Linear Modules

Uncultured Tramp

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$$MAN = \begin{bmatrix} 23 & & & & \\ & 23 & & & \\ & & 23 & & \\ & & & 46 & \\ & & & & \end{bmatrix}$$

Contents

1 Basic Categorical Module theory

1.1 Category of Modules

$$\texttt{LeftModule} :: \prod R \in \mathsf{RING} \;.\; \sum M \in \texttt{Set} \;.\; (M \to M \to M) \times (R \to M \to M)$$

$$M, +, \cdot : \texttt{LeftModule} \iff (M, +) \in \mathsf{ABEL}\ \&$$
$$\&\ \forall a, b \in M \;.\; \forall \alpha \in M \;.\; \alpha(a + b) = \alpha a + \alpha b\ \&$$
$$\&\ \forall a \in M \;.\; \forall \alpha, \beta \in M \;.\; (\alpha + \beta)a = \alpha a + \beta b\ \&$$
$$\&\ \forall a \in M \;.\; \forall \alpha, \beta \in M \;.\; \beta(\alpha a) = (\beta\alpha)a\ \&$$
$$\&\ \forall a \in M \;.\; 1a = a$$

$$\texttt{RightModule} :: \prod R \in \mathsf{RING} \;.\; \sum M \in \mathsf{SET} \;.\; (M \to M \to M) \times (M \to R \to M)$$

$$(M, +, \cdot) : \texttt{RightModule} \iff \Big(M, +, \texttt{swap}(\cdot)\Big) : \texttt{LeftModule}(R^{\mathrm{op}})$$

$\texttt{Assume}\ R : \texttt{Ring},$

$$\texttt{moduleAsGroup} :: \texttt{LeftModule}(R) \to \mathsf{ABEL}$$
$$\texttt{moduleAsSet}\,((M, +, \cdot)) = \texttt{implicit} := (M, +)$$

$$\texttt{scalarMult} :: \prod M : \texttt{LeftModule}(R) \;.\; M \to M \to M$$
$$\texttt{scalarMult}\,(M, +, \cdot) = \cdot_M := \cdot$$

$$\texttt{ZeroMultInModule} :: \forall M : \texttt{LeftModule}(R) \;.\; \forall a \in M \;.\; 0a = 0$$

$\texttt{Proof} =$

$(1) := \eth\texttt{LeftModule}(R)(M)\eth\texttt{Neutral}(R, +)(0)(1)\eth\texttt{LeftModule}(R)(M) :$
$\quad : a = 1a = (1 + 0)a = 1a + 0a = a + 0a,$

$(2) := \eth\texttt{Neutaral}(M, +)(0)(a) : a + 0 = 0,$

$(*) := \texttt{TotalGroupMult}(1)(2) : 0a = 0;$

$\square$

$$\texttt{NegativeMultInModule} :: \forall M : \texttt{LeftModule}(R) \;.\; \forall a \in M \;.\; (-1)a = -a$$

$\texttt{Proof} =$

$(1) := \texttt{ZeroMultInModule}(a)\eth\texttt{inverse}(R, +)(1)\eth\texttt{RightModule}(M) : 0 = 0a = (1 - 1)a = 1 + (-1)a,$

$(2) := \eth\texttt{inverse}(a) : 0 = a + (-a),$

$(*) := \texttt{TotalGroupMult}(1)(2) : (-1)a = -a;$

$\square$

$$\texttt{LeftLinear} :: \prod A, B : \texttt{LeftModule}(R) \;.\; ?(A \xrightarrow{\mathsf{ABEL}} B)$$
$$T : \texttt{LeftLinear} \iff \forall a \in A \;.\; \forall \theta \in R \;.\; T(\theta a) = \theta T(a)$$

$$\texttt{RightLinear} :: \prod A, B : \texttt{LeftModule}(R) \;.\; ?(A \xrightarrow{\mathsf{ABEL}} B)$$
$$T : \texttt{RightLinear} \iff \forall a \in A \;.\; \forall \theta \in R \;.\; T(a\theta) = T(a\theta) = T(a)\theta$$

$\texttt{IdIsiLeftLinear} :: \forall A : \texttt{LeftModule}(R) \,.\, \mathrm{id}_A : \texttt{LeftLinear}(A, A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{IdIsRightLinear} :: \forall A : \texttt{RightModule}(R) \,.\, \mathrm{id}_A : \texttt{RightLinear}(A, A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LeftLinearComp} :: \forall A, B, C : \texttt{LeftModule}(R) \,.\, \forall T : \texttt{LeftLinear}(A, B) \,.\, \forall S : \texttt{LeftLinear}(B, C) \,.$
$\quad .\, S \circ T : \texttt{LeftLinear}(A, C)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ \theta : R,$

$a := \eth\texttt{compose}(S, T)\eth\texttt{LeftLinearComp}(T)\eth\texttt{LeftLinear}(S)\eth^{-1}\texttt{compose}(S, T) :$
$\quad : S \circ T(\theta a) = S\Big(T(\theta a)\Big) = S\Big(\theta T(a)\Big) = \theta S\Big(T(a)\Big) = \theta S \circ T(a);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{RightLinear} : \Big[S \circ T : \texttt{RightLinear}(A, C)\Big];$

$\square$

$\texttt{RightLinearComp} :: \forall A, B, C : \texttt{RightModule}(R) \,.\, \forall T : \texttt{RightLinear}(A, B) \,.\, \forall S : \texttt{RightLinear}(B, C) \,.$
$\quad .\, S \circ T : \texttt{RightLinear}(A, C)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LeftModuleCategory} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{LeftModuleCategory}\,(R) = R\text{-}\mathsf{MOD} := \Big(\texttt{LeftModule}(R), \texttt{LeftLinear}, \mathrm{id}, \circ\Big)$

$\texttt{RightModuleCategory} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{RightModuleCategory}\,(R) = \mathsf{MOD}\text{-}R := \Big(\texttt{RightModule}(R), \texttt{RightLinear}, \mathrm{id}, \circ\Big)$

$\texttt{ABELIsZMOD} :: \mathsf{ABEL} \cong_{\mathsf{CAT}} \mathsf{MOD}\text{-}\mathbb{Z}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CommutativeLeftModuleIsRight} :: \forall R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \cong_{\mathsf{CAT}} \mathsf{MOD}\text{-}R$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LinearMapsAreModule} :: \forall R \in \mathsf{ANN} . \forall A, B \in R\text{-}\mathsf{MOD} . A \xrightarrow{R\text{-}\mathsf{MOD}} B \in R\text{-}\mathsf{MOD}$

$\texttt{Proof} =$

$\texttt{Assume}\ T, S : A \xrightarrow{R\text{-}\mathsf{MOD}} B,$

$\texttt{Assume}\ a, a' : A,$

$() := \eth\texttt{mapOp}(B, +)(T, S)\eth(\text{-}\mathsf{MOD}R)(T, S)\eth^{-1}\texttt{mapOp}(B, +)(T, S) :$

$: (T + S)(a + a') = T(a + a') + S(a + a') = T(a) + T(a') + S(a) + S(a') = (T + S)(a) + (T + S)(a');$

$\rightsquigarrow (1) := \eth\mathsf{ABEL} : \left[T + S : A \xrightarrow{\mathsf{ABEL}} B\right],$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ \theta : R,$

$() := \eth\texttt{mapOp}(B, +)(T, S) :$

$: (T + S)(\theta a) = T(\theta a) + S(\theta a) = \theta T(a) + \theta S(a) = \theta\Big(T(a) + S(a)\Big) = \theta(T + S)(a);$

$\rightsquigarrow () := \eth R\text{-}\mathsf{MOD}(1) : \left[T + S : A \xrightarrow{R\text{-}\mathsf{MOD}} B\right];$

$\rightsquigarrow (1) := I(\forall) : \forall T, S \in (\text{-}\mathsf{MOD}R) . T + S \in (\text{-}\mathsf{MOD}R),$

$\texttt{Assume}\ T : (\text{-}\mathsf{MOD}R),$

$\texttt{Assume}\ \theta : R,$

$\texttt{Assume}\ a, a' : A,$

$() := \eth\texttt{mapOp}(B, \cdot)(\theta, T)\eth(\text{-}\mathsf{MOD}R)(T)\eth^{-1}\texttt{mapOp}(B, \cdot)(\theta, T) :$

$: (\theta T)(a + a') = \theta\Big(T(a + a')\Big)) = \theta\Big(T(a) + T(a') = \theta T(a) + \theta T(a');$

$\rightsquigarrow (2) := \eth\mathsf{ABEL} : \left[\theta T : A \xrightarrow{\mathsf{ABEL}} B\right];$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ \rho : R,$

$() := \eth\texttt{mapOp}(B, \cdot)\eth(\text{-}\mathsf{MOD}T)\eth\mathsf{ANN}(R)\eth^{-1}\texttt{mapOp}(B, \cdot)(\theta, T) :$

$: (\theta T)(\rho a) = \theta(T(\rho a)) = \theta\rho T(a) = \rho\theta T(a);$

$\rightsquigarrow () := \eth R\text{-}\mathsf{MOD}(2) : \left[\theta T : A \xrightarrow{R\text{-}\mathsf{MOD}} B\right];$

$\rightsquigarrow (2) := I(\forall) : \forall T : A \xrightarrow{R\text{-}\mathsf{MOD}} B . \forall\theta \in R . \theta T : A \xrightarrow{R\text{-}\mathsf{MOD}} B,$

$(*) := \eth(\text{-}\mathsf{MOD}R)(1)(2) : \texttt{This};$

$\square$

$\texttt{Submodule} :: \prod M \in R\text{-}\mathsf{MOD} . ??M$

$S : \texttt{Submodule} \iff S \subset_{R\text{-}\mathsf{MOD}} M \iff (S, +_M, \cdot_M) \in R\text{-}\mathsf{MOD}$

$\texttt{LinearImageIsSubmodule} :: \forall A, B \in R\text{-}\mathsf{MOD} . \forall T : A \xrightarrow{R\text{-}\mathsf{MOD}} B . \operatorname{Im} T \subset_{R\text{-}\mathsf{MOD}} B$

$\texttt{Proof} =$

$(1) := \texttt{GroupImage} : \operatorname{Im} T \subset_{\mathsf{GRP}} B,$

$\texttt{Assume}\ y : \operatorname{Im} T,$

$\Big(x, (2)\Big) := \eth\texttt{image}(T) : \sum x \in A . Tx = y,$

$\texttt{Assume}\ \rho : R,$

$(3) := (2)(\rho y)\eth R\text{-}\mathsf{MOD}(A, B)(T)(x, \rho) : \rho y = \rho Tx = T\rho x,$

$() := \eth\texttt{image}(T)(3) : \rho y \in \operatorname{Im} T;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Suvset}(B)(1) : \operatorname{Im} T \subset_{R\text{-}\mathsf{MOD}} B;$

$\square$

$\texttt{LinearConstriction} :: \forall A, B \in R\text{-MOD} \,.\, \forall S \subset_{R\text{-MOD}} A \,.\, \forall T : A \xrightarrow{R\text{-MOD}} B \,.\, T_{|S} : S \xrightarrow{R\text{-MOD}} B$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{SubspacrLinearImageIsSubspace} :: \forall A, B \in R\text{-MOD} \,.\, \forall S \subset_{R\text{-MOD}} A \,.\, \forall T : A \xrightarrow{R\text{-MOD}} B \,.$
$\,.\, T(S) \subset_{R\text{-MOD}} B$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{LinearPreimageiIsSubmodule} :: \forall A, B \in R\text{-MOD} \,.\, \forall S \subset_{R\text{-MOD}} B \,.\, \forall T : A \xrightarrow{R\text{-MOD}} B \,.$
$\,.\, T^{-1}(S) \subset_{R\text{-MOD}} A$

$\texttt{Proof} \ =$

$(1) := \texttt{GroupPreimage}(T, S) : T^{-1}(S) \subset_{\text{GRP}} A,$

$\texttt{Assume}\ x : T^{-1}S,$

$(2) := \eth T^{-1}S(x) : T(x) \in S,$

$\texttt{Assume}\ \rho : R,$

$(3) := \eth R\text{-MOD}(A, B)T(x, \rho)\eth\texttt{Submodule}(B)(S)(2)(\rho, T(x)) : T(\rho x) = \rho T(x) \in S,$

$() := \eth T^{-1}S(3) : \rho x \in T^{-1}S;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Submodule}(A)(1) : T^{-1}S \subset_{R\text{-MOD}} A;$

$\square$

$\texttt{ZeroModule} :: \Big(\{\star\}, (\star, \star) \mapsto \star, (\rho, \star) \mapsto \star\Big) \in R\text{-MOD}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{zeroModule} :: R\text{-MOD}$

$\texttt{zeroModule}\,() = \star := \Big(\{\star\}, (\star, \star) \mapsto \star, (\rho, \star) \mapsto \star\Big)$

$\texttt{ZeroElementStable} :: \forall M \in R\text{-MOD} \,.\, \forall \alpha \in R \,.\, \alpha 0 = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : M \cong_{R\text{-MOD}} \star,$

$(2) := \eth\star : \alpha 0 = 0;$

$\rightsquigarrow (1) := I(\rightarrow) : M \cong_{LMODR} \star \Rightarrow \alpha 0 = 0,$

$\texttt{Assume}\ (2) : M \ncong_{R\text{-MOD}} \star,$

$\Big(m, (3)\Big) := \eth \star (2) : \sum m \in M \,.\, m \neq 0,$

$(4) := \eth R\text{-MOD}\eth\texttt{Neutral}(M, +)(0)(m) : \alpha m + \alpha 0 = \alpha(m + 0) = \alpha m,$

$(5) := \eth\texttt{Neutral}(M, +)(0)(m) : \alpha m + 0 = \alpha m,$

$() := \texttt{GroupTotalMult}(4)(5) : \alpha 0 = 0;$

$\square$

$\texttt{ZeroSubmodule} :: \forall M \in R\text{-MOD} . \{0\} \subset_{R\text{-MOD}} M$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{kerIsSubmodule} :: \forall A, B \in R\text{-MOD} . \forall T : A \xrightarrow{R\text{-MOD}} B . \ker T \subset_{R\text{-MOD}} A$

$\texttt{Proof} =$

$(1) := \eth \ker T : \ker T = T^{-1}\{0\},$

$(*) := \texttt{LinearPreimageIsSubmodule}(T)\texttt{ZeroSubmodule}(B)(1) : \ker T \subset_{R\text{-MOD}} A;$

$\square$

$\texttt{quotScalarMult} :: \prod M \in R\text{-MOD} . \prod S \in S\text{-MOD} . R \to \frac{M}{S} \to \frac{M}{S}$

$\texttt{quotScalarMult}\,(\rho, [m]) = \rho[m] := [\rho m]$

$\texttt{Assume}\ s : S,$

$(*) := \eth\texttt{quotScalarMult}(m+s)\eth R\text{-MOD}(M)\eth\texttt{Submodule}(S)(\rho, s)\eth\texttt{quotientGroup}(M, S) :$

$: \rho[m+s] = \big[\rho(m+s)\big] = [\rho m + \rho s] = [\rho m] = \rho[m];$

$\square$

$\texttt{QuotientModule} :: \forall M \in R\text{-MOD} . \forall S \subset_{R\text{-MOD}} M . \left(\frac{S}{M}, +_{\frac{S}{M}}, \texttt{quotScalarMult}\right)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LinearSubmoduleProjection} :: \forall M \in R\text{-MOD} . \forall S \subset_{R\text{-MOD}} M . \pi_S : M \xrightarrow{R\text{-MOD}} \frac{S}{M}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SubmoduleProjUP} :: \forall M \in R\text{-MOD} . \forall S \subset_{R\text{-MOD}} M . \forall N \in R\text{-MOD} . \forall T : M \xrightarrow{R\text{-MOD}} N .$

$. \forall (0) : S \subset \ker T . \exists! T' : \frac{S}{M} \xrightarrow{R\text{-MOD}} N . \pi_S T' = T$

$\texttt{Proof} =$

$\big(T', (1)\big) := \texttt{SubgroupProjUP} : \sum T' : \frac{M}{S} \xrightarrow{\text{GRP}} N . \pi_S T' = T\ \&$

$\&\ \forall T'' : \frac{M}{S} \xrightarrow{\text{GRP}} N . \pi_S T'' = T' \Rightarrow T' = T'',$

$\texttt{Assume}\ [m] : \frac{M}{S},$

$\texttt{Assume}\ \alpha : R,$

$(2) := \eth^{-1}\pi_S\eth\texttt{quotScalarMult}\eth(1)\eth R\text{-MOD}(M, N)(T)(m, \alpha) : T'\alpha[m] = \pi_S T'(\alpha m) = T(\alpha m) = \alpha T(m),$

$(3) := \eth^{-1}\pi_S(1) : T'[m] = \pi_S T'(m) = T(m),$

$() := (2)(3) : T'\alpha[m] = \alpha T'[m];$

$\rightsquigarrow (2) := \eth R\text{-MOD} : [T' : \frac{M}{S} \xrightarrow{R\text{-MOD}} N],$

$(*) := (1)(2) : \texttt{This};$

$\square$

$\texttt{IntersectionOfSubmodule} :: \forall N \in R\text{-MOD} \,.\, \prod I \in \mathsf{SET} \,.\, \forall S : I \to \texttt{Submodule}(M) \,.\, \bigcap_{i\in I} S_i \subset_{R\text{-MOD}} M$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{SumOfSubmodules} :: \forall M \in R\text{-MOD} \,.\, \prod I \in \mathsf{SET} \,.\, \forall S : I \to \texttt{Submodule}(M) \,.\, \sum_{i\in I} S_i \subset_{R\text{-MOD}} M$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{UnionOfSubmodules} :: \forall M \in R\text{-MOD} \,.\, \prod I : \texttt{Toset} \,.\, \forall S : \texttt{Nondecreasing}(M)(I, \texttt{Submodule}(M)) \,.\, \bigcup_{i\in I} S_i \subset_{R\text{-MOD}} M$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{Simple} :: ?R\text{-MOD}$

$M : \texttt{Simple} \iff \left\{M, \{0\}\right\} = \texttt{Submodule}(M)$

$\texttt{LinearInverse} :: \forall A, B : \texttt{LMOD}R \,.\, \forall T : A \xrightarrow{R\text{-MOD}} B \ \&\ A \xleftrightarrow{\mathsf{SET}} B \,.\, T^{-1} : A \xleftrightarrow{R\text{-MOD}} B$

$\texttt{Proof} \ =$

$\texttt{Assume}\ y : B,$

$\texttt{Assume}\ \alpha : R,$

$x := T^{-1}(y) : A,$

$(1) := \eth R\text{-MOD}(A, B)\eth x : T(\alpha x) = \alpha T(x) = \alpha y,$

$() := \eth\texttt{Inverse}(T)(1)\eth x : T^{-1}(\alpha y) = \alpha x = \alpha T^{-1}(y);$

$\rightsquigarrow (*) := \eth R\text{-MOD} : [T^{-1} : A \xleftrightarrow{R\text{-MOD}} B];$

$\square$

$\texttt{SchurLemma} :: \forall A, B : \texttt{Simple} \,.\, \forall T : A \xrightarrow{R\text{-MOD}} B \,.\, \forall (0) : T \neq 0 \Rightarrow T : A \xleftrightarrow{M\text{-MOD}} B$

$\texttt{Proof} \ =$

$(1) := \texttt{LinearImageIsSubmodule} : T(A) \subset_{R\text{-MOD}} B,$

$(2) := \eth\texttt{Simple}(A, B)(1)(0) : T(A) = B,$

$(3) := \texttt{KerIsSubmodule}(T) : \ker T \subset_{R\text{-MOD}} A,$

$(4) := \eth\texttt{Simple}\eth \ker T(0)(3) : \ker T = \{0\},$

$(*) := \eth^{-1}\texttt{Iso}(R\text{-MOD})\texttt{InjectiveByKernel}(M, N, T)(4)(2) : [T : A \xleftrightarrow{R\text{-MOD}} B];$

$\square$

$\texttt{LinearMapsFromTheRing} :: \forall A \in \mathsf{ANN} \,.\, \forall M \in A\text{-}\mathsf{MOD} \,.\, \mathcal{M}_{A\text{-}\mathsf{MOD}}(A, M) \cong_{A\text{-}\mathsf{MOD}} M$

$\texttt{Proof} \ =$

$\varphi := \Lambda T : A \xrightarrow{A\text{-}\mathsf{MOD}} M \,.\, T(1) : \mathcal{M}_{A\text{-}\mathsf{MOD}}(A, M) \xrightarrow{A\text{-}\mathsf{MOD}} M,$

$\texttt{Assume}\ T : A \xrightarrow{A\text{-}\mathsf{MOD}} M,$

$\texttt{Assume}\ (1) : \varphi(T) = 0,$

$(2) := \eth\varphi(0) : T(1) = 0,$

$\texttt{Assume}\ a : A,$

$() := \eth A\text{-}\mathsf{MOD}(A, M)(T)() : T(a) = aT(1) = a0 = 0;$

$\rightsquigarrow () := E(\rightarrow, =) : T = 0;$

$\rightsquigarrow (1) := \texttt{InjevtiveByKernel} : [T : A \hookrightarrow M],$

$\texttt{Assume}\ m : M,$

$T := \Lambda a \in A \,.\, am : A \xrightarrow{A\text{-}\mathsf{MOD}} M,$

$() := \eth\varphi(T) : \varphi(T) = m;$

$\rightsquigarrow (2 := \eth\texttt{Iso}(R\text{-}\mathsf{MOD})(1) : [\varphi : A \xleftrightarrow{A\text{-}\mathsf{MOD}} M],$

$(*) := \eth\texttt{Isomotphic}(2) : \mathcal{M}_{A\text{-}\mathsf{MOD}}(A, M) \cong_{A\text{-}\mathsf{MOD}} M;$

$\square$

$\texttt{PolynomialModuleStructure} :: \forall A \in \mathsf{RING} \,.\, \forall M \in A\text{-}\mathsf{MOD} \,.\, \Big\{(M, \cdot) : A[\mathbb{Z}_+]\text{-}\mathsf{MOD}\Big\} \cong_{\mathsf{SET}} \mathrm{End}_{A\text{-}\mathsf{MOD}}(M)$

$\texttt{Proof} \ =$

$\odot := \Lambda T : \mathrm{End}_{A\text{-}\mathsf{MOD}}(M) \,.\, \Lambda f \in A\Big[\mathbb{Z}_+\Big] \,.\, \Lambda m \in M \,.\, \sum_{i=0} f_i T^i(m) : \mathrm{End}_{A\text{-}\mathsf{MOD}} M \to \Big\{(M, \cdot) : A[\mathbb{Z}_+]\text{-}\mathsf{MOD}\Big\},$

$\square$

1.2 Limits of Modules

$\texttt{DirectProductOfModulesIsAModule} :: \forall I \in \mathsf{SET} \,.\, \forall M : I \to R\text{-}\mathsf{MOD} \,.\, \prod_{i \in I} M_i \in R\text{-}\mathsf{MOD}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{ProjectionIsLinear} :: \forall I \in \mathsf{SET} \,.\, \forall M : I \to R\text{-}\mathsf{MOD} \,.\, \forall i \in I \,.\, \pi_i : \prod_{j \in I} M_j \xrightarrow{R\text{-}\mathsf{MOD}} M_i$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{DirectProductIsProduct} :: \Big(\texttt{directProduct}, \pi\Big) : \texttt{Product}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ I : \mathsf{SET},$

$\texttt{Assume}\ M : I \to R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ N : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ T : \prod i \in I \,.\, N \xrightarrow{R\text{-}\mathsf{MOD}} M_i,$

$\Big(T', (1)\Big) := \eth\texttt{Product}(\mathsf{ABEL})(I, M, N, T) : \sum T' : N \xrightarrow{\mathsf{ABEL}} \prod_{i \in I} M_i \,.\, \forall i \in I \,.\, T'\pi_i = T_i,$

$\texttt{Assume}\ n : N,$

$\texttt{Assume}\ \alpha : R,$

$\texttt{Assume}\ i : I,$

$(2) := \eth^{-1}\pi_i(1)\eth R\text{-}\mathsf{MOD}(N, M_i)(T_i) : \Big(T'(\alpha n)\Big)_i = T'\pi_i(\alpha n) = T_i(\alpha n) = \alpha T_i(n),$

$(3) := \eth^{-1}\pi_i : \Big(T'(n)\Big)_i = T'\pi_i(n) = T_i(n),$

$() := (2)(3) : \Big(T'(\alpha n)\Big)_i = \alpha\Big(T'(n)\Big)_i;$

$\rightsquigarrow () := \eth\texttt{product} : T'(\alpha n) = \alpha T'(n);$

$\rightsquigarrow (*) := \eth^{-1}R\text{-}\mathsf{MOD} : \left[T' : N \xrightarrow{R\text{-}\mathsf{MOD}} \prod_{i \in I} M_i\right];$

$\square$

$\texttt{DirectSumOfModulesIsAModule} :: \forall I \in \mathsf{SET} \,.\, \forall M : I \to R\text{-}\mathsf{MOD} \,.\, \bigoplus_{i \in I} M_i \in R\text{-}\mathsf{MOD}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{InclusionIsLinear} :: \forall I \in \mathsf{SET} \,.\, \forall M : I \to R\text{-}\mathsf{MOD} \,.\, \forall i \in I \,.\, \iota_i : M_i \xrightarrow{R\text{-}\mathsf{MOD}} \bigoplus_{j \in I} M_j$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DirectSumIsCoproduct} :: \left(\texttt{directSum}, \iota\right) : \texttt{Coproduct}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ I : \mathsf{SET},$

$\texttt{Assume}\ M : I \to R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ N : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ T : \prod i \in I \,.\, M_i \xrightarrow{R\text{-}\mathsf{MOD}} N,$

$\left(T', (1)\right) := \eth\texttt{Product}(\mathsf{ABEL})(I, M, N, T) : \sum T' : \bigoplus i \in I M_i \to N \,.\, \forall i \in I \,.\, \iota_i T' = T_i,$

$\texttt{Assume}\ m : \bigoplus_{i \in I} M_i,$

$\texttt{Assume}\ \alpha : R,$

$\texttt{Assume}\ i : I,$

$(2) := \eth^{-1}\iota_i(1)\eth R\text{-}\mathsf{MOD}(M_i, N)(T_i) : T'(\alpha m_i) = \iota_i T'(\alpha m) = T_i(\alpha m) = \alpha T_i(m),$

$(3) := \eth^{-1}\pi_i : T'(m_i) = \iota_i T'(m) = T_i(m),$

$() := (2)(3) : T'(\alpha m_i) = \alpha T(m_i);$

$\rightsquigarrow () := \eth\texttt{directSum}\eth R\text{-}\mathsf{MOD}\eth R\text{-}\mathsf{MOD}(\iota) :$

$$: T'(\alpha m) = \sum_{i \in I} T'\left(\alpha \iota_i(m_i)\right) = \sum_{i \in I} \alpha T'\left(\iota_i(m_i)\right) = \alpha \sum_{i \in I} T'(\iota_i(m_i)) = \alpha T'(m);$$

$\rightsquigarrow (*) := \eth^{-1} R\text{-}\mathsf{MOD} : \left[T' : \bigoplus_{i \in I} M_i \xrightarrow{R\text{-}\mathsf{MOD}} N\right];$

$\square$

$\texttt{ZeroModuleIsZero} :: \star : \texttt{Zero}\,(R\text{-}\mathsf{MOD})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ M : M\text{-}\mathsf{MOD},$

$0_1 := \star \mapsto 0 : \star \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$0_2 := m \mapsto \star : M \xrightarrow{R\text{-}\mathsf{MOD}} \star,$

$\texttt{Assume}\ T : \star \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$(1) := \texttt{NeutralImage}(T) : T(\star) = 0,$

$() := \eth T(1) : T = 0_1;$

$\rightsquigarrow (1) := I(\forall)(T) : \forall T : \star \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, T = 0_1,$

$\texttt{Assume}\ T : M \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$() := \eth \star (T) : T = 0_2;$

$\rightsquigarrow (2) := I(\forall) : \forall T : M \xrightarrow{R\text{-}\mathsf{MOD}} \star \,.\, T = 0_2,$

$(*) := \eth^{-1}\texttt{Zero}(1)(2) : [\star : \texttt{Zero}(R\text{-}\mathsf{MOD})];$

$\square$

$$\texttt{fibredModule} :: \prod I \in \mathsf{SET} \,.\, \prod M : I \to R\text{-}\mathsf{MOD} \,.\, \prod N \in R\text{-}\mathsf{MOD} \,.\, \left(\prod_{i \in I} M_i \xrightarrow{R\text{-}\mathsf{MOD}} N\right) \to R\text{-}\mathsf{MOD}$$

$$\texttt{fibredModule}\,(\nu) = \prod_{\substack{i \in I \\ N \,\square\, \nu}} M_i := \bigcap_{i,j \in I} \ker\left(\pi_i \nu_i - \pi_j \nu_j\right)$$

$\texttt{FibredModuleIsPullback} :: \texttt{fibredModule} : \texttt{Pullback}\left(R\text{-}\mathsf{MOD}\right)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ I : \mathsf{SET},$

$\texttt{Assume}\ M : I \to R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ N : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ \nu : \prod i \in I \,.\, M_i \xrightarrow{R\text{-}\mathsf{MOD}} N,$

$\texttt{Assume}\ P : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ T : \prod i \in I \,.\, P \xrightarrow{R\text{-}\mathsf{MOD}} M_i,$

$\texttt{Assume}\ (1) : \forall i, j \in I \,.\, T_i \nu_i = T_j \nu_j,$

$\texttt{Assume}\ (i, j) : I,$

$\texttt{Assume}\ p : P,$

$(2) := (1)(\pi_i \nu_i - \pi_j \nu_i) : \forall p \in P \,.\, (\pi_i \nu_i - \pi_j \nu_j) T(p) = T_i \nu_i(p) - T_j \nu_j(p) = 0,$

$() := \eth \ker : T(p) \in \ker T_i \nu_i - T_j \nu_j;$

$$\rightsquigarrow (2) := \eth^{-1} \texttt{fibredModule} : \operatorname{Im} T \subset \prod_{\substack{i \in I \\ N \,\square\, \nu}} M_i,$$

$$T' := T^{|\texttt{fibredProduct}(I,M,N,\nu)} : P \xrightarrow{R\text{-}\mathsf{MOD}} \prod_{\substack{i \in I \\ N \,\square\, \nu}} M_i,$$

$$\texttt{Assume}\ T'' : P \xrightarrow{R\text{-}\mathsf{MOD}} \prod_{\substack{i \in I \\ N \,\square\, \nu}} M_i,$$

$\texttt{Assume}\ (3) : \forall i \in I \,.\, T'' \pi_i = T_i,$

$() := \eth T'(3) : T'' = T';$

$\rightsquigarrow (*) := \eth^{-1} \texttt{Pushout} : \left[\texttt{fibredModule} : \texttt{Pullback}\left(R\text{-}\mathsf{MOD}\right)\right];$

$\square$

$\texttt{span} :: \prod M \in R\text{-}\mathsf{MOD} \,.\, ?M \to \texttt{Submodule}(M)$

$$\texttt{span}\,(X) = \operatorname{span}(X) := \bigcap_{X \subset S \subset_{R\text{-}\mathsf{MOD}} M} S$$

$$\texttt{fibredSum} :: \prod I \in \mathsf{SET} \,.\, \prod M : I \to R\text{-}\mathsf{MOD} \,.\, \prod N \in R\text{-}\mathsf{MOD} \,.\, \left(\prod_{i \in I} M_i \xrightarrow{R\text{-}\mathsf{MOD}} N\right) \to R\text{-}\mathsf{MOD}$$

$$\texttt{fibredSum}\,(\nu) = \bigoplus_{\substack{i \in I \\ N \,\square\, \nu}} M_i := \frac{\bigoplus_{i \in I} M_i}{\operatorname{span}\{\nu_i \iota_i(n) - \nu_j \iota_j(n) | i, j \in I, n \in N\}}$$

$\texttt{FibredSumIsPushout} :: \texttt{fibredModule} : \texttt{Pushout}\Big(R\text{-MOD}\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ I : \text{SET},$

$\texttt{Assume}\ M : I \to R\text{-MOD},$

$\texttt{Assume}\ N : R\text{-MOD},$

$\texttt{Assume}\ \nu : \prod i \in I \,.\, N \xrightarrow{R\text{-MOD}} M_i,$

$\texttt{Assume}\ P : R\text{-MOD},$

$\texttt{Assume}\ T : \prod i \in I \,.\, M_i \xrightarrow{R\text{-MOD}} P,$

$\texttt{Assume}\ (1) : \forall i, j \in I \,.\, \nu_i T_i = \nu_j T_j,$

$\texttt{Assume}\ m : \operatorname{span}\{\nu_i \iota_i(n) - \nu_j \iota_j(n) | i, j \in I, n \in N\},$

$$L, i, j, n, 2) := \eth \operatorname{span}(m) : \sum L \in \mathbb{Z}_+ \,.\, \sum i, j : L \to I \,.\, n : L \to N \,.\, m = \sum_{l=1}^{L} \nu_{i_l}\iota_{i_l}(n_l) - \nu_{j_l}\iota_{j_l}(n_l),$$

$$(3) := (2)(1) : \sum_{i \in I} T_i(m_i) == \sum_{l=1}^{L} \nu_{i_l} T_{i_l}(n_l) - \nu_{j_l} T_{j_l}(n_l) = 0,$$

$$() := \eth \ker(3) : m \in \bigoplus_{I \in i} T_i(m);$$

$$\leadsto (T', ()) := \texttt{SubmoduleUP} : \sum T' : \bigoplus_{\substack{i \in I \\ N \,\square\, \nu}} M_i \to P \,.\, \forall m \in \bigoplus_{i \in I} M_i \,.\, T'[m] = \bigoplus_{i \in I} T_i(m)\ \&$$

$$\&\ \forall T'' : \bigoplus_{\substack{i \in I \\ N \,\square\, \nu}} M_i \to P \,.\, \left(\forall m \in \bigoplus_{i \in I} M_i \,.\, T''[m] = \bigoplus T'(m)\right) \Rightarrow T'' = T';$$

$$\leadsto () := \eth^{-1}\texttt{Pushout} : \left[\bigoplus_{\substack{i \in I \\ N \,\square\, \nu}} M_i : \texttt{Pushout}(R\text{-MOD}, I, M, N, \nu) \right];$$

$$\leadsto (*) := I(=, \to) : \left(\texttt{fibedModule} : \texttt{Pushout}\Big(R\text{-MOD}\Big)\right);$$

$\square$

$\texttt{LeftModulesAreBicomplete} :: R\text{-MOD} : \texttt{Complete}\ \&\ \texttt{Cocomplete}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{InnerDirectSum} :: \prod A \in \textsf{ANN} \,.\, \prod M \in A\text{-}\textsf{MOD} \,.\, ?\texttt{Submodule}^2(M)$

$(X, Y) : \texttt{InnerDirectSum} \iff M = X \oplus Y \iff X + Y = M \,\&\, X \cap Y = \{0\}$

$\texttt{InnerDirectSumIsDirectSum} :: \forall A \in \textsf{ANN} \,.\, \forall M \in A\text{-}\textsf{MOD} \,.\, \forall X, Y \subset_{A\text{-}\textsf{MOD}} M \,.$

$\quad . \, M = X \oplus Y \Rightarrow M \cong_{A\text{-}\textsf{MOD}} X \oplus Y$

$\texttt{Proof} =$

$\varphi := \Lambda(x, y) : X \oplus Y \,.\, x + y : X \oplus Y \xrightarrow{A\text{-}\textsf{MOD}} M,$

$[1] := \eth\texttt{InnerDirectSum}(A, M, X, Y) : X + Y = M \,\&\, X \cap Y = \{0\},$

$[*] := \eth^{-1}\texttt{Iso}[1] : M \cong_{A\text{-}\textsf{MOD}} X \oplus Y;$

$\square$

$\texttt{MultiInnerDirectSum} :: \prod A \in \textsf{ANN} \,.\, \prod M \in \,.\text{-}\textsf{MOD}? \sum I \in \texttt{Set} \,.\, I \to \texttt{Submodule}(M)$

$$(I, X) : \texttt{MultiInnerDirectSum} \iff M = \bigoplus_{i \in I} X_i \iff M = \sum_{i \in I} X_i \,\&\, \forall i \in I \,.\, X_i \cap \sum_{j \in I, j \neq i} X_j = \{0\}$$

$\texttt{MultiInnerDirectSumIsDirectSum} :: \forall A \in \textsf{ANN} \,.\, \forall M \in A\text{-}\textsf{MOD} \,.\, \forall I \in \textsf{SET} \,.\, \forall I : X \to \texttt{Submodule}(M) \,.$

$$. \, M = \bigoplus_{i \in I} X_i \Rightarrow M \cong_{A\text{-}\textsf{MOD}} \bigoplus_{i \in I} X_i$$

$\texttt{Proof} =$

$$\varphi := \Lambda x : \bigoplus_{i \in I} X_i \,.\, \sum_{i \in I} x_i : \bigoplus_{i \in I} X_i \xrightarrow{A\text{-}\textsf{MOD}} M,$$

$$[1] := \eth\texttt{InnerDirectSum}(A, M, X, Y) : M = \bigoplus_{i \in I} X_i \Rightarrow M \cong_{A\text{-}\textsf{MOD}} \bigoplus_{i \in I} X_i,$$

$$[*] := \eth^{-1}\texttt{Iso}[1] : \bigoplus_{i \in I} X_i \cong_{A\text{-}\textsf{MOD}} \bigoplus_{i \in I} X_i;$$

$\square$

1.3 Free Modules and Generation of Submodules

$$\texttt{freeModule} :: \texttt{Covariant}(\mathsf{SET}, R\text{-}\mathsf{MOD})$$
$$\texttt{freeModule}\,(X) = R^{\oplus X} := \bigoplus_{i \in I} R$$
$$\texttt{freeModule}\,(X, Y, f) = R^{\oplus f}_{X,Y} := \Lambda v \in \mathcal{R}^{\oplus f} \ . \ \sum_{x \in X} v_x \iota_{f(x)}(1)$$

$$\texttt{FreeModuleIsAdjoint} :: \texttt{freeModule} \vdash \texttt{forgetful}(R\text{-}\mathsf{MOD}, \mathsf{SET})$$
$\texttt{Proof} \ =$
$\texttt{Assume}\ X : \mathsf{SET},$
$\texttt{Assume}\ M : R\text{-}\mathsf{MOD},$
$$\phi := \Lambda f : X \to M \ . \ \Lambda \sum_{x \in X} v_x \iota_x(1) \in R^{\oplus X} \ . \ \sum_{x \in X} v_x f(x) : (X \to M) \to (R^{\oplus X} \xrightarrow{R\text{-}\mathsf{MOD}})M,$$
$$\psi := \Lambda T : R^{\oplus X} \xrightarrow{R\text{-}\mathsf{MOD}} M \ . \ \Lambda x \in X \ . \ T\big(\iota_x(1)\big) : (R^{\oplus X} \xrightarrow{R\text{-}\mathsf{MOD}} M) \to (X \to M),$$
$\texttt{Assume}\ f : X \to M,$
$$() := \eth\phi\eth\psi : \psi \circ \phi(f) = \psi\left(\Lambda \sum_{x \in X} v_x \iota_x \in R^{\oplus X} \ . \ \sum_{x \in X} v_x f(x)\right) = \Lambda x \in X \ . \ f(x) = f;$$
$$\rightsquigarrow (1) := \eth^{-1}\texttt{RightInverse} : \Big[\psi : \texttt{RightInverse}(\phi)\Big],$$
$\texttt{Assume}\ T : R^{\oplus X} \xrightarrow{R\text{-}\mathsf{MOD}} M,$
$$() := \eth\psi\eth\phi : \phi \circ \psi(T) = \phi\Big(\Lambda x \in X \ . \ T\big(\iota_x(1)\big)\Big) = \Lambda \sum_{x \in X} v_x \iota_x(1) \ . \ \sum_{x \in X} v_x x T\big(\iota_x\big) = T;$$
$$\rightsquigarrow () := \eth^{-1}\texttt{inverse} : \psi = \phi^{-1};$$
$$\rightsquigarrow (*) := \eth^{-1}\texttt{LeftAdjoint} : \texttt{This};$$
$\square$

$$\texttt{basisVector} :: \prod X \in \mathsf{SET} \ . \ X \to R^{\oplus X}$$
$$\texttt{basisVector}\,(x) = e_x := \iota_x(1)$$

$$\texttt{linearCombination} :: \prod X \in \mathsf{SET} \ . \ \prod M \in R\text{-}\mathsf{MOD} \ . \ R^{\otimes X} \times M^X \to M$$
$$\texttt{linearCombination}\,(a, m) = am := \sum_{x \in X} a_x m_x$$

$$\texttt{spanWithFamily} :: \prod X \in \mathsf{SET} \ . \ \prod M \in R\text{-}\mathsf{MOD} \ . \ (X \to M) \to \texttt{Subspace}(M)$$
$$\texttt{spanWithFamily}\,(v) = \operatorname{span}(v_x)_{x \in X} := \left\{ av | a \in R^{\oplus X} \right\}$$

$$\texttt{SpanIsSpan} :: \forall X \in \mathsf{SET} \ . \ \forall M \in R\text{-}\mathsf{MOD} \ . \ \forall m : X \to M \ . \ \operatorname{span}(m_x)_{x \in X} = \operatorname{span} \operatorname{Im} m$$
$\texttt{Proof} \ =$
$\ldots$
$\square$

$\texttt{FinitelyGeneratedModule} :: ?R\text{-MOD}$

$M : \texttt{FinitelyGeneratedModule} \iff \exists F : \texttt{Finite}(M) : M = \operatorname{span}(F)$

$\texttt{Noetherian} :: ?R\text{-MOD}$

$M : \texttt{Noetherian} \iff \forall A \subset_{R\text{-MOD}} M \,.\, A : \texttt{FinitelyGeneratedModule}(R)$

$\texttt{FGMBySubspace} :: \forall M \in R\text{-MOD} \,.\, \forall N \subset_{R\text{-MOD}} M \,.$

$\quad .\, N, \dfrac{M}{N} : \texttt{FinitelyGeneratedModule} \Rightarrow M : \texttt{FinitelyGeneratedModule}$

$\texttt{Proof} =$

$\Big(n, a, (1)\Big) := \eth\texttt{FinitelyGeneratedModule}(R)(N) : \sum n \in \mathbb{N} \,.\, \sum a : n \to N \,.\, N = \operatorname{span}(a_i)_{i=1}^n,$

$\Big(m, [b], (2)\Big) := \eth\texttt{FinitelyGeneratedModule}(R)\, \dfrac{M}{N} : \sum m \in \mathbb{N} \,.\, \sum [b] : m \to \dfrac{M}{N} \,.\, \dfrac{M}{N} = \operatorname{span}\Big([b_i]\Big)_{i=1}^m,$

$\texttt{Assume } x : M,$

$\Big(\alpha, (3)\Big) := \eth\dfrac{M}{N}[x] : \sum \alpha : m \to R \,.\, \sum_{i=1}^m \alpha_i [b_i] = \left[\sum_{i=1}^m \alpha_i b_i\right],$

$(y, 4) := \eth\dfrac{M}{N}(3) : \sum y \in N \,.\, x = y + \sum_{i=1}^m \alpha_i b_i,$

$(\beta, 5) := (2)(y) : \sum \beta : n \to R \,.\, y = \sum_{i=1}^n \beta_i a_i,$

$() := (5)(4) : x = \sum_{i=1}^n \beta_i a_i + \sum_{i=1}^m \alpha_i b_i;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{FinitelyGeneratedModule}(R) : \Big[M : \texttt{FinitelyGeneratedModule}\Big];$

$\square$

$\texttt{NoetherianBySubspace} :: \forall M : \texttt{FinitelyGeneratedModule}(R) \,.\, \forall N \subset_{R\text{-MOD}} M \,.$

$\quad .\, N, \dfrac{M}{N} : \texttt{Noetherian}(R) \iff M : \texttt{Noetherian}(R)$

$\texttt{Proof} =$

$\texttt{Assume } B : \texttt{Submodule}(A),$

$(3) := \texttt{SecondIsomorphism}(M, N, B) : \dfrac{B}{B \cap N} \cong_{R\text{-MOD}} \dfrac{B+N}{N},$

$(4) := \eth\texttt{Noetherian}\, \dfrac{M}{N}(3) : \left[\dfrac{B}{B \cap N} : \texttt{FinitelyGeneratedModule}(R)\right],$

$(5) := \eth\texttt{Noetherian}(N)(N \cap B) : [N \cap B : \texttt{FinitelyGeneratedModule}(R)],$

$() := \texttt{FGMBySubspace}(4)(6) : \Big[B : \texttt{FinitelyGeneratedModule}(R)\Big];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Noetherian} : [M : \texttt{Noetherian}(R)],$

$\square$

$\texttt{NoetherianByNoetherian} :: \forall A : \texttt{Noetherian} \,.\, \forall M : \texttt{FinitelyGeneratedModule}(A) \,.\, M : \texttt{Noetherian}(A)$

$\texttt{Proof} \;=$

$\Big(n, N, (1)\Big) := \eth\texttt{FGM}(A)(M) : \sum n \in \mathbb{N} \,.\, \sum N \subset_{A\text{-MOD}} A^{\oplus n} \,.\, M = \dfrac{A^{\oplus n}}{N},$

$☿ := \Lambda n \in \mathbb{N} \,.\, A^{\oplus n} : \texttt{Noetherian}(n) : \mathbb{N} \to \texttt{Type},$

$(2) := \eth\texttt{Noetherian}(A) : ☿(1),$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ (3) : ☿(n),$

$(4) := \eth\texttt{quotientModule}(A^{\oplus n+1}, A^{\oplus n}) : \dfrac{A^{\oplus n+1}}{A^{\oplus}} \cong_{A\text{-MOD}} A,$

$(5) := \texttt{NoetherianBySubspace}(2)(3)(4) : ☿n + 1,$

$\rightsquigarrow (3) := \eth\texttt{InductiveSet}(\mathbb{N}) : \forall n \in \mathbb{N} \,.\, ☿(n),$

$(4) := (3) : ☿(n),$

$(*) := \texttt{NoetherianBySubspace}(1)(4) : [M : \texttt{Noetherian}];$

$\square$

$\texttt{Generating} :: \prod M \in R\text{-MOD} \,.\, ?\sum X \in \mathsf{SET} \,.\, X \to M$

$m : \texttt{Generating} \iff \operatorname{span}(m_i)_{i\in X} = M$

$\texttt{LinearlyIndependent} :: \prod M \in R\text{-MOD} \,.\, ?\sum X \in \mathsf{SET} \,.\, X \to M$

$m : \texttt{LinearlyIndependent} \iff \forall \alpha \in R^{\oplus X} \,.\, \alpha m = 0 \iff \alpha = 0$

$\mathit{Basis} := \Lambda M \in R\text{-MOD} \,.\, \texttt{Generaring}\ \&\ \texttt{LinearlyIndependent}(M) : R\text{-MOD} \to \texttt{;}$

$\texttt{FreeModule} :: ?R\text{-MOD}$

$M : \texttt{FreeModule} \iff \exists X \in \mathsf{SET} \,.\, M \cong_{R\text{-MOD}} R^{\oplus X}$

$\texttt{BasisIffFree} :: \prod M \in R\text{-MOD} \,.\, \exists (X, m) : \texttt{Basis}(M) \iff M : \texttt{FreeModule}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (X, m) : \texttt{Basis}(M),$

$\varphi := \Lambda \alpha \in A^{\oplus X} \,.\, \sum_{x\in X} \alpha_x m_x : A^{\oplus X} \xrightarrow{R\text{-MOD}} M,$

$(1) := \eth\texttt{Generating}(X, m)\eth^{-1}\texttt{Surjective} : [\varphi : A^{\oplus X} \twoheadrightarrow M],$

$(2) := \eth\texttt{LinearlyIndependent}(X, n)\eth^{-1}\texttt{Injective} : [\varphi : A^{\oplus X} \twoheadrightarrow M],$

$(3) := \texttt{LinearInversion}\eth\texttt{Bijective}(2)(1) : [\varphi : A^{\oplus X} \xleftrightarrow{R\text{-MOD}} M],$

$(4) := \eth\texttt{Isomorphic} : [A^{\oplus X} \cong_{R\text{-MOD}} M],$

$() := \eth^{-1}\texttt{FreeModule} : [M : \texttt{FreeModule}];$

$\rightsquigarrow (1) := I(\Rightarrow) : \exists (X, m) : \texttt{Basis}(M) \Rightarrow M : \texttt{FreeModule},$

$\texttt{Assume}\ (2) : [M : \texttt{FreeModule}],$

$\Big(X, (3)\Big) := \eth\texttt{FreeModule}(X) : [M = R^{\oplus X}],$

$() := \eth^{-1}\texttt{Basis}\eth\texttt{basis} : [e(R, X) : \texttt{Basis}(X)];$

$\square$

$\texttt{LinearlyIndependentSet} :: \prod M : R\text{-}\mathsf{MOD} \,.\, ?M$

$S : \texttt{LinearlyIndependentSet} \iff \exists m : \texttt{LinearlyIndependent}(M) \,.\, S = \operatorname{Im} m$

$\texttt{MaximalLinearIndependentExists} :: \forall M : R\text{-}\mathsf{MOD} \,.\, \forall S : \texttt{LinearlyIndependentSet}(M) \,.$
$\quad .\, \exists S' \in \max \texttt{LinearlyIndependentSet}(M) \,.\, S \subset S'$

$\texttt{Proof} \ =$

Use Zorn Lemma.

□

$\texttt{LinearIndependenceOverFaF} :: \forall A : \texttt{IntegralDomain} \,.\, \forall X, Y \in \mathsf{SET} \,.\, \forall m : Y \to A^{\oplus X} \,.$
$\quad .\, (Y, m) : \texttt{LinearlyIndependent}(A^{\oplus X}) \iff (Y, m) : \texttt{LinearlyIndependent}(\operatorname{Frac} A^{\oplus X})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : [(Y, m) : \texttt{LinearlyIndependent}(A^{\oplus X})],$

$\texttt{Assume}\ \frac{\alpha}{\beta} : Y \to \operatorname{Frac}(A)^{\oplus X},$

$\texttt{Assume}\ (2) : \frac{\alpha}{\beta} m = 0,$

$$(3) := \eth A\text{-}\mathsf{MOD}(2) : 0 = \frac{\alpha}{\beta} m = \sum_{y \in Y} \frac{\alpha_y}{\beta_y} m_y = \frac{\sum_{y \in Y} \alpha_y \left(\prod_{y' \neq y} \beta_y \right) m_y}{\prod_{y \in Y} \beta_y},$$

$$(4) := \eth\texttt{VectorSpace}(\operatorname{Frac}(A))(3) : \sum_{y \in Y} \alpha_y \left(\prod_{y' \neq y} \beta_{y'} \right) m_y = 0,$$

$$(5) := \eth\texttt{LinearlyIndependent}(m)(a)(4) : \forall y \in Y \,.\, \alpha_y \prod_{y' \neq y} \beta_{y'} = 0,$$

$(6) := \eth \operatorname{Frac} A(5) : \forall y \in Y \,.\, \alpha_y = 0,$

$() := \eth \operatorname{Frac} A \left(\frac{\alpha}{\beta} \right) : \frac{\alpha}{\beta} = 0;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{LinearlyIndependent}(\operatorname{Frac}(A)^{\oplus X}) : [m : \texttt{LinearlyIndependent}(\operatorname{Frac}(A)^{\oplus X})];$

□

$\texttt{BasisIso} :: \forall M, N : R\text{-}\mathsf{MOD} \,.\, \forall (X, f) : \texttt{Basis}(M) \,.\, \forall T : M \xleftrightarrow{R\text{-}\mathsf{MOD}} N \,.\, \Big(X, T(f) \Big) : \texttt{Basis}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \alpha : A^{\oplus X},$

$\texttt{Assume}\ (1) : \alpha T(f) = 0,$

$(2) := \eth R\text{-}\mathsf{MOD}(T)(1) : T(\alpha f) = 0,$

$(3) := \eth\texttt{Iso}\Big(R\text{-}\mathsf{MOD}\Big)(T)(2) : \alpha f = 0,$

$() := \eth\texttt{LinearlyIndependent}()(3) : \alpha = 0;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{LinearlyIndependent}(a) : [T(f) : \texttt{LinearlyIndependent}(N)],$

$\texttt{Assume}\ n : N,$

$(m, (2)) := \eth\texttt{Iso}(T)(n) : \sum m \in M \,.\, n = T(m),$

$(\alpha, (3)) := \eth\texttt{Generating}(M)(f)(m) : m = \alpha f,$

$(4) := \eth R\text{-}\mathsf{MOD}(M)(T) : T(m) = T(\alpha f) = \alpha T(f);$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Basis} : [T(f) : \texttt{Basis}(M)],$

□

$\texttt{MaximalLinearIndDominates} :: \forall A : \texttt{IntegralDomain} \,.\, \forall M : \texttt{FreeModule}A \,.$

$\quad .\, \forall E \in \max \texttt{LinearlyIndependentSet}(M) \,.\, \forall S \in \texttt{LinearlyIndependentSet}(M) \,.$

$\quad .\, |E| \leq |S|$

$\texttt{Proof} \ =$

$\Big(X, e\Big) := \eth\texttt{LinearlyIndependentSet}(M)(E) : \sum X \in \mathsf{SET} \,.\, e : X \xleftrightarrow{\mathsf{SET}} E \;\&\; \texttt{LinearlyIndependent}(M),$

$\Big(Y, s\Big) := \eth\texttt{LinearlyIndependentSet}(M)(S) : \sum X \in \mathsf{SET} \,.\, s : Y \xleftrightarrow{\mathsf{SET}} S \;\&\; \texttt{LinearlyIndependent}(S),$

$Y' := Y \sqcup \{0\} : \mathsf{SET},$

$(1) := \texttt{WellOrderingTheorem}(X', 0) : [X' : \texttt{WellOrdered} \;\&\; 0 = \min X'],$

$I_0 := \emptyset : \emptyset \to X,$

$m_0 := e : X \to E,$

$\texttt{Assume}\ y : Y,$

$\Big(\alpha, \beta, (2)\Big) := \eth \max \texttt{LinearlyIndependentSet}(\operatorname{Im} m_{y--})(s_y) : \sum \alpha : A^{\oplus X} \,.\, \sum \beta \in A^* \,.$

$\quad .\, \sum_{x \in X} \alpha_x m_{y--,x} = \beta s_y,$

$\Big(a, b, (3)\Big) := \eth m_{y--} : \sum a : I_{y--}\Big(\texttt{Less}(y)\Big) \to S \,.$

$\quad .\, \sum b; X \setminus I_{y--}\Big(\texttt{Less}(y)\Big) \to E \,.\, m = a \oplus b,$

$\Big(\xi, \zeta, (4)\Big) := \texttt{decomp}(\alpha, \operatorname{Im} I_{y--}) : \sum \xi : \operatorname{Im} I_{y--} \to A \,.$

$\quad \sum X \setminus \in I_{y--} \to \ .\ .\ \alpha = \xi \oplus \zeta,$

$(5) := (3)(4) : \beta s_y = \sum_x \xi_x a_x + \sum_x \zeta_x b_x,$

$\Big(x, (6)\Big) := \eth\texttt{LinearlyIndependentSet}(M)(S)(a, b)(5) : \sum x \in X \,.\, \zeta_x \neq 0,$

$I_y := I_{y--} \oplus (y \mapsto x) : \texttt{Less}(y++) \hookrightarrow X,$

$m_y := \Lambda x \in X \,.\, \texttt{if}\ x \in \operatorname{Im} I\ \texttt{then}\ s\Big(I^{-1}(x)\Big)\ \texttt{else}\ e(x) : X \to M,$

$\texttt{Assume}\ \gamma : A^{\oplus X},$

$\texttt{Assume}\ (7) : \gamma m_y = 0,$

$\texttt{Assume}\ (8) : \gamma \neq 0,$

$(9) := \eth\texttt{LinearlyIndependent}(m_{y--})(8) : \gamma_x \neq 0,$

$(10) := (2)(9) : 0 = \alpha_x^{-1}\beta_x\gamma_x e_x + \sum_{x' \in X : x' \neq x} \alpha_x^{-1}\beta\gamma_x m_{x'} - \gamma_{x'} m_{x'},$

$(11) := \eth\texttt{LinearlyIndependent}(m_{y--})(10) : \alpha_x^{-1}\beta\gamma_x = 0,$

$() := (8)(6)(11) : \perp;$

$\rightsquigarrow (12) := \eth\texttt{LinearlyIndependent}(M) : [m_y : \texttt{LinearlyIndependent}(M)],$

$(13) := \eth m_y(6)(5) : e_x \in \operatorname{span}(m_{y,x'})_{x' \in X},$

$(14) := \eth \max \texttt{LinearlyIndependentSet}(M)(\operatorname{Im} m_{y--})(13) : [\operatorname{Im} m_y \in \max \texttt{LinearlyIndependentSet}(M)];$

$\rightsquigarrow \Big(I, m, (2)\Big) := I\Big(\sum\Big) : \sum I : \prod y \in Y' \,.\, \texttt{Less}(y++) \hookrightarrow X \,.\, \sum m : \prod y \in Y' \,.\, X \to M \,.$

$\quad .\, \operatorname{Im} m \in \max \texttt{LinearlyIndependentSet}(M) \;\&\; \forall x \in \operatorname{Im} I \,.\, m_x = s_{I^{-1}(x)} \;\&\; \forall x \notin \operatorname{Im} I \,.\, m_x = e_x,$

$(I') := \texttt{TransfiniteInduction}(Y')(2) : I' : X \hookrightarrow Y,$

$(*) := \texttt{CardByInclusion}(I')\texttt{CardIso}(e, s) : |S| \leq |E|;$

$\square$

$\texttt{FreeReflectsIso} :: \forall X, Y \in \mathsf{SET} \,.\, \forall A : \texttt{IntegralDomain} \,.\, \forall (0) : A^{\oplus X} \cong A^{\oplus Y} \,.\, X \cong_{\mathsf{SET}} Y$

$\texttt{Proof} \;=$

$T := \eth\texttt{Isomorphic}(0) :: A^{\oplus X} \xleftrightarrow{A\text{-}\mathsf{MOD}} A^{\oplus Y},$

$(1) := \texttt{Isobasis}(T, e(A, X)) : \left[T\big(e(A, X)\big) : \texttt{Basis}(A^{\oplus Y})\right],$

$(2) := \texttt{MaximalLinearIndDomianates}(1)(T\big(e(A, X)\big)) : |X| \le |Y|,$

$(3) := \texttt{MaximalLinearIndDominates}(1)(e(A, Y), T\big(e(A, X)\big)) : |Y| \le |X|,$

$() := \texttt{EqCard}\big((2), (3)\big) : |Y| = |X|;$

$\square$

$\texttt{idealModule} :: \prod A \in \mathsf{ANN} \,.\, \prod M \in A\text{-}\mathsf{MOD} \,.\, \texttt{Ideal}(A) \to \texttt{Subspace}(M)$

$\texttt{idealModule}\,(I) = IM := \operatorname{span}\{am | a \in I, m \in M\}$

$\texttt{QuotientRingModule} :: \forall A \in \mathsf{ANN} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall M \in A\text{-}\mathsf{MOD} \,.\, \forall (0) : IM = 0 \,.\, M \in \left(\frac{A}{I}\right)\text{-}\mathsf{MOD}$

$\texttt{Proof} \;=$

$\odot := \Lambda[a] \in \frac{A}{I} \,.\, \Lambda m \in M \,.\, am : \Lambda\frac{A}{I} \times M \to M,$

$\texttt{Assume}\; a : A,$

$\texttt{Assume}\; i : I,$

$\texttt{Assume}\; m : M,$

$(1) := \eth \odot \eth A\text{-}\mathsf{MOD}(M)(0)\eth^{-1}\odot : [a + i] \odot m = (a + i)m = am + im = am = [a]\odot;$

$\leadsto () := \eth A\text{-}\mathsf{MOD}(M)\eth\odot : \left[(M, \odot) \in \left(\frac{A}{I}\right)\text{-}\mathsf{MOD}\right];$

$\square$

$\texttt{FreeIdealModule} :: \forall A \in \mathsf{ANN} \,.\, \forall X \in \mathsf{SET} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall \frac{A^X}{IA^X} \cong_{\left(\frac{A}{I}\right)\text{-}\mathsf{MOD}} \left(\frac{A}{I}\right)^X$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{InvariantBasisProperty} :: ?\mathsf{RING}$

$R : \texttt{InvariantBasisProperty} \iff \texttt{freeModule}(R) : \texttt{Conservative}\big(\mathsf{SET}, R\text{-}\mathsf{MOD}\big)$

$\texttt{CRingIsIBP} :: \forall A \in \mathsf{ANN} \,.\, A : \texttt{InvariantBasisProperty}$

$\texttt{Proof} \;=$

$I := \texttt{MaximalIdealExists}(A) : \texttt{MaximalIdeal}(A),$

$\texttt{Assume}\; X, Y : \mathsf{SET},$

$\texttt{Assume}\; (1) : A^{\oplus X} \cong_{A\text{-}\mathsf{MOD}} A^{\oplus Y},$

$T := \eth\texttt{Isomorphic} : A^{\oplus X} \cong_{A\text{-}\mathsf{MOD}} A^{\oplus Y},$

$f := T\Big(e(A, X)\Big) : \texttt{Basis}\Big(A^{\oplus Y}\Big),$

$T' := \Lambda \sum_{x \in X} [a_i][e_x] \,.\, \sum_{x \in X} [a_x][f_x] : \left(\frac{A}{I}\right)^{\oplus X} \xleftrightarrow{\left(\frac{A}{I}\right)\text{-}\mathsf{MOD}} \left(\frac{A}{I}\right)^{\oplus Y},$

$(1) := \texttt{FreeModCreate}\eth\texttt{Isomorphic}(A) : |X| = |Y|;$

$\square$

$\texttt{rank} :: \prod A \in \mathsf{ANN} \,.\, \texttt{FinitelyGeneratedModule}(A) \to \mathbb{N}$

$\texttt{rank}\,(M) = \operatorname{rank} M := |X| \quad \texttt{where} \quad A^{\oplus X} \cong_{A\text{-}\mathsf{MOD}} M$

$\texttt{NoetherianACC} :: \forall M : \texttt{Noetherian}(R) \,.\, \forall N : \texttt{Nondescending}\Big(\mathbb{N}, \texttt{Submodule}(M)\Big) \,.$

$\quad .\, \exists n \in \mathbb{N} : \forall k \in \mathbb{N} \,.\, k \geq n \Rightarrow N_k = N_n$

$\texttt{Proof} \;=$

$V := \bigcup_{n=1}^{\infty} N_n : \texttt{Submodule}(R),$

$(m, v, 1) := \eth\texttt{Noetherian}(A)(M)(V) : \sum m \in \mathbb{N} \,.\, \sum v : m \to V \,.\, V = \operatorname{span}(v_i)_{i=1}^{m},$

$(n, 2) := \eth\texttt{Nondescending}(N)\eth\texttt{union}(N)(v) : \exists n \in \mathbb{N} \,.\, N_n = V,$

$(3) := \eth\texttt{Nondescending}(N)\texttt{UnionContains}(N, V) : \texttt{THIS};$

$\square$

1.4 Chain Complexes and Exact Sequences

$$\texttt{MorphismChain} := \Lambda R \in \mathsf{RING} \,.\, \sum V : \mathbb{Z} \to R\text{-}\mathsf{MOD} \,.\, \sum \prod_{i=-\infty}^{\infty} \phi : V_i \xrightarrow{R\text{-}\mathsf{MOD}} V_{i-1} : \mathsf{RING} \to \texttt{Type};$$

$$\texttt{ChainComplex} :: \prod R \in \mathsf{RING} \,.\, ?\sum V : \mathbb{Z} \to R\text{-}\mathsf{MOD} \,.\, \sum \phi : V_i \xrightarrow{R\text{-}\mathsf{MOD}} V_{i-1}$$

$$(V, \phi) : \texttt{ChainComplex} \iff \forall i \in \mathbb{Z} \,.\, \operatorname{Im} \phi_{i+1} \subset \ker \phi_i$$

$$\texttt{Exact} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{ChainComplex}(R)$$

$$(V, \phi) : \texttt{Exact} \iff \forall i \in \mathbb{Z} \,.\, \operatorname{Im} \phi_{i+1} = \ker \phi_i$$

$$\texttt{Finite} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{ChainComplex}(R)$$

$$V, \phi : \texttt{Finite} \iff \exists n \in \mathbb{N} \,.\, \exists m \in \mathbb{N} : \forall i \in \mathbb{Z} \,.\, i < -n \,\&\, i > m \Rightarrow V_i = 0$$

$$\texttt{finiteChain} :: \prod R \in \mathsf{RING} \,.\, \left(\sum n \in \mathbb{N} \,.\, \sum V : n \to R\text{-}\mathsf{MOD} \,.\, \sum \phi : \prod i \in (n-1) \,.\, V_{i+1} \xrightarrow{R\text{-}\mathsf{MOD}} V_i \right) \to$$

$$\to \texttt{MorphismChain}(R)$$

$$\texttt{finiteChain}(V, \phi) = V_n \xrightarrow{\phi_{n-1}} V_{n-1} \xrightarrow{\phi_{n-2}} \dots \xrightarrow{\phi_1} V_1 :=$$

$$:= \Big(\Lambda i \in \mathbb{Z} \,.\, \textbf{if } i \in n \textbf{ then } V_i \textbf{ else } 0, \Lambda i \in \mathbb{Z}. \textbf{if } 2 \le i \le n \textbf{ then } \phi_{i-1} \textbf{ else } 0 \Big)$$

$$\texttt{RightChain} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{MorphismChain}(R)$$

$$V, \phi : \texttt{RightChain} \iff \forall i \in \mathbb{Z} \,.\, i > 0 \Rightarrow V_i = 0$$

$$\texttt{LeftChain} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{MorphismChain}(R)$$

$$V, \phi : \texttt{LeftChain} \iff \forall i \in \mathbb{Z} \,.\, i < 0 \Rightarrow V_i = 0$$

$\texttt{InjectionByRightChain} :: \forall (V, \varphi); \texttt{RightChain} \,\&\, \texttt{Exact}(R) \,.\, \varphi_1 : V_1 \hookrightarrow V_0$

Proof =

$[1] := \texttt{RightChain}(V, \varphi) \eth R\text{-}\mathsf{MOD}(0) : \varphi_1 = 0,$

$[2] := \eth^{-1} \operatorname{Im} \varphi_0 [1] : \operatorname{Im} \varphi_1 = 0,$

$[3] := \eth \texttt{Exact}[2] : \ker \varphi_0 = 0,$

$[*] := \texttt{ZeroKernelTHM}[4] : (\varphi_1 : V_1 \hookrightarrow V_0);$

□

$\texttt{SurjectionByLeftChain} :: \forall (V, \varphi); \texttt{LeftChain} \,\&\, \texttt{Exact}(R) \,.\, \varphi_0 : V_0 \twoheadrightarrow V_{-1}$

Proof =

$[1] := \texttt{LeftChain}(V, \varphi) \eth R\text{-}\mathsf{MOD}(0) : \varphi_1 = 0,$

$[2] := \eth^{-1} \ker \varphi_1 [1] : \ker \varphi_1 = V_0,$

$[3] := \eth \texttt{Exact}[2] : \operatorname{Im} \varphi_0 = V_0,$

$[*] := \eth^{-1} \texttt{Surjective} : (\varphi_0 : V_0 \twoheadrightarrow V\_1);$

□

$\texttt{MorphismOfChains} :: \prod (V,\varphi),(W,\psi) : \texttt{ChainComplex}(R) \;.\; ?\prod_{i=-\infty}^{\infty} V_i \xrightarrow{R\text{-MOD}} W_i$

$f : \texttt{MorphismOfChains} \iff \forall i \in \mathbb{Z} \;.\; f_{i-1}\psi_i = \phi_i f_i$

$\texttt{MorphismOfChainsComposition} :: \prod (V,\varphi),(W,\psi),(U,\eta) : \texttt{ChainComplex}(R) \;.$

$\;.\; \forall f : \texttt{MorphismOfChains}\Big((V,\varphi),(W,\psi)\Big) \;.\; \forall g : \texttt{MorphismOfChains}\Big((W,\psi),(U,\eta)\Big) \;.$

$\;.\; g \circ f : \texttt{MorphismOfChains}\Big((V,\varphi),(U,\eta)\Big)$

Proof $=$

Assume $i : \mathbb{Z}$,

$[i.*] := \eth^2\texttt{MorphismOfChains}\Big((V,\varphi),(W,\psi)\Big)\Big((W,\psi),(U,\eta)\Big)(f)(g) : f_{i-1}g_{i-1}\eta_i = f_{i-1}\psi_i g_i = \varphi_i f_i g_i;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{MorphismOfChains} : \left(g \circ f : \texttt{MorphismOfChains}\Big((V,\varphi),(U,\eta)\Big)\right);$

$\square$

$\texttt{CategoryOfChains} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{CategoryOfChains}\,(R) = R\text{-}\mathsf{CH} := \Big(\texttt{ChainComplex}, \texttt{MorphismOfChains}, (\circ)_*, \Lambda n \in \mathbb{Z} \;.\; \mathrm{id}\Big)$

$\texttt{IsoByExact} :: \forall A \xrightarrow{\varphi} B : \texttt{Exact}(R) \;.\; \varphi : A \overset{R}{\leftrightarrow} B$

Proof $=$

Combine InjectionByRightChain and SurjectionByLeftChain.

$\square$

$\texttt{ShortExact} :: \prod R \in \mathsf{RING} \;.\; ?\texttt{Exact}R$

$(V,\varphi) : \texttt{ShortExact} \iff \exists A,B,C \in R\text{-}\mathsf{MOD} : \exists \alpha : A \xrightarrow{R\text{-MOD}} B : \exists \beta : B \xrightarrow{R\text{-MOD}} C : (V,\varphi) = A \xrightarrow{\alpha} B \xrightarrow{\beta} C$

$\texttt{ShortExactProperty} :: \forall A \xrightarrow{\alpha} B \xrightarrow{\beta} C : \texttt{ShortExact}(R) \;.\; C \cong_{R\text{-MOD}} \dfrac{B}{\operatorname{Im}\alpha}$

Proof $=$

Combine exactens with the first isomorphism theorem.

$\square$

$\texttt{ShortExactOfHomo} :: \forall R \in \mathsf{RING} \;.\; \forall A,B \in R\text{-}\mathsf{MOD} \;.\; \forall T : A \xrightarrow{R\text{-MOD}} B \;.$

$\;.\; \ker T \xrightarrow{\iota} A \xrightarrow{T|^{\operatorname{Im} T}} \operatorname{Im} T : \texttt{ShortExact}(R)$

Proof $=$

$\ldots$

$\square$

$\texttt{ExactIsNaturalForChains1} :: \forall (V,\varphi),(W,\psi) \in R\text{-CH} \,.\, \forall f : (V,\varphi) \xleftrightarrow{R\text{-CH}} (W,\psi) \,.$
$\quad (V,\varphi) : \texttt{Exact}(R) \Rightarrow (W,\varphi) : \texttt{Exact}(R)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \Big((V,\varphi) : \texttt{Exact}(R)\Big),$

$\texttt{Assume}\ i : \mathbb{Z},$

$[2] := \eth\texttt{Iso}(f)\eth R\text{-CH} : f_i : V_i \xleftrightarrow{R\text{-MOD}} W_i \ \&\ f_{i-1} : V_{i-1} \xleftrightarrow{R\text{-MOD}} W_{i-1},$
$[3] := \eth\texttt{Exact}(V,\varphi)(i-1) : \ker\varphi_{i-1} = \operatorname{Im}\varphi_i,$
$[4] := \eth(\text{-CH}R)(f)(i-1) : f_{i-1}\psi_{i-1} = \varphi_{i-1}f_{i-2},$
$[5] := \eth\texttt{Iso}(f)[2][4] : \ker\psi_{i-1} = f_{i-1}(\ker\varphi_{i-1}),$
$[6] := \eth(\text{-CH}R)(f)(i) : f_i\psi_i = \varphi_i f_{i-1},$
$[7] := \eth\texttt{Iso}(f)[2][6] : \operatorname{Im}\psi_i = f_{i-1}(\operatorname{Im}\varphi_i),$
$[1.*] := [3][5][7] : \operatorname{Im}\psi_i = \ker\psi_{i-1};$
$\rightsquigarrow [*] := I(\Rightarrow)\eth^{-1}(\text{-CH}R)(V,\varphi)(W,\psi) : (V,\varphi) : \texttt{Exact}(R) \Rightarrow (W,\psi) : \texttt{Exact}(R);$
$\square$

$\texttt{ExactIsNaturalForChains2} :: \forall (V,\varphi),(W,\psi) \in R\text{-CH} \,.\, \forall f : (V,\varphi) \xleftrightarrow{R\text{-CH}} (W,\psi) \,.$
$\quad (V,\varphi) : \texttt{Exact}(R) \iff (W,\varphi) : \texttt{Exact}(R)$

$\texttt{Proof} \;=$

$[1] := \texttt{ExactIsNaturalForChains}(f) : (V,\varphi) : \texttt{Exact}(R) \Rightarrow (W,\psi) : \texttt{Exact}(R),$
$[2] := \texttt{ExactIsNaturalForChains}(f^{-1}) : (W,\psi) : \texttt{Exact}(R) \Rightarrow (V,\varphi) : \texttt{Exact}(R),$
$[*] := I(\iff)[1][2] : \texttt{This};$
$\square$

1.5 Split Exact Sequences

$\texttt{DirectSumShortExact} :: \forall R \in \mathsf{RING} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B : \texttt{ShortExact}(R)$

$\dots$

$\square$

$\texttt{Split} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{ShortExact}(R)$

$(V, \varphi) : \texttt{Split} \iff \exists A, B \in R\text{-}\mathsf{MOD} \,.\, A \xrightarrow{\iota_1} A \oplus B \xrightarrow{\pi_2} B \cong_{R\text{-}\mathsf{CH}} (V, \varphi)$

$\texttt{LeftInverseBySplit} :: \forall R \in \mathsf{RING} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall \varphi : A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.$

$\quad \varphi : \texttt{LeftInvertible}(A, B) \iff A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker} \varphi : \texttt{Split}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [0] : \Big(\varphi : \texttt{LeftInvertible}(A, B)\Big),$

$\Big(\psi, [1]\Big) := \eth\texttt{LeftInvertible}(\varphi) : \sum \psi : B \xrightarrow{R\text{-}\mathsf{MOD}} A \,.\, \varphi\psi = \mathrm{id},$

$[2] := [1]\texttt{ZeroKernelTHM} : (\varphi : A \hookrightarrow B),$

$[3] := \eth^{-1}\texttt{Iso}[2] : (\varphi^{|\operatorname{Im}\varphi} : A \xleftrightarrow{R\text{-}\mathsf{MOD}} \operatorname{Im}\varphi),$

$T := \Lambda(a, t) \in \operatorname{Im}\varphi \oplus \ker\psi \,.\, a + t : \operatorname{Im}\varphi \oplus \ker\psi \xrightarrow{R\text{-}\mathsf{MOD}} B,$

$S := \Lambda b \in B \,.\, \Big(\psi\varphi(b), b - \psi\varphi(b)\Big) : B \xrightarrow{R\text{-}\mathsf{MOD}} \operatorname{Im}\varphi \oplus \ker\psi,$

$[5] := \forall (b, t) \in \operatorname{Im}\varphi \oplus \ker\psi \,.\, \eth T[1]\eth S : \forall (b, t) \in \operatorname{Im}\varphi \oplus \ker\psi \,.\, ST(a, t) = S\big(a + t\big) = (a, t),$

$[6] := \forall b \in B \,.\, \eth S[1]\eth T : \forall b \in B \,.\, TS(b) = S\big(\psi\varphi(b), b - \psi\varphi(b)\big) = b,$

$[7] := [5][6]\eth\texttt{Iso} : \Big(T : \ker\varphi \oplus \operatorname{Im}\psi \xleftrightarrow{R\text{-}\mathsf{MOD}} A\Big),$

$[8] := \texttt{IsomorphismTHM3}(\operatorname{Im}\varphi)[1] : \Big(\pi^{-1}_{|\ker\psi} : \operatorname{coker}\varphi \xleftrightarrow{R\text{-}\mathsf{MOD}} \ker\psi\Big),$

$[9] := \forall a \in A \,.\, [1](a) : \forall a \in A \,.\, \varphi^{|\operatorname{Im}\varphi}\iota(a) = (\varphi(a), 0) = \big(\varphi\psi\varphi(a), \varphi(a) - \varphi\psi\varphi(a)\big) = \varphi S(a),$

$[10] := I(=, \rightarrow)[9] : \varphi^{|\operatorname{Im}\varphi}\iota = \varphi S,$

$[11] := \forall b \in B \,.\, \dots : \pi_2 S(b) = b - \psi\varphi(b) = \pi_{\operatorname{Im}\varphi}\pi^{-1}_{\ker\psi}(b),$

$[12] := I(=, \rightarrow)[11] : \pi_2 S = \pi_{\operatorname{Im}\varphi}\pi^{-1}_{\ker\psi},$

$[0.*] := \eth^{-1}\texttt{Split}\eth^{-1}R\text{-}\mathsf{CH}[10][12] : A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}\varphi : \texttt{Split};$

$\rightsquigarrow [0] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [1] : \Big(A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}\varphi : \texttt{Split}\Big),$

$\big(X, Y, [2]\big) := \eth\texttt{Split}[1] : \sum X, Y \in R\text{-}\mathsf{MOD} \,.\, X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{\pi_2} Y \cong_{R\text{-}\mathsf{CH}} A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}\varphi,$

$[3] := \eth\texttt{Isomorphic}\big((\text{-}\mathsf{CH}R)\big) : \sum f : X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{\pi_2} Y \xleftrightarrow{R\text{-}\mathsf{CH}} A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}\varphi,$

$[4] := \eth\iota\eth\pi : \iota_1\pi_1 = \mathrm{id},$

$[5] := \eth R\text{-}\mathsf{CH}(f)[4]\eth\texttt{Inverse}(f_0) : \varphi f_{-1}\pi_1 f_0^{-1} = f_0\iota_1\pi_1 f_0^{-1} = \mathrm{id},$

$[1.*] := \eth\texttt{LeftInvertible} : \Big(\varphi : \texttt{LeftInvertible}(A, B)\Big);$

$\rightsquigarrow [1] := I(\Leftarrow)[0]I(\iff) : \texttt{This};$

$\square$

$\texttt{RightInverseBySplit} :: \forall R \in \mathsf{RING} \, . \, \forall A, B \in R\text{-}\mathsf{MOD} \, . \, \forall \varphi : A \xrightarrow{R\text{-}\mathsf{MOD}} B \, .$

$\varphi : \texttt{RightInvertible}(A, B) \iff \ker \varphi \xrightarrow{\iota} A \xrightarrow{\varphi} B : \texttt{Split}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [0] : \Big(\varphi : \texttt{RightInvertible}(A, B)\Big),$

$\Big(\psi, [1]\Big) := \eth\texttt{RightInvertible}(\varphi) : \sum \psi : B \xrightarrow{R\text{-}\mathsf{MOD}} A \, . \, \psi\varphi = \mathrm{id},$

$[2] := [1]\eth^{-1}\texttt{Surjective} : (\varphi : A \twoheadrightarrow B),$

$[3] := \texttt{IsomorphismTHM}[2] : \left(\hat{\varphi} : \frac{A}{\ker \varphi} \xleftrightarrow{R\text{-}\mathsf{MOD}} B\right),$

$f := pi_{\ker \varphi | \operatorname{Im} \psi} \hat{\varphi} : \operatorname{Im} \psi \xleftrightarrow{R\text{-}\mathsf{MOD}} B,$

$T := \Lambda(b, t) \in \ker \varphi \oplus \operatorname{Im} \psi \, . \, b + t : \ker \varphi \oplus \operatorname{Im} \psi \xrightarrow{R\text{-}\mathsf{MOD}} A,$

$S := \Lambda a \in A \, . \, \Big(a - \varphi\psi(a), \varphi\psi(a)\Big) : A \xrightarrow{R\text{-}\mathsf{MOD}} \ker \varphi \oplus \operatorname{Im} \psi,$

$[5] := \forall (a, t) \in \ker \varphi \oplus \operatorname{Im} \psi \, . \, \eth T[1]\eth S : \forall (a, t) \in \ker \varphi \oplus \operatorname{Im} \psi \, . \, ST(a, t) = S\big(a + t\big) = (a, t),$

$[6] := \forall b \in B \, . \, \eth S[1]\eth T : \forall a \in A \, . \, TS(a) = S\big(\varphi\psi(a), a - \varphi\psi(a)\big) = a,$

$[7] := [5][6]\eth\texttt{Iso} : \Big(S : A \xleftrightarrow{R\text{-}\mathsf{MOD}} \ker \varphi \oplus \operatorname{Im} \psi\Big),$

$[8] := \forall a \in ker\varphi \, . \, [1](a) : \forall a \in \ker \varphi \, . \, \iota_1(a) = (a, 0) = \big(a - \varphi\psi(a), \varphi\psi(a)\big) = \iota S(a),$

$[9] := I(=, \rightarrow)[8] : \iota_1 = \iota S,$

$[10] := \forall a \in A \, . \, \ldots : \varphi f^{-1}(a) = \varphi\psi(a) = S\pi_2(b),$

$[11] := I(=, \rightarrow)[10] : \varphi f^{-1} = S\pi_2,$

$[12] := \eth^{-1}\texttt{Split}\eth^{-1}R\text{-}\mathsf{CH}[10][12] : \ker \varphi \xrightarrow{\iota} A \xrightarrow{\varphi} B : \texttt{Split};$

$\rightsquigarrow [0] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume} \ [1] : \Big(\ker \varphi \xrightarrow{\iota} A \xrightarrow{\varphi} B \xrightarrow{\pi} : \texttt{Split}\Big),$

$\big(X, Y, [2]\big) := \eth\texttt{Split}[1] : \sum X, Y \in R\text{-}\mathsf{MOD} \, . \, X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{\pi_2} Y \cong_{R\text{-}\mathsf{CH}} \ker \varphi \xrightarrow{\iota} A \xrightarrow{\varphi} B,$

$[3] := \eth\texttt{Isomorphic}\big((\text{-}\mathsf{CH}R)\big) : \sum f : X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{\pi_2} Y \xleftrightarrow{R\text{-}\mathsf{CH}} \ker \varphi \xrightarrow{\iota} A \xrightarrow{\varphi} B,$

$[4] := \eth\iota\eth\pi : \iota_2\pi_2 = \mathrm{id},$

$[5] := \eth R\text{-}\mathsf{CH}(f)[4]\eth\texttt{Inverse}(f_0) : f_{-2}\iota_2 f_{-1}^{-1}\varphi = f_{-2}\iota_2\pi_2 f_{-2}^{-1} = \mathrm{id},$

$[1.*] := \eth\texttt{RightInvertible} : \Big(\varphi : \texttt{RightInvertible}(A, B)\Big);$

$\rightsquigarrow [1] := I(\Leftarrow)[0]I(\iff) : \texttt{This};$

$\square$

1.6 Homology And Snake Lemma

$$\texttt{homology} :: \texttt{ChainComplex}(R) \to \mathbb{Z} \to R\text{-MOD}$$

$$\texttt{homology}\,((V,\varphi),i) = H_i(V,\varphi) := \frac{\ker \varphi_i}{\operatorname{Im} \varphi_{i+1}}$$

$$\texttt{SnakeLemma} :: \forall L_1 \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} N_1, \forall L_0 \xrightarrow{\alpha_0} M_0 \xrightarrow{\beta_0} N_0 : \texttt{ShortExact}(R)\ .$$

$$.\ \forall(\lambda,\mu,\nu) : L_1 \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} N_1 \xrightarrow{R\text{-CH}} L_0 \xrightarrow{\alpha_0} M_0 \xrightarrow{\beta_0} N_0\ .$$

$$\exists\delta : \ker\nu \xrightarrow{R\text{-MOD}} \operatorname{coker}\lambda : \ker\lambda \xrightarrow{\alpha_{1|\ker\lambda}} \ker\mu \xrightarrow{\beta_{1|\ker\mu}} \ker\nu \xrightarrow{\delta} \operatorname{coker}\lambda \xrightarrow{\hat{\alpha}_0} \operatorname{coker}\mu \xrightarrow{\hat{\beta}_0} \operatorname{coker}\nu : \texttt{Exact}(R)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : \ker\nu,$

$[1] := \eth\texttt{ShortExact}(\beta_1) : \big(\beta_1 : M_1 \twoheadrightarrow N_1,$

$[2] := \eth\texttt{Surjective}(\beta_1) : \{b \in M_1 : \beta_1(b) = a\} \neq \emptyset,$

$\texttt{Assume}\ b : M_1,$

$\texttt{Assume}\ [4] : \beta_1(b) = a,$

$c := \mu(b) : M_0,$

$[5] := \eth(\text{-CH}R)(\mu)\eth\ker\nu : \beta_0(c) = \mu\beta_0(b) = \beta_1\nu(b) = 0,$

$[6] := \eth^{-1}\ker\beta_0 : c \in \ker\beta_0,$

$[7] := \eth\texttt{Exact}[6] : c \in \operatorname{Im}\alpha,$

$\texttt{Assume}\ d, d' : L_0,$

$\texttt{Assume}\ [8] : \alpha_0(d') = c \ \&\ \alpha_0(d) = c,$

$\delta(a) := [d] : \operatorname{coker}\lambda,$

$[9] := \eth\texttt{ShortExact}(\alpha_0) : \Big(\alpha_0 :\hookrightarrow (L_0, M_0)\Big),$

$[d.*] := \eth\texttt{Injective}(\alpha_0)[8] : a = a';$

$\rightsquigarrow (\delta(a),[8]) := \eth\texttt{WellDefined} : \sum \delta(a) \in \operatorname{coker}\lambda\ .\ \exists d \in L_0 : [d] = \delta(a)\alpha_0(d) = c,$

$\texttt{Assume}\ b' : M_1,$

$\texttt{Assume}\ [9] : \beta_1(b') = a,$

$[10] := \eth R\text{-MOD}\beta_1[9] - [4] : 0 = \beta_1(b') - \beta_1(b) = \beta_1(b' - b),$

$[11] := \eth^{-1}\ker\beta_1[10] : b' - b \in \ker\beta_1,$

$[12] := \eth\texttt{Exact} : b' - b \in \operatorname{Im}\alpha_1,$

$(z,[13]) := \eth\operatorname{Im}[12] : \sum z \in L_1\ .\ \alpha_1(z) = b' - b,$

$[14] := \eth R\text{-CH}[13] : \lambda\alpha_0(z) = \alpha_1\mu(z) = \mu(b' - b),$

$[a.*] := \eth^{-1}\operatorname{Im}\lambda[14] : \mu(b' - b) \in \operatorname{Im}\lambda;$

$\rightsquigarrow \big(\delta,[1]\big) := \eth^{-1}\texttt{WellDefined} : \sum \delta : \ker\nu \xrightarrow{R\text{-MOD}} \operatorname{coker}\lambda\ .$

$.\ \forall a \in \ker\nu\ .\ \exists d \in L_0 : \delta(a) = [d]\ \&\ \exists b \in M_1 : \beta_1(b) = a\ \&\ \mu(b) = \alpha_0(d),$

$\texttt{Assume}\ a : \ker\lambda,$

$[2] := \eth R\text{-}\mathsf{CH}(\lambda,\mu,\nu) : \lambda\alpha_0 = \alpha_1\mu,$

$[3] := [2]\eth\ker\lambda(a) : 0 = \lambda\alpha_0(a) = \alpha_1\mu(a),$

$[a.*] := \eth^{-1}\ker\alpha_1 : \alpha_1(a) \in \ker\mu;$

$\rightsquigarrow [2] := \eth^{-1}\texttt{image}\eth^{-1}\texttt{Subset} : \alpha_1(\ker\lambda) \subset \ker\mu,$

$\texttt{Assume}\ a : \ker\mu,$

$[a.2] := \eth R\text{-}\mathsf{CH}(\mu,\mu,\nu) : \mu\beta_0 = \beta_1\nu,$

$[a.3] := [a.2]\eth\ker\mu(a) : 0 = \mu\beta_0(a) = \beta_1\nu(a),$

$[a.*] := \eth^{-1}\ker\beta_1 : \beta_1(a) \in \ker\nu;$

$\rightsquigarrow [3] := \eth^{-1}\texttt{image}\eth^{-1}\texttt{Subset} : \beta_1(\ker\mu) \subset \ker\nu,$

$\texttt{Assume}\ a : \operatorname{Im}\lambda,$

$[a.1] := \eth R\text{-}\mathsf{CH}(\lambda,\mu,\nu) : \lambda\alpha_0 = \alpha_1\mu,$

$(b,[a.2]) := \eth\operatorname{Im}\lambda a : \sum b \in L_1\ .\ a = \lambda(b),$

$[a.3] := [a.2][a.1]\eth\texttt{quotientModule} : \alpha_0\pi_{\operatorname{Im}\mu}(a) = \lambda\alpha_0\pi_{\operatorname{Im}\mu}(b) = \alpha_1\mu\pi_{\operatorname{Im}\mu}(b) = 0,$

$[a.*] := \eth^{-1}\ker[a.3] : a \in \ker\alpha_0\pi_{\operatorname{Im}\mu};$

$\rightsquigarrow \left(\hat\alpha_0,[4]\right) := \texttt{RestrictMorphism} : \sum\hat\alpha_0 : \operatorname{coker}\lambda \xrightarrow{R\text{-}\mathsf{MOD}} \operatorname{coker}\mu\ .\ \hat\alpha_0[a] = [\alpha_0(a)],$

$\texttt{Assume}\ a : \operatorname{Im}\mu,$

$[a.1] := \eth R\text{-}\mathsf{CH}(\lambda,\mu,\nu) : \mu\beta_0 = \beta_1\nu,$

$(b,[a.2]) := \eth\operatorname{Im}\mu a : \sum b \in L_1\ .\ a = \mu(b),$

$[a.3] := [a.2][a.1]\eth\texttt{quotientModule} : \beta_0\pi_{\operatorname{Im}\nu}(a) = \mu\beta_0\pi_{\operatorname{Im}\nu}(b) = \beta_1\nu\pi_{\operatorname{Im}\nu}(b) = 0,$

$[a.*] := \eth^{-1}\ker[a.3] : a \in \ker\beta_0\pi_{\operatorname{Im}\nu};$

$\rightsquigarrow \left(\hat\beta_0,[5]\right) := \texttt{RestrictMorphism} : \sum\hat\beta_0 : \operatorname{coker}\mu \xrightarrow{R\text{-}\mathsf{MOD}} \operatorname{coker}\nu\ .\ \hat\beta_0[a] = [\beta_0(a)],$

$\texttt{Assume}\ a : \ker\nu \cap \beta_1(\ker\mu),$

$(d.b,[6]) := [1](a) : \sum d \in L_0\ .\ \sum b \in M_1\ .\ \delta(a) = [d]\ \&\ \beta_1(b) = a\ \&\ \mu(b) = \alpha_0(d),$

$[7] := [6]\eth a : b \in \ker\mu,$

$[8] := [7]\eth\ker\mu[6] : \alpha_0(d) = 0,$

$[9] := \eth\texttt{Injective}[8] : d = 0,$

$[a.*] := \eth^{-1}\ker\delta[9][6] : a \in \ker\delta;$

$\rightsquigarrow [6] := \eth^{-1}\texttt{Subset} : \operatorname{Im}\beta_{1|\ker\mu} \subset \ker\delta,$

$\texttt{Assume}\ c : \operatorname{Im}\delta,$

$\texttt{Assume}\ (a,[c.1]) : \sum a \in \ker\nu\ .\ a = \delta(c),$

$\left(d,b,[c.2]\right) := [1](a) : \sum d \in L_0\ .\ \sum b \in M_1\ .\ \delta(a) = [d]\ \&\ \beta_1(b) = a\ \&\ \mu(b) = \alpha_0(d),$

$[c.3] := [c.1][c.2][4]\eth\texttt{quotientModule} : \hat\alpha_0(c) = \hat\alpha_0\delta(a) = \hat\alpha_0[d] = [\alpha_0(d)] = [\mu(b)] = 0,$

$[c.*] := \eth^{-1}\ker\hat\alpha_0[c.3] : c \in \ker\hat\alpha_0;$

$\rightsquigarrow [7] := \eth\texttt{Subset} : \operatorname{Im}\alpha_0 \subset \ker\hat\alpha_0,$

$\texttt{Assume}\ a : \ker\delta,$

$\Big(d, b, [a.1]\Big)) := [1](a) : \sum d \in L_0\ .\ \sum b \in M_1\ .\ \delta(a) = [d]\ \&\ \beta_1(b) = a\ \&\ \mu(b) = \alpha_0(d),$

$[a.2] := \eth \ker\delta(a)[a.1] : d \in \operatorname{Im}\lambda,$

$(c, [a.3]) := \eth \ker\delta : \sum c \in L_1\ .\ d = \lambda(c),$

$[a.4] := [a.3]\eth(\text{-CH}R)(\lambda, \mu, \nu) : \mu(b) = \lambda\alpha_0(c) = \alpha_1\mu(c),$

$[a.5] := \ldots : 0 = \mu\big(b - \alpha_1(c)\big),$

$[a.6] := R\text{-MOD}(\beta_1)\eth\texttt{ChainComplex}(\ldots)[a.3] : \beta_1\big(b - \alpha_1(c)\big) = \beta_1(b) = a,$

$[a.*] := [a.6][a.5] : a \in \beta_1(\ker\mu);$

$\rightsquigarrow [8] := \eth^{-1}\texttt{Subset}\eth^{-1}\texttt{SetEq}[6] : \beta_1(\ker\mu) = \ker\delta,$

$\texttt{Assume}\ [d] : \ker\hat{\alpha}_0,$

$[d.1] := \eth \ker\hat{\alpha}_0\eth \operatorname{coker}\mu : \alpha_0(d) \in \operatorname{Im}\mu,$

$(b, [d.2]) := \eth \operatorname{Im}\mu[d.1] : \sum b \in M_1\ .\ \alpha_0(d) = \mu(b),$

$[d.*] := \eth\delta[d.2] : \delta\beta_1(b) = [d];$

$\rightsquigarrow [8] := \eth^{-1}\texttt{Subset}\eth^{-1}\texttt{SetEq}[6] : \beta_1(\ker\mu) = \ker\delta,$

$[*] := \eth^{-1}\texttt{Exact}[2][3][4][5][6][7][8] : \texttt{This};$

$\square$

1.7 Torsion, Presentation and Resolution

$\texttt{Torsion} :: \prod M \in R\text{-MOD} \,.\, ?M$

$m : \texttt{Torsion} \iff \{m\} \,!\, \texttt{LinearlyIndependent}(M)$

$\texttt{Assume } A : \texttt{IntegralDomain},$

$\texttt{torsion} :: \prod M \in A\text{-MOD} \,.\, \texttt{Submod}(M)$

$\texttt{torsion}\,(M) = \operatorname{tor} M := \{m \in M : m : \texttt{Torsion}(M)\}$

$\texttt{TorsionFree} :: ?A\text{-MOD}$

$M : \texttt{TorsionFree} \iff \operatorname{tor} M = \{0\}$

$\texttt{Torsion} :: ?A\text{-MOD}$

$M : \texttt{Toraion} \iff \operatorname{tor} M = M$

$\texttt{TorsionFreeSubmoduleIsTorsionFree} :: \forall M : \texttt{TorsionFree}(A) \,.\, \forall S \subset_{A\text{-MOD}} M \,.\, S : \texttt{TorsionFree}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{TorsionFreeSumIsTorsionFree} :: \forall I \in \mathsf{SET} \,.\, \forall M : I \to \texttt{TorsionFree}(A) \,.\, \bigoplus_{i \in I} M_i : \texttt{TorsionFree}(R)$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FreeModuleIsTorsionFree} :: \forall X \in \mathsf{SET} \,.\, A^{\oplus X} : \texttt{TorsionFree}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Cyclic} :: ?R\text{-MOD}$

$M : \texttt{Cyclic} \iff \exists m \in M \,.\, M = \langle m \rangle$

$\texttt{FieldByCyclic} :: \Big(\forall M : \texttt{Cyclic}(A) \,.\, M : \texttt{TorsionFree}\Big) \Rightarrow A : \texttt{Field}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{annihilator} :: \prod M : R\text{-MOD} \,.\, M \to \texttt{LeftIdeal}(R)$

$\texttt{annihilator}\,(m) = \mathrm{Ann}(m) := \{r \in R : rm = 0\}$

$\texttt{annihilator} :: R\text{-MOD} \to \texttt{LeftIdeal}(R)$

$\texttt{annihilator}\,(M) = \mathrm{Ann}\, M := \bigcap_{m \in M} \mathrm{Ann}(m)$

$\texttt{TorsionHasNontrivalAnnihilator} :: \forall M : \texttt{Torsion} \,\&\, \texttt{FinitelyGeneratedModule}(A) \,.\, \mathrm{Ann}\, M \neq \{0\}$

$\texttt{Proof} =$

$(n, m, 1) := \eth\texttt{FinitelyGeneratedModule}(A)(M) : \sum n \in \mathbb{N} \,.\, m : n \to M \,.\, \mathrm{span}(m_i)_{i=1}^n,$

$(2) := \forall i \in n \,.\, \eth\texttt{Torsion}(A)(M)(a_i)\eth^{-1}\,\mathrm{Ann}(a_i) : \forall i \in n \,.\, \mathrm{Ann}(a_i) \neq \{0\},$

$(3) := \eth\texttt{IntegralDomain}(A)(2) : \prod_{i=1}^n \mathrm{Ann}(a_i) \neq \{0\},$

$(4) := \eth\,\mathrm{Ann}\, M(3) : \{0\} \neq \prod_{i=1}^n \mathrm{Ann}(a_i) \subset \mathrm{Ann}\, M;$

$\square$

$\texttt{Presentation} :: R\text{-MOD} \to ?\texttt{Exact}(R\text{-MOD})$

$\Big([A, B, C, D], [\phi, \pi, z]\Big) : \texttt{Presentation} \iff \Lambda M \in R\text{-MOD} \,.$
$\quad .\, C = M \,\&\, D = 0 \,\&\, A, B : \texttt{FreeModule}(R)$

$\texttt{FinitelyPresented} :: ?R\text{-MOD}$

$M : \texttt{FinitelyPresented} \iff \exists\texttt{Presentation}(M)$

$\texttt{OverNoetherianRingFGMIsFI} :: \forall A : \texttt{Noetherian} \,.\, \forall M : \texttt{FinitelyGeneratedModule}(A) \,.$
$\quad .\, M : \texttt{FinitelyPresented}(A)$

$\texttt{Proof} =$

$\Big(n, m, (1)\Big) := \eth\texttt{FinitelyGeneratedModule}(A)(M) : \sum n \in \mathbb{N} \,.\, m : n \to M \,.\, \mathrm{span}(m_i)_{i=1}^n,$

$T := \Lambda \alpha \in A^n \,.\, \alpha m : A^n \xrightarrow{A\text{-MOD}} M,$

$(2) := \eth T(2) : [T : A^n \twoheadrightarrow M],$

$(3) := \eth\texttt{Noetherian}(A^n, \ker T)\eth\texttt{NoetherianByNoetherian}(A, A^n) : \Big[\ker T : \texttt{FinitelyGeneratedModule}(A)\Big],$

$\Big(k, v, (4)\Big) := \eth\texttt{FinitelyGeneratedModule}(A)(\ker T) : \sum k \in \mathbb{N} \,.\, v : k \to \ker T \,.\, \ker T = \mathrm{span}(v_i)_{i=1}^k,$

$T' := \Lambda \alpha \in A^k \,.\, \alpha v : A^k \to A^n,$

$(*) := \eth^{-1}\texttt{Presentation} : \Big[\Big([A^k, A^n, M, 0], [T', T, 0]\Big) : \texttt{Presentation}(M)\Big];$

$\square$

$\texttt{FreeResolution} :: R\text{-MOD} \to ?\texttt{Exact}(R\text{-MOD})$

$(P, \varphi) : \texttt{FreeResolution} \iff \Lambda M \in R\text{-MOD} . P_{-1} = 0 \ \& \ P_0 = M \ \&$

$\quad \& \ \forall n \in \mathbb{N} . P_n : \texttt{FreeModule}(R)$

$\texttt{FreeResolutionExtends1} :: ?\text{RING}$

$A : \texttt{FreeResolutionExtends1} \iff \forall M : R\text{-MOD} . \forall n \in \mathbb{Z}_+ . \forall T : R^n \xrightarrow{R\text{-MOD}} M . \forall (0) : \operatorname{Im} T = M .$

$\quad . \exists m \in \mathbb{N} . \exists S : R^m \xrightarrow{R\text{-MOD}} R^n : \operatorname{Im} S \subset \ker T \ \& \ \ker S = 0$

$\texttt{FreeResolutionImpliesPID} :: \forall A \in \text{ANN} . \forall (0) : [A : \texttt{FreeResulionExtends1}] .$

$\quad . A : \texttt{PrincipleIdealDomain}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ I : \texttt{Ideal}(A),$

$\Big(m, T, (1)\Big) := \eth\texttt{FreeResolutionExtends1}\left(\frac{A}{I}, 1, \pi_I\right) : \sum m \in \mathbb{N} . \sum T : A^n \xrightarrow{A} A .$

$\quad . \operatorname{Im} T = \ker \pi_I \ \& \ \ker T = \{0\},$

$(2) := \eth\pi_I(1) : \operatorname{Im} T = I,$

$\texttt{Assume} \ (3) : n > 1,$

$(4) := \eth A\text{-MOD}(A^n, A)(T) : T\Big(\big(T(e_2), -T(e_1)\big) \oplus 0\Big) = T(e_1)T(e_2) - T(e_1)T(e_2) = 0,$

$(4) := \eth e(1)_2 : T(e_1) \neq 0 \ \& \ T(e_2) \neq 0,$

$() := (1)_2(5)(4) : \bot;$

$\rightsquigarrow (3) := E(\bot) : n = 1,$

$() := (3)(1)_1 : I = \langle T(1) \rangle;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{PrincipleIdealDomain} : [A : \texttt{PrincipleIdealDomain}];$

$\square$

$\texttt{MapsToTorsionFreeIsTorsionFree} :: \forall M : \texttt{TorsionFree}(A) . \forall N : A\text{-MOD} .$

$\quad . \forall \mathcal{M}_{A\text{-MOD}}(N, M) : \texttt{TorsionFree}(A)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{IncreasingAnnihilator} :: \forall A \in \text{ANN} . \forall M \in A\text{-MOD} . \forall m \in M . \forall a \in A . \ \operatorname{Ann}(m) \subset \operatorname{Ann}(am)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.8 Associated Primes

SubmoduleByIdeal $:: \forall A \in \mathsf{ANN} \,.\, \forall M : A\text{-}\mathsf{MOD} \,.\, \forall I : \mathtt{Ideal}(A) \,.\, \forall m \in M \,.\, \forall (0) : \mathrm{Ann}(m) = I \,.$

$\quad . \exists S \subset_{A\text{-}\mathsf{MOD}} M \,.\, S \cong_{A\text{-}\mathsf{MOD}} \dfrac{M}{I}$

Proof $=$

$(*) := \ldots : \mathrm{span}\{m\} \cong_{A\text{-}\mathsf{MOD}} \dfrac{M}{I};$

□

associated $:: \prod A \in \mathsf{ANN} \,.\, \prod M \in A\text{-}\mathsf{MOD} \,.\, ?\mathtt{Ideal}(A)$

associated $() = \mathrm{Ass}(M) := \{\mathrm{Ann}(m) | m \in M \;\&\; \mathrm{Ann}(m) : \mathtt{Prime}(M)\}$

MaximalAnnihilatorsAreAssociated $:: \forall a \in A \,.\, \forall M : A\text{-}\mathsf{MOD} \,.$

$\quad \max\{\mathrm{Ann}(m) | m \in M \setminus \{0\}\} \subset \mathrm{Ass}(M)$

Proof $=$

Assume $\mathrm{Ann}(m) : \max\{\mathrm{Ann}(m) | m \in M \setminus \{0\}\},$

Assume $a, b : A,$

Assume $(1) : ab \in \mathsf{ANN}(m),$

$(2) := \eth \,\mathrm{Ann}(m) \eth^{-1} \mathtt{Subset}(a, b) : \mathrm{Ann}(m) \subset \mathrm{Ann}(am) \;\&\; \mathrm{Ann}(m) \subset \mathrm{Ann}(bm),$

Assume $(3) : am \neq 0,$

$(4) := (1) \eth \,\mathrm{Ann}(bm) : b \in \mathrm{Ann}(am),$

$() := \eth \mathtt{max}(4)(2) I(|) : b \in \mathrm{Ann}(m) | a \in \mathrm{Ann}(m);$

$\rightsquigarrow (3) := I(\Rightarrow) : am \neq 0 \Rightarrow b \in \mathrm{Ann}(m) | a \in \mathrm{Ann}(m),$

Assume $(4) : am = 0,$

$() := \eth \,\mathrm{Ann}(m)(4) I(|) : a \in \mathrm{Ann}(m) | b \in \mathrm{Ann}(m);$

$\rightsquigarrow () := E(|)(2)(3) I(\Rightarrow) : b \in \mathrm{Ann}(m) | a \in \mathrm{Ann}(m);$

$\rightsquigarrow (1) := \eth^{-1} \mathtt{Prime} : [\mathrm{Ann}(m) : \mathtt{Prime}(M)],$

$() := \eth^{-1} \,\mathrm{Ass}(M) : \mathrm{Ann}(m) \subset \mathrm{Ass}(M);$

$\rightsquigarrow (*) := \eth^{-1} \mathtt{Subset} : \max\{\mathrm{Ann}(m) | m \in M \setminus \{0\}\} \subset \mathrm{Ass}(M);$

□

AssociatedPrimesExistInNoetherian $:: \forall A : \mathtt{Noetherian}(M) \,.$

$\quad . \forall M : A\text{-}\mathsf{MOD} \,.\, \exists I \in \mathrm{Ass}(M)$

Proof $=$

$\ldots$

□

$\texttt{PrimeFactorizationSeria} :: \forall A : \texttt{Noetherian}(M) \,.\, \forall M : \texttt{FinitelyGeneratedModule}(A) \,.\, \exists n \in \mathbb{N} :$

$\quad : \exists (N, P) : n \to \texttt{Submodule}(M) : \exists P : (n-1) \to \texttt{Prime}(n) \,.\, N_n = M \ \&\ N_1 = \{0\}\ \&$

$$\quad \&\ \forall i \in (n-1) \,.\, N_i \subset N_{i+1} \ \&\ \frac{N_{i+1}}{N_i} \cong \frac{A}{P_i}$$

$\texttt{Proof} \ =$

$N'_1 := M : A\text{-MOD},$

$f_1 := \mathrm{id}_M : M \xrightarrow{A\text{-MOD}} N'_1,$

$\texttt{Assume}\ n : \mathbb{N},$

$(1) := \texttt{NoetherianFactor}(N'_n) : [N'_n : \texttt{Noetherian}],$

$\Big(m, P_n, (2)\Big) := \texttt{AssociatedPrimesExistInNoetherian}(A)(N'_n) :$

$\quad : \sum m \in M \,.\, \sum P_n : \texttt{Ideal}(A) \,.\, P_n = \mathrm{Ann}(m)\ \&\ N'_n \neq 0 \Rightarrow P_n : \texttt{Prime},$

$$\Big(S, (3)\Big) := \texttt{SubmoduleByIdeal}\Big(N'_n, P_n, (2)\Big) : \sum S \subset_{A\text{-MOD}} N'_N \,.\, S \cong_{A\text{-MOD}} \frac{A}{P_n},$$

$$N'_{n+1} := \frac{N'_n}{S} : A\text{-MOD},$$

$f_{n+1} := f_n \pi_S : M \xrightarrow{A\text{-MOD}} N'_{n+1};$

$\rightsquigarrow \Big(N', P, f, (1)\Big) := \texttt{IductiveConstr} : \sum N' : \mathbb{N} \to A\text{-MOD} \,.\, \sum P : \mathbb{N} \to \texttt{Ideal}(A) \,.$

$\quad \sum f : \prod n \in \mathbb{N} \,.\, M \xrightarrow{A\text{-MOD}} N'_n \,.\, \forall n \in \mathbb{N} \,.\, \Im f_n = N'_n\ \&\ N'_n \neq 0 \Rightarrow P_n : \texttt{Prime}(A)$

$$\quad \&\ \exists g : N'_n \xrightarrow{A\text{-MOD}} N'_{n+1} : \ker g \cong_{A\text{-MOD}} \frac{A}{P_n}\ \&\ f_{n+1} = f_n g,$$

$N := \ker f : \mathbb{N} \to \texttt{Submodule}(M),$

$(2) := \eth^{-1}\texttt{Nodecreasing}(1) : \Big[N : \texttt{Nondecreasing}\big(\mathbb{N}, \texttt{Submodule}(M)\big)\Big],$

$\Big(n, (3)\big) := \texttt{NoetherianACC}(N)(2) : \sum n \in \mathbb{N} \,.\, \forall k \in \mathbb{N} \,.\, k \geq N \Rightarrow N_k = N_n,$

$(4) := (3)(n+1) : N_n = N_{n+1},$

$(5) := (1)(n) : \mathrm{Im}\, f_{n+1} = N'_{n+1},$

$$\Big(g, (6)\Big) := (1)(n) : \sum g : N'_n \xrightarrow{A\text{-MOD}} N'_{n+1} \,.\, f_n g = f_{n+1}\ \&\ \ker g \cong \frac{A}{P_n},$$

$(7) := (6)_1(1)(3) : \ker g = \{0\},$

$$(8) := (6)_2(7) : \frac{A}{P_n} \cong \{0\},$$

$(9) := \eth\texttt{quotientRing}(A, P_n)(8) : P_n = A,$

$(10) := (1)\eth\texttt{Prime}(A)(9) : N'_n = 0,$

$(11) := \eth N \eth \ker(10) : N_n = M,$

$\texttt{Assume}\ k : (n-1),$

$(k.1) := (3)(k+1) : \mathrm{Im}\, f_{k+1} = N'_{n+1},$

$$\Big(g, (k.2)\Big) := (3)(k) : \sum g : N'_k \xrightarrow{A\text{-MOD}} N'_{k+1} \,.\, f_{k+1} = f_k g\ \&\ \ker g \cong \frac{A}{P_k},$$

$(k.*) := \eth N (k.2)_2 \texttt{CompositionKernel}(f_k, g)(k.1)\eth\texttt{qutientModule}(k.2)_2 :$

$$\quad : \frac{N_{k+1}}{N_k} = \frac{\ker f_{k+1}}{\ker f_k} = \frac{\ker f_k g}{\ker f_k} = \frac{\ker f_k \oplus \ker g}{\ker f_k} \cong \ker f \cong \frac{A}{P_k};$$

$\rightsquigarrow (*) := I(\forall) : \texttt{This}(n, N_{|n}),$

$\square$

$\texttt{listOfAssPrimes} :: \prod A : \texttt{Noetherian} . \prod M : \texttt{FinitelyGeneratedModule}(A) . \sum n \in \mathbb{N} . n \to \mathrm{Ass}(M)$

$\texttt{lisOfAssPrimes}\,() = (n(M), \mathfrak{p}(M)) := \texttt{PrimeFactorizationSeria}(M)$

$\texttt{AllTheAssesAreListed} :: \forall A : \texttt{Noetherian} . \forall M : \texttt{FinitelyGeneratedModule}(A) . \ \mathrm{Im}\,\mathfrak{p}(M) = \mathrm{Ass}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ M : \texttt{FinitelyGeneratedModule}(A),$

$\texttt{Assume}\ P : \mathrm{Ass}(M),$

$Q := \mathfrak{p}_1(M) : \mathrm{Ass}(M),$

$\Big(m, n, (M.P.1)\Big) := \Big(\eth\,\mathrm{Ass}(M)\Big)^2(P)(Q) : \sum m, n \in M \ . \ P = \mathrm{Ann}(m) \ \&\ Q = \mathrm{Ann}(n),$

$\texttt{Assume}\ (M.P.2) : P \neq Q,$

$\texttt{Assume}\ \alpha, \beta : A,$

$\texttt{Assume}\ (M.P.2.1) : \alpha m = \beta n,$

$a := \alpha m : M,$

$(M.P.2.1.1) := \texttt{IncreasingAnnihilator}\eth a(M.P.2.1) : \mathrm{Ann}(m) \subset \mathrm{Ann}(a) \ \&\ \mathrm{Ann}(n) \subset \mathrm{Ann}(a),$

$(M.P.2.1.2) := \eth\texttt{Subset}(M.P.2)(M.P.2.1.1) : \mathrm{Ann}(m) \subsetneq \mathrm{Ann}(a) \ \&\ \mathrm{Ann}(n) \subsetneq \mathrm{Ann}(a),$

$(M.P.2.1.3) := \texttt{MaximalAnnihilatorsAreAssociated}(M.P.2.1.2) : \mathrm{Ann}(a) = \mathrm{Ann}(0) = A,$

$(P.2.*) := \eth\,\mathrm{Ann}(a)\eth A\text{-}\mathsf{MOD}(M)(M.P.2.1.3) : a = 0;$

$\rightsquigarrow (M.*) := \texttt{InverseImplication}I(\Rightarrow)I(\forall)I(\Rightarrow)\eth^{-1}\texttt{span} : P = Q | \,\mathrm{span}(n) \cap \mathrm{span}(m) = \{0\};$

$\rightsquigarrow (*) := \eth(n(M), \mathfrak{p}(M)) : \texttt{This};$

$\square$

$\texttt{FGMHasFiniteAss} :: \forall A : \texttt{Noetherian} . \forall M : \texttt{FinitelyGeneratedModule}(A) . \,|\,\mathrm{Ass}(M)| < \infty$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{UnionOfAnnihilators} :: \forall A \in \mathsf{ANN} . \forall M \in A\text{-}\mathsf{MOD} . \bigcup_{m \in M : m \neq 0} \mathrm{Ann}(m) = \bigcup \mathrm{Ass}(M)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.9 Free Modules over PID

$\texttt{IdealIsLinearlyARingInPID} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall I : \texttt{Proper}(A) \,.\, I \cong_{A\text{-MOD}} A$

$\texttt{Proof} \ =$

$\Big(a, [1]\Big) := \eth\texttt{PrincipleIdealDomain}(A)(I) : \sum a \in A \,.\, I = \langle a \rangle,$

$[2] := \eth\texttt{Proper}(I)\eth\texttt{genIdeal}[1] : a \neq 0,$

$\varphi := \Lambda x \in A \,.\, xa : A \xrightarrow{A\text{-MOD}} I,$

$[3] := [2]\eth\texttt{IntegralDomain}(A) : \ker \varphi = \{0\},$

$[4] := \texttt{ZeroKernelTHM}(2) : (\varphi : A \hookrightarrow I),$

$[5] := \eth\texttt{genIdeal}[1] : (\varphi : A \twoheadrightarrow I),$

$[6] := \eth\mathsf{ANN}(A)[4][5] : (\varphi : A \xleftrightarrow{A\text{-MOD}} I),$

$[*] := \eth^{-1}\texttt{Isomorphic}(\mathsf{ANN})[6] : A \cong_{A\text{-MOD}} I;$

$\square$

$\texttt{NonZeroFunctionalExist} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall M : \texttt{FreeModule}(A) \,.$

$\quad .\, \forall N \subset_{A\text{-MOD}} \,.\, \forall [0] : N \neq \{0\} \,.\, \exists \varphi : M \xrightarrow{A} A \,.\, \varphi(N) \neq 0$

$\texttt{Proof} \ =$

$\Big(n, [1]\Big) := \eth\texttt{Singleton}[0] : \sum n \in N \,.\, n \neq 0,$

$(k, x) := \eth\texttt{FreeModule}(A)(M) : \sum k \in \mathsf{SET} \,.\, M \xleftrightarrow{A\text{-MOD}} A^k,$

$\Big(i, [2] := \eth\texttt{Iso}(x)[1] : \sum i \in k \,.\, x_i(n) \neq 0,$

$[*] := \eth\texttt{image}(x_i, N)[2] : x_i(N) \neq \{0\},$

$\square$

$\texttt{OneDimensionalSubspaceDecomposition} :: \forall A : \texttt{PrincipleIdealDomain} \,.$

$\quad .\, \forall M : \texttt{FreeModule}(A) \,.\, \forall N \subset_{A\text{-MOD}} M \,.\, \forall (0) : N \neq \{0\} \,.$

$\quad .\, \exists a \in A :: \exists m \in M : \exists n \in N : \exists M' \subset_{A\text{-MOD}} M : \exists N' \subset_{A\text{-MOD}} N :$

$\quad : n = am \ \&\ N' = M' \cap N \ \&\ M = \operatorname{span}(m) \oplus M' \ \&\ N = \operatorname{span}(n) \oplus N'$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \varphi : M \xrightarrow{A\text{-MOD}} A,$

$I_\varphi := \varphi(N) : \texttt{Submodule}(A),$

$[\varphi.*] := \eth\mathsf{ANN}(A)(I_\varphi) : \big(I_\varphi : \texttt{Ideal}(A)\big);$

$\rightsquigarrow I := I(\rightarrow) : M \xrightarrow{A\text{-MOD}} A \rightarrow \texttt{Ideal}(A),$

$[1] := \eth(M \xrightarrow{A\text{-MOD}} A)(0) : 0 \in \operatorname{Im} I,$

$[2] := \eth\texttt{NonEmpty}\big([1]\big) : \operatorname{Im} I \neq \emptyset,$

$J := \max I : \texttt{Ideal}(A),$

$\Big(\alpha, [3]\Big) := \eth J : \sum \alpha : M \xrightarrow{A\text{-MOD}} A \rightarrow \texttt{Ideal}(A) \,.\, \alpha(N) = J,$

$[4] := \texttt{NonZeroFunctionalExists}(A, M, N, [0])(\eth\alpha) : \alpha \neq 0,$

$\Big(a, [5]\Big) := \eth\texttt{PrincipleIdealDomain}(A)(J) : \sum a \in A : \langle a \rangle = J,$

$[6] := [3][4][5] : a \neq 0,$

$\Big(n, [7]\Big) := \eth\texttt{genIdeal}[3][4] : \sum n \in N \,.\, a = \alpha(n),$

$\texttt{Assume}\ \varphi : M \xrightarrow{A\text{-MOD}} A,$

$\Big(b, [\varphi.1]\Big) := \eth\texttt{PrincipleIdealDomain}(A)\langle a, \varphi(n)\rangle : \sum b \in A : \langle a, \varphi(n)\rangle = \langle b\rangle,$

$\Big(s, r, [\varphi.2]\Big]) := \eth\texttt{Ideal}[\varphi.1] : \sum s, r \in A \ .\ b = sa + r\varphi(n),$

$\psi := s\alpha + r\varphi : M \xrightarrow{A\text{-MOD}} A,$

$[\varphi.3] := \eth\psi[\varphi.2] : b \in \psi(N),$

$[\varphi.4] := [5][\varphi.2][\varphi.3] : J \subset \psi(N),$

$[\varphi.5] := \eth J[\varphi.4] : J = \psi(N),$

$[\varphi.*] := \eth\texttt{Principle}(J)[\varphi.5] : a|\varphi(n),$

$\rightsquigarrow [8] := I(\forall) : \forall \varphi : M \xrightarrow{A\text{-MOD}} A \ .\ a|\varphi(n),$

$(k, x) := \eth\texttt{FreeModule}(A)(M) : \sum k \in \mathsf{SET}\ .\ M \xleftrightarrow{A\text{-MOD}} A^k,$

$[9] := \forall i \in k \ .\ [8](x_i) : \forall i \in k \ .\ a|x_i(n),$

$\Big(r, [10]\Big) := \eth\texttt{Divides}[9] : \sum r : A^k \ .\ x(n) = ar,$

$m := x^{-1}(r) : M,$

$[11] := \eth m[10] : n = am,$

$[12] := [7][11]\eth A\text{-MOD}(M, A)(\alpha) : a = \alpha(n) = \alpha(am) = a\alpha(m),$

$[13] := \eth\texttt{IntegralDomain}(A)[12] : \alpha(m) = 1,$

$M' := \ker\alpha : \texttt{Submodule}(M),$

$N' := M' \cap N : \texttt{Submodule}(N),$

$\texttt{Assume}\ v : M,$

$[v.1] := \eth\texttt{Functor}\,(inverse, ()\,\alpha(v)m) + v : v = \alpha(v)m + (v - \alpha(v)m),$

$[v.2] := [13]\eth A\text{-MOD}(M, A)\alpha\eth\texttt{Inverse} : \alpha(v - \alpha(v)m) = \alpha(v) - \alpha(v) = 0,$

$[v.*] := [v.1][v.2]\eth^{-1}M' : (v - \alpha(v)m) \in M';$

$\rightsquigarrow [14] := \texttt{DirectDecmposition} : M = \langle m\rangle \oplus M',$

$\texttt{Assume}\ w : N,$

$\Big(c.[w.1]\Big) := \eth\texttt{genIdeal}[5](w) : \sum c \in A \ .\ \alpha(w) = ca,$

$[w.2] := [w.1][11] : \alpha(w)m = cam = cn \in \operatorname{span}(n),$

$[w.3] := \eth\texttt{inverse}(\alpha(w)m) + w : w = \alpha(w)m + (w - \alpha(w)m),$

$[w.4] := [13]\eth A\text{-MOD}(M, A)(\alpha)\eth\texttt{Inverse} : \alpha(w - \alpha(w)m) = \alpha(w) - \alpha(w) = 0,$

$[w.*] := [w.3][w.4]\eth\texttt{Sumodule}(M)(N)\eth M'\eth N' : (w - \alpha(w)m) \in N',$

$\rightsquigarrow [*] := \texttt{DirectDecomposition} : N = \operatorname{span}(n) \oplus N';$

$\square$

$$\texttt{SubmodOfFreeIsFree} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall F : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(A) \,.$$
$$.\, \forall M \subset_{A\text{-MOD}} F \,.\, M : \texttt{FreeModule}(A)$$

$\texttt{Proof} \ =$

$$V_0 := M : \texttt{Submodule}(F),$$
$$[1.0] := \eth V_0 : V_0 = M,$$
$$\texttt{Assume}\ n : \mathbb{N},$$
$$\texttt{Assume}\ [n.1] : V_{n-1} = \{0\},$$
$$v_n := 0 : V_{n-1},$$
$$V_n := V_{n-1} : \texttt{Submodule}(F),$$
$$[1.n] := [1.(n-1)]\eth V_n \eth v_n : M = \bigoplus_{i=1}^{n} \operatorname{span}(v_i) \oplus V_n;$$
$$\rightsquigarrow [n.1] := I(\Rightarrow) : V_{n-1} = \{0\} \Rightarrow (\ldots),$$
$$\texttt{Assume}\ [n.2] : V_{n-1} \neq \{0\},$$
$$\Big(\ldots, V_n, v_n, [n.3]\Big) := \texttt{OneDimensionalSubspaceDecompasition}(F, V_{n-1}, [n.2]) :$$
$$: \ldots \,.\, \sum V_n : \texttt{Submodule}(F) \,.\, \sum v_n \in V_{n-1} \,.\, V_{n-1} = \operatorname{span}(v_i) \oplus V_n,$$
$$[1.n] := [1.(n-1)][n.3] : M = \bigoplus_{i=1}^{n} \operatorname{span}(v_i) \oplus V_n;$$
$$\rightsquigarrow [n.*] := I(\Rightarrow)E(|)[n.1] : (\ldots);$$
$$\rightsquigarrow \big(V, v, [2]\big) := I\left(\sum\right) : \sum V : \mathbb{N} \to \texttt{Submodule}(F) \,.\, \sum v : \mathbb{N} \to M \,.\, \forall n \in \mathbb{N} \,.\, M = \bigoplus_{i=1}^{n} \{v_i\} \oplus V_n,$$
$$[3] := \eth\texttt{FinitelyGeneratedModule}(F)(v) : \exists k \in (\operatorname{rank} F + 1) \,.\, (v_i)_{i=1}^{k} \,!\, \texttt{LinearlyIndependent}(F),$$
$$[4] := \eth\texttt{InnerDirectSum}\eth(v, V, [2])[3] : M = \bigoplus_{i=1}^{k-1} \operatorname{span}(v_i),$$
$$[*] := \eth^{-1}\texttt{FreeModule}(A)\texttt{IdealIsLinearlyARingInPID}[4] : (M : \texttt{FreeModule}(A));$$

$\square$

$$\texttt{Snowball} :: \prod A : \texttt{IntegralDomain} \,.\, ?A^n$$
$$a : \texttt{Snowball} \iff \forall i \in (n-1) \,.\, a_{i-1} | a_i$$

$$\texttt{FreeSubmodBasisTHM} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall F : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(A) \,.$$
$$.\, \forall M \subset_{A\text{-MOD}} F \,.\, \forall [0] : M \neq \{0\} \,.\, \exists (e_i)_{i=1}^{n} : \texttt{Basis}(F) : (f_i)_{i=1}^{m} : \texttt{Basis}(M) \,.\, \exists a : \texttt{Snowball}(m, A) : ae_{|m} = f$$
$$\texttt{where} \quad n = \operatorname{rank} F, \quad m = \operatorname{rank} M$$

$\texttt{Proof} \ =$

$$V_0 := F : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(A),$$
$$W_0 := N : \texttt{Submodule}(V_0),$$
$$[1.0] := \eth V_0 : V_0 = F,$$
$$[2.0] := \eth W_0 : W_0 = M,$$
$$[3.0] := [0]\eth W_0 : W_0 \neq \{0\},$$
$$\texttt{Assume}\ k : m,$$
$$\Big(V_k, W_k, v_k, w_k, a_k, [k.1]\Big) := \texttt{OneDimensionalSubspaceDecomposition}\Big(V_{k-1}, W_{k-1}, [3.(k-1)]\Big) :$$
$$: \sum V_k \subset_{A\text{-MOD}} V_{k-1} \,.\, \sum W_k \subset_{A\text{-MOD}} W_{k-1} \,.\, \sum v_k \in V_{k-1} \,.\, \sum w_k \in W_{k-1} \,.\, \sum a_n \in A \,.$$
$$.\, w_k = a_k v_k \,\&\, \operatorname{span}(w_k) \oplus W_k = W_{k-1} \,\&\, \oplus \operatorname{span}(v_k) \oplus V_k = V_{k-1} \,\&\, W_k = V_k \cap W_{k-1},$$

$[k.2] := \texttt{SubmodOfFreeIsFree}(V_{k-1}, V_k) : (V_k : \texttt{FreeModule}(A)),$

$[1.k] := [k.1][1.(k-1)] : F = \bigoplus_{i=1}^{k} \operatorname{span}(v_i) \oplus V_k,$

$[2.k] := [k.1][2.(k-1)] : M = \bigoplus_{i=1}^{k} \operatorname{span}(w_i) \oplus W_k,$

$[3.k] := \eth m \eth \operatorname{rank}[2.k] : k = m | W_k \neq \{0\};$

$\rightsquigarrow \Big(V, W, v, w, a, [4]\Big) := I\Big(\sum\Big) : \sum V : m \to \texttt{Submodule}(F) \;.\; \sum W : m \to \texttt{Submodule}(M) \;.$

$\;.\; \sum v : \prod k \in m \;.\; V[k-1] \;.\; \sum w : \prod k \in m \;.\; W_{k-1} \;.\; \sum a : m \to A \;.$

$\;.\; w = av \;\&\; \forall k \in m \;.\; F = \bigoplus_{i=1}^{k} \operatorname{span}(v_i) \oplus V_k \;\&\; M = \bigoplus_{i=1}^{k} \oplus W_k \;\&\; W_k = V_k \cap V[k-1],$

$v' := \texttt{FreeHasBasis}(V_m) : \texttt{Basis}(V_m),$

$e := v \oplus v' : \texttt{Basis}(F),$

$f := w : \texttt{Basis}(M),$

$[5] := \eth f \eth 4[4]_1 : ae_{|m} = f,$

$\texttt{Assume}\ k : (m-1),$

$\varphi := \texttt{free}(\{k, n\}, A)(i \mapsto \texttt{if}\ i \leq k+1\ \texttt{then}\ 1\ \texttt{else}\ 0) : V_{i-1} \xrightarrow{A\text{-MOD}} A,$

$[k.1] := \eth \texttt{free} \eth \varphi : \varphi(e_k) = 1 \;\&\; \varphi(e_{k+1}) = 1,$

$[k.2] := [5] \eth A\text{-MOD}(V_{k-1}, A)(\varphi)[k.1] : \varphi(f_k) = \varphi(a_k e_k) = a_k \varphi(e_k) = a_k,$

$[\varphi.3] := \eth a_k \eth \texttt{OneDimensionalSubspaceDecomposition}[\varphi.2] : \varphi(W_{k-1}) = \langle a_k \rangle,$

$[\varphi.4] := [5] \eth A\text{-MOD}(V_{k-1}, A)(\varphi)[k.1] : \varphi(f_{k+1}) = \varphi(a_{k+1} e_{k+1}) = a_{k+1} \varphi(f_{k+1}) = a_{k+1},$

$[k.*] := [k.4][k.3] : a_k | a_{k+1};$

$\rightsquigarrow [*] := \eth^{-1} \texttt{Snowball} : \Big(a : \texttt{Snowball}(A, m)\Big);$

□

$\texttt{PIDExtendsToFreeResolution} :: \forall A : \texttt{PrincipleIdealDomain} \;.\; A : \texttt{FreeResolutionExtends1}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ M : \texttt{FinitelyGeneratedModule}(A),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ \pi : \texttt{Epi}((\text{-MOD}A), A^n, M),$

$[1] := \texttt{SubmodOfFreeIsFree}(A, \ker \pi) : (\ker \pi : \texttt{FreeModule}(A)),$

$(m, x) := \eth \texttt{FreeModule}(A)(\ker \pi) : \sum m \in \mathbb{N} \;.\; x : A^m \overset{\ker}{\longleftrightarrow} \pi,$

$[*] := \eth^{-1} \texttt{FreeResoluton} \eth x : 0 \xrightarrow{0} A^m \xrightarrow{x} A^n \xrightarrow{x} M \xrightarrow{0} 0 : \texttt{FreeResolution}(A, M);$

$\rightsquigarrow [*] := \eth^{-1} \texttt{FreeResolutionExtends1} : (A : \texttt{FreeResolutionExtend1});$

□

1.10 Classification of Finitely Generated Modules over PID

$\texttt{PidClassificationTheorem1} :: \forall A : \texttt{PrincipleIdealDomain} . \forall M : \texttt{FinitelyGeneratedModule}(A) .$

$$. \exists n \in \mathbb{N} : \exists a : \texttt{Snowball}(n, A) . M \cong A^{\operatorname{rank} M} \oplus \bigoplus_{i=1}^{n} \frac{A}{\langle a_i \rangle}$$

$\texttt{Proof} =$

$(m, \pi) := \eth\texttt{FinitelyGeneratedModule}(A)(M) : \sum m \in \mathbb{N} . \pi : \texttt{Epi}(A\text{-MOD}, A^m, M),$

$\left(n, x\right) := \texttt{PIDExtensToFreeResolution}(m, \pi, M) : \sum n \in \mathbb{N} . \sum x : A^n \xleftrightarrow{A\text{-MOD}} \ker \pi,$

$\left(e, a, f, [1]\right) := \texttt{FreeSubmodBasisTHM}(A, M, \ker \pi) : \sum e : \texttt{Basis}(A^m) . \sum a : \texttt{Snowball}(A, n) .$

$. \sum f : \texttt{Basis}(\ker \pi) . f = a e_{|n},$

$$[*] := [1]\eth \ker \pi \eth x \eth^{-1} \texttt{InnerDirectSum} : M = A^{\operatorname{rank} M} \oplus \bigoplus_{i=1}^{n} \frac{A}{\langle a_i \rangle};$$

$\square$

$\texttt{PidClassificationTheorem2} :: \forall A : \texttt{PrincipleIdealDomain} . \forall M : \texttt{FinitelyGeneratedModule}(A) .$

$$. \exists n \in \mathbb{N} : \exists p : n \to \texttt{PrimeElement}(A) : \exists k : n \to \mathbb{N} : \exists t : \prod i \in n . k_i \to \mathbb{N} : M \cong A^{\operatorname{rank} M} \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} \frac{A}{\langle p_i^{t_j} \rangle}$$

$\texttt{Proof} =$

$$a_i = \prod_{j=1}^{n} p_{i,j}^{k_j}$$

$$\frac{A}{\langle a_i \rangle} = \bigoplus_{j=1}^{n} \frac{A}{\langle p_{i,j}^{t_j} \rangle}$$

$\square$

$\texttt{snowballOfMod} :: \prod A : \texttt{PrincipleIdealDomain} . \texttt{FinitelyGeneratedModule}(A) \to$

$\to \sum n \in \mathbb{Z}_+ . \texttt{Snowball}(A, n)$

$\texttt{snowballOfMod}\,(M) := \texttt{PidClassificationTheorem1}(A, M)$

$\texttt{primesOfMod} :: \prod A : \texttt{PrincipleIdealDomain} . \texttt{FinitelyGeneratedModule}(A) \to$

$\to \sum n \in \mathbb{N} : \sum p : n \hookrightarrow \texttt{PrimeElement}(A) : \sum k : n \to \mathbb{N} : \sum t : \prod i \in n . k_i \to \mathbb{N}$

$\texttt{primesOfMod}\,(M) := \texttt{PidClassificationTheorem2}(A, M)$

$\texttt{AnnihilatorBySnowball} :: \forall A : \texttt{PrincipleIdealDomain} . \forall M : \texttt{FinitelyGeneratedModule} \,\&\, \texttt{Torsion}(A) .$

$. \operatorname{Ann}(M) = \langle a_n \rangle \quad \texttt{where} \quad (n, a) = \texttt{snowballOfMod}(M)$

$\texttt{Proof} =$

$\ldots$

$\square$

1.11 Graded Modules

$\texttt{LeftGradedModule} :: \prod \Delta : \texttt{CommutativeMonoid} . \prod (R, H) \in \mathsf{GRING}(\Delta) .$

$\quad . ? \sum M \in R\text{-}\mathsf{MOD} . \Delta \to \texttt{LeftSubmodule}(M)$

$(M, S) : \texttt{LeftGradedModule} \iff (M, \Delta, S) : \texttt{GradedAbelean} \; \& \; \forall a, b \in \Delta . H_a S_b \subset S_{a+b}$

$\texttt{RightGradedModule} :: \prod \Delta : \texttt{CommutativeMonoid} . \prod (R, H) \in \mathsf{GRING}(\Delta) .$

$\quad . ? \sum M \in \mathsf{MOD}\text{-}R . \Delta \to \texttt{RightSubmodule}(M)$

$(M, S) : \texttt{RightGradedModule} \iff (M, \Delta, S) : \texttt{GradedAbelean} \; \& \; \forall a, b \in \Delta . S_b H_a \subset S_{a+b}$

$\texttt{GradedModule} :: \prod \Delta : \texttt{CommutativeMonoid} . \prod (R, H) \in \mathsf{GRING}(\Delta) .$

$\quad . ? \sum M \in R\text{-}\mathsf{MOD} \; \& \; \mathsf{MOD}\text{-}R . \Delta \to \texttt{Submodule}(M)$

$(M, S) : \texttt{GradedModule} \iff (M, \Delta, S) : \texttt{GradedAbelean} \; \& \; \forall a, b \in \Delta . H_a S_b \cup S_b H_a \subset S_{a+b}$

$\texttt{HomogeneousSubmodule} :: \forall (M, S) : \texttt{LeftGadedModule}(\Delta, R, H) . \forall [0] : (\Delta : \texttt{Cancelable}) .$

$\quad . \forall \delta \in \Delta . M_\delta : R_0\text{-}\mathsf{MOD}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{GradedLeftModuleHomo} :: \prod (M, S), (M', S') : \texttt{LeftGradedModule}(\Delta, R, H) . ? M \xrightarrow{R\text{-}\mathsf{MOD}} M'$

$f : \texttt{GradedLeftModuleHomo} \iff \exists \delta \in \Delta : . \forall \alpha \in \Delta . f\Big(S_\alpha\Big) \subset S_{\alpha+\delta}$

$\texttt{GradedRightModuleHomo} :: \prod (M, S), (M', S') : \texttt{RightGradedModule}(\Delta, R, H) . ? M \xrightarrow{\mathsf{MOD}\text{-}R} M'$

$f : \texttt{GradedRightModuleHomo} \iff \exists \delta \in \Delta : . \forall \alpha \in \Delta . f\Big(S_\alpha\Big) \subset S_{\alpha+\delta}$

$\texttt{degreeOfMorphism} :: \prod (M, S), (M', S') : \texttt{GradedLeftModule}(R, \Delta, H) . \texttt{GradedLeftModuleHomo}(R, \Delta, H) \to$

$\texttt{degreeOfMorphism}(\varphi) = \deg \varphi := \eth \texttt{GradedRightModuleHomo} R H \Delta (M, S)(M', S')(\varphi)$

$\texttt{degreeOfMorphism} :: \prod (M, S), (M', S') : \texttt{GradedRightModule}(R, \Delta, H) . \texttt{GradedRightModuleHomo}(R, \Delta, H)$

$\texttt{degreeOfMorphism}(\varphi) = \deg \varphi := \eth \texttt{GradedRightModuleHomo} R H \Delta (M, S)(M', S')(\varphi)$

$\texttt{degreeOfMorphismComp} :: \forall (M, S), (M', S'), (M'', S'') \in \texttt{GradedLeftModule} .$

$\quad . \forall \varphi : \texttt{GradedLeftModuleHomo}(R, \Delta, H)(M, S)(M', S') .$

$\quad . \forall \psi : \texttt{GradedLeftModuleHomo}(R, \Delta, H)(M', S')(M'', S'') . \deg \varphi \psi = \deg \varphi + \deg \psi$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{categoryOfGradedLeftModules} :: \mathsf{GRING} \to \mathsf{CAT}$
$\texttt{categoryOfGradedLeftModules}(R,\Delta,H) = (R,H)\text{-}\mathsf{GMOD}(\Delta) :=$
$$:= \Big(\texttt{GradedLeftModules}(R,\Delta,H),$$
$$,\{f : \texttt{GradedLeftModuleHomo}(R,\Delta,H) : \deg f = 0\}, \circ, \mathrm{id}\Big)$$

$\texttt{categoryOfGradedRightModules} :: \mathsf{GRING} \to \mathsf{CAT}$
$\texttt{categoryOfGradedRightModules}(R,\Delta,H) = \mathsf{GMOD}(\Delta)\text{-}(R,H) :=$
$$:= \Big(\texttt{GradedRightModules}(R,\Delta,H),$$
$$,\{f : \texttt{GradedRightModuleHomo}(R,\Delta,H) : \deg f = 0\}, \circ, \mathrm{id}\Big)$$

$\texttt{GradedSubmodule} :: \prod (M,S) \in (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.\, ?\,(R,H)\text{-}\mathsf{GMOD}(\Delta)$
$(N,Z) : \texttt{GradedSubmodule} \iff (N,Z) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M,S) \iff N \subset M \;\&\; \forall \delta \in \Delta \,.\, Z_\delta = S_\delta \cap N$

$\texttt{GradedSubmoduleByHomogeneousPart} :: \forall (M,S),(N,Z) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.$
$\quad . \,(M,S) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (N,Z) \iff \forall n \in N \,.\, \forall \delta \in \Delta \,.\, n_\delta \in N$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{GeneratedGratedSubmodule} :: \forall (M,S),(N,Z) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.$
$\quad . \,(N,Z) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M,S) \iff \exists A :? \texttt{Homogeneous}(M) \,.\, N = \mathrm{span}(A)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{HomogeneousGeneration} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.\, \forall (N,Z) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M,S) \,.$
$\quad . \,\forall A : \texttt{Generating}(N) \,.\, \{a_\delta | a \in A, \delta \in \Delta\} : \texttt{Generating}(N)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{FiniteHomogeneousGeneration} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.$
$\quad . \,\forall [0] : \big(M : \texttt{FinitelyGeneratedModule}(R)\big) \,.\, \exists F :? \texttt{Homogeneous} \;\&\; \texttt{Finite} \;\&\; \texttt{Generating}(M,\Delta,H)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{GradedQuotient} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \,.\, \forall (N,Z) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M,S) \,.$
$$. \left(\frac{M}{N}, \left(\frac{S_\delta + N}{N}\right)_{\delta \in \Delta}\right) : (R,H)\text{-}\mathsf{GMOD}(\Delta)$$
$\texttt{Proof} =$

$\texttt{GradedImage} :: \forall (M,S),(M',S') : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall \phi : (M,S) \xrightarrow{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M',S') \;.$

$\quad .\; (\phi(M), \phi(S)) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M',S')$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedKernel} :: \forall (M,S),(M',S') : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall \phi : (M,S) \xrightarrow{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M',S') \;.$

$\quad .\; \forall [0] : \deg \phi : \texttt{Cancelable}(\Delta) \;.\; (\ker \varphi, \ker \varphi \cap S) \subset_{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M,S)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedIsomorphismTheorem} :: \forall (M,S),(M',S') : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall \phi : (M,S) \xrightarrow{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (M',S') \;.$

$\quad .\; \forall [0] : \deg \phi = 0 \;.\; \exists \psi : \left(\frac{M}{\ker \phi}, \frac{S + \ker \phi}{\ker \phi} \right) \xleftrightarrow{(R,H)\text{-}\mathsf{GMOD}(\Delta)} (\mathrm{Im}(\phi), \mathrm{Im}(\phi) \cap S') : \forall x \in M \;.\; \psi[x] = \phi(x)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedSum} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall I \in \mathsf{SET} \;.$

$\quad .\; \forall (N,Z) : I \to \texttt{GradedLeftSubmodule}(R,\Delta,H)(M,S) \;.\; \left(\sum_{i \in I} N_i, \sum_{i \in I} Z_i \right) : (R,H)\text{-}\mathsf{GMOD}(\Delta)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedIntersect} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall I \in \mathsf{SET} \;.$

$\quad .\; \forall (N,Z) : I \to \texttt{GradedLeftSubmodule}(R,\Delta,H)(M,S) \;.\; \left(\bigcap_{i \in I} N_i, \bigcap_{i \in I} Z_i \right) : (R,H)\text{-}\mathsf{GMOD}(\Delta)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedAnnihilator} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall \delta : \texttt{Cancelable}(\Delta) \;.\; \forall x \in S_\delta \;.\; \mathrm{Ann}(x) : \texttt{GradedLeftIde}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedAnnihilator} :: \forall (M,S) : (R,H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall \delta : \texttt{Cancelable}(\Delta) \;.\; \forall x \in S_\delta \;.\; \mathrm{Ann}(x) : \texttt{GradedLeftIde}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GradedAnnihilator} :: \forall (M, S) : (R, H)\text{-}\mathsf{GMOD}(\Delta) \;.\; \forall [0] : \big(\Delta : \texttt{Cancelable}(\Delta)\big) \;.$

$\quad . \; \mathrm{Ann}(M) : \texttt{GradedLeftIdeal}(R, \Delta, H)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2 Linear Operators and Matrices

2.1 Matrices as Tables

$\texttt{transpose} :: \prod X, n, m \in \mathsf{SET} \; . \; X^{n\times m} \to X^{m\times n}$

$\texttt{transpose}\,(A) = A^\top := \Lambda(j,i) \in m \times n \; . \; A_{i,j}$

$\texttt{Symmetric} :: \prod X, n \in \mathsf{SET} \; . \; ?X^{n\times n}$

$A : \texttt{Symmetric} \iff A^\top = A$

$\texttt{column} :: \prod X, n, m \in \mathsf{SET} \; . \; X^{n\times m} \to m \to X^n$

$\texttt{column}\,(A, j) = \mathcal{C}_j(A) := \Lambda i \in n \; . \; A_{i,j}$

$\texttt{row} :: \prod X, n, m \in \mathsf{SET} \; . \; X^{n\times m} \to n \to X^m$

$\texttt{row}\,(A, i) = \mathcal{R}_i(A) := \Lambda j \in m \; . \; A_{i,j}$

$\texttt{fromColumns} :: \prod X, n, m \in \mathsf{SET} \; . \; m \to X^n \to X^{n\times m}$

$\texttt{fromColumns}\,(C) := \Lambda(i,j) \in n \times m \; . \; C_i(j)$

$\texttt{fromRows} :: \prod X, n, m \in \mathsf{SET} \; . \; n \to X^m \to X^{n\times m}$

$\texttt{fromRows}\,(R) := \texttt{fromColumns}(R)^\top$

$\texttt{diagonal} :: \prod X, n \in \mathsf{SET} \; . \; X^{n\times n} \to X^n$

$\texttt{diagonal}\,(A) = \operatorname{diag}(A) := \Lambda i \in n \; . \; A_{i,i}$

$\texttt{Submatrix} :: \prod X, n, m, n', m' \in \mathsf{SET} \; . \; X^{n\times m} \to ?X^{n'\times m'}$

$A' : \texttt{Submatrix} \iff \Lambda A \in X^{n\times m} \; . \; \exists k : n' \hookrightarrow n : \exists l : m' \hookrightarrow m \; . \; A' = A_{k,l}$

$\texttt{fromBlocks} :: \prod X, n, m, k, l \in \mathsf{SET} \; . \; \prod p : \texttt{Partition}(n,k) \; . \; \prod q : \texttt{Partition}(m,l) \; .$
$\quad . \left(\prod (i,j) \in k \times l \; . \; X^{p_i \times q_j}\right) \to X^{n\times k}$

$\texttt{fromBlocks}\,(B) := \Lambda(i,j) \in n \times m \; . \; B_{i,j}(i', j')$
$\quad \textbf{where} \quad i' = \eth\texttt{Partition}(n,k)(p)(i), j' = \eth\texttt{Partition}(m,l)(q)(j)$

$\texttt{TransposeIsConvolution} :: \forall X, n, m \in \mathsf{SET} \; . \; \forall A \in A^{n\times m} \; . \; A^{\top\top} = A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{blockDiagonal} :: \prod R \in \mathsf{RING} \; . \; \prod n, k \in \mathsf{SET} \; . \; \prod p : \texttt{Partition}(n,k) \; . \; \left(\prod i \in k \; . \; X^{p_i \times p_i}\right) \to X^{n, \times n}$

$\texttt{blockDiagonal}\,(A) := \texttt{fromBlocks}(X, n, m, k, l)(p,p)(\Lambda(i,j) \in n \times n \; . \; \textbf{if } i == j \textbf{ then } A_{i,j} \textbf{ else } 0)$

2.2 Elementary Matrix algebra

$\texttt{matrixMult} :: \prod R \in \mathsf{RING} \;.\; \prod n, m, k \in \mathbb{Z} \;.\; R^{n\times m} \times R^{m\times k} \to R^{n\times k}$

$\texttt{matrixMult}\,(A, B) = AB := \Lambda(i, j) \in n \times k \;.\; \mathcal{R}_i(A)\mathcal{C}_j(B)$

$\texttt{matrixMultIsAssoc} :: \forall R \in \mathsf{RING} \;.\; \forall n : 4 \to \mathbb{Z} \;.\; \forall A \in R^{n_1\times n_2} \;.\; \forall B \in R^{n_2\times n_3} \;.\; \forall C \in R^{n_3\times n_4} \;.$

$\quad .\; (AB)C = A(BC)$

$\texttt{Proof} \;=$

$\texttt{Assume}\;(i, j) : n_1 \times n_4,$

$[j.*] := \eth\texttt{matixMult}((AB), C)\eth\mathcal{R}_i\eth\mathsf{RING}(R)\eth^{-1}C_j(BC)\eth^{-1}\texttt{matrixMult} :$

$$: \Big((AB)C\Big)_{i,j} = \mathcal{R}_i(AB)\mathcal{C}_j(C) = \sum_{t=1}^{n_3} \mathcal{R}_i(A)\mathcal{C}_t(B)C_{t,j} = \mathcal{R}_i(A)\sum_{t=1}^{n_3} \mathcal{C}_t(B)C_{t,j} =$$

$$= \mathcal{R}_i(A)\mathcal{C}_j(BC) = \Big(A(BC)\Big)_{i,j};$$

$\leadsto [*] := I(=, \to) : (AB)C = A(BC);$

$\square$

$\texttt{MatrixAlgebra} :: \forall R \in \mathsf{RING} \;.\; \forall n \in \mathbb{N} \;.\; \Big(R^{n\times n}, \texttt{matrixMult}\Big) : \texttt{AssociativeAlgebra}(R)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{fromDiag} :: \prod R \in \mathsf{RING} \;.\; \prod n \in \mathbb{N} \;.\; R^n \to R^{n\times n}$

$\texttt{fromDiag}\,(a) := \Lambda(i, j) \in n \times n \;.\; \texttt{if}\; i == j \;\texttt{then}\; a_i \;\texttt{else}\; 0$

$\texttt{identityMatrix} :: \prod R \in \mathsf{RING} \;.\; \prod n \in \mathbb{N} \;.\; R^n$

$\texttt{indentityMatrix}\,() = I := \texttt{fromDiag}(i \mapsto 1)$

$\texttt{IdentityMatrixIsIdentity} :: \forall R \in \mathsf{RING} \;.\; \forall n \in \mathbb{N} \;.\; \forall A \in R^{n\times n} \;.\; IA = A = AI$

$\texttt{Proof} \;=$

$\texttt{Assume}\; i, j : n \times n,$

$[j.*] := \eth\texttt{matrixMult}(i, j)\eth\texttt{identityMatrix} : (IA)_{i,j} = \mathcal{R}_i(I)\mathcal{C}_j(A) = A_{i,j} = \mathcal{R}_i(A)\mathcal{C}_j(I) = (AI)_{i,j};$

$\leadsto [*] := I(=, \to) : IA = A = AI,$

$\square$

$\texttt{MatrixAlgebra2} :: \forall R \in \mathsf{RING} \;.\; \forall n \in \mathbb{N} \;.\; \Big(R^{n\times n}, \texttt{matrixMult}, I\Big) : \texttt{UnitaryAlgebra}(R)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{IdentityIsSymmetric} :: \forall R \in \mathsf{RING} \;.\; \forall n \in \mathbb{N} \;.\; I_{R,n} : \texttt{Symmetric}(R, n, n)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{TransposeIsLinear} :: \forall R \in \mathsf{RING} \,.\, \forall n, m \in \mathbb{N} \,.\, \texttt{transpose}(R, n, m) : R^{n\times m} \xrightarrow{R\text{-}\mathsf{MOD}} R^{m\times n}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{TransposeIsAlgebraAntihomomorphism} :: \forall R \in \mathsf{ANN} \,.$

$\quad .\, \forall n, m, k \in \mathbb{N} \,.\, \forall A \in R^{n\times m} \,.\, \forall B \in R^{m\times k} \,.\, (AB)^\top = B^\top A^\top$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (i, j) : k \times n,$

$j.* := \eth\texttt{transpose}(AB)\eth\texttt{MatrixMult}\eth^{-2}\texttt{transpose}(A)(B)\eth\mathsf{ANN}(R)\eth^{-1}\texttt{matrixMult} :$

$\quad : (AB)^\top_{i,j} = (AB)_{j,i} = \mathcal{R}_j(A)\mathcal{C}_i(B) = \mathcal{C}_j(A^\top)\mathcal{R}_i(B^\top) = \mathcal{R}_i(B^\top)\mathcal{C}_j(A^\top) = \left(B^\top A^\top\right)_{i,j};$

$\rightsquigarrow [*] := I(=m, \to) : (AB)^\top = B^\top A^\top;$

$\square$

$\texttt{rowExchange} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, n \times n \to R^{n\times n}$

$\texttt{rowExchange}\,(i, j) := \texttt{fromRows}(\Lambda k \in n \,.\, \texttt{if}\ k == i\ \texttt{then}\ \delta^j\ \texttt{else if}\ k == j\ \texttt{then}\ \delta^i\ \texttt{else}\ \delta^k)$

$\texttt{rowScalarMult} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, n \to R^* \to R^{n\times n}$

$\texttt{rowScalarMult}\,(i, \alpha) := \texttt{fromRows}(\Lambda k \in n \,.\, \texttt{if}\ k == i\ \texttt{then}\ \alpha\delta^i\ \texttt{else}\ \delta^k)$

$\texttt{rowAddScalarMult} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, n \to n \to R \to R^{n\times n}$

$\texttt{rowAddScalarMult}\,(i, j, \alpha) := I + \texttt{FromRows}(\Lambda k \in n \,.\, \texttt{if}\ k == i\ \texttt{then}\ \alpha\delta^j\ \texttt{else}\ 0)$

$\texttt{ElementaryRowOperation} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?R^{n\times n}$

$E : \texttt{ElementaryRowOperation} \iff \exists i, j \in n : \exists \alpha \in R^* : \exists \beta \in R \,.\, E = \texttt{rowExchange}(i, j)$

$\quad | E = \texttt{rowScalarMult}(i, \alpha)$

$\quad | E = \texttt{rowAddScalarMult}(i, j, \beta)$

$\texttt{RowEquivalent} :: \prod R \in \mathsf{RING} \,.\, \prod n, m \in \mathbb{N} \,.\, ?\left(R^{n\times m} \times R^{n\times m}\right)$

$$A, B : \texttt{RowEquivalent} \iff \exists N \in \mathbb{Z}_+ \,.\, \exists E : N \to \texttt{ElementaryRowOperations}(R, n) : \left(\prod_{i=1}^N E_i\right) A = B$$

$\texttt{ColumnEquivalent} :: \prod R \in \mathsf{RING} \,.\, \prod n, m \in \mathbb{N} \,.\, ?\left(R^{n\times m} \times R^{n\times m}\right)$

$$A, B : \texttt{ColumnEquivalent} \iff \exists N \in \mathbb{Z}_+ \,.\, \exists E : N \to \texttt{ElementaryRowOperations}(R, m) : A\left(\prod_{i=1}^N E_i\right) = B$$

$\texttt{RowColumnEquivalent} :: \prod R \in \mathsf{RING} \,.\, \prod n, m \in \mathbb{N} \,.\, ?\left(R^{n\times m} \times R^{n\times m}\right)$

$A, B : \texttt{RowColumnEquivalent} \iff \exists N, N' \in \mathbb{Z}_+ \,.\, \exists E : N \to \texttt{ElementaryOperations}(R, n) :$

$$: \exists E' : N' \to \texttt{ElementaryOperation}(R, m) : \left(\prod_{i=1}^N E_i\right) A \left(\prod_{i=1}^N E'_i\right) = B$$

$\texttt{SmithNormalForm} :: \prod R \in \texttt{IntegralDomain} \,.\, \prod n, m \in \mathbb{N} \,.\, ?R^{n\times m}$

$A : \texttt{SmithNormalForm} \iff \exists r \in \min(n,m) \,.\, \exists a : \texttt{Snowball}(R, r) \,.$

$\quad A = \texttt{fromColumns}(\Lambda i \in n \,.\, \Lambda j \in m \,.\, \texttt{if}\ i == j\ \&\ i \leq r\ \texttt{then}\ a_i\ \texttt{else}\ 0)$

$\texttt{SmithNormalFormTHM} :: \forall R \in \texttt{EuclideanRing} \,.\, \forall n, m \in \mathbb{N} \,.\, \forall X \in R^{n\times m} \,.\, \exists A : \texttt{SmithNormalForm}(R, n, m) :$

$\quad \Big((X, A) : \texttt{RowColumnEquivalent}(R, n, m)\Big)$

$\texttt{Proof} \ =$

$M^{0,0} := X : R^{n\times m},$

$♂ := \Lambda K \in \min(n,m) \,.\, \forall k \in [1, K]_{\mathbb{N}} \,.\, \forall j \in n \,.\, \forall j' \in m \,.$

$\quad .\, j \neq k \Rightarrow M^{K,0}_{j,k} = 0\ \&\ j' \neq k \Rightarrow M^{K,0}_{k,j'} = 0 : \min(n,m) \to \texttt{Type},$

$♀ := \Lambda K \in \min(n,m) \,.\, \forall i \in k \,.\, \forall j \in i \,.\, M^{K,0}_{j,j} | M^{K,0}_{i,i} : \min(n,m) \to \texttt{Type},$

$1.0 := ð\emptyset ð ♂ : ♂(0),$

$2.0 := ð\emptyset ð ♀ : ♀(0),$

$\texttt{Assume}\ K : \min(n,m),$

$\texttt{Assume}\ [1.(K-1)] : ♂(K-1),$

$\texttt{Assume}\ [2.(K-1)] : ♀(K-1),$

$\texttt{Assume}\ [3.0] : \exists (i,j) \in [k,n] \times [k,m] : M^{(K-1),0} \neq 0,$

$v_0 := +\infty : \mathbb{Z}_+ \cup \{+\infty\},$

$\texttt{Assume}\ k : \mathbb{N},$

$\Big(I, J, [4]\Big) := \texttt{ArgMinExists}([3.k-1])(\texttt{normOfEuclid}(R)(M)) :$

$\quad : \sum (I,J) \in [K,n] \times [K,m] \,.\, M^{K,(k-1)}_{I,J} \neq 0\ \&\ \forall (i,j) \in [K,n] \times [K,m] \,.\, |M^{(K-1),(k-1)}_{I,J}| \leq |M^{(K-1)),(k-1)}_{i,j}|,$

$N := \texttt{RowExchange}(K, I) M^{(K-1),(k-1)} : R^{n\times m},$

$v_k := |N_{K,K}| : \mathbb{Z}_+,$

$\mathcal{I} := \{i \in [K+1, n] : N_{i,K} \neq 0\} :?[K+1, n],$

$\mathcal{J} := \{j \in [K+1, m] : N_{K,j} \neq 0\} :?[K+1, n] \neq 0,$

$\texttt{Assume}\ i : \mathcal{I},$

$\Big(a_i, r_i, [i.*]\Big) := ð\texttt{EuclideanRing}(R)(N_{i,K}, N_{K,K}) : \sum a_i \in R \,.\, \sum r_i \in R \,.\, N_{i,K} = a_i N_{K,K} + r_i |r_i| < N_{K,K};$

$\rightsquigarrow \Big(a, r, [k.1]\Big) := I\left(\sum\right) : \sum (a, r) : \mathcal{I} \to R^2 \,.\, \forall i \in I \,.\, N_{i,K} = a_i N_{K,K} + r_i |r_i| < |N_{K,K}|,$

$\texttt{Assume}\ j : \mathcal{J},$

$\Big(b_j, r'_j, [j.*]\Big) := ð\texttt{EuclideanRing}(R)(N_{K,j}, N_{K,K}) : \sum b_j \in R \,.\, \sum r'_j \in R \,.\, N_{K,j} = b_j N_{K,K} + r'_j |r'_j| < |N_{K,K}|;$

$\rightsquigarrow \Big(b, r', [k.2]\Big) := I\left(\sum\right) : \sum (b, r') : \mathcal{J} \to R^2 \,.\, \forall j \in J \,.\, N_{K,j} = a_j N_{K,K} + r'_j |r'_j| < |N_{K,K}|,$

$\texttt{Assume}\ [5] : \mathcal{I} \neq \emptyset | \mathcal{J} \neq \emptyset,$

$M^{(K-1),k} := \left(\prod_{i\in\mathcal{I}} \texttt{rowAddScalarMult}(i, K, -a_i)\right) N \left(\prod_{j\in\mathcal{J}} \texttt{rowAddScalarMult})(j, K, -b_k) : R^{n\times m},\right.$

$[3.k] := ðNðM^{(K-1),k} : M^{(K-1),k}_{K,K} \neq 0;$

$\texttt{Assume}\ [5] : \mathcal{I} = \emptyset = \mathcal{J},$

$\texttt{Assume}\ (i,j) : [K+1, n] \times [K+1, m],$

$\texttt{Assume}\ [6] : N_{(K-1),K} \nmid N_{i,j},$

$M^{(K-1),k} := \texttt{rowAddScalarMult}(K, i, 1)N : R^{n\times m},$

$[3.k] := \eth N[5]\eth M^{(K-1),k} : M^{(K-1),k}_{K,K} \neq 0;$

$\texttt{Assume}\ [6] : \forall (i, j) \in [K + 1, n] \ .\ N_{K,K}|N_{i,j},$

$M^{(K-1),k} := N : R^{n\times m},$

$[3.k] := \eth N\eth M^{(K-1),k} : M^{(K-1),k}_{K,K} \neq 0;$

$\rightsquigarrow \left(M^{(K-1),v,[K.1]}\right) := I\left(\sum\right) : \sum M^{(K-1)} : \mathbb{N} \to R^{n\times m} \ .\ \sum v : \texttt{Nonincreasing}(\mathbb{N}, \mathbb{Z}_+) \ .\ \ldots,$

$\left(k, [4]\right) := \eth\texttt{WellOrdered}(\mathbb{Z}_+)(v) : \sum k \in \mathbb{N} : \forall k' : \texttt{after}(k) \ .\ v_k = v'_k,$

$[5] := [4][K.1] : \forall k' : \texttt{after}(k) \ .\ M^{(K-1),k'} = M^{(K-1),k')},$

$M^{K,0} := M^{K-1,k} : R^{n\times m},$

$1.K := [5][K.1][1.(K-1)] : ♂(K),$

$2.K := [5][K.1][2.(K-1)] : ♀(K);$

$\texttt{Assume}\ [3.0] : \forall (i, j) \in [K, n] \times [K, m] \ .\ M^{(K-1),(k-1)} = 0,$

$M^{K,0} := M^{K-1,0} : R^{n\times m},$

$1.K := \eth M^{K,0}[3.0][1.K-1] : ♂(K),$

$2.K := \eth M^{K,0}[2.0][2.K-1] : ♀(K);$

$\rightsquigarrow (M, [1], [2]) := I(\left(\sum\right))I\left(\prod\right) : \sum M : \min(n, m) \to R^{n,m} \ .\ \prod k \in \min(n, m) \ .\ ♂(k)♀(k),$

$A := M^{\min(n,m)} : R^{n,m},$

$[3] := \eth^{-1}\texttt{SmithNormalForm}\eth A1.\min(n, m)2.\min(n, m) : (A : \texttt{SmithNormalForm}),$

$[*] := \eth M\eth A : \left((X, A) : \texttt{RowColumnEquivalent}(R, n, m)\right);$

$\square$

2.3 Matrices as Linear Operators, Change of Basis

$$\texttt{matrixOfOperator} :: \prod R \in \mathsf{RING} \,.\, \prod A, B : \texttt{FreeModule}(R) \,.$$
$$.\, (A \xrightarrow{R\text{-}\mathsf{MOD}} B) \to \texttt{Basis}(A) \to \texttt{Basis}(B) \to R^{\dim B \times \dim A}$$
$$\texttt{matrixOfOperator}\,(T, e, f) = T^{e,f} := \Lambda x \in \dim A \,.\, \Lambda y \in \dim B \,.\, r_y \quad \texttt{where} \quad Ae_x = \sum_{y \in Y} r_y f_y$$

$$\texttt{operatorFromMatrix} :: \prod R \in \mathsf{RING} \,.\, \prod A, B : \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$$
$$.\, R^{\dim B \times \dim A} \to \texttt{Basis}(A) \to \texttt{Basis}(B) \to (A \xrightarrow{R\text{-}\mathsf{MOD}} B)$$
$$\texttt{operatorFromMatrix}\,(M, e, f) = M_{e,f} := \Lambda ae \in A \,.\, a_i M_{j,i} f_j$$

$$\texttt{ChoiceOfBasisDefinesIso} :: \forall R \in \mathsf{RING} \,.\, \forall A, B : \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$$
$$.\, \forall e : \texttt{Basis}(A) \,.\, \forall f : \texttt{Basis}(B) \,.\, (\cdot)^{e,f} : R\text{-}\mathsf{MOD}(A, B) \xleftrightarrow{R\text{-}\mathsf{MOD}} R^{\dim B \times \dim A}$$

$\texttt{Proof} \;=$

. . .

□

$$\texttt{MatricesAreLinearMaps} :: \forall R \in \mathsf{RING} \,.\, \forall A, B : \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$$
$$.\, R\text{-}\mathsf{MOD}(A, B) \cong_{R\text{-}\mathsf{MOD}} R^{\dim B \times \dim A}$$

$\texttt{Proof} \;=$

. . .

□

$$\texttt{ChoiceOfBasisDefinesAlgIso} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$$
$$\forall e : \texttt{Basis}(A) \,.\, (\cdot)^{e,e} : \mathrm{End}_{R\text{-}\mathsf{MOD}}(A) \xleftrightarrow{R\text{-}\mathsf{ALG}} R^{\dim A \times \dim A}$$

$\texttt{Proof} \;=$

$n := \operatorname{rank} A : \mathbb{Z}_+,$

$\texttt{Assume}\; T, S : \mathrm{End}_{R\text{-}\mathsf{MOD}}(A),$

$\texttt{Assume}\; i : n,$

$(a_{\cdot,i}, [1.i]) := \eth\texttt{Basis}(e)(Te_i) : \sum a : n \to A \,.\, a_{\cdot,i} e = Te_i;$

$\rightsquigarrow (a, [1]) := I\,(\sim) : \sum a : n \to n \to A \,.\, \forall i \in n \,.\, \mathcal{C}_i(a) e = Te_i,$

$\texttt{Assume}\; i : n,$

$(b_{\cdot,i}, [2.i]) := \eth\texttt{Basis}(e)(Se_i) : \sum b : n \to A \,.\, b_{\cdot,i} e = Se_i;$

$\rightsquigarrow (b, [2]) := I\,(\sim) : \sum b : n \to n \to A \,.\, \forall i \in n \,.\, \mathcal{C}_i(b) e = Se_i,$

$\texttt{Assume}\; i, j : n,$

$[j.1] := [1.j]\texttt{EinsteinSummation}\eth\mathcal{C}_j \forall k \in n \,.\, [2.i]\texttt{EinstenSummation}\eth\mathcal{C}_k(b) :$

$\quad : STe_j = S\mathcal{C}_j(a)e = Sa_{k,j}e_k = a_{k,j}Se_k = a_{k,j}\mathcal{C}_k(b)e = a_{k,j}b_{t,k}e_t,$

$[j.*] := \eth\texttt{matrixOfOperator}[j.1]\eth^{-1}\mathcal{R}_i(b)\mathcal{C}_j(a) :: (ST)^{e,e}_{i,j} = a_{k,j}b_{i,k} = \mathcal{R}_i(b)\mathcal{C}_j(a);$

$\rightsquigarrow [3] := I(\forall) : \forall i, j \in n \,.\, (ST)^{e,e}_{i,j} = \mathcal{R}_i(b)\mathcal{C}_j(a),$

$[*] := [3]\eth a \eth b \eth^{-1}\texttt{matrixOfOperator} : (ST)^{e,e} = S^{e,e}T^{e,e};$

□

$\texttt{MatricesAreOperators} :: \forall R \in \mathsf{RING} \,.\, \forall A : \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad . \operatorname{End}_{R\text{-}\mathsf{MOD}}(A) \cong_{R\text{-}\mathsf{ALG}} R^{\dim A \times \dim A}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{changeOfBasisMatrix} :: \prod R \in \mathsf{ANN} \,.\, \prod A : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad . \texttt{Basis}(A) \to \texttt{Basis}(A) \to R^{\dim A \times \dim A}$

$\texttt{changeOfBasisMatrix}\,(e, f) = C^{e \to f} := \texttt{coordinate}(f, e)$

$\texttt{GeneralLinearGroup} = \mathrm{GL} := \Lambda R \in \mathsf{RING} \,.\, \Lambda n \in \mathbb{N} \,.\, \left(R^{n \times n}\right)^* : \mathsf{RING} \to \mathbb{N} \to \mathsf{GRP};$

$\texttt{BasisMatrixInvertible} :: \forall R \in \mathsf{ANN} \,.\, \forall A : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad . \forall f, e : \texttt{Basis}(A) \,.\, C^{e \to f} \in \mathrm{GL}(R, \operatorname{rank} R)\Big)$

$\texttt{Proof} \;=$

$n := \operatorname{rank} A : \mathbb{Z}_+,$

$E := C^{e \to f} : R^{\operatorname{rank} A \times \operatorname{rank} A},$

$[1] := \eth E : \forall i \in n \,.\, E_{f,f} f_i = e_i,$

$\texttt{Assume}\; v : A,$

$\Big(\alpha, [v.1]\Big) := \eth\texttt{Basis}(A)(e)(v) : \sum \alpha \in R^n \,.\, v = \alpha e,$

$[v.*] := \eth R\text{-}\mathsf{MOD}(A, A)(E_{f,f})(\alpha f)[1][v.1] : E_{f,f}\alpha f = \alpha e = v;$

$\rightsquigarrow [2] := \eth^{-1}\texttt{Surjective} : \Big(E_{f,f} : \texttt{Surjective}(A, A)\Big),$

$\texttt{Assume}\; v : \ker E_{f,f},$

$\Big(\alpha, [v.1]\Big) := \eth\texttt{Basis}(A)(f)(v) : \sum \alpha \in R^n \,.\, v = \alpha f,$

$[v.2] := \eth R\text{-}\mathsf{MOD}(A, A)(E_{f,f})(\alpha f)[1][v.1] : 0 = E_{f,f} v = E_{f,f}\alpha f = \alpha e,$

$[v.*] := \eth\texttt{Basis}(e)[v.2][v.1] : v = 0;$

$\rightsquigarrow [3] := \eth\texttt{Iso}()\text{-}\mathsf{MOD}[2]\texttt{ZeroKernelTHM} : \Big(E_{f,f} : A \xleftrightarrow{R\text{-}\mathsf{MOD}} A\Big),$

$[*] := \texttt{ChoiceOfBasisDefinesAlgIso}[3] : E \in \mathrm{GL}(R, n);$

$\square$

$\texttt{ChangeOfBasisInversion} :: \forall R \in \mathsf{RING} \,.\, \prod A : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad . \forall e, f : \texttt{Basis}(A) \,.\, C^{e \to f} C^{f \to e} = I = C^{f \to e} C^{e \to f}$

$\texttt{Proof} \;=$

$[1] := \eth C \eth^{-1}\texttt{coordinates} : C^{e \to f} C^{f \to e} = \texttt{coordinate}(f, e)\texttt{coordinates}(e, f) = \texttt{coordinates}(e, e) = I,$

$[2] := \eth C \eth^{-1}\texttt{coordinates} : C^{f \to e} C^{e \to f} = \texttt{coordinate}(e, f)\texttt{coordinates}(f, e) = \texttt{coordinates}(f, f) = I,$

$[*] := [1][2] : \texttt{This},$

$\square$

$\texttt{LinearMapCategory} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{LinearMapCategory}\,() = R\text{-}\mathsf{LMAP} :=$

$$:= \left(\sum A \in \texttt{FreeModule}\ \&\ \texttt{FinitelyGeneratedModule}(R)\ .\ \texttt{Basis}(A), R\text{-}\mathsf{MOD}, \mathrm{id}, \circ\right)$$

$\texttt{MatrixCategory} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{MatrixCategory}\,() = R\text{-}\mathsf{MAT} :=$

$$:= \left(\sum A \in \texttt{FreeModule}\ \&\ \texttt{FinitelyGeneratedModule}(R)\ .\ \texttt{Basis}(A),\right.$$

$$\left., (A,e),(B,f) \mapsto R^{\operatorname{rank} B \times \operatorname{rank} A}, I, \texttt{matrixMult}\right)$$

$\texttt{inCoord} :: \prod R \in \mathsf{RING}\ .\ R\text{-}\mathsf{LMAP} \xleftrightarrow{\mathsf{CAT}} R\text{-}\mathsf{MAT}$

$\texttt{inCoord}\,(A,e) := (A,e)$

$\texttt{inCoord}\,((A,e),(B,f),T) := T^{e,f}$

$\texttt{ChangeOfBasis} :: \forall R \in \mathsf{RING}\ .\ \prod A, B : \texttt{FreeModule}\ \&\ \texttt{FinitelyGeneratedModule}(R)\ .$

$\quad .\ \forall T \in A \xrightarrow{R\text{-}\mathsf{MOD}} B\ .\ \forall e, e' : \texttt{Basis}(A)\ .\ \forall f, f' : \texttt{Basis}(B)\ .\ T^{e',f'} = C^{f \to f'} T^{e,f} C^{e' \to e}$

$\texttt{Proof}\ =$

$n := \operatorname{rank} A : \mathbb{Z}_+,$

$m := \operatorname{rank} B : \mathbb{Z}_+,$

$\texttt{Assume}\ i : n,$

$[i.*] := \eth\texttt{CovariantinCoord}(R\text{-}\mathsf{LMAP}, R\text{-}\mathsf{MAT})\eth C^{e' \to e}_{e',e} \eth C^{f \to f'} \eth \texttt{operatorFromMatrix} :$

$\quad : \left(C^{f \to f'} T^{e,f} C^{e' \to e}\right)_{e',f'} e'_i = C^{f \to f'}_{f,f'} T^{e,f}_{e,f} C^{e' \to e}_{e',e} e'_i = C^{f \to f'}_{f,f'} T e'_i = T e'_i;$

$\rightsquigarrow [1] := I(=,\to) : \left(C^{f \to f'} T^{e,f} C^{e' \to e}\right)_{e',f'} = T,$

$[*] := \texttt{ChoiceOfBasisDefinesIso}[1] : C^{f \to f'} T^{e,f} C^{e' \to e} = T^{e',f'};$

$\square$

$\texttt{EquivalentMatrices} :: \prod R \in \mathsf{RING}\ .\ \prod n, m \in \mathbb{N}\ .\ ?\left(R^{n \times m} \times R^{n \times m}\right)$

$A, B : \texttt{EquivalentMatices} \iff A \approx B \iff \exists T : R^m \xrightarrow{R\text{-}\mathsf{MOD}} R^n : \exists e, e' : \texttt{Basis}(R^m) : \exists f, f' : \texttt{Basis}(R^n) :$

$\quad : T^{e,f} = A\ \&\ T^{e',f'} = B$

$\texttt{InvertibleIsChangeOfBasis} :: \forall R \in \mathsf{ANN}\ .\ \forall A \in \texttt{FreeModule}\ \&\ \texttt{FinitelyGeneratedModule}(R)\ .$

$\quad \forall M \in \left(R^{\operatorname{rank} A \times \operatorname{rank} A}\right)^*\ .\ \exists e, f : \texttt{Basis}(A) : A = C^{e \to f}$

$\texttt{Proof}\ =$

$e := \texttt{FreehasBasis}(A) : \texttt{Basis}(A),$

$[1] := \texttt{ChoicOfBasisDefineAlgIso}(e, M) : M_{e,e} \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(A),$

$f := M_{e,e} e : \operatorname{rank} A \to A,$

$[2] := \eth\mathrm{Aut}_{R\text{-}\mathsf{MOD}}(A)(M_{e,e})[1]\eth^{-1}\texttt{Basis} : \left(f : \texttt{Basis}(A)\right),$

$[*] := \eth^{-1} C^{f \to e}[2]\eth f : M = C^{f \to e};$

$\square$

$\texttt{AltMatrixEquivalence} :: \forall R \in \textsf{ANN} \,.\, \forall n, m \in \mathbb{N} \,.\, \forall A, B \in R^{n \times m} \,.$

$A \approx B \iff \exists M \in \mathrm{GL}(R, n) : \exists N \in \mathrm{GL}(R, m) : B = MAN$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{MatrixeEquivalenceisEquivalence} :: \forall n, m \in \mathbb{N} \,.\, \forall R \in \textsf{ANN} \,.$

$.\ \texttt{EquivalentMatrices}(R, n, m) : \texttt{Equivalence}(R^{n \times m})$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A, B, C : R^{n \times m},$

$\texttt{Assume}\ [1] : A \approx B \;\&\; B \approx C,$

$\big(T, e, e', f, f', [1.1]\big) := \eth\texttt{EquivalentMatrices}[1]_1 :$

$: \sum T : R^m \xrightarrow{R\text{-}\mathsf{MOD}} R^n \,.\, \sum e, e' \texttt{Basis}(R^m) \,.\, \sum f, f' : \texttt{Basis}(R^n) \,.\, T^{e,f} = A \;\&\; T^{e',f'} = B,$

$\big(S, e'', e''', f'', f''', [1.2]\big) := \eth\texttt{EquivalentMatrices}[1]_2 :$

$: \sum S : R^m \xrightarrow{R\text{-}\mathsf{MOD}} R^n \,.\, \sum e'', e''' \texttt{Basis}(R^m) \,.\, \sum f'', f''' : \texttt{Basis}(R^n) \,.\, S^{e'',f''} = B \;\&\; S^{e''',f'''} = C,$

$[1.3] := \texttt{ChangeOfBasis}([1.1]) : C^{f \to f'} A C^{e' \to e} = B,$

$[1.4] := \texttt{ChangeOfBasis}([2.2]) : C^{f'' \to f'''} B C^{e''' \to e''} = C,$

$[1.5] := [1.3][1.4] : C^{f'' \to f'''} C^{f \to f'} A C^{e' \to e} C^{e''' \to e''} = C,$

$[*] := \texttt{ChangeOfBasisInversion}(\ldots)\texttt{AltMatixEquivalence}^{-1}(\ldots) : A \approx C;$

$\square$

2.4 Determinant and Trace

$\texttt{matrixDeterminant} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{n \times n} \to R$

$$\texttt{matrixDeterminant}\,(A) = \det A := \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$\texttt{DetTranspose} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A \in R^{n \times n} \,.\, \det A^\top = \det A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{DetMultilinear} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \det \circ \texttt{fromRows}(n) \in \mathcal{L}_R(n; i \mapsto R^n; R)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A : R^{n \times n},$

$\texttt{Assume}\ v : R^n,$

$\texttt{Assume}\ \alpha : R,$

$\texttt{Assume}\ B : R^{n \times n},$

$\texttt{Assume}\ i : n,$

$\texttt{Assume}\ [1] : \mathcal{R}_i(A) + v = \mathcal{R}_i(B)\ \&\ \forall j \in n \,.\, i \neq j \Rightarrow \mathcal{R}_j(A) = \mathcal{R}_j(B),$

$C := \texttt{fromRows}\Big(\Lambda j \in n \,.\, \texttt{if}\ j == i\ \texttt{then}\ v\ \texttt{else}\ \mathcal{R}_j(A)\Big) : R^{n \times n},$

$[1.*] := \eth \det B [1] \eth \mathsf{ANN}(R) \eth^{-1} C :$

$$: \det B = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} B_{i,\sigma(i)} = \sum_{\sigma \in S_n} (-1)^\sigma (A_{i,\sigma(i)} + v_{\sigma(i)}) \prod_{j=1:j \neq i}^{n} A_{j,\sigma(j)} =$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} A_{i,\sigma(i)} + \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(i)} \prod_{j=1:i \neq i}^{n} A_{i,\sigma(i)} = \det A + \det C;$$

$\rightsquigarrow [1] := I(\Rightarrow) : (\ldots),$

$\texttt{Assume}\ [2] : \mathcal{R}_i(B) = \alpha \mathcal{R}_i(B)\ \&\ \forall j \in n \,.\, i \neq j \Rightarrow \mathcal{R}_j(A) = \mathcal{R}_j(B),$

$[2.*] := \eth \det B [2] \eth \, \mathrm{Ann}(A) \eth^{-1} \det A :$

$$: \det B = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} B_{i,\sigma(i)} = \sum_{\sigma \in S_n} (-1)^\sigma \alpha A_i \prod_{j=1:j \neq i}^{n} A_{j,\sigma(j)} \alpha \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} A_{i,\sigma(i)} = \alpha \det A;$$

$\rightsquigarrow [2] := I(\Rightarrow) : (\ldots),$

$[*] := \eth^{-1} \texttt{Multilinear}(R, n, i \mapsto R^n, R)[1][2] : \det \circ \texttt{fromRows} \in \mathcal{L}_R(n; i \mapsto R^n; R);$

$\square$

$\texttt{DetAntisymmetic} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \det \circ \texttt{fromRows}(n) : \texttt{Antisymmetric}(R, n, i \mapsto R^n, R)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A : R^{n\times n},$

$\texttt{Assume}\ \tau : 2\text{-}\texttt{Cycle}(n),$

$B := \texttt{fromRows} \circ \tau\big(\mathcal{R}(A)\big) : R^{n\times n},$

$[\tau.*] := \eth \det B \eth B \eth \texttt{sign} \eth \mathsf{GRP}(S_n) \eth \mathsf{GRP}(S_n, \texttt{Sign}) \texttt{sign} \eth \mathsf{GRP}(S_n) \eth^{-1} \det A :$

$$: \det B = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n B_{i,\sigma(i)} = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n A_{\tau(i),\sigma(i)} - \sum_{\sigma\in S_n} (-1)^\sigma (-1)^\tau \prod_{i=1}^n A_{i,\sigma\tau(i)} =$$

$$= -\sum_{\sigma\in S_n} (-1)^{\sigma\tau} \prod_{i=1}^n A_{i,\sigma\tau(i)} = -\sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n A_{i,\sigma(i)} = -\det A;$$

$\rightsquigarrow [*] := \eth^{-1}\texttt{AntisymmetricByTranspositions} : \big(\det \circ \texttt{fromRows}(n) : \texttt{Antysymmetric}(R, n, i \mapsto R^n, R)\big);$

$\square$

$\texttt{DetHomo1} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A, B \in R^{n\times n} \,.\, \det AB = \det A \det B$

$\texttt{Proof} \;=$

$[*] := \eth \det AB \eth \mathcal{C} \eth \mathcal{R} \eth \mathsf{ANN}(R) \eth \texttt{sign} \eth \mathsf{GRP}(S_n, \texttt{Signs}) \texttt{signMultIsBij}(S_n) \eth \mathsf{ANN}(R) \eth^{-1} \det :$

$$: \det AB = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n \mathcal{R}_i(A)\mathcal{C}_{\sigma(i)}(B) = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n \sum_{j=1}^n A_{i,j} B_{j,\sigma(i)} =$$

$$= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{J:n\to n} \prod_{i=1}^n A_{i,J_i} B_{J_i,\sigma(i)} = \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\tau\in S_n} \prod_{i=1}^n A_{i,\tau(i)} \prod_{i=1}^n B_{\tau(i),\sigma(i)} =$$

$$= \sum_{\sigma\in S_n} \sum_{\tau\in S_n} (-1)^{\sigma\tau^{-1}} (-1)^\tau \prod_{i=1}^n A_{i,\tau(i)} \prod_{i=1}^n B_{i,\sigma\tau^{-1}(i)} = \sum_{\sigma\in S_n} \sum_{\tau\in S_n} (-1)^\sigma (-1)^\tau \prod_{i=1}^n A_{i,\tau(i)} \prod_{i=1}^n B_{i,\sigma(i)} =$$

$$= \left(\sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n A_{i,\sigma(i)}\right) \left(\sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n B_{i,\sigma(i)}\right) = \det A \det B;$$

$\square$

$\texttt{DetHomo2} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \det : \mathrm{GL}(R, n) \xrightarrow{\mathsf{GRP}} R^*$

$\texttt{Proof} \;=$

$[1] := \eth I \eth \det : \det I = 1,$

$[*] := \texttt{detHomo} \eth^{-1} \mathsf{GRP}\big(\mathrm{GL}(R, n), R^*\big) : (\det : \mathrm{GL}(R, n) \xrightarrow{\mathsf{GRP}} R^*);$

$\square$

$\texttt{DetBasisInvariant} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModuleFinitelyGeneratedModule}(R) \,.$

$\quad . \forall T \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \,.\, \forall e, f : \texttt{Basis}(M) \,.\, \det T^{e,e} = \det T^{f,f}$

$\texttt{Proof} \;=$

$[1] := \texttt{ChangeOfBasis}(T, e, f) : T^{f,f} = C^{e\to f} T^{e,e} C^{f\to e},$

$[*] := \det[1] \texttt{DetHomo}(\ldots) \eth \mathsf{ANN}(R) \texttt{ChangeOfBasisInversion}(e, f) \texttt{DetHomo2}(\ldots) :$

$$: \det T^{f,f} = \det C^{e\to f} T^{e,e} C^{f\to e} = \det C^{e\to f} \det T^{e,e} \det C^{f\to e} = \det C^{e\to f} \det C^{f\to e} \det T^{e,e} = \det T^{e,e};$$

$\square$

$\texttt{determinantOfTheOperator} :: \prod R \in \mathsf{ANN} \,.\, \prod M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$
$\quad . \operatorname{End}_{R\text{-}\mathsf{MOD}}(M) \to R$
$\texttt{determinantOfTheOperator}\,(T) = \det T := \det T^{e,e} \quad \texttt{where} \quad e = \texttt{FreeHasBasis}(M)$

$\texttt{DetOperatorHomo} :: \forall R \in \mathsf{ANN} \,.\, \prod M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$
$\quad . \forall A, B \in \operatorname{End}_{R\text{-}\mathsf{MOD}}(M) \,.\, \det AB = \det A \det B$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{DetOperatorHomo1} :: \forall R \in \mathrm{Ann} \,.\, \forall M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$
$\quad . \det : \operatorname{Aut}_{R\text{-}\mathsf{MOD}}(M) \xrightarrow{\mathsf{GRP}} R^*$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{antiindex} :: n \to (n-1) \to n$
$\texttt{antiindex}\,(i) = \hat{i} := \Lambda j \in (n-1) \,.\, \texttt{if } j < i \texttt{ then } j \texttt{ else } j+1$

$\texttt{minor} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{n\times n} \to (n \times n) \to R$
$\texttt{minor}\,(A, i, j) = \Delta_{i,j}(A) := (-1)^{i+j} \det A_{\hat{i},\hat{j}}$

$\texttt{DeterminantComputation} :: \forall R \in \mathrm{Ann} \,.\, \forall n \in \mathbb{N} \,.\, \forall A \in R^{n\times n} \,.\, \forall i \in n \,.\, \det A = \sum_{j=1}^{n} A_{i,j}\Delta_{i,j}(A)$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{adjointMatrix} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{n\times n} \to R^{n\times n}$
$\texttt{adjointMatrix}\,(A) = \operatorname{adj} A := \Delta^{\top}(A)$

$\texttt{CramerMatrixInversion} :: \forall R \in \mathrm{Ann} \; . \; \forall n \in \mathbb{N} \; . \; \forall A \in \mathrm{GL}(R, n) \; . \; A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$

$\texttt{Proof} \; =$

$\texttt{Assume} \; i : n,$

$[1] := \eth \texttt{matrixMult} \eth \operatorname{adj} A \texttt{DeterminantComputation}(A) \eth \texttt{Inverse}(\det A) :$

$$: \left(\frac{A \operatorname{adj} A}{\det A} \right)_{i,i} = \frac{1}{\det A} \sum_{j=1}^{n} A_{i,j} \Delta_{i,j}(A) = \frac{\det A}{\det A} = 1,$$

$\texttt{Assume} \; j : n,$

$\texttt{Assume} \; [2] : j \neq i,$

$C := \texttt{fromRows}(\Lambda k \in i \; . \; \texttt{if} \; k == j \; \texttt{then} \; \mathcal{R}_i(A) \; \texttt{else} \; \mathcal{R}_k(A)) : R^{n \times n},$

$[2.*] := \eth \texttt{matrixMult} \eth \operatorname{adj} A \texttt{DeterminantComputation}(C) \texttt{AntisymmetricZero}(\det C) :$

$$: \left(\frac{A \operatorname{adj} A}{\det A} \right)_{i,j} = \frac{1}{\det A} \sum_{k=1}^{n} A_{i,k} \Delta_{i,k}(A) = \frac{\det C}{\det A} = 0;$$

$\rightsquigarrow [*] := \eth^{-1} I : \dfrac{A \operatorname{adj} A}{\det A} = I;$

$\square$

$\texttt{CramerMatrixInversionCorollary} :: \forall R \in \mathsf{ANN} \; . \; \forall n \in \mathbb{N} \; . \; \forall A \in R^{n \times n} \; . \; \det A \in R^* \Rightarrow A \in \mathrm{GL}(R, n)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{specialLinearGroup} :: \mathsf{ANN} \to \mathbb{N} \to \mathsf{GRP}$

$\texttt{specialLinearGroup}\,(R, n) = \mathrm{SL}(R, n) := \Lambda R \in \mathrm{Ann} \; . \; \Lambda n \in \mathbb{N} \; . \; \det^{-1}_{R,n}(1)$

$\texttt{trace} :: \prod R \in \mathsf{RING} \; . \; \prod n \in \mathbb{N} \; . \; R^{n \times n} R\text{-}\mathsf{MOD} R$

$\texttt{trace}\,(A) = \operatorname{tr} A := A_{i,i}$

$\texttt{TraceTransInvariant} :: \forall R \in \mathsf{RING} \; . \; \forall n \in \mathbb{N} \; . \; \forall A \in R^{n \times n} \; . \; \operatorname{tr} A = \operatorname{tr} A^\top$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{TraceProduct} :: \forall R \in \mathrm{Ann} \; . \; \forall n \in \mathbb{N} \; . \; \forall A, B \in R^{n \times n} \; . \; \operatorname{tr} AB^\top = A_{i,j} B_{i,j}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ShiftInTrace} :: \forall R \in \mathrm{Ann} \; . \; \forall n \in \mathbb{N} \; . \; \forall A, B \in R^{n \times m} \; . \; \operatorname{tr} AB = \operatorname{tr} BA$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{TraceIsBasisInvariant} :: \forall R \in \mathrm{Ann} \; . \forall M : \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \; .$

$\quad . \; \forall T \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \; . \; \forall e, f : \texttt{Basis}(M) \; . \; \operatorname{tr} T^{e,e} = \operatorname{tr} T^{f,f}$

$\texttt{Proof} \;=$

$[1] := \texttt{ChangeOfBasis}(T, e, f) : T^{f,f} = C^{e \to f} T^{e,e} C^{f \to e},$

$[*] := \det[1]\texttt{ShiftInTraceChangeOfBasisInversion}(e, f) :$

$\quad : \operatorname{tr} T^{f,f} = \operatorname{tr} C^{e \to f} T^{e,e} C^{f \to e} = \operatorname{tr} C^{f \to e} C^{e \to f} \det T^{e,e} = \operatorname{tr} T^{e,e};$

$\square$

$\texttt{traceOfTheOperator} :: \prod R \in \mathrm{Ann} \; . \; \prod M : \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \; .$

$\quad . \; \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \xrightarrow{R\text{-}\mathsf{MOD}} R$

$\texttt{traceOfTheOperator}\,(T) = \operatorname{tr} T := \operatorname{tr} T^{e,e} \quad \texttt{where} \quad e = \texttt{FreeHasBasis}(M)$

$\texttt{ShiftInTrace2} :: \forall R \in \mathrm{Ann} \; . \forall M \in \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \; .$

$\quad . \; \forall A, B \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \; . \; \operatorname{tr} AB = \operatorname{tr} BA$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{DetOfBlockDiagonalMatrix} :: \forall R \in \mathrm{Ann} \; . \forall n, k \in \mathbb{N} \; . \forall p : \texttt{Partition}(n, k) \; . \forall A \in R^{n \times n} \; .$

$$. \; \forall B : \prod i \in k \; . \; R^{p_i \times p_i} \; . \; \forall [0] : A = \texttt{blockDiagonal}(n, k, p, B) \; . \; \det A = \prod_{i=1}^{k} \det B_i$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.5 Upper and Lower Triangular Matrices

$\texttt{UpperTriangularMatrix} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?R^{n\times n}$

$A : \texttt{UpperTriangularMatrix} \iff \forall i,j \in n \,.\, i > j \Rightarrow A_{i,j} = 0$

$\texttt{LowerTriangularMatrix} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?R^{n\times n}$

$A : \texttt{LowerTriangularMatrix} \iff \forall i,j \in n \,.\, i < j \Rightarrow A_{i,j} = 0$

$\texttt{DiagonalMatrix} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?R^{n\times n}$

$A : \texttt{DiagonalMatrix} \iff A : \texttt{UpperTriangularMatrix}(R,n) \;\&\; A : \texttt{LowerTriangularMatrix}(R,n)$

$\texttt{DetOfUpperTriangular} :: \forall R \in \mathsf{ANN} \,.\, \forall A : \texttt{UpperTriangularMatrix}(A) \,.\, \det A = \prod_{i=1}^{n} A_{i,i}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{DetOfLowerTriangular} :: \forall R \in \mathsf{ANN} \,.\, \forall A : \texttt{LowerTriangularMatrix}(A) \,.\, \det A = \prod_{i=1}^{n} A_{i,i}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{UpperTriangularizable} :: \prod M \in \texttt{FinitelyGeneratedModule} \;\&\; \texttt{FreeModule}(R) \,.\, ?\mathrm{End}_{R\text{-}\mathsf{MOD}}(M)$

$T : \texttt{UpperTriangularizable} \iff \exists e : \texttt{Basis}(M) : T^{e,e} : \texttt{UpperTriangularMatix}$

$\texttt{LowerTriangularizable} :: \prod M \in \texttt{FinitelyGeneratedModule} \;\&\; \texttt{FreeModule}(R) \,.\, ?\mathrm{End}_{R\text{-}\mathsf{MOD}}(M)$

$T : \texttt{LowerTriangularizable} \iff \exists e : \texttt{Basis}(M) : T^{e,e} : \texttt{LowerTriangularMatix}$

$\texttt{Diagonalizable} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?R^{n\times n}$

$A : \texttt{Diagonalizable} \iff \exists e : \texttt{Basis}(M) : T^{e,e} : \texttt{DiagonalMatix}$

2.6 Eigenelemments and Simmilarity

$\texttt{SimmilarMatrices} :: \prod R \in \mathsf{RING} \,.\, \prod n \in \mathbb{N} \,.\, ?(R^{n\times n} \times R^{n\times n})$

$(A,B) : \texttt{SimmilarMatrices} \iff A \sim B \iff \exists M \in \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$
$\,.\, \exists T \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M) \,.\, \exists e, f : \texttt{Basis}(M) \,.\, A_{e,e} = T = B_{f,f}$

$\texttt{characteristicIdeal} :: \prod R \in \mathsf{ANN} \,.\, R^{n\times n} \to \texttt{Ideal}(R[\mathbb{Z}_+])$

$\texttt{characteristicIdeal}\,(N) = \mathcal{A}(N) := \mathrm{Ann}_{R[\mathbb{Z}_+]}(N)$

$\texttt{characteristicIdealOfSimmilarMatricesAgree} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall X, Y \in R^{n\times n} \,.$
$\,.\, X \sim Y \Rightarrow \mathcal{A}(X) = \mathcal{A}(Y)$

$\texttt{Proof} \ =$

$\big(A, [1]\big) := \texttt{ChangeOfBasis}(\ldots)\eth\texttt{Simmilar}(X,Y) : \sum A \in \mathrm{GL}(R,n) \,.\, Y = AXA^{-1},$

$\texttt{Assume}\ f : \mathcal{A}(X),$

$[f.1] := \eth\texttt{applyPolynimial}(f,Y)[1]\forall i \in \deg f \,.\, \texttt{ConjugatePower}(A,X)\eth(\text{-}\mathsf{ALG}R)(R^{n\times n})$

$$\eth^{-1}\texttt{applyPolynomial}(F,X)\eth\mathcal{A}(X)(f) : f(Y) = \sum_{i=0}^{\deg f} f_i Y^i = \sum_{i=0}^{\deg f} f_i\Big(AXA^{-1})^i = \sum_{i=0}^{\deg f} f_i AX^iA^{-1} =$$

$$= A\left(\sum_{i=0}^{\deg f} f_i X^i\right)A^{-1} = Af^{-1}(X)A^{-1} = 0,$$

$[f.*] := \eth\mathcal{A}(Y)[f.1] : f \in \mathcal{A}(Y);$

$\rightsquigarrow [2] := \eth^{-1}\texttt{Subset} : \mathcal{A}(Y) \subset \mathcal{A}(X),$

$\texttt{Assume}\ f : \mathcal{A}(Y),$

$[f.1] := \eth\texttt{applyPolynomial}(f,X)[1]\forall i \in \deg f \,.\, \texttt{ConjugatePower}(A,Y)\eth(\text{-}\mathsf{ALG}R)(R^{n\times n})$

$$\eth^{-1}\texttt{applyPolynomial}(F,X)\eth\mathcal{A}(Y)(f) : f(X) = \sum_{i=0}^{\deg f} f_i X^i = \sum_{i=0}^{\deg f} f_i\Big(A^{-1}YA)^i = \sum_{i=0}^{\deg f} f_i A^{-1}Y^iA =$$

$$= A^{-1}\left(\sum_{i=0}^{\deg f} f_i Y^i\right)A = A^{-1}f^{-1}(Y)A = 0,$$

$[f.*] := \eth\mathcal{A}(Y)[f.1] : f \in \mathcal{A}(X);$

$\rightsquigarrow [*] := \eth^{-1}\texttt{SetEq}[2] : \mathcal{A}(X) = \mathcal{A}(Y),$

$\square$

$\texttt{Eigenvalue} :: \prod R \in \mathsf{RING} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \to ?R$

$\lambda : \texttt{Eigenvalue} \iff \Lambda T \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \,.\, \exists m \in M \setminus \{0\} \,.\, Tm = \lambda m$

i

$\texttt{Eigenspace} :: \prod R \in \mathsf{RING} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \to R \to ?M$

$m : \texttt{Eigenspace} \iff \Lambda T \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \,.\, \Lambda\lambda \in R \,.\, \ker \lambda\,\mathrm{id} - T$

$\texttt{GeneralizedEigenelement} :: \prod R \in \mathsf{RING} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \to R \to ?M$

$$m : \texttt{GeneralizedEigenelement} \iff \Lambda T \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \,.\, \Lambda\lambda \in R \,.\, \bigcup_{k=1}^{\infty} \ker(\lambda\,\mathrm{id} - T)^k$$

$\texttt{characteristicPolynomial} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{n\times n} \to R\mathbb{Z}_+$

$\texttt{characteristicPolynomial}\,(A) = \chi_A(\lambda) := \det A - \lambda I$

$\texttt{characteristicPolynomial2} :: \prod R \in \mathsf{ANN} \,.\, M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$
$\quad \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \to R\mathbb{Z}_+$

$\texttt{characteristicPolynomial2}\,(A) = \chi_A(\lambda) := \det A - \lambda\,\mathrm{id}$

$\texttt{characteristicPolynomialsOfSimmilarAgree} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall X, Y \in R^{n\times n} \,.$
$\quad .\, X \sim Y \Rightarrow \chi_X(\lambda) = \chi_Y(\lambda)$

$\texttt{Proof} \;=$

$\big(A, [1]\big) := \texttt{ChangeOfBasis}(\ldots)\eth\texttt{Simmilar}(X,Y) : \sum A \in \mathrm{GL}(R,n) \,.\, Y = AXA^{-1},$

$[*] := \eth\chi_X(\lambda)\eth\mathsf{GRP}(\mathrm{GL}(n,R), R^*)(\det)[1]\eth^{-1}\chi_Y(\lambda) :$

$\quad : \chi_X(\lambda) = \det(X - \lambda I) = \det A \det(X - \lambda I) \det A^{-1} = \det\left(AXA^{-1} - \lambda AA^{-1}\right) = \det(Y - \lambda I) = \chi_Y(\lambda);$

$\square$

$\texttt{SimmilarOperators} :: \prod R \in \mathsf{RING} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?\big(\mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \times \mathrm{End}_{R\text{-}\mathsf{MOD}}(M)\big)$

$X, Y : \texttt{SimmilarOperators} \iff X \sum Y \iff \exists A \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M) \,.\, Y = AXA^{-1}$

$\texttt{SimmilarOperatorsHaveSameEigenvalues} :: \forall R \in \mathsf{RING} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (X,Y) : \texttt{Simmilar}(M) \,.$
$\quad .\, \texttt{Eigenvalue}(X) = \texttt{Eigenvalue}(Y)$

$\texttt{Proof} \;=$

$\big(A, [1]\big) := \eth\texttt{Simmilar}(X,Y) : \sum A \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M) \,.\, Y = AXA^{-1},$

$\texttt{Assume}\ \lambda : \texttt{Eigenvalue}(X),$

$\big(m, [\lambda.1]\big) := \eth\texttt{Eigenvalue}(X) : \sum m \in M \,.\, m \neq 0 \,\&\, Xm = \lambda m,$

$[\lambda.2] := \eth\mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M)(A)\texttt{ZeroKerTHM}(A) : Am \neq 0,$

$[\lambda.3] := [1][\lambda.1]\eth R\text{-}\mathsf{ALG}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(M)) : YAm = AXA^{-1}Am = AXm = \lambda Am,$

$[\lambda.*] := \eth^{-1}\texttt{Eigenvalue}[\lambda.1][\lambda.2] : \big(\lambda : \texttt{Eigenvalue}(Y)\big);$

$\leadsto [2] := \eth^{-1}\texttt{Subset} : \texttt{Eigenvalue}(X) \subset \texttt{Eigenvalue}(Y),$

$\texttt{Assume}\ \lambda : \texttt{Eigenvalue}(X),$

$\big(m, [\lambda.1]\big) := \eth\texttt{Eigenvalue}(Y) : \sum m \in M \,.\, m \neq 0 \,\&\, Ym = \lambda m,$

$[\lambda.2] := \eth\mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M)(A^{-1})\texttt{ZeroKerTHM}(A^{-1}) : A^{-1}m \neq 0,$

$[\lambda.3] := [1][\lambda.1]\eth R\text{-}\mathsf{ALG}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(M)) : XA^{-1}m = A^{-1}YA^{-1}Am = A^{-1}Ym = \lambda A^{-1}m,$

$[\lambda.*] := \eth^{-1}\texttt{Eigenvalue}[\lambda.1][\lambda.2] : \big(\lambda : \texttt{Eigenvalue}(X)\big);$

$\leadsto [*] := \eth^{-1}\texttt{SetEq}[2] : \texttt{EigenValue}(X) = \texttt{Eigenvalue}(Y);$

$\square$

$\texttt{PolynomialModuleStructure} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \to R[\mathbb{Z}_+]\text{-}\mathsf{MOD}$

$\texttt{polynomialModuleStructure}\,(T) = M_T := \texttt{PolinimialModuleStructure}(M, R)$

$\texttt{SimmilarityByPolynomialModuleStructure} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall X, Y \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(M) \,.$
$X \sim Y \iff M_X \cong M_Y$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : X \sim Y,$

$\big(A, [1.1]\big) := \texttt{ChangeOfBasis}(\dots)\eth\texttt{Simmilar}(X, Y) : \sum A \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M) \,.\, Y = AXA^{-1},$

$\texttt{Assume}\ f : R[\mathbb{Z}_+],$

$\texttt{Assume}\ m : M,$

$[f.*] := \eth M_X \eth \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M)(A) \forall i \in \deg f \,.\, \texttt{ConjugatePower}(X, A)[1.1]\eth^{-1} M_Y :$

$$: Af \cdot_{M_X} m = A \sum_{i=0}^{\deg f} f_i X^i m = \sum_{i=0}^{\deg f} f_i AX^i A^{-1} Am = \sum_{i=0}^{\deg f} f_i AX^i A^{-1} Am = \sum_{i=0}^{\deg f} f_i \Big(AXA^{-1}\Big)^i Am =$$
$$= \sum_{i=0}^{\deg f} f_i Y^i Am = f \cdot_{M_Y} Am;$$

$\rightsquigarrow [1.2] := \eth^{-1}\texttt{Iso} : \big(A : M_X \xleftrightarrow{R[\mathbb{Z}_+]\text{-}\mathsf{MOD}} M_Y\big),$

$[1.*] := \eth^{-1}\texttt{Isomorphic}[1.2] : M_X \cong M_Y;$

$\rightsquigarrow [1] := I(\Rightarrow) : X \sim Y \Rightarrow M_X \cong M_Y,$

$\texttt{Assume}\ [2] : M_X \cong M_Y,$

$A := \eth\texttt{Isomorphic}[2] : M_X \xleftrightarrow{R[\mathbb{Z}_+]\text{-}\mathsf{MOD}} M_Y,$

$\texttt{Assume}\ m : M,$

$[m.*] := \eth A : AX(m) = Ax \cdot_{M_X} m = x \cdot_{M_Y} Am = YAm;$

$\rightsquigarrow [2.1] := I(=, \rightarrow) : AX = YA,$

$[2.2] := [2.1]A^{-1} : AXA^{-1} = Y,$

$[2.*] := \eth^{-1}\texttt{Simmilar}[2.2] : X \sim Y;$

$\rightsquigarrow [*] := I(\iff)[1] : X \sim Y \iff A_X \cong A_Y;$

$\square$

2.7 Elementary Duality

$\texttt{dual} :: \prod R \in \mathsf{RING} \,.\, \texttt{Contravariant}(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD})$

$\texttt{dual}\,(M) = M^* := \mathcal{M}_{R\text{-}\mathsf{MOD}}(M, R)$

$\texttt{dual}\,(A, B, T) = T^* := \Lambda f \in B^* \,.\, f \circ T$

$\texttt{dualBasis} :: \prod R \in \mathsf{RING} \,.\, \prod M : \texttt{FreeModule}(R) \,.\, \texttt{Basis}(M) \to \operatorname{rank} M \to M^*$

$\texttt{dualBasis}\,(e, i) = e_*^i := \texttt{free}(\Lambda j \in \operatorname{rank} M \,.\, \delta_j^i)$

$\texttt{DualBasisTHM} :: \forall R \in \mathsf{RING} \,.\, \forall M \in \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad .\, \forall e : \texttt{Basis}(M) \,.\, e_* : \texttt{Basis}(M^*)$

$\texttt{Proof} \;=$

$n := \operatorname{rank} M : \mathbb{Z}_+,$

$\texttt{Assume}\; f : M^*,$

$\texttt{Assume}\; m : M,$

$(\alpha, [1]) := \eth\texttt{Generating}(e)(m) : \sum \alpha \in R^n \,.\, m = \alpha e,$

$m.* := [1]\eth M^*\eth e_*\eth M^*[1] : f(m) = f(\alpha e) = \alpha_i f(e_i) = \alpha_i f(e_j) e_*^j(e_i) = f(e_i) e_*^i(m);$

$\rightsquigarrow [f.*] := I(=, to)) : f = f(e) e_*;$

$\rightsquigarrow [1] := \eth^{-1}\texttt{Generatiting} : \big(e_* : \texttt{Generating}(M^*)\big),$

$\texttt{Assume}\; \alpha : R^n,$

$\texttt{Assume}\; [\alpha.1] : \alpha e_* = 0,$

$\texttt{Assume}\; i : n,$

$[i.*] := [\alpha.1]\eth e_* : 0 = \alpha e_*(e_i) = \alpha_i;$

$\rightsquigarrow [\alpha.*] := I(\to, =) : \alpha = 0;$

$\rightsquigarrow [2] := \eth^{-1}\texttt{LinearlyIndependent}(M^*) : \big(e_* : \texttt{LinearlyIndependent}(M^*)\big),$

$[*] := \eth^{-1}\texttt{Basis}[1][2] : \big(e_* : \texttt{Basis}(M)\big);$

$\square$

$\texttt{DualOfFreeModuleIsFree} :: \forall R \in \mathsf{RING} \,.\, \forall M \in \texttt{FiniteGroup} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad M^* : \texttt{FiniteGroup} \,\&\, \texttt{FinitelyGeneratedModule}(R)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RankOfFreeDual} :: \forall R \in \texttt{Ring} \,.\, \forall M \in \texttt{FiniteGroup} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.\, \operatorname{rank} M^* = \operatorname{rank} M$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\color{red}{\texttt{DualMapByTranspose}} :: \forall R \in \mathsf{RING} \,.\, \forall M, N \in \texttt{FreeModule} \ \& \ \texttt{FinitelyGeneratedModule}(R) \,.$

$\quad . \,\forall T : M \xrightarrow{R\text{-}\mathsf{MOD}} N \,.\, \forall e : \texttt{Basis}(M) \,.\, \forall f : \texttt{Basis}(N) \,.\, \left(T^{e,f}\right)^{\top} = \left(T^*\right)^{f_*,e_*}$

$\texttt{Proof} \ =$

$n := \operatorname{rank} N : \mathbb{Z}_+,$

$m := \operatorname{rank} M : \mathbb{Z}_+,$

$\texttt{Assume}\ j : n,$

$\texttt{Assume}\ i : m,$

$[(i,j).*] := \eth T^* \eth^{-1} T^{e,f} \eth f_* \eth \mathcal{C}_i : T^* f^j_*(e_i) = f^j_*(Te_i) = f^j_*\Big(\mathcal{C}_i(T^{e,f})f\Big) = \Big(\mathcal{C}_i(T^{e,f})\Big)_j = T^{e,f}_{i,j};$

$\rightsquigarrow [*] := I(=,\to) : (T^*)^{f_*,e_*} = \left(T^{e,f}\right)^{\top};$

$\square$

$\texttt{Reflexive} :: \prod R \in \mathsf{RING} \,.\, ?R\text{-}\mathsf{MOD}$

$M : \texttt{Reflexive} \iff M \cong_{R\text{-}\mathsf{MOD}} M^*$

$\color{red}{\texttt{FinitelGenerateFreeModuleIsReflexive}} :: \forall R \in \mathsf{RING} \,.$

$\quad . \,\forall M \in \texttt{FreeModule} \ \& \ \texttt{FinitelyGeneratedModule}(R) \,.\, M : \texttt{Reflexive}(R)$

$\texttt{Proof} \ =$

$\texttt{naturalEmbedding} :: \prod R \in \mathsf{RING} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} M^{**}$

$\texttt{naturalEmbedding}\,(m) = \omega_M(m) := \Lambda f \in M^* \,.\, f(m)$

$\color{red}{\texttt{NaturalEmbeddingIsNatural}} :: \forall R \in \mathsf{RING} \,.\, \omega : \texttt{NaturalTransform}\big(\mathrm{id}_{R\text{-}\mathsf{MOD}}, (\cdot)^{**}\big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ M, N : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ T : M \xrightarrow{R\text{-}\mathsf{MOD}} N,$

$\texttt{Assume}\ m : M,$

$\texttt{Assume}\ f : N^*,$

$\big[(m,f).*\big] := \eth\texttt{compose}\eth\omega_N\eth^{-1}T^*\eth^{-1}\omega_M(m)\eth^{-1}T^{**}\eth^{-1}\texttt{compose} : \big(T\omega_N(m)\big)(f) = \omega_N\big(T(m)\big)(f) = f\big(T(m)\big) =$

$\rightsquigarrow \big[(M,N).*,\big] :=: T\omega_N = \omega_M T^* *;$

$\rightsquigarrow [*] := \eth^{-1}\texttt{NaturalTransform} : \Big(\omega : \texttt{NaturalTransform}(\mathrm{id}_{R\text{-}\mathsf{MOD}}, (\cdot)^* *)\Big);$

$\square$

$\color{red}{\texttt{DoubleDualNaturalIsomorphism}} :: \forall R \in \mathsf{RING} \,.\, \forall M : \texttt{FreeModule} \ \& \ \texttt{FinitelyGeneratedModule} \,.$

$\quad \omega_M : M \xleftrightarrow{R\text{-}\mathsf{MOD}} M^{**}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

3 Advanced Categorical Module Theory

3.1 Duality