Random Variables

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Contents

1 Random Variable in Topological Vector Space

1.1 Random Variables And Corresponding Events

$$\texttt{RandomVariable} = \prod (\Omega, \mathcal{F}, P) : \texttt{ProbabilitySpace} \ . \ \prod V : \mathsf{TOPVS}(K) \ . \ (\Omega, \mathcal{F}) \to_{\mathsf{BOR}} (V, \mathcal{B}V)$$

$$\texttt{associatedProbability} :: \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, P), V\Big) \to \texttt{Probability}(V, \mathcal{B}V)$$

$$\texttt{associatedProbability}\,(X) = \mathbb{P}_X := \Lambda B \in \mathcal{B}V \ . \ P(X^{-1}B)$$

$$\texttt{empiricEventProbability} :: \texttt{RandomVariable}(\Omega, V) \to \mathcal{B}V \to [0, 1]$$

$$\texttt{empiricEventProbability}\,(X, B) = \mathbb{P}(X \in B) := \mathbb{P}_X(B)$$

$$\texttt{IndependentEvents} :: \prod (\Omega, \mathcal{F}, P) : \texttt{PobabilitySpace} \ . \ ?\mathcal{F}^2$$

$$A, B : \texttt{IndependantEvents} \iff A \perp B \iff P(A \cap B) = P(A)P(B)$$

$$\texttt{conditionalProbability} :: \prod (\Omega, \mathcal{F}, P) : \texttt{ProbabilitySpace} \ . \ \mathcal{F} \to \mathcal{F} \setminus I^0_P \to [0, 1]$$

$$\texttt{conditionalProbability}\,(A, B) = P(A|B) := \frac{P(A \cap B)}{P(B)}$$

$$\texttt{TotalProbability} :: \forall (\Omega, \mathcal{F}, P) \ . \ \forall A \in \mathcal{F} \ . \ \forall n \in \aleph_0 \ . \ \forall B : \texttt{Disjoint}(n, \mathcal{F} \setminus I^0_P) \ .$$

$$. \ \forall (1) : \bigcup_{i=1}^{n} B_i = \Omega \ . \ P(A) = \sum_{i=1}^{n} P(B_i) P(A|B_i)$$

$\texttt{Proof} =$

$(2) := \forall i \in n \ . \ \texttt{DisjointIntersection}(A, B_i) : A \cap B : \texttt{Disjoint}(n, \mathcal{F}),$

$(3) := \texttt{SetSpliting}(A, B)(1) : A = \bigcup_{i=1}^{n} A \cap B_i,$

$(4) := \eth\texttt{Measure}(\Omega, \mathcal{F})(P)(2) : P(A) = \sum_{i=1}^{n} P(A \cap B_i),$

$\texttt{Assume}\ i : n,$

$(5) := P(B_i)\eth P(A|B_i) : P(A \cap B_i) = P(B_i)P(A|B_i);$

$\rightsquigarrow (5) := I(\forall) : \forall i \in n \ . \ P(A \cap B_i) = P(B_i)P(A \cap B_i),$

$(*) := (5)(4) : P(A) = \sum_{i=1}^{n} P(B_i)P(A \cap B_i);$

$\square$

1.2 Distribution Functions and Densities in Eucledean Space

$$\texttt{ProbabilityDistribution} :: \texttt{RandomVariable}(\Omega, \mathbb{R}^n) \to \mathbb{R}^n \to [0,1]$$
$$\texttt{ProbabilityDistribution}\,(X) = P_X := \Lambda x \in \mathbb{R}^d \,.\, \mathbb{P}_X((-\infty, x])$$

$$\texttt{ProbabilityFunction} :: \texttt{RandomVariable}(\Omega, V) \to V \to [0,1]$$
$$\texttt{ProbabilityFunction}\,(X) = p_X := \frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\#}$$

$$\texttt{ProbabiltiDensityByMeasure} :: \prod X : \texttt{RandomVariable}(\Omega, V) \,.\, \prod \mu : \texttt{Measure}(V, \mathcal{B}V) \,.\, \mathbb{P}_X \ll \mu \to$$
$$\to V \to \mathbb{R}_+$$
$$\texttt{ProbabilityDensityByMeasure}\,(X, \mu) = f_{X,\mu} := \frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\mu}$$

$$\texttt{AbsolutelyContinuous} :: ?\texttt{RandomVariable}(\Omega, \mathbb{R}^d)$$
$$X : \texttt{AbsolutelyContinuous} \iff \mathbb{P}_X \ll \lambda$$

$$\texttt{ProbabilityDensity} :: \prod X : \texttt{AbsolutelyContinuous}(\Omega, \mathbb{R}^d) \,.\, \mathbb{R}^d \to \mathbb{R}_+$$
$$\texttt{ProbabilityDensity}\,(X) = f_X := \frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\lambda}$$

1.3 Independant Random Objects

$$\texttt{RandomObject} = \prod (\Omega, \mathcal{F}, P) : \texttt{ProbabilitySpace} \,.\, \prod (\Omega', \mathcal{F}') \in \mathsf{BOR} \,.\, (\Omega, \mathcal{F}) \to_{\mathsf{BOR}} (\Omega', \mathcal{F}')$$

$$\texttt{associatedProbability} :: \texttt{RandomObject}\Big((\Omega, \mathcal{F}, P), (\Omega', \mathcal{F}')\Big) \to \texttt{Probability}(V, \mathcal{B}V)$$
$$\texttt{associatedProbability}\,(X) = \mathbb{P}_X := \Lambda B \in \mathcal{F}' \,.\, P(X^{-1}B)$$

$$\texttt{empiricEventProbability} :: \texttt{RandomObject}(\Omega, (\Omega', \mathcal{F}')) \to \mathcal{F}' \to [0,1]$$
$$\texttt{empiricEventProbability}\,(X, B) = \mathbb{P}(X \in B) := \mathbb{P}_X(B)$$

$$\texttt{IndependentFamily} :: \prod n \in \mathbb{N} \,.\, ?\prod i \in n \,.\, \texttt{RandomObject}(\Omega, (\Omega'_i, \mathcal{F}'_i))$$
$$X : \texttt{IndependentFamily} \iff \perp(X) \iff \forall B : \prod i \in n \to \mathcal{F}'_i \,.\, \mathbb{P}\left((X_i)_{i=1}^n \in \prod_{i=1}^n B_i\right) = \prod_{i=1}^n P(X_i \in B_i)$$

$\texttt{IndependenceByCDF} :: \forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.\, \forall X : \prod i \in n \,.\, \texttt{RandomVariable}(\Omega, \mathbb{R}^{m_i}) \,.$

$$. \perp(X) \iff P_X = \prod_{i=1}^{m} P_{X_i}$$

$\texttt{Proof} =$

$$N := \sum_{i=1}^{n} m_i : \mathbb{N},$$

$\texttt{Assume}\ L : \perp(X),$

$\texttt{Assume}\ x : \mathbb{R}^n,$

$$(1) := \eth P_X \eth \perp(X) \forall i \in n \,.\, \eth^{-1} P_{X_i} : P_X(x) = \mathbb{P}_X(-\infty, x] = \prod_{i=1}^{n} \mathbb{P}_{X_i}(-\infty, x_i] = \prod_{i=1}^{n} P_{X_i}(x_i);$$

$$\rightsquigarrow (1) := I(\forall) : P_X = \prod_{i=1}^{n} P_{X_i};$$

$$\rightsquigarrow (1) := I(\Rightarrow) : \perp(X) \Rightarrow P_X = \prod_{i=1}^{n} P_{X_i},$$

$$\texttt{Assume}\ R : P_X = \prod_{i=1}^{m} P_{X_i},$$

$$\texttt{Assume}\ A : \left\{ \bigcup_{i=1}^{k} (x_i, y_i] | k \in \mathbb{N}, x, y : k \to \mathbb{R}^n \right\},$$

$\texttt{Assume}\ B : \prod i \in n \,.\, \mathcal{B}\mathbb{R}^{m_i},$

$$Z := \prod_{i=1}^{n} B_i : \mathcal{B}\mathbb{R}^N,$$

$\texttt{Assume}\ (2) : Z \in A,$

$$(k, x, y, 3) := \eth A(Z)(2) : \sum k \in \mathbb{N} \,.\, \sum (x, y) : \mathbb{N} \to \mathbb{R}^n \times \mathbb{R}^n \,.\, \forall i, j \in k \,.\, x_i < y_i \ \&$$

$$\& \ i \neq j \Rightarrow (x_i, y_i] \cap (x_j, y_j) = \emptyset \ \& \ B = \bigcup_{i=1}^{k} (x_i, y_i],$$

$$(4) := \eth \mathbb{P}(X \in Z)(3) \eth P_X (2) \forall i \in n \,.\, \eth^{-1} \mathbb{P}(X \in B_i) :$$

$$: \mathbb{P}(X \in Z) = \sum_{i=1}^{k} \mathbb{P}\Big(X \in (x_i, y_i] \Big) = \sum_{i=1}^{k} \Big(P_X(y_i) - P_X(x_i) \Big) = \sum_{i=1}^{k} \prod_{j=1}^{n} \left(P_{X_j}(y_{i,j}) - P_{X_j}(x_{i,j}) \right) =$$

$$= \prod_{j=1}^{n} \sum_{i=1}^{k} \left(P_{X_j}(y_{i,j}) - P_{X_j}(x_{i,j}) \right) = \prod_{i=1}^{n} \mathbb{P}(X \in B_i);$$

$$\rightsquigarrow (2) := I(\forall) I(\Rightarrow) : \forall B : n \to \mathcal{B}\mathbb{R}^{m_i} \,.\, \prod_{i=1}^{n} B_i \in A \Rightarrow \mathbb{P}\left(X \in \prod_{i=1}^{n} B_i \right) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i),$$

$$Z := \left\{ \prod n \in \mathbb{N} \,.\, \mathcal{B}\mathbb{R}_i^m : \mathbb{P}\left(X \in \prod_{i=1}^{n} B_i \right) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i) \right\} :? \prod i \in n \,.\, \mathcal{B}\mathbb{R}^{m_i},$$

$$(3) := \forall i \in n \,.\, \texttt{MonotoneConvergence}(\mathbb{P}_{X_i}) : \forall i \in n \,.\, \Big(Z(i) : \texttt{MonotoneClass} \Big),$$

$$(4) := \texttt{MonotoneClassTHM}(2)(3) : \Big(Z : \texttt{Universe} \Big);$$

$$\rightsquigarrow (*) := I(\iff)(1) I(\Rightarrow) : \texttt{This};$$

$\square$

$\texttt{IndependenceByDensity} :: \forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.$

$$. \forall X : \texttt{RandomVariable}\left(\Omega, \prod_{i=1}^{n} \mathbb{R}^{m_i}\right) . \forall (1) : \mathbb{P}_X \ll \lambda \,.\, \forall \perp(X) \iff f_X = \prod_{i=1}^{n} f_{X_i}$$

$\texttt{Proof} =$

$$N := \sum_{i=1}^{n} m_i : \mathbb{N},$$

$\texttt{Assume}\ i : n,$

$\texttt{Assume}\ (2) : \mathbb{P}_{X_i} \not\ll \lambda,$

$(B, 3) := \eth(2) : \sum B \in \mathcal{B}\mathbb{R}^{m_i} \,.\, \mathbb{P}_{X_i}(B) > 0 \,\&\, \lambda(B) = 0,$

$$(4) := \texttt{FullProduct}(\mathbb{P}_X, B)(3) : \mathbb{P}_X\left(B \oplus \bigoplus_{j=1:j\neq i}^{n} \mathbb{R}^{m_i}\right) = \mathbb{P}_{X_i}(B) > 0,$$

$$(5) := \texttt{FullProduct}(\lambda, B)(3) : \lambda\left(B \oplus \bigoplus_{j=1:j\neq i}^{n} \mathbb{R}^{m_i}\right) = \lambda(B) = 0,$$

$(6) := \eth\texttt{AbsolutelyContinuous}(4, 5) : \mathbb{P}_X \not\ll \lambda,$

$(7) := I(\perp)(1)(6) : \perp;$

$\rightsquigarrow (2) := I(\forall)E(\perp) : \forall i \in n \,.\, \mathbb{P}_{X_i} \ll \lambda,$

$\texttt{Assume}\ L : \perp(X),$

$\texttt{Assume}\ x : \mathbb{R}^N,$

$(3) := \eth f_X(x)\eth L\texttt{LebesgueIsProduct}(N, m)\,\texttt{CubeSeparation}(N, m)\forall i \in n \,.\, \eth^{-1} f_{X_i} :$

$$f_X(x) = \lim_{r\to 0} \inf_{C:\texttt{Cube}(N,r,x)} \frac{\mathbb{P}_X(C)}{\lambda(C)} = \lim_{r\to 0} \inf_{C:\texttt{Cube}(N,r,x)} \frac{\prod_{i=1}^{n} \mathbb{P}_{X_i}(C_i)}{\prod_{i=1}^{n} \lambda(C_i)} = \prod_{i=1}^{n} \lim_{r\to 0} \inf_{C:\texttt{Cube}(m_i,r,x_i)} \frac{\mathbb{P}_{X_i}(C_i)}{\lambda(C_i)} = \prod_{i=1}^{n} f_{X_i}(x_i);$$

$$\rightsquigarrow (2) := I(\Rightarrow)I(=,\to)I(\forall) : \perp(X) \Rightarrow f_X = \prod_{i=1}^{n} f_{X_i},$$

$$\texttt{Assume}\ R : f_X = \prod_{i=1}^{n} f_{X_i},$$

$\texttt{Assume}\ B : \prod i \in n \,.\, \mathcal{B}\mathbb{R}^{m_i},$

$$A := \prod_{i=1}^{n} B_i : \mathcal{B}\mathbb{R}^N,$$

$(3) := \texttt{DensityProbability}(\mathbb{P}_X, A)\texttt{Fubini}(A, B_i)R\texttt{IntegralHomogen}\forall i \in n \,.\, \texttt{DensityProbability}(\mathbb{P}_{X_i}, B_i) :$

$$: \mathbb{P}(X \in A) = \int_A f_X \, d\lambda = \int_{B_1} \dots \int_{B_n} f_X(x_i)_{i=1}^{n} \, d\lambda(x_1) \dots d\lambda(x_n) =$$

$$= \int_{B_1} \dots \int_{B_n} \prod_{i=1}^{n} f_{X_i}(x_i) \, d\lambda(x_1) \dots d\lambda(x_n) = \prod_{i=1}^{n} \int_{B_i} f_{X_i} d\lambda = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i);$$

$\rightsquigarrow (*) := I(\iff)(2)I(\Rightarrow)\eth\perp(X)I(\forall) : \texttt{This};$

$\square$

$\texttt{AbstractIndepndanceByDensity} :: \forall n \in \mathbb{N} \,.\, \forall(\Omega', \mathcal{F}') : n \to \mathsf{BOR} \,.\, \forall \mu : \prod i \in n \,.\, \texttt{Measure}(\Omega'_i, \mathcal{F}'_i) \,.$

$$. \forall X : \prod i \in \ . \texttt{RandomObject}\Big(\Omega, (\Omega'_i, \mathcal{F}'_i)\Big) \,.\, \forall (1) : X \ll M \,.\, f_{X,M} = \prod_{i=1}^{n} f_{X_i, \mu_i}$$

where

$$M = \prod_{i=1}^{n} \mu_i \iff \perp(X)$$

Proof =

Proof as above: replace λ by M

□

$\texttt{IndependanceIsBOR} :: \forall n \in \mathbb{N} \,.\, \forall(\Omega', \mathcal{F}'), (\Omega'', \mathcal{F}'') : n \to \mathsf{BOR} \,.$

$$. \forall X : \prod i \in n \,.\, \texttt{RandomObject}\Big(\Omega, (\Omega'_i, \mathcal{F}'_i)\Big) \,.\, \forall g : \prod i \in n \,.\, (\Omega'_i, \mathcal{F}'_i) \to_{\mathsf{BOR}} (\Omega''_i, \mathcal{F}''_i) \,.\, \perp(X) \Rightarrow \perp\Big(g(X)\Big)$$

Proof =

$$B : \prod i \in n \,.\, \mathcal{F}_i$$

$$A = \prod_{i=1}^{n} B_I \in \sigma\left(\prod_{i=1}^{n} \mathcal{F}_i\right)$$

$$\mathbb{P}\Big(g(X) \in A\Big) = \mathbb{P}(X \in g^{-1}A) = \prod_{i=1}^{n} \mathbb{P}(X_i \in g_i^{-1} B_i) = \prod_{i=1}^{n} \mathbb{P}\Big(g_i(X_i) \in B_i\Big)$$

□

$\texttt{IndependentClass} :: \prod (\Omega, \mathcal{F}, P) : \texttt{ProbabilitySpace} \,.\, ? \prod n \in \mathbb{N} \,.\, n \to ?\mathcal{F}$

$$X : \texttt{IndependentClass} \iff \perp X \iff \forall B : \prod i \in n \,.\, X_i \,.\, P\left(\bigcap_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} P(B_i)$$

1.4 Pushforward of Density

$\texttt{DensityPushforward} :: \forall X : \texttt{AbsolutelyContinuous}(\Omega, \mathbb{R}^n) \;.\; \forall U, V : \texttt{Open}(\mathbb{R}^n) \;.$

$\;.\; \forall (1) : \operatorname{Im} X \subset U \;.\; \forall g : (\mathbb{R}^n, U) \leftrightarrow_{\mathsf{DIFF}(1)} (\mathbb{R}^n, V) \;.\; \forall x \in V \;.\; f_{g(X)}(x) = \left| \det \mathrm{D} g^{-1}|_x \right| f\left(g^{-1}(x)\right)$

$\texttt{Proof} \;=$

$$\mathbb{P}(g(X) \in A) = \mathbb{P}(X \in g^{-1}A) = \int_{g^{-1}(A)} f \, \mathrm{d}\lambda = \int_A \left| \det \mathrm{D} g^{-1} \right| f \circ g^{-1} \mathrm{d}\lambda$$

$\square$

1.5 Expectation And Variance

$\texttt{ExpectaionExists} :: ?(\texttt{RandomVariable}(\Omega, V))$
$X : \texttt{ExpectationExists} \iff \mathrm{id} : \texttt{PetisIntegrable}(\mathbb{P}_X)$

$\texttt{NthMomentExists} :: \prod V : \texttt{TopologicalAlgbra}(K) \,.\, \mathbb{N} \to ?\texttt{RandomVariable}(\Omega, V)$

$X : \texttt{NthMomentExists}(n) \iff \mathrm{id}^n : \texttt{PetisIntegrable}(\mathbb{P}_X)$

$\texttt{expectation} :: \texttt{NthMomentExists}(\Omega, V)(1) \to V$
$$\texttt{expectation}\,(X) = \mathbb{E}\,X := \int_V x \,\mathrm{d}\mathbb{P}_X$$

$\texttt{ExpectationPushforward} :: \forall X : \texttt{RandomObject}(\Omega, A) \,.\, \forall g : \texttt{PetisIntegrable}(A, V)(\mathbb{P}_X) \,.$
$$.\ \mathbb{E}\,g(X) = \int g \,\mathrm{d}\,\mathbb{P}_X$$

$\texttt{Proof} \ =$
Use simple function approximation □

$\texttt{NthCentralMomentExists} :: ?\texttt{NthMomentExists}(\Omega, \mathbb{R})(1)$
$X : \texttt{NthCentralMomentExists}(n) \iff \left(\mathrm{id} - \mathbb{E}\,X\right)^n : \texttt{Integrable}(\mathbb{P}_X)$

$\texttt{NthAbsoluteCentralMomentExists} :: ?\texttt{NthMomentExists}(\Omega, \mathbb{R})(1)$
$X : \texttt{NthAbsoluteCentralMomentExists}(n) \iff \left|\mathrm{id} - \mathbb{E}\,X\right|^n : \texttt{Integrable}(\mathbb{P}_X)$

$\texttt{variance} :: \texttt{NthCentralMomentExists}(\Omega, \mathbb{R})(2) \to \mathbb{R}_+$
$$\texttt{variance}\,(X) = \mathbb{V}(X) := \int_{-\infty}^{\infty} (x - \mathbb{E}\,X)^2 \,\mathrm{d}\mathbb{P}_X$$

$\texttt{Covariant} :: \prod V : \mathsf{BAN}(K) \,.\, ?\texttt{ExpectationExists}(\Omega, V)(1)$
$X : \texttt{Covariant} \iff \|\mathrm{id}\|^2 : \texttt{Integrable}(\mathbb{P}_X)$

$\texttt{covarianceOperator} :: \prod V : \mathsf{BAN}(K) \,.\, \texttt{Covariant}(\Omega, V) \to \mathcal{B}(V^*; V^{**})$
$$\texttt{covarianceOperator}\,(X) = \mathbb{V}\,X := \Lambda a, b \in V^* \,.\, \int_V a(x - \mathbb{E}\,x) b(x - \mathbb{E}\,x) \,\mathrm{d}\mathbb{P}_X(x)$$

MomentExistanceTheorem $:: \forall X : \texttt{NthMomentExists}(V,\mathbb{R})(n) \;.\; \forall k \in n \;.\; X : \texttt{NthMomentExists}(\Omega,\mathbb{R})(k)$

Proof $=$

$$\int_0^\infty x^k \, \mathrm{d}\mathbb{P}_X = \int_{-1}^1 x^k \, \mathrm{d}\mathbb{P}_X + \int_1^\infty x_k \, \mathrm{d}\mathbb{P}_X \le \mathbb{P}(|X| < 1) + \int_0^\infty x^n < \infty$$

Apply same logic to possibly negative parts.

□

VarianceRepresentation $:: \forall X : \texttt{NthMomentExists}(V,\mathbb{R})(2) \;.\; \mathbb{V}\, X = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2$

Proof $=$

$$\mathbb{V}\, X = \mathbb{E}\left(X - \mathbb{E}\, X\right)^2 = \mathbb{E}\, X^2 - 2X\,\mathbb{E}\, X + (\mathbb{E}\, X)^2 = \mathbb{E}\, X^2 - (\mathbb{E}\, X)^2$$

□

IndepedentExpextion $:: \forall V : \texttt{BanachAlgebra}(K) \;.\; \forall n \in \mathbb{N} \;.\; \forall X : n \to \texttt{RandomVariable}(\Omega, V)$

$$.\; \forall(1) : \perp(X) \;.\; \mathbb{E}\prod_{i=1}^n X_i = \prod_{i=1}^n \mathbb{E}\, X_i$$

Proof $=$

Use Fubbini theorem

□

covariance $:: \texttt{NthMomentExists}^2(\Omega,\mathbb{R})(2) \to \mathbb{R}$

covariance $(X,Y) = \operatorname{Cov}(X,Y) := \mathbb{V}(X,Y)(e_1)(e_2)$

correlation $:: \texttt{NthMomentExists}^2(\Omega,\mathbb{R})(2) \to [-1,1]$

correlation $(X,Y) = \operatorname{Corr}(X,Y) := \dfrac{\operatorname{Cov}(X,Y)}{\sqrt{\mathbb{V}(X)\,\mathbb{V}(Y)}}$

VarianceSum $:: \forall n \in \mathbb{N} \;.\; \forall X : n \to \texttt{NthMomentExists}(\Omega,\mathbb{R})(2) \;.\; \mathbb{V}\left(\sum_{i=1}^n X_i\right) = \sum_{i,j=1}^n \operatorname{Cov}(X_i, X_j)$

Proof $=$

By sum expansion and the additivity of the integral

□

ExpectionIsLinear $:: \mathbb{E} : \mathcal{L}\Big(\texttt{NthMomemntExists}(\Omega, V)(1), V\Big)$

Proof $=$

By linearity of integral

□

CovarianceIsBilliner $:: \operatorname{Cov} : \mathcal{L}\Big(\texttt{NthMomentExists}^2(\Omega,\mathbb{R})(2), \mathbb{R}\Big)$

Proof $=$

As product of two linear maps

□

$\texttt{VarianceIsQuadraticForm} :: \mathbb{V} : \mathcal{Q}\Big(\texttt{NthMomentExists}(\Omega, \mathbb{R})(2)\Big)$

$\texttt{Proof} \ =$

By linearity of expectation

$\mathbb{V}\, cX = \mathbb{E}\, c^2 X^2 + \big(\mathbb{E}\, cX\big)^2 = c^2\Big(\mathbb{E}\, X^2 + \big(\mathbb{E}\, X\big)^2\Big) = c^2\, \mathbb{V}\, X$

$\mathbb{V}\, X + Y = \mathbb{V}\, X + 2\mathrm{Cov}(X, Y)\, \mathbb{V}\, Y$

$\square$

$\texttt{VarianceMatrix} :: \forall X : \texttt{RandomVariable}(\Omega, \mathbb{R}^n) \,.\, \forall A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \,.\, \mathbb{V}\, AX = A(\mathbb{V}\, X)A^*$

$\texttt{Proof} \ =$

$\Sigma := \mathbb{V}\, X : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}),$

$B := A\Sigma A^* : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m; \mathbb{R}),$

$\texttt{Assume}\ i, j : m,$

$$(1) := \eth\, \mathbb{V}\, AX \eth^{-1} B : \Big(\mathbb{V}\, AX\Big)_{i,j} = \exp\left(\sum_{k=1}^{n} A_{i,k}(X_k - \mathbb{E}\, X_k)\right)\left(\sum_{k=1}^{n} A_{j,k}(X_k - \mathbb{E}\, X_k)\right) =$$

$$= \sum_{k,l=1}^{n} A_{i,k} \Sigma_{i,j} A_{j,k} = B_{i,j};$$

$\rightsquigarrow (2) := \texttt{MatrixUnique} : \mathbb{V}\, AX = B;$

$\square$

1.6 Weak Law of large numbers

$\texttt{RandomSequence}(\Omega, V) = \mathbb{N} \to \texttt{RandomVariable}(\Omega, V)$

$\texttt{RandomSequenceIsRandomVariable} :: \forall X : \texttt{RandomSequence}(\Omega, V) \,.\, X : \texttt{RandomVariable}\left(\Omega, \bigoplus_{i=1}^{\infty} V\right)$

$\texttt{Proof} =$

$$X^{-1}A = \bigcup_{n=1}^{\infty} X_n^{-1}\pi_n A \in \mathcal{F}_\Omega$$

$\square$

$\texttt{IndependentSequence} :: ?\texttt{RandomSequence}(\Omega, V)$

$X : \texttt{IndependentSequence} \iff \forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.\, \perp(X_m) \iff$

$\texttt{WeakLawOfLargeNumbers} :: \forall X : \texttt{RandomSequence}(\Omega, \mathbb{R}) \,.\, \forall (1) : \perp(X) \,.$

$. \forall (2) : \forall n \in \mathbb{N} \,.\, X_n : \texttt{NthCentralMomentExists}(\Omega, \mathbb{R})(2) \,.$

$. \forall (M, 3) : \sum M \in \mathbb{R}_{++} \,.\, \forall n \in \mathbb{N} \,.\, \mathbb{V}\, X_n \le M \,.\, \dfrac{\sum_{i=1}^n X_i - \sum_{i=1}^n \mathbb{E}\, X_i}{n} \xrightarrow[n\to\infty]{\mathbb{P}_X} 0$

$\texttt{Proof} =$

Assume $\varepsilon \in \mathbb{R}_{++}$

By Chebyshev Inequelity

$$\mathbb{P}\left(\frac{|\sum_{i=1}^n X_i - \mathbb{E}\, X_i|}{n} > \varepsilon\right) \le \frac{\mathbb{E}\left(\sum_{i=1}^n X_i - \mathbb{E}\, X_i\right)^2}{\varepsilon^2 n^2} = \frac{\mathbb{V}\left(\sum_{i=1}^n X_i\right)}{\varepsilon^2 n^2} = \frac{\sum_{i=1}^n \mathbb{V}(X_i)}{\varepsilon^2 n^2} \le \frac{M}{\varepsilon^2 n} \xrightarrow[n\to\infty]{} 0$$

$\square$

2 Conditional Probability Theory

2.1 Conditional Probability and Excpectation In the Point

$\texttt{ConditionalProbabilityExists} :: \forall X : \texttt{RandomObject}\Big((\Omega, \mathcal{F}, \mathbb{P}), (\Omega', \mathcal{F}')\Big) \ . \ \forall B \in \mathcal{F} \ .$

$$. \ \exists g : (\Omega', \mathcal{F}') \to_{\mathsf{BOR}} (\mathbb{R}, \mathcal{B}\mathbb{R}) \ . \ \forall A \in \mathcal{F}' \ . \ \mathbb{P}\Big(X^{-1}A \cap B\Big) = \int_A g(x) \, \mathrm{d}\mathbb{P}_X$$

$\texttt{Proof} \ =$

We have measure which is a. c. w. r. $\mathbb{P}_X$.

Apply Radon-Nikodym theorem to it.

□

$\texttt{conditionalProbabilityAtAPoint} :: \texttt{RandomObject}\Big((\Omega, \mathcal{F}, \mathbb{P}), (\Omega', \mathcal{F}')\Big) \to \mathcal{F} \to \Omega' \to \mathbb{R}_{++}$

$\texttt{conditionalProbabilityAtAPoint}\,(X, B, x) = \mathbb{P}(B|X = x) := g(x)$

$\quad \texttt{Where}$

$\quad g = \texttt{ConditionalProbabilityExists}(X, B)$

$\texttt{conditionalDensity} :: \texttt{AbsolutelyContinuous}\Big(\Omega, \Omega'^2\Big) \to \Omega' \to \mathbb{R}$

$$\texttt{conditionalDensity}\,(Y|X) = f_{Y|X} := \frac{f_{X,Y}}{f_Y}$$

$\texttt{ConditionalDensityTHM} :: \forall X, Y : \texttt{AbsolutelyContinuous}\Big(\Omega, (\Omega', \mathcal{F}')^2\Big) \ . \ \forall B \in \mathcal{F}$

$$. \ \forall x \in \Omega' \ . \ \mathbb{P}(Y \in B|X = x) = \int_B f_{Y|X}(x, y) \, \mathrm{d}\lambda(y)$$

$\texttt{Proof} \ =$

...

□

$\texttt{ConditionalExpectationExists} :: \forall (X, Y) : \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, \mathbb{P}), \Omega' \times (\mathbb{C}^m, \mathcal{B}\mathbb{C}^n)\Big) \ .$

$$\exists g \in \Omega' \to_{\mathsf{BOR}} (\mathbb{C}^n, \mathcal{B}\mathbb{C}^n) \ . \ \forall A \in \mathcal{B}V \ . \ \int_{X^{-1}A} Y \, \mathrm{d}\mathbb{P} = \int_A g \, \mathrm{d}\mathbb{P}_X \ .$$

$\texttt{Proof} \ =$

Simmilar prove with Radon-Nikodym applied to a complex measures.

□

$\texttt{conditionalExpectationAtAPoint} :: \texttt{RandomVariable}\Big(\Omega, \Omega'\Big) \to \texttt{RandomVariable}\Big(\Omega, (\mathbb{C}^m, \mathcal{B}\mathbb{C}^m)\Big) \to \Omega' \to$

$\texttt{conditionalExpectationAtAPoint}\,(X, Y, x) = \mathbb{E}(Y|X = x) := g(x)$

$\quad \texttt{Where}$

$\quad g = \texttt{ConditionalExpectationExists}(X, Y)$

$K = \mathbb{R} | K = \mathbb{C}$

$\texttt{ConditionalExpectationWithDensity} :: \forall (X,Y) : \texttt{AbsolutelyContinuous}\Big(\Omega, (K^m, \mathcal{B}K^m) \times (K^n, \mathcal{B}K^n)\Big) .$

$\quad . \forall x \in K^m . \mathbb{E}(Y|X = x) = \int_{K^n} y f_{Y|X}(x,y) \, \mathrm{d}y$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : \mathcal{F}_{\Omega'},$

$(1) := \eth^{-1} f_{(X,Y)} \left(\int_{X^{-1}A} Y \, \mathrm{d}\mathbb{P}_\Omega \right) \texttt{Fubbini}(\lambda) \texttt{UnitalMult}(f_X(x)) \eth^{-1} f_{X|Y}(x,y) \eth f_X :$

$\quad : \int_{X^{-1}A} Y \, \mathrm{d}\mathbb{P}_\Omega = \int_{A \times K^n} y f_{(X,Y)}(x,y) \, \mathrm{d}(x,y) =$

$\quad = \int_A \int_{K^n} y f_{(X,Y)}(x,y) \, \mathrm{d}(y) \mathrm{d}(x) = \int_A f_X(x) \int_{K^n} y f_{(Y|X)}(x,y) \, \mathrm{d}(y) \mathrm{d}(x) = \int_A \int_{K^n} y f_{Y|X} \, \mathrm{d}(y) \mathrm{d}\mathbb{P}_X(x);$

$\rightsquigarrow (*) := \eth^{-1} \, \mathbb{E}(Y|X = x) : \mathbb{E}(Y|X = x) = \int_{K^n} y f_{Y|X}(x,y) \, \mathrm{d}y;$

$\square$

$\texttt{ConditionalExpectationPushforward} :: \forall (X,Y) : \texttt{AbsolutelyContinuous}\Big(\Omega, (K^m, \mathcal{B}K^m) \times (K^n, \mathcal{B}K^n)\Big) .$

$\quad . \forall x \in K^m . \forall g : (K^n, \mathcal{B}K^n) \to (K^k, \mathcal{B}K^k) . \mathbb{E}(g(Y)|X = x) = \int_{K^n} g(y) f_{Y|X}(x,y) \, \mathrm{d}y$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : \mathcal{F}_{\Omega'},$

$(1) := \eth^{-1} f_{(X,Y)} \left(\int_{X^{-1}A} g(Y) \, \mathrm{d}\mathbb{P}_\Omega \right) \texttt{Fubbini}(\lambda) \texttt{UnitalMult}(f_X(x)) \eth^{-1} f_{X|Y}(x,y) \eth f_X :$

$\quad : \int_{X^{-1}A} g(Y) \, \mathrm{d}\mathbb{P}_\Omega = \int_{A \times K^n} g(y) f_{(X,Y)}(x,y) \, \mathrm{d}(x,y) =$

$\quad = \int_A \int_{K^n} g(y) f_{(X,Y)}(x,y) \, \mathrm{d}(y) \mathrm{d}(x) = \int_A f_X(x) \int_{K^n} g(y) f_{(Y|X)}(x,y) \, \mathrm{d}(y) \mathrm{d}(x) = \int_A \int_{K^n} g(y) f_{Y|X} \, \mathrm{d}(y) \mathrm{d}\mathbb{P}_X(x);$

$\rightsquigarrow (*) := \eth^{-1} \, \mathbb{E}(g(Y)|X = x) : \mathbb{E}(g(Y)|X = x) = \int_{K^n} g(y) f_{Y|X}(x,y) \, \mathrm{d}y;$

$\square$

2.2 Conditional Expectation Given a Sigma-Field

$$\texttt{inducedSigmaField} :: \texttt{RandomObject}\Big(\Omega, (\Omega', \mathcal{F}')\Big) \to \sigma\texttt{-Algebra}\,(\Omega)$$

$$\texttt{inducedSigmaField}\,(X) = \sigma(X) := \{X^{-1}(A) | A \in \mathcal{F}'\}$$

$$\texttt{MeasurabilityWithInducedSigmaField} :: \forall X : \texttt{RandomObject}\Big((\Omega, \mathcal{F}), (\Omega', \mathcal{F}')\Big) \ .$$

$$. \ \sigma(X) = \bigcap \Big\{ \mathcal{A} : \sigma\texttt{-Algebra}\,(\Omega) : X \in \mathcal{M}_{\mathsf{BOR}}\Big((\Omega, \mathcal{A}), (\Omega', \mathcal{F}')\Big)\Big\}$$

$\texttt{Proof} \ =$

Every set in the intersection contains $\sigma(X)$ and $\sigma(X)$ itself is in the intersection. Result follows

$\square$

$$\texttt{BORCompositionExists} :: \forall X : \texttt{RandomObject}(\Omega, \Omega') \ . \ \forall Z : \texttt{RandomObject}\Big(\sigma(X), (K^n, \mathcal{B}K^n)\Big) \ .$$

$$. \ \exists f : \Omega' \to_{\mathsf{BOR}} \Omega'' \ . \ Z = f \circ X$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : \operatorname{Im} Z,$

$(A, 2) := Z^{-1}a : \sum A \in \sigma(X) \ . \ A \neq \emptyset,$

$\texttt{Assume}\ \omega : A,$

$\texttt{Assume}\ (\omega', 3) : \sum \omega \in \Omega \ . \ X(\omega') = X(\omega),$

$(4) := \eth\sigma(X)(A)(\omega, \omega')(3) : \omega' \in A;$

$\leadsto (3) := I(\forall) : \forall p \in \Omega : X(p) = X(\omega) \ . \ p \in A,$

$f(X(\omega)) := a : K^n;$

$\leadsto f := I(\to) : A \to \{a\};$

$(2) := \texttt{StitchedFunction} : f : \operatorname{Im} X \to K^n;$

$f' := \Lambda x \in O' \ . \ \texttt{if} \quad x \in \operatorname{Im} X \quad \texttt{then} \quad f(x) \quad \texttt{else} \quad 0 : \Omega' \to_{\mathsf{BOR}} (K^n, \mathcal{B}K^n),$

$(*) := \eth f \eth f' : Z = f' \circ X;$

$\square$

$$\texttt{ConditionalExpectationWRSFExists} :: \forall X : \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, \mathbb{P}), (K^n, \mathcal{B}K^n)\Big) \ .$$

$$. \ \forall \mathcal{A} : \sigma\texttt{-Subalgebra}(\mathcal{F}) \ . \ \exists g : (\Omega . \mathcal{A}) \to_{\mathsf{BOR}} (K^n, \mathcal{B}K^n) \ . \ \forall A \in \mathcal{A} \ . \ \int_A X \, d\mathbb{P} = \int_A g \, d\mathbb{P}$$

$\texttt{Proof} \ =$

Apply Radon-Nikodym to a measure defined by the integral relatively to $\mathbb{P}$

$\square$

$$\texttt{conditionalExpectationWRSF} :: \texttt{RandomVariable}\Big(\Omega, (K^n, \mathcal{B}K^n)\Big) \to \sigma\texttt{-Subalgebra}(\mathcal{F}) \to$$

$$\to \texttt{RandomVariable}\Big(\Omega, (K^n, \mathcal{B}K^n)\Big)$$

$$\texttt{conditionalExpectationWRSF}\,(X, \mathcal{A}) = \mathbb{E}(X|\mathcal{A}) := g$$

$\texttt{Where}$

$g = \texttt{ConditionalExpectationWRSFExists}(X, \mathcal{A})$

$\texttt{conditonalExpetationWRRO} :: \texttt{RandomVariable}\Big(\Omega, K^n, \Big) \to \texttt{RandomObject}(\Omega, \Omega') \to$

$\to \texttt{RandomVariable}\Big(\Omega, (K^n, \mathcal{B}K^n)\Big)$

$\texttt{conditionalExpectationWRRO}\,(X, Y) = \mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$

$\texttt{conditionalProbabilityWRSA} :: \prod(\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} \; . \; \sigma\texttt{-subalgebra}(\mathcal{F}) \to \mathcal{F} \to [0, 1]$

$\texttt{conditionalProbabilityWRSA}\,(A) = \mathbb{P}(A|\mathcal{A}) := \mathbb{E}(I_A|\mathcal{A})$

$\texttt{conditionalProbabilityWRRO} :: \prod(\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} \; . \; \texttt{RandomObject}(\Omega, \Omega') \to \mathcal{F} \to [0, 1]$

$\texttt{conditionalProbabilityWRRO}\,(A) = \mathbb{P}(A|X) := \mathbb{P}(A|\sigma(X))$

$\texttt{CEWRSAofAConstant} :: \forall X : \texttt{RandomVariable}\Big(\Omega, (K^n, \mathcal{B}K^n)\Big)\forall(c, 1) : \sum c \in \mathbb{C}^n \; . \; \mathbb{P}(X = x) = 1 \; .$

$\forall \mathcal{A} : \sigma\texttt{-Subalgebra}(\mathcal{F}_\Omega) \; . \; \mathbb{P}\Big(\mathbb{E}(X|\mathcal{A}) = c\Big) = 1$

$\texttt{Proof} \; =$

$(2) := \texttt{MeasurableConstant}(c) : \Big(\Lambda\omega \in \Omega \; . \; c : \Omega \to_{\mathsf{BOR}} (K^n, \mathcal{B}K^n)\Big),$

$E := X^{-1}\{c\} : \mathcal{F}_\Omega,$

$(3) := (1)(E) : \mathbb{P}_\Omega(E) = 1,$

$\texttt{Assume}\; C : \mathcal{A},$

$(4) := \texttt{DisjointMeasure}\left(\int_\bullet X \, d\mathbb{P}, C \cap E, C \cap E^c\right)(3)\texttt{ConstantInegral}(c, C, \mathbb{P}) :$

$: \int_C X \, d\mathbb{P} = \int_{C \cap E} c \, d\mathbb{P} + \int_{C \cap E^\complement} X d\mathbb{P} = c\mathbb{P}(C) = \int_C c \, d\mathbb{P};$

$\rightsquigarrow * := \eth^{-1} \, \mathbb{E} \, ` : (X|\mathcal{A})(2) : \mathbb{P}\Big(\mathbb{E}(X|\mathcal{A}) = c\Big) = 1;$

$\square$

$\texttt{CEWRSAIneq} :: \forall X, Y \in \texttt{RandomVariable}\Big(\Omega, \mathbb{R}\Big) \; . \; \forall \mathcal{A} : \sigma\texttt{-Subalgebra}(\mathcal{F}_\Omega) \; .$

$. \; \forall(1) : X \le Y \; . \; \mathbb{E}(X|\mathcal{A}) \le \mathbb{E}(Y|\mathcal{A}) \quad \mathrm{a\,.\,e\,.}\;[\mathbb{P}_\Omega]$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{CEWRSATriangularIneq} :: \forall X \in \texttt{RandomVariable}\Big(\Omega, K^n\Big) \; . \; \forall \mathcal{A} : \sigma\texttt{-Subalgebra}(\mathcal{F}_\Omega) \; .$

$. \; \big\|\mathbb{E}(X|\mathcal{A})\big\| \le \mathbb{E}\left(\|X\| |\mathcal{A}\right) \quad \mathrm{a\,.\,e\,.}\;[\mathbb{P}_\Omega]$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{CEWRSALinearity} :: \forall X,Y \in \texttt{RandomVariable}\left(\Omega, K^n\right) . \forall \mathcal{A} . \forall a,b \in K .$

$$\mathbb{E}(aX+bY|\mathcal{A}) = a\,\mathbb{E}(X|\mathcal{A}) + b\,\mathbb{E}(Y|\mathcal{A}) \quad \mathrm{a.e.}\ [\mathbb{P}_\Omega]$$

$\texttt{Proof} =$

$\texttt{Assume}\ C : \mathcal{A},$

$$\int_C aX + bY \,\mathrm{d}\mathbb{P}_\Omega = a\int_C X\,\mathrm{d}\mathbb{P}_\Omega + b\int_C Y\,\mathrm{d}\mathbb{P}_\Omega = a\int_C \mathbb{E}(X|\mathcal{A})\mathrm{d}\mathbb{P}_\Omega + b\int_C \mathbb{E}(Y|\mathcal{A})\mathrm{d}\mathbb{P}_\Omega =$$

$$= \int_C a\,\mathbb{E}(X|\mathcal{A}) + b\,\mathbb{E}(Y|\mathcal{A})\mathrm{d}\mathbb{P}_\Omega;$$

$\square$

$\texttt{CEWRSAMonotonicConvergence} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R}) . \forall X : \texttt{RandomVariable}(\Omega, \mathbb{R}) .$

$$. \forall(1) : Y \uparrow X \quad \mathrm{a.e.}\ [\mathbb{P}_\Omega]\ .\ \mathbb{E}(Y|\mathcal{A}) \uparrow \mathbb{E}(X|\mathcal{A}) \quad \mathrm{a.e.}\ [\mathbb{P}_\Omega]$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CEWRSATonneli} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R}) . \forall(1) : Y \geq 0 \qquad \mathrm{a.e.}\ [\mathbb{P}_\Omega]\ .$

$$. \ \mathbb{E}\left(\sum_{n=1}^{\infty} Y_n|\mathcal{A}\right) = \sum_{n=1}^{\infty} \mathbb{E}(Y_n|\mathcal{A}) \qquad \mathrm{a.e.}\ [\mathbb{P}_\Omega]$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CPWRSAMonotonicConvergence} :: \forall(\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} . \forall A : \texttt{Disjoint}(\mathbb{N}, \mathcal{F}) .$

$$. \ \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i|\mathcal{A}\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(A_i|\mathcal{A}\right) \quad \mathrm{a.e.}\ [\mathbb{P}_\Omega]$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CEWRSAExpectation} :: \mathbb{E}\,\mathbb{E}(Y|\mathcal{A}) = \mathbb{E}\,Y$

$\texttt{Proof} =$

$$\mathbb{E}\,\mathbb{E}(Y|\mathcal{A}) = \int_\Omega \mathbb{E}(Y|\mathcal{A})\,\mathrm{d}\mathbb{P} = \int_\Omega Y\,\mathrm{d}\mathbb{P} = \mathbb{E}\,Y$$

$\square$

$\texttt{CEWRSADominatedConvergence} :: \forall Y : n \to \texttt{RandomVariable}(\Omega, K^n)\ .$

$: \forall (Z, 1) : \sum Z : \texttt{RandomVariable}(\Omega.\mathbb{R})\ .\ \forall n \in \mathbb{N}\ .\ \|Y_n\| \leq Z\ .$

$.\ \forall (X, 2) : \sum Y : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ \lim_{n\to\infty} Y_n = X \quad \text{a.e.}\ [\mathbb{P}]\ .\ \lim_{n\to\infty} \mathbb{E}(Y_n|\mathcal{A}) = \mathbb{E}(X|\mathcal{A}) \quad \text{a.e.}\ [\mathbb{P}]$

$\texttt{Proof} =$

$\Delta := \Lambda n \in \mathbb{N}\ .\ \sup_{k\geq n} \|X - Y_n\| : \mathbb{N} \to \texttt{RandomVariable}(\Omega, K^n),$

$(3) := (2)(\eth\Delta) : \Delta_{n\to\infty} \downarrow 0 \quad \text{a.e.}\ [\mathbb{P}],$

$(4) := \texttt{CEWARALinearity}(Y, X)\texttt{CEWSRATriangularIneq}(Y - X)\eth^{-1}\Delta :$

$: \|\mathbb{E}(Y|\mathcal{A}) - \mathbb{E}(X|\mathcal{A})\| \leq \mathbb{E}\left(\|Y - X\|\Big|\mathcal{A}\right) \leq \mathbb{E}(\Delta|\mathcal{A}),$

$(5) := \texttt{CEWRAMonotonicConvergence}(2Z - \Delta)(2) : \lim_{n\to\infty} \mathbb{E}(2Z - \Delta_n|\mathcal{A}) = \mathbb{E}(2Z|\mathcal{A}),$

$(6) := \texttt{CEWRALinearity}(-2Z + \Delta, 2Z)\texttt{LinearLimit}(\ldots)(5) :$

$\lim_{n\to\infty} \mathbb{E}(\Delta_n|\mathcal{A}) = \lim_{n\to\infty} -\mathbb{E}(2Z - \Delta_n|\mathcal{A}) + \mathbb{E}(2Z|\mathcal{A})$

$= \lim_{n\to\infty} \mathbb{E}(2Z|\mathcal{A}) - \lim_{n\to\infty} \mathbb{E}(2Z - \Delta_n|\mathcal{A}) = \mathbb{E}(2Z|\mathcal{A}) - \mathbb{E}(2Z|\mathcal{A}) \quad \text{a.e.}\ [\mathbb{P}] = 0 \quad \text{a.e.}\ [\mathbb{P}],$

$(7) := (4)(6) : \lim_{n\to\infty} \left\|\mathbb{E}(X|\mathcal{A}) - \mathbb{E}(Y_n|\mathcal{A})\right\| = 0 \quad \text{a.e.}\ [\mathbb{P}],$

$(*) := \texttt{DiffferenceLimit}(7) : \lim_{n\to\infty} (Y_n|\mathcal{A}) = \mathbb{E}(X|\mathcal{A}) \quad \text{a.e.}\ [\mathbb{P}];$

$\square$

$\texttt{CEWRSAExtendedMonotoneConvergenceAbove} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R})\ .$

$.\ \forall (Z, 1) : \sum Z : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ \mathbb{E}\, Z > -\infty\ .\ \forall (2) : Y > Z \quad \text{a.e.}\ [\mathbb{P}]\ .$

$.\ \forall (X, 3) : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ Y \uparrow X \quad \text{a.e.}\ [\mathbb{P}]\ .\ \mathbb{E}(Y|\mathcal{A}) \uparrow \mathbb{E}(X|\mathcal{A}) \quad \text{a.e.}\ [\mathbb{P}]$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CEWRSAExtendedMonotoneConvergenceBelow} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R})\ .$

$.\ \forall (Z, 1) : \sum Z : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ \mathbb{E}\, Z < \infty\ .\ \forall (2) : Y < Z \quad \text{a.e.}\ [\mathbb{P}]\ .$

$.\ \forall (X, 3) : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ Y \downarrow X \quad \text{a.e.}\ [\mathbb{P}]\ .\ \mathbb{E}(Y|\mathcal{A}) \downarrow \mathbb{E}(X|\mathcal{A}) \quad \text{a.e.}\ [\mathbb{P}]$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CEWRSAFatouLemmaAbove} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R})\ .$

$.\ \forall (Z, 1) : \sum Z : \texttt{RandomVariable}(\Omega, \mathbb{R})\ .\ \mathbb{E}\, Z > -\infty\ .\ \forall (2) : Y > Z \quad \text{a.e.}\ [\mathbb{P}]\ .$

$.\ \lim_{n\to\infty}\inf \mathbb{E}(Y_n|\mathcal{A}) \geq \mathbb{E}\left(\lim_{n\to\infty}\inf Y_n|\mathcal{A}\right)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CEWRSAFatouLemmaBelow} :: \forall Y : \mathbb{N} \to \texttt{RandomVariable}(\Omega, \mathbb{R})$.

$. \forall (Z, 1) : \sum Z : \texttt{RandomVariable}(\Omega, \mathbb{R}) \ . \ \mathbb{E}\, Z < \infty \ . \ \forall (2) : Y < Z \quad \mathrm{a.e.}\ [\mathbb{P}]$.

$. \limsup_{n\to\infty} \mathbb{E}(Y_n|\mathcal{A}) \le \mathbb{E}\left(\limsup_{n\to\infty} Y_n|\mathcal{A}\right)$

$\texttt{Proof} =$

...

$\square$

$\texttt{TrivialCEWRSA} :: \forall X : \texttt{RandomVariable}(\Omega, K^n) \ . \ \mathbb{E}\left(X|\{\emptyset, \Omega\}\right) = \mathbb{E}\, X$

$\texttt{Proof} =$

The only measurable maps fo this algebra are constant and $\mathbb{E}\, X$ is the only constant which satisfy

$$\int_\Omega \mathbb{E}\, X \,\mathrm{d}\mathbb{P} = \mathbb{E}\, X = \int_\Omega X \,\mathrm{d}\mathbb{P}$$

$$\int_\emptyset \mathbb{E}\, X \mathrm{d}\mathbb{P} = 0 = \int_\emptyset X \,\mathrm{d}\mathbb{P}$$

$\square$

$\texttt{FullCEWRSA} :: \forall X : \texttt{RandomVariable}(\Omega, K^n) \ . \ \mathbb{E}\left(X|\mathcal{F}_\Omega\right) = X \quad \mathrm{a.e.}\ [\mathbb{P}]$

$\texttt{Proof} =$

Integration on Measurable subsets defines random variable almost surely.

$\square$

$\texttt{Atom} :: \prod (\Omega, \mathcal{F}, \mu) : \mathsf{MEAS} \ . \ ?\mathcal{F}$

$A : \texttt{Atom} \iff \mu(A) > 0 \ \& \ \left(\forall B \in \mathcal{F} \ . \ B \subset A \Rightarrow \left(\mu(B) = 0 | \mu(A \setminus B) = 0\right)\right)$

$\texttt{AtomConstantLemma} :: \forall \Omega : \mathsf{MEAS} \ . \ \forall f : \Omega \to_{\mathsf{BOR}} (\overset{\infty}{\mathbb{R}}, \mathcal{B}\, \overset{\infty}{\mathbb{R}}) \ . \ \forall A : \texttt{Atom}(\Omega) \ . \ f_{|A} : \texttt{Constant} \quad \mathrm{a.e.}\ [\mu_\Omega]$

$\texttt{Proof} =$

$\mu := \mu_\Omega : \texttt{Measure}(\Omega)$,

$\texttt{Assume}\ y : f(A)$,

$(B_y, 1) := f^{-1}\{y\} \cap A : \sum B_y :?A \ . \ B_y \ne \emptyset;$

$\leadsto B := I\left(\prod\right) : \prod x \in f(A) \ . \ \sum B_x :?A \ . \ B_y \ne \emptyset,$

$(1) := \texttt{DisjointPreimage}\eth B : \left(B : \texttt{Disjoint}(f(A), \mathcal{F}_\Omega)\right),$

$\texttt{Assume}\ (x, y, 2) : \sum x, y \in f(A) \ . \ x \ne y \ \& \ \mu(A \setminus B_x) = 0 \ \& \ \mu(A \setminus B_y) = 0,$

$(3) := \texttt{SubsetMeasure}(A, B_x \cup B_y)\eth\texttt{Measure}(\Omega)(\mu) :$

$: \mu(A) > \mu(B_x \cup B_y) = \mu(B_x) + \mu(B_y) = \mu(A) - \mu(A \setminus B_y) + \mu(A) - \mu(A \setminus B_x) = 2\mu(A),$

$(4) := \eth\texttt{OrderedField}(\mathbb{R})(3) : \perp;$

$\leadsto (2) := E(\perp) : \forall x, y \in f(A) \ . \ \left(\mu(B_x) = \mu(A) \ \& \ \mu(B_y) = \mu(A)\right) \Rightarrow x = y,$

Assume $n : \mathbb{N}$,

$G := \texttt{grid}\left(\overset{\infty}{\mathbb{R}}, \frac{1}{n}\right) : \texttt{Grid}\left(\overset{\infty}{\mathbb{R}}\right)$,

$(3) := \eth\texttt{Grid}(G)\texttt{MonotoneConvergence} : \mu\left(A \cap \bigcup_{i=1}^{\infty} f^{-1}G_i\right) = \mu(A)$,

$g_n := \bigcap\{G_m : m \in \mathbb{N}, \mu(A) = \mu(A \cap f^{-1}G_m)\} : \texttt{Compact}\left(\overset{\infty}{\mathbb{R}}\right)$,

$(4) := \eth\texttt{Atom}(A)(3)(2)\eth g_n : g_n \neq \emptyset$;

$\rightsquigarrow (g, 3) := I\left(\prod\right) : \prod n \in \mathbb{N} \,.\, \sum g_n : \texttt{Compact}\left(\overset{\infty}{\mathbb{R}}\right) \,.\, \rho(g) \leq \frac{1}{n}$,

$(4) := \eth g(2) : \left(g : \texttt{Decreasing}\right)$,

$(x, 5) := \texttt{NestedCompactIntersection}(g, 4)(3) : \sum x \in \overset{\infty}{\mathbb{R}} \,.\, \{x\} = \bigcap_{n=1}^{\infty} g_n$,

$(6) := \texttt{MonotoneConvergence}(\eth g)(5) : \mu(A) = \lim_{n\to\infty} \mu(A \cap f^{-1}g_n) = \mu(B_x)$,

$(7) := (2)(6) : \forall y \in f(A) : y \neq x \,.\, \mu(B_y) = 0$,

$(*) := \eth B(6)(7) : f_{|A} = x \quad \mathrm{a.e.}\ [\mu]$;

□

$\texttt{AtomSmoothingCEWRSA} :: \forall A : \texttt{Atom}(\mathcal{A}) \,.\, \forall X : \texttt{RandomVariable}(\Omega, \mathbb{R}) \,.$

$$. \ \mathbb{E}(X|\mathcal{A})_{|A} = \frac{1}{\mathbb{P}(A)} \int_A X \, d\mathbb{P} \quad \mathrm{a.e.}\ [\mathbb{P}]$$

Proof $=$

$$\int_A \mathbb{E}(X|\mathcal{A}) \, d\mathbb{P} = \int_A X \, d\mathbb{P}$$

But conditional expectation is a constant almost surely on A, which provides the exact value.

□

$\texttt{CoarserCEWRSA1} :: \forall (\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} \,.\, \forall (\Omega, \mathcal{C}), (\Omega, \mathcal{S}) : \mathsf{BOR} \,.\, \forall (1) : \mathcal{C} \subset \mathcal{S} \subset \mathcal{F} \,.$

$$. \ \forall X : \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, \mathbb{P}), K^n\Big) \,.\, \mathbb{E}\Big(\mathbb{E}(X|\mathcal{S})|\mathcal{C}\Big) = \mathbb{E}(X|\mathcal{C})$$

Proof $=$

$$\int_C \mathbb{E}\Big(\mathbb{E}(X|\mathcal{S})|\mathcal{C}\Big) \, d\mathbb{P} = \int_C \mathbb{E}(X|\mathcal{S}) \, d\mathbb{P} = \int_C Y \, d\mathbb{P} = \int_C \mathbb{E}(X|\mathcal{C})$$

□

$\texttt{CoarserCEWRSA2} :: \forall (\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} \,.\, \forall (\Omega, \mathcal{C}), (\Omega, \mathcal{S}) : \mathsf{BOR} \,.\, \forall (1) : \mathcal{C} \subset \mathcal{S} \subset \mathcal{F} \,.$

$$. \ \forall X : \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, \mathbb{P}), K^n\Big) \,.\, \mathbb{E}\Big(\mathbb{E}(X|\mathcal{C})|\mathcal{S}\Big) = \mathbb{E}(X|\mathcal{C})$$

Proof $=$

$$\mathbb{E}(X|\mathcal{C}) : (\Omega, \mathcal{S}) \to_{\mathsf{BOR}} (K^n, \mathcal{B}K^n)$$

□

$\texttt{CEWRSAMeasurableModularity} :: \forall (\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace} \,.\, \forall (\Omega, \mathcal{C}) : \mathsf{BOR} \,.$

$$. \ \forall (1) : \mathcal{C} \subset \mathcal{F} \,.\, \forall X : \texttt{RandomVariable}\Big((\Omega, \mathcal{F}, \mathbb{P}), K\Big) \,.\, \forall Z : (\Omega, \mathcal{C}) \to_{\mathsf{BOR}} (K, \mathcal{B}K) \,.\, \mathbb{E}(ZX|\mathcal{C}) = Z\,\mathbb{E}(X|\mathcal{C})$$

Proof $=$

Start with indiicators and proceed to measurable functions.

□

2.3 Regular Conditional Probabilities

$\Omega := (\Omega, \mathcal{F}, \mathbb{P}) : \texttt{ProbabilitySpace},$

$\texttt{Assume}\ (\Omega, \mathcal{C}) : \mathsf{BOR},$

$\texttt{Assume}\ (0) : \mathcal{C} \subset \mathcal{F},$

$$\texttt{RegularConditionalDistribution} :: \prod X : \texttt{RandomVariable}(\Omega, \mathbb{R}) \,.\, ?\big(\Omega \to \mathbb{R} \to [0,1]\big)$$

$$F : \texttt{RegualarConditionalDistribution} \iff \Big(\forall \omega \in \Omega \,.\, F(\omega) : \texttt{DistributionFunction}\Big) \,\&$$
$$\& \left(\forall y \in \mathbb{R} \,.\, \Lambda \omega \in \Omega \,.\, F(\omega, y) = \mathbb{P}(X \leq y|\mathcal{A}) \quad \mathrm{a\,.\,e\,.}\ [\mathbb{P}]\right)$$

$$\texttt{RegularConditionalDistributionExists} :: \forall X : \texttt{RandomVariable}(\Omega, \mathbb{R}) \,.$$
$$.\, \exists F : \texttt{RegularConditionalDistribution}(X)$$

$\texttt{Proof}\ =$