Operator Analysis

Uncultured Tramp

July 8, 2017

$$Tf(x) = \int_\Omega f(y) K(x, y)\, \mu\big(\mathrm{d}y\big)$$

Contents

1 General Bounded Operators

1.1 Concept of Operator's Boundedness

$$\texttt{BoundedOperator} :: \prod V, W : \texttt{SeminormedSpace}(K) \;.\; V \to_{\mathsf{VS}(K)} W$$

$$T : \texttt{BoundedOperator} \iff \mathcal{B}(V,W) \iff \exists C \in \mathbb{R}_+ : \forall x \in V \;.\; \|Tx\| \le C\|x\|$$

$V, W : \texttt{SeminormedSpace}(K)$

$\texttt{BoundedSphereDefinition} :: \forall T \in \mathcal{L}(V,W) \;.\; T \in \mathcal{B}(V,W) \iff \sup\{\|Tu\| : u \in \mathbb{S}_V\} < \infty$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T : \mathcal{B}(V,W),$

$C := \eth\mathcal{B}(V,W) : \mathbb{R}_+ : \forall v \in V \;.\; \|Tv\| \le C\|v\|,$

$\texttt{Assume}\; u : \mathbb{B}_V,$

$(1) := \eth C(u)\eth\mathbb{B}_V(u) : \|Tu\| \le C\|u\| \le C;$

$\rightsquigarrow (1) := \texttt{UniversalIntroduction} : \forall u \in \mathbb{B}_V \;.\; \|Tu\| \le C,$

$(2) := \eth^{-1}\sup(1) : \sup_{u \in \mathbb{B}_V} Tu \le \infty;$

$\rightsquigarrow (1) := \texttt{ImplicationIntroduction} : T \in \mathcal{B}(v,W) \Rightarrow \sup\{\|Tu\| : u \in \mathbb{S}_V\} < \infty,$

$\texttt{Assume}\; A : \sup\{\|Tu\| : u \in \mathbb{S}_V\} < \infty,$

$C := \eth\sup(A) : \mathbb{R}_+ : \forall u \in \mathbb{B} \;.\; \|u\| \le C,$

$\texttt{Assume}\; x : V : \|x\| \ne 0,$

$(2) := \eth_2\texttt{Seminorm}(W)(Tx)(\|x\|^{-1})\eth C : \dfrac{\|Tx\|}{\|x\|} = \left\|T\dfrac{x}{\|x\|}\right\| \le C,$

$(3) := \texttt{Ineqer}(2) : \|Tx\| \le C\|x\|;$

$\rightsquigarrow (2) := \texttt{UniversalIntroduction} : \forall x \in V : \|x\| \ne 0 \;.\; \|Tx\| \le C\|x\|,$

$\texttt{Assume}\; x : V : \|x\| = 0,$

$\texttt{Assume}\; B : \|Tx\| > 0,$

$a := \eth B : \mathbb{R}_{++} : \|Tx\| = a,$

$b := \eth\texttt{Archemeadian}(a,C) : \mathbb{N} : ba > C,$

$(3) := \eth_2\texttt{Seminorm}(x)(b)\eth x : \|bx\| = b\|x\| = 0,$

$(4) := A(3)\eth_2\mathcal{L}(V,W)(T)(x)(b)\eth_2\texttt{Seminorm}(W)(Tx)(b)\eth a\eth b : C \ge \|Tbx\| = b\|Tx\| = ba > C,$

$5 := \texttt{SelfIneq}(4) : \bot;$

$\rightsquigarrow (3) := \texttt{Contradiction} : \|Tx\| = 0,$

$(4) := \texttt{AsIneq}(\texttt{UniqueZero}(\eth x, 3)) : \|Tx\| \le C\|x\|;$

$\rightsquigarrow (3) := \texttt{UniversalIntroduction} : \forall x \in V : \|x\| = 0 \;.\; \|Tx\| \le C\|x\|,$

$(4) := \texttt{Synthesis}(2,3) : \forall x \in V \;.\; \|Tx\| \le C\|x\|,$

$(5) := \eth^{-1}\mathcal{B}(V,W)(4) : (T : \mathcal{B}(V,W));$

$\rightsquigarrow (*) := \texttt{IffIntroduction}(1) : T \in \mathcal{B}(v,W) \iff \sup\{\|Tu\| : u \in \mathbb{S}_V\} < \infty;$

$\square$

$\texttt{BoundedSetDefinition} :: \forall T \in \mathcal{L}(V,W) \,.\, T \in \mathcal{B}(V,W) \iff \forall A : \texttt{Bounded}(V) \,.\, T(A) : \texttt{Bounded}(W)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ T : \mathcal{B}(V,W),$

$C := \eth\mathcal{B}(V,W)(T) : \mathbb{R}_+ : \forall x \in V \,.\, \|Tv\| \leq C\|v\|,$

$\texttt{Assume}\ A : \texttt{Bounded}(V),$

$r := \eth\texttt{Boinded}(V)(A) : \mathbb{R}_+ + : A \subset r\mathbb{B}_V,$

$(1) := \texttt{SubsetMap}(\eth r, T)\eth C : TA \subset Tr\mathbb{B}_V \subset rC\mathbb{B}_W,$

$(2) := \eth^{-1}\texttt{Bounded}(W)(2) : (TA : \texttt{Bounded}(W));;$

$\rightsquigarrow (1) := \texttt{ImplicationIntroduction} : T \in \mathcal{B}(V,W) \Rightarrow \forall A : \texttt{Bounded}(V) \,.\, T(A) : \texttt{Bounded}(W),$

$\texttt{Assume}\ B : \forall A : \texttt{Bounded}(V) \,.\, T(A) : \texttt{Bounded}(W),$

$(2) := B(\mathbb{B}_V) : T\mathbb{B}_V : \texttt{Bounded},$

$(3) := \texttt{BoundedBallDefinition}^{-1}(T)(2) : (T : \mathcal{B}(V,W));$

$\rightsquigarrow (2) := \texttt{IffIntroduction}(1) : T \in \mathcal{B}(V,W) \iff \forall A : \texttt{Bounded}(V) \,.\, T(A) : \texttt{Bounded}(W),$

$\square$

$\texttt{ContractionOperator} :: ?\mathcal{B}(V,W)$

$T : \texttt{ContractionOperator} \iff T \in \mathcal{B}_{\circ\to\cdot}(V,W) \iff \forall x \in V \,.\, \|Tx\| \leq \|x\|$

$\texttt{Isometry} :: ?\mathcal{B}_{\circ\to\cdot}(V,W)$

$T : \texttt{Isometry} \iff T \in \mathcal{B}_{\circ\to\circ}(V,W) \iff \forall x \in V \,.\, \|Tx\| = \|x\|$

$\texttt{Coisometry} :: ?\mathcal{B}(V,W)$

$T : \texttt{Coisometry} \iff T \in \mathcal{B}_{\cdot\to\circ}(V,W) \iff \mathbb{B}_W \subset T\mathbb{B}_V$

$\texttt{TopologicalyInjectiveOperator} :: ?\mathcal{B}(V,W)$

$T : \texttt{TopologicalyInjectiveOperator} \iff T : V \leftrightarrow_{\mathsf{TOP}} \operatorname{Im} T$

$\texttt{TopologicalySurjictiveOperator} :: ?\mathcal{B}(V,W)$

$T : \texttt{TopologicalyInjectiveOperator} \iff T : V \twoheadrightarrow_{\mathsf{SET}} W \;\&\; \forall U \subset W : T^{-1}\texttt{Open}(V) \,.\, U : \texttt{Open}(W)$

$\texttt{dual} :: \texttt{SeminormedSpace}(K) \to \texttt{SeminormedSpace}(K)$

$\texttt{dual}\,(V) = V^* := \mathcal{B}(V,K)$

1.2 Operator Norm

$\texttt{OperatorNorm} :: \mathcal{B}(V,W) \to \mathbb{R}_+$

$\texttt{OperatorNorm}\,(T) = \|T\| := \sup_{v \in \mathbb{S}_V} \|T\|$

$\texttt{BoundedAsSubspace} :: \mathcal{B}(V,W) \subset_{\mathsf{VS}(K)} \mathcal{L}(V,W)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T, S : \mathcal{B}(V,W),$

$\texttt{Assume}\; x : V,$

$(1) := \eth +_{\mathcal{L}(V,W)} (T,S)(x)\eth_1\texttt{Seminorm}(W)(Tx,Sx)\eth\texttt{OperatorNorm}(V,W)(T,S) ::$

$\quad \|(T+S)x\| = \|Tx+Sx\| \leq \|Tx\| + \|Sx\| \leq \|T\|\|x\| + \|S\|\|x\| = (\|T\| + \|S\|)\|x\|;$

$\rightsquigarrow (1) := \eth^{-1}\mathcal{B}(V,W) : T + S : \mathcal{B}(V,W);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Additive} : \mathcal{B}(V,W) : \texttt{Additive},$

$\texttt{Assume}\; T : \mathcal{B}(V,W),$

$\texttt{Assume}\; a : K,$

$\texttt{Assume}\; x : V,$

$(2) := \eth_2\texttt{Seminorm}(Tx,a)\eth\texttt{OperatorNorm}(V,W)(T)(x) : \|aTx\| = |a|\|Tx\| \leq |a|\|T\|\|x\|;$

$\rightsquigarrow (2) := \eth^{-1}\mathcal{B}(V,W) : aT \in \mathcal{B}(V,W);$

$(2) := \eth^{-1}\texttt{Subspace}(\mathcal{L}(V,W))(1) : \mathcal{B}(V,W) \subset_{\mathsf{VS}(K)} \mathcal{L}(V,W);$

$\square$

$\texttt{OperatorNormIsSeninorm} :: \texttt{OperatorNorm}(V,W) : \texttt{Seminorm}(\mathcal{B}(V,W))$

$\texttt{Proof} \;=$

$\texttt{Assume}\; S, T : \mathcal{B}(V,W),$

$(1) := \eth\texttt{OperatorNorm}(S+T)\eth_n\texttt{Seminorm}(W)(Sv,Tv)$

$\quad \texttt{SupremumSum}(\Lambda v \in V \,.\, \|Sv\|, \Lambda v \in V \,.\, \|Tv\|)\eth^{-1}\texttt{OperatorNorm}$

$\quad : \|S+T\| = \sup_{v\in\mathbb{S}_V} \|(S+T)v\| \leq \sup_{v\in\mathbb{S}_V} \|Sv\| + \|Tv\| \leq \sup_{v\in\mathbb{S}_V} \|Sv\| + \sup_{v\in\mathbb{S}_V} \|Tv\| = \|S\| + \|T\|;$

$\rightsquigarrow (1) := \texttt{UnivesalIntroduction} : \forall S,T \in \mathcal{B}(V,W) \,.\, \|S+T\| \leq \|S\| + \|T\|,$

$\texttt{Assume}\; T : \mathcal{B}(V,W),$

$\texttt{Assume}\; a : K,$

$(2) := \eth\texttt{OperatorNorm}\eth_1\texttt{Seminorm}\eth^{-1}\texttt{OperatorNorm} : \|aT\| = \sup_{\mathbb{S}_V} \|aTs\| = \sup_{\mathbb{S}_V} |a|\|Ts\| = |a|\|T\|;$

$\rightsquigarrow (2) := \texttt{UniversalIntroduction} : \forall T \in \mathcal{B}(V,W) \,.\, \forall a \in K \,.\, \|aT\| = a\|T\|,$

$(3) := \eth^{-1}\texttt{Seminorm}(\mathcal{B}(C,W)))(2,3) : (\texttt{OperatorNorm}(V,W) : \texttt{Seminorm}(V,W));$

$\square$

$\texttt{OperatorNormIsNorm} :: W : \texttt{NormedSpace}(K) : W \neq 0 \Rightarrow \texttt{OperatorNorm}(V, W) : \texttt{Norm}(\mathcal{B}(V, W))$

$\texttt{Proof} \ =$

$\texttt{Assume}\ T : \mathcal{B}(V, W) : \|T\| = 0,$

$\texttt{Assume}\ x : V : x \neq 0,$

$(1) := \eth\texttt{OperatorNorm} : \|T\| \geq \dfrac{\|Tx\|}{\|x\|},$

$(2) := (1)(\|T\| = 0) : \|Tx\| = 0,$

$(3) := \eth\texttt{NormedSpace}(W)(2) : Tx = 0;$

$\leadsto (1) := \eth\texttt{Zero} : T = 0;$

$\leadsto (1) := \texttt{UniversalIntroduction} : \forall T : \mathcal{B}(V, W) : \|T\| = 0 \ .\ T = 0,$

$(*) := \eth^{-1}\texttt{Norm} : \texttt{OperatorNorm}(V, W) : \texttt{Norm}(\mathcal{B}(V, W));$

$\square$

$\texttt{OperatorNormProduct} :: \forall T \in \mathcal{B}(V, W) \ .\ \forall S \in \mathcal{B}(W, Z) \ .\ \|TS\| \leq \|T\|\|S|$

$\texttt{Proof} \ =$

$(1) := \dots : \|ST\| = \sup_{x \in \mathbb{S}_V} \|STx\| \leq \sup_{x \in \mathbb{S}_W} \|S(\sup_{y \in \mathbb{S}_V} \|Ty\|)x\| = \|T\|\|S\|;$

$\texttt{OperatorNormInIPS} :: \forall H, E : \texttt{PrehilbertSpace}(\mathbb{C}) \ .\ \forall T : \mathcal{B}(H, E) \ .\ \|T\| = \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \langle Tv, w \rangle$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{OperatorNormIdMult}(\sup_{w \in \mathbb{S}_W} \|w\|)\texttt{CouchySwarc}^{-1}(Tv, w) :$

$\quad : \|T\| = \sup_{v \in \mathbb{S}_H} \|Tv\| = \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \|Tv\|\|w\| \geq \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} |\langle Tv, w \rangle|,$

$\texttt{Assume}\ A : \|T\| = 0,$

$(2) := (1)\texttt{LBAbs}(|\langle Tv, w \rangle|)A : \|T\| \geq \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} |\langle Tv, w \rangle| \geq 0 = \|T\|;$

$\leadsto (2) := \texttt{ImplicationIntroduction} : \|T\| = 0 \Rightarrow \|T\| = \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \langle Tv, w \rangle,$

$\texttt{Assume}\ A : \|T\| \neq 0,$

$(3) := \texttt{MultAndDivide}(\|T\|)(\|T\|, A)\eth\texttt{OperatorNorm}(V, W)(T)\texttt{IPAsSeminorm}(Tv)$

$\quad \texttt{Homogenity}_2(W \otimes \overline{W} \to_{\mathsf{VS}(K)} K)(\texttt{innerProduct}(W))(Tv \otimes Tv)(1/\|T\|)\texttt{CircleSup}(\|Tv/\|T\|\| \leq 1) :$

$\quad : \|T\| = \dfrac{\|T\|^2}{\|T\|} = \dfrac{\sup_{v \in \mathbb{S}_H} \|Tv\|^2}{\|T\|} = \dfrac{\sup_{v \in \mathbb{S}_H} |\langle Tv, Tv \rangle|}{\|T\|} = \sup_{v \in \mathbb{S}_H} \left| \left\langle Tv, \dfrac{Tv}{\|T\|} \right\rangle \right| \leq \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} |\langle Tv, w \rangle|;$

$\leadsto (3) := \texttt{ImplicationIntroduction} : \|T\| \neq 0 \Rightarrow \|T\| \leq \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \langle Tv, w \rangle,$

$(4) := \texttt{Synthesis}(2, 3) : \|T\| \leq \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \langle Tv, w \rangle,$

$(*) := \texttt{DoubleIneq}(1, 4) : \|T\| = \sup_{v \in \mathbb{S}_H} \sup_{w \in \mathbb{S}_E} \langle Tv, w \rangle;$

$\square$

1.3 Examples of Operators

$$\texttt{zeroOperator} :: \mathcal{B}(V, W)$$
$$\texttt{zeroOPerator}\,(v) = \mathbf{0}v := 0$$

$$\|\mathbf{0}\| = \sup_{v \in \mathbb{S}_V} \|\mathbf{0}v\| = \sup_{v \in \mathbb{S}_V} 0 = 0$$

$$\texttt{idOperator} :: \mathcal{B}(V, V)$$
$$\texttt{idOPerator}\,(v) = \mathbf{I}v := v$$

$$\|\mathbf{I}\| = \sup_{v \in \mathbb{S}_V} \|\mathbf{I}v\| = \sup_{v \in \mathbb{S}_V} 1 = 1$$

$$\texttt{diagonalOperator} :: l_\infty \to \mathcal{B}(l_p, l_p)$$
$$\texttt{diagonalOperator}\,(\lambda, v) = \operatorname{diag}(\lambda)(v) := (\lambda_i v_i)_{i=1}^\infty$$

$$\|\operatorname{diag}(\lambda)\| = \sup_{v \in \mathbb{S}_V} \|\operatorname{diag}(\lambda)(v)\| \leq \sup_{v \in \mathbb{S}_V} \|\|\lambda\|_\infty v\| = \|\lambda\|_\infty$$

$$\|\operatorname{diag}(\lambda)\| = \sup_{v \in \mathbb{S}_V} \|\operatorname{diag}(\lambda)(v)\| \geq \sup_{n \in \mathbb{N}} \|\operatorname{diag}(\lambda)(e_n)\| = \|\lambda\|_\infty$$

$$\texttt{leftShift} :: \mathcal{B}(l_p, l_p)$$
$$\texttt{leftShift}\,(x) := (x_{i+1})_{i=1}^\infty$$

$$\|\texttt{leftShift}\| = \sup_{v \in \mathbb{S}_{l_p}} \|\texttt{leftshift}v\| \leq \sup_{v \in \mathbb{S}_{l_p}} \|v\| = 1$$

$$\|\texttt{leftShift}(e_2)\| = \|e_1\| = 1$$

$$\|\texttt{leftShift}\| = 1$$

$$\texttt{rightShift} :: \mathcal{B}(l_p, l_p)$$
$$\texttt{rightShift}\,(x) := 0 \oplus x$$

$$\|\texttt{rightShift}\| = \sup_{v \in \mathbb{S}_{l_p}} \|\texttt{rightshift}v\| = \sup_{v \in \mathbb{S}_{l_p}} \|0 \oplus v\| = 1$$

Ω : MEAS

$$\texttt{GeneralDiagonalOperator} :: L_\infty(\Omega) \to \mathcal{B}(L_p(\Omega), L_p(\Omega))$$
$$\texttt{GeneralDiagonalOperator}\,(a, f) = \operatorname{Diag}(a)(f) := af$$

$$\|\mathrm{Diag}(a)\| = \|a\|_\infty$$

$$\texttt{undefiniteIntegral} :: \mathcal{B}(L^2[0,1], L^2[0,1])$$
$$\texttt{undefiniteIntegral}\,(f) = \int f := \Lambda t \in [0,1]\;.\;\int_0^t f(x)\,\mathrm{d}x$$

$$\left\| \int_{|C[0,1]} \right\| = 1$$

$$\left\| \int_{|L_1[0,1]} \right\| = 1$$

$$\texttt{IntegralOperator} :: L_2(\Omega \times \Omega) \to \mathcal{B}(L_2(\Omega), L_2(\Omega))$$
$$\texttt{IntegralOperator}\,(K, f) := \Lambda x \in \Omega\;.\;\int_\Omega K(x,\omega) f(\omega)\,\mathrm{d}\mu(\omega)$$

$$\texttt{TimeShift} :: \mathbb{R} \to \mathcal{B}(L_2(\mathbb{R}), L_2(\mathbb{R}))$$
$$\texttt{TimeShift}\,(a, f) := \Lambda t \in \mathbb{R}\;.\;f(t+a)$$

$$\texttt{CircleShift} :: \mathbb{S}^1 \to \mathcal{B}(L_2(\mathbb{S}^1), L_2(\mathbb{S}^1))$$
$$\texttt{CircleShift}\,(a, f) := \Lambda s \in \mathbb{S}^1\;.\;f(as)$$

$$\texttt{Differentiation} :: \prod k, n \in \mathbb{N}\;.\;\mathcal{B}(C^{n+k}(M), C^n(M))$$
$$\texttt{Differentiation}\,(f) = D^k(f) := \frac{\mathrm{d}^k\, f(x)}{\mathrm{d}\, x^k}$$

1.4 Category Structure

$$\mathsf{PRE} :: \texttt{AVField} \to \texttt{Category}$$
$$\mathcal{O}(\mathsf{PRE}(K)) = \texttt{SeminormedSpace}(K)$$
$$\mathcal{M}_{\mathsf{PRE}(K)}(A,B) = \mathcal{B}(A,B)$$
$$\cdot_{\mathsf{PRE}(K)} = \circ$$

$$\mathsf{PRE}_{\circ\to\cdot} :: \texttt{AVField} \to \texttt{Category}$$
$$\mathcal{O}(\mathsf{PRE}_{\circ\to\cdot}(K)) = \texttt{SeminormedSpace}(K)$$
$$\mathcal{M}_{\mathsf{PRE}_{\circ\to\cdot}(K)}(A,B) = \mathcal{B}_{\circ\to\cdot}(A,B)$$
$$\cdot_{\mathsf{PRE}_{\circ\to\cdot}(K)} = \circ$$

$$\mathsf{NORM} :: \texttt{AVField} \to \texttt{Category}$$
$$\mathcal{O}(\mathsf{NORM}(K)) = \texttt{SeminormedSpace}(K)$$
$$\mathcal{M}_{\mathsf{NORM}(K)}(A,B) = \mathcal{B}(A,B)$$
$$\cdot_{\mathsf{NORM}(K)} = \circ$$

$$\mathsf{NORM}_{\circ\to\cdot} :: \texttt{AVField} \to \texttt{Category}$$
$$\mathcal{O}(\mathsf{NORM}_{\circ\to\cdot}(K)) = \texttt{SeminormedSpace}(K)$$
$$\mathcal{M}_{\mathsf{NORM}_{\circ\to\cdot}(K)}(A,B) = \mathcal{B}_{\circ\to\cdot}(A,B)$$
$$\cdot_{\mathsf{NORM}_{\circ\to\cdot}(K)} = \circ$$

$\texttt{TopologicalIsomorphismCharacteristic} :: \forall T : V \to_{\mathsf{PRE}} W \ \&\ V \leftrightarrow_{\mathsf{SET}} W\ .$
$\quad . \ T : V \leftrightarrow_{\mathsf{PRE}} W \iff \exists c, C \in \mathbb{R}_+ : \forall x \in V\ .\ c\|x\| \le \|Tx\| \le C\|x\|$

$\texttt{Proof} \ =$

$C := \eth\mathcal{B}(V,W)(T) : \mathbb{R}_+ : \forall x \in V\ .\ \|Tx\| \le C\|x\|,$

$\texttt{Assume}\ T : V \leftrightarrow_{\mathsf{PRE}} W,$

$(1) := \eth\texttt{Isomorphism}(V,W)(T) : (T^{-1} : W \to_{\mathsf{PRE}} V),$

$c := \eth\mathcal{B}(W,V)(T^{-1}) : \mathbb{R}_+ : \forall x \in W\ .\ \|T^{-1}x\| \le c\|x\|,$

$(2) := \texttt{Replace}(\eth\texttt{Inverse}(T), \eth c) : \forall x \in V\ .\ \|x\| \le c\|Tx\|,$

$(3) := c^{-1}(2) : \forall x \in V\ .\ c^{-1}\|x\| \le \|Tx\|,$

$(4) := \texttt{Synthesis}(3, \eth C) : \forall x \in V\ .\ c^{-1}\|x\| \le \|Tx\| \le C\|x\|;$

$\rightsquigarrow (1) := \texttt{ImplicationIntroduction}\ \texttt{ExistanceIntroduction}(c^{-1}) : \texttt{LEFT} \Rightarrow \texttt{RIGHT},$

$\texttt{Assume}\ R : \texttt{RIGHT},$

$c := \eth_1 R : \mathbb{R}_+ : \forall x \in V\ .\ c\|x\| \le \|Tx\|,$

$(2) := \texttt{Replace}(\eth\texttt{Inverse}(T), \eth c) : \forall x \in W\ .\ c\|T^{-1}x\| \le \|x\|,$

$(3) := c^{-1}(2) : \|T^{-1}x\| \le c^{-1}\|x\|,$

$(4) := \eth^{-1}\mathcal{B}(V,W)(3) : (T^{-1} : \mathcal{B}(V,W)),$

$(5) := \eth^{-1}\texttt{Isomorphism}(\mathsf{PRE})(4) : (T^{-1} : V \leftrightarrow_{\mathsf{PRE}} W);$

$\rightsquigarrow (*) := \texttt{IffIntroduction} : T : V \leftrightarrow_{\mathsf{PRE}} W \iff \exists c, C \in \mathbb{R}_+ : \forall x \in V\ .\ c\|x\| \le \|Tx\| \le C\|x\|;$

$\square$

$\texttt{IsometricIsomorphismCharacteristicI} :: \forall T : V \to_{\mathsf{PRE}_{\circ\to\cdot}} W \,\&\, W \leftrightarrow_{\mathsf{SET}} V \,.$
$\quad . \, T : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} W \iff T : \mathcal{B}_{\circ\to\circ}(V, W)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{IsometricIsomorphismCharacteristicII} :: \forall T : V \to_{\mathsf{PRE}_{\circ\to\cdot}} W \,\&\, W \leftrightarrow_{\mathsf{SET}} V \,.$
$\quad . \, T : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} W \iff T : \mathcal{B}_{\cdot\to\circ}(V, W)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{IsometryPreservesInnerProduct} :: \forall H, E : \texttt{PrehilbertSpace}(K) \,.\, \forall T : \mathcal{B}_{\circ\to\circ}(V, W) \,.$
$\quad . \, \forall x, y \in H \,.\, \langle Tx, Ty \rangle = \langle x, y \rangle$

$\texttt{Proof} \; =$

$\dots$

$\square$

$X, Y : \mathsf{PRE}$

$\texttt{WeaklyTopologicalyEqualent} :: ?\big(V \to_{\mathsf{PRE}} W \times X \to_{\mathsf{PRE}} Y\big)$
$(f, g) : \texttt{WeaklyTopologicalyEqualent} \iff f \simeq_{\mathsf{PRE}} g \iff \exists \varphi : V \leftrightarrow_{\mathsf{PRE}} X : \exists \psi : W \leftrightarrow_{\mathsf{PRE}} Y : f\psi = \varphi g$

$\texttt{TopologicalyEqualent} :: ?\big(V \to_{\mathsf{PRE}} V \times X \to_{\mathsf{PRE}} V\big)$
$(f, g) : \texttt{TopologicalyEqualent} \iff f \cong_{\mathsf{PRE}} g \iff \exists \varphi : V \leftrightarrow_{\mathsf{PRE}} X : f\varphi = \varphi g$

$V, W, X, Y : \mathsf{PRE}_{\circ\to\cdot}$

$\texttt{WeaklyIsometricalyEqualent} :: ?\big(V \to_{\mathsf{PRE}_{\circ\to\cdot}} W \times X \to_{\mathsf{PRE}_{\circ\to\cdot}} Y\big)$
$(f, g) : \texttt{WeaklyIsometricalyEqualent} \iff f \simeq_{\mathsf{PRE}_{\circ\to\cdot}} g \iff \exists \varphi : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} X :$
$\quad \exists \psi : W \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} Y : f\psi = \varphi g$

$\texttt{IsometricalyEqualent} :: ?\big(V \to_{\mathsf{PRE}_{\circ\to\cdot}} V \times X \to_{\mathsf{PRE}_{\circ\to\cdot}} V\big)$
$(f, g) : \texttt{IsometricalyEqualent} \iff f \cong_{\mathsf{PRE}_{\circ\to\cdot}} g \iff \exists \varphi : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} X : f\varphi = \varphi g$

$\texttt{NaturalInclusion} :: \prod S : \texttt{Subspace}(V) \,.\, S \to_{\mathsf{PRE}_{\circ\to\cdot}} V$
$\texttt{NaturalInclusion}\,(v) = i_S(v) := v$

$\texttt{NaturalProjection} :: \prod S : \texttt{Subspace}(V) \,.\, V \to_{\mathsf{PRE}_{\circ\to\cdot}} \dfrac{V}{S}$
$\texttt{NaturalProjection}\,(v) = \pi_S(v) := [v]$

$\texttt{TopologicalyInjectiveDecomposition} :: \forall T : \texttt{TopologicalyInjictiveOperator}(V, W) .$
$\quad . \exists S : \texttt{Subspace}(W) : \exists I : V \leftrightarrow_{\mathsf{PRE}} S : T = I i_S$

$\texttt{Proof} \ =$

$I := \texttt{ContractToIm}(T) : V \to \operatorname{Im} T,$

$(2) := \eth\texttt{TopologicalyInjictiveOperator}(V, W)(T) : (I : V \leftrightarrow_{\mathsf{PRE}} \operatorname{Im} T),$

$(*) := \eth I : T = I i_{\operatorname{Im} T};$

$\square$

$\texttt{Bicontraction} :: \forall T : V \to_{\mathsf{PRE}} W . \forall S : \texttt{Subspace}(V) . \forall R : \texttt{Closed} \ \& \ \texttt{Subspace}(W) .$
$\quad . \exists! \tilde{T} : \frac{V}{S} \to_{\mathsf{PRE}} \frac{W}{R} . T\pi_R = \pi_S \tilde{T} \ \& \ \|\tilde{T}\| \le \|T\|$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{GeneratedOperator} :: \prod T : V \to_{\mathsf{PRE}} W . \frac{V}{\ker T} \to W$

$\texttt{GeneratedOperator}\ (T) = \tilde{T} := \texttt{Bicontraction}(T, \ker T, \{0\})$

$\texttt{TopologicalySurjectiveDecomposition} :: \texttt{Iff}(T : \texttt{TopologicalySurjectiveOperator}(V, W),$
$\quad \tilde{T} : V \leftrightarrow_{\mathsf{PRE}} W, \exists S : \texttt{Subspace}(V) : \exists I : V \leftrightarrow_{\mathsf{PRE}} W : T = \pi_S I)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CoisometryDecomposition} :: \texttt{Iff}(T : \texttt{Coisometry}(V, W),$
$\quad \tilde{T} : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} W, \exists S : \texttt{Subspace}(V) : \exists I : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} W : T = \pi_S I)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$K = \mathsf{PRE} | \mathsf{PRE}_{\circ\to\cdot} | \mathsf{NORM} | \mathsf{NORM}_{\circ\to\cdot}$

$V, W \in K$

$\texttt{IsomprphismCharacteristic} :: \forall T : V \to_K W . T : V \hookrightarrow_K W \iff T : V \hookrightarrow W$

$\texttt{Proof} \ =$

$\texttt{Assume}\ L : T : V \hookrightarrow_K W,$

$\texttt{Assume}\ B : T : V \not\hookrightarrow W,$

$(1) := \texttt{InjIffTrivialKernel}(T) : \ker T \ne \{0\},$

$(2) := (1)(\eth 0^V_{\ker T}, \eth i_{\ker T}) : 0^V_{\ker T} \ne i_{\ker T},$

$(3) := \eth \ker T \eth 0^V_{\ker T} \eth i_{\ker T} : 0^V_{\ker T} T = 0^W_{\ker T} = i_{\ker T} T,$

$(4) := \eth(V \not\hookrightarrow_K W)(2,3) : T : V \not\hookrightarrow_K W,$

$(5) :=$ AbsurdIntro$(A, 4) : \bot;$

$\rightsquigarrow (1) :=$ ByContradiction $: T : V \hookrightarrow W;$

$\rightsquigarrow L :=$ ImlicationIntro $: T : V \hookrightarrow_K W \Rightarrow T : V \hookrightarrow W,$

...

$\square$

EpimorphismCharacteristicInPRE $:: \forall K \in \{\mathsf{PRE}, \mathsf{PRE}_{\circ\to\cdot}\}\ .$
$\forall T : V \to_K W\ .\ T : V \twoheadrightarrow_K W \iff T : V \twoheadrightarrow W$

Proof $=$

...

$\square$

EpimorphismCharacteristicInNORM $:: \forall K \in \{\mathsf{NORM}, \mathsf{NORM}_{\circ\to\cdot}\}\ .$
$\forall T : V \to_K W\ .\ T : V \twoheadrightarrow_K W \iff T : V \twoheadrightarrow_{\mathsf{TOP}} W$

Proof $=$

...

$\square$

1.5 Operator Sum and Coproduct

$A : \texttt{Set}$

$V, W : A \to \mathsf{PRE}(K)$

$\texttt{UniformlyBoundedFamily} :: ? \prod a \in A \;.\; V_a \to_{\mathsf{PRE}} W_a$

$T : \texttt{UniformlyBoundedFamily} \iff \|T_A\| : \texttt{Bounded}(\mathbb{R}_{++})$

$p \in [1, \infty]$

$$\texttt{indirectOperatorSum} :: \texttt{UniformlyBoundedFamily} \to \bigoplus_{a\in A}^{p} V_a \to_{\mathsf{VS}(K)} \bigoplus_{a\in A}^{p} W_a$$

$$\texttt{inderectOperatorSum}\,(T) = \bigoplus_{a\in A}^{p} T_a := \Lambda v \in \bigoplus_{a\in A}^{p} V_a \;.\; \Lambda a \in A \;.\; T_a(v_a)$$

$$\texttt{indirectOperatorSumIsBounded} :: \forall T : \texttt{UniformlyBoundedFamily} \;.\; \bigoplus_{a\in A}^{p} T_a : \bigoplus_{a\in A}^{p} V_a \to_{\mathsf{PRE}} \bigoplus_{a\in A}^{p} W_a$$

$\texttt{Proof} \;=$

$$\left\|\bigoplus_{a\in A}^{p} T_a(v)\right\| = \sqrt[p]{\sum_{a\in A} \|Tv_n\|^p} \le \sqrt[p]{\sum_{a\in A} C^p \|v_n\|^p} = C\|v\|$$

$\square$

$\texttt{normedSpaceSum} :: \mathsf{PRE} \to \mathsf{PRE} \to \mathsf{PRE}$

$$\texttt{normedSpaceSum}\,(A, B) = A \oplus B := \bigoplus_{i\in\{1,2\}}^{1} [(1, A), (2, B)]_i$$

$\forall a, b \in A \;.\; W_a = W_b = W$

$$\texttt{directOperatorSum} :: \texttt{UniformlyBoundedFamily} \to \bigoplus_{a\in A}^{1} V_a \to_{\mathsf{PRE}} W$$

$$\texttt{directOperatorSum}\,(T) = \sum_{a\in A}^{\oplus} T_a := \Lambda v \in \bigoplus_{a\in A}^{p} V_a \;.\; \sum_{a\in A} T_a(v_a)$$

$$\left\|\sum_{a\in A}^{\oplus} T_a(v)\right\| \le \sum_{a\in A} \|T_a(v_a)\| \le C \sum_{a\in A} \|v_a\| = C\|v\|$$

$$T \oplus S = \sum_{a\in\{1,2\}}^{\oplus} [(1, T), (2, S)]_a$$

$\texttt{PreCoproduct} :: \texttt{normedSpaceSum} : \texttt{Coproduct}(\mathsf{PRE})$

$\texttt{Proof} =$

$\texttt{Assume}\ V, W, X : \mathsf{PRE},$

$\texttt{Assume}\ T : V \to_{\mathsf{PRE}} X,$

$\texttt{Assume}\ S : W \to_{\mathsf{PRE}} X,$

$F := T \oplus S : V \oplus W \to_{\mathsf{PRE}} X,$

$\texttt{Assume}\ v : V,$

$(1) := \eth F \eth \texttt{inclusion} \eth \oplus : F \circ \imath_{V \oplus W}(v) = (T \oplus S)(v, 0) = Tv + S0 = Tv;$

$\rightsquigarrow (1) := \texttt{MapEq} : F \circ \imath_{V \oplus W} = T,$

$\texttt{Assume}\ w : W,$

$(2) := \eth F \eth \texttt{inclusion} \eth \oplus : F \circ \imath^{V}_{V \oplus W}(w) = (T \oplus S)(0, w) = T0 + Sw = Tw;$

$\rightsquigarrow (2) := \texttt{MapEq} : F \circ \imath^{W}_{V \oplus W} = S,$

$\texttt{Assume}\ G : V \oplus W \to_{\mathsf{PRE}} X : F \circ \imath^{W}_{V \oplus W} = T\ \&\ G \circ \imath^{W}_{V \oplus W} = S,$

$\texttt{Assume}\ (v, w) : V \oplus W,$

$(3) := \eth_1 \mathcal{L}(V \oplus W, X)(G)((v, 0), (w, 0)) \eth G(1, 2) \eth_1^{-1} \mathcal{L}(V \oplus W, X)(F)((v, 0), (w, 0)) :$

$\quad : G(v, w) = G(v, 0) + G(0, w) = T(v) + S(w) = F(v, 0) + F(0, w) = F(v, w);;;;$

$(*) := \eth^{-1} \texttt{Coproduct}(\mathsf{PRE}) : \texttt{normedSpaceSum} : \texttt{Coproduct}(\mathsf{PRE});$

$\square$

$V : \mathsf{PRE}$

$A, B : \texttt{Subspace}(V)$

$i := (i^{A}_{A \oplus B}, i^{B}_{A \oplus B})$

$\texttt{CoproductCharacteristic} :: V = A \oplus_{\mathsf{VS}} B\ \&\ \| \cdot \|_V \cong \| \cdot \|_{A \oplus B} \Rightarrow V \cong A \sqcup_{\mathsf{PRE}} B$

$\texttt{Proof} =$

$(*) := \texttt{NormEquevalence}(\| \cdot \|_V \cong \| \cdot \|_{A \oplus B}) \texttt{PreCoproduct} : V \cong A \sqcup_{\mathsf{PRE}} B;$

$\square$

$\texttt{PreCoproductIsomorphism} :: V \cong A \sqcup_{\mathsf{PRE}} B \Rightarrow \Lambda(a, b) \in A \sqcup_{\mathsf{PRE}} B\ .\ a + b : V \leftrightarrow_{\mathsf{PRE}} A \oplus B$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{IsomorphismOfPreCoproduct} :: \Lambda(a, b) \in A \sqcup_{\mathsf{PRE}} B\ .\ a + b : V \leftrightarrow_{\mathsf{PRE}} A \oplus B \Rightarrow$

$\quad \Rightarrow V = A \oplus_{\mathsf{VS}} B\ \&\ \| \cdot \|_V \cong \| \cdot \|_{A \oplus B}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{TopologicalyDirectComplement} :: \texttt{Subspace}(V) \to ?\texttt{Subspace}(V)$

$A : \texttt{TopologicalyDirectComplement(B)} \iff V \cong A \oplus B$

$\texttt{TopologicalyCompletable} :: ?\texttt{Subspace}(V)$

$A : \texttt{TopologicalyCompletable} \iff \exists \texttt{TopologicalyDirectComplement}(A)$

$\texttt{TCIsClosed} :: \forall V : \mathsf{NORM} \,.\, A : \texttt{TopologicalyCompletable}(V) \,.\, A : \texttt{Closed}(V)$

$\texttt{Proof} \ =$

$B := \eth\texttt{TopologicalyCompletable}(A) : \texttt{Subspace}(V) : V \cong A \oplus B,$

$\texttt{Assume}\ x : \mathbb{N} \to A : \texttt{Convergent}(V),$

$T := \texttt{PreCoproductIsomorphism} : A \oplus B \leftrightarrow_{\mathsf{NORM}} V,$

$(1) := \eth T(x) : Tx : \texttt{Convergrnt}(A \oplus B),$

$(a, b) := \lim\limits_{n\to\infty} Tx_n : \texttt{In}(A \oplus B),$

$\texttt{Assume}\ b : \mathbb{N},$

$\alpha := \eth T(x_n, \eth x) : \texttt{In}(A) : Tx_n = (\alpha, 0),$

$(2) := \texttt{EqEl}(\|Tx_n - (a, b)\|, \eth^{-1}\alpha)\eth\| \cdot \|_{A\oplus B}\texttt{NonnegativeSumOrder}(\|\alpha - a\|) :$

$\quad : \|Tx_n - (a, b)\| = \|(\alpha, 0) - (a, b)\| = \|\alpha - a\| + \|b\| \geq \|b\|;$

$\leadsto (2) := \texttt{UniversalIntro} : \forall n \in \mathbb{N} \,.\, \|Tx_n - (a, b)\| \geq \|b\|,$

$(3) := \eth\mathsf{NORM}(V)\eth\texttt{Convergent}(A \oplus B) : b = 0,$

$(4) := \eth\texttt{InProduct}\eth b(3) : \lim\limits_{n\to\infty} Tx_n \in A \times \{0\},$

$(5) := \eth T \eth V \leftrightarrow_{\mathsf{NORM}} A \oplus B(T) : \lim\limits_{n\to\infty} x \in T^{-1}A \times \{0\} = A;$

$\leadsto (*) := \eth\texttt{Closed}(V) : (A : \texttt{Closed}(V)),$

$\texttt{InclusionOfCompletable} :: A : \texttt{TopologicalyCompletable}(V) \iff i_A : \texttt{Coretraction}(A, V)$

$\texttt{Proof} =$

$\texttt{Assume}\ L : A : \texttt{TopologicalyCompletable}(V),$

$B := \eth\texttt{TopologicalyCompletable}(A) : \texttt{Subspace}(V) : V \cong A \oplus B,$

$(T, S) := \texttt{PreCoproductIsomorphism} : V \leftrightarrow_{\mathsf{PRE}} A \oplus B : \forall x \in V\ .\ Tx + Sx = x,$

$P := \Lambda v \in V\ .\ Tv : V \to_{\mathcal{VS}} A,$

$\texttt{Assume}\ x : \texttt{In}(A),$

$(1) := \eth i_A \eth P : P i_A a = P a = a;$

$\rightsquigarrow () := \eth\texttt{RightInverse}(i_A) : P : \texttt{RightInverse}(i_A),$

$C := \texttt{NormEq}(V, A\eth P) : \mathbb{R}_{++} : \forall (a, b) \in A \oplus B\ .\ \|(a, b)\| \le C\|a + b\|,$

$\texttt{Assume}\ x : \texttt{In}(V),$

$(1) := \texttt{EqEl}(\|Px\|, \eth P)\texttt{NonnegativeSumOrder2}(\|Sx\|)\eth^{-1}\| \cdot \|_{A\oplus B}\eth C\eth(S, T) :$

$\quad : \|Px\| = \|Tx\| \le \|Tx\| + \|Sx\| = \|(Tx, Sx)\| \le C\|Tx + Sx\| = C\|x\|;$

$\rightsquigarrow () := \eth\mathcal{B}(V, A) : V \to_{\mathsf{PRE}} A,$

$() := \eth\texttt{Coretraction}(i_A)(P) : (i_A : \texttt{Coretraction}(A, V));$

$\rightsquigarrow L := \texttt{ImplInto} : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : (i_A : \texttt{Coretraction}(A, V)),$

$B := \ker i_A^{-1} : \texttt{Subspace}(V),$

$T := \Lambda(a, b) \in A \oplus B\ .\ a + b : A \oplus B \to_{\mathcal{VS}} V,$

$\texttt{Assume}\ (a, b), (x, y) : \texttt{In}(A \oplus B) : (a, b) \ne (x, y),$

$(1) := \texttt{TupleIneq}\eth((a, b), (x, y)) : a - x \ne 0 | b - y \ne 0,$

$(2) := \eth B \eth i_A : A \cap B = \{0\},$

$(3) := \eth T(x, y)\eth T(a, b)(1, 2) : T(a, b) - T(x, y) = a + b - x - y \ne 0;$

$\rightsquigarrow (1) := \eth A \oplus B \hookrightarrow V : T : A \oplus B \hookrightarrow V,$

$\texttt{Assume}\ x : \texttt{In}(V),$

$y := i_A^{-1}(x) : \texttt{In}(A),$

$(2) := \eth\mathcal{L}(V, A)(x, -y)\eth y \eth i_A^{-1}\eth\texttt{invese}(V)(y) : i_A^{-1}(x - y) = i_A^{-1}(x) + i_B^{-1}(-y) = y - y = 0,$

$(3) := \eth B(2) : x - y \in B,$

$(4) := \eth A \oplus B(y, x - y)) : (y, x - y) \in A \oplus B,$

$(5) := \eth T(y, x - y)\eth\texttt{invese}(V)(y) : T(y, x - y) = y + x - y = x;$

$\rightsquigarrow (2) := \eth A \oplus B \leftrightarrow V : T : \eth A \oplus B \leftrightarrow V,$

$\texttt{Assume}\ (a, b) : \texttt{In}(A \oplus B),$

$(3) := \texttt{EqEl}(\|T(a, b)\|, \eth T(a, b))\texttt{TriangleIneq}(a, b)\eth^{-1}\| \cdot \|_{A\oplus B} : \|T(a, b)\| = \|a + b\| \le \|a\| + \|b\| = \|(a, b)\|;$

$\rightsquigarrow (3) := \eth\mathcal{B}(A \oplus B, V) : (T : A \oplus B \to_{\mathsf{PRE}} V),$

$(4) := \eth T : T^{-1} = (i_A^{-1}, \mathrm{id} - i_A^{-1}),$

$\texttt{Assume}\ x : \texttt{In}(V),$

$(7) := \texttt{EqEl}(\|T^{-1}x\|, 4)\eth\| \cdot \|_{A\oplus B}\eth\texttt{TriangleIneq}(x, i_A^{-1}(x))\eth\mathcal{B}(V, A)(i_A^{-1}) :$

$\quad : \|T^{-1}x\| = \|(i_A^{-1}(x), x - i_A^{-1}(x))\| = \|i_A^{-1}(x)\| + \|x - i_A^{-1}(x)\| \le 2\|i^{-1}(x)\| + \|x\| \le (2C + 1)\|x\|;$

$\rightsquigarrow () := \eth A \oplus B \leftrightarrow_{\mathsf{PRE}} V(1, 2, 3) : T : A \oplus B \leftrightarrow_{\mathsf{PRE}} V,$

$(5) := \eth(\cong_{\mathsf{PRE}})(T) : A \oplus B \cong V,$

$(6) := \eth\texttt{TopologicalyCompletable}(5) : (A : \texttt{TopologicalyCompletable}(V));$

$(*) := \texttt{IffIntro}(L) : A : \texttt{TopologicalyCompletable}(V) \iff i_A : \texttt{Coretraction}(A, V);$

$\square$

$\texttt{ProjectionOfCompletable} :: A : \texttt{TopologicalyCompletable}(V) \iff \pi_A : \texttt{Retraction}(\mathsf{PRE})\left(V, \frac{V}{A}\right)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ L : A : \texttt{TopologicalyCompletable}(V),$

$B := \eth\texttt{TopologicalyCompletable}(A) : \texttt{Subspace}(V) : V \cong A \oplus B,$

$(T, S) := \texttt{PreCoproductIsomorphism} : V \leftrightarrow_{\mathsf{PRE}} A \oplus B : \forall x \in V\ .\ Tx + Sx = x,$

$I := \Lambda[v] \in \frac{V}{A}\ .\ Sv : V \to_{\mathcal{VS}} A,$

$\texttt{Assume}\ [v] : \texttt{In}\left(\frac{V}{A}\right),$

$(1) := \eth I \eth \pi_A \eth S : \pi_A I[v] = \pi_A S v = [Sv] = [v];$

$\rightsquigarrow () := \eth\texttt{LeftInverse}(\pi_A) : (P : \texttt{LeftInverse}(\pi_A)),$

$C := \texttt{NormEq}(V, A\eth P) : \mathbb{R}_{++} : \forall (a,b) \in A \oplus B\ .\ \|(a,b)\| \le C\|a + b\|,$

$\texttt{Assume}\ [v] : \texttt{In}\left(\frac{V}{A}\right),$

$(1) := \texttt{EqEl}(\|Px\|, \eth P)\texttt{NonnegativeSumOrder2}(\|Sx\|)\eth^{-1}\| \cdot \|_{A \oplus B}\eth C \eth(S, T) :$

$: \|I[v]\| = \|Sv\| = \inf_{a \in A} \|Sv\| + \|a\| = \inf_{a \in A} \|(a, Sv)\| \le \inf_{a \in A} C\|Sv + a\| = \inf_{a \in A} C\|v + a\| = C\|[v]\|;$

$\rightsquigarrow () := \eth\mathcal{B}(V, A) : V \to_{\mathsf{PRE}} A,$

$() := \eth\texttt{Retraction}(\mathsf{PRE})(\pi_A)(I) : \left(\pi_A : \texttt{Retraction}(\mathsf{PRE})\left(V, \frac{V}{A}\right)\right);$

$\rightsquigarrow L := \texttt{ImplInto} : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : \left(\pi_A : \texttt{Retraction}(\mathsf{PRE})\left(V, \frac{V}{A}\right)\right),$

$B := \operatorname{Im} \pi_A^{-1} : \texttt{Subspace}(V),$

$T := \Lambda(a,b) \in A \oplus B\ .\ a + b : A \oplus B \to_{\mathcal{VS}} V,$

$\texttt{Assume}\ (a,b), (x,y) : \texttt{In}(A \oplus B) : (a,b) \ne (x,y),$

$(1) := \texttt{TupleIneq}\eth((a,b),(x,y)) : a - x \ne 0 | b - y \ne 0,$

$(2) := \eth B \eth i_A : A \cap B = \{0\},$

$(3) := \eth T(x,y) \eth T(a,b)(1,2) : T(a,b) - T(x,y) = a + b - x - y \ne 0;$

$\rightsquigarrow (1) := \eth A \oplus B \hookrightarrow V : T : A \oplus B \hookrightarrow V,$

$\texttt{Assume}\ x : \texttt{In}(V),$

$y := \pi_A^{-1}\pi_A x : \texttt{In}(B),$

$(2) := \eth\mathcal{L}(V, A)(x, -y)\eth y \eth i_A^{-1}\eth\texttt{invese}(V)(y) : \pi_A^{-1}\pi_A(x - y) = \pi_A^{-1}\pi_A x + \pi_A^{-1}\pi_A(-y) = y - y = 0,$

$(3) := \eth B(2) : x - y \in A,$

$(4) := \eth A \oplus B(y, x - y)) : (y, x - y) \in A \oplus B,$

$(5) := \eth T(y, x - y)\eth\texttt{invese}(V)(y) : T(y, x - y) = y + x - y = x;$

$\rightsquigarrow (2) := \eth A \oplus B \leftrightarrow V : T : \eth A \oplus B \leftrightarrow V,$

$\texttt{Assume}\ (a,b) : \texttt{In}(A \oplus B),$

$(3) := \texttt{EqEl}(\|T(a,b)\|, \eth T(a,b))\texttt{TriangleIneq}(a,b)\eth^{-1}\| \cdot \|_{A \oplus B} : \|T(a,b)\| = \|a + b\| \le \|a\| + \|b\| = \|(a,b)\|;$

$\rightsquigarrow (3) := \eth\mathcal{B}(A \oplus B, V) : (T : A \oplus B \to_{\mathsf{PRE}} V),$

$(4) := \eth T : T^{-1} = (\mathrm{id} - \pi_A\pi_A^{-1}, \pi_A\pi_A^{-1}),$

$\texttt{Assume}\ x : \texttt{In}(V),$

$(7) := \texttt{EqEl}(\|T^{-1}x\|, 4)\eth\| \cdot \|_{A\oplus B}\eth\texttt{TringleIneq}(x, \pi_A^{-1}\pi_A(x)_A(x))\eth\mathcal{B}(V, A)(\pi_A^{-1}\pi_A(x)) :$
$\quad : \|T^{-1}x\| = \|(x - \pi_A^{-1}\pi_A(x), \pi_A^{-1}\pi_A(x))\| = \|x - \pi_A^{-1}\pi_A(x)_A^{-1}(x)\| + \|\pi_A^{-1}\pi_A(x)\| \leq$
$\quad \leq 2\|\pi_A^{-1}\pi_A(x)(x)\| + \|x\| \leq (2C + 1)\|x\|;$

$\rightsquigarrow () := \eth A \oplus B \leftrightarrow_{\mathsf{PRE}} V(1, 2, 3) : T : A \oplus B \leftrightarrow_{\mathsf{PRE}} V,$

$(5) := \eth(\cong_{\mathsf{PRE}})(T) : A \oplus B \cong V,$

$(6) := \eth\texttt{TopologicalyCompletable}(5) : (A : \texttt{TopologicalyCompletable}(V));$

$$(*) := \texttt{IffIntro}(L) : A : \texttt{TopologicalyCompletable}(V) \iff \pi_A : \texttt{Retraction}(\mathsf{PRE})\left(V, \frac{V}{A}\right);$$

$\square$

1.6 Topological Properties

V, W : PRE

$\texttt{BoundedIsUniformlyCont} :: \forall T : \mathcal{B}(V,W) \,.\, T : V \to_{\mathsf{UTOP}} W$

$\texttt{Proof} \;=$

$C := \eth\mathcal{B}(V,W)(T) : \mathbb{R}_+ : \forall v \in V \,.\, \|Tv\| \le C\|v\|,$

$\texttt{Assume}\ \epsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ v, w : V : \|v - w\| \le \dfrac{\epsilon}{C},$

$(1) := \eth_1\mathcal{L}(v,w)(T)\eth C\eth(v,w) : \|Tv - Tw\| = \|T(v-w)\| \le C\|(v-w)\| \le \epsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{UniformlyCont}(V,W) : (T : V \to_{\mathsf{UTOP}} W),$

$\square$

$\texttt{ContAtZeroIsBounded} :: \forall T : \mathcal{L}(V,W)\ \&\ \texttt{ContinuousAt}(V,0) \,.\, T : \mathcal{B}(V,W)$

$\texttt{Proof} \;=$

$\delta := \eth\texttt{ContinuousAt}(V,0)(1) : \mathbb{R}_{++} : \forall v \in V : \|v\| \le \delta \,.\, \|Tv\| \le 1,$

$\texttt{Assume}\ x : V : \|x\| \ne 0,$

$(1) := \eth_2\texttt{Seminorm}\left(Tx, \dfrac{\delta}{\|x\|}\right)\eth_2\mathcal{L}(V,W)(T)\left(x, \dfrac{\delta}{\|x\|}\right)\eth\delta : \dfrac{\delta}{\|x\|}\|Tx\| = \left\|T\dfrac{\delta x}{\|x\|}\right\| \le 1,$

$(2) := \dfrac{\|x\|}{\delta}(1) : \|Tx\| \le \delta^{-1}\|x\|;$

$\rightsquigarrow (1) := \texttt{UniversalInroduction}\ \texttt{ExistanceIntroduction}(\delta^{-1}) :$

$\quad : \exists C \in \mathbb{R}_+ \,.\, \forall v \in V : \|v\| \ne 0 \,.\, \|Tv\| \le C\|v\|,$

$(2) := \eth\texttt{ContinuousAt}(V,0) : \forall v \in V : \|v\| = 0 \,.\, \|Tv\| = \|T0\| = \|0\| = 0 = \|v\|,$

$(*) := \eth^{-1}\mathcal{B}(V,W)(\texttt{Synthesis}(2,3)) : (T : \mathcal{B}(V,W));$

$\square$

$\texttt{TopIsoIsHomeo} :: \forall T : V \leftrightarrow_{\mathsf{PRE}} W \,.\, T : V \leftrightarrow_{\mathsf{TOP}} W$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TopInjCharacteristic} :: \forall T : \mathcal{B}(V,W) \,.\, T : V \hookrightarrow_{\mathsf{SET}} W\ \&\ \exists C \in \mathbb{R}_{++} :$

$\quad : \forall v \in V \,.\, C\|v\| \le \|Tv\| \iff T : \texttt{TopologicalyInjectiveOperator}(V,W)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TopSurjIsOpen} :: \forall T : \texttt{TopolologicalySurjective}(V,W) \,.\, T : \texttt{OpenMap}(V,W)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; U : \texttt{Open} V,$

$\texttt{Assume}\; u : U,$

$\texttt{Assume}\; x : \ker T,$

$(1) := \ldots : T(u+x) = T(u) \in TU;$

$\rightsquigarrow (1) := \texttt{SubsetIntroduction}(V) : U + \ker T \subset T^{-1}TU,$

$\texttt{Assume}\; v : T^{-1}TU,$

$(2) := \eth T^{-1}(v) : Tv \in TU,$

$u := \eth TU(Tv) : U : Tv = Tu,$

$(3) := \eth_1 \mathcal{L}(V,W)(T)(v,u)\eth^{-1}\texttt{Zero}(2) : T(v-u) = Tv - Tu = 0,$

$(4) := \eth \ker(3) : v - u \in \ker T,$

$(5) := \texttt{PlusMinus}(v,u) : v = u + v - u \in U + \ker T;$

$\rightsquigarrow (2) := \texttt{SubsetIntroduction}(V) : T^{-1}TU \subset U + \ker T,$

$(3) := \texttt{SetEq}(1,2) : T^{-1}TU = U + \ker T,$

$(4) := \texttt{AdditionCont}(U + \ker T)(3) : T^{-1}TU : \texttt{Open}(V),$

$() := \eth\texttt{TopolologicalySurjective}(VW)(T)(4) : \texttt{Proves}(TU : \texttt{Open}(V));$

$\rightsquigarrow (*) := \eth^{-1}\texttt{OpenMap}(V,W) : \texttt{Proves}(T : \texttt{OpenMap}(V,W)),$

$\square$

$\texttt{NonColapsing} :: ?V \to_{\mathsf{PRE}} W$

$T : \texttt{NonColapsing} \iff \exists C \in \mathbb{R}_+ : \forall y \in W \,.\, \exists x \in V : y = Tx : \|x\| \le C\|y\|$

$\texttt{NormedNonColapsingCharacteristic} :: \forall V,W : \mathsf{NORM} \,.\, \forall T : V \to_{\mathsf{NORM}} W \,\&\, \texttt{OpenMap}(V,W) \,.$
$\quad .\, T : \texttt{NonColapsing}(V,W)$

$\texttt{Proof} \;=$

$(1) := \eth\mathcal{L}(V,W) : T0 = 0,$

$(2) := \eth\texttt{Image}(T, \mathbb{B}_V, (1), 0 \in \mathbb{B}_V) : 0 \in T\mathbb{B}_V,$

$(3) := \eth\texttt{OpenMap}(V,W)(T)(\mathbb{B}_V) : T\mathbb{B}_V : \texttt{Open}(W),$

$t := \eth\texttt{MetricTopologyTHM}(2,3) : \mathbb{R}_+ : 0 \in \mathbb{B}_W(0,t) \subset T\mathbb{B}_V,$

$\texttt{Assume}\; y : W : y \ne 0,$

$(4) := \eth\texttt{ball}(y,t) : \dfrac{t}{2\|y\|}y \in \mathbb{B}_W(0,t),$

$(5) := \texttt{SubsetTransitivity}(\eth t, 4) : \dfrac{t}{2\|y\|}y \in T\mathbb{B}_V,$

$x := \texttt{InImage}(4) : \texttt{In}(\mathbb{B}_V) : Tx = \dfrac{t}{2\|y\|}y,$

$(6) := \eth_2\mathcal{L}(V,W)\left(x, \dfrac{2\|y\|}{t}\right)\texttt{MultBy}(\eth x)\left(\dfrac{2\|y\|}{t}\right) : T\dfrac{2\|y\|}{t}x = y;$

$(7) := \eth\texttt{ball}(x)(6) : \left\|\dfrac{2\|y\|}{t}x\right\| \le \dfrac{2\|y\|}{t};$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NonColapsing}(V,W) : (T : \texttt{NonColapsing}(V,W));$

$\square$

$\texttt{NonColapsingIsOpen} :: \forall T : \texttt{NonColapsing}(V, W) \,.\, T : \texttt{OpenMap}(V, W)$

$\texttt{Proof} \ =$

$C := \eth\texttt{NonColapsing}(V, W)(T) : \mathbb{R}_+ + : \forall y \in W \,.\, \exists x \in V : y = Tx \;\&\; \|x\| \leq C\|y\|,$

$\texttt{Assume}\ U : \texttt{Open}(V),$

$\texttt{Assume}\ y : TU,$

$x := \eth\texttt{Image}(U, y) : \texttt{In}U : Tx = y,$

$r := \eth\texttt{MetricTopology}(V)(U)(x) : \mathbb{R}_{++} : \mathbb{B}_V(x, r) \subset U,$

$\texttt{Assume}\ w : W : \|w - y\| \leq C^{-1}t,$

$v := \eth C(w - y) : \texttt{In}(V) : \|v\| \leq r \;\&\; Tv = w - y,$

$u := x + v : \texttt{In}(V),$

$(1) := \eth(r)\eth(v)\eth(u) : (u : \texttt{In}(U)),$

$(2) := \eth V \to_{\mathsf{Set}} W(T)(\eth u)\eth V \to_{\mathsf{VS}(K)} W(T)(x, v) \ldots : Tu = T(x + v) = Tx + Tv = y + w - y = w;$

$(3) := \eth^{-1}\texttt{Image}(U)(T)(1, 2) : w \in UT;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Subset}(U) : \mathbb{B}(y, C^{-1}t) \subset U;$

$\rightsquigarrow (1) := \texttt{MetricTopology} : TU : \texttt{Open}(V);$

$\rightsquigarrow (*) := \eth\texttt{OpenMap}(V, W) : T : \texttt{OpenMap}(V, W);$

$\square$

$\texttt{OpenIsTopologicalySurjective} :: \forall T : \texttt{OpenMap}(V, W) \,.\, T : \texttt{TopolologicalySurjective}(V, W)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U : \texttt{Subset}(W) : T^{-1}U : \texttt{Open}(V),$

$(1) := \eth\texttt{OpenMap}(V, W(T)(T^{-1}U) : TT^{-1}U : \texttt{Open}(W),$

$(2) := \texttt{ImagePreimage}(T, U) : TT^{-1}U = U,$

$(3) := \texttt{Synthesis}(1, 2) : (U : \texttt{Open}(V));$

$\rightsquigarrow (1) := \eth V \twoheadrightarrow_{\mathsf{PRE}} W : (T : V \twoheadrightarrow_{\mathsf{PRE}} W);$

$\square$

$\texttt{SeminormPushforward} :: (V \to_{\mathsf{PRE}} W) \to \texttt{SeminormedSpace}$

$\texttt{SeminormPushforward}\,(T) = (V, |||\cdot|||_T) := (V, \Lambda x \in V \,.\, \|Tx\|)$

$\texttt{BoundnesNormCharacteristic} :: \forall T : V \to_{\mathsf{VS}(K)} W \,.\, T : V \to_{\mathsf{PRE}} W \iff \|\cdot\|_V \succeq |||\cdot|||_T$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ClosedOperatorKernel} :: \forall W \in \mathsf{NORM} \,.\, \forall T : V \to_{\mathsf{PRE}} W \,.\, \ker T : \texttt{Closed}(V)$

$\texttt{Proof} \ =$

1.7 Infinite Matrices

$$\texttt{Matrix} :: (V \to_{\mathsf{PRE}} W) \to \texttt{Shauder}(V) \to \texttt{Shauder}(W) \to ?(\mathbb{N} \to \mathbb{N} \to K)$$

$$A : \texttt{Matrix(T, e, h)} \iff \forall n \in \mathbb{N} \,.\, Te_n = \sum_{m=1}^{\infty} A_{mn} h_m$$

$$\texttt{matrix} :: \prod H, E : \texttt{Prehilbet}(K) \,.\, \prod T : E \to_{\mathsf{PRE}} H \,.\, \prod e : \texttt{ShauderHilbert}(E) \,.$$
$$. \prod h : \texttt{ShauderHilbert}(H) \to \texttt{Matrix}(T, e, h)$$
$$\texttt{matrix}\,(H, E, T, e, h) = T_{e,h} := \lambda n, m \in \mathbb{N} \,.\, \langle Te_n, h_m \rangle$$

$$\texttt{MatrixNorm} :: \forall T : l_2 \to_{\mathsf{PRE}} l_2 \,.\, \|T\| = \sup \left\{ \left| \sum_{n,m=1}^{\infty} (A_{e,e})_{n,m} v_n w_m \right| |v, w \in \mathbb{B}_{l_2} \right\}$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$$\texttt{EqMatricesTHM} :: \forall e : \texttt{Shauder}(V) \,.\, \forall S : V \to_{\mathsf{PRE}} V \,.\, \forall T : W \to_{\mathsf{PRE}} W : T \cong_{\mathsf{PRE}} S \,.$$
$$. \exists h : \texttt{Shauder}(W) : \forall A : \texttt{Matrix}(S, e, e) \,.\, \forall B : \texttt{Matrix}(T, h, h) \,.\, A = B$$

$\texttt{Proof} \;=$

$I := \eth S \cong_{\mathsf{PRE}} T : V \leftrightarrow_{\mathsf{PRE}} W : IS = TI,$

$h := Ie : \mathbb{N} \to W,$

$\texttt{Assume}\ y : \texttt{In}(W),$

$x := I^{-1} y : \texttt{In}(Y),$

$$a := \eth\texttt{Shauder}(e) : \texttt{Unique}\left(\mathbb{N} \to K, x = \sum_{n=1}^{\infty} a_n e_n \right),$$

$$(1) := \eth x \eth a \eth \mathcal{L}_1(V, W)(T)(ae)\mathcal{L}_2(V, W)(T)(a, e)\eth^{-1}(h) : y = Tx = T \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^{\infty} a_n T e_n = \sum_{n=1}^{\infty} a_n h_n;$$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Shauder} : \big(h : \texttt{Shauder}(W)\big),$

$\texttt{Assume}\ A : \texttt{Matrix}(S, e, e),$

$\texttt{Assume}\ B : \texttt{Matrix}(T, h, h),$

$\texttt{Assume}\ n : \mathbb{N},$

$(2) := \eth^{-1} B \eth h \eth I \eth A \eth \mathcal{L}(V, W)(I) \eth^{-1} h :$

$$: \sum_{m=1}^{\infty} B_{mn} h_m = T h_n = T I e_n = I S e_n = I \sum_{m=1}^{\infty} A_{mn} e_m = \sum_{m=1}^{\infty} A_{mn} I e_m = \sum_{m=1}^{\infty} A_{mn} h_m,$$

$(3) := \eth\texttt{Shauder}(e)(2) : \forall m \in \mathbb{N} \,.\, A_{mn} = B_{mn};$

$\rightsquigarrow (2) := \texttt{FuncEq} : A = B;$

$\rightsquigarrow (4) := \texttt{UnivIntro} : \forall A : \texttt{Matrix}(S, e, e) \,.\, \forall B : \texttt{Matrix}(T, h, h) \,.\, A = B,$

$\square$

1.8 Bounded Multilinear Operators

$n \in \mathbb{N}$

$X : n \to \mathsf{PRE}$

$$\texttt{JointlyBounded} :: ?\mathcal{L}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right)$$

$$R : \texttt{JointlyBounded} \iff R \in \mathcal{B}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right) \iff \sup\left\{\|Rx\| \,|x \in \prod_{i=1}^{n} \mathbb{B}_{X_i}\right\} < \infty$$

$$\texttt{DisjointlyBounded} :: ?\mathcal{L}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right)$$

$$R : \texttt{DisjointlyBounded} \iff R \in \left[\bigotimes_{i=1}^{n}\right] \mathcal{B}(X_i, V) \iff \forall m \in n \,.\, \forall x \in \prod_{i=1}^{n} X_i \,.$$

$$.\, \Lambda w \in X_m \,.\, R\left(\bigoplus_{i=1}^{m-1} x_i \oplus w \oplus \bigoplus_{i=m+1}^{n} x_i\right) : \mathcal{B}(X_m, V)$$

$$\texttt{JointlyBoundedIsDisjointlyBounded} :: \forall R : \mathcal{B}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right) \,.\, R : \left[\bigotimes_{i=1}^{n}\right] \mathcal{B}(X_i, V)$$

Proof $=$

...

$\square$

$$\texttt{MultioperatorNorm} :: \mathcal{B}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right) \to \mathbb{R}_+$$

$$\texttt{MultioperatorNorm}\,(R) = \|R\| := \sup\left\{\|Rx\| \,|x \in \prod_{i=1}^{n} \mathbb{B}_{X_i}\right\}$$

$$\texttt{MultilinearConvergence} :: \forall R : \mathcal{B}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right) \,.\, \forall x : \mathbb{N} \to \prod_{i=1}^{n} X_i :$$

$$: \forall i \in n \,.\, x^i : \texttt{Convergent}(X_i) \,.\, R(x) : \texttt{Convergent}(V)$$

Proof $=$

...

$\square$

$\texttt{MultilinearContinuity} :: \forall R : \mathcal{B}\left(\left[\bigotimes_{i=1}^{n}\right] X, V\right) \, . \, R : \prod_{i=1}^{n} X_i \rightarrow_{\mathsf{TOP}} V$

$\texttt{Proof} \ =$

...

$\square$

1.9 One-Dimensional Operators

$\texttt{OneDimensionalOperator} :: V^* \to W \to V \to_{\mathsf{PRE}} W$
$\texttt{OneDimensionalOperator}(f, y, x) = (f \otimes y)(x) := \langle f, x \rangle y$

$f \in V^*$

$y \in W$

$y \neq 0 \neq f$
$\dim \operatorname{Im} f \otimes y = \dim \operatorname{span}(y) = 1$

$$\|f \otimes y\| = \sup_{x \in \mathbb{B}_V} \|\langle f, x \rangle y\| = \sup_{x \in \mathbb{B}_V} |\langle f, x \rangle| \|y\| = \|f\| \|y\|$$

$\texttt{OneDimensionalOperatorRepresentation} :: \forall T : V \to_{\mathsf{PRE}} W : \dim \operatorname{Im} T = 1 \,.$
$\quad . \exists f \in V^* : \exists y \in W \,.\, T = f \otimes y$
$\texttt{Proof} =$
$y := \eth \dim \operatorname{Im} T = 1 : \texttt{In}(W) \;\&\; \operatorname{Im} T = \operatorname{span}(y),$
$\texttt{Assume}\ x : \texttt{In}(V),$
$f(x) := \eth y(x) : \texttt{In}(K) : f(x) y = T(x);$
$\rightsquigarrow f := \eth^{-1} \texttt{Dual} : \texttt{In}(V^*),$
$(*) := \eth f : T = f \otimes y;$
$\square$

$U : \mathsf{PRE}$

$\texttt{OneDimensionalMultiplication} :: \forall f \in V^* \,.\, \forall g \in W^* \,.\, \forall w \in W \,.\, \forall u \in U \,.$
$\quad . (g \otimes u)(f \otimes w) = \langle g, w \rangle (f \otimes u)$
$\texttt{Proof} =$
$\texttt{Assume}\ x : V,$
$(g \otimes u)(f \otimes w)x = (g \otimes u)\langle f, x \rangle w = \langle g, \langle f, x \rangle w \rangle u = \langle g, w \rangle \langle f, x \rangle u = \langle g, w \rangle (f \otimes u)(x)$
$\square$

1.10 Projection Operators

$\texttt{Projector(PRE)} :: \texttt{Subspace}(V) \to \mathcal{B} \,\&\, \texttt{Idempotent}(V, V)$
$P : \texttt{Projector(PRE)(S)} \iff \operatorname{Im} P = S$

$\texttt{ProjectorOfTopologicalDirectSum} :: V \cong A \oplus B \Rightarrow \exists P : \texttt{Projector}(\mathsf{PRE})(A)$
$\texttt{Proof} \;=$
$P := \texttt{ProjectorOfDirectProduct}(V) : \texttt{Projector}(\mathsf{VS})(A),$
$C := \texttt{NormEq}(V, A \oplus B) : \mathbb{R}_{+}+ : \forall (a,b) \in A \oplus B \,.\, \|(a,b)\| \le C\|a+b\|,$
$\texttt{Assume } v : \texttt{In}(v),$
$(a,b) := \texttt{PreCoproductIsomorphism} : \texttt{In}(A \oplus B) : v = a + b,$
$(1) := \texttt{EqEl}(\|Pv\|, \eth(a,b))\eth\texttt{Projector}(A)\texttt{NonnegativeSumOrder2}(\|b\|)\eth\| \cdot \|_{A\oplus B}\eth C \eth^{-1}(a,b) :$
$\quad : \|Pv\| = \|P(a+b)\| = \|a\| \le \|a\| + \|b\| = \|(a,b)\| \le C\|a+b\| = C\|v\|;$
$\rightsquigarrow (*) := \eth^{-1}\texttt{Projector}(\mathsf{PRE})(A) : (P : \texttt{Projector}(\mathsf{PRE})(A)),$
$\square$

$\texttt{TopologicalDirectSumOfProjector} :: \forall P : \texttt{Projector}(\mathsf{PRE})(A) \,.\, V \cong A \oplus \ker P$
$\texttt{Proof} \;=$
$T := \Lambda x \in V \,.\, (Px, (\mathrm{id} - P)x) : V \to_{\mathsf{VS}} A \oplus \ker P,$
$1 := \texttt{DirectSumOfProjector}(P) : T : V \leftrightarrow_{\mathsf{VS}} A \oplus \ker P,$
$\texttt{Assume } x : \texttt{In}(V),$
$(2) := \texttt{EqEl}(\|Tx\|, \eth T)\eth\| \cdot \|_{A\oplus \ker P}\texttt{TriangleIneq}(x, -Px)\eth\mathcal{B}(V,V)(P) :$
$\quad : \|Tx\| = \|(Px, x - Px)\| = \|Px\| + \|x - Px\| \le 2\|Px\| + \|x\| \le (1 + 2\|P\|)\|x\|;$
$\rightsquigarrow (2) := \eth\mathcal{B}(V, A \oplus \ker P) : T \in \mathcal{B}(V, A \oplus \ker P),$
$\texttt{Assume } (a,b) : \texttt{In}(A \oplus B),$
$(3) := \texttt{EqEl}(T^{-1}, \eth T)\texttt{TriangleEq}(a,b)\eth^{-1}\| \cdot \|_{A\oplus \ker P} : \|T^{-1}(a,b)\| = \|a+b\| \le \|a\| + \|b\| = \|(a,b)\|;$
$\rightsquigarrow (3) := \eth\mathcal{B}(A \oplus \ker P, V) : T^{-1} \in \mathcal{B}(A \oplus \ker P, V),$
$() := \eth V \leftrightarrow_{\mathsf{PRE}} A \oplus \ker P(1,2,3) : (T : V \leftrightarrow_{\mathsf{VS}} A \oplus \ker P),$
$(*) := \eth(V \cong A \oplus \ker P)(T) : V \cong A \oplus \ker P,$
$\square$

$\texttt{DiagonalSeqProjection} :: \forall a \in l_\infty \,.\, \operatorname{diag}(a) : \texttt{Projector}(l_\infty) \iff \forall n \in \mathbb{N} \,.\, \lambda_n \in \{0,1\}$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{DiagonalFuncProjection} :: \forall f \in L_\infty(\Omega,\mu) \,.\, \operatorname{Diag}(a) : \texttt{Projector}(L_\infty(\Omega,\mu)) \iff \exists X \in \mathcal{F}_\Omega : f = I_X$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{IntegralOperatorProjection} :: \forall n \in \mathbb{N} \,.\, \forall g, f : n \to L(\Omega, \mu) :$

$$: \forall k, l \in n \,.\, \int_\Omega f_k g_l \, \mathrm{d}\mu = \delta_{k,l} \,.\, \texttt{IntegralOperator}(K) : \texttt{Projector}(L_\infty(\Omega, \mu))$$

$\texttt{Where } K = \Lambda s \in \Omega \,.\, \Lambda t \in \Omega \,.\, f(s)g(t)$

$\texttt{Proof} \ =$

$T := \texttt{IntegralOperator}(K) : \mathcal{B}(L_2(\Omega, \mu), L_2(\Omega, \mu)),$

$\texttt{Assume } x : L_2(\Omega, \mu),$

$$Tx = \Lambda s \in \Omega \,.\, \int_\Omega x(t) \sum_{k=1}^n f_k(s) g_k(t) \, \mathrm{d}\mu(t)$$

$\texttt{Assume } r : \Omega,$

$$T^2 x(r) = \int_\Omega \left(\int_\Omega x(t) \sum_{k=1}^n f_k(s) g_k(t) \, \mathrm{d}\mu(t) \right) \sum_{k=1}^n f_k(r) g_k(s) \, \mathrm{d}\mu(s) =$$

$$= \int_\Omega \int_\Omega x(t) \sum_{k,l=1}^n f_k(s) g_k(t) f_l(r) g_k(s) \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) =$$

$$= \int_\Omega x(t) \sum_{k,l=1}^n f_l(r) g_k(t) \left(\int_\Omega f_k g_k \, \mathrm{d}\mu \right) \mathrm{d}\mu = \int_\Omega x(t) \sum_{k=1}^n f_k(r) g_k(t) \, \mathrm{d}\mu(t) = Tx(r);$$

$T^2 x = Tx$

$T : \texttt{Projector}(L_\infty(\Omega, \mu))$

$\square$

1.11 Bounded Functionals

$c_0^* = l_1$

Proof $=$

$x \in c_0$

$y \in l_1$

$a = \sum_{n=1}^{\infty} x_n y_n$

$|a| \le \sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |x_n||y_n| \le \|x\| \sum_{n=1}^{\infty} |y_n| = \|x\|\|y\| < \infty \rightsquigarrow$

$\rightsquigarrow a \in K;$

$\phi :: l_1 \to c_0^*$

$\phi : y \mapsto \left(x \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$

$f \in c_0^*$

$y = \sum_{i=1}^{\infty} f(e_i)$

$\sum_{i=1}^{\infty} |y_i| = \lim_{n\to\infty} f\left(\sum_{k=1}^{n} e_k \mathrm{sign}(y_k) \right) \le \lim_{n\to\infty} \|f\| = \|f\| \rightsquigarrow$

$y \in l_1$

$\phi(y) = f$

$\phi : c_0^* \leftrightarrow_{\mathsf{NORM}_{\circ\to\cdot}} l_1 \square$

$l_1^* = l_\infty$

Proof $=$

$x \in l_1$

$y \in l_\infty$

$a = \sum_{n=1}^{\infty} x_n y_n$

$|a| = \|x\|\|y\| < \infty \rightsquigarrow a \in K$

$f \in l_1^*$

$y = (f(e_i))_{i=1}^{\infty}$

$\|y\| = \sup_{n\in\mathbb{N}} |f(e_n)| \le \sup_{n\in\mathbb{N}} \|f\|\|e_n\| = \sup_{n\in\mathbb{N}} \|f\| = \|f\|$

$\texttt{LpDual} :: \forall p,q \in (1,\infty) : p^{-1}+q^{-1}=1 \,.\, \forall(\Omega,\mathcal{F},\mu) : \mathsf{MEAS} \,.\, L_p^*(\Omega,\mathcal{F},\mu) \cong L_q(\Omega,\mathcal{F},\mu)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ y : L_q(\Omega,\mathcal{F},\mu),$

$\varphi(y) := \Lambda x \in L_p(\Omega,\mathcal{F},\mu) \,.\, \int_\Omega xy \,\mathrm{d}\mu : L_p(\Omega,\mathcal{F},\mu) \to K^\infty,$

$\texttt{Assume}\ x : L_p(\Omega,\mathcal{F},\mu),$

$(1) := \texttt{IntegralTriangleIneq}(yx)\ldots\eth\langle\cdot,\rangle_{L_2}\texttt{HolderIneq}(p,q) :$

$\quad : |\phi(y)(x)| \le \int_\Omega |yx|\,\mathrm{d}\mu = \int_\Omega |y||x|\,\mathrm{d}\mu = \langle |y|,|x|\rangle \le \|y\|_p\|x\|_q;;$

$\rightsquigarrow \phi := \eth\mathcal{B}(L_q(\Omega,\mathcal{F},\mu), L_p^*(\Omega,\mathcal{F},\mu)) : L_q(\Omega,\mathcal{F},\mu) \to_{\mathsf{PRE}_{\circ\to\cdot}} L_p^*(\Omega,\mathcal{F},\mu)),$

$\ldots$

$\square$

$\texttt{DualsHaveClosedKernel} :: \forall f : \mathcal{L}(V,K) \,.\, f \in V^* \iff \ker f : \texttt{Closed}(V)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ L : f \in V^*,$

$(1) := \eth\ker f : \ker f = f^{-1}(k),$

$(2) := \texttt{BoundeIsUniformlyCont}(f) : f : V \to_{\mathsf{TOP}} K,$

$(1) := \texttt{ClosedPreimage}(f,\{0\}) : \ker f : \texttt{Closed}(V);$

$\rightsquigarrow L := \texttt{ImplicationIntroduction} : f \in V^* \Rightarrow \ker f : \texttt{Closed}(V),$

$\texttt{Assume}\ R : (\ker f : \texttt{Closed}(V)),$

$\texttt{Assume}\ A : \ker f = V,$

$(1) := \eth\ker f(A) : f = 0,$

$(1) := \eth 0 : f \in V^*;$

$\rightsquigarrow (1) := \texttt{ImplicationIntroduction} : \ker f = V \Rightarrow f \in V^*,$

$\texttt{Assume}\ A : \ker f \neq V,$

$v := \eth\texttt{NotIn}(\ker f)(A) : \texttt{In}(V)\ \&\ \texttt{NotIn}(\ker f),$

$C := \texttt{ClosedSubspaceRepresentation}(\ker f, v) : \mathbb{R}_+ + : \forall x \in V \,.\, \forall a \in K \,.\, \forall z \in \ker f : x = av + z \,.$

$\quad . \, |a| \le C\|x\|,$

$\texttt{Assume}\ x : V,$

$b := f(x) : \texttt{In}(K),$

$a := \dfrac{b}{f(v)} : \texttt{In}(K) : f(av) = b,$

$(2) := \eth_1\mathcal{L}(V,K)(x,-av)\eth b\eth x\eth\texttt{inverse}(b) : f(x-av) = f(x) + f(-av) = b - b = 0,$

$() := \eth\ker f(2) : x - av \in \ker f,$

$(3) := \texttt{add}(av, x-av)\eth\texttt{inverse}(av) : x = av + x - av,$

$(4) := \texttt{EqEl}(|f(x)|,(3))\eth_1\mathcal{L}(V,K)(av,x-av)\eth\ker f(x-av)\eth_2\mathcal{L}(V,K)(v,a)$

$\quad \eth_2\texttt{AbsValue}(K)(f(v)a)\eth C(3,x,a,x-av) :$

$\quad : |f(x)| = |f(av+x-av)| = |f(av)+f(x-av)| = |f(av)| = |a||f(v)| \le C|f(v)|\|x\|;;$

$\rightsquigarrow (*) := \texttt{IffIntro}(L)\texttt{OrEl}(V = \ker f | V \neq \ker f)(1)\texttt{ImplicationInto} : f \in V^* \iff \ker f : \texttt{Closed}(V),$

1.12 Hahn-Banach Theorem

$\texttt{RealHahnBanach} :: \forall V : \mathsf{PRE}(\mathbb{R}) \,.\, \forall A : \texttt{Subspace}(V) \,.\, \forall f \in A^* \,.\, \exists F \in V^* : F_{|A} = f \;\&\; \|F\| = \|f\|$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : f = 0,$

$F := 0 : \texttt{In}(V^*),$

$(2) := \eth F_{|A}(1) : F_{|A} = 0 = f;$

$(3) := \eth F \eth f : \|F\| = \|0\| = \|f\|;$

$\rightsquigarrow (1) := \texttt{ImplicationInto} : f = 0 \Rightarrow \texttt{RealHahnBanach},$

$\texttt{Assume}\ (2) : f \neq 0,$

$g := \dfrac{f}{\|f\|} : A^* : \|g\| = 1,$

$\texttt{HahnBanachLemma} :: \operatorname{codim}_V A = 1 \Rightarrow \texttt{RealHahnBanach}$

$\texttt{Proof} \;=$

$() := \eth \operatorname{codim}_V A = 1 : A^{\complement} : \texttt{NonEmpty},$

$x := \eth \texttt{NonEmpty}(A^{\complement}) : A^{\complement},$

$\texttt{Assume}\ a, b : A,$

$(3) := \eth \texttt{abs}(g(a-b)) \eth \texttt{operatorNorm}(g)(a-b) \texttt{AddSubstract}(a-b, x) \eth_2 \| \cdot \|((x+a),(x-a)) :$

$\quad : g(a-b) \le |g(a-b)| \le \|a-b\| \le \|(x+a)-(x-b)\| \le \|x+a\| + \|x+b\|,$

$(4) := \texttt{SumIneq}(3, g(a), -g(b), \|x+a\|, \|x+b\|) : -g(b) - \|x+b\| \le \|x+a\| - g(a),$

$X_b := -g(b) - \|x+b\| : \mathbb{R};$

$Y_a := \|x+a\| - g(a) : \mathbb{R};$

$\rightsquigarrow (X, Y)) := \texttt{FuncIntro} : A \times A \to \mathbb{R} \times \mathbb{R} : \forall (a,b) \in A \times A \,.\, X_b \le Y_a,$

$C_x := \inf_{a \in A} Y_a : \mathbb{R},$

$c_x := \sup_{a \in A} X_a : \mathbb{R},$

$(3) := \eth (X, Y) : c_x \le C_x,$

$r := \texttt{IntermidiateReal}(c_x, C_x) : \mathbb{R} : c_x \le r \le C_x,$

$(4) := \eth (X, Y, r) : \forall a \in A \,.\, |r + g(a)| \le \|x+a\|,$

$\texttt{Assume}\ v : V,$

$(a, s) := \eth \operatorname{codim}_V A = 1(v, x) : A \times \mathbb{R} : sx + av = sx + a,$

$G(v) := g(a) + sr : \mathbb{R};$

$\texttt{Assume}\ O : v \in A,$

$(5) := \eth (s, a) O : v = a,$

$(6) := \texttt{EqEl}(|G(v)|, \eth F, (5)) \eth_2 g \eth a : |G(v)| = |g(a)| \le \|a\| = \|v\|;$

$\rightsquigarrow (5) := \texttt{ImplyIntro} : v \in A \Rightarrow |G(v)| \le \|v\|,$

$\texttt{Assume}\ O : v \notin A,$

$(5) := \eth (s, a) O : s \neq 0,$

$(6) := \texttt{EqEl}(|G(v)|, \eth F,)\eth\texttt{AbsVal}(\mathbb{R})(sc + g(a), s)\eth_2\mathcal{L}(A, K)(g)(s^{-1}, a)(4)\left(\frac{a}{s}\right)$

$\eth_2^{-1}\texttt{Norm}(V)(|s|, x + s^{-1}a)\eth^{-1}(a, s) :$

$$: |G(v)| = |sr + g(a)| = |s|\left|r + \frac{g(a)}{s}\right| = |s|\left|r + g\left(\frac{a}{s}\right)\right| \leq |s|\left\|x + \frac{a}{s}\right\| = \|sx + a\| = \|v\|;$$

$\rightsquigarrow (6) := \texttt{ImplyIntro} : v \notin A \Rightarrow |G(v)| \leq \|v\|,$

$(7) := \texttt{OrEl}(v \in A | v \notin A)(5, 6) : |G(v)| \leq \|v\|;$

$\rightsquigarrow G := \texttt{FuncIntro} : V^* : \|G\| \leq 1 \;\&\; G_{|A} = g,$

$(5) := \eth_2 G \eth g : \|G\| \geq \|g\| = 1,$

$(6) := \texttt{TwofoldIneq}\eth_1 G(5) : \|g\| = 1,$

$F := \|f\| G : V^* : \|F\| = \|f\| \;\&\; F_{|A} = f,$

$(*) := \eth\texttt{RealHahnBanach}(F) : \texttt{RealHahnBanach};$

$\square$

$$\mathcal{S} := \Bigg(\left\{ (S, \varphi) : \sum \texttt{Subspace}(V) \;.\; S^* : A \subset S : \varphi_{|A} = f \;\&\; \|\varphi\| = \|f\| \right\},$$

$$, \left\{ \big((S, \varphi), (R, \psi)\big) \in \mathcal{S} \times \mathcal{S} : S \subset R : \psi_{|S} = \varphi \right\} \Bigg) : \texttt{Poset},$$

$\texttt{Assume } \mathcal{C} : \texttt{Chain}(\mathcal{S}),$

$$M := \bigcup_{(S,\varphi) \in \mathcal{C}} S : \texttt{Subspace}(V),$$

$\texttt{Assume } x : M,$

$(3) := \eth M : \exists (S, \varphi) \in \mathcal{C} : x \in S,$

$\Phi(x) := \varphi(x) : \mathbb{R};$

$\rightsquigarrow \Phi := \texttt{FuncIntro} : M^*,$

$(4) := \eth M \eth \Phi : (M, \Phi) \in \mathcal{S},$

$\texttt{Assume } (S, \varphi) : \mathcal{C},$

$(5) := \eth M(S) : S \subset M;$

$(6) := \eth \Phi(\varphi) : \Phi_{|S} = \varphi;$

$(7) := \eth \Phi \eth \mathcal{C}(S, \varphi) : \|\Phi\| = \|\varphi\|;$

$\rightsquigarrow (5) := \eth \preceq_{\mathcal{S}} : (S, \varphi) \preceq (M, \Phi),$

$(6) := \eth^{-1}\texttt{Maximal}(\mathcal{S}, \mathcal{C}) : \big((M, \Phi) : \texttt{Maximal}(\mathcal{S}, \mathcal{C})\big);$

$\rightsquigarrow (7) := \texttt{UniversalIntro} : \forall \mathcal{C} : \texttt{Chain}(\mathcal{S}) \;.\; \exists \texttt{Maximal}(\mathcal{S}, \mathcal{C}),$

$(M, \Phi) := \texttt{ZornLemma}(\mathcal{S}, 7) : \texttt{Maximal}(\mathcal{S}),$

$\texttt{Assume } H : M \neq V,$

$x := H \eth \texttt{NonEmpty}\left(M^{\complement}\right) : x \in M^{\complement},$

$W := M + \texttt{span}\{x\} : \texttt{Subspace}(V) : \operatorname{codim}_W M = 1,$

$F := \texttt{HahnBanachLemma}(M, \Phi)(W) : W^* : \|F\| = \|\varphi\| = \|f\| \;\&\; F_{|M} = \varphi,$

$(8) := \eth \prec_{\mathcal{S}} (\eth W, \eth F) : (M, \Phi) \prec (W, F),$

$(9) := \texttt{Absurd}(\eth\texttt{Maximal}(\mathcal{S})(M, \Phi), (8)) : \perp;$

$\rightsquigarrow (8) := \texttt{ByContradiction} : M = V,$

$(*) := \eth\texttt{RealHahnBanach}(\Phi, \eth\mathcal{S}(V, \Phi)) : \texttt{RealHahnBanach};$

$\square$

$\texttt{ComplexHahnBanach} :: \forall V : \mathsf{PRE}(\mathbb{C}) \,.\, \forall A : \texttt{Subspace}(V) \,.\, \forall f \in A^* \,.\, \exists F \in V^* : F_{|A} = f \ \&\ \|F\| = \|f\|$

$\texttt{Proof} \ =$

$g := \Re(f) : A \to_{\mathsf{PRE}(\mathbb{R})} \mathbb{R},$

$G := \texttt{RealHahnBanach}(V, A, g) : V \to_{\mathsf{PRE}(\mathbb{R})} \mathbb{R} : G_{|A} = g \ \&\ \|G\| = \|g\|,$

$F := g - \mathrm{i}g\mathrm{i} : V \to_{\mathsf{PRE}(\mathbb{R})} \mathbb{C},$

$\texttt{Assume}\ x : V,$

$() := \eth F(\mathrm{i}x)\eth\mathcal{L}_{\mathbb{R}}(V,\mathbb{R})(g)(x,-1)\texttt{MultDivide}\mathrm{i}\eth^{-1}F(x) :$

$\quad : F(\mathrm{i}x) = G(\mathrm{i}x) - \mathrm{i}G(-x) = \mathrm{i}G(x) + G(\mathrm{i}x) = \mathrm{i}(G(x) - \mathrm{i}G(\mathrm{i}x)) = \mathrm{i}F(x);$

$\rightsquigarrow () := \texttt{ComplexLinearity} : (F : V \to_{\mathsf{PRE}(\mathbb{C})} \mathbb{C}),$

$\texttt{Assume}\ x : A,$

$() := \eth F(x)\eth_1 G\eth g\eth_2\mathcal{L}(x,\mathrm{i})\eth^{2,-1}(\Re,\Im)(f(x))\eth\Re\eth^{-1}(\Re,\Im)(f(x)) :$

$\quad : F(x) = G(x) - \mathrm{i}G(\mathrm{i}x) = g(x) - \mathrm{i}g(\mathrm{i}x) = \Re f(x) - \mathrm{i}\Re f(\mathrm{i}x) =$

$\quad = \Re f(x) - \mathrm{i}\Re(\mathrm{i}\Re(x) - \Im f(x)) = \Re f(x) + \mathrm{i}\Im f(x) = f(x);$

$\rightsquigarrow (1) := \eth\texttt{domainConctraction}\,\texttt{EqIntro} : F_{|A} = f,$

$\texttt{Assume}\ x : \mathbb{S}_V,$

$y := \dfrac{\overline{F(x)}}{|F(x)|}x : \mathbb{S}_V,$

$(2) := \eth y(F) : F(y) = |F(y)|,$

$(3) := \eth\texttt{absVal}(2) : F(y) \in \mathbb{R},$

$(4) := \eth_2\texttt{AbsValueEqEl}(|F(y)|\eth F(3))\eth\texttt{operatorNorm}\eth_2 G\texttt{NormExtension}(g,f)\texttt{NormExtension}(f,F) :$

$\quad : |F(x)| = |F(y)| = |G(y)| \le \|G\| = \|g\| \le \|f\| \le \|F\|;$

$\rightsquigarrow 2 := \texttt{TwofoldIneq}\eth^{-1}\texttt{operatorNorm} : \|F\| = \|f\|;$

□

$\texttt{StrongHahnBanach} :: \forall V : \mathsf{PRE}(\mathbb{R}) \,.\, \forall A : \texttt{Subspace}(V) \,.\, \forall \rho : \texttt{ConvexFunction}(V) \,.\, \forall f \in A^* : f \le \rho_{|A} \,.$

$\quad .\, \exists F \in V^* : F_{|A} = f \ \&\ F \le \rho$

$\texttt{Proof} \ =$

$\ldots$

□

$\texttt{GenerateFunctional} :: \forall x \in V \,.\, \exists f \in \mathbb{S}_{V^*} : f(x) = \|x\|$

$\texttt{Proof} \ =$

Use Hahn-Banach with $A = \texttt{span}(\{x\}), f(cx) = c\|x\|$

□

$\texttt{SeparatngFunctionals} :: \forall V \in \mathsf{NORM} \,.\, \forall x, y \in V : x \ne y \,.\, \exists f \in V : f(x) \ne f(y)$

$\texttt{Proof} \ =$

If x and y are linearly independant

Use Hahn-Banach with $A = \texttt{span}(\{x,y\}), f(cx + ay) = c\|x\|$

Otherwise use previous construction.

□

$\texttt{FiniteDimensionIsTopologicalyCompletabe} :: \forall V : \mathsf{NORM}(\mathbb{C}) \,.\, \forall A : \texttt{Subspace}(V) : \dim A < \infty \,.$
$\quad A : \texttt{TopologicalyCompletable}(V) : \eth\texttt{TopologicalyCompletable}(V)(A) : \texttt{Closed}(V)$

$\texttt{Proof} \;=$

$\texttt{induction} :: \mathbb{N} \to \texttt{Type}$

$\texttt{induction}\,(n) = \mathcal{I} := \forall V : \mathsf{NORM}(\mathbb{C}) \,.\, \forall A : \texttt{Subspace}(V) : \dim A \leq n \,.$
$\quad A : \texttt{TopologicalyCompletable}(V) : \eth\texttt{TopologicalyCompletable}(V)(A) : \texttt{Closed}(V)$

$\texttt{Assume}\ V : \mathsf{NORM}(\mathbb{C}),$

$\texttt{Assume}\ A : \texttt{Subspace}(V) : \dim A = 1,$

$a := \eth_1 A : A : A = \texttt{span}\{a\},$

$f := \texttt{GenerateFunctional}(a) : V^* : f(a) = \|a\| \neq 0,$

$B := \ker f : \texttt{Subspace}(V),$

$(1){+} := \eth B\texttt{DualsHaveClosedKernel}(f) : \texttt{Proves}(B : \texttt{Closed}(V));$

$(2) := \eth B\texttt{DualDirectSum}(f, \eth a \eth f) : V = A \oplus B;$

$\rightsquigarrow (1) := \eth^{-1}\mathcal{I}(1)\eth^{-1}\texttt{TopologicalyCompletable}(V) : \mathcal{I}(1),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ I : \mathcal{I}(n),$

$\texttt{Assume}\ A : \texttt{Subspace}(V) : \dim A = n + 1,$

$X := \texttt{DimensionalTower}(A) : \texttt{Subspace}(A) : \dim X = n,$

$Y := I(V)(X) : \texttt{Subspace}\ \&\ \texttt{Closed}(V) : V = X \oplus Y,$

$W := A \cap Y : \texttt{Subspace}(V),$

$() := \eth W\texttt{DimIntersection}(A, Y, \eth Y) : \dim W = 1,$

$() := \texttt{IntersectionSubspace}(W, Y, \eth Y) : \texttt{Proves}(W : \texttt{Subspace}(Y)),$

$B := (1)(Y)(W) : \texttt{Subspace}\ \&\ \texttt{Closed}(Y) : Y = W \oplus B,$

$(2) := \texttt{CodimOneDirectSum}(A, X, W) : A = X \oplus W,$

$(){+} := \eth B\eth\texttt{Associative}(V)(\oplus)(2) : V = X \oplus Y = X \oplus (W \oplus B) = (X \oplus W) \oplus B = A \oplus B;$

$() := \texttt{ClosedInClosed}(V, Y, B) : \texttt{Proves}(B : \texttt{Closed}(V));$

$\rightsquigarrow (*) := \texttt{NatInduction}(\mathcal{I}, 1)\texttt{UniIntro}\ \texttt{ImplyIntroNatExtension}(I) :$
$\quad : \forall V : \mathsf{NORM}(\mathbb{C}) \,.\, \forall A : \texttt{Subspace}(V) : \dim A < \infty \,.$
$\quad A : \texttt{TopologicalyCompletable}(V) : \eth\texttt{TopologicalyCompletable}(V)(A) : \texttt{Closed}(V);$

$\square$

1.13 Hyperplanes

$\texttt{Hyperplane} :: ??V$

$D : \texttt{Hyperplane} \iff \exists y \in V : \exists A : \texttt{Subspace}(V) : \operatorname{codim}_V A = 1 : D = \{y + a : a \in A\}$

$\texttt{hyperplane} :: V^* \setminus \{0\} \to K \to \texttt{Hyperplane}$

$\texttt{hyperplane}(f, c) = D_{f,c} := f^{-1}\{c\}$

$A := \ker f : \texttt{Subspace}(V),$

$v := \eth(f \neq 0) : A^{\complement},$

$(1) := \texttt{DimSumThm}(\eth A) : \operatorname{codim}_V A = 1,$

$s := \dfrac{c}{f(v)} : \texttt{In}(K) : f(sv) = c,$

$\texttt{Assume } a : A,$

$(2) := \eth f \eth A(a) \eth s : f(sv + a) = c,$

$() := \eth D_{c,f}(2) : sv + a \in A;$

$\rightsquigarrow (2) := \eth\texttt{Subset} : \{sv + a : a \in A\} \subset D_{c,f},$

$\texttt{Assume } x : D_{c,f},$

$(z, a) := \eth\operatorname{codim}_V(1)(\eth v, x) : K \times A : x = zv + a,$

$(3) := \eth D_{c,f}(x)\eth(z, a) : zf(v) = c,$

$(4) := \eth c(3) : z = s,$

$() := \eth^{-1}(z, a)(4) : x = sv + a;$

$\rightsquigarrow (3) := \eth\texttt{SetEq}(2)\eth\texttt{Subset} : \{sv + a : a \in A\} = D_{c,f},$

$(*) := \eth^{-1}\texttt{Hyperplane}(D_{c,f})(sv, (A, 1), 3) : \texttt{Proves}(D_{c,f} : \texttt{Hyperplane});$

$\square$

$\texttt{SubspaceAsHyperplane} :: \forall f \in V^* \setminus \{0\} \;.\; \forall c \in K \;.\; D_{f,c} : \texttt{Subspace}(V) \iff c = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{HyperplaneRepresentation} :: \forall H : \texttt{Hyperplane} \;.\; \exists f \in V^* \setminus \{0\} : \exists c \in K : H = D_{f,c}$

$\texttt{Proof} \;=$

$(A, v) := \eth\texttt{Hyperplane}(H) : \texttt{Subspace}(V) : \operatorname{codim}_V A = 1 \times V : H = \{v + a | a \in A\},$

$w := \eth\operatorname{codim}_V(A) : \texttt{In}(A^{\complement}),$

$\texttt{Assume } x : \texttt{In}(V),$

$(s, a) := \eth w \eth A(x) : \texttt{In}(K \times A) : x = sw + a,$

$f(x) := s : \texttt{In}(K);$

$\rightsquigarrow f := \texttt{FuncIntro} : V^*,$

$c := f(v) : \texttt{K},$

$\ldots$

$(*) := \ldots : D_{f,c} = H;$

$\texttt{HyperplaneEq} :: \forall f, g \in V^* \setminus \{0\} \, . \, \forall a, b \in K \, . \, D_{f,a} = F_{g,b} \iff \exists s \in K \setminus \{0\} : sf = g \; \& \; sa = b$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{Support} :: ?V \to ?\texttt{Hyperplane}$

$D_{f,c} : \texttt{Support(X)} \iff c = \inf\{f(x) | x \in X\} | c = \sup\{f(x) | x \in X\}$

$\texttt{BallSupport} :: \forall f \in V^* \setminus 0 \, . \, \|f\| = 1 \iff D_{f,1} : \texttt{Support}(\mathbb{B}_V)$

$\texttt{Proof} \; =$

By definition of operator norm.

$\square$

$\texttt{GeometricHahnBanach} :: \forall A : \texttt{Subspace}(V) \, . \, \forall D : \texttt{Support}(\mathbb{B}_A) . \exists H : \texttt{Support}(\mathbb{B}_V) : D \subset H$

$\texttt{Proof} \; =$

Use Hahn-Banach on functional of hyperplane

$\square$

$\texttt{Separating} :: \texttt{Set}(V) \to \texttt{Set}(V) \to ?\texttt{Hyperplane}$

$D_{f,c} : \texttt{Separating}(A, B) \iff \forall a \in A \, . \, \forall b \in B \, . \, f(a) \leq c \leq f(b)$

$\texttt{RelativelyInteriorPoint} :: \prod A : \texttt{Set}(V) \, . \, ?A$

$p : \texttt{RelativelyInteriorPoint} \iff \forall x \in A \, . \, \exists U \in \mathcal{U}(x) : \forall t \in U \, . \, p + tx \in A$

$\texttt{ConvexHaveSeparatingHyperplane} :: \forall A, B : \texttt{Convex}(V) : A \cap B = \emptyset \, .$
$\quad \forall p : \texttt{RelativelyInteriorPoint}(M) \, . \, \exists \texttt{Separating}(A, B)$

$\texttt{Proof} \; =$

$\square$

1.14 Reflexive Duality

$\texttt{evalOperator} :: V \to_{\mathsf{PRE}} V^{**}$

$\texttt{evalOperator}\,(x, f) = \alpha^V_x(f) = f(x) :=$

$\texttt{CanonicalIsometry} :: (\alpha^V : \texttt{Isometry}(V, V^{**}))$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : V,$

$\texttt{Assume}\ f : V^*,$

$(1) := \texttt{EqEl}(\alpha^V_x(f), \eth\alpha^V)\eth\texttt{operatorNorm}(f) : |\alpha^V_x(f)| = |f(x)| \le \|f\|\|x\|;$

$\rightsquigarrow (1) := \eth\texttt{operatorNorm} : \|\alpha^V_x\| \le \|x\|,$

$f := \texttt{GenerateFunctional}(x) : V^* : \|f\| = 1\ \&\ |f(x)| = \|x\|,$

$(2) := \texttt{EqEl}(\alpha^V_x(f), \eth\alpha^V)\eth_2(x) : |\alpha^V_x(f)| = |f(x)| = \|x\|,$

$() := \eth^{-1}\texttt{OperatorNorm}(1,2) : \|\alpha^V_x\| = \|x\|;$

$\rightsquigarrow (*) := \eth\texttt{Isometry}(V, V^{**}) : (\alpha^V : \texttt{Isometry}(V, V^*{*}));$

$\square$

$\texttt{Reflexive} :: ?\texttt{SeminormedSpace}$

$V : \texttt{Reflexive} \iff \alpha^V : V \leftrightarrow_{\mathsf{PRE}_{\circ\to\cdot}} V^{**}$

$\texttt{LpReflexive} :: \forall(\Omega, \mathcal{F}, \mu) : \mathsf{MEAS}\ .\ \forall p \in (1, \infty)\ .\ L_P(\Omega, \mathcal{F}, \mu) : \texttt{Reflexive}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ReflexiveDual} :: \forall V : \texttt{Reflexive}\ .\ V^* : \texttt{Reflexive}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \varphi : V^{***},$

$\texttt{Assume}\ x : V,$

$f(x) := \varphi(a^V_x) : K;$

$\rightsquigarrow f := \texttt{FuncIntro} : V^*,$

$\texttt{Assume}\ \psi : V^{**},$

$x := \eth\texttt{Reflexive} : \alpha^V_x = \psi,$

$() := \eth^{-1}(x)\eth f\eth^{-1}\alpha^V_x\eth^{-1}\alpha^{V^*}_f\eth x : \varphi(\psi) = \varphi(\alpha^V_x) = f(x) = \alpha^V_x(f) = \alpha^{V^*}_f\alpha^V_x = \alpha^{V^*}_f(\psi);$

$\rightsquigarrow () := \texttt{EqIntro}(V^{**} \to K) : \varphi = \alpha^{V^*}_f;$

$(*) := \eth^{-1}\texttt{Reflexive} : V^* : \texttt{Reflexive};$

$\square$

$\texttt{ReflexiveGeometricInterpretation} :: \forall V : \texttt{NormedSpace}\ .\ V : \texttt{Reflexive} \iff$

$\iff \forall f \in V^*\ .\ \exists x \in \mathbb{S}_V : f(x) = \|f\|$

$\texttt{Proof} \;=$

$\dots$

$\square$

2 Compact Operators

2.1 Compactness in a Normed Space

$\mathtt{SuperboundedSum} :: \forall V : \mathsf{NORM}(K) \,.\, \forall A, B : \mathtt{Superbounded}(V) \,.\, A + B : \mathtt{Superbounded}(V)$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ \varepsilon : \mathtt{In}(\mathbb{R}_{++}),$

$(n, a, 1) := \eth\mathtt{Superbounded}(V)(A)(\varepsilon/2) : \sum n \in \mathbb{N} \,.\, a : n \to A \,.\, \forall v \in A \,.\, \exists i \in n \,.\, \|a_i - v\| \le \frac{\varepsilon}{2},$

$(m, b, 1) := \eth\mathtt{Superbounded}(V)(B)(\varepsilon/2) : \sum m \in \mathbb{N} \,.\, b : m \to B \,.\, \forall v \in B \,.\, \exists i \in m \,.\, \|b_i - b\| \le \frac{\varepsilon}{2},$

$z := \Lambda(i, j) \in n \times m \,.\, a_i + b_j : n \times m \to A + B,$

$(3) := \eth^{-1}\mathtt{FiniteFiniteProductCardinality}(n, m) : |n \times m| = nm < \infty,$

$\mathtt{Assume}\ x + y : \mathtt{In}(A + B),$

$(i, 4) := (1)(x) : \sum i \in n \,.\, \|x - a_i\| \le \frac{\varepsilon}{2},$

$(j, 5) := (2)(y) : \sum j \in m \,.\, \|y - b_j\| \le \frac{\varepsilon}{2},$

$() := \eth z\Big(\|x + y - z_{i,j}\|\Big)\eth_1\mathtt{Seminorm}(V)(x - a_i, y - b_i)(4)(5) :$

$\quad : \|x + y - z_{i,j} = \|x + y - a_i - b_j\| \le \|x - a_i\| + \|y - b_j\| \le \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\mathtt{Superbounded}(V)\Big(I(\forall)(\varepsilon)\big(z, (3), I(\forall)(x + y)(i, j)\big)\Big) : \Big(A + B : \mathtt{Superbounded}(V)\Big);$

$\square$

$\mathtt{SuperboundedDelation} :: \forall V : \mathsf{NORM}(K) \,.\, \forall A : \mathtt{Superbounded}(V) \,.\, \forall k \in K \,.\, kA : \mathtt{Superbounded}(V)$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ \varepsilon : \mathtt{In}(\mathbb{R}_{++}),$

$(n, a, 1) := \eth\mathtt{Superbounded}(V)(A)\left(\frac{\varepsilon}{|k|}\right) : \sum n \in \mathbb{N} \,.\, \sum a : n \to A \,.\, \forall v \in A \,.\, \exists i \in n \,.\, \|v - a_i\| \le \frac{\varepsilon}{|k|},$

$\mathtt{Assume}\ kx : \mathtt{In}(kA),$

$(i, 2) := (1)(x) : \sum i \in n \,.\, \|v - a_i\| \le \frac{\varepsilon}{|k|},$

$(3) := \eth_2\mathtt{Seminorm}(k, x - a_i)(2) : \|kx - ka_i\| = |k|\|x - a_i\| \le \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\mathtt{Superbounded}(V)\bigg(I(\forall)(\varepsilon)\Big(I(\exists)(ka)I(\forall)(kx)\big(I(\exists)(i)(3)\big)\Big)\bigg) : \Big(kA : \mathtt{Superbounded}(V)\Big);$

$\square$

$\texttt{AlmostOrthogonalLemma} :: \forall V : \mathsf{NORM}(K) \,.\, \forall H \subsetneq_{\mathsf{NORM}} (V) \,.\, \forall \varepsilon \in \mathbb{R}_{++} \,.\, \exists x \in \mathbb{B}_V : d(x,H) > 1-\varepsilon$

$\texttt{Proof} =$

$(1) := \eth\texttt{Proper}(V)(H) : H^{\complement} \neq \emptyset,$

$(y,2) := \eth\mathsf{VS}\left(\Lambda y \in H^{\complement} \,.\, \dfrac{y}{d(y,H)}\right)\eth\texttt{NonEmpty}(1) : \sum y \in H^{\complement} \,.\, d(y,H) = 1,$

$(\delta,3) := \texttt{LimitMajorization}([0,\infty],(0,1))\left(\Lambda x \in [0,\infty] \,.\, \dfrac{1}{1+x}\right)(1-\epsilon) : \sum \delta \in \mathbb{R}_{++} \,.\, \dfrac{1}{1+\delta} > 1-\epsilon,$

$(z,4) := \eth\texttt{distanceToSet}(y,H)(2)\eth\inf(\delta) : \sum z \in H \,.\, d(z,y) < 1+\delta,$

$X := y - z : \texttt{In}\left(H^{\complement}\right),$

$x := \dfrac{X}{\|x\|} : \texttt{In}(\mathbb{B}_v),$

$:= \eth\texttt{distanceToSet}(x,H)\eth x\eth_2\texttt{Seminorm}(\|y-z\|^{-}1)\eth\inf(\eth\mathsf{VS}(K)(V))\eth^{-1}\texttt{distanceToSet}(y,H)(4)(2) :$

$: d(x,H) = \inf_{h\in H}\|x-h\| = \inf_{h\in H}\left\|\dfrac{y-z}{\|y-z\|} - h\right\| == \left|\dfrac{1}{\|y-z\|}\right| \inf_{h\in H}\|y-h\| > (1-\varepsilon)d(y,H) = (1-\varepsilon);$

$\square$

$\texttt{RiezCompactness} :: \forall V : \mathsf{NORM}(K) \,.\, V : \texttt{LocallyCompact} \iff \dim V < \infty$

$\texttt{Proof} =$

$\texttt{Assume}\ L : \left(V : \texttt{LocallyCompact}\right),$

$(1) := \eth\mathsf{NORM}(L) : \left(\mathbb{B}_V : \texttt{Superbounded}(V)\right),$

$\texttt{Assume}\ d : \dim V = \infty,$

$x_0 := \eth\mathsf{NORM}(K)(V)(1)\eth\texttt{NonTrivial}(d) : \texttt{In}(\mathbb{S}_V),$

$\texttt{Assume}\ n : \mathbb{N},$

$H_n := \operatorname{span}\left(\{x_{i-1} | i \in n\}\right) : \texttt{Subspace}(\mathsf{NORM}(K), V),$

$(x_n, 2) := \texttt{AlmostOrthogonalLemma}(V, H_n, 1/2) : \sum x_n \in \mathbb{B}_V \,.\, d(x_n, H_n) > 1/2,$

$(3) := \eth H_n(2) : \forall i \in n \,.\, d(x_n, x_{i-1}) > 1/2;$

$\rightsquigarrow (x,3) := \texttt{PrimitiveRecursion} : \sum x : \mathbb{N} \to \mathbb{B}_V \,.\, \forall n,m \in \mathbb{N} : n \neq m \,.\, d(x_n, x_m) > 1/2,$

$\rightsquigarrow (2) := \texttt{NoDistantSeq}(1,(x,3)) : \perp;$

$\rightsquigarrow (1) := I(\to)E(\perp)(d) : \left(V : \texttt{LocallyCompact} \Rightarrow \dim V < \infty\right),$

$\ldots$

$\square$

2.2 Compact Operators on Normed Space

$\texttt{CompactOperator} :: \prod V, W : \mathsf{NORM}(K) \,.\, ?\mathcal{L}(V, W)$

$T : \texttt{CompactOperator} \iff T \in \mathcal{K}(V, W) \iff \forall A : \texttt{Bounded}(V) \,.\, TA : \texttt{Superbounded}(W)$

$\texttt{CompactAltDefinition} :: \forall V, W : \mathsf{NORM}(K) \,.\, \forall T : \mathcal{L}(V, W) \,.$
$\quad .\, T : \mathcal{K}(V, W) \iff T\mathbb{B}_V : \texttt{Superbounded}(W)$

$\texttt{Proof} =$

...

$\square$

$\texttt{FiniteDimIsCompact} :: \forall V, W : \mathsf{NORM}(K) \,.\, \forall T : \mathcal{B}(V, W) \,.\, \forall d : \dim W < \infty \,.\, T : \mathcal{K}(V, W)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ProjectorIsNotCompact} :: \forall V : \mathsf{NORM}(K) \,.\, \forall T : \texttt{Projector}(V) \,.\, \forall d : \dim \operatorname{Im} T = \infty \,.\, T \,!\, \mathcal{K}(V, W)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompactOperatorsAsSubspace} :: \forall V, W : \mathsf{NORM}(K) \,.\, \mathcal{K}(V, W) \subset_{\mathsf{NORM}} \mathcal{B}(V, W)$

$\texttt{Proof} =$

$\texttt{Assume}\ A, B : \mathcal{K}(V, W),$

$\texttt{Assume}\ H : \texttt{Bounded}(V),$

$(1) := \texttt{SuperboundedSum}(AH, BH) : \big(AH + BH : \texttt{Superbounded}(W)\big),$

$(2) := \eth\texttt{SetSum}(AH, BH)\eth\texttt{mapSum}(A, B)\eth\texttt{SetMap}(A + B)(H) : (A + B)(H) \subset A(H) + B(H),$

$(3) := \texttt{SuperboundedSubset}(1, 2) : (A + H)(B) : \texttt{Superbounded}(W);$

$\rightsquigarrow (1) := I(\forall)\Big(\eth^{-1}\mathcal{K}(V, W)\big(I(\forall)(3)(H)\big)\Big)(A, B) : \forall A, B \in \mathcal{K}(V, W) \,.\, A + B \in \mathcal{K}(V, W),$

$\texttt{Assume}\ T : \mathcal{K}(V, W),$

$\texttt{Assume}\ k : K,$

$\texttt{Assume}\ H : \texttt{Bounded}(V),$

$(2) := \texttt{SuperboundedDelation}(kTH) : kTH : \texttt{Superbounded}(W);$

$\rightsquigarrow (2) := I(\forall)\left(I(\forall)\Big(\eth^{-1}\mathcal{K}(V, W)\big(I(\forall)(2)(H)\big)\Big)(k)\right)(T) : \forall A \in \mathcal{K}(V, W) \,.\, \forall k \in K \,.\, KT \in \mathcal{K}(V, W),$

$(3) := \eth^{-1}\texttt{Subspace}(\mathsf{VS}, \mathcal{B}(V, W))(1, 2) : \mathcal{K}(V, W) \subset_{\mathsf{VS}} \mathcal{B}(V, W),$

$\texttt{Assume}\ A : \mathbb{N} \to \mathcal{K}(V, W),$

$\texttt{Assume}\ (T, 4) : \sum T \in \mathcal{K}(V, W) \,.\, \lim_{n\to\infty} A_n = T,$

$\texttt{Assume}\ \varepsilon : \texttt{In}\big(\mathbb{R}_{++}\big),$

$(N,5) := (4)(\varepsilon/3) : \forall n \in \mathbb{N} : n \geq N \;.\; \|A_n - T\| \leq \varepsilon/3,$
$(m,w,6) := \eth\texttt{Superbounded}(W)(A_N\mathbb{B}_V)(\varepsilon/3) :$
$\quad : \sum m \in \mathbb{N} \;.\; \sum w : m \to A_N\mathbb{B}_V \;.\; \forall y \in A_N\mathbb{B}_V \;.\; \exists i \in m \;.\; \|y - w_i\| < \frac{\varepsilon}{3},$
$(v,7) := \eth\texttt{Image}(A_N\mathbb{B}_V)(v) : \sum v : m \to \mathbb{B}_V \;.\; A_N v = w,$
$\texttt{Assume}\ y : T\mathbb{B}_V,$
$(x,7) := \eth\texttt{Image}(T\mathbb{B}_V)(w) : \sum x \in \mathbb{B}_V \;.\; Tx = y,$
$(i,8) := (7)(6)(A_N x) : \sum i \in m \;.\; \|A_N(x - v_i)\| < \frac{\varepsilon}{3},$
$(9) := \eth^{-1}x\big(y - Tv\big)\eth_1\texttt{Seminorm}(W)(Tx - A_N x, A_N x - A_N v_i, A_N v_i - T v_i)$
$\quad \eth^2\texttt{OperatorNorm}(T - A_N)\eth\texttt{unitBall}(V)(8)(5) :$
$\quad : \|y - Tv_i\| \leq \big\|(T - A_N)x\big\| + \big\|A_N(x - v_i)\big\| + \big\|(T - A_N)(v_i)\big\| \leq 2\|T - A_N\| + \big\|A_N(x - v_i)\big\| < \varepsilon;$
$\rightsquigarrow (5) := \eth^{-1}\texttt{Superbounded}(W) : \Big(T\mathbb{B}_V : \texttt{Superbounded}(W)\Big),$
$(6) := \texttt{CompactAltDefinition}(5) : \Big(T : \mathcal{K}(V,W)\Big);$
$\rightsquigarrow (4) := \texttt{ClosedBySeq} : \Big(\mathcal{K}(V,W) : \texttt{Closed}\big(\mathcal{B}(V,W)\big)\Big),$
$(*) := \eth^{-1}\texttt{Subspace}\big(\mathsf{NORM}, \mathcal{B}(V,W)\big) : \mathcal{K}(V,W) \subset_{\mathsf{NORM}} \mathcal{B}(V,W);$
$\square$

$\texttt{CompactOperatorsAreBanach} :: \forall V \in \mathsf{NORM}(K) \;.\; \forall W \in \mathsf{BAN}(K) \;.\; \mathcal{K}(V,W) : \mathsf{BAN}(K)$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{FiniteRank} :: \prod V, W : \texttt{TopologicalVectorSpace}(K) \;.\; ?\mathcal{B}(V,W)$
$T : \texttt{FiniteRank} \iff \dim \operatorname{Im} T < \infty$

$\texttt{LimitOfFIniteDimIsCompact} :: \forall V, W \in \mathsf{NORM}(K) \;.\; \forall T \in \mathcal{B}(V,W) \;.$
$\quad . \forall A : \mathbb{N} \to \texttt{FiniteDimensional}(V,W) \;.\; \forall L : \lim_{n\to\infty} A_n = T \;.\; T : \mathcal{K}(V, W$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{ProductAsCompact} :: \forall V, W, U \in \mathsf{NORM}(K) . \forall T \in \mathcal{B}(V, W) . \forall S \in \mathcal{B}(W, U) .$
$. \forall a : T \in \mathcal{K}(V, W) | S \in \mathcal{K}(W, U) . ST : \mathcal{K}(V, U)$

$\texttt{Proof} =$

$\texttt{Assume } L : T \in \mathcal{K}(W, U),$

$\texttt{Assume } B : \texttt{Bounded}(V),$

$(1) := \eth\mathcal{K}(V, W)(T)(B) : \Big(TB : \texttt{Superbounded}(W)\Big),$

$\texttt{Assume } O : S = 0,$

$(2) := \texttt{CompactOperatorsAsSubspace}(W, U)\big(\eth\texttt{Subspace}(\mathsf{NORM}(K)), O\big) : S \in \mathcal{K}(W, U);$

$\rightsquigarrow (2) := I(\forall) : \forall O : S = 0 . S \in \mathcal{K}(W, U),$

$\texttt{Assume } O : S \neq 0,$

$(3) := \eth\texttt{OperatorNorm}(O) : \|S\| \neq 0,$

$\texttt{Assume } \varepsilon : \mathbb{R}_{++},$

$(n, w, 4) := \eth\texttt{Superbounded}(W)(TB)\left(\frac{\varepsilon}{\|B\|}\right) :$
$: \sum n \in \mathbb{N} . \sum w : n \to V . \forall x \in TB . \exists i \in n . \|x - w_i\| < \frac{\varepsilon}{\|S\|},$

$\texttt{Assume } y : STB,$

$(x, 5) := \eth\texttt{Image}(STB, S, y) : \sum x \in TB . y = Sx,$

$(i, 6) := (4)(x) : \sum i \in n . \|x - w_i\| < \frac{\varepsilon}{\|S\|},$

$(7) := \eth x\big(\|y - Sw_i\|\big)\eth\texttt{BoundedOperator}(S) : \|y - Sw_i\| = \|Sx - Sw_i\| \leq \|S\|\|x - w_i\| < \varepsilon;$

$\rightsquigarrow (2) := I(\forall)\Bigg(E\Big(\texttt{Choice}(S = 0), 2, I(\forall)\big(\eth^{-1}\mathcal{K}\eth^{-1}\texttt{Superbounded}(U)\big)(O)\Big)(L)\Bigg) :$
$: \forall L : T \in \mathcal{K}(V, W) . ST : \mathcal{K}(V, U),$

$\texttt{Assume } R : S \in \mathcal{K}(W, U),$

$\texttt{Assume } B : \texttt{Bounded}(V),$

$(3) := \eth^{-1}\texttt{Bounded}(W)\eth\mathcal{B}(V, W)(T) : \big(TB : \texttt{Bounded}(W)\big),$

$(4) := \eth\mathcal{K}(W, U)(S)(TB) : \big(STB : \texttt{Superbounded}(U)\big);$

$\rightsquigarrow (5) := E(|)\Big(a, (2)\big(I(\forall)(\eth K)(R)\big)\Big) : ST \in \mathcal{K}(V, U),$

$\square$

$\texttt{ConjugateCompactness} :: \forall V, W \in \mathsf{NORM}(K) . \forall T \in \mathcal{B}(V, W) . T \in \mathcal{K}(V, W) \iff T^* \in \mathcal{K}(W^*, V^*)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ L : T \in \mathcal{K}(V, W),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(n, w, 1) := \eth\texttt{Superbounded}(W)(T\mathbb{B}_V)(\varepsilon/4) :$

$$: \sum n \in \mathbb{N} . \sum w : n \to T\mathbb{B}_V . \forall y \in T\mathbb{B}_V . \exists i \in n . \|y - w_i\| \le \frac{\varepsilon}{4},$$

$S := \Lambda f \in W^* . (f\, w_i)_{i=1}^n : \mathcal{B}(W^*, K^n),$

$(2) := \texttt{FiniteDimIsCompact}(S) : S \in \mathcal{B}(W^*, K^n),$

$(m, v, 3) := \eth\texttt{Superbounded}(K^n)(S\mathbb{B}_{W^*})(\varepsilon/2) :$

$$: \sum m \in \mathbb{N} . \sum v : m \to S\mathbb{B}_{W^*} . \forall y \in S\mathbb{B}_{W^*} . \exists i \in m . \|y - v_i\| \le \frac{\varepsilon}{2},$$

$(f, 4) := \eth\texttt{Image}(S\mathbb{B}_{W^*}) : \sum f : m \to \mathbb{B}_{W^*} . v = Sf,$

$\texttt{Assume}\ y : T^*\mathbb{B}_{W^*},$

$(x, 5) := \eth\texttt{Image}(T^*\mathbb{B}_{W^*})(y) : \sum x \in \mathbb{B}_{W^*} . y = T^*x,$

$(i, 6) := (3)(Sx) : \sum i \in m . \|Sx - v_i\| \le \frac{\varepsilon}{2},$

$() := (5)\big(\|y - T^* f_i\|\big)\eth\texttt{dualOperator}\eth\texttt{OperatorNorm}\min_{j\in m}\eth_1\texttt{Seminorm}((x - f_i)w_j, (x - f_i)\,(Tu - w_j))$

$\quad \texttt{ReplaceSummandByPositiveSum}(\|(x - f_i\, w_j\|, m)\eth\mathcal{B}(x - f_i)\Big(\big\|(x - f_i)\,Tu - w_j\big\|\Big)$

$\quad \eth(\mathbb{B}_{W^*} + \mathbb{B}_{W^*})(x - f_i)\eth^{-1}(S)(1)(6) :$

$$: \|y - T^* f_i\| = \|T^*(x - f_i)\| = \sup_{u\in\mathbb{B}_V} \|(x - f_i)\,Tu\| \le \sup_{u\in\mathbb{B}_V}\min_{j\in n} \|(x - f_i)\,w_j\| + \|(x - f_i)\,(Tu - w_j)\| \le$$

$$\le \sup_{u\in\mathbb{B}_V}\min_{j\in n}\sum_{k=1}^{n} \|(x - f_i)\,w_k\| + \|(x - f_i)\|\|Tu - w_j)\| \le \sup_{u\in\mathbb{B}_V}\min_{j\in n}\|S(x - f_i)\| + 2\|Tu - w_j\| < \varepsilon;$$

$\leadsto (5) := \eth^{-1}\texttt{Superbounded}(V^*) : \Big(T^*\mathbb{B}_{W^*} : \texttt{Superbounded}(W)\Big);$

$(1) := I(\Rightarrow)\,\texttt{CompactAltDef}(5) : T \in \mathcal{K}(V, W) \Rightarrow T^* \in \mathcal{K}(W^*, V^*);$

$\texttt{Assume}\ R : T^* \in \mathcal{K}(W^*, V^*),$

$(2) := (1)(R) : T^{**} \in \mathcal{K}(V^{**}, W^{**}),$

$(3) := \eth\texttt{dualOperator}\ \texttt{SuperboundedSubset}(2) : T = T^{**}_{|V} \in \mathcal{K}(V, W);$

$\leadsto (*) := I(\iff)(1)\big(I(\Rightarrow)\big) : T \in \mathcal{K}(V, W) \iff T^* \in \mathcal{K}(V^*, W^*),$

$\square$

2.3 Approximation Property

$\texttt{ApproximationProperty} :: ?\mathsf{BAN}(K)$

$V : \texttt{ApproximationProperty} \iff \forall W : \mathsf{NORM}(K)\ .\ \texttt{FiniteRank}(W,V) : \texttt{Dense}\big(\mathcal{K}(W,V)\big)$

$\texttt{ApproximationInHilbertSpace} :: \forall H : \mathsf{HIL}(K)\ .\ H : \texttt{ApproximationProperty}$

$\texttt{Proof} =$

$\texttt{Assume}\ W : \mathsf{NORM}(K),$

$\texttt{Assume}\ T : \mathcal{K}(W,H),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(n,v,1) := \eth\texttt{Superbounded}(H)(T\mathbb{B}_W)(\varepsilon/2) :$

$: \sum n \in \mathbb{N}\ .\ \sum v : n \to T\mathbb{B}_V\ .\ \forall y \in T\mathbb{B}_W\ .\ \exists i \in n\ .\ \|y - v_i\| < \frac{\varepsilon}{2},$

$(V,2) := \operatorname{span}\{v_i | i \in n\} : \sum V \subset_{\mathsf{HIL}} H\ .\ \dim V = n,$

$P := \texttt{OrthoprojectorExists}(H,V) : \texttt{Orthoprojector}(H,V),$

$(3) := \eth^{-1}\texttt{FiniteRank}(W,H)(PT)(2) :$

$: \Big(PT : \texttt{FiniteRank}(W,H)\Big),$

$(4) := \eth\texttt{operatorNorm}(T - PT) \min_{i \in n} \eth_1\texttt{Seminorm}(H)(Tw - v_i, v_i - PTw)$

$\eth\texttt{Projector}(H,V)(P)(\eth V)\eth\mathcal{B}(P)\texttt{NormOfOrthoprojector}(1) :$

$: \|T - PT\| = \sup_{w \in \mathbb{B}_W} \|Tw - PTw\| \le \sup_{w \in \mathbb{B}_W} \min_{i \in n} \|Tw - v_i\| + \|v_i - PTw\| \le$

$\le \sup_{w \in \mathbb{B}_W} \min_{i \in n} \|Tw - v_i\| + \|P\|\|v_i - Tw\| = \sup_{w \in \mathbb{B}_W} \min_{i \in n} 2\|Tw - v_i\| < \varepsilon;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{ApproximationProperty} : \big(H : \texttt{ApproximationProperty}\big);$

$\square$

$\texttt{PiProperty} :: ?\mathsf{BAN}(K)$

$V : \texttt{PiProperty} \iff \exists E : \mathbb{N} \to \texttt{Subspace}(V)\ \&\ \texttt{Increasing} : \forall n \in \mathbb{N}\ .\ \dim E_n < \infty\ \&$

$\&\ \exists P : \prod n \in \mathbb{N}\ .\ \texttt{Projector}(V, E_n)\ \&\ \texttt{UniformlyBoundedOperatorFamily}(\mathbb{N}, V, V)\ \&$

$\&\ \bigcup_{n=1}^{\infty} E_n : \texttt{Dense}(V)$

$\texttt{SchauderImpliesPiProperty} :: \forall V \in \mathsf{BAN}(K)\ .\ \forall e : \texttt{Schauder}\ .\ V : \texttt{PiProperty}$

$\texttt{Proof} =$

$\texttt{Assume}\ n : \mathbb{N},$

$(E_n, 1) := \operatorname{span}\{e_i | i \in n\} : \sum E_n \subset_{\mathsf{NORM}} V\ .\ \dim E_n = n,$

$P_n := \Lambda \sum_{i=1}^{\infty} v_i e_i : \prod n \in \mathbb{N}\ .\ \texttt{Projector}(V, E_n);$

$\rightsquigarrow (E, P, 1) := I(\Pi) : \prod n \in \mathbb{N}\ .\ \sum E_n : \texttt{Subspace}(\mathsf{NORM}, V)\ .\ .$

$(1_n, P) : \dim E_n < \infty \times \texttt{Projector}(V, E_n),$

$(2) := \eth E \eth\texttt{Schauder}(V)(e) : \left(\bigcup_{n=1}^{\infty} E_n : \texttt{Dense}(V)\right),$

Assume $v : V$,

Assume $(\infty) : \sup_{n\in\mathbb{N}} \|P_n v\| = \infty$,

$(3) := \texttt{WellOrderedOfflimit}(\infty) : \lim_{n\to\infty} \|P_n v\| = \infty$,

$(4) := \eth\texttt{Shauder}(V)(e)(\eth P)(v)\texttt{NormIsContinuous}(V)\texttt{ContLimit}(\texttt{norm}(V)) :$

$: \|v\| = \left\|\lim_{n\to\infty} P_n v\right\| = \lim_{n\to\infty} \|P_n v\|$,

$(5) := \texttt{FiniteInfinity}(3,3) : \perp$;

$\rightsquigarrow (3) := \eth^{-1}\texttt{PointwiseBoundedOperatorFamily}\big(E(\perp)\big) :$

$: \Big(P : \texttt{PointwiseBoundedOperatorFamily}(\mathbb{N}, V, V)\Big)$,

$(4) := \texttt{BanachSteinhaus}(3) : \Big(B : \texttt{UniformlyBoundedOperatorFamily}(\mathbb{N}, V, V)\Big)$,

$(*) := \eth^{-1}\texttt{PiProperty}(2,4) : (V : \texttt{PiProperty})$;

$\square$

$\texttt{ApproximationByPiProperty} :: \forall V : \texttt{PiProperty} \, . \, V : \texttt{ApproximationProperty}$

Proof $=$

$(P, V) := \eth\texttt{PiProperty}(V) : \dots,$

$(C, 1) := \eth TYPEUniformlyBoundedOperatorFamily(\mathbb{N}, V, V)(P) : \sum C \in \mathbb{R}_{++} \, . \, \forall n \in \mathbb{N}\|P_n\| \le C$,

Assume $W : \mathsf{NORM}(K)$,

Assume $T : \mathcal{K}(W, V)$,

Assume $\varepsilon : \mathbb{R}_{++}$,

$(n, v, 2) := \eth\texttt{Superbounded}(V)(T\mathbb{B}_W)\left(\frac{\varepsilon}{2(1+C)}\right) :$

$: \sum n \in \mathbb{N} \, . \, \sum v : n \to T\mathbb{B}_W \, . \, \forall y \in T\mathbb{B}_W \, . \, \exists i \in n \, . \, \|y - v_i\| < \frac{\varepsilon}{2(1+C)}$,

Assume $i : n$,

$(3) := \eth\texttt{PiProperty}(V)\eth P(v_i) : v_i = \lim_{n\to\infty} P_n v_i$,

$(N_i, 4) := \eth\texttt{Limit}(3)\left(\frac{\varepsilon}{2}\right) : \sum N_i \in \mathbb{N} \, . \, \forall m \in \mathbb{N} \, . \, \forall b : m \ge N_i \, . \, \|P_m v_i - v_i\| < \frac{\varepsilon}{2}$;

$\rightsquigarrow (N_i, 3_i) := I(\prod) : \prod i \in n \, . \, \dots,$

$(M, 4) := M = \max_{i\in n} N_i : \sum M \in \mathbb{N} \, . \, \forall i \in n \, . \, \|v_i - P_M v_i\| \le \frac{\varepsilon}{2}$,

Assume $w : W$,

$(i, 5) := (4)(T_M w) : \sum i \in n \, . \, \|T_M w - v_i\| \le \varepsilon$,

$(6) := \eth_1\texttt{Seminorm}(V)(Tw - P_M Tw - v_i + P_M v_i, v_i - P_M v_i)\eth\mathcal{B}(W, V)(I - P_M)(Tw - v_i)\eth$
$\texttt{Seminorm}(\mathcal{B}(V))(I, P_M)(1)(4)(i, M)(5) :$

$: \|Tw - P_M Tw\| \le \|(I - P_M)(Tw - v_i)\| + \|v_i - P_M v_i\| \le \|I - P_M\|\|Tw - v_i\| + \|v_i - P_M v_i\| < \varepsilon$;

$\rightsquigarrow (*) := \eth^{-1}\texttt{ApproximationProperty} : (V : \texttt{ApproximationProperty})$;

$\square$

2.4 Singular Form

$\texttt{CompactOperatorNormAttained} :: \forall H, G : \mathsf{HIL}(K) \,.\, \forall T : \mathcal{K}(H,G) \,.\, \exists h \in \mathbb{S}_H \,.\, \|Th\| = \|T\|$

$\texttt{Proof} \;=$

$(1) := \eth\mathcal{K}(V,W)(T)(\mathbb{S}_H) : \big(T\mathbb{S}_H : \texttt{Superbounded}(G)\big),$

$(x,2) := \eth\texttt{OperatorNorm}\eth\sup : \sum x : \mathbb{N} \to \mathbb{B}_H \,.\, \lim_{n\to\infty} \|Tx_n\| = \|T\|,$

$(m,3) := \texttt{CompactConvergence}(1, Tx) : \sum m : \texttt{Subsequencer} \,.\, Tx_m : \texttt{Convergent}(G),$

$y := \lim_{n\to\infty} Tx_{m_n} : G,$

$v := x_{m_n} : \mathbb{N} \to \mathbb{B}_W,$

$\texttt{Assume}\ k, l : \mathbb{N},$

$(4) := \texttt{ParalellogramLaw}(v_l, v_k)\eth\mathbb{B}_H : \|v_l - v_k\|^2 \le 2\|v_l\| + 2\|v_l\| - \|v_l + v_k\| \le 4 - \|v_l + v_k\|;$

$\leadsto (4) := \lim_{l,k\to\infty}(\cdot) : \lim_{l,k\to\infty} \|v_l - v_k\|^2 \le 4 - \lim_{l,k\to\infty} \|v_l + v_k\|,$

$\texttt{Assume}\ k, l : \mathbb{N},$

$(5) := \eth\mathcal{B}(H,G)(T)(v_l + v_k) : \|T(v_l + v_K)\| \le \|T\|\|v_l + v_k\|,$

$(6) := (5)/\|T\| : \|v_l + v_k\| \ge \dfrac{\|T(v_l) + T(v_k)\|}{\|T\|};$

$\leadsto (5) := \lim_{l,k\to\infty}(\cdot)\texttt{NormIsContinuousContLimit}\eth v\eth y^{-1}(2) :$

$$: \lim_{l,k\to\infty} \|v_l + v_k\| \ge \lim_{l,k\to\infty} \frac{\|T(v_l) + T(v_k)\|}{\|T\|} = \frac{\|\lim_{l\to\infty} T(v_l) + \lim_{k\to\infty} T(v_k)\|}{\|T\|} = \frac{\|2y\|}{\|T\|} = 2\frac{\|y\|}{\|T\|} = 2,$$

$(6) := \texttt{ContLimit}(4,5) : \lim_{k,l\to\infty} \|v_k - v_l\| = 0,$

$(7) := \eth^{-1}\texttt{Cauchy}(6) : \big(v : \texttt{Cauchy}(V)\big),$

$(h,8) := \eth\texttt{Complete}(H) : \sum h \in \mathbb{S}_W \,.\, \lim_{n\to\infty} v_n = h,$

$(*) := (8)\big(\|Th\|\big)\texttt{ContLimit}\big(\big\|T(\cdot)\big\|\big)\eth v\texttt{SubLimitAgrees}(2) : \|Th\| = \left\|T \lim_{n\to\infty} v_n\right\| = \lim_{n\to\infty} \|Tv_n\| = \|T\|;$

$\square$

$\texttt{OrthogonalImage} :: \forall H, G : \mathsf{HIL}(K) \,.\, \forall T : \mathcal{K}(H,G) \,.\, \forall h \in \mathbb{S}_H \,.\, \forall E : \|Th\| = \|T\| \,.\, \forall x \in \{h\}^\perp \,.\, Th \perp Tx$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A : \langle Th, Tx\rangle \ne 0,$

$s := |\langle Th, Tx\rangle| \langle Th, Tx\rangle^{-1} : \mathbb{S}_K,$

$(1) := \eth s(\langle Th, Tsx\rangle) : \langle Th, Tsx\rangle = s\langle Th, Tx\rangle > 0,$

$\texttt{Assume}\ t : \mathbb{R}_{++},$

$(2) := \Big(\eth\mathcal{B}(H,G)(T)(h + tsx)\Big)^{-2}\texttt{InnerProductAsNorm}(G) :$

$$: \|T\|^2\|h + tsx\|^2 \ge \|Th + tsTx\| = \|Th\|^2 + 2t\langle Th, Tsx\rangle + t^2\|Tsx\|^2 = \|T\| + 2t\langle Th, Tsx\rangle + t^2\|Tsx\|,$$

$(3) := \texttt{Pythagorus}(h, tsx) : \|T\|^2\|h + tsx\| = \|T\|^2 + t^2\|T\|^2\|sx\|,$

$(4) := (3)(2) - \|T\|^2 : t^2\left(\|T\|^2\|sx\| - \|Tsx\|^2\right) \ge 2t\langle Th, Tsx\rangle;$

$\leadsto (2) := I(\forall) : \forall t \in \mathbb{R}_{++} \,.\, \exists a, b \in \mathbb{R}_{++} \,.\, t^2 a \ge tb,$

$(3) := \eth^{-1}\texttt{InversO}(2) : \dfrac{1}{t} \ne O\left(\dfrac{1}{t^2}\right),$

$(4) := \texttt{QuadraticConverganceFaser}(3) : \bot;$

$\leadsto (*) := \eth^{-1}\texttt{Orthogonal} : Th \perp Tx;$

$\square$

$\texttt{SchmidtTheorem} :: \forall H, G : \mathsf{HIL}(K) \,.\, \forall T : \mathcal{K}(H,G) \,.\, \exists N : \texttt{Range}(\mathbb{N}) : \exists e : \texttt{Orthonormal}(N,H) :$

$\quad : \exists e' : \texttt{Orthonormal}(N,G) : \exists s : N \to \mathbb{R}_{++} \;\&\; \texttt{Nonincreasing} \,.\, T = \sum_{n\in N} s_n e_n \otimes e'_n$

$\texttt{Proof} \;=$

$T_1 := T : \mathcal{K}(H,E),$

$V_1 := H : \mathsf{HIL}(K),$

$(e_1, E_1) := \texttt{CompactOperatorNormAttained}(T_1) : \sum e_1 \in \mathbb{S}_H \,.\, \|T_1 e_1\| = \|T\|,$

$\texttt{Iterate} \quad e_n, E_n, E_n^\perp, e'_n, E'_n, E'^\perp_n, s_n, S_n \quad \texttt{on} \quad n \in \mathbb{N} \quad \texttt{until} \quad T_{n|(Ke_n)^{\perp(V_n)}} = 0$

$e'_n := \dfrac{T_n e_n}{\|T_n\|} : G,$

$E'_n := E_n(\eth e'_n) : \|e'_n\| = \dfrac{\|T_n e_n\|}{\|T\|} = 1,$

$E'^\perp_n := \forall i \in n-1 \,.\, \texttt{OrthogonalImage}\big(H, G, T, e_n, (e_i, E_n^\perp)\big) : \forall i \in n-1 \,.\, e'_i \perp e'_n,$

$s_n := \|T_n e_n\| : \mathbb{R}_{++},$

$S_n := \texttt{NormOfContracted}(T)(\eth s, \eth s_n) : \forall i \in n-1 \,.\, s_n \le s_i,$

$A_n := \eth s_n \eth e_n : s_n e'_n = T e_n,$

$V_{n+1} := \big(Ke_n\big)^{\perp(V_n)} : \mathsf{HIL}(K),$

$T_{n+1} := T_{n|V_{n+1}} : \mathcal{K}(V_{n+1}, G),$

$(e_{n+1}, E_{n+1}) := \texttt{CompactOperatorNormAttained}(T_{n+1}) : \sum e_{n+1} \in \mathbb{S}_{V_{n+1}} \,.\, \|T_{n+1} e_{n+1}\| = \|T_{n+1}\|,$

$E_{n+1}^\perp := \eth\texttt{orthogonalComplement}(\eth V_{n+1}, \eth e) : \forall i \in n \,.\, e_i \perp e_{n+1};$

$\rightsquigarrow (N, e, e', s, 1) := \texttt{PrimitiveRecursion} : \sum N : \texttt{Range}(\mathbb{N}) \,.\, \prod n \in N \,.$

$\quad . \left(\sum (e_n, e'_n, e_n) \in \mathbb{S}_H \times \mathbb{S}_G \times \mathbb{R}_{++} s_n e'_n T_n \;\&\; \forall i \in (n-1) \,.\, s_n \le s_i \;\&\; e_i \perp e_n \;\&\; e'_i \perp e_n\right) \;\&$

$\quad \&\; \ker T = \big(\operatorname{span}\{e_n | n \in N\}\big)^\perp,$

$(2*) := \eth^{-1}\texttt{Orthonormal}(1) : \Big(e : \texttt{Orthonormal}(N,H) \;\&\; e' : \texttt{Orthonormal}(N,G)\Big),$

$(3*) := \eth^{-1}\texttt{NonDeacreasing}(1) : \Big(s : \texttt{NonIncreasing}(N, \mathbb{R}_{++})\Big),$

$\texttt{Assume}\; h : H,$

$(x, v, 4) := \texttt{OrthogonalComplementDecomposion}(h, \operatorname{span}\{e_n | n \in N\}) :$

$\quad : \sum (x, v) \in (N \to K) \times \{e_n | n \in N\}^\perp \,.\, h = v + \sum_{n\in N} x_n e_n,$

$(5) := T(4)(1)\eth\texttt{Orthonormal}(N,H)(e,h)\eth^{-1}\texttt{OneDimensionalOperator}(H,G)(e,e')\eth\texttt{mapSum}(N, se \otimes e') :$

$\quad : Th = Tv + \sum_{n\in N} x_n T e_n = \sum_{n\in N} x_n s_n e'_n \sum_{n\in N} s_n \langle h, e_n\rangle e'_n = \sum_{n\in N} s_n e_n \otimes e'_n(h) = \left(\sum_{n\in N} s_n e_n \otimes e'_n\right) h;$

$\rightsquigarrow (*) := I(=_{H\to G}) : T = \sum_{n\in N} s_n e_n \otimes e'_n;$

$\square$

$\texttt{SingularAmount} :: \forall H, G : \mathsf{HIL}(K) \,.\, \forall T : \mathcal{K}(H,G) \,.\, \forall N : \texttt{Range}(N) : [N, \ldots] = \texttt{SchmidtTheorem}(T) \,.$

$\quad . \; N = \texttt{range}(\operatorname{rank} T)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SingularNumbersUnique} :: \forall H, G : \mathsf{HIL}(K) \,.\, \forall T : \mathcal{K}(H, G) \,.\, \forall N : \texttt{Range}(\mathbb{N}) \,.$

$\quad . \forall e, f : \texttt{Orthonormal}(N, H) \,.\, \forall e', f' : \texttt{Orthonormal}(N, G) \,.\, \forall s, z : N \to \mathbb{R}_{++} \,.$

$\quad . \forall A : [e, e', s] = \texttt{SchmidtTheorem}(H, G, T) \;\&\; [f, f', z] = \texttt{SchmidtTheorem}(H, G, T) \,.\, s = z$

$\texttt{Proof} \;=$

$S_1 := \{s_n | n \in N\} \cup \{z_n | n \in N\} : \texttt{Subset}(\mathbb{R}_{++}),$

$\texttt{Iterate} \quad r_k, E_k, F_k, I_k, J_k \quad \texttt{on} \quad k \in |S_1| \quad \texttt{Until} \quad S_n \neq \emptyset$

$r_k := \sup S_k : \mathbb{R}_+,$

$I_k := \{n \in N : s_i = r_k\} : \texttt{Subset}(N),$

$J_k := \{n \in N : z_i = r_k\} : \texttt{Subset}(N),$

$E_k := \operatorname{span}\{e_n | n \in I_k\} : \texttt{Subspace}(\mathsf{HIL}(K), H),$

$F_k := \operatorname{span}\{f_n | n \in J_k\} : \texttt{Subspace}(\mathsf{HIL}(K), H),$

$\texttt{Assume}\ v : E_k,$

$(x, 1) := \eth E_k(\eth \operatorname{span})(v) : \sum x : I_k \to K \,.\, v = \sum_{n \in I_k} x_n e_n,$

$(2) := (1)\texttt{Pythagorus}(A, xTe)\eth I_k \texttt{HilbertNorm} : \|Tv\|^2 = \left\| \sum_{n \in I_k}^{\infty} x_i T e_n \right\| = \sum_{n \in I_k} r_k^2 |x_n|^2 = r_k^2 \|v\|^2,$

$(3) := \sqrt{(2)} : Tv = r_k \|v\|^2;$

$\rightsquigarrow B_K^E := I(\forall) : \forall v \in E_k \,.\, \|Tv\| = r_k \|v\|,$

$\texttt{Assume}\ v : F_k,$

$(x, 1) := \eth F_k(\eth \operatorname{span})(v) : \sum x : J_k \to K \,.\, v = \sum_{n \in J} x_n f_n,$

$(2) := (1)\texttt{Pythagorus}(A, xTe)\eth J_k \texttt{HilbertNorm} : \|Tv\|^2 \leq \sum_{n \in J_k} \|Tf_n\|^2 |\langle v, f_n \rangle|^2 = \sum_{n \in J_K} r_k^2 |x_n|^2 = r_k^2 \|v\|^2,$

$(3) := \sqrt{(2)} : Tv \leq r_k \|v\|^2;$

$\rightsquigarrow B_K^F := I(\forall) : \forall v \in F_k \,.\, \|Tv\| \leq r_k \|v\|,$

$S_{k+1} := S_k \setminus \{r_k\} : \texttt{Subset}(\mathbb{R}_{++});$

$\rightsquigarrow (\kappa, r, E, F, I, J, 1) := \texttt{PrimitiveRecursion} : \sum \kappa : \texttt{range}(|S_1|) \,.\, \prod k \in \kappa \,.$

$\quad . (r_k, E_k, F_k, I_k, J_k) : \mathbb{R}_{++} \times \texttt{Subspace}^2(\mathsf{HIL}(K), H) \times \texttt{Subset}(N) \,.\, (\forall v \in E_k \,.\, \|Tv\| = \|r_k\| \|v\|) \;\&$

$\quad \& \; (\forall v \in F_k \,.\, \|Tv\| = \|r_k\| \|v\|) \;\&\; \forall i \in I_k \,.\, \|Te_i| = r_k \;\&\; \forall j \in J_k \,.\, \|Tf_j\| = r_k \;\&\; r : \kappa \twoheadrightarrow S_1,$

$\texttt{Assume}\ (h, 2) : \sum v \in V \,.\, \|Tv\| = \|r_1\| \|v\|,$

$(x, v, 3) := \texttt{OrthogonalRepressentation}(H, \{e_n : n \in N\}, h) :$

$\quad : \sum (x, v) : (N \to K) \times \{e_n : n \in N\}^\perp \,.\, h = v + \sum_{n \in N} x_n e_n,$

$(y, v, 4) := \texttt{OrthogonalRepressentation}(H, \{f_n : n \in N\}, h) :$

$\quad : \sum (y, v) : (N \to K) \times \{e_n : n \in N\}^\perp \,.\, h = v + \sum_{n \in N} x_n y_n,$

$(5) := (2)^2(3)\texttt{AddNonNeg}(A, \|v\|^2)\texttt{HilbertNorm} :$

$\quad : r_1^2 \|h\|^2 = \|Th\| = \sum_{n \in N} s_n^2 |x_n|^2 \leq r_1^2 \sum_{n \in N} \|x_i\|^2 + \|v\|^2 = r_1^2 \|h\|^2,$

$(6) := \texttt{DoubleIneq}(5) - \sum_{n \in N} s_n \|x_n\|^2 : 0 = \|v\|^2 + \sum_{n \in I_1^\complement} (r_1^2 - s_n^2) |x_n|^2,$

$(7) := \eth E_1(6) : h \in E_1,$

$(8) := (2)^2(4)\texttt{AddNonNeg}(A, \|v\|^2)\texttt{HilbertNorm}:$

$$: r_1^2\|h\|^2 = \|Th\| = \sum_{n\in N} z_n^2|y_n|^2 \le r_1^2 \sum_{n\in N} \|y_i\|^2 + \|w\|^2 = r_1^2\|h\|^2,$$

$$(9) := \texttt{DoubleIneq}(8) - \sum_{n\in N} z_n\|y_n\|^2 : 0 = \|w\|^2 + \sum_{n\in J_1^{\complement}} (r_1^2 - z_n^2)|y_n|^2,$$

$(10) := \eth F_1(9) : h \in F_1;$

$\rightsquigarrow (2) := \eth E_1 \eth F_1 \eth \texttt{SetEq} : F_1 = E_1,$

$(\bullet) := \texttt{takePoint}(2) : \texttt{TakePoint},$

$V := \{v \in H : \|Tv\| = \|T\|\|v\|\}^\perp : \texttt{Subspace}(\mathsf{HIL}(K), H),$

$(3) := \texttt{recurse}\Big((\bullet)(V, G, T_{|V}), T_V \neq 0\Big) : \forall k \in \kappa \; . \; E_k = F_k,$

$\texttt{Assume}\ k : \kappa,$

$(4) := \dim(3)(k) : \dim E_k = \dim F_k,$

$\texttt{Assume}\ (i, j, 5) : \sum i, j \in I_k \; . \; i \neq j,$

$(6) := \big(\texttt{NormAsMetric}(G)(Te_j, Te_i)\big)^2 \texttt{Pythagoras} \eth I_k :$

$$: d(Te_j, Te_i) = \sqrt{\|Te_j - Te_i\|^2} = \sqrt{\|Te_j\|^2 + \|Te_i\|^2} = \sqrt{2}r_k;$$

$\rightsquigarrow (6) := \eth^{-1}\texttt{Equidistant} : \Big(Te_{I_k} : \texttt{Equidistant}(T\mathbb{S}_H)\Big),$

$(7) := \eth\mathcal{K}(H, G)(\mathbb{S}_H) : \Big(T\mathbb{S}_H : \texttt{Superbounded}(G)\Big),$

$(8) := \texttt{EquidistantIsFinite}(6, 7) : |I_k| < \infty,$

$(9) := \eth E_k \eth I_k(3) \eth F_k \eth J_k(8) : |I_k| = |J_k| < \infty;$

$\rightsquigarrow (4) := I(\forall) : \forall k \in \kappa \; . \; |I_k| = |J_k|,$

$(*) := \eth\texttt{NonIncreasing}(z, s)(4, \eth I, \eth J, A) : z = s;$

$\square$

$\texttt{singularNumbers} :: \prod H, G \in \mathsf{HIL}(K) \; . \; \prod Ti \in \mathcal{K}(H, G) \; . \; \texttt{range}(\operatorname{rank} T) \to \mathbb{R}_{++}$

$\texttt{singularNumbers}\,(T) = \mathbf{s}^T := s$

$\quad \texttt{where} \quad (e, e', s) = \texttt{SchmidtTheorem}(H, G, T)$

$\texttt{CompactIsCompactInHS} :: \forall H, G \in \mathsf{HIL}(K) \;.\; \forall T \in \mathcal{K}(H, G) \;.\; T\mathbb{B}_H : \texttt{Compact}(G)$

$\texttt{Proof} \;=$

$N := \operatorname{rank} T : \texttt{In}(\aleph_1),$

$(e, e', s, 1) := \texttt{SchmidtTheorem} : \sum (e, e', s) :$

$\quad : \texttt{Orthonormal}(N, H) \times \texttt{Orthonormal}(N, G) \times \texttt{Nonincreasing}(N, \mathbb{R}_{++}) \;.$

$$\quad .\; T = \sum_{n \in N} s_n e_n \otimes e'_n,$$

$\texttt{Assume}\; g : \texttt{LimitPoint}(T\overline{\mathbb{B}}_H),$

$(2) := \texttt{DistantSubspace}(g, T\mathbb{B}_H) : g \in \operatorname{span}\{e'_n | n \in N\},$

$$(y, 3) := \eth \operatorname{span}(2) : \sum x : N \to K \;.\; g = \sum_{n \in N} y_n e'_n,$$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$(w, 4) := \eth\texttt{LimitPoint}(T\overline{\mathbb{B}}_H)(g) : \sum w \in T\overline{\mathbb{B}}_H \;.\; \|w - g\| \le \varepsilon,$

$$(x, 5) := \eth\texttt{Image}(1) : \sum x \in \mathbb{B}_{l_2^N} \;.\; w = \sum_{n \in N} x_n s_n e'_n,$$

$$(6) := (4)^2(3)(5)\texttt{Pythagorus} : \varepsilon^2 < \sum_{n \in N} |y_n - s_n x_n|^2,$$

$\texttt{Assume}\; n : N,$

$(7) := \texttt{SummandOfPositiveBoundedSum}(6, n) : |y_n - s_n x_n|^2 < \varepsilon^2,$

$(8) := \sqrt{(7)}\texttt{NormDifference} : \varepsilon \ge |y_n - s_n x_n| \ge -s_n|x_n| + |y_n|,$

$$(9) := s_n^{-1}((8) + s_n|x_n|) : \frac{|y_n|}{s_n} \le |x_n| + \frac{\varepsilon}{s_n};$$

$$\rightsquigarrow (7) := I(\forall) : \forall n \in N \;.\; \frac{|y_n|}{s_n} \le |x_n| + \frac{\varepsilon}{s_n},$$

$\texttt{Assume}\; m : N,$

$$(8) := (7)\left(\sqrt{\sum_{n=1}^{m} \frac{|y_n|^2}{s_n^2}}\right)\texttt{SumOfSqueresIneq}\eth x :$$

$$: \sqrt{\sum_{n=1}^{m} \frac{|y_n|^2}{s_n^2}} \le \sqrt{\sum_{n=1}^{m} \left(|x_n| + \frac{\varepsilon}{s_n}\right)^2} \le \sum_{n=1}^{m} |x_n|^2 + \frac{m\varepsilon^2}{s_n^2} \le 1 + \frac{m\varepsilon^2}{s_n^2};$$

$$\rightsquigarrow (8) := I(\forall) : \forall m \in N \;.\; \sqrt{\sum_{n=1}^{m} \frac{|y_n|^2}{s_n^2}} \le 1 + \frac{m\varepsilon^2}{s_n^2};$$

$$\rightsquigarrow (4) := \lim_{m \to N} \lim_{\varepsilon \to 0}(\cdot) : \sqrt{\sum_{n \in N} \frac{|y_n|^2}{s_n^2}} \le 1,$$

$(5) := \eth\mathbb{B}_H(4)(1) : g \in T\mathbb{B}_H;$

$\rightsquigarrow (2) := \texttt{ClosedByLimits} : \left(T\overline{\mathbb{B}}_H : \texttt{Closed}(G)\right),$

$(3) := \eth\mathcal{K}(H, G)(T)(\mathbb{B}_T) : \left(T\overline{\mathbb{B}}_H : \texttt{Superbounded}(G)\right),$

$(*) := \texttt{SuperBoundedAndClosedIsCompact} : \left(T\overline{\mathbb{B}}_H : \texttt{Compact}(G)\right);$

$\square$

2.5 Hilbert-Schmidt Operators

$$\texttt{hilbertSchmidtOperators} :: \mathsf{HIL}(K) \to \mathsf{HIL}(K) \to \mathsf{NORM}(K)$$
$$\texttt{hilbertSchmidtOperators}\,(V,W) = \mathcal{S}(V,W) :=$$
$$:= \left(\left\{T \in \mathcal{K}(V,W) : \sum_{n \in \operatorname{rank} T} \left(\mathbf{s}_n^T\right)^2 < \infty\right\}, T \mapsto \sum_{n \in \operatorname{rank} T} \left(\mathbf{s}_n^T\right)^2\right)$$

$$\texttt{hilbertSchmidtAltDefs} :: \forall T : \mathcal{K}(H,G) \ .$$
$$(I) \quad T : \mathcal{S}(H,G) \iff$$
$$(II) \quad \forall f : \texttt{Orthonormal} \ \& \ \texttt{Total}(H) \ . \sum_{n=1} \|Tf_n\|^2 < \infty \iff$$
$$(III) \quad \exists f : \texttt{Orthonormal} \ \& \ \texttt{Total}(H) \ . \sum_{n=1} \|Tf_n\|^2 < \infty \iff$$
$$(IV) \quad \exists h : \texttt{Orthonormal} \ \& \ \texttt{Total}(H) : \exists g : \texttt{Orthonormal} \ \& \ \texttt{Total}(G) \ . \sum_{n,m=1} (T_{h,g})^2_{n,m} \iff$$
$$(V) \quad \forall h : \texttt{Orthonormal} \ \& \ \texttt{Total}(H) \ . \ \forall g : \texttt{Orthonormal} \ \& \ \texttt{Total}(G) \ . \sum_{n,m=1} (T_{h,g})^2_{n,m}$$

$\texttt{Proof} \ =$

$r := \operatorname{rank} T : \texttt{Less}(\aleph_1)$,

$d := \dim_{\mathsf{HIL}} H : \texttt{Cardinal}$,

$d' := \dim_{\mathsf{HIL}} H : \texttt{Cardinal}$,

$(e, e', s, 1) := \texttt{SchmidtTheorem}(T) : \sum(e, e', s) :$

$: \texttt{Orthonormal}(r, H) \times \texttt{Orthonormal}(r, G) \times \texttt{Nonincreasing}(r, \mathbb{R}_{++}) \ .$

$$T = \sum_{n \in r} s_n e_n \otimes e'_n,$$

$\texttt{Assume} \ f : \texttt{Orthonormal} \ \& \ \texttt{Total}(H)$,

$\texttt{Assume} \ m : d$,

$\texttt{Assume} \ n : r$,

$(2) := (1)(\langle Tf_m, e'_n\rangle)\eth\forall k \in r \ . \ \texttt{OneDimensionalOperator}(e_k, e'_k)\eth\texttt{Orthonormal}(r, G)(e') :$

$$: \langle Tf_m, e'_n\rangle = \left\langle \sum_{k \in r} s_n e_k \otimes e'_k(f_m), e'_n \right\rangle = \left\langle \sum_{k \in r} s_k \langle e_k, f_n\rangle e'_k, e'_n \right\rangle = s_n\langle e_n, f_n\rangle;$$

$\rightsquigarrow (2) := I(\forall) : \forall n \in r \ . \ \langle Tf_m, e'_n\rangle = s_n\langle f_n, e_n\rangle$,

$(3) := \texttt{FurieSeria}(Tf_m, e')\left(\|Tf_m\|^2\right)\texttt{Pythagorus}(2) :$

$$: \|Tf_m\|^2 = \left\|\sum_{n \in r}\langle Tf_m, e'_n\rangle e'_n\right\|^2 = \sum_{n \in r} \|\langle Tf_m, e'_n\rangle e'_n\|^2 = \sum^{n \in r} \left|s_n\langle f_m, e'_n\rangle\right|^2;$$

$$\rightsquigarrow (2) := I^2(\forall) : \forall f : \texttt{Orthonormal} \ \& \ \texttt{Total}(H) \ . \ \forall m \in d \ . \ \|Tf_m\|^2 = \sum_{n \in r} \left|s_n\langle f_m, e'_n\rangle\right|^2,$$

$\texttt{Assume}\ (IT) : (I),$

$\texttt{Assume}\ f : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$(3) := \eth\mathcal{S}(H,G)(T,1)\forall n \in r\ .\ \texttt{MultByUnity}(\|e\|)\texttt{Parceval}(e,f)\texttt{Fubbini}(2)(f) :$

$$: \infty > \sum_{n\in r} s_n^2 = \sum_{n\in r} s_n^2\|e_n\| = \sum_{n\in r} s_n^2 \sum_{m\in d} |\langle e_n, f_m\rangle|^2 = \sum_{m\in d}\sum_{n\in r} s_n^2|\langle e_n, f_m\rangle|^2 = \sum_{m\in d}\|Tf_m\|^2;$$

$\rightsquigarrow (1') := I(\Rightarrow)\eth(II)I(\forall) : (I) \Rightarrow (II),$

$\texttt{Assume}\ (IIIT) : (III),$

$(f,3) := \eth(III)(IIIT) : \sum f : \texttt{Orthonormal}\ \&\ \texttt{Total}(H)\ .\ \sum_{n\in d}\|Tf_n\|^2 < \infty,$

$(4) := (3)(2)(f)\texttt{Fubbini Parceval}(e,f)\eth\texttt{Orthonormal}(e) :$

$$: \infty > \sum_{n\in d}\|TE_n\|^2 = \sum_{m\in d}\sum_{n\in r}\left|s_n\langle f_m, e'_n\rangle\right|^2 = \sum_{n\in r} s_n \sum_{m\in d}\left|\langle f_m, e'_n\rangle\right|^2 = \sum_{n\in r} s_n^2\|e_n\|^2 = \sum_{n\in r}^{\infty} s_n^2,$$

$(5) := \eth^{-1}\mathcal{S}(V,W) : \big(T \in \mathcal{S}(V,W)\big);$

$\rightsquigarrow (2') := I(\Rightarrow)\eth(I) : (III) \to (I),$

$\texttt{Assume}\ (IIT) : (II),$

$\texttt{Assume}\ h : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$\texttt{Assume}\ g : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$(3) := \eth\texttt{matrix}(T,h,g)\texttt{Perceval}(Th,g)(IIT)(h) :$

$$: \sum_{i\in d'}\sum_{j\in d}(T_{h,g})_{i,j}^2 = \sum_{i\in d'}\sum_{j\in d}\left|\langle Th_j, g_i\rangle\right|^2 = \sum_{j\in d}\|Th_j\|^2 < \infty;$$

$\rightsquigarrow (3') := I(\Rightarrow)I^2(\forall) : (II) \Rightarrow (V),$

$(4') := I(\exists)E(\forall)(V,(e,\ldots),(e',\ldots)) : (V) \Rightarrow (IV),$

$(5') :=$ [Inverse Proof Of $(3')$] $: (IV) \Rightarrow (III),$

$(*) := \texttt{circle}(1',3',4',4',6') : \texttt{This};$

□

$\texttt{HilbertSchmidtAreSubspace} :: \mathcal{S}(H,G) \subset_{\mathsf{BAN}} \mathcal{K}(H,G)$

$\texttt{Proof}\ =$

$d := \dim_{\mathsf{HIL}} H : \texttt{Cardinal},$

$d' := \dim_{\mathsf{HIL}} H : \texttt{Cardinal},$

$(h,1) := \texttt{HilbertBasisExists}(H) : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$(g,1) := \texttt{HilbertBasisExists}(G) : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$\beta := \Lambda T : \mathcal{S}(H,G)\ .\ \Lambda(i,j) \in d \times d' : (T_{h,g})_{i,j},$

$\beta := \eth\beta\texttt{HilbertSchmidtAltDefs}(H,G) : \beta : \mathcal{S}(H,G) \leftrightarrow_{\mathsf{BAN}} l_2(d \times d'),$

$(2) := \eth\mathcal{S}(H,G) \leftrightarrow_{\mathsf{BAN}} l_2(d\times d') : \mathcal{S}(H,G) \subset_{\mathsf{BAN}} \mathcal{K}(H,G);$

□

$\texttt{HilbertSchmidtAreHilbert} :: \mathcal{S}(H,G) \in \mathsf{HIL}_K$

$\texttt{Proof}\ =$

Same proof is above with β being isomorphism.

□

2.6 Trace Class

$\texttt{traceClass} :: \mathsf{HIL}(K) \to \mathsf{HIL}(K) \to \mathsf{NORM}(K)$

$$\texttt{traceClass}\,(H,G) = \mathcal{N}(H,G) := \left(\left\{T \in \mathcal{K}(H,K) : \sum_{n\in\operatorname{rank}T} \mathbf{s}_n^T < \infty\right\}, T \mapsto \sum_{n\in\operatorname{rank}T} \mathbf{s}_n^T\right)$$

$\texttt{Assume}\ T : \mathcal{N}(H,G),$

$\texttt{Assume}\ k : K,$

$$(1) := \eth\texttt{This}(\|T\|_\mathcal{N})\eth\texttt{singularNumbers}(kT)\eth^{-1}\texttt{This}(\|T\|_\mathcal{N}) : \|kT\|_\mathcal{N} = \sum_{n\in\operatorname{rank}T} |k|\mathbf{s}_n^T = |k|\|T\|_\mathcal{N};$$

$\rightsquigarrow (1) := I^2(\forall) : \forall T \in \mathcal{N}(H,G)\ .\ \forall k \in K\ .\ \|kT\|_\mathcal{N} = |k|\|T\|_\mathcal{N},$

$\texttt{Assume}\ T,S : \mathcal{N}(H,G),$

$(h,2) := \eth\texttt{This}\|t+s\|_N\eth\texttt{singularNumbers}\eth_1\texttt{Seminorm}(G) :$

$: \sum N : \texttt{Range}(\mathbb{N})\ .\ \sum h|h' : \texttt{Orthomormal}(N, H|G)\ .$

$$.\ \|T+S\|_N = \sum_{n\in N}\langle (T+S)h_n, h'_n\rangle \le \sum_{n\in N}\langle Th_n, h'_n\rangle + \langle Sh_n, h'_n\rangle \le\le \sum_{n\in N}|\langle Th_n, h'_n\rangle| + \sum_{n\in N}|\langle Tf_n, f'_n\rangle|,$$

$r := \operatorname{rank}T : \texttt{Less}(\aleph_1),$

$(e,e',s,3) := \texttt{SchmidtTheorem}(T) : \sum(e,e',s) :$

$: \texttt{Orthonormal}(r,H) \times \texttt{Orthonormal}(r,H) \times \texttt{Nonincreasing}(r, \mathbb{R}_{++})\ .$

$$.\ T = \sum_{n\in r} s_n e_n \otimes e'_n,$$

$$(4) := (3)\left(\sum_{n\in N}|\langle Th_n, h'_n\rangle|\right)\eth\texttt{InnerProduct}(G)\eth\texttt{AbsVal}(K)\texttt{Tonneli}(3)$$

$\texttt{CauchySchwartz}(l_2(N))\eth\texttt{unitBall}(l_2(N))\eth^{-1}\texttt{This} ::$

$$\sum_{n\in N}|\langle Th_n, h'_n\rangle| = \sum_{n\in N}\sum_{m\in r}\mathbf{s}_m^T\Big|\langle\langle e_m, h_n\rangle e'_m, h'_m\rangle\Big| \le \sum_{n\in N}\sum_{m\in r}\mathbf{s}_m^T|\langle e_m, h_n\rangle||\langle e'_m, h'_m\rangle| =$$

$$= \sum_{n\in N}\mathbf{s}_m^T\sum_{m\in r}|\langle e_m, h_n\rangle||\langle e'_m, h'_m\rangle| \le \sum_{n\in r}\mathbf{s}_m^T = \|T\|_\mathcal{N},$$

$r' := \operatorname{rank}S : \texttt{Less}(\aleph_1),$

$(f,f',z,5) := \texttt{SchmidtTheorem}(s) : \sum(f,f',z) :$

$: \texttt{Orthonormal}(r',H) \times \texttt{Orthonormal}(r',H) \times \texttt{Nonincreasing}(r', \mathbb{R}_{++})\ .$

$$.\ S = \sum_{n\in r'} z_n f_n \otimes f'_n,$$

$$(6) := (5)\left(\sum_{n\in N}|\langle Sf_n, f'_n\rangle|\right)\eth\texttt{InnerProduct}(G)\eth\texttt{AbsVal}(K)\texttt{Tonneli}(3)$$

$\texttt{CauchySchwartz}(l_2(N))\eth\texttt{unitBall}(l_2(N))\eth^{-1}\texttt{This} ::$

$$\sum_{n\in N}|\langle Th_n, h'_n\rangle| = \sum_{n\in N}\sum_{m\in r'}\mathbf{s}_m^S\Big|\langle\langle f_m, h_n\rangle f'_m, h'_m\rangle\Big| \le \sum_{n\in N}\sum_{m\in r'}\mathbf{s}_m^S|\langle f_m, h_n\rangle||\langle f'_m, h'_m\rangle| =$$

$$= \sum_{n\in N}\mathbf{s}_m^S\sum_{m\in r'}|\langle f_m, h_n\rangle||\langle f'_m, h'_m\rangle| \le \sum_{n\in r'}\mathbf{s}_m^S = \|S\|_\mathcal{N},$$

$(7) := (2)(4,6) : \|T+S\|_\mathcal{N} \le \|T\|_\mathcal{N} + \|S\|_\mathcal{N};$

$\rightsquigarrow (2) := I^2(\forall) : \forall T,S \in \mathcal{N}(H,F)\ .\ \|T+S\|_\mathcal{N} \le \|T\|_\mathcal{N} + \|S\|_\mathcal{N},$

$\texttt{Assume}\ T : \mathcal{N}(V,W),$

$\texttt{Assume}\ o : T \neq 0,$

$(3) := \eth\texttt{singularValue}(o) : \mathbf{s}^T \neq \emptyset,$

$(4) := \eth\texttt{This}\,\texttt{NonNegSum} : \|T\|_{\mathcal{N}} = \sum_{n \in \operatorname{rank} T} \mathbf{s}^T_n > 0;$

$\rightsquigarrow (3) := I^2(\forall) : \forall T \in \mathcal{N}(H,G)\ .\ \forall o : T \neq 0\ .\ \|T\|_N > 0,$

$(4) := \eth^{-1}\mathsf{NORM}(K) : \mathcal{N}(H,G) \in \mathsf{NORM}(K),$

$\square$

$\texttt{ProductIsTraceClass} :: \forall A \in \mathcal{S}(H,G)\ .\ \forall B : \mathcal{S}(G,E)\ .\ AB \in \mathcal{N}(H,E)$

$\texttt{Proof}\ =$

$r := \operatorname{rank} AB : \texttt{Less}(\aleph_1),$

$(e,e',s,1) := \texttt{SchmidtTheorem}(AB) : \sum(e,e',s) :$

$: \texttt{Orthonormal}(r,H) \times \texttt{Orthonormal}(r,E) \times \texttt{Nonincreasing}(r,\mathbb{R}_{++})\ .$

$.\ T = \sum_{n \in r} s_n e_n \otimes e'_n,$

$g := \texttt{HilbertBasisExist}(G) : \texttt{Orthonormal}\ \&\ \texttt{Total}(G),$

$(2) := \eth\|AB\|_{\mathcal{N}}(1)\eth^{-1}\texttt{matrix}\eth\texttt{matrixProduct}\eth^{-1}\texttt{InnerProduct}(l_2(\mathbb{N}\times\mathbb{N})) :$

$: \|AB\|_{\mathcal{N}} = \sum_{n \in r} s_n = \sum_{n \in r} \big((AB)_{e,e'}\big)_{n,n} = \langle A_{e,f}, B_{f,e'}\rangle < \infty,$

$(*) := \eth^{-1}\mathcal{N}(H,E)(2) : AB \in \mathcal{N}(H,E);$

$\square$

$\texttt{TraceClassIsIdeal} :: \mathcal{N}(H) : \texttt{TwoSidedIdeal}\big(\mathcal{B}(H)\big)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ S : \mathcal{N}(H),$

$\texttt{Assume}\ T : \mathcal{B}(H),$

$r := \operatorname{rank} T : \texttt{Less}(\aleph_1),$

$\tau := \operatorname{rank} TS : \texttt{Less}(\aleph_1),$

$\rho := \operatorname{rank} ST : \texttt{Less}(\aleph_1),$

$(e,e',s,1) := \texttt{SchmidtTheorem}(T) : \sum(e,e',s) :$

$: \texttt{Orthonormal}(r,H) \times \texttt{Orthonormal}(r,H) \times \texttt{Nonincreasing}(r,\mathbb{R}_{++})\ .$

$.\ T = \sum_{n \in r} s_n e_n \otimes e'_n,$

$(e,e',s,3) := \texttt{SchmidtTheorem}(TS) : \sum(f,f',z) :$

$: \texttt{Orthonormal}(\tau,H) \times \texttt{Orthonormal}(\tau,H) \times \texttt{Nonincreasing}(\tau,\mathbb{R}_{++})\ .$

$.\ TS = \sum_{n \in \tau} z_n f_n \otimes f'_n,$

$(e,e',s,3) := \texttt{SchmidtTheorem}(ST) : \sum(h,h',z) :$

$: \texttt{Orthonormal}(\rho,H) \times \texttt{Orthonormal}(\rho,H) \times \texttt{Nonincreasing}(\rho,\mathbb{R}_{++})\ .$

$.\ ST = \sum_{n \in r} q_n h_n \otimes h'_n,$

$$(4) := (2)\left(\sum_{n\in\tau} z_n\right)(1)\texttt{AbsoluteDominatesSum}\eth^{-1}\texttt{operatorNorm}(T)\texttt{CauchySchwartz}(l_2(\dim H))\eth\mathbb{B}_{l_2(\dim H)}$$

$$\eth^{-1}\|S\|_{\mathcal{N}} : \sum_{n\in\tau} z_n = \sum_{n\in\tau}\langle STf_n, f'_n\rangle = \sum_{m\in r} s_m \sum_{n\in\tau}\langle Tf_n, e_m\rangle\langle e'_m, f'_n\rangle \leq$$

$$\leq \|T\|\sum_{m\in r} s_m \sum_{n\in\tau}|\langle f_n, e_m\rangle||\langle f'_n, e'_m\rangle| \leq \|T\|\sum_{m\in r} s_m = \|T\|\|S\|_{\mathcal{N}},$$

$$(5) := (3)\left(\sum_{n\in\rho} q_n\right)(1)\texttt{AbsoluteDominatesSum}\eth^{-1}\texttt{operatorNorm}(T)\texttt{CauchySchwartz}(l_2(\dim H))\eth\mathbb{B}_{l_2(\dim H)}$$

$$\eth^{-1}\|S\|_{\mathcal{N}} : \sum_{n\in\rho} q_n = \sum_{n\in\rho}\langle TSh_n, h'_n\rangle = \sum_{m\in r} s_m \sum_{n\in\rho}\langle h_n, e_m\rangle\langle Te'_m, h'_n\rangle \leq$$

$$\leq \|T\|\sum_{m\in r} s_m \sum_{n\in\rho}|\langle h_n, e_m\rangle||\langle h'_n, e'_m\rangle| \leq \|T\|\sum_{m\in r} s_m = \|T\|\|S\|_{\mathcal{N}},$$

$(6) := \eth^{-1}\mathcal{N}(H)(4\ \&\ 5) : ST \in \mathcal{N}(H)\ \&\ TS \in \mathcal{N}(H);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{TwoSidedIdeal} : \mathcal{N}(H,G) : \texttt{TwoSidedIdeal}\big(\mathcal{B}(H,G)\big);$

$\square$

LeftBTCTC $:: \forall H, G, F : \mathsf{HIL}(K)\ .\ \forall S : \mathcal{N}(G,F)\ .\ \forall T : \mathcal{B}(H,G)\ .\ TS : \mathcal{N}(H,F)$

Proof $=$

$\ldots$

$\square$

RightBTCTC $:: \forall H, G, G : \mathsf{HIL}(K)\ .\ \forall S : \mathcal{N}(H,G)\ .\ \forall T : \mathcal{B}(G,H)\ .\ ST : \mathcal{N}(H,F)$

Proof $=$

$\ldots$

$\square$

$\texttt{TraceIsCoordinateFree} :: \forall T : \mathcal{N}(H,H) \,.\, \forall g, f : \texttt{Orthonormal} \,\&\, \texttt{Total}(H) \,.$

$$. \sum_{i \in \dim_{\mathsf{HIL}} H} (T_g)_{i,i} = \sum_{i \in \dim_{\mathsf{HIL}} H} (T_f)_{i,i}$$

$\texttt{Proof} \;=$

$d := \dim_{\mathsf{HIL}} H : \texttt{Cardinal},$

$r := \operatorname{rank} T : \texttt{Less}(\aleph_1),$

$(e, e', s, 1) := \texttt{SchmidtTheorem}(T) : \sum(e, e', s) :$

$\quad : \texttt{Orthonormal}(r, H) \times \texttt{Orthonormal}(r, H) \times \texttt{Nonincreasing}(r, \mathbb{R}_{++}) \,.$

$$\quad . \, T = \sum_{n \in r} s_n e_n \otimes e'_n,$$

$\texttt{Assume } f : \texttt{Orthonormal} \,\&\, \texttt{Total}(H),$

$\texttt{Assume } m : d,$

$(1)* := \eth\texttt{matrix}(T, f, f)(i)(1)\eth\texttt{OneDimensionalOperator}(e, e')\eth\texttt{ScalarProduct}(H) :$

$$\quad : (T_f)_i = \langle T f_i, f_i \rangle = \left\langle \sum_{n \in r} s_n \langle f_m, e_n \rangle e'_n, f_m \right\rangle = \sum_{n \in r} s_n \big\langle e'_n, \langle e_n, f_m \rangle f_m \big\rangle,$$

$$\rightsquigarrow (2) := I(\forall) : \forall m \in d \,.\, (T_f)_i = \sum_{n \in r} s_n \big\langle e'_n, \langle e_n, f_m \rangle f_m \big\rangle \,\&\, (T_g)_i = \sum_{n \in r} s_n \big\langle e'_n, \langle e_n, g_m \rangle g_m \big\rangle,$$

$$(3) := \forall n \in r \,.\, \texttt{FurieSummable}(e_n \,\&\, e'_n)(f) : \forall n \in r \,.\, \Big(\big| \langle e_n, f_m \rangle \big| \Big)_{m \in d} \Big(\big| \langle e'_n, f_m \rangle \big| \Big)_{m \in d} \in l_2(d),$$

$(4) := \forall n \in r \,.\, \eth\texttt{InnerProduct}(l_2(d))(3)\texttt{CouchySchwartz}\eth\texttt{Orthonormal}(e, e') :$

$$\quad : \forall n \in r \,.\, \sum_{m \in d} \big| \langle e_n, f_m \rangle \big| \big| \langle e'_n, f_m \rangle \big| = \left\langle \Big(\big| \langle e_n, f_m \rangle \big| \Big)_{m \in d}, \big| \langle e'_n, f_m \rangle \big| \Big)_{m \in d} \right\rangle \le$$

$$\quad \le \left\| \Big(\big| \langle e_n, f_n \rangle \big| \Big)_{m \in d} \right\| \left\| \Big(\big| \langle e'_n, f_n \rangle \big| \Big)_{m \in d} \right\| = 1,$$

$$(5) := \ldots (4)\eth\mathcal{N}(H,H)(T)(1) : \sum_{n \in r} s_n \sum_{m \in d} \big| \langle e_n, f_m \rangle \big| \big| \langle e'_n, f_m \rangle \big| \le \sum_{n \in r} s_n < \infty,$$

$$(6) := (2) \left(\sum_{m \in d} (T_f)_m \right) \texttt{FubbiniToneli}(5)\eth\texttt{InnerProduct}(H)\forall n \in r \,.\, \texttt{FurieSerias}(e_n, f) :$$

$$\quad : \sum_{m \in d} (T_f)_m = \sum_{m \in d} \sum_{n \in r} s_n \big\langle e'_n, \langle e_n, f_m \rangle f_m \big\rangle = \sum_{n \in r} s_n \left\langle e'_n, \sum_{m \in d} \langle e_n, f_m \rangle f_m \right\rangle = \sum_{n \in r} s_n \langle e'_n, e_n \rangle;$$

$$\rightsquigarrow (2) := I(\forall) : \forall f : \texttt{Orthonormal} \,\&\, \texttt{Total}(H) \,.\, \sum_{m \in d} (T_f)_m = \sum_{n \in r} s_n \langle e_n, e'_n \rangle,$$

$$(3) := (2)(f) : \sum_{m \in d} (T_f)_m = \sum_{n \in r} s_n \langle e_n, e'_n \rangle,$$

$$(4) := (2)(g) : \sum_{m \in d} (T_g)_m = \sum_{n \in r} s_n \langle e_n, e'_n \rangle,$$

$$(*) := (3)(4) : \sum_{m \in d} (T_f)_m = \sum_{m \in d} (T_g)_m;$$

$\square$

$\texttt{trace} :: \prod H : \mathsf{HIL}(K) \,.\, \mathcal{N}(H) \to K$

$\texttt{trace}\,(T) = \operatorname{tr} T := \sum_{i \in \dim_{\mathsf{HIL}} H} (T_f)_i$

$\quad \texttt{where} \quad f = \texttt{HilbertBasisExists}(\mathsf{HIL}(K))$

$\texttt{CommuteInTrace} :: \forall H \in \mathsf{HIL}(K) \,.\, \forall S \in \mathcal{N}(H) \,.\, \forall T \in \mathcal{B}(H) \,.\, \forall \pi : TS \in \mathcal{N}(H) \,.\; \operatorname{tr} TS = \operatorname{tr} ST$

$\texttt{Proof} \;=$

$d := \dim_{\mathsf{HIL}} H : \texttt{Cardinal},$

$r := \operatorname{rank} T : \texttt{Less}(\aleph_1),$

$(e, e', s, 1) := \texttt{SchmidtTheorem}(S) : \sum (e, e', s) :$

$\quad : \texttt{Orthonormal}(r, H) \times \texttt{Orthonormal}(r, H) \times \texttt{Nonincreasing}(r, \mathbb{R}_{++}) \,.$

$\quad .\; S = \sum_{n \in r} s_n e_n \otimes e'_n,$

$\texttt{Assume}\; m : r,$

$(2) := (1)\big(\langle TSe_m, e_m\rangle\big)\eth\texttt{Orthogonal}(e)\eth\texttt{InnerProduct}(H)\eth\texttt{Orthogonal}(e')(1) :$

$$: \langle TSe_m, e_m\rangle = \left\langle \sum_{n \in r} s_n \langle e_m, e_n\rangle Te'_n, e_m \right\rangle = s_m \langle e'_m, e_m\rangle \langle Te'_m, e_m\rangle =$$

$$= \big\langle s_m \langle Te'_m, e_m\rangle e'_m, e'_m \big\rangle = \left\langle \sum_{n \in r} s_n \langle Te'_m, e_m\rangle e'_n, e'_m \right\rangle = \langle STe'_m, e'_m\rangle;$$

$\rightsquigarrow (2) := I(\forall) : \forall m \in r \,.\, \langle TSe_m, e_m\rangle = \langle STe'_m, e'_m\rangle,$

$(E, 3) := \texttt{ExtendToBasis}(e) : \sum E : \texttt{Orthonormal} \;\&\; \texttt{Total}(H) \,.\, \{e_n | n \in r\} \subset \{E_n | n \in d\},$

$(E', 4) := \texttt{ExtendToBasis}(e') : \sum E' : \texttt{Orthonormal} \;\&\; \texttt{Total}(H) \,.\, \{e'_n | n \in r\} \subset \{E'_n | n \in d\},$

$(*) := \eth \operatorname{tr} TS \eth\texttt{matrix}(1)\eth\texttt{Orthonormal}(e)(1)(2)(1)\eth\texttt{Orhonormal}(e')(1)\eth^{-1}\texttt{matrix}\eth^{-1} \operatorname{tr} ST :$

$$: \operatorname{tr} TS = \sum_{m \in d} (TS_E)_{m,m} = \sum_{m \in d} \langle TSE_m, E_m\rangle = \sum_{m \in d} \left\langle \sum_{n \in r} s_n \langle E_m, e_n\rangle Te'_n, E_m \right\rangle =$$

$$= \sum_{m \in r} \left\langle \sum_{n \in r} s_n \langle e_m, e_n\rangle Te'_n, e_m \right\rangle = \sum_{m \in r} \langle TSe_m, e_m\rangle = \sum_{m \in r} \langle STe'_m, e'_m\rangle = \sum_{m \in d} \left\langle \sum_{n \in r} s_n \langle Te'_m, e_n\rangle e'_n, e'_m \right\rangle =$$

$$= \sum_{m \in r} \left\langle \sum_{n \in r} s_n \langle TE'_m, e_n\rangle e'_n, E'_m \right\rangle = \sum_{m \in d} \langle STE'_m, E'_m\rangle = \sum_{m \in d} (ST_{E'})_{m,m} = \operatorname{tr} ST;$$

$\square$

$\texttt{TraceIsContinuous} :: \forall H \in \mathsf{HIL}(K)\ .\ \mathrm{tr} \in \mathcal{N}^*(H)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ T : \mathcal{N}(H),$

$r := \operatorname{rank} T : \texttt{Less}(\aleph_1),$

$(e, e', s, 1) := \texttt{SchmidtTheorem}(T) : \sum(e, e', s) :$

$\quad : \texttt{Orthonormal}(r, H) \times \texttt{Orthonormal}(r, H) \times \texttt{Nonincreasing}(r, \mathbb{R}_{++})\ .$

$\quad .\ T = \sum_{n \in r} s_n e_n \otimes e'_n,$

$(2) := |\eth\, \mathrm{tr}\, T|\, \eth \texttt{Seminorm}(H) \eth \texttt{Orthonormal}(e, e') \eth^{-1} \|T\|_{\mathcal{N}} :$

$\quad : |\, \mathrm{tr}\, T| = \left| \sum_{n \in r} s_n \langle e_n, e'_n \rangle \right| \le \sum_{n \in r} s_n |\langle e_n, e'_n \rangle| \le \sum_{n \in r} s_n = \|T\|_{\mathcal{N}};$

$\rightsquigarrow (*) := \eth^{-1} \mathcal{B}\big(\mathcal{N}(H), K\big) : \mathrm{tr} \in \mathcal{N}^*(H);$

$\square$

$\texttt{TracePropertyIsUnique} :: \forall f \in \mathcal{N}^*(H)\ .\ \Big(\forall A, B \in \mathcal{N}(H)\ .\ f(AB) = f(BA)\Big) \Rightarrow f \in K\, \mathrm{tr}$

$\texttt{Proof} \ =$

$d := \dim H : \texttt{Cardinal},$

$e := \texttt{HibertBasisExists} : \texttt{Orthonormal}\ \&\ \texttt{Total}(H),$

$\texttt{Assume}\ i, j : d,$

$\texttt{Assume}\ o : i \ne j,$

$(1*) := \ldots : f(e_i \otimes e_i) = f\Big((e_i \otimes e_j)(e_j \otimes e_i)\Big) = f\Big((e_j \otimes e_i)(e_i \otimes e_j)\Big) = f(e_j \otimes e_j),$

$(2*) := \ldots : f(e_i \otimes e_j) = f\Big((e_i \otimes e_i)(e_i \otimes e_j)\Big) = f\Big((e_i \otimes e_j)(e_i \otimes e_i)\Big) = f(0) = 0;$

$\rightsquigarrow (1) := I^2\forall : \forall i, j \in d : i \ne j\ .\ f(e_i \otimes e_i) = f(e_j \otimes e_j)\ \&\ f(e_j \otimes e_i) = 0,$

$* := \eth\, \mathrm{tr}\, \texttt{BasisDefinesOperator}(1) : f = f(e_1 \otimes e_1)\, \mathrm{tr};$

$\square$

2.7 Schatten-Von Neuman Theory[*!]

2.8 Fredholm Operators and Index

$\texttt{Fredholm} :: \forall V, W : \mathsf{BAN}(K) \,.\, ?\mathcal{B}(V, W)$

$T : \texttt{Fredholm} \iff T \in \Phi(V, W) \iff \dim\ker T < \infty \;\&\; \operatorname{codim}\operatorname{Im} T < \infty$

$\texttt{index} :: \texttt{Fredholm}(V, W) \to \mathbb{Z}$

$\texttt{index}\,(T) = \operatorname{ind} T := \dim\ker T - \operatorname{codim}\operatorname{Im} T$

$\texttt{ClosedImageTHM} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}(V, W) \,.\, \forall c : \operatorname{codim}\operatorname{Im} T < \infty \,.\, \operatorname{Im} T : \texttt{Closed}(W)$

$\texttt{Proof} \;=$

$(F, 1) := \eth\operatorname{codim}(c) : \sum F \subset_{\mathsf{BAN}} W \,.\, W \cong_{\mathsf{VS}(K)} \operatorname{Im} T \oplus F,$

$V' := \dfrac{V}{\ker T} : \mathsf{BAN}(K),$

$(2) := \eth V'(1) : \operatorname{Im} T \oplus F \cong_{\mathsf{BAN}(K)} V' \oplus F,$

$S := \Lambda(v, f) \in V' \oplus F \,.\, \tilde{T}v + f : \mathcal{L}(V' \oplus F, W),$

$\texttt{Assume}\;(v, f) : V' \oplus F,$

$(3) := \|\eth S(v, f)\|\eth_1\texttt{Seminorm}(W)\eth\texttt{operatorNorm}(T)\texttt{HomogenizeIneqWithMax}\eth^{-1}\texttt{SumNorm} :$

$\quad : \|S(v, f)\| = \left\|\tilde{T}v + f\right\| \le \left\|\tilde{T}v\right\| + \|f\| \le \|T\|\|v\| + \|f\| \le \max\left(\|T\|, 1\right)\left(\|v\| + \|f\|\right) \le$

$\quad \le \max\left(\|T\|, 1\right)\left\|(v, f)\right\|;$

$\rightsquigarrow (3) := \eth\mathcal{B}(V' \oplus F, W) : S \in \mathcal{B}(V' \oplus F, W),$

$(4) := \eth S(1)(2) : \left(S : V' \oplus F \leftrightarrow_{\mathsf{VS}} W\right),$

$(5) := \texttt{InverseMappingTHM}(3)(4) : \left(S : V' \oplus F \leftrightarrow_{\mathsf{VS}} W\right),$

$(*) := (2)(5) : \operatorname{Im} T : \texttt{Closed}(W);$

$\square$

$\texttt{FredholmIsCategory} :: \forall A : \texttt{Fredholm}(V, W) \,.\, \forall B : \texttt{Fredholm}(W, U) \,.\, BA : \texttt{Fredholm}(V, U)$

$\texttt{Proof} \;=$

$F := \ker B \cap \operatorname{Im} A : \texttt{subspace}(\mathsf{vs}, w),$

$(1) := \texttt{intersectionissubset}(\ker B, \eth F) : F \subset \ker B,$

$(2) := \texttt{subsetdimension}(1)\eth\texttt{fredholm}(W, U)(B) : \dim F \le \dim\ker B < \infty,$

$(3) := \texttt{ProductKernel}(BA)\eth\texttt{Fredholm}(V, W)(A)(2) : \dim\ker BA = \dim\ker A + \dim F < \infty,$

$(G, 4) := \eth\operatorname{codim}\eth\texttt{Fredholm}(V, W)(A) : \sum G : \texttt{Subspace}(\mathsf{BAN}, W) \,.\, W = \operatorname{Im} A \oplus G \;\&\; \dim G < \infty,$

$(5) := \texttt{ProductImage}(BA)\eth\texttt{Fredholm}(W, U)(B)(4) : \operatorname{codim} BA \le \operatorname{codim} B + \dim G < \infty,$

$(1) := \eth^{-1}\texttt{Fredholm}(V, U)(3, 5) : \Big(BA : \texttt{Fredholm}(V, W)\Big);$

$\square$

$\texttt{IndexIsHomomorph} :: \forall A : \texttt{Fredholm}(V,W) \ . \ \forall B : \texttt{Fredholm}(W,U) \ . \ \operatorname{ind}(AB) = \operatorname{ind}(A) + \operatorname{ind}(B)$

$\texttt{Proof} \ =$

$F := \ker B \cap \operatorname{Im} A : \texttt{subspace}(\mathsf{vs}, w),$

$(1) := \texttt{intersectionissubset}(\ker B, \eth F) : F \subset \ker B,$

$(2) := \texttt{subsetdimension}(1)\eth\texttt{fredholm}(W,U)(B) : \dim F \le \dim\ker B < \infty,$

$(3) := \texttt{ProductKernel}(BA) : \dim\ker BA = \dim\ker A + \dim F,$

$(G,4) := \eth \operatorname{codim} \eth\texttt{Fredholm}(V,W)(A) : \sum G : \texttt{Subspace}(\mathsf{BAN}, W) \ . \ W = \operatorname{Im} A \oplus G \ \& \ \dim G < \infty,$

$Y := \{y \in G : \exists x \in \operatorname{Im} A \ . \ Bx = By\} : \texttt{Subspace}(\mathsf{BAN}, G),$

$(5)) := \texttt{ProductImage}\eth \operatorname{codim} \eth Y : \operatorname{codim} BA = \operatorname{codim} B + \dim G - \dim Y,$

$\texttt{Assume} \ y : Y,$

$(x,6) := \eth Y(y) : \sum x \in \operatorname{Im} A \ . \ BU = Bx,$

$(7) := \eth\mathcal{L}(W,U)B(y-x)(5) : B(y-x) = By - Bx = 0,$

$(8) := \eth^{-1}\ker(6) : y \in \ker B + \operatorname{Im} A,$

$(9) := \eth G(8) : y \in \ker B,$

$\rightsquigarrow (6) := \eth Y \eth^{-1}\texttt{Subset} : Y = \ker B \cap G,$

$(*) := \eth \operatorname{ind} BA(3)(5)\eth F(6)\texttt{DijointSumDimension}(\ker B)\eth^{-1}\operatorname{ind} :$

$$\begin{aligned}&\operatorname{ind} BA = \dim\ker BA - \operatorname{codim}\operatorname{Im} BA = \dim\ker A + \dim F - \operatorname{codim} B - \operatorname{codim} A + \dim Y = \\ &= \dim\ker A - \operatorname{codim}\operatorname{Im} A + \dim\ker B \cap \operatorname{Im} A + \dim\ker B \cap G - \operatorname{codim}\operatorname{Im} B = \\ &= \dim\ker A - \operatorname{codim}\operatorname{Im} A + \dim\ker B - \operatorname{codim}\operatorname{Im} B = \operatorname{ind} B + \operatorname{ind} A;\end{aligned}$$

$\square$

$\texttt{FredholmTHM} :: \forall T \in \mathcal{K}(V) \ . \ I - T : \texttt{Fredholm}(V,V)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ x : \ker I - T,$

$(1) := \eth\ker(x) : Tx = x;$

$\rightsquigarrow (1) := \eth\texttt{identity}(\ker I - T) : T_{\ker I - T} = I,$

$(2) := \eth\mathcal{K}(V)(1) : \dim\ker I - T < \infty,$

$\texttt{Assume} \ y : \texttt{Converging}(\operatorname{Im} I - T),$

$Y := \lim_{n\to\infty} y_n : V,$

$(x,3) := \eth \operatorname{Im} y : \sum x : \mathbb{N} \to V \ . \ y = x - Tx,$

$(G,4) := \texttt{algebraicComplement}(\ker I - T) : \sum G : \texttt{Subspace}(\mathsf{VS}, V) \ . \ V = G \oplus \ker I - T,$

$P := \texttt{ProjectorAlongFiniteDim}(2,4) : \texttt{Projector}(V,G) \ \& \ \mathcal{B}(V),$

$x' := Px : \mathbb{N} \to V,$

$(5) := (3)\eth P : x' - Tx' = y,$

$X := \{x'_n | n \in \mathbb{N}\} : \texttt{Subset}(V),$

$\texttt{Assume} \ X : \texttt{Unbounded}(V),$

$(m,6) := \eth\texttt{Unbounded} : \sum m : \texttt{Subsequencer} \ . \ \lim_{n\to\infty} \|x_{m_n}\| = \infty,$

$z := \dfrac{x'_m}{\|x'_m\|} : \mathbb{N} \to \mathbb{S}_V,$

$(7) := \eth\mathcal{K}(V)(T)(\mathbb{B}_V) : \Big(T\mathbb{S}_V : \texttt{Superbounded}(V)\Big),$

$(k,8) := \texttt{AlmostCompact} : \sum k : \texttt{Subsequencer} \;.\; Tz_k : \texttt{Convergent}(T\mathbb{S}_V),$

$(9) := \eth\mathcal{B}(V)(I-T)(z)\eth y\eth z\eth x : \lim_{n\to\infty} z_n - Tz_n = 0,$

$Z := \lim_{n\to\infty} Tz_{k_n} : V,$

$(10) := (9)(\eth Z) : \lim_{n\to\infty} z_{k_n} = Z,$

$(11) := (10)\eth x' : Z \in G,$

$(12) := (I-T)(Z)\eth Z(9) : Z \in \ker I - T,$

$(13) := (4)(11)(12) : Z = 0,$

$(14) := (\eth z) : \|Z\| = 1,$

$(15) := \eth_2\texttt{Seminorm}(V)(13)(14) : \perp;$

$\rightsquigarrow (6) := \texttt{Contradiction} : \Big(X : \texttt{Bounded}(V)\Big),$

$(m,7)) := \eth\mathcal{K}(V)(T) : \sum m : \texttt{Subseqer}Tx'_m : \texttt{Convergent}(V),$

$a := \lim_{n\to\infty} Tx'_{m_n} : V,$

$(8) := \eth Y\eth^{-1} : Y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n - Tx_n = -a + \lim_{n\to\infty} x_n,$

$(9) := (8) + a : \lim_{n\to\infty} x_n = Y + a,$

$(10) := \eth a(Y + a - TY - Ta)\eth\mathcal{L}(V)(T)\eth x_n\eth^{-1}Y :$

$: Y + a - TY - Ta = Y - TY + \lim_{n\to\infty} Tx_n - T^2x_n = Y - TY + T(\lim_{n\to\infty} x_n - Tx_n) =$

$= Y - TY + TY = Y,$

$(11) := \eth^{-1}\operatorname{Im} I - T(10) : Y \in \operatorname{Im} I - T;$

$\rightsquigarrow (3) := \operatorname{Im} I - T : \texttt{Closed}(V),$

$V' := \dfrac{V}{\operatorname{Im} I - T} : \mathsf{BAN}(K),$

$(4) := \eth V'(T) : \pi_{V'}T = I,$

$(5) := \eth\mathcal{K}(V')(\pi_{V'}) : \dim V' < \infty,$

$(6) := \eth\operatorname{codim}\eth V' : \operatorname{codim} I - T < \infty,$

$(*) := \eth^{-1}\texttt{Fredholm}(V,V)(2)(6) : (I - T : \texttt{Fredholm}(V,V));$

$\square$

$\texttt{FredholmAlternative} :: \forall T : \mathcal{K}(V) \;.\; I - T : V \hookrightarrow V \iff I - T : V \twoheadrightarrow V$

$\texttt{Proof} \;=$

$\texttt{Assume}\; L : \Big(I - T : V \hookrightarrow V\Big),$

$\texttt{Assume}\; C : \operatorname{codim} \operatorname{Im} I - T > 0,$

$E_0 := V : \texttt{Subspace}(\mathsf{BAN}, V),$

$\texttt{Assume}\; n : \mathbb{N},$

$E_n := (I - T)E_{n-1} :\subset_{\mathsf{BAN}} V,$

$(1) := L(\eth E_n) : E_n \neq \{0\},$

$(2) := C(\eth E_n) : E_{n-1} \subsetneq E_n,$

$(x_n, 3) := \texttt{AlmostOrthogonal}(E_{n-1}, E_n, 1/2) : \sum y_n \in \mathbb{B}_{E_{n-1}} \;.\; d(x_n, E_n) > 1/2,$

$\texttt{Assume}\; m : \texttt{Less}(n),$

$(4) := \Big(\eth(I - T)(x_m - x_n)\Big)(Tx_m - Tx_n) : Tx_m - Tx_n = (I - T)(x_n - x_m) - x_n + x_m,$

$z := (I - T)(x_m - x_n) + x_n : E_m,$

$(5) := \texttt{NormAsMetric}(V)(Tx_m, Tx_n)(4)\eth z \eth^{-1}\texttt{distanceToSet}(3_m) :$

$\quad : d(Tx_m, Tx_n) = \|Tx_n - Tx_n\| = \|x_m - z\| \geq d(x_m, E_m) > 1/2;$

$\rightsquigarrow (4) := I(\forall) : \forall m : \texttt{Less}(n) \;.\; d(Tx_n, Tx_m) > 1/2;$

$\rightsquigarrow (x, 1) := \texttt{PrimitiveRecursion} : \sum x : \mathbb{N} \to \mathbb{B}_V \;.\; Tx : \texttt{Equidistant}(T\mathbb{B}_V),$

$(2) := \texttt{NoEquidistant}(T\mathbb{B}_V, Tx) : \bot;$

$\rightsquigarrow (1) := I(\Rightarrow)\eth^{-1} V \twoheadrightarrow V \eth \operatorname{codim} \texttt{Negation} : I - T : V \hookrightarrow V \Rightarrow I - T : V \twoheadrightarrow V,$

$\texttt{Assume}\; R : \Big(I - T : V \twoheadrightarrow V\Big),$

$\texttt{Assume}\; C : \dim \ker I - T > 0,$

$E_0 := \{0\} : \texttt{Subspapace}(\mathsf{BAN}, V),$

$\texttt{Assume}\; n : \mathbb{N},$

$E_n := \ker(I - T)^n : \texttt{Subspace}(\mathsf{BAN}, V),$

$(2) := R(\eth E_n) C(\eth \ker) : E_{n-1} \subsetneq E_n,$

$(x_n, 3) := \texttt{AlmostOrthogonal}(E_n, E_{n-1}, 1/2) : \sum x_n \in \mathbb{B}_{E_n} \;.\; d(x_n, E_{n-1}) > 1/2,$

$\texttt{Assume}\; m : \texttt{Less}(n),$

$(4) := \ldots : Tx_n - Tx_m = (I - T)(x_m - x_n) + x_m - x_n,$

$z := (I - T)(x_m - x_n) + x_m : E_{n-1},$

$(5) := (4)\eth^{-1}\texttt{distanceToSet}(3_n) :$

$\quad : \|Tx_n - Tx_m\| = \|z - x_n) \geq d(x_n, E_{n-1}) > 1/2;$

$\rightsquigarrow (4) := I(\forall) : \forall m : \texttt{Less}(n) \;.\; d(Tx_n, Tx_m) > 1/2;$

$\rightsquigarrow (x, 1) := \texttt{PrimitiveRecursion} : \sum x : \mathbb{N} \to \mathbb{B}_V \;.\; Tx : \texttt{Equidistant}(T\mathbb{B}_V),$

$(2) := \texttt{NoEquidistant}(T\mathbb{B}_V, Tx) : \bot;$

$\rightsquigarrow (*) := I(\iff)(1)\eth^{-1} V \hookrightarrow V \eth \dim \texttt{Negation} : I - T : V \hookrightarrow V \iff I - T : V \twoheadrightarrow V,$

$\square$

$\texttt{FredholmIndex} :: \forall T : \mathcal{K}(V) \;.\; \operatorname{ind} I - T = 0$

$\texttt{Proof} \;=$

$V' := \texttt{witness}(\operatorname{coker} I - T) : \texttt{Subset}(\mathsf{BAN}, V),$

$\texttt{Assume}\ R : \mathcal{B}(\ker I - T, V'),$

$S := \Lambda x \in V \;.\; \pi_{V'}x - \tilde{T}\pi_{V'}x + [R\pi_{\ker I - T}x] : \mathcal{B}(V, V),$

$A := \Lambda x \in V \;.\; \tilde{T}\pi_{V'}x - R\pi_{\ker I - T}x : \mathcal{K}(V, V),$

$(1) := \eth^{-1}A\eth S : S = I - A,$

$(2) := \eth V'\eth \ker I - T\eth R\eth S : \ker S = \ker R,$

$(3) := \eth^{-1}\texttt{directSum}\eth \ker \eth V'\eth \ker I - T\eth R\eth R\eth S : \operatorname{Im} S \cong \operatorname{Im} I - T \oplus \operatorname{Im} R,$

$\texttt{Assume}\ C : \Big(R : \ker I - T \twoheadrightarrow V' \;\&\; !\; \ker I - T \hookrightarrow V'\Big),$

$(4) := \texttt{LinearKernel}(C)(2) : \ker S \neq 0,$

$(5) := \texttt{LinearKernel}(4) : S\, !\, V \hookrightarrow V,$

$(6) := \eth R(C) : \operatorname{Im} R = V',$

$(7) := (3)(6) : \operatorname{Im} S \cong \operatorname{Im} I - T \oplus V',$

$(8) := \eth V'(7) : S : V \twoheadrightarrow V,$

$(9) := \texttt{FredholAlternative}(1)(5, 8) : \bot;$

$\rightsquigarrow (4)* := \texttt{Negation} : \Big(R\, !\, \ker I - T \twoheadrightarrow V' | R : \ker I - T \hookrightarrow V'\Big),$

$\texttt{Assume}\ C : R : \ker I - T \hookrightarrow V' \;\&\; !\; \ker I - T \twoheadrightarrow V',$

$(5) := \texttt{LinearKernel}^2(C)(2) : S : V \hookrightarrow V,$

$(6) := (3)\eth R(C) : S\, !\, V \twoheadrightarrow V,$

$(7) := \texttt{FredholmAlternative}(1)(5, 6) : \bot;$

$\rightsquigarrow (1) := \texttt{FiniteDimOperatorStructureNegation} : \dim V' = \dim \ker I - T,$

$(*) := \eth^{-1} \operatorname{ind} \eth V'\eth \ker I - T(1) : \operatorname{ind} I - T = 0;$

$\square$

$\texttt{Nikolski} :: \forall V, W \in \mathsf{BAN}(K) \;.\; \forall S : \mathcal{B}(V, W) \;.\; S : \texttt{Fredholm}(V, W) \iff$
$\iff \exists T \in \mathcal{K}(V) : \exists T' \in \mathcal{K}(W) : \exists A \in \mathcal{B}(W, V) : AS = I - T \;\&\; SA = I - T'$

$\texttt{Proof} \;=$

$\texttt{Assume}\ L : \big(S : \texttt{Fredholm}(V, W)\big),$

$E := \eth\texttt{Frdeholm}(V, W)(S)\texttt{FinDimComplement}(\ker S) : \sum E \subset_{\mathsf{BAN}} V \;.\; V = \ker S \oplus E,$

$F := \eth\texttt{Frdeholm}(V, W)(S)\texttt{FinDimComplement}(\operatorname{Im} S) : \sum F \subset_{\mathsf{BAN}} W \;.\; W = \operatorname{Im} S \oplus F,$

$:= \texttt{InverseImageTHM}\Big(S_{|E}^{|\operatorname{Im} S}\Big) : \Big(S_{|E}^{\operatorname{Im} S} : E \twoheadrightarrow_{\mathsf{BAN}(K)} \operatorname{Im} S\Big),$

$(2*) := \eth E\eth F : \pi_E(S_{|E}^{|\operatorname{Im} S})^{-1}\pi_{\operatorname{Im} S}S = I - \pi_{\ker S},$

$(3*) := \eth F\eth E : S\pi_E(S_{|E}^{|\operatorname{Im} S})^{-1}\pi_{\operatorname{Im} S} = I - \pi_F;$

$\rightsquigarrow (1) := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : \texttt{Right},$

$(T, T', A, 2) := \eth R : \sum (T, T', A) : \mathcal{K}(V) \times \mathcal{K}(W) \times \mathcal{B}(W, V)\ .\ AS = I - T\ \&\ SA = I - T',$

$(3) := (2)\texttt{FredholmTHM}(I - T') : \Big(SA : \texttt{Fredholm}(V, V)\Big),$

$(4) := \texttt{ProductImage}\eth\texttt{Fredholm}(V, V)(SA) : \operatorname{codim} \operatorname{Im} S \leq \operatorname{codim} \operatorname{Im} SA < \infty,$

$(5) := (2)\texttt{FredholmTHM}(I - T) : \big(SA : \texttt{Fredholm}(W, W)\big),$

$(6) := \texttt{ProductKernel}\eth\texttt{Fredholm}(W, W)(AS) : \dim \ker S \leq \dim \ker AS < \infty,$

$(8) := \eth^{-1}\texttt{Fredholm}(V, W) : \Big(S : \texttt{Fredholm}(V, W)\Big);$

$\rightsquigarrow (*) := I(\iff)(1) : \texttt{This};$

$\square$

$\texttt{CompactPerturbations} :: \forall S : \texttt{Fredholm}(V, W)\ .\ \forall T : \mathcal{K}(V, W)\ .\ S + T : \texttt{Fredholm}(V, W)\ \&\ \operatorname{ind}(S + T) = \operatorname{ind}$

$\texttt{Proof} =$

$(T, T', A, 1) := \texttt{Nikolski}(S) : \sum (T, T', A) : \mathcal{K}(V) \times \mathcal{K}(W) \times \mathcal{B}(W, V)\ .\ AS = I - T\ \&\ SA = I - T',$

$(2) := (1)\big(A(S + T)\big) : A(S + T) = I - T' + AT,$

$(3) := (1)\big(A(S + T)\big) : (S + T) = I - T + TA,$

$(4) := \texttt{Nikolski}(2)(3) : \Big(S + T : \texttt{Fredholm}(V, W)\Big),$

$(4) := \texttt{Nikolski}(1) : \Big(A : \texttt{Fredholm}(V, W)\Big)\ \&\ \operatorname{ind} A = -\operatorname{ind} S,$

$(5) := \texttt{FredholmIndex}(I - T' + AT)(2)\texttt{IndexHomomorph} : 0 = \operatorname{ind}(A(S + T)) = \operatorname{ind}(A) + \operatorname{ind}(S + T),$

$(*) := \big((5) - \operatorname{ind}(A)\big)(4) : \operatorname{ind} S = \operatorname{ind}(S + T);$

$\square$

$\texttt{FredholmIsomorphism} :: \forall S : \texttt{Fredholm}(V, W)\ .\ \operatorname{ind} S = 0 \iff \exists A : V \leftrightarrow_{\mathsf{BAN}} W : \exists T \in \mathcal{K}(V, W)\ .\ S = A + T$

$\texttt{Proof} =$

$(\Rightarrow)$

There is a map $B : \ker T \leftrightarrow_{\mathsf{BAN}} [\operatorname{coker} T]$,then $A = S + B\pi_{\ker S}$ is an isomorphism.

$(\Leftarrow)$

A is Fredholm and $A^{-1}S = I + AT$.

We know that $0 = \operatorname{ind}(I + AT) = \operatorname{ind}(A^{-1}S) = \operatorname{ind} A^{-1} + \operatorname{ind} S = \operatorname{ind} S$.

$\square$

$\texttt{FredholmAdjoint} :: \forall V, W \in \mathsf{BAN}(K)\ .\ \forall T \in \mathcal{B}(V, W)\ .\ T : \texttt{Fredholm}(V, W) \iff T^* : \texttt{Fredholm}(W^*, V^*)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SmallPerturbations} :: \Phi(V,W) : \texttt{Open}\Big(\mathcal{B}(V,W)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ S : \Phi(V,W),$

$(R,T,T',1) := \texttt{Nikolski}(S) : \sum(R,T,T') : \Phi(W,V) \times \mathcal{K}(V) \times \mathcal{K}(W) \;.\; RS = I - T \ \&\ SR = I - T',$

$\texttt{Assume}\ A : \mathcal{B}(V,W),$

$\texttt{Assume}\ r : \|A\| < \|R\|,$

$(2) := (1)\big(R(S+A)\big) : R(S+A) = I - T + RA,$

$(3) := (1)\big((S+A)R\big) : (S+A)R = I - T' + AR,$

$(4) := \texttt{InvertibleAreOpen}(2)(r) : I + RA : W \leftrightarrow_{\mathsf{BAN}} W,$

$(5) := \texttt{InvertibleAreOpen}(3)(r) : I + AR : V \leftrightarrow_{\mathsf{BAN}} V,$

$(6) := \texttt{Nikolski}(4)(5) : S + A : \Phi(V,W);$

$\rightsquigarrow (1) := \texttt{OpenContainsBall}I^3(\forall) : \Phi(V,W) : \texttt{Open}\Big(\mathcal{B}(V,W)\Big);$

□

$\texttt{IndexIsContinuous} :: \mathrm{ind} : C\Big(\Phi(V,W),\mathbb{Z}\Big)$

$\texttt{Proof} \;=$

By previous theorem $\mathrm{ind}(S+A) = \mathrm{ind}(S)$ for small enaugh A

□

$\texttt{FredholmLayer} :: \mathbb{N} \to ?\Phi(V,W)$

$\texttt{FredholLayer}\,(n) = \Phi_n := \{S \in \Phi(V,W) | \,\mathrm{ind}\, S = n\}$

$\texttt{FredholmHilbertGeometry} :: \forall H,G \in \mathsf{HIL}(K) \;.\; \forall n \in \mathbb{N} \;.\; \Phi_n(H,G) : \texttt{NonEmpty}\ \&\ \texttt{LinearlyConnected}$

$\texttt{Proof} \;=$

$\ldots$

□

2.9 Integral Operators

$\texttt{Assume}\ \Omega : \texttt{Compact}\ \&\ \texttt{Hausdorff}\ \&\ \texttt{Separable},$

$\texttt{Assume}\ \mu : \texttt{BorelMeasure}(\Omega),$

$\texttt{Assume}\ \phi : \mu < \infty,$

$\texttt{BasisOfKernels} :: \forall e : \texttt{Orthonormal}\ \&\ \texttt{Total}\big(L_2(\mu)\big)\ .\ e \otimes e : \texttt{Orthonormal}\ \&\ \texttt{Total}\big(L_2(\mu \times \mu)\big)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ (a, b, 1) : \sum (a, b) \in \Big(\dim L_2(\mu) \times \dim L_2(\mu)\Big)^2\ .\ a \neq b,$

$(n, k, 2) := \eth a : \sum (n, k) \in \dim L_2(\mu)\ .\ a = (n, k),$

$(m, l, 3) := \eth b : \sum (m, l) \in \dim L_2(\mu)\ .\ b = (m, l),$

$(4) := \eth \texttt{InnerProduct}\Big(L_2(\mu \times \mu)\Big)\big(e_n \otimes e_k, e_m \otimes e_k\big)\texttt{Fubbini}(2)(3)(1)\eth\texttt{Orthonormal}(e) :$

$$: \langle e_n \otimes e_k, e_m \otimes e_l \rangle = \int_{\Omega \times \Omega} e_n(x)\bar{e}_m(x)e_l(y)\bar{e}_k(y)\, \mu \times \mu(\mathrm{d}x\mathrm{d}y) =$$

$$= \int_\Omega e_n(x)\bar{e}_m(x)\, \mu(\mathrm{d}x) \int_\Omega e_k(x)\bar{e}_l(x)\, \mu(\mathrm{d}x) = 0;$$

$\leadsto (1) := \eth^{-1}\texttt{Orthonormal}\big(L_2(\mu \times \mu)\big) : \Big(e \otimes e : \texttt{Orthonormal}\big(L_2(\mu \times \mu)\big)\Big),$

$\texttt{Assume}\ (f, 2) : \sum f \in L_2(\mu \times \mu)\ .\ f \perp e \otimes e,$

$\texttt{Assume}\ i : \dim L_2(\mu),$

$\texttt{Assume}\ j : \dim L_2(\mu),$

$(3) := (2)\eth\texttt{InnerProduct}\, L_2(\mu \times \mu)\texttt{Fubbini}\eth^{-1}\texttt{InnerProduct}\, L_2(\mu) :$

$$: 0 = \langle e_i \otimes e_j, f \rangle \int_{\Omega \times \Omega} e_i \otimes e_j \bar{f} \mathrm{d}\mu \times \mu = \int_\Omega e_i \int_\Omega f e_j \mathrm{d}\mu \mathrm{d}\mu = \left\langle e_j, \int f\bar{e}_i \right\rangle;$$

$\leadsto (3) := I(\forall) : \forall j \in \dim L_2(\mu)\ .\ \left\langle e_j, \int f\bar{e}_i \right\rangle,$

$(4) := \texttt{TotalitySign}(3) : \int f\bar{e}_i = 0,$

$(5) := \overline{(4)} : \int \bar{f} e_i = 0;$

$\leadsto (3) := I(\forall) : \forall i \in \dim L_2(\mu)\ .\ \int \bar{f} e_i = 0,$

$(4) := \texttt{BasisDefinesOperator}(3)\eth\texttt{integralOperator} : f = 0;$

$\leadsto (5) := \texttt{TotalitySign} : \Big(e \otimes e : \texttt{Total}\ L_2(\mu \times \mu)\Big);$

$\square$

$\texttt{IntegralsAreCompact} :: \forall K : L_2(\mu \times \mu) \;.\; \int K : \mathcal{K}(L_2(\mu))$

$\texttt{Proof} \;=$

$e := \texttt{HilbertBasisExists}\Big(L_2(\mu)\Big) : \texttt{Orthonormal} \;\&\; \texttt{Total}\; L_2(\mu),$

$(1) := \texttt{BasisOfKernels} : \bigg(e \otimes e : \texttt{Total}\Big(\mathbb{N}, L_2(\mu \times \mu)\Big)\bigg),$

$(u, 2) := \texttt{FurieSeria}(K, e \otimes e) : \sum u : \mathbb{N} \times \mathbb{N} \to \texttt{scalars}\Big(L_2(\mu)\Big) \;.\; K = \sum_{n,m=1}^{\infty} u_{n,m} e_n \otimes e_m,$

$\texttt{Assume}\; n : \mathbb{N},$

$\rightsquigarrow T_n := \sum_{i,j}^{n} \int u_{n.m} e_n \otimes e_m : \texttt{FiniteDimensionalOperator}\Big(L_2(\mu), L_2(\mu)\Big),$

$T_n := I(\to) : \mathbb{N} \to \texttt{FiniteDimensionalOperator}\Big(L_2(\mu), L_2(\mu)\Big),$

$(3) := (2)(\eth T) : \int K = \lim_{n \to \infty} T_n,$

$(*) := \texttt{LimitOfFiniteDimIsCompact}(3) : \int K : \mathcal{K}\Big(L_2(\mu)\Big);$

$\square$

$\texttt{HilbertSchmidtIffIntegral} :: \forall T : \mathcal{B}\big(L_2(\mu)\big) \;.\; T : \mathcal{S}\big(L_2(\mu)\big) \iff T : \texttt{IntegralOperator}(\mu)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; L : T \in \mathcal{S}\big(L_2(\mu)\big),$

$(e, e', s, 1) := \texttt{SchmidtTHM}(T) :$

$: \sum e, e' : \texttt{Orthogonal} \;\&\; \texttt{Total}\Big(L_2(\mu)\Big) \;.\; \sum \mathbb{N} \to \mathbb{R}_{++} \;.\; T = \sum_{n=1}^{\infty} s_n e_n \otimes e'_n,$

$\texttt{Assume}\; f : L_2(\mu),$

$(2) := (1)(Tf)\eth\texttt{InnerProduct}\big(L_2(\mu)\big) : Tf = \sum_{n=1}^{\infty} s_n \langle f, e_n \rangle e'_n = \sum_{n=1}^{\infty} s_n \int_{\Omega} f(y) e'_n \bar{e}_n(y) \mathrm{d}y,$

$(3) := \eth\texttt{Orthonormal}(e)(\ldots)\eth\texttt{InnerProduct}(l_2) :$

$: \sum_{n=1}^{\infty} s_n \|e_n\| |\langle f, \bar{e}_n \rangle| \le \sum_{n=1}^{\infty} s_n |\langle f, e_n \rangle| = \Big\langle s, \big(|\langle f, e_n \rangle|\big)_{n=1}^{\infty} \Big\rangle < \infty,$

$(4) := (2)\texttt{FubbiniTonneli}(3) : Tf = \int_{\Omega} f(y) \sum_{n=1}^{\infty} s_n e'_n \bar{e}_n(y) \mathrm{d}y;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{IntegralOperator} : T : \texttt{IntegralOperator}(\mu);$

$\rightsquigarrow (1) := I(\Rightarrow) : T \in \mathcal{S}\Big(L_2(\mu)\Big) \Rightarrow T : \texttt{IntegralOperator}(\mu),$

$\texttt{Assume}\; R : \Big(T : \texttt{IntegralOperator}(\mu)\Big),$

$(K, 2) := \eth\texttt{IntegralOperatot}(\mu)(T) : \sum K \in L_2(\mu \times \mu) \;.\; T = \int K,$

$e := \texttt{HilbertBasisExists}\big(L_2(\mu)\big) : \texttt{Orthonormala} \;\&\; \texttt{Total}\big(L_2(\mu)\big),$

$(3) := \texttt{BasisOfKernels}(e) : \Big(e \otimes e : \texttt{Orthonormal} \ \& \ \texttt{Total}\big(L_2(\mu \times \mu)\big)\Big),$

$(u, 4) := \texttt{FurieSeria}(3)(K) : \sum u : \mathbb{N} \times \mathbb{N} \to \mathbb{C} \ . \ K = \sum_{n,m=1}^{\infty} u_{n,m} e_n \otimes e_m,$

$a := \langle K, u \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{C},$

$(5) := \eth \texttt{matrixNorm}(T_{e,e}) \texttt{PercevalEqualinty} \eth a : \|T_{e,e}\| = \|K\| = \sum_{n,m=1}^{\infty} |a_{n,m}|^2 < \infty,$

$(5) := \texttt{HilbertSchmidtAltDefs}(5) : T \in \mathcal{S}\Big(L_2(\mu)\Big);$

$\rightsquigarrow (*) := I(\iff)(1) : \texttt{This};$

$\square$

$\texttt{IntegralReprezentation} :: \forall T : \mathcal{S}(H) \ . \ \exists K : L_2[0,1]^2 \ . \ T \cong_{\mathsf{HIL}} \int K$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{WeakIntegralReprezentation} :: \forall T : \mathcal{S}(H, G) \ . \ \exists K : L_2[0,1]^2 \ . \ T \approx_{\mathsf{HIL}} \int K$

$\texttt{Proof} =$

$\ldots$

$\square$

3 Spectral Theory Of Bounded Operators

3.1 Spectres Of Operators

3.2 Hilbert Adjointnes

3.3 Self-Adjoint Oprators

3.4 Hilbert-Schmidt Theorem

3.5 Second Order Integral Equations

3.6 Continuous Functional Calculus

3.7 Positive-Definite Operators

3.8 Borel Functional Calculus

3.9 Spectral Measure

3.10 Spectral Theorem

4 Unbounded Operators[!]