMaxPrinciple}(4) : \|B_f^n - f\|_\infty = \sup_{x \in [0,1]} |B_f^n(x) - f(x)| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1} B_f \rightrightarrows f : B_f \rightrightarrows f;$

$\square$

$\texttt{PiecewiseLinearApproximation} :: \forall f \in C\Big([0,1],\mathbb{R}\Big) . \exists L : \mathbb{N} \to \texttt{Piecewise Linear}\,([0,1],\mathbb{R}) \;. L \rightrightarrows f$

$\texttt{Proof} \;=$

$(0) := \texttt{CompactUCriterion}(f) : \Bigg(f : UC\Big([0,1],\mathbb{R}\Big)\Bigg),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta, 1) := \eth UC\Big([0,1],\mathbb{R}\Big)\left(\frac{\varepsilon}{2}\right) : \sum \delta \in \mathbb{R}_{++} . \forall x, y \in [0,1] : |x-y| < \delta . |f(x) - f(y)| < \frac{\varepsilon}{2},$

$(n, t, 2) := \texttt{mesh}[0,1](\delta) : \sum n \in \mathbb{N} . \sum t : \texttt{Increasing}\Big(n, [0,1]\Big) .$

$. [0,1] = \bigcup_{i=1}^{n-1} [t_i, t_{i+1}] \;\&\; \forall i \in \mathbb{N} : i < n . t_i - t_{i-1} < \delta,$

$\texttt{Assume}\ x : [0,1],$

$(i, 3) := (2_1)(1) : \sum i \in n-1 . x \in [t_i, t_{i+1}],$

$L(x) := \frac{t_{i+1} - x}{t_{i+1} - t_i} f(t_i) + \frac{x - t_i}{t_{i+1} - t_i} f(t_{i+1}) : \mathbb{R},$

$(4) := \eth^{-1}\Big[f(t_i), f(t_{i+1})\Big] \text{þ} L(x) : L(x) \in \Big[f(t_i), f(t_{i+1})\Big],$

$() := (4)(1)(2)(3)(x) : |L(x) - f(x)| \le \max\Big(|f(t_i) - f(x)|, |f(x) - f(t_{i+1})|\Big) < \varepsilon;$

$\rightsquigarrow (1) := I(\forall)\eth^{-1}\texttt{Piecewise Linear}I(\forall) : \forall \varepsilon \in \mathbb{R}_{++} . \exists L : \texttt{Piecewise Linear}\Big([0,1],\mathbb{R}\Big) . \|L - f\|_\infty < \varepsilon,$

$(2) := \eth^{-1}\texttt{Dense}(1) : \Bigg(\texttt{Piecewie Linear}\Big([0,1],\mathbb{R}\Big) : \texttt{Dense}\Big(C[0,1](\mathbb{R}), \|\cdot\|_\infty\Big)\Bigg),$

$(*) := \eth^{-1}\texttt{Limit}\eth\texttt{Dense}(2) : \texttt{This};$

□

5.5 Power Series

$$\texttt{radiOfConvergence} :: (\mathbb{N} \to \mathbb{R}) \to \overset{\infty}{\mathbb{R}}$$

$$\texttt{radiOfConvergence}\,(a) = R(a) := \limsup_n \left(\sqrt[n]{|a_n|}\right)^{-1}$$

$$\texttt{powerSeria} :: (\mathbb{N} \to \mathbb{R}) \to \mathbb{N} \to \mathbb{R} \to \mathbb{R}$$

$$\texttt{powerSeria}\,(a, n, x) = F_a^n(x) := \sum_{i=0}^{n} a_{n-1} x^n$$

$\texttt{PowerSeriaConvergence} :: \forall a : \mathbb{Z}_+ \to \mathbb{R} \,.\, \forall r < R(a) \,.\, \exists f : (-r, r) : F_a \rightrightarrows f$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \beta : (r, R(a)),$

$(N, 1) := \eth(1) : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} : n \geq N \,.\, \sqrt[n]{a_n} < \dfrac{1}{\beta},$

$\texttt{Assume}\ x : \texttt{In}[-r, r],$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (2) : n \geq N,$

$() := (1)(n)\eth x \text{þ} \beta : \left|a_n x^n\right| \leq \left(\dfrac{r}{\beta}\right)^n;$

$\rightsquigarrow (2) := I^3(\forall) : \forall x \in [-r, r]. \forall n \in \mathbb{N} : n \geq N \,.\, \left|a_n x^n\right| \leq \left(\dfrac{r}{\beta}\right)^n,$

$(*) := \texttt{ComparissonTest}(2)\texttt{InfiniteGeometricSum}(r/\beta)\eth^{-1} F_a : F_a \rightrightarrows \displaystyle\sum_{n=0}^{\infty} a_n x^n;$

$\square$

6 Applications of Differential Analysis

6.1 Mean Value Theorems

$\texttt{DifferentiableAtInterval} :: \prod [a,b] : \texttt{ClosediInerval}(\mathbb{R}) \,.\, ?C\Big([a,b],\mathbb{R}\Big)$

$f : \texttt{DifferentiableAtInterval} \iff f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \iff f_{|(a,b)} : (a,b) \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R}$

$\texttt{RolleLemma} :: \forall f : C[a,b] : f(a) = f(b) \,.\, \exists x : \texttt{Optimizer}(f) : x \in (a,b)$

$\texttt{Proof} \ =$

$(x,1) := \texttt{CompactMax}(f) : \sum x \in [a,b] \,.\, \forall y \in [a,b] \,.\, |f(x)| \geq |f(y)|,$

$\texttt{Assume}\ (2) : \forall y \in [a,b] \,.\, f(x) \geq f(y),$

$\texttt{Assume}\ (3) : x \in \{a,b\},$

$(4) := (2)(3)\eth f : \forall y \in [a,b] \,.\, f(a) = f(b) \geq f(x),$

$() := \eth^{-1} \arg\min f(4) : \arg\min f \in (a,b);$

$\rightsquigarrow (3) := I(\Rightarrow)I(\exists)(\arg\min f) : x \in \{a,b\} \Rightarrow \exists y : \texttt{Optimizer}(f) : y \in (a,b),$

$() := \texttt{LEM}(x \in \{a,b\})(3) : \exists y : \texttt{Optimizer}(f) : y \in (a,b);$

$\rightsquigarrow (2) := I(\Rightarrow) : \Big(x : \texttt{Maximizer}(f) \Rightarrow \texttt{This}(f)\Big),$

$\texttt{Assume}\ (3) : \forall y \in [a,b] \,.\, f(x) \leq f(y),$

$\texttt{Assume}\ (4) : x \in \{a,b\},$

$(5) := (3)(4)\eth f : \forall y \in [a,b] \,.\, f(a) = f(b) \leq f(x),$

$() := \eth^{-1} \arg\min f(5) : \arg\max f \in (a,b);$

$\rightsquigarrow (4) := I(\Rightarrow)I(\exists)(\arg\max f) : x \in \{a,b\} \Rightarrow \exists y : \texttt{Optimizer}(f) : y \in (a,b),$

$() := \texttt{LEM}(x \in \{a,b\})(3) : \exists y : \texttt{Optimizer}(f) : y \in (a,b);$

$\rightsquigarrow (3) := I(\Rightarrow) : \Big(x : \texttt{Minimizer}(f) \Rightarrow \texttt{This}(f)\Big),$

$(*) := E(|)\eth\texttt{Optimizer}(x)(2)(3) : \texttt{This}(x);$

$\square$

$\texttt{LagrangeMeanValueTheorem} :: \forall f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \,.\, \exists x \in [a,b] : \dfrac{f(b)-f(a)}{a-b} = f'(x)$

$\texttt{Proof} \ =$

$F := \Lambda x \in [a,b] \,.\, f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a) : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R},$

$(x,1) := \texttt{RolleLemma}\text{þ}F : \sum x \in (a,b) \,.\, x : \texttt{Optimizer}(F),$

$(2) := \texttt{FirstDerivativeOptimumCriterion}(1) : F'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a} = 0,$

$(*) := -\Big((2) - f'(x)\Big) : \dfrac{f(b)-f(a)}{b-a} = f'(x);$

$\square$

$\texttt{IncreasingByPositiveDerivative} :: \forall f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \, . \, \forall (0) : f' > 0 \, . \, f : \texttt{Increasing}(a,b)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; x, y : (a,b),$

$\texttt{Assume}\; (1) : x < y,$

$$(t,2) := \texttt{LagrandgeMeanValueTHM}(f_{|[x,y]}(0)(t) : \sum t \in (x,y) \, . \, \frac{f(y)-f(x)}{y-x} = f'(t) > 0,$$

$() := (2)(y-x) + f(x) : f(y) > 0;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Increasing}I(\forall)I(\Rightarrow) : \texttt{This}(f);$

$\square$

$\texttt{DecreasingByNegativeDerivative} :: \forall f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \, . \, \forall (0) : f' < 0 \, . \, f : \texttt{Decreasing}(a,b)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{NonDecreasingByNonNegativeDerivative} :: \forall f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \, . \, \forall (0) : f' \geq 0 \, . \, f : \texttt{NonDecreasing}(a,b)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{NonIncreasingByNonPositveDifferential} :: \forall f : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \, . \, \forall (0) : f' \leq 0 \, . \, f : \texttt{NonIncreasing}(a,b)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{CauchyMeanValueTheorem} :: \forall f, g : [a,b] \to_{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \, . \, \exists t \in (a,b) \, . \, \big(f(b) - f(a)\big) g'(t) = \big(g(b) - g(a)\big) f'(t)$

$\texttt{Proof} \; =$

$F := \Lambda x \in [a,b] \, . \, f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big) : [a,b] \xrightarrow{\mathsf{DIFF}(\mathbb{R})} \mathbb{R},$

$(1) := \text{þ}F(a) : F(a) = f(a)g(b) - f(b)g(a),$

$(2) := \text{þ}F(b) : F(b) = f(a)g(b) - f(b)g(a),$

$(x,3) := \texttt{RolleLemma}(1,2) : \sum x \in (a,b) \, . \, x : \texttt{Optimizer}(F),$

$(*) := \texttt{FirstDerivativeOptimumCriterion}(f) : 0 = f'(t) = f'(x)\big(g(b) - g(a)\big) - g'(x)\big(f(b) - f(a)\big);$

$\square$

6.2 L'hopital Rule

$$\texttt{ZeroLhopitalRule} :: \forall U : \texttt{Open}\ \&\ \texttt{Connected}(\mathbb{R}) \,.\, \forall f, g : U \xrightarrow{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \,.\, \forall V : \texttt{Open}(U) : g'\left(\dot{V}\right) \neq 0 \,.$$

$$.\ \forall a \in \mathbb{R} \,.\, \forall L \in \mathbb{R} \,.\, \forall (0) : \lim_{x \to a} f(x) = 0\ \&\ \lim_{x \to a} f(x) = 0 \,.\, \forall (00) : \lim_{x \to a} \frac{f'(x)}{g'(x)} = L \,.\, \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

$\texttt{Proof} =$

$\texttt{Assume}\ x, y : V,$

$\texttt{Assume}\ (1) : y > x,$

$(t, 2) := \texttt{CauchyMeanValueTheorem}(f_{[|x,y]}, g_{|[x,y]}) : \dfrac{f(y) - f(x)}{g(y) - g(x)} = \dfrac{f'(t)}{g'(t)},$

$() := (2) \left(1 - \dfrac{g(x)}{g(y)}\right) : \dfrac{f'(t)}{g'(t)} \left(1 - \dfrac{g(x)}{g(y)}\right) = \dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)};$

$\rightsquigarrow (1) := I(\forall) : \forall x, y \in V : y > x \,.\, \exists t \in (x, y) : \dfrac{f'(t)}{g'(t)} \left(1 - \dfrac{g(x)}{g(y)}\right) = \dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)},$

$(*) := (00)(0) \lim\limits_{y \to a} \lim\limits_{x \to a} (1)(x, y)(0) : L = \lim\limits_{y \to a} \dfrac{f'(t(y))}{g'(t(y))} = \lim\limits_{y \to a} \lim\limits_{x \to a} \dfrac{f'(t)}{g'(t)} \left(1 - \dfrac{g(x)}{g(y)}\right) =$

$\quad = \lim\limits_{y \to a} \lim\limits_{x \to a} \dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)} = \lim\limits_{y \to a} \dfrac{f(y)}{g(y)};$

$\square$

$$\texttt{InftyLhopitalRule} :: \forall U : \texttt{Open}\ \&\ \texttt{Connected}(\mathbb{R}) \,.\, \forall f, g : U \xrightarrow{\mathsf{DIFF}(\mathbb{R})} \mathbb{R} \,.\, \forall V : \texttt{Open}(U) : g'\left(\dot{V}\right) \neq 0 \,.$$

$$.\ \forall a \in \mathbb{R} \,.\, \forall L \in \mathbb{R} \,.\, \forall (0) : \lim_{x \to a} f(x) = \infty\ \&\ \lim_{x \to a} f(x) = \infty \,.\, \forall (00) : \lim_{x \to a} \frac{f'(x)}{g'(x)} = L \,.\, \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

$\texttt{Proof} =$

$\dots$

$\square$

6.3 Analytic Functions[!]

7 Riemann-Stieltjes Integral

7.1 Riemann Integrable Functions

$$\texttt{RSIntegrable} :: \prod E \in \mathsf{BAN}(K) \,.\, ([a,b] \to \mathbb{R}) \to ?\Big([a,b] \to E\Big)$$

$$f : \texttt{RSIntegrable} \iff f \in \mathcal{R}\Big([a,b], \varphi\Big) \iff \Lambda\varphi[a,b] \to \mathbb{R} \,.\, \exists I \in E \,.$$

$$.\, \forall x : \prod [t,s] : \texttt{ClosedInterval}[a,b] \,.\, [t,s] \,.$$

$$.\, \lim_{(n+1,t)\in\mathfrak{P}[a,b]} \sum_{i=1}^{n} f\Big(x[t_i, t_{i+1}]\Big)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) = I$$

$$\texttt{definiteRSIntegral} :: \mathcal{R}(E)\Big([a,b], \varphi\Big) \to E$$

$$\texttt{definiteRSIntegral}\,(f) = \int_a^b f \mathrm{d}\,\varphi := \eth\mathcal{R}\Big([a,b], \varphi\Big)(f)$$

$$\prod E : \mathsf{BAN}(K) \,.\, \mathcal{R}(E)[a,b] := \prod E : \mathsf{BAN}(K) \,.\, \mathcal{R}(E)\Big([a,b], \varphi\Big) :?\Big([a,b] \to E\Big),$$

$$\texttt{RSIntegrableIsBounded} :: \forall \varphi : \texttt{StrictMonotonic}\Big([a,b], \mathbb{R}\Big) \,.\, \forall f \in (E)\Big([a,b], \varphi\Big) \,.\, f : \texttt{Bounded}\Big([a,b], E\Big)$$

$\texttt{Proof} \;=$

$$I := \int_a^b f \mathrm{d}\,\varphi : \texttt{In}(E),$$

$\texttt{Assume}\ (1) : f \ !\ \texttt{Bounded},$

$\texttt{Assume}\ P : \mathfrak{P}[a,b],$

$\texttt{Assume}\ (n+1,t) : \texttt{In}(P),$

$$(i,2) := \eth\texttt{Bounded}(1)\eth(n+1,t)\eth P \eth\mathfrak{P}[a,b] : \sum i \in n \,.\, \forall \eta \in \mathbb{R}_{++} \,.\, \exists x \in [t_i, t_{i+1}] \,.\, \|f(x)\| \geq \eta,$$

$$(3) := \eth\texttt{StrictlyMonotonic}(\varphi)(n+1,t)(i) : \varphi(t_{i+1}) - \varphi(t_i),$$

$$(x,4) := \texttt{FiniteSelection}\eth\texttt{Norm}\eth I(2)(3) : \sum x : \prod [u,v] : \texttt{ClosedInterval}[a,b] \,.\, [u,v] \,.$$

$$.\, \left\| I - \sum_{i=1}^{n} f\Big(x[t_i, t_{i+1}]\Big)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) \right\| > 1;$$

$$\leadsto (2) := \eth^{-1}\texttt{definiteIntegral}\eth\texttt{NetLimit} : \int_a^b f \mathrm{d}\,\varphi \neq I,$$

$$(3) := (2)\eth I : \bot;$$

$$\leadsto (4) := E(\bot) : \left(f : \texttt{Bounded}\Big([a,b], \mathbb{R}\Big)\right);$$

$\square$

$$\texttt{SummableVariation} :: \prod E \in \mathsf{BAN}(K) \,.\, \Big([a,b] \to \mathbb{R}\Big) \to ?\Big([a,b] \to E\Big)$$

$$f : \texttt{SummableVariation} \iff \Lambda\varphi : [a,b] \to \mathbb{R} \,.\, \lim_{(n+1,t)\in\mathfrak{P}[a,b]} \sum_{i=1}^{n} \omega\Big(f, [t_i, t_{i+1}]\Big)\big|\varphi(t_{i+1}) - \varphi(t_i)\big| = 0$$

$\texttt{RiemannIntegrabilityCriterion} :: \forall \varphi : [a,b] \to \mathbb{R} \,.\, \forall f : \texttt{SummableVariation}(E)\Big([a,b],\varphi\Big) \,.$

$\quad . \; f \in \mathcal{R}(E)\Big([a,b],\varphi\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : \prod [t,s] : \texttt{ClosedInterval}[a,b] \,.\, [t,s],$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$(\delta, 1) := \eth\texttt{SummablleVariation}(f)(\varepsilon) :$

$$: \sum \delta \in \mathbb{R}_{++} \,.\, \forall (n+1,t) : \delta\text{-}\texttt{Mesh}[a,b] \,.\, \sum_{i=1}^{n} \omega\Big(f, [t_i, t_{i+1}]\Big)\big|\varphi(t_i) - \varphi(t_{i+1})\big| < \varepsilon,$$

$\texttt{Assume}\; (n+1,t), (m+1,s) : \delta\text{-}\texttt{Mesh}[a,b],$

$(k+1,u) := (n+1,t) \cup (m+1,s) : \dfrac{\delta}{2}\text{-}\texttt{Mesh}[a,b],$

$(i,2) := \flat(k+1,u)(n+1,t) : \sum i : k \to n : \forall l \in k \,.\, t_{i(l)} \le u_l < t_{i(l)+1},$

$(j,3) := \flat(k+1,u)(m+1,s) : \sum j : k \to m : \forall l \in k \,.\, s_{j(l)} \le u_l < s_{j(l)+1},$

$() := \eth^{-1}(k+1,u)\texttt{DistributiveScalarMult}(E)\texttt{TriangleIneq}(E)\texttt{AbsHomogen}(E)$

$\quad \eth^{-1}\omega(f,\cdot)(1)\Big(\eth(n-1,t)\eth(m+1,s)\Big) :$

$$: \left\| \sum_{i=1}^{n} f\Big(x[t_i,t_{i+1}]\Big)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) - \sum_{j=1}^{m} f\Big(x[s_j,s_{j+1}]\Big)\big(\varphi(s_{j+1}) - \varphi(s_j)\big) \right\| =$$

$$= \left\| \sum_{l=1}^{k} f\Big(x[t_{i(l)},t_{i(l)+1}]\Big)\big(\varphi(u_{l+1}) - \varphi(u_l)\big) - \sum_{l=1}^{k} f\Big(x[s_{j(l)},s_{j(l)+1}]\Big)\big(\varphi(u_{l+1}) - \varphi(u_l)\big) \right\| =$$

$$= \left\| \sum_{l=1}^{k} \Big(f\Big(x[t_{i(l)},t_{i(l)+1}]\Big) - f\Big(x[s_{j(l)},s_{j(l)+1}]\Big)\Big)\big(\varphi(u_{l+1}) - \varphi(u_l)\big) \right\| \le$$

$$\le \sum_{l=1}^{k} \left\| f\Big(x[t_{i(l)},t_{i(l)+1}]\Big) - f\Big(x[s_{j(l)},s_{j(l)+1}]\Big) \right\| \big|\varphi(u_{l+1}) - \varphi(u_l)\big| \le$$

$$\le \sum_{l=1}^{k} \omega\Big(f, \big[\min(t_{i(l)}, s_{j(l)}, \max(t_{i(l)+1}, s_{j(l)})\big]\Big)\big|\varphi(u_{l+1}) - \varphi(u_l)\big| < \varepsilon;$$

$\rightsquigarrow (*) := \eth^{-1}\mathcal{R}(E)\Big([a,b],\varphi\Big)\eth\texttt{Complete}(E)\eth^{-1}\texttt{NetCauchy}(E) : f \in \mathcal{R}(E)\Big([a,b],\varphi\Big);$

$\square$

$\texttt{ContinuousIsRiemannIntegrable} :: \forall f \in C\Big([a,b],E\Big) \,.\, f \in \mathcal{R}\Big([a,b],E\Big)$

$\texttt{Proof} \;=$

$(1) := \texttt{CompactUCCriterion}(\eth f) : \left(f : UC\Big([a,b],E\Big)\right),$

$\texttt{Assume}\ \varepsilon : \mathbb{R},$

$(\delta, 2) := \eth\texttt{UC}(f,1)\left(\frac{\varepsilon}{b-a}\right) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall x,y \in [a,b] : |x-y| < \delta \,.\, \|f(x)-f(y)\| < \frac{\varepsilon}{b-a},$

$\texttt{Assume}\ (n+1,t) : \delta\text{-}\texttt{Mesh}[a,b],$

$() := (2)(\eth\delta\text{-}\texttt{Mesh})\texttt{DistributiveScalarMult}(E) :$

$$: \sum_{i=1}^{n} \omega\Big(f,[t_i,t_{i+1}]\Big)(t_{i+1}-t_i) < \frac{\varepsilon}{b-a}\sum_{i=1}^{n} t_{i+1}-t_i = \frac{\varepsilon(b-a)}{b-a} = \varepsilon;$$

$\rightsquigarrow (2) := \eth^{-1}\texttt{SummableVariation} : \Big(f : \texttt{SummableVariation}([a,b],\mathrm{id})\Big),$

$(*) := \texttt{RiemannIntegrabilityCriterion}(2) : f \in \mathcal{R}\Big([a,b],E\Big);$

$\square$

$\texttt{MonotonicIsRiemannIntegrable} :: \forall f : \texttt{Monotonic}\Big([a,b],\mathbb{R}\Big) \,.\, \forall \varphi \in C\Big([a,b],\mathbb{R}\Big) \,.\, f \in \mathcal{R}(\mathbb{R})\Big([a,b],\varphi\Big) \,.$

$\texttt{Proof} \;=$

$(1) := \texttt{CompactUCCriterion}(\eth\varphi) : \left(\varphi : UC\Big([a,b],\mathbb{R}\Big)\right),$

$\texttt{Assume}\ \varepsilon : \mathbb{R},$

$(\delta, 2) := \eth\texttt{UC}(\varphi,1)\left(\frac{\varepsilon}{f(b)-f(a)}\right) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall x,y \in [a,b] : |x-y| < \delta \,.\, |\varphi(x)-\varphi(y)| < \frac{\varepsilon}{f(b)-f(a)},$

$\texttt{Assume}\ (n+1,t) : \delta\text{-}\texttt{Mesh}[a,b],$

$() := (2)(\eth\delta\text{-}\texttt{Mesh})\texttt{DistributiveScalarMult}(E)\eth\omega(f,\cdot)\eth\texttt{Monotonic}(f) :$

$$: \sum_{i=1}^{n} \omega\Big(f,[t_i,t_{i+1}]\Big)(\varphi(t_{i+1})-\varphi(t_i)) < \frac{\varepsilon}{f(b)-f(a)}\sum_{i=1}^{n} t_{i+1}-t_i = \frac{\varepsilon(f(b)-f(a))}{f(b)-(a)} = \varepsilon;$$

$\rightsquigarrow (2) := \eth^{-1}\texttt{SummableVariation} : \Big(f : \texttt{SummableVariation}([a,b],\varphi)\Big),$

$(*) := \texttt{RiemannIntegrabilityCriterion}(2) : f \in \mathcal{R}(\mathbb{R})\Big([a,b],\varphi\Big);$

$\square$

$\texttt{RiemannIntegrableFormVS} :: \forall E \in \mathsf{BAN}(k) \,.\, \forall [a,b] : \texttt{Type}\mathit{ClosedInterval}(\mathbb{R}) \,.\, \forall \varphi : [a,b] \to \mathbb{R} \,.$

$: \mathcal{R}(E)\Big([a,b],\varphi\Big) \in \mathsf{VS}(k)$

$\texttt{Proof} \;=$

Follows from continuity of addition and scalar multiplication in E.

$\square$

$\texttt{RiemannIntegralIsFunctional} :: \forall E \in \mathsf{BAN}(k) \,.\, \forall [a,b] : \texttt{Type}\mathit{closedinterval}(\mathbb{R}) \,.\, \forall \varphi : [a,b] \to \mathbb{R} \,.$

$: \texttt{definiteRSInegral} : \mathcal{R}(E)\Big([a,b],\varphi\Big) \xrightarrow{\mathsf{VS}(k)} E$

$\texttt{Proof} \;=$

Follows from continuity of addition and scalar multiplication in E.

$\square$

7.2 Darbuex Lore

$$\mathcal{R}\Big([a.b],\varphi\Big) := \mathcal{R}(\mathbb{R})[a,b] :?\Big([a,b]\to\mathbb{R}\Big),$$

$$\texttt{lowerDarbuexSum} :: \prod [a,b] : \texttt{ClosedInerval}(\mathbb{R}) \ . \ \texttt{NonDecreasing}\Big([a,b],\mathbb{R}\Big) \to \Big([a,b]\to\mathbb{R}\Big) \to$$
$$\to \texttt{Mesh}[a,b] \to \mathbb{R}$$

$$\texttt{lowerDarbuexSum}\,(\varphi,f,(n+1,t)) = s\Big(\varphi,f,(n+1,t)\Big) := \sum_{i=1}^{n} \inf_{x\in[t_i,t_{i+1}]} f(x)\big(\varphi(t_{i+1})-\varphi(t_i)\big)$$

$$\texttt{upperDarbuexSum} :: \prod [a,b] : \texttt{ClosedInerval}(\mathbb{R}) \ . \ \texttt{upperDecreasing}\Big([a,b],\mathbb{R}\Big) \to \Big([a,b]\to\mathbb{R}\Big) \to$$
$$\to \texttt{Mesh}[a,b] \to \mathbb{R}$$

$$\texttt{LowerDarbuexSum}\,(\varphi,f,(n+1,t)) = S\Big(\varphi,f,(n+1,t)\Big) := \sum_{i=1}^{n} \sup_{x\in[t_i,t_{i+1}]} f(x)\big(\varphi(t_{i+1})-\varphi(t_i)\big)$$

$$\texttt{LowerDarbuexLemmaI} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \ . \ \varphi : \texttt{NonDecreasing}\Big([a,b]\Big) \ . \ \forall f : [a,b]\to\mathbb{R} \ .$$
$$. \ \forall (n+1,t) : \texttt{Mesh}\big([a,b]\big) \ . \ s\Big(\varphi,f,(n+1,t)\Big) =$$
$$= \inf\left\{\sum_{i=1}^{n} f\Big(x[t_i,t_{i+1}]\Big)\big(\varphi(t_{i+1})-\varphi(t_i)\big)\middle| x : \prod [u,v] : \texttt{ClosedInterval}[a,b] \ . \ [u,v]\right\}$$

$\texttt{Proof} \ =$

Trivially, compute infimums.

□

$$\texttt{UpperDarbuexLemmaI} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \ . \ \varphi : \texttt{NonDecreasing}\Big([a,b]\Big) \ . \ \forall f : [a,b]\to\mathbb{R} \ .$$
$$. \ \forall (n+1,t) : \texttt{Mesh}\big([a,b]\big) \ . \ S\Big(\varphi,f,(n+1,t)\Big) =$$
$$= \sup\left\{\sum_{i=1}^{n} f\Big(x[t_i,t_{i+1}]\Big)\big(\varphi(t_{i+1})-\varphi(t_i)\big)\middle| x : \prod [u,v] : \texttt{ClosedInterval}[a,b] \ . \ [u,v]\right\}$$

$\texttt{Proof} \ =$

Trivially, compute supremums.

□

$$\texttt{DarbuexIneq} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \ . \ \varphi : \texttt{NonDecreasing}\Big([a,b]\Big) \ . \ \forall f : [a,b]\to\mathbb{R} \ .$$
$$. \ \forall P,Q : \texttt{Mesh}[a,b] \ . \ s(\varphi,f,P) \le S(\varphi,f,Q)$$

$\texttt{Proof} \ =$

$R := P\cap Q : \texttt{Mesh}[a,b],$

$(*) := \eth s(\varphi,f,\cdot)\eth S(\varphi,f,\cdot) : s(\varphi,f,P) \le s(\varphi,f,R) \le S(\varphi,f,R) \le s(\varphi,f,Q);$

$\texttt{lowerDarbuexLemmaII} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \,.\, \forall \varphi : C \ \&\ \texttt{Increasing}\Big([a,b], \mathbb{R}\Big) \,.$

$\quad .\ \forall f : \texttt{Bounded}\Big([a,b], \mathbb{R}\Big) \,.\, \lim_{P \in \mathfrak{P}[a,b]} s(f, \varphi, P) = \lim_{\varepsilon \to 0} \sup \Big\{ s(\varphi, f, P) | P : \varepsilon\text{-}\texttt{Mesh}[a,b] \Big\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ P : \varepsilon\text{-}\texttt{Mesh},$

$() := \eth s(\varphi, f, P) : \sup_{x \in [a,b]} f(x)\big(\varphi(b) - \varphi(a)\big) \geq s(\varphi, f, P) \geq \inf_{x \in [a,b]} f(x)\big(\varphi(b) - \varphi(a)\big);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Bounded} : \Big(s(\varphi, f, P) : \texttt{Bounded}(\mathbb{R}) \Big),$

$m(\varepsilon) := \sup \Big\{ s(\varphi, f, P) | P : \varepsilon\text{-}\texttt{Mesh}[a,b] \Big\} : \mathbb{R};$

$\rightsquigarrow m := I(\rightarrow) : \mathbb{R}_{++} \to \mathbb{R},$

$(2) := \eth m \eth \texttt{supremum} : \Big(m : \texttt{Increasing} \Big) \ \&\ m \geq \inf_{x \in [a,b]} f(x)\Big(\varphi(b) - \varphi(a)\Big),$

$\underline{I} := \lim_{\varepsilon \to 0} m(\varepsilon) : \mathbb{R},$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(P, 3) := \eth \underline{I} \left(\frac{\varepsilon}{2}\right) : \sum (t, (n+1)) : \texttt{Mesh}[a,b] \,.\, \underline{I} - s\Big(\varphi, f, (t, n-1)\Big) < \frac{\varepsilon}{2},$

$(\delta, 4) := \eth^{-1} UC(\varphi) \left(\frac{\varepsilon}{2n\omega\Big(f, [a,b]\Big)} \right) : \sum \delta \in \mathbb{R} \,.\, \forall x, y \in [a,b] \,.\, |\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2n\omega\Big(f, [a,b]\Big)},$

$\lambda := \min_{i \in n} t_{i+1} - t_i : \mathbb{R}_{++},$

$r := \min(\lambda, \delta) : \mathbb{R}_{++},$

$\texttt{Assume}\ Q : r\text{-}\texttt{Mesh}[a,b],$

$() := \eth s(f, \varphi, Q) \flat r \eth Q(3)(4) : \underline{I} - s(f, \varphi, Q) < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NetLimit} : \lim_{P \in \mathfrak{P}[a,b]} S(\varphi, f, P) = \underline{I};$

$\square$

$\texttt{lowerDarbuexLemmaII} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \,.\, \forall \varphi : C \ \&\ \texttt{Increasing}\Big([a,b], \mathbb{R}\Big) \,.$

$\quad .\ \forall f : \texttt{Bounded}\Big([a,b], \mathbb{R}\Big) \,.\, \lim_{P \in \mathfrak{P}[a,b]} S(f, \varphi, P) = \lim_{\varepsilon \to 0} \inf \Big\{ S(\varphi, f, P) | P : \varepsilon\text{-}\texttt{Mesh}[a,b] \Big\}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{lowerDarbuexIntegral} :: \prod [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \,.$

$\quad .\ \texttt{NonDecreasing}\Big([a,b], \mathbb{R}\Big) \to \texttt{Bounded}\Big([a,b], \mathbb{R}\Big) \to \mathbb{R}$

$\texttt{lowerDarbuexIntegral}\,(\varphi, f) = \underline{I}(\varphi, f) := \lim_{P \in \mathfrak{P}[a,b]} s(\varphi, f, P)$

$\texttt{upperDarbuexIntegral} :: \prod [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \,.$

$\quad .\ \texttt{NonDecreasing}\Big([a,b], \mathbb{R}\Big) \to \texttt{Bounded}\Big([a,b], \mathbb{R}\Big) \to \mathbb{R}$

$\texttt{upperDarbouxIntegral}\,(\varphi, f) = \overline{I}(\varphi, f) := \lim_{P \in \mathfrak{P}[a,b]} S(\varphi, f, P)$

$\texttt{DarbuexCriterion} :: \forall [a,b] : \texttt{ClosedInterval}(\mathbb{R}) \;.\; \forall \varphi \in C \;\&\; \texttt{Increasing}(\mathbb{R}) \;.\; \forall f : [a,b] \to \mathbb{R} \;.$

$$f \in \mathcal{R}\Big([a,b], \varphi\Big) \iff \overline{I}(\varphi, f) = \underline{I}(\varphi, f)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ R : \overline{I}(\varphi, f) = \underline{I}(\varphi, f),$

$\texttt{Assume}\ (n+1, t) : \texttt{Mesh}[a,b],$

$() := \eth\texttt{infimum}\eth\texttt{supremum}\texttt{LowerDarbuexLemmaI} \otimes \texttt{UpperDarbuexLemmaII}\Big([a,b], \varphi, f, (n+1, t)\Big) :$

$$: \forall x : \prod [u,v] : \texttt{ClosedInterval}[a,b] \;.\; [u,v] \;.$$

$$.\; s\Big(\varphi, f, (n+1,t)\Big) \leq \sum_{i=1}^{n} f\Big(x[t_i, t_{i+1}]\Big)\big(\varphi(t_i) - \varphi(t_{i+1})\big) \leq S\Big(\varphi, f(n+1,t)\Big);$$

$\rightsquigarrow () := \eth^{-1}\texttt{definiteRSintegral}R\texttt{DoubleIneqLimit}I(\forall) : \displaystyle\int_a^b f \; \mathrm{d}\varphi = \underline{I}(\varphi, f);$

$\rightsquigarrow R := I(\Leftarrow) : \texttt{Left} \Leftarrow \texttt{Right},$

$\texttt{Assume}\ L : f \in \mathcal{R}\Big([a,b], \varphi\Big),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_+ +,$

$(\delta_0, 1) := \eth\mathcal{R}\Big([a,b],\varphi\Big)(f)\left(\dfrac{\varepsilon}{4}\right) : \sum \delta_0 \in \mathbb{R}_{++} \;.\; \forall (n+1, t, x) : \delta_0\text{-}\texttt{PointedMesh} \;.\; \left|\displaystyle\sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_{i+1})\big) -\right.$

$(\delta_-, 2) := \eth\underline{I}(\varphi, f)\left(\dfrac{\varepsilon}{6}\right) :$

$$: \sum \delta_- \in \mathbb{R}_{++} \;.\; \forall P : \delta_-\text{-}\texttt{Mesh}[a,b] \;.\; |\underline{I}(\varphi, f) - s(\varphi, f, P)| < \frac{\varepsilon}{6},$$

$(\delta_+, 3) := \eth\overline{I}(\varphi, f)\left(\dfrac{\varepsilon}{6}\right) : \sum \delta_+ \in \mathbb{R}_{++} \;.\; \forall P : \delta_+\text{-}\texttt{Mesh}[a,b] \;.\; \left|\overline{I}(\varphi, f) - S(\varphi, f, P)\right| < \dfrac{\varepsilon}{6},$

$\delta := \min(\delta_-, \delta_0, \delta_+) : \mathbb{R}_{++},$

$\texttt{Assume}\ (n+1, t) : \delta\text{-}\texttt{Mesh}[a,b],$

$(x, 4) := \eth\texttt{infimum}\eth s\Big(\varphi, f, (n+1,t)\Big)\left(\dfrac{\varepsilon}{6}\right) :$

$$: \sum x : \prod i \in n \;.\; [t_i, t_{i+1}] \;.\; \left|s\Big(\varphi, f, (n+1,t)\Big) - \sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_{i+1})\big)\right| < \frac{\varepsilon}{6},$$

$(y, 5) := \eth\texttt{supremum}\eth S\Big(\varphi, f, (n+1,t)\Big)\left(\dfrac{\varepsilon}{6}\right) :$

$$: \sum y : \prod i \in n \;.\; [t_i, t_{i+1}] \;.\; \left|S\Big(\varphi, f, (n+1,t)\Big) - \sum_{i=1}^{n} f(y_i)\big(\varphi(t_{i+1}) - \varphi(t_{i+1})\big)\right| < \frac{\varepsilon}{6},$$

$() := \texttt{TriangleIneq}(1)(2)(3)(4)(5)\text{þ}\delta\eth(n+1,t)\eth x\eth y : |\underline{I}(\varphi, f) - \overline{I}(\varphi, f)| \leq$

$$\leq \left|\underline{I}(\varphi, f) - s\Big(\varphi, f, (n+1,t)\Big)\right| + \left|s\Big(\varphi, f, (n+1,t)\Big) - \sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big)\right| +$$

$$+ \left|\sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) - \int_a^b f \; \mathrm{d}\varphi\right| + \left|\sum_{i=1}^{n} f(y_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) - \int_a^b f \; \mathrm{d}\varphi\right| +$$

$$+ \left|S\Big(\varphi, f, (n+1,t)\Big) - \sum_{i=1}^{n} f(y_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big)\right| + \left|\overline{I}(\varphi, f) - S\Big(\varphi, f, (n+1,t)\Big)\right| < \varepsilon;$$

$\rightsquigarrow () := \eth_1\texttt{AbsValue}(\texttt{absValue}(\mathbb{R}))\lim\limits_{\varepsilon \to 0} I(\forall) : \underline{I}(\varphi, f) = \overline{I}(\varphi, f);$

$\rightsquigarrow (*) := I(\iff)RI(\Rightarrow) : \texttt{This};$

$\square$

$\varphi : C \ \&\ \texttt{Increasing}\Big([a,b],\mathbb{R}\Big)$

$\texttt{RiemannIntegrableHaveSummableVariation} :: \forall f \in \mathcal{R}\Big([a,b],\varphi\Big) \ . \ f : \texttt{SummableVariation}(\mathbb{R})\Big([a,b],\varphi\Big)$

$\texttt{Proof} \ =$

$I := \displaystyle\int_a^b f \mathrm{d}\,\varphi : \mathbb{R},$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta_-,1) := \texttt{DarbuexCriterion}(f)\eth \underline{I}\left(\frac{\varepsilon}{2}\right) : \exists \delta_- \in \mathbb{R}_{++} \ . \ \forall P : \delta_-\text{-}\texttt{Mesh}[a,b] \ . \ \left|\displaystyle\int_a^b f \ \mathrm{d}\varphi - s(\varphi,f,P)\right| < \frac{\varepsilon}{2},$

$(\delta_+,2) := \texttt{DarbuexCriterion}(f)\eth \overline{I}\left(\frac{\varepsilon}{2}\right) : \exists \delta_+ \in \mathbb{R}_{++} \ . \ \forall P : \delta_+\text{-}\texttt{Mesh}[a,b] \ . \ \left|\displaystyle\int_a^b f \ \mathrm{d}\varphi - S(\varphi,f,P)\right| < \frac{\varepsilon}{2},$

$\delta := \min(\delta_-,\delta_+) : \mathbb{R}_{++},$

$\texttt{Assume}\ (n+1,t) : \delta\text{-}\texttt{Mesh}[a,b],$

$() := \eth\omega(f,\cdot)\eth^{-1}S\Big(\varphi,f,(n+1,t)\Big)\eth^{-1}s\Big(\varphi,f,(n+1,t)\Big)\flat(\delta)\eth(n+1,t)(1)(2) :$

$\quad : \displaystyle\sum_{i=1}^{n}\omega\Big(f,[t_i,t_{i+1}]\Big)(\varphi(t_i)-\varphi(t_{i+1})) = S\Big(\varphi,f,(n+1,t)\Big) - s\Big(\varphi,f,(n+1,t)\Big) < \varepsilon;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{NetLimit}I(\forall)I(\exists)(\delta)I(\forall) : \displaystyle\lim_{(n+1,t)\in\mathfrak{P}[a,b]}\sum_{i=1}^{n}\omega\Big(f,[t_i,t_{i+1}]\Big)(\varphi(t_i)-\varphi(t_{i+1})) = 0,$

$(*) := \eth^{-1}\texttt{SummableVariation} : \texttt{This};$

$\square$

$\texttt{ConctractionIsIntegrable} :: \forall f \in \mathcal{R}\Big([a,b],\varphi\Big) \ . \ \forall [c,d] : \texttt{ClosedInterval}[a,b] \ . \ f_{|[c,d]} \in \mathcal{R}\Big([c,d],\varphi\Big)$

$\texttt{Proof} \ =$

Sums of variations of contraction can be bounded by sums of variations of f.

$\square$

$\texttt{AbsValIsIntegrable} :: \forall f \in \mathcal{R}\Big([a,b],\varphi\Big) \ . \ |f|_{|[c,d]} \in \mathcal{R}\Big([c,d],\varphi\Big)$

$\texttt{Proof} \ =$

Sums of variations of absolute values can be bounded by sums of variations of f.

$\square$

$\texttt{SquareIsIntegrable} :: \forall f \in \mathcal{R}\Big([a,b],\varphi\Big) \ . \ f^2 \in \mathcal{R}\Big([a,b],\varphi\Big)$

$\texttt{Proof} \ =$

Use estimate for variation

$$\omega\Big(f^2,A\Big) = \sup_{x,y\in A}\|f^2(x)-f^2(y)\| = \sup_{x,y\in A}\Big\|\big(f(x)+f(y)\big)\big(f(x)-f(y)\big)\Big\| \le 2\|f\|_\infty \sup_{x,y\in A}\big|f(x)-f(y)\big| =$$

$$= 2\|f\|_\infty\omega(f,A)$$

$\square$

$\texttt{RiemannIntegrableFormAlgebra} :: \mathcal{R}\Big([a,b],\varphi\Big) : \texttt{Algebra}(\mathbb{R})$

$\texttt{Proof} \ =$

Use representation for $f, g \in \mathcal{R}\Big([a,b],\varphi\Big)$

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

$\square$

$\texttt{IntegralDecomposition} :: \forall f \in \mathcal{R}\Big([a,b],\varphi\Big) \ . \ \forall c \in [a,b] \ . \ \int_a^b f \ \mathrm{d}\varphi = \int_a^c f \ \mathrm{d}\varphi + \int_c^b f \ \mathrm{d}\varphi$

$\texttt{Proof} \ =$

$$\int_a^b f \ \mathrm{d}\varphi = \int_a^b I_{[a,c]} f + I_{[c,b]} f \ \mathrm{d}\varphi = \int_a^b I_{[a,c]} f \ \mathrm{d}\varphi + \int_a^b I_{[c,b]} f \mathrm{d}\varphi = \int_a^c f \ \mathrm{d}\varphi + \int_c^b f \ \mathrm{d}\varphi$$

$\square$

$\texttt{inverseIntegral} :: \mathcal{R}\Big([a,b],\varphi\Big) \to \texttt{ClosedInterval}[a,b] \to \mathbb{R}$

$\texttt{inverseIntegral}\,(f,[c,d]) = \int_c^d f \ \mathrm{d}\varphi := -\int_d^c f \ \mathrm{d}\varphi$

$\texttt{generalAntiderivative} :: \mathcal{R}\Big([a,b],\varphi\Big) \to [a,b] \to \mathbb{R}$

$\texttt{generalAntiderivative}\,(f) = \int f := \Lambda x \in [a,b] \ . \ \int_a^x f \ \mathrm{d}\varphi$

7.3 Integral Estimates

IntegralIsMonotonic $:: \forall f, g \in \mathcal{R}\Big([a,b], \varphi\Big) \,.\, \forall (0) : f \leq g \,.\, \displaystyle\int_a^b f \, \mathrm{d}\varphi \leq \int_a^b g \, \mathrm{d}\varphi$

Proof $=$

For every pointed mesh $(n+1, t, x)$ it holds

$$\sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big) \leq \sum_{i=1}^{n} g(x_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big),$$

and taking limit over $\mathfrak{P}[a,b]$ delievers the result.

$\square$

IntegralTriangleIneq $:: \forall f \in \mathcal{R}\Big([a,b], \varphi\Big) \,.\, \left|\displaystyle\int_a^b f \, \mathrm{d}\varphi\right| \leq \int_a^b |f| \, \mathrm{d}\varphi$

Proof $=$

For every pointed mesh $(n+1, t, x)$ it holds

$$\left|\sum_{i=1}^{n} f(x_i)\big(\varphi(t_{i+1}) - \varphi(t_i)\big)\right| \leq \sum_{i=1}^{n} \Big|f(x_i)\Big|\big(\varphi(t_{i+1}) - \varphi(t_i)\big),$$

and taking limit over $\mathfrak{P}[a,b]$ with continuity of $|\cdot|$ delievers the result.

$\square$

BasicIntegralEstimate $:: \forall f \in \mathcal{R}\Big([a,b], \varphi\Big) \,.$

$$. \left(\inf_{x \in [a,b]} f(x)\right) (\varphi(b) - \varphi(a)) \leq \int_a^b f \, \mathrm{d}\varphi \leq \left(\sup_{x \in [a,b]} f(x)\right) (\varphi(b) - \varphi(a))$$

Proof $=$

By definition of infimum and supremum

$$\inf_{x \in [a,b]} f(x) \leq f \leq \sup_{x \in [a,b]} f(x),$$

So by IntegralMonotonic this estimte holds.

$\square$

BasicMeanValueIntegral $:: \forall f \in \mathcal{R}\Big([a,b], \varphi\Big) \,.\, \exists \mu \in \left[\inf_{x \in [a,b]} f(x), \sup_{x \in [a,b]} f(x)\right] \,.\, \displaystyle\int_a^b f \, \mathrm{d}\varphi = \mu\big(\varphi(b) - \varphi(a)\big)$

Proof $=$

$$\mu = \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f \, \mathrm{d}\varphi$$

By BasicIntegralEstimate $\quad \mu \in \left[\inf_{x \in [a,b]} f(x), \sup_{x \in [a,b]} f(x)\right]$

$\square$

ContBasicMeanValueIntegral $:: \forall f \in C[a,b] \,.\, \exists x \in [a,b] \,.\, \int_a^b f \,\mathrm{d}\varphi = f(x)\big(\varphi(b) - \varphi(a)\big)$

Proof $=$

By IntermidiateValueTHM there exists $x : f(x) = \mu$, where μ selected as in the privious theorem.

□

MeanValueIntegralI $:: \forall f, g \in \mathcal{R}\Big([a,b], \varphi\Big) \,.\, \forall(0) : g \geq 0 \,.\, \exists \mu \in \left[\inf_{x\in[a,b]} f(x), \sup_{x\in[a,b]} f(x)\right] :$

$$: \int_a^b fg\mathrm{d}\varphi = \mu \int_a^b g\mathrm{d}\varphi$$

Proof $=$

By initial assumptions

$\inf_{x\in[a,b]} f(x)g \leq fg \leq \sup_{x\in[a,b]} f(x)g,$

then the proof follows as with the basic estimate.

□

ContMeanValueIntegralI $:: \forall g \in \mathcal{R}\Big([a,b], \varphi\Big) \,.\, \forall f \in C[a,b] \,.\, \forall(0) : g \geq 0 \,.\, \exists x \in [a,b] :$

$$: \int_a^b fg\mathrm{d}\varphi = f(x) \int_a^b g\mathrm{d}\varphi$$

Proof $=$

...

□

AbelTransform $:: \forall a, b : \mathbb{N} \to \mathbb{R} \,.\, \forall n \in \mathbb{N} \,.\, \sum_{i=1}^{n} a_i b_i = b_n S_n(a) + \sum_{i=i}^{n-1} S_i(a)(b_i - b_{i+1})$

Proof $=$

$(*) := \eth\texttt{sum}\eth^{-1} S_1(a) :$

$$: \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} \Big(S_i(a) - S_{i-1}(a)\Big) b_i = \sum_{i=1}^{n} S_i(a) b_i - \sum_{i=1}^{n-1} S_i(a) b_{i+1} = S_n(a) b_n + \sum_{i=1}^{n-1} S_i(a)(b_i - b_{i+1});$$

□

AbelTransformIneq $:: \forall a, b : \mathbb{N} \to \mathbb{R} \,.\, \forall m, M \in \mathbb{R} \,.\, \forall(0) : \forall n \in \mathbb{N} \,.\, m \leq S_n(a) \leq M \,.$

$$. \forall(00) : b \geq 0 \,.\, \forall(000) : \Big(b : \texttt{NonIncreaisng}\Big) \,.\, \forall n \in \mathbb{N}\, b_1 m \leq \sum_{i=1}^{n} a_i b_i \leq b_1 M$$

Proof $=$

$(1) :=$ AbelTransform$(0)(00)(000)(M)\eth$Distributive(mult($\mathbb{R}$)) :

$$: \sum_{i=1}^{n} a_i b_i = S_n(a) b_n + \sum_{i=1}^{n-1} S_i(a)(b_i - b_{i+1}) \leq M b_n + \sum_{i=1}^{n=1} M b_i = M b_1,$$

$(2) :=$ AbelTransform$(0)(00)(000)(m)\eth$Distributive(mult($\mathbb{R}$)) :

$$: \sum_{i=1}^{n} a_i b_i = S_n(a) b_n + \sum_{i=1}^{n-1} S_i(a)(b_i - b_{i+1}) \geq m b_n + \sum_{i=1}^{n=1} m b_i = m b_1,$$

□

$\texttt{ContinuousAntiderivative} :: \forall f \in \mathcal{R}[a,b] \,.\, \int f \in C[a,b]$

$\texttt{Proof} \; =$

$\texttt{Assume}\ x : \texttt{In}(a,b),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\delta := \min\left(\frac{\varepsilon}{\|f\|_\infty}, x-a, b-x\right) : \mathbb{R}_{++},$

$\texttt{Assume}\ h : \texttt{In}(-\delta,\delta),$

$() := \eth\texttt{generalAntiderivative}(f)\texttt{IntegralDecomposition}\ \texttt{IntegralTriangleIneq}$
$\quad \texttt{BasicIntegralEstimate}(|f|)\eth h \text{þ} \delta :$

$$: \left|\int f(x) - \int f(x+h)\right| = \left|\int_a^x f(x)\ \mathrm{d}x - \int_a^{x+h} f(x)\ \mathrm{d}x\right| = \left|\int_x^{x+h} f(x)\ \mathrm{d}x\right| \le$$

$$\le \left|\int_x^{x+h} |f|(x)\mathrm{d}x\right| \le \|f\|_\infty |h| < \varepsilon;$$

$\rightsquigarrow (*) := \eth^{-1} C[a,b] : \left(\int f \in C[a,b]\right),$

$\square$

$\texttt{IntegralMeanValueLemma} :: \forall f,g \in \mathcal{R}[a,b] \,.\, \forall (0) : g > 0 \,.\, \forall g : \texttt{NonDecreasing} \,.\, \exists x \in [a,b] \,.$

$$.\ \int_a^b f(x)g(x)\ \mathrm{d}x = g(a)\int_a^x f(x)\ \mathrm{d}x$$

$\texttt{Proof} \; =$

$\texttt{Assume}\ (n+1,t) : \texttt{Mesh}[a,b],$

$() := \texttt{RiemannIntegralIsFunctional}(f)\texttt{IntegralDecomposition}\Big(f,(n+1,t)\Big) :$

$$\int_a^b f(x)g(x)\ \mathrm{d}x = \sum_{i=1}^n \int_{t_i}^{t_{i+1}} f(x)g(x)\ \mathrm{d}x = \sum_{i=1}^n g(t_i)\int_{t_i}^{t_{i+1}} f(x)\ \mathrm{d}x + \int_{t_i}^{t_{i+1}} f(x)\Big(g(x)-g(t_i)\Big)\ \mathrm{d}x;$$

$\rightsquigarrow (1) := I(\forall) : \forall (n+1,t) : \texttt{Mesh}[a,b] \,.$

$$.\ \int_a^b f(x)g(x)\ \mathrm{d}x = \sum_{i=1}^n g(t_i)\int_{t_i}^{t_{i+1}} f(x)\ \mathrm{d}x + \int_{t_i}^{t_{i+1}} f(x)\Big(g(x)-g(t_i)\Big)\ \mathrm{d}x,$$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(\delta, 2) := \eth\texttt{SummableVariation}(g)\frac{\varepsilon}{\|f\|_\infty} :$

$$: \sum \delta \in \mathbb{R}_{++} \,.\, \forall (n+1,t) : \delta\text{-}\texttt{Mash}[a,b] \,.\, \sum_{i=1}^n \omega\Big(g,[t_i,t_{i+1}]\Big)(t_{i+1}-t_i) < \frac{\varepsilon}{\|f\|_\infty},$$

$\texttt{Assume}\ (n+1,t) : \delta\text{-}\texttt{Mesh},$

$() := \texttt{TriangleIIneq}\ \texttt{IntegralTriangleIneq}\ \texttt{BasicIntegralEstimate}\eth(n+1,t)(2) :$

$$: \left|\sum_{i=1}^n \int_{t_i}^{t_{i+1}} f(x)\big(g(x)-g(t_i)\big)\ \mathrm{d}x\right| \le \sum_{i=1}^n \int_{t_i}^{t_{i+1}} |f(x)||g(x)-g(t_i)|\ \mathrm{d}x \le$$

$$\le \|f\|_\infty \sum_{i=1}^n \omega\Big(f,[t_i,t_{i+1}]\Big))(t_{i+1}-t_i) < \varepsilon;$$

$$\rightsquigarrow (2) := \eth^{-1}\texttt{NetLimit} : \lim_{(n+1,t)\in\mathfrak{P}[a,b]} \sum_{i=1}^n \int_{t_i}^{t_{i+1}} f(x)\big(g(x)-g(t_u)\big)\ \mathrm{d}x = 0,$$

$m := \min_{x\in[a,b]} \int f(x) : \mathbb{R},$

$M := \max_{x\in[a,b]} \int f(x) : \mathbb{R},$

$\texttt{Assume}\ (n, t+1) : \texttt{Mesh}[a,b],$

$() := \texttt{AbelTrandformIneq}(g(t), \dots)\text{þ}m\text{þ}M(0)(00) : g(a)m \le \sum_{i=1}^{n} g(t_i) \int_{t_i}^{t_{i+1}} \le g(a)M;$

$\rightsquigarrow (3) := \lim_{P\in\mathfrak{p}[a,b]} (1)(2)(P) : g(a)m \le \int_a^b f(x)g(x)\mathrm{d}x \le g(a)M,$

$\mu := \frac{\int_a^b f(x)g(x)\,\mathrm{d}x}{g(a)} : \mathbb{R},$

$(4) := (3)\text{þ}\mu : \mu \in m, M,$

$(x, 5) := \texttt{IntermidiateValueTHM}\left(\int f\right)(4) : \sum x \in [a,b] \,.\, \mu = \int_a^x f(x)\,\mathrm{d}x,$

$(6) := (5)\text{þ}\mu : \int_a^b f(x)g(x)\,\mathrm{d}x = g(a) \int_a^x f(x)\,\mathrm{d}x;$

$\square$

$\texttt{IntegralMeanValueTHMII} :: \forall f, g \in \mathcal{R}[a,b] \,.\, \forall(0) : \Big(g : \texttt{Monotonic}\Big) \,.\, \exists s \in [a,b] \,.$

$: \int_a^b g(x)f(x)\,\mathrm{d}x = g(a) \int_a^s f(x)\,\mathrm{d}x - g(b) \int_s^b f(x)\,\mathrm{d}x$

$\texttt{Proof}\ =$

$G := \Lambda x \in [a,b] \,.\, \max_{i\in\{1,-1\}} i(g(b) - g(x)) : \texttt{NonIncreasing}\Big([a,b], \mathbb{R}\Big),$

$(1) := \text{þ}G : G > 0,$

$(s, 2) := \eth G\texttt{IntegralMeanValueLeamma}(f, G, 1)\eth G :$

$: \sum s \in [a,b] \,.\, g(b) \int_b^a f(x)\,\mathrm{d}x - \int_b^a f(x)g(x)\,\mathrm{d}x = \pm \int_b^a G(x)f(x)\,\mathrm{d}x =$

$\pm G(a) \int_a^s f(x)\,\mathrm{d}x = g(b) \int_a^s f(x)\,\mathrm{d}x - g(a) \int_a^s f(x)\,\mathrm{d}x,$

$(*) := \texttt{IntegralDecomposition}\left(-\left((2) - g(b)\int_a^b f(x)\,\mathrm{d}x\right)\right) :$

$: \int_a^b f(x)g(x)\,\mathrm{d}x = g(a) \int_a^s f(x)\mathrm{d}x - g(b) \int_s^b f(x)\mathrm{d}x;$

$\square$

7.4 Fundamental Theorem of Calculus

DifferentiableAntiderivative :: $\forall f \in \mathcal{R}[a,b] \,.\, \forall x \in (a,b)(0) : f \in C\Big([a,b],\mathbb{R},x\Big) \,.\, \left(\int f\right)'(x) = f(x)$

Proof $=$

$c := \min(x-a, b-x) : \mathbb{R}_{++},$

Assume $h : (-c,c),$

$(1^*) :=$ ð generalAntiderivative ð IntegralDecomposition $ð^{-1} f(a)$:

$$: \int f(x+h) - \int f(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t = \int_x^{x+h} f(t)\,\mathrm{d}t =$$

$$= \int_x^{x+h} f(x) + \int_x^{x+h} f(t) - f(x)\,\mathrm{d}t = hf(x) + \int_x^{x+h} f(t) - f(x)\,\mathrm{d}t,$$

$() :=$ IntegralTriangleIneq$(f - f(a))$BasicIntegralEstimate$(|f(t) - f(a)|)ð^{-1}\omega$:

$$: \left|\int_x^{x+h} f(t) - f(x)\,\mathrm{d}t\right| \le \int_x^{x+h} |f(t) - f(x)|\,\mathrm{d}t \le \omega\Big(f, (x \pm h, x \mp h)\Big)|h|;$$

$$\rightsquigarrow (1) := I(\forall) : \forall h \in (-c,c) \,.\, \int f(x+h) - \int f(x) = hf(x) + \int_x^{x+h} f(t) - f(x)\,\mathrm{d}t\ \&$$

$$\&\ \left|\int_x^{x+h} f(t) - f(a)\,\mathrm{d}t\right| < \omega\Big(f, (x \pm h, x \mp h)\Big)|h|,$$

$$(2) := (0) \lim_{h\to 0}(1_2)(h) : \lim_{h\to 0} \frac{\int_x^{x+h} f(t) - f(x)\,\mathrm{d}t}{h} = 0,$$

$(*) := ð^{-1}$Differential$(1_1)(2)$: $\left(\int f\right)'(x) = f'(x);$

$\square$

Antiderivative :: $\mathcal{R}[a,b] \to ?C[a,b]$

F : Antiderivative $\iff \Lambda f \in \mathcal{R}[a,b] \,.\, \exists X :$ Finite$[a,b] \,.\, \forall x \in X^{\complement} \,.\, F'(x) = f(x)$

$\texttt{straightPath} :: \Big([a.b] \to \mathbb{R}\Big) \to \mathbb{R}$

$\texttt{straightPath}\,(F) = F|_a^b := F(b) - F(a)$

$\texttt{FundamentalTheoremOfCalculus} :: \forall f \in \texttt{Piecewise}\ C[a,b] \,.\, \forall F : \texttt{Antiderivative}(f) \,.$

$\quad . \displaystyle\int_a^b f(t)\ \mathrm{d}t = F|_a^b$

$\texttt{Proof}\ =$

$(X,1) := ð\texttt{Antiderivative}(F)ð\texttt{Piecewise}\ C[a,b] : \sum X : \texttt{Finite} \,.\, \forall x \in X^{\complement} \,.$

$\quad . f \in C\Big([a,b], \mathbb{R}, x\Big)\ \&\ F'(x) = f(x),$

$\texttt{Assume}\ x : \texttt{In}\Big(X^{\complement}\Big),$

$(2) := \texttt{DifferentiableAndtiderivative}(f,x)ðF(x) : F(x) = \displaystyle\int_a^x f(t)\ \mathrm{d}t + F(a),$

$(*) := (2) - F(a) : F(x) - F(a) = \displaystyle\int_a^x f(t)\ \mathrm{d}t;$

$\rightsquigarrow (2) := I(\forall) : \forall x \in X^{\complement} \,.\, \displaystyle\int_a^x f(t)\ \mathrm{d}t = F(x) - F(a),$

$(*) := \lim_{x \to b}(2)(b) : \displaystyle\int_a^b f(t)\ \mathrm{d}t = F(b) - F(a);$

$\square$

7.5 Theorems of Integral Calculus [!!]

$$\texttt{IntegrationByParts} :: \forall v, u : [a,b] \xrightarrow{\text{DIFF}} \mathbb{R} \; . \; \int_a^b v(x)u'(x)\,\mathrm{d}x = vu|_a^b - \int_a^b v'(x)u(x)\,\mathrm{d}x$$

$\texttt{Proof} \; =$

$(1) := \texttt{ProductDifferential}(v,u) : (vu)' = v'u + vu',$

$(2) := \texttt{FundamentalTheoremOfCalculus}(1) : \displaystyle\int_a^b v'(x)u(x) + v(x)u'(x)\,\mathrm{d}x = vu|_a^b,$

$(3) := (2) - \displaystyle\int_a^b v'(x)u(x)\,\mathrm{d}x : \int_a^b v(x)u'(x)\,\mathrm{d}x = vu|_a^b - \int_a^b v'(x)u(x)\,\mathrm{d}x;$

$\square$

$\texttt{IntegralReminderTaylorSeria} :: \forall U : \texttt{Open}(\mathbb{R}) \; . \; \forall f \in C^n(U) \; . \; \forall [a,x] : \texttt{ClosedInterval}(U) \; .$

$$. \; f(x) - f(a) = \sum_{k=1}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!} + \int_a^x \frac{f^{(n)}(t)\big(x-t\big)^{n-1}}{(n-1)!}\,\mathrm{d}t$$

$\texttt{Proof} \; =$

$A(1) := \texttt{FundamentalTheoremOfCalculus}(f', f) : f(x) - f(a) = \displaystyle\int_a^x f'(t)\mathrm{d}t,$

$\texttt{Assume}\; m : \texttt{In}(n-1),$

$\texttt{Assume}\; A(m) : f(x) - f(a) = \displaystyle\sum_{k=1}^{m-1} \frac{f^{(k)}(a)(x-a)^n}{k!} + \int_a^x \frac{f^{(m)}(t)(x-t)^{m-1}}{(m-1)!}\,\mathrm{d}t,$

$A(m+1) := \texttt{IntegrationByParts}\Big(f^(m), (x-t)^{m-1}\Big)A(m) :$

$$: f(x) - f(a) = \sum_{k=1}^{m} \frac{f^{(k)}(a)(x-a)^n}{k!} + \int_a^x \frac{f^{(m+1)}(t)(x-t)^m}{m!}\,\mathrm{d}t;$$

$\rightsquigarrow R := I(\forall) : \forall m \in n-1 \; . \; \texttt{This}(m) \Rightarrow \texttt{This}(m+1),$

$(*) := E(n)A(1)R : \texttt{This};$

$\square$

$$\texttt{ChangeOfVariableInIntegral} :: \forall f \in C[\alpha,\beta] \; . \; \forall \varphi : [a,b] \xleftrightarrow{\text{DIFF}} [\alpha,\beta] \; . \; \int_a^b f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t = \int_\alpha^\beta f(x)\,\mathrm{d}x$$

$\texttt{Proof} \; =$

$F := \Lambda x \in [\alpha,\beta] \; . \; \displaystyle\int_\alpha^x f(t)\,\mathrm{d}t : [\alpha,\beta] \xrightarrow{DIFF} \mathbb{R},$

$(1) := \texttt{CpmpositionDiff}(F(\varphi)) : \Big(F(\varphi)\Big)' = \varphi' f(\varphi),$

$(*) := \texttt{FuncdamentalTheoremOfCalculus}(1) : \displaystyle\int_a^b f(\varphi(t))\varphi'(t)\,\mathrm{d}t = \int_\alpha^\beta f(x)\,\mathrm{d}x;$

$\square$

7.6 Improper Integral[!]

7.7 Additive Functions of Intervals[!]

8 Lebesgue Measure on the Interval

8.1 Measure of Open Sets

$$\texttt{length} :: \texttt{OpenInterval}(\mathbb{R}) \to \mathbb{R}_+$$
$$\texttt{length}\,(a,b) = \lambda(a,b) := b - a$$

$\texttt{FiniteOuterIntervalBound} ::$

$$\forall (A,B) : \texttt{OpenInterval}(\mathbb{R}) \;.\; \forall n \in \mathbb{N} \;.\; \forall (a,b) : \texttt{DisjointFamily}\Big(\{1,\dots,n\}, \texttt{OpenInterval}(\mathbb{R})\Big) \;.$$
$$.\; \forall \sqsupset : \forall k \in \{1,\dots,n\} \;.\; (a_k,b_k) \subset (A,B) \;.\; \sum_{k=1}^{n} \lambda(a_k,b_k) \le \lambda(A,B)$$

$\texttt{Proof} \;=$

$$\Big((a,b), \sqsupset, [1]\Big) := \texttt{sort}\Big((a,b), \Lambda(c,d) : \texttt{OpenInterval}(\mathbb{R}) \,.\, c\Big) :$$
$$: \sum (a,b) : \texttt{DisjointFamily}\Big(\{1,\dots\}, \texttt{OpenInterval}(\mathbb{R})\Big) \;.\; \forall k \in \{1,\dots,n\} \;.\; (a_k,b_k) \subset (A,B)\ \&$$
$$\&\ \forall k \in \{1,\dots,n-1\} \;.\; a_k \le a_{k+1},$$
$$[2] := \texttt{E}\,\texttt{DisjointFamily}\Big(\{1,\dots,n\}, \texttt{OpenInterval}(\mathbb{R}), (a,b)\Big)[1] : \forall k \in \{1,\dots,n-1\} \;.\; a_k \le b_k < a_{k+1},$$
$$[3] := \texttt{E}\,\sqsupset\texttt{E}\,\texttt{OpenInterval}(\mathbb{R}) : \forall k \in \{1,\dots,n\} \;.\; A \le a_k\ \&\ b_k \le B,$$
$$[4] := [3.1](1) : A \le a_1,$$
$$[5] := [3.2](n) : b_n \le B,$$
$$[6] := [2]\texttt{E}\,\texttt{OrderedField}(\mathbb{R}) : \forall k \in \{1,\dots,n-1\} \;.\; b_k - a_{k+1} < 0,$$
$$[*] := \Lambda k \in \{1,\dots,n\}\texttt{E}\ \lambda(a_k,b_k)\texttt{E}\ \texttt{sum}[6][4][5]\texttt{I}\ \lambda(A,B) :$$
$$: \sum_{k=1}^{n} \lambda(a_k,b_k) = \sum_{k=1}^{n} b_k - a_k = b_n + \left(\sum_{k=1}^{n-1} b_k - a_{k+1}\right) - a_1 \le b_n - a_1 \le B - A = \lambda(A,B);$$

□

$\texttt{CountableOuterIntervalBound} ::$

$$\forall (A,B) : \texttt{OpenInterval}(\mathbb{R}) \;.\; \forall (a,b) : \texttt{DisjointSequence}\Big(\texttt{OpenInterval}(\mathbb{R})\Big) \;.$$
$$.\; \forall \sqsupset : \forall k \in \{1,\dots,n\} \;.\; (a_k,b_k) \subset (A,B) \;.\; \sum_{n=1}^{\infty} \lambda(a_n,b_n) \le \lambda(A,B)$$

$\texttt{Proof} \;=$

$$\sum_{n=1}^{\infty} \lambda(a_n,b_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(a_k,b_k) \le \lim_{n\to\infty} \lambda(A,B) = \lambda(A,B) \;.$$

□

$$\texttt{openLebesgueMeasure} :: \mathcal{T}(\mathbb{R}) \to \overset{\infty}{\mathbb{R}}_+$$
$$\texttt{openLebesgueMeasure}\,(U) = \lambda U := \sum_{i\in I} \lambda(a_i,b_i) \quad \texttt{where} \quad \Big(I,(a,b)\Big) = \texttt{OpenRealStructure}(U)$$

OpenOuterIntervalBound ::

$: \forall U \in \mathcal{T}(\mathbb{R}) \;.\; \forall (A,B) : \texttt{OpenInterval}(\mathbb{R}) \;.\; \forall \sqsupset : U \subset (A,B) \;.\; \lambda U \le \lambda(A,B)$

Proof =

$U = \bigcup_{i \in I} (a_i, b_i)$ by property of real line, where each (a_i, b_i) is disjoint.

$\sqsupset$ says that $(a_i, b_i) \subset (A, B)$ for each $i \in I$.

So by definition and previously proved theorems $\lambda U \le \lambda(A, B)$.

$\square$

OpenOuterOpenBound ::

$: \forall U, V \in \mathcal{T}(\mathbb{R}) \;.\; \forall \sqsupset : U \subset V \;.\; \lambda U \le \lambda V$

Proof =

$U = \bigcup_{i \in I} (a_i, b_i)$ by property of real line, where each (a_i, b_i) is disjoint.

Also $V = \bigcup_{i \in J} (c_i, d_i)$, where each (c_i, d_i) is disjoint.

By definition of open interval $\sqsupset$ witnesses partion $E : J \to 2^I$ of I such that $i \in E_j \iff (a_i, b_i) \subset (c_j, d_j)$.

Then, $\lambda U = \sum_{i \in I} \lambda(a_i, b_i) = \sum_{j \in J} \sum_{i \in E_j} \lambda(a_i, b_i) \le \sum_{j \in J} \lambda(c_j, d_j) = \lambda V$.

$\square$

OpenLebesgueMeasureAsInf :: $\forall U \in \mathcal{T}(\mathbb{R}) \;.\; \lambda U = \min \left\{ \lambda V \middle| V \in \mathcal{T}(\mathbb{R}), U \subset V \right\}$

Proof =

Obvious.

$\square$

OpenLebesgueAdditivity :: $\forall U : \texttt{DisjointSequence}\Big(\mathcal{T}(\mathbb{R})\Big) \;.\; \lambda \bigcup_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \lambda U_n$

Proof =

$U_n = \bigcup_{i \in I_n} (a_{n,i}, b_{n,i})$ by property of real line, where each $(a_{n,i}, b_{n,i})$ is disjoint.

But, for distinct n, m set U_n, U_m are disjoint.

So, each $(a_{n,i}, b_{n,i})$ and $(a_{m,j}, b_{m,j})$ are disjoint for each $i \in I_n$ and each $j \in J_n$.

Hence, $\left(\sum_{n=1}^{\infty} I_n, (a, b) \right)$ is a suitable representation for $\bigcup_{n=1}^{\infty} U_n$.

By partition of the sum, the result follows .

$\square$

ClosedIntervalBound ::

$:: \forall A, B \in \mathbb{R} . \forall \aleph : A \leq B . \forall n \in \mathbb{N} . \forall (a, b) : \{1, \ldots, n\} \to \texttt{OpenInterval}(\mathbb{R}) .$

$. \forall \beth : [A, B] \subset \bigcup_{k=1}^{n} (a_k, b_k) . B - A < \sum_{k=1}^{n} \lambda(a_k, b_k)$

Proof $=$

$\alpha_1 := A \in [A, B]$,

Assume $m \in \mathbb{N}$,

$\Big(k_m, [1]\Big) := \mathtt{E}\, \beth(\alpha_m) : \sum k_m \in \{1, \ldots, n\} . \alpha_m \in (a_{k_m}, b_{k_m})$,

$[2] := \mathtt{E}\, \texttt{OpenInterval}(a_{k_m}, b_{k_m})[1] : A \leq \alpha_m < b_{k_m}$,

$\alpha_{m+1} := \texttt{if } B \in (a_{k_m}, b_{k_m}) \texttt{ then } \alpha_m \texttt{ else } b_{k_m} : \mathbb{R}$,

$[m.*] := \mathtt{E}\, \alpha_{m+1}[2] : \alpha_{m+1} \in [A, B]$;

$\leadsto \Big(m, k, [1]\Big) := \texttt{FiniteRecursion} :$

$: \sum_{m=1}^{n} \sum k : \{1, \ldots, m\} \to \{1, \ldots, n\} . a_{k_1} < A \,\&\, B < b_{k_m} \,\&\, \forall l \in \{1, \ldots, m-1\} . a_{k_{l+1}} < b_{k_l} < b_{k_{l+1}}$,

$[*] := [1.2]\mathtt{E}\, \texttt{sum}(\mathbb{R})[1.2][1.1] :$

$: \sum_{k=1}^{n} \lambda(a_k, b_k) \geq \sum_{l=1}^{m} \lambda(a_{k_l}, b_{k_l}) = \sum_{l=1}^{m} b_{k_l} - a_{k_l} = b_{k_m} + \left(\sum_{l=1}^{m-1} b_{k_l} - a_{k_{l+1}} \right) - a_{k_1} > b_{k_m} - a_{k_1} > B - A;$

$\square$

OpenIntervalSubbaditivity ::

$:: \forall (A, B) : \texttt{OpenInterval}(\mathbb{R}) . \forall I : \texttt{Countable} . \forall U : I \to \mathcal{T}(\mathbb{R}) . \forall \aleph : (A, B) = \bigcup_{i \in I} U_i . \lambda(A, B) \leq \sum_{i \in i} \lambda\, U_i$

Proof $=$

$\Big(J, (a, b), [1]\Big) := \texttt{OpenRealsStrucure}(U) :$

$: \sum J : I \to \texttt{Countable} . (a, b) : \prod_{i \in I} J_i \to \texttt{OpenInterval}(\mathbb{R}) . \forall i \in I . U_i = \bigcup_{j \in J_i} (a_{i,j}, b_{i,j})$,

Assume $\varepsilon \in \mathbb{R}_{++}$,

$[2] := \mathtt{E}\, \aleph \mathtt{E}\, \texttt{ClosedIntervals}\Big(\mathbb{R}, [A, B]\Big) : [A, B] \subset (A - \varepsilon, A + \varepsilon) \cup (B - \varepsilon, B + \varepsilon) \cup \bigcup_{i \in I} \bigcup_{j \in J_i} (a_{i,j}, b_{i,j})$,

$\Big(n, k, [3]\Big) := \mathtt{E}\, \texttt{CompactSubset}\Big(\mathbb{R}, [A, B]\Big)[2] :$

$: \sum_{n=1}^{\infty} \sum k : \{1, \ldots, n\} \to \sum_{i \in i} J_i . [A, B] \subset (A - \varepsilon, A + \varepsilon) \cup (B - \varepsilon, B + \varepsilon) \cup \bigcup_{l=1}^{n} (a_{k_l}, b_{k_l})$,

$[\varepsilon.*] := \mathtt{E}\, \lambda(A, B)\texttt{ClosedIntervalBound}[3]\mathtt{E}\, {}^2\texttt{length}\mathtt{E}\, k \Lambda i \in I . \mathtt{I}\, \lambda(U_i) :$

$: \lambda(A, B) = B - A < \lambda(A - \varepsilon, A + \varepsilon) + \lambda(B - \varepsilon, B + \varepsilon) + \sum_{l=1}^{n} \lambda(a_{k_l}, b_{k_l}) = 4\varepsilon + \sum_{l=1}^{n} \lambda(a_{k_l}, b_{k_l}) \leq$

$\leq 4\varepsilon + \sum_{i \in I} \sum_{j \in J_i} \lambda(a_{i,j}, b_{i,j}) = 4\varepsilon + \sum_{i \in I} \lambda(U_i);$

$\leadsto [*] := \texttt{LimitIneq} : \lambda(A, B) \leq \sum_{i \in I} \lambda(U_i);$

$\square$

$\texttt{OpenSubbaditivity} ::$

$$:: \forall V \in \mathcal{T}(\mathbb{R}) \,.\, \forall I : \texttt{Countable} \,.\, \forall U : I \to \mathcal{T}(\mathbb{R}) \,.\, \forall \aleph : V = \bigcup_{i \in I} U_i \,.\, \lambda\, V \leq \sum_{i \in i} \lambda\, U_i$$

$\texttt{Proof} =$

$$\Big(J, (a,b), [1]\Big) := \texttt{OpenRealsStrucure}(V) :$$

$$: \sum J : \texttt{Countable} \,.\, (a,b) : \texttt{DisjointFamily}\Big(J, \texttt{OpenInterval}(\mathbb{R})\Big) \,.\, V = \bigcup_{j \in J} (a_j, b_j),$$

$$W := \Lambda i \in I \,.\, \Lambda j \in J \,.\, U_i \cap (a_j, b_j) : I \to J \to \mathcal{T}(\mathbb{R}),$$

$$[2] := \texttt{E}\, W \texttt{E}\, \texttt{DisjointFamily}\Big(J, \texttt{OpenInterval}(\mathbb{R}), (a,b)\Big) : \forall i \in I \,.\, \texttt{DisjointFamily}\Big(J, \mathcal{T}(\mathbb{R}), W_i\Big),$$

$$[3] := \texttt{E}\, W \texttt{E}\, \aleph : \forall j \in J \,.\, (a_j, b_j) = \bigcup_{i \in I} W_{i,j},$$

$$[4] := \texttt{E}\, \aleph \texttt{E}\, W : \forall i \in I \,.\, U_i = \bigcup_{j \in J} W_{i,j},$$

$$[*] := \texttt{E}\, \lambda\, V [1] \texttt{OpenIntervalSubbaditivity} [3] \texttt{NonNegSumExchange}(\lambda W) \texttt{OpenLebesgueAdditivity} [2][4] :$$

$$: \lambda\, V = \sum_{j \in J} \lambda(a_j, b_j) \leq \sum_{j \in J} \sum_{i \in I} \lambda\, W_{i,j} = \sum_{i \in I} \sum_{j \in j} \lambda\, W_{i,j} = \sum_{i \in I} \lambda\, U_i;$$

□

8.2 Outer Measure and Measurabilty

$\texttt{outerMeasureOfLebesgue} :: 2^{\mathbb{R}} \to \overset{\infty}{\mathbb{R}}_+$

$\texttt{outerMeasureOfLebesgue}\,(A) = \lambda^\star(A) := \inf \left\{ \lambda\, U | U \in \mathcal{T}(\mathbb{R}), A \subset U \right\}$

$\texttt{OuterMeasureOpenValue} :: \forall U \in \mathcal{T}(X) \,.\, \lambda(U) = \lambda^\star(U)$

$\texttt{Proof} =$

Use open Lebesgue measure as inf.

□

$\texttt{OuterMeasure} :: \texttt{OuterMeasure}(\mathbb{R}, \lambda^\star)$

$\texttt{Proof} =$

$[1] := \texttt{OuterMeasureOpenValue}(\emptyset) : \lambda^\star\big(\emptyset\big) = \lambda(\emptyset) = 0,$

$[2] := \Lambda A, B \subset \mathbb{R} \,.\, \Lambda \aleph : A \subset B \,.\, \texttt{E}\ \lambda^\star(A)\texttt{InfIsAntitone}(\aleph)\texttt{I}\ \lambda^\star(B) : \lambda^\star(A) \le \lambda^\star(B),$

$\texttt{Assume}\ A : \mathbb{N} \to 2^{\mathbb{R}},$

$\texttt{Assume}\ \aleph : \sum_{n=1}^{\infty} \lambda^\star(A_n) < \infty,$

Otherwise the bound is trivial.

$\big(V, [3]\big) := \Lambda n \in \mathbb{N} \,.\, \texttt{E}\ \lambda^\star(A_n)\texttt{E}\ \mathbb{R} :$

$\quad \sum V : \mathbb{N}^2 \to \mathcal{T}(\mathbb{R}) \,.\, \forall n \in \mathbb{N} \,.\, \lambda(V_n) \downarrow \lambda^\star(A_n)\ \&\ \forall m \in \mathbb{N} \,.\, \lambda(V_{n,m}) \le \lambda^\star(A_n) + \frac{1}{2^n}\ \&\ A_n \subset V_{n,m},$

$[4] := \texttt{E}\ \aleph\texttt{PowerSeriesConvergence} : \sum_{n=1}^{\infty} \lambda^\star(A_n) + \frac{1}{2^{-n}} < \infty,$

$[A.*] := \texttt{E}\ \lambda^\star \left(\bigcup_{n=1}^{\infty} A_n \right) \Lambda m \in \mathbb{N} \,.\, \texttt{E}\ \inf \left(\bigcup_{n=1}^{\infty} V_{n,m} \right) [3.3]\texttt{LimitIneq}\Lambda m \in \mathbb{N} \,.\, \texttt{OpenSubbaditivity}(V_{\bullet,m})$

$\quad \texttt{DominatedConvergenceTHM}\Big(\lambda V_n, \Lambda n \in \mathbb{N} \,.\, \lambda^\star(A_n) + 2^{-n}, [3.2], [4]\Big)[3.1] :$

$$: \lambda^\star \left(\bigcup_{n=1}^{\infty} A_n \right) = \inf \left\{ \lambda\, U \middle| U \in \mathcal{T}(\mathbb{R}), \bigcup_{n=1}^{\infty} A \subset U \right\} \le \lim_{m\to\infty} \lambda \bigcup_{n=1}^{\infty} V_{n,m} \le \lim_{m\to\infty} \sum_{n=1}^{\infty} \lambda\, V_{n,m} =$$

$$= \sum_{n=1}^{\infty} \lim_{m\to\infty} \lambda\, V_{n,m} = \sum_{n=1}^{\infty} \lambda^\star(A_n);$$

$\rightsquigarrow [3] := \texttt{I}\ \forall : \forall A : \mathbb{N} \to 2^{\mathbb{R}} \,.\, \lambda^\star \left(\bigcap_{n=1}^{\infty} A_n \right) \le \sum_{n=1}^{\infty} \lambda^\star(A_n),$

$[*] := \texttt{I}\ \texttt{OuterMeasure}[1][2][3] : \texttt{OuterMeasure}\,(\mathbb{R}, \lambda^\star)\,;$

□

$\texttt{measurableSetsOfLebesgue} :: \sigma\texttt{-Algebra}(\mathbb{R})$

$\texttt{measurableSetsOfLebesgue}\,() = \Lambda := \Sigma_{\lambda^\star}$

$\texttt{measureOfLebesgue} :: \texttt{Measure}(\mathbb{R})$

$\texttt{measureOfLebesgue}\,() = \lambda := \lambda^\star_{|\Lambda}$

$\texttt{OpenHalfIntervalsAreLebesgueMeasurable} :: \forall \alpha \in \mathbb{R} \,.\, (\alpha, +\infty) \in \Lambda$

$\texttt{Proof} \; =$

$\texttt{Assume}\ A \in 2^{\mathbb{R}},$

$[1] := \Lambda \varepsilon \in \mathbb{R}_{++} \,.\, \texttt{E}\ {}^2\lambda^\star \texttt{InfSum}(\overset{\infty}{\mathbb{R}})\texttt{OpenSubbaditivity}(\ldots)\texttt{E}\ \lambda\texttt{E}\ \inf \texttt{OpenAdditivity}(\ldots)$
$\quad \texttt{OpenOuterOpenBound}(\ldots)\texttt{I}\ \lambda^\star(A) :$

$$: \forall \varepsilon \in \mathbb{R}_{++} \,.\, \lambda^\star\Big(A \cap (\alpha, +\infty)\Big) + \lambda^\star\Big(A \setminus (\alpha, +\infty)\Big) =$$
$$= \inf\Big\{\lambda\, U \Big| U \in \mathcal{T}(\mathbb{R}), A \cap (\alpha, +\infty) \subset U\Big\} + \inf\Big\{\lambda\, U \Big| U \in \mathcal{T}(\mathbb{R}), A \setminus (\alpha, +\infty) \subset U\Big\} =$$
$$= \inf\Big\{\lambda\, U + \lambda\, V \Big| U, V \in \mathcal{T}(\mathbb{R}), A \cap (\alpha, +\infty) \subset U, A \cap (-\infty, \alpha]\Big\} \leq$$
$$\leq \inf\Big\{\lambda\, U + \lambda\, V + \lambda(\alpha - \varepsilon, \alpha + \varepsilon) \Big| U, V \in \mathcal{T}(\mathbb{R}), A \cap (\alpha, +\infty) \subset U, A \cap (-\infty, \alpha) \subset V\Big\} =$$
$$= \inf\Big\{\lambda(V \cup U) \Big| U \in \mathcal{T}(\alpha, +\infty), V \in \mathcal{T}(-\infty, \alpha), A \cap (\alpha, +\infty) \subset U, A \cap (-\infty, \alpha) \subset V\Big\} + 2\varepsilon \leq$$
$$\leq \inf\Big\{\lambda(U) \Big| U \in \mathcal{T}(\mathbb{R}), A \subset U\Big\} + 2\varepsilon = \lambda^\star(A) + 2\varepsilon,$$

$[2] := \texttt{LimIneq}[1] : \lambda^\star\Big(A \cap (\alpha, +\infty)\Big) + \lambda^\star\Big(A \setminus (\alpha, +\infty)\Big) \leq \lambda(A),$

$[A.*] := \texttt{E}\ {}_3\texttt{OuterMeasure}(\mathbb{R}, \lambda^\star)[2] : \lambda^\star\Big(A \cap (\alpha, +\infty)\Big) + \lambda^\star\Big(A \setminus (\alpha, +\infty)\Big) = \lambda(A);$

$\rightsquigarrow [*] := \texttt{E}\ \Lambda : (a, +\infty) \in \Lambda;$

$\square$

$\texttt{BorelSetsAreLebesgueMeasurable} :: \mathcal{B}(\mathbb{R}) \subset \Lambda$

$\texttt{Proof} \; =$

Closed rays of form $(-\infty, \alpha]$ are complements of open rays of form $(\alpha, +\infty$.

Represent half open intervals as intersections $(\alpha, \beta] = (\alpha, +\infty) \cap (+\infty, \beta]$.

Represent open intervals $(\alpha, \gamma) = \bigcap_{n=1}^{\infty} (\alpha, \beta_n]$, where $\beta_n = \gamma + \frac{1}{n}$ for example.

Open intervals generate topology of $\mathbb{R}$, so every Borel set is measurable.

$\square$

8.3 Measuring with Closed Sets

$\texttt{MeasureOfClosedInterval} :: \forall [A,B] : \texttt{ClosedInterval}(\mathbb{R}) \; . \; \lambda[A,B] = B - A$

$\texttt{Proof} \; =$

$\texttt{Assume}\ U \in \mathcal{T}(\mathbb{R}),$

$\texttt{Assume}\ \aleph : [A,B] \subset A,$

$\Big(I,(a,b),[1]\Big) := \texttt{OpenRealStrucute}(U) :$

$\quad : \sum I : \texttt{Countable} \; . \; (a,b) : \texttt{DisjointFamily}\Big(I, \texttt{OpenInterval}(\mathbb{R})\Big) \; . \; U = \bigcup_{i \in i} (a_i, b_i),$

$[2] := \texttt{E}\ \texttt{ClosedInterval}\Big(\mathbb{R},[A,B]\Big) : [A,B] \neq \emptyset,$

$\leadsto \Big(i,[3]\Big) := \texttt{SubsetOfUnionIntersection} : \sum i \in I \; . \; \exists [A,B] \cap (a_i,b_i),$

$t := \texttt{E}\ \exists[3] \in [A,B] \cap (a_i,b_i),$

$[4] := \Lambda \beth : A \leq a_i \; . \; \texttt{E}\ \beth \texttt{E}\ t \texttt{E}\ \texttt{ClosedInterval}[A,B] \texttt{E}\ \aleph \texttt{E}\ \texttt{Open}(a_i,b_i) \texttt{E}\ \texttt{OpenInterval}(a_i,b_i)$

$\quad \texttt{E}\ \texttt{DisjointFamily}\Big(I, \texttt{OpenInterval}(\mathbb{R}), (a_i,b_i)\Big) :$

$\quad : A \leq a_i \Rightarrow A \leq a_i \leq t \leq B \Rightarrow a_i \in [A,B] \Rightarrow \exists j \in I \; . \; a_i \in (a_j,b_j) \Rightarrow$

$\quad \Rightarrow \exists j \in I \; . \; j \neq I \; \& \; \exists (a_j,b_j) \cap (a_i,b_i) \Rightarrow \perp,$

$[5] := \texttt{E}\ \perp[4] : a_i < A,$

$[6] := \Lambda \beth : B \geq a_i \; . \; \texttt{E}\ \beth \texttt{E}\ t \texttt{E}\ \texttt{ClosedInterval}[A,B] \texttt{E}\ \aleph \texttt{E}\ \texttt{Open}(a_i,b_i) \texttt{E}\ \texttt{OpenInterval}(a_i,b_i)$

$\quad \texttt{E}\ \texttt{DisjointFamily}\Big(I, \texttt{OpenInterval}(\mathbb{R}), (a_i,b_i)\Big) :$

$\quad : B \geq b_i \Rightarrow B \geq b_i \geq t \geq A \Rightarrow b_i \in [A,B] \Rightarrow \exists j \in I \; . \; b_i \in (a_j,b_j) \Rightarrow$

$\quad \Rightarrow \exists j \in I \; . \; j \neq I \; \& \; \exists (a_j,b_j) \cap (a_i,b_i) \Rightarrow \perp,$

$[7] := \texttt{E}\ \perp[6] : B < b_j,$

$[8] := \texttt{E}\ \lambda(a_i,b_i)[5][7] : \lambda(a_i,b_i) = b_i - a_i > B - A,$

$[U.*] := \texttt{E}\ \lambda U[1] \texttt{NonegSumIneq}\Big(I,(a,b),i\Big)[8] : \lambda\, U = \sum_{j \in I} \lambda(a_j,b_h) \geq \lambda(a_i,b_i) > B - A;$

$\leadsto [1] := \texttt{I}\ {}^2\forall : \forall U \in \mathcal{T}(\mathbb{R}).[A,B] \subset U \Rightarrow \lambda(U) > B - A,$

$[2] := \texttt{E}\ \lambda[A,B][1] : \lambda[A,B] \geq B - A,$

$[3] := \Lambda \varepsilon \in \mathbb{R}_{++} \; . \; \texttt{E}\ \texttt{ClosedInterval}[A,B] \texttt{E}\ \texttt{OpenInterval}(A-\varepsilon, B+\varepsilon) \texttt{I}\ \subset :$

$\quad : \forall \varepsilon > 0 \; . \; [A,B] \subset (A-\varepsilon, B-\varepsilon),$

$[4] := \Lambda \varepsilon \in \mathbb{R}_{++} \; . \; \texttt{MeasureMonotonicity}(\mathbb{R},\lambda)[3](\varepsilon) \texttt{E}\ \lambda(A-\varepsilon, B-\varepsilon) :$

$\quad : \forall \varepsilon > 0 \; . \; \lambda[A,B] \leq \lambda(A-\varepsilon, B-\varepsilon) = B - A + 2\varepsilon,$

$[5] := \texttt{LimitIneq}[4] : \lambda[A,B] \leq B - A,$

$[*] := \texttt{E}\ (\leq)[2][5] : \lambda[A,B] = B - A;$

$\square$

$\texttt{CantorSetHasMeasureZero} :: \lambda(\mathcal{C}) = 0$

$\texttt{Proof} =$

Cantor's set $\mathcal{C}$ is constructed as $[0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{k_n} \Delta_{n,i}$.

Here each $\Delta_{n,i}$ represent evenly spaced disjoint open intervals of length 3^{-n} .

$k^n = 2^{n-1}$ represents quantity of intervals of fixed length.

So the measure of sum of $\Delta_{n,i}$ equels to $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1$.

By basic property of measure the result follows.

$\square$

$\texttt{BoundedInteriorMeasure} ::$

$:: \forall E \in \Lambda \,.\, \texttt{Bounded}(\mathbb{R}, E) \Rightarrow \lambda\ E = \sup\{\lambda\ K | K : \texttt{Closed}(\mathbb{R}), K \subset E\}$

$\texttt{Proof} =$

$\Big((A,B),[1]\Big) := \texttt{E}\ \texttt{Bounded}(\mathbb{R}, E) : \sum (A,B) : \texttt{OpenInterval}(\mathbb{R}) \,.\, E \subset R,$

$F := [A,B] \setminus E \in \Lambda,$

$\Big(U,[2]\Big) := \texttt{E}\ \lambda(F)\texttt{E}\ \inf : \sum U : \mathbb{N} \downarrow \mathcal{T}(\mathbb{R}) \,.\, \forall n \in \mathbb{N} \,.\, F \subset U_n\ \&\ \lambda\ F = \lim_{n\to\infty} \lambda(U_n)\ \&\ \forall n \in \mathbb{N} \,.\, \lambda(U_n \setminus [A,B]) <$

$K := \Lambda n \in \mathbb{N} \,.\, [A,B] \setminus U_n : \mathbb{N} \to \texttt{Closed}(\mathbb{R}),$

$[3] := \texttt{E}\ K\texttt{E}\ F[2.1] : \forall n \in \mathbb{N} \,.\, K_n \subset E,$

$[4] := \texttt{Difference}(\mathbb{R}, \Lambda, \lambda)[2.2]\texttt{ContinuousAddition}\Big(\lambda[A,B]\Big)\texttt{E}\ \Lambda\Big([A,B]\Big)\texttt{Difference}(\mathbb{R}, \Lambda, \lambda)[2.3]$

$\texttt{ContinuousAddition}\Big(\lambda\ K, \lambda n \in \mathbb{N} \,.\, 1/n\Big)\texttt{ReductioInfima} :$

$: \lambda\ E = \lambda[A,B] - \lambda(F) = \lambda[A,B] - \lim_{n\to\infty} \lambda(U_n) = \lim_{n\to\infty} \Lambda[A,B] - \lambda(U_n) =$

$= \lim_{n\to\infty} \Lambda[A,B] - \lambda\Big(U_n \cap [A,B]\Big) + \lambda\Big(U_n \setminus [A,B]\Big) \le \lim_{n\to\infty} \lambda\ K_n + \frac{1}{n} =$

$= \lim_{n\to\infty} \lambda\ K_n + \lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \lambda\ K_n,$

$[5] := \texttt{E}\ {}_2\texttt{Measure}(\mathbb{R}, \Lambda, \lambda)[3] : \forall n \in \mathbb{N} \,.\, \lambda(K_n) \le \lambda\ E,$

$[6] := \texttt{LimitIneq}[5]\texttt{E}\ (\le)[4] : \lambda(E) = \lim_{n\to\infty} K_n,$

$[*] := \texttt{E}\ {}_2\texttt{Measure}(\mathbb{R}, \Lambda, \lambda)[5]\texttt{I}\ \sup : \lambda\ E = \sup\{\lambda\ K | K : \texttt{Closed}(\mathbb{R}), K \subset E\};$

$\square$

$\texttt{InteriorMeasure} ::$

$:: \forall E \in \Lambda \,.\, \lambda\ E = \sup\{\lambda\ K | K : \texttt{Closed}(\mathbb{R}), K \subset E\}$

$\texttt{Proof} \ =$

$F := \Lambda n \in \mathbb{Z} \,.\, E \cap [n, n+1] : \texttt{DisjointFamily}(\mathbb{Z}, \Lambda),$

$\Big(K, [1]\Big) := \texttt{BoundedInteriorMeasure}(F)\texttt{E}\ \sup :$

$: \sum K : \mathbb{Z} \times \mathbb{N} \to \texttt{Closed}(\mathbb{R}) \,.\, \forall n \in \mathbb{Z} \,.\, \texttt{Increasing}\Big(K_n\Big)\ \&\ \lim_{m\to\infty} \lambda(K_{n,m}) = \lambda(F_n)\ \&$

$\&\ \forall m \in \mathbb{N} \,.\, K_{n,m} \subset F_n,$

$[2] := \texttt{E}\ F : E = \bigcup_{n=-\infty}^{\infty} F_n,$

$[3] := \texttt{E}\ F[1.3] : \forall m \in \mathbb{N} \,.\, \texttt{LocallyFinite}(K_{\bullet,m}),$

$G := \Lambda m \in \mathbb{N} \,.\, \bigcup_{n=-\infty}^{\infty} K_{n,m} : \mathbb{N} \to \texttt{Closed}(\mathbb{R}),$

$[4] := \texttt{E}\ G[1.3]\texttt{SubsetOfUnion}[2] : \forall m \in \mathbb{N} \,.\, G_m \subset E,$

$[5] := \texttt{E}\ \texttt{DisjointFamily}(\mathbb{Z}, \Lambda)[1.3] : \forall m \in \mathbb{N} \,.\, \texttt{DisjointFamily}(\mathbb{Z}, \Lambda, K_{\bullet,m}),$

$[6] := \texttt{E}\ _3\texttt{Measure}(\mathbb{R}, \Lambda, \lambda)[1.2]\texttt{MonotonicConvergenceTHM}(\#, \lambda\ K)[1.1]\texttt{E}\ _3\texttt{Measure}(\mathbb{R}, \Lambda, \lambda)[5]\texttt{I}\ G :$

$: \lambda(E) = \sum_{n=-\infty}^{\infty} \lambda(F_n) = \sum_{n=-\infty}^{\infty} \lim_{m\to\infty} \lambda\ K_{n,m} = \lim_{m\to\infty} \sum_{n=-\infty}^{\infty} \lambda\ K_{n,m} = \lim_{m\to\infty} \lambda \bigcup_{n=-\infty}^{\infty} K_{n,m} = \lim_{m\to\infty} \lambda\ G_m,$

$[*] := \texttt{Monotonicity}\Big(\mathbb{R}, \Lambda, \lambda\Big)[4][6] : \lambda\ E = \sup\{\lambda\ K | K : \texttt{Closed}(\mathbb{R}), K \subset E\};$

□

$\texttt{PointHasZeroMeasure} :: \forall t \in \mathbb{R} \,.\, \lambda\{t\} = 0$

$\texttt{Proof} \ =$

$\lambda\{t\} = \lambda[t, t] = t - t = 0 \,.$

□

$\texttt{CountableHasZeroMeasure} :: \forall C : \texttt{Countable}(\mathbb{R}) \,.\, \lambda\ C = 0$

$\texttt{Proof} \ =$

$\lambda\ = \lambda \bigcup_{x\in C} \{x\} = \sum_{x\in C} \lambda\{x\} = \sum_{x\in C} 0 = 0 \,.$

□

8.4 Motion Invariance

ShiftPreservesIntervalLength :: $\forall (a,b) : \texttt{OpenInterval}(\mathbb{R}) \ . \ \forall t \in \mathbb{R}\lambda\Big((a,b)+t\Big) = \lambda(a,b)$

Proof =

$\lambda\Big((a,b)+t\Big)\lambda\Big(a+t, b+t\Big) = (b+t)-(a+t) = b-a = \lambda(a,b).$

□

ReflectionPreservesIntervalLength :: $\forall (a,b) : \texttt{OpenInterval}(\mathbb{R})\lambda(a,b) = \lambda(-b,-a)$

Proof =

$\lambda(a,b) = b-a = (-a)-(-b) = \lambda(-b,-a)$.

□

MotionPreservesIntervalLength :: $\forall (a,b) : \texttt{OpenInterval}(\mathbb{R}) \ . \ \forall \phi \in \mathbf{E}(1) \ . \ \lambda\phi(a,b) = \lambda(a,b)$

Proof =

Use the fact that every $\phi \in \mathbf{E}(1)$ can be represented as composition of shifts and reflections .

□

MotionPreservesMeasureOfOpenSets :: $\forall U \in \mathcal{T}(\mathbb{R}) \ . \ \forall \phi \in \mathbf{E}(1) \ . \ \lambda\ \phi(U) = \lambda\ U$

Proof =

$U = \bigcup_{i\in I}(a_i, b_i)$ for some countable set I by property of Reals, and (a,b) are pairwise disjoint.

Obviously ϕ maps open intervals into open intervals.

And the image also pairwise disjoint as ϕ is bijection .

So, $\lambda\ \phi(U) = \lambda\bigcup_{i\in I}\phi(a_i,b_i) = \sum_{i\in i}\lambda\phi(a_i,b_i) = \sum_{i\in I}\lambda(a_i,b_i) = \lambda U.$

□

MotionPreservesOuterMeasure :: $\forall A \subset \mathbb{R} \ . \ \forall \phi \in \mathbf{E}(1) \ . \ \lambda^*\ \phi(A) = \lambda^*\ A$

Proof =

Let U be some open set with $A \subset U$.

Then $\phi(A) \subset \phi(U)$.

But $\lambda\ \phi(U) = \lambda\ U$.

So, $\lambda^*\phi(A) \leq \lambda^*\ A$.

On the other hand, ϕ^{-1} is also a motion so simmilar derivations witness that $\lambda^\star A \leq \lambda^*\phi(A)$.

□

MotionPreservesLebesgueMeasure :: $\forall E \in \Lambda \ . \ \forall \phi \in \mathbf{E}(1) \ . \ \lambda\ \phi(E) = \lambda\ E$

Proof =

Obvious at this stage .

□

`MotionPreservesInnerMeasure` :: $\forall A \subset \mathbb{R} \; . \; \forall \phi \in \mathbf{E}(1) \; . \; \lambda_* \, \phi(A) = \lambda_* \, A$

`Proof` $=$

Simmilar proof as with outer measure.

But instead of open U use measurable E with $E \subset A$.

$\square$

`LengthScaling` :: $\forall (a,b) : \texttt{Interval}(\mathbb{R}) \; . \; \forall t \in \mathbb{R}_+ \; . \; \lambda\sigma\left(\frac{a+b}{2}, t, (a,b)\right) = t\lambda(a,b)$

`Proof` $=$

$$\lambda\sigma\left(\frac{a+b}{2}, t, (a,b)\right) = \lambda\left(t\left(a - \frac{a+b}{2}\right) + \frac{a+b}{2}, t\left(a - \frac{a+b}{2}\right) + \frac{a+b}{2}\right) =$$

$$t\left(b - \frac{a+b}{2}\right) + \frac{a+b}{2} - t\left(a - \frac{a+b}{2}\right) - \frac{a+b}{2} = tb - ta = t(b-a) = t\lambda(a,b)$$

$\square$

8.5 Vitali's Theorem

$$\texttt{VitalisCover} :: \prod A \subset \mathbb{R} \,.\, ?\texttt{Cover}\Big(A, \texttt{ClosedInterval}(\mathbb{R})\Big)$$

$$\mathcal{V} : \texttt{VitalisCover} \iff \forall a \in A \,.\, \forall t \in \mathbb{R}_{++} \,.\, \exists I \in \mathcal{V} \,.\, \lambda I < t \;\&\; a \in I \;\&\; \forall I \in \mathcal{V} \,.\, \lambda I > 0$$

$$\texttt{VitaliCoveringTHM} :: \forall A : \texttt{Bounded}(\mathbb{R}) \,.\, \forall \mathcal{V} : \texttt{VitalisCover}(A) \,.$$

$$.\, \exists \mathcal{V}' : \texttt{Countable} \;\&\; \texttt{PairwiseDisjoint}(\mathcal{V}) \,.\, \lambda^\star \left(A \setminus \bigcup_{V \in \mathcal{V}'} V \right) = 0$$

$\texttt{Proof} \;=$

$\Delta := [\inf A, \sup A] : \texttt{ClosedInterval}(\mathbb{R})$,

Without loss of generality assume that $\forall I \in \mathcal{V} \,.\, I \subset \Delta$.

$\mathcal{V}''_0 := \emptyset : \texttt{Finite}(\mathcal{V})$,

$\texttt{Assume}\; n \in \mathbb{N}$,

$$\texttt{Assume}\; \aleph : \lambda^\star \left(A \setminus \bigcup_{V \in \mathcal{V}'} V \right) > 0,$$

Otherwise just set $\mathcal{V}''_n = \mathcal{V}''_{n+1}$.

$\mathcal{A} := \{I \in \mathcal{V} : \forall J \in \mathcal{V}''_{n-1} \,.\, I \cap J = \emptyset\} :?\mathcal{V}$,

$$K := \bigcup_{I \in \mathcal{V}''_{n-1}} I : \texttt{Closed}(\mathbb{R}),$$

$U := K^\complement \in \mathcal{T}(\mathbb{R})$,

$$\Big(X, (a,b), [1]\Big) := \texttt{OpenRealsStrucuture}(U) :$$

$$: \sum X : \texttt{Countable} \,.\, \sum (a,b) : \texttt{DisjointFamily}\Big(I, (a,b)\Big) \,.\, U = \bigcup_{i \in X} (a_i, b_i),$$

$[2] := \mathtt{E}\;{}_1\texttt{OuterMeasure}(\mathbb{R}, \lambda^\star)\mathtt{E}\;\aleph : \exists A \setminus K$,

$t := \mathtt{E}\;\exists[2] \in A \setminus K$,

$\Big(i, [2]\Big) := \mathtt{E}\; U[1] : \sum i \in I \,.\, t \in (a_i, b_i)$,

$\Big(I, [3]\Big) := \mathtt{E}\;\texttt{VitalisCover}(A, \mathcal{V})[2] : \sum I \in \mathcal{V} \,.\, t \in I \subset (a_i, b_i)$,

$[4] := \mathtt{E}\; U[3]\mathtt{I}\; I : \mathcal{A} \neq \emptyset$,

$$c_n := \sup_{I \in \mathcal{A}} \lambda\, I : \mathbb{R}_+ +,$$

$$I, [5] := \mathtt{E}\; c_n \mathtt{E}\; \sup : \sum I \in \mathcal{A} \,.\, \lambda\, I > \frac{c_n}{2},$$

$\mathcal{V}''_n := V''_{n+1} \cup \{I\} : \texttt{Finite}(\mathcal{V})$;

$$\rightsquigarrow \Big(V'', c, [1]\Big) := \mathtt{I}\; \sum : \sum \mathcal{V}'' : \mathbb{Z}_+ \to \texttt{Finite}(\mathcal{V}) \,.\, \sum c : \mathbb{N} \to \mathbb{R}_{++} \,.\, \forall n \in \mathbb{Z}_+ \,.$$

$$.\, |\mathcal{V}''_n| = n \;\&\; \texttt{PairwiseDisjoint}(\mathcal{V}''_n) \;\&$$

$$\&\; \forall n \in \mathbb{N} \,.\, \forall I : \texttt{ClosedInterval}(\mathbb{R}) \,.\, \forall \aleph : \{I\} = \mathcal{V}''_n \setminus \mathcal{V}''_{n+1} \,.\, \lambda\, I \geq \frac{c_n}{2},$$

$$\mathcal{V}' := \bigcup_{n=1}^{\infty} \mathcal{V}''_n : \texttt{Countable} \;\&\; \texttt{PairwiseDisjoint}(\mathcal{V}),$$

$B := \bigcup \mathcal{V}' \in \mathcal{B}(\mathbb{R}),$

$J := \Lambda[a,b] \in \mathcal{V}'' \;.\; \texttt{scale}\left(\frac{a+b}{2}, 5, [a,b]\right) : \mathcal{V}'' \to \texttt{ClosedInterval}(\mathbb{R}),$

$[2] := \texttt{E}\; J_I \texttt{LengthScaling}(I,5)\texttt{E}\;_3\texttt{Measure}(\mathbb{R},\Lambda,\lambda)\texttt{MonotonicityE}\;\lambda\Delta :$

$: \sum_{I\in\mathcal{V}'} \lambda\; J_I = \sum_{I\in\mathcal{V}'} 5\lambda\; I = 5\lambda \bigcup_{I\in\mathcal{V}'} I \le 5\lambda\Delta < \infty,$

$I := \texttt{enumerate}(\mathcal{V}') : \texttt{Surjective}\Big(\mathbb{N}, \mathcal{V}'\Big),$

$[3.1] := \texttt{AbsoluteConvergence}[3] : \lim_{n\to\infty} \lambda J_{I_n} = 0,$

$[3.2] := \texttt{MeasureMonotonicity}(\mathbb{R},\Lambda,\lambda)[3.1] : \lim_{n\to\infty} \lambda I_n = 0,$

$K := \Lambda n \in \mathbb{N} \;.\; \bigcup_{k=1}^{n} I_k : \mathbb{N} \uparrow \texttt{Closed}(\mathbb{R}),$

$U := K^{\complement} : \mathbb{N} \downarrow \texttt{Open}(\mathbb{R}),$

$\texttt{Assume}\; n \in \mathbb{N},$

$\texttt{Assume}\; x \in A \setminus B,$

$[4] := \texttt{E}\; Ux : \forall n \in \mathbb{N} \;.\; x \in U_n,$

$\Big(C, [5]\Big) := \texttt{E}\;\texttt{VitalisCover}(\mathcal{V})[4] : \sum C : \mathbb{N} \to \mathcal{V} \;.\; \forall n \in \mathbb{N} \;.\; x \in C_n \subset U_n,$

$[6] := \texttt{E}\; c[5]\texttt{E}\; U[1] : \forall n \in \mathbb{N} \;.\; \lambda\; C_n \le c_n \le 2\lambda\; I_n,$

$[7] := [3.2][6] : \lim_{n\to\infty} \lambda C_n = 0,$

$m := \min\{m \in \mathbb{N} : \exists F_m \cap C_n\} \in \mathbb{N},$

$[8] := \texttt{E}\; m\texttt{E}\; F_n : \exists C \cap I_m,$

$[9] := \texttt{E}\; m\texttt{E}\; F_n[1]\texttt{E}\; c : \lambda C_n \le c_m \le 2\lambda I_m,$

$[10] := \texttt{MeasureOfClosedInterval}[9]\texttt{E}\;\texttt{ClosedInterval}(C_m)\texttt{I}\; J_m : C_m \subset J_m,$

$[11] := [5]\texttt{E}\; m : m > n,$

$[x.*] := [5][10] : x \in J_m;$

$\leadsto [n.*] := \texttt{I}\; \exists\texttt{I} \bigcup : A \setminus B \subset \bigcup_{m=n+1}^{\infty} J_m;$

$\leadsto [4] := \texttt{I}\; \forall : \forall n \in \mathbb{N} \;.\; A \setminus B \subset \bigcup_{m=n+1}^{\infty} J_m,$

$[*] := \texttt{E}\;\texttt{OuterMeasure}(\mathbb{R},\lambda^\star)[4]\texttt{Subbaditivity}[3.1] :$

$: \lambda^\star(A \setminus B) \le \lim_{n\to\infty} \lambda\left(\bigcup_{m=n+1}^{\infty} J_m\right) \le \lim_{n\to\infty} \sum_{m=n+1}^{\infty} \lambda\; J_m = 0;$

$\square$

$\texttt{ApproximateVitaliCoveringTHM} :: \forall A : \texttt{Bounded}(\mathbb{R}) \;.\; \forall \mathcal{V} : \texttt{VitalisCover}(A) \;.$

$. \forall \varepsilon \in \mathbb{R}_{++} \;.\; \exists \mathcal{V}' : \texttt{Finite} \;\&\; \texttt{PairwiseDisjoint}(\mathcal{V}) \;.\; \lambda^\star\left(A \setminus \bigcup_{V\in\mathcal{V}'} V\right) < \varepsilon$

$\texttt{Proof} \;=$

Find large finite sum instead of the infinite one as in the proof above.

$\square$

8.6 Measurable Wonders

MeasurableSetsCardinality :: $|\Lambda| = 2^{|\mathbb{R}|}$

Proof =

$\Lambda \subset 2^{\mathbb{R}}$, so $|\Lambda| \le 2^{|\mathbb{R}|}$.

But as $\lambda\, \mathcal{C} = 0$, every subset of $\mathcal{C}$ is measurable.

However, $|\mathcal{C}| = |\mathbb{R}|$, so $|\Lambda| = 2^{|\mathbb{R}|}$.

□

NonMeasurableSetExists :: $\Lambda \subsetneq 2^{\mathbb{R}}$

Proof =

Compute quotient $X = \dfrac{[0,1]}{\mathbb{Q}}$ by $x \sim y$ if $x - y \in \mathbb{Q}$.

By axiom of choice sellect set of representatives $E = \left\{ x \middle| [x] \in X \right\}$.

Let q be anumeration of $\mathbb{Q} \cap [0,1]$.

Let $E_k = (E + q_k) \mod 1$.

Then, if E is measurable, then $2 = \lambda[-1,1] = \lambda \bigcup_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \lambda\, E_n$, where we used that E_n are disjoint. .

So, there must be some E_n with $\lambda\, E_n > 0$.

But by translation invariance for each $\lambda E_n = \lambda E_m$.

So, $\sum_{n=1}^{\infty} \lambda\, E_n = \infty$, a contradiction.

□

LebesgueMeasurableAreMoreThenBorel :: $\mathcal{B}(\mathbb{R}) \subsetneq \Lambda$

Proof =

Let A be a Borel non-measurable subset of $\mathbb{R}$.

But $\mathbb{R}$ is Borel-isomorphic to Cantor set $\mathcal{C} \subset \mathbb{R}$.

So, let φ be a corresponding isomorphism.

Then $\varphi(A)$ must also be non-Borel in $\mathcal{C}$.

So, $\varphi(A)$ is also non-Borel in $\mathbb{R}$ as $\mathcal{C}$ has subset topology.

But $\lambda(\mathcal{C}) = 0$, so $\varphi(A)$ must be Lebesgue measurable as $(\mathbb{R}, \Lambda, \lambda)$ is complete.

As measure space produced by outer measures are complete.

□

8.7 Lebesgue-Steltjes Measures and Distributions and Distributions

$\texttt{LebesgueStieltjes} :: ?\texttt{Measure}\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$

$\mu : \texttt{LebesgueStieltjes} \iff \forall I \in \mathcal{B}(\mathbb{R}) \,.\, \texttt{Bounded}(\mathbb{R}, U) \Rightarrow \mu(I) < \infty$

$\texttt{DistirbutionFunction} :: ?\left(\texttt{RightContinuous\&Increasing}\left(\overset{\infty}{\mathbb{R}}, \overset{\infty}{\mathbb{R}}\right)\right)$

$F : \texttt{DistirbutionFunction} \iff F(\infty) > -\infty$

$\texttt{MeasureAsDistribution} :: \forall \mu : \texttt{LebesgueStieltjes}(\mathbb{R}) \,.\, \forall x, c \in \mathbb{R} \,.$

$\quad \exists F : \texttt{DistirbutionFunction}(\mathbb{R}) : F(x) = c : \forall (a,b] : \texttt{SemiClosed}(\mathbb{R}) \,.\; .\, \mu(a,b] = F(b) - F(a)$

$\texttt{Proof} \; =$

$F := \Lambda t \in \mathbb{R} \,.\, \texttt{if } t = x \texttt{ then } c \texttt{ else if } t < x \texttt{ then } c - \mu(a,x] \texttt{ else } \mu(x,a] - c : \mathbb{R} \to \mathbb{R},$

$[1] := \texttt{E } F \texttt{E Measure}\Big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu\Big) : \forall (a,b] : \texttt{SemiClosed}(\mathbb{R}) \,.\, F(b) - F(a) = \mu(a,b],$

$\texttt{Assume } a, b \in \mathbb{R},$

$\texttt{Assume } \aleph : b > a,$

$[2] := [1](a,b]\texttt{E Measure}\Big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu\Big) : F(b) - F(a) = \mu(a,b] \geq 0,$

$\Big[(a,b).*\Big] := [1] + F(a) : F(b) \geq F(a);$

$\leadsto [2] := \texttt{I Increasing} : \texttt{Increasing}(\mathbb{R}, \mathbb{R}, F),$

$\texttt{Assume } a : \mathbb{N} \to \mathbb{R},$

$\texttt{Assume } A \in \mathbb{R},$

$\texttt{Assume } \aleph : a \downarrow A,$

$[3] := \texttt{E } \aleph \texttt{I } \emptyset : (A, a] \downarrow \emptyset,$

$[4] := \Lambda n \in \mathbb{N} \,.\, [1](A, a_n)\texttt{UpperContinuity}\Big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu\Big)\texttt{E Measure}\Big(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu\Big) :$

$$: \lim_{n\to\infty} \big(F(a_n) - F(A)\big) = \lim_{n\to\infty} \mu(A, a_n) = \mu \bigcap_{n=1}^{\infty} \mu(A, a_n) = \mu(\emptyset) = 0,$$

$[a.*] := \texttt{ConstantLimit}\Big([4] + F(A)\Big) : \lim_{n\to\infty} F(a_n) = F(A);$

$\leadsto [3] := \texttt{I RightContinuous} : \texttt{RightContinuous}(F),$

$[*] := \texttt{I DistirbutionFunction}[2][3] : \texttt{DistirbutionFunction}(F);$

$\square$

$\texttt{toDistribution} :: \texttt{LebesgueStieltjes} \to \texttt{DistirbutionFunction}$

$\texttt{toDistribution}\,(\mu) = F_\mu := \texttt{MeasureAsDistribution}(\mu, 0, 0)$

$\texttt{DistributionAsMeasure} :: \forall F : \texttt{DistirbutionFunction}(\mathbb{R}) \;.\; \exists! \mu : \texttt{LebesgueStieltjes}(\mathbb{R}) \;.$
$\;.\; \texttt{MeasureAsDistribution}(\mu, 0, F(0)) = F$

$\texttt{Proof} \; =$

$$\mu^* := \Lambda A \subset \mathbb{R} \;.\; \inf \left\{ \sum_{n=1}^{\infty} F(b_n) - F(a_n) \middle| (a_n, b_n] : \mathbb{N} \to \texttt{SemiClosed}(\mathbb{R}), A \subset \bigcup_{n=1}^{\infty} (a_n, b_n] \right\} : 2^{\mathbb{R}} \to \overset{\infty}{\mathbb{R}}_+,$$

Then mimic the construcion of the Lebesgue measure.

□

$\texttt{measureFromDistribution} :: \texttt{DistirbutionFunction} \to \texttt{LebesgueStieltjes}(\mathbb{R})$

$\texttt{measureFromDistribution}\,(F) = \mu_F := \texttt{DistributionAsMeasure}(F)$

9 Lebesgue Integration on the Real Line

9.1 Integration over Intervals

9.2 Laplace Transform

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