Algebraic Measure Theory

Uncultured Tramp

October 23, 2022

Contents

Intro

1 Measure Algebras

1.1 Subject

1.1.1 Definition and Basic Property

$\texttt{MeasureAlgebra} :: ?\sum A : \sigma\texttt{-DedekindComplete} \,.\, A \to \overset{\infty}{\mathbb{R}}_+$

$(A, \mu) : \texttt{MeasureAlgebra} \iff \forall a \in A \,.\, \mu(a) = 0 \iff a = 0\ \&$

$\quad \& \ \forall a : \texttt{PairwiseDisjointElements}(\mathbb{N}, A) \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$

$\texttt{measureAlgebraCategory} :: \mathsf{CAT}$

$\texttt{measureAlgebraCategory}\,() = \mathsf{MA} := \Big(\texttt{MeasureAlgebra}, \mathsf{BOOL}, \circ, \mathrm{id}\Big)$

$\texttt{MeasureMonotonicity} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, a \leq b \Rightarrow \mu(a) \leq \mu(b)$

$\texttt{Proof} =$

Write $\mu(b) = \mu(a) + \mu(b \setminus a) \geq \mu(a)$.

$\square$

$\texttt{MeasureStrictMonotonicity} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, a > b \Rightarrow \mu(a) > \mu(b)$

$\texttt{Proof} =$

Definition of measure algebra implies that $\mu(b \setminus a) > 0$.

Write $\mu(b) = \mu(a) + \mu(b \setminus a) > \mu(a)$.

$\square$

$\texttt{MinkovskyIneq} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, \mu(a \vee b) \leq \mu(a) + \mu(b)$

$\texttt{Proof} =$

Write $\mu(a) + \mu(b) = \mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab) + \mu(ab) \geq mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab = \mu(a \vee b)$.

$\square$

$\texttt{MonotonicSupremumAsLimit} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \uparrow A \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n \to \infty} \mu(a_n)$

$\texttt{Proof} =$

Construct disjoint sequence $b_n = a_n \setminus \bigvee_{k=1}^{n-1} a_k$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} \mu(b_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(b_n) = \lim_{n\to\infty} \mu\left(\bigvee_{k=1}^{n} b_k\right) = \lim_{n\to\infty} \mu(a_n)$.

$\square$

SupremumIneq :: $\forall (A,\mu) :$ MeasureAlgebra . $\forall a : \mathbb{N} \to A \, . \, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) \le \sum_{n=1}^{\infty} \mu(a_n)$

Proof =

Construct increasing sequence $b_n = \bigvee_{k=1}^{n} a_n$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) = \lim_{n\to\infty} \mu\left(\bigvee_{k=1}^{n} a_k\right) \le \lim_{n\to\infty} \sum_{k=1}^{n} \mu(a_k) = \sum_{n=1}^{\infty} \mu(a_n)$.

□

MonotonicInfimumAsLimit ::

:: $\forall (A,\mu) :$ MeasureAlgebra . $\forall a : \mathbb{N} \downarrow A \, . \, \forall \aleph : \inf_{n\in\mathbb{N}} \mu(a_n) < \infty \, . \, \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$

Proof =

Without loss of generality assume that $\mu(a_1) < \infty$.

Then construc the increasing sequence $b_n = a_1 \setminus a_n$.

Then $\mu(a_1) - \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(a_1 \setminus \bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} a_1 \setminus a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) =$

$= \lim_{n\to\infty} \mu\left(a_1 \setminus a_n\right) = \lim_{n\to\infty} \mu(a_1) - \mu(a_n) = \mu(a_1) - \lim_{n\to\infty} \mu(a_n)$.

So basic algebraic manipulations $\mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$.

□

SupremumExistance ::

:: $\forall (A,\mu) :$ MeasureAlgebra . $\forall C :$ UpwardsDirected(A) . $\forall \aleph : \sup_{c\in C} \mu(c) < \infty \, . \, \exists a \in A : a = \sup C$

Proof =

1 Assume $\gamma = \sup_{c\in C} \mu(c)$.

2 Then there exists a sequrnce of $a : \mathbb{N} \to C$ such that $\mu(a_n) \ge \gamma - 2^{-n}$.

3 As C is upwards closed, it is possible to find $c : \mathbb{N} \to C$ such that $c_{n+1} \ge a_n \vee c_n$.

4 Then c is monotonic-nondecreasing and so it has $\mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \lim_{n\to\infty} \mu(c_n) = \gamma$.

4.1 Note that $\gamma \ge \mu(c_n) \ge \gamma - 2^{-n}$.

5 let $d = \bigvee_{n=1}^{\infty} c_n$.

6 $d \ge f$ for everty $f \in C$.

6.1 Assume this is false.

6.2 Then $f \setminus d \ne 0$ and so $\alpha = \mu(f \setminus d) > 0$.

6.3 Then there exists n such that $\gamma - \mu(c_n) < \alpha$.

6.4 As C is upwards derected there is $g \in C$ such that $g \ge f \vee c_n$.

6.5 But $\mu(g) \ge \mu(f \vee c_n) = \mu(c_n) + \mu(f \setminus c_n) \ge \mu(c_n) + \mu(f \setminus d) > \gamma$ which is impossible.

7 If there is another f with the property (6), then $d = \bigvee_{n=1}^{\infty} c_n \le f$ as $c_n \le f$ for each $n \in \mathbb{N}$.

□

UpperContinuity ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall C : \texttt{UpwardsDirected}(A) \, . \, \forall \aleph : \exists a \in A : a = \sup C \, . \, \sup_{c \in C} \mu(c) = \mu(\sup C)$

Proof $=$

Case $\sup_{c \in C} \mu(c) = \infty$ is trivial.

Finite case follows from the cconstruction in the previous theorem.

$\square$

DisjointUpperContinuity ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall C : \texttt{PairwiseDisjointElements}(A) \, . \, \forall \aleph : \exists a \in A : a = \sup C \, .$

$. \, \mu(\sup C) = \sum_{c \in C} \mu(c)$

Proof $=$

Construct a new set $D = \left\{ \bigvee_{n=1}^{\infty} c_k \middle| c : \mathbb{N} \to C \right\}$.

Then D is upwards directed and $\sup C = \sup D$.

But this is evedent that $\mu(\sup D) = \sup_{d \in D} \mu(d) = \sup_{c : \mathbb{N} \to C} \mu\left(\bigvee_{n=1} c_n \right) = \sup_{n \in \mathbb{N}, c : \{1, \dots, n\} \to C} \sum_{k=1}^{n} \mu(c_k) = \sum_{c \in C} \mu(c)$.

$\square$

InfimumExistance ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall C : \texttt{DownwaedDirected}(A) \, . \, \forall \aleph : \inf_{c \in C} \mu(c) < \infty \, . \, \exists a \in A : a = \inf C$

Proof $=$

1 There exists some $a \in C$ such that $\mu(a) < \infty$.

2 Construct another set $D = a \setminus C$.

3 Then D is upwards directed and $\sup_{d \in D} \mu(d) \leq \mu(a) < \infty$.

4 So there is $d = \sup d$.

5 Define $f = a \setminus d$.

6 $f \leq c$ for any $c \in C$ as $a \setminus f \geq a \setminus c$.

7 if some g has property (6) then $a \setminus g \geq d$ and so $g \leq f$.

$\square$

LowerContinuity ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall C : \texttt{DownwardsDirected}(A) \, . \, \forall \aleph : \exists a \in A : a = \inf C \, .$

$. \, \forall \beth : \inf_{c \in C} \mu(c) < \infty \, . \, \inf_{c \in C} \mu(c) = \mu(\inf C)$

Proof $=$

Use the construction in the previous theorem.

$\square$

1.1.2 Measure Algebras Generated by Measure Spaces

measureAlgebra :: MEAS $\to$ MeasureAlgebra

measureAlgebra $(X, \Sigma, \mu) = (A_\mu, \bar{\mu}) := \left(\frac{\Sigma}{\Sigma \cap \mathcal{N}_\mu}, [E] \mapsto \mu(E) \right)$

This is obviously well defined as $[E] = [F]$ iff $\mu(E \triangle F) = 0$.

canononicalProjection :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \sigma\text{-}\mathsf{BOOL}(\Sigma, A_\mu)$

canonicalProjection $(E) = \pi_\mu(E) := [E]$

1 The algebraic properites are obvious as $\Sigma \cap \mathcal{N}_\mu$ is an ideal.

2 In order to prove sigma-continuity assume $E : \mathbb{N} \to \Sigma$.

2.1 Let $Z : \mathbb{N} \to \Sigma \cap \mathcal{N}_\mu$.

2.2 Then $F_Z = \bigvee_{n=1}^{\infty} (E_n \triangle Z_n) = \left(\bigvee_{n=1}^{\infty} E_n \right) \triangle \left(\bigvee_{n=1}^{\infty} Z_n \right)$.

2.3 Note that $\mu \left(\bigvee_{n=1}^{\infty} Z_n \right) \leq \sum_{n=1}^{\infty} \mu(Z_n) = 0$.

2.4 So $\bigvee_{n=1}^{\infty} Z_n \in \Sigma \cap \mathcal{N}_\mu$ as $\mu \geq 0$.

2.5 Thus $[F_Z] = \left[\bigcap_{n=1}^{\infty} E_n \right]$ for any selection of Z.

2.6 This means that $\pi_\mu \left(\bigcap_{n=1}^{\infty} E_n \right) = \bigvee_{n=1}^{\infty} \pi_\mu(E_n)$.

□

MeasureAlgebraMonotonicity :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall T \subset_\sigma \Sigma . \pi_\mu(T) \subset_\sigma A_\mu$

Proof =

1 Clearly $B = \pi_\mu(T) \subset A_\mu$.

2 Also as T is σ-algebra and $\pi - \mu$ is a σ-continuous homomorphism B is again.

□

MeasureAlgebraInverseMonotonicity :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall B \subset_\sigma A_\mu . \pi_\mu^{-1}(B) \subset_\sigma \Sigma$

Proof =

1 Clearly $T = \pi_\mu^{-1}(B) \subset \Sigma$.

2 Assume F is a set constructed by applying σ-algebra operations to setes $E_1, E_2, \ldots \in T$.

3 Then $\pi_\mu(F)$ can be constructed by applying same operations to $\pi(E_1), \pi(E_2), \ldots$.

4 This implies that $\pi_\mu(F) \in B$ and reciprorary $F \in T$.

5 Thus T is a σ-algebra.

□

1.1.3 Stone Representation Theorem

StoneRepresentationTheorem :: $\forall (A,\mu)$: MeasureAlgebra . $\exists (X,\Sigma,\nu) \in$ MEAS . $(A,\mu) = (A_\nu, \bar{\nu})$

Proof $=$

1 By Loomis-Sikorski representation there exists a set X with a sigma-algebra Σ and sigma-ideal I such that $\frac{\Sigma}{I} \cong_{\mathsf{BOOL}} A$.

2 Then there is a canonical projetion $\pi_I \in \mathsf{BOOL}(\Sigma, A)$.

3 Define $\nu = \pi_I \mu$.

4 ν is measure on Σ.

4.1 $\nu(\emptyset) = \mu(0) = 0$.

4.2 Assume E is a disjoint sequence in Σ.

4.3 Then $\pi_I(E_n)\pi_I(E_m) = \pi_i(E_n \cap E_m) = \pi_i(\emptyset) = 0$, so $\pi_I(E)$ is disjoint in A.

4.4 Thus, $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \pi_I\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigvee_{n=1}^{\infty} \pi_I(E_n)\right) = \sum_{n=1}^{\infty} \pi_I\mu(E_n) = \sum_{n=1}^{\infty} \nu(E_n)$.

5 Also by consytuction $\mathcal{N}_\nu \cap \Sigma = I$, so $(A,\mu) = (A_\nu, \bar{\nu})$.

$\square$

spaceOfStone :: MeasureAlgebra $\to$ MEAS

SpaceOfStone $(A,\mu) = (Z_A, \dot{\Sigma}_\mu, \dot{\mu}) :=$ StoneRepresentationTheorem(A,μ)

1.1.4 Ideals

$\texttt{PrincipleIdealRestriction} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a \in A \,.\, \texttt{MeasureAlgebra}\Big((a), \mu_{|(a)}\Big)$

$\texttt{Proof} \ =$

This is obvious.

$\square$

$\texttt{measureQuotient} ::$

$$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall [a] \in \frac{A}{I} \,.\, \exists \gamma \in \overset{\infty}{\mathbb{R}}_{++} \,.\, \gamma = \min\{\mu(b) | b \in A, \pi_I(b) = [a]\}$$

$\texttt{Proof} \ =$

1 $\gamma = \inf\{\mu(b)|b \in A, \pi_I(b) = [a]\}$ exists as a set is bounded by below by 0.

2 If $\gamma = \infty$ then the result is obvious.

3 Otherwise there is a decreasing sequence $b : \mathbb{N} \to A$ such that $\pi_I(b_n) = [a]$ for any n and $\lim_{n\to\infty} \mu(b_n) = \gamma$.

4 Then $c = \bigwedge_{n=1}^{\infty} b_n$ is such that $\mu(c) = \gamma$ and $\pi_I(c) = a$.

4.1 Clearly $\pi_I\left(\bigwedge_{n=1}^{\infty} b_n\right) = \bigwedge_{n=1}^{\infty} \pi_I(b_n) = \bigwedge_{n=1}^{\infty} [a] = [a]$.

5 So the infimum is atteined.

$\square$

$\texttt{measureQuotient} :: \prod (A, \mu) : \texttt{MeasureAlgebra} \,.\, \prod I : \texttt{Ideal}(A) \,.\, \frac{A}{I} \to \mathbb{R}_{++}$

$\texttt{measureQuotient}\,(a) = \mu_I(a) := \min\{\mu(b)|b \in A, \pi_I(b) = a\}$

$\texttt{finiteElementsIdeal} :: \prod (A, \mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Ideal}(A)$

$\texttt{finiteElementsIdeal}\,() = A^f := \{a \in A|\mu(a) < \infty\}$

$\texttt{MeasureIdealQuotient} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \texttt{MeasureAlgebra}\left(\frac{A}{I}, \mu_I\right)$

$\texttt{Proof} \ =$

1 Clearly $\mu_I(0) = 0$.

2 Assume that $[a] \neq 0$.

2.1 Then there exists $b \in A$ such that $\pi_I(a) = [a]$ and $\mu(b) = \mu_I[a]$.

2.2 As $[a] \neq 0$, then $b \neq 0$, and henceforth $\mu(b) \neq 0$.

2.3 Thus, $\mu_I[a] \neq 0$.

3 Assume $[a] : \mathbb{N} \to \dfrac{A}{I}$ is disjoint.

3.1 It is possible to select representatives b_n for each $[a_n]$ such that $\mu(b_n) = \mu_I[a_n]$.

3.2 Then $b_n b_m \in I$ if $n \neq m$.

3.3 Construct a new sequence $c_n = b_n + \sum_{k=1}^{n-1} b_n b_k$ is a disjoint represintative sequance for $[a_n]$.

3.3.1 In fact $c = b$.

3.4 $\bigvee_{n=1}^{\infty} c_n$ is the minimal representative of $\bigvee_{n=1}^{\infty} [a_n]$.

3.4.1 Assume d is a representative for $\bigvee_{n=1}^{\infty} a_n$.

3.4.2 If $\mu(d) < \mu\left(\bigvee_{n=1}^{\infty} c_n\right)$ then we may construct $c_n \wedge d$ which is smaller then c.

3.4.3 But this is a contradiction.

3.5 So $\mu_I\left(\bigvee_{n=1}^{\infty} [a_n]\right) = \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \mu_I[a_n]$.

□

1.1.5 Measure Properties

$\texttt{ProbabilityAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \pi) : \texttt{ProbabilityAlgebra} \iff \pi(e) = 1$

$\texttt{FiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \mu) : \texttt{FiniteMeasureAlgebra} \iff \mu(e) < \infty$

$\sigma\texttt{-FiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$

$$(A, \mu) : \sigma\texttt{-FiniteMeasureAlgebra} \iff \exists a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, \mu(a_n) < \infty \,\&\, \bigvee_{n=1}^{\infty} a_n = e$$

$\texttt{SemifiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \mu) : \texttt{SemifiniteMeasureAlgebra} \iff \forall a \in A \,.\, \mu(a) = \infty \Rightarrow \exists b \in A \,.\, b < a \,\&\, 0 < \mu(b) < \infty$

$\texttt{LocalizableMeasureAlgebra} := \texttt{OrderDedekindComplete} \,\&\, \texttt{SemifiniteMeasureAlgebra} : \texttt{Type};$

$\texttt{ProbabilityConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Probability}(X, \Sigma, \mu) \iff \texttt{ProbabilityAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
□

$\texttt{FiniteConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Finite}(X, \Sigma, \mu) \iff \texttt{FiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
□

$\texttt{SigmaFiniteConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \sigma\texttt{-Finite}(X, \Sigma, \mu) \iff \sigma\texttt{-FiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
□

$\texttt{SemifiniteConstruction} ::$
$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Semifinite}(X, \Sigma, \mu) \iff \texttt{SemifiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
□

$\texttt{LocalizableConstruction} ::$
$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Localizable}(X, \Sigma, \mu) \iff \texttt{LocalizableMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
□

AtomInConstruction ::

:: $\forall(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;\forall E\in\Sigma\;.\;E\in\mathrm{Atom}(X,\Sigma,\mu)\iff[E]\in\mathrm{Atom}(A_\mu,\bar\mu)$

Proof =

This is obvious.

□

AtomlessConstruction ::

:: $\forall(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;\forall E\in\Sigma\;.\;E\in\mathtt{Atomless}(X,\Sigma,\mu)\iff[E]\in\mathtt{Atomless}(A_\mu,\bar\mu)$

Proof =

This is obvious.

□

PurelyAtomicConstruction ::

:: $\forall(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;\forall E\in\Sigma\;.\;E\in\mathtt{PurelyAtomic}(X,\Sigma,\mu)\iff[E]\in\mathtt{PurelyAtomic}(A_\mu,\bar\mu)$

Proof =

This is obvious.

□

FinitenessPropertiesIerarchy ::

:: $\forall(A,\mu):\mathtt{MeasureAlgebra}\;.\;\mathtt{PobabilityAlgebra}(A,\mu)\Rightarrow\mathtt{FiniteMeasureAlgebra}(A,\mu)\Rightarrow$

$\Rightarrow\sigma\text{-}\mathtt{FiniteMeasureAlgebra}(A,\mu)\Rightarrow\mathtt{LocalizableMeasureAlgebra}(A,\mu)\Rightarrow\mathtt{Semifinite}(A,\mu)$

Proof =

1 Most implications here are obvious expect the one deriving Localizability from σ-finiteness.

2 So assume that (A,μ) is σ-finite .

2.1 Then the corresponding Stone space $(\mathsf{Z}A,\Sigma_\mu,\bar\mu)$ is σ-finite.

2.2 But then $(\mathsf{Z}A,\Sigma_\mu,\bar\mu)$ is localizable .

2.3 So (A,μ) is also localizable.

□

MeasureAlgebraOfCompletion :: $\forall(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;A_\mu\cong_{\mathsf{BOOL}}A_{\hat\mu}$

Proof =

This is basically follows from definitions.

□

MeasureAlgebraOfLocallyDeterminedCompletion ::

:: $\forall(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;\exists A_\mu\xrightarrow{\phi}A_{\tilde\mu}:\mathsf{BOOL}\;.\;\forall a\in A_{\tilde\mu}\;.\;\tilde\mu(a)<\infty\Rightarrow\exists b\in A_\mu\;.\;\phi(b)=a\;\&$

$\&\;\forall b\in A_\mu\;.\;\hat\mu(b)<\infty\Rightarrow\tilde\mu(\phi(b))=\hat\mu(b)$

Proof =

. . .

□

localDeterminationMorphism :: $\prod(X,\Sigma,\mu)\in\mathsf{MEAS}\;.\;\mathsf{BOOL}(A_\mu,A_{\tilde\mu})$

localDeterminationMorphism $()=\phi_\mu:=$ MeasureAlgebraOfLocallyDeterminedCompletion

$\texttt{localDeterminationMorhismInjectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \ . \ \texttt{Semifinite}(X, \Sigma, \mu) \iff \texttt{Injective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{localDeterminationMorhismBijectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \ . \ \texttt{Localizable}(X, \Sigma, \mu) \iff \texttt{Bijective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{SemifinitenessCriterion} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \ .$

$. \ \texttt{SemifiniteMeasureAlgebra}(A, \mu) \iff \exists P : \texttt{PartitionOfUnity}(A) \ . \ \forall p \in P \ . \ \mu(p) < \infty$

$\texttt{Proof} \ =$

1 $(\Rightarrow)$ assume first that (A, μ) is semifinite.

1.1 Then A^f is order dense in A.

1.2 By order density theorem there is a desired partition of unity.

2 $(\Leftarrow)$ Let P be the partition of unity.

2.1 Assume $a \in A$ is such that $\mu(a) = \infty$.

2.2 Then there exists $p \in P$ such that $pa \neq 0$.

2.3 Note that this means that $\mu(pa) > 0$.

2.4 Also it is clear that $\mu(pa) \leq \mu(p) < \infty$.

$\square$

$\texttt{SemifiniteneSupElementExpression} ::$

$$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \ . \ \forall a \in A \ . \ a = \bigvee \{b \in A : b \leq a, \mu(b) < \infty\}$$

$\texttt{Proof} \ =$

This follows from the previous theorem.

$\square$

$\texttt{SemifiniteneSupMeasureComputation} ::$

$$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \ . \ \forall a \in A \ . \ \mu(a) = \bigvee \{\mu(b) \in A : b \leq a, \mu(b) < \infty\}$$

$\texttt{Proof} \ =$

This follows from the previous theorem.

$\square$

1.1.6 Connections with other Boolean Properties

SemifiniteIsWeaklyDistributive ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra(A,μ) . (σ,∞)-WeaklyDistributive(A,μ)

Proof =

1 Assume $X : \mathbb{N} \to 2^A$ is a sequence of downwards selected sets with $\inf X_n = 0$ for every $n \in \mathbb{N}$.

2 Let $C = \{a \in A : \forall n \in \mathbb{N} \,.\, \exists x \in X_n \,.\, a \geq x\}$.

3 Assume $d \in A$ is such that $d \neq 0$.

4 Then there is an element $d' \leq d$ such that $0 < \mu(d') < 0$.

5 $\inf\limits_{x \in X} d'x = 0$ for each $n \in N$.

6 Select a sequence $x : \prod\limits_{n=1}^{\infty} X_n$ suc that $\mu(d'x_n) \leq 2^{-n-2}\mu(d')$.

7 Define $c = \sup\limits_{n=1} a_n \in C$.

8 Then $\mu(d'c) \leq \sum\limits_{n=0}^{\infty} \mu(cx_n) < \mu(d')$.

9 This means that $d \not\leq c$.

10 And as d was arbitrary $\inf C = 0$.

□

SemifiniteIffCCC :: $\forall (A,\mu)$: SemifiniteMeasureAlgebra(A,μ) .

. σ-FiniteMeasureAlgebra$(A,\mu) \iff$ WithCountableChainCondition(A)

Proof =

1 $(\Leftarrow)$ assume that A has ccc.

1.1 Then there is a partition of unitity P in A consisting of finite elements as A is semifinite.

1.2 But as A has ccc P must be atmost countable.

1.3 This proves that A is σ-finite.

2 $(\Rightarrow)$ assume that (A,μ) is σ-finite .

2.1 Then there exists a countable partition of unity P of A wirh finite elements.

2.2 If A is not ccc, then there exists an uncountable refinement Q of A with finite elements.

2.3 Then by pigionhole principle there exists $p \in P$

such that set $Q' = \{q \in Q : q \subset p\}$ such that Q' is uncountable.

2.4 as for $\mu(q) > 0$ for any $q \in Q'$ by pigionhole principle there exists some $n \in \mathbb{Z}$

such that there are an infinite number of $q \in Q'$ with $\mu(q) \in [2^{-n-1}, 2^{-n}]$.

2.5 So $\mu(p) \geq \sum\limits_{q \in Q'} \mu(q) = \infty$, but this is a contradiction.

□

$\texttt{SemifiniteIffProbabilityRenormalizationExists} ::$

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \ . \ A \neq \{0\} \Rightarrow$

$\Rightarrow \exists \pi : A \xrightarrow{\infty} \mathbb{R}_+ \ . \ \texttt{ProbabilityAlgebra}(A, \pi)$

$\texttt{Proof} \ =$

1 Corresponding Stone space is σ-finite.

2 So there exists a proper renormalization of $\bar{\mu}$ to a probability π with the same sets of measure zero.

3 Then the measure algebra of $(\mathsf{Z}A, \pi)$ is a probability algebra and $A_\pi \cong_{\mathsf{BOOL}} A$.

$\square$

1.1.7 Subspace Measures and Indefinite Integrals

MeasurableEnvelopePrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \; . \; \forall Y \subset X \; . \; \forall E : \texttt{MeasurableEnvelope}(X, \Sigma, \mu, Y) \; . \; (A_{\mu|Y}, \widehat{\mu|Y}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

This result is technically convoluted but actually is pretty intuituve.

□

PrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \; . \; \forall E \in \Sigma \; . \; (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

A straightforward application of a previous theorem.

□

ThickEquivalence ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \; . \; \forall Y : \texttt{Thick}(X, \Sigma, \mu) \; . \; (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} (X, \hat{\mu})$$

Proof =

A straightforward application of a previous theorem.

□

IndefiniteIntegralPrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \; . \; \forall f \in \mathsf{I}_+(X, \Sigma, \mu) \; . \; \exists E \in \Sigma \; . \; A_{f \; d\mu} \cong_{\mathsf{BOOL}} ([E])$$

Proof =

We may assume that $\operatorname{supp} f$ has a measurable envelope E.

Then the result is obvious as $\mathcal{N}_\mu \subset \mathcal{N}_{f \; d\mu}$.

□

1.1.8 Simple Products

$$\texttt{simpleProduct} :: \prod_{I \in \mathsf{SET}} (I \to \texttt{MeasureAlgebra}) \to \texttt{MeasureAlgebra}$$

$$\texttt{simpleProduct}\,(A, \mu) = \prod_{i \in I} (A_i, \mu_i) := \left(\prod_{i \in I} A_i, \sum_{i \in I} \mu_i \right)$$

Obviously $\sum_{i \in I} \mu_i(0) = \sum_{i \in I} 0 = 0.$

Also assume $a : \mathbb{N} \to \prod_{i \in I} A_i$ is disjoint.

Then $\sum_{i \in I} \mu_i \left(\bigvee_{n=1}^{\infty} a_n \right) = \sum_{i \in I} \sum_{n=1}^{\infty} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty} \sum_{i \in I} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty} \sum_{i \in I} \mu_i(a_n).$

□

$\texttt{PrincipleIdealsInMeasureAlgebras}$ ::

$$:: \forall I \in \mathsf{SET} \;.\; \forall (A, \mu) : I \to \texttt{MeasureAlgebra} \;.\; (A_i, \mu_i) \cong_{\mathsf{MA}} \left((e_i), \left(\sum_{i \in I} \mu_i \right)_{|(e_i)} \right)$$

$\texttt{Proof}$ =

This is pretty ovious.

□

$\texttt{SimpleProductCoproductCorrespondance}$ ::

$$:: \forall I \in \mathsf{SET} \;.\; \forall (X, \Sigma, \mu) : I \to \mathsf{MEAS} \;.\; \prod_{i \in I} (A_{\mu_i}, \hat{\mu}_i) \cong \texttt{measureAlgebra} \coprod_{i \in I} (X_i, \Sigma_i, \mu_i)$$

$\texttt{Proof}$ =

Obvious by Stone Theory.

□

$\texttt{SimpleProductOfSemifinite}$ ::

$$:: \forall I \in \mathsf{SET} \;.\; \forall (A, \mu) : I \to \texttt{SemifiniteMeasureAlgebra} \;.\; \texttt{SemifiniteMeasureAlgebra} \left(\prod_{i \in I} (A, \mu) \right)$$

$\texttt{Proof}$ =

Assume a has infinite measure in (A, μ).

Then there exists $i \in I$ such that $a_i \neq 0$.

As (A_i, μ_i) is semifinite there is $b \leq a_i$ such that $0 < \mu_i(b) < \infty$.

Then $be_i \leq a$ and $0 < \sum_{j \in I} \mu_j(be_i) = \mu_i(b) < \infty$.

□

SimpleProductOfLocalizable ::

$$:: \forall I \in \mathsf{SET} . \forall (A,\mu) : I \to \texttt{LocalizableMeasureAlgebra} . \texttt{LocalizableMeasureAlgebra}\left(\prod_{i\in I}(A,\mu)\right)$$

Proof =

Let J be a set and $a : J \to \prod_{i\in I}(A_i,\mu_i)$.

Then $\sup_{j\in J} a_j = (\sup_{j\in J} a_{j,i})_{i\in I}$.

□

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} . \forall (e_n)_{n=1}^{\infty} : \texttt{PartitionOfUnity}(A) . (A,\mu) \cong_{\mathsf{MA}} \prod_{n=1}^{\infty}\Big((e_n), \mu_{|(e_m)}\Big)$$

Proof =

This is pretty obvious.

□

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} . \exists I \in \mathsf{SET} . \exists (B,\nu) : I \to \texttt{FiniteMeasureAlgebra} .$$
$$. (A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$$

Proof =

It is possible to select a partition of unity P of A consisting of finite elements.

Then by previous theorem $(A,\mu) \cong \prod_{p\in P}\Big((p), \mu_{|(p)}\Big)$.

And each $\Big((p), \mu_{|(p)}\Big)$ are obviously finite.

□

LocalizableMeasureAlgebrasHasLocallyDeterminedRepresentations ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} . \exists (X,\Sigma,\nu) : \texttt{LocallyDetermined} . (A,\mu) \cong_{\mathsf{MA}} (A_\nu, \hat{\nu})$$

Proof =

Represent $(A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$.

Then Stone's spaces $\mathsf{Z}\ B_i$ correspond to finite measure spaces.

And Stone's space of product correspond to a disjoint union of $\mathsf{Z}\ B_i$.

But such spaces are trivially locally determined.

□

1.1.9 Strictly Localizable Spaces

$\texttt{StrictlyLocalizableSpacePoU} ::$

$:: \forall (X, \Sigma, \mu) : \texttt{StrictlyLocalizable} \, . \, \forall P : \texttt{PartitionOfUnity}(A_\mu) \, .$

$. \, \exists E : P \to \Sigma \, . \, \forall p \in P \, . \, [E_p] = p \,\&\, \texttt{Decomposition}(X, \Sigma, \mu, \operatorname{Im} E)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.1.10 Subalgebras

$\texttt{SubalgebaMeasureAlgebra} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall B \subset_\sigma A \, . \, \texttt{MeasureAlgebra}(B, \mu_{|B})$

Proof $=$

This is obvious.

□

$\texttt{SubalgebaFinifteMeasureAlgebra} ::$

$:: \forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \, . \, \forall B \subset_\sigma A \, . \, \texttt{FiniteMeasureAlgebra}(B, \mu_{|B})$

Proof $=$

This is obvious.

□

$\texttt{SigmaFiniteSubalgebraMeasureAlgebra} ::$

$:: \forall (A, \mu) : \sigma\texttt{-FiniteMeasureAlgebra} \, . \, \forall B \subset_\sigma A \, .$

$. \, \texttt{SemifiniteMeasureAlgebra}(B, \mu_{|B}) \Rightarrow \sigma\texttt{-FiniteMeasureAlgebra}(B, \mu_{|B})$

Proof $=$

1 The set B^f is order-dense in B.

2 But then B^f is also order-dense in A.

3 Select a finite-measured countable partition of unity P in A.

4 If B is not σ-finite, then there is a subordinate uncountale partition of unity Q.

5 Then there would exist a uncountable refinement of P subordinate to Q.

6 Then P must contain an infinite element, but this is imposible!.

7 So Q must be countable, and so $(B, \mu_{|B})$ must be countable.

□

$\texttt{FinifteMeasureAlgebraBySubalgebra} ::$

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall B \subset_\sigma A \, . \, \texttt{FiniteMeasureAlgebra}(B, \mu_{|B}) \Rightarrow \texttt{FiniteMeasureAlgebra}(A, \mu)$

Proof $=$

This is obvious.

□

$\texttt{ProbabilityAlgebraBySubalgebra} ::$

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall B \subset_\sigma A \, .$

$. \, \texttt{ProbabilityAlgebra}(B, \mu_{|B}) \Rightarrow \texttt{ProbabilityAlgebra}(A, \mu)$

Proof $=$

This is obvious.

□

$\texttt{SigmaFiniteAlgebraBySubalgebra} ::$

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \, . \, \forall B \subset_\sigma A \, .$

$. \, \sigma\text{-}\texttt{Finite}(B, \mu_{|B}) \Rightarrow \sigma\text{-}\texttt{Finite}(A, \mu)$

$\texttt{Proof} \; =$

This is obvious.

$\square$

1.1.11 Localization

MeasureAlgebraCompletion ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra $. \exists! \hat{\mu} : \tau(A) \to \overset{\infty}{\mathbb{R}}_{++}$.

$. \hat{\mu}_{|A} = \mu$ & LocalizableMeasureAlgebra$(\tau(A), \hat{\mu})$

Proof =

1 Define $\hat{\mu}(t) = \sup\{\mu(a) | a \in A, a \leq t\}$.

2 As A is order dense in $\tau(A)$, it holds that $\hat{\mu}(a) = 0 \iff a = 0$ for any $a \in \tau(A)$.

3 If $t : \mathbb{N} \to \tau(A)$ is disjoint then $\hat{\mu}\left(\bigvee_{n=1}^{\infty} t_n\right) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

3.1 Write $S = \{a \in A : \exists c : \mathbb{N} \to A \, . \, a = \lim_{n \to \infty} c_n \,\&\, c \leq t\}$.

3.2 Then there is $s = \sup S \in \tau(A)$.

3.3 We write $\hat{\mu}(s) = \sup_{c \leq t} \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sup_{c \leq t} \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \sup_{c \leq t_n} \mu(c) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

4 Obviously $(\tau(A), \hat{\mu})$ is semifinite and order-complete, and hence Localizable.

□

localization :: SemifiniteMeasureAlgebra → LocalizableMeasureAlgebra

localization $(A, \mu) = \Big(\tau(A), \tau(\mu)\Big)$:= MeasureAlgebraCompletion

LocalizationFiniteEmbedding ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra $. \iota_\tau(A^f) = \tau^f(A)$

Proof =

1 Assume $t \in \tau(A)$ such that $\hat{\mu}(t) < \infty$.

2 Note, $\hat{\mu}(t) = \sup_{a \leq t} \mu(a)$.

3 So we may select an increasing $a : \mathbb{N} \to A$ such that $\lim_{n \to \infty} \mu(a_n) = \hat{\mu}(t)$.

4 Then $b = \bigvee_{n=1}^{\infty} a_n \in A$ and $\hat{\mu}(b) = \mu(b) = \hat{\mu}(t)$.

5 So $\mu(t \setminus b) = 0$, and so $t = b \in A$ as clearly $b < t$.

□

1.1.12 Stone Spaces

LocallalizableMeasureAlgebraHasStrictlyLocalizableStoneSpace ::

:: $\forall (A, \mu)$: LocalizableMeasureAlgebra . StrictlyLocalizable($\mathsf{Z}\ A, \Sigma_\mu, \bar{\mu}$)

Proof =

1 We already proved that $\bar{\mu}$ is locally determined.

2 As (A, μ) is semifinite there is a partition of unity P consisting of finite elements.

3 Use Stone representation $S_A(P)$ to construct a corresponding set in $\mathsf{Z}\ A$.

4 Assume $E \in \Sigma_\mu$ such that $\bar{\mu}(E) > 0$.

5 By definition of Stone's Space there is a clopen set $F \in \mathsf{Z}\ A$ such that $E \triangle F$ is meager.

6 And there is a Stone representation $a \in A$ such that $F = S_A(a)$.

7 Then $\mu(a) = \nu(S_A(a)) = \nu(E) > 0$.

8 So, there exists $p \in P$ such that $ap \neq 0$.

9 Ths means that $\nu(E \cap S_A(p)) > 0$.

10 As E was arbitrary this means that $S_A(P)$ provides a strict localization for $\bar{\mu}$.

□

MeagerSetsAreNowhereDense ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall M \in \mathbf{MGR}(\mathsf{Z}\ A)$. NowhereDense($\mathsf{Z}\ A, M$)

Proof =

1 As it was shown A is (σ, ∞)-WeaklyDistributive boolean algebra.

2 And this is a property of (σ, ∞)-WeaklyDistributive boolean algebra.

□

StoneSpaceMeasurableExpression ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\exists U$: Clopen($\mathsf{Z}\ A$) . $\exists F$: NowhereDense($\mathsf{Z}\ A$) . $E = U \cap F$

Proof =

1 This is clear from the previous theorem.

□

StoneSpaceMeasureComputation ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\bar{\mu}(E) = \sup \left\{ \mu(U) \middle| U : \text{Clopen}(\mathsf{Z}\ A), U \subset E \right\}$

Proof =

1 This is clear from the previous theorem.

□

StoneSpaceCLDIsStrictlyLocalizable ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . StrictlyLocalizable($\mathsf{Z}\ A, \bar{\Sigma}_\mu, \bar{\bar{\mu}}$)

Proof =

...

□

$\texttt{StoneSpaceCLDZeroSets}$ ::

:: $\forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \mathcal{N}_{\bar{\bar{\mu}}} = \mathcal{N}_{\bar{\mu}}$

$\texttt{Proof}$ =

...

$\square$

$\texttt{FiniteStoneSpaceMeasureComputation}$ ::

:: $\forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \,.\, \forall E \in \Sigma_\mu \,.$

$. \ \bar{\mu}(E) = \inf \left\{ \mu(U) \middle| U : \texttt{Clopen}(\mathsf{Z}\ A), E \subset U \right\}$

$\texttt{Proof}$ =

1 This is clear from the previous theorem.

$\square$

1.1.13 Purely Infinite Elements

$$\texttt{purelyInfiniteElements} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \sigma\texttt{-Ideal}(A)$$

$$\texttt{purelyInfiniteElements}\,() = I_\infty(\mu := \{a \in A : \forall b \in A \,.\, b \leq a \;\&\; \mu(b) < \infty \Rightarrow b = 0\}$$

$$\texttt{semifiniteMeasure} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \frac{A}{I_\infty(\mu)} \xrightarrow{\infty} \mathbb{R}_+$$

$$\texttt{semifiniteMeasure}\,([a]) = \mu_{\mathrm{sf}} := \sup\{\mu(b) | b \in A : b \leq a \;\&\; \mu(b) < \infty\}$$

If $[a] = [b]$, then $a \triangle b \in I_\infty(\mu)$.

So μ_{sf} is well-defined.

$\texttt{SemifiniteMeasureIsMeasure}$ ::

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{SemifiniteMeasureAlgebra}\left(\frac{A}{I}, \mu_{\mathrm{sf}}\right)$$

Proof =

1 If $\mu_{\mathrm{sf}}[a] = 0$, then clearly $a \in I_\infty$.

2 Assume $[a] : \mathbb{N} \to A$ is disjoint.

2.1 Then $a_n a_m \in I_\infty$ if $n \neq m$.

2.2 Select increasing $b : \mathbb{N} \to A^f$ such that $b_n \leq \bigvee_{k=1}^{\infty} a_k$ and $\lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}}\left[\bigvee_{k=1}^{\infty} a_k\right] = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k]$.

2.3 By (2.1) we mat assert that ab_n is disjoint and then $\bigvee_{k=1}^{\infty} a_k b_n = b_n$ for any $n \in \mathbb{N}$.

2.4 So $\mu(b) = \sum_{k=1}^{\infty} \mu(a_k b_n)$.

2.5 By taking limits and using monotonic convergence theorem

$$\sum_{k=1}^{\infty} \mu_{\mathrm{sf}}[a_k] = \sum_{k=1}^{\infty} \lim_{n\to\infty} \mu(a_k b_n) = \lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k].$$

3 Clearly $\mu_{\mathrm{sf}}[a] < \mu(a)$.

3.1 If $\mu_{\mathrm{sf}}[a] = \infty$, then $a \notin I_\infty$.

3.2 So it is possible to select $b \in A$ such that $b \leq a$ and $0 < \mu(b) \leq a$.

3.3 $0 < \mu_{\mathrm{sf}}[b] \leq \mu(b) < \infty$.

3.4 This proves that $\left(\frac{A}{I}, \mu_{\mathrm{sf}}\right)$ is semifinite.

□

1.2 Topology

1.2.1 Subject

$\texttt{measureAlgebraAsTopologicalSpace} :: \texttt{MeasureAlgebra} \to \mathsf{TOP}$

$\texttt{measureAlgebraAsTopologicalSpace}\,((A,\mu)) = (A,\mu) :=$

$$:= \Big(A, \mathcal{W}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f \,.\, \Lambda b \in A^f \,.\, \Lambda c \in A \,.\, \mu(ac + ab)\big)\Big)$$

$\texttt{measureAlgebraAsUniformlSpace} :: \texttt{MeasureAlgebra} \to \mathsf{UNI}$

$\texttt{measureAlgebraAsUniformSpace}\,((A,\mu)) = (A,\mu) :=$

$$:= \Big(A, \mathcal{I}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f \,.\, \Lambda b \in A^f \,.\, \Lambda c \in A \,.\, \mu(ac \triangle ab)\big)\Big)$$

$\texttt{metricOfFrechetNikodym} :: \prod (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Metric}(A^f)$

$\texttt{metricOfFrechetNikodym}\,() = \rho_\mu := \Lambda a, b \in A^f \,.\, \mu(a \triangle b)$

$\texttt{BooleanOperationsAreUniformlyContinuous} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, (*), (\backslash), (\vee), (\wedge) \in \mathsf{UNI}(A \times A, A)$

$\texttt{Proof} =$

1 Let $\circ$ stay for any binary operation above.

2 Select $c, d \in A$.

3 Then $\mu\Big(a(c \circ d) + b\Big) \le \mu\Big(a(c \vee d) + b\Big) \le \mu(ac + d) + \mu(ad + b)$.

4 So μ is bounded by the sum of uniform functions and is uniformly continuous.

$\square$

$\texttt{FiniteElementsAreDense} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Dense}(A, A^f)$

$\texttt{Proof} =$

1 Select $c \in A$.

2 Then c has a base of neighborhoods of form $U = \{u \in A : \mu(au + ac) \le r\}$ with $a \in A^f, r \in \mathbb{R}_{++}$.

3 But then $ac \in U$ and $ac \in A^f$.

$\square$

$\texttt{FiniteMeasureAlgebraHasUniformlyContinuousMeasure} ::$

$\forall (A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

$\texttt{Proof} =$

This is pretty obvious as $\mu = \rho_\mu(0, a)$.

$\square$

FiniteMeasureAlgebraHasUniformlyContinuousMeasure ::

$\forall(A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

Proof =

This is pretty obvious as $\mu = \rho_\mu(0, a)$.

□

SemifinitMeasureAlgebraHasLowerSemicontinuousMeasure ::

$\forall(A,\mu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \mu \in \texttt{LowerSemicontinuous}(A, \overset{\infty}{\mathbb{R}}_{++})$

Proof =

1 Assume $a \in A$ and $\alpha \in \mathbb{R}_+$ such that $\mu(a) > \alpha$.

2 As A is semifinite there exists $b \leq a$ such that $\infty > \mu(b) > \alpha$.

3 Assume $c \in A$ is such that $\mu(b + cb) < \mu(b) - \alpha$.

4 Then $\mu(c) \geq \mu(cb) = \mu(b) - \mu(b(a \setminus c)) = \mu(b) - \mu(b + cb) > \alpha$.

□

MeasureAlgebraHasUniformlyContinuousFinitisedMeasure ::

$\forall(A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a \in A^f \,.\, (\Lambda c \in A \,.\, \mu(ac)) \in \mathsf{UNI}(A, \mathbb{R}_{++})$

Proof =

This is simmilar to the case of finite measure space.

□

finiteElementMetric :: $\prod A : \texttt{MeasureAlgebra} \,.\, A^f \to \texttt{Semimetric}(A)$

finiteElementMetric $(a) = \rho_a := \Lambda x, y \in A \,.\, \mu(ax + ay)$

MeasurAlgebraProductTopology ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall(A,\mu) : I \to \texttt{MeasureAlgebra} \,.\, \prod_{i \in I}(A,\mu) =_{\mathsf{TOP}} \left(\prod_{i\in I} A_i, \sum_{i \in I} \mu_i\right)$$

Proof =

...

□

1.2.2 Relations with an Order Structure

upwardDirectedFilter ::

$:: \prod(A,\mu) : \texttt{MeasureAlgebra} . \texttt{NonEmpty} \,\&\, \texttt{UpwardsDirected}(A) \to \texttt{CauchyFilerbase}(A)$

$\texttt{upwardDirectedFilter}\,(D) = \mathcal{F}(\uparrow D) := \Big\{\{c \in D : d \le c\}\Big| d \in D\Big\}$

1 Write $F_d = \{c \in D : d \le c\}$.

2 $\mathcal{F}(\uparrow D)$ is a filter.

2.1 As D is non empty, $\mathcal{F}(\uparrow D)$ is also non-empty.

2.2 $d \in F_d$, so $F_d \ne \emptyset$ and henceforth $\emptyset \notin \mathcal{F}(\uparrow D)$.

2.3 Assume $F_d, F_f \in \mathcal{F}(\uparrow D)$.

2.3.1 Then there is an element $g \in D$ such that $g \ge f \vee d$.

2.3.2 Note, that $F_g \subset F_d \cap F_f$ and $F_g \in \mathcal{F}(\uparrow D)$.

3 $\mathcal{F}(\uparrow D)$ is Cauchy.

3.1 Assume U is some measure connector for (A,μ).

3.2 then there is an element $a \in A^f$ and $r \in \mathbb{R}_{++}$ such that $\{(f,g) \in A \times A : \mu(af + ag) < r\} \subset U$.

3.3 The set $\{\mu(ad)|d \in D\}$ is bounded by $\mu(a)$, so supremum is attained.

3.4 So there is $f \in D$, so $\mu(ad) < \mu(af) + r/2$ for any $d \in D$.

3.5 Assume $g, h \in F_f$.

3.5 Then $\mu(ag + ah) \le \mu(ag \setminus af) + \mu(ah \setminus af) = \Big(\mu(ag) - \mu(af)\Big) + \Big(\mu(ah) - \mu(af)\Big) < r$.

3.6 Thus, $(g,h) \in U$ and $F_f \times F_f \subset U$.

□

UpwardsDirectedSup ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} . \forall D : \texttt{UpwardsDirected}(A) \to \texttt{CauchyFilerbase}(A) . \forall a \in A .$

$. \, a = \sup D \Rightarrow a = \lim \mathcal{F}(\uparrow D)$

Proof $=$

1 Assume $a = \sup D$.

2 Assume U is an uniformity fo (A,μ).

3 then there is an element $c \in A^f$ and $r \in \mathbb{R}_{++}$ such that $\{g \in A \times A : \mu(ca + cg) < r\} \subset U(a)$.

4 Consider set $M = \{\mu(cd)|d \in D\}$.

5 Then $\sup M = \mu(ca)$.

6 So there is $d \in D$ such that $\mu(ca + cd) < r$.

7 But $d \le f \le a$for any $f \in F_d$.

8 Thus $\mu(cf + cd) < r$ and $F_d \subset U(a)$.

9 Thus, $da = \lim \mathcal{F}(\uparrow D)$.

□

UpwardsDirectedLimit ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \;.\; \forall D : \texttt{NonEmpty} \;\&\; \texttt{UpwardsDirected}(A) \;.\; \forall a \in A \;.$

$. \; a = \sup D \Rightarrow a \in \operatorname{cl}_A D$

Proof =

...

□

UpwardsDirectedFilterLimit ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \;.\; \forall D : \texttt{NonEmpty} \;\&\; \texttt{UpwardsDirected}(A) \;.\; \forall a \in A \;.$

$. \; a = \lim \mathcal{F}(\uparrow D) \iff a = \sup D$

Proof =

1 $(\Rightarrow)$ $\quad a = \lim \mathcal{F}(\uparrow D)$.

1.1 Then for any connector U of (A,μ) There is some $F \in \mathcal{F}(\uparrow F)$ such that $F \subset U(a)$.

1.2 Assume $d \in D$.

1.3 Assume $d \not\leq a$.

1.4 Then there is $f \in A$ such that $f \leq d \setminus a$ and $0 < \mu(f) < \infty$.

1.5 Thus $\mu(fh + fa) \geq \mu(f)$ for every $h \in F_s$.

1.6 But $G \cap F_d \neq \emptyset$ for any $G \in \mathcal{F}(\uparrow D)$ so this contradicts (1.1).

□

lowerDirectedFilter ::

$:: \prod (A,\mu) : \texttt{MeasureAlgebra} \;.\; \texttt{NonEmpty} \;\&\; \texttt{LowerDirected}(A) \to \texttt{CauchyFilerbase}(A)$

$\texttt{loweDirectedFilter}\,(D) = \mathcal{F}(\uparrow D) := \left\{ \{c \in D : d \geq c\} \middle| d \in D \right\}$

LowerDirectedInf ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \;.\; \forall D : \texttt{NonEmpty} \;\&\; \texttt{LowerDirected}(A) \;.\; \forall a \in A \;.$

$. \; a = \inf D \Rightarrow a = \lim \mathcal{F}(\uparrow D)$

Proof =

By duality.

□

UpwardsDirectedLimit ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \;.\; \forall D : \texttt{NonEmpty} \;\&\; \texttt{LowerDirected}(A) \;.\; \forall a \in A \;.$

$. \; a = \inf D \Rightarrow a \in \operatorname{cl}_A D$

Proof =

By duality.

□

UpwardsDirectedFilterLimit ::

$:: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \;.\; \forall D : \texttt{NonEmpty} \;\&\; \texttt{LowerDirected}(A) \;.\; \forall a \in A \;.$

$. \; a = \lim \mathcal{F}(\uparrow D) \iff a = \inf D$

Proof =

By duality.

□

$\texttt{ClosedSetsAreOrderClosed} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall F : \texttt{Closed}(A) \,.\, \texttt{OrderClosed}(A,F)$

Proof $=$

Follows from previous theorems in this chapter.

□

$\texttt{DenseSetsAreOrderDense} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall \texttt{Dense}(A,D) \,.\, \texttt{OrderDense}(A,D) \,.$

Proof $=$

Follows from previous theorems in this chapter.

□

$\texttt{ClosedRays} :: \forall (A,\mu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \forall a \in A \,.\, \texttt{Closed}\Big(A, \{c \in A : c \le a\} \ \&\ \{c \in A : c \ge a\}\Big)$

Proof $=$

1 Let $F = \{c \in A : c \le a\}$.

2 Assume $d \in F^{\complement}$.

3 Then $d \setminus a \ne 0$.

4 As A is semifinite there is an $g \in A^f$ such that $g \le d \setminus a$ and $0 < \mu(g)$.

5 $\rho_g(d,f) \ge \mu(g)$ fo any $f \in F^{\complement}$.

6 And this means that $F^{\complement}$ and F is closed.

□

$\texttt{SupremumConvergence} :: \forall A : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \uparrow A \,.\, \forall s \in A \,.\, s = \sup_{n=1} a_n \Rightarrow s = \lim_{n=1} a_n$

Proof $=$

This is obvious now.

□

$\texttt{InfimumConvergence} :: \forall A : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \downarrow A \,.\, \forall s \in A \,.\, s = \inf_{n=1} a_n \Rightarrow s = \lim_{n=1} a_n$

Proof $=$

This is obvious now.

□

$\texttt{SummableIncrements} :: \prod A : \texttt{MeasureAlgebra} \,.\, ?(\mathbb{N} \to A)$

$$a : \texttt{SummableIncrements} \iff \forall n \in \mathbb{N} \,.\, \sum_{n=1}^{\infty} \mu(a_n + a_{n+1}) < \infty$$

`SummableIncrementsLimSupLimInfEq` ::

:: $\forall A :$ `MeasureAlgebra` . $\forall a :$ `SummableIncrements`(A) . $\inf_{n=1} \sup_{m=n} a_n = \sup_{n=1} \inf_{m=n} a_n$

`Proof` =

1 Let $\alpha_n = \mu(a_n + a_{n+1}), \beta_n = \sum_{m=n}^{\infty} \alpha_n$.

2 As a has summable increments this means $\beta \downarrow 0$.

3 Let $b_n = \sup_{m \geq n} a_m + a_{m+1} = \bigvee_{m=n}^{\infty} a_m + a_{m+1}$.

4 Then $\mu(b_n) \leq \sum_{m=n}^{\infty} \mu(c_m + c_{m+1}) = \beta_n$.

5 Assume $m \leq n$.

6 And also $a_m + a_n \leq \sup_{m \leq k \leq n} a_k + a_{k+1} \leq b_n$.

7 So $a_n \setminus b_n \leq a_m \leq a_n \vee b_n$.

8 Thus $a_n \setminus b_n \leq \inf_{k \geq m} a_k \leq \sup_{k \geq m} a_k \leq a_n \vee b_n$.

9 By taking limits in m one gets $a_n \setminus b_n \leq \inf_{m=1} \sup_{k=n} a_k \leq \sup_{m=1} \inf_{k=m} a_k \leq a_n \vee b_n$.

10 $a_n + \inf_{m=1} \sup_{k=m} a_k \leq b_n$.

11 $a_n + \sup_{m=1} \inf_{k=m} a_k \leq b_n$.

12 From (10) and (11) $\inf_{m=1} \sup_{k=m} a_k \setminus \sup_{m=1} \inf_{k=m} a_k \leq b_n$.

13 But $\lim_{n \to \infty} b_n = 0$.

14 So $\inf_{m=1} \sup_{k=m} a_k = \sup_{m=1} \inf_{k=m} a_k$.

□

`SummableIncrementsLim` ::

:: $\forall A :$ `MeasureAlgebra` . $\forall a :$ `SummableIncrements`(A) . $\forall x \in A$.

. $x = \lim_{n \to \infty} a_n \Rightarrow \inf_{n=1} \sup_{m=n} a_n = x = \sup_{n=1} \inf_{m=n} a_n$

`Proof` =

This follows from the previous proof.

□

1.2.3 Classification Theorems

SemifiniteIffHausdorff :: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{SemifiniteMeasureAlgebra}(A,\mu) \iff \texttt{T2}(A)$

Proof $=$

1 ($\Rightarrow$) assume that (A,μ) is semifinite.

1.1 Take $x, y \in A$ such that $x \neq y$.

1.2 Then $x + y \neq 0$ so there is $a \in A^f$ such that $\mu(a) > 0$ and $a < x + y$.

1.3 So $\rho_a(x,y) = \mu(a) > 0$.

1.4 And cells of form $\mathbb{B}_{\rho_a}(x, \mu(a)/2)$ and $\mathbb{B}_{\rho_a}(y, \mu(a)/2)$ produce the separation.

2 ($\Leftarrow$) assume that A is Hausdorff in the topology of (A,μ).

2.1 Assume $x \in A$ such that $\mu(x) = \infty$.

2.2 Then $x \neq 0$.

2.3 Assume $a \in A^f$.

2.4 If $xa = 0$ then $\rho_a(x,0) = 0$.

2.5 So, as A is Hausdorff there mustb some $a \in A^f$ such that $xa \neq 0$.

2.6 But this means that (A,μ) is semifinite.

□

SigmaFiniteIffMetrizable ::

:: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \sigma\texttt{-FiniteMeasureAlgebra}(A,\mu) \iff \texttt{Metrizable}(A)$

Proof $=$

1 ($\Rightarrow$) assume that (A,μ) is σ-finite.

1.1 Then there is a countable partition of unity a with finite elements.

1.2 define $\sigma : A^2 \to \mathbb{R}_{++}$ as $\sigma(x,y) = \displaystyle\sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{a_n}(x,y)}{\mu(a_n)}$.

1.3 Then σ is a metic for A .

1.4 So the topology of (A,μ) is metrizable.

2 ($\Leftarrow$) assume that (A,μ) is metrizable.

2.1 Let σ be an metrizing metric.

2.2 Then there exists a system of elements $k : \mathbb{N} \to \mathbb{N}, a : \displaystyle\prod_{n=1}^{\infty} \{1,\dots,k_n\} \to A^f$ and $\delta : \mathbb{N} \to \mathbb{R}_{++}$

such that $\rho_{a_{n,i}}(b,e)$ for any $1 \leq i \leq k_n$ imply that $\sigma(b,e) < 2^{-n}$ for any $b \in A$.

2.3 Then $e = \displaystyle\bigvee_{n=1}^{\infty} \bigvee_{i=1}^{k_n} a_{n,i}$.

2.4 So (A,μ) is σ-finite.

□

LocalizableIffComplete ::

:: $\forall(A,\mu)$: MeasureAlgebra . LocalizableMeasureAlgebra$(A,\mu) \iff$ T2 & Complete(A)

Proof =

1 $(\Rightarrow)$ Assume (A,μ) is localizable measure algebra.

1.2 Then A is Hausdorff as (A,μ) is semifinite.

1.3 Assume $\mathcal{F}$ is a Cauchy filter in A.

1.4 Take $a \in A^f$.

1.5 Then there is $d_a \le a$ and a cauchy sequence c_a subordinate to $\mathcal{F}$ such that $\lim_{n\to\infty} \rho_a(d_a, c_{a,n}) = 0$.

1.5.1 select a sequence $F_a : \mathbb{N} \to \mathcal{F}$ such that $\rho_a(x,y) \le 2^{-n}$ for $x,y \in F_{a,n}$ and $n \in \mathbb{N}$.

1.5.2 Then select a sequence $c_{a,n} \in \bigcap_{k=1}^{n} F_{a,k}$.

1.5.3 Then $\rho(c_{a,n}, c_{a,n+1}) \le 2^{-n}$.

1.5.4 Construct $d_a = \liminf ac_a$.

1.5.5 Then $\lim_{n\to\infty} \rho_a(d_a, c_{an}) = \lim_{n\to\infty} \mu(d_a + ac_a) = 0$.

1.6 Assume $a,b \in A^f$ are such that $a \le b$.

1.7 Then $d_a = ad_b$.

1.7.1 $F_{n,a} \cap F_{n,b} \neq \emptyset$.

1.7.2 So select $f \in F_{n,a} \cap F_{n,b}$.

1.7.3 Then $\rho_a(d_a,d_b) \le \rho_a(d_a,c_{a,n}) + \rho_a(c_{a,n},f) + \rho_a(f,c_{b,n}) + \rho_a(c_{b,n},d_b) \le$

$\le \rho_a(d_a,c_{a,n}) + 2^{-n} + 2^{-n} + \rho_a(c_{b,n},d_b) \to 0$ as $n \to \infty$.

1.8 Let $f = \bigvee_{a\in A^f} d_a$.

1.9 Then $\lim \mathcal{F} = f$.

1.9.1 $ad_a = af$ for any $a \in A^f$.

1.9.2 and there is a $\mathcal{F}$ subordinate Cauchy sequence c_a such that $\rho_a(f,c_a) = \rho_a(d_a,c_a) \to 0$.

1.9.3 Then there is $n \in \mathbb{N}$ such that $\rho_a(d_a,c_{a,n}) + 2^{-n} < \varepsilon$.

1.9.4 Take any $g \in F_{a,n}$.

1.9.5 But $\rho_a(f,g) \le \rho_a(f,c_{a,n}) + \rho_{c_{a,n}} \le \rho_a(d_a,c_{a,n}) + 2^{-n} < \varepsilon$.

1.9.6 This $F_{a,n} \subset \mathbb{B}_{\rho_a}(f,\varepsilon)$.

2 $(\Leftarrow)$ now Assume that A is Hausdorff and complete.

2.1 As A is Hausdorff (A,μ) must be semifinite.

2.2 As A is complete (A,μ) is order complete and hence localizable.

2.2.1 Think about order filters $\mathcal{F}(\uparrow D)$ and $\mathcal{F}(\downarrow D)$.

□

LessRelationIsClosed :: $\forall(A,\mu)$: SemifiniteMeasureAlgebra . Closed$\Big(A^2, \{(a,b) \in A^2 : a \le b\}\Big)$

Proof =

1 As (A,μ) is a semifinite measure algebra A must be Hausdorff.

2 So singleton $\{0\}$ is closed.

3 Then $\{(a,b) \in A^2 : a \le b\} = (\backslash)^{-1}\{0\}$ is closed.

□

1.2.4 Closed Subalgebras

ClosedSubalgebraTHM ::

:: $\forall (A, \mu)$: LocalizableMeasureAlgebra . $\forall B \subset_{\mathsf{RING}} A$. Closed$(A, B) \iff$ OrderClosed(A, B)

Proof =

1 ($\Rightarrow$) follows from the general theory.

2 ($\Leftarrow$) Assume now that B is order-closed.

2.1 Assume $g \in \mathrm{cl}_A B$.

2.2 Assume $a \in A^f$ and $n \in \mathbb{N}$.

2.3 Then there exists a sequence $c_a : \mathbb{N} \to B$ such that $\rho_a(c_{a,n}, g) < 2^{-n}$.

2.4 Note, $\sum_{n=1}^{\infty} \mu(ac_{a,n} + ac_{a,n+1}) \le \sum_{n=1}^{\infty} \mu(ac_{a,n} + ag) + \mu(ag + ac_{a,n+1}) < \sum_{n=1}^{\infty} 2^{-n} + 2^{-n-1} = \frac{3}{2}$.

2.5 So, sequence ac_a has summable increments .

2.6 Define $d_a = \liminf c_a$.

2.7 As ac_a has finite increments $\lim_{n\to\infty} \rho_a(c_{a,n}, d_n) = 0$.

2.8 Furthermore, $\rho_a(d_a, g) = 0$, so $ag = d_a$.

2.9 As B is order-closed $d_a \in B$ for each $a \in A^f$.

2.10 Set $d'_a = \inf\{d_b : b \in A^f, a \le b\} \in B$.

2.11 $d'_a a = \bigwedge_{a\le b} d_b a = \bigwedge_{a\le b} d_b b a = \bigwedge_{a\le b} g b a = g a$.

2.12 Let $D = \{d'_a | a \in A\}$.

2.13 Clearly D is upwards directed as $d'_a \vee d'_b = d'_{a\wedge b}$.

2.14 Then $\sup D = \{ad'_a | a \in A\} = \{ag | a \in A\} = g$ as (A, μ) is semifinite.

2.15 so $g \in B$ as B is order-closed.

2.16 Thus B is closed.

□

SubalgebraClosure :: $\forall (A, \mu)$: LocalizableMeasureAlgebra . $\forall B \subset_{\mathsf{RING}} A$. $\overline{B} = \tau(B)$

Proof =

1 Note that $\overline{B}$ is a subgroup of A.

2 Also it must be order-closed as $\overline{B}$ is closed.

3 Also $\tau(B)$ is an order-closed subalgebra, and hence a closed subalgebra.

4 So both objects can be realized as intersections of closed subalgebras containing B, and hence they are equal.

□

ClosedMeasureSubalgebra :: $\prod (A, \mu)$: MeasureAlgebra . Subalgebra(A)

B : ClosedMeasureSubalgebra $\iff B \subset_{\mathsf{MA}} A \iff$ Closed(A, B)

$\texttt{OrderClosedExtension}$ ::

:: $\forall (A, \mu) : \texttt{LocalizableMeasureAlgebra} \,.\, \forall B \subset_{\mathsf{MA}} A \,.\, \forall a \in A \,.\, \langle B \cup \{a\} \rangle_{\mathsf{BOOL}} \subset_{\mathsf{MA}} A$

$\texttt{Proof}$ =

This follows from order-closed subalgebra extension theorem for boolean algebras.

□

$\texttt{SigmaFiniteSigmaSubalgebraIsClosed}$ :: $\forall (X, \Sigma, \mu) : \sigma\texttt{-Finite} \,.\, \forall T \subset_\sigma \Sigma \,.\, \pi_\mu(T) \subset_{\mathsf{MA}} A_\mu$

$\texttt{Proof}$ =

1 As (X, Σ, μ) is σ-finite A_μ is also σ-finite.

2 So $A_m u$ is actually metrizable with a metric σ.

3 In a metric space set is closed iff it is sequence-closed.

4 Consider a sequence $a : \mathbb{N} \to \pi_\mu(T)$ with a limit x.

5 Then there is a sequence $E : \mathbb{N} \to T$ such that $a = [E]$.

7 Then $\limsup E = \liminf E \in T$, but also $[\limsup E] = x$.

8 Thus $x \in \pi_\mu(T)$.

□

$\texttt{SigmaFiniteSigmaSubalgebraIsClosed2}$ :: $\forall (X, \Sigma, \mu) : \sigma\texttt{-Finite} \,.\, \forall B \subset_{\mathsf{MA}} A_\mu \,.\, \pi_\mu^{-1}(B) \subset_\sigma A_\mu$

$\texttt{Proof}$ =

Inverse argument.

□

$\texttt{OrderClosedSetsAreClosedInLocalizableAlgebra}$ ::

:: $\forall (A, \mu) : \texttt{LocalizableMeasureAlgebra} \,.\, \forall C : \texttt{OrderClosed}(A) \,.\, \texttt{Closed}(A, C)$

$\texttt{Proof}$ =

1 Same proof as with closed algebras.

□

$\texttt{SubalgebraClosureIsSubalgebra}$ ::

:: $\forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall B \subset_{\mathsf{RING}} A \,.\, \overline{B} \subset_{\mathsf{RING}} A$

$\texttt{Proof}$ =

1 B is a topological subgroup of A.

2 So by general theory of topological groups $\overline{B}$ is a subgroup of A again.

3 So $\overline{B}$ is closed under operation $(+)$.

4 Also $\overline{B}$ is closed and hence order-closed.

5 But then it is closed under operations $(\vee), (\wedge)$.

6 And being closed under operations $(\vee), (\wedge), (+)$ is enough to be a boolean algebra.

□

1.2.5 Metric Space of Finite Elements

BooleanOperationsAreUniformlyContinuous ::

:: $\forall (A, \mu)$: MeasureAlgebra . $(*), (\setminus), (\vee), (\wedge) \in \mathsf{UNI}(A^f \times A^f, A^f)$

Proof =

This is obvious.

$\square$

MeasureIs1Lip ::

:: $\forall (A, \mu)$: MeasureAlgebra . $\mu_{|A^f} \in$ 1-Lip(A^f)

Proof =

This is obvious.

$\square$

FiniteElementsAreComplete ::

:: $\forall (A, \mu)$: MeasureAlgebra . Complete(A^f)

Proof =

1 Assume a is a cauchy sequence in A^f.

2 without loss of generality we may assume that a has summable differences .

2.1 Just select a subsequence.

3 Define $x = \liminf a \in A$.

4 Then $\lim_{n \to \infty} a_n = x$.

5 So, there is some $n \in \mathbb{N}$ such that $\mu(x \setminus a_n) < \infty$.

6 Thus $\mu(x) < \infty$ and $x \in A^f$.

$\square$

1.2.6 Relation with Convergence In Measure

$$\texttt{indicatorFunctionRepresentation} :: \prod (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, A_\mu \to \mathbf{L}^0(X,\Sigma,\mu)$$

$$\texttt{indicatorFunctionRepresentation}\,(a) = \chi_a := [\chi_E] \quad \texttt{where} \quad a = [E]$$

1 This is well defined.
2 Assume that $a = [E] = [F]$ for some $E, F \in \Sigma$.
3 Then $\mu(E \triangle F) = 0$.
4 Hence, $\chi_E =_\mu \chi_F$ and $[\chi_E] = [\chi_F]$.
□

$\texttt{IndicatorFunctionRepresentationIsHomeo}$ ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \texttt{Homeomorphism}\Big(A_\mu, \chi_{A_\mu}, \chi_\bullet\Big)$$

Proof =
1 Here we always assume that $\mathbf{L}^0(X,\Sigma,\mu)$ is equiped with a convergenve in measure topology.
2 Clearly $\chi_\bullet$ is injective.
2.1 Assume $\chi_a = \chi_b$.
2.2 Then there is common representative $E \in \Sigma$ such that $a = [E] = b$.
3 Also $\chi_\bullet$ is trivially sirjective.
4 $\chi_\bullet$ is homeomorphism.
4.1 This can be seen by direct corespondence between semimetrics ρ_a
4.2 and $\rho_E = \inf_{t \in \mathbb{R}_{++}} t + \mu\Big\{x \in E : |f(x) - g(x)| > t\Big\}$.
4.3 where corespondence is between finite $a \in A^f_\mu$ and $E \in \Sigma^f$ such that $a = [E]$.
□

$\texttt{FiniteIndicatorEmbeddingL1Isometri}$ ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \texttt{Isometry}\Big(A_\mu, \chi_{A_\mu}, \chi_\bullet\Big)$$

Proof =
This is obvious as difference of indicators are measure of difference of sets.
□

1.2.7 Localization

LocalizationIsCompletion :: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . Completion$(A, \tau(A), \iota_\tau)$

Proof $=$

1 $\iota_\tau(A)$ is order dense in $\tau(A)$.

2 So its order-closure is $\tau(A)$.

3 $\tau(A)$ is localizable and $\iota_\tau(A)$ is a subalgebra, so the closure of $\iota_\tau(A)$ is equal to the order closure.

$\square$

1.2.8 Metric Space of Pobability Subalgebras

$\texttt{metricSpaceOfProbabilitySubalgebra} :: \texttt{ProbabilityAlgebra} \to \texttt{CompleteMetricSpace}$

$\texttt{metricSpaceOfProbabilitySubalgebra}\,(A,\pi) = \mathsf{FB}(A,\pi) :=$

$$:= \left(\texttt{Closed}\ \&\ \texttt{Subring}(A), \Lambda B, C \subset_{\mathsf{MA}} A\ .\ \max\left(\sup_{b\in B}\inf_{c\in C}\rho_\pi(b,c), \sup_{c\in C}\inf_{b\in B}\rho_\pi(b,c)\right)\right)$$

1 Note, that indicator representation maps such closed subalgebras into closed uniformly integrable subsets of $\mathbf{L}^1(\mathsf{Z}\ A, \Sigma_\pi, \bar{\pi})$.

2 Then there is a natural isometric inclusion $\chi\Big(\mathbf{FB}(A,\pi)\Big) \subset \mathbf{F}\Big(\mathbf{L}^1(\mathsf{Z}\ A, \Sigma_\pi, \bar{\pi})\Big)$, which can be equiped with a Hausdorff metric d.

3 Now consider an boolean binary operation $\circ$.

4 Assume $C : \mathbb{N} \to \mathsf{FB}(A,\pi)$ is a converging sequence with a limit L.

5 Then clearly $e, 0 \in L$ as $e, 0 \in C_n$ for every $n \in \mathbb{N}$.

6 Now assume $x, y \in L$.

7 Then there exists a sequences $u, v : \prod_{n=1}^{\infty} C_n$ such that $x = \lim_{n\to\infty} u_n$ and $y = \lim_{n\to\infty} v_n$.

8 But Then $u_n \circ v_n \in C_n$ and $x \circ y = \lim_{n\to\infty} u_n \circ \lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n \circ v_n \in L$.

9 So $L \in \mathsf{FB}(A,\pi)$.

10 As C and L were arbitraty $\chi\Big(\mathbf{FB}(A,\pi)\Big)$ must be a closed subset of $\mathsf{F}\Big(\mathbf{L}^1(\mathsf{Z}\ A, \Sigma_\pi, \bar{\pi})\Big)$.

But $\mathsf{F}\Big(\mathbf{L}^1(\mathsf{Z}\ A, \Sigma_\pi, \bar{\pi})\Big)$ as complete $\mathbf{L}^1(\mathsf{Z}\ A, \Sigma_\pi, \bar{\pi})$ is complete, so $\mathsf{FB}(A,\pi)$ is complete.

□

1.2.9 Topology of the Lebesgue Algebra

$\texttt{algebraOfLebesgue} :: \sigma\texttt{-Finite}$

$\texttt{algebraOfLebesgue}\,() = \Lambda := \mathcal{B}(\mathbb{R})_\lambda$

$\texttt{LebesgueAlgebraIsSeparable} :: \texttt{Separable}(\Lambda)$

$\texttt{Proof} \ =$

1 consider $\mathcal{A}$ to be an algebra generated by open intervals with rational endpoints.

2 Then $|\mathcal{A}| = \aleph_0$ as $\mathbb{Q}$ are countable.

3 As Λ is localizable $\Lambda = \pi_\lambda\Big(\mathcal{B}(\mathbb{R})\Big) = \pi_\lambda\Big(\tau_{\mathcal{B}(\mathbb{R})}(\mathcal{A})\Big) = \tau\Big(\pi_\lambda(\mathcal{A})\Big) = \overline{\pi_\lambda(\mathcal{A})}$.

4 So Λ is separable.

$\square$

1.3 Category

1.3.1 Measure Algebra Functor

NullIdealPreservingMapToHomomorphism ::

:: $\forall (X,\Sigma,\mu),(Y,T,\nu) \in \mathsf{MEAS}\ .\ \forall D : \mathtt{Thick}(X,\Sigma,\mu)\ .\ \forall f : D \to Y\ .$

$.\ \forall \aleph : \forall E \in T\ .\ f^{-1}(E) \in (\hat{\Sigma}|D)\ .\ \forall \beth : \forall E \in \mathcal{N}_\nu \cap T\ f^{-1}(E) \in \mathcal{N}_\mu\ .$

$.\ \exists \phi \in \mathsf{MA}\ \&\ \mathtt{SequentiallyOrderClosed}(A_\nu, A_m u)\ .\ \forall E \in T\ .\ \forall F \in \Sigma\ .$

$.\ \phi[E] = [F] \iff f^{-1}(F) \triangle (E \cap D) \in \mathcal{N}_\mu$

Proof =

1 Define $\phi[E] = \left[f^{-1}(E)\right]$.

2 ϕ is well defined.

2.1 Assume $E, F \in T$ are such that $[E] = [F]$.

2.2 Then $\nu(E \triangle F) = 0$.

2.3 So $\mu\Big(f^{-1}(E \triangle F)\Big) = 0$.

2.4 Write $\phi[E] = \left[f^{-1}(E)\right] = \left[f^{-1}(E \triangle F \triangle F)\right] = \left[f^{-1}(F)\right] + \left[f^{-1}(E \triangle F)\right] = \left[f^{-1}(F)\right] = \phi[F]$.

3 ϕ is a boolean morphism.

3.1 $\phi(1) = \left[f^{-1}(X)\right] = \left[f^{-1}(Y)\right] = 1$.

3.2 The rest is obvious from properties of $f^{-1} : 2^Y \to 2^D$.

3.3 As measures are σ-additive the σ-continuity follows by simmilar arguments.

4 The final property is also obvious by construction.

□

measureAlgebraFunctor :: $\mathtt{Contravariant}(\mathsf{BOR}_0, \mathtt{MeasureAlgebra})$

measureAlgebraFunctor $((X,\Sigma,\mu)) = \mathsf{MA}(X,\Sigma,\mu) := (A_\mu, \hat{\mu})$

measureAlgebraFunctor $(X,Y,f) = \mathsf{MA}_{X,Y}(f) :=$ NullIdealPreservingMapToHomomorphism

1.3.2 Stone Space Functor

$\texttt{spaceOfStoneFunctor} :: \texttt{Contravariant}(\texttt{MeasureAlgebra}, \mathsf{BOR}_0)$

$\texttt{spaceOfStoneFunctor}\,((A,\mu)) = \mathsf{Z}(A,\mu)) := (\mathsf{Z}A, \Sigma_\mu, \bar{\mu})$

$\texttt{spaceOfStoneFunctor}\,(X,Y,f) = \mathsf{Z}_{X,Y}(f) := \mathsf{Z}_{X,Y}(f)$

1 Assume E is nowhere dense in $\mathsf{Z}X$.

1.2 Then $\left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(E)$ is nowhere dense in $\mathsf{Z}Y$.

1.3 But this means that $\left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(E)$ is meager and has measure zero.

2 Now assume E has $\bar{\mu}$-measure zero.

2.1 Then E must be meager.

2.2 So write $E = \bigcap_{n=1}^{\infty} N_n$, where each N is nowhere dense.

2.3 By elementary set theory $\left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(E) = \bigcup_{n=1}^{\infty} \left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(N_n)$.

2.3 As each $\left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(N_n)$ has measure 0, $\left(\mathsf{Z}_{X,Y}(f)\right)^{-1}(E)$ also has measure 0.

□

1.3.3 Order Continuous Homomorphism

OrderContinuousByCodomain ::

:: $\forall (A, \mu) \in \mathsf{MA} \,.\, \forall (B, \nu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \forall \phi \in \mathsf{MA}\Big((A, \mu), (B, \nu)\Big) \,.$

$. \, \phi \in \mathsf{TOP}(A, B) \Rightarrow \texttt{OrderContinuous}(A, B, \phi)$

Proof =

1 Assume D is downwards directed subset of A such that $\inf D = 0$.

2 Then $0 \in \overline{D}$.

3 As ϕ is continuous $0 \in \overline{\phi(D)}$.

4 $\inf \phi(D) = 0$.

4.1 Assume $\inf \phi(D) = b > 0$.

4.2 As ν is semifinite, where is $c \in B^f$ such that $c \leq b$ and $\nu(b) > 0$.

4.3 Then $\rho_c(\phi(a), 0) = \nu(\phi(a)c) = \nu(c) > 0$ for any $a \in D$.

4.4 So $0 \notin \overline{\phi(D)}$, a contradiction!

5 Then ϕ must be order-continuous.

□

ContinuousByDomain ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall (B,\nu) \in \mathsf{MA}$. $\forall \phi \in \mathsf{MA}\Big((A,\mu),(B,\nu)\Big)$.

. OrderContinuous$(A,B,\phi) \Rightarrow \phi \in \mathsf{TOP}(A,B)$

Proof =

1 It is enough to prove that ϕ is continuous at zero.

2 Assume $b \in B^f$ and $\varepsilon \in \mathbb{R}_{++}$.

2.1 Assume that for any $a \in A^f$ and $\delta \in \mathbb{R}_{++}$ where is some $c \in A$ such that $\rho_a(c,0) < \delta$ but $\rho_b(\phi(c),0) \geq \varepsilon$.

2.1.1 Then it is possible to construct a system of elements $c : A^f \times \mathbb{N} \to A$
such that $\rho_a(c_{a,n},0) < 2^{-n}$ and $\rho_b(\phi(c_{a,n}),0) \geq \varepsilon$.

2.1.2 Set $d_a = \liminf c_a$.

2.1.3 Then $\rho_a(d_a,0) = 0$.

2.1.4 Thus, $d_a a = 0$.

2.1.5 As ϕ is order continuous $\phi(d_a) = \limsup \phi(c_a)$.

2.1.6 So, $\rho_b(\phi(d_a),0) \geq \varepsilon$.

2.1.7 This implies that $\rho_b(\phi(\bar{a}),0) \geq \varepsilon$.

2.1.8 Now consider set $D = \{\bar{a} | a \in A^f\}$.

2.1.8.1 Then D is downwards directed.

2.1.8.1.1 If $c,d \in A^f$ then $c \vee d \in A^f$ also.

2.1.8.1.2 So by De Muavre law if $\bar{c},\bar{d} \in D$, then $\bar{a} \wedge \bar{b} = \overline{a \vee b} \in D$.

2.1.8.2 As μ is semifinite $\inf D = 0$.

2.1.8.2.1 There is dense subset consisting of elements of A^f.

2.1.9 So $0 \in \overline{D}$.

2.1.10 But (2.1.9) is in contradiction with (2.1.7)!

2.2 So we showed that there is always some δ and $a \in A^f$ such that $\rho_b(\phi(c),0) < \varepsilon$ for any $c \in \mathbb{B}_a(0,\delta)$.

3 But as b and ε were arbitrary, the homomorphism ϕ must be continuous.

□

ContinuoutyEquivalence ::

:: $\forall (A,\mu),(B,\nu)$: SemifiniteMeasureAlgebra . $\forall \phi \in \mathsf{MA}\Big((A,\mu),(B,\nu)\Big)$.

. OrderContinuous$(A,B,\phi) \iff \phi \in \mathsf{TOP}(A,B)$

Proof =

Combine two previous results.

□

UniformEquivalencse ::

:: $\forall A \in \mathsf{BOOL}$. $\forall \mu,\nu$: SemifiniteMeasureAlgebra(A) . $\mathcal{U}_\nu = \mathcal{U}_\mu$

Proof =

1 Identity mapping is always order-continuous.

2 But by previous theorem it must be a homeomorphism.

3 A homomorphism whis is also a homeomorphism must be a unimorphism.

□

1.3.4 Measure Preserving Homomorphism

$\texttt{MeasurePreservingHomomorphism} :: \prod(A,\mu),(B,\nu) \in \mathsf{MA} \,.\, ?\mathsf{BOOL}(A,B)$

$\phi : \texttt{MeasurePreservingHomomorphism} \iff \forall a \in A \,.\, \mu(a) = \nu\Big(\phi(a)\Big)$

$\texttt{measurePreservingMeasureAlgebraCategory} :: \mathsf{LSCAT}$

$\texttt{measurePreservingMeasureAlgebraCategory}\,() = \mathsf{MA}_{\#} := \Big(\mathsf{MA}, \texttt{MeasurePreservingHomomorphism}, \circ, \mathrm{id}\Big)$

$\texttt{MPHIsInjective} :: \forall (A,\mu),(B,\nu) \in \mathsf{MA} \,.\, \forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big) \,.\, \forall \phi \in \texttt{Injective}(A,B)$

$\texttt{Proof} =$

1 If ϕ is not injective then it has nontrivial kernel.

2 Select $a \in \ker \phi$ such that $a \neq 0$.

3 Then $\mu(a) > 0$ but $\nu(\phi(a)) = \nu(0) = 0$, a contradiciton!

$\square$

$\texttt{MPHFiniteness} ::$

$:: \forall (A,\mu),(B,\nu) \in \mathsf{MA} \,.\, \forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big) \,.$

$. \, \texttt{FiniteMeasureAlgebra}(A,\mu) \iff \texttt{FiniteMeasureAlgebra}(B,\nu)$

$\texttt{Proof} =$

1 For any boolean homomorphism $\phi(e_A) = e_B$.

2 So finiteness follows by measure preservation.

$\square$

$\texttt{FiniteMPHIsContinuous} ::$

$:: \forall (A,\mu),(B,\nu) : \texttt{FiniteMeasureAlgebra} \,.\, \forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big) \,.\, \phi \in \mathsf{TOP}(A,B)$

$\texttt{Proof} =$

ϕ is an isometry with respect to natural metrics ρ_μ and ρ_μ.

$\square$

$\texttt{FiniteMPHIsOrderContinuous} ::$

$:: \forall (A,\mu),(B,\nu) : \texttt{FiniteMeasureAlgebra} \,.\, \forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big) \,.$

$. \, \texttt{OrderContinuous}(A,B,\phi)$

$\texttt{Proof} =$

This follows from the previous chapter and previous theorem.

$\square$

SigmaFiniteMPH1 :: $\forall(A,\mu)$: SemifiniteMeasureAlgebra . $\forall(B,\nu)$: σ-FiniteMeasureAlgebra .

. $\forall\phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$. σ-Finite(A,μ)

Proof =

1 As μ is semifenit there is a partition of unity of finite elements D.

2 $|\phi(D)| = |D|$ as ϕ is injective.

3 $\phi(D)$ is disjoint.

3 As ν is σ-finite $\phi(D)$ can be embeded into a countable partition of unity, so $|D| \le \aleph_0$.

4 This means that μ is σ-finite.

□

SigmaFiniteMPH2 :: $\forall(A,\mu)$: σ-FiniteMeasureAlgebra . $\forall(B,\nu) \in \mathsf{MA}$.

. $\forall\phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$ & σ-Continuous(A,B) . σ-FiniteMeasureAlgebra(B,ν)

Proof =

1 There is countable partition of unity P consisting of finite measure elements in A.

2 Then $\phi(P)$ is a countable disjoint subset of B consisting of finite measure elements.

3 But $\sup\phi(P) = \phi(\sup P) = \phi(e_A) = e_B$.

4 Thus, $\phi(P)$ is also a countable partition of unity in (B,ν) consisting of finite measure elements.

5 So (B,ν) is σ-finite.

□

SemifiniteMPH :: $\forall(A,\mu)$: SemifiniteMeasureAlgebra . $\forall(B,\nu) \in \mathsf{MA}$.

. $\forall\phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$ & OrderContinuous(A,B) . SemifiniteMeasureAlgebra(B,ν)

Proof =

1 There is partition of unity P consisting of finite measure elements in A.

2 Then $\phi(P)$ is a disjoint subset of B consisting of finite measure elements.

3 But $\sup\phi(P) = \phi(\sup P) = \phi(e_A) = e_B$.

4 Thus, $\phi(P)$ is also a partition of unity in (B,ν) consisting of finite measure elements.

5 So (B,ν) is semifinite.

□

AtomlessMPH :: $\forall(A,\mu)$: SemifiniteMeasureAlgebra & Atomless . $\forall(B,\nu) \in \mathsf{MA}$.

. $\forall\phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$ & OrderContinuous(A,B) . Atomless(B)

Proof =

1 There is partition of unity P consisting of finite measure elements in A.

2 Then $\phi(P)$ is a disjoint subset of B consisting of finite measure elements.

3 But $\sup\phi(P) = \phi(\sup P) = \phi(e_A) = e_B$.

4 Thus, $\phi(P)$ is also a partition of unity in (B,ν) consisting of finite measure elements.

5 Now assume b is an atom in B.

5.1 Then there is an element $a \in P$ such that $\phi(a)b \neq 0$.

5.2 But as b is an atom this means that $b = \phi(a)$.

5.3 A is atomless so there are some c such that $0 < c < a$.

5.4 So $0 < \phi(c) < \phi(a) = b$.

5.5 But this means that b is not an atom, a contradiction!!

□

PurelyAtomicMPH ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall (B,\mu)$: PurelyAtomicMeasureAlgebra .

. $\forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$. PurelyAtomic(A)

Proof =

1 Assume $a \in A$ is such that $a \neq 0$.

1.1 Assume that a do not contain any atoms.

1.2 As A is semifinite there is a $c \in A^f$ such that $0 < c \leq a$.

1.3 Then there exist a sequnce of partitions $d : \mathbb{B}^* \to A^f$ such that

such that $c = \bigvee_{t \in \mathbb{B}^n}^{2^n} d_t$ and $d_t \neq 0$ for any $t \in \mathbb{B}^*$and $d_t d_s = d_s$ iff $t \sqsubset s$ and $d_t d_s = 0$ iff $|s| = |t|$ and $t \neq s$

and $\mu(d_t) \to 0$ as $|t| \to \infty$.

1.3 Then $\phi(d)$ has all same properties .

1.4 Moreover $\nu(\phi(c)) = \nu\phi\left(\bigvee_{t \in \mathbb{B}^n}^{2^n} d_t\right) = \sum_{t \in \mathbb{B}^n}^{2^n} \nu(\phi(d_t))$.

1.5 So $\phi(c) = \bigvee_{t \in \mathbb{B}^n}^{2^n} \phi(d_t)$ as $\phi(d_t)$ must be disjoint.

1.6 So $\phi(c)$ can't contain atoms.

1.7 But B is purely atomic, so we have a contradiction!

□

GeneratedSigmaSubalgebraImage ::

:: $\forall (A,\mu),(B,\nu)$: FiniteMeasureAlgebra . $\forall \phi \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$. $\forall C \subset A$. $\phi\langle C\rangle_\sigma = \langle \phi(C)\rangle_\sigma$

Proof =

This follows from previous theorems about finite measure algebras.

□

MeasurePreservingMeasureAlgebra ::

:: $\forall (X,\Sigma,\mu),(Y,T,\nu) \in \mathsf{MEAS}$. $\forall f \in \mathsf{MEAS}^{\#}\Big((X,\Sigma,\mu),(Y,T,\nu)\Big)$. $\mathsf{MA}_{X,Y}(f) \in \mathsf{MA}_{\#}(T_\nu, \Sigma_\mu)$

Proof =

This is obvious.

□

MeasurePreservingZeroSpace ::

:: $\forall (A\mu),(B,\nu) \in$ MeasureAlgebra . $\forall f \in \mathsf{MA}_{\#}\Big((A,\mu),(B,\nu)\Big)$.

. $\mathsf{Z}_{A,B}(f) \in \mathsf{MEAS}^{\#}\Big((\mathsf{Z}\ B, \Sigma_\nu, \bar\nu), (\mathsf{Z}\ A, \Sigma_\mu, \bar\mu)\Big)$

Proof =

This is obvious.

□

MeasurePresevingHomomorphismExtensionFromSubalgebra ::

$:: \forall (A,\mu),(B,\nu) : \texttt{FiniteMeasureAlgebra} \;.\; \forall C \subset_{\mathsf{BOOL}} A \;.\; \forall \aleph : \texttt{Dense}(A,C) \;.$

$. \;\forall \phi \in \texttt{MeasureAlgebra}_{\#}(C,B) \;.\; \exists \Phi \in \texttt{MeasureAlgebra}_{\#}(A,B) \;.\; \Phi_{|C} = \phi$

Proof =

1 obviously ϕ is an isometry.

2 So there exists a uniqui iometry extrnsion Φ of ϕ by $\aleph$.

3 Φ is a homomorphism.

3.1 This holds as boolean operations are continuous and ϕ is also continuous.

3.2 Let $\circ$ be some binary boolean operation and $u, v \in A$.

3.3 Then there are sequences $x, y : \mathbb{N} \to C$ such that $u = \lim x$ and $v = \lim y$.

3.4 $\Phi(v) \circ \Phi(u) = \lim_{n\to\infty} \phi(x_n) \circ \phi(y_n) = \lim_{n\to\infty} \phi(x_n \circ y_n) = \Phi(v \circ u)$.

4 Φ is measure preserving.

4.1 Assume $a \in A$.

4.2 just note $\nu(\Phi(a)) = \rho_\nu(\Phi(a), 0) = \rho_\nu(\Phi(a), \Phi(0)) = \rho_\mu(a, 0) = \mu(a)$.

□

MeasurePresevingHomomorphismExtensionFromSubset ::

$:: \forall (A,\mu),(B,\nu) : \texttt{FiniteMeasureAlgebra} \;.\; \forall C \subset A \;.\; \forall f : C \to A \;.$

$. \;\forall \aleph : \forall c : \mathbb{N} \to C \;.\; \nu(\inf f(c)) = \mu(\inf c) \;.\; \exists \Phi \in \mathsf{MA}_{\#}\Big(\langle C \rangle_{\mathsf{MA}}, B\Big) \;.\; \Phi_{|C} = f$

Proof =

...

□

1.3.5 Example

Let $A = 2^{\mathbb{N}}$ with $\mu = \#$.
The elements of A can be identified with sequences $\mathbb{N} \to \mathsf{BOOL}$.
Let $\phi(a)$ be defined as right shift padded by 0 if a is finite.
Let $\phi(a)$ be defined as right shift padded by 1 if a is cofinite.
Otherwis let $\phi(a) = a$.
Then as finite sets form an and $0 + 0 = 0$ and $0 \wedge t = 0$ it is clear ϕ that preserves their structure.
Also as cofinite sets are their complement and $1 + 1 = 0$ and $1 \wedge t = t$
it is clear that ϕ is an algebra morphism.
Clearly ϕ preserves cardinality.
On the other hand consider a sequence $f_n = \{2, \ldots, 2n\}$.

Then $\displaystyle\bigvee_{n=1}^{\infty} f_n = 2\mathbb{N}$.

But $\displaystyle 2\mathbb{N} = \phi(2\mathbb{N}) = \phi\left(\bigvee_{n=1}^{\infty} f_n\right) \neq \bigvee_{n}^{\infty} = 1\phi(f_n) = 2\mathbb{N} + 1$.

□

1.3.6 Tensor Products

$$\texttt{measureAlgebraTensorProduct} :: \prod I : \texttt{Finite} \,.\, (I \to \mathsf{MA}) \to \mathsf{MA}$$

$$\texttt{measureAlgebraTensorProduct}\,(A, \mu) = \left(\bigotimes_{i \in I} A_i, \prod_{i \in I} \mu_i\right) := \mathsf{MA}\left(\bigotimes_{i \in \mathcal{I}} \mathsf{Z}(A_i, \mu_i)\right)$$

$$\texttt{measureAlgebraTensorProductEmbedding} ::$$

$$:: \prod I : \texttt{Finite} \,.\, \prod (A, \mu) : I \to \mathsf{MA} \,.\, \prod_{i \in I} \texttt{OrderContinuous}\left(A_i, \bigotimes_{j \in I} A_j\right)$$

$$\texttt{measureAlgebraTensorProductEmbedding}\,() = \iota_i := \mathsf{MA}_{\mathsf{Z}(A_i,\mu_i), \bigotimes_{i \in I} \mathsf{Z}(A_i,\mu_i)}(\pi_i)$$

1 ι_i is well defined.

1.1 Assume $E \in \sigma_{\mu_i}$ is such that $\bar{\mu}_i(E) = 0$.

1.2 Then $\bigotimes_{j \in I} \bar{\mu}_j\left(\pi_i^{-1}(E)\right) = \bigotimes_{j \in I} \bar{\mu}_j \prod_{k \in I}\left(\widehat{\mathsf{Z}A}_i(E)\right)_k = \sup\left\{\prod_{j \in I} \bar{\mu}_j(F) \middle| F : \prod_{j \in I} \Sigma_{\mu_i}, F_i \subset E\right\} = 0.$

1.3 So $\pi_i \in \mathsf{BOR}_0\left(\bigotimes_{j \in I} \mathsf{Z}(A_j, \mu_j), \mathsf{Z}(A_i, \mu_i)\right)$.

2 ι_i is order-continuous.

2.1 Assume $D \subset A_i$ is downwards closed with $\inf D = 0$.

2.2 Also assume $0 \neq u = \inf \iota_i(D)$.

2.3 Then $\prod_{i \in I} \mu_i(u) > 0$.

2.4 By definition there is $E : \prod_{i \in I} \Sigma^f_{\bar{\mu}_i}$ and $F \in \bigotimes_{j \in I} \Sigma_{\bar{\mu}_j}$ such that $u = [F]$ and $\bigotimes_{i \in I} \bar{\mu}_i\left(F \cap \prod_{j \in I} E_j\right) > 0$.

2.5 But $\inf_{d \in D} d[E_i] = 0$, so $\inf_{d \in D} \mu_i\left(d[E_i]\right) = 0$.

2.6 So there exists $d \in D$ such that $\mu_i\left(d[E_i]\right) \prod_{j \in \{i\}^\complement} \bar{\mu}_j(E_j) < \bigotimes_{i \in I} \bar{\mu}_i\left(F \cap \prod_{j \in I} E_j\right)$.

2.7 Also there is $G \in \Sigma$ such that $d = [G]$.

2.8 Thus, $\bigotimes_{i \in I} \bar{\mu}_i\left(F \setminus \prod_{j \in I}\left(\widehat{E}_i(G)\right)_j\right) = 0$.

2.9 Then $\bigotimes_{i \in I} \bar{\mu}_i\left(F \cap \prod_{j \in I} E_j\right) \leq \bigotimes_{j \in J} \bar{\mu}_i\left(\prod_{j \in I}\left(\widehat{E}_i(G \cap E_i)\right)_j\right) = \bar{\mu}_i(G \cap E_i) \prod_{j \in \{i\}^\complement} \mu_j(E_j) =$
$\mu_i\left(d[E_i]\right) \prod_{j \in \{i\}^\complement} \mu_j(E_j)$.

2.10 A contradiction with (2.5)!

$\square$

$\texttt{measureAlgebraTensorRepresentation} ::$

$$:: \prod I : \texttt{Finite} \,.\, \prod (A, \mu) : I \to \mathsf{MA} \,.\, \mathsf{BOOL}\left(\bigotimes_{i \in I} A_i, \bigotimes_{i \in I} (A_i, \mu_i)\right)$$

$$\texttt{measureAlgebraTensorRepresentation}\,() = \Psi_{A,\mu} := \texttt{tensor}\left(\Lambda[E] \in \prod_{i \in I} A_i \,.\, \left[\prod_{i \in I} E_i\right]\right)$$

$\texttt{TensorRepresentationsAreDense} :: \forall I : \texttt{Finite} \,.\, \forall (A, \mu) : I \to \mathsf{MA} \,.\, \texttt{Dense}\left(\bigotimes_{i \in i} (A, \mu_i), \Psi_{A,\mu}\left(\bigotimes_{i \in I} A_i\right)\right)$

$\texttt{Proof} =$

1 Assume $s \in \bigotimes_{i \in I} (A_i, \mu_i)$ and $f \in \left(\bigotimes_{i \in I} (A_i, \mu_i)\right)^f$ and $\varepsilon \in \mathbb{R}_{++}$.

2 Then there is $S, F \in \bigotimes_{i \in I} \mathsf{Z}(A_i, \mu_i)$ such that $s = [E]$ and $f = [F]$.

3 We show that there is $t \in \Psi_{A,\mu}\left(\bigotimes_{i \in I} A_i\right)$ such that $\rho_f(t, s) < \varepsilon$.

3.1 As sf is finite there must exist a natural number n and a system $E : \{1, \ldots, n\} \to \prod_{i \in I} \Sigma_\mu$

such that $\bigotimes_{i \in I} \hat{\mu}_i \left(S \cap F \bigtriangleup \bigcup_{k=1}^n \prod_{i \in I} E_i\right) < \varepsilon$.

3.2 But then $\rho_f\left(s, \bigvee_{k=1}^n \Psi_{A,\mu}\left(\bigotimes_{i \in I} [E_i]\right)\right) < \varepsilon$.

□

Write just $\bigotimes_{i \in I} a_i$ for $\Psi_{A,\mu}\left(\bigotimes_{i \in I} a_i\right)$.

$\texttt{TensorMeasureComputation} :: \forall I : \texttt{Finite} \,.\, \forall (A, \mu) : I \to \mathsf{MA} \,.\, \forall t \in \bigotimes_{i \in I} (A, \mu) \,.$

$$.\ \prod_{i \in I} \mu_i(t) = \sup\left\{\prod_{i \in I} \mu_i\left(t \bigotimes_{i \in I} a_i\right) \middle| a \in \prod_{i \in I} A_i^f\right\}$$

$\texttt{Proof} =$

This follos by the definition of the cld product.

□

$\texttt{TensorRepresentationComputation} :: \forall I : \texttt{Finite} \,.\, \forall (A,\mu) : I \to \texttt{SemifiniteMeasureAlgebra} \,.$

$$. \forall a \in \prod_{i \in I} A_i \,.\, \prod_{i \in I} \mu_i \left(\bigotimes_{i \in I} a_i \right) = \prod_{i \in I} \mu(a_i)$$

Proof =

This is pretty obvious.

□

$\texttt{TensorRepresentationUniqueness} ::$

$$. \forall I : \texttt{Finite} \forall (A,\mu) : I \to \texttt{SemifiniteMeasureAlgebra} \,.$$

$$. \texttt{Injective} \left(\bigotimes_{i \in I} A_i, \bigotimes_{i \in I} (A_i, \mu_i), \Psi_{A,\mu} \right)$$

Proof =

This follows from the previous result.

□

$\texttt{MeasureSpaceCLDProductUniversalProperty} ::$

$$:: \forall I : \texttt{Finite} \,.\, \forall (X,\Sigma,\mu) : I \to \texttt{Semifinite} \,.\, \forall (A,\nu) : \texttt{LocalizableMeasureAlgebra} \,.$$

$$. \forall \phi : \prod_{i \in I} \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}\left(\mathsf{MA}(X_i, \Sigma_i, \mu_i), A\right) \,.$$

$$. \forall \aleph : \forall x \in \prod_{i \in I} \mathsf{MA}(X_i, \Sigma_i, \mu_i) \,.\, \nu \left(\bigwedge_{i \in I} \phi_i(x_i) \right) = \prod_{i \in I} \mu_i(x_i) \,.$$

$$. \exists! \psi : \texttt{MeasurePreservingHomomorphism} \left(\mathsf{MA} \left(\bigotimes_{i \in I} (X,\Sigma,\mu) \right), (A,\nu) \right) \,.\, \psi \left(\bigotimes_{i \in I} x_i \right) = \bigwedge_{i \in I} \phi_i(x_i)$$

Proof =

...

□

$\texttt{LocalizableTensorProductUniversalProperty} ::$

$$:: \forall I : \texttt{Finite} \,.\, \forall (A,\mu) : I \to \texttt{SemifiniteMeasureAlgebra} \,.\, \forall (B,\eta) : \texttt{LocalizableMeasureAlgebra} \,.$$

$$. \forall \phi : \prod_{i \in I} \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}(A_i, B) \,.\, \forall \aleph : \forall a : \prod_{i \in I} (A_i) \,.\, \eta \left(\bigvee_{i \in I} \phi_i(a_i) \right) = \prod_{i \in I} \mu_i(a_i) \,.$$

$$. \exists! \psi : \texttt{MeasurePreservingHomomorphism} \,\&\, \texttt{OrderContinuous} \left(\bigotimes_{i \in I} (A, \mu_i), B \right) \,.\, \iota\psi = \phi$$

Proof =

...

□

1.3.7 Independent Process Algebra

$$\texttt{independentProcessAlgebra} :: \prod_{I \in \mathsf{SET}} (I \to \texttt{ProbabilityAlgrbra}) \to \texttt{ProbabilityAlgebra}$$

$$\texttt{randomProcessAlgebra}\,(A,p) = \left(\bigotimes_{i\in I} A_i, \prod_{i\in I} p_i\right) := \mathsf{MA}\left(\bigotimes_{i\in\mathcal{I}} \mathsf{Z}(A_i,p_i)\right)$$

$$\texttt{independentAlgebraTensorProductEmbedding} ::$$

$$:: \prod_{I\in\mathsf{SET}} \prod (A,\mu) : I \to \mathsf{MA} \,.\, \prod_{i\in I} \texttt{OrderContinuous}\left(A_i, \bigotimes_{j\in I} A_j\right)$$

$$\texttt{independetAlgebraTensorProductEmbedding}\,() = \iota_i := \mathsf{MA}_{\mathsf{Z}(A_i,p_i),\bigotimes_{i\in I}\mathsf{Z}(A_i,p_i)}(\pi_i)$$

$$\texttt{independentProcessUniversalProperty} ::$$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,p) : I \to \texttt{ProbabilityAlgebra} \,.\, \forall (B,q) : \texttt{ProbabilityAlgebra} \,.$$

$$.\, \forall \phi : \prod_{i\in I} \texttt{OrderContinuous} \;\&\; \mathsf{BOOL}\left(A_i, \bigotimes_{i\in I}(A_i,p_i)\right) \,.$$

$$.\, \forall \aleph : \forall J : \texttt{Finite}(I) \,.\, \forall a : \prod_{j\in J}(A_j) \,.\, \eta\left(\bigvee_{j\in J} \phi_j(a_j)\right) = \prod_{j\in J} \mu_j(a_j) \,.$$

$$.\, \exists! \psi : \texttt{MeasurePreservingHomomorphism} \;\&\; \texttt{OrderContinuous}\left(\bigotimes_{i\in I}(A_i,\mu_i), B\right) \,.\, \iota\psi = \phi$$

Proof =

...

□

$$\texttt{measureAlgebraTensorRepresentation} ::$$

$$:: \prod I \in \mathsf{SET} \,.\, \prod (A,p) : I \to \texttt{ProbabilityAlgebra} \,.\, \mathsf{BOOL}\left(\bigotimes_{i\in I} A_i, \bigotimes_{i\in I}(A_i,p_i)\right)$$

$$\texttt{measureAlgebraTensorRepresentation}\,() = \Psi_{A,\mu} := \texttt{tensor}\left(\Lambda[E] \in \prod_{i\in I} A_i \,.\, \left[\prod_{i\in I} E_i\right]\right)$$

$$\texttt{TensorRepresentationsAreDense} ::$$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,p) : I \to \texttt{ProbabilityAlgebra} \,.\, \texttt{Dense}\left(\bigotimes_{i\in i}(A,p_i), \Psi_{A,\mu}\left(\bigotimes_{i\in I} A_i\right)\right)$$

Proof =

...

□

1.3.8 independent Subalgebras

$$\texttt{StochasticalyIndependent} :: \prod (A,p) : \texttt{ProbabilityAlgebra} \,.\, \prod I \in \mathsf{SET} \,.\, ?(I \to \texttt{Subring}(A))$$

$$C : \texttt{StochasticalyIndependent} \iff \forall J : \texttt{Finite}(I) \,.\, \forall c : \prod_{j \in J} A_j \,.\, p\left(\bigvee_{j \in J} c_j\right) = \prod_{j \in J} p(c_j)$$

$\texttt{StochasticalyIndependentGeneration}$ ::

$$:: \forall (A,p) : \texttt{ProbabilityAlgebra} \,.\, \forall I \in \mathsf{SET} \,.\, \forall C : \texttt{StochasticalyIndependent}(A,p,I) \,.$$

$$.\, \forall \aleph : \forall i \in I \,.\, C_i \subset_{\mathsf{MA}} (A,p) \,.\, \bigotimes_{i \in I} (C_i, p) \cong_{\mathsf{MA}} \left\langle \bigcup_{i \in I} C_i \right\rangle_{\mathsf{MA}} \subset_{\mathsf{MA}} (A,p)$$

$\texttt{Proof}$ =

This is obvious.

□

$\texttt{StochasticalyIndependentInProcessAlgebra}$ ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,p) : I \to \texttt{ProbabilityAlgebra} \,.\, \texttt{StochasticalyIndependent}\left(\bigotimes_{i \in I} (A_i, p_i), I, (A,p)\right)$$

$\texttt{Proof}$ =

This is obvious.

□

1.3.9 Coordinate Determination

$$\texttt{coordinateSubalgebra} :: \prod_{I \in \mathsf{SET}} (I \to \texttt{ProbabilityAlgebra}) \to ?I \to \texttt{ProbabilityAlgebra}$$

$$\texttt{coordinateSubalgebra}\,((C,p),J) = C_J := \bigvee_{j \in J} \iota_j(C_j)$$

$\texttt{ProcessAlgebraRepresentation}$ ::

$$:: \forall I \in \mathsf{SET}\ .\ \forall (C,p) : I \to \texttt{ProbabilityAlgebra}\ .\ \forall J \subset I\ .\ C_J \cong_{\mathsf{MA}} \bigotimes_{j \in J} (C_j, p_j)$$

$\texttt{Proof}$ =

This is obvious.

□

$\texttt{CoordinateDeterminationExists}$ ::

$$:: \forall i \in \mathsf{SET} \forall (C,p) : I \to \texttt{ProbabilityAlgebra}\ .\ \forall c \in C\ .\ \exists! \min \left\{ J : \texttt{Countable}(I) \middle| c \in C_J \right\}$$

$\texttt{Proof}$ =

1 Let $\mathcal{J} = \left\{ J : \texttt{Countable}(I) \middle| c \in C_J \right\}$.

2 $\mathcal{J} \neq \emptyset$.

2.1 Note that $\bigotimes_{i \in I} C_i$ is dense in $\bigotimes_{i \in I} (C_i, p_i)$.

2.2 So there exists a sequence of natural numbers $n : \mathbb{N} \to \mathbb{N}$,

a system of finite subsets $i \in \prod_{k=1}^{\infty} \{1, \dots, n_l\} \times \{1, \dots, n_k\} \to I$ and $t \in \prod_{k=1}^{\infty} \prod_{l=1}^{k} \prod_{h=1}^{n_k} C_{i_{k,l,h}}$

such that $c = \lim_{k \to \infty} \sum_{l=1}^{k} \bigotimes_{h=1}^{n_k} t_{k,t,h}$, where are all missing slots are filled by e.

2.3 Then $J = \operatorname{Im} i \in \mathcal{J}$, so $\mathcal{J} \neq \emptyset$.

3 $\mathcal{J}$ has a minimal element.

3.1 Assume $\mathcal{C}$ is a chain in $\mathcal{J}$.

3.2 Then $c \in C_J$ for any $J \in \mathcal{C}$.

3.3 So $c \in \bigcap_{J \in \mathcal{C}} C_J = C_{\bigcap_{J \in \mathcal{C}} J}$.

3.3.1 Here we used the fact that $\mathcal{C}$ is decreasing.

3.3.2 C_J Form a sequence of decreasing closed subalgebras.

3.4 So $\bigcap_{J \in \mathcal{C}} J \in \mathcal{J}$ and the lower bound is atained.

4 The minimum Is unique.

4.1 Assume that $I, J \in \mathcal{J}$.

4.2 Then $c \in C_I \cap C_J$.

. . .

□

$\texttt{coordinateDetermination} :: \prod_{I \in \mathsf{SET}} \prod (C,p) : I \to \texttt{ProbabilityAlgebra} \,.\, \bigotimes_{i \in I} (C_i, p_i) \to \texttt{Countable}(I)$

$\texttt{coordinateDetermination}\,(c) = J_c := \texttt{CoordinateDeterminationExists}$

$\texttt{MidElementCoordinatesDetermination} ::$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (C,p) : I \to \texttt{PurelyAtomic} \,.\, \forall a, c \in \bigotimes_I (C_i, p_i) \,.\, \forall \aleph : a \le c \,.\, \exists b \in C_{J_a \cap J_c} \,.\, a \le b \le c$$

$\texttt{Proof} \ =$

This follows from Fubbini Theorem!

...

□

$\texttt{MidElementCoordinatesDetermination} ::$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (C,p) : I \to \texttt{PurelyAtomic} \,.\, \forall a, c \in \bigotimes_I (C_i, p_i) \,.\, \forall \aleph : a \le c \,.\, \exists b \in C_{J_a \cap J_c} \,.\, a \le b \le c$$

$\texttt{Proof} \ =$

This follows from Fubbini Theorem!

...

□

$\texttt{CoordinatesDetermination} ::$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (C,p) : I \to \texttt{PurelyAtomic} \,.\, \forall \mathcal{J} :??I \,.\, \bigcap C_{\mathcal{J}} = C_{\bigcap \mathcal{J}}$$

$\texttt{Proof} \ =$

Part of the previous Theorem.

...

□

Note: It may be interesting to prove this results independently of abstract measure theory, and then prove Fubbini therorem and related results from coordinate Determination.

1.3.10 Reduced Products

1.4 Radon-Nikodym Parallels

1.4.1 Finitely Additive Functionals

$\texttt{finitelyAdditiveFunctionals} :: \texttt{Contravariant}(\mathsf{BOOL}, \mathbb{R}\text{-}\mathsf{VS})$

$\texttt{finitlyAdditiveFunctionals}\,(A) = \mathsf{a}(A) :=$

$$:= \Big\{ f : A \to \mathbb{R} : \forall (a,b) : \texttt{DisjointPair}(A) \,.\, f(a \vee b) = f(a) + f(b) \Big\}$$

$\texttt{finitelyAdditiveFunctionals}\,(A, B, \phi) = \mathsf{a}_{A,B}(\phi) := \phi_*$

$\texttt{boundedAdditiveFunctionals} :: \texttt{Contravariant}(\mathsf{BOOL}, \mathbb{R}\text{-}\mathsf{VS})$

$\texttt{boundedAdditiveFunctionals}\,(A) = \mathsf{ba}(A) := \Big\{ f \in \mathsf{a}(A) : \exists r \in \mathbb{R}_+ \,.\, \forall a \in a \,.\, |f(a)| < r \Big\}$

$\texttt{boundedAdditiveFunctionals}\,(A, B, \phi) = \mathsf{ba}_{A,B}(\phi) := \phi_*$

$\texttt{Zero} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, f(0) = 0$

$\texttt{Proof} =$

1 $(0, 0)$ is a disjoint pair as $0 \cdot 0 = 0$.

2 So $f(0) = f(0 \vee 0) = f(0) + f(0)$.

3 Which can be rewritten as $f(0) = 0$.

$\square$

$\texttt{Restriction} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, \forall a \in A \,.\, \Lambda c \in A \,.\, f(ac) \in \mathsf{a}(A)$

$\texttt{Proof} =$

1 Defin $g(c) = f(ab)$.

2 Assume (c, d) is a disjoint pair.

3 Then $(ac)(ad) = acd = 0$.

4 (ac, ad) is a disjoint pair also.

5 So $g(c \vee d) = f\Big(a(c \vee d)\Big) = f\Big(ac \vee ad\Big) = f(ac) + f(ad) = g(c) + g(d)$.

6 Thus, $g \in \mathsf{a}(A)$.

$\square$

$\texttt{PositiveIffMonotonic} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, f \geq 0 \iff \texttt{Monotonic}(A, \mathbb{R}, f)$

$\texttt{Proof} =$

1 Assume $f > 0$.

1.1 Asume $a, b \in A$ is such that $a > b$.

1.2 Then $f(a) = f(ab \vee b \setminus a) = f(ab) + f(b \setminus a) = f(a) + f(b \setminus a) \geq f(a)$.

2 Assume that f is monotonic.

2.1 Assume $a \in A$.

2.2 Note, that $f(0) = 0$.

2.3 So, as $a \geq 0$ then $f(a) \geq 0$.

$\square$

JordanDecomposition ::

$:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, f \in \mathsf{ba}(A) \iff \exists g, h \in \mathsf{a}(A) \,.\, g, h \geq 0 \,\&\, f = g - h$

Proof =

1 ($\Rightarrow$) Assume f is bounded.

1.1 Define $g(a) = \sup\{f(c)|c \in A, c \leq a\}$.

1.2 g is finitely addive.

1.2.1 Assume $a, b \in A$ are such that $ab = 0$.

1.2.2 Then $g(a \vee b) = \sup\{f(c)|c \in A, c \leq a \vee b\} = \sup\left\{f\Big(c(a \vee b)\Big)\Big|c \in A, c \leq a \vee b\right\} =$

$= \sup\{f(ca \vee cb)|c \in A, c \leq a \vee b\} = \sup\{f(ca) + f(cb)|c \in A, c \leq a \vee b\} =$

$= \sup\{f(c) + f(d)|c, d \in A, c \leq a, d \leq b\} = \sup\{f(c)|c \in A, c \leq a\} + \{f(c)|c \in A, c \leq b\} = g(a) + g(b).$

1.3 Then h can be defined in a simmilar manner but for $-f$.

1.4 $f = g - h$.

1.4.1 $g(a) - h(a) = \sup\{f(c)|c \in A, c \leq a\} - \sup\{-f(c)|c \in A, c \leq a\} =$

$= \sup\{f(c)|c \in A, c \leq a\} + \inf\{f(c)|c \in A, c \leq a\}$.

1.4.2 Then we may select some $c : \mathbb{N} \to (a)$ such that $g(a) = \lim\limits_{n\to\infty} f(c_n)$.

1.4.3 Then $-h(a) = \lim\limits_{n\to\infty} f(a \setminus c_n)$ from (1.4.1).

1.4.4 Thus $g(a) - h(a) = \lim\limits_{n\to\infty} f(c_n) + \lim\limits_{n\to\infty} f(a \setminus c_n) = \lim\limits_{n\to\infty} f(c_n) + f(a \setminus c_n) = \lim\limits_{n\to\infty} f(a) = f(a).$

2 ($\Leftarrow$) Assume there are $g, h \in \mathsf{a}_+(A)$ such that $f = g - h$.

2.1 Assume $a : \mathbb{N} \to A$ is a disjoint sequence.

2.2 Then $\sum\limits_{n=1}^{\infty} g(a_n) = \lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} g(a_n) = \lim\limits_{k\to\infty} g\left(\bigvee\limits_{n=1}^{k} a_n\right) \leq g(a) < \infty.$

2.3 g is bounded.

2.3.1 Assume now that g is unbounded.

2.3.2 Then there exists a sequence c such that $\lim\limits_{n\to\infty} g(c_n) = 0.$

2.3.3 Define $a_n = c_n \setminus \bigvee\limits_{k=1}^{n-1} a_k.$

2.3.4 Then $\sum\limits_{n=1}^{\infty} g(a_n) = \lim\limits_{n\to\infty} g\left(\bigvee\limits_{k=1}^{n} c_k\right) \geq \lim\limits_{n\to\infty} g(c_k) = \infty$.

2.3.5 But this contradicts (2.2)!

2.4 The same is true about h.

2.5 So f is bounded as linear comination of bounded functionals.

□

decompositionOfJordan $:: \prod A \in \mathsf{BOOL}\mathsf{ba}(A) \to \mathsf{ba}^2_+(A)$

decompositionOfJordan $(f) = (f_+, f_-) :=$ JordanDecomposition(A, f)

$\texttt{cilindersElements} :: \prod I \in \mathsf{SET} \,.\, (I \to \mathsf{BOOL}) \to \texttt{Monoid}$

$$\texttt{cilindersElements}\,(A) = C(I,A) := \left\{ \bigwedge_{j \in J} \iota_j(a_j) \middle| J : \texttt{Finite}(I), a \in \prod_{j \in J} A_j \right\}$$

$\texttt{CoproductExtension} :: \forall I \in \mathsf{SET} \,.\, \forall A : I \to \mathsf{BOOL} \,.\, \forall \theta : C(I,A) \to \mathbb{R} \,.$

$$.\, \forall \aleph : \forall c \in C(I,A) \,.\, \forall i \in I \,.\, \forall a \in A_i \,.\, \theta(c) = \theta\Big(c\iota_i(a)\Big) + \theta\Big(c\overline{\iota_i(a)}\Big) \,.\, \exists f \in \mathsf{a}\left(\bigotimes_{i \in I} A_i\right) \,.\, f_{|C} = \theta$$

$\texttt{Proof} \;=$

. . .

$\square$

1.4.2 Properly Atomless Functionals

$$\texttt{ProperlyAtomless} :: \prod_{A \in \mathsf{BOOL}} ?\mathsf{a}(A)$$

$$f : \texttt{ProperlyAtomless} \iff$$
$$\iff \forall \varepsilon \in \mathbb{R}_{++} \,.\, \exists P : \texttt{PartitionOfUnity}(A) \,.\, |P| < \infty \;\&\; \forall p \in P \,.\, \forall a \in (p) \,.\, |f(a)| \le \varepsilon$$

$\texttt{VectorSubspace} :: \forall A \in \mathsf{BOOL} \,.\, \texttt{ProperlyAtomless}(A) \subset_{\mathbb{R}\text{-}\mathsf{VS}} \mathsf{ba}(A)$

$\texttt{Proof} =$

1 Assume f is properly Atomless.

2 Then f is bounded.

2.1 There is a finite partition of unity P such that $|f(a)| < 1$ for any $p \in P$ and $a \in (p)$.

2.2 Then $|f(a)| = \left|f\left(a \bigvee P\right)\right| = \left|\sum_{p \in P} f(ap)\right| \le \sum_{p \in P} |f(ap)| \le |P| < \infty$.

3 Then αP may use simmilar partitions as P fo $\dfrac{\varepsilon}{|\alpha|}$.

4 And a sum $f + g$ may use intermeshes of f and g.

$\square$

ContinuousPartitioningTHM1 ::

:: $\forall A : \sigma\text{-Algebra} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to \mathsf{a}_+(A) \,.$

$. \forall \aleph : \forall a \in A \,.\, \exists \alpha \in \left[\frac{1}{3}, \frac{2}{3}\right] \,.\, \exists a' \in (a) \,.\, \forall i \in I \,.\, \alpha f_i(a) = f_i(a') \,.$

$. \forall a \in A \,.\, \exists u : [0,1] \uparrow (a) \,.\, u_0 = 0 \;\&\; u_1 = a \;\&\; \forall \tau \in [0,1] \,.\, \forall i \in I \,.\, f_i(u_\tau) = \tau f_i(a)$

Proof $=$

1 Assume that there is $k \in I$ such that $f_k(a) > 0$.

1.1 Otherwise set $u_1 = a$ and $u_\tau = 0$.

2 Define $\gamma_i = \dfrac{f_i(a)}{f_k(a)}$.

3 Define sets $D : \mathbb{Z}_+ \to 2^{(a)}$ recursevely in a such way that D is increasing, and each D_n is finite and ordered with $a, 0 \in D_n$ and $f_i(d) = \gamma_i f_k(d)$ for every $i \in I$.

3.1 Let $D_0 = \{0, a\}$.

3.2 Then assume $m = |D_n|$ and let $d : \{1, \ldots, m\} \to D_n$ be an an order-preserving enumeration.

3.3 Then by $\aleph$ there is $c : \{1, \ldots, m-1\} \to (a)$ such that $c_l \leq d_{l+1} \setminus d_l$m

And a sequence $\alpha : \{1, \ldots, m-1\} \to \left[\frac{1}{3}, \frac{2}{3}\right]$ such that $f_i(c_l) = \alpha_l f_i(d_{l+1} \setminus d_l)$ for any $i \in I$.

3.4 Define $D_{n+1} = D_n \cup \left\{ d_l \vee c_l \middle| l \in \{1, \ldots, m-1\} \right\}$.

3.5 The it is obvious that D_{n+1} is finite and ordered.

3.6 So we constructed and increasing D with a property $f_i(d_{l+1} \setminus d_l) \leq \left(\frac{2}{3}\right)^n f_i(a)$

for any $i \in I$ and d being enumeration of D_n as above.

4 Set $C = \bigcup_{n=1}^{\infty} D_n$.

5 Then C is countale totally ordered set with $0, a \in C$ and $f_k(C)$ is dense in $[0, f_k(a)]$.

6 Define $u_\tau = \sup\{c \in C, f_k(c) \leq \tau f_k(a)\}$.

6.1 This supremum has to exists.

6.2 As $f_k(C)$ is dense in $[0, f_k(a)]$ there is a sequence $c : \mathbb{N} \to C$ with $\lim_{n\to\infty} c_n = \tau f_k(a)$.

6.3 Without loss of generality we may assume that c is non-decreasing.

6.4 And we may define $u_\tau = \bigvee_{n=1}^{\infty} c_n$.

7 Then $u_0 = 0$ and $u_1 = a$ and $f_i(u_\tau) = \tau f_i(a)$ for any $i \in I$.

$\square$

ContinuousPartitioningTHM2 ::

$:: \forall A : \sigma\text{-Algebra} \,.\, \forall n \in \mathbb{N} \,.\, \forall f : \{1,\dots,n\} \to \text{ProperlyAtomless}(A) \,.$

$. \forall \aleph : \forall i \in \{1,\dots,n\} \,.\, 0 \le f_i \le f_1 \,.$

$. \forall a \in A \,.\, \exists u : [0,1] \uparrow (a) \,.\, u_0 = 0 \;\&\; u_1 = a \;\&\; \forall \tau \in [0,1] \,.\, \forall i \in \{1,\dots,n\} \,.\, f_i(u_\tau) = \tau f_i(a)$

Proof =

1 We prove that conditions of Previous theorem are satisfied with $I = \{1,\dots,n\}$.

2 At first consider the case $I = \{1\}$.

2.1 We seek to prove $a \in A$ there is an $\alpha \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $a' \in (a)$ such that $f_1(a') = \alpha f_1(a)$.

2.2 Then there is finite partition of unity P such that $|f_1(c)| < \frac{1}{3} f_1(a)$ for any $p \in P$ and for any $c \le p$.

2.3 Then it must be possible to sample $Q \subset P$ in such a way that $a' = a \bigvee_{p \in Q} p$ and $\frac{f_1(a')}{f_1(a)} \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

2.3.1 We know that $a = a \bigvee_{p \in P} p$ and $|f_1(ap)| \le \frac{f_1(a)}{3}$.

2.3.2 So if $f_1(ap) < \frac{f_1(a)}{3}$ for some $p \in P$ there is also some $q \in P$ such that $f_1(a(p \vee q)) \le \frac{2 f_1(a)}{3}$.

2.3.3 This Process must stop as $f_1(a) = f_1\left(a \bigvee_{p \in P} p\right) = \sum_{p \in P} f_1(ap)$.

3 We follow by induction.

3.1 Assume the theorem holds for all $i \in I$ with $i < m$ and we have corresponding u for $\{1,\dots,m-1\}$.

3.2 $|f_m(u_t) - f_m(u_s)| = f_m(u_t \setminus u_s) \le f_0(u_t \setminus u_s) = (t-s) f_0(a)$ for $0 \le s \le t \le 1$.

3.3 So $\phi(t) = f_m(u_t)$ is continuous.

3.4 and the function $\psi : \left[0, \frac{1}{2}\right] \to \mathbb{R}_+$ defined by $\psi(t) = f_m(u_{t+\frac{1}{2}}) - f_m(u_t)$ is continuous.

3.5 Note that $\psi(0) + \psi\left(\frac{1}{2}\right) = f_m(a)$.

3.6 So by the intermidiate value theorem there must be some $t \in \left[0, \frac{1}{2}\right]$ such that $\psi(t) = \frac{1}{2} f_m(a)$.

3.7 Define $u' = u_{t+\frac{1}{2}} \setminus u_t$.

3.8 Then $f_i(u') = \frac{1}{2} f_i(a)$ for all $i \in \{1,\dots,m\}$.

3.9 But this means that that the assertion holds fo $\{1,\dots,m\}$ And we can use the previous theorem.

$\square$

1.4.3 Liapounoff's Convexity Theorem

$$\texttt{vectorValuedFinitelyAdditiveFunctionals} :: \prod V : \mathbb{R}\text{-}\mathsf{BAN} \,.\, \texttt{Contravariant}(\mathsf{BOOL}, \mathbb{R}\text{-}\mathsf{VS})$$

$$\texttt{finitlyAdditiveFunctionals}\,(A) = \mathsf{a}(A, V) :=$$

$$:= \Big\{ f : A \to V : \forall (a,b) : \texttt{DisjointPair}(A) \,.\, f(a \vee b) = f(a) + f(b) \Big\}$$

$$\texttt{finitelyAdditiveFunctionals}\,(A, B, \phi) = \mathsf{a}_{A,B}(\phi) := \phi_*$$

$$\texttt{vectorValuedBoundedAdditiveFunctionals} :: \prod V : \mathbb{R}\text{-}\mathsf{BAN} \,.\, \texttt{Contravariant}(\mathsf{BOOL}, \mathbb{R}\text{-}\mathsf{VS})$$

$$\texttt{boundedAdditiveFunctionals}\,(A) = \mathsf{ba}(A, V) := \Big\{ f \in \mathsf{a}(A) : \exists r \in \mathbb{R}_+ \,.\, \forall a \in a \,.\, \|f(a)\| < r \Big\}$$

$$\texttt{boundedAdditiveFunctionals}\,(A, B, \phi) = \mathsf{ba}_{A,B}(\phi) := \phi_*$$

$$\texttt{ProperlyAtomless} :: \prod_{A \in \mathsf{BOOL}} \prod_{V \in \mathbb{R}\text{-}\mathsf{BAN}} ?\mathsf{a}(A)$$

$$f : \texttt{ProperlyAtomless} \iff$$

$$\iff \forall \varepsilon \in \mathbb{R}_{++} \,.\, \exists P : \texttt{PartitionOfUnity}(A) \,.\, |P| < \infty \;\&\; \forall p \in P \,.\, \forall a \in (p) \,.\, \|f(a)\| \leq \varepsilon$$

$$\texttt{LiapounoffsConvexityTHM} ::$$

$$:: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall n \in \mathbb{N} \,.\, \forall f : \texttt{ProperlyAtomless}(A, \mathbb{R}^n) \,.\, \texttt{Convex}(\mathbb{R}^n, f(A))$$

Proof $=$

This is an application of continuous decomposition theorems.

...

□

Note this is an additional problem then this theorem hold for infinite-dimensional vector spaces.

1.4.4 Countably Additive Functionals

$\texttt{countablyAdditiveFunctionals} :: \texttt{Contravariant}(\mathsf{BOOL}_\sigma, \mathbb{R}\text{-}\mathsf{VS})$

$\texttt{countablyAdditiveFunctionals}\,(A) = \mathsf{ca}(A) :=$

$$:= \left\{ f \in \mathsf{a}(A) : \forall a : \texttt{DisjointSequence}(A) \exists \bigvee_{n=1}^{\infty} a_n \Rightarrow f\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} f(a_n) \right\}$$

$\texttt{countablyAdditiveFunctionals}\,(A, B, \varphi) = \mathsf{ca}_{A,B}(\varphi) := \varphi_*$

$\texttt{IncreasingExpression} ::$

$$:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{ca}(A) \,.\, \forall a : \mathbb{N} \uparrow A \,.\, \exists \bigvee_{n=1}^{\infty} a_n \Rightarrow f\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} f(a_n)$$

$\texttt{Proof} \; =$

1 Note that $a_n \setminus a_{n-1}$ is a disjoint sequence with $a_0 = 0$.

2 Then $f\left(\bigvee_{n=1}^{\infty} a_n\right) = f\left(\bigvee_{n=1}^{\infty} a_n \setminus a_{n-1}\right) = \sum_{n=1}^{\infty} f(a_n \setminus a_{n-1}) = \lim_{n\to\infty} \sum_{k=1}^{n} f(a_k \setminus a_{k-1}) = \lim_{n\to\infty} f(a_n).$

$\square$

$\texttt{DecreasingExpression} ::$

$$:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{ca}(A) \,.\, \forall a : \mathbb{N} \downarrow A \,.\, \exists \bigwedge_{n=1}^{\infty} a_n \Rightarrow f\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} f(a_n)$$

$\texttt{Proof} \; =$

1 Note that $a_1 \setminus a_n$ is increasing.

2 Then $f\left(\bigwedge_{n=1}^{\infty} a_n\right) = f\left(a_1 \setminus \bigvee_{n=1}^{\infty} (a_1 \setminus a_n)\right) = f(a_1) - f\left(\bigvee_{n=1}^{\infty} (a_1 \setminus a_n)\right) = f(a_1) - \lim_{n\to\infty} f(a_1 \setminus a_n) =$

$= f(a_1) - \lim_{n\to\infty} f(a_1) - f(a_n) = \lim_{n\to\infty} f(a_n).$

$\square$

$\texttt{Restriction} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{ca}(A) \,.\, \forall a \in A \,.\, \Lambda c \in A \,.\, f(ac) \in \mathsf{ca}(A)$

$\texttt{Proof} \; =$

...

$\square$

CAFByLimits :: $\forall A \in \mathsf{BOOL} . \forall f \in \mathsf{a}(A) . \forall \aleph : \forall a : \mathbb{N} \downarrow A . \bigwedge_{n=1}^{\infty} a_n = 0 \Rightarrow \lim_{n\to\infty} f(a_n) = 0 . f \in \mathsf{ca}(A)$

Proof $=$

1 Assume $a : \mathbb{N} \to A$ is a disjoint sequence with $\bigvee_{n=1}^{\infty} a_n$ existing.

2 Then $\bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} a_n = 0$.

3 So, $\lim_{n\to\infty} f\left(\bigvee_{m=n}^{\infty} a_n\right) = 0$ by $\aleph$.

4 Then for any $m \in \mathbb{N}$ there is a rewrite $f\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{k=1}^{m} f(a_k) + f\left(\bigvee_{n=m+1}^{\infty} a_n\right)$.

5 Taking a limit $m \to \infty$ produces the desired result $f\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} f(a_n)$.

$\square$

DominatedCAF :: $\forall A \in \mathsf{BOOL} . \forall f \in \mathsf{a}(A) . \forall g \in \mathsf{ca}(A) . |f| \leq g \Rightarrow f \in \mathsf{ca}(A)$

Proof $=$

1 Assume $a : \mathbb{N} \downarrow A$ such that $\bigwedge_{n=1}^{\infty} a_n = 0$.

2 $g \in \mathsf{ca}(A)$ imply that $\lim_{n\to\infty} g(a_n) = 0$.

3 But then domination Imply that $\lim_{n\to\infty} f(a_n) = 0$.

4 By previous theorem this means that $f \in \mathsf{ca}(A)$.

$\square$

JordanDecomposition :: $\forall A \in \mathsf{BOOL} . \forall f \in \mathsf{ca}(A) . f \in \mathsf{ba}(A) \iff \exists g, h \in \mathsf{ca}_+(A) . f = g - h$

Proof $=$

1 ($\Rightarrow$) Assume f is bounded.

1.1 Then $f = f_+ - f_-$ by simple Jordan's decomposition for finitely additive functionals.

1.2 We may write $f_+(a) = \sup\{f(c) | c \in A, c \leq a\}$.

1.3 Assume $a : \mathbb{N} \to A$ is a disjoint sequence such that $\bigvee_{n=1}^{\infty} a_n$ exists.

1.4 Then $f_+\left(\bigvee_{n=1}^{\infty} a_n\right) = \sup\left\{f(c) \middle| c \in A, c \leq \bigvee_{n=1}^{\infty} a\right\} = \sup\left\{f\left(c \bigvee_{n=1}^{\infty} a_n\right) \middle| c \in A, c \leq \bigvee_{n=1}^{\infty} a\right\} =$

$$= \sup\left\{f\left(\bigvee_{n=1}^{\infty} ca_n\right) \middle| c \in A, c \leq \bigvee_{n=1}^{\infty} a\right\} = \sup\left\{\sum_{n=1}^{\infty} f(ca_n) \middle| c \in A, c \leq \bigvee_{n=1}^{\infty} a\right\} =$$

$$= \sup\left\{\sum_{n=1}^{\infty} f(c_n) \middle| c : \mathbb{N} \to A, \forall n \in \mathbb{N} . c_n \leq a_n\right\} = \sum_{n=1}^{\infty} \sup\{f(c) | c \in A, c \leq a_n\} = \sum_{n=1}^{\infty} f_+(a_n)$$

1.4.1 the sum $\sum_{n=1}^{\infty} f(c_n)$ must exist as $\sum_{n=1}^{\infty} |f(c_n)| \leq \sum_{n=1}^{\infty} f_+(c_n)$.

1.4.2 And if $\sum_{n=1}^{\infty} f_+(c_n)$ diverges then the sequence $\phi_n = f_+\left(\bigvee_{k=1}^{n} c_k\right)$ must be unbounded.

1.4.3 But f_+ must bounded by basic Jordan decomposition theorem, a contradiciton!

1.4.4 So $\sum_{n=1}^{\infty} f(c_n)$ exists as absolutely converging series.

2 $(\Leftarrow)$ This direction follows from basic Jordan Decomposition.

$\square$

HahnDecomposition1 :: $\forall A \in \sigma\text{-Algebra} \,.\, \mathsf{ca}(A) \subset_{\mathbb{R}\text{-VS}} \mathsf{ba}(A)$

Proof =

1 Assume $f \in \mathsf{ba}(A)$.

2 Let $\gamma = \sup_{a \in A} f(a)$.

3 Then there is a disjoint sequence $a : \mathbb{N} \uparrow A$ such that $\gamma = \lim_{n=1} f(a_n) = f\left(\bigvee_{n=1}^{\infty} a_n\right) < \infty$.

3.1 Clearly there is a sequence $c : \mathbb{N} \to A$ such that $\lim_{n=1} f(c_n) = \gamma$.

3.2 Without loss of generality it may be assumed that $f(c_n) > 0$.

3.2.1 Otherwise $\sup_{a \in A} f(a) = f(0) = 0$.

3.3 So $\gamma = \lim_{n \to \infty} f(c_n) \leq \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) = \lim_{n \to \infty} f\left(\bigvee_{k=1}^{n} c_k\right) \leq \gamma$.

3.4 So $\gamma = \lim_{n \to \infty} f\left(\bigvee_{k=1}^{n} c_k\right)$.

3.5 Just define $a_n = \bigvee_{k=1}^{n} c_k$.

$\square$

HahnDecomposition2 :: $\forall A \in \sigma\text{-Algebra} \,.\, \forall f \in \mathsf{ca}(A) \,.\, \exists d \in A \,.\, \forall c \in A \,.\, f(c\bar{d}) \leq 0 \;\&\; f(cd) \geq 0$

Proof =

1 Let γ and a be as above.

2 Let $d = \bigvee_{n=1}^{\infty} a_n$.

3 Assume $c \in (\bar{d})$.

4 If $f(c) \geq 0$ then $f(c \vee d) > \gamma$ but this is impossible.

5 Otherwise if $c \leq d$ and $f(c) < 0$ then $f(d \setminus c) > \gamma$ which is impossible.

$\square$

1.4.5 Completely Additive Functionals

$\texttt{completelyAdditiveFunctionals} :: \texttt{Contravariant}(\mathsf{BOOL}_\tau, \mathbb{R}\text{-}\mathsf{VS})$
$\texttt{completelyAdditiveFunctionals}\,(A) = \tau\text{-}\mathsf{ca}(A) :=$

$$:= \left\{ f \in \mathsf{a}(A) : \forall D : \texttt{DownwardsDirected}(A) \;.\; \bigvee_{d \in D} d = 0 \Rightarrow \inf_{d \in D} |f(d)| = 0 \right\}$$

$\texttt{CountaleAdditivity} :: \forall A \in \mathsf{BOOL} \;.\; \tau\text{-}\mathsf{ca}(A) \subset_{\mathbb{R}\text{-}\mathsf{VS}} \mathsf{ca}(A)$

$\texttt{Proof}\ =$

1 Assume $f \in \tau\text{-}\mathsf{ca}(A)$.

2 Also Assume $a : \mathbb{N} \to A$ is a decreasing with $\bigwedge_{n=1}^{\infty} a_n = 0$.

3 Then $\lim_{n=1} f(a_1) = 0$.

4 Hence f is countably additive.

$\square$

$\texttt{InfimumLocalization} ::$
$:: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \tau\text{-}\mathsf{ca}(A) \;.\; \forall \varepsilon \in \mathbb{R}_{++} \;.\; \forall D : \texttt{DownwardsDirected}(A) \;.\; \forall \aleph : \inf D = 0 \;.$
$.\; \exists d \in D \;.\; \forall c \in (d) \;.\; |f(c)| < \varepsilon$

$\texttt{Proof}\ =$

1 Assume otherwise.

2 Let $C = \{c \in A : |f(c)| \geq \varepsilon, \exists d : d \leq c\}$.

3 Every member of A includes some membebr of C.

3.1 Assume $d \in D$.

3.2 Then by (1) there is $c \in A$ such that $c \leq d$ and $|f(c)| \geq \varepsilon$.

3.3 Let $D'_d = \{d' \setminus c | d' \in D, d' \leq d\}$.

3.4 Then D'_d is downwards directed and $\lim D'_d = 0$.

3.5 So there is a d' such that $|f(d' \setminus c)| < |f(c)| - \varepsilon$.

3.6 Let $c' = d' \vee c$.

3.7 Then $c' \leq d$.

3.8 Also $|f(c')| = |f(d' \setminus c) + f(c)| \geq |f(c)| - |f(d' \setminus c)| \geq \varepsilon$.

3.9 So $c' \in C$.

4 Since every member of C includes a member A it must be the case that C is downwards directed and $\lim C$

5 On the other hand $\lim_{c \in C} |f(c)| \geq \varepsilon$.

6 And this contradicts the fact of $f \in \tau\text{-}\mathsf{ca}(A)$.

$\square$

$\texttt{Continuity} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \tau\text{-}\mathsf{ca}_+(A) \;.\; \texttt{OrderContinuous}(A, \mathbb{R}, f)$

$\texttt{Proof}\ =$

...

$\square$

Restriction :: $\forall A \in \mathsf{BOOL} \,.\, \forall f \in \tau\text{-}\mathsf{ca}(A) \,.\, \forall a \in A \,.\, \Lambda c \in A \,.\, f(ac) \in \tau\text{-}\mathsf{ca}(A)$

Proof =

...

□

DominatedCAF :: $\forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, \forall g \in \tau\text{-}\mathsf{ca}(A) \,.\, |f| \leq g \Rightarrow f \in \tau\text{-}\mathsf{ca}(A)$

Proof =

...

□

CCCUpgrade :: $\forall A :$ WithCountableChainCondition $.\ \mathsf{ca}(A) = \tau\text{-}\mathsf{ca}(A)$

Proof =

1 Take $f \in \mathsf{ca}(A)$.

2 Assume D is downwards directed in A with $\inf D = 0$.

3 Then there is a countable $C \subset D$ with $\inf C = 0$ as A is CCC.

4 Let c be an enumeration of C with $\lim_{n\to\infty} c_n = 0$.

5 Then it is possible to construct a sequence $d \in D$ such that $d_n \leq \bigvee_{k=1}^{n} c_k$.

6 Thus $\inf f(D) \leq \lim_{n\to infty} f(d_n) = 0$.

7 and so $f \in \tau\text{-}\mathsf{ca}(A)$.

□

BoundedBySup :: $\forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{a}(A) \,.\, \forall \aleph : \forall d :$ DisjointSequence(A) $.\ \sup_{m\in\mathbb{N}} |f(d_m)| < \infty \,.\, f \in \mathsf{ba}(A)$

Proof =

1 Assume f is not bounded.

2 Then we can construct recursevely a countale partition of unity such that $\sup |f(p)| > \infty$.

2.1 Select $p_{0,1} = e$.

2.2 On the step n there can be we seek elemen a with $|f(a)| \geq n + |f(p_{n,n})|$.

2.3 Then we can assert that $a \leq p_{n,n}$.

2.3.1 Define $p_{n+1,k} = p_{n,k}$ fo $k < n$.

2.3.2 Define $p_{n+1,n} = p_{n,n} \setminus a$ and $p_{n+1,n+1} = a$.

2.3.3 Then $\bigvee_{k=1}^{n+1} p_{n+1,k} = \bigvee_{k=1}^{n-1} p_{n,k} \vee (p_{n,n} \setminus a) \vee (a) = \bigvee_{k=1}^{n} p_{n,k}$.

2.3.4 Also $|f(p_{n+1,n})| \geq n > n - 1$.

2.3.5 Then either $\sup |f(p_{n+1,n})| = \infty$ or $\sup |f(p_{n+1,n+1})| = \infty$.

2.3.6 In the first case swap $p_{n+1,n}$ and $p_{n+1,n+1}$.

2.4 As every element $p_{\bullet,k}$ of the fixed index k changes atmost 2 times we can construct an infinite disjoint sequ

2.5 Then $|f(d_n)| \geq n - 1$, so $\sup_{m\in\mathbb{N}} |f(d_m)| = \infty$.

3 This contradicts $(\aleph)$.

□

JordanDecomposition1 :: $\forall A \in \mathsf{BOOL} \,.\, \tau\text{-}\mathsf{ca}(A) \subset_{\mathbb{R}\text{-}\mathsf{VS}} \mathsf{ba}(A)$

Proof $=$

1 Assume that d is a disjoint sequence in A.

2 Define $D = \{a \in A : \exists N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow d_n \leq a\}$.

3 Then D is downwards directed.

4 Also as $d_n^{\complement} \in D$ and d is disjoint it follows that $\bigwedge D = 0$.

5 So it follows that there is $a \in D$ such that $|f(c)| \leq 1$ for all $c \leq a$.

6 But this means that $|f(d_n)| \leq 1$ for a cofinite set of indexes.

7 Thus, d is bounded.

8 So, as d was arbitrary, by the previous theorem f is also bounded.

$\square$

JordanDecomposition2 :: $\forall A \in \mathsf{BOOL} \,.\, \forall f \in \tau\text{-}\mathsf{ca}(A) \,.\, f_+, f_- \in \tau\text{-}\mathsf{ca}(A)$

Proof $=$

1 Write $f_+(a) = \sup\{f(c) | c \in A, c \leq a\}$.

2 Assume D is a downwards directed set with $\bigwedge D = 0$.

3 Also assume $\varepsilon \in \mathbb{R}_{++}$.

4 We know that there is $d \in D$ such that $|f(c)| \leq \varepsilon$ for all $c \leq d$.

5 So $\inf_{u \in D} f_+(u) \leq f_+(d) \leq \varepsilon$.

6 Thus, $\inf_{d \in D} f_+(d) = 0$ and $f_+ \in \tau\text{-}\mathsf{ca}(A)$.

7 The same argument holds for f_-.

$\square$

UnitySummability ::

$$:: \forall A \in \mathsf{BOOL} \,.\, \forall f : A \to \mathbb{R} \,.\, f \in \tau\text{-}\mathsf{ca}(A) \iff \forall P : \mathtt{PartitionOfUnity}(A) \,.\, f(e) = \sum_{p \in P} f(p)$$

Proof =

1 ($\Rightarrow$) Assume $f \in \tau\text{-}\mathsf{ca}(A)$.

1.1 Transfinite induction on $|J|$ with trivial base $f(e) = f(e)$.

1.1.1 Assume that the result holds for some non-limit ordinal κ.

1.1.2 Consider an ordering p of P with cardinality equivalent to $\kappa + 1$.

1.1.3 Let $f' \in \tau\text{-}\mathsf{ca}(A)$ be a restiction of f to $p^{\complement}_{\kappa+1}$.

1.1.4 Also define Q to be qual to P but with $p_{\kappa+1}$ and p_κ replaced by $p_\kappa \vee p_{\kappa+1}$.

1.1.5 Then by induction hypothesis

$$f(e) = f'(e) + f(p_{\kappa+1}) = f(p_{\kappa+1}) + \sum_{q \in Q} f'(q) = f(p_{\kappa+1}) + \sum_{\tau \le \kappa} f(p_\tau) = \sum_{q \in P} f(q).$$

1.2 Now let κ be a limit cardinal and that induction hypothesis holds for all $\tau < \kappa$.

1.2.1 Note that $\sum_{p \in P} f(p)$ converges unconditionally to $f(e)$ iff for any $\varepsilon \in \mathbb{R}_{++}$ there is finite $F \subset P$ such that $\left| f(e) - \sum_{p \in G} f(p) \right| \le \varepsilon$ for any finite G with $F \subset G$.

1.2.2 Consider a set $D = \left\{ e \setminus \bigvee_{p \in F} p \,\middle|\, F : \mathtt{Finite}(P) \right\}$.

1.2.3 Then, as P is a partition of unity $\bigwedge D = 0$.

1.2.4 Also D is downwards directed as $\bar{a} \wedge \bar{b} = \overline{a \vee b}$.

1.2.5 So there is $d \in D$ such that $|f(c)| \le \varepsilon$ for all $c \le d$.

1.2.6 Represent $d = e \setminus \bigvee_{p \in F} p$ for some finite $F \subset P$.

1.2.7 Take some finite $G \subset P$ such that $F \subset G$.

1.2.8 Then $\left| f(e) - \sum_{p \in G} f(p) \right| = \left| f(e \setminus \bigvee_{p \in G} p \right| < \varepsilon$ as $e \setminus \bigvee_{p \in G} p \le e \setminus \bigvee_{p \in F} p$.

2 ($\Leftarrow$) Now consider the case then the second condition holds.

2.1 for any disjoint $D \subset A$ with $\bigvee D = a$ it holds that $f(a) = \sum_{d \in D} f(d)$.

2.1.1 Consider a a partiotion of unity $P = D \cup \{\bar{a}\}$.

2.1.2 Then $\sum_{p \in P} f(p) = f(e) = f(a) + f(\bar{a})$.

2.1.3 By substraction $f(\bar{a})$ one gets $\sum_{d \in D} f(d) = f(a)$.

2.2 $f \in \mathsf{a}(A)$.

2.2.1 Consider $a, c \in A$ such that $ac = 0$.

2.2.2 Then $f(a \vee c) = f(a) + f(c)$ by (2.1).

2.3 $f \in \mathsf{ba}(A)$.

2.3.1 Assume $d : \mathbb{N} \to A$ is disjoint.

2.3.2 Let $\mathcal{D}$ be a set of all disjoint sets D with $\operatorname{Im} d \subset D$.

2.3.3 Then By Zorn Lemma there is an upper bound P which must be a partition of unity.

2.3.4 Then $f(e) = \sum\limits_{p \in P} f(p)$.

2.3.5 But this means the $\lim\limits_{n \to \infty} f(d) = 0$.

2.3.6 As d was arbitrary $f(d)$ is bounded.

2.4 Now it is possible to write $f = f_+ - f_-$.

2.5 Then $\sup\limits_{d \in D} f_+(d) = f(a)$ for a disjoint set D with $\bigwedge D = a$.

2.5.1 Assume D is such disjoint set.

2.5.2 Then $f(b) = \sum\limits_{d \in D} f(bd) \leq \sum\limits_{d \in D} f_+(d)$ for any $b \leq a$.

2.5.3 So by taking supremum $f_+(a) \leq \sum\limits_{d \in D} f_+(d)$.

2.5.4 But also $\sum\limits_{d \in D} f_+(d) = \sup \left\{ \sum\limits_{d \in F} f(d) \middle| F : \mathtt{Finite}(D) \right\} = \sup \left\{ f\left(\bigvee F\right) \middle| F : \mathtt{Finite}(D) \right\} \leq f_+(a)$.

So $\sum\limits_{d \in D} f_+(d) = f_+(a)$.

2.6 $f_+ \in \tau\text{-}\mathsf{ca}(A)$.

2.6.1 Assume D is a downwards directed set with $\bigwedge D = 0$.

2.6.2 Let $C = \{a \in A : \exists d \in D : da = 0\}$.

2.6.3 Then C is order dense.

2.6.4 So it is possible to extract a partition of Unity $P \subset C$.

2.6.5 $\sum\limits_{p \in P} f_+(p) = f_+(e)$ by (2.5).

2.6.6 So for any $\varepsilon \in \mathbb{R}_{++}$ there is some finite $F \subset C$ such that $f_+\left(e \setminus \bigvee F\right) = f_+(e) - \sum\limits_{p \in F} f_+(p) < \varepsilon$.

2.6.7 By construction of C there is a $d \in D$ such that $d \leq e \setminus \bigvee F$.

2.6.8 Therefore $f_+(d) < \varepsilon$.

2.6.9 $\inf\limits_{d \in D} f_+(d) = 0$ as ε was arbitrary.

2.7 $f_- \in \tau\text{-}\mathsf{ca}(A)$ by simmilar arguments.

2.8 So $f \in \tau\text{-}\mathsf{ca}(A)$.

□

`Summability` ::

:: $\forall A \in \mathsf{BOOL} \,.\, \forall f \in \tau\text{-}\mathsf{ca}(A) \,.\, \forall D : \mathtt{Disjoint}(A) \,.\, \forall a \in A\ a = \bigvee D \Rightarrow f(a) = \sum\limits_{d \in D} f(d)$

`Proof` =

This is a part of the previous theorem.

□

StrictHahnDecomposition ::

$:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \tau\text{-}\mathsf{ca}(A) \,.\, \exists! q \in A \,.\, \forall c \in C \,.\, 0 < c \leq q \Rightarrow f(c) > 0 \;\&\; c \leq \bar{(q)} \Rightarrow f(c) \leq 0$

Proof $=$

1 Define $C_+ = \{a \in A : 0 < c \leq a \Rightarrow f(c) > 0\}$ and $C_- = \{a \in A : c \leq a \Rightarrow f(c) \leq 0\}$.

2 $C_+ \cup C_-$ is order dense.

2.1 By ordinary Hahn decomposition there is $a' \in A$ such that $f(c) \geq 0$ for all $c \leq a'$ and $f(c) \leq 0$ for all $c \leq \overline{a'}$.

2.2 $a\overline{a'} \in C_-$ for any $a \in A$ such that $a\, neq 0$.

2.3 In case $a\overline{a'} = 0$ it must be the case that $a \leq a'$.

2.4 If $a \notin C_+$ there must be some $d \leq a$ such that $f(d) \leq 0$.

2.5 But $d \leq a'$, so $f(d) = 0$ and $f(c) = 0$ for any $c \leq d$.

2.6 So $d \in C_-$ and $ad \neq 0$.

3 So there is a partition of unity $P \subset C_+ \cup C_-$.

4 $P \cap C_+$ is countable.

4.1 The series $\sum_{p \in P} f(p)$ must be absolutely convergent.

4.2 So any subseries of $\sum_{p \in P} f(p)$ must be strictly convergent.

4.3 This includes $\sum_{p \in P \cap C_+} f(p)$.

4.4 But $f(p) > 0$ any element $P \cap C_+$, so there can be atmost countable number of such elements.

5 Element $q = \bigvee (P \cap C_+)$ exists.

6 Clearly $f(a) = f\left(\bigvee_{p \in P \cap C_+} ap\right) = \sum_{p \in p \cap C_+} f(ap) > 0$ for any $a \leq q$ such that $a \neq 0$.

7 Then $f(a) \leq 0$ if $a \leq \bar{q}$.

8 q is unique.

8.1 Assume p has same properties as q.

8.2 But then $f(p \setminus q) \leq 0$ and $f(q \setminus p) = 0$ meaning that $p \setminus q = q \setminus p = 0$.

8.3 Thus $p = q$.

□

saturation :: $\prod_{A \in \mathsf{BOOL}} \tau\text{-}\mathsf{ca}^2(A) \to A$

saturation $(f, g) = [f > g]_A :=$ StrictHahnDecomposition$(A, f - g)$

1.4.6 Absolutely Continuous Additive Functionals

AbsolutelyContinuousAdditiveFunctional ::

$$:: \prod (A,\mu) \in \mathsf{MA} \ . \ ?\mathsf{a}(A)$$

f : AbsolutelyContinuousAdditiveFunctional $\iff f \in \mathsf{ac}(A,\mu) \iff$

$$\iff \forall \varepsilon \in \mathbb{R}_{++} \ . \ \exists \delta \in \mathbb{R}_{++} \ . \ \forall a \in A \ . \ \mu(a) \leq \delta \Rightarrow |f(a)| \leq \varepsilon$$

ContinuousIsCompletelyAdditive :: $\forall (A,\mu) \in \mathsf{MA} \ . \ \forall f \in \mathsf{a}(A) \ . \ f \in C_0(A) \Rightarrow f \in \tau\text{-}\mathsf{ca}(A)$

Proof $=$

1 Assume D is a downwards directed in A with $\bigwedge D = 0$.

2 Then $\lim_{d \in D} d = 0$ in a measure topology of A.

3 So by continuity $\lim_{d \in D} |f(d)| = 0$ so $\inf_{d \in D} |f(d)| = 0$ also.

4 But this means that $f \in \tau\text{-}\mathsf{ca}(A)$.

□

CountablyAdditive :: $\forall (A,\mu) \in \mathsf{MA} \ . \ \mathsf{ca}(A) \subset \mathsf{ac}(A,\mu)$

Proof $=$

1 Take $f \in \mathsf{ca}(A) \setminus \mathsf{ac}(A,\mu)$.

2 Then there exists $\varepsilon \in \mathbb{R}_{++}$ such that for all $\delta \in \mathbb{R}_{++}$ there is an element $a \in A$ with $\mu(a) \leq \delta$ anf $|f(a)| \geq \varepsilon$.

3 Select a sequence $a : \mathbb{N} \to A$ with $|f(a_n)| \geq \varepsilon$ and $\mu(a_n) \leq 2^{-n}$.

4 Define a decreasing sequence $c_n = \bigvee_{k=n}^{\infty} a_k$.

5 Then $\mu(c_n) = 2^{1-n} \to 0$.

6 So $\lim_{n \to \infty} c_n = 0$ and $\bigvee_{n=1}^{\infty} c_n = 0$.

7 Thus, $\inf_{n \in \mathbb{N}} |f(c_n)| = 0$.

8 on the other hand $f(c_n) \geq \varepsilon$ which leads to a contradiction.

□

QuasiSemifinite :: $\forall (A,\mu) \in \mathsf{MA} \ . \ ?(A \to \mathbb{R})$

φ : QuasiSemifinite $\iff \forall a \in A \ . \ \varphi(a) \neq 0 \Rightarrow \exists c \in A^f \ . \ \varphi(ac) \neq 0$

ContinuousIsQuasiSemifinite ::

:: $\forall (A, \mu) \in \mathsf{MA} \,.\, \forall f \in \mathsf{a}(A)$.

. $f \in C_0(A) \Rightarrow f \in \mathsf{ca}$ & QuasiSemifinite(A)

Proof =

1 We know that $f \in \tau\text{-}\mathsf{ca}(A)$, so $f \in \mathsf{ca}(A)$.

2 Assume $a \in A$ such that $f(a) \neq 0$.

3 Then $a \neq 0$ and $\mu(a) \neq 0$.

4 Assume $\mu(a) = \infty$.

5 If $\mu(c) = \infty$ for any $c \in A$ such that $c \leq a$ and $c \neq 0$ Then $\lim a = 0$.

6 And so $f(a) = \lim f(a) = 0$, which is impossible.

7 Therefore, $\{0\} \subsetneq C = A^f \cap (a)$.

8 Let $D = \{d \in A : d \leq a \,\&\, \forall u \in A \,.\, 0 < u \leq d \Rightarrow \mu(u) = \infty\}$.

9 Then $C \cup D$ is dense in (a).

10 Let $P \subset C \cup D$ be a partition of unity.

11 Note that $f(d) = 0$ by arguments simmilar to (5) and (6) for any $d \in D$.

12 Thus, $0 \neq f(a) = \sum_{p \in P} f(p) = \sum_{p \in P \cap C} f(p)$.

13 Therefore, there exists $c \in C$ such that $f(ac) = f(c) \neq 0$.

$\square$

SigmaAdditiveAndQuasiSemifiniteIsUniformlyContinuous ::

$$:: \forall (A,\mu) \in \mathsf{MA} \,.\, \forall f \in \mathsf{a}(A) \,.\, f \in \mathsf{ca} \;\&\; \texttt{QuasiSemifinite}(A) \Rightarrow f \in \mathsf{UNI}\Big((A,\mu),\mathbb{R}\Big)$$

Proof =

1 f is bounded, so there is a Jordan decomposition $f = f_+ - f_-$.

2 Define $g = f_+ + f_-$ and $\gamma = \sup\{g(a) | a \in A^f\}$.

3 Then there is a sequence of elements $a : \mathbb{N} \to A^f$ such that $\gamma = \lim_{n\to\infty} g(a_n)$.

4 Let $a^* = \bigvee_{n=1}^{\infty} a_n$.

5 If $d \in A$ and $a^* d = 0$ then $f(d) = 0$.

5.1 Assume $d \in A$ is such that $a^* d = 0$ and $c \in A^f$.

5.2 Then $|f(cd)| \le g(cd) \le g(c \setminus a_n) = g(a_n \vee c) - g(a_n) \le \gamma - g(a_n)$.

5.3 By taking the limit we see that $|f(cd)| = 0$ and hence $f(cd) = 0$.

5.4 As c was arbitrary as f is quasi-semifinite $f(d) = 0$.

6 Construct the sequence $c^*_n = \bigvee_{k=n}^{\infty} a_n$.

7 Then $\lim_{n\to\infty} g(a^* \setminus c^*_n) = 0$.

8 As f is countably additive it must be absolutely continuous.

9 Assume $\varepsilon \in \mathbb{R}_{++}$, then there is δ such that $|f(a)| \le \varepsilon$ having $\mu(a) < \delta$ for all $a \in A$.

10 Assume $n \in \mathbb{N}$ is such that $|g(a^* \setminus c^*_n)| < \varepsilon$.

11 Then $|f(a)| \le |f(ac^*_n)| + |f(a(a^* \setminus c^*_n))| + |f(a \setminus a^*)| \le |f(ac^*_n)| + g(a^* \setminus c^*_n) \le |f(ac^*_n)| + \varepsilon$ for any $a \in A$.

12 Assume $a, c \in A$ such that $\mu((b + c)c^*_n) < \delta$.

13 Then $|f(a) - f(c)| \le |f(a \setminus c)| + |f(c \setminus a)| \le |f((a \setminus c)c^*_n)| + |f((a \setminus d)c^*_n)| + 2\varepsilon \le 4\varepsilon$.

14 But this means that f is uniformly continuous.

□

AdditiveFunctionalContinuity :: $\forall (A,\mu) \in \mathsf{MA} \,.\, \forall f \in \mathsf{a}(A) \,.\, f \in C_0(A) \iff f \in \mathsf{UNI}(A,\mathbb{R})$

Proof =

This follows from the previous theorems.

□

SemifiniteAdditiveFunctionalContinuity ::

$:: \forall (A,\mu) :$ Semifinite $. \forall f \in \mathsf{a}(A) \ . \ f \in \tau\text{-}\mathsf{ca}(A) \iff f \in \mathsf{UNI}(A,\mathbb{R})$

Proof $=$

1 $(\Leftarrow)$ is obvious.

1.1 f is continuous in zero an hence completely additive.

2 $(\Rightarrow)$ consider $f \in \tau\text{-}\mathsf{ca}(A)$.

2.1 I will show that f is quasi-Semifinite.

2.1.1 Assume $a \in A$ is such that $f(a) \neq 0$.

2.1.2 Then $a \neq 0$.

2.1.3 $\{0\} \neq C = (a) \cap A^f$ is dense in (a) as μ is semifinite.

2.1.4 Let $P \subset C$ be a partiotion of unity for (a).

2.1.5 Then $0 \neq f(a) = \sum_{p \in P} f(p)$.

2.1.6 So there must be $c \in C$ such that $f(c) \neq 0$.

2.2 As f is also countably additive it must be uniformly continuous.

□

SigmaFiniteAdditiveFunctionalContinuity ::

$:: \forall (A,\mu) :$ σ-Finite $. \forall f \in \mathsf{a}(A) \ . \ f \in \tau\text{-}\mathsf{ca}(A) \iff f \in \mathsf{UNI}(A,\mathbb{R}) \iff f \in \mathsf{ca}(A)$

Proof $=$

1 σ-finite measure algebras are CCC.

2 So every countably additive functional must be completely additive.

□

FiniteAdditiveFunctionalContinuity ::

$:: \forall (A,\mu) :$ Finite $. \forall f \in \mathsf{a}(A)$.

$. \ f \in \tau\text{-}\mathsf{ca}(A) \iff f \in \mathsf{UNI}(A,\mathbb{R}) \iff f \in \mathsf{ca}(A) \iff f \in \mathsf{ac}(A,\mu)$

Proof $=$

1 Assume f is absolutely continuous with repsect to μ.

2 Also assume D is downwards directed in A with $\bigvee D = 0$.

3 So $\inf_{d \in D} \mu(d) = 0$.

3.1 This argument requires μ to be finite.

4 By absolute continuity $\inf_{d \in D} |f(d)| = 0$.

5 As D was arbitraty this means that f is completely continuous .

□

zeroIdealRespectingAdditiveFunctionals $:: \prod (X,\Sigma,\mu) \in \mathsf{MEAS}$. VectorSubspace$(\mathsf{a}(X,\Sigma,\mu))$

zeroIdealRespectingAddivieFunctionals $() = \mathsf{a}_0(X,\Sigma,\mu) := \{f \in \mathsf{a}(X,\Sigma,\mu) : \forall Z \in \mathcal{N}_\mu \ . \ f(Z) = 0\}$

$\texttt{canonicalAdditiveFunctionalsIsomorphism} :: \prod (X,\Sigma,\mu) \in \mathsf{MEAS}$.

. $\texttt{Isomorphism}\Big(\mathbb{R}\text{-}\mathsf{VS}, \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big), \mathsf{a}_0(X,\Sigma,\mu)\Big)$

$\texttt{canonicalAdditiveFunctionalsIsomorphism}\,(f) = \varphi(f) := f \circ \pi_{\mathcal{N}_\mu}$

1 The Mapping φ is cleary injective.

2 It is also bijective.

2.1 Assume $f \in \mathsf{a}_0(X,\Sigma,\mu)$.

2.2 Then there is an auxilary functional $\bar{f}$ defined by $\bar{f}[E] = E$.

2.3 This is well defined as f respects the ideal od zero sets.

2.4 Then obviously $\varphi(\bar{f}) = f$.

□

$\texttt{Isomorphism} :: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} . \forall f \in \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big) . f \in \mathsf{ac}\big(\mathsf{MA}(X,\Sigma,\mu)\big) \iff \varphi(f) \in \mathsf{ac}(X,\Sigma,\mu)$

$\texttt{Proof} =$

This is obvious .

□

$\texttt{IsomorphismCountablyAdditive} ::$

$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} . \forall f \in \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big) . f \in \mathsf{ca}\big(\mathsf{MA}(X,\Sigma,\mu)\big) \iff \varphi(f) \in \mathsf{ca}(X,\Sigma,\mu)$

$\texttt{Proof} =$

This is obvious .

□

$\texttt{IsomorphismCountablyAdditive} ::$

$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} . \forall f \in \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big) . f \in \mathsf{ca}\big(\mathsf{MA}(X,\Sigma,\mu)\big) \iff \varphi(f) \in \mathsf{ac}(X,\Sigma,\mu) \cap \mathsf{ca}(X,\Sigma,\mu)$

$\texttt{Proof} =$

This is obvious.

□

$\texttt{IsomorphismTruelyContinuous} ::$

$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} . \forall f \in \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big) . f \in C\big(\mathsf{MA}(X,\Sigma,\mu)\big) \iff \varphi(f) \in \mathsf{tc}(X,\Sigma,\mu)$

$\texttt{Proof} =$

This is obvious .

□

$\texttt{SemifiniteIsomorphismTruelyContinuous} ::$

$:: \forall (X,\Sigma,\mu) : \texttt{Semifinite} . \forall f \in \mathsf{a}\big(\mathsf{MA}(X,\Sigma,\mu)\big) . f \in \tau\text{-}\mathsf{ca}\big(\mathsf{MA}(X,\Sigma,\mu)\big) \iff \varphi(f) \in \mathsf{tc}(X,\Sigma,\mu)$

$\texttt{Proof} =$

This is obvious .

□

1.4.7 Radon-Nikodym's Isomorphism

$$\texttt{isomorphismOfRadonNikodym} :: \prod (X, \Sigma, \mu) : \texttt{Semifinite} \, .$$
$$. \, \texttt{Isomorphism}\Big(\mathsf{OVS}, \mathbf{L}^1(X, \Sigma, \mu), \tau\text{-}\mathsf{ca}\big(\mathsf{MA}(X, \Sigma, \mu)\big)\Big)$$
$$\texttt{isomorphismOfRadonNikodym}([f]) = \rho\nu[f] := \Lambda[E] \in \Sigma_\mu \, . \int_E f \, d\mu$$

1 The expression $\int_E f \, d\mu$ above is clearly well defined for an integrable f as $[E]$ is defined up to a set of the measure zero.

2 $[f]$ Is also defined up to function g equal to 0 almost everywhere μ so the whole operator $\rho\nu$ is well defined.

3 $\rho\nu$ is invertible.

3.1 Assume $f \in \tau\text{-}\mathsf{ca}(\Sigma_\mu, \bar{\mu})$.

3.2 Then $\varphi(f)$ is truely continuous additive functional on (X, Σ, μ) as this space is semifinite.

3.3 So by classical Radon-Nikodym's theorem there is $\dfrac{d\varphi(f)}{d\mu} \in L^1(X, \Sigma, \mu)$ such that $\varphi(f)(E) = \displaystyle\int_E \frac{d\varphi(f)}{d\mu} \, d\mu$ for any $E \in \Sigma$.

3.4 So define $(\rho\nu)^{-1}(f) = \left[\dfrac{d\varphi(f)}{d\mu}\right] \in \mathbf{L}^1(X, \Sigma, \mu)$.

4 The linearity and order preservation is pretty obvious for $\rho\nu$.

□

Question: Is this a natural equivalence of functors?

1.4.8 Standard Extension

`CompleteleAdditiveRestriction` ::

:: $\forall (A, \mu)$: `MeasureAlgebra` . $\forall a \in A^f$. $(\Lambda c \in A \,.\, \mu(ac)) \in \tau\text{-ca}(A)$

`Proof` =

1 As μ is additive its restriction is also additive.

2 Assume D is a downwards directed in A with $\bigwedge D = 0$.

3 Then aD is still downwards directed in A^f and $\bigwedge aD = 0$.

4 Note, that by choice of a the restriction is finitely additive.

5 Then $\inf\limits_{d \in D} \mu(ad) = \inf\limits_{d \in aD} \mu(d) = 0$.

$\square$

StandardExtensionLemma ::

$:: \forall (A,\mu) \in \texttt{FiniteMeasureAlgebra} \,.\, \forall (C,\nu) \subset_{\mathsf{MA}} (A,\mu) \,.\, \exists! \mathsf{ca}(C,\nu) \xrightarrow{R} \mathsf{ca}(A,\mu) : \mathsf{OVS} \,.$

$. \forall f \in \mathsf{ca}(C,\nu) \,.\, \forall \alpha \in \mathbb{R} \,.\, R(f)_{|C} = f \;\&\; [R(f) > \alpha\mu] = [f > \alpha\nu]$

Proof =

1 Represent (A,μ) as a measure algebra of the measure space $(X,\Sigma,\hat{\mu})$.

2 Then (C,ν) can be seen as a measure algebra of the measure space $(X,T,\hat{\nu})$ with $T \subset \Sigma$ and $\hat{\nu} = \hat{\mu}_{|T}$.

3 $\varphi(f) \in \mathsf{tc}(X,T,\hat{\nu})$ for any $f \in \mathsf{ca}(C,\nu)$.

4 So there is a Radon-Nikodym presentation $\phi = \rho\nu^{-1}(f)$ such that $f[E] = \int_E \phi \, d\hat{\nu}$ for any $E \in T$.

5 Define $R(f)[E] = \int_E \phi \, d\hat{\mu}$ for any $E \in \Sigma$.

6 $[R(f) > \alpha\mu] \in C$.

6.1 Define level sets $H_\alpha = \{x \in X : \phi(x) > \alpha\} \in T$.

6.2 $\int_E \phi \, d\mu > \alpha\hat{\mu}(E)$ if $E \subset H_\alpha$ and $\hat{\mu}(E) > 0$ for any $E \in \Sigma$.

6.3 $\int_E \phi \, d\mu \le \alpha\hat{\mu}(E)$ if $E \cap H_\alpha = \emptyset$ for any $E \in \Sigma$.

6.4 This can be rewritten in terms of measur algebras.

6.5 $R(f)(a) > \alpha\mu(a)$ if $a \le [H_\alpha]$ and $a \ne 0$ for any $a \in A$.

6.6 And $R(f)(a) \le \alpha\mu(a)$ if $a[H_\alpha] = 0$ for any $a \in A$.

6.7 Thus, $[R(f) > \alpha\mu] = [H_\alpha] \in C$.

7 Clearly $R(f)_{|C} = f$.

8 Note, that $\mu_{|C} = \nu$.

9 Therefore, $[R(f) > \alpha\mu] = [f > \alpha\nu]$ for all $\alpha \in \mathbb{R}$.

10 R is uinquily detrermined.

10.1 Assume g has all required properties.

10.2 Then $[g > \alpha\mu] = [f > \alpha\nu] = [R(f) > \alpha\mu]$ for all $\alpha \in \mathbb{R}$.

10.3 Then there is a measurable function $\gamma : X \to \mathbb{R}$ such that $\int_E \gamma \, d\mu = g[E]$ for any $E \in \Sigma$.

10.4 But then level sets of γ equal to level sets of ϕ modulo μ-measure zero.

10.5 So $\gamma = \phi$ μ-almost everywhere.

10.6 Thus $R(f) = g$.

□

standardExtension :: $\prod (A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \prod (C,\nu) \subset_{\mathsf{MA}} (A,\mu) \,.\, \mathsf{OVS}\Big(\mathsf{ca}(C,\nu), \mathsf{ca}(A,\mu)\Big)$

standardExtension $(f) = R(f) :=$ StandardExtensionLemma$\Big((A,\mu),(C,\nu),f\Big)$

Measure :: $\forall (A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \forall (C,\nu) \subset_{\mathsf{MA}} (A,\mu) \,.\, R(\nu) = \mu$

Proof =

In this case $\phi(x) = 1$.

□

$\texttt{PartitionOfUnity} ::$

$:: \forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \ . \ \forall (C, \nu) \subset_{\mathsf{MA}} (A, \mu) \ . \ \forall f : \mathbb{N} \to \mathsf{ca}_+(C, \nu) \ .$

$$. \ \forall \aleph : \forall c \in C \ . \ \nu(c) = \sum_{n=1}^{\infty} f_n(c) \ . \ \forall a \in A \ . \ \mu(a) = \sum_{n=1}^{\infty} R(f_n)(a)$$

$\texttt{Proof} \ =$

1 Let $\phi : \mathbb{N} \to L^1(X, T, \hat{\nu})$ be functional representations for f_n as in the previous theorem .

2 Define $\gamma_n = \displaystyle\sum_{k=0}^{n} \phi_k$.

3 Then γ is an increasing sequence such that $\lim\limits_{n \to \infty} \gamma_n = 1$ almost everywhere μ.

4 But then $\displaystyle\sum_{n=1}^{\infty} R(f_n)[E] = \sum_{n=1}^{\infty} \int_E \phi_n \, d\hat{\mu} = \lim_{n \to \infty} \int_E \gamma_n \, d\hat{\mu} = \int_E \lim_{n \to \infty} \gamma_n \, d\hat{\mu} = \int_E d\hat{\mu} = \mu[E]$.

4.1 Here we used monotonic convergence theorem.

$\square$

2 Maharam's Theory

3 Abstract Ergodic Theory

4 Measurable Algebras

Sources

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