General Topology

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1 Basics

1.1 Topological Sets

1.1.1 Topology and Topological Spaces

$$\texttt{Topology} :: \prod X \in \mathsf{SET} \,.\, ???X$$

$$T : \texttt{Topology} \iff X, \emptyset \in T \;\&$$
$$\forall A, B \in T \,.\, A \cap B \in T$$
$$\forall I \in: \mathsf{SET} \,.\, \forall U : I \to T \,.\, \bigcup_{i \in I} U_i \in T$$

$$\texttt{TopologicalSpace} ::? \sum_{X \in \mathsf{SET}} ??X$$

$$(X, T) : \texttt{ToplogogicalSpace} \iff T : \texttt{Topology}$$

$$\texttt{topologicalSpaceAsSet} :: \texttt{TopologicalSpace} \to \mathsf{SET}$$
$$\texttt{topologicalSpaceAsSet}\,(X, T) = \texttt{implicit}(X, T) := X$$

$$\texttt{topology} :: \prod (X, T) : \texttt{TopologicalSpace} \,.\, \texttt{Topology}(X)$$
$$\texttt{topology}\,() = \mathcal{T}(X, T) := T$$

$$\texttt{Open} :: \prod X : \texttt{TopologicalSpace} \,.\, ??X$$
$$U : \texttt{Open} \iff U \in \mathcal{T}(X)$$

$$\texttt{Closed} :: \prod X : \texttt{TopologicalSpace} \,.\, ??X$$
$$A : \texttt{Closed} \iff A^{\complement} \in \mathcal{T}(X)$$

$$\texttt{AlwaysClosed} :: \forall X : \texttt{TopologicalSpace} \,.\, \emptyset, X : \texttt{Closed}(X)$$

Proof =

...

□

$$\texttt{ClosedIntersection} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall I \in \mathsf{SET} \,.\, \forall A : I \to \texttt{Closed}(X) \,.\, \bigcap_{i \in I} A_i : \texttt{Closed}(X)$$

Proof =

...

□

$$\texttt{ClosedUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to \texttt{Closed}(X) \,.\, \bigcup_{i=1}^{n} A_i : \texttt{Closed}(X)$$

Proof =

...

□

$\texttt{AlwaysOpen} :: \forall X : \texttt{TopologicalSpace} \,.\, \emptyset, X : \texttt{Open}(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{OpenUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall I \in \mathsf{SET} \,.\, \forall U : I \to \texttt{Open}(X) \,.\, \bigcup_{i \in I} U_i : \texttt{Closed}(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{OpenIntersection} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall n \in \mathbb{N} \,.\, \forall U : n \to \texttt{Open}(X) \,.\, \bigcap_{i=1}^{n} U_i : \texttt{Open}(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{BaseOfTopology} :: \prod X : \texttt{TopologicalSpace} \,.\, ?\mathcal{T}(X)$

$\mathcal{B} : \texttt{BaseOfTopology} \iff \forall U \in \mathcal{T} \,.\, \exists I \in \mathsf{SET} : \exists B : I \hookrightarrow \mathcal{B} : U = \bigcup_{i \in I} B_i$

$\texttt{oprnNeighbourhood} :: \prod X : \texttt{TopologicalSpace} \,.\, X \to ?\mathcal{T}(X)$

$\texttt{openNeighbourhood}\,(x) = \mathcal{U}(x) := \{U \in \mathcal{T}(X) | x \in U\}$

$\texttt{neighbourhood} :: \prod X : \texttt{TopologicalSpace} \,.\, X \to ?X$

$\texttt{neighbourhood}\,(x) = \mathcal{N}(x) := \Big\{A \subset X | \exists U \in \mathcal{U}(x) \,.\, U \subset A\Big\}$

$\texttt{BasisEqDef} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{B} \in ?\mathcal{T}(X) \,.\, \forall \mathcal{B} : \texttt{Base}(X) \iff$

$\iff \forall x \in X \,.\, \forall U \in \mathcal{U}(x) \exists B \in \mathcal{B} : x \in B \subset U$

$\texttt{Proof} \; =$

$\texttt{Assume}\ [1] : \Big(\mathcal{B} : \texttt{Base}\Big),$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$\Big(\mathcal{I}, B, [2]\Big) := \eth\texttt{Base}(B)(U) : \sum \mathcal{I} : \mathsf{SET} \,.\, \sum I \hookrightarrow \mathcal{B} \,.\, U = \bigcup_{i \in I} B_i,$

$[3] := \eth\mathcal{U}(x) : x \in U,$

$\Big(i, [1.*.1]\Big) := \texttt{UnionElement}(I, B)[1][2] : \sum i \in I \,.\, x \in B_i,$

$[1.*.2] := \texttt{UnionSubset}(I, B)(B_i)[1] : B_i \subset U,$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big(\mathcal{B} : \texttt{Base}(X)\Big) \Rightarrow \forall x \in X \,.\, \forall U \in \mathcal{U}(x) \,.\, \exists B \in \mathcal{B} : x \in B \subset U,$

$\texttt{Assume}\ [2] : \forall x \in X \,.\, \forall U \in \mathcal{U}(x) \,.\, \exists B \in \mathcal{B} : x \in B \subset U,$

$\texttt{Assume}\ U : \mathcal{T}(X),$

$(B, [3]) := \Lambda x \in U \,.\, [2](x, U) : \prod_{x \in U} \sum_{B_x \in \mathcal{B}} x \in B_x \subset U,$

$[4] := \texttt{UnionElement}[3]\eth^{-1}\texttt{Subset} : U \subset \bigcup_{x\in U} B_x,$

$[5] := \texttt{SubsetUnion} : \bigcup_{x\in U} B_x \subset U,$

$[U.*] := \eth^{-1}\texttt{SetEq}[4][5] : U = \bigcup_{x\in U} B_x;$

$\leadsto [3] := \eth^{-1}\texttt{Base}(X) : \Big(\mathcal{B} : \texttt{Base}(X)\Big);$

$\leadsto [*] := I(\Rightarrow)[1]I(\iff) : \Big(\mathcal{B} : \texttt{Base}(X)\Big) \iff \forall x \in X \,.\, \forall U \in \mathcal{U}(x) \,.\, \exists B \in \mathcal{B} : x \in B \subset U;$

$\square$

$\texttt{weightOfTopology} :: \texttt{TopologicalSpace} \to \mathsf{CARD}$

$\texttt{weightofTopology}\,(X) = w(X) := \min\Big\{|\mathcal{B}|\Big|\mathcal{B} : \texttt{Base}(X)\Big\}$

$\texttt{PotentialBase} :: \prod X \in \mathsf{SET} \,.\, ??X$

$\mathcal{B} : \texttt{PotentialBase} \iff \forall x \in X \,.\, \exists B \in \mathcal{B} : x \in B\ \&$

$\quad \& \ \forall B, B' \in \mathcal{B} \,.\, x \in B \cap B' \Rightarrow \exists B'' \in \mathcal{B} : x \in B'' \subset B' \cap B''$

$\texttt{generateTopologyByBase} :: \prod X \in \mathsf{SET} \,.\, \texttt{PotentialBase}(X) \to \texttt{TopologicalSpace}$

$\texttt{generateTopologyByBase}\,(\mathcal{B}) = \langle\mathcal{B}\rangle_{\mathsf{TOP}} := \left(X, \left\{\bigcup_{B\in\mathcal{B}'} B \middle| \mathcal{B}' \subset B\right\}\right)$

$\texttt{Subbase} :: \prod X : \texttt{TopologicalSpace} \,.\, ?\mathcal{T}(X)$

$\mathcal{B} : \texttt{Subbase} \iff \forall U \in \mathcal{T}(X) \,.\, \exists \mathcal{I} \in \mathsf{SET} : \exists n : \mathcal{I} \to \mathbb{N} \,.\, \exists B : \prod_{i\in\mathcal{I}} n_i \to \mathcal{B} : U = \bigcup_{i\in I}\bigcap_{j=1}^{n_i} B_{i,j}$

$\texttt{PotentialSubbase} :: \prod X \in \mathsf{SET} \,.\, ??(X)$

$\mathcal{B} : \texttt{PotentialSubbase} \iff \forall x \in X \,.\, \exists B \in \mathcal{B} \,.\, x \in \mathcal{B}$

$\texttt{generateTopologyBySybbase} :: \prod X \in \mathsf{SET} \,.\, \texttt{PotentialSubbase}(X) \to \texttt{TopologicalSpace}$

$\texttt{generateTopologyBySubbase}\,(\mathcal{B}) = \langle\langle\mathcal{B}\rangle\rangle_{\mathsf{TOP}} := \left\langle\left\{\bigcap_{B\in\mathcal{B}'} B \middle| \mathcal{B}' : \texttt{Finite}(\mathcal{B})\right\}\right\rangle_{\mathsf{TOP}}$

$\texttt{BaseAt} :: \prod X : \texttt{TopologicalSpace} \,.\, \prod x \in X \,.\, ?\mathcal{U}(x)$

$\mathcal{B} : \texttt{BaseAt} \iff \forall U \in \mathcal{U} \,.\, \exists B \in \mathcal{B} : U \subset B$

$\texttt{BaseLocalization} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{B} : \texttt{Base}(X) \,.\, \forall x \in X \,.\, \mathcal{U}(x) \cap \mathcal{B} : \texttt{BaseAt}(x)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{discreteSpace} :: \mathsf{SET} \to \texttt{TopologicalSpace}$

$\texttt{discreteSpace}\,(X) := (X, 2^X)$

$\texttt{BaseFromLocals} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{B} : \prod_{x \in X} \texttt{BaseAt}(x) \,.\, \bigcup_{x \in X} \mathcal{B}(x) : \texttt{Base}(X)$

Proof $=$

...

$\square$

$\texttt{characterOfPoint} :: \prod X : \texttt{TopologicalSpace} \,.\, X \to \mathsf{CARD}$

$\texttt{characterOfPoint}\,(x) = \chi(x) := \min \left\{ |\mathcal{B}| \middle| \mathcal{B} : \texttt{BaseAt}(x) \right\}$

$\texttt{characterOfSpace} :: \texttt{TopologicalSpace} \to \mathsf{CARD}$

$\texttt{characterOfPoint}\,(X) = \chi(X) := \sup_{x \in X} \chi(x)$

$\texttt{FirstCountable} :: ?\texttt{TopologicalSpace}$

$X : \texttt{FirstCountable} \iff \chi(X) \leq \aleph_0$

$\texttt{SecondCountable} :: ?\texttt{TopologicalSpace}$

$X : \texttt{SecondCountable} \iff w(X) \leq \aleph_0$

$\texttt{OpenByInnerCover} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall U \in ?X \,.\, \Big(\forall u \in U \,.\, \exists O \in \mathcal{U}(x) : O \subset U \Big) \Rightarrow U \in \mathcal{T}(X)$

Proof $=$

...

$\square$

$\texttt{SimplifyOpenUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall c \in \mathsf{CARD} \,.\, \forall [0] : w(X) \leq c \,.\, \forall I \in \mathsf{SET} \,.$
$\quad .\, \forall U : I \to \mathcal{T}(X) \,.\, \exists J \subset I : |J| \leq c \,\&\, \bigcup_{i \in I} U_i = \bigcup_{j \in J} U_j$

$\texttt{Proof} \;=$

$\Big(\mathcal{B}, [1]\Big) := \eth w(X)[0] : \sum \mathcal{B} : \texttt{Base}(X) \,.\, |\mathcal{B}| \leq c,$

$\mathcal{B}' := \Big\{ B \in \mathcal{B} : \exists i \in I : B \subset U_i \Big\} :?\mathcal{T}(X),$

$\alpha := \jmath \mathcal{B}' : \sum B \in \mathcal{B}' \,.\, \sum i(B) \in I \,.\, B \subset U_{\alpha(B)},$

$J := \alpha(\mathcal{B}) :?I,$

$[2] := \texttt{ImageCardinality}(J)\texttt{SubsetCardinality}(\mathcal{B}') : |J| < c,$

$\texttt{Assume}\ x : \bigcup_{i \in I} U_i,$

$\Big(i, [3]\Big) := \eth\texttt{union}(x) : \sum i \in I \,.\, x \in U_i,$

$\Big(\mathcal{B}'', [4]\Big) := \eth\texttt{Base}(\mathcal{B})(U_i) : \sum \mathcal{B}'' \subset \mathcal{B} \,.\, U_i = \bigcup \mathcal{B}'',$

$[5] := \jmath \mathcal{B}' \texttt{UnionSubset}[4] : \mathcal{B}'' \subset \mathcal{B}',$

$[x.*] := [3][4]\jmath\mathcal{B}'\texttt{LargerUnion}[5] : x \in \bigcup_{B \in \mathcal{B}''} U_{\alpha(B)} \subset \bigcup_{j \in J} U_j;$

$\rightsquigarrow [3] := \eth^{-1}\texttt{Subset} : \bigcup_{i \in I} U_i \subset \bigcup_{j \in J} U_j,$

$[4] := \texttt{LargerUnion}(J) : \bigcup_{j \in J} U_j \subset \bigcup_{i \in I} U_i,$

$[*] := \eth^{-1}\texttt{SetEq}[3][4] : \bigcup_{i \in I} U_i = \bigcup_{j \in J} U_j;$

$\square$

$\texttt{SimplifyBase} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall c \in \mathsf{CARD} \,.\, \forall [0] : w(X) \leq c \,.\, \forall \mathcal{B} : \texttt{Base}(X) \,.$
$\quad .\, \exists \mathcal{B}' \subset \mathcal{B} : |\mathcal{B}| \leq c \,\&\, \mathcal{B}' : \texttt{Base}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : c \geq \aleph_0,$

$\Big(\mathcal{A}, [2]\Big) := \eth w(X)[0] : \sum \mathcal{A} : \texttt{Base}(X) \,.\, |\mathcal{A}| \leq c,$

$\beta := \Lambda A \in \mathcal{A} \,.\, \{B \in \mathcal{B} : A \subset B\} : \mathcal{A} \to ?\mathcal{B},$

$\texttt{Assume}\ A : \mathcal{A},$

$\Big(I, B, [3]\Big) := \eth\texttt{Base}(X)(\mathcal{B}) : \sum I \in \mathsf{SET} \,.\, \sum B : I \to \mathcal{B} \,.\, A = \bigcup_{i \in I} B_i,$

$\Big(J_A, [A.1]\Big) := \texttt{SimplifyOpwnUnion}(X, x, [0]I, B) : \sum J_A \subset I \,.\, |J_A| \leq c \,\&\, \bigcup_{j \in J_A} B_j = A,$

$B^A := B_{|J} : J \to \mathcal{B};$

$\rightsquigarrow (J, B, [3]) := I\left(\prod\right)) : \prod A \in \mathcal{A} \,.\, \sum J_A \subset I \,.\, \sum B^A : J_A \to \mathcal{B} \,.\, |J| \leq c \,\&\, A = \bigcup_{j \in J_A} B^A_j,$

$\mathcal{B}' := \Big\{ B^A_j \Big| A \in \mathcal{A}, j \in J_A \Big\} :?\mathcal{B},$

$[4] := \texttt{InfiniteProductCard}\jmath\mathcal{B}'[1][2][3] : |\mathcal{B}'| \leq c,$

$\texttt{Assume}\ x : U,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$\Big(A, [5]\Big) := \texttt{BaseLocalization}(\mathcal{A})\eth\texttt{BaseAt}(x, U) : \sum A \in \mathcal{A} \ . \ x \in A \subset U,$

$\Big(B, [6]\Big) := [3]\jmath\beta\texttt{UnionSubset}(A)\eth\texttt{union} : \sum B \in \mathcal{B}' \ . \ B \in \beta(A) \ \& \ x \in B,$

$[x.*] := \texttt{SubsetTransitivity}\jmath\beta[6] : B \subset U;$

$\rightsquigarrow [1.*] := \eth^{-1}\texttt{BaseAt}(X, x)\texttt{BaseFromLocals} : \Big(\mathcal{B}' : \texttt{Base}\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : c \geq \aleph_0 \Rightarrow \exists \mathcal{B}' \subset \mathcal{B} \ . \ |\mathcal{B}'| \leq c \ \& \ \mathcal{B}' : \texttt{Base}(X),$

$\texttt{Assume}\ [2] : c < \aleph_0,$

$\Big(\mathcal{A}, [3]\Big) := \eth w(X)[0] : \sum \mathcal{A} : \texttt{Base}(X) \ . \ |\mathcal{A}| = w(X),$

$\texttt{Assume}\ B : \mathcal{B},$

$(I_B, A^B, [3]) := \eth\texttt{Base}(\mathcal{A})(B) : \sum I_B \in \mathsf{SET} \ . \ A^B : I_B \hookrightarrow \mathcal{A} \ . \ B = \bigcup_{i \in I_A} A^B_i;$

$\rightsquigarrow (I, A, [4]) := I\Big(\prod\Big) : \prod B \in \mathcal{B} \ . \ \sum I_B \in \mathsf{SET} \ . \ \sum A^B : I_B \hookrightarrow \mathcal{A} \ . \ B = \underline{,}$

$\texttt{Assume}\ A : \mathcal{A},$

$(J_A, B^A, [5]) := \eth\texttt{Base}(\mathcal{B})(A) : \sum J_A \in \mathsf{SET} \ . \ B^A : J_A \hookrightarrow \mathcal{B} \ . \ A = \bigcup_{j \in J_B} B^A_j,$

$[6] := [5][4] : A = \bigcup_{j \in J_A} \bigcup_{i \in I_{B_j}} A^{B_j}_i,$

$\mathcal{A}' := \Big\{A^{B_j}_i | j \in J_A, i \in I_{B_i}\Big\} :?\mathcal{A},$

$[7] := \jmath\mathcal{A}'[6] : A = \bigcup \mathcal{A}',$

$[8] := \eth w(X)\eth\texttt{min}[0][2][3][7] : A \in \mathcal{A},$

$[9] := [6]\texttt{SubsetUnion} : \forall a \in \mathcal{A}' \ . \ a \subset A,$

$[A.*] := \jmath\mathcal{A}[9][5] : A \in \mathcal{B};$

$\rightsquigarrow [2.*] := \eth^{-1}\texttt{Subset} : \mathcal{A} \subset \mathcal{B};$

$\rightsquigarrow [*] := I(\Rightarrow)[1]\texttt{LETrichtomy}(\mathsf{CARD}) : \sum \mathcal{B}' \subset \mathcal{B} \ . \ |\mathcal{B}'| \leq c \ \& \ \mathcal{B}' : \texttt{Base}(X);$

$\square$

$\texttt{FinerTopologyExists} :: \forall X : \mathsf{SET} \ . \ \forall T :?\texttt{Topology}(X) \ . \ \exists t : \texttt{Topology}(X) : t = \sup T$

$\texttt{Proof} =$

$\square$

$\texttt{CoarsestTopologyExists} :: \forall X : \mathsf{SET} \ . \ \forall T :?\texttt{Topology}(X) \ . \ \exists t : \texttt{Topology}(X) : t = \inf T$

$\texttt{Proof} =$

$\square$

1.1.2 Closure and Interior

$$\texttt{closure} :: \prod X : \texttt{TopologicalSpace} \,.\, ?X \to \texttt{Closed}(X)$$

$$\texttt{closure}\,(A) = \overline{A} = \mathrm{cl}_X\ A := \bigcap \left\{ K : \texttt{Closed}(X) : A \subset K \right\}$$

$$\texttt{EquivalentClosure1} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \in ?X \,.\, \overline{A} = \left\{ x \in X : \forall U \in \mathcal{U}(x) \,.\, U \cap A \neq \emptyset \right\}$$

$\texttt{Proof}\ =$

$Z := \left\{ x \in X : \forall U \in \mathcal{U}(x) \,.\, U \cap A \neq \emptyset \right\} :?X,$

$\texttt{Assume}\ x : Z^{\complement},$

$\left(U, [1]\right) := \eth\texttt{complement}\jmath Z : \sum U \in \mathcal{U}(x) \,.\, U \cap A = \emptyset,$

$[x.*] := [1]\jmath Z : U \subset Z^{\complement};$

$\rightsquigarrow [1] := \texttt{OpenByInnerCover} : Z^{\complement} \in \mathcal{T}(X),$

$[2] := \eth^{-1}\texttt{Closed}(X) : \left(Z : \texttt{Closed}\right),$

$[3] := \eth^{-1}\texttt{clusereIntersectionSubset} : \overline{A} \subset Z,$

$\texttt{Assume}\ x : Z,$

$\texttt{Assume}\ K : \texttt{Closed}(X),$

$\texttt{Assume}\ [4] : A \subset K,$

$\texttt{Assume}\ [5] : x \notin K,$

$[6] := \eth\texttt{compliment}[5]\eth\texttt{Closed}(X)\eth^{-1}\mathcal{U}(x) : K^{\complement} \in \mathcal{U}(x),$

$[7] := \texttt{ComplementSubset}[4] : A \cap K^{\complement} = \emptyset,$

$[8] := \jmath Z[7] : x \notin Z,$

$[K.*] := \texttt{InAndNotIn}[4] : \bot;$

$\rightsquigarrow [4] := E(\bot)I(\Rightarrow)I(\forall) : \forall K : \texttt{Closed}(X) \,.\, A \subset K \Rightarrow x \in K,$

$[x.*] := \eth^{-1}\texttt{closure}[4] : x \in \overline{A};$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Subset}[3]\eth\texttt{SetEq} : \overline{A} = Z;$

$\square$

$$\texttt{EquivalentClosure2} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \in ?X \,.$$
$$.\, \overline{A} = \left\{ x \in X : \exists \mathcal{B} : \texttt{BaseAt}(x) \,.\, \forall B \in \mathcal{B} \,.\, B \cap A \neq \emptyset \right\}$$

$\texttt{Proof}\ =$

$\dots$

$\square$

$$\texttt{PotentiallyClosedSet} :: \prod X : \mathsf{SET} \,.\, ???X$$

$$\mathcal{A} : \texttt{PotentiallyClosedSet} \iff \forall \emptyset, X \in \mathcal{A} \,.\ \&$$
$$\&\ \forall A, A' \in \mathcal{A} \,.\, A \cup A' \in \mathcal{A}\ \&$$
$$\&\ \forall \mathcal{A}' \subset \mathcal{A} \,.\, \bigcap \mathcal{A}' \in \mathcal{A}$$

$\texttt{PotentialClosure} :: \prod X : \mathsf{SET}\ .\ ?\left(?X \to ?X\right)$

$c : \texttt{PotentialClosure} \iff c(\emptyset) = \emptyset\ \&$
$\quad \&\ \forall A, B \subset X\ .\ A \subset c(A)\ \&$
$\quad \&\ c^2(A) = c(A)\ \&$
$\quad \&\ c(A \cup B) = c(A) \cup c(B)$

$\texttt{CloserIsPotentialClosure} :: \forall X : \texttt{TopologicalSpace}\ .\ \mathrm{cl}_X : \texttt{PotentialClosure}$

$\texttt{Proof} =$

$[1] := \texttt{AlwaysClosed}(X)\texttt{ClosedClosure}\eth\texttt{closure}(X) : \overline{\emptyset} = \emptyset,$

$\texttt{Assume}\ A, B :?X,$

$\left[(A, B).*.1\right] := \texttt{IntersectSubset}\eth\texttt{closure}(X) : A \subset \overline{A},$

$\left[(A, B).*.2\right] := \texttt{ClosedClosure}\eth\texttt{closure}(X) : \overline{\overline{A}} = \overline{A},$

$[2] := \dots\texttt{UnionSubset} : A \subset \overline{A} \cup \overline{B},$

$[3] := \dots\texttt{UnionSubset} : B \subset \overline{A} \cup \overline{B},$

$[4] := \texttt{SubsetUnion}[2][3] : A \cap B \subset \overline{A} \cup \overline{B},$

$[5] := \texttt{ClosedUnion}[4] : \overline{A} \cup \overline{B} : \texttt{Closed}(X),$

$[6] := \texttt{IntersectSubset}\eth\texttt{closure}[5] : \overline{A \cup B} \subset \overline{A} \cup \overline{B},$

$[7] := \texttt{UnionSubset}\big(A, (A, B)\big)\texttt{IntersectSubset}\eth\texttt{closure}(A) : \overline{A} \subset \overline{A \cup B},$

$[8] := \texttt{UnionSubset}\big(B, (A, B)\big)\texttt{IntersectSubset}\eth\texttt{closure}(B) : \overline{B} \subset \overline{A \cup B},$

$[9] := \texttt{SubsetUnion}(A, B)[7][8] : \overline{A} \cup \overline{B} \subset \overline{A \cup B},$

$\left((A, B).*.3\right) := \eth\texttt{SetEq}[9][6] : \overline{A \cup B} = \overline{A} \cup \overline{B};$

$\rightsquigarrow [*] := \eth\texttt{PotentialClosure} : \left(\mathrm{cl}_X : \texttt{PotentialClosure}(X)\right);$

$\square$

$\texttt{PotentialClosureOperatorIsMonotonic} :: \forall X \in \mathsf{SET}\ .\ \forall c : \texttt{PotentialClosure}(X)\ .\ c \in \mathrm{End}_{\mathsf{CAT}}\left(\mathsf{P}(?X)\right)$

$\texttt{Proof} =$

$\texttt{Assume}\ A, B : \mathsf{P}(?X),$

$\texttt{Assume}\ [1] : A \subset B,$

$[2] := \texttt{UnionWithSubset} : A \cup B = B,$

$[3] := [2]\eth\texttt{PotentialClosure}(c) : c(B) = c(A \cup B) = c(A) \cup c(B),$

$\left[(A, B).*\right] := \texttt{UnionWithSubset}[3] : c(A) \subset c(B);$

$\rightsquigarrow [*] := \eth\mathsf{P}(?X) : c \in \mathrm{End}_{\mathsf{CAT}}\left(\mathsf{P}(?X)\right);$

$\square$

$\texttt{ImageOfClosureOperator} :: \forall X \in \mathsf{SET}\ .\ \forall c : \texttt{PotentialClosure}(X)\ .\ \mathrm{Im}\, c : \texttt{PotentialClosedSets}(X)$

$\texttt{Proof} =$

$[1] := \eth\texttt{PotentialClosure}\eth\,\mathrm{Im}\, c : c(\emptyset) = \emptyset \in \mathrm{Im}\, c,$

$[2] := \eth\texttt{PotentialClosure}\eth\,\mathrm{Im}\, C : A \subset c(A) = A \in \mathrm{Im}\, c,$

$\texttt{Assume}\ A, B : \mathrm{Im}\, C,$

$(A', B', [3]) := \eth\,\mathrm{Im}\, C(A, B) : \sum A', B' \in 2^X\ .\ A = c(A')\ \&\ B = c(B'),$

$\left[(A, B).*\right] := [3]\eth\texttt{PotentialClosure}\eth\,\mathrm{Im}\, c : A \cup B = c(A') \cup c(B') = c(A' \cup B') \in \mathrm{Im}\, c;$

$\rightsquigarrow [3] := I(\forall) : \forall A, B \in \operatorname{Im} c \,.\, A \cap B = \operatorname{Im} c,$

$\texttt{Assume}\ \mathcal{A} :?(\operatorname{Im} c),$

$[5] := \texttt{PotentialClosureIsMonotonic}(c)(THMIntersectSubset(\mathcal{A}') : \forall A \in \mathcal{A} \,.\, c\left(\bigcap \mathcal{A}\right) \subset c(A),$

$[6] := \texttt{SubsetInersect}[4] : c\left(\bigcap \mathcal{A}'\right) \subset \bigcap c(\mathcal{A}) = \bigcap \mathcal{A},$

$[7] := \eth\texttt{PotentialClosure}(A) : \bigcap \mathcal{A} \subset c\left(\bigcap \mathcal{A}\right),$

$[\mathcal{A}.*] := \eth\texttt{SetEq}[6][7]\eth \operatorname{Im} c : \bigcap \mathcal{A} = c\left(\bigcap \mathcal{A}\right) \in \operatorname{Im} c);$

$\rightsquigarrow [*] := [1][2][3]\eth\texttt{PotentialClosedSets}(X) : \left(\operatorname{Im} C : \texttt{PotentialClosedSets}(X)\right);$

$\square$

$\texttt{generateTopologyByClosedSets} :: \prod X \in \mathsf{SET} \,.\, \texttt{PotentialClosedSets} \to \texttt{TopologicalSpace}$

$\texttt{generateTopologyByClosedSets}\,(\mathcal{A}) := \left(X, \{A^\complement | A \in \mathcal{A}\}\right)$

$\texttt{generateTopologyByClosure} :: \prod X \in \mathsf{SET} \,.\, \texttt{PotentialClosure} \to \texttt{TopologicalSpace}$

$\texttt{generateTopologyByClosure}\,(c) := \texttt{generateTopologyByClosedSets}(\operatorname{Im} c)$

$\texttt{interior} :: \prod X : \texttt{TopologicalSpace} \,.\, 2^X \to \mathcal{T}(X)$

$\texttt{interior}\,(A) = \operatorname{int} A := \bigcup \{U \in \mathcal{T}(X) | U \subset A\}$

$\texttt{EquivalentInterior} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \in 2^X \,.\, \forall x \in A \,.\, x \in \operatorname{int} A \iff \exists U \in \mathcal{U}(x) \,.\, U \subset A$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{InteriorAsDifference} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \in 2^X \,.\, \operatorname{int} A = X \setminus \operatorname{cl} A^\complement$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{PotentialInterior} :: \prod X \in \mathsf{SET} \,.\, ?\left(?X \to ?X\right)$

$i : \texttt{PotentialInterior} \iff i(X) = X\ \&$

$\quad \&\ \forall A, B \subset X \,.\, i(A) \subset A\ \&$

$\quad \&\ i^2(A) = i(A)\ \&$

$\quad \&\ i(A \cap B) = i(A) \cap i(B)$

$\texttt{InteriorIsPotentialInterior} :: \forall X : \texttt{TopologicalSpace} \,.\, \operatorname{int}_X : \texttt{PotentialInterior}(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{InteriorIsMonotonic} :: \forall X \in \mathsf{SET} \,.\, \forall i : \texttt{PotentialInterior} \,.\, i \in \mathrm{End}_{\mathsf{POSET}}(?X)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{InteriorImageIsTopology} :: \forall X \in \mathsf{SET} \,.\, \forall i : \texttt{PotentialInterior} \,.\; \mathrm{Im}\, i : \texttt{Topology}(X)$

$\texttt{Proof} \; =$

$\texttt{generateTopologyByInterior} :: \prod X \in \mathsf{SET} \,.\, \texttt{PotentialInterior}(X) \to \texttt{TopologicalSpace}$

$\texttt{generateTopologyByInterior}\,(i) := (X, \mathrm{Im}\, i)$

$\texttt{ClosureUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{A} : \texttt{Finite}(?X) \,.\, \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}}$

$\texttt{Proof} \; =$

$\texttt{LocallyFinite} :: \prod X : \mathsf{SET} \,.\, ???X$

$\mathcal{A} : \texttt{LocallyFinite} \iff \forall x \in X \,.\, \exists U \in \mathcal{U}(x) : \Big|\big\{A \in \mathcal{A} : U \cap \mathcal{A} = \emptyset\big\}\Big| < \infty$

$\texttt{Discrete} :: \prod X : \mathsf{SET} \,.\, ???X$

$\mathcal{A} : \texttt{Discrete} \iff \forall x \in X \,.\, \exists U \in \mathcal{U}(x) : \Big|\big\{A \in \mathcal{A} : U \cap A \neq \emptyset\big\}\Big| = 1$

$\texttt{LocallyFiniteUnionClosure} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{A} : \texttt{LocallyFinite}(X) \,.\, \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}}$

$\texttt{Proof} \; =$

$[1] := \Lambda A \in \mathcal{A} \,.\, \texttt{UnionSubset}(A, \mathcal{A})\texttt{ClosureIsMonotonic}(\mathrm{cl}_X) : \forall A \in \mathcal{A} \,.\, \overline{A} \subset \overline{\bigcup \mathcal{A}},$

$[2] := \texttt{UnionSubset}[1] : \bigcup \overline{\mathcal{A}} \subset \overline{\bigcup \mathcal{A}},$

$\texttt{Assume}\; x : \overline{\bigcup \mathcal{A}},$

$\Big(U, [3]\Big) := \eth\texttt{LocallyFinite}(X)(\mathcal{A}) : \sum U \in \mathcal{U}(x) \,.\, \Big|\big\{A \in \mathcal{A} : A \cap U \neq \emptyset\big\}\Big| < \infty,$

$\mathcal{A}' := \big\{A \in \mathcal{A} : A \cap U \neq \emptyset\big\} :?\mathcal{A},$

$[4] := \texttt{EquivavalentClosure1}\jmath\mathcal{A} : x \notin \overline{\bigcup \mathcal{A} \setminus \mathcal{A}'},$

$[5] := \eth x\texttt{ClosureUnion} : x \in \overline{\bigcup \mathcal{A}} = \overline{\bigcup \mathcal{A}'} \cup \overline{\bigcup \mathcal{A} \setminus \mathcal{A}'},$

$[*.x] := [4][5]\texttt{ClosureUnion}(\mathcal{A}')[3]\texttt{LargerUnion}(\mathcal{A}', \mathcal{A}) : x \in \overline{\bigcup \mathcal{A}'} = \bigcup \overline{\mathcal{A}'} \subset \bigcup \overline{\mathcal{A}};$

$\rightsquigarrow [3] := \eth^{-1}\texttt{Subset} : \overline{\bigcup \mathcal{A}} \subset \bigcup \overline{\mathcal{A}},$

$[*] := \eth^{-1}\texttt{SetEq} : \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}};$

$\square$

$\texttt{LocallyFiniteClosedUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall \mathcal{A} : \texttt{LocallyFinite}(X) \,.$

$\quad .\, \forall [0] : \forall A \in \mathcal{A} \,.\, A : \texttt{Closed}(X) \,.\, \bigcup \mathcal{A} : \texttt{Closed}(X)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{LocallyFiniteClosureIsLocallyFinite} :: \forall X : \texttt{TopologicalSpace} . \forall \mathcal{A} : \texttt{LocallyFinite}(X) .$
$. \overline{\mathcal{A}} : \texttt{LocallyFinite}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : X,$

$\Big(U, [1]\Big) := \eth\texttt{LocallyFinite}(X) : \sum U \in \mathcal{U}(x) . \Big|\big\{A \in \mathcal{A} : A \cap U \neq \emptyset\big\}\Big| < \infty,$

$\mathcal{A}' := \big\{A \in \mathcal{A} : A \cap U \neq \emptyset\big\} : \texttt{Finite}(\mathcal{A}),$

$\texttt{Assume}\ A : \mathcal{A}'^{\complement},$

$\texttt{Assume}\ [2] : \overline{A} \cap U \neq \emptyset,$

$y := \eth\texttt{NonEmpty} : \overline{A} \cap U,$

$[3] := \texttt{EquivalentClosure}(y) : \forall O \in \mathcal{U}(y) . O \cap A \neq \emptyset,$

$[4] := [3](U) : U \cap A \neq \emptyset,$

$[5] := \jmath\mathcal{A}'[4] : A \in \mathcal{A}',$

$[A.*] := \eth\texttt{complement}\eth A\texttt{InAndNotIn}[5] : \bot;$

$\rightsquigarrow [x.*] := E(\bot)\jmath\mathcal{A}'[1] : \Big|\big\{A \in \mathcal{A} : \overline{A} \cap U \neq \emptyset\big\}\Big| < \infty;$

$\rightsquigarrow [*] := \eth\texttt{LocallyFinite}(A) : \Big(\overline{\mathcal{A}} : \texttt{LocallyFinite}(X)\Big);$

$\square$

$\texttt{DiscreteClosureIsDiscrete} :: \forall X : \texttt{TopologicalSpace} . \forall \mathcal{A} : \texttt{Discrete}(X) . \overline{\mathcal{A}} : \texttt{Discrete}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ClosureIntersection} :: \forall X : \texttt{TopologicalSpace} . \forall A, B \subset X . \overline{A \cap B} \subset \overline{A} \cap \overline{B}$

$\texttt{Proof} \ =$

$[1] := \texttt{ClosureIsMonotonic}(A) : A \subset \overline{A},$

$[2] := \texttt{ClosureIsMonotonic}(B) : B \subset \overline{B},$

$[3] := \texttt{SubsetIntersect}[1][2] : A \cap B \subset \overline{A} \cap \overline{B},$

$[4] := \texttt{ClosedIntersection}(\overline{A}, \overline{B}) : \overline{A} \cap \overline{B} : \texttt{Closed}(X),$

$[*] := \eth\texttt{closure}[3][4] : \overline{A \cap B} \subset \overline{A} \cap \overline{B};$

$\square$

$\texttt{ClosureOfDifference} :: \forall X : \texttt{TopologicalSpace} . \forall A, B \subset X . \overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \overline{A} \setminus \overline{B},$

$[1] := \texttt{AlternativeClosure1}(A)(x) : \forall U \in \mathcal{U}(x) . U \cap A \neq \emptyset,$

$\Big(U, [2]\Big) := \texttt{AlternativeClosure}(B)(x) : \sum U \in \mathcal{U}(x) . U \cap B = \emptyset,$

$\texttt{Assume}\ W : \mathcal{U}(x),$

$V := W \cap U : \mathcal{U}(x),$

$[3] := \texttt{IntersectSubset}(V) : V \subset U,$

$[4] := [2][3]\texttt{SubsetIntersect} : V \cap B = \emptyset,$

$[5] := [1](V) : V \cap A \neq \emptyset,$

$[6] := [2] : V \cap (A \setminus B) \neq \emptyset,$

$[W.*] := \texttt{SupersectIntersect}[3][6] : W \cap (A \setminus B) \neq \emptyset;$

$\rightsquigarrow [x.*] := \texttt{AlternativeClosure2} : x \in \overline{A \setminus B};$

$\rightsquigarrow [*] := I\texttt{Subset} : \overline{A} \setminus \overline{B} \subset \overline{A \cap B};$

$\ldots$

$\square$

$\texttt{InfiniteClosureUnion} :: \forall X \in \texttt{TopologicalSpace} . \forall A : \mathbb{N} \to ?X . \overline{\bigcup_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n} \cup \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=1}^{\infty} A_{n+m}}$

$\texttt{Proof} =$

$\texttt{Assume}\ x : \overline{\bigcup_{n=1}^{\infty} A_n},$

$\texttt{Assume}\ [1] : x \notin \bigcap_{n=1}^{\infty} \overline{A_n},$

$\big(U, [2]\big) := \texttt{EquivalentClosure}[1] : \forall n \in \mathbb{N} . \exists U \in \mathcal{U}(x) . \forall U \cap A_n = \emptyset,$

$[3] := \texttt{EquivalntClosure}(x) : \forall V \in \mathcal{U}(x) . V \cap \bigcup_{n=1}^{\infty} A_n \neq \emptyset,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ W : \mathcal{U}(x),$

$V := W \cap \bigcap_{i=1}^{n} U_i : \mathcal{U}(x),$

$[4] := [3](V) : V \cap \bigcup_{n=1}^{\infty} A_n \neq \emptyset,$

$[5] := [4][2]\jmath V : V \cap \bigcup_{i=n+1}^{\infty} A_i \neq \emptyset,$

$[W.*] := \texttt{SubsetIntersect}(V)\texttt{SupersetIntersect}[5] : W \cap \bigcup_{i=n+1}^{\infty} A_i \neq \emptyset;$

$\rightsquigarrow [n.*] := \texttt{EquivalentClosure}(x) : x \in \overline{\bigcup_{i=1}^{n} A_i};$

$\rightsquigarrow [1.*] := \eth^{-1}\texttt{intersect} : x \in \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=1}^{n} A_i};$

$\rightsquigarrow [x.*] := \texttt{InOrNotIn}(x)\eth^{-1}\texttt{Union} : x \in \bigcup_{n=1}^{\infty} \overline{A_n} \cap \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i};$

$\rightsquigarrow [1] := \eth\texttt{Subset} : \overline{\bigcup_{i=1}^{n} A_i} \subset \bigcup_{n=1}^{\infty} \overline{A_n} \cap \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i},$

$[2] := \eth\texttt{SubsetUnion}(A) : \forall n \in \mathbb{N} . A_n \subset \bigcup_{i=1}^{n} A_i,$

$[3] := \eth\texttt{ClosureIsMonotonic}(A)\texttt{SubssetUnion}\overline{A}[2] : \bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$

$[4] := \Lambda n \in \mathbb{N} . \texttt{LargerUnion}(A, n)\texttt{ClosureIsMonotionic} : \forall n \in \mathbb{N} . \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$

$$[5] := \texttt{IntersectSubset}[4] : \bigcup_{\infty=1}^{n} \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[6] := \texttt{SubsetUnion}[4] : \bigcup_{n=1}^{\infty} \overline{A_i} \cup \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[*] := \eth^{-1}\texttt{SetEq}[1][5] : \bigcup_{n=1}^{\infty} \overline{A_i} \cup \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i} = \overline{\bigcup_{n=1}^{\infty} A_n};$$

$\square$

1.1.3 Open and Closed Domains

$\texttt{OpenDomain} :: \prod X : \texttt{TopologicalSpace} \,.\, ?X$

$A : \texttt{OpenDomain} \iff A = \operatorname{int} \overline{A}$

$\texttt{ClosedSetInteriorIsOpenDomain} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A : \texttt{Closed}(X) \,.\, \operatorname{int} A : \texttt{OpenDomain}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U : \texttt{Open}(X),$

$\texttt{Assume}\ [1] : U \subset \overline{\operatorname{int} A},$

$[2] := \eth\texttt{PotentialInteriot}(\operatorname{int})\texttt{ClosureIsMonotonic}(\operatorname{cl}_X)\eth\texttt{PotentialClosure}(\operatorname{cl}_X) : U \subset \overline{\operatorname{int} A} \subset \overline{A} = A,$

$[U.*] := \eth(\operatorname{int} A)[2] : U \subset \operatorname{int} U;$

$\rightsquigarrow [1] := \eth\texttt{interior} : \operatorname{int} \overline{\operatorname{int} A} \subset \operatorname{int} A,$

$\texttt{Assume}\ U : \texttt{Open}(X),$

$\texttt{Assume}\ [2] : U \subset A,$

$[3] := \eth\texttt{interiorSubsetUnion}\eth\texttt{PotentialClosure}(\operatorname{cl}_X) : U \subset \operatorname{int} A \subset \overline{\operatorname{int} A},$

$[U.*] := \eth(\operatorname{int})[3] : U \subset \operatorname{int} \overline{\operatorname{int} A};$

$\rightsquigarrow [3] := \eth\texttt{interior} : \operatorname{int} A \subset \operatorname{int} \overline{\operatorname{int} A},$

$[5] := \eth\texttt{SetEq}[2][3] : \operatorname{int} A = \operatorname{int} \overline{\operatorname{int} A},$

$[*] := \eth^{-1}\texttt{OpenDomain} : \Big(U : \texttt{OpenDomain}(X)\Big);$

$\square$

$\texttt{OpenDomainIntesection} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B : \texttt{OpenDomain}(X) \,.\, A \cap B : \texttt{OpenDomain}$

$\texttt{Proof} \ =$

$[1] := \texttt{closureIntersection}(A, B)\eth\texttt{PotentialInterior}(\operatorname{int}_X)\eth^2\texttt{OpenDomain}(A)(B) :$

$\quad : \operatorname{int} \overline{A \cap B} \subset \operatorname{int} \overline{A} \cap \operatorname{int} \overline{B} = A \cap B,$

$[2] := \eth^2\texttt{OpenDomain}(A)(B)\texttt{OpenIntersection}(A, B) : A \cap B : \texttt{Open}(X),$

$[3] := \eth\texttt{interior}\texttt{interiorIsMonotonic}(A \cap B) : A \cap B = \operatorname{int} A \cap B \subset \operatorname{int} \overline{A \cap B},$

$[4] := \eth^{-1}\texttt{SetEq}[1][3] : A \cap B = \operatorname{int} \overline{A \cap B},$

$[*] := \eth^{-1}\texttt{OpenDomain} : \Big(A \cap B : \texttt{OpenDomain}(X)\Big);$

$\square$

$\texttt{OpenDomaSubset} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B : \texttt{OpenDomain}(X) \,.\, A \subset B \iff \overline{A} \subset \overline{B}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : A \subset B,$

$[1.*] := \texttt{ClosureIsMonotonic}[1] : \overline{A} \subset \overline{B};$

$\rightsquigarrow [1] := I(\Rightarrow) : A \subset B \Rightarrow \overline{A} \subset \overline{B},$

$\texttt{Assume}\ [2] : \overline{A} \subset \overline{B},$

$[3] := \texttt{InteriorIsMonotonic} : \operatorname{int} \overline{A} \subset \operatorname{int} \overline{B},$

$[4] := \eth^2\texttt{OpenDomain}(A)(B)[3] : A \subset B;$

$\rightsquigarrow [2] := I(\Rightarrow) : \overline{A} \subset \overline{B} \Rightarrow A \subset B,$

$[*] := I(\iff)[1][2] : A \subset B \iff \overline{A} \subset \overline{B};$

$\square$

$\texttt{UnionClosureInteriorAsSup} :: \forall X : \texttt{TopologicalSpace} \;.\; \forall I \in \mathsf{SET} \;.\; \forall U : I \to \texttt{OpenDomain}(X) \;.$

$$. \operatorname{int}\overline{\bigcup_{i\in I} U_i} = \min\left\{O : \texttt{OpenDomain}(X) \middle| \forall i \in I \;.\; U_i \subset O\right\}$$

$\texttt{Proof} \;=$

$$[1] := \texttt{ClosedInteriorIsOpenDomain}\left(\operatorname{int}\overline{\bigcup_{i\in I} U_i}\right) : \left(\operatorname{int}\overline{\bigcup_{i\in I} U_i} : \texttt{OpenDomain}(X)\right),$$

$\texttt{Assume}\; O : \texttt{OpenDomain}(X),$

$\texttt{Assume}\; [2] : \forall i \in \mathcal{I} \;.\; U_i \subset O,$

$$[3] := \texttt{UnionSuperset}[2] : \bigcup_{i\in\mathcal{I}} U_i \subset O,$$

$$[4] := \texttt{ClosureIsMonotonic}(\mathrm{cl})\texttt{InteriorIsMonotonic}(\operatorname{int})[1] : \operatorname{int}\overline{\bigcup_{i\in\mathcal{I}} U_i} \subset \operatorname{int}\overline{O}[3],$$

$$[i.*] := \eth\texttt{OpenDomain}[4] : \operatorname{int}\overline{\bigcup_{i\in\mathcal{I}} U_i} \subset O;$$

$$\rightsquigarrow [2] := I(\forall) : \forall O : \texttt{OpenDomain}(X) \;.\; \Big(\forall i \in \mathcal{I} \;.\; U_i \subset O\Big) \Rightarrow \operatorname{int}\overline{\bigcup_{i\in\mathcal{I}} U_i} \subset O,$$

$\texttt{Assume}\; i : I,$

$$[3] := \texttt{UnionSubset}(i, U) : U_i \subset \bigcup_{i\in I} U_i \subset \overline{\bigcup_{i\in I} U_i},$$

$$[*] := \eth\texttt{interior}[3] : U_i \subset \operatorname{int}\overline{\bigcup_{i\in I} U_i};$$

$$\rightsquigarrow [3] := I(\forall) : \forall i \in \mathcal{I} \;.\; U_i \subset \operatorname{int}\overline{\bigcup_{i\in I} U_i},$$

$$[*] := \eth^{-1}\min[3][2][1] : \operatorname{int}\overline{\bigcup_{i\in I} U_i} = \min\left\{O : \texttt{OpenDomain}(X) \middle| \forall i \in I \;.\; U_i \subset O\right\};$$

$\square$

$\texttt{IntersectInteriorAsInf} :: \forall X : \texttt{TopologicalSpace} \;.\; \forall I \in \mathsf{SET} \;.\; \forall U : I \to \texttt{OpenDomain}(X) \;.$

$$. \operatorname{int}\bigcap_{i\in I} U_i = \max\left\{O : \texttt{OpenDomain}(X) \middle| \forall i \in I \;.\; O \subset U_i\right\}$$

$\texttt{Proof} \;=$

$$[1] := \eth\texttt{PotentialClisure}(\mathrm{cl}_X) : \operatorname{int}\bigcap_{i\in I} U_i \subset \overline{\operatorname{int}\bigcap_{i\in I} U_i},$$

$$[2] := \texttt{MonotonicInterior}(\operatorname{int})\eth\texttt{PotentialInterior}(\operatorname{int}) : \operatorname{int}\bigcap_{i\in I} U_i \subset \operatorname{int}\operatorname{int}\bigcap_{i\in I} U_i \subset \operatorname{int}\overline{\operatorname{int}\bigcap_{i\in I} U_i},$$

$\texttt{Assume}\; i : I,$

$$[*.i] := \texttt{SubsetIntersect}(U_i, U)\texttt{ClosureIsMonotonic}(\mathrm{cl}_X)\texttt{InteriorIsMonotonic}(\operatorname{int})\eth\texttt{OpenDomain}(U) :$$

$$: \operatorname{int}\overline{\operatorname{int}\bigcap_{i\in I} U_i} \subset \operatorname{int}\overline{U_i} = U_i;$$

$$\rightsquigarrow [3] := \texttt{IntersectSubset} : \operatorname{int}\overline{\operatorname{int}\bigcap_{i\in I} U_i} \subset \bigcap_{i\in I} U_i,$$

$$[4] := \eth\operatorname{int}[3] : \operatorname{int}\overline{\operatorname{int}\bigcap_{i\in I} U_i} \subset \operatorname{int}\bigcap_{i\in I} U_i,$$

$$[5] := \eth^{-1}\texttt{SetEq}[3][4]\eth^{-1}\texttt{OpenDomain} : \left(\operatorname{int}\bigcap_{i\in I} U_i : \texttt{OpenDomain}\right),$$

$\texttt{Assume}\ O : \texttt{OpenDomain}(X),$

$\texttt{Assume}\ [6] : \forall i \in I\ .\ O \subset U_i,$

$$[7] := \eth\texttt{OpenDomain}(U) : O \subset \bigcap_{i\in I} U_i,$$

$$[O.*] := \eth\,\mathrm{int}[7] : O \subset \mathrm{int} \bigcap_{i\in I} U_i;$$

$$\rightsquigarrow [6] := I(\forall)I(\Rightarrow) : \forall O : \texttt{OpenDomain}(X)\ .\ (\forall i \in I\ .\ O \subset U_i) \Rightarrow O \subset \mathrm{int} \bigcap_{i\in I} U_i,$$

$$[7] := \Lambda i \in I\ .\ \eth\texttt{PotentialInterior}(\mathrm{int})\texttt{IntersectionSubset} : \forall i \in I\ .\ \mathrm{int} \bigcap_{i\in I} U_i \subset \bigcap_{i\in I} U_i \subset U_i,$$

$$[*] := \eth^{-1} \max[5][6][7] : \mathrm{int} \bigcap_{i\in I} U_i = \max \left\{ O : \texttt{OpenDomain}(X) : \forall i \in I\ .\ O \subset U_i \right\};$$

□

$\texttt{ClosedDomain} :: \prod X : \texttt{TopologicalSpace}\ .\ ???X$

$A : \texttt{ClosedDomain} \iff A = \overline{\mathrm{int}\, A} \iff$

$\texttt{ClosedOpenDomainDuality} :: \forall X : \texttt{TopologicalSpace}\ .$

$\quad .\ \texttt{complement} : \texttt{OpenDomain}(X) \xleftrightarrow{\mathsf{SET}} \texttt{ClosedDomain}(X)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ U : \texttt{OpenDomain},$

$[1] := \eth\texttt{OpenDomain} : U = \mathrm{int}\, \overline{U},$

$$[2] := [1]^{\complement}\texttt{ClosureAsComplement}(X, \overline{U})\Big(\texttt{DoubleComplement}(X)\Big)^2(\overline{U})(U)\texttt{InteriorAsComplement}(X, U) :$$

$$: U^{\complement} = \Big(\mathrm{int}(\overline{U})^{\complement\complement}\Big)^{\complement} = \overline{(\overline{U}^{\complement\complement})^{\complement}} = \overline{\mathrm{int}\, U^{\complement}},$$

$$[U.*] := \eth^{-1}\texttt{OpenDomain}[3] : \Big(U^{\complement} : \texttt{ClosedDomain}(X)\Big);$$

$$\rightsquigarrow [1] := I(\forall) : \forall U : \texttt{OpenDomain}(X)\ .\ U^{\complement} : \texttt{ClosedDomain}(X),$$

$\texttt{Assume}\ A : \texttt{ClosedDomain}(X),$

$[2] := \eth\texttt{ClosedDomain} : A = \overline{\mathrm{int}\, A},$

$$[3] := [2]^{\complement}\texttt{InteriorAsComplement}(X, \mathrm{int}\, A)\Big(\texttt{DoubleComplement}(X)\Big)^2(\mathrm{int}\, A))(A)$$

$$\texttt{ClosureAsComplement}(X, A) :$$

$$: A^{\complement} = \overline{(\mathrm{int}\, A)}^{\complement\complement\complement} = \mathrm{int}\,\Big(\mathrm{int}(A^{\complement\complement})\Big)^{\complement} = \mathrm{int}\, \overline{A^{\complement}} =,$$

$$[U.*] := \eth^{-1}\texttt{ClosedDomain}[3] : \Big(A^{\complement} : \texttt{OpenDomain}(X)\Big);$$

$$\rightsquigarrow [2] := I(\forall) : \forall A : \texttt{ClosedDomain}(X)\ .\ A^{\complement} : \texttt{OpenDomain}(X),$$

$$[3] := \texttt{DoubleComplement}[1][2] : \Big(\texttt{complement} : \texttt{ClosedDomain}(X) \xleftrightarrow{(} \mathsf{SET})\texttt{ClosedDomain}(X)\Big);$$

□

$\texttt{OpenSetClosureIsClosedDomain} :: \forall X : \texttt{TopologicalSpace}\ .\ \forall U : \texttt{Open}(X)\ .\ \overline{U} : \texttt{ClosedDomain}(X)$

$\texttt{Proof}\ =$

$\ldots$

□

$\texttt{ClosedDomainUnion} :: \forall X : \texttt{TopologicalSpace} . \forall A, B : \texttt{ClosedDomain}(X) . A \cup B : \texttt{ClosedDomain}(X)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ClosedDomainSubset} :: \forall X : \texttt{TopologicalSpace} . \forall A, B : \texttt{OpenDomain}(X) . A \subset B \iff \overline{A} \subset \overline{B}$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{IntersectionInterioClosureAsSup} :: \forall X : \texttt{TopologicalSpace} . \forall I \in \mathsf{SET} . \forall A : I \to \texttt{ClosedDomain}(X) .$

$$. \overline{\operatorname{int} \bigcap_{i \in I} U_i} = \max \left\{ B : \texttt{ClosedDomain}(X) \middle| \forall i \in I . B \subset A_i \right\}$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{UnionClosureAsSup} :: \forall X : \texttt{TopologicalSpace} . \forall I \in \mathsf{SET} . \forall A : I \to \texttt{ClosedDomain}(X) .$

$$. \overline{\bigcup_{i \in I} U_i} = \min \left\{ B : \texttt{ClosedDomain}(X) \middle| \forall i \in I . A_i \subset B \right\}$$

$\texttt{Proof} \;=$

...

$\square$

1.1.4 Boundary Operator

$$\texttt{boundary} :: \prod X : \texttt{TopologicalSpace} \,.\, ?X \to \texttt{Closed}(X)$$
$$\texttt{boundary}\,(A) = \partial A := \overline{A} \setminus \operatorname{int} A$$

$\texttt{BoundaryCondition} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \forall x \in X \,.\, x \in \partial A \iff$
$\iff \forall U \in \mathcal{U}(x) \,.\, U \neq U \cap A \neq \emptyset$

Proof $=$

...

$\square$

$\texttt{InteriorByBoundary} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \operatorname{int} A = A \setminus \partial A$

Proof $=$

...

$\square$

$\texttt{ClosureByBoundary} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \overline{A} = A \cup \partial A$

Proof $=$

...

$\square$

$\texttt{BoundaryOfUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B \subset X \,.\, \partial(A \cup B) \subset \partial A \cup \partial B$

Proof $=$

...

$\square$

$\texttt{BoundaryOfIntersection} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B \subset X \,.\, \partial(A \cap B) \subset (\overline{A} \cap \partial B) \cup (\partial A \cap \overline{B})$

Proof $=$

...

$\square$

$\texttt{BoundaryComplement} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \partial(X \setminus A) = \partial A$

Proof $=$

...

$\square$

$\texttt{BoundaryDecomposition} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, X = (\operatorname{int} A) \cup \partial A \cup \left(\operatorname{int} A^{\complement}\right)$

Proof $=$

...

$\square$

$\texttt{ClosureBoundary} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \partial \overline{A} \subset \partial A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InteriorBoundary} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \partial \operatorname{int} A \subset \partial A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{BoundarySetUnionIsEqual} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B \subset X \,.\, \forall [0] : A \cap \overline{B} = \emptyset \,\&\, \overline{A} \cap B = \emptyset \,.$
$\quad .\ \partial(A \cup B) = \partial A \cup \partial B$

$\texttt{Proof} \ =$

$[1] := \texttt{BoundaryOfUnion}(A, B) : \partial(A \cup B) \subset \partial A \cup \partial B,$

$\texttt{Assume}\ x : \partial A,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$[2] := \texttt{UnionSubset}(A, B)\texttt{IntersectionSubset}(U, A \cup B, B)\texttt{BoundaryCondition}(A, x) :$
$\quad : \emptyset \neq (U \cap A) \subset U \cap (A \cup B),$

$[3] := \eth x \eth \partial A[0] : x \notin B,$

$\texttt{Assume}\ [4] : (A \cup B) \cap U = U,$

$[5] := [3][4] : x \in A,$

$\Big(V, [6]\Big) := [0][5]\texttt{EquivalentClosure} : \sum V \in \mathcal{U}(x) \,.\, V \cap B = \emptyset,$

$W := V \cap U : \mathcal{U}(x),$

$[7] := \texttt{IntersectionSubsect}[5]\jmath W : W \cap B = \emptyset,$

$[8] := \texttt{BoundaryCondition}(x)(W) : W \cap A \neq W,$

$[9] := [7][8] : W \cap (A \cup B) \neq W,$

$[10] := \jmath W[9] : U \cap (A \cup B) \neq U,$

$[11] := [10][3] : \bot;$

$\rightsquigarrow [3] := E(\bot) : (A \cup B) \cap U \neq Y,$

$[x.*] := \texttt{BoundaryCondition}[3][2] : x \in \partial(A \cup B);$

$\rightsquigarrow [2] := \eth^{-1}\texttt{Subset} : \partial A \subset \partial(A \cup B),$

$[3] := \texttt{Symmetric}[2](A, B) : \partial B \subset \partial(A \cup B),$

$[4] := \texttt{UnionSubset}[2][3] : \partial A \cup \partial B \subset \partial(A \cup B),$

$[*] := \eth\texttt{SetEq}[2][3] : \partial A \cup \partial B = \partial(A \cup B);$

$\square$

$\texttt{LocalyFiniteUnionBoundary} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A : \texttt{LocallyFinite}(X) \,.\, \partial \bigcup A \subset \bigcup \partial A$

$\texttt{Proof} \ =$

$[*] := \eth \partial \bigcup A \texttt{LocallyFiniteYnionBoundary}(A)\texttt{DifferenceUnion}\left(\overline{A}, \operatorname{int} \bigcup A\right)\texttt{UnionRule}(A)$
$\quad \texttt{InteriorIsMonotonic}(\operatorname{int})\texttt{CoincreaingDifference}\eth^{-1}\partial A :$
$\quad : \partial \bigcup A = \overline{\bigcup A} \setminus \operatorname{int} \bigcup A = \bigcup \overline{A} \setminus \operatorname{int} \bigcup A = \bigcup \left(\overline{A} \setminus \operatorname{int} \bigcup A\right) \subset \bigcup \overline{A} \setminus \operatorname{int} A = \bigcup \partial A;$

$\square$

$\texttt{ClosureOfIntersectWithOpenSet} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall U \in \mathcal{T}(x) \,.\, \forall A \subset X \,.\, \overline{A \cap U} = \overline{\overline{A} \cap U}$

$\texttt{Proof} \ =$

$[1] := \texttt{IntersectionSubset}\ \texttt{ClosureIsMonotonic}(X) : \overline{A \cap U} \subset \overline{\overline{A} \cap U},$

$\texttt{Assume}\ x : \overline{\overline{A} \cap U},$

$[2] := \texttt{EquivalentClosure1}(x) : \forall V \in \mathcal{U}(x) \,.\, V \cap \overline{A} \cap U \neq \emptyset,$

$\texttt{Assume}\ V : \mathcal{U}(x),$

$\texttt{Assume}\ [3] : V \cap A \cap U = \emptyset,$

$[4] := [3][2]\texttt{ClosureByBoundary} : (V \cap U) \cap \partial A \neq \emptyset,$

$[5] := \texttt{BoundaryCondition}[4] : (V \cap U \cap A) \neq \emptyset,$

$[V.*] := E(=)[3][5]I(\bot) : \bot;$

$\rightsquigarrow [3] := E(\bot)I(\rightarrow) : \forall V \in \mathcal{U}(x) \,.\, V \cap A \cap U \neq \emptyset,$

$[x.*] := \texttt{EquivalentClosure}[2] : x \in \overline{A \cap U};$

$\rightsquigarrow [2] := \eth\texttt{Subset} : \overline{\overline{A} \cap U} \subset \overline{A \cap U},$

$[*] := \eth\texttt{SetEq}[1][2] : \overline{\overline{A} \cap U} = \overline{A \cap U};$

$\square$

$\texttt{InteriorOfUnionWithClosedSet} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall C : \texttt{Closed}(X) \,.\, \forall A \subset X \,.$

$\quad .\ \operatorname{int}(A \cup C) = \operatorname{int}\Big((\operatorname{int} A) \cap U\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.1.5 Accumulation and Isolated Points

$$\texttt{derivedSet} :: \prod X : \texttt{TopologicalSpace} \,.\, ?X \to \texttt{Closed}(X)$$

$$\texttt{derivedSet}\,(A) = A^{\mathrm{d}} := \left\{x \in X : x \in \overline{A \setminus \{x\}}\right\}$$

$$\texttt{IsolatedPoint} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, ?A$$

$$x : \texttt{IsolatedPoint} \iff x \in (A \setminus A^{\mathrm{d}})$$

$$\texttt{IsolatedPointProperty} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \forall x \in X \,.$$
$$.\, x \in A^{\mathrm{d}} \iff \forall U \in \mathcal{U}(x) \,.\, \exists y \in U \cap A \,.\, y \neq x$$

Proof $=$

...

$\square$

$$\texttt{ClosureByDerivedSet} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A \subset X \,.\, \overline{A} = A \cup A^{\mathrm{d}}$$

Proof $=$

...

$\square$

$$\texttt{DerivedSetIsMonotonic} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B \subset X \,.\, A \subset B \iff A^{\mathrm{d}} \subset B^{\mathrm{d}}$$

Proof $=$

...

$\square$

$$\texttt{DerivedFiniteUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A, B \subset X \,.\, (A \cup B)^{\mathrm{d}} = A^{\mathrm{d}} \cup B^{\mathrm{d}}$$

Proof $=$

...

$\square$

$$\texttt{DerivedUnion} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall I \in \mathsf{SET} \,.\, \forall A : I \to ?X \,.\, \bigcup_{i \in I} A^{\mathrm{d}} \subset \left(\bigcup_{i \in I} A\right)^{\mathrm{d}}$$

Proof $=$

...

$\square$

1.1.6 Dense Sets

$\texttt{Dense} :: \prod X : \texttt{TopologicalSpace} . ??X$

$A : \texttt{Dense} \iff \overline{A} = X$

$\texttt{Codense} :: \prod X : \texttt{TopologicalSpace} . ??X$

$A : \texttt{Codense} \iff A^{\complement} : \texttt{Dense}(X)$

$\texttt{NowhereDense} :: \prod X : \texttt{TopologicalSpace} . ??X$

$A : \texttt{NowhereDense} \iff \overline{A} : \texttt{Codense}(X)$

$\texttt{DenseInItself} :: \prod X : \texttt{TopologicalSpace} . ??X$

$A : \texttt{DenseInItself} \iff A \subset A^{\mathrm{d}}$

$\texttt{DenseByOpenSets} :: \forall X : \texttt{TopologicalSpace} . \forall A \subset X .$

$. A : \texttt{Dense}(X) \iff \forall x \in X . \forall U \subset \mathcal{U}(x) . U \cap A \neq \emptyset$

$\texttt{Proof} =$

$\eth\texttt{DenseEquivalentClosure1} \ \square$

$\texttt{CodenseByOpenSets} :: \forall X : \texttt{TopologicalSpace} . \forall A \subset X .$

$. A : \texttt{Codense}(X) \iff \forall x \in X . \forall U \subset \mathcal{U}(x) . U \cap A^{\complement} \neq \emptyset$

$\texttt{Proof} =$

$\eth\texttt{CodenseDenseByOpenSet} \ \square$

$\texttt{NowhereDenseByOpenSets} :: \forall X : \texttt{TopologicalSpace} . \forall A \subset X .$

$. A : \texttt{NowherDense}(X) \iff \forall x \in \mathcal{U}(x) . \forall U \in \mathcal{U}(x) . \exists V \in \mathcal{T}(X) : V \neq \emptyset \ \& \ V \cap A = \emptyset \ \& \ V \subset U$

$\texttt{Proof} =$

$\eth\texttt{NowhereDense}\eth\texttt{derivedSet} \ \square$

$\texttt{DenseClosure} :: \forall X : \texttt{TopologicalSpace} . \forall A : \texttt{Dense}(X) . \forall U \in \mathcal{T}(X) . \overline{U \cap A} = \overline{U}$

$\texttt{Proof} =$

$[1] := \texttt{SubsetIntersection} : U \cap A \subset U,$

$[2] := \texttt{ClosureIsMonotonic}[1] : \overline{U \cap A} \subset \overline{U},$

$\texttt{Assume } x : \overline{U},$

$[3] := \texttt{EquivalentClosure1}(U)(x) : \forall V \in \mathcal{U}(x) . V \cap U \neq \emptyset,$

$[4] := \texttt{DenseByOpenSets}(A)(x) : \forall V \in \mathcal{U}(x) . V \cap A \cap U \neq \emptyset,$

$[x.*] := \texttt{EquivalenitClosure1}(U \cap A)(x) : x \in U \cap A;$

$\rightsquigarrow [3] := \eth^{-1}\texttt{Subset} : \overline{U} \subset \overline{U \cap A},$

$[*] := \eth\texttt{SetEq}[2][3] : \overline{U} = \overline{U \cap A};$

$\square$

$\texttt{densityCardinal} :: \texttt{TopologicalSpace} \to \mathsf{CARD}$

$\texttt{densityCardinal}\,(X) = d(X) := \min \texttt{Dense}(X)$

$\texttt{Separable} :: ?\texttt{TopologicalSpace}$

$X : \texttt{Separable} \iff d(X) < \aleph_0$

$\texttt{DensityBound} :: \forall X : \texttt{TopologicalSpace} \,.\, d(X) \leq w(X)$

$\texttt{Proof} \;=$

$\Big(\mathcal{B}, [1]\Big) := \eth w(X) : \sum \mathcal{B} : \texttt{Base}(X) \,.\, |\mathcal{B}| = w(X),$

$\texttt{Assume}\ B : \mathcal{B},$

$\texttt{Assume}\ [2] : B \neq \emptyset,$

$q(B) := \eth\texttt{NonEmpty} : B;$

$\rightsquigarrow q := I\Big(\prod\Big) I\Big(\sum\Big) : \prod B \in \mathcal{B} \,.\, \prod B \neq \emptyset \,.\, B \neq \;.\, q(B) \in B,$

$Q := \operatorname{Im} q : ?X,$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ U : \mathcal{U}(X),$

$\Big(I, B, [3]\Big) := \eth\texttt{Base}(\mathcal{B})(U) : \sum I : \texttt{NonEmpty} \,.\, B : I \to \mathcal{B} \,.\, U = \bigcup_{i \in I} B_i,$

$[4] := \eth q[3]\texttt{UnionSubset} : \forall i \in I \,.\, q(B) \in U,$

$[*] := \jmath Q[4]\eth\texttt{NonEmpty}(I) : Q \cap U \neq \emptyset;$

$\rightsquigarrow [3] := \texttt{DenseByOpenSets} : \Big(U : \texttt{Dense}(X)\Big),$

$[4] := \texttt{ImageCardinality}\jmath Q : |Q| \leq |\mathcal{B}|,$

$[*] := [1][4]\eth\texttt{density} : d(X) \leq w(X);$

$\square$

$\texttt{SecondCountableIsSimmilar} :: \forall X : \texttt{SecondCountable} \,.\, X : \texttt{Separable}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LocallyFiniteNowhereDense} :: \forall X : \texttt{TopologicalSpace} \,.\, \forall A : \texttt{LocallyFinite} \,\&\, \texttt{NowhereDense}(X) \,.$

$\quad \bigcup A : \texttt{NowhereDense}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : X,$

$\Big(U, [1]\Big) := \eth\texttt{LocallyFinite}(X)(A)(x) : \sum U \in \mathcal{U}(X) \,.\, \Big|\{a \in A : a \cap U \neq \emptyset\}\Big| < \infty,$

$\mathcal{A} := \{a \in A : a \cap U \neq \emptyset\} : \texttt{Finite}(A),$

$\texttt{Assume}\ V : \mathcal{U}(x),$

$\Big(W, [2]\Big) := \texttt{NowhereDenseByOpenSets}(\mathcal{A}) : \sum W : \mathcal{A} \to \mathcal{U}(x) \,.\, \prod_{a \in \mathcal{A}} \,.\, W_a \subset U \cap V \neq \;\&\; W_a \cap A = \emptyset,$

$O := \bigcap_{a \in \mathcal{A}} W_a : \mathcal{U}(x),$

$[x.*.1] := [2]\jmath O : O \subset V,$

$[x.*.2] := \jmath\mathcal{A}[2]\jmath O : O \cap \bigcup A = \emptyset;$

$\rightsquigarrow [*] := \texttt{NowhereDenseByOpenSets} : \Big(\bigcup A : \texttt{NowhereDense}(X)\Big);$

$\square$

$\texttt{CodenseUnionWithNowhereDenceIsCodence} :: \forall X : \texttt{TopologicalSpace} . \forall A : \texttt{Codense}(X) . \forall B : \texttt{NowhereDen}$

$\texttt{Proof} =$

$[1] := \eth\texttt{NowhereDense}(B)\texttt{UniversumIntersect}(\overline{B}^{\complement})\eth\texttt{Codense}(A)$

$\texttt{ClosureOfIntersectWithOpenSet}(X, A^{\complement}, \overline{B}^{\complement})\texttt{ClosureIsNonotonic}(\underset{X}{\mathrm{cl}})$

$\texttt{ComplementIsComonotonic}(X, B, \overline{V})\eth\texttt{PotentialClosureOperator}(\mathrm{cl}_X)(B)$

$\texttt{ClosureIsMonotonic}(\mathrm{cl}_X)\texttt{DeMorganeLaw}(X, A, B) :$

$: X = \overline{\overline{B}^c} = \overline{X \cap \overline{B}^{\complement}} = \overline{A^{\complement}} \cap \overline{B}^{\complement} = \overline{A^{\complement} \cap \overline{B}^{\complement}} \subset \overline{A^{\complement} \cap B^{\complement}} = \overline{(A \cup B)^{\complement}},$

$[*] := \eth^{-1}\texttt{Codense}[1] : \Big(A \cup B : \texttt{Codense}(X)\Big);$

$\square$

$\texttt{OpenDenseInItself} :: \forall X : \texttt{TopologicalSpace} . \forall U \in \mathcal{T}(X) .$

$. X : \texttt{DenseInItself}(X) \Rightarrow U : \texttt{DenseInItself}(X)$

$\texttt{Proof} =$

$\texttt{Assume}\ u : U,$

$[1] := \texttt{DenseInItself}(X)(X)(u) : u \in X^{\mathrm{d}},$

$[2] := \eth\texttt{derivedSet}[1] : u \in \overline{X \setminus \{u\}},$

$[3] := \texttt{EquivalentClosure}[2] : \forall V \in \mathcal{U}(u) . V \cap X \setminus \{u\} \neq \emptyset,$

$\texttt{Assume}\ V : \mathcal{U}(u),$

$[V.*] := [3](V \cap U)\texttt{IntersecttionDifference}(X)\texttt{UniversumIntersection}(X) :$

$: V \cap (U \setminus \{u\}) = V \cap (U \setminus \{u\}) \cap X = V \cap U \cap (X \setminus \{u\}) \neq \emptyset;$

$\leadsto [4] := \texttt{EquivalentClosure} : u \in \overline{U \setminus \{u\}},$

$[u.*] := \eth^{-1}\texttt{derivedSet}[4] : u \in U^{\mathrm{d}};$

$\leadsto [1] := \eth^{-1}\texttt{Subset} : U \subset U^{\mathrm{d}},$

$[*] := \eth^{-1}\texttt{DenseInItself}[1] : \Big(U : \texttt{DenseInItself});$

$\square$

$\texttt{ClosureOfDenseInItself} :: \forall X : \mathsf{TOP} . \forall A : \texttt{DenseInItself}(X) . \overline{A} : \texttt{DenseInItself}(X)$

$\texttt{Proof} =$

$\texttt{Assume}\ x : \overline{A},$

$\texttt{Assume}\ [1] : x \in A,$

$[2] := \eth\texttt{DenseInItself}(X)(A)[1] : x \in A^{\mathrm{d}},$

$[3] := \eth\texttt{derivedSet}(A)[2]\texttt{MonotonicClosure} : x \in \overline{A \setminus \{x\}} \subset \overline{\overline{A} \setminus \{x\}},$

$[1.*] := \eth^{-1}\texttt{derivedSet}[3] : x \in \overline{A}^{\mathrm{d}};$

$\leadsto [1] := I(\Rightarrow) : x \in A \Rightarrow \overline{A}^{\mathrm{d}},$

$\texttt{Assume}\ [2] : x \notin A,$

$[3] := \eth x\texttt{MonotonicClosure}[2] : x \in \overline{A} \subset \overline{\overline{A} \setminus \{x\}},$

$[4] := \eth^{-1}\texttt{DerivedSet} : x \in \overline{A}^{\mathrm{d}};$

$\leadsto [2] := I(\Rightarrow) : x \notin A \Rightarrow x \in \overline{A}^{\mathrm{d}},$

$[x.*] := E(|)\texttt{InOrNotIn}(x)[1][2] : x \in \overline{A}^{\mathrm{d}};$

$\leadsto [1] := \eth^{-1}\texttt{Subset} : \overline{A},$

$[*] := \eth^{-1}\texttt{DenseInItself}[1] : \Big(\overline{A} : \texttt{DensInItself}\Big);$

$\square$

1.1.7 Separation Axioms

$\texttt{T0} :: ?\mathsf{TOP}$

$X : \texttt{T0} \iff \forall a, b \in X \,.\, \exists U \in \mathcal{T}(X) \,.\, \Big|U \cap \{a, b\}\Big| = 1$

$\texttt{T0CardinalityBound} :: \forall X : \texttt{T0} \,.\, |X| \le \exp w(X)$

$\texttt{Proof} \; =$

$\Big(\mathcal{B}, [1]\Big) := \eth\texttt{weight} : \sum \mathcal{B} : \texttt{Base}(X) \,.\, w(X) = |\mathcal{B}|,$

$\mathcal{A} := \Lambda x \in X \,.\, \mathcal{U}(X) \cap \mathcal{B} : X \to ?\mathcal{B},$

$\texttt{Assume}\; x, y : X,$

$\texttt{Assume}\; [2] : (x \ne y),$

$\Big(U, [3]\Big) := \eth\texttt{T0}(X)(x, y)[2] : \sum U \in \mathcal{T}(X) \,.\, \Big|U \cap \{x, y\}\Big| = 1,$

$[4] := \eth\mathcal{U}[3] : \Big(\exists V \in \mathcal{U}(x) : y \notin V \Big| \exists V \in \mathcal{U}(y) : x \notin V\Big),$

$[5] := \eth\texttt{Base}(X)(\mathcal{B})\jmath^{-1}\mathcal{A}[4] : \Big(\exists V \in \mathcal{A}(x) : y \notin V \Big| \exists V \in \mathcal{A}(y) : x \notin V\Big),$

$[*] := \jmath\mathcal{A}[5] : \mathcal{A}(x) \ne \mathcal{A}(y);$

$\rightsquigarrow [2] := \eth^{-1}\texttt{Injection} : \mathcal{A} : X \hookrightarrow ?\mathcal{B},$

$[*] := \texttt{CardinalityInjectionBound}[2] : |X| \le \exp w(X);$

$\square$

$\texttt{T1} :: ?\mathsf{TOP}$

$X : \texttt{T1} \iff \forall a, b \in X \,.\, \exists U \in \mathcal{T}(a) \,.\, b \notin U$

$\texttt{T1Singelton} :: \forall X : \texttt{T1} \,.\, \forall x \in X \,.\, \{x\} \in G_\delta(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{T1BySingeltons} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{T1} \iff \forall x \in X \,.\, \{x\} : \texttt{Closed}(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{SeparationHierarchy1} :: \texttt{T0} \subsetneq \texttt{T1}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{Separated} :: \prod X \in \mathsf{TOP} \,.\, ?X \times ?X$

$(A, B) : \texttt{Separated} \iff \exists U, V \in \mathcal{T}(X) \,.\, A \subset U \,\&\, B \subset V \,\&\, U \cap V = \emptyset$

$\texttt{T2} :: ?\mathsf{TOP}$

$x : \texttt{T2} \iff x : \texttt{Hausdorff} \iff \forall x, y \in X \,.\, x \ne y \Rightarrow \exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(x) : U \cap V = \emptyset$

$\texttt{SeparationHierarchy2} :: \texttt{T1} \subsetneq \texttt{T2}$

$\texttt{Proof} =$

...

$\square$

$\texttt{T2BySingletons} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{T2} \iff \forall x \in X \,.\, \{x\} = \bigcap_{U \in \mathcal{U}(x)} \overline{U}$

$\texttt{Proof} =$

...

$\square$

$\texttt{T2CardinalityBound1} :: \forall X : \texttt{T2} \,.\, |X| \leq \exp\exp d(X)$

$\texttt{Proof} =$

$\Big(D, [1]\Big) := \eth d(X) : \sum D : \texttt{Dense}(X) \,.\, |D| = d(X),$

$\mathcal{A} := \Lambda x \in X \,.\, \{U \cap D | U \in \mathcal{U}(x)\} : X \to ??D,$

$[2] := \eth\texttt{Dense}(X)(D)\jmath\mathcal{A}\texttt{T2BySingletons}(X) : \forall x \in X \,.\, \bigcap_{A \in \mathcal{A}(x)} \overline{A} = \{x\},$

$[3] := \texttt{InjectiveByMapping}[2] : \Big(\mathcal{A} : X \hookrightarrow ??D\Big),$

$[*] := \texttt{CardinalityByInjectionBound}[2][3] : |X| \leq \exp\exp d(X);$

$\square$

$\texttt{T2CardinalityBound2} :: \forall X : \texttt{T2} \,.\, |X| \leq \Big(\chi(X)\Big)^{d(X)}$

$\texttt{Proof} =$

...

$\square$

$\texttt{ClosedEqualityInT2Space} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{T2} \,.\, \forall f, g : X \xrightarrow{\mathsf{TOP}} Y \,.\, \Big\{x \in X : f(x) = g(x)\Big\} : \texttt{Closed}(X)$

$\texttt{Proof} =$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ [1] : f(x) \neq g(x),$

$\Big(U, V, [2]\Big) := \eth\texttt{T2}\Big(f(x), g(x)\Big)[1] : \sum U \in \mathcal{U}\Big(f(x)\Big) \,.\, \sum V \in \mathcal{U}\Big(g(x)\Big) \,.\, U \cap V = \emptyset,$

$W_x := f^{-1}(U) \cap g^{-1}(W) : \mathcal{U}(x),$

$[x.*] := [2]\eth\texttt{preimage}\jmath W_x : \forall w \in W_x \,.\, f(w) \neq g(w);$

$\rightsquigarrow W := I\left(\prod\right) : \prod_{x \in X} f(x) \neq g(x) \to \sum U \in \mathcal{U}(x) \,.\, \forall u \in U \,.\, f(u) \neq g(u),$

$[1] := \eth W : \{x \in X : f(x) = g(x)\}^{\complement} = \bigcup W,$

$[*] := \eth^{-1}\texttt{Closed}[1] : \Big(\{x \in X : f(x) = g(x)\}^{\complement} : \texttt{Closed}(X)\Big);$

$\square$

$\texttt{setNeighborhood} :: \prod_{X \in \mathsf{TOP}} . ?X \to ?\mathcal{T}(X)$

$\texttt{setNeighborhood}(A) = \mathcal{U}(A) := \Big\{ U \in \mathcal{T}(X) : A \subset U \Big\}$

$\texttt{T3} :: ?\texttt{T1}$

$X : \texttt{T3} \iff X : \texttt{Regular} \iff \forall A : \texttt{Closed}(X) \,.\, \forall x \in X \,.\, \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(x) : V \cap U$

$\texttt{RegularityCriterion} :: \forall X \in \mathcal{T}1 \,.\, X : \texttt{T3} \iff \forall x \in X \,.\, \forall V \in \mathcal{U}(x) \,.\, \exists U \in \mathcal{U}(x) : \overline{U} \subset V$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : (X : \texttt{T3}),$

$\texttt{Assume}\ x : X,$

$\texttt{Assume}\ V : \mathcal{U}(x),$

$\Big(U, W, [2]\Big) := \eth\texttt{T3}(x, V^{\complement}) : \sum U \in \mathcal{U}(x) \,.\, \sum W \in \mathcal{U}(V^{\complement}) \,.\, W \cap U = \emptyset,$

$[1.*] := \eth\texttt{closure}(X)[2]\eth\mathcal{U}(V^{\complement})\texttt{ComplementSubset} : \overline{U} \subset W^{\complement} \subset V;$

$\leadsto [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \forall x \in X \,.\, \forall V \in \mathcal{U}(x) \,.\, \exists U \in \mathcal{U}(x) : \overline{U} \subset V,$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$\texttt{Assume}\ x : A^{\complement},$

$\Big(U, [3]\Big) := [2](A^{\complement}) : \sum U \in \mathcal{U}(x) \,.\, \overline{U} \subset A^{\complement},$

$V := \overline{U}^{\complement} : \mathcal{U}(A),$

$[A.*] := \jmath(V) : V \cap U = \emptyset;$

$\leadsto [4] := \eth^{-1}\texttt{T3} : (X : \texttt{T3});$

$\square$

$\texttt{SeparationHierarchy3} :: \texttt{T2} \subsetneq \texttt{T3}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{T3WeightBound} :: \forall X : \texttt{T3} \,.\, w(X) \le \exp d(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{T4} :: ?\texttt{T1}$

$X : \texttt{T4} \iff X : \texttt{Normal} \iff \forall A, B : \texttt{Closed}(X) \,.\, A \cap B = \emptyset \Rightarrow \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

$\texttt{SeparationHierarchy4} :: \texttt{T3} \subsetneq \texttt{T4}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{T4ByOpenCover} :: \forall X : \texttt{T1} . \forall [0] : \forall A : \texttt{Closed}(X) . \forall U \in \mathcal{U}(A) .$

$: \exists W : \mathbb{N} \to \mathcal{T}(X) : A \subset \bigcup_{i \in I} W_i \;\&\; \forall i \in \mathbb{N} . \overline{W_i} \subset U . X : \texttt{T4}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A, B : \texttt{Closed}(X),$

$\texttt{Assume}\ [1] : A \cap B = \emptyset,$

$\Big(W, [2]\Big) := [0](A, B^\complement) : \sum_{W : \mathbb{N} \to \mathcal{T}(X)} A \subset \bigcup_{i \in I} W_i \;\&\; \forall i \in \mathbb{N} . \overline{W_i} \subset B^\complement,$

$\Big(V, [3]\Big) := [0](B, A^\complement) : \sum_{V : \mathbb{N} \to \mathcal{T}(X)} B \subset \bigcup_{i \in I} V_i \;\&\; \forall j \in \mathbb{N} . \overline{V_i} \subset A^\complement,$

$H := \Lambda i \in \mathbb{N} . W_i \setminus \bigcup_{j=1}^{i} \overline{V_i} : \mathbb{N} \to \mathcal{T}(X),$

$G := \Lambda i \in \mathbb{N} . V_i \setminus \bigcup_{j=1}^{i} \overline{W_i} : \mathbb{N} \to \mathcal{T}(X),$

$[4] := \jmath H[3] : A \subset \bigcup_{n=1}^{\infty} H_n,$

$[5] := \jmath G[2] : B \subset \bigcup_{n=1}^{\infty} G_n,$

$O := \bigcup_{n=1}^{\infty} H_n : \mathcal{U}(A),$

$Q := \bigcup_{n=1}^{\infty} G_n : \mathcal{U}(B),$

$[6] := \jmath G \jmath H : \forall i \in \mathbb{N} . \forall j \in i . H_i \cap G_j = \emptyset,$

$[7] := \jmath H \jmath G : \forall i \in \mathbb{N} . \forall j \in i . G_i \cap H_j = \emptyset,$

$[8] := [6][7] : \forall i, j \in \mathbb{N} . H_i \cap G_j = \emptyset,$

$\Big[(A, B) . *\Big] := \jmath O \jmath Q[8] : O \cap Q = \emptyset;$

$\leadsto [*] := \eth^{-1}\texttt{T4} : (X : \texttt{T4}),$

$\square$

$\texttt{SecondCountableRegularIsNormal} :: \forall X : \texttt{T3} \;\&\; \texttt{SecondCountable} . X : \texttt{T4}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CountableRegularIsNormal} :: \forall X : \texttt{T3} . |X| \le \aleph \Rightarrow X : \texttt{T4}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{Cover} :: \prod_{X \in \mathsf{TOP}} ???X$$

$$\mathcal{A} : \texttt{Cover} \iff \bigcup \mathcal{A} = X$$

$$\texttt{OpenCover} :: \prod_{X \in \mathsf{TOP}} ??\mathcal{T}(X)$$

$$\mathcal{O} : \texttt{OpenCover} \iff \bigcup \mathcal{O} = X$$

$$\texttt{PointFiniteCover} :: \prod_{X \in \mathsf{TOP}} ?\texttt{OpenCover}(X)$$

$$\mathcal{O} : \texttt{PointFiniteCover} \iff \forall x \in X \,.\, \Big|\{O \in \mathcal{O} : x \in O\}\Big| < \infty$$

$\texttt{NormalPointFiniteCoverRefinement} :: \forall X : \texttt{T4} \,.\, \forall \mathcal{O} : \texttt{PointFiniteCover}(X) \exists \mathcal{V} : \texttt{OpenCover}(X) :$
$\quad .\, \exists V : \mathcal{O} \leftrightarrow \mathcal{V} : \forall O \in \mathcal{O} \,.\, \overline{V_O} \subset O$

$\texttt{Proof} =$

$\mathcal{G} := \Big\{V : \mathcal{O} \to \mathcal{T}(X) : \forall O \in \mathcal{O} \,.\, V_O = O | \overline{V_O} \subset O \;\&\; \bigcup_{O \in \mathcal{O}} V_O = X\Big\} :?\Big(\mathcal{O} \to \mathcal{T}(X)\Big),$

$[1] := \jmath\mathcal{G} : \mathcal{O} \in \mathcal{G},$

$[2] := \eth\texttt{NonEmpty}[1] : \mathcal{G} \neq \emptyset,$

$\texttt{Assume}\ G : \mathcal{G},$

$\texttt{Assume}\ O : \mathcal{O},$

$\texttt{Assume}\ [3] : G_O = O,$

$U := \bigcup_{V \in \mathcal{O} : V \neq O} O : \mathcal{T}(X),$

$A := U^{\complement} : \texttt{Closed}(X),$

$\Big(W, [4]\Big) := \texttt{NormalCriterion}(A, O) : \sum W \in \mathcal{T}(X) \,.\, A \subset \overline{W} \subset O,$

$[5] := [4]\eth\texttt{ReplaceValue} : \widehat{G}_O(W) \in \mathcal{G},$

$[G.*] := \eth \leq_{\mathcal{G}} [4][5] : \widehat{G}_O(W) \leq G;$

$\rightsquigarrow [4] := I(\forall) : \forall G \in \mathcal{G} \,.\, \forall O \in \mathcal{O} \,.\, G_O \Rightarrow \exists G' \in \mathcal{G} : G' \leq G,$

$\texttt{Assume}\ \mathcal{G}' : \texttt{Chain}(\mathcal{G}'),$

$G := \Lambda O \in \mathcal{O} \,.\, \bigcap_{G' \in \mathcal{G}} G'_O : \mathcal{O} \to ?X,$

$[5] := \jmath\mathcal{G} : \forall O \in \mathcal{O} \,.\, G_O = O | \overline{G_O} \subset O,$

$\texttt{Assume}\ x : X,$

$\mathcal{O}' := \{O \in \mathcal{O} : x \in O\} : \texttt{Finite}(\mathcal{O}),$

$\Big(O, [6]\Big) := \texttt{PigionholePrinciple}\eth\texttt{Chain}(\mathcal{G}')\jmath\mathcal{O}' : \sum O \in \mathcal{O}' \,.\, x \in \bigcap_{G' \in \mathcal{G}'} G'_O,$

$[x.*] := [6]\texttt{UnionSubset} : x \in \bigcup_{O \in \mathcal{O}} G_O;$

$\rightsquigarrow [6] := I\texttt{Subset} : X = \bigcup_{O \in \mathcal{O}} G_O,$

$[7] := \texttt{LocallyFiniteClosedUnion}\jmath : \forall O \in \mathcal{O} \,.\, G_O \in \mathcal{T}(X),$

$[8] := \jmath\mathcal{G}[7][6][5] : G \in \mathcal{G},$

$[\mathcal{G}'.*] := \jmath G : \forall G' \in \mathcal{G}' \,.\, G \le G';$

$\rightsquigarrow \Big(G, [5]\Big) := \texttt{ZornLemma}[2] : \sum G \in \mathcal{G} \,.\, G = \min \mathcal{G},$

$[*] := [4][5] : \forall O \in \mathcal{O} \,.\, \overline{G_O} \subset O;$

$\square$

$\texttt{T1Invariance} :: \forall X : \texttt{T1} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f : \texttt{Closed}(X, Y) \,.\, f(X) : \texttt{T1}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{T4Invariance} :: \forall X : \texttt{T4} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f : \texttt{Closed}(X, Y) \,.\, f(X) : \texttt{T4}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{T0BySingletonClosures} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{T0} \iff \forall x, y \in X \,.\, x \ne y \Rightarrow \overline{\{x\}} \ne \overline{\{y\}}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{T1DoubleDerivedSet} :: \forall X : \texttt{T1} \,.\, \forall A \subset X \,.\, A^{\mathrm{dd}} \subset A^{\mathrm{d}}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : A^{\mathrm{dd}},$

$[1] := \eth\texttt{derivedSet}(x, A^{\mathrm{dd}}) : x \in \overline{A^{\mathrm{d}} \setminus \{x\}},$

$[2] := \texttt{ClosureEquivalent}[1] : \forall U \in \mathcal{U}(x) \,.\, U \cap (A^{\mathrm{d}} \setminus \{x\}) \ne \emptyset,$

$[3] := \eth\texttt{derivedSet}[2] : \forall U \in \mathcal{U}(x) \,.\, \exists y \in U : y \in \overline{A \setminus \{y\}} \setminus \{x\},$

$[4] := \texttt{ClosureEquivalent}[3] : \forall U \in \mathcal{U}(x) \,.\, \exists y \in U : \forall V \in \mathcal{U}(y) \,.\, V \cap A \setminus \{y\} \ne \emptyset \;\&\; y \ne x \ne \emptyset,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$\Big(y, [5]\Big) := [4](U) : \sum y \in U \,.\, \forall V \in \mathcal{U}(y) \,.\, V \cap A \setminus \{y\} \ne \emptyset \;\&\; y \ne x,$

$[6] := \eth\texttt{T1}(X)[5] : \forall V \in U(y) \,.\, V \cap A \setminus \{y\} \setminus \{x\} \ne \emptyset,$

$[7] := [6](U) : U \cap A \setminus \{y\} \setminus \{x\} \ne \emptyset,$

$[U.*] := \texttt{DecreasingSetminus}(A, \{y, x\}, \{x\})\texttt{MonotonicIntersect}[7] : U \cap (A \setminus \{x\}) \ne \emptyset;$

$\rightsquigarrow [5] := \texttt{ClosureEquivalent} : x \in \overline{A \setminus \{x\}},$

$[x.*] := \eth^{-1}\texttt{derivedSet} : x \in A^{\mathrm{d}};$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Subset} : A^{\mathrm{dd}} \subset A^{\mathrm{d}};$

$\square$

$\texttt{T1DerivedSetIsClosed} :: \forall X : \texttt{T1} . \forall A \subset X . A^{\mathrm{d}} : \texttt{Closed}(A)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : A^{\mathrm{d}\complement},$

$\texttt{Assume}\ [1] : \forall U \in \mathcal{U}(x) . U \cap A^{\mathrm{d}} \neq \emptyset,$

$[2] := \eth\texttt{derivedSet}[1] : \forall U \in \mathcal{U}(x) . \exists y \in U : y \in \overline{A \setminus \{x\}},$

$[3] := \texttt{ByAnalogy}(\texttt{proof T1DoubleDerivedSet})[2] : x \in A^{\mathrm{d}},$

$[4] := \texttt{InAndNotIn}[3] : \perp;$

$\rightsquigarrow [1] := \texttt{OpenByInnerCover} : A^{\mathrm{d}\complement} \in \mathcal{T}(X),$

$[*] := \eth^{-1}\texttt{Closed} : A^{\mathrm{d}} \in \mathcal{T}(X);$

$\square$

$\texttt{T1DerivedSetClosure} :: \forall X : \texttt{T1} . \forall A \subset X . \overline{A}^{\mathrm{d}} = A^{\mathrm{d}}$

$\texttt{Proof} \ =$

$[1] := \eth\texttt{PotentialClosure}(\underset{X}{\mathrm{cl}})(A) : A \subset \overline{A},$

$[2] := \texttt{DervivedSetIsMonotonic}[1] : A^{d} \subset \overline{A}^{\mathrm{d}},$

$\texttt{Assume}\ x : \overline{A}^{\mathrm{d}},$

$[1] := \eth\texttt{derivedSet} : x \in \overline{\overline{A} \setminus \{x\}},$

$[2] := \texttt{EquivalentClosure}[1] : \forall U \in \mathcal{U}(x) . U \cap \overline{A} \setminus \{x\} \neq \emptyset,$

$[3] := \texttt{ClosureEquivalent}[2] : \forall U \in \mathcal{U}(x) . \exists y \in U : \forall V \in \mathcal{U}(y) . V \cap A \neq \emptyset \ \&\ y \neq x \neq \emptyset,$

$\texttt{Assume}\ U : \mathcal{U}(x),$

$\Big(y, [4]\Big) := [3](U) : \sum y \in U . \forall V \in \mathcal{U}(y) . V \cap A \neq \emptyset \ \&\ y \neq x,$

$[5] := [4](U) : U \cap A \neq \emptyset,$

$[U.*] := \eth\texttt{T1}[5][4] : U \cap A \setminus \{x\} \neq \emptyset;$

$\rightsquigarrow [4] := \texttt{EquivalentClosure}[3] : x \in \overline{A \setminus \{x\}},$

$[x.*] := \eth^{-1} A^{\mathrm{d}} : x \in A^{\mathrm{d}};$

$\rightsquigarrow [3] := \eth^{-1}\texttt{Subset} : \overline{A}^{\mathrm{d}} \subset A^{\mathrm{d}},$

$[*] := \eth^{-1}\texttt{SetEq}[2][3] : \overline{A}^{\mathrm{d}} = A^{\mathrm{d}};$

$\square$

$\texttt{T1FiniteDerivedSet} :: \forall X : \texttt{T1} . \forall A : \texttt{Finite}(X) . A^{\mathrm{d}} = \emptyset$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{Retraction} :: \prod X : \texttt{TopologicalSpace} . ?\mathrm{End}_{\mathsf{TOP}}(X) .$

$f : \texttt{Retractrion} \iff f^2 = f$

$\texttt{Retract} :: \prod X : \texttt{TopologicalSpace} . ??X$

$R : \texttt{Retract} \iff \exists f : \texttt{Retraction}(X) . f(X) = R$

$\texttt{HausdorfRetractIsClosed} :: \forall X : \texttt{T2} . \forall R : \texttt{Retract}(X) . R : \texttt{Closed}(X)$

$\texttt{Proof} \ =$

$\Big(f, [1]\Big) := \eth\texttt{Retract} : \sum f : \texttt{Retraction}(X) . f(X) = R,$

$[2] := \eth\texttt{Retraction}(f)[1] : R = \{x \in X : f(x) = x\},$

$[*] := \texttt{ClosedEqualityInT2Space} : R : \texttt{Closed}(X);$

$\square$

$\texttt{NormalIteration} :: \forall X : \texttt{T4} . \forall I : \texttt{Finite} . \forall A : \texttt{Disjoint}\Big(I, \texttt{Closed}(X)\Big) . \exists U : \texttt{Disjoint}\Big(I, \mathcal{U}(A)\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{HausdorffIteration} :: \forall X : \texttt{T4} . \forall I : \texttt{Finite} . \forall A : \texttt{Disjoint}\Big(I, \texttt{Finite}(X)\Big) . \exists U : \texttt{Disjoint}\Big(I, \mathcal{U}(A)\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.2 Convergence

1.2.1 Convergence in Nets

$$\texttt{Net} := \prod D : \texttt{DirectedSet} \,.\, \prod X : \mathsf{TOP} \,.\, D \to X : \texttt{DirectedSet} \to \mathsf{TOP} \to \mathsf{SET};$$

$\texttt{Limit} :: \prod X : \mathsf{TOP} \,.\, \prod D : \texttt{DirectedSet} \,.\, \texttt{Net}(D, X) \to ?X$

$L : \texttt{Limit} \iff x \mapsto L = \lim_{n \in D} x_n \iff x \mapsto \forall U \in \mathcal{U}(L) \,.\, \exists N \in D : \forall n : \texttt{NotLessThen}(N) \,.\, x_n \in U$

$\texttt{Cluster} :: \prod X : \mathsf{TOP} \,.\, \prod D : \texttt{DirectedSet} \,.\, \texttt{Net}(D, X) \to ?X$

$C : \texttt{Cluster} \iff x \mapsto C \in \overline{x} \iff x \mapsto \forall U \in \mathcal{U}(L) \,.\, \forall N \in D : \exists n : \texttt{NotLessThan}(N) \,.\, x_n \in U$

$\texttt{Finer} :: \prod X : \mathsf{TOP} \,.\, \prod D, D' : \texttt{DirectedSet} \,.\, ?\Big(\texttt{Net}(X, D) \,\&\, \texttt{Net}(X, D')\Big)$

$x, y : \texttt{Finer} \iff x \to y \iff \exists \phi : D \to D' : \Big(\forall N' \in D' \,.\, \exists N \in D : \forall n : \texttt{NotLessThen}(N) \,.$

$\quad .\, \phi(n) : \texttt{NotLessThan} N'\Big) \,\&$

$\quad \&\, \forall n \in D \,.\, x_n = y_{\phi(n)}$

$\texttt{ClusterOfFiner} :: \forall X : \mathsf{TOP} \,.\, \forall D, D' : \texttt{DirectedSet}(X) \,.\, \forall x \xrightarrow{D, D'} y \,.\, \forall C = \overline{x} \,.\, C = \overline{y}$

$\texttt{Proof} =$

$\Big(\phi, [1]\Big) := \eth\texttt{Finer}(x, y) : \sum \phi : D \to D' : \Big(\forall N' \in D' \,.\, \exists N \in D : \forall n : \texttt{NotLessThen}(N) \,.$

$\quad .\, \phi(n) : \texttt{NotLessThan} N'\Big) \,\&\, \forall n \in D \,.\, x_n = y_{\phi(n)},$

$\texttt{Assume}\ U : \mathcal{U}(C),$

$[2] := \eth\texttt{Cluser}(x)(C) : \forall N \in D \,.\, \exists n : \texttt{NotLessThen}(N) \,.\, x_n \in U,$

$\texttt{Assume}\ N' : D',$

$\Big(N, [3]\Big) := [1](N') : \sum N \in D \,.\, \Big(\forall n : \texttt{NotLessThen}(N) \,.\, \phi(n) \geq N',$

$\Big(n, [4]\Big) := [2](N) : \sum n \in D \,.\, n \geq N \,\&\, x_n \in U,$

$[5] := [3](n) : \phi(n) \geq N',$

$[6] := [1](n) : y_{\phi(n)} = x_n,$

$[N.*] := [4][6] : y_{\phi(n)} \in U;$

$\rightsquigarrow [U.*] := I(\forall) : \forall N' \in D' \,.\, \exists n' \geq N' \,.\, y_{n'} \in U;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Cluster} : C = \overline{y};$

$\square$

$\texttt{LimitOfMeager} :: \forall X : \mathsf{TOP} \;.\; \forall D, D' : \texttt{DirectedSet}(X) \;.\; \forall x \xrightarrow{D,D'} y \;.\; \forall L = \lim_{n \in D} x_n \;.\; L = \lim_{n \in D'} y_n$

$\texttt{Proof} \;=$

$\Big(\phi, [1]\Big) := \eth\texttt{Finer}(x, y) : \sum \phi : D \to D' : \Big(\forall N' \in D' \;.\; \exists N \in D : \forall n : \texttt{NotLessThen}(N) \;.$

$\quad . \; \phi(n) : \texttt{NotLessThan}N'\Big) \;\&\; \forall n \in D \;.\; x_n = y_{\phi(n)},$

$\texttt{Assume}\; U : \mathcal{U}(L),$

$\Big(N'.[2]\Big) := \eth\texttt{Limit}(y)(L) : \sum N' \in D' \;.\; \forall n' \geq N' \;.\; y_n \in U,$

$\Big(N, [3]\Big) := [1](N') : \sum N \in D \;.\; \Big(\forall n \geq N \;.\; \phi(n) \geq N',$

$\texttt{Assume}\; n : D,$

$\texttt{Assume}\; [4] : n \geq N,$

$[5] := [3][4] : \phi(n) \geq N',$

$[6] := [1](n) : y_{\phi(n)} = x_n,$

$[7] := [2][5] : y_{\phi(n)} \in U,$

$[n.*] := [6][7] : x_n \in U;$

$\leadsto [U.*] := I(\forall) : \forall n \geq N \;.\; x_n \in U;$

$\leadsto [*] := \eth^{-1}\texttt{Limit} : L = \lim_{n \in D} x_n;$

$\square$

$\texttt{FromClusterToLimit} :: \forall X : \mathsf{TOP} \;.\; \forall D : \texttt{DirectedSet} \;.\; \forall x : \texttt{Net}(X, D) \;.\; \forall C = \overline{x} \;.$

$\quad . \; \exists D' : \texttt{DirectedSet} : \exists y : \texttt{Net}(X, D) : C = \lim_{n \in D} y_n \;\&\; y \to x$

$\texttt{Proof} \;=$

$D' := \Big\{(n, U) \in D \times \mathcal{U}_{\geq}(C) : x_n \in U\Big\} : \texttt{PartiallyOrderedSet},$

$\texttt{Assume}\; (n, U), (m, V) : D',$

$W := U \cap V : \mathcal{U}(C),$

$\Big(k, [1]\Big) := \eth\texttt{Cluster}(x)(C)\Big(W, \max(n, m)\Big) : \sum k \in D \;.\; k \geq \max \;\&\; x_k \in W,$

$[\ldots *] := \jmath D' \jmath W \texttt{IntersectionSubset}(U, V) \jmath k[1] : (k, W) \geq (n, U) \;\&\; (k, W) \geq (m, V);$

$\leadsto [1] := \eth^{-1}\texttt{DirectedSet} : \Big(D' : \texttt{DirectedSet}\Big),$

$\phi := \Lambda(n, U) \in D' \;.\; n : D' \to D,$

$y := \Lambda(n, U) \in D' \;.\; x_n : \texttt{Net}(D', X),$

$[2] := \jmath y \jmath \phi : y \to x,$

$\texttt{Assume}\; U : \mathcal{U}(C),$

$N := \eth\texttt{NonEmpty}(D) : D,$

$\Big(N', [3]\Big) := \eth\texttt{Cluster}(x)(C)(U, N) : \sum N' \in D \;.\; N' \geq N \;\&\; x_{N'} \in U,$

$\texttt{Assume}\; (n, V) : D',$

$\texttt{Assume}\; [4] : (n, V) \geq (N', U),$

$[5] := \jmath D'[4] : V \subset U,$

$[U.*] := \jmath D' \texttt{SubsetTransitivity}[5] \jmath y_{(n,V)} : y_{(n,V)} \in U;$

$\leadsto [*] := \eth^{-1}\texttt{Limit} : \lim_{n \in D'} y = C;$

$\square$

$\texttt{ClosureByConvergence} :: \forall X \in \mathsf{TOP} . \forall A \subset X . \forall p \in \overline{A} . \exists x : \texttt{Net}(D, X) : p = \lim_{n \in D} x_n \ \& \ \forall n \in D . x_n \in A$

$\texttt{Proof} \ =$

$D := \mathcal{U}(p) : \texttt{DirectedSet},$

$\texttt{Assume} \ U : D,$

$[1] := \texttt{ClosureEquivalent}(A)(x)(U) : U \cap A \neq \emptyset,$

$x_U := \texttt{NonEmpty} : U \cap A;$

$\rightsquigarrow x := I(\rightarrow) : \texttt{Net}(D, X),$

$[1] := \jmath x \texttt{IntersectSubset} \eth \texttt{Subset} : \forall n \in D . x_n \in A,$

$\texttt{Assume} \ U : \mathcal{U}(p),$

$N := U : D,$

$\texttt{Assume} \ n : D,$

$\texttt{Assume} \ [2] : n \geq N,$

$[3] := \jmath D[2] : n \subset N,$

$[U.*] := \jmath x \eth \texttt{Subset}[2] \texttt{IntersectSubset} \eth \texttt{Subset} \jmath N : x_n \in U;$

$\rightsquigarrow [*] := \eth^{-1} \texttt{Limit} : p = \lim_{n \in D} x_n;$

$\square$

$\texttt{ClosedByLimits} :: \forall X \in \mathsf{TOP} . \forall A \subset X . A : \texttt{Closed}(X) \iff$

$\iff \forall x : \texttt{Net}(D, X) . x(D) \subset A \Rightarrow \forall L = \lim_{n \in D} x_n . L \in A$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DerivedSetByConvergence} :: \forall X \in \mathsf{TOP} . \forall A \subset X . \forall p \in A^{\mathrm{d}} . \exists x : \texttt{Net}(D, X) : p = \lim_{n \in D} x_n \ \&$

$\& \ \forall n \in D . x_n \in A \ \& \ x_n \neq p$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ClusterAsIntersect} :: \forall X : \mathsf{TOP} . \forall x : \texttt{Net}(D, X) . \overline{x} = \bigcap_{N \in D} \overline{\{x_n | n \geq N\}}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{HausdorffByLimits} :: \forall X : \mathsf{TOP} . X : \texttt{T2} \iff \forall x : \texttt{Net}(D, X) . |\lim_{n \in D} x_n| \leq 1$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.2.2 Filters

$$\texttt{IntersectionClosed} :: \prod X : \mathsf{SET} \; . \; ???X$$
$$\mathcal{A} : \texttt{IntersectionClosed} \iff \forall A, B \in \mathcal{A} \; . \; A \cap B \in \mathcal{A}$$

$$\texttt{Filter} :: \prod X : \mathsf{SET} \; . \; \prod \mathcal{A} : \texttt{IntersectionClosed}(X) \; . \; ??\mathcal{A}$$
$$\mathcal{F} : \texttt{Filter} \iff \mathcal{F} \neq \emptyset \;\&\; \emptyset \notin \mathcal{F} \;\&\; \forall A, B \in \mathcal{F} \; . \; A \cap B \;\&\; \forall A \in \mathcal{F} \; . \; \forall B \in \mathcal{A} \; . \; A \subset B \Rightarrow B \in \mathcal{F}$$

$$\texttt{Ultrafilter} :: \prod X : \mathsf{SET} \; . \; \prod \mathcal{A} : \texttt{IntersectionClosed}(X) \; . \; ?\texttt{Filter}(\mathcal{A})$$
$$\mathcal{F} : \texttt{Ultrafilter} \iff \forall \mathcal{F}' : \texttt{Filter}(\mathcal{A}) \; . \; \mathcal{F} \subset \mathcal{F}' \Rightarrow \mathcal{F} = \mathcal{F}'$$

$$\texttt{FilterBase} :: \prod X : \mathsf{SET} \; . \; \prod \mathcal{A} : \texttt{IntersectionClosed}(X) \; . \; ??\mathcal{A}$$
$$\mathcal{B} : \texttt{FilterBase} \iff \mathcal{B} \neq \emptyset \;\&\; \emptyset \notin \mathcal{B} \;\&\; \forall A, B \in \mathcal{B} \; . \; \exists C \in \mathcal{B} : C \subset A \cap B$$

$$\texttt{generateFilter} :: \prod X : \mathsf{SET} \; . \; \prod \mathcal{A} : \texttt{IntersectionClosed}(X) \; . \; \texttt{FilterBase}(\mathcal{A}) \to \texttt{Filter}(\mathcal{A})$$
$$\texttt{generateFilter}\,(\mathcal{B}) = \langle \mathcal{B} \rangle := \{A \in \mathcal{A} : \exists B \in \mathcal{B} : B \subset A\}$$

$$\texttt{FilterLimit} :: \prod X : \mathsf{TOP} \; . \; \texttt{Filter}\; \mathcal{T}(X) \to ?X$$
$$L : \texttt{FilterLimit} \iff \mathcal{F} \mapsto L = \lim \mathcal{F} \iff \mathcal{F} \mapsto \mathcal{U}(L) \subset \mathcal{F}$$

$$\texttt{FilterBaseLimit} :: \prod X : \mathsf{TOP} \; . \; \texttt{FilterBase}\; \mathcal{T}(X) \to ?X$$
$$L : \texttt{FilterLimit} \iff \mathcal{B} \mapsto L = \lim \mathcal{B} \iff \mathcal{B} \mapsto L = \lim \langle \mathcal{B} \rangle$$

$$\texttt{FilterCluster} :: \prod X : \mathsf{TOP} \; . \; \texttt{Filter}\; \mathcal{T}(X) \to ?X$$
$$C : \texttt{FilterCluster} \iff \mathcal{F} \mapsto C = \overline{\mathcal{F}} \iff \mathcal{F} \mapsto \forall U \in \mathcal{F} \; . \; C \in \overline{U}$$

$$\texttt{FilterBasCluster} :: \prod X : \mathsf{TOP} \; . \; \texttt{FilterBase}\; \mathcal{T}(X) \to ?X$$
$$C : \texttt{FilterBaseCluster} \iff \mathcal{B} \mapsto C = \overline{\mathcal{B}} \iff \mathcal{B} \mapsto C = \overline{\langle \mathcal{B} \rangle}$$

$$\texttt{Finer} :: \prod X : \mathsf{TOP} \; . \; \Big(\texttt{Filter}\mathcal{T}(X) \times \texttt{Filter}\mathcal{T}(X)\Big)$$
$$(\mathcal{F}, \mathcal{F}') : \texttt{Finer} \iff \mathcal{F}' \subset \mathcal{F}$$

$$\texttt{netAsFilter} :: \prod X : \mathsf{TOP} \; . \; \texttt{Net}(D, X) \to \texttt{Filter}\; \mathcal{T}(X)$$
$$\texttt{netAsFilter}\,(x) = \mathcal{F}_x := \Big\{ U \in \mathcal{T}(X) : \exists N \in D : \forall n \geq N \; . \; x_n \in U \Big\}$$

$$\texttt{filterAsNet} :: \prod X : \mathsf{TOP} \; . \; \texttt{Net}(D, X) \to \texttt{Filter}\; \mathcal{T}(X)$$
$$\texttt{filterAsNet}\,(\mathcal{F}) = x^{\mathcal{F}} := \Lambda(x, U) \in D \; . \; x \in \quad \texttt{where} \quad D = \Big\{ (x, U) \Big| x \in X, U \in F : x \in U \Big\}$$

$\texttt{FilterNetLimitsEquivalence} :: \forall X \in \mathsf{TOP} \,.\, \forall x : \texttt{Net}(D, X) \,.\, \lim_{n \in D} x_n = \lim \mathcal{F}_x$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{NetFilterLimitsEquivalence} :: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{F} : \texttt{Filter}\ \mathcal{T}(X) \,.\, \lim_{n \in D} x_n^{\mathcal{F}} = \lim \mathcal{F}$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.2.3 Sets with Convergent Sequences

$$\texttt{Subsequence} :: \prod X \in \mathsf{SET} \,.\, (\mathbb{N} \to X) \to ?(\mathbb{N} \to X)$$

$$y : \texttt{Subsequence} \iff \Lambda x : \mathbb{N} \to X \,.\, y \subset x \iff \Lambda x : \mathbb{N} \to X \,.\, \exists k : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) \,.\, y = x_k$$

$$\texttt{WithConvergent} :: ?\left(\sum X \in \mathsf{SET} \,.\, \sum \mathcal{C} :?(\mathbb{N} \to X) \,.\, \mathcal{C} \to X\right)$$

$$(X, \mathcal{C}, L) : \texttt{WithConvergent} \iff \Big(\forall x \in X \,.\, (\Lambda n \in \mathbb{N} \,.\, x) \in \mathcal{C} \;\&\; L(\cdot \mapsto x) = x\Big) \;\&$$

$$\& \Big(\forall x \in \mathcal{C} \,.\, \forall y \subset x \,.\, y \in \mathcal{C} \;\&\; L(x) = L(y)\Big) \;\&$$

$$\& \Big(\forall x \notin \mathcal{C} \,.\, \exists y \subset x : \forall z \subset y \,.\, z \notin \mathcal{C}\Big)$$

$$\texttt{closure} :: \prod (X, \mathcal{C}, L) : \texttt{WithConvergent} \,.\, ?X \to ?X$$

$$\texttt{closure}\,(A) = \overline{A} := \{x \in X : \exists a \in \mathcal{C} : \operatorname{Im} a \subset A \;\&\; L(a) = x\}$$

$$\texttt{PropertiesOfClosure} :: \forall (X, \mathcal{C}, L) : \texttt{WithConvergent} \,.\, \overline{\emptyset} = \emptyset \;\&\; \forall A, B \subset X \,.\, A \subset \overline{A} \;\&\; \overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof $=$

...

□

$$\texttt{DiagonalProperty} :: ?\texttt{WithConvergent}(X)$$

$$(X, \mathcal{C}, L) : \texttt{DiagonalProperty} \iff \forall x \in \mathcal{C} \,.\, \forall y : \mathbb{N} \to \mathcal{C} \,.\, \Big(\forall n \in \mathbb{N} \,.\, L(y_n) = x_n\Big) \Rightarrow$$

$$\Rightarrow \exists i, j : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) : L(y_{i,j}) = L(x)$$

$$\texttt{ClosureAndDiagonalProperty} :: \forall (X, \mathcal{C}, L) : \texttt{WithConvergent} \,.\, (X, \mathcal{C}, L) : \texttt{DiagonalProperty} \iff$$

$$\iff \forall A \subset X \,.\, \overline{\overline{A}} = \overline{A}$$

Proof $=$

...

□

$$\texttt{topologyOfFrechet} :: \prod (X, \mathcal{C}, L) : \texttt{DiagonalProperty} \,.\, \texttt{Topology}(X)$$

$$\texttt{topologyOfFrechet}\,() = F(X, \mathcal{C}, L) := \Big\langle \texttt{closure}(X, \mathcal{C}, L) \Big\rangle_{\mathsf{TOP}}$$

$$\texttt{withDiagonalPropertyAsTopologicalSpace} :: \texttt{DiagonalProperty} \to \mathsf{TOP}$$

$$\texttt{withDiagonalPropertyAsTopologicalSpace}\,(X, \mathcal{C}, L) = \texttt{synecdoche} := \Big(X, F(X, \mathcal{C}, L)\Big)$$

$$\texttt{FrechetConvergenceConsistancy} :: \forall (X, \mathcal{C}, L) : \texttt{DiagonalProperty} \,.\, \forall x : \texttt{Convergent}(\mathbb{N}, X) \,.$$

$$.\, x \in \mathcal{C} \;\&\; \lim_{n \to \infty} x_n = L(x)$$

Proof $=$

...

□

1.3 Category of Topological Spaces

1.3.1 Continuous Morphisms

$$\texttt{ContinuousMap} :: \prod X, Y : \texttt{TopologicalSpace} \;.\; ?(X \to Y)$$
$$f : \texttt{ContinuousMap} \iff f \in C(X,Y) \iff \forall U \in \mathcal{T}(Y) \;.\; f^{-1}(U) \in \mathcal{T}(X)$$

$$\texttt{ContinuosBySubbase} :: \forall X, Y : \texttt{TopologicalSpace} \;.\; \forall \mathcal{B} : \texttt{Subbase}(Y) \;.\; \forall f : X \to Y \;.$$
$$.\; f \in C(X,Y) \iff \forall B \in \mathcal{B} \;.\; f^{-1}(B) \in \mathcal{T}(X)$$
$\texttt{Proof} =$

...

$\square$

$$\texttt{ContinuosByBase} :: \forall X, Y : \texttt{TopologicalSpace} \;.\; \forall \mathcal{B} : \texttt{Base}(Y) \;.\; \forall f : X \to Y \;.$$
$$.\; f \in C(X,Y) \iff \forall B \in \mathcal{B} \;.\; f^{-1}(B) \in \mathcal{T}(X)$$
$\texttt{Proof} =$

...

$\square$

$$\texttt{ContinuosByNeighbourhoods} :: \forall X, Y : \texttt{TopologicalSpace} \;.\; \forall \mathcal{B} : \texttt{Base}(Y) \;.\; \forall f : X \to Y \;.$$
$$.\; f \in C(X,Y) \iff \forall x \in X \;.\; \forall U \in \mathcal{U}\Big(f(x)\Big) \;.\; \exists V \in \mathcal{U}\Big(x\Big) : f(V) \subset U$$
$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \forall x \in X \;.\; \forall U \in \mathcal{U}\Big(f(x)\Big) \;.\; \exists V \in \mathcal{U}(x) : f(V) \subset U,$

$\texttt{Assume}\ U : \mathcal{T}\Big(f(x)\Big),$

$\texttt{Assume}\ x : f^{-1}(U),$

$\Big(V, [2]\Big) := [1](x, U) : \sum V \in \mathcal{U}(x) \;.\; f(V) \subset U,$

$[x.*] := f^2[2] : V \subset f^{-1}(U);$

$\rightsquigarrow [U.*] := \texttt{OpenByCover} : f^{-1}(U) \in \mathcal{T}(X);$

$\rightsquigarrow [*] := \eth^{-1} C(X,Y) : f \in C(X,Y);$

$\square$

$$\texttt{ContinuosByClosedSets} :: \forall X, Y : \texttt{TopologicalSpace} \;.\; \forall f : X \to Y \;.$$
$$.\; f \in C(X,Y) \iff \forall A : \texttt{Closed}(X) \;.\; f^{-1}(A) : \texttt{Closed}(X)$$
$\texttt{Proof} =$

...

$\square$

$$\texttt{ContinuousByClosure1} :: \forall X, Y : \texttt{TopologicalSpace} \;.\; \forall f : X \to Y \;.$$
$$.\; f \in C(X,Y) \iff \forall A \subset X \;.\; f(\overline{A}) \subset \overline{f(A)}$$
$\texttt{Proof} =$

...

$\square$

$\texttt{ContinuousByClosure2} :: \forall X, Y : \texttt{TopologicalSpace} \,.\, \forall f : X \to Y \,.$

$\quad .\, f \in C(X,Y) \iff \forall A \subset Y \,.\, \overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ContinuousByIntetior} :: \forall X, Y : \texttt{TopologicalSpace} \,.\, \forall f : X \to Y \,.$

$\quad .\, f \in C(X,Y) \iff \forall A \subset Y \,.\, f^{-1}(\operatorname{int} A) \subset \operatorname{int} f^{-1}(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{categoryOfTopoplogicalSpaces} :: \mathsf{CAT}$

$\texttt{categoryOfTopologicalSpaces}\,() = \mathsf{TOP} := \Big(\texttt{TopologicalSpace}, C, \circ, \mathrm{id}\Big)$

$\texttt{Homeo} := \texttt{Iso}(\mathsf{TOP}) : \mathsf{TOP}^2 \to \texttt{Type};$

$\texttt{ContinuousAtAPoint} :: \prod X, Y : \texttt{TopologicalSpace} \,.\, X?(X \to Y)$

$f : \texttt{ContinuousAtAPoint} \iff \Lambda x \in X \,.\, \forall U \in \mathcal{U}\Big(f(x)\Big) \,.\, f^{-1}(U) \in \mathcal{U}(x)$

$\texttt{ContinuousImageOfLimits} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, \forall x : \texttt{Net}(D, X) \,.\, f\Big(\lim_{n \in D} x_n\Big) \subset \lim_{n \in D} f(x_n)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; B : f(\lim_{n \in D} x_n),$

$\Big(A, [1]\Big) := \eth\texttt{image} : \sum A = \lim_{n \in D} x_n \,.\, f(A) = B,$

$\texttt{Assume}\; U : \mathcal{U}(B),$

$V := f^{-1}(U) : \mathcal{U}(A),$

$\Big(N, [2]\Big) := \eth\texttt{Limit}(A) : \sum N \in D \,.\, \forall n \geq N \,.\, x_n \in V,$

$[U.*] := \jmath V \eth\texttt{preimage}[2] : \forall n \geq N \,.\, f(x_n) \in U;$

$\rightsquigarrow [B.*] := \eth^{-1}\texttt{Limit} : B = \lim_{n \in D} f(x_n);$

$\rightsquigarrow [*] := I\texttt{Subset} : f(\lim_{n \in D} x_n) \subset \lim_{n \in D} f(x_n);$

$\square$

$\texttt{SeparableByContinuousMap} :: \forall Y \in \mathsf{TOP} \,.\, \forall X : \texttt{Separable} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, \operatorname{Im} f : \texttt{Separable}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OpenMap} :: \forall X, Y \in \mathsf{TOP} \,.\, ?(X \to Y)$

$f : \texttt{OpenMap} \iff \forall U \in \mathcal{U}(X) \,.\, f(U) \in \mathcal{T}(Y)$

$\texttt{ClosedMap} :: \forall X, Y \in \mathsf{TOP} \,.\, ?(X \to Y)$

$f : \texttt{ClosedMap} \iff \forall A : \texttt{Closed}(X) \,.\, f(A) : \texttt{Closed}(Y)$

$\texttt{ClosedMapEquivalent} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \to Y \,.\, f : \texttt{ClosedMap} \iff \forall B \subset Y \,.\, \forall U \in \mathcal{T}(X) \,.$
$\quad . \forall [0] : f^{-1}(B) \subset U \,.\, \exists V \in \mathcal{T}(Y) : B \subset V \;\&\; f^{-1}(V) \subset U$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \forall B \subset Y \,.\, \forall U \in \mathcal{T}(X) \,.\, \forall [0] : f^{-1}(B) \subset U \,.\, \exists V \in \mathcal{T}(Y) : B \subset V \;\&\; f^{-1}(V) \subset U,$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$U := A^{\complement} : \mathcal{T}(X),$

$B := f^{\complement}(A) :?Y,$

$[2] := \jmath(U)\jmath(B) : f^{-1}(B) = U,$

$\Big(V, [3]\Big) := [1]\Big(B, U, [2]\Big) : \sum V \in T(Y) \,.\, B \subset V \;\&\; f^{-1}(V) \subset U,$

$[3] := \jmath U \jmath B[2] : f^{\complement}(A) \subset V \;\&\; f^{-1}(V) \subset A^{\complement},$

$[4] := \texttt{PreimageDisjoint}[3] : V \cap f(A) = \emptyset,$

$[5] := \texttt{SubsetAndDisjointDecompositon}[3][4] : f^{\complement}(A) = V,$

$[1.*] := \eth^{-1}\texttt{Closed}(X)[5] : \Big[f(A) : \texttt{Closed}(X)\Big];$

$\rightsquigarrow [*] := \eth^{-1}\texttt{ClosedMap} : \Big(f : \texttt{ClosedMap}(X, Y)\Big);$

$\square$

$\texttt{OpenMapEquivalent} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \to Y \,.\, f : \texttt{OpenMap} \iff \forall B \subset Y \,.\, \forall A : \texttt{Closed}(X) \,.$
$\quad . \forall [0] : f^{-1}(B) \subset A \,.\, \exists C : \texttt{Closed}(Y) : B \subset C \;\&\; A \subset f^{-1}(C)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ClosedMapCondition} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f : \texttt{ClosedMap}(X, Y) \iff \forall y \in Y \,.\, \forall U \in \mathcal{T}(X) \,.$
$\quad . \forall f^{-1}\{y\} \subset U \,.\, \exists V \in \mathcal{U}(y) : f^{-1}(V) \subset U$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OpenMapCondition} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f : \texttt{ClosedMap}(X, Y) \iff \forall y \in Y \,.\, \forall U \in \mathcal{T}(X) \,.$
$\quad . \forall f^{-1}\{y\} \subset U \,.\, \exists V \in \mathcal{U}(y) : f^{-1}(V) \subset U$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OpenBijectionIsHomeo} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \leftrightarrow Y \,.\, f : \texttt{Open}(X,Y) \;\&\; C(X,Y) \Rightarrow X \xleftrightarrow{\mathsf{TOP}} Y$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OpenBijectionIsHomeo} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \leftrightarrow Y \,.\, f : \texttt{Open}(X,Y) \;\&\; C(X,Y) \Rightarrow X \xleftrightarrow{\mathsf{TOP}} Y$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ClosedMappingClosure} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f : \texttt{Closed}(X,Y) \iff \forall A \subset X \,.\, f\left(\overline{A}\right) = \overline{f(A)}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \Big(f : \texttt{Closed}(X,Y)\Big),$

$\texttt{Assume}\ A :?X,$

$[1] := \eth\texttt{PotentialClosure}(\underset{X}{\mathrm{cl}})(A) : A \subset \overline{A},$

$[2] := \texttt{SubsetImage}(f, A) : f(A) \subset f(\overline{A}),$

$[3] := \eth\texttt{Closure} : \overline{f(A)} \subset f(\overline{A}),$

$[4] := \eth\texttt{PotentialClosure} : f(A) \subset \overline{f(A)},$

$[5] := \texttt{SubsetPreimage} : A \subset f^{-1}(\overline{f(A)}),$

$[6] := \texttt{ClosureIsMonotonic}[5] : \overline{A} \subset f^{-1}(\overline{f(A)}),$

$[7] := \texttt{MonotonicImage}[6] : f(\overline{A}) \subset f f^{-1}(\overline{f(A)}),$

‘$[8] := \texttt{ImageOfPreimage}[7] : f(\overline{A}) \subset \overline{f(A)},$

$[1.*] := \eth^{-1}\texttt{SetEq} : f(\overline{A}) = \overline{f(A)};$

$\leadsto [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \forall A \subset X \,.\, f(\overline{A}) = \overline{f(A)},$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$[3] := \texttt{ClosedClosure}[2] : f(A) = f\left(\overline{A}\right) = \overline{f(A)},$

$[A.*] := \eth\texttt{closure}[3] : \Big(f(A) : \texttt{Closed}(Y)\Big);$

$\leadsto [2.*] := \eth^{-1}\texttt{Closed} : \Big(f : \texttt{Closed}(X,Y)\Big);$

$\leadsto [*] := I(\iff) : f : \texttt{Closed}(X,Y) \iff \forall A \subset X \,.\, f\left(\overline{A}\right) = \overline{f(A)};$

$\square$

$\texttt{OpendMappingInrerior} :: \forall X, Y : \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f : \texttt{Open}(X,Y) \iff \forall A \subset X \,.$
$\quad .\, f\left(\mathrm{int}\, A\right) \subset \mathrm{int}\, f(A)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\mathtt{OpenByInteriorPreimage} :: \forall X, Y : \mathsf{TOP} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} \;.\; f : \mathtt{Open}(X, Y) \iff$
$\quad \iff \forall A \subset Y \;.\; f^{-1}(\operatorname{int} A) = \operatorname{int} f^{-1}(A)$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ [1] : f : \mathtt{Open}(X, Y),$

$\mathtt{Assume}\ A :?Y,$

$[2] := \eth\mathtt{PotentialInterior}(\underset{X}{\mathrm{cl}})(A) : \operatorname{int} A \subset A,$

$[3] := \mathtt{PreimageSubset}[2] : f^{-1}(\operatorname{int} A) \subset f^{-1}(A),$

$[4] := \mathtt{InteriorIsMonotonic}[3] : f^{-1}(\operatorname{int} A) \subset \operatorname{int} f^{-1}(A),$

$[5] := \mathtt{ImageOfPreImage} : f(\operatorname{int} f^{-1}(A)) \subset A,$

$[7] := \eth\mathtt{Interior} : f(\operatorname{int} f^{-1}(A)) \subset \operatorname{int} A,$

$[8] := \eth\mathtt{PreimageSubset} : f^{-1} f\Big(\operatorname{int} f^{-1}(A)\Big) \subset f^{-1}(\operatorname{int} A),$

$[9] := \mathtt{ImagePreimage}(f)\eth\mathtt{interiorSetEq} : \operatorname{int} f^{-1}(A) \subset f^{-1}(\operatorname{int} A),$

$[1.*] := [2][9] : \operatorname{int} f^{-1}(A) = f^{-1}(\operatorname{int} A);$

$\rightsquigarrow [1] := I(\Rightarrow) : \mathtt{Left} \Rightarrow \mathtt{Right},$

$\mathtt{Assume}\ [2] : \mathtt{Right},$

$\mathtt{Assume}\ A :?X,$

$[3] := [2]\Big(f(A)\Big)\mathtt{PreimageOfImage} : \operatorname{int} A \subset \operatorname{int} f^{-1} f(A) f^{-1}\Big(\operatorname{int} f(A)\Big),$

$[A.*] := \mathtt{SubsetImage}[3]\mathtt{ImageOfPreimage} : f(\operatorname{int} A) \subset f f^{-1}\Big(\operatorname{int} f(A)\Big) \subset \operatorname{int} f(A);$

$\rightsquigarrow [2.*] := \mathtt{OpenMappingInterior} : \Big(f : \mathtt{Open}(X, Y)\Big);$

$\rightsquigarrow [*] := I(\iff) : \mathtt{This};$

$\square$

$\mathtt{OpenByInteriorPreimage} :: \forall X, Y : \mathsf{TOP} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} \;.\; f : \mathtt{Open}(X, Y) \iff$
$\quad \iff \forall A \subset Y \;.\; f^{-1}(\operatorname{int} A) = \operatorname{int} f^{-1}(A)$

$\mathtt{Proof} \;=$

$\dots$

$\square$

$\mathtt{Clopen} :: \prod X, Y : \mathsf{TOP} \;.\; X \xrightarrow{\mathsf{TOP}} Y$

$f : \mathtt{Clopen} \iff f : \mathtt{Open}(X, Y) \;\&\; \mathtt{Closed}(X, Y)$

$\mathtt{ClopenMappingOfClosedDomain} :: \forall X, Y : \mathsf{TOP} \;.\; \forall f : \mathtt{Clopen}(X, Y) \;.\; \forall A : \mathtt{ClosedDomain}(X) \;.$
$\quad .\; f(A) : \mathtt{ClosedDomain}(Y)$

$\mathtt{Proof} \;=$

$[1] := \eth\mathtt{ClosedDomain}(A)\mathtt{ClosedMappingClosure}(f)\mathtt{OpenMappingIntereior}(f) :$
$\quad : f(A) f\Big(\overline{(\operatorname{int} A)}\Big) \overline{f(\operatorname{int} A)} =\subset \overline{\operatorname{int} f(A)},$

$[2] := \eth\mathtt{InteriorIsSubset} : \operatorname{int} f(A) \subset f(A),$

$[3] := \mathtt{MonotonicClosure}(\operatorname{int})\eth\mathtt{Closed}(X, Y)(f) : \overline{\operatorname{int} f(A)} \subset f(A),$

$[4] := \mathtt{SetEq}[1][3] : f(A) = \overline{\operatorname{int} f(A)},$

$[*] := \eth^{-1}\mathtt{ClosedDomain} : \Big(f(A) : \mathtt{ClosedDomain}(Y)\Big);$

$\square$

$\texttt{OpenClosedDomainPreimage} :: \forall X, Y : \mathsf{TOP} . \forall f : \texttt{Open}(X, Y) . \forall A : \texttt{ClosedDomain}(Y) .$
$\quad . f^{-1}(A) : \texttt{ClosedDomain}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{OpenOpenDomainPreimage} :: \forall X, Y : \mathsf{TOP} . \forall f : \texttt{Clopen}(X, Y) . \forall A : \texttt{OpenDomain}(Y) .$
$\quad . f^{-1}(A) : \texttt{OpenDomain}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{BorelPreimage} :: \forall X, Y : \mathsf{TOP} . \forall f : X \xrightarrow{\mathsf{TOP}} Y . \forall B \in \mathcal{B}(Y) . \forall f^{-1}(X)$

$\texttt{Proof} =$

...

$\square$

1.3.2 Subspaces

$\texttt{subspaceTopology} :: \prod X \in \mathsf{SET} \,.\, \prod Y \subset X \,.\, \texttt{Topology}(X) \to \texttt{Topology}(Y)$

$\texttt{subspaceTopology}\,(T) := \{U \cap Y | U \in T\}$

$\texttt{topologicalSubspace} :: \prod X \in \mathsf{TOP} \,.\, ?X \to \mathsf{TOP}$

$\texttt{topologicalSubspace}\,(Y) = \texttt{synecdoche} := \Big(Y, \texttt{subspaceTopology}(X, Y)\Big)$

$\texttt{ClosedSetsInASubspace} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \subset X \,.\, \texttt{Closed}(Y) = \Big\{Y \cap A | A : \texttt{Closed}(X)\Big\}$

$\texttt{Proof} =$

...

$\square$

$\texttt{ClosureInASubspace} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \subset X \,.\, \forall A \subset Y \,.\, \underset{Y}{\mathrm{cl}}(A) = \underset{X}{\mathrm{cl}}(A) \cap Y$

$\texttt{Proof} =$

...

$\square$

$\texttt{ContinuousEmbedding} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \subset X \,.\, \iota_{Y,X} : Y \xrightarrow{\mathsf{TOP}} X$

$\texttt{Proof} =$

...

$\square$

$\texttt{ClosedEmbeddingCriterion} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \subset X \,.\, \iota_{Y,X} : \texttt{Closed}(Y, X) \iff Y : \texttt{Closed}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{OpendEmbeddingCriterion} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \subset X \,.\, \iota_{Y,X} : \texttt{Open}(Y, X) \iff Y : \texttt{Open}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ContinuousRestriction} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f_{|A} : A \xrightarrow{\mathsf{TOP}} Y$

$\texttt{Proof} =$

...

$\square$

$\texttt{ContinuousCorestriction} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall A \subset Y \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, \forall [0] : f(X) \subset A \,.\, f^{|A} : X \xrightarrow{\mathsf{TOP}} A$

$\texttt{Proof} =$

...

$\square$

$\texttt{HomeoEmbedding} :: \prod X, Y \in \mathsf{TOP} \, . \, ?(X \xrightarrow{\mathsf{TOP}} Y)$

$f : \texttt{HomeoEmbedding} \iff \exists A \subset Y : \exists \varphi : X \xleftrightarrow{\mathsf{TOP}} A \, . \, f = \varphi \iota_{A,Y}$

$\texttt{Hereditary} :: ??\mathsf{TOP}$

$P : \texttt{Hereditary} \iff \forall X \in \mathsf{TOP} \, . \, X : P \Rightarrow \forall A \subset X \, . \, A : P$

$\texttt{hereditary} :: ?\mathsf{TOP} \to ?\mathsf{TOP}$

$\texttt{hereditary}\,(P) := \Lambda X : P \, . \, \forall A \subset X \, . \, A : P$

$\texttt{SeparationIsHereditary} :: \texttt{T0}, \texttt{T1}, \texttt{T2}, \texttt{T3} : \texttt{Hereditary}$

$\texttt{Proof} =$

...

□

$\texttt{UrysohnIsHereditary} :: \texttt{Urysohn} : \texttt{Hereditary}$

$\texttt{Proof} =$

...

□

$\texttt{PerfectNormalityIsHereditary} :: \texttt{PerfectlyNormal} : \texttt{Hereditary}$

$\texttt{Proof} =$

...

□

$\texttt{HereditaryNormallityCondition1} :: \forall X \in \mathsf{TOP} \, . \, X : \texttt{hereditary}\ \texttt{T4} \iff \forall U \in \mathcal{T}(X) \, . \, U : \texttt{T4}$

$\texttt{Proof} =$

...

□

$\texttt{Separated} :: \prod X \in \mathsf{TOP} \, . \, ?(?X \times ?X)$

$A, B : \texttt{Separated} \iff \overline{A} \cap B = \emptyset \ \& \ A \cap \overline{B} = \emptyset$

$\texttt{HereditaryNormallityCondition2} :: \forall X \in \mathsf{TOP} \, . \, X : \texttt{hereditary}\ \texttt{T4} \iff \forall (A, B) : \texttt{Separated}(X) \, .$
$\quad . \, \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

$\texttt{Proof} =$

...

□

$\texttt{T5} := \texttt{hereditary}\ \texttt{Normal} : \texttt{Type};$

$\texttt{T6} := \texttt{PerfectlyNormal} : \texttt{Type};$

$\texttt{SeparationHierarchy6} :: \texttt{T6} \subset \texttt{T5} \subset \texttt{T4}$

$\texttt{Proof} =$

...

□

$\texttt{Extendable} :: \prod X, Y \in \mathsf{TOP} \;.\; \prod A \subset X \;.\; ?(A \xrightarrow{\mathsf{TOP}} Y)$

$f : \texttt{Extendable} \iff \exists F : X \xrightarrow{\mathsf{TOP}} Y \;.\; f = F_{|A}$

$\texttt{TietzeLemma} :: \forall X : \texttt{T4} \;.\; \forall A : \texttt{Closed}(X) \;.\; \forall c \in \mathbb{R} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} [-c, c] \;.$

$\;.\; \exists F : X \xrightarrow{\mathsf{TOP}} \frac{1}{3}[-c, c] : \forall a \in A \;.\; |f(a) - F(a)| \le \frac{2c}{3}$

$\texttt{Proof} \;=$

$B := f^{-1}\left[-c, -\frac{c}{3}\right] : \texttt{Closed}(A),$

$C := f^{-1}\left[\frac{c}{3}, c\right] : \texttt{Closed}(A),$

$[1] := \texttt{ClosedSetsOfASubset}\jmath B, C : \Big(B, C : \texttt{Closed}(X)\Big),$

$\Big(g, [2]\Big) := \texttt{UrysohnLemma}(B, C) : \sum g : X \xrightarrow{\mathsf{TOP}} [0, 1] \;.\; g(B) = \{0\} \;\&\; g(C) = \{1\},$

$F := \frac{2c}{3}\left(g - \frac{1}{2}\right) : X \xrightarrow{\mathsf{TOP}} \frac{1}{3}[-c, c],$

$[*] := [2]\jmath B \jmath C : \forall a \in A \;.\; |F(a) - f(a)| \le \frac{2c}{3};$

$\square$

$\texttt{TietzeUrysohnExtension} :: \forall X : \texttt{T4} \;.\; \forall A : \texttt{Closed}(X) \;.\; \forall f : A \xrightarrow{\mathsf{TOP}} [-1, 1] \;.\; f : \texttt{Extendable}\Big(X, [-1, 1]\Big)$

$\texttt{Proof} \;=$

$\Big(g_1, \male_1\Big) := \texttt{TietzeLemma}(X) : \sum g_1 : X \xrightarrow{TOP} \frac{1}{3}[-1, 1] \;.\; \forall a \in A \;.\; |g_1(a) - f(a)| \le \frac{2}{3},$

$\texttt{Assume}\; n : \mathbb{N},$

$h := f - \sum_{i=1}^{n} g_n : A \xrightarrow{\mathsf{TOP}} \mathbb{R},$

$[2] := \jmath h \male_n : \operatorname{Im} h \subset \left(\frac{2}{3}\right)^n [-1, 1],$

$\Big(g_{n+1}, [3]\Big) := \texttt{TietzeLemma}(h, [2]) : \sum g_{n+1} : X \xrightarrow{\mathsf{TOP}} \frac{1}{3^{n+1}}[-1, 1] \;.\; \forall a \in A \;.\; |g_{n+1}(a) - h(n)| \le \left(\frac{2}{3}\right)^{n+1},$

$\male_{n+1} := \male_n \eth h[3] : \forall a \in A \;.\; \left|f(a) - \sum_{i=1}^{n+1} g_i(a)\right| \le \left(\frac{3}{2}\right)^{n+1};$

$\rightsquigarrow \Big(g, [2]\Big) := I\left(\sum\right) :$

$\;: \sum g : \mathbb{N} \to X \xrightarrow{\mathsf{TOP}} \mathbb{R} \;.\; \forall n \in \mathbb{N} \;.\; \operatorname{Im} g \subset \frac{1}{3^n}[-1, 1] \forall a \in A \;.\; \left|f(a) - \sum_{i=1}^{n} g_i(a)\right| \le \left(\frac{2}{3}\right)^n,$

$F := \sum_{n=1}^{\infty} g_n : X \xrightarrow{\mathsf{TOP}} \mathbb{R},$

$[3] := [2]\jmath F : F_{|A} = f,$

$[*] := \eth^{-1}\texttt{Extendable}[3] : \Big(f : \texttt{Extendavle}(X, [-1, 1])\Big);$

$\square$

$\texttt{DiscreteSubsetBound} :: \forall X : \texttt{T4} \ \& \ \texttt{Separable} \ . \ \forall A : \texttt{Discrete} \ \& \ \texttt{Closed}(X) \ . \ |A| \leq \aleph_0$

$\texttt{Proof} \ =$

...

$\square$

$$\texttt{Compatible} :: \prod X, Y, I \in \mathsf{SET} \ . \ ?\left(\sum S : \texttt{Cover}(I, X) \ . \ \prod_{i \in I} S_i \to T\right)$$

$$(S, f) : \texttt{Compatible} \iff \forall i, j \in I \ . \ f_{i|S_i \cap S_j} = f_{j|S_i \cap S_J}$$

$$\texttt{combination} :: \prod X, Y, I \in \mathsf{SET} \ . \ \prod (S, f) : \texttt{Compatible}(X, Y, I) \ . \ \prod J \subset I \ . \ \bigcup_{j \in J} S_j \to Y$$

$$\texttt{combination}\,() = \nabla_{j \in J} f_j := \Lambda x \in \bigcup_{j \in J} S_j \ . \ f_j(x) \quad \texttt{where} \quad x \in S_j$$

$$\texttt{ContinuousOpenCombination} :: \forall X, Y \in \mathsf{TOP} \ . \ \forall I \in \mathsf{SET} \ . \ \forall U : \texttt{OpenCover}(I, X) \ . \ \forall f : \prod_{i \in I} U_i \xrightarrow{\mathsf{TOP}} Y \ .$$

$$. \ \forall [0] : \Big((U, f) : \texttt{Compatible}\Big) \ . \ \forall J \subset I \ . \ \nabla_{i \in I} f_j : X \xrightarrow{\mathsf{TOP}} Y$$

$\texttt{Proof} \ =$

...

$\square$

$$\texttt{LocalContinuityCriterion} :: \forall X, Y \in \mathsf{TOP} \ . \ \forall f : X \to Y \ . \ f : X \xrightarrow{\mathsf{TOP}} Y \iff$$

$$\iff \forall U \in \mathcal{T}(X) \ . \ f_{|U} : U \xrightarrow{\mathsf{TOP}} Y$$

$\texttt{Proof} \ =$

...

$\square$

$$\texttt{NormalInduction} :: \forall : \texttt{T4} \ . \ \forall A : \texttt{Discrete}\Big(\mathbb{N}, \texttt{Closed}(X)\Big) \ .$$

$$. \ \exists U : \prod_{n=1}^{\infty} \mathcal{U}(A) : \forall n, m \in \mathbb{N} \ . \ n \neq m \Rightarrow \overline{U_n} \cap \overline{U_m} = \emptyset$$

$\texttt{Proof} \ =$

...

$\square$

$$\texttt{LocallyClosed} :: \prod X \in \mathsf{TOP} \ . \ ??X$$

$$A : \texttt{LocallyClosed} \iff \forall a \in A \ . \ \exists U \in \mathcal{U}(a) : U \cap A : \texttt{Closed}(U)$$

$\mathtt{EquivalentLocallyClosedSet} :: \forall X \in \mathsf{TOP} \;.\; \forall A \subset X \;.\; A : \mathtt{LocallyClosed}(X) \iff \exists B, C : \mathtt{Closed}(X) : A$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [1] : \Big(A : \mathtt{LocallyClosed}(X)\Big),$

$B := \overline{A} : \mathtt{Closed}(X),$

$C := \overline{A} \setminus A :?X,$

$[2] := \mathtt{MonotonicClosure}(A)\mathtt{DoubleClosure} : \overline{\overline{A} \setminus A} \subset \overline{A},$

$[3] := \mathtt{MonotonicClosure}(A)\mathtt{DoubleClosure} : \overline{\overline{A} \setminus A} \cap \overline{A}^{\complement} = \emptyset,$

$\mathtt{Assume}\ x : \overline{\overline{A} \setminus A},$

$[4] := [3](x) : x \in \overline{A},$

$\mathtt{Assume}\ [5] : x \in A,$

$\Big(U, [6]\Big) := [1]\eth\mathtt{LcallyClosed}[1](X)(A)(x) : \sum U \in \mathcal{U}(x) \;.\; U \cap A : \mathtt{Closed}(U),$

$[7] := \mathtt{EquivalentClosure}(\overline{A} \setminus A)(x)(U) : U \cap (\overline{A} \setminus A) \neq \emptyset,$

$[8] := \mathtt{SubsetClosure}[7] : \overline{U \cap A} \neq U \cap A,$

$[9] := \mathtt{ClosedClosure}[6] : \overline{U \cap A} = U \cap A,$

$[5.*] := I(\bot)[8][9] : \bot;$

$\rightsquigarrow [5] := E(\bot) : x \notin A,$

$[x.*] := \eth^{-1}\mathtt{complement}[4][5] : x \in \overline{A} \setminus A;$

$\rightsquigarrow [4] := \eth^{-1}\mathtt{Subset} : \overline{\overline{A} \setminus A} \subset \overline{A} \setminus A,$

$[5] := \mathtt{ClosureSubset}(\overline{A} \setminus A) : (\overline{A} \setminus A) \subset \overline{\overline{A} \setminus A},$

$[6] := \eth^{-1}\mathtt{SetEq} : \overline{A} \setminus A = \overline{\overline{A} \setminus A},$

$[1.*] := E(=)[6]\eth\mathtt{closure}_{\jmath} : \Big(C : \mathtt{Closed}(X)\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : \mathtt{Left} \Rightarrow \mathtt{Right},$

$\mathtt{Assume}\ B, C : \mathtt{Closed}(X),$

$\mathtt{Assume}\ [2] : A = B \setminus C,$

$\mathtt{Assume}\ x : \mathtt{In}(A),$

$[3] := \eth\mathtt{compliment}[2](x) : x \notin C,$

$\Big(U, [4]\Big) := \mathtt{OpenByInnerCover}[3] : \sum U \in \mathcal{U}(x) \;.\; U \cap C = \emptyset,$

$[5] := [2][4] : U \cap A = U \cap B,$

$[x.*] := \mathtt{ClosedInSubspace}(X, A)(B)[5] : \Big(U \cap A : \mathtt{Closed}(U)\Big);$

$\rightsquigarrow \Big[(B, C).*\Big] := \eth^{-1}\mathtt{LocallyClosed} : \Big(A : \mathtt{LocallyClosed}(X)\Big);$

$\rightsquigarrow [*] := I(\iff)[1] : \mathtt{This};$

$\square$

1.3.3 Weak and Strong Topology

$$\texttt{supTopology} :: \prod_{X,I \in \mathsf{SET}} \Big(I \to \texttt{Topology}(X)\Big) \to \texttt{Topology}(X)$$

$$\texttt{supTopolgy}\,(\tau) = \bigvee_{i\in I} \tau_i := \left(X, \left\langle \bigcup_{i \in I} \tau_i \right\rangle_{\mathsf{TOP}}\right)$$

$\texttt{SupPoperty} :: \forall X, I \in \mathsf{SET} \,.\, \forall \tau : I \to \texttt{Topology}(X) \,.\, \forall \sigma : \texttt{Topology}(X) \,.\, \forall \aleph : \forall i \in I \,.\, \tau_i \subset \sigma \,.\, \bigvee_{i \in I} \tau_i \subset \sigma$

$\texttt{Proof} \;=$

Every open set $V \in \bigvee_{i\in I} \tau_i$ can be represented as $V = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} U_{j,k}$, where each $U_{j,k} \in \bigcup_{i \in I} \tau_i$ and $n_j \in \mathbb{Z}_+$.

But each $U_{j,k} \in \sigma$, so also $V \in \sigma$ by definition of topology.

$\square$

$\texttt{SupTopologyConvergenceInNets} ::$

$:: \forall X, I \in \mathsf{SET} \,.\, \forall \tau : I \to \texttt{Topology}(X) \,.\, \forall (\Delta, x) : \texttt{Net}(X) \,.\, \forall L \in X \,.$

$. \lim_{\delta \in \Delta} x_\delta =_{X, \bigvee_{i\in I} \tau_i} L \iff \forall i \in I \,.\, \lim_{\delta\in\Delta} x_\delta =_{X,\tau_i} L$

$\texttt{Proof} \;=$

$(\Rightarrow)$: This implication is obvious as $\bigcup_{i\in I} \tau_i \subset \bigvee_{i \in I} \tau_i$.

$(\Leftarrow)$: Assume that $U \in \mathcal{U}(L)$ in $\bigvee_{i\in I} \tau_i$ topology .

Then there exists a number $n \in \mathbb{N}$ and an index $i : \{1, \dots, n\} \to I$ such that $L \in \bigcap_{k=1}^{n} V_k \subset U$,

where each $V_k \in \tau_{i_k}$.

By convergence hypothesis we can find a collection of elements $\delta : \{1, \dots, n\} \to \Delta$ such tat $x_\alpha \in V_k$ for any $\alpha \geq \delta_k$.

As Δ is directed set there is some γ such that $\gamma \geq \delta_k$ for any $k \in \{1, \dots, n\}$.

Thus, $x_\alpha \in U$ for any $\alpha \geq \gamma$.

As U was arbitrary this means that the sequence converges in sup topology. .

$\square$

$\texttt{SupTopologyConvergenceInFilters} ::$

$:: \forall X, I \in \mathsf{SET} \,.\, \forall \tau : I \to \texttt{Topology}(X) \,.\, \forall \mathcal{F} : \texttt{Filter}(X) \,.\, \forall L \in X \,.$

$. \lim \mathcal{F} =_{X, \bigvee_{i\in I} \tau_i} L \iff \forall i \in I \,.\, \lim \mathcal{F} =_{X,\tau_i} L$

$\texttt{Proof} \;=$

This is true as convergence in nets and filters is equivalent.

$\square$

$$\texttt{infTopology} :: \prod_{X,I\in\mathsf{SET}} \Big(I \to \texttt{Topology}(X)\Big) \to \texttt{Topology}(X)$$

$$\texttt{infTopology}\,(\tau) = \bigwedge_{i\in I} \tau_i := \bigvee \Big\{\sigma : \texttt{Topology}(X), \forall i \in I\ .\ \sigma \subset \tau_i\Big\}$$

$$\texttt{InfTopologyExpression} :: \forall X, I \in \mathsf{SET}\ .\ \forall \tau : I \to \texttt{Topology}(X)\ .\ \bigwedge_{i\in I} \tau_i = \bigcap_{i\in I} \tau_i$$

Proof $=$

Write $\Big\{\sigma : \texttt{Topology}(X), \forall i \in I\ .\ \sigma \subset \tau_i\Big\} = \Upsilon$, then $\bigwedge_{i\in I} \tau_i = \bigvee \Upsilon$.

Then each $\sigma \subset \bigcap_{i\in I} \tau_i$ for each $\sigma \in \Upsilon$.

So, by sup property $\bigwedge_{i\in I} \tau_i \subset \bigcap_{i\in I} \tau_i$.

But, note that $\bigcap_{i\in I} \tau_i \in \Upsilon$, so $\bigcap_{i\in I} \tau_i = \bigwedge_{i\in I} \tau_i$.

□

$$\texttt{weakTopology} :: \prod_{X,I\in\mathsf{SET}} \left(I \to \sum_{Y_i\in\mathsf{TOP}} \mathsf{SET}(X, Y_i)\right) \to \texttt{Topology}(X)$$

$$\texttt{weakTopology}\,(Y, f) = \mathcal{W}_X(I, Y, f) := \bigwedge \Big\{\tau : \texttt{Topology}(X), \forall i \in I\ .\ f_i \in \mathsf{TOP}\big((X,\tau), Y_i\big)\Big\}$$

$$\texttt{strongTopology} :: \prod_{Z,I\in\mathsf{SET}} \left(\prod_{i\in I} \sum_{Y_i\in\mathsf{TOP}} \mathsf{SET}(Y_i, Z)\right) \to \texttt{Topology}(Z)$$

$$\texttt{strongTopology}\,(Y, f) = \mathcal{S}_Z(I, Y, f) := \bigvee \Big\{\tau : \texttt{Topology}(Z), \forall i \in I\ .\ f_i \in \mathsf{TOP}\big(Y_i, (Z,\tau)\big)\Big\}$$

$\texttt{WeakTopologyConvergenceInNets}$ ::

$$:: \forall X, I \in \mathsf{SET}\ .\ \forall (Y, f) : \prod_{i\in I} \sum_{Y_i\in\mathsf{TOP}} X \xrightarrow{f_i} Y_i : \mathsf{SET}\ .$$

$$.\ \forall (\Delta, x) : \texttt{Net}(X)\ .\ \forall L \in X\ .\ \lim_{\delta\in\Delta} x_\delta =_{X,\mathcal{W}(Y,f)} L \iff \forall i \in I\ .\ \lim_{\delta\in\Delta} f_i(x_\delta) = f_i(L)$$

Proof $=$

write $\mathcal{W}(Y, f) = \bigwedge \Big\{\tau : \texttt{Topology}(X), \forall i \in I\ .\ f_i \in \mathsf{TOP}\big((X,\tau), Y_i\big)\Big\} =$

$$= \bigvee \bigg\{\sigma : \texttt{Topology}(X), \forall \tau : \texttt{Topology}(X)\ .\ \Big(\forall i \in I\ .\ f_i \in \mathsf{TOP}\big((X,\tau), Y_i\big)\Big) \Rightarrow \sigma \subset \tau\bigg\} =$$

$$= \bigvee_{i\in I} f_i^{-1}\Big(\mathcal{T}(Y_i)\Big).$$

So we need to proof the result for the case $I = \{i\}$.

$(\Rightarrow)$ follows from the continuity of f_i in weak topology.

$(\Leftarrow)$: Let U be an open neighborhood of L in weak topology..

Then there are $V \in \mathcal{T}(Y_i)$ such that $U = f^{-1}(V)$. .

As V is an open neighborhood of $f_i(L)$, by convergence hypothesis there is $\gamma \in \Delta$ such that $f(x_\delta) \in V$ for each $\delta \geq \gamma$.

So $x_\delta \in U$ for each $\delta \geq \gamma$.

And as U was arbitrary the convergence holds.

□

`WeakTopologyConvergenceInNets` ::

$$:: \forall X, I \in \mathsf{SET} \;.\; \forall (Y, f) : \prod_{i \in I} \sum_{Y_i \in \mathsf{TOP}} X \xrightarrow{f_i} Y_i : \mathsf{SET} \;.$$

$$. \forall \mathcal{F} : \mathtt{Filter}(X) \;.\; \forall L \in X \;.\; \lim \mathcal{F} =_{X, \mathcal{W}(Y,f)} L \iff \forall i \in I \;.\; \lim f_i(\mathcal{F}) = f_i(L)$$

`Proof` =

By equivalence of convergence in nets and in filters.

□

`WeakTopologyContinuity` ::

$$:: \forall X, I \in \mathsf{SET} \;.\; \forall (Y, f) : \prod_{i \in I} \sum_{Y_i \in \mathsf{TOP}} X \xrightarrow{f_i} Y_i : \mathsf{SET} \;.$$

$$. \forall Z \in \mathsf{TOP} \;.\; \forall g : Z \to X \;.\; g \in \mathsf{TOP}\Big(Z, \big(X, \mathcal{W}(Y, f)\big)\Big) \iff \forall i \in I \;.\; g f_i \in \mathsf{TOP}(Z, Y_i)$$

`Proof` =

$(\Rightarrow)$: This follows from continuous composition.

$(\Leftarrow)$: Let U be an open in the weak topology .

We can assume that $U = \bigcap_{k=1}^{n} f_{i_k}^{-1}(V_k)$, where $i : \{1, \dots, n\} \to I$ and each V_k is open in Y_{i_k}.

Then $g^{-1}(U) = \bigcap_{k=1}^{n} \big(g f_{i_k}\big)^{-1}(V_k)$ is open .

As sets of this form generate weak topology g must be continuous.

□

`StrongTopologyContinuity` ::

$$:: \forall Y, I \in \mathsf{SET} \;.\; \forall (X, f) : \prod_{i \in I} \sum_{X_i \in \mathsf{TOP}} X_i \xrightarrow{f_i} Y : \mathsf{SET} \;.$$

$$. \forall Z \in \mathsf{TOP} \;.\; \forall g : Y \to Z \;.\; g \in \mathsf{TOP}\Big(\big(Y, \mathcal{W}(Y, f)\big), Z\Big) \iff \forall i \in I \;.\; f_i g \in \mathsf{TOP}(X_i, Z)$$

`Proof` =

$(\Rightarrow)$: This follows from continuous composition.

$(\Leftarrow)$: Let U be open in Z .

Then $(f_i g)^{-1}(U)$ is open in X_i.

But this means that $g^{-1}(U)$ has open preimage under f_i for each $i \in I$.

But this means that U is open in strong topology.

As set U was arbitrary g must be continuous.

□

1.3.4 Sums

$\texttt{sumTopology} :: \prod I \in \mathsf{SET} \, . \, (I \to \mathsf{TOP}) \to \mathsf{TOP}$

$$\texttt{sumTopology}\,(X) = \coprod_{i \in I} X_i := \left(\bigsqcup_{i \in I} X_i, \mathcal{S}(X, \iota) \right)$$

$\texttt{SumIsCoproduct} :: \Big(\texttt{sumTopology} : \texttt{Coproduct}(X) \Big)$

$\texttt{Proof} \; =$

Let $Y \in \mathsf{TOP}$ and $f_i \in \mathsf{TOP}(X_i, Y)$.

Then by universal property in SET there is unique $h : \coprod_{i \in I} X_i \to Y$ such that $\iota_i h = f_i$.

But as each f_i is continuous the h also mut be cintinuous .

$\square$

$\texttt{SumIsCompatibleWithSubspace} :: \forall I \in \mathsf{SET} \, . \, \forall X : I \to \mathsf{TOP} \, . \, \forall i \in I \, . \, X_i \cong_{\mathsf{TOP}} \iota_{X,i}(X_i)$

$\texttt{Proof} \; =$

From the definition each ι_i is injective.

So $\iota_i^{-1} \iota_i(A) = A$.

But with strong topology this means that ι_i is an open mapping.

As it both open and continuous (by definition) ι_i is a homeomorphic embedding.

$\square$

$\texttt{ClopenSummands} :: \forall I \in \mathsf{SET} \, . \, \forall X : I \to \mathsf{TOP} \, . \, \forall i \in I \, . \, \texttt{Clopen} \left(\coprod_{i \in I} X_i, \iota_{X,i}(X_i) \right)$

$\texttt{Proof} \; =$

By definition of strong topology each X_i is open in $\coprod_{i \in I} X_i$.

But its complement $X_i^{\complement} = \bigcup_{j \neq I} X_j$ is also open as union of open sets (each open by simmilar considirations).

So X_i must be clopen.

$\square$

$\texttt{SumPreservesSeparation} :: \forall I \in \mathsf{SET} \, . \, \forall n \in \{1, \ldots, 6\} \, . \, \forall X : I \to \mathtt{T}n \, . \, \coprod_{i \in I} X_i : \mathtt{T}n$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.3.5 Products

$$\texttt{productTopology} :: \prod I \in \mathsf{SET} \,.\, (I \to \mathsf{TOP}) \to \mathsf{TOP}$$

$$\texttt{productTopology}\,(X) = \prod_{i \in I} X_i := \left(\prod_{i \in I} X_i, \mathcal{W}(X, \pi) \right)$$

$$\texttt{ProductOfTopologicalSpaces} :: \Big(\texttt{productTopology} : \texttt{Product}(\mathsf{TOP}) \Big)$$

Proof $=$

Let $Y \in \mathsf{TOP}$ and $f_i \in \mathsf{TOP}(Y, X_i)$.

Then by universal property in SET there is unique $h : Y \to \prod_{i \in I} X_i$ such that $h\pi_i = f_i$.

But as each f_i is continuous the h also mut be cintinuous .

$\square$

$$\texttt{ProductTopologyBase} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.$$
$$.\, \left\{ \prod_{i \in I} U_i \middle| U \in \prod_{i \in I} \mathcal{T}(X_i) : \Big| \{ i \in I : U_i \neq X_i \} \Big| < \infty \right\} : \texttt{Base}\left(\prod_{i \in I} X_i \right)$$

Proof $=$

This follows from the definition of the weak topology.

$\square$

$$\texttt{ProductOfClosedSets} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \forall A : \prod_{i \in I} \texttt{NonEmpty}(X_i) \,.$$
$$.\, \prod_{i \in I} A_i : \texttt{Closed}\left(\prod_{i \in I} X_i \right) \iff \forall i \in I \,.\, A_i : \texttt{Closed}(X_i)$$

Proof $=$

Firstly assum that if A is closed in X_i .

Then $\left(\prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^{\complement}} X_j \right)^{\complement} = \prod_{j \in \{i\}} A^{\complement} \times \prod_{j \in \{i\}^{\complement}} X_j$ is open by the product topology base. .

So $\prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^{\complement}} X_j$ is closed.

Now let $A : \prod_{i \in I} \texttt{Closed}(X_i)$ be a family of closed set.

Then $\prod_{i \in I} A_i = \bigcap_{i \in I} \prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^{\complement}} X_j$ is closed as an intersections of closed sets.

$\square$

$\texttt{ProductClosure} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \forall A : \prod_{i\in I} ?X_i \,.\, \overline{\prod_{i\in I} A_i} = \prod_{i\in I} \overline{A_i}$

$\texttt{Proof} \; =$

By previous theorem $\prod_{i\in I} \overline{A_i}$ is closed and evedently $\prod_{i\in I} A_i \subset \prod_{i\in I} \overline{A_i}$.

So $\overline{\prod_{i\in I} A_i} \subset \prod_{i\in I} \overline{A_i}$.

Assume $p \in \prod_{i\in I} \overline{A_i}$ And Let $U = \prod_{i\in I} V_i$ to be a base neighborhood of p with $V_i \in \mathcal{T}(X_i)$.

Then each V_i is a neighborhood of $\pi_i(p) \in \overline{A_i}$, so $V_i \cap A_i \neq \emptyset$ by alternative definition of closure.

Thus, $U \cap \prod_{i\in I} A_i \neq \emptyset$.

As p and U was arbitrary by alternative definition of closure $\prod_{i\in I} \overline{A_i} \subset \overline{\prod_{i\in I} A_i}$.

Hence $\overline{\prod_{i\in I} A_i} = \prod_{i\in I} \overline{A_i}$.

$\square$

$\texttt{ProjectionIsOpen} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \forall i \in I \,.\, \pi_{X,i} : \texttt{Open}\left(\prod_{i\in I} X_i, X_i\right)$

$\texttt{Proof} \; =$

Asumme U is open in $\prod_{i\in I} X_i$.

Then it can be represented as $U = \bigcup_{j\in J} \prod_{i\in I} V_{j,i}$, where each $V_{j,i}$ is open X_i.

We have $\pi_i(U) = \bigcap_{j\in J} V_{j,i}$ which must be open as union of open sets.

$\square$

$\texttt{diagonalProduct} :: \prod I \in \mathsf{SET} \,.\, \forall X \in \mathsf{TOP} \,.\, \prod Y : I \to \mathsf{TOP} \,.\, \left(\prod_{i\in I} X \xrightarrow{\mathsf{TOP}} Y_i\right) \to X \xrightarrow{\mathsf{TOP}} \prod_{i\in I} Y_i$

$\texttt{diagonalProduct}\,(f) = \triangle_{i\in I} f_i := \Lambda x \in X \,.\, \Lambda i \in I \,.\, f_i(x)$

$\texttt{ClosedDiagonal} :: \forall I \in \mathsf{SET} . \forall X : I \to \texttt{T2} . \texttt{Closed}\left(\prod_{i \in I} X_i, \triangle \prod_{i \in I} X_i :\right)$

$\texttt{Proof} =$

...

$\square$

$\texttt{Multiplicative} :: ??\mathsf{TOP}$

$$P : \texttt{Multiplicative} \iff \forall I \in \mathsf{SET} . \forall X : I \to P . \prod_{i \in I} X_i : P$$

$\texttt{CardinalMultiplicative} :: \mathsf{CARD} \to ??\mathsf{TOP}$

$$P : \texttt{CardinalMultiplicative} \iff \Lambda k \in \mathsf{CARD} . P : k\text{-}\texttt{Multiplicative} \iff$$
$$: \Lambda k : \mathsf{CARD} . \forall I \in \mathsf{SET} . |I| \le k \Rightarrow \forall X : I \to P . \prod_{i \in I} X_i : P$$

$\texttt{FinitelyMultiplicative} :: ??\mathsf{TOP}$

$$P : \texttt{CardinalMultiplicative} \iff P : \texttt{FinitlyMultiplicative} \iff$$
$$: \forall I \in \mathsf{SET} . |I| < \infty \Rightarrow \forall X : I \to P . \prod_{i \in I} X_i : P$$

$\texttt{CountabilityIsCountablyMultiplicative} :: \texttt{FirstCountable}, \texttt{SecondCountable} : \aleph_0\text{-}\texttt{Multiplicative}$

$\texttt{Proof} =$

...

$\square$

$\texttt{CountabilityIsCountablyMultiplicative} :: \texttt{FirstCountable}, \texttt{SecondCountable} : \aleph_0\text{-}\texttt{Multiplicative}$

$\texttt{Proof} =$

...

$\square$

$\texttt{SeparabilityIsContinuumMultiplicative} :: \texttt{Separable} : \exp(\aleph_0)\text{-}\texttt{Multiplicative}$

$\texttt{Proof} =$

...

$\square$

$\texttt{SeparatePoints} :: \prod X \in \mathsf{TOP} \,.\, \prod I \in \mathsf{SET} \,.\, \prod Y : I \to \mathsf{TOP} \,.\, ?\prod_{i \in I} X \xrightarrow{\mathsf{TOP}} Y_i$

$f : \texttt{SeparatePoints} \iff \forall x, x' \in X \,.\, x \neq x' \Rightarrow \exists i, j \in I : f_i(x) \neq f_j(x')$

$\texttt{SeparatePointsAndClosedSets} :: \prod X \in \mathsf{TOP} \,.\, \prod I \in \mathsf{SET} \,.\, \prod Y : I \to \mathsf{TOP} \,.\, ?\prod_{i \in I} X \xrightarrow{\mathsf{TOP}} Y_i$

$f : \texttt{SeparatePointsAndClosed} \iff \forall x \in X \,.\, \forall A : \texttt{Closed}(X) \,.\, x \notin A \Rightarrow \exists i, j \in I \,.\, f_i(x) \notin \overline{f_j(A)}$

$\texttt{SPaCIsEmbedding} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, (1 \mapsto f) : \texttt{SeparatePointsAndClosedSets}(X, 1, Y) \,.$
$\quad . \, f : \texttt{HomeomorphicEmbedding}(X, Y)$

$\texttt{Proof} \ =$

$F := f^{|\operatorname{Im} f} : X \xrightarrow{\mathsf{TOP}} \operatorname{Im} f,$

$[1] := \jmath F \texttt{SubspaceClosure} \eth^{-1} \texttt{SeparatePointsAndClosedSets} :$
$\quad : \Big((1 \mapsto f) : \texttt{SeparatePointsAndClosedSets}(X, 1, \operatorname{Im} f) \Big),$

$\texttt{Assume}\ U : \texttt{Open}(X),$

$[2] := \eth^{-1} \texttt{Closed}(U) : \Big(U^{\complement} : \texttt{Closed}(X) \Big),$

$\texttt{Assume}\ y : f(U),$

$\Big(x, [3]\Big) := \eth \texttt{Image} : \sum x \in f,$

$[4] := \eth \texttt{complement}(x) : x \notin U,$

$[5] := \eth^{-1} \texttt{SeparatePointsAndClosedSets}(X, 1, \operatorname{Im} f) : \Big(f(x) \notin \overline{f(U^{\complement})} \Big),$

$[*] := \texttt{EquivalenClosure}[3]\texttt{DoubleComplement}(U) : \exists V \in \mathcal{U}(f(x)) : V \subset f(U);$

$\rightsquigarrow [3] := I(\forall) : \forall y \in f(U) \,.\, \exists V \in \mathcal{U}(y) \,.\, y \in V \subset f(U),$

$[U.*] := \texttt{OpenByInnerCover}[3] : f(U) \in \mathcal{T}(\operatorname{Im} f);$

$\rightsquigarrow [2] := \eth^{-1} \texttt{Open} \jmath^{-1} F : \Big(F : \texttt{Open}(X, \operatorname{Im} f) \Big),$

$[3] := \eth \texttt{SeparatePoints}(f) : \Big(f : X \hookrightarrow Y \Big),$

$[*] := \eth^{-1} \texttt{HomeomotphicEmbedding} \jmath F [2][3] : \Big(f : \texttt{HomeomorphicEmbedding} \Big);$

$\square$

$\texttt{DiagonalTheorem} :: \forall X \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall Y : I \to \mathsf{TOP} \,.$

$\quad \forall f : \texttt{SepareatePointsAndClosedSets}(X, I, Y) \,.\, \triangle_{i \in I} f_i : \texttt{HomeamorpohicEmbedding}\left(X, \prod_{i \in I} Y_i \right)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DiagonalTheorem2} :: \forall X \in \mathsf{TOP} \,.\, \forall n \in \mathsf{CARD} \,.\, \triangle(X^n) \cong_{\mathsf{TOP}} X$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{GraphHomeo} :: \forall X, Y \in \mathsf{TOP} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} Y \;.\; X \cong_{\mathsf{TOP}} G(f)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ClosedGraphTheorem} :: \forall X \in \mathsf{TOP} \;.\; \forall Y \in \texttt{T2} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} Y \;.\; G(f) : \texttt{Closed}(X \times Y)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TopologicalSpacesAreComplete} :: \texttt{Bicomplete}(\mathsf{TOP})$

$\texttt{Proof} \;=$

Construct limits or colimits in SET.

Then endow it with weak or strong topology respectively .

$\square$

$$\texttt{ProductPreservesSeparation} :: \forall I \in \mathsf{SET} \;.\; \forall n \in \{1, \ldots, 6\} \;.\; \forall X : I \to \texttt{T}n \;\&\; \texttt{NonEmpty} \;.\; \prod_{i=1}^{n} X_i : \texttt{T}n$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

1.3.6 Quotients

$\texttt{quotientSpace} :: \prod X \in \mathsf{TOP} \,.\, \texttt{Equivalence}(X) \to \mathsf{TOP}$

$\texttt{quotinentSpace}\,(\sim) = \dfrac{X}{(\sim)} := \left(\dfrac{X}{(\sim)}, \mathcal{S}(X, \pi_\sim)\right)$

$\texttt{QuotientMap} :: \prod X, Y \in \mathsf{TOP} \,.\, ?\Big(\mathsf{TOP}\ \&\ \texttt{Surjective}(X,Y)\Big)$

$f : \texttt{QuotientMap} \iff Y \cong_{\mathsf{TOP}} \dfrac{X}{\sim_f}$

$\texttt{OpenSurjectiveMapIsQuotient} ::$

$:: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : \texttt{Surjective}\ \&\ \mathsf{TOP}\ \&\ \texttt{Open}(X,Y) \,.\, \texttt{QuotientMap}(X,Y)$

$\texttt{Proof} =$

Assume U is open in Y.

Then $f^{-1}(U)$ is open in X by continuity of f.

Now assume that $U \subset Y$ is such that $f^{-1}(U)$ is open in X.

Then $ff^{-1}(U)$ is open in Y as f is open.

But $ff^{-1}(U) = U$ as f is surjective, so U is open.

□

$\texttt{ClosedSurjectiveMapIsQuotient} ::$

$:: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : \texttt{Surjective}\ \&\ \mathsf{TOP}\ \&\ \texttt{Closed}(X,Y) \,.\, \texttt{QuotientMap}(X,Y)$

$\texttt{Proof} =$

Assume U is open in Y.

Then $f^{-1}(U)$ is open in X by continuity of f.

Now assume that $U \subset Y$ is such that $f^{-1}(U)$ is open in X.

Then $\left(f^{-1}(U)\right)^{\complement}$ is closed in Y .

So $f\left(f^{-1}(U)\right)^{\complement} = \left(ff^{-1}(U)\right)^{\complement} = U^{\complement}$ is closed as f is surjective.

Thus, U is open.

□

$\texttt{QuotientContinuity} :: \forall X, Y, Z \in \mathsf{TOP} \,.\, \forall f : \texttt{QuotientMap}(X,Y) \,.\, \forall g : Y \to Z \,.\, fg \in \mathsf{TOP}(X,Z) \iff g \in$

$\texttt{Proof} =$

This Follows from the definition of strong topology.

□

1.4 Regularity as Separation

1.4.1 Functional Separation

$\texttt{Tychonoff} :: ?\texttt{T1}$

$X : \texttt{Tychonoff} \iff \forall A : \texttt{Closed}X \; . \; \forall x \in A^{\complement} \; . \; \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] : f(A) = \{1\} \;\&\; f(x) = 0$

$\texttt{SeparationHierarchyTychonff1} :: \texttt{T3} \subsetneq \texttt{Tychonoff}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{TychonoffCriterion} :: \forall X : \texttt{T1} \; . \; X : \texttt{Tychonoff} \iff \forall x \in X \; . \; \forall U \in \mathcal{U}(x) \; .$

$\quad . \; \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] \; . \; f(x) = 0 \;\&\; f\left(U^{\complement}\right) = \{1\}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{UrysohnsLemma} :: \forall X : \texttt{T4} \; . \; \forall A, B : \texttt{Closed}(X) \; . \; \forall [0] : A \cap B = \emptyset \; . \; \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] :$

$\quad : f(A) = \{0\} \;\&\; f(B) = \{1\}$

$\texttt{Proof} \; =$

$\Big(q, [1]\Big) := \texttt{enumerate}(\mathbb{Q} \cap [0,1], \mathbb{Z}_+, 0, 1) : \sum q : \mathbb{Z}_+ \leftrightarrow \Big((0,1) \cap \mathbb{Q}\Big) \; . \; q(0) = 0 \;\&\; q(1) = 1,$

$\Big(U, [2]\Big) := \texttt{T4Critetion}(B^{\cap}) : \sum U \in \mathcal{U}(A) \; . \; \overline{U} \subset B^{\complement},$

$W_0 := U : \mathcal{U}(A),$

$W_1 := B^{\complement} : \mathcal{U}(A),$

$♂_1 := \jmath W_0 \jmath W_1 [2] : \overline{W_0} \subset W_1,$

$\texttt{Assume} \; n : \mathbb{N},$

$t := q_{n+1} : \mathbb{Q} \cap (0,1),$

$a := \max\Big\{q_i : q_i < t | i \in [n]_{\mathbb{Z}_+}\Big\} : \mathbb{Q} \cap [0, t),$

$b := \min\Big\{q_i : q_i > t | i \in [n]_{\mathbb{Z}_+}\Big\} : \mathbb{Q} \cap (t, 1],$

$i := q^{-1}(a) : [n]_{\mathbb{Z}_+},$

$j := q^{-1}(b) : [n]_{\mathbb{Z}_+},$

$[3] := ♂_n(i,j) : \overline{W_i} \subset W_j,$

$\Big(W_{n+1}, ♂_{n+1}(i, n+1), ♂_{n+1}(n+1, j)\Big) := ð\texttt{T4}(X)(\overline{W_i}, W_j^{\complement})[3]♂_n :$

$\quad : \sum W_{n+1} \in \mathcal{U}(A) \; . \; \overline{W_i} \subset W_{n+1} \;\&\; \overline{W_{n+1}} \subset W_j;$

$\rightsquigarrow \Big(W, ♂\Big) := I\Big(\sum\Big) \, I\Big(\prod\Big) : \sum W : \mathbb{Z}_+ \hookrightarrow \mathcal{U}(A) \; . \; \prod i, j \in \mathbb{Z}_+ \; . \; q_i < q_j \Rightarrow \overline{W_i} \subset W_j,$

$O_t := \Lambda t \in [0,1] \; . \; \bigcup_{i : q(i) < t} W_i : \mathcal{U}(A),$

$f := \Lambda x \in X \; . \; \texttt{if} \; x \in B \; \texttt{then} \; 1 \; \texttt{else} \; \inf\Big\{t \in [0,1] : x \in \mathcal{O}_t\Big\} : X \to [0,1],$

$\square$

$\texttt{Assume}\ t : (0,1)$,

$[t.*.1] := \jmath f : f^{-1}[0,t) = \bigcap_{s<t} O_s \in \mathcal{T}(X)$,

$[t.*.2] := \jmath f[] : f^{-1}(t,1] = X \setminus \bigcap_{s>t} \overline{O_s} \in \mathcal{T}(X)$;

$\rightsquigarrow [3] := \texttt{RealContinuityCriterion} : f \in C\Big(X, [0,1]\Big)$,

$[*] := \jmath f : f(A) = \{0\} \ \&\ f(B) = \{1\}$;

$\square$

$\texttt{SeparationHierarchyTychonff1} :: \texttt{Tychonoff} \subsetneq \texttt{T4}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{GDeltaNormalCriterion} :: \forall X : \texttt{T4} \ .\ \forall A : \texttt{Closed}(X) \ .\ A \in G_\delta(X) \iff \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] \ .\ f^{-1}\{0\} = A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FSigmaNormalCriterion} :: \forall X : \texttt{T4} \ .\ \forall A : \texttt{Open}(X) \ .\ A \in F_\sigma(X) \iff \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] \ .\ f^{-1}(0,1] = A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{Separator} :: \prod X \in \mathsf{TOP} \ .\ (?X \times ?X) \to ?\Big(X \xrightarrow{\mathsf{TOP}} [0,1]\Big)$

$f : \texttt{Separator} \iff f(A) = \{0\} \ \&\ f(B) = \{1\}$

$\texttt{CompletelySeparated} :: \prod X \in \mathsf{TOP} \ .\ ?(?X \times ?X)$

$A, B : \texttt{CompletelySeparated} \iff \exists \texttt{Separator}(A, B)$

$\texttt{FunctionalyClosed} :: \prod_{X \in \mathsf{TOP}} ??X$

$A : \texttt{FunctionalyClosed} \iff \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] \ .\ A = f^{-1}\{0\}$

$\texttt{FunctionallyClosed} :: \prod_{X \in \mathsf{TOP}} ??X$

$U : \texttt{FunctionallyOpen} \iff \exists A : \texttt{FunctionallyClosed} \ .\ U = A^{\complement}$

$\texttt{FunctionallyClosedIsClosed} :: \forall X \in \mathsf{TOP} \ .\ \forall A : \texttt{FunctionallyClosed}(X) \ .\ A : \texttt{Closed}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FunctionallyOpenIsOpen} :: \forall X \in \mathsf{TOP} \,.\, \forall U : \texttt{FunctionallyOpen}(X) \,.\, U : \texttt{Open}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{UnionOfFunctionallyClosed} :: \forall X \in \mathsf{TOP} \,.\, \forall A, B : \texttt{FunctionallyClosed}(X) \,.$

$\quad .\, A \cup B : \texttt{FunctionallyClosed}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{IntersectionOfFunctionallyClosed} :: \forall X \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall A : I \to \texttt{FunctionallyClosed}(X) \,.$

$\quad .\, \bigcap_{i \in I} A_i : \texttt{FunctionallyClosed}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CompleteSeparationOfFunctionallyClosed} :: \forall X \in \mathsf{TOP} \,.\, \forall A, B : \texttt{FunctionallyClosed} \,.$

$\quad \forall [0] : A \cap B = \emptyset \,.\, (A, B) : \texttt{CompletelySeparated}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{continuousFunctions} :: \mathsf{TOP} \to \mathsf{SET}$

$\texttt{continuousFunctions}\,(X) = C(X) := C(X, \mathbb{R})$

$\texttt{T1TopologyGeneratedByRealFunctionIsNormal} :: \forall X : \texttt{T1} \,.\, \forall f : \texttt{NonEmpty}\Big(C(X)\Big) \,.$

$\quad .\, X = \langle f \rangle_{\mathsf{TOP}} \Rightarrow X : \texttt{Tychonoff}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TychonoffFunctioanalEquivalence} :: \forall X \in \mathsf{SET} \,.\, \forall T, T' : \texttt{Topology}(X) \,.$

$\quad .\, \forall [0] : \Big((X, T), (X, T') : \texttt{Tychonoff}\Big) \,.\, (X, T) \cong_{TOP} (X, T') \iff C(X, T) = C(X, T')$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.4.2 Perfectly Normal Spaces

$\texttt{PerfectlyNormal} :: ?\texttt{T4}$

$X : \texttt{PerfectlyNormal} \iff \forall A : \texttt{Closed}(X) . A \in G_\delta(X)$

$\texttt{AlternativePerfectlyNormal} :: \forall X : \texttt{T4} . X : \texttt{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) . U \in F_\sigma(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{VedenissoffTHM!} :: \forall X : \texttt{T1} . X : \texttt{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) . U : \texttt{FunctionallyOpen}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{VedenissoffTHM2} :: \forall X : \texttt{T1} . X : \texttt{PerfectlyNormal} \iff \forall A : \texttt{Closed}(X) . A : \texttt{FunctionallyClosed}(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{VedenissoffTHM3} :: \forall X : \texttt{T1} . X : \texttt{PerfectlyNormal} \iff \forall A, B : \texttt{Closed}(X) . A \cap B = \emptyset \Rightarrow$

$\Rightarrow \exists f : X \xrightarrow{\mathsf{TOP}} [0,1] . f^{-1}\{0\} = A \ \& \ f^{-1}\{1\} = B$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{PerfectlyNormalInvariance} :: \forall X : \texttt{PerfectlyNormal} . \forall Y \in \mathsf{TOP} . \forall f : \texttt{Closed}(X, Y) .$

$. f(X) : \texttt{PerfectlyNormal}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{PerfectlyNormalCondition} :: \forall X : \texttt{T1} . X : \texttt{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) .$

$$. \exists V : \mathbb{N} \to \mathcal{T}(X) : U = \bigcup_{n=1}^{\infty} V_i \ \& \ \forall i \in \mathbb{N} . \overline{V_i} \subset U$$

$\texttt{Proof} =$

$\dots$

$\square$

1.4.3 Normally Placed Sets

$\texttt{NormallyPlaced} :: \prod X : \mathsf{TOP} \,.\, ??X$

$A : \texttt{NormallyPlaced} \iff \forall U \in \mathcal{U}(A) \,.\, \exists H : F_\sigma(X) : A \subset H \subset U$

$\texttt{NormallyPlacedInNormal} :: \forall X : \texttt{T4} \,.\, \forall A : \texttt{NormallyPlaced} \,.\, \forall U \in \mathcal{U}(A) \,.\, \exists V : F_\sigma \,\&\, \texttt{Open}(X) \,.\, A \subset V \subset U$

$\texttt{Proof} \;=$

$\Big(H, [1]\Big) := \eth\texttt{NormallyPlaced}(X)(A)(U) : \sum H \in F_\sigma(X) \,.\, A \subset H \subset U,$

$\Big(K, [2]\Big) := \eth F_\sigma(X) : \sum K : \texttt{Increasing}\Big(\mathbb{N}, \texttt{Closed}(X)\Big) \,.\, H = \bigcup_{i \in I} K_i,$

$\Big(V, [3]\Big) := \eth\texttt{T4}(X)(K, U^\complement) : \sum V : \texttt{Increasing}(\mathbb{N}, \texttt{Open}(X)) \,.\, \forall n \in \mathbb{N} \,.\, V_n \subset U \,\&\, \overline{V_n} \cup A_{n+1} \subset V_{n+1},$

$W := \bigcup_{i=1}^{\infty} V_i : \mathcal{T}(X),$

$[4] := \texttt{UnionOfSubsets}[3] : W \subset U,$

$[5] := \texttt{UnionOgSupersets}[3][1] : A \subset H \subset W,$

$[6] := \eth\texttt{union}[3] : W = \bigcup_{i=1}^{\infty} \overline{V_i},$

$[7] := \eth^{-1} F_\sigma(X)[6] : W \in F_\sigma(X),$

$[*] := [4][5][7] : \texttt{This};$

$\square$

$\texttt{PerfectlyNormallyPlaced} :: \forall X : \texttt{PerfectlyNormal} \,.\, \forall A \subset X \,.\, A : \texttt{NormallyPlaced}(X)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{NormallyPlacedUnion} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \mathbb{N} \to \texttt{NormallyPlaced}(X) \,.\, \bigcup_{i=1}^{\infty} A_i : \texttt{NormallyPlaced}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ U : \mathcal{U}\left(\bigcup_{i=1}^{\infty} A_i\right),$

$\Big(f, [1]\Big) := \Lambda i \in \mathbb{N} \,.\, \eth\texttt{NormallyPlaced}(A_i)(U) : \prod_{i=1}^{\infty} F_\sigma(X) \,.\, A_i \subset f_i \subset U,$

$F := \bigcup_{i=1}^{\infty} f_i : F_\sigma(X),$

$[U.*] := \jmath F\texttt{UnionSubset}[1]\texttt{SubsetUnion}[1] : \bigcup_{i=1}^{\infty} A_i \subset F \subset U;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{NormallyPlaced} : \left(\bigcup_{i=1}^{\infty} \texttt{NormalyPlaced}(X)\right);$

$\square$

$\texttt{NormallyPlacedSubspace} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \texttt{NormallyPlacedSet}(X) \,.\, \forall B : F_\sigma(A)(X) \,.\, B : \texttt{NormallyPlaced}$

$\texttt{Proof} \ =$

$$\Big(K, [1]\Big) := \eth F_\sigma(A)(X)(B) : \sum K : n \to \texttt{Closed}(A) \,.\, B = \bigcup_{n=1}^{\infty} K_n,$$

$$\Big(K', [2]\Big) := \texttt{ClosedInSubspace}(X, B, K) : \sum K' : n \to \texttt{Closed}(A) \,.\, \forall n \in \mathbb{N} \,.\, K = K' \cap A,$$

$\texttt{Assume}\ U : \mathcal{U}_A(B),$

1.4.4 Urysohn and Semiregular Spaces

$\texttt{Urysohn} :: ?\mathsf{TOP}$

$X : \texttt{Urysohn} \iff \forall x, y \in X \;.\; x \neq y \Rightarrow \exists U \in \mathcal{U}(x) \;.\; \exists V \in \mathcal{U}(y) : \overline{U} \cap \overline{V} = \emptyset$

$\texttt{Semireguar} :: ?\texttt{T2}$

$X : \texttt{Urysohn} \iff \texttt{OpenDomain}(X) : \texttt{Base}(X)$

$\texttt{UrysohnSeparationHierarchy} :: \texttt{T2} \subset \texttt{Urysohn} \subset \texttt{T3}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SemiregularSeparationHierarchy} :: \texttt{T2} \subset \texttt{Semiregular} \subset \texttt{T3}$

$\texttt{Proof} =$

$\ldots$

$\square$

1.4.5 Semicontinuous Functions

$$\texttt{UpperSemicontinuous} :: \prod X : \mathsf{TOP} \,.\, ?(X \to \mathbb{R})$$

$$f : \texttt{UpperSemicontinuous} \iff f \in C_{1/2}(X) \iff \forall x \in X \,.\, \forall r \in \mathbb{R} \,.\, f(x) > r \Rightarrow$$
$$\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U \,.\, f(u) > r$$

$$\texttt{LowerSemicontinuous} :: \prod X : \mathsf{TOP} \,.\, ?(X \to \mathbb{R})$$

$$f : \texttt{LowerSemicontinuous} \iff f \in C^{1/2}(X) \iff \forall x \in X \,.\, \forall r \in \mathbb{R} \,.\, f(x) < r \Rightarrow$$
$$\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U \,.\, f(u) < r$$

$$\texttt{UpperSemicontinuous} :: \prod X : \mathsf{TOP} \,.\, \prod R : \texttt{Poset} \,.\, ?(X \to R)$$

$$f : \texttt{UpperSemicontinuous} \iff f \in C^{1/2}(X, R) \iff \forall x \in X \,.\, \forall r \in R \,.\, f(x) > r \Rightarrow$$
$$\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U \,.\, f(u) > r$$

$$\texttt{EquivalentUpperSemicontinuous} :: \forall X \in \mathsf{TOP} \,.\, \forall f : C \to \mathbb{R} \,.\, f \in C^{1/2}(X) \iff \forall r \in \mathbb{R} \,.$$
$$.\, \Big\{x \in X : f(x) \le r\Big\} : \texttt{Closed}(X)$$

$\texttt{Proof} =$

$A := \Big\{x \in X : f(x) \le r\Big\} :?X,$

$\texttt{Assume } x : X,$

$\texttt{Assume } [1] : f(x) > r,$

$\Big(U_x, [x.*]\Big) := \eth C^{1/2}(X)[1] : \sum U_x \in \mathcal{U}(x) \,.\, \forall u \in U_x \,.\, f(u) > r;$

$\rightsquigarrow [1] := \texttt{OpenByInnerCover} : A^{\complement} \in \mathcal{T}(X),$

$[*] := \eth\texttt{Closed}[1] : \Big(A : \texttt{Closed}(X)\Big);$

$\square$

$$\texttt{EquivalentLowerSemicontinuous} :: \forall X \in \mathsf{TOP} \,.\, \forall f : C \to \mathbb{R} \,.\, \forall f \in C_{1/2}(X) \iff \forall r \in \mathbb{R} \,.$$
$$.\, \Big\{x \in X : f(x) \ge r\Big\} : \texttt{Closed}(X)$$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{ContinuousByLoweAndUpperSemicontinuity} :: \forall X \in \mathsf{TOP} \,.\, C^{1/2}(X) \cap C_{1/2}(X) = C(X)$$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{SemicomtinuousReversion} :: \forall X \in \mathsf{TOP} \,.\, C^{1/2}(X) = -C_{1/2}(X)$$

$\texttt{Proof} =$

$\dots$

$\square$

UpperSemicontinuousAlgebra :: $\forall X \in \mathsf{TOP} \,.\, \forall f, g \in C^{1/2}(X) \,.\, f + g, \max(f, g), \min(f, g) \in C^{1/2}(X)$

Proof $=$

...

$\square$

UpperSemicontinuousAlgebra2 :: $\forall X \in \mathsf{TOP} \,.\, \forall f, g \in C^{1/2}(X) \,.\, f, g > 0 \Rightarrow f * g \in C^{1/2}(X)$

Proof $=$

...

$\square$

LoweSemicontinuousInfimum :: $\forall X \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to C_{1/2}(X) \,.\, \forall b : X \to \mathbb{R} \,.$
$\quad . \,\forall[0] : \forall i \in I \,.\, f_i \geq b \,.\, \inf_{i \in I} f \in C_{1/2}(X)$

Proof $=$

...

$\square$

UpperSemicontinuousSupremum :: $\forall X \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to C^{1/2}(X) \,.\, \forall b : X \to \mathbb{R} \,.$
$\quad . \,\forall[0] : \forall i \in I \,.\, f_i \geq b \,.\, \sup_{i \in I} f \in C^{1/2}(X)$

Proof $=$

...

$\square$

ForteTheorem :: $\forall X \in \mathsf{TOP} \,.\, \forall f \in C^{1/2}(X) \,.\, \exists U : \mathbb{N} \to \mathcal{T} \,\&\, \mathtt{Dense}(X) \,.\, \forall x \in \bigcap_{n=1}^{\infty} U_i \,.\, f : \mathtt{ContinuousAt}(x)$

Proof $=$

...

$\square$

TychonoffBySemicontinuousApproximation :: $\forall X : \mathtt{T1} \,.\, X : \mathtt{Tychonoff} \iff \forall f \in C^{1/2}(X) \,.$
$\quad . \,\exists I \in \mathsf{SET} : \exists g : I \to C(X) : f = \sup_{i \in I} g_i$

Proof $=$

...

$\square$

NormalBySemicontinuousMidpoint :: $\forall X : \mathtt{T1} \,.\, X : \mathtt{T4} \iff \forall f \in C_{1/2}(X) \,.\, \forall g \in C^{1/2}(X) \,.$
$\quad . \, f \leq g \Rightarrow \exists h \in C(X) : f \leq h \leq g$

Proof $=$

...

$\square$

$\texttt{PerfeclyNormalBySemicontinuousApproximation} :: \forall X : \texttt{T1} \,.\, X : \texttt{PerfectlyNormal} \iff \forall f \in C^{1/2} \,.$
$\,.\, \exists g : \mathbb{N} \to C(X) \,.\, g \uparrow f$

$\texttt{Proof} =$

...

$\square$

$\texttt{PerfeclyNormalBySemicontinuousMidpoint} :: \forall X : \texttt{T1} \,.\, X : \texttt{PerfectlyNormal} \iff \forall f \in C_{1/2} \,.\, \forall g \in C^{1/2} \,.$
$\,.\, f \leq g \Rightarrow \exists h \in C(X) \,.\, f \leq g \leq g \,\&\, \forall x \in X \,.\, f(x) < g(x) \Rightarrow f(x) < h(x) < g(x)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LowerSemicontinuousSubspaceValued} :: \prod X, Y \in \mathsf{TOP} \,.\, ?(Y \to \texttt{Closed}(X))$

$F : \texttt{LowerSemicontinuousSubspaceValued} \iff F \in \mathcal{C}^{1/2}(X, Y) \iff$
$\iff \forall U \in \mathcal{T}(X) \,.\, \{y \in Y : F(y) \cap U \neq \emptyset\} \in \mathcal{T}(Y)$

$\texttt{UpperSemicontinuousSubspaceValued} :: \prod X, Y \in \mathsf{TOP} \,.\, ?(Y \to \texttt{Closed}(X))$

$F : \texttt{UpperSemicontinuousSubspaceValued} \iff F \in \mathcal{C}_{1/2}(X, Y) \iff$
$\iff \forall U \in \mathcal{T}(X) \,.\, \{y \in Y : F(y) \subset U\} \in \mathcal{T}(Y)$

$\texttt{ContinuousBySubspaceSemicontinuity} :: \forall X : \texttt{T1} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f : Y \to X \,.\, f : Y \xrightarrow{\mathsf{TOP}} X \iff$
$\iff \Lambda y \in Y \,.\, \{f(y)\} \in \mathcal{C}^{1/2} \cap \mathcal{C}_{1/2}(X, Y)$

$\texttt{Proof} =$

...

$\square$

$\texttt{OpenBySubspaceSemicontinuity} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{T1} \,.\, \forall f : X \twoheadrightarrow Y \,.\, f : \texttt{Open} \iff$
$\iff \Lambda y \in Y \,.\, f^{-1}\{y\} \in \mathcal{C}^{1/2}(X, Y)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ClosedBySubspaceSemicontinuity} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{T1} \,.\, \forall f : X \twoheadrightarrow Y \,.\, f : \texttt{Closed} \iff$
$\iff \Lambda y \in Y \,.\, f^{-1}\{y\} \in \mathcal{C}_{1/2}(X, Y)$

$\texttt{Proof} =$

...

$\square$

$\texttt{OpenBySubspaceSemicontinuity} :: \forall Y \in \mathsf{TOP} \,.\, \forall f : Y \to \mathbb{R} \,.\, f \in C^{1/2}(Y) \iff$
$\iff \Lambda y \in Y \,.\, (-\infty, f(y)] \in \mathcal{C}^{1/2}(\mathbb{R}, Y)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ClosedBySubspaceSemicontinuity} :: \forall Y \in \mathsf{TOP} \,.\, \forall f : Y \to \mathbb{R} \,.\, f \in C_{1/2}(Y) \iff$
$\iff \Lambda y \in Y \,.\, (-\infty, f(y)] \in \mathcal{C}_{1/2}(\mathbb{R}, Y)$

Proof $=$

$\dots$

$\square$

$\texttt{LowerSemicontinuousUnion} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall F : I \to \mathcal{C}^{1/2}(X, Y) \,.$
$. \left(\Lambda y \in Y \,.\, \overline{\bigcup_{i \in I} F_i(y)}\right) \in \mathcal{C}^{1/2}(X, Y)$

Proof $=$

$\dots$

$\square$

$\texttt{UpperSemicontinuousUnion} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall F, G \in \mathcal{C}_{1/2}(X, Y) \,.\,.\left(\Lambda y \in Y \,.\, F(y) \cup G(y)\right) \in \mathcal{C}_{1/2}(X, Y)$

Proof $=$

$\dots$

$\square$

$\texttt{UpperSemicontinuousIntersect} :: \forall X : \texttt{T4} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall F, G \mathcal{C}_{1/2}(X, Y) \,.$
$. \left(\Lambda y \in Y \,.\, F(y) \cap G(y)\right) \in \mathcal{C}_{1/2}(X, Y)$

Proof $=$

$\dots$

$\square$

1.5 Properties Preserved by Continuous Transformations

1.5.1 Compact Sets

$$\texttt{Compact} :: \prod X \in \mathsf{TOP} \,.\, ??X$$

$$K : \texttt{Compact} \iff \forall \mathcal{O} : \texttt{OpenCover}(X, K) \,.\, \exists \mathcal{O}' \subset \mathcal{O} : \mathcal{O}' : \texttt{Finite} \,\&\, \texttt{OpenCover}(X, K)$$

$$\texttt{FiniteIntersectionProperty} :: \prod X :??\mathsf{SET} \,.\, ?X$$

$$A : \texttt{FiniteIntersectionProperty} \iff \forall B : \texttt{Finite}(A) \,.\, \bigcap_{b \in B} b \neq \emptyset$$

$$\texttt{CompactByFiniteIntersecton} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{Compact}(X) \iff$$
$$\iff \forall A : \texttt{FiniteIntersectionProperty}\ \texttt{Closed}(X) \,.\, \bigcup_{a \in A} a \neq \emptyset$$

Proof $=$

...

$\square$

$$\texttt{CompactAsSubset} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, A : \texttt{Compact}(A) \Rightarrow A : \texttt{Compact}(X)$$

Proof $=$

...

$\square$

$$\texttt{CompactSubset} :: \forall X \in \mathsf{TOP} \,.\, \forall [0] : (X : \texttt{Compact}(X)) \,.\, \forall A : \texttt{Closed}(X) \,.\, A : \texttt{Compact}(X)$$

Proof $=$

...

$\square$

$$\texttt{CompactAsSubspace} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \texttt{Compact}(X) \,.\, A : \texttt{Compact}(A)$$

Proof $=$

...

$\square$

$$\texttt{CompactaUnion} :: \forall X \in \mathsf{TOP} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to \texttt{Compact}(X) \,.\, \bigcup_{i=1}^{n} A_i : \texttt{Compact}(X)$$

Proof $=$

...

$\square$

$$\texttt{CompactaIntersection} :: \forall X \in \mathsf{TOP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall A : I \to \texttt{Compact}(X) \,.\, \bigcap_{i \in I} A_i : \texttt{Compact}(X)$$

Proof $=$

...

$\square$

$\texttt{FiniteCompactaIntersection} :: \forall X \in \mathsf{TOP} \,.\, \forall U \in \mathcal{T}(X) \,.\, \forall I \in \mathsf{SET} \,.\, \forall A : I \to \texttt{Closed}(X) \,.\, \forall i \in I \,.$

$\quad . \forall [0] : \Big(A_i : \texttt{Compact}\Big) \,.\, \forall [00] : \bigcap_{i \in I} \subset U \,.\, \exists F : \texttt{Finite}(I) : \bigcup_{i \in F} A_i \subset U$

$\texttt{Proof} \; =$

$J := I \setminus \{i\} : \texttt{Subset}(J),$

$V := \Lambda j \in J \,.\, A_j^{\complement} : \texttt{Subset}(I),$

$[1] := \jmath V[00] : \Big(\operatorname{Im} V : \texttt{Type}OpenCover(A_i \cap U^{\complement})\Big),$

$\Big(V', [2]\Big) := \eth\texttt{Compact}(X) : \sum V' : \texttt{Finite}(\operatorname{Im} F) \,.\, V' : \texttt{OpenCover}(A_i \cap U^{\complement}),$

$\Big(K, [3]\Big) := \eth\texttt{image}(V') : \sum K : \texttt{Finite}(J) \,.\, V' = V(K),$

$F := K \cup \{i\} : \texttt{Finite}(I),$

$[4] := \eth\texttt{OpenCover}[2]\jmath F : \bigcup_{i \in F} A_i \subset U;$

$\square$

$\texttt{CompactSeparation1} :: \forall X : \texttt{T3} \,.\, \forall A : \texttt{Compact}(X) \,.\, \forall B : \texttt{Closed}(A) \,.$

$\quad . \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{CompactSeparation2} :: \forall X : \texttt{T2} \,.\, \forall A : \texttt{Compact}(X) \,.\, \forall B : \texttt{Compact}(A) \,.$

$\quad . \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{CampactIsNormal} :: \forall X : \texttt{T2} \;\&\; \texttt{Compact} \,.\, X : \texttt{T4}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{TychonoffComapctSeparation} :: \forall X : \texttt{Tychonoff} \,.\, \forall A : \texttt{Compact}(X) \,.\, \forall B : \texttt{Closed}(X) \,.$

$\quad . \exists f \in C(X) : f(A) = \{0\} \;\&\; f(B) = \{1\}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{FiniteIsCompact} :: \forall X \in \mathsf{TOP} \,.\, \forall F : \texttt{Finite}$

$\texttt{Proof} \; =$

$\texttt{CompactIsClosed} :: \forall X : \texttt{T2} \,.\, \forall A : \texttt{Compact}(X) \,.\, A : \texttt{Closed}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : A^{\complement},$

$[1] := \texttt{FiniteIsCompact}(X)(\{x\}) : \Big(\{x\} : \texttt{Compact}(X)\Big),$

$\Big(U, V, [2]\Big) := \texttt{CompactSeparation2}(X, A, \{x\})[1] : \sum U \in \mathcal{U}(A) \,.\, \sum V \in \mathcal{U}(x) \,.\, V \cap U = \emptyset,$

$[3] := \texttt{ClosureEquivalent}(A)[2] : x \notin \overline{A};$

$\rightsquigarrow [1] := \eth^{-1}\texttt{SetEq} : A = \overline{A},$

$[*] := \texttt{ClosedByClosure}[1] : \Big(A : \texttt{Closed}(X)\Big);$

$\square$

$\texttt{CompactImage} :: \forall X : \texttt{Compact} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f \in C \,\&\, \texttt{Surjective}(X, Y) \,.\, Y : \texttt{Compact}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CompactImageClosure} :: \forall X : \texttt{Compact} \,.\, \forall Y : \texttt{T2} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, \forall A \subset X \,.\, f(\overline{A}) = \overline{f(A)}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CompactClosedMap} :: \forall X : \texttt{Compact} \,.\, \forall Y : \texttt{T2} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Y \,.\, f : \texttt{Closed}(X, Y)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CompactHomeo} :: \forall X : \texttt{Compact} \,.\, \forall Y : \texttt{T2} \,.\, \forall f \in C \,\&\, \texttt{Bijective}(X, Y) \,.\, f : X \xleftrightarrow{\mathsf{TOP}} Y$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{KuratowskiLemma} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall A : \texttt{Compact}(X) \,.\, \forall y \in Y \,.\, \forall W \in \mathcal{U}(A \times \{y\}) \,.$
$\quad .\, \exists U \in \mathcal{U}(A) \,.\, \exists V \in \mathcal{U}(y) \,.\, U \times V \subset W$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{KuratowskiTHM1} :: \forall X : \texttt{Compact} \,\&\, \texttt{T2} \,.\, \forall Y \in \mathsf{TOP} \,.\, \pi_2 : \texttt{Closed}(X \times Y, Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : \texttt{Closed}(X \times Y),$

$U := A^{\complement} : \texttt{Open}(X \times Y),$

$B := \pi_1(A) :?X,$

$[1] := \texttt{CompactClosedMap} : \Big(B : \texttt{Closed}(X)\Big),$

$\texttt{Assume}\ y : \Big(\pi_2(A)\Big)^{\complement},$

$[2] := \eth\texttt{compliment}\eth\pi_2\jmath B : B \times y \subset U,$

$\Big(W, V, [3]\Big) := \texttt{KuratowskiLemma}[2] : \sum W \in \mathcal{U}(B) \,.\, \sum V \in \mathcal{U}(y) \,.\, V \times W \subset U,$

$[4] := [3]\jmath U\eth\pi_2 : U \cap A = \emptyset;$

$\rightsquigarrow [2] := \texttt{OpenByInnerCover} : \Big(\pi_2(A)\Big)^{\complement} \in \mathcal{T}(Y),$

$[3] := \eth^{-1}\texttt{Closed}[2] : \Big(\pi_2 A : \texttt{Closed}(X)\Big);$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Closed} : \Big(\pi_2 : \texttt{Closed}(X \times Y, Y)\Big);$

$\square$

$\texttt{KuratowskiProperty} :: ?\mathsf{TOP}$

$X : \texttt{KuratowskiProperty} \iff \forall Y : \texttt{T4} \,.\, \pi_2 : \texttt{Closed}(X, X \times Y)$

$\texttt{KuratowskiTHM2} :: \forall X : \texttt{KuratowskiProperty} \,.\, X : \texttt{Compact} \,\&\, \texttt{T2}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CompactGraphTheorem} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{Compact} \,\&\, \texttt{T2} \,.\, \forall f : X \to Y \,.\, f \in C(X, Y) \iff G(f) : \texttt{Closed}(X,$

$\texttt{Proof} \ =$

$[1] := \texttt{ClosedGraphThm}(X, Y)(f) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \Big(G(f) : \texttt{Closed}(X, Y)\Big),$

$\texttt{Assume}\ A : \texttt{Closed}(Y),$

$B := X \times A : \texttt{Closed}(X \times Y),$

$C := B \cap G(f) : \texttt{Closed}(X \times Y),$

$[3] := \eth G(f)\jmath C\eth^{-1}\pi_{:}f^{-1}(A) = \pi_1(C),$

$[A.*] := \texttt{KuratowskiTHM}[3] : \Big(f^{-1}(A) : \texttt{Closed}(X)\Big);$

$\rightsquigarrow [2.*] := \eth^{-1}\texttt{Continuous} : f \in C(X, Y);$

$\rightsquigarrow [*] := I(\iff)I(\Rightarrow)[1] : f \in C(X, Y) \iff G(f) : \texttt{Closed}(X, U);$

$\square$

$\texttt{CompactLimitTheorem} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{Compact} \iff \forall D : \texttt{DirectedSet} \,.\, \forall x : \texttt{Net}(X, D) \,.\, \exists \texttt{Cluster}(x)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : X : \texttt{Compact},$

$\texttt{Assume}\ D : \texttt{DirectedSet},$

$\texttt{Assume}\ x : \texttt{Net}(X, D),$

$A := \Lambda n \in D \,.\, \overline{\{x_i | i \geq n\}} : D \to \texttt{Closed}(X),$

$[2] := \eth\texttt{DirectedSet}\jmath A \eth^{-1}\texttt{FiniteIntersectionProperty} : (A : \texttt{FiniteIntersectionProperty}\ \texttt{Closed}(X)),$

$[3] := \texttt{CompactByFiniteIntersection}([2]) : \bigcap_{n \in D} A_n \neq \emptyset,$

$B := \bigcap_{n \in D} A_n : \texttt{Closed}(X),$

$\Big(c, [5]\Big) := \eth\texttt{Nonempty} : \sum c \in X \,.\, c \in B,$

$\texttt{Assume}\ U : \mathcal{U}(c),$

$\texttt{Assume}\ n : D,$

$[6] := \jmath B[5]\eth\texttt{intersection} : c \in A_n,$

$[U.*] := \jmath A_n\texttt{AlternativeClosure} : \exists m \in D : m \geq n \ \&\ x_m \in U;$

$\rightsquigarrow [1.*] := \eth^{-1}\texttt{Cluseter}(x) : \Big(c : \texttt{Cluster}\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{LEFT} \Rightarrow \texttt{RIGHT},$

$\texttt{Assume}\ [2] : \texttt{Right},$

$\texttt{Assume}\ A : \texttt{FiniteIntersectionProperty}(X),$

$D := \left\{ \bigcap F \middle| F : \texttt{Finite}(A) \right\} :??X,$

$[3] := \eth\texttt{FiniteIntersectionProperty}\jmath D : \forall d \in D \,.\, d \neq \emptyset,$

$\Big(x, [4]\Big) := \eth\texttt{NonEmpty}[3] : \sum x : D \to X \,.\, \forall d \in D \,.\, x \in d,$

$c := [2](x) : \texttt{Cluster}(x),$

$[5] := \eth\texttt{Cluster}(x)[4] : c \in \bigcap A,$

$[A.*] := \eth^{-1}\texttt{NonEmpty}[5] : \bigcap A \neq \emptyset;$

$\rightsquigarrow [2.*] := \texttt{CompampByFiniteIntersection} : \Big(X : \texttt{Compact}\Big);$

$\rightsquigarrow [*] := I(\Rightarrow)I(\iff)[1] : \texttt{THIS};$

$\square$

$\texttt{CompactQuotientMap} ::$

$:: \forall X : \texttt{Compact} \,.\, \forall Y : \texttt{T2} \,.\, \forall f \in \mathsf{TOP} \ \&\ \texttt{Surjective}(X, Y) \,.\, \texttt{QuotientMap}(X, Y, f)$

$\texttt{Proof} \ =$

Mapping f must be closed, so it is a quotient map.

$\square$

1.5.2 Connected Spaces

$\texttt{Connected} :: ?\mathsf{TOP}$

$X : \texttt{Connected} \iff \texttt{Clopen}(X) = \{\emptyset, X\}$

$\texttt{ConnectedProduct} :: \forall I \in \mathsf{SET} . \forall X : I \to \texttt{Connected} . \prod_{i \in I} X_i$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U : \texttt{Clopen}(X),$

$\texttt{Assume}\ i : I,$

$\texttt{Assume}\ p : \prod_{j \in \{i\}^\complement} X_j,$

$[1] := \texttt{ProjectionHomeo}(I, X, i, p) : \{p\} \times X_i \cong_{\mathsf{TOP}} X_i,$

$[2] := \eth : U \cap \{p\} \times X_i : \texttt{Clopen}(X),$

$[i.*] :=: U \cap \{p\} \times X_i = \emptyset | U \cap \{p\} \times X_i = \{p\} \times X_i;$

$\rightsquigarrow [1] := I(\forall) : \forall i \in I . \exists E \subset \prod_{j \in \{i\}^\complement} X_j . U = X_i \times E,$

$[U.*] := \texttt{Choice}(X) : U = \prod_{i \in I} X_i | U = \emptyset;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Connected} : \texttt{Connected}\left(\prod_{i \in I} X_i\right);$

$\square$

$\texttt{MainTheoremOfConnectedSpace} :: \forall X : \texttt{Connected} . \forall Y \in \mathsf{TOP} . \forall f \in C(X, Y) . \texttt{Connected}(Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConnectedAltDef} :: \forall X \in \mathsf{TOP} . \texttt{Connected}(X) \iff \forall f \in C(X, 2) . \texttt{Constant}(X, 2, f)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConnectedSubset} :: \prod X : \mathsf{TOP} . ??X$

$A : \texttt{ConnectedSubset} \iff \texttt{Connected}(X, A) \iff \texttt{Connected}(A)$

$\texttt{ConnectedUnion1} :: \forall X \in \mathsf{TOP} . \forall I \in \mathsf{SET} . \forall A : I \to \texttt{Connected}(X) .$

$\quad \texttt{PairwiseIntersecting}(X, I, A) \Rightarrow \texttt{Connected}\left(X, \bigcup_{i \in I} A_i\right)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConnectedUnion2} :: \forall X \in \mathsf{TOP} \, . \, \forall I \in \mathsf{SET} \, . \, \forall A : I \to \texttt{Connected}(X) \, .$

$$(\exists i \in I \, . \, \forall j \in I \, . \, A_i \cap A_j \neq \emptyset) \Rightarrow \texttt{Connected}\left(X, \bigcup_{i \in I} A_i\right)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{IntermidiateValueTheorem} :: \forall X : \texttt{Connected} \, . \, \forall f \in C(X) \, . \, a, b \in X \, . \, f(a) < 0 \;\&\; f(b) > 0 \Rightarrow$
$\Rightarrow \exists c \in X : f(c) = 0$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ClosureOfConnectedIsConnected} :: \forall X \in \mathsf{TOP} \, . \, \forall A : \texttt{Connected}(X) \, . \, \operatorname{cl} A : \texttt{Connected}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : C(\operatorname{cl} A, 2),$

$[1] := \texttt{ContinuousRestriction}\Big(f, A, \texttt{ClosureIsSuper}(A)\Big) : f_{|A} \in C(A, 2),$

$[2] := \texttt{AltConnectedDef}(A, f) : \texttt{Constant}(X, 2, f_{|A}),$

$[3] := \eth\texttt{Constant}(X, 2, f)\texttt{ClosureContinuation} : \texttt{Constant}(\operatorname{cl} A, 2);$

$\rightsquigarrow [*] := \texttt{AltConnectedDef} : \texttt{Connected}(X, \operatorname{cl} A);$

...

$\square$

$\texttt{connectedComponents} :: \prod X \in \mathsf{TOP} \, . \, ??X$

$\texttt{connectedComponents}\,() = CC(X) := \sup\Big\{A \subset X : \texttt{Connected}(X)\Big\}$

$\texttt{ConnectedComponentsAreClosed} :: \forall X \in \mathsf{TOP} \, . \, \forall A \in CC(X) \, . \, \texttt{Closed}(X, A)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ConnectedComponentsDisjointCover} :: \forall X \in \mathsf{TOP} \, . \, \bigsqcup_{A \in CC(X)} A = X$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{connectedComponentsOf} :: \prod X \in \mathsf{TOP} \, . \, x \to CC(X)$

$\texttt{connectedComponentsOf}\,(x) = CC(x) := \texttt{ConnectedComponentsDisjointCover}(X, A$

$\texttt{LocallyConnectededConnctedComponentsAreClopen} :: \forall X : \texttt{Locally Connected} \,.\, \forall A \in CC(X) \,.$

$\quad . \, \texttt{Clopen}(X, A)$

$\texttt{Proof} =$

$\ldots$

$\square$

1.5.3 Path-Connected Spaces

$$\texttt{pathSpace} :: \prod X \in \mathsf{TOP} \,.\, X^2 \to ?C\Big([0,1], X\Big)$$

$$\texttt{pathSpace}\,(x,y) = \Omega(x,y) := \Big\{\gamma \in C\Big([0,1], X\Big)\Big\}$$

$$\texttt{joinPaths} :: \prod X \in \mathsf{TOP} \,.\, \prod x,y,z \in X \,.\, \Omega(x,y) \times \Omega(y,z) \to \Omega(x,z)$$

$$\texttt{joinPaths}\,(\alpha,\beta) = \alpha\beta := \Lambda t \in [0,1] \,.\, \texttt{if}\ t \le \frac{1}{2}\ \texttt{then}\ \alpha(2t)\ \texttt{else}\ \beta\,(2t-1)$$

$$\texttt{pathCategory} :: \mathsf{TOP} \to \mathsf{SCAT}$$

$$\texttt{pathCategory}\,(X) = \omega(X,X) := \Big(X, \Omega, \texttt{joinPaths}, \texttt{constant}\big([0,1], X\big)\Big)$$

$$\texttt{reversePath} :: \prod X \in \mathsf{TOP} \,.\, \prod x,y \in X \,.\, \uparrow\Omega(x,y) \to \uparrow\Omega(y,x)$$

$$\texttt{reversePath}\,(\gamma) = \gamma^{\frown} := \Lambda t \in [0,1] \,.\, \gamma(1-t)$$

$$\texttt{Subpath} :: \prod X \in \mathsf{TOP} \,.\, \uparrow\Omega(X,X) \to ?\uparrow\Omega(X,X)$$

$$\alpha : \texttt{Subpath} \iff \Lambda\gamma \in \Omega(X,X) \,.\, \alpha \subset \gamma \iff \Lambda\gamma \in \Omega(X,X) \,.\, \exists \phi : \texttt{Nondeacrizing}\Big([0,1],[0,1]\Big) \,.\, \alpha = \phi\gamma$$

$$\texttt{pathMesh} :: \prod X \in \mathsf{TOP} \,.\, \uparrow\Omega(X,X) \to \mathsf{SET}$$

$$\texttt{pathMesh}\,(\gamma) = M(\gamma) := \left\{(n,\alpha) : \texttt{Chain}\ \Omega(X,X) : \prod_{i=1}^{n} \alpha_i = \gamma\right\}$$

$$\texttt{PathMeshLess} :: \prod X \in \mathsf{TOP} \,.\, \prod \gamma \in\uparrow\Omega(X,X) \,.\, ?\Big(M(\omega) \times M(\omega)\Big)$$

$$(n,\alpha),(m,\beta) : \texttt{PathMeshLess} \iff (n,\alpha) \le (m,\beta) \iff \forall i \in n \,.\, \exists j \in m : \alpha_i \subset \beta_j$$

$\texttt{PathMeshIsDirected} :: \forall X \in \mathsf{TOP} \,.\, \forall \gamma \in\uparrow \Omega(X,X) \,.\, M(\gamma) : \texttt{DirectedSet}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (n,\alpha),(m,\beta) : M(\gamma),$

$[1] := \eth M(\gamma)(n,\alpha) : \forall i \in n \,.\, \alpha_i \subset \gamma,$

$[2] := \eth M(\gamma)(m,\beta) : \forall i \in m \,.\, \beta_i \subset \gamma,$

$\Big(\phi,[3]\Big) := \eth\texttt{Subpath}[1] : \prod_{i=1}^{n} \sum_{\phi:[0,1]\uparrow[0,1]} \,.\, \alpha = \phi_i\gamma,$

$\Big(\psi,[4]\Big) := \eth\texttt{Subpath}[2] : \prod_{i=1}^{m} \sum_{\phi:[0,1]\uparrow[0,1]} \,.\, \beta = \psi_i\gamma,$

$T := \phi(n)(0) \cup \phi(n)(1) \cup \psi(m)(0) \cup \psi(m)(1) :?[0,1],$

$(N,t) := \texttt{sort}(T) : \sum_{N=2}^{\infty} t : \texttt{Increasing}\ \&\ \texttt{Bijection}(N,T),$

$M := N - 1 : \mathbb{N},$

$\omega := \Lambda i \in M \,.\, \Lambda\lambda \in [0,1] \,.\, \gamma\Big((1-\lambda)t_i + \lambda t_{i+1}\Big) : M(\omega),$

$[\ldots *] := \jmath\omega : (M,\omega) \le (n,\alpha)\ \&\ (M,\omega) \le (m,\beta);$

$\rightsquigarrow [*] := \eth^{-1}\texttt{DirectedSet} : \texttt{DirectedSet}\Big(M(\omega), \texttt{PathMehLess}(\omega)\Big);$

$\square$

$\texttt{PathConnected} :: ?\mathsf{TOP}$

$X : \texttt{PathConnected} \iff \forall x,y \in X \,.\, \Omega(x,Y) \ne \emptyset$

$\texttt{PathConnectedIsConnected} :: \forall X : \texttt{PathConnected} \,.\, \texttt{Connected}(X)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PathConnectedSubset} :: \prod X \in \mathsf{TOP} \,.\, ?\texttt{Connected}(X)$

$A : \texttt{PathConnectedSubset} \iff A : \texttt{PathConnected}(X) \iff \Big(A, \texttt{subsetTopology}(X,A)\Big) : \texttt{PathConected}$

$\texttt{MainPathConnectedSpaceTHM} :: \forall X : \texttt{PathConnected} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f \in C(X,Y) \,.\, \texttt{PathConnected}(Y, f(X))$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PathConnectedPair} :: \prod X \in \mathsf{TOP} \,.\, ?(X \times X)$

$(x,y) : \texttt{PathConnectedPair} \iff \Omega(x,y) \ne \emptyset$

$\texttt{PathConnectedPairIsEquivalence} :: \forall X \in \mathsf{TOP} \,.\, \texttt{PathConnectedPairIsEquivalence}(X)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{pathConnectedComponents} :: \prod X \in \mathsf{TOP} \,.\, ??X$

$\texttt{pathConnectedComponents}\,() = \mathrm{PCC}(X) := \texttt{classes}\ \texttt{PathConnectedPair}(X)$

$\texttt{LocallyPathConnectedProperty} :: \forall X : \texttt{LocallyPathConnected} \,.\, \forall U \in \mathrm{PCC}(X) \,.\, U : \texttt{Clopen}(X)$

$\texttt{Proof} \ =$

...

$\square$

1.5.4 Totally Disconected Spaces

$\texttt{TotallyDisconected} :: ?\mathsf{TOP}$

$X : \texttt{TotallyDisconnectedSpace} \iff \forall A \in \mathrm{CC}(X) \;.\; A : \texttt{Singleton}$

$\texttt{TotallyDisconectesByBase} :: \forall X \in \mathsf{TOP} \;.\; \langle \texttt{CLopen}(X) \rangle_{\mathsf{TOP}} = X \iff \texttt{TotallyBounded}(X)$

$\texttt{Proof} =$

1.5.5 Sequential Spaces

$$\texttt{limitOfSequences} :: \prod X : \mathsf{TOP} \,.\, (X \to \mathbb{N}) \to ?X$$
$$\texttt{limitOfSequences}\,(x) = \lim_{n\to\infty} x_n := \lim_{n\in\mathbb{N}} x_n$$

$$\texttt{SequentialSpace} :: ?\mathsf{TOP}$$
$$X : \texttt{SequentialSpace} \iff \forall A \subset X \,.\, A : \texttt{Closed} \iff \forall x : \texttt{Net}(\mathbb{N}, X) \,.\, x_{\mathbb{N}} \subset A \Rightarrow \overline{x} \subset A$$

$$\texttt{FrechetSpace} :: ?\mathsf{TOP}$$
$$X : \texttt{FrechetSpace} \iff \forall A \subset X \,.\, \forall p \in \overline{A} \,.\, \exists x : \texttt{Net}(\mathbb{N}, X) : x_{\mathbb{N}} \subset A : p = \lim_{n\to\infty} x_n$$

$$\texttt{FirstCountableIsFrechetSpace} :: \forall X : \texttt{FirstCounable} \,.\, X : \texttt{FrechetSpace}$$

$\texttt{Proof}$ =

...

□

$$\texttt{FrechetSpaceIsSequential} :: \forall X : \texttt{FrechetSpace} \,.\, X : \texttt{SequentialSpace}$$

$\texttt{Proof}$ =

...

□

$$\texttt{ContinuousByLimits} :: \forall X : \texttt{SequentialSpace} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f : X \to Y \,.\, f \in C(X,Y) \iff$$
$$\iff \forall x : \mathbb{N} \to X \,.\, \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$$

$\texttt{Proof}$ =

...

□

$$\texttt{T1ByLimitNumber} :: \forall X \in \mathsf{TOP} \,.\, \forall [0] : \forall x : \mathbb{N} \to X \,.\, \left|\lim_{n\to\infty} x_n\right| \le 1 \,.\, X : \texttt{T1}$$

$\texttt{Proof}$ =

...

□

$$\texttt{T2ByLimitNumber} :: \forall X : \texttt{FirstCountable} \,.\, X : \texttt{T2} \iff \forall x : \mathbb{N} \to X \,.\, \left|\lim_{n\to\infty} x_n\right| \le 1$$

$\texttt{Proof}$ =

...

□

2 Compacta

In the previous chapter I itroduced compactness on the elementary rather. The chapter is fully devoted to developing these chapters to the level required by advanced abstract analysis. Three sections of this chapter are independent on conceptual level, but latter fragments may reference earlier ones ocasionally.

2.1 Genera

As well as the separation axioms, the compactness may be introduced as a form o regularity, a property which makes space easy to handle. As it was shown during our previous encounter with compacts, they, indeed, may have nice properties. However, being compact is too restrictive, as many common natural spaces like real numbers $\mathbb{R}$ are not compact. In this section I investigate some other predicates which in may ways simmilar to compactness, in the sence of making spaces into handy stuctures of comprehandable size, on the other hand still far more common then the compactness itself.

2.1.1 Lindelöf Spaces

First of all I investigate Lindelöf spaces. Which are spaces which are more general than compacts as only countable subcovers may be extracted from the arbitrary covers. It is obvious that compact spaces are Lindelöf. Also as with compacts I show that this properties is inhereted by all closed sets. In difference with compactness which allows elvating T2 to T4 separation axioms, the Lindelöf property allows only elevating T3 to T4.

$\texttt{Lindelöf} :: ?\mathsf{TOP}$

$X : \texttt{Lindelöf} \iff \forall \mathcal{O} : \texttt{OpenCover}(X) \,.\, \exists \mathcal{O}' \subset \mathcal{O} \,.\, |\mathcal{O}'| \le \aleph_0 \;\&\; \mathcal{O}' : \texttt{OpenCover}(X)$

$\texttt{CompactIsLindelöf} :: \forall X : \texttt{Compact} \,.\, \texttt{Lindelöf}(X)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{SecondCountableIsLindelöf} :: \forall X : \texttt{SecondCountable} \,.\, \texttt{Lindelöf}(X)$

$\texttt{Proof} =$

Let $\mathcal{B}$ be a countable base for X.

As $\mathcal{O}$ is a cover of X it is possible to construct a function $O : X \to \mathcal{O}$ such that $x \in O_x$.

By definition of the base it must be possible to construct a function $B : X \to \mathcal{B}$ such that $x \in B_x \subset O_x$.

Then $\operatorname{Im} B \subset \mathcal{B}$ must be countable.

Then for each $U \in \operatorname{Im} B$ select O'_U with $U \subset O'_U$, which is possible for by construction of B.

Then $\operatorname{Im} O'$ must be countable as an image of countable set.

$x \in U_x$ for any $x \in X$, so $x \in O'_{U_x}$.

Thus, $\operatorname{Im} O'$ forms a countable subcover of $\mathcal{O}$.

□

$\texttt{ClosedLindelöfSubspaceIsLindelöf} :: \forall X : \texttt{Lindelöf} \,.\, \forall F : \texttt{Closed}(X) \,.\, \texttt{Lindelöf}(F)$

$\texttt{Proof} =$

Assume $\mathcal{O}$ is an open cover of F.

Then $\mathcal{O}' = \left\{ O \cup F^{\complement} \middle| O \in \mathcal{O} \right\}$ is an open cover of X.

Select a countable subcover $\mathcal{C}$ of $\mathcal{O}'$ by Lindelöf property.

Then $\mathcal{C}' = \{O \cap F | O \in \mathcal{C}\}$ is a countable subcover of $\mathcal{O}$.

□

CoverSeparationLemma ::

$$:: \forall X \in \mathsf{TOP} \,.\, \forall A, B \subset X \,.\, \forall V, W : \mathbb{N} \to \mathcal{T}(X) \,.\, \forall \aleph : A \subset \bigcup_{n=1}^{\infty} V_n \,.\, \forall \beth : \forall n \in \mathbb{N} \,.\, \overline{V} \cap B = \emptyset \,.$$

$$.\, \forall \gimel : B \subset \bigcup_{n=1}^{\infty} W_n \,.\, \forall \daleth : \forall n \in \mathbb{N} \,.\, \overline{W} \cap A = \emptyset \,.\, \mathtt{Separated}(X, A, B)$$

Proof $=$

Without loss of generality assume that V and W are increasing.

Otherwise construct $V'_n = \bigcup_{k=1}^{n} V_k$ and $W'_n = \bigcup_{k=1}^{n} W_k$.

Clearly, these constructions inherit propertirs $\aleph', \beth', \gimel', \daleth'$.

Also without loss of generality assume that $V_n \cap \overline{W}_n = \emptyset$ and $W_n \cap \overline{V}_n = \emptyset$.

Otherwise construct $V'_n = V_n \setminus \overline{W}_n$ and $W'_n = W_n \setminus \overline{V}_n$.

Clearly, these constructions inherit propertirs $\aleph', \beth', \gimel', \daleth'$.

Define open sets $G = \bigcap_{n=1}^{\infty} V_n$ and $H = \bigcap_{n=1}^{\infty} W_n$.

Then $A \subset G$ and $B \subset H$ by $\aleph$ and $\gimel$.

Take any $x \in G \cap H$.

Then there exists some $n, m \in \mathbb{N}$ such that $x \in V_n \cap W_n$.

Take $k = \max(m, n)$.

Then $x \in V_k \cap W_k$ as both V and W are increasing.

But $V_k \cap \overline{W}_k = \emptyset$, so this is a contradiction!

Thus, $H \cap G = \emptyset$ and H and G provide the separation of A and B.

□

LindelöfRegularity :: $\forall X : \mathtt{Lindelöf}$ & $\mathtt{T3}$. $\mathtt{T4}$ & $\mathtt{Tychonoff}(X)$

Proof $=$

Assume A and B are both closed sets in X.

As X is T3 for every $a \in A$ it must be possible to select open U_a such that $a \in U_a$ and $\overline{U}_a \cap B = \emptyset$.

As A itself is Lindel´of it must be possible to select a countable subcover $\mathcal{U}$ from U_A.

By the simmilar process we may select a countable cover $\mathcal{V}$ for B.

But this leads us to open separation lemma, so A and B can be separated.

Thus X is T4 and hence T3.5 or Tychonoff.

□

LindelöfFSigmaIsLinelof :: $\forall X : \mathtt{Compact} \,.\, \forall A \in F_\sigma(X) \,.\, \mathtt{Lindelöf}(A)$

Proof $=$

Repesent $A = \bigcup_{n=1}^{\infty} C_n$, where each C_n is closed.

Then each C_n is Lindelöf, and if $\mathcal{O}$ is an open cover for A, then $\mathcal{O}$ is open cover for each C_n.

Hence open countable subcovers $\mathcal{O}'_n$ can be selected for each C_n.

Then $\mathcal{O}'' = \bigcup_{n=1}^{\infty} \mathcal{O}'_n$ is still countable and is a cover of A.

□

2.1.2 Locally Compact Spaces

Another intuitive approach on how the niceness of compactness can be extended to the broader class of spaces is the Locall Compactness. So, not the whole space is comprehandable anymore, but every point has a comprehandable neighborhood. In this case a T2 separation axiom can only be lifted to the separation axiom T3.5.

$\texttt{LocallyCompact} :: ?\texttt{TOP}$

$X : \texttt{LocallyCompact} \iff \forall x \in X \,.\, \exists U \in \mathcal{U}(x) : \overline{U} : \texttt{Compact}(X)$

$\texttt{LocallyCompactIsTychonoff} :: \forall X : \texttt{LocallyCompact} \;\&\; \texttt{T2} \,.\, X : \texttt{Tychonoff}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactSeparation} :: \forall X : \texttt{LocallyCompact} \,.\, \forall A : \texttt{Compact}(X) \,.\, \forall U \in \mathcal{U}(A) \,.$

$: \exists V \in \mathcal{U}(A) : \overline{V} : \texttt{Compact}(X) \;\&\; \overline{V} \subset U$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactSubset1} :: \forall X : \texttt{LocallyCompact} \,.\, \forall A \subset X \,.\, \forall U : \texttt{Open}(X) \,.\, \forall K : \texttt{Closed}(X) \,.$

$.\, A = U \cap K \Rightarrow A : \texttt{LocallyCompact}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactSubset2} :: \forall X : \texttt{T2} \;\&\; \texttt{LocallyCompact} \,.\, \forall A \subset X \,.$

$.\, A : \texttt{LocallyCompact} \Rightarrow \exists U : \texttt{Open}(X) : \exists K : \texttt{Closed}(X) :$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactRepresentation} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{LocallyCompact} \iff \exists K : \texttt{Compact} : \exists U \in \mathcal{T}(K) \,.$

$.\, U \cong_{\mathsf{TOP}} X$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactSum} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \coprod_{i \in I} X_i : \texttt{LocallyCompact} \iff$

$\iff \forall i \in I \,.\, X_i : \texttt{LocallyCompact}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LocallyCompactProduct} :: \forall I \in \mathsf{SET} . \forall X : I \to \mathsf{TOP} . \prod_{i \in I} X_i : \texttt{LocallyCompact} \iff$

$\iff \Big(\forall i \in I . X_i : \texttt{LocallyCompact} \Big) \ \& \ \Big| \{ i \in I : X_i \ ! \ \texttt{Compact} \} \Big| < \infty$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{LocallyCompactByMapping} :: \forall X : \texttt{LocallyCompact} . \forall Y : \texttt{T2} . \forall f : \texttt{Surjection} \ \& \ C(X, Y) .$

$. \ Y : \texttt{LocallyCompact}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{WhiteheadQuotientTheorem} :: \forall X : \texttt{LocallyCompact} . \forall Y, Z \in \mathsf{TOP} . \forall \pi : \texttt{QuotientMapping}(\mathsf{TOP}; Y, Z) .$

$. \ \mathrm{id}_X \times \pi : \texttt{QuotientMapping}(\mathsf{TOP}; X \times Y, X \times Z)$

$\texttt{Proof} \ =$

$\dots$

$\square$

2.1.3 Countably Compact Spaces

The predicate dual to being Lindelöf is countable compactness. This duality is present in the sense that both combined form true compactness. The duality suggest that countable compactness allows elevating T2 separation axiom to T3 if the space is first countable.

$\texttt{CountablyCompact} :: ?\mathsf{TOP}$

$X : \texttt{CountablyCompact} \iff \forall \mathcal{O} : \texttt{OpenCover}(X) \,.\, |\mathcal{O}| \leq \aleph_0 \Rightarrow \exists \mathcal{O}' \subset \mathcal{O} : |\mathcal{O}'| < \infty \,.\, \& \mathcal{O}' : \texttt{OpenCover}(X)$

$\texttt{CompactnessDecomposition} :: \forall X \in \mathsf{TOP} \,.\, \forall \texttt{CountableCompact} \,\&\, \texttt{Lindelöf}(X) \iff \texttt{Compact}(X)$

$\texttt{Proof} =$

This is obvious.

□

$\texttt{CondensationPoint} :: \prod_{X \in \mathsf{TOP}} ?X \to ?X$

$x : \texttt{CondensationPoint} \iff \Lambda A \subset X \,.\, \forall U \in \mathcal{U}(x) \,.\, |A \cap U| = \infty$

$\texttt{CountableCompactAltDef} ::$

$:: \forall X \in \mathsf{TOP} \,.\, \texttt{CountableCompact}(X) \iff$

$\iff \forall x : \mathbb{N} \to X \,.\, \exists \texttt{Cluster}(X, x) \iff$

$\iff \forall A : \texttt{Infinite}(X) \,.\, \exists \texttt{CondensationPoint}(X, A)$

$\texttt{Proof} =$

2.1.4 Sequentially Compact Spaces

As it is now evident that countably compact need to be first countable to lift T2 to T3. The first countable countably compact space is in fact sequetially compact. So this concept seems to be more wholesome as convergnece is an such important part of analysis.

$$\texttt{SequentiallyCompact} :: ?\mathsf{TOP}$$
$$X : \texttt{SequentiallyCompact} \iff \forall x : \mathbb{N} \to X \,.\, \exists y \subset x : y : \texttt{Convergent}(X)$$

2.1.5 Pseudocompact Spaces

An important property of compacts is that every continuous function is bounded. Spaces which share this property are called pseudocompact.

$$\texttt{boundedFunctions} :: \mathsf{TOP} \to \mathsf{SET}$$

$$\texttt{boundedFunctions}\,(X) = C_b(X) := \Big\{ f \in C(X) : \exists a, b \in \mathbb{R} : f(X) \subset [a, b] \Big\}$$

$$\texttt{Pseudocompact} :: ?\mathsf{TOP}$$

$$X : \texttt{Pseudocompact} \iff C(X) = C_b(X)$$

2.2 Category of Compact Spaces

Compactness can be studied in two distinct ways. Firstly, it is possible to look at compact subsets of general topological sets. On the other hand, one may investigate how compact sets interact with each other. The later topic leads to the language of category theory naturally.

2.2.1 Filters in categories

One way to apply this lanuage in the smart way is the use of the filter monad. Deep study of this subject is leading to the area known as monoidal topology. However, here I only scrath the surface with the basic results and definitions. Note, that there some complications in the interpretation of $\mathsf{F}(\mathcal{C})(\mathcal{F},\mathfrak{F})$

$\texttt{filterFunctor} :: \texttt{Covariant}(\mathsf{SET},\mathsf{SET})$

$\texttt{filterFunctor}\,(X) = \mathsf{F}(X) := \texttt{Filter}(X)$

$\texttt{filterFunctor}\,(X,Y,f) = \mathsf{F}_{X,Y}(f) := \Lambda\mathcal{F} \in \texttt{Filter}(X)\ .\ \{f(A)|A \in \mathcal{F}\}$

Obviosly, as $\mathcal{F} \neq \emptyset$ then $\mathsf{F}_{X,Y}(f)(\mathcal{F}) \neq \emptyset$.

Also, as $\emptyset \notin \mathcal{F}$ then $f \notin \mathsf{F}_{X,Y}(f)(\mathcal{F})$.

All this is true as an image non-emptyset can't be empty.

Now assume $f(A), f(B) \in \mathsf{F}_{X,Y}(f)(\mathcal{F})$.

Then there is $\emptyset \neq C \in \mathcal{F}$ such that $C \subset A \cap B$.

Thus $f(C) \subset f(A \cap B) \subset f(A) \cap f(B)$.

And $f(C) \in \mathsf{F}_{X,Y}(f)(\mathcal{F})$, so $\mathsf{F}_{X,Y}(f)(\mathcal{F})$ is a filterbase.

□

$$\texttt{pointFilter} :: \prod_{X \in \mathsf{SET}} X \to \mathsf{F}(X)$$

$\texttt{pointFilter}\,(x) = \dot{x} := \{A \subset X : A(x)\}$

$$\texttt{sumOfKowalsky} :: \prod_{X \in \mathsf{SET}} \mathsf{F}^2(X) \to \mathsf{F}(X)$$

$\texttt{sumOfKovalsky}\,(\mathfrak{F}) = m_X(\mathfrak{F}) := \Big\{A \subset X : \{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}\} \in \langle\mathfrak{F}\rangle\Big\}$

As $\emptyset \notin \mathcal{F}$ for any $\mathcal{F} \in \mathsf{F}(X)$ it follows that $\emptyset \notin m_X(\mathfrak{F})$.

Also note that $\{\mathcal{F} \in \mathsf{F}(X)|X \in \mathcal{F}\} = \mathsf{F}(X)$.

And $\mathsf{F}(X) \in \mathfrak{F}$, so $X \in m_X(\mathfrak{F})$.

Whence $m_X(\mathfrak{F}) \neq \emptyset$.

If $A, B \in m_X(\mathfrak{F})$ then $\{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}\}, \{\mathcal{F} \in \mathsf{F}(X)|B \in \mathcal{F}\} \in \mathfrak{F}$.

But $\{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}\} \cap \{\mathcal{F} \in \mathsf{F}(X)|B \in \mathcal{F}\} = \{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}, B \in \mathcal{F}\} = \{\mathcal{F} \in \mathsf{F}(X)|A \cap B \in \mathcal{F}\} \in \mathfrak{F}$.

Hence $A \cap B \in m_X(\mathfrak{F})$.

Now assume that $A \in m_X(\mathfrak{F})$ and $A \subset B$.

Then $\{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}\} \in \mathfrak{F}$.

Also by property of filters being upward closed $\{\mathcal{F} \in \mathsf{F}(X)|A \in \mathcal{F}\} \subset \{\mathcal{F} \in \mathsf{F}(X)|B \in \mathcal{F}\}$.

But as $\mathfrak{F}$ is also filter, then $\{\mathcal{F} \in \mathsf{F}(X)|B \in \mathcal{F}\} \in \mathfrak{F}$ and hence $B \in \mathfrak{F}$.

And hence $m_X(\mathfrak{F})$ is a filter.

□

$\texttt{FilterConvergence} :: \prod_{X \in \mathsf{SET}} ?\Big(X \times \mathsf{F}(X)\Big)$

$\mathcal{C} : \texttt{FilterConvergence} \iff \forall x \in X \,.\, \mathcal{C}\Big(x, \dot{x}\Big) \,\&$

$\quad \& \, \forall \mathfrak{F} \in \mathsf{F}^2(X) \,.\, \exists \mathcal{F} \in \mathsf{F}(X) \,.\, \mathsf{F}(\mathcal{C})(\mathcal{F}, \mathfrak{F}) \,\&\, \mathcal{C}(x, \mathcal{F}) \Rightarrow \mathcal{C}\Big(x, m_X(\mathfrak{F})\Big)$

$\texttt{FilterConvergneceIsTopology} ::$

$\quad :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{C} : \texttt{FilterConvergence}(X) \,.\, \exists! \tau : \texttt{Topology}(X) \,.\, \forall (x, \mathcal{F}) \in X \times \mathsf{F}(X) \,.$

$\quad .\, (x, \mathcal{F}) \in \mathcal{C} \iff x \in \lim_{(X,\tau)} \mathcal{F}$

$\texttt{Proof} \; =$

Let $\mathcal{C}$ be a filter convergence in the sense defined above.

Define $\tau = \Big\{U \subset X : \forall (x, \mathcal{F}) \in \mathcal{C} \,.\, U(x) \Rightarrow \langle \mathcal{F} \rangle(U)\Big\}$.

Then $\emptyset$ is trivally in τ.

If $(x, \mathcal{F}) \in \mathcal{C}$, then $\mathcal{F} \neq \emptyset$ so there are $A \subset \mathcal{F}$.

This means that $A \subset X \in \mathcal{F}$ as $\langle \mathcal{F} \rangle$ must be upward closed.

Assume $U : I \to \tau$ is the collection of sets, and $(x, \mathcal{F}) \in \mathcal{C}$ is such that $\bigcup_{i \in I} U_i(x)$.

Then there is $i \in I$ such that $U_i(x)$, so $\langle \mathcal{F} \rangle(U_i)$.

But as $\langle \mathcal{F} \rangle$ is upwards closed $\langle \mathcal{F} \rangle \left(\bigcup_{i \in I} U_i\right)$.

Thus $\bigcup_{i \in I} U_i \in \tau$.

Now take $n \in \mathbb{N}$ and $U : \{1, \dots, n\} \to \tau$ and also $(x, \mathcal{F}) \in C$ such that $\bigcap_{i=1}^{n} U_i(x)$.

Then $U_i(x)$ for any $i \in \{1, \dots, n\}$, so $\langle \mathcal{F} \rangle(U_i)$.

But $\langle \mathcal{F} \rangle$ is intersection closed, so $\langle \mathcal{F} \rangle \left(\bigcap_{i=1}^{n} U_i\right)$.

Thus $\bigcup_{i \in I} U_i \in \tau$ and we proved that τ is topology.

Then by construction $(x, \mathcal{F}) \in \mathcal{C} \Rightarrow x \in \lim_{(X,\tau)} \mathcal{F}$.

Now take $(x, \mathcal{F}) \in X \times \mathsf{F}(X)$, such that $x \in \lim_{(X,\tau)} \mathcal{F}$.

This means that $\mathcal{F}(U)$ for all $U \in \mathcal{U}(x)$.

The idea is to show that $\mathcal{F} = m_X(\mathfrak{F})$ for some filter of filters $\mathfrak{F}$ and $\mathsf{F}(\mathcal{C})(\dot{x}, \mathfrak{F})$.

So define $\mathfrak{F} = \Big\{\{\langle \mathcal{G} \rangle | \mathcal{G} \in \mathsf{F}(X), A \in \mathcal{G}\} \Big| A \in \mathcal{F}\Big\}$.

$\mathfrak{F}$ is a filterbase: it is nonempty as $\mathcal{F}$ is and any element of $\mathfrak{F}$ conatains $\mathcal{F}$.

It is also closed by intersections as

$\{\langle \mathcal{G} \rangle | \mathcal{G} \in \mathsf{F}(X), A \in \mathcal{G}\} \cap \{\langle \mathcal{G} \rangle | \mathcal{G} \in \mathsf{F}(X), B \in \mathcal{G}\} = \{\langle \mathcal{G} \rangle | \mathcal{G} \in \mathsf{F}(X), A \cap B \in \mathcal{G}\}$

as filters are upward and intersection closed.

Then $\mathcal{F} \subset m_x(\mathfrak{F})$.

Assume $A \in m_x(\mathfrak{F})$.

Then there is $B \in \mathcal{F}$ such that $\{\langle\mathcal{G}\rangle | \mathcal{G} \in \mathsf{F}(X), B \in \mathcal{G}\} \subset \{\langle\mathcal{G}\rangle | \mathcal{G} \in \mathsf{F}(X), A \in \mathcal{G}\}$.

But this means that $A \in \langle\mathcal{F}\rangle$.

So we may say that $m_x(\mathfrak{F}) = \langle\mathcal{F}\rangle$ as filter.

By construction $\dot{x} \in \{\langle\mathcal{G}\rangle | \mathcal{G} \in \mathsf{F}(X), N \in \mathcal{G}\} \in \mathfrak{F}$ for every $N \in \mathcal{N}_\tau(x)$.

This must be enough to shaw that $\mathsf{F}(\mathcal{C})(\dot{x}, \mathfrak{F})$.

And as $\mathcal{C}(x, \dot{x})$ by definition, it follows that $\mathcal{C}(x, \mathcal{F})$.

If σ and τ are two topologies with the desired property, then $\mathcal{N}_\tau(x) = \mathcal{N}_\sigma(x)$ for all $x \in X$.

So $\tau = \sigma$.

□

$$\texttt{convergenceAsTopology} :: \prod_{X \in \mathsf{SET}} \texttt{FilterConvergnce}(X) \to \texttt{Topology}(X)$$

$$\texttt{convergenceAsTopology}\,(\mathcal{C}) = \tau_{\mathcal{C}} := \texttt{FilterConvergenceIsTopology}$$

□

$\texttt{CompactTopologyByConvergenve}$ ::

:: $\forall X \in \mathsf{SET} \,.\, \forall \mathcal{C} : \texttt{FilterConvergence}(X) \,.\, \texttt{Compact}(X, \tau_{\mathcal{C}}) \iff$

$\iff \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \exists x \in X \,.\, (x, \mathcal{F}) \in \mathcal{C}$

Proof =

($\Rightarrow$) Firstly, assume that $(X, \tau_{\mathcal{C}})$ is compact.

Take some $\mathcal{F} \in \mathsf{F}(X)$.

If $\mathcal{F}$ has no limits, then for any $x \in X$ we may select $U \in \mathcal{U}(x)$ such that $U \notin \mathcal{F}$.

Then there exists a finite subcover $\mathcal{O}$ of $\operatorname{Im} U$.

There is a set $A \in \mathcal{F}$ as $\mathcal{F}$ is a filter.

But as $\mathcal{O}$ is a cover there must be $O \in \mathcal{O}$ such that $O \cap A \in \mathcal{F}$.

Furthermore, as $\mathcal{F}$ is upward closed, $O \in \mathcal{F}$, which produces a contradiction.

($\Leftarrow$) Now assume that condion on $\mathcal{C}$ holds.

Then every ultrafilter has a limit.

But this also means that every net has a convergent subnet .

But by $\texttt{CompactLimitTHM}$ this means that $(X, \tau_{\mathcal{C}})$ is compact .

□

$\texttt{HausdorffTopologyByConvergenve}$ ::

:: $\forall X \in \mathsf{SET} \,.\, \forall \mathcal{C} : \texttt{FilterConvergence}(X) \,.\, \texttt{Hausdorff}(X, \tau_{\mathcal{C}}) \iff$

$\iff \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \left|\left\{\exists x \in X \,.\, (x, \mathcal{F}) \in \mathcal{C}\right\}\right| \le 1$

Proof =

($\Rightarrow$) Assume $x, y \in X$ are two distinct limits of the ultrafilter $\mathcal{F}$.

Then there are disjoint $U \in \mathcal{U}(x)$ and $V \in \mathcal{U}(y)$.

But then $U, V \in \mathcal{F}$ but $U \cap V = \emptyset \in \mathcal{F}$, which is a contradiction.

($\Leftarrow$) For $x, y \in X$ there are filters $\mathcal{N}(x), \mathcal{N}(y) \in \mathsf{F}(X)$ such that $x = \lim \mathcal{N}(x)$ and $y = \lim \mathcal{N}(y)$.

There are ultrafilters $\mathcal{F}_x, \mathcal{F}_y$ such that $\mathcal{N}(x) \subset \mathcal{F}_x$ and $\mathcal{N}(y) \subset \mathcal{F}_y$.

If for any pair of open neighborhoods $(U, V) \in \mathcal{U}(x) \times \mathcal{U}(y)$, then $U \in \mathcal{N}(y)$ and $V \in \mathcal{N}(x)$.

But this means that $x, y \in \lim \mathcal{F}_x$ and $x, y \in \lim \mathcal{F}_y$, which contradicts the assumption.

□

2.2.2 Compact Hausdorff Spaces

In Last section it was shown that the FilterConvergence relation $\mathcal{C}$ becomes a function $\lim : \mathsf{UF}(X) \to X$ if and only if the space $(X, \tau_{\mathcal{C}})$ is compact and Hausdorff. Thus, it makes compact Hausdorff spaces of a particular interst as a category. In particular in this category any invertible continuous bijection will have a continuous inverse. Here I aslo prove the Tychonoff theorem, which is a very important result such that any product of compacts is compact.

$\texttt{CategoyOfCompacta} :: \mathsf{CAT}$

$\texttt{CategoryOfCompacta}\,() = \mathsf{HC} := \Big(\texttt{Compact}\ \&\ \texttt{T2}, C, \circ, \mathrm{id}\Big)$

$\texttt{CompactExtensionCriterion} :: \forall X \in \mathsf{TOP}\ .\ \forall Y \in \mathsf{HC}\ .\ \forall D : \texttt{Dense}(X)\ .\ \forall f : D \xrightarrow{\mathsf{TOP}} X\ .$

$.\ \Big(\exists F : X \xrightarrow{\mathsf{TOP}} Y : F_{|D} = f\Big) \iff \forall A, B : \texttt{Closed}(Y)\ .\ A \cap B = \emptyset \Rightarrow \Big(\operatorname*{cl}_x f^{-1}(A)\Big) \cap \Big(\operatorname*{cl}_y f^{-1}(B)\Big) = \emptyset$

$\texttt{Proof}\ =$

$\texttt{Assume}\ F : X \xrightarrow{\mathsf{TOP}} Y,$

$\texttt{Assume}\ [1] : F_{|D} = f,$

$\texttt{Assume}\ A, B : \texttt{Closed}(Y),$

$\texttt{Assume}\ [2] : A \cap B = \emptyset,$

$[3] := [1]\texttt{ClosedContainsLimits} : \operatorname*{cl}_X f^{-1}(A) = F^{-1}(A),$

$[4] := [2]\texttt{ClosedContainsLimits}(B) : \operatorname*{cl}_X f^{-1}(B) = F^{-1}(B),$

$[F.*] := \texttt{DisjointPreimage}[2][3][4] : \operatorname*{cl}_X f^{-1}(A) \cap \operatorname*{cl}_X f^{-1}(B) = \emptyset,$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{LEFT} \Rightarrow \texttt{RIGHT},$

$\texttt{Assume}\ [2] : \texttt{Right}$,

$\texttt{Assume}\ x : X$,

$\texttt{Assume}\ a, b : \texttt{Net}\Big(\mathcal{U}_X(x), D\Big)$,

$\texttt{Assume}\ [3] : \forall U \in \mathcal{U}_X(x)\ .\ a_U, b_U \in U$,

$\texttt{Assume}\ A : \texttt{Cluster}\Big(f(a)\Big)$,

$\texttt{Assume}\ B : \texttt{Cluster}\Big(f(b)\Big)$,

$\texttt{Assume}\ [4] : A \neq B$,

$[5] := \texttt{CompactIsNormal}(Y) : \Big(Y : \texttt{T4}\Big)$,

$\Big(U, V, [6]\Big) := \eth\texttt{Urysohn}(Y)(A, B)[4][5] : \sum U \in \mathcal{U}(A)\ .\ \sum V \in \mathcal{U}(B)\ .\ \overline{U} \cap \overline{B} = \emptyset$,

$[7] := [2][6] : \operatorname*{cl}_X f^{-1}\overline{U} \cap \operatorname*{cl}_X f^{-1}\overline{V} = \emptyset$,

$[8] := \texttt{MonotonicClosure} : \operatorname*{cl}_X f^{-1}(U) \cap \operatorname*{cl}_X f^{-1}(V) = \emptyset$,

$[9] := [8](x) : x \notin \operatorname*{cl}_X f^{-1}(U) \Big| x \notin \operatorname*{cl}_X f^{-1}(V)$,

$[10] := \eth\texttt{Cluster}(f(a))(A)(U)[3]\texttt{ClosureEqual}(X) : x \in \operatorname*{cl}_X f^{-1}(U)$,

$[11] := \eth\texttt{Cluster}(f(b))(B)(V)[3]\texttt{ClosureEqual}(X) : x \in \operatorname*{cl}_X f^{-1}(V)$,

$[(a, b).*] := [9][10][11] : \perp$;

$\rightsquigarrow [3] := I(\forall)I(\Rightarrow)I(\forall)E(\perp) : \forall a, b : \texttt{Net}(\mathcal{U}_X(x), D)\ .$

$\quad .\ \Big(\forall U \in \mathcal{U}_X(x)\ .\ a_U, b_U \operatorname{Im} U\Big) \Rightarrow \forall A : \texttt{Cluster}\Big(f(a)\Big)\ .\ \forall B : \texttt{Cluster}\Big(f(b)\Big)\ .\ A = B$;

$\Big(a, [4]\Big) := \eth\texttt{Dense}(X)(D) : \sum a : \texttt{Net}(\mathcal{U}_X(x), D)\ .\ \forall U \in \mathcal{U}(x)\ .\ a_U \in U$,

$F(x) := \lim_{U \in \mathcal{U}(x)} f(a_U) : Y$;

$\rightsquigarrow F := I(\rightarrow) : X \rightarrow Y$,

$[3] := \jmath F : F_{|D} = f$,

$\texttt{Assume}\ A : \texttt{Closed}(Y)$,

$[4] := \jmath F(A) : F^{-1}(A) = \operatorname*{cl}_X f^{-1}(A)$,

$[*.A] := \eth \operatorname*{cl}_X [4] : F^{-1}(A) : \texttt{Closed}(X)$;

$\rightsquigarrow [2.*] := \eth^{-1} C : F \in C(X, Y)$;

$[*] := I(\iff)[1]I(\Rightarrow) : \texttt{This}$;

$\square$

$\texttt{CompactCoproduct} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \prod_{i \in I} X_i \in \mathsf{HC} \iff \operatorname{Im} X \subset \mathsf{HC} \,\&\, I : \texttt{Finite}$

$\texttt{Proof} =$

...

$\square$

$\texttt{TychonoffTheorem} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \prod_{i \in I} X_i \in \mathsf{HC} \iff \operatorname{Im} X \subset \mathsf{HC}$

$\texttt{Proof} =$

...

$\square$

$\texttt{TychonoffUniversality} :: \forall \kappa : \texttt{InfiniteCardinal} \,.\, [0,1]^\kappa : \texttt{Universal}\left\{X \in \mathsf{HC} : w(X) = \kappa\right\}$

$\texttt{Proof} =$

...

$\square$

$\texttt{TychonoffCriterion} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{Tychonoff} \iff \exists K \in \mathsf{HC} : \exists \texttt{HomeomorphicEmbedding}(X, K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{WallaceTheorem} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{TOP} \,.\, \forall K : \prod_{i \in I} \mathsf{HC} \,\&\, \texttt{Subspace}(X_i) \,.\, \forall W : \texttt{Open} \prod_{i \in I} X_i \,.$

$. \forall [0] : \prod_{i \in I} K_i \subset W \,.\, \exists U : \prod_{i \in I} \texttt{Open}(X_i) : \prod_{i \in I} K_i \subset \prod_{i \in I} U_i \subset W \,\&\, \Big|\{i \in I : U_i \neq X_i\}\Big| < \infty$

$\texttt{Proof} =$

...

$\square$

$\texttt{AlexandroffTheorem} :: \forall X \in \mathsf{HC} \,.\, \forall E : \texttt{Equivalence}(X) \,.\, E : \texttt{Closed} \iff \exists Y : \texttt{T2} : \exists f : X \xrightarrow{\mathsf{TOP}} Y :$

$: \frac{X}{E} \cong_{\mathsf{TOP}} \frac{X}{f}$

$\texttt{Proof} =$

...

$\square$

2.2.3 Compactly Generated Spaces

One issue with compact Hausdorff spaces is that it do not have all categorical limits and colimits. This problem may be overcomed by introducing notion of compactly generated spaces.

$\texttt{CompactlyGenerated} :: ?\texttt{T2}$

$X : \texttt{CompactlyGenerated} \iff \exists Q : \texttt{LocallyCompact} : \exists \pi : \texttt{QuotientMapping}(\mathsf{TOP}, Q, X)$

$\texttt{compactlyGenerated} :: \mathsf{CAT}$

$\texttt{compactlyGenerated}() = \mathsf{CG} := \Big(\texttt{CompactlyGenerated}, C, \circ, \mathrm{id}\Big)$

$\texttt{LocallyCompactIsCompactlyGenerated} :: \forall X : \texttt{LocallyCompact} . \forall X \in \mathsf{CG}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompactlyGeneratedAlternativeDefinition} :: \forall X : \texttt{T2} . X \in \mathsf{CG} \iff \forall A \subset X .$

$. A : \texttt{Closed}(X) \iff \forall K : \texttt{Compact}(X) . K \cap A : \texttt{Closed}(K)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{DualCompactlyGeneratedAlternativeDefinition} :: \forall X : \texttt{T2} . X \in \mathsf{CG} \iff \forall A \subset X .$

$. A \in \mathcal{T}(X) \iff \forall K : \texttt{Compact}(X) . K \cap A \in \mathcal{T}(K)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SequentialHausdorffIsCompactlyGenerated} :: \forall X : \texttt{T2} \,\&\, \texttt{Sequential} . X \in \mathsf{CG}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompactlyGeneratedContinuousMapping} :: \forall X \in \mathsf{CG} . \forall Y \in \mathsf{TOP} . \forall f : X \to Y . f \in C(X,Y) \iff \forall K : \texttt{Co}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompactlyGeneratedClosedMapping} :: \forall X \in \mathsf{TOP} . \forall Y \in \mathsf{CG} . \forall f : X \to Y .$

$. f : \texttt{Closed} \iff \forall K : \texttt{Compact}(Y) . f_{|f^{-1}K} : \texttt{Closed}(f^{-1}(K), K)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompactlyGeneratedOpentMapping} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,.\, \forall f : X \to Y \,.$

$\quad .\, f : \texttt{Open} \iff \forall K : \texttt{Compact}(Y) \,.\, f_{|f^{-1}K} : \texttt{Open}(f^{-1}(K), K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactlyGeneratedQuotientMapping} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,.\, \forall f : X \to Y \,.$

$\quad .\, f : \texttt{QuotientMapping} \iff \forall K : \texttt{Compact}(Y) \,.\, f_{|f^{-1}K} : \texttt{QuotientMapping}(f^{-1}(K), K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactlyGeneratedTransition} :: \forall X \in \mathsf{CG} \,.\, \forall Y : \texttt{T2} \,.\, \forall \pi : \texttt{QuotientMapping}(X, Y) \,.\, Y \in \mathsf{CG}$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactlyGeneratedSpacesHaveFiniteProducts} :: \mathsf{CG} : \texttt{HasFiniteProducts}$

$\texttt{Proof} =$

...

$\square$

$\texttt{spaceKaonization} :: \mathsf{TOP} \to \mathsf{CG}$

$\texttt{spaceKaonization}\,(X) = kX := \Big(X, \big\{U \subset X : \forall K : \texttt{Compact}(X) \,\&\, \texttt{T2} \,.\, K \cap U \in \mathcal{T}(K)\big\}\Big)$

$\texttt{kaonizationFunctor} :: \mathsf{TOP} \xrightarrow{\mathsf{CAT}} \mathsf{CG}$

$\texttt{kaonizationFunctor}\,() = k := (\texttt{spaceKaonization}, \mathrm{id})$

2.2.4 Compact-Open Topology

Here we try to set some notion of exponential objects for topological spaces. The problem comes from the fact that in case of pointwise topology, the currying operation is not continuous. The compact-open topology is acceptable in the sense, that currying is continuous. It turns out that compact-open topology is a generalization of uniform convergence. So many familiar results hold here, including Ascoli-theorem.

$$\texttt{DomainImageSet} :: \prod X, Y \in \mathsf{TOP} \,.\, ?X \to ?Y \to ?C(X,Y)$$

$$\texttt{DomainImageSet}\,(A,B) = M(A,B) := \Big\{ f \in C(X,Y) : f(A) \subset B \Big\}$$

$$\texttt{compactOpenTopology} :: \mathsf{TOP} \to \mathsf{TOP} \to \mathsf{TOP}$$

$$\texttt{compactOpenTopology}\,(X,Y) = \mathcal{C}(X,Y) := \Big\langle\Big\langle \Big\{ M(K,U) \Big| K : \texttt{Compact}(K), U \in \mathcal{T}(Y) \Big\} \Big\rangle\Big\rangle_{\mathsf{TOP}}$$

$$\texttt{RightCompositionIsContinuous} :: \forall X, Y, Z \in \mathsf{TOP} \,.\, \forall g \in C(Y,Z) \,.\, \rho_g \in C\Big(\mathcal{C}(X,Y), \mathcal{C}(Y,Z)\Big)$$

Proof =

...

□

$$\texttt{LetCompositionIsContinuous} :: \forall X : \texttt{T2} \,.\, \forall\forall g \in C(Y,Z) \,.\, \rho_g \in C\Big(\mathcal{C}(X,Y), \mathcal{C}(Y,Z)\Big)$$

Proof =

...

□

$$\texttt{CompactOpenTopologyIsProper} :: \forall X, Y \in \mathsf{TOP} \,.\, \mathcal{TC}(X,Y) : \texttt{Proper}(X,Y)$$

Proof =

...

□

$$\texttt{CompositionIsContinuous} :: \forall X, Z \in \mathsf{TOP} \,.\, \forall Y : \texttt{LocallyCompact} \,.\, \circ_{X,Y,Z} \in C\Big(\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z), \mathcal{C}(X,Z)\Big)$$

Proof =

...

□

$$\texttt{CompactOpenTopologyIsAcceptable} :: \forall X : \texttt{LocallyCompact} \,.\, \forall Y \in \mathsf{TOP} \,.\, \mathcal{TC}(X,Y) : \texttt{Acceptable}$$

Proof =

...

□

$$\texttt{CurryHomeomorphicEmbedding} :: \forall Y \in \mathsf{TOP} \,.\, \forall X, Z : \texttt{T2} \,.$$
$$.\, \Lambda : \texttt{HomeomorphicEmbedding}\Big(\mathcal{C}(X \times Z, Y), \mathcal{C}\big(X, \mathcal{C}(Z,Y)\big)\Big)$$

Proof =

...

□

$\texttt{LocallyCompactCurryHomeo} :: \forall Y \in \mathsf{TOP} . \forall Z : \texttt{T2} . \forall X : : \texttt{ocallyCompact} .$

$. \Lambda : \texttt{Homeo}\Big(\mathcal{C}(X \times Z, Y), \mathcal{C}\big(X, \mathcal{C}(Z, Y)\big)\Big)$

Proof $=$

...

$\square$

$\texttt{CompactlyGeneratedCurryHomeo} :: \forall X, Y, Z \in \mathsf{TOP} . \forall [0] : X \times Z \in \mathsf{CG} .$

$. \Lambda : \texttt{Homeo}\Big(\mathcal{C}(X \times Z, Y), \mathcal{C}\big(X, \mathcal{C}(Z, Y)\big)\Big)$

Proof $=$

...

$\square$

$\texttt{ExponentsOfCompactlyGeneratedSpace} :: \mathcal{C} : \texttt{Exponent}(\mathsf{CG})$

Proof $=$

...

$\square$

$\texttt{CompactOpenTopologyPreservesRegularity} :: \forall i \in \{1, 2, 3, 3.5\} . \forall X \in \mathsf{TOP} . \forall Y \in T(i) . \mathcal{C}(X, Y) \in T(i)$

Proof $=$

...

$\square$

$\texttt{ContinuousSupremum} :: \forall X \in \mathsf{TOP} . \forall K : \texttt{Compact}(X) . \Lambda f \in C(X, \mathbb{R}) \sup_{x \in K} f(x) \in C\Big(\mathcal{C}(X, \mathbb{R})\Big)$

Proof $=$

...

$\square$

$\texttt{CompactOpenTopologyPreservesWeight} :: \forall X, Y \in \mathsf{TOP} . \forall \kappa : \texttt{InfiniteCardinal} . [1] : w(X), w(Y) \le \kappa .$

$. w\mathcal{C}(X, Y) \le \kappa$

Proof $=$

...

$\square$

$\texttt{EvenlyContinuous} :: \forall X, Y \in \mathsf{TOP} . ??C(X, Y)$

$F : \texttt{EvenlyContinuous} \iff \forall x \in X . \forall y \in Y . \forall V \in \mathcal{U}(y) . \exists U \in \mathcal{U}(x) : \exists W \in \mathcal{U}(y) : \big(F \cap M(x, W)\big) \subset V$

$\texttt{EvenlyContinuousClosure} :: \forall X \in \mathsf{TOP} . \forall Y : \texttt{T3} . \forall F : \texttt{EvenlyContinuous}(X, Y) .$

$. \operatorname*{cl}_{\mathcal{C}(X,Y)} F : \texttt{EvenlyContinuous}(X, Y)$

Proof $=$

...

$\square$

$\texttt{AscoliTheorem} :: \forall X \in \mathsf{CG} \,.\, \forall Y : \texttt{T3} \,.\, \forall F \subset \mathcal{C}(X,Y)$

$. \, F : \texttt{EvenlyContinuous} \,\&\, \forall[0] : \forall x \in X \,.\, \overline{F(x)} : \texttt{Compact}(Y) \iff F : \texttt{Compact}\Big(\mathcal{C}(X,Y)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{LocalAscoliTheorem} :: \forall X \in \mathsf{CG} \,.\, \forall Y : \texttt{T3} \,.\, \forall F \subset \mathcal{C}(X,Y)$

$. \, \forall K : \texttt{Compact}(X) \,.\, F_{|K} : \texttt{EvenlyContinuous} K, Y) \,\&\, \forall[0] : \forall x \in X \,.\, \overline{F(x)} : \texttt{Compact}(Y) \iff F : \texttt{Compact}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{DiniTheorem} :: \forall X : \texttt{Compact} \,.\, \forall f : \texttt{Monotonic}(C(X)) \,.\, \forall F : X \to \mathbb{R} \,.$

$\Big(\forall x \in X \,.\, \lim_{n\to\infty} f_i(x) = F(x) \,.\Big) \Rightarrow f \rightrightarrows F$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{StoneWeierstassTheorem} :: \forall X : \texttt{Compact} \,.\, \forall P : \texttt{Subalgebra}\Big(\mathbb{R}, C(X)\Big) \,.$

$. \, \texttt{SeparatesPoints}(X)(P) \Rightarrow P : \texttt{Dense}\Big(C(X), \texttt{uniformTopology}(X,\mathbb{R})\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

2.3 Compactifications

2.3.1 Subject

$$\texttt{Compactification} :: \prod X : \mathsf{TOP} \,.\, ?\sum K \in \mathsf{HC} \,.\, \texttt{HomeomorphicEmbedding}(X, K)$$

$$(K, \iota) : \texttt{Compactification} \iff \operatorname*{cl}_K \iota(X) = K$$

$\texttt{CompactificationIfTychonoff} :: \forall X \in \mathsf{TOP} \,.\, X : \texttt{Tychonoff} \iff \exists \texttt{Compactification}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactificationWeight} :: \forall X : \texttt{Tychonoff} \,.\, \exists (K, \iota) : \texttt{Compactification} : w(K) = w(X)$

$\texttt{Proof} =$

...

$\square$

$$\texttt{compactificationCategory} :: \mathsf{TOP} \to \mathsf{CAT}$$

$$\texttt{compactificationCategory}\,(X) = \mathcal{C}(X) :=$$

$$= \left(\texttt{Compactification}, (A, \alpha), (B, \beta) \mapsto \left\{ f : A \xrightarrow{\mathsf{TOP}} B \,.\, \alpha f = \beta \right\}, \circ, \mathrm{id}\right)$$

$\texttt{CompactifaticationCardinalityBound} :: \forall X \in \mathsf{TOP} \,.\, \forall (K, \iota) \in \mathcal{C}(X) \,.\, |K| \le \exp \exp d(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactificationWeightBound} :: \forall X \in \mathsf{TOP} \,.\, \forall (K, \iota) \in \mathcal{C}(X) \,.\, w(K) \le \exp d(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactificationCategoryIsPoset} :: \forall X \in \mathsf{TOP} \,.\, \mathcal{C}(X) : \texttt{Poset}$

$\texttt{Proof} =$

...

$\square$

$$\texttt{EquivalentCompactificationCriterion} :: \forall X \in \mathsf{TOP} \,.\, \forall (A, \alpha), (B, \beta) \in \mathcal{C}(X) \,.\, (A, \alpha) \cong_{\mathcal{C}(X)} (B, \beta) \iff$$

$$\iff \forall x, y : \texttt{Closed}(X) \,.\, \overline{\alpha x} \cap \overline{\alpha y} \iff \overline{\beta x} \cap \overline{\beta y} \,.$$

$\texttt{Proof} =$

...

$\square$

$\texttt{reminder} :: \prod X \in \mathsf{TOP} \,.\, \prod (K, \iota) : \texttt{Compactification}(X) \,.\, ?K$

$\texttt{reminder}\,() = \operatorname{rem}\iota := K \setminus \iota(X)$

$\texttt{CompactificationReminderTheorem} :: \forall X \in \mathsf{TOP} \,.\, \forall (A, \alpha), (B, \beta) \in \mathcal{C}(X) \,.\, \forall f : (A, \alpha) \xrightarrow{\mathcal{C}(X)} (B, \beta) \,.$
$\quad . \; f(\operatorname{rem}\alpha) = \operatorname{rem}\beta$

$\texttt{Proof} =$

...

$\square$

$\texttt{LocallyCompactCompactification1} :: \forall X : \texttt{Tychonoff} \,.\, X : \texttt{LocallyComapct} \iff \forall (K, \iota) \in \mathcal{C}(X) \,.$
$\quad . \; \operatorname{rem}\iota : \texttt{Closed}(K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LocallyCompactCompactification2} :: \forall X : \texttt{Tychonoff} \,.\, X : \texttt{LocallyComapct} \iff \exists (K, \iota) \in \mathcal{C}(X) :$
$\quad . \; \operatorname{rem}\iota : \texttt{Closed}(K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactificationLeastUpperBoundProperty} :: \forall X \in \mathsf{TOP} \,.\, \mathcal{C}(X) : \texttt{LUBProperty}$

$\texttt{Proof} =$

...

$\square$

$\texttt{compactificationOfStoneAndChech} :: \prod X : \texttt{Tychonoff} \,.\, \mathcal{C}(X)$

$\texttt{compatificationOfStoneAndChech}\,() = \beta X := \sup \mathcal{C}(X)$

$\texttt{AlexadroffCompactificationTHM} :: \forall X : \texttt{LocallyCompact} \,.\, \exists (K, \iota) \in \mathcal{C}(X) \,.\, |\operatorname{rem}\iota| = 1$

$\texttt{Proof} =$

...

$\square$

$\texttt{onePointCompactification} :: \prod X : \texttt{LocallyCompact} \,.\, \mathcal{C}(X)$

$\texttt{onePointCompactification}\,() = \omega X := \texttt{AexandroffCompactificationTheorem}(X)$

$\texttt{OnePointIsInf} :: \forall X : \texttt{LocallyComapact} \,!\, \texttt{Compact}\omega X = \inf \mathcal{C}(X)$

$\texttt{Proof} =$

...

$\square$

`LocallyCompactByOneMinimalCompactification` $:: \forall X :$ `Tychonoff` $. \forall (K, \iota) \in \mathcal{C}(X)$.
$(K, \iota) = \inf \mathcal{C}(X) \Rightarrow X :$ `LocallyCompact`

`Proof` $=$

$\ldots$

$\square$

`ReconstructionByReminder` $:: \forall Y \in \mathsf{HC} . \forall X :$ `LocallyCompact` $. \forall (K, \iota) \in \mathcal{C}(X) . \forall f : \operatorname{rem} \iota \xrightarrow{\mathsf{TOP}} Y$.
$f(\operatorname{rem} \iota) = Y \Rightarrow \exists (Y', \gamma) : \operatorname{rem} \gamma = Y$

`Proof` $=$

$\ldots$

$\square$

2.3.2 Stone-Čech Functor

$\texttt{StoneCechSpace} :: \prod X \in \mathsf{TOP} \,.\, \sum \Omega : \mathsf{HC} \,.\, X \xrightarrow{\mathsf{TOP}} \Omega$

$(\Omega, \varphi) : \texttt{StoneCechSpace} \iff \forall K \in \mathsf{HC} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} K \,.\, \exists g : \Omega \xrightarrow{\mathsf{HC}} K : f = \varphi g$

$\texttt{StoneCechSpaceExists} :: \forall X \in \mathsf{TOP} \,.\, \exists \texttt{StoneCechSpace}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechSpaceHomeo} :: \forall X \in \mathsf{TOP} \,.\, \forall (A, \varphi), (B, \psi) : \texttt{StoneCechSpace}(X) \,.\, A \cong_{\mathsf{HC}} B$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechFunctor} :: \mathsf{TOP} \xrightarrow{\mathsf{CAT}} \mathsf{HC}$

$\texttt{StoneCechFunctor}\,(X) = \beta X := \texttt{StonCechSpaceExists} \,\&\, \texttt{StoneCechSpaceHome}(X)$

$\texttt{StoneCechFunctor}\,(X, Y, f) = \beta f := \eth\texttt{StoneCechSpace}(\omega X)(f \varphi_Y)$

$\texttt{StoneCechConsistancy} :: \forall X : \texttt{Tychonoff} \,.\, \texttt{CompacticationOfStoneAndChech}(X) : \texttt{StoneCechSpace}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechAdjoint} :: \beta : \texttt{LeftAdjoint}(U_{\mathsf{HC},\mathsf{TOP}})$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompleteSeparationInStoneCech} :: \forall X \in \mathsf{TOP} \,.\, \forall (A, B) : \texttt{CompletelySeparated}(X) \,.$

$. \, (\overline{\varphi_X A}, \overline{\varphi_X B}) : \texttt{CompletelySeparated}(\beta X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechByCompleteSeparation} :: \forall X \in \mathsf{TOP} \,.\, \forall (K, \iota) \in \mathcal{C}(X) \,.$

$. \, \forall [1] : \forall (A, B) : \texttt{CompletelySeparated}(X) \,.\, (\overline{\iota A}, \overline{\iota B}) : \texttt{CompletelySeparated}(K) \,.\, (K, \iota) \cong_{\mathcal{C}(X)} (\beta X, \varphi_X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechClopenSubset} :: \forall X : \texttt{Tychonoff} \,.\, \forall A : \texttt{Clopen}(X) \,.\, \overline{\varphi_X A} : \texttt{Clopen}(\beta X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneCechSubspaceCompactification} :: \forall X : \texttt{Tychonoff} . \forall A \subset X .$

$. \forall [0] : \forall f : A \xrightarrow{\mathsf{TOP}} [0,1] . \exists F : X \xrightarrow{\mathsf{TOP}} [0,1] : F_{|A} = f . (\operatorname*{cl}_{\beta X} A, \varphi_x) \in \mathcal{C}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NormalStoneCechSubspaceCompactification} :: \forall X : \texttt{T4} . \forall A \subset X \ \operatorname*{cl}_{\beta X} A \cong_{\mathsf{TOP}} \beta A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{StoneCechSuperspaceCompactification} :: \forall X : \texttt{Tychonoff} . \forall A \subset \beta X . X \subset A \Rightarrow \beta A = \beta X$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{DiscreteStoneCechCardinality} :: \forall X \in \mathsf{SET} . |\beta\ D\ X| = \exp\exp|X|$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{DiscreteStoneCechWeight} :: \forall X \in \mathsf{SET} . w(\beta\ D\ X) = \exp|X|$

$\texttt{Proof} =$

$\texttt{ClopenSubsetInDiscreteStoneCech} :: \forall X \in \mathsf{SET} . \forall x \in \beta\ D\ X . \forall U \in \mathcal{U}_{\beta\ D\ X}(x) .$

$. \exists V : \texttt{Clopen}(\beta\ D\ X) : V \subset U$

$\texttt{Proof} =$

$\Big(W, [1]\Big) := \texttt{AltT4}(\beta\ D\ X, U) : \sum W : \texttt{Open}(\beta\ D\ X) . \overline{W} \subset U,$

$A := \varphi_{DX}\varphi^{-1}_{DX}(W) :?\beta\ D\ X,$

$[2] := \eth\texttt{StoneCechClopenSubset}(X, A) : \Big(\overline{A} : \texttt{Clopen}(\varphi_X X)\Big),$

$[3] := \jmath A\eth\texttt{preimage} : A \subset W,$

$[4] := \texttt{MonotonicClosure}(\beta\ D\ X)[3][1] : \overline{A} \subset \overline{W} \subset U;$

$\square$

$\texttt{NaturalNumbersStoneCechSelfsimmilarity} :: \forall A : \texttt{Closed}\ \&\ \texttt{Infinite}(\beta\mathbb{N}) . \exists B \subset A : B \cong_{\mathsf{TOP}} \beta\mathbb{N}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NaturalStoneCechConvergentSequences} :: \forall x : \texttt{Convergent}(\mathbb{N}, \beta\mathbb{N}) . x : \texttt{FinallyConstant}$

$\texttt{Proof} =$

$\dots$

$\square$

2.3.3 Wallman Extension

$\texttt{FilterDisownesEmptySet} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{X} \in ??X \,.\, \forall F : \texttt{Filter}(\mathcal{X}) \,.\, \emptyset \notin F$

$\texttt{Proof} \ =$

...

□

$\texttt{FilterIntersectionClosed} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{X} \in ??X \,.\, \forall F : \texttt{Filter}(\mathcal{X}) \,.\, \forall A, B \in F \,.\, A \cap B \in F$

$\texttt{Proof} \ =$

...

□

$\texttt{GreedyUltrafilters} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{X} \in ??X \,.\, \forall F : \texttt{Ultrafilter}(\mathcal{X}) \,.\, \forall A \in \mathcal{X} \,.\, \forall [1] : \forall B \in F \,.\, A \cap B \neq \emptyset \,.$

$\texttt{Proof} \ =$

...

□

$\texttt{UltrafilterSupercomplete} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{X} \in ??X \,.\, \forall F : \texttt{Ultrafilter}(\mathcal{X}) \,.\, \forall A \in F \,.\, \forall B \in \mathcal{X} \,.\, A \subset B \Rightarrow$

$\texttt{Proof} \ =$

...

□

$\texttt{DifferentUltrafilters} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{X} \in ??X \,.\, \forall F, G : \texttt{Ultrafilter}(\mathcal{X}) F \neq G \iff \exists A \in F : \exists B \in G :$

$\texttt{Proof} \ =$

$\texttt{PrincipleUltrafilter} :: \prod X \in \mathsf{SET} \,.\, \prod \mathcal{X} \in ??X \,.\, ?\texttt{Ultrafilter}(\mathcal{X})$

$$F : \texttt{PrincipleUltrafilter} \iff \exists x \in X : \bigcap_{A \in F} A = \{x\}$$

$\texttt{NonPrincipleUltrafilter} :: \prod X \in \mathsf{SET} \,.\, \prod \mathcal{X} \in ??X \,.\, ?\texttt{Ultrafilter}(\mathcal{X})$

$$F : \texttt{NonPrincipleUltrafilter} \iff \bigcap_{A \in F} A = \emptyset$$

$\texttt{T1UltrafilterClassification} :: \forall X : \texttt{T1} \,.\, \forall F : \texttt{Ultrafilter}\ \texttt{Closed}(X) \,.$

$\quad .\, F : \texttt{PrincipleUltrafilter}\ \texttt{Closed}(X) \Big| F : \texttt{NonPrincipleUltrafilter}\ \texttt{Closed}(X)$

$\texttt{Proof} \ =$

...

□

$\texttt{WallmanExtension} :: \texttt{T1} \to \texttt{T1} \,\&\, \texttt{Compact}$

$$\texttt{WallmanExtension}\,(X) = W(X) := \left\langle \left\{ \{F : \texttt{Ultrafilter Closed}(X) : \exists A \in F : A \subset U\} \middle| U \in \mathcal{T}(X) \right\} \right\rangle_{\mathsf{TOP}}$$

$\texttt{WallmanEmbedding} :: \prod X \in \texttt{T1} \,.\, X \xrightarrow{\mathsf{TOP}} W(X)$

$$\texttt{WallmanEmbedding}\,(X) = w_X := \left\{ A : \texttt{Closed}(X) : x \in X \right\}$$

$\texttt{WallmanExtensionTheorem} :: \forall X : \texttt{T1} \,.\overline{w(X)} = W(X)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{WallmanExtensionUniversality} :: \forall X : \texttt{T1} \,.\, \forall Z : \texttt{Compact} \,.\, \forall f : X \xrightarrow{\mathsf{TOP}} Z \,.\, \exists g : W(X) \xrightarrow{\mathsf{TOP}} Z : f = wg$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{WallmanExtensionRegularity} :: \forall X : \texttt{T1} \,.\, W(X) : \texttt{T2} \iff X : \texttt{T4}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{WallmanStoneCechEquivalence} :: \forall X : \texttt{T4} \,.\, W(X) \cong_{\mathsf{TOP}} \beta X$

$\texttt{Proof} \; =$

$\dots$

$\square$

2.3.4 Perfect Mappings

$$\texttt{Perfect} :: \prod X : \texttt{T2} \, . \, \prod Y \in \mathsf{TOP} \, . \, ?\texttt{Closed}(X, Y)$$
$$f : \texttt{Perfect} \iff \forall y \in Y \, . \, f^{-1}(y) : \texttt{Compact}$$

$\texttt{PerfectInjection} :: \forall X : \texttt{T2} \, . \, \forall Y \in \mathsf{TOP} \, . \, \forall f : \texttt{Injection}(X, Y) \, .$
$\quad . \, f : \texttt{Perfect}(X, Y) \iff f : \texttt{Closed}(X, Y)$
$\texttt{Proof} \; =$
$\ldots$
$\square$

$\texttt{PerfectInclusion} :: \forall X : \texttt{T2} \, . \, \forall A \subset X \, . \, \iota_A : \texttt{Perfect}(A, X) \iff A : \texttt{Closed}(X)$
$\texttt{Proof} \; =$
$\ldots$
$\square$

$\texttt{PerfectProjection} :: \forall X \in \mathsf{HC} \, . \, \forall Y : \texttt{T2} \, . \, \pi_Y : \texttt{Perfect}(X \times Y, Y)$
$\texttt{Proof} \; =$
$\ldots$
$\square$

$\texttt{CompactPerfectPreimage} :: \forall X : \texttt{T2} \, . \, \forall Y \in \mathsf{TOP} \, . \, \forall f : \texttt{Perfect}(X, Y) \, . \, \forall K \subset Y \, . \, \forall [0] : K \in \mathsf{HC} \, .$
$\quad . \, f^{-1}(K) \in \mathsf{HC}$
$\texttt{Proof} \; =$
$[1] := \texttt{HausdorffSubset}(X, f^{-1}(K)) : \left(f^{-1}(K) : \texttt{T2}\right),$
$\texttt{Assume} \; A : \texttt{Filter} \; \texttt{Closed}(f^{-1}(K)),$
$[2] := \eth\texttt{Closed}(X, Y)(f) : f(A) \in ?\texttt{Closed}(Y),$
$[3] := \texttt{ImageIntersection}(f) : \left(f(A) : \texttt{FiniteIntersectionProperty}(K)\right),$
$[4] := \texttt{CompactByFiniteIntersection}[3] : \bigcap_{a \in A} f(a) \neq \emptyset,$
$y) := \eth\texttt{NonEmpty}[4] : \bigcap_{a \in A} f(a),$
$[5] := \eth\texttt{Perfect}(X, Y)(f) : \left(f^{-1}(y) : \texttt{Compact}\right),$
$B := A \cap f^{-1}(y) : ?\texttt{Closed}(f^{-1}(x)),$
$[6] := \jmath B \jmath y : \forall b \in B \, . \, b \neq \emptyset,$
$[7] := \texttt{FilterRestriction}[6](A, B) : \left(B : \texttt{Filter}\left(\texttt{Closed}(f^{-1}y)\right)\right),$
$[8] := \eth\texttt{Filter}(B)\eth^{-1}\texttt{FiniteIntersection} : \left(B : \texttt{FiniteIntersectionProperty}\left(\texttt{Closed}(f^{-1}y)\right)\right),$
$[9] := \texttt{CompactByFiniteIntersection}(B) : \bigcap B \neq \emptyset,$
$[*] := \texttt{SubsetIntersectionNonEmpty}(A, B)[9] : \bigcap A \neq \emptyset;$
$\rightsquigarrow [*] := \texttt{CompactByFilterPrincipality} : \left(f^{-1}K : \texttt{Compact}\right);$
$\square$

$\texttt{PerfectComposition} :: \forall X, Y : \texttt{T2} . \forall Z \in \mathsf{TOP} . \forall f : \texttt{Perfect}(X, Y) . \forall g : \texttt{Perfect}(Y, X) .$
$\quad . fg : \texttt{Perfect}(X, Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ z : Z,$

$[1] := \eth\texttt{Perfect}(g)(z) : \left(g^{-1}(z) : \texttt{Compact}(Y)\right),$

$[2] := \texttt{CompactPerfectPreimage}(f)[1] : \left(f^{-1}g^{-1}(z) : \texttt{Compact}(X)\right),$

$[z.*] := \texttt{PreimageComposition}(f, g, z)[2]) : \left((fg)^{-1}(z) : \texttt{Compact}(X)\right);$

$\leadsto [*] := \eth^{-1}\texttt{Perfect}(X, Z) : \left(fg : \texttt{Perfect}\right);$

$\square$

$\texttt{PerfectRestriction} :: \forall X : \texttt{T2} . \forall Z \in \mathsf{TOP} . \forall f : \texttt{Perfect}(X, Y) .$
$\quad . \forall A : \texttt{Closed}(X) . f_{|A} : \texttt{Perfect}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{PerfectProduct} :: \forall I \in \mathsf{SET} . \forall X : I \to \texttt{T2} . \forall Y : I \to \mathsf{TOP} .$
$\quad . \forall f : \prod_{i \in I} \to X_i \xrightarrow{\mathsf{TOP}} Y_i .$
$\quad . \prod_{i \in I} f_i : \texttt{Perfect}\left(\prod_{i \in I} X_i, \prod_{i \in I} Y_i\right) \iff \forall i \in I . f_i : \texttt{Perfect}(X_i, Y_i)$

$\texttt{Proof} \ =$

$\dots$

$\square$

2.3.5 Abstract Baire Theory

$\texttt{CechCompleteSpace} :: ?\texttt{Tychonoff}$

$X : \texttt{CechCompleteSpace} \iff \forall (K, \gamma) \in \mathcal{K}(X) \ . \ \mathrm{rem}\, \gamma : F_\sigma(K)$

$\texttt{BaireSpace} :: ?\mathsf{TOP}$

$$X : \texttt{BaireSpace} \iff \forall A : \mathbb{N} \to \texttt{NowhereDense}(X) \ . \ \bigcup_{i=1}^{\infty} A_i : \texttt{Codense}(X)$$

$\texttt{BaireCategoryTheorem} :: \forall X : \texttt{CechCompleteSpace} \ . \ X : \texttt{BaireSpace}$

$\texttt{Proof} =$

...

$\square$

$$\texttt{DualBaireProperty} :: \forall X : \texttt{BaireSpace} \ . \ \forall U : \mathbb{N} \to \mathcal{T} \ \& \ \texttt{Dense}(X) \ . \ \bigcap_{i=1}^{\infty} U_i : \texttt{Dense}(X)$$

$\texttt{Proof} =$

...

$\square$

2.3.6 Realcompact Spaces

$\texttt{Realcompact} :: ?\mathsf{TOP}$

$X : \texttt{Realcompact} \iff \exists \kappa \in \mathsf{CARD} : \exists A : \texttt{Closed}(\mathbb{R}^\kappa) : \exists \varphi : \texttt{HomeomorphicEmbedding}(X, A)$

3 Cardinal Functions