Convex Analysis

Uncultured Tramp

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Contents

1 Convex Functions

1.1 Subject

$\texttt{epigraph} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \left(D \to \overset{\infty}{\mathbb{R}}\right) \to ?(V \oplus \mathbb{R})$

$\texttt{epigraph}\,(f) = \operatorname{epi} f := \{(x, \phi) | x \in D, \phi \in \mathbb{R}, \phi \geq f(x)\}$

$\texttt{Convex} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . ?\left(D \to \overset{\infty}{\mathbb{R}}\right)$

$f : \texttt{Convex} \iff \texttt{Convex}(V \oplus \mathbb{R}, \operatorname{epi} f)$

$\texttt{effectiveDomain} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \texttt{Convex}(V, D) \to ?D$

$\texttt{effectiveDomain}\,(f) = \operatorname{dom} f := \pi_1 \operatorname{epi} f$

$\texttt{DomainIsConvex} :: \forall V \in \mathbb{R}\text{-VS} . \forall D \subset V . \forall f : \texttt{Convex}(V, D) . \texttt{Convex}(V, \operatorname{dom} f)$

Proof =

As a linear image of convex set.

□

$\texttt{ProperConvexFunction} :: \prod V : \mathbb{R}\text{-VS} . ?\texttt{Convex}(V, V) .$

$f : \texttt{ProperConvexFunction} \iff \forall x \in V . f(x) > -\infty \;\&\; \exists x \in V . f(x) < +\infty$

$\texttt{InterpolationProperty} ::$

$:: \forall V : \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$

$. \texttt{Convex}(V, C, f) \iff \forall x, y \in C . \forall \lambda \in [0, 1] .$

$. f\Big(\lambda x + (1 - \lambda) y\Big) \leq \lambda f(x) + (1 - \lambda) f(y)$

Proof =

$(\Rightarrow)$: assume that f is convex.

Then f has convex epigraph.

Take arbitrary $x, y \in C$ and $\lambda \in [0, 1]$.

If f takes value $+\infty$ either in x or y, then the inequality follows, so assume the contrary.

Then $\Big(x, f(x)\Big), \Big(y, f(y)\Big)$ trivially belong to the epigraph,

so by convexity $\Big(\lambda x + (1 - \lambda) y, \lambda f(x) + (1 - \lambda) f(y)\Big)$ is also in epigraph.

The definition of epigraph produces the inequality $f\Big(\lambda x + (1 - \lambda) y\Big) \leq \lambda f(x) + (1 - \lambda) f(y)$.

$(\Leftarrow)$: now assume that inequality always hold.

Assume $(x, \phi), (y, \psi)$ belong to the epigraph and $\lambda \in [0, 1]$.

Then $\lambda \phi + (1 - \lambda) \psi \geq \lambda f(x) + (1 - \lambda) f(y) \geq f\Big(\lambda x + (1 - \lambda) y\Big)$.

So $\lambda(x, \phi) + (1 - \lambda)(y, \psi)$ belong to the epigraph.

Thus, epigraph is convex and f is convex.

JensensIneq ::

$:: \forall V : \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$

$$. \forall n \in \mathbb{N} . \forall \lambda \in \mathbb{R}^n_+ . \forall \aleph : \sum_{k=1}^{n} \lambda_k = 1 . \forall v \in V^n . f\left(\sum_{k=1}^{n} \lambda_k v_k\right) \leq \sum_{k=1}^{n} \lambda_k f(v_k)$$

Proof =

Iterate the interpolation property.

□

SecondDerivativeConvexityTest :: $\forall I : \texttt{OpenInterval}(\mathbb{R}) . \forall f \in C^2(I) .$

$. \texttt{Convex}(\mathbb{R}, I, f) \iff f'' \geq 0$

Proof =

$(\Rightarrow)$: assume there is a $t \in I$ such that $f''(t) < 0$.

As f'' must be continous there is whole open interval (a, b) such that $f''(j) < 0$ for all $j \in (a, b)$.

Take some $x, y \in (a, b)$ with $x < y$ and define$z = \lambda x + (1 - \lambda)y$ for siome $\lambda \in (0, 1)$.

Then $f(z) - f(x) = \displaystyle\int_x^z f'(t)\, dt > f'(z)(z - x)$ and $f(y) - f(z) = \displaystyle\int_z^y f'(t)\, dt < f'(z)(y - z)$.

Then from definiton of z we get $f(z) > f(x) - (1 - \lambda)f'(z)(y - x)$ and $f(z) > f(y) + \lambda f'(z)(y - x)$.

By adding two inequalities with multipliers λ and $(1 - \lambda)$ one gets $f\Big(\lambda x + (1 - \lambda)y\Big) > \lambda f(x) + (1 - \lambda)f(y)$.

But this contradicts a convexity. .

$(\Rightarrow)$: use same inequalities but with different sign to prove the convexity.

□

ExponentIsConvexity :: $\forall \alpha \in \mathbb{R} . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}, \Lambda t \in \mathbb{R} . e^{\alpha t}\Big)$

Proof =

write $f(t) = e^{\alpha t}$.

Then $f''(t) = \alpha^2 e^{\alpha t} \geq 0$.

□

MonomialConvexity1 :: $\forall p \in [1, +\infty) . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . t^p\Big)$

Proof =

Write $f(t) = t^p$.

Then $f''(t) = p(p - 1)t^{p-2} \geq 0$ for $t > 0$.

□

MonomialConvexity2 :: $\forall p \in [0, 1) . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . - t^p\Big)$

Proof =

Write $f(t) = t^p$.

Then $f''(t) = p(1 - p)t^{p-2} \geq 0$ for $t > 0$.

□

$\texttt{MonomialConvexity3} :: \forall p \in (-\infty, 0] \;.\; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t^p\Big)$

$\texttt{Proof} \; =$

write $f(t) = t^p$.

Then $f''(t) = p(p-1)t^{p-2} \geq 0$ for $t > 0$.

$\square$

$\texttt{GeneralizedArcsinDerivativeIsConvex} :: \forall \alpha \in \mathbb{R}_{++} \;.\; \texttt{Convex}\left(\mathbb{R}, (-\alpha, \alpha), \Lambda t \in \mathbb{R} \;.\; \frac{1}{\sqrt{\alpha^2 - t^2}}\right)$

$\texttt{Proof} \; =$

Write $f(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}}$.

Then $f'(t) = \dfrac{t}{\sqrt{\alpha^2 - t^2}^3}$.

And $f''(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}^3} + \dfrac{3t^2}{\sqrt{\alpha^2 - t^2}^5} > 0$ for $t \in (-\alpha, \alpha)$.

$\square$

$\texttt{NegativeLogIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; -\ln(t)\right)$

$\texttt{Proof} \; =$

Write $f(t) = -\ln(t)$.

Then $f''(t) = \dfrac{1}{t^2} > 0$ for $t > 0$.

$\square$

$\texttt{NegativeEntropyIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t\ln(t)\right)$

$\texttt{Proof} \; =$

Write $f(t) = t\ln(t)$.

Then $f'(t) = \ln(t) + 1$.

And $f''(t) = \dfrac{1}{t} > 0$ for $t > 0$.

$\square$

$\texttt{Concave} :: \prod V : \mathbb{R}\text{-VS} \;.\; \prod D \subset V \;.\; ?\Big(D \to \overset{\infty}{\mathbb{R}}\Big)$

$f : \texttt{Concave} \iff \texttt{Convex}(V, D, -f)$

$\texttt{SecondDerivativeConvexityTest2} :: \forall V : \texttt{EucledeanSpace} \;.\; \forall U : \texttt{Open} \,\&\, \texttt{Convex}(V) \;.\; \forall f \in C^2(U) \;.$
$\;.\; \texttt{Convex}(\mathbb{R}, U, f) \iff \mathbf{D}^2 f \geq 0$

$\texttt{Proof} \; =$

For $x \in U$ and $v \in V \setminus \{0\}$ define $\phi_{x,v}(t) = f(x + tv)$ with a domain $I_{x,v} = \{t \in \mathbb{R} | x + tv \in C\}$.

Then f is convex iff every $\phi_{x,v}$ does.

But $\phi''_{x,v}(t) = \langle v, \mathbf{D}^2 f|_y v\rangle$, where $y = x + tv$.

So f is convex iff $\mathbf{D}^2 f$ is positive-semidefinite.

$\square$

GeometricMeanIsConcave ::

$$:: \forall V : \texttt{EucledeanSpace} \,.\, \texttt{Concave}\left(V, V_{++}, \Lambda x \in V \,.\, \prod_{k=1}^{n} \sqrt[n]{x_k}\right) \quad \texttt{where} \quad n = \dim V$$

Proof =

write $f(x) = \prod_{k=1}^{n} \sqrt[n]{x_k}$.

Then $\nabla f|_x = \left(\frac{1}{n\sqrt[n]{x_i}^{n-1}} \prod_{j\neq i}^{n} \sqrt[n]{x_j}\right)_{i=1}^{n}$.

And $\mathbf{D}^2_{i,j} f|_x = \frac{1}{n^2 \sqrt[n]{x_i x_j}^{n-1}} \prod_{k\neq i,j}^{n} \sqrt[n]{x_k}$ when $i \neq j$, and $\mathbf{D}^2_{i,i} f|_x = -\frac{n-1}{n^2 \sqrt[n]{x_i}^{2n-1}} \prod_{j\neq i}^{n} \sqrt[n]{x_j}$.

So, $\mathbf{D}^2 f|_x(v,v) = -\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{\sqrt[n]{x_i}^{2n-1}} \prod_{j\neq i}^{n} \sqrt[n]{x_j} + \frac{1}{n^2} \sum_{i\neq j}^{n} \frac{v_i v_j}{\sqrt[n]{x_i x_j}^{n-1}} \prod_{k\neq i,j}^{n} \sqrt[n]{x_k} =$

$$= f(x)\left(-\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} + \frac{1}{n^2} \sum_{i\neq j}^{n} \frac{v_i v_j}{x_i x_j}\right) = -\frac{f(x)}{n^2}\left(n \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} - \left(\sum_{i=1}^{n} \frac{v_i}{x_i}\right)^2\right) \leq 0 \,.$$

This follows from obvious matching schema.

□

NormsAreConvex $:: \forall V : \mathbb{R}\text{-VS} \,.\, \forall \eta : \texttt{Norm}(V)\texttt{Convex}(V, V, \eta)$

Proof =

Write $\eta(v) = \|v\|$.

Just use triangle inequality $\left\|\lambda x + (1-\lambda)y\right\| \leq \|\lambda x\| + \left\|(1-\lambda)y\right\| = \lambda\|x\| + (1-\lambda)\|y\|$.

□

convexIndicator $:: \forall V : \mathbb{R}\text{-VS} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

convexIndicator $(C) = \Lambda x \in V \,.\, \chi(x|A) := \Lambda x \in V \,.\, \infty\left[x \in C^{\complement}\right]$

supportFunction $:: \forall V : \mathbb{R}\text{-HIL} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

supportFunction $(C) = \Lambda x \in V \,.\, \chi^*(x|A) := \sup_{y\in C}\langle x, y\rangle$

gauge $:: \forall V : \mathbb{R}\text{-VS} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

gauge $(C) = \Lambda x \in V \,.\, \gamma(x|A) := \Lambda x \in V \,.\, \inf\{\lambda \in \mathbb{R}_{++} | x \in \lambda C\}$

ConvexFunctionHasConvexLevelSets ::

$$:: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall f : \texttt{Convex}(V, V) \,.\, \forall \alpha \in \overset{\infty}{\mathbb{R}} \,.\, \texttt{Convex}\Big(V, \{v \in V : f(v) \geq \alpha\}\Big)$$

Proof =

...

□

ConvexFunctionHasConvexStrictLevelSets ::

$$:: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall f : \texttt{Convex}(V, V) \,.\, \forall \alpha \in \overset{\infty}{\mathbb{R}} \,.\, \texttt{Convex}\Big(V, \{v \in V : f(v) > \alpha\}\Big)$$

Proof =

...

□

ConvexlyBoundedRegionIsConvex ::

$$:: \forall V \in \mathbb{R}\text{-VS} . \forall I \in \mathsf{SET} . \forall \alpha : I \to \overset{\infty}{\mathbb{R}} . \forall f : I \to \texttt{Convex}(V, V) . \texttt{Convex}\Big(V, \{v \in V : \forall i \in I . f_i(v) > \alpha_i\}\Big)$$

Proof =

GeneralizedAMGMIneq :: $\forall n \in \mathbb{N} . \forall \lambda : \mathbb{R}^n_+ . \forall x : \mathbb{R}^n_{++} . \forall \aleph : \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$

Proof =

By Jensen inequality for natural logarithm $\ln\left(\sum_{i=1}^n \lambda_i x_i\right) \geq \sum_{i=1}^n \lambda_i \ln(x_i)$.

Then by exponentiating both parts $\sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$.

□

PositivelyHomogeneous :: $\prod V : \mathbb{R}\text{-VS} . ?\Big(V \to (-\infty, +\infty]\Big)$

$f : \texttt{PositivelyHomogeneous} \iff \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . f(\alpha v) = \alpha f(v)$

PositiveHomogeneousZeroPositivity :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \texttt{PositivelyHomogeneous}(V) . f(0) \geq 0$

Proof =

Note that $f(0) = f(t0) = tf(0)$ for all $t \in \mathbb{R}_{++}$.

This means that $f(0)$ is either 0 or $+\infty$.

□

PositiveHomogeneousConvexity :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \texttt{PositivelyHomogeneous}(V)$.

$. \texttt{Convex}(V, V, f) \iff \forall x, y \in V . f(x + y) \leq f(x) + f(y)$

Proof =

$(\Rightarrow)$: assume f is convex.

Then $f(x + y) = f\left(\frac{2}{2}x + \frac{2}{2}y\right) \leq \frac{1}{2}f(2x) + \frac{1}{2}f(2y) = f(x) + f(y)$ for any $x, y \in V$.

$(\Leftarrow)$: assume the inequality holds .

Then $f\Big(\lambda x + (1 + \lambda)y\Big) \leq f(\lambda x) + f\Big((1 - \lambda)y\Big) = \lambda f(x) + (1 - \lambda)f(y)$ when $\lambda \in (0, 1)$ and $x, y \in V$.

Otherwise, when $\lambda = 0, 1$, convexity condition holds trivially.

□

Conic $:= \lambda V \in \mathbb{R}\text{-VS} . \texttt{Convex}(V, V) \times \texttt{PositivelyHomogeneous}(V) : \mathbb{R}\text{-VS} \to \texttt{Type}$;

ConicIneq :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) \,\&\, \texttt{PositivelyHomogeneous}(V) . \forall n \in \mathbb{N} . \forall x \in V^n$.

$$. \forall \lambda \in \mathbb{R}^n_{++} . f\left(\sum_{i=1}^n \lambda_i x\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

Proof =

Iterate previous theorem.

□

`ConicEpigraph` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : V \to (-\infty, +\infty) . \texttt{Conic}(V, f) \iff \texttt{ConvexCone}(V, \operatorname{epi} f)$

`Proof` =

...

$\square$

`ConicIsSupersymmetric` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . \forall v \in V . f(v) \geq -f(-v)$

`Proof` =

Write $f(x) + f(-x) \geq f(x - x) = f(x) \geq 0$.

So $f(x) \geq -f(-x)$.

$\square$

`ConicIsLinearIffsymmetric` :: $\forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . f \in V^* \iff \forall v \in V . f(-v) = -f(v)$

`Proof` =

$(\Rightarrow)$: this is trival.

$(\Leftarrow)$: assume that the property holds .

Let $x, y \in V$.

Then $f(x) + f(y) \geq f(x + y) \geq -f(-x - y) \geq -f(-x) - f(-y) = f(x) + f(y)$.

This mean $f(x) + f(y) = f(x + y)$.

But as x and y were arbitrary f must be additive and hence linear.

$\square$

1.2 Convexity Preserving Operations

1.3 Closures

1.4 Continuity

2 Duality

3 (Sub)differential Calculus

4 From Optimization to Convex Algebra

Sources

1. Convex Analysis — R. T. Rockaffeler 1972