Group Theory

Uncultured Tramp

October 29, 2018

Contents

1 Group Structure

1.1 Definitions And Examples

$\texttt{Group} :: ?\texttt{Monoid}$

$(G, \cdot) : \texttt{Group} \iff \forall g \in G \,.\, g : \texttt{Invertible}(G, \cdot)$

$\texttt{Abelean} :: ?\texttt{Group}$

$(G, +) : \texttt{Abelean} \iff (+) : \texttt{Commutative}$

$\texttt{AutomorphismsIsGroup} :: \forall \mathcal{C} : \texttt{Category} \,.\, \forall O \in \mathcal{C} \,.\, \Big(\mathrm{AUT}(O), \circ\Big) : \texttt{Group}$

$\texttt{Proof} =$

$(1) := \eth\texttt{Category}(\mathcal{C}) : \Big((\circ) : \texttt{Associative}\big(\mathrm{AUT}(O), \mathrm{AUT}(O)\big)\Big),$

$(2) := \eth\mathcal{M}_{\mathcal{C}}(O, O)\eth\mathrm{AUT}(O) : \underset{O}{\mathrm{id}} \in \mathrm{AUT}(O),$

$(3) := \eth\underset{O}{\mathrm{id}} : \Big(\underset{O}{\mathrm{id}} : \texttt{Identity}(\mathrm{AUT}(O), \circ)\Big),$

$(4) := \eth\mathrm{AUT}(O) : \forall f \in \mathrm{AUT}(O) \,.\, f : \texttt{Invertible}\Big(\mathrm{AUT}(O), \circ\Big),$

$(*) := \eth^{-1}\texttt{Group}(1-4) : \Big(\big(\mathrm{AUT}(O), \circ\big) : \texttt{Group}\Big);$

$\square$

$\texttt{IntegerIsAbelean} :: (\mathbb{Z}, +) : \texttt{Abelean}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{directProduct} :: \texttt{Group} \to \texttt{Group} \to \texttt{Group}$

$\texttt{directProduct}\,(G, H) = G \times H := \Big(G \times H, \Lambda\big((a, b), (s, r)\big) \in (G \times H) \times (G \times H) \,.\, (as, br)\Big)$

$\texttt{directFuncProduct} :: \prod X : \texttt{Set} \,.\, (X \to \texttt{Group}) \to \texttt{Group}$

$\texttt{directFuncProduct}\,(G) = \prod_{x \in X} G_x := \Big(\prod x \,.\, G_x, \Lambda f, g : \prod x \,.\, G_x \,.\, \lambda x \in X \,.\, f(x)g(x)\Big)$

$\texttt{Subgroup} :: \prod G : \texttt{Group} \,.\, ??G$

$H : \texttt{Subgroup} \iff H \subset_{\mathsf{GRP}} G \iff (H, \cdot_G) : \texttt{Group}$

$\texttt{TrivialSubgroups} :: \forall G : \texttt{Group} \,.\, \{e_G\}, G : \texttt{Subgroup}(G)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ProperSubgroup} :: \prod G : \texttt{Group} . ?\texttt{Subgroup}(G)$

$H : \texttt{ProperSubgroup} \iff H \subset'_{\textsf{GRP}} G \iff H \neq G$

$\texttt{Nontrivial} :: \prod G : \texttt{Group} . ?\texttt{Subgroup}(G)$

$H : \texttt{Nontrivial} \iff H \subset^*_{\textsf{GRP}} G \iff H \neq \{e_G\}$

$\texttt{SubgroupInrsect} :: \forall X : \texttt{Set} . \forall G : \texttt{Group} . \forall H : X \to \texttt{Subgroup}(G) . \bigcap_{x \in X} H_x \subset_{\textsf{GRP}} G$

$\texttt{Proof} =$

$(1) := \eth H \eth \texttt{Subgroup}(G) : \forall x \in X . e_G \in H_x,$

$(2) := \eth \texttt{intersect}(1) : e_G \in \bigcap_{x \in X} H_x,$

$\texttt{Assume}\ a, b : \texttt{In} \bigcap_{x \in X} H_x,$

$(3) := \eth \texttt{intersect}(H) \eth (a, b) : \forall x \in X . a, b \in H_x,$

$(4) := \eth H \eth \texttt{Subgroup}(G) : \forall x \in X . ab \in H_x,$

$(5) := \eth \texttt{Intersect}(G)(4) : ab \in \bigcap_{x \in X} H_x;$

$\rightsquigarrow (3) := I(\forall) : \forall a, b \in \bigcap_{x \in X} H_x . ab \in \bigcap_{x \in X} H_x,$

$\texttt{Assume}\ a : \texttt{In} \bigcap_{x \in X} H_x,$

$(4) := \eth \texttt{intersect}(H) \eth (a) : \forall x \in X . a \in H_x,$

$(5) := \eth H \eth \texttt{Subgroup}(G) : \forall x \in X . a^{-1} \in H_x,$

$(6) := \eth \texttt{Intersect}(G) : a^{-1} \in \bigcap_{x \in X} H_x;$

$\rightsquigarrow (4) := I(\forall) : \forall a \in \bigcap_{x \in X} H_x . a^{-1} \in \bigcap_{x \in X} H_x,$

$(*) := \eth^{-1} \texttt{Subgroup}(G)(2-4) : \bigcap_{x \in X} H_x \subset_{\textsf{GRP}} G;$

$\square$

$\texttt{Generates} :: \prod G : \texttt{Group} . ??G$

$S : \texttt{Generates} \iff \forall a \in G . \exists n \in \mathbb{N} : \exists s : n \to S \cup S^{-1} . a = \prod_{i=1}^{n} s_i$

$\texttt{Cyclic} :: ?\texttt{Group}$

$G : \texttt{Cyclic} \iff \exists S : \texttt{Generates}(G) : |S| = 1$

$\texttt{generate} :: \prod G : \texttt{Group} . ?G \to \texttt{Subgroup}(G)$

$\texttt{generate}\,(S) = \langle S \rangle := \bigcap \{H \subset_{\textsf{GRP}} G : S \subset H\}$

$\texttt{genCycle} :: \prod G : \texttt{Group} \,.\, G \to \texttt{Subgroup}(G)$

$\texttt{genCycle}\,(g) = \langle g \rangle := \Big(\{g^n : n \in \mathbb{Z}\}, \cdot_G\Big)$

$\texttt{CyclicAbelean} :: \forall G : \texttt{Cyclic} \,.\, G : \texttt{Abelean}$

$\texttt{Proof} \;=$

$(g, 1) := \eth\texttt{Cyclic}(G) : \sum g \in G \,.\, G = \langle g \rangle,$

$\texttt{Assume}\; a, b : G,$

$(n, m, 2) := (1)(a, b) : \sum n, m \in \mathbb{Z} \,.\, a = g^n \,\&\, b = g^m,$

$(*) := (2)(ab)\texttt{PowerProduct}\eth\texttt{Coomutativeadd}(\mathbb{Z})(2)(a)(2)(b) : ab = g^n g^m = g^{n+m} = g^{m+n} = g^m g^n = ba;$

$\leadsto (2) := \eth^{-1}\texttt{Commutative} : \Big((\cdot) : \texttt{Commutative}\Big),$

$(*) := \eth^{-1}\texttt{Abelean} : (G : \texttt{Abelean});$

$\square$

$\texttt{Homomorphism} :: \prod G, H : \texttt{Group} \,.\, ?(G \to H)$

$f : \texttt{Homomorphism} \iff \forall a, b \in G\; f(ab) = f(a)f(b)$

$\texttt{HomoComposition} :: \forall A, B, C : \texttt{Group} \,.\, \forall f : \texttt{Homomorphism}(A, B) \,.\, \forall g : \texttt{Homomorphism}(B, C) \,.$
$\quad g \circ f : \texttt{Homomorphism}(A, C)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y : A,$

$() := \eth\texttt{compose}(g, f)(xy)\eth\texttt{Homomrphism}(f)\eth\texttt{Homorphism}(g)\eth^{-1}\texttt{compose} :$
$\quad : g \circ f(xy) = g(f(xy)) = g(f(x)f(y)) = g(f(x))g(f(y)) = g \circ f(x) g \circ f(y);$

$\leadsto (*) := \eth^{-1}\texttt{Homomorphism} : \Big(g \circ f : \texttt{Homomorphism}(A, C)\Big);$

$\square$

$\texttt{HomoId} :: \forall G : \texttt{Group} \,.\, \underset{G}{\mathrm{id}} : \texttt{Homomotphism}(G, G)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a, b : G,$

$() := \eth\,\mathrm{id}\,ab\eth^{-1}\,\mathrm{id}\,a\eth^{-1}\,\mathrm{id}\,b : \mathrm{id}\,ab = ab = \mathrm{id}\,a\,\mathrm{id}\,b;$

$\leadsto (*) := \eth^{-1}\texttt{Homorphism} : \Big(\mathrm{id} : \texttt{Homomorphism}(G, G)\Big),$

$\square$

$\texttt{GroupsCat} :: \texttt{Category}$

$\texttt{GroupsCat}\,() = \mathsf{GRP} := \Big($
$\quad \mathcal{O}(\mathsf{GRP}) = \texttt{Group},$
$\quad \mathcal{M}_{\mathsf{GRP}}(A, B) = \texttt{Homomorphism}(A, B),$
$\quad fg = g \circ f\Big)$

$\texttt{HomoOnId} :: \forall G, H : \texttt{Group} \,.\, \forall f : G \xrightarrow{\mathsf{GRP}} H \,.\, f(e_G) = e_H$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : G,$

$() := \eth f \eth e_G : f(e_G) f(a) = f(e_G a) = f(a);$

$\rightsquigarrow () := \eth^{-1} \texttt{Identity}(H) \texttt{UniqueId}(H) : f(e_G) = e_H;$

$\square$

$\texttt{HomoOnInv} :: \forall G, H : \texttt{Group} \,.\, \forall f : G \xrightarrow{\mathsf{GRP}} H \,.\, \forall a \in G \,.\, f(a^{-1}) = (f(a))^{-1}$

$\texttt{Proof} \ =$

$(1) := \eth f \eth \texttt{inv}(a) \texttt{HomoOnId} : f(a) f(a^{-1}) = f(aa^{-1}) = f(e_G) = e_H,$

$(*) := \eth \texttt{Inverse}(1) : (f(a))^{-1} = f(a^{-1});$

$\square$

$\texttt{kernel} :: \prod G, H : \texttt{Group} \,.\, \mathcal{M}_{\mathsf{GRP}}(G, H) \to ?G$

$\texttt{kernel}\,(f) = \ker f := f^{-1}(e_H)$

$\texttt{KernelIsGroup} :: \forall G, H : \texttt{Group} \,.\, \forall f : G \xrightarrow{\mathsf{GRP}} H \,.\, \ker f \subset_{\mathsf{GRP}} G$

$\texttt{Proof} \ =$

$(1) := \texttt{HomOId}(f) \eth \ker f : e_G \in \ker f,$

$\texttt{Assume}\ a, b : \ker f,$

$() := \eth f(a, b) \eth (a, b) \eth e_H : f(ab) = f(a) f(b) = e_H e_H = e_H;$

$\rightsquigarrow (2) := I(\forall) : \forall a, b \in \ker f \,.\, ab \in \ker f,$

$\texttt{Assume}\ a : \ker f,$

$() := \eth \texttt{Inverse}(f(a^{-1})) \texttt{HomoOnInv}(a^{-1}) \texttt{InvInv} \eth a \eth e_H :$

$\quad : e_H = f(a^{-1})(f(a^{-1}))^{-1} = f(a^{-1}) f(a) = f(a^{-1}) e_H = f(a^{-1});$

$\rightsquigarrow (3) := I(\forall) : \forall a \in \ker f \,.\, a^{-1} \in \ker f,$

$(*) := \eth^{-1} \texttt{Subgroup}(1, 2, 3) : \ker f \subset_{\mathsf{GRP}} G;$

$\square$

$\texttt{ImageIsGroup} :: \prod G, H : \texttt{Group} \,.\, \forall f : G \xrightarrow{\mathsf{GRP}} H \,.\, \operatorname{Im} f \subset_{\mathsf{GRP}} H$

$\texttt{Proof} \ =$

$(1) := \texttt{HomoOnId} : f(e_G) = e_H \in \operatorname{Im} f,$

$\texttt{Assume}\ y, b : \operatorname{Im} f,$

$(x, a, 2) := \eth \operatorname{Im} f \eth (y, b) : \sum x, a \in G \,.\, y = f(x) \,\&\, b = f(a),$

$() := (2) \eth f \eth^{-1} \operatorname{Im} f : yb = f(x) f(a) = f(xa) \in \operatorname{Im} f;$

$\rightsquigarrow (2) := I(\forall) : \forall a, b \in \operatorname{Im} f \,.\, ab \in \operatorname{Im} f,$

$\texttt{Assume}\ y : \operatorname{Im} f,$

$(x, 3) := \eth \operatorname{Im} f : \sum x \in G \,.\, f(x) = y,$

$() := \texttt{HomoOnInv}(x) : y^{-1} = f(x^{-1}) \in \operatorname{Im} f;$

$\rightsquigarrow (3) := I(\forall) : \forall x \in \operatorname{Im} f \,.\, x^{-1} \in \operatorname{Im} f,$

$(*) := \eth^{-1} \texttt{Subgroup}(1, 2, 3 : \operatorname{Im} f \subset_{\mathsf{GRP}} H;$

$\square$

$\texttt{TrivialKernelTHM} :: \forall G, H : \texttt{Group} \,.\, \forall f : G \xrightarrow{\mathsf{GRP}} H \,.\, \forall (0) : \ker f = \{e_G\} \,.\, f : G \hookrightarrow H$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a, b : G,$

$\texttt{Assume}\ (1) : f(a) = f(b),$

$(2) := \eth \texttt{inv}(f(a))(1)\texttt{HomoInInv}\eth\texttt{Homomorphism}(f) : e_H = f(a)\Big(f(b)\Big)^{-1} = f(a)f(b^{-1}) = f(ab^{-1}),$

$(3) := (0)(2) : ab^{-1} = e_G,$

$() := \texttt{UniqueInverse}(3) : a = b;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Injective} : \Big(f : G \hookrightarrow H\Big);$

$\square$

$\texttt{ProductCondition} :: \forall G : \texttt{Group} \,.\, \forall A, B \subset_{\mathsf{GRP}} G \,.\, \forall (0) : G = AB \,.\, \forall (00) : \forall a \in A \,.\, \forall b \in B \,.\, ab = ba \,.$

$\,.\, \forall (000) : A \cap B = \{e_G\} \,.\, \Lambda(a, b) \in A \times B \,.\, ab : A \times B \xleftrightarrow{\mathsf{GRP}} G$

$\texttt{Proof} \ =$

$f := \Lambda(a, b) \in A \times B \,.\, ab : A \times B \to G,$

$\texttt{Assume}\ (a, b), (c, d) : \texttt{In}(A \times B),$

$() := \eth(a, b)(c, d)\eth f(00)(c, b)\eth^{-1} f : f\Big((a, b)(c, d)\Big) = f(ac, bd) = acbd = abcd = f(a, b)f(c, d);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Homomorphism} : \Big(f : A \times B \xrightarrow{\mathsf{GRP}} G\Big),$

$(2) := (0)(\eth^{-1} f) : (f : A \times B \twoheadrightarrow G),$

$\texttt{Assume}\ (a, b) : A \times B,$

$\texttt{Assume}\ (3) : f(a, b) = ab = e,$

$(4) := \eth\texttt{Inverse}(3) : b = a^{-1},$

$() := \eth\texttt{Subgroup}(G)(A, B)(000)(4) : (a, b) = (e, e);$

$\rightsquigarrow (3) := \eth^{-1} \ker f : \ker f = \{e\},$

$(4) := \texttt{TrivialKernelTHM}(3) : (f : A \times B \hookrightarrow G),$

$(*) := \eth\texttt{Bijection}(1, 2, 4) : \Big(f : A \times B \xleftrightarrow{\mathsf{GRP}} G\Big);$

$\square$

$\texttt{powerMap} :: \prod G : \mathsf{GRP} \,.\, \mathbb{Z} \to G \to G$

$\texttt{powerMap}\,(n, g) = F_n(g) := g^n$

$\texttt{PowerMapHomo} :: \forall G : \texttt{Abelean} \,.\, \forall n \in \mathbb{Z} \,.\, F_n : G \xrightarrow{\mathsf{GRP}} G$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a, b : G,$

$() := \eth F_n(ab)\eth\texttt{Abelean}(G)(a, b)\eth^{-1} F_n : F_n(ab) = (ab)^n = a^n b^n = F_n(a)F_n(b);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Homomorphism} : \Big(F_n : G \xrightarrow{\mathsf{GRP}} G\Big);$

$\square$

$\texttt{TotalGroupMult} :: \forall G \in \mathsf{GRP} \;.\; \forall a, b \in G \;.\; \exists c \in G \;.\; ac = b$

$\texttt{Proof} \;=$

$(*) := \eth\texttt{Inverse}(a)(aa^{-1}b) : aa^{-1}b = b;$

$\square$

$\texttt{index} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Subgroup}(G) \to \mathsf{CARD}$

$\texttt{index}\,(H) = [G : H] := \#\{gH | g \in G\}$

$\texttt{IndexTHM} :: \forall G \in \texttt{Group} \;\&\; \texttt{Finite} \;.\; \forall H \subset_{\mathsf{GRP}} G \;.\; |H|[G : H] = |G|$

$\texttt{Proof} \;=$

$(1) := \texttt{TotalGroupMult}(G) : \bigcup\{gH|g \in G\} = G,$

$(2) := \eth\texttt{Group}(G) : \forall g \in G \;.\; |gH| = |H|,$

$\texttt{Assume}\; a, b : G,$

$\texttt{Assume}\; c : \texttt{In}(aH \cap bH),$

$(h, g, 3) := \eth c : \sum h, g \in H \;.\; c = ah \;\&\; c = bg,$

$\texttt{Assume}\; y : \texttt{In}(aH),$

$(x, 4) := \eth y : \sum x \in H \;.\; ax = y,$

$(5) := (3)\eth\texttt{inv}(h)(4) : bgh^{-1}x = ahh^{-1}x = ax = y,$

$() := \eth(g, h, x)\eth H(5) : y \in bH;$

$\leadsto () := \texttt{EqSizeSubset}(2)\eth\texttt{Subset} : aH = bH;$

$\leadsto (3) := \eth^{-1}\texttt{Disjoint} : \Big(\{gH|g \in G\} : \texttt{Disjoint}\Big),$

$(*) := \texttt{DisjointSum}(1)(2)\eth^{-1}[G : h] : |G| = \sum_{X \in \{gH|g \in G\}} |X| = \Big|\{gH|g \in G\}\Big||H| = [G : H]|H|;$

$\square$

$\texttt{leftCosets} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Subgroup}(G) \to ??G$

$\texttt{leftCosets}\,(H) = G/H := \{gH : g \in G\}$

$\texttt{rightCosets} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Subgroup}(G) \to ??G$

$\texttt{rightCosets}\,(H) = G \setminus H := \{Hg : g \in G\}$

1.2 Normal Subgroups

$\texttt{Normal} :: \prod G : \texttt{Group} \, . \, ?\texttt{Subgroup}(G)$

$H : \texttt{Normal} \iff H \lhd G \iff \forall h \in H \, . \, \forall g \in G \, . \, ghg^{-1} \in H$

$\texttt{NormalPropertyI} :: \forall G : \texttt{Group} \, . \, \forall N \lhd H \, . \, \forall g \in G \, . \, gN = Ng$

$\texttt{Proof} =$

$\texttt{Assume } a : \texttt{In}(N),$

$b := gag^{-1} : \texttt{In}(N),$

$(1) := \eth b(bg)\eth\texttt{inverse}(g) : bg = gag^{-1}g = ga,$

$(2) := \eth Ng(1) : ga \in Ng;$

$\leadsto (*) := \texttt{EqSizeSubset} : gN = Ng,$

$\square$

$\texttt{NormalQuetient} :: \forall N \lhd G \, . \, \Big(\{gN : g \in N\}, \cdot \Big) \in \mathsf{GRP}$

$\texttt{Proof} =$

$\texttt{Assume } a, b : G,$

$(1) := \texttt{NormalPropertyI}(N)(B) : aNbN = abNN = abN;$

$\leadsto (2) := I(\forall) : \forall a, b \in G \, . \, aNbN = abN,$

$\dots$

$\square$

$\texttt{quetientGroup} :: \prod G \in \mathsf{GRP} \, . \, \texttt{Normal}(G) \to \mathsf{GRP}$

$\texttt{quetientGroup}\,(N) = \dfrac{G}{N} := \Big(\{gN | g \in G\}, \cdot \Big)$

$\texttt{naturalProjection} :: \prod G \in \mathsf{GRP} \, . \, \prod N \lhd G \, . \, G \xrightarrow{\mathsf{GRP}} \dfrac{G}{N}$

$\texttt{naturalProjection}\,(g) = \pi_N(g) := gN$

$\texttt{NormalKernel} :: \forall G, H \in \mathsf{GRP} \, . \, \forall f : G \xrightarrow{\mathsf{GRP}} H \, . \, \ker f \lhd G$

$\texttt{Proof} =$

$\texttt{Assume } a : \ker f,$

$\texttt{Assume } g : G,$

$(1) := \eth f(gag^{-1})\eth a\texttt{HomoOnInv}(f)(g)\eth\eth\texttt{Identity}(H)(e_H)\texttt{inverse}(f(g)) :$
$\quad : f(gag^{-1}) = f(g)f(a)f(g^{-1}) = f(g)e_H(f(g))^{-1} = e_H,$

$() := \eth^{-1} \ker f(1) : gag^{-1} \in \ker f;$

$\leadsto (*) := \eth^{-1}\texttt{Normal} : \ker f \lhd G;$

$\square$

$\texttt{KernelOfProjection} :: \forall G \in \textsf{GRP} \, . \, \forall N \lhd G \, . \; \ker \pi_N = N$

$\texttt{Proof} \; =$

$\texttt{Assume} \; a : N,$

$(1) := \eth \pi_N(a) \eth \texttt{Subgroup}(N) : \pi_N(a) = aN = N,$

$() := \eth^{-1} \ker \pi_N : a \in \ker \pi_N;$

$\rightsquigarrow (1) := \eth^{-1} \texttt{Subset} : N \subset \ker \pi_N,$

$\texttt{Assume} \; a : \ker \pi_N,$

$(2) := \eth a \eth \ker \pi_N : aN = \pi_N(a) = N,$

$(b, 3) := \eth \texttt{Subgroup}(N)(2) : \sum b \in N \, . \, ab = e,$

$(4) := \eth^{-1} \texttt{Inverse}(3) : a = b^{-1},$

$(5) := \eth \texttt{Subgroup}(N)(4) : a \in N;$

$\rightsquigarrow (*) := \eth \texttt{SetEq}(1) \eth \texttt{Subset} : \ker \pi_N = N;$

$\square$

$\texttt{AbeleanAllNormal} :: \forall G : \texttt{Abelean} \, . \, \forall H \subset_{\textsf{GRP}} G \, . \, H \lhd G$

$\texttt{Proof} \; =$

$\texttt{Assume} \; h : H,$

$\texttt{Assume} \; g : G,$

$() := \eth \texttt{Abelean}(G) : ghg^{-1} = gg^{-1}h = h;$

$\rightsquigarrow (*) := \eth \texttt{Normal}(G) : H \lhd G;$

$\square$

$\texttt{NormalIntersect} :: \forall G \in \textsf{GRP} \, . \, \forall X \in \textsf{SET} \, . \, \forall N : X \to \texttt{Normal}(G) \, . \, \bigcap_{x \in X} N_x \lhd G$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{normalizer} :: \prod G \in \textsf{GRP} \, . \, \texttt{Set}(G) \to \texttt{Subgroup}(G)$

$\texttt{normalizer}\,(X) = N(X) := \{g \in G : \forall x \in X \, . \, gxg^{-1} \in X\}$

$\texttt{centralizer} :: \prod G \in \textsf{GRP} \, . \, G \to \texttt{Subgroup}(G)$

$\texttt{centralizer}\,(g) = C(g) := N(\{g\})$

$\texttt{setCentralizer} :: \prod G \in \textsf{GRP} \, . \, G \to \texttt{Subgroup}(G)$

$\texttt{setCentralizer}\,(X) = C(X) := \bigcap_{x \in X} C(x)$

$\texttt{NormalizerContains} :: \forall G \in \mathsf{GRP} \;.\; \forall H, K \subset_{\mathsf{GRP}} G \;.\; \forall (0) : H \lhd K \;.\; K \subset N(H)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a : K,$

$\texttt{Assume}\; h : H,$

$() := (0)(aha^{-1}) : aha^{-1} \in G,$

$\rightsquigarrow () := \eth^{-1} N(H) : a \in N(H);$

$\rightsquigarrow (*) := \eth\texttt{Subset} : K \subset N(H);$

$\square$

$\texttt{NormalizerIsAGroup} :: \forall G \in \mathsf{GRP} \;.\; \forall H \subset_{\mathsf{GRP}} G \;.\; N(H) \in \mathsf{GRP}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a, b : N(H),$

$\texttt{Assume}\; h : H,$

$(1) := \eth N(H)(b, h) : bhb^{-1} \in H,$

$() := \eth N(H)(a, bhb^{-1})(1) : abhb^{-1}a^{-1} \in H;$

$\rightsquigarrow () := \eth^{-1} N(H) : ab \in N(H);$

$\rightsquigarrow (1) := I(\forall) : \forall a, b \;.\; ab \in N(H),$

$\texttt{Assume}\; a : N(H),$

$\texttt{Assume}\; h : H,$

$(2) := \eth N(H)(a, h^{-1}) : ah^{-1}a^{-1} \in H,$

$() := \eth\texttt{Subgroup(G)(H)} : \left(ah^{-1}a^{-1}\right) = a^{-1}ha \in H;$

$\rightsquigarrow () := \eth N(H) : \forall a \in N(H) \;.\; a^{-1} \in N(H);$

$\rightsquigarrow (*) := \eth\texttt{Grouup}(1)I(\forall) : N(H) \in \mathsf{GRP};$

$\square$

$\texttt{NormalizerAsLargest} :: \forall G \in \mathsf{GRP} \;.\; \forall H \subset_{\mathsf{GRP}} G \;.\; N(H) = \bigcup \{K \subset_{\mathsf{GRP}} G : H \lhd K\}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; K : \texttt{Subgroup}(K),$

$\texttt{Assume}\; (1) : H \lhd K,$

$() := \eth^{-1} N(H) \eth\texttt{Normal}(K)(H) : K \subset N(H);$

$\rightsquigarrow (1) := I(\forall) : \forall K \subset_{\mathsf{GRP}} H : H \lhd K \;.\; K \subset N(H),$

$(2) := \eth N(H) : N(H) \in \{K \subset_{\mathsf{GRP}} G : H \lhd k\},$

$(*) := \texttt{MaxSet}(1)(2) : N(H) = \bigcup \{K \subset_{\mathsf{GRP}} G : H \lhd K\};$

$\square$

$\texttt{ProductOfSubgroups} :: \forall G \in \mathsf{GRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \forall K \subset_{\mathsf{GRP}} N(H) \,.\, H \lhd KH$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a, b : K,$

$\texttt{Assume}\ h, g : H,$

$(1) := \texttt{NormalPropertyI}(N(H), H) : hb \in bH,$

$(f, 2) := \eth bH(1) : \sum f \in H \,.\, hb = bf,$

$() := (2)\eth^{-1}KH : ahbg = abfg \in KH;$

$\rightsquigarrow (1) := I(\forall) : \forall x, y \in KH \,.\, xy \in KH,$

$\texttt{Assume}\ a : K,$

$\texttt{Assume}\ h : H,$

$(2) := \texttt{NormalPropertyI}(N(H), H) : h^{-1}a^{-1} \in a^{-1}H,$

$(g, 3) := \eth a^{-1}H(2) : \sum g \in H : h^{-1}a^{-1} = a^{-1}g,$

$() := \texttt{InverseMult}(2)\eth^{-1}KH : (ah)^{-1} = h^{-1}a^{-1} = a^{-1}g \in KH;$

$\rightsquigarrow (0) := \eth^{-1}\texttt{Group}(1)I(\forall) : KH \in \mathsf{GRP},$

$\texttt{Assume}\ a : K,$

$\texttt{Assume}\ h, g : H,$

$(2) := \texttt{NormalPropertyI}(N(H), H) : ha^{-1}, ga^{-1}, h^{-1}a^{-1} \in a^{-1}H,$

$(u, v, s, 3) := \eth a^{-1}H : \sum u, v, s \in H \,.\, a^{-1}u = ha^{-1} \;\&\; a^{-1}v = ga^{-1} \;\&\; a^{-1}s = h^{-1}a^{-1},$

$() := (3)\eth\texttt{inverse}\eth H\eth u, v, h, s : ahgh^{-1}a^{-1} = ahga^{-1}s = aha^{-1}us = aa^{-1}vus = vus \in H;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Normal} : H \lhd KH;$

$\square$

$\texttt{isomorphismTHMI} :: \forall A, B \in \mathsf{GRP} \,.\, \forall f : A \xrightarrow{\mathsf{GRP}} B \,.\, \exists! \varphi : \dfrac{A}{\ker f} \xleftrightarrow{\mathsf{GRP}} \operatorname{Im} f : \pi_{\ker f}\varphi\iota_{\operatorname{Im} f} = f$

$\texttt{Proof} \;=$

$\texttt{Assume}\ X : \dfrac{A}{\ker f},$

$(a, 1) := \eth\dfrac{A}{\ker f} : \sum a \in A \,.\, a \in X,$

$\varphi(X) := f(a) : \operatorname{Im} f;$

$\texttt{Assume}\ b : X,$

$(x, 2) := \eth\dfrac{A}{\ker f} : \sum x \in \ker f \,.\, b = xa,$

$() := 2)(f(b))\eth f\eth x\eth^{-1}\varphi(X) : f(b) = f(xa) = f(x)f(a) = f(a) = \varphi(X);$

$\rightsquigarrow \varphi := \texttt{WellDefined}I(\to) : \dfrac{A}{\ker f} \to \operatorname{Im} f,$

$\texttt{Assume}\ X, Y : \dfrac{A}{\ker \varphi},$

$(a, b, 1) := \eth X, Y : \sum a, b \in A : a \in X \,.\, a \in b,$

$(2) := \eth^{-1}XY(1) : ab \in XY,$

$:= \eth\varphi(XY)(2)\eth f\eth^{-1}\varphi : \varphi(XY) = f(ab) = f(a)f(b) = \varphi(X)\varphi(Y);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Homomorphism} : \left(\varphi : \dfrac{A}{\ker f} \xrightarrow{\mathsf{GRP}} \operatorname{Im} f\right),$

$\texttt{Assume}\ y : \operatorname{Im} f,$

$(x,2) := \eth \operatorname{Im} f \eth y : \sum x \in A \,.\, f(x) = y,$

$() := \eth\varphi(x \ker f)\eth x \ker f : \varphi(x\ker f) = f(x) = y;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Surjection} : \left(\varphi : \dfrac{A}{\ker f} \twoheadrightarrow \operatorname{Im} f\right),$

$\texttt{Assume}\ X : \ker\varphi,$

$(x,3) := \eth\varphi\eth X : \sum x \in X \,.\, \varphi(X) = f(x) = e,$

$(4) := \eth^{-1}\ker f(3) : x \in \ker f,$

$(5) := \texttt{IndexTHM}(4) : X = \ker f;$

$\rightsquigarrow (3) := \eth^{-1}\texttt{Bijection}(\varphi)\texttt{TrivialKernelTHM}\eth^{-1}\texttt{Singleton} : \left(\varphi : \dfrac{A}{\ker f} \xleftrightarrow{\mathsf{GRP}} \operatorname{Im} f\right),$

$\texttt{Assume}\ a : A,$

$() := \eth \ldots : (a)\pi_{\ker f}\varphi\iota_{\operatorname{Im} f} = (a\ker f)\varphi\iota_{\operatorname{Im} f} = f(a)\iota_{\operatorname{Im} f} = f(a);$

$\rightsquigarrow (4) := I(\rightarrow \,.\, =) : \pi_{\ker}\varphi\iota_{\operatorname{Im} f};$

$\texttt{Assume}\ (\psi,5) : \sum \psi : \dfrac{A}{\ker f} \xleftrightarrow{\mathsf{GRP}} \operatorname{Im} f \,.\, \pi\psi\iota = f,$

$\texttt{Assume}\ X : \dfrac{A}{\ker f},$

$(a,6) := \eth X : \sum a \in A : a \in X,$

$(7) := \eth\pi(6)(5) : (X)\psi\iota = (a)\pi\psi\iota = f(a),$

$(8) := \eth\pi(6)(4) : (X)\varphi\iota = (a)\pi\varphi\iota = f(a),$

$() := \eth\texttt{Injective}(\iota)(7,8) : \varphi(X) = \psi(X);$

$\rightsquigarrow (*) := I(\exists!) : \texttt{This},$

$\square$

$\texttt{inducedIsomorphism} :: \prod A, B \in \mathsf{GRP} \,.\, \prod f : A \xrightarrow{\mathsf{GRP}} B \,.\, \dfrac{A}{\ker f} \xleftrightarrow{\mathsf{GRP}} \operatorname{Im} f$

$\texttt{inducedIsomorphism}\,() = f_* := \texttt{IsomorphismTHMI}(f)$

$\texttt{IsomorphismTHMII} :: \forall G \in \mathsf{GRP} \,.\, \forall K, H \lhd G \,.\, \forall (0) : K \lhd H \,.\, \dfrac{G}{K}\Big/\dfrac{H}{K} \cong_{\mathsf{GRP}} \dfrac{G}{H}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ (aK) : \dfrac{G}{K},$

$\varphi(aK) := aH : \dfrac{G}{H},$

$\texttt{Assume}\ b : aK,$

$(k,1) := \eth aH(b) : \sum k \in K \,.\, b = ah,$

$() := (1)(0)\eth^{-1}\varphi(aK) : bH = akH = aH = \varphi(aK);$

$\rightsquigarrow \varphi := \texttt{WellDefined}I(\rightarrow)\texttt{NormalPropertyI} : \dfrac{G}{K} \xrightarrow{\mathsf{GRP}} \dfrac{G}{H},$

$\texttt{Assume}\ aK : \dfrac{G}{K},$

$\texttt{Assume}\ (1) : \varphi(aK) = H,$

$(2) := \eth\varphi(1)\eth H : a \in H,$

$(2) := \eth^{-1}\dfrac{H}{K}(2) : aK \in \dfrac{H}{K};$

$\rightsquigarrow (1) := \eth^{-1}\ker\varphi : \ker\varphi = \dfrac{H}{K},$

$(2) := \eth\varphi_* E(=)(1) : \left(\varphi_* : \dfrac{G}{K}/\dfrac{H}{K} :\xleftrightarrow{\mathsf{GRP}} \dfrac{G}{H}\right),$

$(*) := \eth^{-1}\texttt{Isomorphic}(\mathsf{GRP}) : \texttt{This};$

$\square$

$\texttt{IsomorphismTHMIII} :: \forall G \in \mathsf{GRP}\ .\ \forall H, K :\lhd G\ .\ \dfrac{H}{H \cap K} \cong \dfrac{HK}{K}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ h : H,$

$\varphi(h) := hK : \dfrac{HK}{K};$

$\rightsquigarrow \varphi := I(\rightarrow) : \varphi : H \rightarrow \dfrac{HK}{H},$

$\texttt{Assume}\ a, b : H,$

$(1) := \eth H \eth^{-1} N(K) : H \subset_{\mathsf{GRP}} G = N(K),$

$(2) := \texttt{ProductOfSubgroups}(K, H)(1) : K \lhd HK,$

$() := \texttt{NormalPropertyI}(b, K)\eth K : \varphi(a)\varphi(b) = aKbK = abKK = abK = \varphi(ab);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Homomorphism} : \left(\varphi : H \xrightarrow{\mathsf{GRP}} \dfrac{HK}{K}\right),$

$(2) := \eth\varphi : \ker\varphi = K,$

$(3) := \eth\varphi : \operatorname{Im}\varphi = \dfrac{HK}{K},$

$(4) := \texttt{IsomorphismTHMI}(1, 2, 3) : \left(\varphi_* : \dfrac{H}{H \cap K} \xleftrightarrow{\mathsf{GRP}} \dfrac{HK}{K}\right),$

$(*) := \eth^{-1}\texttt{Isomorphic}(\mathsf{GRP}) : \dfrac{H}{H \cap K} \cong \dfrac{HK}{K};$

$\square$

$\texttt{NormalPullback} :: \forall A, B \in \mathsf{GRP}\ .\ \forall f : A \xrightarrow{\mathsf{GRP}} B\ .\ \forall N \lhd B\ .\ f^{-1}(N) \lhd A$

$\texttt{Proof}\ =$

$\texttt{Assume}\ x : f^{-1}(N),$

$(1) := \eth\texttt{Preimage} : f(x) \in N,$

$\texttt{Assume}\ a : A,$

$(2) := \eth f\texttt{HomoOnInv}(1)\eth\texttt{Normal}(B)N : f(axa^{-1}) = f(a)f(x)(f(a))^{-1} \in N,$

$() := \eth^{-1}\texttt{Preimage}(f, N)(2) : axa^{-1} \in f^{-1}(N);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Normal} : f^{-1}(N) \lhd B;$

$\square$

1.3 Solvable Groups

$$\texttt{Tower} :: \prod G \in \mathsf{GRP} \ . \ \sum n \in \mathbb{Z}_+ \ . \ n \to \texttt{Subgroup}(G)$$
$$H : \texttt{Tower} \iff H_0 = G \ \& \ \forall i \in n-1 \ . \ H_{i+1} \subset_{\mathsf{GRP}} H_i$$

$$\texttt{NormalTower} :: ?\texttt{Tower}(G)$$
$$(n+1, H) : \texttt{NormalTower} \iff \forall i \in n \ . \ H_{i+1} \lhd H_i$$

$$\texttt{TowerType} :: \prod T :?\mathsf{GRP} \ . \ ?\texttt{NormalTower}(G)$$
$$(n+1, H) : \texttt{TowerType} \iff (n+1, H) : T\text{-}\texttt{Tower} \iff \forall i \in n \ . \ \frac{H_i}{H_{i+1}} : T$$

$$\texttt{Solvable} :: ?\mathsf{GRP}$$
$$G : \texttt{Solvable} \iff \exists (n, H) : \texttt{Abelean-Tower}(G) \ . \ H_n = \{e\}$$

$$\texttt{Refinement} :: \prod G \in \mathsf{GRP} \ . \ ?(\texttt{Tower}(G) \times \texttt{Tower}(G))$$
$$\Big((n, A), (m, B)\Big) : \texttt{Refinement} \iff (n, A) \le (m, B) \iff$$
$$\iff n \le m \ \& \ \exists j : \texttt{Increasing}(n, m) : \forall i \in n \ . \ A_i = B_{j(i)}$$

$$\texttt{FiniteGroupsAbeleanTowerAdmitsCyclicRefinement} :: \forall G \in \mathsf{GRP} \ . \ \forall (n, H) : \texttt{Abelean-Tower}(G) \ .$$
$$. \ \forall (0) : |G| < \infty \ . \ \exists (m, Z) : \texttt{Cyclic-Tower}(G) : (n, H) \le (m, Z)$$

$\texttt{Proof} =$

$(n_{0,0} + 1, H^{0,0}) := (n, H) : \texttt{Abelean-Tower},$

$$\kappa_0 := \left|\left\{k \in n_{0,0} : \frac{H^{0,0}_k}{H^{0,0}_{k+1}} \ ! \ \texttt{Cyclic}\right\}\right| : \mathbb{N},$$

$\texttt{Assume}\ i : \mathbb{N},$

$\texttt{Assume}\ (00) : \kappa_{i-1} \neq 0,$

$$k := \min\left\{k \in n_{i-1,0} : \frac{H^{i-1,0}_k}{H^{i-1,0}_{k+1}} \ ! \ \texttt{Cyclic}\right\} : n_{i-1,0},$$

$\texttt{Assume}\ j : \mathbb{N},$

$$\texttt{Assume}\ (1) : \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} \ ! \ \texttt{Cyclic},$$

$$(2) := \eth\texttt{Abelean-Tower}(G)\left(H^{i-1,j-1}\right)(k) : \left(\frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} : \texttt{Abelean}\right),$$

$$(y, 3) := (1)\eth\texttt{Trivial} : \sum y \in \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} \ . \ y \neq e,$$

$$(4) := \texttt{AbeleanAllNormal}(2)(\langle y\rangle) : \left(\langle y\rangle \lhd \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}}\right),$$

$X := \pi^{-1}\langle y\rangle : \texttt{Subgroup}(H^{i-1,j-1}_k),$

$(5) := \texttt{NormalPullback}(4)\eth X : X \lhd H^{i-1,j-1}_k,$

$(6) := \texttt{SubgroupProduct}(5) : H^{i-1,j-1}_{k+1} \lhd X H^{i-1,j-1}_{k+1} \lhd H^{i-1,j-1}_k,$

$(n_{i-1,j}+1, H^{i-1,j}) := (n_{i-1,j-1}+2, H^{i-1,j-1}_{1,\dots,k} - XH^{i-1,j-1}_{k+1} - H^{i-1,j-1}_{k+1,\dots,n_{i-1,j-1}+1}) : \texttt{Abelean-Tower}(G)$,

$\eta_{i,j} := \eth^{-1}\texttt{Refinement}\eth(n_{i-1,j}+1, H^{i-1,j}) : (n_{i-1,j-1}+1, H^{i-1,j-1}) \le (n_{i-1,j}+1, H^{i-1,j})$,

$\zeta_{i,j} := \eth(n_{i-1,j}+1, H^{i-1,j}) : \left(\left| \frac{H^{i-1,j}_k}{H^{i-1,j}_{k+1}} \right| < \left| \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} \right| \right)$,

$(7) := \eth X \texttt{IsomorphismTHMIII}\eth(n_{i-1,j}+1, H^{i-1,j}) : \frac{H^{i-1,j}_{k+1}}{H^{i-1,j}_{k+2}} = \frac{XH^{i-1,j-1}_{k+1}}{H^{i-1,j-1}_{k+1}} \cong \frac{X}{X \cap H^{i-1,j-1}_{k+1}} = \langle y \rangle : \texttt{Cyclic}$,

$\xi_{i,j} := (7) \prod_{l=1}^{j-1} \xi_{i,l} : \forall l \in n_{i-1,j} - n_{i-1,0} \,.\, \frac{H^{i-1,j}_{k+l}}{H^{i-1,j}_{k+l+1}} : \texttt{Cyclic}$;

$\texttt{Assume}\ (1) : \left(\frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} : \texttt{Cyclic} \right)$,

$(n_{i-1,j}+1, H^{i-1,j}) := (n_{i-1}+1, H^{i-1,j}) : \texttt{Abelean-Tower}(G)$,

$(\eta, \zeta, \xi) := ((), (), ()) : \top^3$;

$\rightsquigarrow \Big((n_{i-1}, H^{i-1}), \eta_i, \zeta_i, \xi_i) \Big) := I \left(\sum \right) I \left(\prod \right) :$

$\quad : \sum (n^{i-1}, H^{i-1}) : \mathbb{Z}_+ \to \texttt{Abelean-Tower}(G) \,.\, \prod j \in \mathbb{N} \,.\, \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} \,!\, \texttt{Cyclic} \Rightarrow$

$\quad \Rightarrow (n_{i-1,j-1}+1, H^{i-1,j-1}) < (n_{i-1,j}, H^{i-1,j})\ \&\ \left| \frac{H^{i-1,j}_k}{H^{i-1,j}_{k+1}} \right| < \left| \frac{H^{i-1,j-1}_k}{H^{i-1,j-1}_{k+1}} \right|\ \&$

$\quad \&\ \forall l \in n_{i-1,j} - n_{i-1,0} \,.\, \frac{H^{i-1,j}_{k+l}}{H^{i-1,j}_{k+l+1}} : \texttt{Cyclic}$,

$(j,2) := (0) \prod_{j=1}^{\infty} \zeta_{i,j} : \sum j \in \mathbb{N} \,.\, \frac{H^{i-1,j}_k}{H^{i-1,j}_{k+1}} : \texttt{Cyclic}$,

$(n_{i,0}+1, H^{i,0}) := (n^{i-1,j}+1, H^{i,0}) : \texttt{Abelean-Tower}$,

$\kappa_i := \left| \left\{ k \in n_{i,0} : \frac{H^{i,0}_k}{H^{i-1,j}_{k+1}} \,!\, \texttt{Cyclic} \right\} \right| : \mathbb{N}$,

$\mathcal{H}_i := \mathcal{H}_{i-1} \prod_{j=1}^{\infty} \eta_{i,j} : (n,H) < (n_{i,0}, H^{i,0})$,

$\Xi_i := \Xi_{i-1} \eth(n_{i,0}, H^{i,0})(3) \prod_{l=0}^{j} \xi_{i,j} \eth^{-1} \kappa : \kappa_i < \kappa_{i-1}$;

$\texttt{Assume}\ (00) : \kappa_{i-1} = 0$,

$(n_{i,0}, H^{i,0}) := (n_{i-1,0}, H^{i,0}) : \texttt{Abelean-Tower}$,

$(\mathcal{H}_i, \Xi_i) := (\mathcal{H}_{i-1}, ()) : (n,H) < (n_{i,0}, H_{i,0}) \times \top$;

$\rightsquigarrow \Big((n,H), \mathcal{H}, \Xi \Big) := I \left(\sum \right) I \left(\prod \right) : \sum (n,H) : \mathbb{Z}^2_+ \to \texttt{Abelean-Tower} \,.$

$\quad . \prod i \in \mathbb{N} \,.\, (n,H) < (n_{i,0}, H^{i,0})\ \&\ \kappa_{i-1} \ne 0 \Rightarrow \kappa_i < \kappa_{i-1}$,

$(i,1) := \eth n \prod_{i=1}^{\infty} \Xi_i : \sum i \in \mathbb{N} : \kappa_i = 0$,

$(m, Z) := (n_{i,0}, H^{i,0}) : \texttt{Cyclic-Tower}$,

$(*) := \eth(m,Z)\mathcal{H}_i : (n,H) < (m,Z)$;

□

$\texttt{SolvableSubgroup} :: \forall G : \texttt{Solvable} \,.\, \forall N \lhd G \,.\, N : \texttt{Solvable}$

$\texttt{Proof} \;=$

$\Big((n+1, H), 0\Big) := \eth\texttt{Solvable}(G) : \sum(n+1, H) : \texttt{Abelean-Tower}(G) \,.\, H_{n+1} = \{e\},$

$H' := \Lambda i \in n+1 \,.\, H_i \cap N : (n+1)_{\mathbb{Z}_+} \to \texttt{Subgroup}(N),$

$(1) := \Big(\eth H'\Big)_0 \eth N : H'_0 = G \cap N = N,$

$(2) := \Big(\eth H'\Big)_{n+1}(0)\eth N : H'_{n+1} = \{e\} \cap N = \{e\},$

$(3) := \eth(H, n+1)\eth H' \eth N : \forall i \in (n)_{\mathbb{Z}_+} \,.H'_{i+1} \lhd H'_i,$

$(4) := \eth^{-1}\texttt{NormalTower}(1,3) : \Big((H_i, n+1) : \texttt{NormalTower}(N)\Big),$

$\texttt{Assume}\ i : (n)_{\mathbb{Z}_+},$

$(5) := \eth H' \eth\texttt{Tower}(H)\eth^{-1} H_{i+1} \cap (H_i \cap N)\texttt{IsomorphismTHMIII}(H_{i+1}(H_i \cap N), H_{i+1})\eth^{-1}\texttt{Subset} :$

$\quad : \dfrac{H'_i}{H'_{i+1}} = \dfrac{H_i \cap N}{H_{i+1} \cap N} = \dfrac{H_i \cap N}{H_{i+1} \cap (H_i \cap N)} \cong \dfrac{H_{i+1}(H_i \cap N)}{H_{i+1}} \subset \dfrac{H_i}{H_{i+1}},$

$() := \eth\texttt{Abelean-Tower}(H, n)(5) : \left(\dfrac{H'_i}{H'_{i+1}} : \texttt{Abelean}\right);$

$\rightsquigarrow (5) := I(\forall) : \forall i \in n \,.\, \dfrac{H'_i}{H'_{i+1}} : \texttt{Abelean},$

$(6) := \eth^{-1}\texttt{Abelean-Tower}(4,5) : \Big((H, n) : \texttt{Abelean-Tower}(N)\Big),$

$(*) := \eth^{-1}\texttt{Solvable}(2,6) : \Big(N : \texttt{Solvable}\Big);$

$\square$

$\texttt{SolvableQuetient} :: \forall G : \texttt{Solvable} \,.\, \forall N \lhd G \,.\, \dfrac{G}{N} : \texttt{Solvable}$

$\texttt{Proof} \;=$

$\Big((n+1, H), 0\Big) := \eth\texttt{Solvable}(G) : \sum(n+1, H) : \texttt{Abelean-Tower}(G) \,.\, H_{n+1} = \{e\},$

$H' := \Lambda i \in (n+1)_{\mathbb{Z}_+} \,.\, \dfrac{H_i}{H_i \cap N} : (n+1)_{\mathbb{Z}_+} \to \mathsf{GRP},$

$(1) := (\eth H')_0 \eth N : H'_0 = \dfrac{G}{N \cap G} = \dfrac{G}{N},$

$(2) := (\eth H')_{n+1}(0) : H'_{n+1} = \dfrac{\{e\}}{\{e\} \cap N} = \{e\},$

$(3) := \eth(H, n+1)\eth H' \eth N : \forall i \in (n)_{\mathbb{Z}_+} \,.H'_{i+1} \lhd H'_i,$

$(4) := \eth^{-1}\texttt{NormalTower}(1,3) : \Big((H_i, n+1) : \texttt{NormalTower}(N)\Big),$

$\texttt{Assume}\ i : (n)_{\mathbb{Z}_+},$

$(5) := \eth H' \texttt{IsomorphisTHMIII}(H_{i+1}, H_i \cap N)\texttt{IsomorphismTHMII}(H_i, H_{i+1}, H_i \cap N)$

$\quad \texttt{IsomorphismTHMII}(H_i, H_{i+1}(H_i \cap N), H_{i+1}) :$

$\quad : \dfrac{H'_i}{H'_{i+1}} = \left(\dfrac{H_i}{H_i \cap N}\right) / \left(\dfrac{H_{i+1}}{H_{i+1} \cap N}\right) = \left(\dfrac{H_i}{H_i \cap N}\right) / \left(\dfrac{H_{i+1}}{H_{i+1} \cap (H_i \cap N)}\right) \cong$

$\quad \cong \left(\dfrac{H_i}{H_i \cap N}\right) / \left(\dfrac{H_{i+1}(H_i \cap N)}{H_i \cap N}\right) \cong \dfrac{H_i}{H_{i+1}(H_i \cap N)} \cong \left(\dfrac{H_i}{H_{i+1}}\right) / \left(\dfrac{H_{i+1}(H_i \cap N)}{H_{i+1}}\right),$

$() := \eth\texttt{Abelean-Tower}(H)\eth\texttt{quetientGroup}(5)\eth^{-1}\texttt{Abelean} : \left(\frac{H'_i}{H'_{i+1}} : \texttt{Abelean}\right);$

$\leadsto (5) := I(\forall) : \forall i \in n \,.\, \frac{H'_i}{H'_{i+1}} : \texttt{Abelean},$

$(6) := \eth^{-1}\texttt{Abelean-Tower}(4,5) : \Big((H', n) : \texttt{Abelean-Tower}(N)\Big),$

$(*) := \eth^{-1}\texttt{Solvable}(2,6) : \left(\frac{G}{N} : \texttt{Solvable}\right);$

$\square$

$\texttt{SolvabilityCriterion} :: \forall G \in \mathsf{GRP} \,.\, \forall N \lhd G \,.\, \forall (0) : N, \frac{G}{N} : \texttt{Solvable} \,.\, G : \texttt{Solvable}$

$\texttt{Proof} \; =$

$(1) := \eth\texttt{quetientGroup}(G, N) : G \cong N \times \frac{G}{N},$

$\Big((n+1, A), 2\Big) := \eth\texttt{Solvable}(N) : \sum(n+1, A) : \texttt{Abelean-Tower}(N) \,.\, A_{n+1} = \{e\},$

$\Big((m+1, B), 3\Big) := \eth\texttt{Solvable}(G/N) : \sum(m+1, A) : \texttt{Abelean-Tower}\, \frac{G}{N} \,.\, B_{m+1} = \{e\},$

$\texttt{Assume}\ (4) : n = m,$

$H := \Lambda i \in (n+1)_{\mathbb{Z}_+} \,.\, A_i \times B_i : (n+1)_{\mathbb{Z}_+} \to \texttt{Subgroup}\left(N \times \frac{G}{N}\right),$

$(5) := (1)(\eth H)_0 : H_0 \cong G,$

$(6) := (2)(3)(\eth H)_{n+1} : H_{n+1} = \{e\} \times \{e\} \cong \{e\},$

$\ldots$

$\square$

1.4 Commutator Subgroup

$$\texttt{commutator} :: \prod G \in \mathsf{GRP} \,.\, \texttt{Subgroup}(G)$$

$$\texttt{commutator}\,() = G^c := \left\langle \{aba^{-1}b^{-1} | a, b \in G\} \right\rangle$$

$$\texttt{CommutatorIsNormal} :: \forall G \in \mathsf{GRP} \,.\, G^c \lhd G$$

$\texttt{Proof} =$

$\texttt{Assume}\ g : G,$

$\texttt{Assume}\ h : G^c,$

$$(n, a, b, 1) := \eth G^c(h) : \sum n \in \mathbb{N} \,.\, \sum a, b : n \to G \,.\, h = \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1},$$

$$() := (1)(ghg^{-1})\eth^{-1}(ga_ig^{-1}, gb_ig^{-1})\eth^{-1}G^c :$$

$$: ghg^{-1} = g\left(\prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}\right) g^{-1} = \prod_{i=1}^{n} (ga_ig^{-1})(gb_ig^{-1})(ga_ig^{-1})^{-1}(gb_ig^{-1})^{-1} \in G^c;$$

$$\rightsquigarrow (*) := \eth^{-1}\texttt{Normal} : G^c \lhd G;$$

$\square$

$$\texttt{CommutatorQuetientIsAbelean} :: \forall G \in \mathsf{GRP} \,.\, \frac{G}{G^c} : \texttt{Abelean}$$

$\texttt{Proof} =$

$\texttt{Assume}\ aG^c, bG^c : \dfrac{G}{G^c},$

$$() := \texttt{NormalProperty}(G^c)(a, b)\eth G^c\texttt{NormalProperty}(G^c)(a, b) :$$

$$: aG^cbG^c = abG^c = abb^{-1}a^{-1}baG^c = baG^c = bG^caG^c;$$

$$\rightsquigarrow (*) := \eth^*\texttt{Abelean} : \texttt{Abelean}\ \frac{G}{G^c};$$

$\square$

$$\texttt{Simple} :: ?\mathsf{GRP}$$

$$G : \texttt{Simple} \iff \Big\{H \lhd G : \big(H : \texttt{NonTrivial}(G)\big)\Big\} = \emptyset$$

1.5 Jordan-Hölder Theory

$\texttt{PreButterflyLemmaI} :: \forall G \in \mathsf{GRP} \,.\, \forall A, B \subset_{\mathsf{GRP}} G \,.\, \forall a \lhd A \,.\, \forall b \lhd B \,.\, a(A \cap b) \lhd a(A \cap B)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ c, d : a,$

$\texttt{Assume}\ x : A \cap B,$

$\texttt{Assume}\ y : A \cap b,$

$(z, 1) := \texttt{NormalPropertyI}(y, x^{-1}) : \sum z \in A \cap b \,.\, x^{-1}z = yx^{-1},$

$(s, 2) := \texttt{NormalPropertyI}(c^{-1}, y) : \sum s \in a \,.\, zc^{-1} = sz,$

$(3) := \eth\texttt{Normal}(A)(a)(d, x) : xdx^{-1} \in a,$

$() := (1)(2)(3) : cxdyx^{-1}c^{-1} = cxdx^{-1}zc^{-1} = cxdx^{-1}sz \in a(A \cap b);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Normal} : a(A \cap b) \lhd a(A \cap B);$

$\square$

$\texttt{PreButterflyLemmaII} :: \forall G \in \mathsf{GRP} \,.\, \forall A, B \subset_{\mathsf{GRP}} G \,.\, \forall a \lhd A \,.\, \forall b \lhd B \,.\, (a \cap B)b \lhd (A \cap B)b$

$\texttt{Proof} \ =$

$\texttt{Assume}\ c, d : b,$

$\texttt{Assume}\ x : A \cap B,$

$\texttt{Assume}\ y : a \cap B,$

$(z, 2) := \texttt{NormalPropertyI}(c, y) : \sum z \in b \,.\, yz = cy,$

$(s, 3) := \texttt{NormalPropertyI}(c^{-1}, y) : \sum s \in (a \cap B) \,.\, xy = sx,$

$(4) := \eth\texttt{Normal}(B)(b)(zdc^{-1}, x) : xzdc^{-1}x^{-1} \in b,$

$() := (2)(3)(4) : xcydc^{-1}x^{-1} = xyzdc^{-1}x^{-1} = sxzdc^{-1}x^{-1} \in (a \cap B)b;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Normal} : (a \cap B)b \lhd (A \cap B)b;$

$\square$

$\texttt{ButterflyLemma} :: \forall G \in \mathsf{GRP} \,.\, \forall A, B \subset_{\mathsf{GRP}} G \,.\, \forall a \lhd A \,.\, \forall b \lhd B \,.\, \dfrac{a(A \cap B)}{a(A \cap b)} \cong \dfrac{(A \cap B)b}{(a \cap B)b}$

$\texttt{Proof} \ =$

$(3) := \texttt{IntersectProduct}(A, B, a, b)\texttt{IsomorphismTHMIII}\Big(a(A \cap B), ab\Big)$

$\quad \texttt{NormalPropertyI}^2(A, a)(B, b)\texttt{IsomorphismTHMIII}\Big((A \cap B)b, ab\Big)\texttt{IntersectionProduct}(A, B, a, b) :$

$$\frac{a(A \cap B)}{a(A \cap b)} = \frac{a(A \cap B)}{a(A \cap B) \cap ab} \cong \frac{a(A \cap B)ab}{ab} = \frac{ab(A \cap B)b}{ab} \cong \frac{(A \cap B)b}{ab \cap (A \cap B)b} = \frac{(A \cap B)b}{(a \cap B)b};$$

$\square$

$\texttt{EqNormalTowers} :: \prod G \in \mathsf{GRP} \;.\; ?\texttt{NormalTower}^2(G)$

$\Big((r+1,H),(s+1,E)\Big) : \texttt{EqNormalTowers} \iff (r+1,H) \sim (s+1,E) \iff r = s \;\&$

$\quad \& \; \exists \sigma \in \mathrm{Aut}_{\mathsf{SET}}(r) \;.\; \forall i \in r \;.\; \dfrac{H_i}{H_{i+1}} \cong \dfrac{E_{\sigma(i)}}{E_{\sigma(i)+1}}$

$\texttt{PotentialSolution} :: \prod G \in \mathsf{GRP} \;.\; ?\texttt{NormalTower}(G)$

$(r,H) : \texttt{PotentialSolution} \iff H_r = \{e\}$

$\texttt{SchreierTHM} :: \forall G \in \mathsf{GRP} \;.\; \forall (H,r+1),(E,s+1) : \texttt{PotentialSolution}(G) \;.$

$\quad .\; \exists (H',t+1),(E',t+1) : \texttt{NormalTower}(G) : (H,r+1) \le (H',t+1) \;\&\; (E,s+1) \le (E',t+1) \;\&$

$\quad \& \; (H,r+1) \sim (E,t+1)$

$\texttt{Proof} \;=$

$t := rs : \mathbb{N},$

$X := \Lambda(i,j) \in r \times s \;.\; H_{i+1}(H_i \cap E_j) : r \times s \to \texttt{Subgroup}(G),$

$Y := \Lambda i,j \in s \times r \;.\; (H_j \cap E_i)E_{i+1} : s \times r \to \texttt{Subgroup}(G),$

$\texttt{Assume}\; k : t+1,$

$\texttt{Assume}\; (1) : k = t+1,$

$H'_k := \{e\} : \texttt{Subgroup}(G);$

$\texttt{Assume}\; (i,1) : \sum i \in \mathbb{N} \;.\; k = is,$

$H'_k := E_{i+1} : \texttt{Subgroup}(G);$

$\texttt{Assume}\; (i,j,1) : \sum i \in \mathbb{N} \;.\; \sum j \in s-1 \;.\; k = is + j,$

$H'_k := X_{i+1,j+1} : \texttt{Subgrouo}(G);$

$\rightsquigarrow \Big((t,H'),1\Big) := I\left(\sum\right)\texttt{PreButterflyLemmaI}\eth X : \sum (t,H') : \texttt{NormalTower}(G) \;.\; (r,H) \le (t,H'),$

$\texttt{Assume}\; k : t+1,$

$\texttt{Assume}\; (1) : k = t+1,$

$E'_k := \{e\} : \texttt{Subgroup}(G);$

$\texttt{Assume}\; (i,1) : \sum i \in \mathbb{N} \;.\; k = ir,$

$H'_k := E_{i+1} : \texttt{Subgroup}(G);$

$\texttt{Assume}\; (i,j,1) : \sum i \in \mathbb{N} \;.\; \sum j \in r-1 \;.\; k = ir + j,$

$H'_k := Y_{i+1,j+1} : \texttt{Subgroup}(G);$

$\rightsquigarrow \Big((t,H'),2\Big) := I\left(\sum\right)\texttt{PreButterflyLemmaII}\eth Y : \sum (t,H') : \texttt{NormalTower}(G) \;.\; (r,H) \le (t,H'),$

$\texttt{Assume}\; i : r,$

$\texttt{Assume}\; j : s,$

$(2) := \eth H' \eth X \texttt{ButterflyLemma} \eth^{-1} Y \eth^{-1} :$

$$: \frac{H'_{(i-1)s+j}}{H'_{(i-1)s+j+1}} = \frac{X_{i,j}}{X_{i,j+1}} = \frac{H_{i+1}(H_i \cap E_j)}{H_{i+1}(H_i \cap E_{j+1})} \cong \frac{(H_i \cap E_j)E_{j+1}}{(H_{i+2} \cap E_{j+1})E_{j+1}} = \frac{Y_{j,i}}{Y_{j,i+1}} = \frac{E'_{(j-1)r+i}}{E'_{(j-1)r+i+1}};$$

$\rightsquigarrow (*) := \eth^{-1}\texttt{EqNormalTower} : (H',t+1) \sim (E',t+1);$

$\square$

$\texttt{JordanHölderTHM} :: \prod G \in \mathsf{GRP} \ . \ \forall (r,H),(s,E) : \texttt{Simple-Tower}(G) \ . \ (r,H) \sim (s,E)$

$\texttt{Proof} \ =$

$\Big((t,H'),(s,E'),1\Big) := \texttt{SchreirTHM}\Big((r,H),(s,E)\Big) : \sum (t,H'),(t,E') : \texttt{NormalTower}(G) \ .$

$\quad . \ (t,H') \sim (s,E') \ \& \ (r,H) \leq (t,H') \ \& \ (s,E) \leq (t,E'),$

$(2) := (1)ð(t,H')ð\texttt{Simple-Tower}(r,H) : (r,H) = (t,H'),$

$(3) := (1)ð(t,E')ð\texttt{Simple-Tower}(s,E) : (s,E) = (t,E'),$

$(4) := (1)(2)(3) : (r,H) \sim (t,E');$

$\square$

1.6 Semidirect Product

$\texttt{SemidirectProductBijection} :: \forall G \in \mathsf{GRP} . \forall N \lhd G . \forall H \subset_{\mathsf{GRP}} G . \forall (0) : H \cap N = \{e\} .$
$. \Lambda(a,b) \in N \times H . ab : N \times H \leftrightarrow NH$

$\texttt{Proof} =$

$\mu := \Lambda(a,b) \in N \times H . ab : N \times H \to NH,$

$\texttt{Assume}\ x : NH,$

$(a,b,1) := \eth\texttt{product} : \sum(a,b) \in N \times H . x = ab,$

$() := \eth\mu(a,b)(1) : \mu(a,b) = ab = x;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Surjection} : \mu : N \times H :\twoheadrightarrow NH,$

$\texttt{Assume}\ (a,b),(c,d) : N \times H,$

$\texttt{Assume}\ (2) : ab = cd,$

$(3) := \eth\texttt{Normal}(G)(N)(a,b^{-1}) : b^{-1}ab \in N,$

$(4) := (3)(2) : b^{-1}cd \in N,$

$(5) := \eth\texttt{Normal}(G)(N)(4,d) : db^{-1}c \in N,$

$(6) := \texttt{TotalGroupMult}(N)(5) : db^{-1} \in N,$

$(7) := \eth\texttt{Subgroup}(G,H)(d,b^{-1} : db^{-1} \in H,$

$(8) := \eth(0)(5,7) : db^{-1} = e,$

$(9) := (8)b : d = b,$

$(10) := \texttt{TotalGroupMult}(G)(9)(2) : a = c,$

$() := I(=,\times)(9,10) : (a,b) = (c,d);$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Injective} : \mu : N \times H \hookrightarrow NH,$

$(*) := \eth^{-1}\texttt{Bijective}(1,2) : \mu N \times H \leftrightarrow NH;$

$\square$

$\texttt{InnerSemidirectProduct} :: \prod G \in \mathsf{GRP} . ?(\texttt{Normal} \times \texttt{Subgroup}(G))$

$(N,H) : \texttt{InnerSemidirectProduct} \iff G = N \leftthreetimes H \iff N \cap H = \{e\}\ \&\ NH = G$

$\texttt{SemidirectProductAsDirectProduct} :: \forall (0) : G = N \leftthreetimes H . \forall (00) : \gamma_{|H|N} = \mathrm{id}_N . G \cong_{\mathsf{GRP}} N \times H$

$\texttt{Proof} =$

$\texttt{Assume}\ a : H,$

$\texttt{Assume}\ b : N,$

$(1) := (00)(a,b) : aba^{-1} = b,$

$() := (1)(a) : ab = ba;$

$\rightsquigarrow (1) := I(\forall) : \forall a \in H . b \in N . ab = ba,$

$\texttt{Assume}\ h : H,$

$\texttt{Assume}\ x : G,$

$(a,b,2) := (0)_2(x) : \sum a \in N . \sum b \in H . x = ab,$

$(0) := (2)(1)(a,b)(1)(a,h)\eth\texttt{Subgroup}(H) : xhx^{-1} = abhb^{-1}a^{-1} = baha^{-1}b^{-1} = bhb^{-1} \in H;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Normal}(G) : \Big[H : \texttt{Normal}(G)\Big],$

$(*) := \texttt{ProductCondition}(1,2) : H \times N \cong_{\mathsf{GRP}} G;$

$\square$

$\texttt{SemidirectQuotient} :: \forall (0) : G = N \leftthreetimes H \;.\; \frac{G}{N} \cong_{\mathsf{GRP}} H$

$\texttt{Proof} \;=$

$(*) := \texttt{SemodirectProductBijection}(0) : \pi_{|H} : H \xleftrightarrow{\mathsf{GRP}} \frac{G}{N};$

$\square$

$\texttt{OuterSemidirectProductIsGroup} :: \forall G, H \in \mathsf{GRP} \;.\; \forall \varphi : G \xrightarrow{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{GRP}}(H) \;.$

$\quad . \forall (\odot) : (G \times H)^2 \to G \times H \;.\; \forall (0) : \forall (a, b), (c, d) \in (G \times H)^2 \;.\; (a, b) \odot (c, d) = \Big(ac, b\phi(a)(d)\Big) \;.$

$\quad . \Big(G \times H, \odot\Big) \in \mathsf{GRP}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (a, b)(a', b')(a'', b'') : G \times H,$

$() := (0)\eth^2\texttt{Homo}(\phi)(\phi(a)) : ((a, b) \odot (a', b')) \odot (a'', b'') = (aa', b\phi(a)(b')) \odot (a'', b'') =$

$\quad = \Big(aa'a'', b\phi(a)(b')\phi(aa')(b'')\Big) = \Big(aa'a'', b\phi(a)(b')\phi(a)\big(\phi(a')(b'')\big)\Big) = \Big(aa'a'', b\phi(a)\big(b'\phi(a')(b'')\big)\Big) =$

$\quad = (a, b) \odot (a'a'', b'(\phi(a')(b'')) = (a, b) \odot ((a', b') \odot (a'', b''));$

$\leadsto (1) := \eth^{-1}\texttt{Semigroup} : \Big[(G \times H, \odot) : \texttt{Semigroup}\Big],$

$\texttt{Assume}\ (a, b) : G \times H,$

$()_1 := (0)\eth\texttt{Homo}(\phi) : (e, e) \odot (a, b) = (a, \phi(e)(b)) = (a, b),$

$()_2 := (0)\eth\texttt{Homo}(\phi(a)) : (a, b) \odot (e, e) = (a, b\phi(a)(e)) = (a, b);$

$\leadsto (2) := \eth^{-1}\texttt{Monoid} : \Big[(G \times H, \odot) : \texttt{Monoid}\Big],$

$\texttt{Assume}\ (a, b) : G \times H,$

$(c, 3) := \eth\texttt{Auto}(\phi(a)) : \sum c \in H \;.\; \phi(a)(c) = b^{-1},$

$()_1 := (0)(3)\eth\texttt{Inverse} : (a, b) \odot (a^{-1}, c) = (aa^{-1}, b\phi(a)(c)) = (aa^{-1}, bb^{-1}) = (e, e),$

$()_2 := (0)\eth\texttt{homo}(\phi)(\phi(a))(3)\eth\texttt{Inverse} : (a^{-1}, c) \odot (a, b) = \Big(aa^{-1}c\phi(a^{-1})(b)\Big) =$

$\quad = \Big(a^{-1}a, c\phi^{-1}(a)(b)\Big) = (a^{-1}a, cc^{-1}) = (e, e);$

$\leadsto (*) := \eth^{-1}\mathsf{GRP} : (G \times G, \odot) \in \mathsf{GRP},$

$\square$

$\texttt{outerSemidirectProduct} :: \prod G, H \in \mathsf{GRP} \;.\; G \xrightarrow{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{GRP}}(H) \to \mathsf{GRP}$

$\texttt{outerSemidirectProduct}\,(\phi) = G \leftthreetimes_\phi H := \Big(G \times H, \Lambda(a, b), (c, d) \in G \times H \;.\; (ac, b\phi(a)(d))\Big)$

$\texttt{InnerAsOuter} :: \forall (0) : G = N \leftthreetimes H \;.\; G \cong_{\mathsf{GRP}} H \leftthreetimes_\gamma N$

$\texttt{Proof} \;=$

$\mu := \Lambda(a, b) : H \leftthreetimes_\gamma N \;.\; ba : H \leftthreetimes_\gamma N \to G,$

$\texttt{Assume}\ (a, b), (a', b') : H \leftthreetimes_\gamma N,$

$() := \eth H \leftthreetimes_\gamma N(a, b)(a', b')bd\mu\eth\texttt{Inverse}(a)\eth^{-1}\mu :$

$\quad : \mu\Big((a, b)(a', b')\Big) = \mu\Big(aa', bab'a^{-1}\Big) = bab'a^{-1}aa' = bab'a' = \mu(a, b)\mu(a', b');$

$\leadsto (1) := \eth^{-1}\texttt{Homo} : \Big[\mu : H \leftthreetimes_\gamma N \xrightarrow{\mathsf{GRP}} G\Big],$

$(*) := \texttt{SemidirectProductBijection}(0)\eth\mu(1) : \Big[\mu : H \leftthreetimes_\gamma B \xleftrightarrow{\mathsf{GRP}} G\Big];$

$\square$

1.7 Group Extension[!]

1.8 Nillpotent Groups[!]

2 Finite Groups

2.1 Cyclic Groups

$$\texttt{exponent} :: \prod G \in \mathsf{GRP} \,.\, G \to \mathbb{Z} \xrightarrow{\mathsf{GRP}} G$$
$$\texttt{exponent}\,(g, n) = \exp_g(n) := g^n$$

$$\texttt{FiniteGroup} :: ?\texttt{Group}$$
$$G : \texttt{FiniteGroup} \iff |G| < \infty$$

$$\texttt{OrderIsWellDefined} :: \forall G : \texttt{FiniteGroup} \,.\, \forall g \in G \,.\, \exists n \in \mathbb{N} \,.\, g^n = e$$
$$\texttt{Proof} =$$
$$(1) := \eth\texttt{FiniteGroup}(G)(g) : \langle g\rangle < |G| < \infty,$$
$$(n, m, 2) := \eth g(1) : \sum n, m \in \mathbb{N} \,.\, g^n = g^m \;\&\; n < m,$$
$$(*) := (2)g^{-1} : e = g^{n-m};$$
$$\square$$

$$\texttt{order} :: \prod G : \texttt{FiniteGroup} \,.\, G \to \mathbb{N}$$
$$\texttt{order}\,(g) = o(g) := \min \ker \exp_{\mathrm{g}} \cap \mathbb{N}$$

$$\texttt{OrderDivides} :: \forall G : \texttt{FiniteGroup} \,.\, \forall g \in G \,.\, o(g) \,\vdots\, |G|$$
$$\texttt{Proof} =$$
$$(1) := \eth o(g) : o(g) = \big|\langle g\rangle\big|,$$
$$(2) := \texttt{IndexTHM}\big(\langle g\rangle\big) : \big|\langle g\rangle\big|\big[G : \langle g\rangle\big] = |G|,$$
$$(*) := \eth^{-1}\texttt{Divisor}\big(|G|\big)(1)(2) : o(g) \,\vdots\, |G|;$$
$$\square$$

$$\texttt{PrimeOederIsCyclic} :: \forall G : \texttt{FiniteGroup} \,.\, |G| : \texttt{Prime} \Rightarrow G : \texttt{Cyclic}$$
$$\texttt{Proof} =$$
$$(1) := \eth\texttt{Prime}(|G|) : |G| \neq 1,$$
$$(g, 2) := \eth\texttt{Group}(G)\eth\texttt{Cardinality}(1)(e) : \sum g \in G \,.\, g \neq e,$$
$$(3) := \eth\texttt{generate}(2) : \Big|\langle g\rangle\Big| \neq 1,$$
$$(4) := \texttt{OrderDivides}(g)\eth o(g) : \Big|\langle g\rangle\Big| \,\vdots\, |G|,$$
$$(5) := \eth\texttt{Prime}\;|G|(3)(4) : \Big|\langle g\rangle\Big| = |G|,$$
$$(6) := \texttt{FiniteSubsetTHM}(G, \langle g\rangle)\;\texttt{FiniteSetIsoTHM}(G, \langle g\rangle)(5) : \langle g\rangle = G,$$
$$(*) := \eth^{-1}\texttt{Cyclic}(6) : \texttt{This};$$
$$\square$$

$\texttt{PrimeOrderSamePeriod} :: \forall G : \texttt{FiniteGroup} \,.\, \forall (0) : \Big(|G| : \texttt{Prime}\Big) \,.\, \forall g \in G \,.\, \forall (00) : g \neq e \,.\, o(g) = |G|$

$\texttt{Proof} \ =$

$(1) := \texttt{OrderDivides}(g) : o(g) \,\vdots\, |G|,$

$(*) := \eth\texttt{Prime}\, |G|(1)(0)(00) : o(g) = |G|;$

$\square$

$\texttt{IntIsCyclic} :: (\mathbb{Z} : \texttt{Cyclic})$

$\texttt{Proof} \ =$

$(1) := \eth\mathbb{Z} : \mathbb{Z} = \langle 1 \rangle,$

$(*) := \eth^{-1}\texttt{Cyclic}(1) : (\mathbb{Z} : \texttt{Cyclic});$

$\square$

$\texttt{IntSubgroupIsCyclic} :: \forall N \lhd \mathbb{Z} \,.\, N : \texttt{Cyclic}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : N \neq \{0\},$

$n := \min N \cap \mathbb{N} : \mathbb{N},$

$\texttt{Assume}\ a, b : N,$

$(s, r, 2) := \texttt{GCDAlgorithm}\Big(|a|, |b|\Big) : \sum s, r \in \mathbb{Z} \,.\, \gcd\Big(|a|, |b|\Big) = s|a| + r|b|,$

$() := \eth\texttt{Subgroup}(\mathbb{Z})(N)(2) : \gcd\Big(|a|, |b|\Big) \in N;$

$\rightsquigarrow (2) := I(\forall) : \forall a, b \in N \,.\, \gcd\Big(|a|, |b|\Big) \in N,$

$\texttt{Assume}\ g : N,$

$(3) := (2)(n, g) : \gcd\Big(n, |g|\Big) \in N,$

$(4) := \eth\gcd\Big(n, |g|\Big)\texttt{DivisorSize}(n) : 0 < \gcd\Big(n, |g|\Big) \leq n,$

$(5) := \eth n(4) : \gcd\Big(n, |g|\Big) = n,$

$(6) := \eth\gcd\Big(n, |g|\Big)(5) : n \,\vdots\, |g|;$

$\rightsquigarrow (3) := I(\forall) : \forall g \in N \,.\, n \,\vdots\, |g|,$

$(4) := \eth\mathbb{Z}(3)\eth n : N = \langle n \rangle,$

$(5) := \eth^{-1}\texttt{Cyclic} : (N : \texttt{Cyclic});$

$\ldots$

$\square$

$\texttt{ImageOfCyclicIsCyclic} :: \forall G \in \mathsf{GRP} \,.\, \forall Z : \texttt{Cyclic} \,.\, \forall f : Z \xrightarrow{\mathsf{GRP}} G \,.\, f(Z) : \texttt{Cyclic}$

$\texttt{Proof} \ =$

$(z, 1) := \eth\texttt{Cyclic} : \sum z \in Z \,.\, Z = \langle z \rangle,$

$(2) := \eth\texttt{Image}\eth f \eth\texttt{generate}(z) : f(Z) = \langle f(z) \rangle,$

$(*) := \eth^{-1}\texttt{Cyclic} : f(Z) : \texttt{Cyclic};$

$\square$

$\texttt{QuetientOfCyclicIsCyclic} :: \forall Z : \texttt{Cyclic} \,.\, \forall N \subset_{\mathsf{GRP}} Z \,.\, \frac{Z}{N} : \texttt{Cyclic}$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{quetientGroup}(Z, N)\eth^{-1}\pi_N : \frac{Z}{N} = \pi_N(Z),$

$(2) := \texttt{ImageOfCyclicIsCyclic}(1) : \frac{Z}{N} = \pi_N(Z);$

$\square$

$\texttt{nElementCyclic} :: \mathbb{N} \to \texttt{Cyclic}$

$\texttt{nElementCyclic}\,(n) = Z_n := \frac{\mathbb{Z}}{n\mathbb{Z}}$

$\texttt{SubgroupOfCyclicIsCyclic} :: \forall Z : \texttt{Cyclic} \,.\, \forall N \subset_{\mathsf{GRP}} Z \,.\, N : \texttt{Cyclic}$

$\texttt{Proof} \;=$

$(z, 1) := \eth\texttt{Cyclic}(Z) : \forall z \in Z \,.\, \langle z \rangle = Z,$

$f := \exp_z : \mathbb{Z} \xrightarrow{\mathsf{GRP}} Z,$

$M := f^{-1}(N) : \texttt{Subgroup}(Z),$

$(2) := \texttt{IntSubgroupIsCyclic}(M) : (M : \texttt{Cyclic}),$

$(3) := \eth M \eth\texttt{Preimage} : f(M) = N,$

$(*) := \texttt{ImageOfCyclicIsCyclic}(2)(3) : (N : \texttt{Cyclic});$

$\square$

$\texttt{Generator} :: \prod G : \mathsf{GRP} \,.\, ?G$

$g : \texttt{Generator} \iff \langle g \rangle = G$

$\texttt{GeneratorsOfIntegers} :: \texttt{Generator}(\mathbb{Z}) = \{1, -1\}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; n : \texttt{Generator}Int,$

$(1) := \eth\texttt{Generator}(\mathbb{Z})(n) : \mathbb{Z} = \langle n \rangle,$

$(k, 2) := \texttt{genGroup}(1)[1] : \sum k \in \mathbb{Z} \,.\, 1 = kn,$

$(*) := \texttt{UnitDivisor}(2) : n \in \{1, -1\};$

$\square$

$\texttt{Integerlike} :: \forall Z : \texttt{Cyclic} \,.\, \forall(0) : |Z| = \infty \,.\, Z \cong_{\mathsf{GRP}} \mathbb{Z}$

$\texttt{Proof} \;=$

$g := \eth\texttt{Cyclic}(Z) : \texttt{Generator}(Z),$

$f := \exp_g : \mathbb{Z} \to_{\mathsf{GRP}} Z,$

$(1) := \eth\texttt{Generator}(Z)(g)\eth f(0) : (f : \texttt{Injective}),$

$(2) := \eth\texttt{Generator}(Z)(g)\eth f : (f : \texttt{Surjective}),$

$(3) := \eth^{-1}\texttt{Bijection}(1)(2) : (f : \texttt{Bijection}),$

$(4) := \eth\texttt{Isomorphic}(\eth f)(3) : Z \cong \mathbb{Z};$

$\square$

$\texttt{GeneratorsImageIsGenerator} :: \forall A, B \in \mathsf{GRP} . \forall \varphi : A \leftrightarrow_{\mathsf{GRP}} B .$
$\forall a : \texttt{Generator}(A) . \varphi(a) : \texttt{Generator}(B)$

$\texttt{Proof} \ =$

Same proof as in ImageOfCyclicIsCyclic

$\square$

$\texttt{GeneratorsOfInfiniteCyclic} :: \forall Z : \texttt{Cyclic} . \forall (0) : |Z| = \infty . \#\texttt{Generator}(Z) = 2$

$\texttt{Proof} \ =$

Combine GeneratorsImageIsGenerator, Integerlike and GeneratorsOfIntegers

$\square$

$$\texttt{OrderOfPower} :: \forall G : \texttt{FiniteGroup} . \forall g \in G . \forall n \in \mathbb{N} . o(g^n) = \frac{o(g)}{\gcd(o(g), n)}$$

$\texttt{Proof} \ =$

$m := \dfrac{o(g)}{\gcd(o(g), n)} : \mathbb{N},$

$(k, 1) := \eth\texttt{Divisor}(n)\ \texttt{GCDIsDivisor}\eth m : \sum k \in \mathbb{N} . mn = ko(g),$

$(2) := \eth o(g)(1)(g^{mn}) : e = g^{ko(g)} = g^m n,$

$\texttt{Assume}\ l : \mathbb{N},$

$\texttt{Assume}\ (3) : l < m,$

$(u, r, 4) := \texttt{ReminderDivision}(nl, o(g)) : \sum u, r \in \mathbb{Z}_+ . nl = uo(g) + r \ \& \ r < o(g),$

$(5) := (3)\eth m\eth\texttt{Divisor} : nl \not| \ \texttt{Divisor}(o(g)),$

$(6) := \eth\texttt{Divisor}(4)(5) : r \neq 0,$

$() := (4)(g^{nl})(5)\eth o(g)(4) : g^{nl} = g^{uo(g)} g^r = g^r \neq e;$

$\leadsto (*) := \eth^{-1} o(g^n)(2) : o(g^n) = m;$

$\square$

$\texttt{GeneratorsByCoprime} :: \forall G \in \mathsf{GRP} . \forall n \in \mathbb{N} . \forall (0) : |G| = n . \forall g : \texttt{Generator}(G) .$
$. \texttt{Generator}(G) = \left\{ g^k \middle| k : \texttt{Coprime}(n) \right\}$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{Generator}(G)(g)\eth^{-1}(o(g))(0) : o(g) = n,$

$\texttt{Assume}\ k : \texttt{Coprime}(n),$

$(2) := \texttt{OrderOfPower}(g, k)\eth\texttt{Coprime}(n)(k)(1) : o(g^k) = \dfrac{o(g)}{\gcd(o(g), k)} = n,$

$() := \eth^{-1}\texttt{Generator}(G)(g)\eth o(g^k)(2) : \left(g^k : \texttt{Generator}(G) \right);$

$\leadsto (2) := \eth\texttt{Subset} : \left\{ g^k \middle| k : \texttt{Coprime}(n) \right\} \subset \texttt{Generator}(G),$

$\texttt{Assume}\ h : \texttt{Generator}(G),$

$(3) := \eth\texttt{Generator}(G))(h)\eth^{-1}(o(h))(0) : o(h) = n,$

$(k, 4) := \eth\texttt{Generator}(G)(g)(h) : \sum k \in \mathbb{N} . g^k = h,$

$() := \eth^{-1}\texttt{Coprime}(n)\texttt{NeutralDivision}\ \texttt{OrderOfPower}(4)(3) : \left(k : \texttt{Coprime}(n) \right);$

$\leadsto (*) := \eth^{-1}\texttt{SetEq}(2) : \texttt{This};$

$\square$

$\texttt{CyclicAuto} :: \forall G \in \textsf{GRP} \,.\, \forall a, b : \texttt{Generator}(G) \,.\, \exists! \varphi \in \mathrm{Aut}_{\textsf{GRP}}(G) \,.\, \varphi(a) = b$

$\texttt{Proof} \;=$

Trivially define $\varphi(a^k) = b^k$

$\square$

$\texttt{CyclicDividingSubgroup} :: \forall G : \texttt{Cyclic} \,.\, \forall n \in \mathbb{N} \,.\, \forall (0) : |G| = n \,.$

$\quad .\, \forall d : \texttt{Divisor}(n) \,.\, \exists! N \subset_{\textsf{GRP}} G : |N| = d$

$\texttt{Proof} \;=$

$m := \dfrac{n}{d} : \mathbb{N},$

$g := \eth\texttt{Cyclic} : \texttt{Generator}(G),$

$N := \langle g^m \rangle : \texttt{Subgroup}(G),$

$(1) := \texttt{OrderOfPower}(\eth N) : |N| = d,$

$\texttt{Assume}\ H : H \subset_{\textsf{GRP}} G,$

$\texttt{Assume}\ (2) : |H| = d,$

$(3) := \texttt{CyclicSubgroup} : \Big(H : \texttt{Cyclic}\Big),$

$(k, 4) := \eth\texttt{Cyclic}(H) : \sum k \in \mathbb{N} \,.\, \langle g^k \rangle = H,$

$(5) := (2)(4)\eth o(g^k) : o(g^k) = d,$

$(6) := \texttt{OrderOfPower}(H)(4)(5) : \gcd(k, d) = m,$

$(l, 7) := \eth\gcd(k, m)(6) : \sum l : \texttt{Coprime}(n) \,.\, k = ml,$

$(8) := (7)\eth^7 H : g^k = (g^m)^l \in H,$

$(6) := \eth\texttt{generateGroup}\eth N(1)(2)(5) : H = N;$

$\rightsquigarrow (*) := I\texttt{Unique} : \mathit{This},$

$\square$

$\texttt{CyclicProduct} :: \forall A, B : \texttt{Cyclic} \,.\, \forall (a, b) : \texttt{Coprime}(\mathbb{Z}) \,.\, \forall (0) : |A| = a \,\&\, |B| = b \,.\, A \times B : \texttt{Cyclic}$

$\texttt{Proof} \;=$

Apply $\texttt{ChineseReminder}(\mathbb{Z})$

$\square$

$\texttt{CyclicByNumberOfSubgroups} :: \forall G \in \textsf{ABEL} \,.\, \forall n \in \mathbb{N} \,.\, \forall (0) : |G| = n \,.$

$\quad .\, \forall (00) : \forall d : \texttt{Divisor}(n) \,.\, \#\{H \subset_{\textsf{GRP}} G : |H| = d\} \le 1 \,.\, G : \texttt{Cyclic}$

$\texttt{Proof} \;=$

$a := \Lambda m \in \mathbb{N} \,.\, \#\{g \in G : o(g) = m\} : \mathbb{N} \to \mathbb{N},$

$u := \Lambda m \in \mathbb{N} \,.\, \#\{H \subset_{\textsf{GRP}} G : |H| = n \,\&\, H : \texttt{Cyclic}\} : \mathbb{N} \to \mathbb{N},$

$(1) := \texttt{DivisorSum}(n)(0) : \sum_{d|n} \phi(d) = n = |G|,$

$(2) := \texttt{GeneratorByCoprime} : \forall m \in \mathbb{N} \,.\, a_m = \phi(m) u_m,$

$(3) := \eth a \eth o \texttt{OrderDivides}(2)(00)(1) : |G| = \sum_{d|n} a_n = \sum_{d|n} u_n \phi(n) \le \sum \phi(n) = |G|,$

$(4) := \texttt{MaximalSum}(00)(3) : \forall d \in \texttt{Divisor}(n) \,.\, u_d = 1,$

$(5) := (4)(n) : u_n = 1,$

$\eth u_n(5) := \Big[G : \texttt{Cyclic}\Big] :$ $\square$

;

2.2 Symmetric Group

$\texttt{symmetricGroup} :: \mathbb{N} \to \texttt{FiniteGroup}$
$\texttt{symmetricGroup}\,(n) = S_n := \mathrm{Aut}_{\mathsf{SET}}(n)$

$\texttt{KCycle} :: \prod n \in \mathbb{N} \,.\, n \to ?S_n$
$\sigma : \texttt{KCycle}(k) \iff \sigma : k\text{-}\texttt{Cycle} \iff \exists t : k \to n \,.\, \forall i \in k \,.\, \sigma(t_i) = \sigma(t_{i+1})\ \&$
$\quad \&\ \forall j \in (\mathrm{Im}\, t)^{\complement} \,.\, \sigma(j) = j$

$\texttt{Cycle} :: \prod n \in \mathbb{N} \,.\, ?S_n$
$\sigma : \texttt{Cycle} \iff \exists k \in n \,.\, \sigma : k\text{-}\texttt{Cycle}$

$\texttt{support} :: \prod n \in \mathbb{N} \,.\, S_n \to ?n$
$\texttt{support}\,(\sigma) = \mathrm{supp}(\sigma) := \Big(\texttt{fixedPoints}(\sigma)\Big)^{\complement}$

$\texttt{NonIntersecting} :: \prod n \in \mathbb{N} \,.\, ?\texttt{Cycle} \times \texttt{Cycle}(n)$
$(\sigma, \tau) : \texttt{NonIntersecting} \iff \mathrm{supp}\,\sigma \cap \mathrm{supp}\,\tau = \emptyset$

$\texttt{PairwiseNonIntersecting} :: \prod \in \mathbb{N} \,.\, ??\texttt{Cycle}(n)$
$A : \texttt{PairwiseNonIntersecting} \iff \forall \sigma, \tau \in A \,.\, \sigma \neq \tau \Rightarrow (\sigma, \tau) : \texttt{NonIntersecting}(n)$

$\texttt{cycle} :: \prod n \in \mathbb{N} \,.\, \prod k \in \mathbb{N} \,.\, (k \to n) \to k\text{-}\texttt{Cycle}(n)$
$\texttt{cycle}\,(a) = (a_1 a_2 \dots a_n) := \Lambda i \in n \,.\, \textbf{if } i = a_j \textbf{ then } a_{j+1} \textbf{ else } i$

$\texttt{NonIntersectingCyclesCommute} :: \forall n \in \mathbb{N} \,.\, \forall (\sigma, \tau) : \texttt{PairwiseNonIntersecting}(n) \,.\, \sigma\tau = \tau\sigma$
$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NIProduct} :: \forall n, t \in \mathbb{N} \,.\, \forall z : t \to \texttt{Cycle}(n) \,.\, \forall (0) : \mathrm{Im}\, z : \texttt{PairwiseNonIntersecting}(n) \,.\, \forall i \in t \,.$
$$.\ \prod_{j=1}^{t} z_j = \left(\prod_{j=1 : j \neq i}^{t} z_j\right) z_i$$
$\texttt{Proof} =$

Apply Commutativity

$\square$

$\texttt{NonIntersectingCycleDecomposition} :: \forall n \in \mathbb{N} . \forall \sigma \in S_n .$

$$. \exists! t \in \mathbb{N} : \exists! Z : \texttt{PairwiseNonIntersecting}(n) : \exists z : t \leftrightarrow_{\mathsf{SET}} Z : \sigma = \prod_{i=1}^{t} z_i$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ i : n,$

$F := \{k \in \mathbb{N} : \sigma^k(i) = i\} :?\mathbb{N},$

$(1) := \eth\mathrm{Aut}_{\mathsf{SET}}(n)(\sigma)(\eth F) : F \neq \emptyset,$

$k := \min F : \mathbb{N},$

$t := \Lambda j \in k . \sigma^j(i) : k \to n,$

$z_i := (t_1, \dots, t_k) : k\text{-}\texttt{Cycle}(n);$

$\rightsquigarrow z := I(\to) : n \to \texttt{Cycle}(n),$

$Z := \{z_i | i \in n\} :?\texttt{Cycle}(n),$

$t := |Z| : \mathbb{N},$

$\texttt{Assume}\ \alpha, \beta : Z,$

$\texttt{Assume}\ (1) : \alpha \neq \beta,$

$(i, 2) := E(\#, \to)(1) : \sum i \in n . \alpha(i) \neq \beta(i),$

$\texttt{Assume}\ (3) : \alpha(i) \neq i,$

$(4) := \eth(\alpha)\eth(Z)\eth(z) : \operatorname{supp} \alpha = \{\sigma^k(i) | k \in \mathbb{N}\},$

$\texttt{Assume}\ (5) : \operatorname{supp} \alpha \cap \operatorname{supp} \beta \neq \emptyset,$

$j := \eth\texttt{NonEmpty}(5) : j \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta,$

$(k, 6) := (4)\eth\texttt{intersect}(\operatorname{supp} \alpha, \operatorname{supp} \beta)(\eth j) : \sum k \in \mathbb{N} . j = \sigma^k(i),$

$(7) := bd\texttt{Cycle}(n)(\beta)(6)(\eth j) : i \in \operatorname{supp} \beta,$

$(8) := (4)\eth\beta\eth Z(7) : \alpha(i) = \sigma(i) = \beta(i),$

$() := (8)(3) : \bot;$

$\rightsquigarrow (6) := \eth^{-1}\texttt{NonInterseting}(n)E(\bot) : (\alpha, \beta) : \texttt{NonIntersecting}(n);$

$\dots$

$\rightsquigarrow (2) := \eth^{-1}\texttt{PairwiseNonIntersecting}I(\forall)I(\Rightarrow)E(|)(\dots) : \Big(Z : \texttt{PairwiseNonIntersecting}(n)\Big),$

$z' := \texttt{enumerate}(Z) : t \leftrightarrow_{\mathsf{SET}} Z,$

$$(0^*) := \eth z' \eth Z : \prod_{i=1}^{t} z'_i = \sigma,$$

$$\texttt{Assume}\ (s, Q, q, 3) : \sum s \in \mathbb{N} . \sum Q : \texttt{PairwiseNonIntersecting}(n) . \sum q : s \leftrightarrow_{\mathsf{SET}} Q . \sigma = \prod_{i=1}^{s} q_i,$$

$\texttt{Assume}\ i : s,$

$(j, k, 4)) := \eth\texttt{Cycle}(3)(i) : \sum j, k \in n . \operatorname{supp} q_i = \{\sigma^l(j) | l \in k\},$

$(5) := \eth\texttt{PairwiseNonIntersecting}(n)(Q)(3)(4) : q_i(\sigma^k(j)) = j,$

$(6) := \eth Z(5) : q_i \in Z,$

$(5) := \eth\texttt{Bijection}(q)(4) : Z \setminus \{e\} = Q \setminus \{e\} \ \& \ t = s;$

$\rightsquigarrow (*) := I(\texttt{Unique}) : \texttt{This},$

$\square$

$\texttt{kSign} :: \prod n \in \mathbb{N} \,.\, \prod k \in n \,.\, k\text{-}\texttt{Cycle}(n) \to \texttt{Sign}$

$\texttt{kSign}\,(z) := (-1)^{k+1}$

$\texttt{sign} :: \prod n \in \mathbb{N} \,.\, S_n \to \texttt{Sign}$

$$\texttt{sign}\,(\sigma) = (-1)^\sigma := \prod_{i=1}^{t} \texttt{kSign}(z_i)$$

where

$(t, Z, z) = \texttt{NonIntersectingCycleDecomposition}(n, \sigma)$

$\texttt{SignByTranspositions} :: \forall n \in \mathbb{N} \,.\, \forall k \in \mathbb{Z}_+ \,.\, \forall \sigma \in S_n \,.\, \forall \tau : k \to 2\text{-}\texttt{Cycle}(n) \,.$

$$.\, \forall(0) : \sigma = \prod_{i=1}^{k} \tau_i \,.\, (-1)^\sigma = (-1)^k$$

$\texttt{Proof} =$

$$R := \Lambda k \in \mathbb{Z}_+ \,.\, \forall \sigma \in S_n \,.\, \forall \tau : k \to 2\text{-}\texttt{Cycle}(n) \,.\, \forall(0) : \sigma = \prod_{i=1}^{k} \tau_i \,.\, (-1)^\sigma = (-1)^k : \mathbb{Z}_+ \to \texttt{Type},$$

$\texttt{Assume}\ (1) : k = 0,$

$(2) := (0)(1) : \sigma = e,$

$() := (2)\eth\texttt{sign}(e)\texttt{PowerZero} : (-1)^\sigma = (-1)^e = (-1)^2 = 1 = (-1)^0;$

$\rightsquigarrow (2) := \eth R : R(0),$

$\texttt{Assume}\ k : \mathbb{N},$

$\texttt{Assume}\ (3) : R(k-1),$

$$\sigma' := \prod_{i=1}^{k-1} \tau_i : S_n,$$

$(4) := R(k-1)(\sigma', \tau_{|k-1}, \eth\sigma') : (-1)^{\sigma'} = (-1)^{k-1},$

$(t, Z, z, 5) := \texttt{NonIntersectingCycleDecomposition}(n, \sigma') : \sum t \in n \,.$

$$.\, \sum Z : \texttt{NonIntersectingCycle}(n) \,.\, z : t \leftrightarrow_{\mathsf{SET}} Z \,.\, \sigma' = \prod_{i=1}^{t} z_i,$$

$$(6) := (0)\eth\sigma'(5) : \sigma = \left(\prod_{i=1}^{t} z_i\right) \tau_k,$$

$(a, b, 7) := \eth 2\text{-}\texttt{Cycle}(n)(\tau_k) : \tau_k = (ab),$

$\texttt{Assume}\ (8) : \forall i \in t \,.\, a, b \notin \operatorname{supp} z_i,$

$$() := \eth\texttt{sign}(4)(6)(8) : (-1)^\sigma = \texttt{sign}\left(\prod_{i=1}^{t} z_i\right) \texttt{sign}(\tau_k) = -(-1)^{k-1} = (-1)^k;$$

$\rightsquigarrow (8) := I(\Rightarrow) : \texttt{sign}(\sigma) = (-1)^k,$

$\texttt{Assume}\ (9) : \forall i \in t \,.\, a \notin \operatorname{supp} z_i \ \&\ \exists i \in t : b \in \operatorname{supp} z_i,$

$(i, 10)) := E(\exists) : \sum i \in t \,.\, b \in \operatorname{supp} z_i,$

$$(11) := \texttt{NIProduct}(n, t)(z)(i) : \sigma' = \left(\prod_{j=1, j \neq i}^{t} z_j\right) z_i,$$

$(l,c,12) := \eth\mathtt{Cycle}(z_i) : \sum l \in n \,.\, \sum c : l \to n \,.\, z_i = (c_1 \dots c_l b),$

$(13) := \eth S_n(12)(7) z_i \tau_k : z_i \tau_k = (c_1 \dots c_l ba),$

$() := \eth\mathtt{Sign}(6)\eth\mathtt{kSign}(z_i)\tau_k(13)(4) : (-1)^\sigma = \left(\prod_{j=1:j\neq i}^{t} \mathtt{kSign}(z_j)\right) \mathtt{kSign}(c_1 \dots c_l ba) = -(-1)^{k-1} = (-1)^k;$

$\rightsquigarrow (9) := I(\Rightarrow) : \forall i \in t \,.\, b \notin \operatorname{supp} z_i \;\&\; \exists a \in \operatorname{supp} z_i \Rightarrow (-1)^\sigma = (-1)^k,$

$\mathtt{Assume}\ (10) : \exists i,j \in t \,.\, a \in \operatorname{supp} z_i \;\&\; b \in \operatorname{supp} z_j,$

$\mathtt{Assume}\ (11) : z_i = z_j,$

$(l,m,s,c,12) := \eth\mathtt{Cycle}(z_i) : \sum l \in n \,.\, \sum s,m \in l \,.\, \sum c : l \to n \,.\, z_i = (c_1 \dots c_s a c_{s+1} \dots c_m b c_{m+1} \dots c_l),$

$(13) := \mathtt{NIProduct}(n,t)(z)(i) : \sigma' = \left(\prod_{j=1,j\neq i}^{t} z_j\right) z_i,$

$(14) := \eth S_n(12)(7) : z_i \tau_k = (c_1 \dots c_s a c_{m+1} \dots c_l)(c_{s+1} \dots c_m b),$

$() := \eth\mathtt{sign}(\sigma)(6)(13)\eth\mathtt{kSign}(14) : (-1)^\sigma = (-1)^k;$

$\rightsquigarrow (11) := I(\Rightarrow) : z_i = z_j \Rightarrow (-1)^\sigma = (-1)^k,$

$\mathtt{Assume}\ (12) : z_i \neq z_j,$

$(l,c,13) := \eth\mathtt{Cycle}(z_i) : \sum l \in n \,.\, \sum c : l \to n \,.\, z_i = (c_1 \dots c_l a),$

$(l',c',14) := \eth\mathtt{Cycle}(z_j) : \sum l' \in n \,.\, \sum c' : l' \to n \,.\, z_j = (bc'_1 \dots c'_{l'} b),$

$(15) := \left(\mathtt{NIProduct}(n,t)(z)\right)^2 (i)(j) : \sigma' = \left(\prod_{u=1,u\notin\{i,j\}}^{t} z_u\right) z_i z_j,$

$(16) := \eth S_n(13)(14)(7) : z_i z_j \tau_k = (c1 \dots c_l a b c'_1 \dots c'_l),$

$() := \eth\mathtt{sign}(\sigma)\eth\mathtt{kSign}(15)(16) : (-1)^\sigma = (-1)^k;$

$\rightsquigarrow (12) := I(\Rightarrow) : z_i \neq z_j \Rightarrow (-1)^\sigma = (-1)^k,$

$(13) := \mathtt{LEM}(z_i = z_j) : z_i \neq z_j | z_i = z_j,$

$() := E(|)(13)(12,11) : (-1)^\sigma = (-1)^k;$

$\rightsquigarrow (10) := I(\Rightarrow) : \exists i,j \in t \,.\, a \in \operatorname{supp} z_i \;\&\; b \in \operatorname{supp} z_j \Rightarrow (-1)^\sigma = (-1)^k,$

$() := E(|)\mathtt{LEM}(\forall i \in t \,.\, a,b \notin \operatorname{supp} z_i)(8,9,10) : (-1)^\sigma = (-1)^k;$

$\rightsquigarrow (3) := I(\forall)I(\Rightarrow) : \forall k \in \mathbb{N} \,.\, R(k-1) \Rightarrow R(k),$

$(*) := \mathtt{Induction}\eth\mathtt{WellFounded}(\mathbb{Z}_+)(2,3) : \mathtt{This};$

$\square$

$\mathtt{CycleByTransposition} :: \forall n \in \mathbb{N} \,.\, \forall z : \mathtt{Cycle}(n) \,.\, \exists t \in \mathbb{N} : \exists \tau : t \to 2\text{-}\mathtt{Cycle}(n) \,.\, z = \prod_{i=1}^{t} \tau_i$

$\mathtt{Proof}\ =$

Write $(a_1 \dots a_k) = \prod_{i=2}^{k} (a_{i-1} a_i)$

$\square$

$\texttt{PermutationByTransposition} :: \forall n \in \mathbb{N} . \forall \sigma \in S_n . \exists t \in \mathbb{N} : \exists \tau : t \to 2\text{-}\texttt{Cycle}(n) . \sigma = \prod_{i=1}^{t} \tau_i$

$\texttt{Proof} =$

Combine $\texttt{NonIntersectingCycleDecomposition}$ and $\texttt{CycleByTransposition}$.

□

$\texttt{TranspositionsGenPermutations} :: \forall n \in \mathbb{N} . S_n = \langle 2\text{-}\texttt{Cycle}(n) \rangle$

$\texttt{Proof} =$

Direct consequence of $\texttt{PermutationByTransposition}(n)$.

□

$\texttt{SignIsHomomorphism} :: \forall n \in \mathbb{N} . \texttt{sign}_{S_n} : S_n \to_{\mathsf{GRP}} \texttt{Sign}$

$\texttt{Proof} =$

$\texttt{Assume}\ \sigma, \tau : S_n,$

$(s, z, 1) := \texttt{TransmutationByTransposition}(n)(\sigma) : \prod s \in \mathbb{N} . \prod z : s \to 2\text{-}\texttt{Cycle} . \sigma = \prod_{i=1}^{s} z_i,$

$(t, z', 2) := \texttt{TransmutationByTransposition}(n)(\tau) : \prod t \in \mathbb{N} . \prod z' : t \to 2\text{-}\texttt{Cycle} . \tau = \prod_{i=1}^{t} z'_i,$

$(3) := (1, 2)(\texttt{sign}(\sigma\tau))\texttt{SignByTransmutations}\eth\texttt{SignSignByTransmutations}\eth^{-1}\texttt{sign}^2(\sigma)(\tau) :$

$$: \texttt{sign}\,(\tau\sigma) = \texttt{sign}\left(\prod_{i=1}^{t} z_i \prod_{i=1}^{s} z'_i\right) = (-1)^{t+s} = (-1)^t(-1)^s =$$

$$= \texttt{sign}\left(\prod_{i=1}^{t} z_i\right) \texttt{sign}\left(\prod_{i=1}^{s} z'_i\right) = \texttt{sign}(\sigma)\texttt{sign}(\tau);$$

$\leadsto (4) := \eth^{-1}\mathcal{M}_{\mathsf{GRP}}(S_n, \texttt{Sign}) : (\texttt{sign} : S_n \to_{\mathsf{GRP}} \texttt{Sign}),$

□

$\texttt{AlternatingGroup} :: \prod n \in \mathbb{N} . \texttt{Normal}(S_n)$

$\texttt{AlternatingGroup}\,() = A_n := \ker \texttt{sign}_{S_n}$

$\texttt{cycleStructure} :: \prod n \in \mathbb{N} . S_n \to n \to \mathbb{Z}_+$

$\texttt{cycleStructure}\,(\sigma, k) := \texttt{if}\ k == 1\ \texttt{then}\ |\texttt{fixedPoints}(\sigma)|\ \texttt{else}\ \left|\left\{i \in t \middle| |\operatorname{supp} z_i| = k\right\}\right|$

$\texttt{where}$

$(t, Z, z) = \texttt{NonIntersectingCycleDecomposition}(n, \sigma)$

$\texttt{CycleStructureIsPreservedByConjugationForCycles} :: \forall n \in \mathbb{N} . \forall z : \texttt{Cycle}(n) \forall \sigma \in S_n .$

$\texttt{cycleStructure}\left(\sigma z \sigma^{-1}\right) = \texttt{cycleStructure}(z)$

$\texttt{Proof} =$

$\sigma(a_1 \dots a_n)\sigma^{-1} = (\sigma(a_1) \dots \sigma(a_n))$

□

$\texttt{CycleStructureIsPreservedUnderConcjugation} :: \forall n \in \mathbb{N} . \forall \sigma, \alpha \in S_n .$
$\quad \texttt{cycleStructure}(\sigma\alpha\sigma^{-1}) = \texttt{cycleStructure}(\alpha)$
$\texttt{Proof} \; =$

From $\texttt{NonIntersectingCycleDecomposition}$ take $\alpha = \prod_{i=1}^{t} z_i$,

then $\sigma \left(\prod_{i=1}^{t} z_i \right) \sigma^{-1} = \prod_{i=1}^{t} \sigma z_i \sigma^{-1}$,

and apply $\texttt{CycleStructurePreservedByCinjugationForCycles}$ to get result.
$\square$

$\texttt{AlternatingGroupBy3Cycles} :: \forall n \in \mathbb{N} . \forall (0) : n > 4 . A_n = \langle 3\text{-}\texttt{Cycle}(n) \rangle$
$\texttt{Proof} \; =$
$\texttt{Assume } \sigma : \texttt{In}(A_n)$,

$(k, z, 1) := \texttt{PermutataionByTranspositions}(n, \sigma) : \sum k \in \mathbb{Z}_+ . z : k \to 2\text{-}\texttt{Cycle}(n) . \sigma = \prod_{i=1}^{k}$,

$(2) := \eth A_n \texttt{SignByTransposition}(1) : \Big(2 : \texttt{Divides}(k)\Big)$,

$k(\sigma) := \frac{k}{2} : \mathbb{Z}_+$;

$\rightsquigarrow k := I \left(\prod\right) I \left(\sum\right) : \prod \sigma \in A_n . \sum k \in \mathbb{Z}_+ . \exists z : 2k \to 2\text{-}\texttt{Cycle}(n) : \sigma = \prod_{i=1}^{2k} z_i$,

$♀ := \Lambda m \in \mathbb{Z}_+ . \forall \sigma \in A_n . \forall ♂ : k(\sigma) = m . \exists l \in \mathbb{Z}_+ . \exists \tau : l \to 3\text{-}\texttt{Cycle}(n) . \sigma = \prod_{i=1}^{l} \tau_i : \mathbb{Z}_+ \to \texttt{Type}$,

$\texttt{Assume } \sigma : A_n$,
$\texttt{Assume } ♂ : k(\sigma) = 0$,
$(1) := \eth k(\sigma) : \sigma = e$;
$\rightsquigarrow (1) := \eth^{-1} ♀ : ♀(0)$,
$\texttt{Assume } m : \mathbb{Z}$,
$\texttt{Assume } (2) : ♀(m-1)$,
$\texttt{Assume } \sigma : A_n$,
$\texttt{Assume } ♂ : k(\sigma) = m$,

$(z, 3) := \eth k ♂ : \sum z : 2k \to 2\text{-}\texttt{Cycle}(n) . \sigma = \prod_{i=1}^{2k} z_i$,

$\sigma' := \prod_{i=1}^{2(k-1)} z_i : \texttt{In}(S_n)$,

$(4) := \texttt{SignByTranspositions}(\sigma') \eth \sigma' : \sigma' \in A_n$,

$(l, \tau, 5) := \eth ♀(2)(\sigma') : \sum l \in \mathbb{Z}_+ . \sum \tau : l \to 3\text{-}\texttt{Cycle}(n) . \sigma' = \prod_{i=1}^{l} \tau_i$,

$(6) := (3) \eth \sigma' (5) : \sigma = \left(\prod_{i=1}^{l} \tau_i \right) z_{2k-1} z_{2k}$,

Assume $(7) : \operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k} \neq \emptyset$,
Assume $(8) : |\operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k}| = 2$,
$() := \eth \operatorname{supp} \eth 2\text{-}\mathtt{Cycle}(n)(z_{2k-1}z_{2k})(8) : z_{2k-1}z_{2k} = e$;
$\rightsquigarrow (8) := I(\Rightarrow)\eth^{-1}♀(m)(\sigma)(6) : |\operatorname{supp} z_{2-1} \cap \operatorname{supp} z_{2k}| = 2 \Rightarrow ♀(m)(\sigma)$,
Assume $(9) : |\operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k}| = 1$,
$(a, b, c, 10) := \eth \operatorname{supp} \eth 2\text{-}\mathtt{Cycle}(n)(z_{2k}, z_{2k-1}) : \sum a, b, c \in n \,.\, z_{2k-1} = (ab) \,\&\, z_{2k} = (ac)$,
$() := \eth S_n(z_{2k-1}z_{2k})(10) : z_{2k-1}z_{2k} = (ab)(ac) = (acb)$;
$\rightsquigarrow (9) := I(\Rightarrow)\eth^{-1}♀(m)(6) : |\operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k}| = 1 \Rightarrow ♀(m)(\sigma)$,
$:= \eth^{-1} \operatorname{supp} \eth 2\text{-}\mathtt{Cycle}(z_{2k-1}z_{2k})E(|)(8, 9) : ♀(m)(\sigma)$;
$\rightsquigarrow (7) := I(\Rightarrow) : \operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k} \neq \emptyset \Rightarrow ♀(m)(\sigma)$,
Assume $(8) : \operatorname{supp} z_{2k-1} \cap z_{2k} = \emptyset$,
$(a, b, c, d, 9) := \eth 2\text{-}\mathtt{Cycle}(z_{2k}, z_{2k-1}) : \sum a, b, c, d \in n \,.\, z_{2k-1} = (ab) \,\&\, z_{2k} = (cd)$,
$(f, 10) := (5)(a, b, c, d) : \exists f \in n \,.\, f \notin \{a, b, c, d\}$,
$() := \eth S_n(9)(10 : z_{2k-1}z_{2k} = (ab)(cd) = (fab)(fcb)(fcd)$;
$\rightsquigarrow (8) := I(\Rightarrow)\eth^{-1}♀(m)(\sigma)(6) : \operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k} = \emptyset \Rightarrow ♀(m)(\sigma)$,
$() := \mathtt{LEM}(\operatorname{supp} z_{2k-1} \cap \operatorname{supp} z_{2k} = \emptyset)E(|)(7)(6) : ♀(m)(\sigma)$;
$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)I(\forall) : \forall m \in \mathbb{N} \,.\, ♀(m-1) \Rightarrow ♀(m)$,
$(*) := \eth♀\mathtt{Induction}\eth\mathtt{WellFounded}(\mathbb{Z}_+)(1)(2) : \mathtt{This}$;
$\square$

NormalAlternating3CycleLemma $:: \forall n \in \mathbb{N} \,.\, \forall(0) : n \geq 5 \,.\, \forall N \lhd A_n \,.$
$\quad . \forall \tau : 3\text{-}\mathtt{Cycle}(n) \,.\, \forall(00) : \tau \in N \,.\, N = A_n$
Proof $=$
Assume $\theta : 3\text{-}\mathtt{Cycle}(n)$,
$(\sigma, 1) := \eth S_n(\operatorname{supp}\theta, \operatorname{supp}\tau) : \sum \sigma \in S_n \,.\, \sigma\tau\sigma^{-1} = \theta$,
Assume $(2) : \mathtt{sign}(\sigma) = -1$,
$(a, b, 3) := (0)\eth 3\text{-}\mathtt{Cycle} : a, b \notin \operatorname{supp}\tau$,
$(4) := \eth S_n(1)(3) : \sigma(ab)\tau(ab)\sigma^{-1} = \theta$,
$(5) := \eth^{-1}A_n(2)\mathtt{SignByTranspositions} : \sigma(ab) \in A_n$,
$() := \eth\mathtt{Normal}(A_n)(N)(4)(5) : \theta \in N$;
$\rightsquigarrow (2) := I(\Rightarrow) : \mathtt{sign}(\sigma) = -1 \Rightarrow \theta \in N$,
Assume $(3) : \mathtt{sign}(\sigma) = 1$,
$() := \eth\mathtt{Normal}(A_n, n)\eth^{-1}A_n(1)(3) : \theta \in N$;
$\rightsquigarrow (3) := I(\Rightarrow) : \mathtt{sign}(\sigma) = 1 \Rightarrow \theta \in N$,
$(4) := \mathtt{FiniteSelection}(\mathtt{Sign})((-1)^\sigma) : (-1)^\sigma = 1 | (-1)^\sigma = -1$,
$() := E(|)(4)(2, 3) : \theta \in N$;
$\rightsquigarrow (1) := \eth^{-1}\mathtt{Subset}I(\forall) : 3\text{-}\mathtt{Cycle}(n) \subset N$,
$(*) := \mathtt{AlternatingGroupBy3Cycles}\eth\mathtt{Subgroup}(A_n)(N)(1) : A_n = N$;
$\square$

$\texttt{AlternatingGroupIsSimple} :: \forall n \in \mathbb{N} \,.\, \forall (0) : n \geq 5 \,.\, A_n : \texttt{Simple}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ N : \texttt{Nontrivial} \ \& \ (A_n),$

$\sigma := \arg \min_{\sigma \in N : \sigma \neq e} |\operatorname{supp} \sigma| : \texttt{In}(N),$

$(t, Z, z, 1) := \texttt{NonIntersectingCycleDecomposion}(n)(\sigma) :$

$$: \sum t \in \mathbb{Z}_+ \,.\, \sum Z : \texttt{PairwiseNonIntersecting}(n) \,.\, \sum z : t \leftrightarrow Z \,.\, \sigma = \sum_{i=1}^{t} z_i,$$

$\texttt{Assume}\ (2) : \forall i \in t \,.\, o(z_i) = 2,$

$(3) := (2)(1)\eth A_n \texttt{SignByTransposition}(\sigma) : t \geq 2,$

$(a, b, c, d, 4) := \eth 2\text{-}\texttt{Cycle}(z_1, z_2) : \sum a, b, c, d \in n \,.\, z_1 = (ab) \ \& \ z_2 = (cd),$

$(f, 5) := (0)(a, b, c, d) : \sum f \in n \,.\, f \notin \{a, b, c, d\},$

$\tau := (cdf) : \texttt{In}(A_n),$

$(6) := \eth S_n \eth \tau (3)(5) : 0 < |\operatorname{supp}(\tau \sigma \tau^{-1} \sigma)| < |\operatorname{supp}(\sigma)|,$

$() := \eth \sigma (6) : \bot;$

$\rightsquigarrow (i, 2) := \eth \sigma E(\bot) : \sum i \in t \,.\, o(z_i) > 2,$

$(\{a, b, c\}, 3) := \eth \texttt{Cycle}(n)(2) : \sum \{a, b, c\} : \texttt{Pairwise NonEq}(n) \,.\, a, b, c \in \operatorname{supp} z_i,$

$\texttt{Assume}\ (4) : \operatorname{supp} \sigma \neq \{a, b, c\},$

$(x, y, 5) := \eth A_n (2)(4) : \sum x, y \in \operatorname{supp} \sigma \,.\, x \neq y \ \& \ x, y \notin \{a, b, c\},$

$\tau := (cxy) : \texttt{A}_\texttt{n},$

$(6) := \eth S_n (5)(2) : 0 < |\operatorname{supp} \tau \sigma \tau^{-1} \sigma^{-1}| < |\operatorname{supp} \sigma|,$

$() := \eth \sigma (6) : \bot;$

$\rightsquigarrow (3) := E(\bot) : \operatorname{supp} \sigma = \{a, b, c\},$

$(4) := \eth^{-1} 3\text{-}\texttt{Cycle}(n) : (\sigma : 3\text{-}\texttt{Cycle}(n)),$

$() := \texttt{NormalAlternating3CycleLemma}(n, 0, N, \sigma) : N = A_n;$

$\rightsquigarrow (*) := \eth^{-1} \texttt{Simple} : (A_4 : \texttt{Simple});$

$\square$

2.3 Group Action

$$\texttt{Action} := \Lambda G \in \mathsf{GRP} \;.\; \Lambda X \in \mathsf{SET} \;.\; G \to_{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{SET}}(X) : \mathsf{GRP} \to \mathsf{SET} \to \mathsf{SET};$$

$$\texttt{Faithful} :: \prod G \in \mathsf{GRP} \;.\; \prod X \in \mathsf{SET} \;.\; ?\texttt{Action}(X, G)$$
$$\alpha : \texttt{Faithful} \iff \ker \alpha = \{e\}$$

$$\texttt{orbit} :: \prod \alpha : \texttt{Action}(G, X) \;.\; X \to ?X$$
$$\texttt{orbit}\,(x) = O_\alpha(x) := \{\alpha(g)(x) | g \in G\}$$

$$\texttt{Isotropy} :: \prod \alpha : \texttt{Action}(G, X) \;.\; X \to \texttt{Subgroup}(X)$$
$$\texttt{Isotropy}\,(x) = \mathrm{Stab}_\alpha(x) := \{g \in G : \alpha(g)(x) = x\}$$

$$\texttt{ActionCounting} :: \forall \alpha : \texttt{Action}(G, X) \;.\; \forall x \in X \;.\; \Big[G : \mathrm{Stab}_\alpha(x)\Big] = \Big|O_\alpha(x)\Big|$$

$\texttt{Proof} \;=$

$\texttt{Assume}\; g, h : G,$

$\texttt{Assume}\; (1) : \alpha(g)(x) = \alpha(h)(x),$

$s := g^{-1}h : G,$

$(2) := (1)\eth s : s \in \mathrm{Stab}_\alpha(x),$

$() := (2)\eth\texttt{Subgroup}()(\mathrm{Stab}_\alpha(x)) : g\mathrm{Stab}_\alpha(x) = h\mathrm{Stab}_\alpha(x);$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall g, h \in G \;.\; \alpha(g)(x) = \alpha(h)(x) \Rightarrow g\mathrm{Stab}_\alpha(x) = h\mathrm{Stab}_\alpha(x),$

$\texttt{Assume}\; g, h : G,$

$\texttt{Assume}\; (2) : g\mathrm{Stab}_\alpha(x) = h\mathrm{Stab}_\alpha(x),$

$(s, 3) := \eth\texttt{Subgroup}(G)(\mathrm{Stab}_\alpha(x))(2) : \sum s \in \mathrm{Stab}_\alpha(x) \;.\; g = hs,$

$() := \eth\mathrm{Stab}_\alpha(x)(s)(3) : \alpha(g)(x) = \alpha(h)(x);$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall g, h \in G \;.\; g\mathrm{Stab}_\alpha(x) = h\mathrm{Stab}_\alpha(x) \Rightarrow \alpha(g)(x) = \alpha(h)(x),$

$f := \Lambda g\mathrm{Stab}_\alpha(x) \in G\mathrm{Stab}_\alpha(x) \;.\; \alpha(g)(x) : G\mathrm{Stab}_\alpha(x) \hookrightarrow X,$

$(3) := \eth f(1) : \mathrm{Im}\, f = O_\alpha(x),$

$(*) := (3)\eth^{-1}[G : \mathrm{Stab}_\alpha(x)] : [G : \mathrm{Stab}_\alpha(x)] = O_\alpha(x);$

$\square$

$$\texttt{actionByConjugation} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Action}(G, G)$$
$$\texttt{actionByConjugation}\,(g) = \gamma_G(g) := \Lambda h \in G \;.\; ghg^{-1}$$

$$\texttt{actionByClassConjugation} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Action}(G, \texttt{Subgroup}(G))$$
$$\texttt{actionByClassConjugation}\,(g) = \Gamma_G(g) := \Lambda H \subset_{\mathsf{GRP}} G \;.\; gHg^{-1}$$

$$\texttt{Inner} :: \prod G \;.\; ?\mathrm{Aut}_{\mathsf{GRP}}(G)$$
$$\phi : \texttt{Inner} \iff \exists g \in G \;.\; \phi = \gamma_G(g)$$

$\texttt{leftTranslation} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Action}(G, G)$

$\texttt{leftTranslation}\,(g) = \lambda_G(g) := \Lambda h \in G \;.\; gh$

$\texttt{rightTranslation} :: \prod G \in \mathsf{GRP} \;.\; \texttt{Action}(G.G)$

$\texttt{rightTranslation}\,(g) = \rho_G(g) := \Lambda h \in G \;.\; hg$

$\texttt{leftCosetTranslation} :: \prod G \in \mathsf{GRP} \;.\; \prod H \subset_{\mathsf{GRP}} G \;.\; \texttt{Action}(G, G/H)$

$\texttt{leftCosetTranslation}\,(g) = \Lambda_{G,H}(g) := \lambda A \in G/H \;.\; gA$

$\texttt{rightCosetTranslation} :: \prod G \in \mathsf{GRP} \;.\; \prod H \subset_{\mathsf{GRP}} G \;.\; \texttt{Action}(G.G \setminus H)$

$\texttt{rightCosetTranslatrion}\,(g) = \mathcal{R}_{G,H}(g) := \lambda A \in G \setminus H \;.\; Ag$

$\texttt{ConjugateClassCounting} :: \forall G \in \mathsf{GRP} \;.\; \forall H \subset_{\mathsf{GRP}} G \;.\; \Big|\Gamma(G)(H)\Big| = \Big[G : N(H)\Big]$

$\texttt{Proof} =$

Apply $\texttt{ActionCounting}$ with $|O_\Gamma(H)| = |\Gamma(G)(H)|, \mathrm{Stab}_\Gamma(H) = N(H)$

□

$\texttt{OrbitalDecomposition} :: \forall \alpha : \texttt{Action}(G, X) \;.\; \exists I : \texttt{Set} : \exists x : I \to X \;.\; X = \bigsqcup_{i \in I} O_\alpha(x_i)$

$\texttt{Proof} =$

$\ldots$

□

$\texttt{StabDecomposition} :: \forall X : \texttt{Finite} \;.\; \forall \alpha : \texttt{Action}(G, X) \;.\; \exists I : \texttt{Set} : \exists x : I \to X \;.$

$$. \; |X| = \sum_{i=1}^{n} [G : \mathrm{Stab}_\alpha(x_i)]$$

$\texttt{Proof} =$

Combine $\texttt{ActionCountiong}$ and $\texttt{OrbitalDecomposition}$ with prpoerties of disjoint union.

□

$\texttt{ClassFormula} :: \forall G : \texttt{FiniteGroup} \;.\; \exists n \in \mathbb{N} : \exists g : n \to G \;.\; |G| = \sum_{i=1}^{n} [G : \mathrm{Stab}_\gamma(g_i)]$

$\texttt{Proof} =$

$\ldots$

□

$$\texttt{GSet} := \prod G \in \mathsf{GRP} \; . \; \sum X \in \mathsf{SET} \; . \; \alpha : \texttt{Action}(G, X) : \texttt{Type},$$
$$G\text{-}\texttt{Set} := \texttt{GSet}(G) : \texttt{Type};$$

$$\texttt{GMap} :: \prod (X, \alpha), (Y, \beta) : G\text{-}\texttt{Set} \; . \; ?(X \to Y)$$
$$f : \texttt{GMap} \iff f : G\text{-}\texttt{Map}\Big((X, \alpha), (Y, \beta)\Big) \iff \forall x \in X \; . \; \forall g \in G \; . \; f\Big(\alpha(g)(x)\Big) = \beta(g)\Big(f(x)\Big)$$

$$\texttt{gSetCat} :: \mathsf{GRP} \to \texttt{Category}$$
$$\texttt{gSetCat}\,(G) = G\text{-}\mathsf{SET} := (G\text{-}\texttt{Set}, G\text{-}\texttt{Map}, \circ)$$

2.4 Sylow Theory

$\texttt{PGroup} :: \texttt{Prime} \to ?\texttt{FiniteGroup}$

$G : \texttt{PGroup} \iff p\text{-}\texttt{Group} \iff \Lambda p : \texttt{Prime} \,.\, \exists n \in \mathbb{N} \,.\, |G| = p^n$

$\texttt{PSylow} :: \prod G : \texttt{FiniteGroup} \,.\, \texttt{Prime} \to ?\texttt{Subgroup}(G)$

$H : \texttt{PSylow} \iff H : p\text{-}\texttt{Sylow}(G) \iff \Lambda p : \texttt{Prime} \,.\, \exists m : \texttt{Coprime}(p) : \exists n \in \mathbb{Z}_+ \,.\, |H| = p^n \,\&\, |G| = mp^n$

$\texttt{PSylowLemma} :: \forall G : \texttt{Abelean} \,\&\, \texttt{FiniteGroup} \,.\, \forall p : \texttt{Prime} \,.\, \forall (0) : \texttt{Divides}(p, |G|) \,.\, \exists H \lhd G : |H| = p$

$\texttt{Proof} \; =$

$☿ := \Lambda n \in \mathbb{N} \,.\, \forall G : \texttt{Abelean} \,.\, \forall (0) : |G| \le n \,.\, \forall p : \texttt{Prime} \,.\, \forall (00) : \texttt{Divides}(p, |G|) \,.\, \exists H \lhd G : |H| = p :$

$\quad : \mathbb{N} \to \texttt{Type},$

$(1) := ð☿ð\texttt{Empty}ð\texttt{Prime} : ☿(1),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ (2) : ☿(n),$

$\texttt{Assume}\ G : \texttt{Abelean},$

$\texttt{Assume}\ (0) : |G| = n + 1,$

$\texttt{Assume}\ p : \texttt{Prime} \,\&\, \texttt{Divisor}(n),$

$(g, 3) := ð^{-1}\texttt{Group}(0)(e) : \sum g \in G \,.\, g \ne e,$

$\texttt{Assume}\ (4) : \texttt{Divides}(p, o(g)),$

$(m, 5) := ð\texttt{Divides}(p, o(g))(4) : \sum m \in \mathbb{N} \,.\, o(g) = mp,$

$() := ð^{-1}\texttt{generateGroup}(g^m)ð\texttt{order}5 : \left|\langle g^m \rangle\right| = p;$

$\leadsto (4) := I(\Rightarrow) : \texttt{Divides}(p.o(g)) \Rightarrow ☿(n+1)(G, (0), p),$

$\texttt{Assume}\ (5) : !\texttt{Divides}(p, o(g)),$

$(6) := (3)ð\texttt{quatientGroup} : \left|\dfrac{G}{\langle g \rangle}\right| \le n,$

$(7) := \texttt{DivisionTransition}\ \texttt{IndexTHM}ðp(5) : \texttt{Divides}\left(p, \left|\dfrac{G}{\langle g \rangle}\right|\right),$

$(H, 8) := ☿(n)(G, (7), p) : \sum H \lhd \dfrac{G}{\langle h \rangle} \,.\, |H| = p,$

$(h, 9) := ð^{-1}\texttt{CyclicPrimeOrderIsCyclic}(8, ðp) : \sum h \in G \,.\, H = \langle [h] \rangle,$

$() := (9)(8)(5) : |\langle h^{o(g)} \rangle| = n;$

$\leadsto (5) := I(\Rightarrow) : !\texttt{Divides}(p, o(g)) \Rightarrow ☿(n+1)(G, (0), p),$

$() := E(|)\texttt{LEM}(\texttt{Divides}(p, o(g)))(4, 5) : ☿(n+1)(G, (0), p);$

$\leadsto (2) := I(\forall)I(\Rightarrow)I(\forall) : \forall n \in \mathbb{N} \,.\, ☿(n) \Rightarrow ☿(n+1),$

$(*) := E(\mathbb{N})(1)(2) : \texttt{This};$

$\square$

$\texttt{SylowTHMI} :: \forall G : \texttt{FiniteGroup} \,.\, \forall p : \texttt{PrimeDivisor}(|G|) \,.\, \exists H : p\text{-}\texttt{Sylow}(G)$
$\texttt{Proof} \;=$
$☿ := \Lambda n \in \mathbb{N} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall (0) : |G| \le n \,.\, \forall p : \texttt{PrimeDivisor}(|G|) \,.\, \exists H : p\text{-}\texttt{Sylow}(G) : \mathbb{N} \to \texttt{Type},$
$(1) := ð☿ð\texttt{EmptyprimeDivisor}(1) : ☿(1),$
$\texttt{Assume}\; n : \mathbb{N},$
$\texttt{Assume}\; (2) : ☿(n),$
$\texttt{Assume}\; G : \mathsf{GRP},$
$\texttt{Assume}\; (0) : |G| = n + 1,$
$\texttt{Assume}\; p : \texttt{PrimeDivisor}(n+1),$
$(t, s, 3) := ðp : \sum t, s \in \mathbb{N} \,.\, |G| = sp^t,$
$\texttt{Assume}\; (4) : \forall H : \texttt{ProperSubgroup}(G) \,.\, [G : H] : \texttt{DivisibleBy}(p),$
$$(k, g, 5) := \texttt{ClassFormula}(G) : \sum k \in \mathbb{N} \,.\, g : k \to G \,.\, |G| = \sum_{i=1}^{k} [G : \mathrm{Stab}_\gamma(g_i)],$$
$(6) := ð\gamma(G)(e)ðe : \gamma(G)(e) = \{e\},$
$(7) := ð\mathrm{Stab}_\gamma(e)(6) : \mathrm{Stab}_\gamma(e) = G,$
$m := |Z(G)| : \mathbb{N},$
$$(8) := \texttt{OrbitalDecomposition}(G)(5)(6) : |G| = |Z(G)| + \sum_{i=m+1}^{k} [G : \mathrm{Stab}_\gamma(g_i)] \;\&$$
$$\& \; \forall i \in k \,.\, (00) : i > m \,.\, g_i \notin Z(G),$$
$$x := \sum_{i=m+1}^{k} [G : \mathrm{Stab}_\gamma(g_i)] : \mathbb{N},$$
$(9) := (4)(8)_2 ðx : x : \texttt{DivisibleBy}(p),$
$(10) := ðp(9)(8) : |Z(G)| : \texttt{DivisibleBy}(p),$
$(H, 11) := \texttt{PSylowLemma}(Z(G), (10), p) : \sum H \lhd C(G) \,.\, |H| = p,$
$(12) := ð^{-1}\texttt{Normal}ð\texttt{center}(Z(G))(H) : H \lhd G,$
$$(13) := \texttt{IndexTHM}(G, H)(0) : \left|\frac{G}{H}\right| \le n,$$
$$(14) := \texttt{IndexTHM}(H, G)(3) : \left|\frac{G}{H}\right| = mp^{t-1},$$
$$(K, 15) := ☿(n)(G, p) : \sum K \subset_{\mathsf{GRP}} \frac{G}{H} \,.\, |K| = p^{t-1},$$
$K' := \pi^{-1}(K) : \texttt{Subgroup}(G),$
$() := (11)(15)ðK' : |K'| = p^t;$
$\rightsquigarrow (3) := I(\Rightarrow) : \forall H : \texttt{ProperSubgroup}(G) \,.\, [G : H] : \texttt{Divisible}(p) \Rightarrow ☿(n+1)(G, p),$
$\texttt{Assume}\; (H, 4) : \sum H : \texttt{ProperSubgroup}(G) \,.\, [G : H] : \texttt{Divisible}(p),$
$(5) := (4)ð\texttt{index}(G, H)(3) : |H| : \texttt{DivisibleBy}(p^t),$
$(K, 6) := ☿(n)(H) : \sum K \subset_{\mathsf{GRP}} H \,.\, |K| = p^t;$
$\rightsquigarrow (4) := I(\exists)I(\Rightarrow) : \exists H \subset_{\mathsf{GRP}} G : [G : H] : \texttt{Divisible}(p) \Rightarrow ☿(n+1)(G, p),$
$(5) := E(|)\texttt{LEM}(3)(4) : ☿(n+1)(G, p);$
$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)I(\forall) : \forall n \in \mathbb{N} \,.\, ☿(n) \Rightarrow ☿(n+1),$
$(*) := E(\mathbb{N})(1)(2) : \texttt{This};$
$\square$

$\texttt{FixedPoints} :: \prod X \in \mathsf{SET} \,.\, \texttt{Action}(G, X) \to ?X$

$x : \texttt{FixedPoints} \iff \Lambda\alpha : \texttt{Action}(G, X) \,.\, \forall g \in G \,.\, \alpha(g)(x) = x$

$\texttt{FixedPointsOfPGroups} :: \forall G : p\text{-}\texttt{Group} \,.\, \forall X : \texttt{Finite} \,.\, \forall\alpha : \texttt{Action}(G, X) \,.\, \Big|\texttt{FixedPoints}(\alpha)\Big| =_{Z_p} |X|$

$\texttt{Proof} \ =$

$$(n, g, 1) := \texttt{StabDecompositon}(\alpha) : \sum n \in \mathbb{N} \,.\, \sum x : n \to X \,.\, |X| = \sum_{i=1}^{n} [G : \mathrm{Stab}_\alpha(x_i)],$$

$$m := \Big|\texttt{FixedPoints}(\alpha)\Big| : \mathbb{N},$$

$$(2) := \eth\texttt{StabDecomposition}\eth^{-1} m : |X| = m + \sum_{i=m+1}^{n} [G : \mathrm{Stab}_\alpha(x_i)] \ \&$$

$\quad \& \ \forall i \in [m+1, n]_\mathbb{N} \,.\, x_i \notin \texttt{Fixedpoints}(\alpha),$

$(3) := \forall i \in [m+1, n]_\mathbb{N} \,.\, \eth^{-1}\mathrm{Stab}_\alpha(x_i)\texttt{SubgroupDivides}\eth p\text{-}\texttt{Group}(G) :$

$\quad : \forall i \in [m+1, n]_\mathbb{N} \,.\, [G : \mathrm{Stab}_\alpha(x_i)] : \texttt{DivisibleBy}(p),$

$(*) := ((2) \mod p)(3) : |X| =_{Z_p} m;$

$\square$

$\texttt{SylowLemmaII} :: \forall G : \texttt{FiniteGroup} \,.\, \forall H : \texttt{Subgroup(G)} \ \& \ p\text{-}\texttt{Group} \,.\, \forall K : p\text{-}\texttt{Sylow}(G) \,.$

$\quad . \forall (0) : H \subset N(K) \,.\, H \subset K$

$\texttt{Proof} \ =$

$$\pi := \texttt{projection}(N(K), K) : N(K) \twoheadrightarrow_{\mathsf{GRP}} \frac{N(K)}{K},$$

$(n, t, 1) := \eth\texttt{PrimeDivisor}(|G|)(p) : \sum t \in \mathbb{N} \,.\, \sum n : \texttt{Coprime}(p) \,.\, |G| = np^t,$

$(m, 2) := \eth p\text{-}\texttt{Sylow}(G)(K)\texttt{SubgroupDivides}\eth N(K) : \sum m \in \texttt{Divisor}(n) \,.\, |N(K)| = mp^t,$

$$(3) := \eth p\text{-}\texttt{Sylow}(G)(K)\texttt{IndexTHM}(2)(3) : \left|\frac{N(K)}{K}\right| = m,$$

$\texttt{Assume}\ h : H,$

$(k, 4) := \eth p\text{-}\texttt{Group}(H)\texttt{OrderDivides}(h, H)\eth o(h) : \sum k \in \mathbb{N} \,.\, h^{p^k} = e,$

$(5) := \pi(4) : \pi(h^{p^k}) = K,$

$$(6) := (3)\texttt{OrderDivides}\left(\frac{N(K)}{K}, \pi(h)\right) : m | o(\pi\ h),$$

$(7) := \eth m(6) : o(\pi\ h) = 1,$

$(8) := \eth o(7) : \pi\ h = K,$

$(9) := \eth\pi(8) : h \in K;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Subset}I(\forall) : H \subset K;$

$\square$

$\texttt{SylowGroupMaximizePGroups} :: \forall G : \texttt{FiniteGroup} \,.\, \forall H : \texttt{Subgroup}(G) \,\&\, p\text{-}\texttt{Group} \,.$
$\exists K : p\text{-}\texttt{Sylow}(G) \,.\, H \subset K$

$\texttt{Proof} \;=$

$(1) := \texttt{SubgroupDivides}(G,H)\eth p\text{-}\texttt{Group}(H) : |G| \mid p,$

$K := \texttt{SylowTHMI}(G,p)(1) : p\text{-}\texttt{Sylow}(G),$

$S := \Gamma(H)(K) :?p\text{-}\texttt{Sylow}(G),$

$\alpha := \Lambda h \in H \,.\, \Lambda A \in S \,.\, hAh^{-1} : \texttt{Action}(H,S),$

$(n,t,2) := \eth\texttt{PrimeDivisor}(|G|)(p) : \sum t \in \mathbb{N} \,.\, \sum n : \texttt{Coprime}(p) \,.\, |G| = np^t,$

$(3) := \eth p\text{-}\texttt{Sylow}(G)(K)(2) : |K| = p^t,$

$(4) := (3)\eth N(K)\eth^{-1}\mathrm{Stab}_\Gamma(K) : \mathrm{Stab}_\Gamma(K) : \texttt{DivisibleBy}(p^t),$

$(5) := \eth^{-1}(S)\eth n(2)\texttt{ActionCounting}(4) : S \,!\, \texttt{DivisibleBy}(p),$

$(6) := \texttt{FixedPointsOfPGroups}(\alpha)(5) : |\texttt{FixedPoints}(\alpha)| \,!\, \texttt{DivisibleBy}(p),$

$(7) := \texttt{ZeroIsDivisible}(6) : \texttt{FixedPoints}(\alpha) \neq \emptyset,$

$F := \eth\texttt{Nonempty}(7) : \texttt{FixedPoinint}(\alpha),$

$(8) := \eth^{-1}N(F)\eth\alpha\eth\texttt{FixesPoints}(\alpha)(F) : H \subset N(F),$

$(*) := \texttt{SelowLemmaII}(8) : H \subset F;$

$\square$

$\texttt{SylowTHMII} :: \forall H, K : p\text{-}\texttt{Sylow}(G) \,.\, \exists g \in G \,.\, K = gHg^{-1}$

$\texttt{Proof} \;=$

$S := \Gamma(H)(K) :?p\text{-}\texttt{Sylow}(G),$

$\alpha := \Lambda h \in H \,.\, \Lambda A \in S \,.\, hAh^{-1} : \texttt{Action}(H,S),$

$(n,t,2) := \eth\texttt{PrimeDivisor}(|G|)(p) : \sum t \in \mathbb{N} \,.\, \sum n : \texttt{Coprime}(p) \,.\, |G| = np^t,$

$(3) := \eth p\text{-}\texttt{Sylow}(G)(K)(2) : |K| = p^t,$

$(4) := (3)\eth N(K)\eth^{-1}\mathrm{Stab}_\Gamma(K) : \mathrm{Stab}_\Gamma(K) : \texttt{DivisibleBy}(p^t),$

$(5) := \eth^{-1}(S)\eth n(2)\texttt{ActionCounting}(4) : S \,!\, \texttt{DivisibleBy}(p),$

$(6) := \texttt{FixedPointsOfPGroups}(\alpha)(5) : |\texttt{FixedPoints}(\alpha)| \,!\, \texttt{DivisibleBy}(p),$

$(7) := \texttt{ZeroIsDivisible}(6) : \texttt{FixedPoints}(\alpha) \neq \emptyset,$

$F := \eth\texttt{Nonempty}(7) : \texttt{FixedPoinint}(\alpha),$

$(8) := \eth^{-1}N(F)\eth\alpha\eth\texttt{FixesPoints}(\alpha)(F) : H \subset N(F),$

$(9) := \texttt{SelowLemmaII}(8) : H \subset F,$

$(10) := \texttt{EqByCardinality}(9)\eth p\text{-}\texttt{Sylow}(G)(H,F) : H = F,$

$(g,11) := \eth S\eth F(10) : \sum g \in G \,.\, H = gKg^{-1};$

$\square$

$\texttt{numberOfSylow} :: \prod G : \texttt{FiniteGroup} \,.\, \texttt{PrimeDivisor}(|G|) \to \mathbb{N}$

$\texttt{numberOfSylow}\,(p) = n(G,p) := \Big|p\text{-}\texttt{Sylow}(G)\Big|$

$\texttt{SylowLemmaIII} :: \forall G : \texttt{FiniteGroup} . \forall H : \texttt{Subgroup}(G) \;\&\; p\text{-}\texttt{Group} . [G : H] =_{Z_p} [N(H) : H]$

$\texttt{Proof} \;=$

$\alpha := \Lambda_{G,H|H} : \texttt{Action}(H, G/H),$

$\texttt{Assume}\ gH : \texttt{FixedPoints}(\alpha),$

$\texttt{Assume}\ h : H,$

$(1) := \eth\texttt{FixedPoints}(\alpha)(gH, h) : hgH = gH,$

$(x, 2)) := (1)\eth\texttt{LeftCoset} : \sum x \in H \,.\, hgx = g,$

$(3) := g^{-1}h^{-1}(2) : x = g^{-1}h^{-1}g,$

$() := \eth\texttt{Group}(H)(3) : ghg^{-1} \in H;$

$\rightsquigarrow (1) := I(\forall) : \forall h \in H \,.\, ghg^{-1} \in H,$

$() := \eth^{-1}N(H)(1) : gH \subset N(H);$

$\rightsquigarrow (1) := \eth\texttt{LeftCoset}(N(H)) : \texttt{FixedPoints}(\alpha) = N(H)/H,$

$(*) := \texttt{FixedPointsOfPGroup}(\alpha)(1) : [G : H] =_{Z_p} [N(H) : H];$

$\square$

$\texttt{SylowTHMIII} :: \forall G : \texttt{FiniteGroup} . \forall p : \texttt{PrimeDivisor}(|G|) . \forall m : \texttt{Coprime}(p) . \forall t \in \mathbb{N} .$

$\quad . \forall (0) : |G| = mp^t \,.\, n(G, p) : \texttt{Divisor}(m) \;\&\; n(G, p) =_{Z_p} 1$

$\texttt{Proof} \;=$

$K := \texttt{SylowTHMI}(G, p) : p\text{-}\texttt{SylowTHMI}(G),$

$(2) := \texttt{IndexTHM}(G, K)(0, \eth p\text{-}\texttt{Sylow}(G)(K)) : m = [G : K],$

$(3) := \texttt{SylowTHMII}(G, p)\eth^{-1}n(G, p)\texttt{ActionCounting}(\Gamma)\eth^{-1}N(K) : m(G, p) = [G : N(K)],$

$(4) := \texttt{IndexTHM}(N(K), K)(2)\left(\frac{|N(K)|}{|N(K)|}\right)\eth\texttt{index}(3) : m = n(G, p)[N(K) : K],$

$(5^*) := \eth^{-1}\texttt{Divsor}(m)(4) : \Big(n(G, p) : \texttt{Divisor}(m)\Big),$

$(6) := \texttt{SylowLemmaIII}(G, K)(2) : m =_{Z_p} [N(K) : K],$

$(*) := (4)(6) : n(G, p) =_{Z_p} 1;$

$\square$

$\texttt{ExtSylowTHM} :: \forall G : \texttt{FiniteGroup} . \forall p : \texttt{PrimeDivisor}(|G|) . \forall t : \texttt{DivisorExponent}(|G|, p) .$

$\quad . \exists H \subset_{\mathsf{GRP}} G : |H| = p^t$

$\texttt{Proof} \;=$

Simillar proof as $\texttt{SylowTHMI}$

$\square$

$\texttt{NormalByPrimeIndex} :: \forall G : \texttt{FiniteGroup} \,.\, \forall p : \texttt{Min}\Big(\texttt{PrimeDivisor}(|G|)\Big) \,.$

$\quad .\, \forall H \subset_{\mathsf{GRP}} G \,.\, \forall (0) : [G : H] = p \,.\, H \lhd G$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : N(H) = H,$

$S := O_\Gamma(H) :??G,$

$(2) := \texttt{ActionCountong}(\Gamma, H)(1) : |S| = p,$

$s := \texttt{enumerate}(S, 2) : p \leftrightarrow S,$

$\varphi := \Lambda g \in G \,.\, \Lambda i \in p \,.\, s^{-1}(gs(i)g^{-1}) : G \to p \to p,$

$(3) := \eth s \eth \varphi : \operatorname{Im} \varphi = S_i,$

$\texttt{Assume}\ a, b : G,$

$\texttt{Assume}\ i :,$

$() := \eth\varphi(b)(i)\eth\texttt{Inverse}(s)\eth\varphi(a)\eth^{-1}\varphi(ab) : \varphi(a)\varphi(b)(i) = \varphi(a)s^{-1}\Big(bs(i)b^{-1}\Big) = s^{-1}(abs(i)b^{-1}a^{-1}) = \varphi(ab)(i);$

$\rightsquigarrow () := I(=, \to) : \varphi(a)\varphi(b) = \varphi(a)(b);$

$\rightsquigarrow (4) := \eth^{-1}\texttt{Homomorphism} : \Big(\varphi : G \to_{\mathsf{GRP}} S_p\Big),$

$K := \ker \phi : \texttt{Normal}(G),$

$(5) := (1)\eth K : K \subset H,$

$\texttt{Assume}\ (6) : K \neq H,$

$(7) := \texttt{IndexTHM}(G, K)(G, H)(H, K)(0) : [G : K] = p[H : K],$

$(8) := \texttt{IsomorphismTHMI}(G, S_n)(\varphi)\eth K(7) : |\operatorname{Im} \varphi| = p[H : K],$

$(6') := \Big(|S_2| = 2\Big)\eth\varphi\texttt{NatMultInc}(8) : p \neq 2,$

$(9) := \texttt{SoubgroupDivide}(\operatorname{Im} \varphi) : |\operatorname{Im} \phi| \Big| p!,$

$(10) := \eth\texttt{index}(H, K)(6) : [H : K] \neq 1,$

$(11) := (10)(9)(8) : \Big(|H| : \texttt{NontrivialDivisor}((p-1)!)\Big),$

$(q, 12) := \texttt{FactorialPrimeDivisor}\eth\texttt{factorial}(p-1))(11)(6')\eth\texttt{NontrivialDivisor} :$

$\quad : \sum q : \texttt{PrimeDivisor}(|H|) \,.\, q < p,$

$(13) := \texttt{SubgroupDivides}(G, H)(\eth q) : q||G|,$

$() := \eth p(12)(13) : \bot;$

$\rightsquigarrow (6) := E(\bot) : K = H,$

$(7) := E(=)(6)\eth\texttt{Normal}(G)(K)\eth^{-1}N(K) : N(H) = N(K) = G,$

$() := (0)\eth\texttt{index}(G, H))I(\#, \to)(1)(7) : \bot;$

$\rightsquigarrow (1) := E(\bot)(0)\eth\texttt{index}(G, H)\texttt{SubsetOfEqFiniteCard} : N(H) = G,$

$(*) := \eth^{-1}\texttt{Normal}(1) : H \lhd G;$

$\square$

$\texttt{PGroupIsSolvable} :: \forall G : p\text{-}\texttt{Group} \,.\, G : \texttt{Solvable}$

$\texttt{Proof} \;=$

$☿ := \Lambda n \in \mathbb{N} \,.\, \forall G : p\text{-}\texttt{Group} \,.\, \forall (0) : |G| \leq p^n \,.\, G : \texttt{Cyclic} : \mathbb{N} \to \texttt{Type},$

$(1) := \eth^{-1}\texttt{Solvalble}\,\texttt{PrimeIsCyclic}\eth^{-1}☿ : ☿(1),$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; (2) : ☿(n),$

$\texttt{Assume}\; G : p\text{-}\texttt{Group},$

$\texttt{Assume}\; (3) : |G| = p^{n+1},$

$(H, 4) := \texttt{ExtSylowTHM}(G, p, n) : \sum H \subset_{\mathsf{GRP}} G : |H| = p^n,$

$(5) := \texttt{IndexTHM}(G, H)(3)(4) : [G : H] = p,$

$(6) := ☿(n)(H) : (H : \texttt{Solvable}),$

$(7) := \texttt{NormalByPrimeIndex}(G, H) : H \lhd G,$

$(8) := \texttt{PrimeIsCyclic}(5) : \left(\frac{G}{H} : \texttt{Cyclic}\right),$

$() := \eth^{-1}\texttt{Solvable}(8)(6) : (G : \texttt{Solvable});$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)I(\forall)\eth^{-1}☿) : \forall n \in \mathbb{N} \,.\, ☿(n) \Rightarrow ☿(n+1),$

$(3) := E(\mathbb{N})(1)(2) : \texttt{This};$

$\square$

$\texttt{PGroupHasCyclicTower} :: \forall G : p\text{-}\texttt{Group} \,.\, \exists (n, H) : \texttt{Cyclic-Tower}(G)$

$\texttt{Proof} \;=$

Evident from the previous proof

$\square$

$\texttt{PQSolvabale} :: \forall p, q : \texttt{Prime} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall (0) : |G| = pq \,.\, G : \texttt{Solvable}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.5 Additional Tools[!]

3 Abelean Groups

3.1 Direct sums

$\texttt{AbeleanCategory} :: \texttt{Category}$

$\texttt{AbeleanCategory}\,() = \mathsf{ABEL} := (\texttt{Abelean}, \texttt{Homomorphism}, \circ)$

$\texttt{directSum} :: \mathsf{ABEL} \times \mathsf{ABEL} \to \mathsf{ABEL}$

$\texttt{directSum}\,(A, B) = A \oplus B := A \otimes B$

$\texttt{directSummation} :: \left(\sum I \in \mathsf{SET} \,.\, I \to \mathsf{ABEL}\right) \to \mathsf{ABEL}$

$$\texttt{directSummation}\,(I, G) = \bigoplus_{i \in I} G_i := \left\{ g \in \bigotimes_{i=1} G_i : \Big|\{i \in I : g_i \neq 0\}\Big| < \infty \right\}$$

$\texttt{Insertion} :: \prod I \in \mathsf{SET} \,.\, \prod G : I \to \mathsf{ABEL} \,.\, \prod i \in I \,.\, G_i \hookrightarrow_{\mathsf{ABEL}} \bigoplus_{i \in I} G_i$

$\texttt{Insertion}\,(g) = \iota_{I,G,i}(g) := \Lambda j \in I \,.\, \texttt{if } i == j \texttt{ then } g \texttt{ else } 0$

$\texttt{AbeleanCoproducts} :: (\texttt{directSummation}, \texttt{insertion}) : \texttt{Coproduct}(\mathsf{ABEL})$

$\texttt{Proof} \;=$

$\texttt{Assume } I : \texttt{Set},$

$\texttt{Assume } G : I \to \mathsf{ABEL},$

$\texttt{Assume } H : \mathsf{ABEL},$

$\texttt{Assume } \varphi : \prod i \in I \,.\, G_i \to_{\mathsf{ABEL}} H,$

$$\psi := \Lambda x \in \bigoplus_{i \in I} G_i \,.\, \sum_{i \in I} \varphi_i(x_i) : \bigoplus_{i \in I} G_i \to H,$$

$\ldots$

$\square$

$\texttt{AbeleanProducts} :: (\texttt{groupProduct}, \texttt{projection}) : \texttt{Product}(\mathsf{ABEL})$

$\texttt{Proof} \;=$

$\texttt{Assume } I : \texttt{Set},$

$\texttt{Assume } G : I \to \mathsf{ABEL},$

$\texttt{Assume } H : \mathsf{ABEL},$

$\texttt{Assume } \varphi : \prod i \in I \,.\, H \to_{\mathsf{ABEL}} G_i,$

$$\psi := \Lambda h \in H \,.\, \Lambda i \in I \,.\, \varphi_i(h) : H \to_{\mathsf{ABEL}} \bigotimes_{i \in I} G_i,$$

$\ldots$

$\square$

$\texttt{freeAbelean} :: \mathsf{SET} \to \mathsf{ABEL}$

$$\texttt{freeAbelean}\,(X) = \mathbb{Z}\langle X\rangle := \bigoplus_{x\in X}\mathbb{Z}$$

$\texttt{Basis} :: \prod G \in \mathsf{ABEL}\ .\ ?\texttt{NonEmpty}(G)$

$$V : \texttt{Basis} \iff \forall g \in G\ .\ \exists! z \in \mathbb{Z}\langle V\rangle\ .\ g = \sum_{v\in V} z_v v$$

$\texttt{FreeAbelean} :: ?\mathsf{ABEL}$

$G : \texttt{FreeAbelean} \iff \exists V : \texttt{Basis}(G)$

$\texttt{AltFreeAbelean} :: \forall G \in \mathsf{ABEL}\ .\ G : \texttt{FreeAbelean} \iff \exists S \in \mathsf{SET}\ .\ G \cong_{\mathsf{ABEL}} \mathbb{Z}\langle S\rangle$

$\texttt{Proof} =$

...

$\square$

$\texttt{freeAbeleanPushforward} :: \prod X, Y \in \mathsf{SET}\ .\ (X \to Y) \to (\mathbb{Z}\langle X\rangle \to \mathbb{Z}\langle Y\rangle)$

$$\texttt{freeAbeleanPushforward}\,(f) = f_* := \Lambda v \in \mathbb{Z}\langle X\rangle\ .\ \Lambda y \in Y\ .\ \sum_{x\in f^{-1}(y)} v_x$$

$\texttt{FreeAbeleanFunctoriality} :: (\texttt{freeAbelean}, \texttt{freeAbeleanPushforwatd}) : \texttt{Functor}\,(\mathsf{SET}, \mathsf{ABEL})$

$\texttt{Proof} =$

...

$\square$

$F_{\mathsf{ABEL}} := (\texttt{freeAbelean}, \texttt{freeAbeleanPushforward}) : \texttt{Functor}\,(\mathsf{SET}, \mathsf{ABEL})\,;$

$\texttt{groupOfGrothendieck} :: \texttt{Commutative}\ \&\ \texttt{Monoid} \to \mathsf{ABEL}$

$$\texttt{groupOfGrothendieck}\,(M) = K(M) := \frac{F_{\mathsf{ABEL}}(M)}{E}$$

where

$$E = \Big\langle \{F_{\mathsf{ABEL}}(a+b) - F_{\mathsf{ABEL}}(a) - F_{\mathsf{ABEL}}(b) | a, b \in M\} \Big\rangle \lhd F_{\mathsf{ABEL}}(M)$$

$\texttt{insertionOfGrothendieck} :: \prod M : \texttt{Commutative}\ \&\ \texttt{Monoid}\ .\ K(M)$

$$\texttt{insertionOfGrothendieck}\,(m) = \kappa(m) := \Big[\Lambda n \in M\ .\ \delta_{n,m}\Big]$$

$\texttt{GrothendieckGroupTHM} :: \forall M : \texttt{Commutative}\ \&\ \texttt{Monoid} . \forall A \in \mathsf{ABEL} . \forall \varphi : \texttt{Homomorphism}(M, A) .$
$. \exists! \psi : K(M) \to_{\mathsf{ABEL}} A : \kappa\psi = \phi$

$\texttt{Proof} =$

$(1) := \eth A \eth K(A) : K(A) = A,$

$(2) := \eth^{-1}\texttt{Functor}\,(\mathsf{SET}, \mathsf{ABEL})\,(F_{\mathsf{ABEL}}) : \varphi F_{\mathsf{ABEL}} = F_{\mathsf{ABEL}}\varphi_*,$

$E := \left\langle \{F_{\mathsf{ABEL}}(a+b) - F_{\mathsf{ABEL}}(a) - F_{\mathsf{ABEL}}(b) | a, b \in M\} \right\rangle : \texttt{Normal}(F_{\mathsf{ABEL}}(M)),$

$B := \left\langle \{F_{\mathsf{ABEL}}(a+b) - F_{\mathsf{ABEL}}(a) - F_{\mathsf{ABEL}}(b) | a, b \in A\} \right\rangle : \texttt{Normal}(F_{\mathsf{ABEL}}(A)),$

$\texttt{Assume}\ a, b : M,$

$(3) := \eth\texttt{Homomorphism}(K(M).K(A))(\phi_*)(2)(1)\eth\texttt{Homomorphism}(M, A)(\phi) :$
$: \pi_B\Big(\varphi_*(F_{\mathsf{ABEL}}(a+b) - F_{\mathsf{ABEL}}(a) - F_{\mathsf{ABEL}}(b))\Big) = \pi_B\Big(\varphi_* F_{\mathsf{ABEL}}(a+b) - \varphi_* F_{\mathsf{ABEL}}(a) - \varphi_* F_{\mathsf{ABEL}}(b)\Big) =$
$= \pi_B\Big(F_{\mathsf{ABEL}}\varphi(a+b) - F_{\mathsf{ABEL}}\varphi(a) - F_{\mathsf{ABEL}}\varphi(b)\Big) = \varphi(a+b) - \varphi(a) - \varphi(b) = 0,$

$() := \eth^{-1} \ker \varphi_*\pi_B(3) : F_{\mathsf{ABEL}}(a+b) - F_{\mathsf{ABEL}}(a) - F_{\mathsf{ABEL}}(b) \in \ker \varphi_*\pi_B;$

$\rightsquigarrow (3) := \eth^{-1} \ker \pi_E \eth^{-1}\texttt{Subset}\, I(\forall) : \ker \pi_E \subset \ker \varphi_*\pi_B,$

$\psi := \Lambda[x] \in K(M) . \pi_B\varphi_*(x) : K(M) \to_{\mathsf{ABEL}} A,$

$\texttt{Assume}\ m : M,$

$() := \eth\texttt{freeAbeleanPushforward}(\varphi)\eth\psi\eth\kappa(m) : \psi\kappa(m) = \pi_B\varphi_*(\delta_m) = \varphi(m);$

$\rightsquigarrow (4) := I(=, \to) : \kappa\psi = \varphi,$

$\texttt{Assume}\ \phi : K(M) \to_{\mathsf{ABEL}} M,$

$\texttt{Assume}\ (5) : \kappa\phi = \varphi,$

$\texttt{Assume}\ x : K(M),$

$(k, m, z, 6) := \eth F_{\mathsf{ABEL}} \eth K(M)(x) : \sum k \in \mathbb{N} . \sum m : k \to M . \sum z : k \to \mathbb{Z} . x = \sum_{i=1}^{k} z_i [F_{\mathsf{ABEL}}(m_i)],$

$(7) := (5)(6) : \phi(x) = \sum_{i=1}^{k} z_i \phi(m_i),$

$(8) := (4)(6) : \psi(x) = \sum_{i=1}^{k} z_i \phi(m_i),$

$() := I(=)(7, 8) : \phi(x) = \psi(x);$

$\rightsquigarrow () := I(=, \to) : \phi = \psi;$

$\rightsquigarrow () := \eth^{-1}\texttt{Unique} : \texttt{This};$

$\square$

$\texttt{GrothendieckGroupWithCancelation} :: \forall M : \texttt{Monoid} \ \& \ \texttt{Commutative} \ . \ M : \texttt{CancelationLaw} \Rightarrow$
$\Rightarrow \kappa : M \hookrightarrow K(M)$

$\texttt{Proof} \ =$

$E := \Big\{\big((a,b),(c,d)\big) | a,b,c,d \in M : a+d = c+b\Big\} :?M^4,$

$(1) := \eth\texttt{CanclationLaw}(M)\eth^{-1}\texttt{Equalence}(M^2)(E) : \Big(E : \texttt{Equivalence}(M \times M)\Big),$

$A := \left(\dfrac{M}{E}, \Big([(a.b)],[c,d)]\Big) \mapsto [(a+b,c+d)]\right) : \mathsf{ABEL},$

$\varphi := \Lambda m \in M \ . \ [(0,m)] : \texttt{Injective} \ \& \ \texttt{Homomorphism}(M,A),$

$(*) := \texttt{InjectiveByComposition}(\kappa)\texttt{GrothendiekGroupTHM}(M)(\varphi) : \Big(\kappa : M \hookrightarrow K(M)\Big);$

$\square$

$\texttt{AbeleanGroupsAreZModules} :: \mathsf{ABEL} \cong_{\mathsf{CAT}} \mathbb{Z}\text{-}\mathsf{MOD}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

3.2 Classification of Finetly Generated Abelean Groups

$$\texttt{torsion} :: \prod A \in \mathsf{ABEL} \,.\, \texttt{Normal}(A)$$
$$\texttt{torsion}\,(A) = \operatorname{tor} A := \{a \in A : \exists n \in \mathbb{N} : na = 0\}$$

$$\texttt{Torsion} :: ?\mathsf{ABEL}$$
$$A : \texttt{Torsion} \iff \operatorname{tor} A = A$$

$$\texttt{TorsionFree} :: ?\mathsf{ABEL}$$
$$A : \texttt{TorsionFree} \iff \operatorname{tor} A = \{0\}$$

$$\texttt{pTorsion} :: \prod A \in \mathsf{ABEL} \,.\, \texttt{Prime}(\mathbb{Z}) \to \texttt{Normal}(A)$$
$$\texttt{pTorsion}\,(A, p) = p\text{-tor}(A) := \{a \in A : \exists k \in \mathbb{N} : p^k a = 0\}$$

$$\texttt{PTorsionIsAPGroup} :: \forall A \in \mathsf{ABEL} \,.\, \forall p : \texttt{Prime}(\mathbb{Z}) \,.\, \forall (0) : |p\text{-tor}\, A| < \infty \,.\, p\text{-tor}\, A : p\text{-}\texttt{Group}$$
$$\texttt{Proof} \;=$$
$$\texttt{Assume}\; q : \texttt{PrimeDivisor}\Big(p\text{-tor}\, A\Big),$$
$$\texttt{Assume}\; (1) : q \neq p,$$
$$(H, 1) := \texttt{PSylowLemma}(p\text{-tor}\, A, q) : \sum H \lhd p\text{-!}\operatorname{tor} A \,.\, |H| = q,$$
$$(2) := \eth\texttt{Prime}(q) : q \neq 1,$$
$$(h, 3) := \eth\texttt{NonTrivial}(G)(H)(1, 2) : \sum h \in H \,.\, h \neq 0,$$
$$(4) := \texttt{OrderDivides}(H, h)(2, 3) : o(h) = q,$$
$$(5) := \eth\texttt{order}(4)\texttt{PrimePowersCoprime}(p, q) : \forall t \in \mathbb{N} \,.\, p^t h \neq 0,$$
$$(6) := \texttt{orderDivides}(p\text{-tor}\, A, h)\eth H(5) : \bot;$$
$$\rightsquigarrow (*) := \eth^{-1} p\text{-}\texttt{Group} : \Big(p\text{-tor}\, A : p\text{-}\texttt{Group}\Big);$$
$$\square$$

$\texttt{TorsionDecomposition} :: \forall A : \texttt{Torsion} \, . \, A \cong_{\mathsf{ABEL}} \bigoplus_{p:\texttt{Prime}(\mathbb{Z})} p\text{-tor}\, A$

$\texttt{Proof} \; =$

$P := \bigoplus_{p:\texttt{Prime}(\mathbb{Z})} p\text{-tor}\, A : \mathsf{ABEL},$

$\varphi := \Lambda x \in P \, . \, \sum_{i=1}^{\infty} x_i : P \to_{\mathsf{ABEL}} A,$

$\texttt{Assume}\; a : A,$

$(1) := \eth\texttt{Torsion}(A)(a) : \{n \in \mathbb{N} : na = 0\} \neq \emptyset,$

$n := \min\{n \in \mathbb{N} : na = 0\} : \mathbb{N},$

$\texttt{Assume}\; (2) : n \neq 1,$

$(k, t, p, 3) := \texttt{PrimeFactorization}(n, 2) : \sum k \in \mathbb{N} \, . \, \sum p : k \hookrightarrow \texttt{Prime}(\mathbb{Z}) \, . \, \sum t : k \to \mathbb{N} \, . \, n = \prod_{i=1}^{k} p_i^{t_i},$

$(s, 4) := \eth^{-1} n \texttt{IterativeMainTHMOfEucledeanDivision}(k, p^t) : \sum s : k \to \mathbb{Z} \, . \, \sum_{n=1}^{k} s_k = 1 \,\&\, p_k^{t_k} s_k : \texttt{DivisiblyB}$

$(5) := \eth^{-1} p\text{-tor}\, A(4) : \forall i \in k \, . \, s_i x \in p_i\text{-tor}\, A,$

$x := \Lambda q : \texttt{Prime}(\mathbb{Z}) \, . \, \texttt{if}\; q = p_i \;\texttt{then}\; s_k x \;\texttt{else}\; 0 : P,$

$() := \eth\phi(x)\eth A(4)_1 : \phi(x) = \sum_{i=1}^{k} s_i x = \left(\sum_{i=1}^{k} s_i\right) x = x;$

$\rightsquigarrow (2) := I(\Rightarrow)I(\exists) : n \neq 1 \Rightarrow \exists x \in P : \varphi(x) = a,$

$\texttt{Assume}\; (3) : n = 1,$

$(2) := (3)\eth n : a = 0,$

$() := \eth\varphi(a)(2) : \varphi(0) = a;$

$\rightsquigarrow (3) := I(\Rightarrow)I(\exists) : n = 1 \Rightarrow \exists x \in P : \varphi(x) = a,$

$() := E(|)\texttt{LEM}(n = 1)(2, 3) : \exists x \in P : \varphi(x) = a;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Surjective} I(\forall) : \varphi : P \twoheadrightarrow A,$

$\texttt{Assume}\; x : \ker\varphi,$

$\texttt{Assume}\; p : \texttt{Prime}(\mathbb{Z}),$

$\texttt{Assume}\; (2) : x_p \neq 0,$

$Q := \{q : \texttt{prime}(\mathbb{Z}) : x_q \neq 0\} :?\texttt{Prime}(\mathbb{Z}),$

$(3) := \eth x(2)\eth Q\eth P(x) : 1 < |Q| < \infty,$

$(t, 4) := \eth P\eth x\forall q \in Q \, . \, \eth q\text{-tor}\, A\eth^{-1}Q : \sum t : Q \to \mathbb{N} \, . \, \forall q \in Q \, . \, q^{t_q} x_q = 0,$

$n := \prod_{q \in Q} q^{t_q} : \mathbb{N},$

$m := \frac{n}{p^{t_p}} : \mathbb{N},$

$(5) := \eth\varphi(x)\eth^{-1}Q\eth m(4)\eth p\texttt{CoprimePeriods}(2) : m\varphi(x) = \sum_{q \in Q} m x_q = m x_p \neq 0,$

$(6) := \eth\texttt{unity}(A)(5) : \varphi(x) \neq 0,$

$(7) := \eth x(6) : \bot;$

$\rightsquigarrow (2) := I(\forall)E(\bot) : \forall p : \texttt{Prime}(\mathbb{Z}) \, . \, x_p = 0,$

$() := I(=, \to)(2) : x = 0;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Singleton} : \ker\varphi(x) = \{0\},$

$(3) := \texttt{TrivialKernelTHM}(2) : \Big(\varphi : P \twoheadrightarrow A\Big),$

$(*) := \eth^{-1}\texttt{Isomorphic}(\mathsf{ABEL})\eth^{-1}\texttt{Isomorphism}(\mathsf{ABEL})(P, A)(1)(3) : A \cong_{\mathsf{ABEL}} P;$

$\square$

$\texttt{PType} :: \prod p : \texttt{Prime}(\mathbb{Z}) \ . \ \prod n \in \mathbb{N} \ . \ (n \to \mathbb{N}) \to ?(\mathsf{ABEL} \ \& \ p\text{-}\texttt{Group})$

$$A : \texttt{PType} \iff A : p\text{-}\texttt{Type} \iff \Lambda t : n \to \mathbb{N} \ . \ A \cong \bigoplus_{i=1}^{n} \frac{\mathbb{Z}}{p^{t_i}\mathbb{Z}}$$

$\texttt{AbeleanPGroupOrderLemma} :: \forall A : \mathsf{ABEL} \ \& \ p\text{-}\texttt{Group} \ . \ \forall s, t \in \mathbb{N} \ . \ \forall a \in A \ . \ \forall (0) : o(a) = p^t \ .$

$\quad . \ \forall (00) : o(A) \le \{p^t\} \ . \ \forall [b] \in \dfrac{A}{\langle a \rangle} \ . \ \forall (000) : o\Big([b]\Big) = p^s \ . \ \exists c \in [b] : o(c) = p^s$

$\texttt{Proof} =$

$(1) := \eth\texttt{order}(000)\eth\texttt{factorGroup}\Big(A, \langle a \rangle\Big) : p^s b \in \langle a \rangle,$

$(n, 2) := \eth\texttt{Cyclic}(a)(1) : \sum n \in \mathbb{N} \ . \ p^s b = na,$

$(k, m, 3) := \texttt{PrimeDivisor}(n, p) : \sum k \in \mathbb{Z}_+ \ . \ \sum m : \texttt{Coprime}(p) \ . \ n = p^k m,$

$(4) := \eth\texttt{order}(0)(3)(2) : k \le t,$

$(5) := (00)(4)\eth p\text{-}\texttt{Group}(A)\texttt{OrderDivides}(0) : \sum l \in t - k \ . \ p^{s+l} b = 0,$

$(6) := (5)(3)(2) : 0 = p^{k+l} ma,$

$(7) := \eth l(0)\eth m\texttt{GeneratorsByCoprime}\Big(\langle a \rangle\Big)(p^t)(0)(a) : l = t - k,$

$(8) := (5)(7)\eth^{-1} o(b) : o(b) = t + s - k,$

$(9) := (00)(8) : s \le k,$

$c := b - p^{k-s} ma : [b],$

$(10) := \eth c(p^s c)(2)(3)\eth\texttt{inverse}(p^k ma) : p^s c = p^s b - p^k ma = p^k ma - p^k ma = 0,$

$(11) := \eth c\eth[b](000) : \forall r \in \mathbb{N} \ . \ r < s \ . \ p^r c \ne 0,$

$(*) := \eth^{-1} o(c)(10, 11) : o(c) = p^s;$

$\square$

$\texttt{AbeleanPGroupHasPType} :: \forall A : \mathsf{ABEL} \ \& \ p\text{-}\texttt{Group} \ . \ \exists! n \in \mathbb{N} : \exists! t : \texttt{NonIncreasing}(n, \mathbb{N}) : A : p\text{-}\texttt{Type}(n)(t)$

$\texttt{Proof} =$

$♂ := \Lambda k \in \mathbb{N} \ . \ \forall A : \mathsf{ABEL} \ \& \ p\text{-}\texttt{Group} \ . \ \forall (0) : |A| \le p^k \ .$

$\quad . \ \exists k \in \mathbb{N} \ . \ \exists! n \in \mathbb{N} : \exists! t : \texttt{NonIncreasing}(n, \mathbb{N}) : A : p\text{-}\texttt{Type}(n)(t) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume} \ A : \mathsf{ABEL} \ \& \ p\text{-}\texttt{Group},$

$\texttt{Assume} \ (1) : |A| = p,$

$(2) := \texttt{PrimeIsCyclic} : \Big(A : \texttt{Cyclic}\Big),$

$() := \eth^{-1} p\text{-}\texttt{Type}(A, (2)) : \Big(A : p\text{-}\texttt{Type}(1, (1))\Big);$

$\rightsquigarrow (1) := \eth^{-1}♂ : ♂(1),$

$\texttt{Assume } k : \mathbb{N},$

$\texttt{Assume } (2) : \female(k),$

$\texttt{Assume } A : \textsf{ABEL} \ \& \ p\text{-}\texttt{Group},$

$\texttt{Assume } (3) : |A| = p^{k+1},$

$\texttt{Assume } (4) : A \ ! \ \texttt{Cyclic},$

$a := \arg\max_{a \in A} o(A) : A,$

$r := \log_p o(a) : \mathbb{N},$

$(5) := (4)(a) : \langle a \rangle \neq A,$

$$(n, t, 6) := \female(k)\left(\frac{A}{\langle a \rangle}\right) : \sum n \in \mathbb{N} \ . \ t : \texttt{NonDecreasing}(n, \mathbb{N}) \ . \ \left(\frac{A}{\langle a \rangle} : p\text{-}\texttt{Type}(n, t)\right),$$

$$(B, 7) := \eth p\text{-}\texttt{Type}\left(\frac{A}{\langle a \rangle}\right)(6) : \sum B : n \to \texttt{Normal}\left(\frac{A}{\langle a \rangle}\right) \ . \ \frac{A}{\langle a \rangle} = \bigoplus_{i=1}^{n} B_i \ \& \ \forall i \in n \ . \ B_i \cong_{\textsf{ABEL}} \frac{\mathbb{Z}}{p^{t_i}\mathbb{Z}},$$

$\texttt{Assume } i : n,$

$$(b, 8) := \eth\texttt{Cyclic}(7)_2(i) : \sum b \in A \ . \ B_i = \Big\langle [b] \Big\rangle,$$

$$(9) := (7)_2(i)(8) : o\Big([b]\Big) = p^{t_i},$$

$$(c_i, 10) := \texttt{AbeleanPGRoupOrderLemma}(A, r, t_i, a, \eth r, \eth a, [b], (9)) : \sum c_i \in [b] \ . \ o(c_i) = p^{t_i};$$

$$\rightsquigarrow (c, 8) := I\left(\prod\right) : \prod i \in n \ . \ \sum c_i \in A \ . \ B_i = \Big\langle [c_i] \Big\rangle \ \& \ o(c_i) = p^{t_i},$$

$C := \langle c \rangle : n \to \texttt{Normal}(A),$

$$(9) := \eth C(8) : \forall i \in n \ . \ C_i \cong \frac{\mathbb{Z}}{p^{t_i}\mathbb{Z}},$$

$$G := \langle a \rangle \oplus \bigoplus_{i=1}^{n} C_i : \textsf{ABEL},$$

$$\varphi := \Lambda x \in G \ . \ \sum_{i=1}^{n+1} x_i : G \to_{\textsf{ABEL}} A,$$

$\texttt{Assume } y : A,$

$$(z_i, 10) := (7)_1(8)([y]) : \sum z : n \to \mathbb{Z} \ . \ [y] = \sum_{i=1}^{n} z_i[c_i],$$

$$(z', 11) := \eth[c_i](10) : \sum z' \in \mathbb{Z} \ . \ y = z'_1 a + \sum_{i=1}^{n} z_i c_i,$$

$x := \Lambda i \in n + 1 \ . \ \texttt{if } i = 1 \texttt{ then } z' a \texttt{ else } z_{i-1} c_{i-1} : G,$

$() := \eth x(11) : \varphi(x) = y;$

$\rightsquigarrow (10) := \eth^{-1}\texttt{Surjective} : (\varphi : G \twoheadrightarrow A),$

$\texttt{Assume } x : \ker \varphi,$

$$(11) := \eth \ker \varphi(x) \eth \varphi(x) : 0 = \varphi(x) = \sum_{i=1}^{n+1} x_i,$$

$$(12) := \pi_{\langle a \rangle}(11) : 0 = \sum_{i=2}^{n+1} [x_i],$$

$(13) := (7)(8)(12) : \forall i \in n \ . \ x_{i+1} = 0,$

$(14) := (11)(13) : x_1 = 0,$

$() := (13)(14) : x = 0;$

$\rightsquigarrow (11) := \eth^{-1}\texttt{Singleton} : \ker\varphi = \{0\}$,

$(12) := \texttt{TrivialKernelTHM}(11) : (\varphi : G \hookrightarrow A)$,

$(13) := \eth^{-1}\texttt{Issomrphic}(\mathsf{ABEL})\eth^{-1}\texttt{Isomorphism}(G, A)(10, 12) : G \cong A$,

$(N, \tau) := (n+1, (r;t)) : \sum N \in \mathbb{N} \,.\, N \to \mathbb{N}$,

$(14) := \eth G(13)\eth^{-1}(N, \tau) : \Big(A : p\text{-}\texttt{Type}(N, \tau)\Big)$,

$\texttt{Assume}\ (M, \sigma) : \sum M \in \mathbb{N} \,.\, M \to \mathbb{N}$,

$\texttt{Assume}\ (15) : \Big(A : p\text{-}\texttt{Type}(M, \sigma)\Big)$,

$(16) := \eth a\eth^{-1}pA : a \notin pA$,

$(17) := \texttt{StrictSubsetCard}(A, \eth p\text{-}\texttt{Group}(A), 16) : |pA| < |A|$,

$(\nu.\mu, 18) := \eth pA\eth^{-1}\texttt{Unique}♂(k)(pA)(17) : \sum \nu \in N \,.\, \sum \mu \in M \,.\, (N - \nu, \tau_{|N-\nu}) = (M - \mu, \sigma_{|M-\mu})\ \&$

$\quad \&\ \Big(\forall i \in N \,.\, i \geq N - \nu \Rightarrow \tau_i = 1\Big)\ \&\ \Big(\forall i \in M \,.\, i \geq M - \mu \Rightarrow \sigma_i = 1\Big)$,

$(19) := E(=, \&)(18)_1 : N - \nu = M - \mu\ \&\ \tau_{|N-\nu} = \sigma_{|M-\mu}$,

$(20) := \texttt{ProductCardinality}(14, 15) : p^{\mu} \prod_{i=1}^{M-\mu} p^{\sigma_i} = |A| = p^{\nu} \prod^{N-\nu} p^{\tau_i}$,

$(21) := (20)(19) : \nu = mu$,

$() := (21)(19) : (N, \tau) = (M, \sigma)$;

$\rightsquigarrow (4) := I(\Rightarrow)I(\exists!) : A\,!\,\texttt{Cyclic} \Rightarrow ♂(n+1)(A)$,

$\texttt{Assume}\ (5) : (A : \texttt{Cyclic})$,

$() := \eth^{-1}p\text{-}\texttt{Type}\eth\texttt{Cyclic} : \Big(A : p\text{-}\texttt{Type}(1, \log_p |A|)\Big)$;

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)I(\forall)E(|)\texttt{LEM}(A : \texttt{Cyclic})I(\Rightarrow)(4) : \forall n \in \mathbb{N} \,.\, ♂(n) \Rightarrow ♂(n+1)$,

$(*) := E(\mathbb{N})(1, 2) : \texttt{This}$;

$\square$

$\texttt{FinitelyGeneratedAbelean} := \mathsf{ABEL}\ \&\ \texttt{FinetelyGenerated} :?\mathsf{ABEL}$;

$\texttt{FinitelyGeneratedTorsionIsFinite} :: \forall A : \texttt{FinitelyGeneratedAbelean}\ \&\ \texttt{Torsion} \,.\, |A| < \infty$

$\texttt{Proof}\ =$

$(F, 1) := \eth\texttt{FinitelyGeneratedAbelean}(A) : \sum F : \texttt{Finite} \,.\, A = \langle F \rangle$,

$(n, 2) := \forall f \in F \,.\, \eth\texttt{Torsion}(n) : \sum n : F \to \mathbb{N} \,.\, \forall f \in F \,.\, n_f f = 0$,

$(3) := \eth\texttt{FinitelyGeneratedAbelean}(A)\texttt{ProductCardinality}(2) : |A| \leq \prod_{f \in F} n_f \leq \infty$,

$\square$

$\texttt{TorsionOfFGAIsFinite} :: \forall A : \texttt{FinitelyGeneratedAbelean} \,.\, |\operatorname{tor} A| < \infty$

$\texttt{Proof} \;=$

$(n, a, 1) := \eth\texttt{Finite}\eth\texttt{FinitelyGeneratedAbelean}(A) : \sum n \in \mathbb{N} \,.\, \sum a : n \to A \,.\, A = \left\langle \{a_i | i \in n\} \right\rangle,$

$\varphi := \Lambda z \in \mathbb{Z}^n \,.\, \sum_{i=1}^{n} z_i a_i : \mathbb{Z}^n \twoheadrightarrow_{\mathsf{ABEL}} A,$

$(2) := \texttt{FreeSubspace}(\mathbb{Z}^n, \phi^{-1}(\operatorname{tor} A)) : (\phi^{-1}(\operatorname{tor} A) : \texttt{FinitelyGeneratedAbelean}),$

$(3) := \texttt{GeneratorsPushforward}(\phi, \phi^{-1}(\operatorname{tor} A)) : (\operatorname{tor} A : \texttt{FinitelyGeneratedAbelean}),$

$(*) := \texttt{FinitelyGeneratedTorsionIsFinite}(\operatorname{tor} A) : |\operatorname{tor} A| < \infty;$

$\square$

$\texttt{TorsionFreeFGAIsFree} :: \forall A : \texttt{FinitelyGeneratedAbelean} \;\&\; \texttt{TorsionFree} \,.\, A : \texttt{Free}$

$\texttt{Proof} \;=$

$(F, 1) := \eth\texttt{FinitelyGeneratedAbelean}(A) : \sum F : \texttt{Finite} \,.\, A = \langle F \rangle,$

$E := \max\{E \subset F : E : \texttt{LinearlyIndependent}(\mathbb{Z})\} : \texttt{LinearlyIndependent}(\mathbb{Z})(A) \;\&\; \texttt{Subset}(F),$

$B := \langle E \rangle : \texttt{Normal}(A),$

$(2) := \texttt{FreeIsGenByLinearlyInd}(B, E) : (B : \texttt{Free}),$

$\texttt{Assume}\ f : F \cap E^{\complement},$

$(\alpha, z, 3) := \eth f \eth E \eth\texttt{LinearlyIndependent}(\mathbb{Z}) : \sum \alpha \in \mathbb{Z} \,.\, z : E \to \mathbb{Z} \,.\, 0 = \alpha f + \sum_{a \in E} z_a a \;\&\; \alpha \neq 0 | z \neq 0,$

$\texttt{Assume}\ (4) : \alpha \neq 0,$

$(5) := \eth f(4) : \alpha f \neq 0,$

$() := (3)(5) \;\&\; (4) : \alpha f \in B \;\&\; \alpha \neq 0;$

$\rightsquigarrow (4) := I(\Rightarrow) : \alpha \neq 0 \Rightarrow (\alpha f \in B \;\&\; \alpha \neq 0),$

$\texttt{Assume}\ (5) : z \neq 0,$

$(6) := \eth\texttt{LinearlyIndependent}(\mathbb{Z})(E)(5) : \sum_{a \in E} z_a a \neq 0,$

$() := (6)(3)_1 : \alpha f \in B \;\&\; \alpha \neq 0;$

$\rightsquigarrow (6) := E(|)(3)_2(4)I(\Rightarrow) : \alpha f \in B \;\&\; \alpha \neq 0,$

$n_f := |\alpha| : \mathbb{N};$

$\rightsquigarrow (n, 3) := I\left(\prod\right) : \prod f \in F \cap E^{\complement} \,.\, \sum n_f \in \mathbb{N} \,.\, n_f f \in B,$

$m := \prod_{f \in F \cap E^{\complement}} n_f : \mathbb{N},$

$(4) := (3)\eth m(1) : mA \subset B,$

$(5) := \texttt{FreeSubspace}(4) : mA : \texttt{Free},$

$\varphi := \Lambda a \in A \,.\, ma : A \to_{\mathsf{ABEL}} mA,$

$(6) := \eth\texttt{TorsionFree}(A)\eth\varphi : \varphi : A \leftrightarrow_{\mathsf{ABEL}} mA,$

$(*) := \texttt{GeneratorsPushforward}(\phi^{-1}, mA) : (A : \texttt{Free});$

$\square$

$\texttt{TorsionFactorIsTorsionFree} :: \forall A \in \mathsf{ABEL} \;.\; \dfrac{A}{\operatorname{tor} A} : \texttt{TorsionFree}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [x] : \operatorname{tor} \dfrac{A}{\operatorname{tor} A},$

$(n, 2) := \eth \operatorname{tor} \dfrac{A}{\operatorname{tor} A} [x] : \sum n \in \mathbb{N} \;.\; n[x] = \operatorname{tor} A,$

$(3) := \eth[x](2) : nx \in \operatorname{tor} A,$

$(m, 4) := \eth \operatorname{tor} A(nx) : \sum m \in \mathbb{N} \;.\; mnx = 0,$

$(5) := \eth^{-1} \operatorname{tor} A(m) : x \in \operatorname{tor} A,$

$() := \eth[x](5) : [x] = \operatorname{tor} A;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Singleton} : \operatorname{tor} \dfrac{A}{\operatorname{tor} A} = \{\operatorname{tor} A\},$

$(*) := \eth^{-1}\texttt{TorsionFree} : \left(\dfrac{A}{\operatorname{tor} A} : \texttt{TorsionFree}\right);$

$\square$

$\texttt{FGAClassification} :: \forall A : \texttt{FinitelyGeneratedAbelean} \;.\; \exists! n, m \in \mathbb{Z}_+ :$

$: \exists! p : \texttt{NonIncreasing}(m, \texttt{Prime}(\mathbb{Z})) : \exists! k : m \to \mathbb{N} \;.\; \exists! t : \prod i \in m \;.\; \texttt{NonIncreasing}(k_i, \mathbb{N}) :$

$: A \cong_{\mathsf{ABEL}} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \dfrac{\mathbb{Z}}{p^{t_i^j}\mathbb{Z}}$

$\texttt{Proof} \;=$

write A as $A \cong \dfrac{A}{\operatorname{tor} A} \oplus \operatorname{tor} A$;

Combine theorems $\texttt{TorsionFactorIsTorsionFree}$, $\texttt{TorsionFreeFGAIsFree}$, $\texttt{GeneratorsPushforward}$ and $\texttt{AltFree}$ to get $\dfrac{A}{\operatorname{tor} A} \cong \mathbb{Z}^n$;

Use $\texttt{TorsionDecomposion}$ to write $\operatorname{tor} A \cong \bigoplus_{i=1}^{m} p_i\text{-}\operatorname{tor} A$ for a unique collection of primes $(p_i)_{i=1}^m$;

for each $i \in m$ the theorem $\texttt{TorsionOfFGAIsFinite}$ ensures that p_i-tor A is also Finite;

Hence, it is a p-group,so use $\texttt{AbeleanPGroupHasPType}$ to write $p_i\text{-}\operatorname{tor} A = \bigoplus_{j=1}^{k_i} \dfrac{\mathbb{Z}}{p_i^{t_i^j}\mathbb{Z}}$;

Combining all together completes the proof.

$\square$

3.3 Dual Groups

$\texttt{Exponent} :: \mathbb{N} \to ?\mathsf{ABEL}$

$A : \texttt{Exponent} \iff \Lambda n \in \mathbb{N} \,.\, nA = \{0\}$

$\texttt{dualGroup} :: \texttt{Exponent}(n) \to \texttt{Exponent}(n)$

$\texttt{dualGroup}\,(A) = A^\wedge := \mathcal{M}_{\mathsf{ABEL}}(A, Z_n)$

$\texttt{dualHomomorphism} :: \prod A, B : \texttt{Exponent}(n) \,.\, (A \to_{\mathsf{ABEL}} B) \to (B^\wedge \to_{\mathsf{ABEL}} A^\wedge)$

$\texttt{dualHomomorphism}\,(\varphi) = \varphi^\wedge := \Lambda f \in B^\wedge \,.\, f \circ \varphi$

$\texttt{DualId} :: \forall A : \texttt{Exponent}(n) \,.\, \underset{A}{\mathrm{id}}^\wedge = \underset{A^\wedge}{\mathrm{id}}$

$\texttt{Proof} =$

$f \circ \mathrm{id} = f = \mathrm{id}(f)$

$\square$

$\texttt{ContravariantDual} :: \forall A, B, C : \texttt{Exponent}(n) \,.\, \forall \varphi : A \to_{\mathsf{ABEL}} B \,.\, \forall \psi : B \to_{\mathsf{ABEL}} C \,.\, (\phi\psi)^\wedge = \psi^\wedge\phi^\wedge$

$\texttt{Proof} =$

$\texttt{Assume}\ f : C^\wedge,$

$(\phi\psi)^\wedge(f) = f \circ \psi \circ \phi = \phi^\wedge(f \circ \psi) = \psi^\wedge\phi^\wedge(f)$

$\square$

$\texttt{DualProduct} :: \forall A, B : \texttt{Exponent}(n) \,.\, (A \times B)^\wedge = A^\wedge \times B^\wedge$

$\texttt{Proof} =$

$\varphi := \Lambda f \in (A \times B)^\wedge \,.\, \Big(\Lambda a \in A \,.\, f(a, 0), \Lambda b \in B \,.\, f(0, b)\Big) : (A \times B)^\wedge \to_{\mathsf{ABEL}} A^\wedge \times B^\wedge,$

$\psi := \Lambda(f, g) \in A^\wedge \times B^\wedge \,.\, \Lambda(a, b) \in A \times B \,.\, f(a) + g(b) : A^\wedge \times B^\wedge \to_{\mathsf{ABEL}} (A \times B)^\wedge,$

$(1) := \eth\varphi\eth\psi : \varphi\psi = \mathrm{id},$

$(2) := \eth\psi\eth\varphi : \psi\varphi = \mathrm{id},$

$(*) := \eth\texttt{Isomorphic}(\mathsf{ABEL})\eth\texttt{Isomorphism}(1, 2) : A^\wedge \times B^\wedge \cong (AB)^\wedge;$

$\square$

$\texttt{DualOfCyclic} :: \forall m : \texttt{Divisor}(n) \,.\, Z_m^{\wedge(n)} \cong_{\mathsf{ABEL}} Z_m$

$\texttt{Proof} =$

$(N, 1) := \texttt{CyclicDividingSubgroup}(Z_n, m, n)) : \sum N \lhd Z_n \,.\, N \cong Z_m \;\&\; N : \texttt{Cyclic},$

$\varphi := \Lambda k \in N \,.\, l \in Z_m \,.\, lk : N \to_{\mathsf{ABEL}} Z_m^\wedge,$

$\psi := \Lambda f \in Z_m^\wedge \,.\, f(1) : Z_m^\wedge \to_{\mathsf{ABEL}} Z_m,$

$(2) := \eth\psi\eth\varphi : \psi\varphi = \mathrm{id},$

$(3) := \eth\varphi\eth\psi : \varphi\psi = \mathrm{id},$

$(*) := \eth\texttt{Isomorphic}(\mathsf{ABEL})\eth\texttt{Isomorphism}(2, 3) : Z_m^\wedge \cong_{\mathsf{ABEL}} Z_m;$

$\square$

$\texttt{DualOfFiniteGroup} :: \forall A : \texttt{Exponent}(n) \,\&\, \texttt{FiniteGroup} \,.\, A^\wedge \cong A$

Proof $=$

The torsion of A having exponent n is the A itself, so by FGAClassification A is a product of cyclic groups, and combinig dualProduct and DualOfCyclic provides result.

□

$\texttt{BillinearBalanceTHM} :: \forall A, B \in \mathsf{ABEL} \,.\, \forall T : \mathcal{L}_{\mathbb{Z}}(A, B; Z_m) \,.\, \forall (0) : \left|\frac{A}{l\text{-}\ker T}\right| < \infty \,.$

$. \frac{A}{l\text{-ker}\, T} \cong \frac{B}{r\text{-ker}\, T}$

Proof $=$

$(1) := \eth Z_m \eth \frac{A}{l\text{-ker}\, T} : \left(\frac{A}{l\text{-ker}\, T} : \texttt{Exponent}(m)\right),$

$(2) := \eth Z_m \eth \frac{B}{r\text{-ker}\, T} : \left(\frac{B}{r\text{-ker}\, T} : \texttt{Exponent}(m)\right),$

$(C, 3) := \eth \frac{B}{r\text{-ker}\, T} \eth^{-1} \left(\frac{A}{r\text{-ker}\, T}\right)^\wedge : \sum C \lhd \left(\frac{A}{r\text{-ker}\, T}\right)^\wedge \,.\, C \cong \frac{B}{l\text{-ker}\, T},$

$(4) := \texttt{DualOfFiniteGroup}(0) : \left(\frac{A}{l\text{-ker}\, T}\right)^\wedge \cong \frac{A}{l\text{-ker}\, T},$

$(5) := \texttt{SubsetCardinality}(3)(4) : \left|\frac{B}{r\text{-ker}\, T}\right| < \infty,$

$(D, 6) := \eth \frac{A}{l\text{-ker}\, T} \eth^{-1} \left(\frac{B}{r\text{-ker}\, T}\right)^\wedge : \sum D \lhd \left(\frac{B}{l\text{-ker}\, T}\right)^\wedge \,.\, D \cong \frac{A}{r\text{-ker}\, T},$

$(7) := \texttt{DualOfFiniteGroup}(0) : \left(\frac{B}{r\text{-ker}\, T}\right)^\wedge \cong \frac{B}{r\text{-ker}\, T},$

$(*) := (4)(3)(6)(7) : \frac{A}{l\text{-}\ker T} \cong \frac{B}{r\text{-}\ker T};$

□

$\texttt{OrthogonalComplement} :: \forall A : \mathsf{ABEL} \,\&\, \texttt{FiniteGroup} \,.\, \forall B \lhd A \,.\, \frac{A^\wedge}{B^\perp} \cong B^\wedge$

Proof $=$

Apply privious theorem with

$T = \Lambda b \in B \,.\, \Lambda f \in A^\wedge \,.\, f(b)$

□

3.4 Miscoleneus Facts

$\texttt{TwoGroupIsAbelean} :: \forall G : 2\text{-}\texttt{Group} \,.\, G \in \mathsf{ABEL}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a, b : G,$

$(1) := \eth^2 2\text{-}\texttt{Group}(G)\Big((ab)(ba)\Big) : (ab)(ba) = ab^2a = a^2 = e,$

$(2) := \eth^{-1}\texttt{Inverse}(1) : (ab)^{-1} = ba,$

$(3) := \eth 2\text{-}\texttt{Group}(ab)^2 : (ab)^2 = e,$

$(4) := \eth^{-1}\texttt{Inverse} : (ab)^{-1} = ab,$

$() := (3)(4) : ba = ab;$

$\rightsquigarrow (1) := \eth^{-1}\mathsf{ABEL} : G \in \mathsf{ABEL},$

$\square$

3.5 Hildebrandt Theory[!]

4 Global Group Theory

4.1 Groups as Category

$\texttt{GroupsHaveProducts} :: \mathsf{GRP} : \texttt{WithProducts}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ I : \mathsf{SET},$

$\texttt{Assume}\ G : I \to \mathsf{GRP},$

$\texttt{Assume}\ H : \mathsf{GRP},$

$\texttt{Assume}\ \phi : \prod i \in I \;.\; H \xrightarrow{\mathsf{GRP}} G_i,$

$\psi := \Lambda h \in H \;.\; (\phi_i(h))_{i\in I} : H \to \prod_{i\in I} G_i,$

$(1) := \eth\psi : \forall i \in I \;.\; \pi_i\psi = \phi_i,$

$(2) := \eth^{-1} \int \mathrm{Cone}_G(1) : \left[\psi : (H,\phi) \xrightarrow{\int \mathrm{Cone}_G} \left(\prod_{i\in I} G_i, \pi\right)\right],$

$\texttt{Assume}\ \alpha : \psi : H \xrightarrow{\int \mathrm{Cone}_G} \prod_{i\in I}(G_i, \pi),$

$(3) := \eth \int \mathrm{Cone}_G(\alpha) : \forall i \in I \;.\; \pi_i\psi = \phi_i,$

$() := I(=,\to)(2)(3) : \psi = \alpha;$

$\leadsto () := \eth^{-1}\texttt{Limit}(I,G) : \left[\left(\prod_{i\in I} G_i, \pi\right) : \texttt{Limit}(I,G)\right];$

$(*) := \eth^{-1}\texttt{WithProducts} : [\mathsf{GRP} : \texttt{WithProducts}];$

$\square$

$\texttt{GroupsWithEq} :: \mathsf{GRP} : \texttt{WithEqualizers}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ G, H : \mathsf{GRP},$

$\texttt{Assume}\ \phi, \psi : G \xrightarrow{\mathsf{GRP}} H,$

$E := \{g \in G : \phi(g) = \psi(g)\} : \texttt{Subset}(G),$

$\texttt{Assume}\ a, b : E,$

$(1) := \eth\phi\eth a\eth b\eth\psi : \phi(ab) = \phi(a)\phi(b) = \psi(a)\psi(b) = \psi(ab),$

$() := \eth E(1) : ab \in E;$

$\leadsto (1) := I(\forall) : \forall a, b \in E \;.\; ab \in E,$

$\texttt{Assume}\ a : E,$

$(2) := \texttt{HomoInv}(\phi)\eth a\texttt{HomoInv}(\psi) : \phi(a^{-1}) = \phi^{-1}(a) = \psi^{-1}(a) = \psi(a^{-1}),$

$() := \eth E(2) : a^{-1} \in E;$

$(2) := \eth E(2) : a^{-1} \in E,$

$(3) := \texttt{HomoId}^2(\phi)(\psi)E(=) : \phi(e) = e = \psi(e),$

$(4) := \eth E(3) : e \in E,$

$(5) := \eth^{-1}\mathsf{GRP}\eth^{-1}\texttt{Group}(1,2,4) : E \in \mathsf{GRP},$

$\dots$

$\square$

$\texttt{GroupCatIsComplete} :: \mathsf{GRP} : \texttt{Complete}$

$\texttt{Proof} =$

$(*) := \texttt{CompleteByProductsAndEqualizers}(\mathsf{GRP}) : \left[\mathsf{GRP} : \texttt{Complete}\right];$

$\square$

$\texttt{ForgetGroupStructure} :: \mathsf{GRP} \xrightarrow{\mathsf{CAT}} \mathsf{SET}$

$\texttt{ForgetGroupStructure}\,(G, \cdot) = U_{\mathsf{GRP}}(G, \cdot) := G$

$\texttt{ForgetGroupStructure}\,(G, H, f) = U_{\mathsf{GRP}}(G, H, f) := f$

$\texttt{ForgettingGroupStructureIsContinuous} :: U_{\mathsf{GRP}} : \texttt{Continuous}$

$\texttt{Proof} =$

$$(1) := \ldots : U_{\mathsf{GRP}}\left(\prod_{i \in I} G_i\right) = \prod_{i \in I} U_{\mathsf{GRP}}(G_i),$$

$$(2) := \ldots : U_{\mathsf{GRP}}(\mathtt{eq}(f, g)) = \mathtt{eq}(U_{\mathsf{GRP}}(f), U_{\mathsf{GRP}}(g)),$$

$\square$

$\texttt{GroupsAreIntersectable} :: \mathsf{GRP} : \texttt{Intersectable}$

$\texttt{Proof} =$

$\texttt{Assume}\ G : \mathsf{GRP},$

$\texttt{Assume}\ (H, i) : \texttt{Subobject}(G),$

$(S, 1) := \eth\texttt{Subobject}(G) : i(H) \cong_{\mathsf{GRP}} H;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Intersectable} : \mathsf{GRP} : \texttt{Intersectable},$

$\square$

$\texttt{GroupIsCocomplete} :: \mathsf{GRP} : \texttt{Cocomplete}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mathcal{I} : \mathsf{SCAT},$

$\texttt{Assume}\ G : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathsf{GRP},$

$A := \bigsqcup_{i \in \mathcal{I}} : \mathsf{SET},$

$\kappa := \max(|A|, \aleph_0) : \texttt{Cardinal},$

$V := \prod \left\{ H \middle| [H] : \texttt{Isoclass}(\mathsf{GRP}) : |H| < \kappa \right\} : \mathsf{GRP},$

$I := \left\{ \phi \middle| \prod i \in \mathcal{I} \,.\, G_i \to V \right\} : \texttt{Set}\left(\prod i \in \mathcal{I} \,.\, G_i \to V \right),$

$A := \Lambda \phi \in I \,.\, \left\langle \bigcup_{i \in \mathcal{I}} \operatorname{Im} \phi_i \right\rangle : I \to \mathsf{GRP},$

$X := \Lambda \phi \in I \,.\, A_\phi / N \left\{ \phi_i(x) \phi_j^{-1}\big(G_{i,j}(h)(x)\big) \middle| i, j \in \mathcal{I}, x \in G_i, h : i \xrightarrow{\mathcal{I}} j \right\} : I \to \mathsf{GRP},$

$f := \Lambda \phi \in I \,.\, \Lambda j \in \mathcal{I} \,.\, \phi_j^{|A_i} \pi_{X_\phi} : \sum \phi \in I \,.\, \sum j \in \mathcal{I} \,.\, G_j \xrightarrow{\mathsf{GRP}} X(\phi),$

$\texttt{Assume}\ \phi : I,$

$\texttt{Assume}\ j, k : \mathcal{I},$

$\texttt{Assume}\ h : j \xrightarrow{\mathcal{I}} k,$

$() := \eth X \eth f : \phi_k = G_{j,k}(h) \phi_j;$

$\rightsquigarrow (1) := I(\forall) \eth^{-1} \texttt{NaturalTransform} : \forall \phi \in I \,.\, f : G \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} \Delta X(phi),$

$\texttt{Assume}\ H : \mathsf{GRP},$

$\texttt{Assume}\ \psi : G \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} \Delta H,$

$F := \left\langle \bigcup_{i \in \mathcal{I}} \psi_i(G_i) \right\rangle : \mathsf{GRP},$

$(2) := \eth \kappa \texttt{UnionCardinalityGeneratedCardinality} : |F| \le \kappa,$

$(\phi, \varphi, 3) := \eth V \eth I \eth F(2) : \sum \phi \in I \,.\, \sum \varphi : A_\phi \xleftrightarrow{\mathsf{GRP}} F \,.\, \forall i \in \mathcal{I} \,.\, \psi_i = \phi_i \varphi \iota_H,$

$(4) := \eth \texttt{NaturalTransform}(\psi) \eth X \eth \phi(3) : A_\phi \cong_{\mathsf{GRP}} X_\phi,$

$() := \eth f(4)(3) : \psi = f_\phi \Delta(\varphi \iota_H);$

$\rightsquigarrow (2) := I(\forall) I^3(\exists) I^2(\forall) I^2(\exists) :$

$: \forall G : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathsf{GRP} \,.\, \exists I : \mathsf{SET} : \exists X : I \to \mathsf{GRP}^{\mathcal{I}} \,.\, \exists f : \sum i \in I \,.\, G \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} \Delta(X_\phi) : \forall H : \mathsf{GRP} \,.\, \forall \psi : G \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} \Delta$

$. \exists i \in I \,.\, \exists h : X_i \xrightarrow{\mathsf{GRP}} H \,.\, \psi = f_i \Delta(h),$

$(L, 3) := \texttt{GeneralAdjointFunctorTheorem}(\Delta_{\mathcal{I}}, (2)) : \sum L : \mathsf{GRP}^{\mathcal{I}} \xrightarrow{\mathsf{CAT}} \mathsf{GRP} \,.\, L \vdash \Delta_{\mathcal{I}},$

$() := \texttt{ColimitAsAdjoint}(3) : \forall G : \mathsf{GRP}^{\mathcal{I}} \,.\, \operatorname*{colim}_{i \in \mathcal{I}} G_i = L(G);$

$\rightsquigarrow (*) := \eth^{-1} \texttt{Cocomplete} I(\forall) : [\mathsf{GRP} : \texttt{Cocomplete}];$

$\square$

4.2 Free Products

$\texttt{testSet} :: \prod I : \mathsf{SET} \, . \, (I \to \mathsf{GRP}) \to \mathsf{SET}$

$\texttt{testSet}\,(G) = T(G) := \left\{ a : \left(\bigsqcup_{i \in I} G_i \setminus \{e\} \right)^* : \forall n \in \mathbb{N} \, . \, a_{n,1} \neq a_{n+1,1} \right\}$

$\texttt{testAction} :: \prod I : \mathsf{SET} \, . \, \prod G : I \to \mathsf{GRP} \, . \, \left(\sum i \in I \, . \, G_i \right) \to \mathrm{End}_{\mathsf{SET}} T(G)$

$\texttt{testAction}\,((i,e)) = \iota_i(e) := \mathrm{id}$

$\texttt{testAction}\,((i,g)) = \iota_i(g) := \Lambda a \in T(G) \, . \, \texttt{if } a = \epsilon \texttt{ then } g \texttt{ else if } a_1 = g^{-1} \texttt{ then } a_{+1} \texttt{ else}$
$\texttt{else if } a_1 \in G_i \texttt{ then } g a_1 \oplus a_{2,\ldots} \texttt{ else } g \oplus a$

$\texttt{testActionIsHomo} :: \forall I : \mathsf{SET} \, . \, \forall (0) : |I| > 1 \, . \, \forall G : I \to \mathsf{GRP} \, . \, \forall i \in I \, . \, \forall g, h \in G_i \, . \, \iota_i(g)\iota_i(h) = \iota_(gh)$

$\texttt{Proof} \ =$

$(00) := \eth T(G)(0) : T(G) \neq \emptyset,$

$\texttt{Assume } a : T(G),$

$\texttt{Assume } (1) : a_1 \in G_i,$

$\texttt{Assume } (2) : a_1 \neq h^{-1},$

$\texttt{Assume } (3) : h a_1 \neq g^{-1},$

$() := \eth \mathsf{GRP} \eth \iota_i(2)(3) : \iota_i(g)\iota_i(h)a = \iota_i(g) h a_1 \oplus a_{+1} = g h a_1 \oplus a_{+1} = \iota_i(gh)(a);$

$\rightsquigarrow (3) := I(\Rightarrow) : h a_1 \neq g^{-1} \Rightarrow \texttt{This},$

$\texttt{Assume } (4) : h a_1 = g^{-1},$

$(5) := h^{-1}(4)\texttt{InverseProduct} : a_1 = h^{-1} g^{-1} = (hg)^{-1},$

$() := (1)(2)(4)(5) : \iota_i(g)\iota_i(h)(a) = \iota_i(g) h a_1 \oplus a_{+1} = a_{+1} = \iota_i(gh)a;$

$\rightsquigarrow () := I(\Rightarrow)E(|)\texttt{LEM}(3) : \texttt{This};$

$\rightsquigarrow (2) := I(\Rightarrow) : a_1 \neq h^{-1} \Rightarrow \texttt{This},$

$\texttt{Assume } (3) : a_1 = h^{-1},$

$() := (1)(3) : \iota_i(g)\iota_i(h)a = \iota_i(g)a_{+1} = g \oplus a_{+1} = (g h a_1) \oplus a_{+1} = \iota_i(gh)a;$

$\rightsquigarrow () := I(\Rightarrow)E(|)(2)\texttt{LEM} : \texttt{This};$

$\rightsquigarrow (1) := I(\Rightarrow) : a_1 \in G_1 \Rightarrow \texttt{This},$

$\texttt{Assume } (2) : a_1 \notin G_1,$

$\texttt{Assume } (3) : g \neq h^{-1},$

$() := (2)(3) : \iota_i(g)\iota_i(h)a = \iota_i(g) h \oplus a = (gh) \oplus a = \iota_i(gh)a;$

$\rightsquigarrow (3) := I(\Rightarrow) : g \neq h^{-1} \Rightarrow \texttt{This},$

$\texttt{Assume } (4) : g = h^{-1},$

$() := (2)(4)\eth^{-1}\iota_i(e)(4) : \iota_i(g)\iota_i(h)a = \iota_i(g) h \oplus a = a = \iota_i(e)a = \iota_i(gh)a;$

$\rightsquigarrow () := I(\Rightarrow)E(|)(3)\texttt{LEM} : \texttt{This};$

$\rightsquigarrow (*) := I(\Rightarrow)E(|)(1)\texttt{LEM} : \texttt{This};$

$\square$

$\texttt{testActionIsAuto} :: \forall I : \mathsf{SET} \, . \, \forall (0) : |I| > 1 \, . \, \forall G : I \to \mathsf{GRP} \, . \, \forall i \in I \, . \, \forall g \in G_i \, . \, \iota_i(g) \in \mathrm{Aut}_{\mathsf{SET}} T(G)$

$\texttt{Proof} \ =$

$(1) := \texttt{TestActionIsHomo}(g, g^{-1}) : \iota_i(g)\iota_i(g^{-1}) = \iota_i(e) = \mathrm{id},$

$(1) := \texttt{TestActionIsHomo}(g^{-1}, g) : \iota_i(g^{-1})\iota_i(g) = \iota_i(e) = \mathrm{id},$

$(*) := \eth\texttt{Automorphisms}(\mathsf{SET})(1)(2)\texttt{InvertibleBijection} : \iota_i(g) \in \mathrm{Aut}_{\mathsf{SET}} T(G);$

$\texttt{freeProduct} :: \prod I : \mathsf{SET} \,.\, (I \to \mathsf{GRP}) \to \mathsf{GRP}$

$\texttt{freeProduct}\,((\{i\}, G)) = \coprod_{i\in I} G_i := G_i$

$\texttt{freeProduct}\,((I, G)) = \coprod_{i\in I} G_i := \left\langle \left\{ \iota_i(g) | i \in \mathcal{I}, g \in G_i : g \neq e \right\} \right\rangle$

$\texttt{LengthType} :: \prod I \in \mathsf{SET} \,.\, \prod G : I \to \mathsf{GRP} \,.\, \mathbb{N} \to ? \coprod_{i\in I} G_i$

$x : \texttt{LengthType} \iff \Lambda n \in \mathbb{N} \,.\, \exists i : n \to \mathcal{I} \,.\, \exists g : \prod j \in n \,.\, G_{i_j} \,.\, x = \prod_{i=1}^{n} \iota_{i_j}(n)$

$\texttt{FreeProductStructure} :: \forall I \in \mathsf{SET} \,.\, \forall G : I \to \mathsf{GRP} \,.\, \forall x \in \coprod_{i\in I} \,.\, \exists n \in \mathbb{Z}_+ \,.\, \exists i : n \to I \,.\, \exists g : \prod j \in n \,.\, G_{i_j} \,.\, x($

$\texttt{Proof} =$

$(n, i, g, 1) := \eth \coprod : \sum n \in \mathbb{Z}_+ \,.\, \sum i : n \to I \,.\, \sum g : \prod j \in n \,.\, G_{i_j} \,.\, x = \prod_{j=1}^{n} \iota_{i_j}(g_j),$

$\texttt{Assume}\ (2) : n = 0,$

$(3) := (1)(2) : x = e,$

$(4) := (3)(2) : x(\epsilon) = e(\epsilon) = \epsilon = \prod_{n=1}^{0} g_i,$

$\rightsquigarrow (2) := I(\Rightarrow) : n = 0 \Rightarrow \texttt{This}(x),$

$\texttt{Assume}\ (3) : n > 1,$

$\texttt{Assume}\ (4) : \forall y : \texttt{LengthType}(n-1) \,.\, \texttt{This}(y),$

$(m, i, g, 5, 6) := (4)(1) : \sum m \in \mathbb{Z}_+ \,.\, \sum j : m \to I \,.\, \sum h : \prod k \in m \,.\, G_{j_k} \,.\, \prod_{j=1}^{m} \iota_{j_k}(h_k) = \prod_{j=1}^{n} \iota_{i_j}(g_j)\ \&$

$\quad \& \prod_{j=1}^{m} h_k = \prod_{j=1}^{n} \iota_{i_j}(g_j)(\epsilon),$

$(7) := (1)(5) : x = \iota_{i_n}(g_n) \prod_{k=1}^{m} \iota_{j_k}(h_{j_k}),$

$(8) := (1)(5) : x(\epsilon) = \iota_{i_n}(g_n) \prod_{k=1}^{m} h_{j_k},$

$\texttt{Assume}\ (9) : h_{j_m} \in G_{i_n},$

$10 := (9)(7) : x = \iota_{i_n}(g_n h_n) \prod_{k=1}^{m-1} \iota_{j_k}(h_{j_k}),$

$11 := (9)(8) : x(\epsilon) = g_n h_n \prod_{k=1}^{m-1} \iota_{j_k}(h_{j_k}),$

$() := (10)(11) : \texttt{This};$

$\rightsquigarrow (9) := I(\Rightarrow) : h_{j_m} \in G_{i_n} \Rightarrow \texttt{This}(x),$

$(10) := I(\Rightarrow)\eth T(G)(5)(6)(7)(8) : h_{j_m} \in G_{i_n} \Rightarrow \texttt{This}(x),$

$(11) := E(|)(10)(9)\texttt{LEM} : \texttt{This};$

$\rightsquigarrow (12) := \texttt{Induction}(\mathbb{Z}_+)(2) : \texttt{This},$

$\square$

$\mathtt{FreeProductIsCoproduct} :: \forall I \in \mathsf{SET} \,.\, \forall G : I \to \mathsf{GRP} \,.\, \coprod_{i \in \mathcal{I}} G_i : \mathtt{Coproduct}(\mathsf{GRP}, G)$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ H : \mathsf{GRP},$

$\mathtt{Assume}\ \phi : \prod i \in \mathcal{I} \,.\, G_i \xrightarrow{\mathsf{GRP}} H,$

$\mathtt{Assume}\ x : \coprod_{i \in I} G_i,$

$(n, i, g, 1) := \eth T(G) x(\varepsilon) : \sum n \in n \,.\, \sum i : n \to I \,.\, \sum g : \prod j \in n \,.\, G_{j_k} \,.\, x(\varepsilon) = \prod_{j=1}^{n} g_j,$

$\psi(x) := \prod_{j=1}^{n} \phi_{i_j}(g_j) : H;$

$\rightsquigarrow \psi := I(\to) : \coprod_{i \in I} G_i \xrightarrow{\mathsf{GRP}} H,$

$\mathtt{Assume}\ i : I,$

$\mathtt{Assume}\ a : G_i,$

$() := \eth \psi_i(a) : \psi \iota_i(a) = \phi_i(a);$

$\rightsquigarrow (1) := I(\forall) : \forall i \in I \,.\, \forall a \in G_i \,.\, \psi \iota_i(a) = \phi_i(a),$

$\mathtt{Assume}\ \psi' : \coprod_{i \in I} G_i \xrightarrow{\mathsf{GRP}} H,$

$\mathtt{Assume}\ (2) : \forall i \in I \,.\, \forall a \in G_i \,.\, \psi' \iota_i(a) = \phi_i(a),$

$\mathtt{Assume}\ x : \coprod_{i \in I} G_i,$

$(n, i, g, 3) := \mathtt{FreeProductStructure}(x) : \sum n \in \mathbb{N} \,.\, \sum i : n \to I \,.\, \sum g : \prod j \in n \,.\, G_j \,.$

$\quad .\ x = \prod_{j=1}^{n} \iota_{i_j}(g_j) \ \&\ x(\varepsilon)) = \prod_{j=1}^{n} g_j,$

$() := (1)(3)(2) : \psi(x) = \prod_{j=1}^{n} \phi_{j_i}(g_j) = \psi'(x);$

$\rightsquigarrow (*) := \eth^{-1} \mathtt{Coproduct} : \coprod_{i \in \mathcal{I}} X_i;$

$\square$

4.3 Free Groups and Presentation

$\texttt{freeGroup} :: \mathsf{SET} \to \mathsf{GRP}$

$\texttt{freeGroup}\,(A) = F_{\mathsf{GRP}}(A) := \coprod_{a \in A} \mathbb{Z}$

$\texttt{freeHomo} :: \prod A, B\mathsf{SET} \;.\; A \xrightarrow{\mathsf{SET}} B \to F(A) \xrightarrow{\mathsf{GRP}} F(B)$

$\texttt{freeHomo}\,(f) = F_{A,B}(f) := \Lambda \prod_{i=1}^{n} (a_i, m_i) \in F(A) \;.\; \prod_{i=1}^{n} (f(a_i), m_i)$

$\texttt{ForgetfulFunctorAdmitsLeftAdjoint} :: \exists F : \mathsf{SET} \xrightarrow{\mathsf{CAT}} \mathsf{GRP} \;.\; F \dashv U$

$\texttt{Proof} \;=$

$\texttt{Assume}\; \mathcal{I} : \mathsf{SCAT},$

$\texttt{Assume}\; A : \mathsf{SET},$

$\kappa := \max(|A|, \aleph_0) : \texttt{Cardinal},$

$V := \prod \left\{ H \middle| [H] : \texttt{Isoclass}(\mathsf{GRP}) : |H| < \kappa \right\} : \mathsf{GRP},$

$I := \left\{ \phi \middle| A \to V \right\} : \texttt{Set}\,(A \to V)\,,$

$X := \Lambda \phi \in I \;.\; \langle \operatorname{Im} \phi \rangle : I \to \mathsf{GRP},$

$f := \Lambda \phi \in I \;.\; \phi : \sum \phi \in I \;.\; A \xrightarrow{\mathsf{SET}} U(X_\phi),$

$\texttt{Assume}\; H : \mathsf{GRP},$

$\texttt{Assume}\; \psi : A \xrightarrow{\mathsf{SET}} U(H),$

$H' := \langle \psi_i(A) \rangle : \mathsf{GRP},$

$(2) := \eth\kappa\texttt{GeneratedCardinality} : |H'| \le \kappa,$

$(\phi, \varphi, 3) := \eth V \eth I \eth F (2) : \sum \phi \in I \;.\; \sum \varphi : X_\phi \xleftrightarrow{\mathsf{GRP}} H' \;.\; \forall i \in \mathcal{I} \;.\; \psi_i = \phi_i \varphi \iota_H,$

$() := \eth f (4)(3) : \psi = f_\phi U(\varphi \iota_H);$

$\leadsto (2) := I(\forall) I^3(\exists) I^2(\forall) I^2(\exists) :$

$: \forall A : \mathsf{SET} \;.\; \exists I : \mathsf{SET} : \exists X : I \to \mathsf{GRP} \;.\; \exists f : \sum i \in I \;.\; A \xrightarrow{SET} U(X_i) : \forall H : \mathsf{GRP} \;.\; \forall \psi : G \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} U(H) \;.$

$.\; \exists i \in I \;.\; \exists h : X_i \xrightarrow{\mathsf{GRP}^{\mathcal{I}}} H \;.\; \psi = f_i \phi,$

$(F, 3) := \texttt{GeneralAdjointFunctorTheorem}(\Delta_{\mathcal{I}}, (2)) : \sum L : \mathsf{GRP}^{\mathcal{I}} \xrightarrow{\mathsf{CAT}} \mathsf{GRP} \;.\; L \vdash \Delta_{\mathcal{I}},$

$\ldots$

$\square$

$\texttt{freeGroup2} :: \mathsf{SET} \xrightarrow{\mathsf{CAT}} \mathsf{GRP}$

$\texttt{freeGroup2}\,() = F'_{\mathsf{GRP}} := \texttt{ForgetfulFunctorAdmitsLeftAdjoint}$

$\texttt{FreeGroupOfSingleton} :: F'_{\mathsf{GRP}}(\mathbf{1}) \cong_{\mathsf{GRP}} \mathbb{Z}$

$\texttt{Proof} =$

$C := \Lambda 1 \in \mathbf{1} \; . \; n \; \texttt{as} \; \mathbb{Z} : \mathbf{1} \xrightarrow{\mathsf{SET}} \mathbb{Z},$

$(\varphi, a, 1) := \eth F'_{\mathsf{GRP}}(\mathbf{1})(C) : \sum \varphi : F'_{\mathsf{GRP}}(\mathbf{1}) \xrightarrow{\mathsf{GRP}} \mathbb{Z} \; . \; \sum a : \mathbf{1} \to\in F'_{\mathsf{GRP}}(\mathbf{1}) \; . \; aU(\varphi) = C,$

$(2) := (1)\eth \exp_a : \exp_a \varphi(a) = a,$

$(3) := \eth \exp_a(1) : \varphi \exp_a(1) = 1,$

$(4) := \eth F'_{\mathsf{GRP}}(2) : \exp_a \varphi = \underset{F'_{\mathsf{GRP}}()}{\mathrm{id}},$

$(5) := \texttt{CyclicEndomorph}(3) : \varphi \exp_a = \mathrm{id},$

$(*) := \eth^{-1}\texttt{Iso}(4)(5) : F'_{\mathsf{GRP}}(\mathbf{1}) \cong_{\mathsf{GRP}} \mathbb{Z};$

$\square$

$\texttt{FreeGroupStrtucture} :: F_{\mathsf{GRP}} = F'_{\mathsf{GRP}}$

$\texttt{Proof} =$

$(*) := \texttt{LeftAdointCommutesWithLimits}(F'_{\mathsf{GRP}})\texttt{FreeGroupOfSingleton}\eth F_{\mathsf{GRP}} : F_{\mathsf{GRP}} = F'_{\mathsf{GRP}};$

$\square$

$\texttt{Free} :: ?\mathsf{GRP}$

$G : \texttt{Free} \iff \exists A : \mathsf{SET} \; . \; G \cong_{\mathsf{GRP}} F(G)$

$\texttt{boolenization} :: \mathsf{GRP} \to \mathsf{VS}(\mathbb{F}_2)$

$\texttt{boolenization}\,(G) = \mathrm{bool}(G) := \dfrac{G}{\{x^2 | x \in G\}}$

$\texttt{DimensionOfFreeBoolenization} :: \forall A : \mathsf{SET} \; . \; \dim_{\mathbb{F}_2} \mathrm{bool} \; F(A) = |A|$

$\texttt{Proof} =$

$\texttt{Assume} \; b : A \to \mathbb{F}_2,$

$\texttt{Assume} \; (1) : \Big|\{a \in A | b(a) = 1\}\Big| < \infty,$

$() := \eth \,\mathrm{bool} : \sum_{a \in A} b(a)[a] \neq 0;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{LinearlyIndependent} : \left[\Big([a]\Big)_{a \in A} : \texttt{LinearlyIndependent}(\mathrm{bool}\, F(A))\right],$

$(2) := \eth \,\mathrm{bool}\, \eth F(A) : \left[\Big([a]\Big)_{a \in A} : \texttt{Generating}(\mathrm{bool}\, F(A))\right],$

$(3) := \eth^{-1}\texttt{Basis}(1, 2) : \left[\Big([a]\Big)_{a \in A} : \texttt{Basis}(\mathrm{bool}\, F(A))\right],$

$(*) := \eth \dim(3) : \dim F(A) = |A|;$

$\square$

$\texttt{FreeGroupRankIsWellDefined} :: \forall A, B : \mathsf{SET} \ . \ \forall (0) : F(A) \cong_{\mathsf{GRP}} F(B) \ . \ A \cong_{\mathsf{SET}} B$

$\texttt{Proof} \ =$

$(1) := \eth \operatorname{bool}(0) : \operatorname{bool} F(A) \cong_{\mathsf{VS}(\mathbb{F}_2)} \operatorname{bool} F(B),$

$(2) := \dim(1) : \dim \operatorname{bool} F(A) = \dim \operatorname{bool} F(B),$

$(3) := \texttt{DimensionOfFreeBoolenization}(2) : |A| = |B|,$

$(*) := \eth\texttt{EquallCard}(3)\eth^{-1}\texttt{Iso}(\mathsf{SET}) : A \cong_{\mathsf{SET}} B;$

$\square$

$\texttt{freeGroupRank} :: \texttt{Free} \to \texttt{Card}$

$\texttt{freeGroupRank}\,(G) = \operatorname{rank} G := \#\{\eth\texttt{Free}(g)\}$

$\texttt{GroupRelation} := \Lambda X \in \mathsf{SET} \ . \ \sum n \in \mathbb{N} \ . \ n \to (X \times \mathbb{Z}) : \mathsf{SET} \to \texttt{Type};$

$\texttt{presentation} :: \prod X \in \mathsf{SET} \ . \ ?\texttt{GroupRelation}(X) \to \mathsf{GRP}$

$$\texttt{presentation}\,(R) = \langle X \mid R \rangle := \frac{F(X)}{N\left\{\prod_{i=1}^{n} \eta^{p_i}(x_i) \middle| (n, x, p) \in R\right\}}$$

$\texttt{PresentationHolds} :: \prod X : \mathsf{SET} \ . \ \prod G : \mathsf{GRP} \ . \ ?\Big((?\texttt{GroupRelation}) \times (X \to U(G))\Big)$

$$(R, f) : \texttt{PresentationHods} \iff \forall (n, x, p) \in R \ . \ \prod_{i=1}^{n} f^{p_i}(x_i) = e$$

$\texttt{UniversalPropertyOfPresentation} :: \forall X : \mathsf{SET} \ . \ \forall G : \mathsf{GRP} \ . \ \forall (R, f) : \texttt{PresentationHolds} \ .$

$\quad . \ \exists! \varphi : \langle X | R \rangle \xrightarrow{\mathsf{GRP}} G \ . \ f = \eta U(\pi\varphi)$

$\texttt{Proof} \ =$

$(\psi, 1) := \eth\texttt{Unit}(\eta_F)(f) : \sum \psi : F(X) \xrightarrow{\mathsf{GRP}} G \ . \ f = \eta U(\psi),$

$$A := \left\{ \prod_{i=1}^{n} \eta^{p_i}(x_i) | (n, x, p) \in R \right\} :?F(X),$$

$N := N(A) : \texttt{Normal}\Big(F(X)\Big),$

$(2) := \eth\texttt{PresentationHolds}\eth A \eth\psi : A \subset \ker \psi,$

$(3) := \eth\texttt{KernelIsNormal}(2)\eth N : N \subset \ker \psi,$

$(4) := \eth \ker \eth\pi(3) : \forall a \in F(X) \ . \ \forall b \in \pi(a) \ . \ \psi(a) = \psi(b),$

$\varphi := \Lambda[a] \in \langle X | R \rangle \ . \ \psi(a) : \langle X | R \rangle \xrightarrow{\mathsf{GRP}} G,$

$(*, 1) := \eth\varphi(1) : f = \eta U(\pi\varphi),$

$(*, 2) := \eth\langle G | T \rangle \eth f : \forall \varphi' : \langle X | R \rangle \xrightarrow{\mathsf{GRP}} G \ . \ \forall (0) : \eta U(\pi\varphi') = f \ . \ \varphi = \varphi';$

$\square$

4.4 Groups as Categories

$\texttt{groupCategory} :: \mathsf{GRP} \to \mathsf{SCAT}$

$\texttt{groupCategory}\,(G) = \mathsf{G}G := \Big(\{*\}, * \mapsto G, (a,b) \mapsto ab, * \mapsto e\Big)$

$\mathcal{C}\text{-}\texttt{Action} := \Lambda G \in \mathsf{GRP}\ .\ \Lambda\mathcal{C} \in \mathsf{CAT}\ .\ \mathsf{G}G \xrightarrow{\mathsf{CAT}} \mathcal{C} : \mathsf{GRP} \times \mathsf{CAT} \to \texttt{Type};$

$\texttt{FixedPointAsLimit} :: \forall \alpha : \mathsf{SET}\text{-}\texttt{Action}(G)\ .\ \lim \alpha \cong_{\mathsf{SET}} \mathrm{Fix}(\alpha)$

$\texttt{Proof} =$

$I := \Lambda a \in \mathrm{Fix}(\alpha)\ .\ a : \mathrm{Fix}(\alpha) \to \alpha(*),$

$\texttt{Assume}\ g : G,$

$\texttt{Assume}\ a : \mathrm{Fix}(\alpha),$

$() := \eth I \eth \mathrm{Fix}(\alpha) \eth^{-1} I : \alpha(g)\Big(I(a)\Big) = \alpha(g)(a) = a = I(a);$

$\rightsquigarrow () := I(=,\to) : I\alpha(g) = I;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : (\mathrm{Fix}(\alpha), * \mapsto I) : \texttt{Cone}(\mathsf{G}G, \alpha),$

$\texttt{Assume}\ (C, f) : \texttt{Cone}(\mathsf{G}G, \alpha),$

$(2) := \eth\texttt{Cone}(\mathsf{G}G, \alpha)(Cf)) : \mathrm{Im}\, f \subset \mathrm{Fix}(\alpha),$

$g := \Lambda x \in C\ .\ f(x) : C \to \mathrm{Fix}(\alpha),$

$(3) := \eth g \eth I : f = gI,$

$\texttt{Assume}\ g' : C \to \mathrm{Fix}(\alpha),$

$\texttt{Assume}\ (4) : f = g'I,$

$() := \eth I(4) : g = g';$

$\rightsquigarrow () := \eth^{-1}\texttt{Limit} : \lim \alpha = \mathrm{Fix}(\alpha);$

$\square$

$\texttt{OrbitsAsColimit} :: \forall \alpha : \mathsf{SET}\text{-}\texttt{Action}(G)\ .\ \mathrm{colim}\, \alpha \cong_{\mathsf{SET}} O(\alpha)$

$\texttt{Proof} =$

$\texttt{Assume}\ g : G,$

$\texttt{Assume}\ a : \alpha(*),$

$() := \eth O_\alpha(\alpha(g)) : O_\alpha(\alpha(g)(a)) = O_\alpha(a);$

$\rightsquigarrow () := I(=,\to) : \alpha(g) O_\alpha = O_\alpha;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cocone} : (O(\alpha), O_\alpha) : \texttt{Cocone}(\mathsf{G}G, \alpha),$

$\texttt{Assume}\ (C, f) : \texttt{Cocone}(\mathsf{G}G, \alpha),$

$(2) := \eth\texttt{Cocone}(\mathsf{G}F, \alpha)(C, f) : \forall a \in \alpha(*)\ .\ \forall b \in O_\alpha(a)\ .\ f(a) = f(b),$

$(g, 3) := \texttt{ClassExtension}(2) : \sum g : O(\alpha) \to C\ .\ f = gO_\alpha,$

$\texttt{Assume}\ g' : O(\alpha) \to C,$

$\texttt{Assume}\ (4) : f = g'O_\alpha,$

$() := \eth O_\alpha(2) : g = g';$

$\rightsquigarrow () := \eth^{-1}\texttt{Colimit} : \mathrm{colim}\, \alpha = O(\alpha);$

$\square$

$G\text{-}\mathcal{C} := \Lambda G \in \mathsf{GRP}\ .\ \Lambda\mathcal{C} \in \texttt{Category}\ .\ \mathcal{C}^{\mathsf{G}G} : \mathsf{GRP} \times \mathsf{CAT} \to \mathsf{CAT};$

4.5 Direct Limits

$\texttt{PiDivisible} :: \prod p : \texttt{Prime} \,.\, ?\mathsf{ABEL}$

$G : \texttt{PiDivisible} \iff G : p\text{-}\texttt{Divisible} \iff \Lambda g \in G \,.\, pG : \texttt{Surjective}$

$\texttt{piMult} :: \prod p : \texttt{Prime} \,.\, \prod G : p\text{-}\texttt{Divisible} \,.\, \mathrm{End}_{\mathsf{ABEL}}(G)$

$\texttt{piMult}\,(a) = p(a) := pa$

$\texttt{TateDiagramm} :: \prod p : \texttt{Prime} \,.\, \prod G : p\text{-}\texttt{Divisible} \,.\, \mathsf{P}\mathbb{N} \xrightarrow{\mathsf{CAT}} \mathsf{GRP}$

$\texttt{TateDiagramm}\,(n, m, n \le m) = A_{n,m} := \left(\ker p^n, \ker p^m, p^{m-n}\right)$

$\texttt{TateGroup} :: \prod p : \texttt{Prime} \,.\, p\text{-}\texttt{Divisible} \to \mathsf{GRP}$

$\texttt{TateGroup}\,(G) = T_p(G) := \lim_{n \in \mathsf{P}\mathbb{N}} A_n(p, G)$

$\texttt{Profinite} :: ?\mathsf{GRP}$

$G : \texttt{Profinite} \iff \exists (\mathcal{I}, F) : \texttt{DirectedDiagramm}(\mathsf{GRP}) \,.\, \lim_{i \in \mathcal{I}} F(i) = G \;\&\; \forall i \in \mathcal{I} \,.\, |F(i)| < \infty$

$\texttt{DirectedNormality} :: \prod G \in \mathsf{GRP} \,.\, ?\Big(\mathbb{N} \to \texttt{Normal}(G)\Big)$

$N : \texttt{DirectedNormality} \iff \forall n, m \in \mathbb{N} \,.\, m \ge n \Rightarrow N_m \subset N_n$

$\texttt{Cauchy} :: \prod G \in \mathsf{GRP} \,.\, \prod N : \texttt{DirectedNormality}(G) \,.\, ?(\mathbb{N} \to G)$

$g : \texttt{Cauchy} \iff \forall m \in \mathbb{N} \,.\, \exists M \in \mathbb{N} \,.\, \forall t, s \in \mathbb{N} \,.\, t, s \ge M \Rightarrow g_t g_s^{-1} \in N_m$

$\texttt{Null} :: \prod G \in \mathsf{GRP} \,.\, \prod N : \texttt{DirectedNormality}(G) \,.\, ?(\mathbb{N} \to G)$

$g : \texttt{Null} \iff \forall m \in \mathbb{N} \,.\, \exists M \in \mathbb{N} \,.\, \forall t \in \mathbb{N} \,.\, t \ge M \Rightarrow g_t \in N_m$

$\texttt{CauchyIsGroup} :: \forall G \in \mathsf{GRP} \,.\, \forall N : \texttt{DirectedNormality}(G) \,.\, \texttt{Cauchy}(G, N) \in \mathsf{GRP}$

$\texttt{Proof} =$

$\texttt{Assume}\; a, b : \texttt{Cauchy}(G, N),$

$\texttt{Assume}\; m : \mathbb{N},$

$(M, 1) := \eth\texttt{Cauchy}(G, N)(b) : \sum M \in \mathbb{N} \,.\, \forall t, s \in \mathbb{N} \,.\, t, s \ge M \Rightarrow b_t b_s^{-1} \in N_m,$

$\texttt{Assume}\; t, s : \mathbb{N},$

$\texttt{Assume}\; (2) : t, s \ge M,$

$() := (1)(t, s)\eth\texttt{Normal}(G)(N_m) : a_t b_t b_s^{-1} a_s^{-1} \in N_m;$

$\leadsto (1) := I(\forall) : \forall a, b \in \texttt{Cauchy}(G, N) \,.\, ab \in \texttt{Cauchy}(G, N),$

$(2) := \eth\texttt{Subgroup}(G)(N) : \forall a \in \texttt{Cauchy}(G, N) \,.\, a^{-1} : \texttt{Cauchy}(G, N),$

$(*) := \eth\mathsf{GRP}(1, 2) : \texttt{Cauchy}(G, N) \in \mathsf{GRP};$

$\square$

$\texttt{NullIsNormal} :: \forall G \in \mathsf{GRP} \,.\, \forall N : \texttt{DirectedNormality}(G) \,.\, \texttt{Null}(G, N) \lhd \texttt{Cauchy}(G, N)$

$\texttt{Proof} \; =$

$(1) := \eth\texttt{Subgroup}(N) : \texttt{Null}(G, N) \subset \texttt{Cauchy}(G, N),$

$(*) := \eth\texttt{Normal}(N) : \texttt{Null}(G, N) \lhd \texttt{Cauchy}(G, N);$

$\square$

$\texttt{completion} :: \prod G \in \mathsf{GRP} \,.\, \texttt{DirectedNormality}(G) \to \mathsf{GRP}$

$$\texttt{completion}\,(N) = C(G, N) := \frac{\texttt{Cauchy}(G, N)}{\texttt{Null}(G, N)}$$

$$\texttt{CompletionAsLimit} :: \forall G \in \mathsf{GRP} \,.\, \forall N \in \texttt{DirectedNormality}(G) \,.\, C(G, N) \cong_{\mathsf{GRP}} \lim_{n \in \mathsf{P}\mathbb{N}} \frac{G}{N_n}$$

$\texttt{Proof} \; =$

$\dots$

$\square$

4.6 Primitive Groups

4.7 Pullbacks and Pushouts[!]

4.8 Nielson-Schrier Theory [!]

4.9 Group objects [!]

5 Virtual Groups [!!]