Topological Rings And Fields

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Contents

1 Subject

1.1 Subject

$$\texttt{TopologicalRing} :: ? \sum_{R \in \mathsf{RING}} \texttt{Topology}(R)$$

$$(R, \tau) : \texttt{TopologicalRing} \iff (R, +, \tau) \in \mathsf{TGRP} \;\&\; (\cdot) \in \mathsf{TOP}\Big((R, \tau) \times (R, \tau), (R, \tau)\Big)$$

1.2 Fields

$\texttt{TopologicalField} :: ?\sum k : \texttt{Field} \,.\, \texttt{Topology}(R)$

$(k,\tau) : \texttt{TopologicalRing} \iff \texttt{TopologicalRing}(k,\tau) \;\&\; \mathrm{inv} \in \mathsf{TOP}\Big((k_*,\tau),(k_*,\tau)\Big)$

$\texttt{Value} :: \prod k : \texttt{Field} \,.\, ?(k \to \mathbb{R}_{++})$

$v : \texttt{Value} \iff \forall a,b \in k \,.\, v(ab) = v(a)v(b) \;\&\; \forall a,b \in k \,.\, v(a)+v(b) \leq v(a)+v(b) \;\&\; \forall a \in k \,.\, v(a) = 0 \iff a =$

$\texttt{valueMetrization} :: \prod k : \texttt{Field} \,.\, \texttt{Value}(k) \to \texttt{Metric}(k)$

$\texttt{valueMetrization}\,(v) = d_v := \Lambda a,b \in k \,.\, v(a-b)$

$\texttt{UnitValue} :: \forall k : \texttt{Field} \,.\, \forall v : \texttt{Value}(k) \,.\, v(1) = 1$

$\texttt{Proof} =$

1 $v(1) = v(1*1) = v^2(1)$.

2 As $v(1) \neq 0$ it must be the case $v(1) = 1$.

2.1 $1 \neq 0$ in any field.

2.2 And $v(a) = 0 \iff a = 0$.

$\square$

$\texttt{NegValue} :: \forall k : \texttt{Field} \,.\, \forall v : \texttt{Value}(k) \,.\, v(-1) = 1$

$\texttt{Proof} =$

1 $1 = v(1) = v\Big((-1)*(-1)\Big) = v^2(-1)$.

2 As $v(-11) \neq 0$ it must be the case $v(1) = 1$.

2.1 $1 \neq 0$ in any field.

2.2 And $v(a) = 0 \iff a = 0$.

$\square$

2 Non-Archimedean Fields

2.1 Ultravalue

$\texttt{Ultravalue} :: \prod k : \texttt{Field} \;.\; ?\texttt{Value}(k)$

$v : \texttt{Ultravalue} \iff \forall a, b \in k \;.\; v(a+b) \leq \max\Big(v(a), v(b)\Big)$

$\texttt{UltravaluedField} := \sum k : \texttt{Field} \;.\; \texttt{Ultravalue}(k) : \texttt{Type};$

$\texttt{UltravalueDefinesUltrametric} :: \forall k : \texttt{Field} \;.\; \forall v : \texttt{Ultravalue}(k) \;.\; \texttt{Ultrametric}(k, d_v)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{AddingLessPreservesUltravalue} :: \forall k : \texttt{UltravaluedField} \;.\; \forall a, b \in k \;.\; |a| > |b| \Rightarrow |a+b| = |a|$

$\texttt{Proof} =$

1 $|a+b| \leq \max\Big(|a|, |b|\Big) = |a|$.

2 $|a| = \Big|b - (a+b)\Big| \leq \max\Big(|b|, |a+b|\Big) = |a+b|$.

2.1 This must be the case as $|b| < |a|$.

3 $|a| = |a+b|$.

$\square$

Sources

1. Naricci L. ; Beckenstein E. - Topological Vector Spaces I (2010)

2. Angel Barria Comicheo ; Khodr Shamseddine - Summary on non-Archimedean valued fields (2018)