

# General Topology

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# Contents

<b>1</b>	<b>Basics</b>	<b>3</b>
1.1	Topological Sets . . . . .	3
1.1.1	Topology and Topological Spaces . . . . .	3
1.1.2	Closure and Interior . . . . .	9
1.1.3	Open and Closed Domains . . . . .	16
1.1.4	Boundary Operator . . . . .	20
1.1.5	Accumulation and Isolated Points . . . . .	23
1.1.6	Dense Sets . . . . .	24
1.1.7	Separation Axioms . . . . .	27
1.2	Convergence . . . . .	35
1.2.1	Convergence in Nets . . . . .	35
1.2.2	Filters . . . . .	38
1.2.3	Sets with Convergent Sequences . . . . .	40
1.3	Category of Topological Spaces . . . . .	41
1.3.1	Continuous Morphisms . . . . .	41
1.3.2	Subspaces . . . . .	47
1.3.3	Weak and Strong Topology . . . . .	52
1.3.4	Sums . . . . .	55
1.3.5	Products . . . . .	56
1.3.6	Quotients . . . . .	61
1.4	Regularity as Separation . . . . .	62
1.4.1	Functional Separation . . . . .	62
1.4.2	Perfectly Normal Spaces . . . . .	65
1.4.3	Normally Placed Sets . . . . .	66
1.4.4	Urysohn and Semiregular Spaces . . . . .	68
1.4.5	Semicontinuous Functions . . . . .	69
1.5	Properties Preserved by Continuous Transformations . . . . .	73
1.5.1	Compact Sets . . . . .	73
1.5.2	Connected Spaces . . . . .	78
1.5.3	Path-Connected Spaces . . . . .	81
1.5.4	Totally Disconnected Spaces . . . . .	84
1.5.5	Sequential Spaces . . . . .	85
<b>2</b>	<b>Compacta</b>	<b>86</b>
2.1	General . . . . .	86
2.1.1	Lindelöf Spaces . . . . .	86
2.1.2	Locally Compact Spaces . . . . .	88
2.1.3	Countably Compact Spaces . . . . .	90
2.1.4	Sequentially Compact Spaces . . . . .	91
2.1.5	Pseudocompact Spaces . . . . .	92
2.2	Category of Compact Spaces . . . . .	93
2.2.1	Filters in categories . . . . .	93
2.2.2	Compact Hausdorff Spaces . . . . .	96
2.2.3	Compactly Generated Spaces . . . . .	99
2.2.4	Compact-Open Topology . . . . .	101
2.3	Compactifications . . . . .	104
2.3.1	Subject . . . . .	104
2.3.2	Stone-Čech Functor . . . . .	107
2.3.3	Wallman Extension . . . . .	109
2.3.4	Perfect Mappings . . . . .	111
2.3.5	Abstract Baire Theory . . . . .	113
2.3.6	Realcompact Spaces . . . . .	114
<b>3</b>	<b>Cardinal Functions</b>	<b>115</b>

# 1 Basics

## 1.1 Topological Sets

### 1.1.1 Topology and Topological Spaces

$\text{Topology} :: \prod X \in \text{SET} . ???X$

$T : \text{Topology} \iff X, \emptyset \in T \ \&$

$\forall A, B \in T . A \cap B \in T$

$\forall I \in \text{SET} . \forall U : I \rightarrow T . \bigcup_{i \in I} U_i \in T$

$\text{TopologicalSpace} :: ? \sum_{X \in \text{SET}} ??X$

$(X, T) : \text{TopologicalSpace} \iff T : \text{Topology}$

$\text{topologicalSpaceAsSet} :: \text{TopologicalSpace} \rightarrow \text{SET}$

$\text{topologicalSpaceAsSet}(X, T) = \text{implicit}(X, T) := X$

$\text{topology} :: \prod (X, T) : \text{TopologicalSpace} . \text{Topology}(X)$

$\text{topology}() = \mathcal{T}(X, T) := T$

$\text{Open} :: \prod X : \text{TopologicalSpace} . ??X$

$U : \text{Open} \iff U \in \mathcal{T}(X)$

$\text{Closed} :: \prod X : \text{TopologicalSpace} . ??X$

$A : \text{Closed} \iff A^c \in \mathcal{T}(X)$

$\text{AlwaysClosed} :: \forall X : \text{TopologicalSpace} . \emptyset, X : \text{Closed}(X)$

$\text{Proof} =$

...

□

$\text{ClosedIntersection} :: \forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{Closed}(X) . \bigcap_{i \in I} A_i : \text{Closed}(X)$

$\text{Proof} =$

...

□

$\text{ClosedUnion} :: \forall X : \text{TopologicalSpace} . \forall n \in \mathbb{N} . \forall A : n \rightarrow \text{Closed}(X) . \bigcup_{i=1}^n A_i : \text{Closed}(X)$

$\text{Proof} =$

...

□

**AlwaysOpen** ::  $\forall X : \text{TopologicalSpace} . \emptyset, X : \text{Open}(X)$

**Proof** =

...

□

**OpenUnion** ::  $\forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall U : I \rightarrow \text{Open}(X) . \bigcup_{i \in I} U_i : \text{Closed}(X)$

**Proof** =

...

□

**OpenIntersection** ::  $\forall X : \text{TopologicalSpace} . \forall n \in \mathbb{N} . \forall U : n \rightarrow \text{Open}(X) . \bigcap_{i=1}^n U_i : \text{Open}(X)$

**Proof** =

...

□

**BaseOfTopology** ::  $\prod X : \text{TopologicalSpace} . ?\mathcal{T}(X)$

$\mathcal{B} : \text{BaseOfTopology} \iff \forall U \in \mathcal{T} . \exists I \in \text{SET} : \exists B : I \hookrightarrow \mathcal{B} : U = \bigcup_{i \in I} B_i$

**oprnnNeighbourhood** ::  $\prod X : \text{TopologicalSpace} . X \rightarrow ?\mathcal{T}(X)$

**openNeighbourhood**  $(x) = \mathcal{U}(x) := \{U \in \mathcal{T}(X) | x \in U\}$

**neighbourhood** ::  $\prod X : \text{TopologicalSpace} . X \rightarrow ?X$

**neighbourhood**  $(x) = \mathcal{N}(x) := \{A \subset X | \exists U \in \mathcal{U}(x) . U \subset A\}$

**BasisEqDef** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{B} \in ?\mathcal{T}(X) . \forall \mathcal{B} : \text{Base}(X) \iff$   
 $\iff \forall x \in X . \forall U \in \mathcal{U}(x) \exists B \in \mathcal{B} : x \in B \subset U$

**Proof** =

**Assume** [1] :  $(\mathcal{B} : \text{Base})$ ,

**Assume**  $x : X$ ,

**Assume**  $U : \mathcal{U}(x)$ ,

$(\mathcal{I}, B, [2]) := \mathfrak{d}\text{Base}(B)(U) : \sum \mathcal{I} : \text{SET} . \sum I \hookrightarrow \mathcal{B} . U = \bigcup_{i \in I} B_i$ ,

[3] :=  $\mathfrak{d}\mathcal{U}(x) : x \in U$ ,

$(i, [1.*.1]) := \text{UnionElement}(I, B)[1][2] : \sum i \in I . x \in B_i$ ,

$[1.*.2] := \text{UnionSubset}(I, B)(B_i)[1] : B_i \subset U$ ,

$\leadsto [1] := I(\Rightarrow) : (\mathcal{B} : \text{Base}(X)) \Rightarrow \forall x \in X . \forall U \in \mathcal{U}(x) . \exists B \in \mathcal{B} : x \in B \subset U$ ,

**Assume** [2] :  $\forall x \in X . \forall U \in \mathcal{U}(x) . \exists B \in \mathcal{B} : x \in B \subset U$ ,

**Assume**  $U : \mathcal{T}(X)$ ,

$(B, [3]) := \Lambda x \in U . [2](x, U) : \prod_{x \in U} \sum_{B_x \in \mathcal{B}} x \in B_x \subset U$ ,

$$[4] := \text{UnionElement}[3] \bar{\partial}^{-1} \text{Subset} : U \subset \bigcup_{x \in U} B_x,$$

$$[5] := \text{SubsetUnion} : \bigcup_{x \in U} B_x \subset U,$$

$$[U.*] := \bar{\partial}^{-1} \text{SetEq}[4][5] : U = \bigcup_{x \in U} B_x;$$

$$\leadsto [3] := \bar{\partial}^{-1} \text{Base}(X) : (\mathcal{B} : \text{Base}(X));$$

$$\leadsto [*] := I(\Rightarrow)[1]I(\Leftarrow) : (\mathcal{B} : \text{Base}(X)) \iff \forall x \in X . \forall U \in \mathcal{U}(x) . \exists B \in \mathcal{B} : x \in B \subset U;$$

□

$$\text{weightOfTopology} :: \text{TopologicalSpace} \rightarrow \text{CARD}$$

$$\text{weightofTopology}(X) = w(X) := \min \left\{ |\mathcal{B}| \mid \mathcal{B} : \text{Base}(X) \right\}$$

$$\text{PotentialBase} :: \prod X \in \text{SET} . ??X$$

$$\mathcal{B} : \text{PotentialBase} \iff \forall x \in X . \exists B \in \mathcal{B} : x \in B \ \& \\ \& \forall B, B' \in \mathcal{B} . x \in B \cap B' \Rightarrow \exists B'' \in \mathcal{B} : x \in B'' \subset B' \cap B''$$

$$\text{generateTopologyByBase} :: \prod X \in \text{SET} . \text{PotentialBase}(X) \rightarrow \text{TopologicalSpace}$$

$$\text{generateTopologyByBase}(\mathcal{B}) = \langle \mathcal{B} \rangle_{\text{TOP}} := \left( X, \left\{ \bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subset \mathcal{B} \right\} \right)$$

$$\text{Subbase} :: \prod X : \text{TopologicalSpace} . ?\mathcal{T}(X)$$

$$\mathcal{B} : \text{Subbase} \iff \forall U \in \mathcal{T}(X) . \exists \mathcal{I} \in \text{SET} : \exists n : \mathcal{I} \rightarrow \mathbb{N} . \exists B : \prod_{i \in \mathcal{I}} n_i \rightarrow \mathcal{B} : U = \bigcup_{i \in \mathcal{I}} \bigcap_{j=1}^{n_i} B_{i,j}$$

$$\text{PotentialSubbase} :: \prod X \in \text{SET} . ??(X)$$

$$\mathcal{B} : \text{PotentialSubbase} \iff \forall x \in X . \exists B \in \mathcal{B} . x \in B$$

$$\text{generateTopologyBySybbase} :: \prod X \in \text{SET} . \text{PotentialSubbase}(X) \rightarrow \text{TopologicalSpace}$$

$$\text{generateTopologyBySubbase}(\mathcal{B}) = \langle \langle \mathcal{B} \rangle \rangle_{\text{TOP}} := \left\langle \left\{ \bigcap_{B \in \mathcal{B}'} B \mid \mathcal{B}' : \text{Finite}(\mathcal{B}) \right\} \right\rangle_{\text{TOP}}$$

$$\text{BaseAt} :: \prod X : \text{TopologicalSpace} . \prod x \in X . ?\mathcal{U}(x)$$

$$\mathcal{B} : \text{BaseAt} \iff \forall U \in \mathcal{U} . \exists B \in \mathcal{B} : U \subset B$$

$$\text{BaseLocalization} :: \forall X : \text{TopologicalSpace} . \forall \mathcal{B} : \text{Base}(X) . \forall x \in X . \mathcal{U}(x) \cap \mathcal{B} : \text{BaseAt}(x)$$

Proof =

...

□

$$\text{discreteSpace} :: \text{SET} \rightarrow \text{TopologicalSpace}$$

$$\text{discreteSpace}(X) := (X, 2^X)$$

**BaseFromLocals** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{B} : \prod_{x \in X} \text{BaseAt}(x) . \bigcup_{x \in X} \mathcal{B}(x) : \text{Base}(X)$

**Proof** =

...

□

**characterOfPoint** ::  $\prod X : \text{TopologicalSpace} . X \rightarrow \text{CARD}$

**characterOfPoint**  $(x) = \chi(x) := \min \left\{ |\mathcal{B}| \mid \mathcal{B} : \text{BaseAt}(x) \right\}$

**characterOfSpace** ::  $\text{TopologicalSpace} \rightarrow \text{CARD}$

**characterOfPoint**  $(X) = \chi(X) := \sup_{x \in X} \chi(x)$

**FirstCountable** ::  $? \text{TopologicalSpace}$

$X : \text{FirstCountable} \iff \chi(X) \leq \aleph_0$

**SecondCountable** ::  $? \text{TopologicalSpace}$

$X : \text{SecondCountable} \iff w(X) \leq \aleph_0$

**OpenByInnerCover** ::  $\forall X : \text{TopologicalSpace} . \forall U \in ?X . \left( \forall u \in U . \exists O \in \mathcal{U}(x) : O \subset U \right) \Rightarrow U \in \mathcal{T}(X)$

**Proof** =

...

□

**SimplifyOpenUnion** ::  $\forall X : \text{TopologicalSpace} . \forall c \in \text{CARD} . \forall [0] : w(X) \leq c . \forall I \in \text{SET} .$

$$. \forall U : I \rightarrow \mathcal{T}(X) . \exists J \subset I : |J| \leq c \ \& \ \bigcup_{i \in I} U_i = \bigcup_{j \in J} U_j$$

**Proof** =

$$\left( \mathcal{B}, [1] \right) := \mathfrak{d}w(X)[0] : \sum \mathcal{B} : \text{Base}(X) . |\mathcal{B}| \leq c,$$

$$\mathcal{B}' := \left\{ B \in \mathcal{B} : \exists i \in I : B \subset U_i \right\} : ?\mathcal{T}(X),$$

$$\alpha := j\mathcal{B}' : \sum B \in \mathcal{B}' . \sum i(B) \in I . B \subset U_{\alpha(B)},$$

$$J := \alpha(\mathcal{B}) : ?I,$$

$$[2] := \text{ImageCardinality}(J) \text{SubsetCardinality}(\mathcal{B}') : |J| < c,$$

$$\text{Assume } x : \bigcup_{i \in I} U_i,$$

$$\left( i, [3] \right) := \mathfrak{d}\text{union}(x) : \sum i \in I . x \in U_i,$$

$$\left( \mathcal{B}'', [4] \right) := \mathfrak{d}\text{Base}(\mathcal{B})(U_i) : \sum \mathcal{B}'' \subset \mathcal{B} . U_i = \bigcup \mathcal{B}'',$$

$$[5] := j\mathcal{B}' \text{UnionSubset}[4] : \mathcal{B}'' \subset \mathcal{B}',$$

$$[x.*] := [3][4]j\mathcal{B}' \text{LargerUnion}[5] : x \in \bigcup_{B \in \mathcal{B}''} U_{\alpha(B)} \subset \bigcup_{j \in J} U_j;$$

$$\leadsto [3] := \mathfrak{d}^{-1} \text{Subset} : \bigcup_{i \in I} U_i \subset \bigcup_{j \in J} U_j,$$

$$[4] := \text{LargerUnion}(J) : \bigcup_{j \in J} U_j \subset \bigcup_{i \in I} U_i,$$

$$[*] := \mathfrak{d}^{-1} \text{SetEq}[3][4] : \bigcup_{i \in I} U_i = \bigcup_{j \in J} U_j;$$

□

**SimplifyBase** ::  $\forall X : \text{TopologicalSpace} . \forall c \in \text{CARD} . \forall [0] : w(X) \leq c . \forall \mathcal{B} : \text{Base}(X) .$

$$. \exists \mathcal{B}' \subset \mathcal{B} : |\mathcal{B}'| \leq c \ \& \ \mathcal{B}' : \text{Base}(X)$$

**Proof** =

$$\text{Assume } [1] : c \geq \aleph_0,$$

$$\left( \mathcal{A}, [2] \right) := \mathfrak{d}w(X)[0] : \sum \mathcal{A} : \text{Base}(X) . |\mathcal{A}| \leq c,$$

$$\beta := \Lambda A \in \mathcal{A} . \{ B \in \mathcal{B} : A \subset B \} : \mathcal{A} \rightarrow ?\mathcal{B},$$

$$\text{Assume } A : \mathcal{A},$$

$$\left( I, B, [3] \right) := \mathfrak{d}\text{Base}(X)(\mathcal{B}) : \sum I \in \text{SET} . \sum B : I \rightarrow \mathcal{B} . A = \bigcup_{i \in I} B_i,$$

$$\left( J_A, [A.1] \right) := \text{SimplifyOpwnUnion}(X, x, [0]I, B) : \sum J_A \subset I . |J_A| \leq c \ \& \ \bigcup_{j \in J_A} B_j = A,$$

$$B^A := B|_J : J \rightarrow \mathcal{B};$$

$$\leadsto (J, B, [3]) := I \left( \prod \right) : \prod A \in \mathcal{A} . \sum J_A \subset I . \sum B^A : J_A \rightarrow \mathcal{B} . |J| \leq c \ \& \ A = \bigcup_{j \in J_A} B_j^A,$$

$$\mathcal{B}' := \left\{ B_j^A \mid A \in \mathcal{A}, j \in J_A \right\} : ?\mathcal{B},$$

$$[4] := \text{InfiniteProductCard}j\mathcal{B}'[1][2][3] : |\mathcal{B}'| \leq c,$$

Assume  $x : U$ ,

Assume  $U : \mathcal{U}(x)$ ,

$(A, [5]) := \text{BaseLocalization}(\mathcal{A}) \text{BaseAt}(x, U) : \sum A \in \mathcal{A} . x \in A \subset U$ ,

$(B, [6]) := [3]_j \beta \text{UnionSubset}(A) \text{union} : \sum B \in \mathcal{B}' . B \in \beta(A) \ \& \ x \in B$ ,

$[x.*] := \text{SubsetTransitivity}_j \beta [6] : B \subset U$ ;

$\leadsto [1.*] := \text{BaseAt}(X, x) \text{BaseFromLocals} : (\mathcal{B}' : \text{Base})$ ;

$\leadsto [1] := I(\Rightarrow) : c \geq \aleph_0 \Rightarrow \exists \mathcal{B}' \subset \mathcal{B} . |\mathcal{B}'| \leq c \ \& \ \mathcal{B}' : \text{Base}(X)$ ,

Assume  $[2] : c < \aleph_0$ ,

$(\mathcal{A}, [3]) := \text{Base}(X)[0] : \sum \mathcal{A} : \text{Base}(X) . |\mathcal{A}| = w(X)$ ,

Assume  $B : \mathcal{B}$ ,

$(I_B, A^B, [3]) := \text{Base}(\mathcal{A})(B) : \sum I_B \in \text{SET} . A^B : I_B \hookrightarrow \mathcal{A} . B = \bigcup_{i \in I_A} A_i^B$ ;

$\leadsto (I, A, [4]) := I \left( \prod \right) : \prod B \in \mathcal{B} . \sum I_B \in \text{SET} . \sum A^B : I_B \hookrightarrow \mathcal{A} . B = \text{ }_2$

Assume  $A : \mathcal{A}$ ,

$(J_A, B^A, [5]) := \text{Base}(\mathcal{B})(A) : \sum J_A \in \text{SET} . B^A : J_A \hookrightarrow \mathcal{B} . A = \bigcup_{j \in J_B} B_j^A$ ,

$[6] := [5][4] : A = \bigcup_{j \in J_A} \bigcup_{i \in I_{B_j}} A_i^{B_j}$ ,

$\mathcal{A}' := \left\{ A_i^{B_j} \mid j \in J_A, i \in I_{B_j} \right\} : ?\mathcal{A}$ ,

$[7] := \mathcal{A}'[6] : A = \bigcup \mathcal{A}'$ ,

$[8] := \text{Base}(X) \text{min}[0][2][3][7] : A \in \mathcal{A}$ ,

$[9] := [6] \text{SubsetUnion} : \forall a \in \mathcal{A}' . a \subset A$ ,

$[A.*] := \mathcal{A}[9][5] : A \in \mathcal{B}$ ;

$\leadsto [2.*] := \text{Subset} : \mathcal{A} \subset \mathcal{B}$ ;

$\leadsto [*] := I(\Rightarrow)[1] \text{LETrichotomy}(\text{CARD}) : \sum \mathcal{B}' \subset \mathcal{B} . |\mathcal{B}'| \leq c \ \& \ \mathcal{B}' : \text{Base}(X)$ ;

□

**FinerTopologyExists** ::  $\forall X : \text{SET} . \forall T : ?\text{Topology}(X) . \exists t : \text{Topology}(X) : t = \sup T$

**Proof** =

□

**CoarsestTopologyExists** ::  $\forall X : \text{SET} . \forall T : ?\text{Topology}(X) . \exists t : \text{Topology}(X) : t = \inf T$

**Proof** =

□



### 1.1.2 Closure and Interior

$\text{closure} :: \prod X : \text{TopologicalSpace} . ?X \rightarrow \text{Closed}(X)$

$\text{closure}(A) = \overline{A} = \text{cl}_X A := \bigcap \{K : \text{Closed}(X) : A \subset K\}$

$\text{EquivalentClosure1} :: \forall X : \text{TopologicalSpace} . \forall A \in ?X . \overline{A} = \{x \in X : \forall U \in \mathcal{U}(x) . U \cap A \neq \emptyset\}$

**Proof** =

$Z := \{x \in X : \forall U \in \mathcal{U}(x) . U \cap A \neq \emptyset\} : ?X,$

**Assume**  $x : Z^c,$

$(U, [1]) := \text{compliment}_j Z : \sum U \in \mathcal{U}(x) . U \cap A = \emptyset,$

$[x.*] := [1]_j Z : U \subset Z^c;$

$\leadsto [1] := \text{OpenByInnerCover} : Z^c \in \mathcal{T}(X),$

$[2] := \text{Closed}(X) : (Z : \text{Closed}),$

$[3] := \text{closureIntersectionSubset} : \overline{A} \subset Z,$

**Assume**  $x : Z,$

**Assume**  $K : \text{Closed}(X),$

**Assume**  $[4] : A \subset K,$

**Assume**  $[5] : x \notin K,$

$[6] := \text{compliment}[5] \text{Closed}(X) \text{Closed}(X) : K^c \in \mathcal{U}(x),$

$[7] := \text{ComplementSubset}[4] : A \cap K^c = \emptyset,$

$[8] := jZ[7] : x \notin Z,$

$[K.*] := \text{InAndNotIn}[4] : \perp;$

$\leadsto [4] := E(\perp)I(\Rightarrow)I(\forall) : \forall K : \text{Closed}(X) . A \subset K \Rightarrow x \in K,$

$[x.*] := \text{closure}[4] : x \in \overline{A};$

$\leadsto [*] := \text{Subset}[3] \text{SetEq} : \overline{A} = Z;$

□

$\text{EquivalentClosure2} :: \forall X : \text{TopologicalSpace} . \forall A \in ?X .$

$\overline{A} = \{x \in X : \exists \mathcal{B} : \text{BaseAt}(x) . \forall B \in \mathcal{B} . B \cap A \neq \emptyset\}$

**Proof** =

...

□

$\text{PotentiallyClosedSet} :: \prod X : \text{SET} . ???X$

$\mathcal{A} : \text{PotentiallyClosedSet} \iff \forall \emptyset, X \in \mathcal{A} . \&$

$\& \forall A, A' \in \mathcal{A} . A \cup A' \in \mathcal{A} \&$

$\& \forall \mathcal{A}' \subset \mathcal{A} . \bigcap \mathcal{A}' \in \mathcal{A}$

**PotentialClosure** ::  $\prod X : \text{SET} . ? \left( ?X \rightarrow ?X \right)$

$c : \text{PotentialClosure} \iff c(\emptyset) = \emptyset \ \&$   
 $\& \forall A, B \subset X . A \subset c(A) \ \&$   
 $\& c^2(A) = c(A) \ \&$   
 $\& c(A \cup B) = c(A) \cup c(B)$

**CloserIsPotentialClosure** ::  $\forall X : \text{TopologicalSpace} . \text{cl}_X : \text{PotentialClosure}$

**Proof** =

[1] := **AlwaysClosed**(X) **ClosedClosure**  $\bar{\bar{c}}$ closure(X) :  $\bar{\bar{\emptyset}} = \emptyset$ ,

**Assume**  $A, B : ?X$ ,

$\left[ (A, B).*.1 \right] := \text{IntersectSubset} \bar{\bar{c}}$ closure(X) :  $A \subset \bar{A}$ ,

$\left[ (A, B).*.2 \right] := \text{ClosedClosure} \bar{\bar{c}}$ closure(X) :  $\bar{\bar{\bar{A}}} = \bar{A}$ ,

[2] := ... **UnionSubset** :  $A \subset \bar{A} \cup \bar{B}$ ,

[3] := ... **UnionSubset** :  $B \subset \bar{A} \cup \bar{B}$ ,

[4] := **SubsetUnion**[2][3] :  $A \cap B \subset \bar{A} \cup \bar{B}$ ,

[5] := **ClosedUnion**[4] :  $\bar{A} \cup \bar{B} : \text{Closed}(X)$ ,

[6] := **IntersectSubset**  $\bar{\bar{c}}$ closure[5] :  $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ ,

[7] := **UnionSubset**(A, (A, B)) **IntersectSubset**  $\bar{\bar{c}}$ closure(A) :  $\bar{A} \subset \overline{A \cup B}$ ,

[8] := **UnionSubset**(B, (A, B)) **IntersectSubset**  $\bar{\bar{c}}$ closure(B) :  $\bar{B} \subset \overline{A \cup B}$ ,

[9] := **SubsetUnion**(A, B)[7][8] :  $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$ ,

$\left( (A, B).*.3 \right) := \bar{\bar{\text{SetEq}}}$ [9][6] :  $\overline{A \cup B} = \bar{A} \cup \bar{B}$ ;

$\leadsto [*] := \bar{\bar{\text{PotentialClosure}}} : \left( \text{cl}_X : \text{PotentialClosure}(X) \right)$ ;

□

**PotentialClosureOperatorIsMonotonic** ::  $\forall X \in \text{SET} . \forall c : \text{PotentialClosure}(X) . c \in \text{End}_{\text{CAT}} \left( \text{P}(?X) \right)$

**Proof** =

**Assume**  $A, B : \text{P}(?X)$ ,

**Assume** [1] :  $A \subset B$ ,

[2] := **UnionWithSubset** :  $A \cup B = B$ ,

[3] := [2]  $\bar{\bar{\text{PotentialClosure}}}$ (c) :  $c(B) = c(A \cup B) = c(A) \cup c(B)$ ,

$\left[ (A, B).* \right] := \text{UnionWithSubset}$ [3] :  $c(A) \subset c(B)$ ;

$\leadsto [*] := \bar{\bar{\text{P}}}(?X) : c \in \text{End}_{\text{CAT}} \left( \text{P}(?X) \right)$ ;

□

**ImageOfClosureOperator** ::  $\forall X \in \text{SET} . \forall c : \text{PotentialClosure}(X) . \text{Im } c : \text{PotentialClosedSets}(X)$

**Proof** =

[1] :=  $\bar{\bar{\text{PotentialClosure}}}$   $\bar{\bar{\text{Im}}}$  c :  $c(\emptyset) = \emptyset \in \text{Im } c$ ,

[2] :=  $\bar{\bar{\text{PotentialClosure}}}$   $\bar{\bar{\text{Im}}}$  C :  $A \subset c(A) = A \in \text{Im } c$ ,

**Assume**  $A, B : \text{Im } C$ ,

$(A', B', [3]) := \bar{\bar{\text{Im}}} C(A, B) : \sum A', B' \in 2^X . A = c(A') \ \& \ B = c(B')$ ,

$\left[ (A, B).* \right] := [3] \bar{\bar{\text{PotentialClosure}}}$   $\bar{\bar{\text{Im}}}$  c :  $A \cup B = c(A') \cup c(B') = c(A' \cup B') \in \text{Im } c$ ;

$\leadsto [3] := I(\forall) : \forall A, B \in \text{Im } c . A \cap B = \text{Im } c,$

**Assume**  $\mathcal{A} : ?(\text{Im } c),$

$[5] := \text{PotentialClosureIsMonotonic}(c)(\text{THMIntersectSubset}(\mathcal{A}') : \forall A \in \mathcal{A} . c \left( \bigcap \mathcal{A} \right) \subset c(A),$

$[6] := \text{SubsetInersect}[4] : c \left( \bigcap \mathcal{A}' \right) \subset \bigcap c(\mathcal{A}) = \bigcap \mathcal{A},$

$[7] := \text{PotentialClosure}(A) : \bigcap \mathcal{A} \subset c \left( \bigcap \mathcal{A} \right),$

$[\mathcal{A}.*] := \text{SetEq}[6][7] \text{Im } c : \bigcap \mathcal{A} = c \left( \bigcap \mathcal{A} \right) \in \text{Im } c;$

$\leadsto [*] := [1][2][3] \text{PotentialClosedSets}(X) : \left( \text{Im } C : \text{PotentialClosedSets}(X) \right);$

□

$\text{generateTopologyByClosedSets} :: \prod X \in \text{SET} . \text{PotentialClosedSets} \rightarrow \text{TopologicalSpace}$

$\text{generateTopologyByClosedSets}(\mathcal{A}) := \left( X, \{A^c \mid A \in \mathcal{A}\} \right)$

$\text{generateTopologyByClosure} :: \prod X \in \text{SET} . \text{PotentialClosure} \rightarrow \text{TopologicalSpace}$

$\text{generateTopologyByClosure}(c) := \text{generateTopologyByClosedSets}(\text{Im } c)$

$\text{interior} :: \prod X : \text{TopologicalSpace} . 2^X \rightarrow \mathcal{T}(X)$

$\text{interior}(A) = \text{int } A := \bigcup \{U \in \mathcal{T}(X) \mid U \subset A\}$

**EquivalentInterior** ::  $\forall X : \text{TopologicalSpace} . \forall A \in 2^X . \forall x \in A . x \in \text{int } A \iff \exists U \in \mathcal{U}(x) . U \subset A$

**Proof** =

...

□

**InteriorAsDifference** ::  $\forall X : \text{TopologicalSpace} . \forall A \in 2^X . \text{int } A = X \setminus \text{cl } A^c$

**Proof** =

...

□

**PotentialInterior** ::  $\prod X \in \text{SET} . ? \left( ?X \rightarrow ?X \right)$

$i : \text{PotentialInterior} \iff i(X) = X \ \&$

$\& \forall A, B \subset X . i(A) \subset A \ \&$

$\& i^2(A) = i(A) \ \&$

$\& i(A \cap B) = i(A) \cap i(B)$

**InteriorIsPotentialInterior** ::  $\forall X : \text{TopologicalSpace} . \text{int}_X : \text{PotentialInterior}(X)$

**Proof** =

...

□

**InteriorIsMonotonic** ::  $\forall X \in \text{SET} . \forall i : \text{PotentialInterior} . i \in \text{End}_{\text{POSET}}(?X)$

**Proof** =

...

□

**InteriorImageIsTopology** ::  $\forall X \in \text{SET} . \forall i : \text{PotentialInterior} . \text{Im } i : \text{Topology}(X)$

**Proof** =

**generateTopologyByInterior** ::  $\prod X \in \text{SET} . \text{PotentialInterior}(X) \rightarrow \text{TopologicalSpace}$   
**generateTopologyByInterior** ( $i$ ) :=  $(X, \text{Im } i)$

**ClosureUnion** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{A} : \text{Finite}(?X) . \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}}$

**Proof** =

**LocallyFinite** ::  $\prod X : \text{SET} . ???X$

$\mathcal{A} : \text{LocallyFinite} \iff \forall x \in X . \exists U \in \mathcal{U}(x) : \left| \{A \in \mathcal{A} : U \cap A = \emptyset\} \right| < \infty$

**Discrete** ::  $\prod X : \text{SET} . ???X$

$\mathcal{A} : \text{Discrete} \iff \forall x \in X . \exists U \in \mathcal{U}(x) : \left| \{A \in \mathcal{A} : U \cap A \neq \emptyset\} \right| = 1$

**LocallyFiniteUnionClosure** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{A} : \text{LocallyFinite}(X) . \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}}$

**Proof** =

[1] :=  $\Lambda A \in \mathcal{A} . \text{UnionSubset}(A, \mathcal{A}) \text{ClosureIsMonotonic}(\text{cl}_X) : \forall A \in \mathcal{A} . \overline{A} \subset \overline{\bigcup \mathcal{A}}$ ,

[2] :=  $\text{UnionSubset}[1] : \bigcup \overline{\mathcal{A}} \subset \overline{\bigcup \mathcal{A}}$ ,

**Assume**  $x : \overline{\bigcup \mathcal{A}}$ ,

$(U, [3]) := \text{dLocallyFinite}(X)(\mathcal{A}) : \sum U \in \mathcal{U}(x) . \left| \{A \in \mathcal{A} : A \cap U \neq \emptyset\} \right| < \infty$ ,

$\mathcal{A}' := \{A \in \mathcal{A} : A \cap U \neq \emptyset\} : ?\mathcal{A}$ ,

[4] :=  $\text{EquivalentClosure1} \mathcal{A} : x \notin \overline{\bigcup \mathcal{A} \setminus \mathcal{A}'}$ ,

[5] :=  $\text{dxClosureUnion} : x \in \overline{\bigcup \mathcal{A}} = \overline{\bigcup \mathcal{A}' \cup \bigcup \mathcal{A} \setminus \mathcal{A}'}$ ,

$[*.x] := [4][5] \text{ClosureUnion}(\mathcal{A}')[3] \text{LargerUnion}(\mathcal{A}', \mathcal{A}) : x \in \overline{\bigcup \mathcal{A}'} = \bigcup \overline{\mathcal{A}'} \subset \bigcup \overline{\mathcal{A}}$ ;

$\leadsto [3] := \text{d}^{-1} \text{Subset} : \overline{\bigcup \mathcal{A}} \subset \bigcup \overline{\mathcal{A}}$ ,

$[*] := \text{d}^{-1} \text{SetEq} : \overline{\bigcup \mathcal{A}} = \bigcup \overline{\mathcal{A}}$ ;

□

**LocallyFiniteClosedUnion** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{A} : \text{LocallyFinite}(X) .$   
 $. \forall [0] : \forall A \in \mathcal{A} . A : \text{Closed}(X) . \bigcup \mathcal{A} : \text{Closed}(X)$

**Proof** =

...

□

**LocallyFiniteClosureIsLocallyFinite** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{A} : \text{LocallyFinite}(X) .$   
 $\quad . \overline{\mathcal{A}} : \text{LocallyFinite}(X)$

**Proof** =

**Assume**  $x : X$ ,

$(U, [1]) := \text{dLocallyFinite}(X) : \sum U \in \mathcal{U}(x) . \left| \{A \in \mathcal{A} : A \cap U \neq \emptyset\} \right| < \infty,$

$\mathcal{A}' := \{A \in \mathcal{A} : A \cap U \neq \emptyset\} : \text{Finite}(\mathcal{A}),$

**Assume**  $A : \mathcal{A}'^c,$

**Assume**  $[2] : \overline{A} \cap U \neq \emptyset,$

$y := \text{dNonEmpty} : \overline{A} \cap U,$

$[3] := \text{EquivalentClosure}(y) : \forall O \in \mathcal{U}(y) . O \cap A \neq \emptyset,$

$[4] := [3](U) : U \cap A \neq \emptyset,$

$[5] := \text{J}\mathcal{A}'[4] : A \in \mathcal{A}',$

$[A.*] := \text{dcomplement}\text{dAInAndNotIn}[5] : \perp;$

$\sim [x.*] := E(\perp)\text{J}\mathcal{A}'[1] : \left| \{A \in \mathcal{A} : \overline{A} \cap U \neq \emptyset\} \right| < \infty;$

$\sim [*] := \text{dLocallyFinite}(A) : (\overline{\mathcal{A}} : \text{LocallyFinite}(X));$

□

**DiscreteClosureIsDiscrete** ::  $\forall X : \text{TopologicalSpace} . \forall \mathcal{A} : \text{Discrete}(X) . \overline{\mathcal{A}} : \text{Discrete}(X)$

**Proof** =

...

□

**ClosureIntersection** ::  $\forall X : \text{TopologicalSpace} . \forall A, B \subset X . \overline{A \cap B} \subset \overline{A} \cap \overline{B}$

**Proof** =

$[1] := \text{ClosureIsMonotonic}(A) : A \subset \overline{A},$

$[2] := \text{ClosureIsMonotonic}(B) : B \subset \overline{B},$

$[3] := \text{SubsetIntersect}[1][2] : A \cap B \subset \overline{A} \cap \overline{B},$

$[4] := \text{ClosedIntersection}(\overline{A}, \overline{B}) : \overline{A} \cap \overline{B} : \text{Closed}(X),$

$[*] := \text{dclosure}[3][4] : \overline{A \cap B} \subset \overline{A} \cap \overline{B};$

□

**ClosureOfDifference** ::  $\forall X : \text{TopologicalSpace} . \forall A, B \subset X . \overline{A \setminus B} \subset \overline{A} \setminus \overline{B}$

**Proof** =

**Assume**  $x : \overline{A} \setminus \overline{B},$

$[1] := \text{AlternativeClosure1}(A)(x) : \forall U \in \mathcal{U}(x) . U \cap A \neq \emptyset,$

$(U, [2]) := \text{AlternativeClosure}(B)(x) : \sum U \in \mathcal{U}(x) . U \cap B = \emptyset,$

**Assume**  $W : \mathcal{U}(x),$

$V := W \cap U : \mathcal{U}(x),$

$[3] := \text{IntersectSubset}(V) : V \subset U,$

$[4] := [2][3]\text{SubsetIntersect} : V \cap B = \emptyset,$

$[5] := [1](V) : V \cap A \neq \emptyset,$

$[6] := [2] : V \cap (A \setminus B) \neq \emptyset,$

$$[W.*] := \text{SupersectIntersect}[3][6] : W \cap (A \setminus B) \neq \emptyset;$$

$$\leadsto [x.*] := \text{AlternativeClosure2} : x \in \overline{A \setminus B};$$

$$\leadsto [*] := \text{ISubset} : \overline{A} \setminus \overline{B} \subset \overline{A \cap B};$$

...

□

$$\text{InfiniteClosureUnion} :: \forall X \in \text{TopologicalSpace} . \forall A : \mathbb{N} \rightarrow ?X . \overline{\bigcup_{n=1}^{\infty} A_n} = \bigcup_{n=1}^{\infty} \overline{A_n} \cup \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=1}^{\infty} A_{n+m}}$$

Proof =

$$\text{Assume } x : \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$\text{Assume } [1] : x \notin \bigcap_{n=1}^{\infty} \overline{A_n},$$

$$(U, [2]) := \text{EquivalentClosure}[1] : \forall n \in \mathbb{N} . \exists U \in \mathcal{U}(x) . \forall U \cap A_n = \emptyset,$$

$$[3] := \text{EquivalntClosure}(x) : \forall V \in \mathcal{U}(x) . V \cap \bigcup_{n=1}^{\infty} A_n \neq \emptyset,$$

$$\text{Assume } n : \mathbb{N},$$

$$\text{Assume } W : \mathcal{U}(x),$$

$$V := W \cap \bigcap_{i=1}^n U_i : \mathcal{U}(x),$$

$$[4] := [3](V) : V \cap \bigcup_{n=1}^{\infty} A_n \neq \emptyset,$$

$$[5] := [4][2]_j V : V \cap \bigcup_{i=n+1}^{\infty} A_i \neq \emptyset,$$

$$[W.*] := \text{SubsetIntersect}(V) \text{SupersectIntersect}[5] : W \cap \bigcup_{i=n+1}^{\infty} A_i \neq \emptyset;$$

$$\leadsto [n.*] := \text{EquivalentClosure}(x) : x \in \overline{\bigcup_{i=1}^n A_i};$$

$$\leadsto [1.*] := \text{d}^{-1} \text{intersect} : x \in \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=1}^n A_i};$$

$$\leadsto [x.*] := \text{InOrNotIn}(x) \text{d}^{-1} \text{Union} : x \in \bigcup_{n=1}^{\infty} \overline{A_n} \cap \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i};$$

$$\leadsto [1] := \text{d} \text{Subset} : \overline{\bigcup_{i=1}^n A_i} \subset \bigcup_{n=1}^{\infty} \overline{A_n} \cap \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i},$$

$$[2] := \text{d} \text{SubsetUnion}(A) : \forall n \in \mathbb{N} . A_n \subset \bigcup_{i=1}^n A_i,$$

$$[3] := \text{d} \text{ClosureIsMonotonic}(A) \text{SubssetUnion} \overline{A}[2] : \bigcup_{n=1}^{\infty} \overline{A_n} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[4] := \Lambda n \in \mathbb{N} . \text{LargerUnion}(A, n) \text{ClosureIsMonotonic} : \forall n \in \mathbb{N} . \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[5] := \text{IntersectSubset}[4] : \bigcup_{\infty=1}^n \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[6] := \text{SubsetUnion}[4] : \bigcup_{n=1}^{\infty} \overline{A_i} \cup \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i} \subset \overline{\bigcup_{n=1}^{\infty} A_n},$$

$$[*] := \mathfrak{d}^{-1}\text{SetEq}[1][5] : \bigcup_{n=1}^{\infty} \overline{A_i} \cup \bigcup_{n=1}^{\infty} \overline{\bigcup_{i=n+1}^{\infty} A_i} = \overline{\bigcup_{n=1}^{\infty} A_n};$$

□

### 1.1.3 Open and Closed Domains

$\text{OpenDomain} :: \prod X : \text{TopologicalSpace} . ?X$

$A : \text{OpenDomain} \iff A = \text{int } \overline{A}$

$\text{ClosedSetInteriorIsOpenDomain} :: \forall X : \text{TopologicalSpace} . \forall A : \text{Closed}(X) . \text{int } A : \text{OpenDomain}(X)$

**Proof** =

**Assume**  $U : \text{Open}(X)$ ,

**Assume**  $[1] : U \subset \overline{\text{int } A}$ ,

$[2] := \text{PotentialInterior}(\text{int})\text{ClosureIsMonotonic}(\text{cl}_X)\text{PotentialClosure}(\text{cl}_X) : U \subset \overline{\text{int } A} \subset \overline{A} = A$ ,

$[U.*] := \text{PotentialInterior}(\text{int})[2] : U \subset \text{int } U$ ;

$\leadsto [1] := \text{interior} : \text{int } \overline{\text{int } A} \subset \text{int } A$ ,

**Assume**  $U : \text{Open}(X)$ ,

**Assume**  $[2] : U \subset A$ ,

$[3] := \text{interiorSubsetUnion}\text{PotentialClosure}(\text{cl}_X) : U \subset \text{int } A \subset \overline{\text{int } A}$ ,

$[U.*] := \text{PotentialInterior}(\text{int})[3] : U \subset \text{int } \overline{\text{int } A}$ ;

$\leadsto [3] := \text{interior} : \text{int } A \subset \overline{\text{int } A}$ ,

$[5] := \text{SetEq}[2][3] : \text{int } A = \overline{\text{int } A}$ ,

$[*] := \text{OpenDomain}^{-1} : (U : \text{OpenDomain}(X))$ ;

□

$\text{OpenDomainIntesection} :: \forall X : \text{TopologicalSpace} . \forall A, B : \text{OpenDomain}(X) . A \cap B : \text{OpenDomain}$

**Proof** =

$[1] := \text{closureIntersection}(A, B)\text{PotentialInterior}(\text{int}_X)\text{OpenDomain}(A)(B) :$

$: \text{int } \overline{A \cap B} \subset \text{int } \overline{A} \cap \text{int } \overline{B} = A \cap B$ ,

$[2] := \text{OpenDomain}(A)(B)\text{OpenIntersection}(A, B) : A \cap B : \text{Open}(X)$ ,

$[3] := \text{interiorinteriorIsMonotonic}(A \cap B) : A \cap B = \text{int } A \cap B \subset \overline{A \cap B}$ ,

$[4] := \text{SetEq}[1][3] : A \cap B = \overline{A \cap B}$ ,

$[*] := \text{OpenDomain}^{-1} : (A \cap B : \text{OpenDomain}(X))$ ;

□

$\text{OpenDomaSubset} :: \forall X : \text{TopologicalSpace} . \forall A, B : \text{OpenDomain}(X) . A \subset B \iff \overline{A} \subset \overline{B}$

**Proof** =

**Assume**  $[1] : A \subset B$ ,

$[1.*] := \text{ClosureIsMonotonic}[1] : \overline{A} \subset \overline{B}$ ;

$\leadsto [1] := I(\Rightarrow) : A \subset B \Rightarrow \overline{A} \subset \overline{B}$ ,

**Assume**  $[2] : \overline{A} \subset \overline{B}$ ,

$[3] := \text{InteriorIsMonotonic} : \text{int } \overline{A} \subset \text{int } \overline{B}$ ,

$[4] := \text{OpenDomain}(A)(B)[3] : A \subset B$ ;

$\leadsto [2] := I(\Rightarrow) : \overline{A} \subset \overline{B} \Rightarrow A \subset B$ ,

$[*] := I(\iff)[1][2] : A \subset B \iff \overline{A} \subset \overline{B}$ ;

□



**UnionClosureInteriorAsSup** ::  $\forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall U : I \rightarrow \text{OpenDomain}(X) .$

$$. \text{int} \overline{\bigcup_{i \in I} U_i} = \min \left\{ O : \text{OpenDomain}(X) \mid \forall i \in I . U_i \subset O \right\}$$

**Proof** =

$$[1] := \text{ClosedInteriorIsOpenDomain} \left( \text{int} \overline{\bigcup_{i \in I} U_i} \right) : \left( \text{int} \overline{\bigcup_{i \in I} U_i} : \text{OpenDomain}(X) \right),$$

**Assume**  $O : \text{OpenDomain}(X),$

**Assume**  $[2] : \forall i \in I . U_i \subset O,$

$$[3] := \text{UnionSuperset}[2] : \bigcup_{i \in I} U_i \subset O,$$

$$[4] := \text{ClosureIsMonotonic}(\text{cl}) \text{InteriorIsMonotonic}(\text{int})[1] : \text{int} \overline{\bigcup_{i \in I} U_i} \subset \text{int} \overline{O}[3],$$

$$[i.*] := \text{OpenDomain}[4] : \text{int} \overline{\bigcup_{i \in I} U_i} \subset O;$$

$$\leadsto [2] := I(\forall) : \forall O : \text{OpenDomain}(X) . \left( \forall i \in I . U_i \subset O \right) \Rightarrow \text{int} \overline{\bigcup_{i \in I} U_i} \subset O,$$

**Assume**  $i : I,$

$$[3] := \text{UnionSubset}(i, U) : U_i \subset \bigcup_{i \in I} U_i \subset \overline{\bigcup_{i \in I} U_i},$$

$$[*] := \text{Interior}[3] : U_i \subset \text{int} \overline{\bigcup_{i \in I} U_i};$$

$$\leadsto [3] := I(\forall) : \forall i \in I . U_i \subset \text{int} \overline{\bigcup_{i \in I} U_i},$$

$$[*] := \text{Interior}^{-1} \min[3][2][1] : \text{int} \overline{\bigcup_{i \in I} U_i} = \min \left\{ O : \text{OpenDomain}(X) \mid \forall i \in I . U_i \subset O \right\};$$

□

**IntersectInteriorAsInf** ::  $\forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall U : I \rightarrow \text{OpenDomain}(X) .$

$$. \text{int} \bigcap_{i \in I} U_i = \max \left\{ O : \text{OpenDomain}(X) \mid \forall i \in I . O \subset U_i \right\}$$

**Proof** =

$$[1] := \text{PotentialClisure}(\text{cl}_X) : \text{int} \bigcap_{i \in I} U_i \subset \overline{\text{int} \bigcap_{i \in I} U_i},$$

$$[2] := \text{MonotonicInterior}(\text{int}) \text{PotentialInterior}(\text{int}) : \text{int} \bigcap_{i \in I} U_i \subset \text{int} \text{int} \bigcap_{i \in I} U_i \subset \overline{\text{int} \text{int} \bigcap_{i \in I} U_i},$$

**Assume**  $i : I,$

$$[*.*] := \text{SubsetIntersect}(U_i, U) \text{ClosureIsMonotonic}(\text{cl}_X) \text{InteriorIsMonotonic}(\text{int}) \text{OpenDomain}(U) :$$

$$: \text{int} \text{int} \overline{\bigcap_{i \in I} U_i} \subset \text{int} \overline{U_i} = U_i;$$

$$\leadsto [3] := \text{IntersectSubset} : \text{int} \text{int} \overline{\bigcap_{i \in I} U_i} \subset \bigcap_{i \in I} U_i,$$

$$[4] := \text{Interior}[3] : \text{int} \text{int} \overline{\bigcap_{i \in I} U_i} \subset \text{int} \bigcap_{i \in I} U_i,$$

$$[5] := \text{Interior}^{-1} \text{SetEq}[3][4] \text{OpenDomain} : \left( \text{int} \bigcap_{i \in I} U_i : \text{OpenDomain} \right),$$

Assume  $O : \text{OpenDomain}(X)$ ,

Assume  $[6] : \forall i \in I . O \subset U_i$ ,

$[7] := \text{OpenDomain}(U) : O \subset \bigcap_{i \in I} U_i$ ,

$[O.*] := \text{int}[7] : O \subset \text{int} \bigcap_{i \in I} U_i$ ;

$\leadsto [6] := I(\forall)I(\Rightarrow) : \forall O : \text{OpenDomain}(X) . (\forall i \in I . O \subset U_i) \Rightarrow O \subset \text{int} \bigcap_{i \in I} U_i$ ,

$[7] := \Lambda i \in I . \text{PotentialInterior}(\text{int})\text{IntersectionSubset} : \forall i \in I . \text{int} \bigcap_{i \in I} U_i \subset \bigcap_{i \in I} U_i \subset U_i$ ,

$[*] := \text{OpenDomain}^{-1} \max[5][6][7] : \text{int} \bigcap_{i \in I} U_i = \max \left\{ O : \text{OpenDomain}(X) : \forall i \in I . O \subset U_i \right\}$ ;

□

$\text{ClosedDomain} :: \prod X : \text{TopologicalSpace} . ???X$

$A : \text{ClosedDomain} \iff A = \overline{\text{int } A} \iff$

$\text{ClosedOpenDomainDuality} :: \forall X : \text{TopologicalSpace} .$

$. \text{complement} : \text{OpenDomain}(X) \xleftrightarrow{\text{SET}} \text{ClosedDomain}(X)$

Proof =

Assume  $U : \text{OpenDomain}$ ,

$[1] := \text{OpenDomain} : U = \text{int } \overline{U}$ ,

$[2] := [1]^{\text{C}} \text{ClosureAsComplement}(X, \overline{U}) \left( \text{DoubleComplement}(X) \right)^2 (\overline{U})(U) \text{InteriorAsComplement}(X, U) :$   
 $: U^{\text{C}} = \left( \text{int}(\overline{U})^{\text{CC}} \right)^{\text{C}} = \overline{(\overline{U}^{\text{CC}})^{\text{C}}} = \overline{\text{int } U^{\text{C}}}$ ,

$[U.*] := \text{OpenDomain}^{-1}[3] : \left( U^{\text{C}} : \text{ClosedDomain}(X) \right)$ ;

$\leadsto [1] := I(\forall) : \forall U : \text{OpenDomain}(X) . U^{\text{C}} : \text{ClosedDomain}(X)$ ,

Assume  $A : \text{ClosedDomain}(X)$ ,

$[2] := \text{ClosedDomain} : A = \overline{\text{int } A}$ ,

$[3] := [2]^{\text{C}} \text{InteriorAsComplement}(X, \text{int } A) \left( \text{DoubleComplement}(X) \right)^2 (\text{int } A)(A)$

$\text{ClosureAsComplement}(X, A) :$

$: A^{\text{C}} = \overline{(\text{int } A)^{\text{CC}}} = \text{int} \left( \text{int}(A^{\text{CC}}) \right)^{\text{C}} = \text{int } \overline{A^{\text{C}}} =$ ,

$[U.*] := \text{ClosedDomain}^{-1}[3] : \left( A^{\text{C}} : \text{OpenDomain}(X) \right)$ ;

$\leadsto [2] := I(\forall) : \forall A : \text{ClosedDomain}(X) . A^{\text{C}} : \text{OpenDomain}(X)$ ,

$[3] := \text{DoubleComplement}[1][2] : \left( \text{complement} : \text{ClosedDomain}(X) \xleftrightarrow{\text{SET}} \text{ClosedDomain}(X) \right)$ ;

□

$\text{OpenSetClosureIsClosedDomain} :: \forall X : \text{TopologicalSpace} . \forall U : \text{Open}(X) . \overline{U} : \text{ClosedDomain}(X)$

Proof =

...

□

**ClosedDomainUnion** ::  $\forall X : \text{TopologicalSpace} . \forall A, B : \text{ClosedDomain}(X) . A \cup B : \text{ClosedDomain}(X)$

**Proof** =

...

□

**ClosedDomainSubset** ::  $\forall X : \text{TopologicalSpace} . \forall A, B : \text{OpenDomain}(X) . A \subset B \iff \overline{A} \subset \overline{B}$

**Proof** =

...

□

**IntersectionInteriorClosureAsSup** ::  $\forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{ClosedDomain}(X) .$

$$. \overline{\text{int} \bigcap_{i \in I} U_i} = \max \left\{ B : \text{ClosedDomain}(X) \mid \forall i \in I . B \subset A_i \right\}$$

**Proof** =

...

□

**UnionClosureAsSup** ::  $\forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{ClosedDomain}(X) .$

$$. \overline{\bigcup_{i \in I} U_i} = \min \left\{ B : \text{ClosedDomain}(X) \mid \forall i \in I . A_i \subset B \right\}$$

**Proof** =

...

□

### 1.1.4 Boundary Operator

**boundary** ::  $\prod X : \text{TopologicalSpace} . ?X \rightarrow \text{Closed}(X)$

**boundary**  $(A) = \partial A := \overline{A} \setminus \text{int } A$

**BoundaryCondition** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \forall x \in X . x \in \partial A \iff$   
 $\iff \forall U \in \mathcal{U}(x) . U \neq U \cap A \neq \emptyset$

**Proof** =

...

□

**InteriorByBoundary** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \text{int } A = A \setminus \partial A$

**Proof** =

...

□

**ClosureByBoundary** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \overline{A} = A \cup \partial A$

**Proof** =

...

□

**BoundaryOfUnion** ::  $\forall X : \text{TopologicalSpace} . \forall A, B \subset X . \partial(A \cup B) \subset \partial A \cup \partial B$

**Proof** =

...

□

**BoundaryOfIntersection** ::  $\forall X : \text{TopologicalSpace} . \forall A, B \subset X . \partial(A \cap B) \subset (\overline{A} \cap \partial B) \cup (\partial A \cap \overline{B})$

**Proof** =

...

□

**BoundaryComplement** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \partial(X \setminus A) = \partial A$

**Proof** =

...

□

**BoundaryDecomposition** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . X = (\text{int } A) \cup \partial A \cup \left( \text{int } A^c \right)$

**Proof** =

...

□

**ClosureBoundary** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \partial \overline{A} \subset \partial A$

**Proof** =

...  
□

**InteriorBoundary** ::  $\forall X : \text{TopologicalSpace} . \forall A \subset X . \partial \text{int } A \subset \partial A$

**Proof** =

...  
□

**BoundarySetUnionIsEqual** ::  $\forall X : \text{TopologicalSpace} . \forall A, B \subset X . \forall [0] : A \cap \overline{B} = \emptyset \ \& \ \overline{A} \cap B = \emptyset .$   
 $\partial(A \cup B) = \partial A \cup \partial B$

**Proof** =

[1] := **BoundaryOfUnion**(A, B) :  $\partial(A \cup B) \subset \partial A \cup \partial B$ ,

**Assume**  $x : \partial A$ ,

**Assume**  $U : \mathcal{U}(x)$ ,

[2] := **UnionSubset**(A, B)**IntersectionSubset**(U, A ∪ B, B)**BoundaryCondition**(A, x) :  
 $\emptyset \neq (U \cap A) \subset U \cap (A \cup B)$ ,

[3] :=  $\exists x \exists \partial A [0] : x \notin B$ ,

**Assume** [4] :  $(A \cup B) \cap U = U$ ,

[5] := [3][4] :  $x \in A$ ,

$(V, [6]) := [0][5]$ **EquivalentClosure** :  $\sum V \in \mathcal{U}(x) . V \cap B = \emptyset$ ,

$W := V \cap U : \mathcal{U}(x)$ ,

[7] := **IntersectionSubsect**[5] $jW : W \cap B = \emptyset$ ,

[8] := **BoundaryCondition**(x)(W) :  $W \cap A \neq W$ ,

[9] := [7][8] :  $W \cap (A \cup B) \neq W$ ,

[10] :=  $jW[9] : U \cap (A \cup B) \neq U$ ,

[11] := [10][3] :  $\perp$ ;

$\leadsto [3] := E(\perp) : (A \cup B) \cap U \neq Y$ ,

$[x.*] := \text{BoundaryCondition}[3][2] : x \in \partial(A \cup B)$ ;

$\leadsto [2] := \exists^{-1}$ **Subset** :  $\partial A \subset \partial(A \cup B)$ ,

[3] := **Symmetric**[2](A, B) :  $\partial B \subset \partial(A \cup B)$ ,

[4] := **UnionSubset**[2][3] :  $\partial A \cup \partial B \subset \partial(A \cup B)$ ,

[\*] :=  $\exists$ **SetEq**[2][3] :  $\partial A \cup \partial B = \partial(A \cup B)$ ;

□

**LocallyFiniteUnionBoundary** ::  $\forall X : \text{TopologicalSpace} . \forall A : \text{LocallyFinite}(X) . \partial \bigcup A \subset \bigcup \partial A$

**Proof** =

[\*] :=  $\exists \partial \bigcup A$ **LocallyFiniteYnionBoundary**(A)**DifferenceUnion**  $(\overline{A}, \text{int } \bigcup A)$  **UnionRule**(A)

**InteriorIsMonotonic**(int)**CoincreaingDifference** $\exists^{-1} \partial A :$

$\partial \bigcup A = \overline{\bigcup A} \setminus \text{int } \bigcup A = \bigcup \overline{A} \setminus \text{int } \bigcup A = \bigcup (\overline{A} \setminus \text{int } \bigcup A) \subset \bigcup \overline{A} \setminus \text{int } A = \bigcup \partial A$ ;

□

**ClosureOfIntersectWithOpenSet** ::  $\forall X : \text{TopologicalSpace} . \forall U \in \mathcal{T}(X) . \forall A \subset X . \overline{A \cap U} = \overline{\overline{A} \cap U}$

**Proof** =

[1] := **IntersectionSubset ClosureIsMonotonic**(X) :  $\overline{A \cap U} \subset \overline{\overline{A} \cap U}$ ,

**Assume**  $x : \overline{\overline{A} \cap U}$ ,

[2] := **EquivalentClosure1**(x) :  $\forall V \in \mathcal{U}(x) . V \cap \overline{A} \cap U \neq \emptyset$ ,

**Assume**  $V : \mathcal{U}(x)$ ,

**Assume** [3] :  $V \cap A \cap U = \emptyset$ ,

[4] := [3][2]**ClosureByBoundary** :  $(V \cap U) \cap \partial A \neq \emptyset$ ,

[5] := **BoundaryCondition**[4] :  $(V \cap U \cap A) \neq \emptyset$ ,

[V.\*] :=  $E(=)$ [3][5] $I(\perp) : \perp$ ;

$\leadsto$  [3] :=  $E(\perp)I(\rightarrow) : \forall V \in \mathcal{U}(x) . V \cap A \cap U \neq \emptyset$ ,

[x.\*] := **EquivalentClosure**[2] :  $x \in \overline{A \cap U}$ ;

$\leadsto$  [2] := **Subset** :  $\overline{\overline{A} \cap U} \subset \overline{A \cap U}$ ,

[\*] := **SetEq**[1][2] :  $\overline{\overline{A} \cap U} = \overline{A \cap U}$ ;

□

**InteriorOfUnionWithClosedSet** ::  $\forall X : \text{TopologicalSpace} . \forall C : \text{Closed}(X) . \forall A \subset X .$   
 $\quad . \text{int}(A \cup C) = \text{int} \left( (\text{int } A) \cap U \right)$

**Proof** =

...

□

### 1.1.5 Accumulation and Isolated Points

$\text{derivedSet} :: \prod X : \text{TopologicalSpace} . ?X \rightarrow \text{Closed}(X)$

$\text{derivedSet}(A) = A^d := \left\{ x \in X : x \in \overline{A \setminus \{x\}} \right\}$

$\text{IsolatedPoint} :: \forall X : \text{TopologicalSpace} . \forall A \subset X . ?A$

$x : \text{IsolatedPoint} \iff x \in (A \setminus A^d)$

$\text{IsolatedPointProperty} :: \forall X : \text{TopologicalSpace} . \forall A \subset X . \forall x \in X .$

$. x \in A^d \iff \forall U \in \mathcal{U}(x) . \exists y \in U \cap A . y \neq x$

$\text{Proof} =$

...

□

$\text{ClosureByDerivedSet} :: \forall X : \text{TopologicalSpace} . \forall A \subset X . \overline{A} = A \cup A^d$

$\text{Proof} =$

...

□

$\text{DerivedSetIsMonotonic} :: \forall X : \text{TopologicalSpace} . \forall A, B \subset X . A \subset B \iff A^d \subset B^d$

$\text{Proof} =$

...

□

$\text{DerivedFiniteUnion} :: \forall X : \text{TopologicalSpace} . \forall A, B \subset X . (A \cup B)^d = A^d \cup B^d$

$\text{Proof} =$

...

□

$\text{DerivedUnion} :: \forall X : \text{TopologicalSpace} . \forall I \in \text{SET} . \forall A : I \rightarrow ?X . \bigcup_{i \in I} A^d \subset \left( \bigcup_{i \in I} A \right)^d$

$\text{Proof} =$

...

□

### 1.1.6 Dense Sets

$\text{Dense} :: \prod X : \text{TopologicalSpace} . ??X$

$A : \text{Dense} \iff \overline{A} = X$

$\text{Codense} :: \prod X : \text{TopologicalSpace} . ??X$

$A : \text{Codense} \iff A^c : \text{Dense}(X)$

$\text{NowhereDense} :: \prod X : \text{TopologicalSpace} . ??X$

$A : \text{NowhereDense} \iff \overline{A} : \text{Codense}(X)$

$\text{DenseInItself} :: \prod X : \text{TopologicalSpace} . ??X$

$A : \text{DenseInItself} \iff A \subset A^d$

$\text{DenseByOpenSets} :: \forall X : \text{TopologicalSpace} . \forall A \subset X .$

$. A : \text{Dense}(X) \iff \forall x \in X . \forall U \subset \mathcal{U}(x) . U \cap A \neq \emptyset$

$\text{Proof} =$

$\text{DenseEquivalentClosure1} \square$

$\text{CodenseByOpenSets} :: \forall X : \text{TopologicalSpace} . \forall A \subset X .$

$. A : \text{Codense}(X) \iff \forall x \in X . \forall U \subset \mathcal{U}(x) . U \cap A^c \neq \emptyset$

$\text{Proof} =$

$\text{CodenseDenseByOpenSet} \square$

$\text{NowhereDenseByOpenSets} :: \forall X : \text{TopologicalSpace} . \forall A \subset X .$

$. A : \text{NowherDense}(X) \iff \forall x \in \mathcal{U}(x) . \forall U \in \mathcal{U}(x) . \exists V \in \mathcal{T}(X) : V \neq \emptyset \ \& \ V \cap A = \emptyset \ \& \ V \subset U$

$\text{Proof} =$

$\text{NowhereDenseDderivedSet} \square$

$\text{DenseClosure} :: \forall X : \text{TopologicalSpace} . \forall A : \text{Dense}(X) . \forall U \in \mathcal{T}(X) . \overline{U \cap A} = \overline{U}$

$\text{Proof} =$

$[1] := \text{SubsetIntersection} : U \cap A \subset U,$

$[2] := \text{ClosureIsMonotonic}[1] : \overline{U \cap A} \subset \overline{U},$

$\text{Assume } x : \overline{U},$

$[3] := \text{EquivalentClosure1}(U)(x) : \forall V \in \mathcal{U}(x) . V \cap U \neq \emptyset,$

$[4] := \text{DenseByOpenSets}(A)(x) : \forall V \in \mathcal{U}(x) . V \cap A \cap U \neq \emptyset,$

$[x.*] := \text{EquivalenitClosure1}(U \cap A)(x) : x \in U \cap A;$

$\leadsto [3] := \text{D}^{-1}\text{Subset} : \overline{U} \subset \overline{U \cap A},$

$[*] := \text{DSetEq}[2][3] : \overline{U} = \overline{U \cap A};$

$\square$

$\text{densityCardinal} :: \text{TopologicalSpace} \rightarrow \text{CARD}$

$\text{densityCardinal}(X) = d(X) := \min \text{Dense}(X)$



**Separable** :: ?TopologicalSpace

$X : \text{Separable} \iff d(X) < \aleph_0$

**DensityBound** ::  $\forall X : \text{TopologicalSpace} . d(X) \leq w(X)$

**Proof** =

$(\mathcal{B}, [1]) := \delta w(X) : \sum \mathcal{B} : \text{Base}(X) . |\mathcal{B}| = w(X),$

**Assume**  $B : \mathcal{B},$

**Assume**  $[2] : B \neq \emptyset,$

$q(B) := \delta \text{NonEmpty} : B;$

$\leadsto q := I \left( \prod \right) I \left( \sum \right) : \prod B \in \mathcal{B} . \prod B \neq \emptyset . B \neq . q(B) \in B,$

$Q := \text{Im } q : ?X,$

**Assume**  $x : X,$

**Assume**  $U : \mathcal{U}(X),$

$(I, B, [3]) := \delta \text{Base}(\mathcal{B})(U) : \sum I : \text{NonEmpty} . B : I \rightarrow \mathcal{B} . U = \bigcup_{i \in I} B_i,$

$[4] := \delta q[3] \text{UnionSubset} : \forall i \in I . q(B) \in U,$

$[*] := jQ[4] \delta \text{NonEmpty}(I) : Q \cap U \neq \emptyset;$

$\leadsto [3] := \text{DenseByOpenSets} : (U : \text{Dense}(X)),$

$[4] := \text{ImageCardinality} jQ : |Q| \leq |\mathcal{B}|,$

$[*] := [1][4] \delta \text{density} : d(X) \leq w(X);$

□

**SecondCountableIsSimmilar** ::  $\forall X : \text{SecondCountable} . X : \text{Separable}$

**Proof** =

...

□

**LocallyFiniteNowhereDense** ::  $\forall X : \text{TopologicalSpace} . \forall A : \text{LocallyFinite} \ \& \ \text{NowhereDense}(X) .$

$\bigcup A : \text{NowhereDense}(X)$

**Proof** =

**Assume**  $x : X,$

$(U, [1]) := \delta \text{LocallyFinite}(X)(A)(x) : \sum U \in \mathcal{U}(X) . \left| \{a \in A : a \cap U \neq \emptyset\} \right| < \infty,$

$\mathcal{A} := \{a \in A : a \cap U \neq \emptyset\} : \text{Finite}(A),$

**Assume**  $V : \mathcal{U}(x),$

$(W, [2]) := \text{NowhereDenseByOpenSets}(\mathcal{A}) : \sum W : \mathcal{A} \rightarrow \mathcal{U}(x) . \prod_{a \in \mathcal{A}} . W_a \subset U \cap V \neq \emptyset \ \& \ W_a \cap A = \emptyset,$

$O := \bigcap_{a \in \mathcal{A}} W_a : \mathcal{U}(x),$

$[x.*.1] := [2] jO : O \subset V,$

$[x.*.2] := j\mathcal{A}[2] jO : O \cap \bigcup A = \emptyset;$

$\leadsto [*] := \text{NowhereDenseByOpenSets} : \left( \bigcup A : \text{NowhereDense}(X) \right);$

□

**CodenseUnionWithNowhereDenceIsCodence** ::  $\forall X : \text{TopologicalSpace} . \forall A : \text{Codense}(X) . \forall B : \text{NowhereDense}(X) .$

**Proof** =

[1] :=  $\text{NowhereDense}(B) \text{UniversumIntersect}(\overline{B}^c) \text{Codense}(A)$   
 $\text{ClosureOfIntersectWithOpenSet}(X, A^c, \overline{B}^c) \text{ClosureIsNonotonic}(\text{cl}_X)$   
 $\text{ComplementIsComonotonic}(X, B, \overline{V}) \text{PotentialClosureOperator}(\text{cl}_X)(B)$   
 $\text{ClosureIsMonotonic}(\text{cl}_X) \text{DeMorganeLaw}(X, A, B) :$   
 $: X = \overline{\overline{B}^c} = \overline{X \cap \overline{B}^c} = \overline{A^c \cap \overline{B}^c} = \overline{A^c \cap \overline{B}^c} \subset \overline{A^c \cap \overline{B}^c} = \overline{(A \cup B)^c},$   
[\*] :=  $\text{NowhereDense}^{-1} \text{Codense}[1] : (A \cup B : \text{Codense}(X));$   
□

**OpenDenseInItself** ::  $\forall X : \text{TopologicalSpace} . \forall U \in \mathcal{T}(X) .$

$. X : \text{DenseInItself}(X) \Rightarrow U : \text{DenseInItself}(X)$

**Proof** =

**Assume**  $u : U,$   
[1] :=  $\text{DenseInItself}(X)(X)(u) : u \in X^d,$   
[2] :=  $\text{derivedSet}[1] : u \in \overline{X \setminus \{u\}},$   
[3] :=  $\text{EquivalentClosure}[2] : \forall V \in \mathcal{U}(u) . V \cap X \setminus \{u\} \neq \emptyset,$   
**Assume**  $V : \mathcal{U}(u),$   
[V.\*] :=  $[3](V \cap U) \text{IntersectionDifference}(X) \text{UniversumIntersection}(X) :$   
 $: V \cap (U \setminus \{u\}) = V \cap (U \setminus \{u\}) \cap X = V \cap U \cap (X \setminus \{u\}) \neq \emptyset;$   
 $\leadsto [4] := \text{EquivalentClosure} : u \in \overline{U \setminus \{u\}},$   
 $[u.*] := \text{derivedSet}^{-1}[4] : u \in U^d;$   
 $\leadsto [1] := \text{Subset}^{-1} : U \subset U^d,$   
[\*] :=  $\text{DenseInItself}^{-1}[1] : (U : \text{DenseInItself});$   
□

**ClosureOfDenseInItself** ::  $\forall X : \text{TOP} . \forall A : \text{DenseInItself}(X) . \overline{A} : \text{DenseInItself}(X)$

**Proof** =

**Assume**  $x : \overline{A},$   
**Assume** [1] :  $x \in A,$   
[2] :=  $\text{DenseInItself}(X)(A)[1] : x \in A^d,$   
[3] :=  $\text{derivedSet}(A)[2] \text{MonotonicClosure} : x \in \overline{A \setminus \{x\}} \subset \overline{\overline{A \setminus \{x\}}},$   
[1.\*] :=  $\text{derivedSet}^{-1}[3] : x \in \overline{A}^d;$   
 $\leadsto [1] := I(\Rightarrow) : x \in A \Rightarrow \overline{A}^d,$   
**Assume** [2] :  $x \notin A,$   
[3] :=  $\text{MonotonicClosure}[2] : x \in \overline{A} \subset \overline{\overline{A \setminus \{x\}}},$   
[4] :=  $\text{DerivedSet}^{-1} : x \in \overline{A}^d;$   
 $\leadsto [2] := I(\Rightarrow) : x \notin A \Rightarrow x \in \overline{A}^d,$   
 $[x.*] := E(|) \text{InOrNotIn}(x)[1][2] : x \in \overline{A}^d;$   
 $\leadsto [1] := \text{Subset}^{-1} : \overline{A},$   
[\*] :=  $\text{DenseInItself}^{-1}[1] : (\overline{A} : \text{DenseInItself});$   
□

### 1.1.7 Separation Axioms

**T0** :: ?TOP

$$X : \mathbf{T0} \iff \forall a, b \in X . \exists U \in \mathcal{T}(X) . \left| U \cap \{a, b\} \right| = 1$$

**T0CardinalityBound** ::  $\forall X : \mathbf{T0} . |X| \leq \exp w(X)$

**Proof** =

$$(\mathcal{B}, [1]) := \mathfrak{d}\mathbf{weight} : \sum \mathcal{B} : \mathbf{Base}(X) . w(X) = |\mathcal{B}|,$$

$$\mathcal{A} := \Lambda x \in X . \mathcal{U}(X) \cap \mathcal{B} : X \rightarrow ?\mathcal{B},$$

**Assume**  $x, y : X$ ,

**Assume** [2] :  $(x \neq y)$ ,

$$(U, [3]) := \mathfrak{d}\mathbf{T0}(X)(x, y)[2] : \sum U \in \mathcal{T}(X) . \left| U \cap \{x, y\} \right| = 1,$$

$$[4] := \mathfrak{d}\mathcal{U}[3] : \left( \exists V \in \mathcal{U}(x) : y \notin V \mid \exists V \in \mathcal{U}(y) : x \notin V \right),$$

$$[5] := \mathfrak{d}\mathbf{Base}(X)(\mathcal{B})j^{-1}\mathcal{A}[4] : \left( \exists V \in \mathcal{A}(x) : y \notin V \mid \exists V \in \mathcal{A}(y) : x \notin V \right),$$

$$[*] := j\mathcal{A}[5] : \mathcal{A}(x) \neq \mathcal{A}(y);$$

$$\leadsto [2] := \mathfrak{d}^{-1}\mathbf{Injection} : \mathcal{A} : X \hookrightarrow ?\mathcal{B},$$

$$[*] := \mathbf{CardinalityInjectionBound}[2] : |X| \leq \exp w(X);$$

□

**T1** :: ?TOP

$$X : \mathbf{T1} \iff \forall a, b \in X . \exists U \in \mathcal{T}(a) . b \notin U$$

**T1Singelton** ::  $\forall X : \mathbf{T1} . \forall x \in X . \{x\} \in G_\delta(X)$

**Proof** =

...

□

**T1BySingeltons** ::  $\forall X \in \mathbf{TOP} . X : \mathbf{T1} \iff \forall x \in X . \{x\} : \mathbf{Closed}(X)$

**Proof** =

...

□

**SeparationHierarchy1** ::  $\mathbf{T0} \subsetneq \mathbf{T1}$

**Proof** =

...

□

**Separated** ::  $\prod X \in \mathbf{TOP} . ?X \times ?X$

$$(A, B) : \mathbf{Separated} \iff \exists U, V \in \mathcal{T}(X) . A \subset U \ \& \ B \subset V \ \& \ U \cap V = \emptyset$$

**T2** :: ?TOP

$$x : \mathbf{T2} \iff x : \mathbf{Hausdorff} \iff \forall x, y \in X . x \neq y \Rightarrow \exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : U \cap V = \emptyset$$

**SeparationHierarchy2** ::  $\mathbf{T1} \subsetneq \mathbf{T2}$

**Proof** =

...

□

**T2BySingletons** ::  $\forall X \in \mathbf{TOP} . X : \mathbf{T2} \iff \forall x \in X . \{x\} = \bigcap_{U \in \mathcal{U}(x)} \overline{U}$

**Proof** =

...

□

**T2CardinalityBound1** ::  $\forall X : \mathbf{T2} . |X| \leq \exp \exp d(X)$

**Proof** =

$(D, [1]) := \mathfrak{d}d(X) : \sum D : \mathbf{Dense}(X) . |D| = d(X),$

$\mathcal{A} := \Lambda x \in X . \{U \cap D \mid U \in \mathcal{U}(x)\} : X \rightarrow ??D,$

$[2] := \mathfrak{d}\mathbf{Dense}(X)(D)_j \mathbf{AT2BySingletons}(X) : \forall x \in X . \bigcap_{A \in \mathcal{A}(x)} \overline{A} = \{x\},$

$[3] := \mathbf{InjectiveByMapping}[2] : (\mathcal{A} : X \hookrightarrow ??D),$

$[*] := \mathbf{CardinalityByInjectionBound}[2][3] : |X| \leq \exp \exp d(X);$

□

**T2CardinalityBound2** ::  $\forall X : \mathbf{T2} . |X| \leq \left(\chi(X)\right)^{d(X)}$

**Proof** =

...

□

**ClosedEqualityInT2Space** ::  $\forall X \in \mathbf{TOP} . \forall Y : \mathbf{T2} . \forall f, g : X \xrightarrow{\mathbf{TOP}} Y . \left\{x \in X : f(x) = g(x)\right\} : \mathbf{Closed}(X)$

**Proof** =

**Assume**  $x : X,$

**Assume**  $[1] : f(x) \neq g(x),$

$(U, V, [2]) := \mathfrak{d}\mathbf{T2}(f(x), g(x))[1] : \sum U \in \mathcal{U}(f(x)) . \sum V \in \mathcal{U}(g(x)) . U \cap V = \emptyset,$

$W_x := f^{-1}(U) \cap g^{-1}(V) : \mathcal{U}(x),$

$[x.*] := [2]\mathfrak{d}\mathbf{preimage}_j W_x : \forall w \in W_x . f(w) \neq g(w);$

$\leadsto W := I\left(\prod\right) : \prod_{x \in X} f(x) \neq g(x) \rightarrow \sum U \in \mathcal{U}(x) . \forall u \in U . f(u) \neq g(u),$

$[1] := \mathfrak{d}W : \left\{x \in X : f(x) = g(x)\right\}^c = \bigcup W,$

$[*] := \mathfrak{d}^{-1}\mathbf{Closed}[1] : \left(\left\{x \in X : f(x) = g(x)\right\}^c : \mathbf{Closed}(X)\right);$

□

$\text{setNeighborhood} :: \prod_{X \in \text{TOP}} . ?X \rightarrow ?\mathcal{T}(X)$

$\text{setNeighborhood}(A) = \mathcal{U}(A) := \left\{ U \in \mathcal{T}(X) : A \subset U \right\}$

$\text{T3} :: ?\text{T1}$

$X : \text{T3} \iff X : \text{Regular} \iff \forall A : \text{Closed}(X) . \forall x \in X . \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(x) : V \cap U$

$\text{RegularityCriterion} :: \forall X \in \mathcal{T}1 . X : \text{T3} \iff \forall x \in X . \forall V \in \mathcal{U}(x) . \exists U \in \mathcal{U}(x) : \overline{U} \subset V$

$\text{Proof} =$

$\text{Assume } [1] : (X : \text{T3}),$

$\text{Assume } x : X,$

$\text{Assume } V : \mathcal{U}(x),$

$(U, W, [2]) := \delta \text{T3}(x, V^c) : \sum U \in \mathcal{U}(x) . \sum W \in \mathcal{U}(V^c) . W \cap U = \emptyset,$

$[1.*] := \delta \text{closure}(X)[2] \delta \mathcal{U}(V^c) \text{ComplementSubset} : \overline{U} \subset W^c \subset V;$

$\leadsto [1] := I(\Rightarrow) : \text{Left} \Rightarrow \text{Right},$

$\text{Assume } [2] : \forall x \in X . \forall V \in \mathcal{U}(x) . \exists U \in \mathcal{U}(x) : \overline{U} \subset V,$

$\text{Assume } A : \text{Closed}(X),$

$\text{Assume } x : A^c,$

$(U, [3]) := [2](A^c) : \sum U \in \mathcal{U}(x) . \overline{U} \subset A^c,$

$V := \overline{U}^c : \mathcal{U}(A),$

$[A.*] := j(V) : V \cap U = \emptyset;$

$\leadsto [4] := \delta^{-1} \text{T3} : (X : \text{T3});$

□

$\text{SeparationHierarchy3} :: \text{T2} \subsetneq \text{T3}$

$\text{Proof} =$

...

□

$\text{T3WeightBound} :: \forall X : \text{T3} . w(X) \leq \exp d(X)$

$\text{Proof} =$

...

□

$\text{T4} :: ?\text{T1}$

$X : \text{T4} \iff X : \text{Normal} \iff \forall A, B : \text{Closed}(X) . A \cap B = \emptyset \Rightarrow \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

$\text{SeparationHierarchy4} :: \text{T3} \subsetneq \text{T4}$

$\text{Proof} =$

...

□

**T4ByOpenCover** ::  $\forall X : \mathbf{T1} . \forall [0] : \forall A : \mathbf{Closed}(X) . \forall U \in \mathcal{U}(A) .$

$: \exists W : \mathbb{N} \rightarrow \mathcal{T}(X) : A \subset \bigcup_{i \in I} W_i \ \& \ \forall i \in \mathbb{N} . \overline{W_i} \subset U . X : \mathbf{T4}$

**Proof** =

**Assume**  $A, B : \mathbf{Closed}(X),$

**Assume**  $[1] : A \cap B = \emptyset,$

$(W, [2]) := [0](A, B^c) : \sum_{W : \mathbb{N} \rightarrow \mathcal{T}(X)} A \subset \bigcup_{i \in I} W_i \ \& \ \forall i \in \mathbb{N} . \overline{W_i} \subset B^c,$

$(V, [3]) := [0](B, A^c) : \sum_{V : \mathbb{N} \rightarrow \mathcal{T}(X)} B \subset \bigcup_{i \in I} V_i \ \& \ \forall j \in \mathbb{N} . \overline{V_i} \subset A^c,$

$H := \Lambda i \in \mathbb{N} . W_i \setminus \bigcup_{j=1}^i \overline{V_j} : \mathbb{N} \rightarrow \mathcal{T}(X),$

$G := \Lambda i \in \mathbb{N} . V_i \setminus \bigcup_{j=1}^i \overline{W_j} : \mathbb{N} \rightarrow \mathcal{T}(X),$

$[4] := jH[3] : A \subset \bigcup_{n=1}^{\infty} H_n,$

$[5] := jG[2] : B \subset \bigcup_{n=1}^{\infty} G_n,$

$O := \bigcup_{n=1}^{\infty} H_n : \mathcal{U}(A),$

$Q := \bigcup_{n=1}^{\infty} G_n : \mathcal{U}(B),$

$[6] := jGjH : \forall i \in \mathbb{N} . \forall j \in i . H_i \cap G_j = \emptyset,$

$[7] := jHjG : \forall i \in \mathbb{N} . \forall j \in i . G_i \cap H_j = \emptyset,$

$[8] := [6][7] : \forall i, j \in \mathbb{N} . H_i \cap G_j = \emptyset,$

$\left[ (A, B) . * \right] := jOjQ[8] : O \cap Q = \emptyset;$

$\leadsto [*] := \mathfrak{d}^{-1} \mathbf{T4} : (X : \mathbf{T4}),$

□

**SecondCountableRegularIsNormal** ::  $\forall X : \mathbf{T3} \ \& \ \mathbf{SecondCountable} . X : \mathbf{T4}$

**Proof** =

...

□

**CountableRegularIsNormal** ::  $\forall X : \mathbf{T3} . |X| \leq \aleph \Rightarrow X : \mathbf{T4}$

**Proof** =

...

□

$$\text{Cover} :: \prod_{X \in \text{TOP}} ???X$$

$$\mathcal{A} : \text{Cover} \iff \bigcup \mathcal{A} = X$$

$$\text{OpenCover} :: \prod_{X \in \text{TOP}} ??\mathcal{T}(X)$$

$$\mathcal{O} : \text{OpenCover} \iff \bigcup \mathcal{O} = X$$

$$\text{PointFiniteCover} :: \prod_{X \in \text{TOP}} ?\text{OpenCover}(X)$$

$$\mathcal{O} : \text{PointFiniteCover} \iff \forall x \in X . \left| \{O \in \mathcal{O} : x \in O\} \right| < \infty$$

$$\text{NormalPointFiniteCoverRefinement} :: \forall X : \text{T4} . \forall \mathcal{O} : \text{PointFiniteCover}(X) \exists \mathcal{V} : \text{OpenCover}(X) : \\ . \exists V : \mathcal{O} \leftrightarrow \mathcal{V} : \forall O \in \mathcal{O} . \overline{V_O} \subset O$$

$$\text{Proof} =$$

$$\mathcal{G} := \left\{ V : \mathcal{O} \rightarrow \mathcal{T}(X) : \forall O \in \mathcal{O} . V_O = O | \overline{V_O} \subset O \ \& \ \bigcup_{O \in \mathcal{O}} V_O = X \right\} : ?(\mathcal{O} \rightarrow \mathcal{T}(X)),$$

$$[1] := j\mathcal{G} : \mathcal{O} \in \mathcal{G},$$

$$[2] := \text{NonEmpty}[1] : \mathcal{G} \neq \emptyset,$$

$$\text{Assume } G : \mathcal{G},$$

$$\text{Assume } O : \mathcal{O},$$

$$\text{Assume } [3] : G_O = O,$$

$$U := \bigcup_{V \in \mathcal{O} : V \neq O} O : \mathcal{T}(X),$$

$$A := U^c : \text{Closed}(X),$$

$$(W, [4]) := \text{NormalCriterion}(A, O) : \sum W \in \mathcal{T}(X) . A \subset \overline{W} \subset O,$$

$$[5] := [4] \text{ReplaceValue} : \widehat{G}_O(W) \in \mathcal{G},$$

$$[G.*] := \text{d} \leq_{\mathcal{G}} [4][5] : \widehat{G}_O(W) \leq G;$$

$$\sim [4] := I(\forall) : \forall G \in \mathcal{G} . \forall O \in \mathcal{O} . G_O \Rightarrow \exists G' \in \mathcal{G} : G' \leq G,$$

$$\text{Assume } \mathcal{G}' : \text{Chain}(\mathcal{G}'),$$

$$G := \bigwedge O \in \mathcal{O} . \bigcap_{G' \in \mathcal{G}'} G'_O : \mathcal{O} \rightarrow ?X,$$

$$[5] := j\mathcal{G} : \forall O \in \mathcal{O} . G_O = O | \overline{G_O} \subset O,$$

$$\text{Assume } x : X,$$

$$\mathcal{O}' := \{O \in \mathcal{O} : x \in O\} : \text{Finite}(\mathcal{O}),$$

$$(O, [6]) := \text{PigionholePrinciple} \text{dChain}(\mathcal{G}') j\mathcal{O}' : \sum O \in \mathcal{O}' . x \in \bigcap_{G' \in \mathcal{G}'} G'_O,$$

$$[x.*] := [6] \text{UnionSubset} : x \in \bigcup_{O \in \mathcal{O}} G_O;$$

$$\sim [6] := I \text{Subset} : X = \bigcup_{O \in \mathcal{O}} G_O,$$

$$[7] := \text{LocallyFiniteClosedUnion} j : \forall O \in \mathcal{O} . G_O \in \mathcal{T}(X),$$

$$[8] := j\mathcal{G}[7][6][5] : G \in \mathcal{G},$$

$[\mathcal{G}'.*] := jG : \forall G' \in \mathcal{G}' . G \leq G';$   
 $\leadsto (G, [5]) := \text{ZornLemma}[2] : \sum G \in \mathcal{G} . G = \min \mathcal{G},$   
 $[*] := [4][5] : \forall O \in \mathcal{O} . \overline{G_O} \subset O;$   
 $\square$

**T1Invariance** ::  $\forall X : \text{T1} . \forall Y \in \text{TOP} . \forall f : \text{Closed}(X, Y) . f(X) : \text{T1}$

**Proof** =

...

$\square$

**T4Invariance** ::  $\forall X : \text{T4} . \forall Y \in \text{TOP} . \forall f : \text{Closed}(X, Y) . f(X) : \text{T4}$

**Proof** =

...

$\square$

**T0BySingletonClosures** ::  $\forall X \in \text{TOP} . X : \text{T0} \iff \forall x, y \in X . x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$

**Proof** =

...

$\square$

**T1DoubleDerivedSet** ::  $\forall X : \text{T1} . \forall A \subset X . A^{\text{dd}} \subset A^{\text{d}}$

**Proof** =

**Assume**  $x : A^{\text{dd}},$

$[1] := \text{derivedSet}(x, A^{\text{dd}}) : x \in \overline{A^{\text{d}} \setminus \{x\}},$

$[2] := \text{ClosureEquivalent}[1] : \forall U \in \mathcal{U}(x) . U \cap (A^{\text{d}} \setminus \{x\}) \neq \emptyset,$

$[3] := \text{derivedSet}[2] : \forall U \in \mathcal{U}(x) . \exists y \in U : y \in \overline{A \setminus \{y\}} \setminus \{x\},$

$[4] := \text{ClosureEquivalent}[3] : \forall U \in \mathcal{U}(x) . \exists y \in U : \forall V \in \mathcal{U}(y) . V \cap A \setminus \{y\} \neq \emptyset \ \& \ y \neq x \neq \emptyset,$

**Assume**  $U : \mathcal{U}(x),$

$(y, [5]) := [4](U) : \sum y \in U . \forall V \in \mathcal{U}(y) . V \cap A \setminus \{y\} \neq \emptyset \ \& \ y \neq x,$

$[6] := \text{derivedSet}(X)[5] : \forall V \in U(y) . V \cap A \setminus \{y\} \setminus \{x\} \neq \emptyset,$

$[7] := [6](U) : U \cap A \setminus \{y\} \setminus \{x\} \neq \emptyset,$

$[U.*] := \text{DecreasingSetminus}(A, \{y, x\}, \{x\})\text{MonotonicIntersect}[7] : U \cap (A \setminus \{x\}) \neq \emptyset;$

$\leadsto [5] := \text{ClosureEquivalent} : x \in \overline{A \setminus \{x\}},$

$[x.*] := \text{derivedSet}^{-1} : x \in A^{\text{d}};$

$\leadsto [*] := \text{Subset}^{-1} : A^{\text{dd}} \subset A^{\text{d}};$

$\square$



**T1DerivedSetIsClosed** ::  $\forall X : \mathbf{T1} . \forall A \subset X . A^d : \mathbf{Closed}(A)$

**Proof** =

**Assume**  $x : A^{dc}$ ,

**Assume** [1] :  $\forall U \in \mathcal{U}(x) . U \cap A^d \neq \emptyset$ ,

[2] := **derivedSet**[1] :  $\forall U \in \mathcal{U}(x) . \exists y \in U : y \in \overline{A \setminus \{x\}}$ ,

[3] := **ByAnalogy**(**proof T1DoubleDerivedSet**)[2] :  $x \in A^d$ ,

[4] := **InAndNotIn**[3] :  $\perp$ ;

$\leadsto$  [1] := **OpenByInnerCover** :  $A^{dc} \in \mathcal{T}(X)$ ,

[\*] :=  $\mathfrak{D}^{-1}\mathbf{Closed} : A^d \in \mathcal{T}(X)$ ;

□

**T1DerivedSetClosure** ::  $\forall X : \mathbf{T1} . \forall A \subset X . \overline{A}^d = A^d$

**Proof** =

[1] := **potentialClosure**<sub>X</sub>(A) :  $A \subset \overline{A}$ ,

[2] := **DerivedSetIsMonotonic**[1] :  $A^d \subset \overline{A}^d$ ,

**Assume**  $x : \overline{A}^d$ ,

[1] := **derivedSet** :  $x \in \overline{A \setminus \{x\}}$ ,

[2] := **EquivalentClosure**[1] :  $\forall U \in \mathcal{U}(x) . U \cap \overline{A \setminus \{x\}} \neq \emptyset$ ,

[3] := **ClosureEquivalent**[2] :  $\forall U \in \mathcal{U}(x) . \exists y \in U : \forall V \in \mathcal{U}(y) . V \cap A \neq \emptyset \ \& \ y \neq x \neq \emptyset$ ,

**Assume**  $U : \mathcal{U}(x)$ ,

( $y, [4]$ ) := [3](U) :  $\sum y \in U . \forall V \in \mathcal{U}(y) . V \cap A \neq \emptyset \ \& \ y \neq x$ ,

[5] := [4](U) :  $U \cap A \neq \emptyset$ ,

[U.\*] := **T1**[5][4] :  $U \cap A \setminus \{x\} \neq \emptyset$ ;

$\leadsto$  [4] := **EquivalentClosure**[3] :  $x \in \overline{A \setminus \{x\}}$ ,

[x.\*] :=  $\mathfrak{D}^{-1}A^d : x \in A^d$ ;

$\leadsto$  [3] :=  $\mathfrak{D}^{-1}\mathbf{Subset} : \overline{A}^d \subset A^d$ ,

[\*] :=  $\mathfrak{D}^{-1}\mathbf{SetEq}$ [2][3] :  $\overline{A}^d = A^d$ ;

□

**T1FiniteDerivedSet** ::  $\forall X : \mathbf{T1} . \forall A : \mathbf{Finite}(X) . A^d = \emptyset$

**Proof** =

...

□

**Retraction** ::  $\prod X : \mathbf{TopologicalSpace} . \mathbf{?End}_{\mathbf{TOP}}(X) .$

$f : \mathbf{Retractrion} \iff f^2 = f$

**Retract** ::  $\prod X : \mathbf{TopologicalSpace} . \mathbf{??}X$

$R : \mathbf{Retract} \iff \exists f : \mathbf{Retraction}(X) . f(X) = R$

**HausdorffRetractIsClosed** ::  $\forall X : \mathbf{T2} . \forall R : \mathbf{Retract}(X) . R : \mathbf{Closed}(X)$

**Proof** =

$(f, [1]) := \mathfrak{d}\mathbf{Retract} : \sum f : \mathbf{Retraction}(X) . f(X) = R,$

$[2] := \mathfrak{d}\mathbf{Retraction}(f)[1] : R = \{x \in X : f(x) = x\},$

$[*] := \mathbf{ClosedEqualityInT2Space} : R : \mathbf{Closed}(X);$

□

**NormalIteration** ::  $\forall X : \mathbf{T4} . \forall I : \mathbf{Finite} . \forall A : \mathbf{Disjoint}(I, \mathbf{Closed}(X)) . \exists U : \mathbf{Disjoint}(I, \mathcal{U}(A))$

**Proof** =

...

□

**HausdorffIteration** ::  $\forall X : \mathbf{T4} . \forall I : \mathbf{Finite} . \forall A : \mathbf{Disjoint}(I, \mathbf{Finite}(X)) . \exists U : \mathbf{Disjoint}(I, \mathcal{U}(A))$

**Proof** =

...

□

## 1.2 Convergence

### 1.2.1 Convergence in Nets

$$\mathbf{Net} := \prod D : \mathbf{DirectedSet} . \prod X : \mathbf{TOP} . D \rightarrow X : \mathbf{DirectedSet} \rightarrow \mathbf{TOP} \rightarrow \mathbf{SET};$$

$$\mathbf{Limit} :: \prod X : \mathbf{TOP} . \prod D : \mathbf{DirectedSet} . \mathbf{Net}(D, X) \rightarrow ?X$$

$$L : \mathbf{Limit} \iff x \mapsto L = \lim_{n \in D} x_n \iff x \mapsto \forall U \in \mathcal{U}(L) . \exists N \in D : \forall n : \mathbf{NotLessThan}(N) . x_n \in U$$

$$\mathbf{Cluster} :: \prod X : \mathbf{TOP} . \prod D : \mathbf{DirectedSet} . \mathbf{Net}(D, X) \rightarrow ?X$$

$$C : \mathbf{Cluster} \iff x \mapsto C \in \bar{x} \iff x \mapsto \forall U \in \mathcal{U}(L) . \forall N \in D : \exists n : \mathbf{NotLessThan}(N) . x_n \in U$$

$$\mathbf{Finer} :: \prod X : \mathbf{TOP} . \prod D, D' : \mathbf{DirectedSet} . ?(\mathbf{Net}(X, D) \& \mathbf{Net}(X, D'))$$

$$x, y : \mathbf{Finer} \iff x \rightarrow y \iff \exists \phi : D \rightarrow D' : \left( \forall N' \in D' . \exists N \in D : \forall n : \mathbf{NotLessThan}(N) . \right. \\ \left. . \phi(n) : \mathbf{NotLessThan} N' \right) \& \\ \& \forall n \in D . x_n = y_{\phi(n)}$$

$$\mathbf{ClusterOfFiner} :: \forall X : \mathbf{TOP} . \forall D, D' : \mathbf{DirectedSet}(X) . \forall x \xrightarrow{D, D'} y . \forall C = \bar{x} . C = \bar{y}$$

**Proof** =

$$\left( \phi, [1] \right) := \mathfrak{d}\mathbf{Finer}(x, y) : \sum \phi : D \rightarrow D' : \left( \forall N' \in D' . \exists N \in D : \forall n : \mathbf{NotLessThan}(N) . \right. \\ \left. . \phi(n) : \mathbf{NotLessThan} N' \right) \& \forall n \in D . x_n = y_{\phi(n)},$$

**Assume**  $U : \mathcal{U}(C)$ ,

$$[2] := \mathfrak{d}\mathbf{Cluster}(x)(C) : \forall N \in D . \exists n : \mathbf{NotLessThan}(N) . x_n \in U,$$

**Assume**  $N' : D'$ ,

$$\left( N, [3] \right) := [1](N') : \sum N \in D . \left( \forall n : \mathbf{NotLessThan}(N) . \phi(n) \geq N' \right),$$

$$\left( n, [4] \right) := [2](N) : \sum n \in D . n \geq N \& x_n \in U,$$

$$[5] := [3](n) : \phi(n) \geq N',$$

$$[6] := [1](n) : y_{\phi(n)} = x_n,$$

$$[N.*] := [4][6] : y_{\phi(n)} \in U;$$

$$\rightsquigarrow [U.*] := I(\forall) : \forall N' \in D' . \exists n' \geq N' . y_{n'} \in U;$$

$$\rightsquigarrow [*] := \mathfrak{d}^{-1}\mathbf{Cluster} : C = \bar{y};$$

□

**LimitOfMeager** ::  $\forall X : \text{TOP} . \forall D, D' : \text{DirectedSet}(X) . \forall x \xrightarrow{D, D'} y . \forall L = \lim_{n \in D} x_n . L = \lim_{n \in D'} y_n$

**Proof** =

$(\phi, [1]) := \mathfrak{d}\text{Finer}(x, y) : \sum \phi : D \rightarrow D' : \left( \forall N' \in D' . \exists N \in D : \forall n : \text{NotLessThan}(N) . \right.$   
 $\left. \phi(n) : \text{NotLessThan} N' \right) \ \& \ \forall n \in D . x_n = y_{\phi(n)},$

**Assume**  $U : \mathcal{U}(L),$

$(N', [2]) := \mathfrak{d}\text{Limit}(y)(L) : \sum N' \in D' . \forall n' \geq N' . y_{n'} \in U,$

$(N, [3]) := [1](N') : \sum N \in D . \left( \forall n \geq N . \phi(n) \geq N', \right.$

**Assume**  $n : D,$

**Assume**  $[4] : n \geq N,$

$[5] := [3][4] : \phi(n) \geq N',$

$[6] := [1](n) : y_{\phi(n)} = x_n,$

$[7] := [2][5] : y_{\phi(n)} \in U,$

$[n.*] := [6][7] : x_n \in U;$

$\leadsto [U.*] := I(\forall) : \forall n \geq N . x_n \in U;$

$\leadsto [*] := \mathfrak{d}^{-1}\text{Limit} : L = \lim_{n \in D} x_n;$

□

**FromClusterToLimit** ::  $\forall X : \text{TOP} . \forall D : \text{DirectedSet} . \forall x : \text{Net}(X, D) . \forall C = \bar{x} .$   
 $\left. \exists D' : \text{DirectedSet} : \exists y : \text{Net}(X, D) : C = \lim_{n \in D} y_n \ \& \ y \rightarrow x \right.$

**Proof** =

$D' := \left\{ (n, U) \in D \times \mathcal{U}_{\geq}(C) : x_n \in U \right\} : \text{PartiallyOrderedSet},$

**Assume**  $(n, U), (m, V) : D',$

$W := U \cap V : \mathcal{U}(C),$

$(k, [1]) := \mathfrak{d}\text{Cluster}(x)(C) \left( W, \max(n, m) \right) : \sum k \in D . k \geq \max \ \& \ x_k \in W,$

$[\dots *] := jD' jW \text{IntersectionSubset}(U, V) jk[1] : (k, W) \geq (n, U) \ \& \ (k, W) \geq (m, V);$

$\leadsto [1] := \mathfrak{d}^{-1}\text{DirectedSet} : \left( D' : \text{DirectedSet} \right),$

$\phi := \Lambda(n, U) \in D' . n : D' \rightarrow D,$

$y := \Lambda(n, U) \in D' . x_n : \text{Net}(D', X),$

$[2] := jyj\phi : y \rightarrow x,$

**Assume**  $U : \mathcal{U}(C),$

$N := \mathfrak{d}\text{NonEmpty}(D) : D,$

$(N', [3]) := \mathfrak{d}\text{Cluster}(x)(C)(U, N) : \sum N' \in D . N' \geq N \ \& \ x_{N'} \in U,$

**Assume**  $(n, V) : D',$

**Assume**  $[4] : (n, V) \geq (N', U),$

$[5] := jD'[4] : V \subset U,$

$[U.*] := jD' \text{SubsetTransitivity}[5] jy_{(n, V)} : y_{(n, V)} \in U;$

$\leadsto [*] := \mathfrak{d}^{-1}\text{Limit} : \lim_{n \in D'} y = C;$

□

**ClosureByConvergence** ::  $\forall X \in \mathbf{TOP} . \forall A \subset X . \forall p \in \overline{A} . \exists x : \mathbf{Net}(D, X) : p = \lim_{n \in D} x_n \ \& \ \forall n \in D . x_n \in A$

**Proof** =

$D := \mathcal{U}(p) : \mathbf{DirectedSet},$

**Assume**  $U : D,$

$[1] := \mathbf{ClosureEquivalent}(A)(x)(U) : U \cap A \neq \emptyset,$

$x_U := \mathbf{NonEmpty} : U \cap A;$

$\leadsto x := I(\rightarrow) : \mathbf{Net}(D, X),$

$[1] := jx \mathbf{IntersectSubset} \mathfrak{d} \mathbf{Subset} : \forall n \in D . x_n \in A,$

**Assume**  $U : \mathcal{U}(p),$

$N := U : D,$

**Assume**  $n : D,$

**Assume**  $[2] : n \geq N,$

$[3] := jD[2] : n \in N,$

$[U.*] := jx \mathfrak{d} \mathbf{Subset}[2] \mathbf{IntersectSubset} \mathfrak{d} \mathbf{Subset} jN : x_n \in U;$

$\leadsto [*] := \mathfrak{d}^{-1} \mathbf{Limit} : p = \lim_{n \in D} x_n;$

□

**ClosedByLimits** ::  $\forall X \in \mathbf{TOP} . \forall A \subset X . A : \mathbf{Closed}(X) \iff$

$\iff \forall x : \mathbf{Net}(D, X) . x(D) \subset A \Rightarrow \forall L = \lim_{n \in D} x_n . L \in A$

**Proof** =

...

□

**DerivedSetByConvergence** ::  $\forall X \in \mathbf{TOP} . \forall A \subset X . \forall p \in A^d . \exists x : \mathbf{Net}(D, X) : p = \lim_{n \in D} x_n \ \&$

$\ \& \ \forall n \in D . x_n \in A \ \& \ x_n \neq p$

**Proof** =

...

□

**ClusterAsIntersect** ::  $\forall X : \mathbf{TOP} . \forall x : \mathbf{Net}(D, X) . \bar{x} = \bigcap_{N \in D} \overline{\{x_n | n \geq N\}}$

**Proof** =

...

□

**HausdorffByLimits** ::  $\forall X : \mathbf{TOP} . X : \mathbf{T2} \iff \forall x : \mathbf{Net}(D, X) . |\lim_{n \in D} x_n| \leq 1$

**Proof** =

...

□

### 1.2.2 Filters

$\text{IntersectionClosed} :: \prod X : \text{SET} . ???X$

$A : \text{IntersectionClosed} \iff \forall A, B \in \mathcal{A} . A \cap B \in \mathcal{A}$

$\text{Filter} :: \prod X : \text{SET} . \prod \mathcal{A} : \text{IntersectionClosed}(X) . ??\mathcal{A}$

$\mathcal{F} : \text{Filter} \iff \mathcal{F} \neq \emptyset \ \& \ \emptyset \notin \mathcal{F} \ \& \ \forall A, B \in \mathcal{F} . A \cap B \in \mathcal{F} \ \& \ \forall A \in \mathcal{F} . \forall B \in \mathcal{A} . A \subset B \Rightarrow B \in \mathcal{F}$

$\text{Ultrafilter} :: \prod X : \text{SET} . \prod \mathcal{A} : \text{IntersectionClosed}(X) . ?\text{Filter}(\mathcal{A})$

$\mathcal{F} : \text{Ultrafilter} \iff \forall \mathcal{F}' : \text{Filter}(\mathcal{A}) . \mathcal{F} \subset \mathcal{F}' \Rightarrow \mathcal{F} = \mathcal{F}'$

$\text{FilterBase} :: \prod X : \text{SET} . \prod \mathcal{A} : \text{IntersectionClosed}(X) . ??\mathcal{A}$

$\mathcal{B} : \text{FilterBase} \iff \mathcal{B} \neq \emptyset \ \& \ \emptyset \notin \mathcal{B} \ \& \ \forall A, B \in \mathcal{B} . \exists C \in \mathcal{B} : C \subset A \cap B$

$\text{generateFilter} :: \prod X : \text{SET} . \prod \mathcal{A} : \text{IntersectionClosed}(X) . \text{FilterBase}(\mathcal{A}) \rightarrow \text{Filter}(\mathcal{A})$

$\text{generateFilter}(\mathcal{B}) = \langle \mathcal{B} \rangle := \{A \in \mathcal{A} : \exists B \in \mathcal{B} : B \subset A\}$

$\text{FilterLimit} :: \prod X : \text{TOP} . \text{Filter} \mathcal{T}(X) \rightarrow ?X$

$L : \text{FilterLimit} \iff \mathcal{F} \mapsto L = \lim \mathcal{F} \iff \mathcal{F} \mapsto \mathcal{U}(L) \subset \mathcal{F}$

$\text{FilterBaseLimit} :: \prod X : \text{TOP} . \text{FilterBase} \mathcal{T}(X) \rightarrow ?X$

$L : \text{FilterLimit} \iff \mathcal{B} \mapsto L = \lim \mathcal{B} \iff \mathcal{B} \mapsto L = \lim \langle \mathcal{B} \rangle$

$\text{FilterCluster} :: \prod X : \text{TOP} . \text{Filter} \mathcal{T}(X) \rightarrow ?X$

$C : \text{FilterCluster} \iff \mathcal{F} \mapsto C = \overline{\mathcal{F}} \iff \mathcal{F} \mapsto \forall U \in \mathcal{F} . C \in \overline{U}$

$\text{FilterBasCluster} :: \prod X : \text{TOP} . \text{FilterBase} \mathcal{T}(X) \rightarrow ?X$

$C : \text{FilterBaseCluster} \iff \mathcal{B} \mapsto C = \overline{\mathcal{B}} \iff \mathcal{B} \mapsto C = \overline{\langle \mathcal{B} \rangle}$

$\text{Finer} :: \prod X : \text{TOP} . \left( \text{Filter} \mathcal{T}(X) \times \text{Filter} \mathcal{T}(X) \right)$

$(\mathcal{F}, \mathcal{F}') : \text{Finer} \iff \mathcal{F}' \subset \mathcal{F}$

$\text{netAsFilter} :: \prod X : \text{TOP} . \text{Net}(D, X) \rightarrow \text{Filter} \mathcal{T}(X)$

$\text{netAsFilter}(x) = \mathcal{F}_x := \left\{ U \in \mathcal{T}(X) : \exists N \in D : \forall n \geq N . x_n \in U \right\}$

$\text{filterAsNet} :: \prod X : \text{TOP} . \text{Net}(D, X) \rightarrow \text{Filter} \mathcal{T}(X)$

$\text{filterAsNet}(\mathcal{F}) = x^{\mathcal{F}} := \Lambda(x, U) \in D . x \in \quad \text{where} \quad D = \left\{ (x, U) \mid x \in X, U \in \mathcal{F} : x \in U \right\}$

**FilterNetLimitsEquivalence** ::  $\forall X \in \mathbf{TOP} . \forall x : \mathbf{Net}(D, X) . \lim_{n \in D} x_n = \lim \mathcal{F}_x$

**Proof** =

...

□

**NetFilterLimitsEquivalence** ::  $\forall X \in \mathbf{TOP} . \forall \mathcal{F} : \mathbf{Filter} \mathcal{T}(X) . \lim_{n \in D} x_n^{\mathcal{F}} = \lim \mathcal{F}$

**Proof** =

...

□

### 1.2.3 Sets with Convergent Sequences

**Subsequence** ::  $\prod X \in \text{SET} . (\mathbb{N} \rightarrow X) \rightarrow ?(\mathbb{N} \rightarrow X)$

$y : \text{Subsequence} \iff \Lambda x : \mathbb{N} \rightarrow X . y \subset x \iff \Lambda x : \mathbb{N} \rightarrow X . \exists k : \text{Increasing}(\mathbb{N}, \mathbb{N}) . y = x_k$

**WithConvergent** ::  $?(\sum X \in \text{SET} . \sum \mathcal{C} : ?(\mathbb{N} \rightarrow X) . \mathcal{C} \rightarrow X)$

$(X, \mathcal{C}, L) : \text{WithConvergent} \iff \left( \forall x \in X . (\Lambda n \in \mathbb{N} . x) \in \mathcal{C} \ \& \ L(\cdot \mapsto x) = x \right) \ \&$   
 $\ \& \left( \forall x \in \mathcal{C} . \forall y \subset x . y \in \mathcal{C} \ \& \ L(x) = L(y) \right) \ \&$   
 $\ \& \left( \forall x \notin \mathcal{C} . \exists y \subset x : \forall z \subset y . z \notin \mathcal{C} \right)$

**closure** ::  $\prod (X, \mathcal{C}, L) : \text{WithConvergent} . ?X \rightarrow ?X$

$\text{closure}(A) = \overline{A} := \{x \in X : \exists a \in \mathcal{C} : \text{Im } a \subset A \ \& \ L(a) = x\}$

**PropertiesOfClosure** ::  $\forall (X, \mathcal{C}, L) : \text{WithConvergent} . \overline{\emptyset} = \emptyset \ \& \ \forall A, B \subset X . A \subset \overline{A} \ \& \ \overline{A \cup B} = \overline{A} \cup \overline{B}$

**Proof** =

...

□

**DiagonalProperty** ::  $? \text{WithConvergent}(X)$

$(X, \mathcal{C}, L) : \text{DiagonalProperty} \iff \forall x \in \mathcal{C} . \forall y : \mathbb{N} \rightarrow \mathcal{C} . \left( \forall n \in \mathbb{N} . L(y_n) = x_n \right) \Rightarrow$   
 $\Rightarrow \exists i, j : \text{Increasing}(\mathbb{N}, \mathbb{N}) : L(y_{i,j}) = L(x)$

**ClosureAndDiagonalProperty** ::  $\forall (X, \mathcal{C}, L) : \text{WithConvergent} . (X, \mathcal{C}, L) : \text{DiagonalProperty} \iff$   
 $\iff \forall A \subset X . \overline{\overline{A}} = \overline{A}$

**Proof** =

...

□

**topologyOfFrechet** ::  $\prod (X, \mathcal{C}, L) : \text{DiagonalProperty} . \text{Topology}(X)$

$\text{topologyOfFrechet}() = F(X, \mathcal{C}, L) := \left\langle \text{closure}(X, \mathcal{C}, L) \right\rangle_{\text{TOP}}$

**withDiagonalPropertyAsTopologicalSpace** ::  $\text{DiagonalProperty} \rightarrow \text{TOP}$

$\text{withDiagonalPropertyAsTopologicalSpace}(X, \mathcal{C}, L) = \text{synecdoche} := (X, F(X, \mathcal{C}, L))$

**FrechetConvergenceConsistancy** ::  $\forall (X, \mathcal{C}, L) : \text{DiagonalProperty} . \forall x : \text{Convergent}(\mathbb{N}, X) .$   
 $\ . \ x \in \mathcal{C} \ \& \ \lim_{n \rightarrow \infty} x_n = L(x)$

**Proof** =

...

□



## 1.3 Category of Topological Spaces

### 1.3.1 Continuous Morphisms

**ContinuousMap** ::  $\prod X, Y : \text{TopologicalSpace} . ?(X \rightarrow Y)$

$f : \text{ContinuousMap} \iff f \in C(X, Y) \iff \forall U \in \mathcal{T}(Y) . f^{-1}(U) \in \mathcal{T}(X)$

**ContinuosBySubbase** ::  $\forall X, Y : \text{TopologicalSpace} . \forall \mathcal{B} : \text{Subbase}(Y) . \forall f : X \rightarrow Y .$   
 $. f \in C(X, Y) \iff \forall B \in \mathcal{B} . f^{-1}(B) \in \mathcal{T}(X)$

**Proof** =

...

□

**ContinuosByBase** ::  $\forall X, Y : \text{TopologicalSpace} . \forall \mathcal{B} : \text{Base}(Y) . \forall f : X \rightarrow Y .$   
 $. f \in C(X, Y) \iff \forall B \in \mathcal{B} . f^{-1}(B) \in \mathcal{T}(X)$

**Proof** =

...

□

**ContinuosByNeighbourhoods** ::  $\forall X, Y : \text{TopologicalSpace} . \forall \mathcal{B} : \text{Base}(Y) . \forall f : X \rightarrow Y .$   
 $. f \in C(X, Y) \iff \forall x \in X . \forall U \in \mathcal{U}(f(x)) . \exists V \in \mathcal{U}(x) : f(V) \subset U$

**Proof** =

**Assume** [1] :  $\forall x \in X . \forall U \in \mathcal{U}(f(x)) . \exists V \in \mathcal{U}(x) : f(V) \subset U,$

**Assume**  $U : \mathcal{T}(f(x)),$

**Assume**  $x : f^{-1}(U),$

$(V, [2]) := [1](x, U) : \sum V \in \mathcal{U}(x) . f(V) \subset U,$

$[x.*] := f^2[2] : V \subset f^{-1}(U);$

$\leadsto [U.*] := \text{OpenByCover} : f^{-1}(U) \in \mathcal{T}(X);$

$\leadsto [*] := \text{f}^{-1}C(X, Y) : f \in C(X, Y);$

□

**ContinuosByClosedSets** ::  $\forall X, Y : \text{TopologicalSpace} . \forall f : X \rightarrow Y .$   
 $. f \in C(X, Y) \iff \forall A : \text{Closed}(X) . f^{-1}(A) : \text{Closed}(X)$

**Proof** =

...

□

**ContinuousByClosure1** ::  $\forall X, Y : \text{TopologicalSpace} . \forall f : X \rightarrow Y .$   
 $. f \in C(X, Y) \iff \forall A \subset X . f(\overline{A}) \subset \overline{f(A)}$

**Proof** =

...

□

**ContinuousByClosure2** ::  $\forall X, Y : \text{TopologicalSpace} . \forall f : X \rightarrow Y .$

$$. f \in C(X, Y) \iff \forall A \subset Y . \overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$$

**Proof** =

...

□

**ContinuousByIntetior** ::  $\forall X, Y : \text{TopologicalSpace} . \forall f : X \rightarrow Y .$

$$. f \in C(X, Y) \iff \forall A \subset Y . f^{-1}(\text{int } A) \subset \text{int } f^{-1}(A)$$

**Proof** =

...

□

**categoryOfTopoplogicalSpaces** :: CAT

**categoryOfTopologicalSpaces** () = TOP :=  $(\text{TopologicalSpace}, C, \circ, \text{id})$

**Homeo** := Iso(TOP) :  $\text{TOP}^2 \rightarrow \text{Type}$ ;

**ContinuousAtAPoint** ::  $\prod X, Y : \text{TopologicalSpace} . X \rightarrow Y$

$$f : \text{ContinuousAtAPoint} \iff \Lambda x \in X . \forall U \in \mathcal{U}(f(x)) . f^{-1}(U) \in \mathcal{U}(x)$$

**ContinuousImageOfLimits** ::  $\forall X, Y \in \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} Y . \forall x : \text{Net}(D, X) . f\left(\lim_{n \in D} x_n\right) \subset \lim_{n \in D} f(x_n)$

**Proof** =

**Assume**  $B : f(\lim_{n \in D} x_n),$

$$(A, [1]) := \text{Image} : \sum A = \lim_{n \in D} x_n . f(A) = B,$$

**Assume**  $U : \mathcal{U}(B),$

$$V := f^{-1}(U) : \mathcal{U}(A),$$

$$(N, [2]) := \text{Limit}(A) : \sum N \in D . \forall n \geq N . x_n \in V,$$

$$[U.*] := \text{preimage}[2] : \forall n \geq N . f(x_n) \in U;$$

$$\leadsto [B.*] := \text{Limit} : B = \lim_{n \in D} f(x_n);$$

$$\leadsto [*] := \text{Subset} : f(\lim_{n \in D} x_n) \subset \lim_{n \in D} f(x_n);$$

□

**SeparableByContinuousMap** ::  $\forall Y \in \mathbf{TOP} . \forall X : \mathbf{Separable} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . \text{Im } f : \mathbf{Separable}$

**Proof** =

...

□

**OpenMap** ::  $\forall X, Y \in \mathbf{TOP} . ?(X \rightarrow Y)$

$f : \mathbf{OpenMap} \iff \forall U \in \mathcal{U}(X) . f(U) \in \mathcal{T}(Y)$

**ClosedMap** ::  $\forall X, Y \in \mathbf{TOP} . ?(X \rightarrow Y)$

$f : \mathbf{ClosedMap} \iff \forall A : \mathbf{Closed}(X) . f(A) : \mathbf{Closed}(Y)$

**ClosedMapEquivalent** ::  $\forall X, Y : \mathbf{TOP} . \forall f : X \rightarrow Y . f : \mathbf{ClosedMap} \iff \forall B \subset Y . \forall U \in \mathcal{T}(X) . \forall[0] : f^{-1}(B) \subset U . \exists V \in \mathcal{T}(Y) : B \subset V \ \& \ f^{-1}(V) \subset U$

**Proof** =

**Assume** [1] :  $\forall B \subset Y . \forall U \in \mathcal{T}(X) . \forall[0] : f^{-1}(B) \subset U . \exists V \in \mathcal{T}(Y) : B \subset V \ \& \ f^{-1}(V) \subset U$ ,

**Assume**  $A : \mathbf{Closed}(X)$ ,

$U := A^c : \mathcal{T}(X)$ ,

$B := f^c(A) : ?Y$ ,

[2] :=  $j(U)j(B) : f^{-1}(B) = U$ ,

$(V, [3]) := [1](B, U, [2]) : \sum V \in \mathcal{T}(Y) . B \subset V \ \& \ f^{-1}(V) \subset U$ ,

[3] :=  $jUjB[2] : f^c(A) \subset V \ \& \ f^{-1}(V) \subset A^c$ ,

[4] := **PreimageDisjoint**[3] :  $V \cap f(A) = \emptyset$ ,

[5] := **SubsetAndDisjointDecompositon**[3][4] :  $f^c(A) = V$ ,

[1.\*] :=  $\mathfrak{d}^{-1}\mathbf{Closed}(X)[5] : [f(A) : \mathbf{Closed}(X)]$ ;

$\leadsto [*] := \mathfrak{d}^{-1}\mathbf{ClosedMap} : (f : \mathbf{ClosedMap}(X, Y))$ ;

□

**OpenMapEquivalent** ::  $\forall X, Y : \mathbf{TOP} . \forall f : X \rightarrow Y . f : \mathbf{OpenMap} \iff \forall B \subset Y . \forall A : \mathbf{Closed}(X) . \forall[0] : f^{-1}(B) \subset A . \exists C : \mathbf{Closed}(Y) : B \subset C \ \& \ A \subset f^{-1}(C)$

**Proof** =

...

□

**ClosedMapCondition** ::  $\forall X, Y : \mathbf{TOP} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . f : \mathbf{ClosedMap}(X, Y) \iff \forall y \in Y . \forall U \in \mathcal{T}(X) . \forall f^{-1}\{y\} \subset U . \exists V \in \mathcal{U}(y) : f^{-1}(V) \subset U$

**Proof** =

...

□

**OpenMapCondition** ::  $\forall X, Y : \mathbf{TOP} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . f : \mathbf{ClosedMap}(X, Y) \iff \forall y \in Y . \forall U \in \mathcal{T}(X) . \forall f^{-1}\{y\} \subset U . \exists V \in \mathcal{U}(y) : f^{-1}(V) \subset U$

**Proof** =

...

□

**OpenBijectionIsHomeo** ::  $\forall X, Y : \text{TOP} . \forall f : X \leftrightarrow Y . f : \text{Open}(X, Y) \ \& \ C(X, Y) \Rightarrow X \xrightarrow{\text{TOP}} Y$

**Proof** =

...

□

**OpenBijectionIsHomeo** ::  $\forall X, Y : \text{TOP} . \forall f : X \leftrightarrow Y . f : \text{Open}(X, Y) \ \& \ C(X, Y) \Rightarrow X \xrightarrow{\text{TOP}} Y$

**Proof** =

...

□

**ClosedMappingClosure** ::  $\forall X, Y : \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} Y . f : \text{Closed}(X, Y) \iff \forall A \subset X . f(\overline{A}) = \overline{f(A)}$

**Proof** =

**Assume** [1] :  $(f : \text{Closed}(X, Y))$ ,

**Assume** A : ?X,

[1] :=  $\text{PotentialClosure}(\text{cl})(A) : A \subset \overline{A}$ ,

[2] :=  $\text{SubsetImage}(f, A) : f(A) \subset f(\overline{A})$ ,

[3] :=  $\text{Closure} : \overline{f(A)} \subset f(\overline{A})$ ,

[4] :=  $\text{PotentialClosure} : f(A) \subset \overline{f(A)}$ ,

[5] :=  $\text{SubsetPreimage} : A \subset f^{-1}(\overline{f(A)})$ ,

[6] :=  $\text{ClosureIsMonotonic}[5] : \overline{A} \subset f^{-1}(\overline{f(A)})$ ,

[7] :=  $\text{MonotonicImage}[6] : f(\overline{A}) \subset f f^{-1}(\overline{f(A)})$ ,

[8] :=  $\text{ImageOfPreimage}[7] : f(\overline{A}) \subset \overline{f(A)}$ ,

[1.\*] :=  $\text{SetEq} : f(\overline{A}) = \overline{f(A)}$ ;

$\leadsto$  [1] :=  $I(\Rightarrow) : \text{Left} \Rightarrow \text{Right}$ ,

**Assume** [2] :  $\forall A \subset X . f(\overline{A}) = \overline{f(A)}$ ,

**Assume** A :  $\text{Closed}(X)$ ,

[3] :=  $\text{ClosedClosure}[2] : f(A) = f(\overline{A}) = \overline{f(A)}$ ,

[A.\*] :=  $\text{closure}[3] : (f(A) : \text{Closed}(Y))$ ;

$\leadsto$  [2.\*] :=  $\text{Closed} : (f : \text{Closed}(X, Y))$ ;

$\leadsto$  [\*] :=  $I(\iff) : f : \text{Closed}(X, Y) \iff \forall A \subset X . f(\overline{A}) = \overline{f(A)}$ ;

□

**OpenMappingInterior** ::  $\forall X, Y : \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} Y . f : \text{Open}(X, Y) \iff \forall A \subset X .$

$f(\text{int } A) \subset \text{int } f(A)$

**Proof** =

...

□

**OpenByInteriorPreimage** ::  $\forall X, Y : \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} . f : \text{Open}(X, Y) \iff$   
 $\iff \forall A \subset Y . f^{-1}(\text{int } A) = \text{int } f^{-1}(A)$

**Proof** =

**Assume** [1] :  $f : \text{Open}(X, Y)$ ,

**Assume**  $A : ?Y$ ,

[2] :=  $\text{PotentialInterior}(\text{cl}_X)(A) : \text{int } A \subset A$ ,

[3] :=  $\text{PreimageSubset}[2] : f^{-1}(\text{int } A) \subset f^{-1}(A)$ ,

[4] :=  $\text{InteriorIsMonotonic}[3] : f^{-1}(\text{int } A) \subset \text{int } f^{-1}(A)$ ,

[5] :=  $\text{ImageOfPreImage} : f(\text{int } f^{-1}(A)) \subset A$ ,

[7] :=  $\text{Interior} : f(\text{int } f^{-1}(A)) \subset \text{int } A$ ,

[8] :=  $\text{PreimageSubset} : f^{-1}f(\text{int } f^{-1}(A)) \subset f^{-1}(\text{int } A)$ ,

[9] :=  $\text{ImagePreimage}(f)\text{InteriorSetEq} : \text{int } f^{-1}(A) \subset f^{-1}(\text{int } A)$ ,

[1.\*] := [2][9] :  $\text{int } f^{-1}(A) = f^{-1}(\text{int } A)$ ;

$\leadsto$  [1] :=  $I(\Rightarrow) : \text{Left} \Rightarrow \text{Right}$ ,

**Assume** [2] :  $\text{Right}$ ,

**Assume**  $A : ?X$ ,

[3] := [2]  $\left( f(A) \right) \text{PreimageOfImage} : \text{int } A \subset \text{int } f^{-1}f(A)f^{-1}(\text{int } f(A))$ ,

[A.\*] :=  $\text{SubsetImage}[3]\text{ImageOfPreImage} : f(\text{int } A) \subset f f^{-1}(\text{int } f(A)) \subset \text{int } f(A)$ ;

$\leadsto$  [2.\*] :=  $\text{OpenMappingInterior} : \left( f : \text{Open}(X, Y) \right)$ ;

$\leadsto$  [\*] :=  $I(\iff) : \text{This}$ ;

□

**OpenByInteriorPreimage** ::  $\forall X, Y : \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} . f : \text{Open}(X, Y) \iff$   
 $\iff \forall A \subset Y . f^{-1}(\text{int } A) = \text{int } f^{-1}(A)$

**Proof** =

...

□

**Clopen** ::  $\prod X, Y : \text{TOP} . X \xrightarrow{\text{TOP}} Y$

$f : \text{Clopen} \iff f : \text{Open}(X, Y) \ \& \ \text{Closed}(X, Y)$

**ClopenMappingOfClosedDomain** ::  $\forall X, Y : \text{TOP} . \forall f : \text{Clopen}(X, Y) . \forall A : \text{ClosedDomain}(X) .$   
 $. f(A) : \text{ClosedDomain}(Y)$

**Proof** =

[1] :=  $\text{ClosedDomain}(A)\text{ClosedMappingClosure}(f)\text{OpenMappingIntereior}(f) :$   
 $: f(A)f(\overline{(\text{int } A)}) \overline{f(\text{int } A)} = \overline{\text{int } f(A)}$ ,

[2] :=  $\text{InteriorIsSubset} : \text{int } f(A) \subset f(A)$ ,

[3] :=  $\text{MonotonicClosure}(\text{int})\text{Closed}(X, Y)(f) : \overline{\text{int } f(A)} \subset f(A)$ ,

[4] :=  $\text{SetEq}[1][3] : f(A) = \overline{\text{int } f(A)}$ ,

[\*] :=  $\text{ClosedDomain} : \left( f(A) : \text{ClosedDomain}(Y) \right)$ ;

□

**OpenClosedDomainPreimage** ::  $\forall X, Y : \text{TOP} . \forall f : \text{Open}(X, Y) . \forall A : \text{ClosedDomain}(Y) .$   
 $. f^{-1}(A) : \text{ClosedDomain}(X)$

**Proof** =

...

□

**OpenOpenDomainPreimage** ::  $\forall X, Y : \text{TOP} . \forall f : \text{Clopen}(X, Y) . \forall A : \text{OpenDomain}(Y) .$   
 $. f^{-1}(A) : \text{OpenDomain}(X)$

**Proof** =

...

□

**BorelPreimage** ::  $\forall X, Y : \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} Y . \forall B \in \mathcal{B}(Y) . \forall f^{-1}(B)$

**Proof** =

...

□

### 1.3.2 Subspaces

$\text{subspaceTopology} :: \prod X \in \text{SET} . \prod Y \subset X . \text{Topology}(X) \rightarrow \text{Topology}(Y)$

$\text{subspaceTopology}(T) := \{U \cap Y \mid U \in T\}$

$\text{topologicalSubspace} :: \prod X \in \text{TOP} . ?X \rightarrow \text{TOP}$

$\text{topologicalSubspace}(Y) = \text{synecdoche} := (Y, \text{subspaceTopology}(X, Y))$

$\text{ClosedSetsInASubspace} :: \forall X \in \text{TOP} . \forall Y \subset X . \text{Closed}(Y) = \{Y \cap A \mid A : \text{Closed}(X)\}$

Proof =

...

□

$\text{ClosureInASubspace} :: \forall X \in \text{TOP} . \forall Y \subset X . \forall A \subset Y . \text{cl}_Y(A) = \text{cl}_X(A) \cap Y$

Proof =

...

□

$\text{ContinuousEmbedding} :: \forall X \in \text{TOP} . \forall Y \subset X . \iota_{Y,X} : Y \xrightarrow{\text{TOP}} X$

Proof =

...

□

$\text{ClosedEmbeddingCriterion} :: \forall X \in \text{TOP} . \forall Y \subset X . \iota_{Y,X} : \text{Closed}(Y, X) \iff Y : \text{Closed}(X)$

Proof =

...

□

$\text{OpenEmbeddingCriterion} :: \forall X \in \text{TOP} . \forall Y \subset X . \iota_{Y,X} : \text{Open}(Y, X) \iff Y : \text{Open}(X)$

Proof =

...

□

$\text{ContinuousRestriction} :: \forall X, Y \in \text{TOP} . \forall A \subset X . \forall f : X \xrightarrow{\text{TOP}} Y . f|_A : A \xrightarrow{\text{TOP}} Y$

Proof =

...

□

$\text{ContinuousCorestriction} :: \forall X, Y \in \text{TOP} . \forall A \subset Y . \forall f : X \xrightarrow{\text{TOP}} Y . \forall [0] : f(X) \subset A . f|_A : X \xrightarrow{\text{TOP}} A$

Proof =

...

□

`HomeoEmbedding` ::  $\prod X, Y \in \text{TOP} . ?(X \xrightarrow{\text{TOP}} Y)$

$f : \text{HomeoEmbedding} \iff \exists A \subset Y : \exists \varphi : X \xrightarrow{\text{TOP}} A . f = \varphi \iota_{A,Y}$

`Hereditary` ::  $??\text{TOP}$

$P : \text{Hereditary} \iff \forall X \in \text{TOP} . X : P \Rightarrow \forall A \subset X . A : P$

`hereditary` ::  $? \text{TOP} \rightarrow ? \text{TOP}$

$\text{hereditary}(P) := \Lambda X : P . \forall A \subset X . A : P$

`SeparationIsHereditary` :: `T0, T1, T2, T3` : `Hereditary`

`Proof` =

...

□

`UrysohnIsHereditary` :: `Urysohn` : `Hereditary`

`Proof` =

...

□

`PerfectNormalityIsHereditary` :: `PerfectlyNormal` : `Hereditary`

`Proof` =

...

□

`HereditaryNormalityCondition1` ::  $\forall X \in \text{TOP} . X : \text{hereditary T4} \iff \forall U \in \mathcal{T}(X) . U : \text{T4}$

`Proof` =

...

□

`Separated` ::  $\prod X \in \text{TOP} . ?(?X \times ?X)$

$A, B : \text{Separated} \iff \overline{A} \cap B = \emptyset \ \& \ A \cap \overline{B} = \emptyset$

`HereditaryNormalityCondition2` ::  $\forall X \in \text{TOP} . X : \text{hereditary T4} \iff \forall (A, B) : \text{Separated}(X) . \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$

`Proof` =

...

□

`T5` := `hereditary Normal` : `Type`;

`T6` := `PerfectlyNormal` : `Type`;

`SeparationHierarchy6` :: `T6`  $\subset$  `T5`  $\subset$  `T4`

`Proof` =

...

□



$$\text{Extendable} :: \prod X, Y \in \text{TOP} . \prod A \subset X . ?(A \xrightarrow{\text{TOP}} Y)$$

$$f : \text{Extendable} \iff \exists F : X \xrightarrow{\text{TOP}} Y . f = F|_A$$

$$\begin{aligned} \text{TietzeLemma} &:: \forall X : \text{T4} . \forall A : \text{Closed}(X) . \forall c \in \mathbb{R} . \forall f : X \xrightarrow{\text{TOP}} [-c, c] . \\ & . \exists F : X \xrightarrow{\text{TOP}} \frac{1}{3}[-c, c] : \forall a \in A . |f(a) - F(a)| \leq \frac{2c}{3} \end{aligned}$$

**Proof** =

$$B := f^{-1} \left[ -c, -\frac{c}{3} \right] : \text{Closed}(A),$$

$$C := f^{-1} \left[ \frac{c}{3}, c \right] : \text{Closed}(A),$$

$$[1] := \text{ClosedSetsOfASubset} B, C : \left( B, C : \text{Closed}(X) \right),$$

$$(g, [2]) := \text{UrysohnLemma}(B, C) : \sum g : X \xrightarrow{\text{TOP}} [0, 1] . g(B) = \{0\} \ \& \ g(C) = \{1\},$$

$$F := \frac{2c}{3} \left( g - \frac{1}{2} \right) : X \xrightarrow{\text{TOP}} \frac{1}{3}[-c, c],$$

$$[*] := [2] B C : \forall a \in A . |F(a) - f(a)| \leq \frac{2c}{3};$$

□

$$\text{TietzeUrysohnExtension} :: \forall X : \text{T4} . \forall A : \text{Closed}(X) . \forall f : A \xrightarrow{\text{TOP}} [-1, 1] . f : \text{Extendable}(X, [-1, 1])$$

**Proof** =

$$(g_1, \sigma_1) := \text{TietzeLemma}(X) : \sum g_1 : X \xrightarrow{\text{TOP}} \frac{1}{3}[-1, 1] . \forall a \in A . |g_1(a) - f(a)| \leq \frac{2}{3},$$

**Assume**  $n : \mathbb{N}$ ,

$$h := f - \sum_{i=1}^n g_n : A \xrightarrow{\text{TOP}} \mathbb{R},$$

$$[2] := h \sigma_n : \text{Im } h \subset \left( \frac{2}{3} \right)^n [-1, 1],$$

$$(g_{n+1}, [3]) := \text{TietzeLemma}(h, [2]) : \sum g_{n+1} : X \xrightarrow{\text{TOP}} \frac{1}{3^{n+1}}[-1, 1] . \forall a \in A . |g_{n+1}(a) - h(a)| \leq \left( \frac{2}{3} \right)^{n+1},$$

$$\sigma_{n+1} := \sigma_n \delta h[3] : \forall a \in A . \left| f(a) - \sum_{i=1}^{n+1} g_i(a) \right| \leq \left( \frac{3}{2} \right)^{n+1};$$

$$\rightsquigarrow (g, [2]) := I \left( \sum \right) :$$

$$: \sum g : \mathbb{N} \rightarrow X \xrightarrow{\text{TOP}} \mathbb{R} . \forall n \in \mathbb{N} . \text{Im } g \subset \frac{1}{3^n}[-1, 1] \forall a \in A . \left| f(a) - \sum_{i=1}^n g_i(a) \right| \leq \left( \frac{2}{3} \right)^n ,$$

$$F := \sum_{n=1}^{\infty} g_n : X \xrightarrow{\text{TOP}} \mathbb{R},$$

$$[3] := [2] F : F|_A = f,$$

$$[*] := \delta^{-1} \text{Extendable}[3] : (f : \text{Extendable}(X, [-1, 1]));$$

□

**DiscreteSubsetBound** ::  $\forall X : \mathbf{T4} \ \& \ \mathbf{Separable} . \forall A : \mathbf{Discrete} \ \& \ \mathbf{Closed}(X) . |A| \leq \aleph_0$

**Proof** =

...

□

**Compatible** ::  $\prod X, Y, I \in \mathbf{SET} . ? \left( \sum S : \mathbf{Cover}(I, X) . \prod_{i \in I} S_i \rightarrow T \right)$

$(S, f) : \mathbf{Compatible} \iff \forall i, j \in I . f_{i|S_i \cap S_j} = f_{j|S_i \cap S_j}$

**combination** ::  $\prod X, Y, I \in \mathbf{SET} . \prod (S, f) : \mathbf{Compatible}(X, Y, I) . \prod J \subset I . \bigcup_{j \in J} S_j \rightarrow Y$

**combination** () =  $\nabla_{j \in J} f_j := \Lambda x \in \bigcup_{j \in J} S_j . f_j(x) \quad \text{where} \quad x \in S_j$

**ContinuousOpenCombination** ::  $\forall X, Y \in \mathbf{TOP} . \forall I \in \mathbf{SET} . \forall U : \mathbf{OpenCover}(I, X) . \forall f : \prod_{i \in I} U_i \xrightarrow{\mathbf{TOP}} Y .$

$. \forall [0] : \left( (U, f) : \mathbf{Compatible} \right) . \forall J \subset I . \nabla_{i \in I} f_j : X \xrightarrow{\mathbf{TOP}} Y$

**Proof** =

...

□

**LocalContinuityCriterion** ::  $\forall X, Y \in \mathbf{TOP} . \forall f : X \rightarrow Y . f : X \xrightarrow{\mathbf{TOP}} Y \iff$

$\iff \forall U \in \mathcal{T}(X) . f|_U : U \xrightarrow{\mathbf{TOP}} Y$

**Proof** =

...

□

**NormalInduction** ::  $\forall : \mathbf{T4} . \forall A : \mathbf{Discrete}(\mathbb{N}, \mathbf{Closed}(X)) .$

$. \exists U : \prod_{n=1}^{\infty} \mathcal{U}(A) : \forall n, m \in \mathbb{N} . n \neq m \Rightarrow \overline{U_n} \cap \overline{U_m} = \emptyset$

**Proof** =

...

□

**LocallyClosed** ::  $\prod X \in \mathbf{TOP} . ??X$

$A : \mathbf{LocallyClosed} \iff \forall a \in A . \exists U \in \mathcal{U}(a) : U \cap A : \mathbf{Closed}(U)$

**EquivalentLocallyClosedSet** ::  $\forall X \in \text{TOP} . \forall A \subset X . A : \text{LocallyClosed}(X) \iff \exists B, C : \text{Closed}(X) : A$

**Proof** =

**Assume** [1] :  $\left( A : \text{LocallyClosed}(X) \right),$

$B := \overline{A} : \text{Closed}(X),$

$C := \overline{A} \setminus A : ?X,$

[2] := **MonotonicClosure**(A)**DoubleClosure** :  $\overline{\overline{A} \setminus A} \subset \overline{A},$

[3] := **MonotonicClosure**(A)**DoubleClosure** :  $\overline{\overline{A} \setminus A} \cap \overline{A}^c = \emptyset,$

**Assume**  $x : \overline{\overline{A} \setminus A},$

[4] := [3](x) :  $x \in \overline{A},$

**Assume** [5] :  $x \in A,$

$\left( U, [6] \right) := [1] \text{d}\text{LocallyClosed}[1](X)(A)(x) : \sum U \in \mathcal{U}(x) . U \cap A : \text{Closed}(U),$

[7] := **EquivalentClosure**( $\overline{A} \setminus A$ )(x)(U) :  $U \cap (\overline{A} \setminus A) \neq \emptyset,$

[8] := **SubsetClosure**[7] :  $\overline{U \cap \overline{A}} \neq U \cap A,$

[9] := **ClosedClosure**[6] :  $\overline{U \cap A} = U \cap A,$

[5.\*] :=  $I(\perp)[8][9] : \perp;$

$\leadsto [5] := E(\perp) : x \notin A,$

[x.\*] :=  $\text{d}^{-1} \text{complement}[4][5] : x \in \overline{A} \setminus A;$

$\leadsto [4] := \text{d}^{-1} \text{Subset} : \overline{\overline{A} \setminus A} \subset \overline{A} \setminus A,$

[5] := **ClosureSubset**( $\overline{A} \setminus A$ ) :  $(\overline{A} \setminus A) \subset \overline{\overline{A} \setminus A},$

[6] :=  $\text{d}^{-1} \text{SetEq} : \overline{A} \setminus A = \overline{\overline{A} \setminus A},$

[1.\*] :=  $E(=)[6] \text{d}\text{closure}_j : \left( C : \text{Closed}(X) \right);$

$\leadsto [1] := I(\Rightarrow) : \text{Left} \Rightarrow \text{Right},$

**Assume**  $B, C : \text{Closed}(X),$

**Assume** [2] :  $A = B \setminus C,$

**Assume**  $x : \text{In}(A),$

[3] :=  $\text{d}\text{compliment}[2](x) : x \notin C,$

$\left( U, [4] \right) := \text{OpenByInnerCover}[3] : \sum U \in \mathcal{U}(x) . U \cap C = \emptyset,$

[5] := [2][4] :  $U \cap A = U \cap B,$

[x.\*] := **ClosedInSubspace**(X, A)(B)[5] :  $\left( U \cap A : \text{Closed}(U) \right);$

$\leadsto \left[ (B, C). * \right] := \text{d}^{-1} \text{LocallyClosed} : \left( A : \text{LocallyClosed}(X) \right);$

$\leadsto [*] := I(\iff)[1] : \text{This};$

□

### 1.3.3 Weak and Strong Topology

$$\text{supTopology} :: \prod_{X, I \in \text{SET}} (I \rightarrow \text{Topology}(X)) \rightarrow \text{Topology}(X)$$

$$\text{supTopology}(\tau) = \bigvee_{i \in I} \tau_i := \left( X, \left\langle \bigcup_{i \in I} \tau_i \right\rangle_{\text{TOP}} \right)$$

$$\text{SupPoperty} :: \forall X, I \in \text{SET} . \forall \tau : I \rightarrow \text{Topology}(X) . \forall \sigma : \text{Topology}(X) . \forall \mathbb{N} : \forall i \in I . \tau_i \subset \sigma . \bigvee_{i \in I} \tau_i \subset \sigma$$

**Proof** =

Every open set  $V \in \bigvee_{i \in I} \tau_i$  can be represented as  $V = \bigcup_{j \in J} \bigcap_{k=1}^{n_j} U_{j,k}$ , where each  $U_{j,k} \in \bigcup_{i \in I} \tau_i$  and  $n_j \in \mathbb{Z}_+$ .

But each  $U_{j,k} \in \sigma$ , so also  $V \in \sigma$  by definition of topology.

□

**SupTopologyConvergenceInNets** ::

$$\begin{aligned} &:: \forall X, I \in \text{SET} . \forall \tau : I \rightarrow \text{Topology}(X) . \forall (\Delta, x) : \text{Net}(X) . \forall L \in X . \\ & . \lim_{\delta \in \Delta} x_\delta =_{X, \bigvee_{i \in I} \tau_i} L \iff \forall i \in I . \lim_{\delta \in \Delta} x_\delta =_{X, \tau_i} L \end{aligned}$$

**Proof** =

$(\Rightarrow)$  : This implication is obvious as  $\bigcup_{i \in I} \tau_i \subset \bigvee_{i \in I} \tau_i$ .

$(\Leftarrow)$  : Assume that  $U \in \mathcal{U}(L)$  in  $\bigvee_{i \in I} \tau_i$  topology .

Then there exists a number  $n \in \mathbb{N}$  and an index  $i : \{1, \dots, n\} \rightarrow I$  such that  $L \in \bigcap_{k=1}^n V_k \subset U$ ,

where each  $V_k \in \tau_{i_k}$ .

By convergence hypothesis we can find a collection of elements  $\delta : \{1, \dots, n\} \rightarrow \Delta$  such that  $x_\alpha \in V_k$  for any  $\alpha \geq \delta_k$ .

As  $\Delta$  is directed set there is some  $\gamma$  such that  $\gamma \geq \delta_k$  for any  $k \in \{1, \dots, n\}$ .

Thus,  $x_\alpha \in U$  for any  $\alpha \geq \gamma$ .

As  $U$  was arbitrary this means that the sequence converges in sup topology. .

□

**SupTopologyConvergenceInFilters** ::

$$\begin{aligned} &:: \forall X, I \in \text{SET} . \forall \tau : I \rightarrow \text{Topology}(X) . \forall \mathcal{F} : \text{Filter}(X) . \forall L \in X . \\ & . \lim \mathcal{F} =_{X, \bigvee_{i \in I} \tau_i} L \iff \forall i \in I . \lim \mathcal{F} =_{X, \tau_i} L \end{aligned}$$

**Proof** =

This is true as convergence in nets and filters is equivalent.

□

$$\text{infTopology} :: \prod_{X, I \in \text{SET}} \left( I \rightarrow \text{Topology}(X) \right) \rightarrow \text{Topology}(X)$$

$$\text{infTopology}(\tau) = \bigwedge_{i \in I} \tau_i := \bigvee \left\{ \sigma : \text{Topology}(X), \forall i \in I . \sigma \subset \tau_i \right\}$$

$$\text{InfTopologyExpression} :: \forall X, I \in \text{SET} . \forall \tau : I \rightarrow \text{Topology}(X) . \bigwedge_{i \in I} \tau_i = \bigcap_{i \in I} \tau_i$$

**Proof** =

Write  $\left\{ \sigma : \text{Topology}(X), \forall i \in I . \sigma \subset \tau_i \right\} = \Upsilon$ , then  $\bigwedge_{i \in I} \tau_i = \bigvee \Upsilon$ .

Then each  $\sigma \subset \bigcap_{i \in I} \tau_i$  for each  $\sigma \in \Upsilon$ .

So, by sup property  $\bigwedge_{i \in I} \tau_i \subset \bigcap_{i \in I} \tau_i$ .

But, note that  $\bigcap_{i \in I} \tau_i \in \Upsilon$ , so  $\bigcap_{i \in I} \tau_i = \bigwedge_{i \in I} \tau_i$ .

□

$$\text{weakTopology} :: \prod_{X, I \in \text{SET}} \left( I \rightarrow \sum_{Y_i \in \text{TOP}} \text{SET}(X, Y_i) \right) \rightarrow \text{Topology}(X)$$

$$\text{weakTopology}(Y, f) = \mathcal{W}_X(I, Y, f) := \bigwedge \left\{ \tau : \text{Topology}(X), \forall i \in I . f_i \in \text{TOP}((X, \tau), Y_i) \right\}$$

$$\text{strongTopology} :: \prod_{Z, I \in \text{SET}} \left( \prod_{i \in I} \sum_{Y_i \in \text{TOP}} \text{SET}(Y_i, Z) \right) \rightarrow \text{Topology}(Z)$$

$$\text{strongTopology}(Y, f) = \mathcal{S}_Z(I, Y, f) := \bigvee \left\{ \tau : \text{Topology}(Z), \forall i \in I . f_i \in \text{TOP}(Y_i, (Z, \tau)) \right\}$$

**WeakTopologyConvergenceInNets** ::

$$:: \forall X, I \in \text{SET} . \forall (Y, f) : \prod_{i \in I} \sum_{Y_i \in \text{TOP}} X \xrightarrow{f_i} Y_i : \text{SET} .$$

$$. \forall (\Delta, x) : \text{Net}(X) . \forall L \in X . \lim_{\delta \in \Delta} x_\delta =_{X, \mathcal{W}(Y, f)} L \iff \forall i \in I . \lim_{\delta \in \Delta} f_i(x_\delta) = f_i(L)$$

**Proof** =

write  $\mathcal{W}(Y, f) = \bigwedge \left\{ \tau : \text{Topology}(X), \forall i \in I . f_i \in \text{TOP}((X, \tau), Y_i) \right\} =$

$= \bigvee \left\{ \sigma : \text{Topology}(X), \forall \tau : \text{Topology}(X) . \left( \forall i \in I . f_i \in \text{TOP}((X, \tau), Y_i) \right) \Rightarrow \sigma \subset \tau \right\} =$

$= \bigvee_{i \in I} f_i^{-1}(\mathcal{T}(Y_i))$ .

So we need to proof the result for the case  $I = \{i\}$ .

$(\Rightarrow)$  follows from the continuity of  $f_i$  in weak topology.

$(\Leftarrow)$  : Let  $U$  be an open neighborhood of  $L$  in weak topology..

Then there are  $V \in \mathcal{T}(Y_i)$  such that  $U = f^{-1}(V)$ .

As  $V$  is an open neighborhood of  $f_i(L)$ , by convergence hypothesis there is  $\gamma \in \Delta$  such that  $f(x_\delta) \in V$  for each  $\delta \geq \gamma$ .

So  $x_\delta \in U$  for each  $\delta \geq \gamma$ .

And as  $U$  was arbitrary the convergence holds.

□

**WeakTopologyConvergenceInNets** ::

$$\begin{aligned} &:: \forall X, I \in \mathbf{SET} . \forall (Y, f) : \prod_{i \in I} \sum_{Y_i \in \mathbf{TOP}} X \xrightarrow{f_i} Y_i : \mathbf{SET} . \\ &. \forall \mathcal{F} : \mathbf{Filter}(X) . \forall L \in X . \lim \mathcal{F} =_{X, \mathcal{W}(Y, f)} L \iff \forall i \in I . \lim f_i(\mathcal{F}) = f_i(L) \end{aligned}$$

**Proof** =

By equivalence of convergence in nets and in filters.

□

**WeakTopologyContinuity** ::

$$\begin{aligned} &:: \forall X, I \in \mathbf{SET} . \forall (Y, f) : \prod_{i \in I} \sum_{Y_i \in \mathbf{TOP}} X \xrightarrow{f_i} Y_i : \mathbf{SET} . \\ &. \forall Z \in \mathbf{TOP} . \forall g : Z \rightarrow X . g \in \mathbf{TOP}\left(Z, (X, \mathcal{W}(Y, f))\right) \iff \forall i \in I . gf_i \in \mathbf{TOP}(Z, Y_i) \end{aligned}$$

**Proof** =

( $\Rightarrow$ ) : This follows from continuous composition.

( $\Leftarrow$ ) : Let  $U$  be an open in the weak topology .

We can assume that  $U = \bigcap_{k=1}^n f_{i_k}^{-1}(V_k)$ , where  $i : \{1, \dots, n\} \rightarrow I$  and each  $V_k$  is open in  $Y_{i_k}$ .

Then  $g^{-1}(U) = \bigcap_{k=1}^n (gf_{i_k})^{-1}(V_k)$  is open .

As sets of this form generate weak topology  $g$  must be continuous.

□

**StrongTopologyContinuity** ::

$$\begin{aligned} &:: \forall Y, I \in \mathbf{SET} . \forall (X, f) : \prod_{i \in I} \sum_{X_i \in \mathbf{TOP}} X_i \xrightarrow{f_i} Y : \mathbf{SET} . \\ &. \forall Z \in \mathbf{TOP} . \forall g : Y \rightarrow Z . g \in \mathbf{TOP}\left((Y, \mathcal{W}(Y, f)), Z\right) \iff \forall i \in I . f_i g \in \mathbf{TOP}(X_i, Z) \end{aligned}$$

**Proof** =

( $\Rightarrow$ ) : This follows from continuous composition.

( $\Leftarrow$ ) : Let  $U$  be open in  $Z$  .

Then  $(f_i g)^{-1}(U)$  is open in  $X_i$ .

But this means that  $g^{-1}(U)$  has open preimage under  $f_i$  for each  $i \in I$ .

But this means that  $U$  is open in strong topology.

As set  $U$  was arbitrary  $g$  must be continuous.

□

### 1.3.4 Sums

$\text{sumTopology} :: \prod I \in \text{SET} . (I \rightarrow \text{TOP}) \rightarrow \text{TOP}$

$$\text{sumTopology}(X) = \prod_{i \in I} X_i := \left( \bigsqcup_{i \in I} X_i, \mathcal{S}(X, \iota) \right)$$

$\text{SumIsCoproduct} :: (\text{sumTopology} : \text{Coproduct}(X))$

**Proof** =

Let  $Y \in \text{TOP}$  and  $f_i \in \text{TOP}(X_i, Y)$ .

Then by universal property in  $\text{SET}$  there is unique  $h : \prod_{i \in I} X_i \rightarrow Y$  such that  $\iota_i h = f_i$ .

But as each  $f_i$  is continuous the  $h$  also must be continuous.

□

$\text{SumIsCompatibleWithSubspace} :: \forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \forall i \in I . X_i \cong_{\text{TOP}} \iota_{X,i}(X_i)$

**Proof** =

From the definition each  $\iota_i$  is injective.

So  $\iota_i^{-1} \iota_i(A) = A$ .

But with strong topology this means that  $\iota_i$  is an open mapping.

As it both open and continuous (by definition)  $\iota_i$  is a homeomorphic embedding.

□

$\text{ClopenSummands} :: \forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \forall i \in I . \text{Clopen} \left( \prod_{i \in I} X_i, \iota_{X,i}(X_i) \right)$

**Proof** =

By definition of strong topology each  $X_i$  is open in  $\prod_{i \in I} X_i$ .

But its complement  $X_i^c = \bigcup_{j \neq i} X_j$  is also open as union of open sets (each open by similar considerations).

So  $X_i$  must be clopen.

□

$\text{SumPreservesSeparation} :: \forall I \in \text{SET} . \forall n \in \{1, \dots, 6\} . \forall X : I \rightarrow \mathbf{T}n . \prod_{i \in I} X_i : \mathbf{T}n$

**Proof** =

...

□

### 1.3.5 Products

$$\text{productTopology} :: \prod I \in \text{SET} . (I \rightarrow \text{TOP}) \rightarrow \text{TOP}$$

$$\text{productTopology}(X) = \prod_{i \in I} X_i := \left( \prod_{i \in I} X_i, \mathcal{W}(X, \pi) \right)$$

$$\text{ProductOfTopologicalSpaces} :: \left( \text{productTopology} : \text{Product}(\text{TOP}) \right)$$

**Proof** =

Let  $Y \in \text{TOP}$  and  $f_i \in \text{TOP}(Y, X_i)$ .

Then by universal property in **SET** there is unique  $h : Y \rightarrow \prod_{i \in I} X_i$  such that  $h\pi_i = f_i$ .

But as each  $f_i$  is continuous the  $h$  also must be continuous.

□

$$\text{ProductTopologyBase} :: \forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} .$$

$$. \left\{ \prod_{i \in I} U_i \mid U \in \prod_{i \in I} \mathcal{T}(X_i) : \left| \{i \in I : U_i \neq X_i\} \right| < \infty \right\} : \text{Base} \left( \prod_{i \in I} X_i \right)$$

**Proof** =

This follows from the definition of the weak topology.

□

$$\text{ProductOfClosedSets} :: \forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \forall A : \prod_{i \in I} \text{NonEmpty}(X_i) .$$

$$. \prod_{i \in I} A_i : \text{Closed} \left( \prod_{i \in I} X_i \right) \iff \forall i \in I . A_i : \text{Closed}(X_i)$$

**Proof** =

Firstly assume that if  $A$  is closed in  $X_i$ .

Then  $\left( \prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^c} X_j \right)^c = \prod_{j \in \{i\}} A^c \times \prod_{j \in \{i\}^c} X_j$  is open by the product topology base.

So  $\prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^c} X_j$  is closed.

Now let  $A : \prod_{i \in I} \text{Closed}(X_i)$  be a family of closed set.

Then  $\prod_{i \in I} A_i = \bigcap_{i \in I} \prod_{j \in \{i\}} A \times \prod_{j \in \{i\}^c} X_j$  is closed as an intersection of closed sets.

□



**ProductClosure** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \forall A : \prod_{i \in I} ?X_i . \overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$

**Proof** =

By previous theorem  $\prod_{i \in I} \overline{A_i}$  is closed and evedently  $\prod_{i \in I} A_i \subset \prod_{i \in I} \overline{A_i}$ .

So  $\overline{\prod_{i \in I} A_i} \subset \prod_{i \in I} \overline{A_i}$ .

Assume  $p \in \prod_{i \in I} \overline{A_i}$  And Let  $U = \prod_{i \in I} V_i$  to be a base neighborhood of  $p$  with  $V_i \in \mathcal{T}(X_i)$ .

Then each  $V_i$  is a neighborhood of  $\pi_i(p) \in \overline{A_i}$ , so  $V_i \cap A_i \neq \emptyset$  by alternative definition of closure.

Thus,  $U \cap \prod_{i \in I} A_i \neq \emptyset$ .

As  $p$  and  $U$  was arbitrary by alternative definition of closure  $\prod_{i \in I} \overline{A_i} \subset \overline{\prod_{i \in I} A_i}$ .

Hence  $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$ .

□

**ProjectionIsOpen** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \forall i \in I . \pi_{X,i} : \text{Open} \left( \prod_{i \in I} X_i, X_i \right)$

**Proof** =

Asumme  $U$  is open in  $\prod_{i \in I} X_i$ .

Then it can be represented as  $U = \bigcup_{j \in J} \prod_{i \in I} V_{j,i}$ , where each  $V_{j,i}$  is open  $X_i$ .

We have  $\pi_i(U) = \bigcap_{j \in J} V_{j,i}$  which must be open as union of open sets.

□

**diagonalProduct** ::  $\prod I \in \text{SET} . \forall X \in \text{TOP} . \prod Y : I \rightarrow \text{TOP} . \left( \prod_{i \in I} X \xrightarrow{\text{TOP}} Y_i \right) \rightarrow X \xrightarrow{\text{TOP}} \prod_{i \in I} Y_i$

**diagonalProduct** ( $f$ ) =  $\Delta_{i \in I} f_i := \Lambda x \in X . \Lambda i \in I . f_i(x)$

**ClosedDiagonal** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \mathbf{T2} . \text{Closed} \left( \prod_{i \in I} X_i, \Delta \prod_{i \in I} X_i : \right)$

**Proof** =

...

□

**Multiplicative** :: ??TOP

$P : \text{Multiplicative} \iff \forall I \in \text{SET} . \forall X : I \rightarrow P . \prod_{i \in I} X_i : P$

**CardinalMultiplicative** ::  $\text{CARD} \rightarrow \text{??TOP}$

$P : \text{CardinalMultiplicative} \iff \Lambda k \in \text{CARD} . P : k\text{-Multiplicative} \iff$   
 $: \Lambda k : \text{CARD} . \forall I \in \text{SET} . |I| \leq k \Rightarrow \forall X : I \rightarrow P . \prod_{i \in I} X_i : P$

**FinitelyMultiplicative** :: ??TOP

$P : \text{CardinalMultiplicative} \iff P : \text{FinitlyMultiplicative} \iff$   
 $: \forall I \in \text{SET} . |I| < \infty \Rightarrow \forall X : I \rightarrow P . \prod_{i \in I} X_i : P$

**CountabilityIsCountablyMultiplicative** :: **FirstCountable**, **SecondCountable** :  $\aleph_0\text{-Multiplicative}$

**Proof** =

...

□

**CountabilityIsCountablyMultiplicative** :: **FirstCountable**, **SecondCountable** :  $\aleph_0\text{-Multiplicative}$

**Proof** =

...

□

**SeparabilityIsContinuumMultiplicative** :: **Separable** :  $\exp(\aleph_0)\text{-Multiplicative}$

**Proof** =

...

□

$$\text{SeparatePoints} :: \prod X \in \text{TOP} . \prod I \in \text{SET} . \prod Y : I \rightarrow \text{TOP} . ? \prod_{i \in I} X \xrightarrow{\text{TOP}} Y_i$$

$$f : \text{SeparatePoints} \iff \forall x, x' \in X . x \neq x' \Rightarrow \exists i, j \in I : f_i(x) \neq f_j(x')$$

$$\text{SeparatePointsAndClosedSets} :: \prod X \in \text{TOP} . \prod I \in \text{SET} . \prod Y : I \rightarrow \text{TOP} . ? \prod_{i \in I} X \xrightarrow{\text{TOP}} Y_i$$

$$f : \text{SeparatePointsAndClosed} \iff \forall x \in X . \forall A : \text{Closed}(X) . x \notin A \Rightarrow \exists i, j \in I . f_i(x) \notin \overline{f_j(A)}$$

$$\text{SPaCIsEmbedding} :: \forall X, Y \in \text{TOP} . \forall f : X \xrightarrow{\text{TOP}} Y . (1 \mapsto f) : \text{SeparatePointsAndClosedSets}(X, 1, Y) .$$

$$. f : \text{HomeomorphicEmbedding}(X, Y)$$

**Proof** =

$$F := f|_{\text{Im } f} : X \xrightarrow{\text{TOP}} \text{Im } f,$$

$$[1] := jF \text{SubspaceClosure} \bar{\partial}^{-1} \text{SeparatePointsAndClosedSets} :$$

$$: \left( (1 \mapsto f) : \text{SeparatePointsAndClosedSets}(X, 1, \text{Im } f) \right),$$

$$\text{Assume } U : \text{Open}(X),$$

$$[2] := \bar{\partial}^{-1} \text{Closed}(U) : \left( U^c : \text{Closed}(X) \right),$$

$$\text{Assume } y : f(U),$$

$$\left( x, [3] \right) := \bar{\partial} \text{Image} : \sum x \in f,$$

$$[4] := \bar{\partial} \text{complement}(x) : x \notin U,$$

$$[5] := \bar{\partial}^{-1} \text{SeparatePointsAndClosedSets}(X, 1, \text{Im } f) : \left( f(x) \notin \overline{f(U^c)} \right),$$

$$[*] := \text{EquivalenClosure}[3] \text{DoubleComplement}(U) : \exists V \in \mathcal{U}(f(x)) : V \subset f(U);$$

$$\leadsto [3] := I(\forall) : \forall y \in f(U) . \exists V \in \mathcal{U}(y) . y \in V \subset f(U),$$

$$[U.*] := \text{OpenByInnerCover}[3] : f(U) \in \mathcal{T}(\text{Im } f);$$

$$\leadsto [2] := \bar{\partial}^{-1} \text{Open} j^{-1} F : \left( F : \text{Open}(X, \text{Im } f) \right),$$

$$[3] := \bar{\partial} \text{SeparatePoints}(f) : \left( f : X \hookrightarrow Y \right),$$

$$[*] := \bar{\partial}^{-1} \text{HomeomotphicEmbedding} jF[2][3] : \left( f : \text{HomeomorphicEmbedding} \right);$$

□

$$\text{DiagonalTheorem} :: \forall X \in \text{TOP} . \forall I \in \text{SET} . \forall Y : I \rightarrow \text{TOP} .$$

$$\forall f : \text{SepareatePointsAndClosedSets}(X, I, Y) . \triangle_{i \in I} f_i : \text{HomeamorpohicEmbedding} \left( X, \prod_{i \in I} Y_i \right)$$

**Proof** =

...

□

$$\text{DiagonalTheorem2} :: \forall X \in \text{TOP} . \forall n \in \text{CARD} . \triangle(X^n) \cong_{\text{TOP}} X$$

**Proof** =

...

□

**GraphHomeo** ::  $\forall X, Y \in \mathbf{TOP} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . X \cong_{\mathbf{TOP}} G(f)$

**Proof** =

...

□

**ClosedGraphTheorem** ::  $\forall X \in \mathbf{TOP} . \forall Y \in \mathbf{T2} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . G(f) : \mathbf{Closed}(X \times Y)$

**Proof** =

...

□

**TopologicalSpacesAreComplete** ::  $\mathbf{Bicomplete}(\mathbf{TOP})$

**Proof** =

Construct limits or colimits in **SET**.

Then endow it with weak or strong topology respectively .

□

**ProductPreservesSeparation** ::  $\forall I \in \mathbf{SET} . \forall n \in \{1, \dots, 6\} . \forall X : I \rightarrow \mathbf{T}n \ \& \ \mathbf{NonEmpty} . \prod_{i=1}^n X_i : \mathbf{T}n$

**Proof** =

...

□

### 1.3.6 Quotients

$\text{quotientSpace} :: \prod X \in \text{TOP} . \text{Equivalence}(X) \rightarrow \text{TOP}$

$\text{quotientSpace}(\sim) = \frac{X}{(\sim)} := \left( \frac{X}{(\sim)}, \mathcal{S}(X, \pi_{\sim}) \right)$

$\text{QuotientMap} :: \prod X, Y \in \text{TOP} . ?(\text{TOP} \ \& \ \text{Surjective}(X, Y))$

$f : \text{QuotientMap} \iff Y \cong_{\text{TOP}} \frac{X}{\sim_f}$

$\text{OpenSurjectiveMapIsQuotient} ::$

$:: \forall X, Y \in \text{TOP} . \forall f : \text{Surjective} \ \& \ \text{TOP} \ \& \ \text{Open}(X, Y) . \text{QuotientMap}(X, Y)$

**Proof** =

Assume  $U$  is open in  $Y$ .

Then  $f^{-1}(U)$  is open in  $X$  by continuity of  $f$ .

Now assume that  $U \subset Y$  is such that  $f^{-1}(U)$  is open in  $X$ .

Then  $ff^{-1}(U)$  is open in  $Y$  as  $f$  is open.

But  $ff^{-1}(U) = U$  as  $f$  is surjective, so  $U$  is open.

□

$\text{ClosedSurjectiveMapIsQuotient} ::$

$:: \forall X, Y \in \text{TOP} . \forall f : \text{Surjective} \ \& \ \text{TOP} \ \& \ \text{Closed}(X, Y) . \text{QuotientMap}(X, Y)$

**Proof** =

Assume  $U$  is open in  $Y$ .

Then  $f^{-1}(U)$  is open in  $X$  by continuity of  $f$ .

Now assume that  $U \subset Y$  is such that  $f^{-1}(U)$  is open in  $X$ .

Then  $\left(f^{-1}(U)\right)^c$  is closed in  $Y$ .

So  $f\left(f^{-1}(U)\right)^c = \left(ff^{-1}(U)\right)^c = U^c$  is closed as  $f$  is surjective.

Thus,  $U$  is open.

□

$\text{QuotientContinuity} :: \forall X, Y, Z \in \text{TOP} . \forall f : \text{QuotientMap}(X, Y) . \forall g : Y \rightarrow Z .$

$. fg \in \text{TOP}(X, Z) \iff g \in \text{TOP}(Y, Z)$

**Proof** =

This Follows from the definition of strong topology.

□

## 1.4 Regularity as Separation

### 1.4.1 Functional Separation

**Tychonoff** :: ?T1

$X : \text{**Tychonoff**} \iff \forall A : \text{**Closed**} X . \forall x \in A^c . \exists f : X \xrightarrow{\text{TOP}} [0, 1] : f(A) = \{1\} \ \& \ f(x) = 0$

**SeparationHierarchyTychonff1** :: T3  $\subsetneq$  **Tychonoff**

**Proof** =

...

□

**TychonoffCriterion** ::  $\forall X : \text{**T1**} . X : \text{**Tychonoff**} \iff \forall x \in X . \forall U \in \mathcal{U}(x) .$

$\exists f : X \xrightarrow{\text{TOP}} [0, 1] . f(x) = 0 \ \& \ f(U^c) = \{1\}$

**Proof** =

...

□

**UrysohnsLemma** ::  $\forall X : \text{**T4**} . \forall A, B : \text{**Closed**}(X) . \forall [0] : A \cap B = \emptyset . \exists f : X \xrightarrow{\text{TOP}} [0, 1] :$   
 $f(A) = \{0\} \ \& \ f(B) = \{1\}$

**Proof** =

$(q, [1]) := \text{enumerate}(\mathbb{Q} \cap [0, 1], \mathbb{Z}_+, 0, 1) : \sum q : \mathbb{Z}_+ \leftrightarrow ((0, 1) \cap \mathbb{Q}) . q(0) = 0 \ \& \ q(1) = 1,$

$(U, [2]) := \text{**T4Critetion**}(B^c) : \sum U \in \mathcal{U}(A) . \overline{U} \subset B^c,$

$W_0 := U : \mathcal{U}(A),$

$W_1 := B^c : \mathcal{U}(A),$

$\sigma_1 := jW_0jW_1[2] : \overline{W_0} \subset W_1,$

**Assume**  $n : \mathbb{N},$

$t := q_{n+1} : \mathbb{Q} \cap (0, 1),$

$a := \max \{q_i : q_i < t | i \in [n]_{\mathbb{Z}_+}\} : \mathbb{Q} \cap [0, t),$

$b := \min \{q_i : q_i > t | i \in [n]_{\mathbb{Z}_+}\} : \mathbb{Q} \cap (t, 1],$

$i := q^{-1}(a) : [n]_{\mathbb{Z}_+},$

$j := q^{-1}(b) : [n]_{\mathbb{Z}_+},$

$[3] := \sigma_n(i, j) : \overline{W_i} \subset W_j,$

$(W_{n+1}, \sigma_{n+1}(i, n+1), \sigma_{n+1}(n+1, j)) := \delta \text{**T4**}(X)(\overline{W_i}, W_j^c)[3]\sigma_n :$

$\sum W_{n+1} \in \mathcal{U}(A) . \overline{W_i} \subset W_{n+1} \ \& \ \overline{W_{n+1}} \subset W_j;$

$\leadsto (W, \sigma) := I \left( \sum \right) I \left( \prod \right) : \sum W : \mathbb{Z}_+ \hookrightarrow \mathcal{U}(A) . \prod i, j \in \mathbb{Z}_+ . q_i < q_j \Rightarrow \overline{W_i} \subset W_j,$

$\mathcal{O}_t := \Lambda t \in [0, 1] . \bigcup_{i: q(i) < t} W_i : \mathcal{U}(A),$

$f := \Lambda x \in X . \text{if } x \in B \text{ then } 1 \text{ else } \inf \{t \in [0, 1] : x \in \mathcal{O}_t\} : X \rightarrow [0, 1],$

□

Assume  $t : (0, 1)$ ,

$$[t.*.1] := jf : f^{-1}[0, t) = \bigcap_{s < t} O_s \in \mathcal{T}(X),$$

$$[t.*.2] := jf[] : f^{-1}(t, 1] = X \setminus \bigcap_{s > t} \overline{O_s} \in \mathcal{T}(X);$$

$$\leadsto [3] := \text{RealContinuityCriterion} : f \in C(X, [0, 1]),$$

$$[*] := jf : f(A) = \{0\} \ \& \ f(B) = \{1\};$$

□

**SeparationHierarchyTychonff1** :: **Tychonoff**  $\subsetneq$  **T4**

**Proof** =

...

□

**GDeltaNormalCriterion** ::  $\forall X : \mathbf{T4} . \forall A : \mathbf{Closed}(X) . A \in G_\delta(X) \iff \exists f : X \xrightarrow{\text{TOP}} [0, 1] . f^{-1}\{0\} = A$

**Proof** =

...

□

**FSigmaNormalCriterion** ::  $\forall X : \mathbf{T4} . \forall A : \mathbf{Open}(X) . A \in F_\sigma(X) \iff \exists f : X \xrightarrow{\text{TOP}} [0, 1] . f^{-1}(0, 1] = A$

**Proof** =

...

□

**Separator** ::  $\prod X \in \mathbf{TOP} . (?X \times ?X) \rightarrow ?(X \xrightarrow{\text{TOP}} [0, 1])$

$$f : \mathbf{Separator} \iff f(A) = \{0\} \ \& \ f(B) = \{1\}$$

**CompletelySeparated** ::  $\prod X \in \mathbf{TOP} . ?(?X \times ?X)$

$$A, B : \mathbf{CompletelySeparated} \iff \exists \mathbf{Separator}(A, B)$$

**FunctionallyClosed** ::  $\prod_{X \in \mathbf{TOP}} ??X$

$$A : \mathbf{FunctionallyClosed} \iff \exists f : X \xrightarrow{\text{TOP}} [0, 1] . A = f^{-1}\{0\}$$

**FunctionallyClosed** ::  $\prod_{X \in \mathbf{TOP}} ??X$

$$U : \mathbf{FunctionallyOpen} \iff \exists A : \mathbf{FunctionallyClosed} . U = A^c$$

**FunctionallyClosedIsClosed** ::  $\forall X \in \mathbf{TOP} . \forall A : \mathbf{FunctionallyClosed}(X) . A : \mathbf{Closed}(X)$

**Proof** =

...

□

**FunctionallyOpenIsOpen** ::  $\forall X \in \text{TOP} . \forall U : \text{FunctionallyOpen}(X) . U : \text{Open}(X)$

**Proof** =

...

□

**UnionOfFunctionallyClosed** ::  $\forall X \in \text{TOP} . \forall A, B : \text{FunctionallyClosed}(X) .$

$. A \cup B : \text{FunctionallyClosed}(X)$

**Proof** =

...

□

**IntersectionOfFunctionallyClosed** ::  $\forall X \in \text{TOP} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{FunctionallyClosed}(X) .$

$. \bigcap_{i \in I} A_i : \text{FunctionallyClosed}(X)$

**Proof** =

...

□

**CompleteSeparationOfFunctionallyClosed** ::  $\forall X \in \text{TOP} . \forall A, B : \text{FunctionallyClosed} .$

$\forall[0] : A \cap B = \emptyset . (A, B) : \text{CompletelySeparated}$

**Proof** =

...

□

**continuousFunctions** ::  $\text{TOP} \rightarrow \text{SET}$

**continuousFunctions**  $(X) = C(X) := C(X, \mathbb{R})$

**T1TopologyGeneratedByRealFunctionIsNormal** ::  $\forall X : \text{T1} . \forall f : \text{NonEmpty}(C(X)) .$

$. X = \langle f \rangle_{\text{TOP}} \Rightarrow X : \text{Tychonoff}$

**Proof** =

...

□

**TychonoffFunctionoanalEquivalence** ::  $\forall X \in \text{SET} . \forall T, T' : \text{Topology}(X) .$

$. \forall[0] : ((X, T), (X, T') : \text{Tychonoff}) . (X, T) \cong_{\text{TOP}} (X, T') \iff C(X, T) = C(X, T')$

**Proof** =

...

□



### 1.4.2 Perfectly Normal Spaces

`PerfectlyNormal :: ?T4`

`X : PerfectlyNormal  $\iff \forall A : \text{Closed}(X) . A \in G_\delta(X)$`

`AlternativePerfectlyNormal ::  $\forall X : \text{T4} . X : \text{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) . U \in F_\sigma(X)$`

`Proof =`

`...`

`□`

`VedenisoffTHM! ::  $\forall X : \text{T1} . X : \text{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) . U : \text{FunctionallyOpen}(X)$`

`Proof =`

`...`

`□`

`VedenisoffTHM2 ::  $\forall X : \text{T1} . X : \text{PerfectlyNormal} \iff \forall A : \text{Closed}(X) . A : \text{FunctionallyClosed}(X)$`

`Proof =`

`...`

`□`

`VedenisoffTHM3 ::  $\forall X : \text{T1} . X : \text{PerfectlyNormal} \iff \forall A, B : \text{Closed}(X) . A \cap B = \emptyset \Rightarrow$`

`$\Rightarrow \exists f : X \xrightarrow{\text{TOP}} [0, 1] . f^{-1}\{0\} = A \ \& \ f^{-1}\{1\} = B$`

`Proof =`

`...`

`□`

`PerfectlyNormalInvariance ::  $\forall X : \text{PerfectlyNormal} . \forall Y \in \text{TOP} . \forall f : \text{Closed}(X, Y) .$`

`$f(X) : \text{PerfectlyNormal}$`

`Proof =`

`...`

`□`

`PerfectlyNormalCondition ::  $\forall X : \text{T1} . X : \text{PerfectlyNormal} \iff \forall U \in \mathcal{T}(X) .$`

`$\exists V : \mathbb{N} \rightarrow \mathcal{T}(X) : U = \bigcup_{n=1}^{\infty} V_n \ \& \ \forall i \in \mathbb{N} . \overline{V_i} \subset U$`

`Proof =`

`...`

`□`

### 1.4.3 Normally Placed Sets

**NormallyPlaced** ::  $\prod X : \text{TOP} . ??X$

$A : \text{NormallyPlaced} \iff \forall U \in \mathcal{U}(A) . \exists H : F_\sigma(X) : A \subset H \subset U$

**NormallyPlacedInNormal** ::  $\forall X : \text{T4} . \forall A : \text{NormallyPlaced} . \forall U \in \mathcal{U}(A) . \exists V : F_\sigma \ \& \ \text{Open}(X) . A \subset V \subset U$

**Proof** =

$(H, [1]) := \partial \text{NormallyPlaced}(X)(A)(U) : \sum H \in F_\sigma(X) . A \subset H \subset U,$

$(K, [2]) := \partial F_\sigma(X) : \sum K : \text{Increasing}(\mathbb{N}, \text{Closed}(X)) . H = \bigcup_{i \in I} K_i,$

$(V, [3]) := \partial \text{T4}(X)(K, U^0) : \sum V : \text{Increasing}(\mathbb{N}, \text{Open}(X)) . \forall n \in \mathbb{N} . V_n \subset U \ \& \ \overline{V_n} \cup A_{n+1} \subset V_{n+1},$

$W := \bigcup_{i=1}^{\infty} V_i : \mathcal{T}(X),$

$[4] := \text{UnionOfSubsets}[3] : W \subset U,$

$[5] := \text{UnionOgSupersets}[3][1] : A \subset H \subset W,$

$[6] := \partial \text{union}[3] : W = \bigcup_{i=1}^{\infty} \overline{V_i},$

$[7] := \partial^{-1} F_\sigma(X)[6] : W \in F_\sigma(X),$

$[*] := [4][5][7] : \text{This};$

□

**PerfectlyNormallyPlaced** ::  $\forall X : \text{PerfectlyNormal} . \forall A \subset X . A : \text{NormallyPlaced}(X)$

**Proof** =

...

□

**NormallyPlacedUnion** ::  $\forall X \in \text{TOP} . \forall A : \mathbb{N} \rightarrow \text{NormallyPlaced}(X) . \bigcup_{i=1}^{\infty} A_i : \text{NormallyPlaced}(X)$

**Proof** =

**Assume**  $U : \mathcal{U} \left( \bigcup_{i=1}^{\infty} A_i \right),$

$(f, [1]) := \Lambda i \in \mathbb{N} . \partial \text{NormallyPlaced}(A_i)(U) : \prod_{i=1}^{\infty} F_\sigma(X) . A_i \subset f_i \subset U,$

$F := \bigcup_{i=1}^{\infty} f_i : F_\sigma(X),$

$[U.*] := jF \text{UnionSubset}[1] \text{SubsetUnion}[1] : \bigcup_{i=1}^{\infty} A_i \subset F \subset U;$

$\leadsto [*] := \partial^{-1} \text{NormallyPlaced} : \left( \bigcup_{i=1}^{\infty} \text{NormallyPlaced}(X) \right);$

□

**NormallyPlacedSubspace** ::  $\forall X \in \mathbf{TOP} . \forall A : \mathbf{NormallyPlacedSet}(X) . \forall B : F_\sigma(A)(X) . B : \mathbf{NormallyPlacedSubspace}(X, A, B)$   
**Proof** =

$$\left(K, [1]\right) := \exists F_\sigma(A)(X)(B) : \sum K : n \rightarrow \mathbf{Closed}(A) . B = \bigcup_{n=1}^\infty K_n,$$
$$\left(K', [2]\right) := \mathbf{ClosedInSubspace}(X, B, K) : \sum K' : n \rightarrow \mathbf{Closed}(A) . \forall n \in \mathbb{N} . K = K' \cap A,$$

**Assume**  $U : \mathcal{U}_A(B)$ ,

#### 1.4.4 Urysohn and Semiregular Spaces

`Urysohn` :: ?TOP

$X : \text{Urysohn} \iff \forall x, y \in X . x \neq y \Rightarrow \exists U \in \mathcal{U}(x) . \exists V \in \mathcal{U}(y) : \overline{U} \cap \overline{V} = \emptyset$

`Semireguar` :: ?T2

$X : \text{Urysohn} \iff \text{OpenDomain}(X) : \text{Base}(X)$

`UrysohnSeparationHierarchy` :: T2  $\subset$  `Urysohn`  $\subset$  T3

`Proof` =

...

□

`SemiregularSeparationHierarchy` :: T2  $\subset$  `Semiregular`  $\subset$  T3

`Proof` =

...

□

### 1.4.5 Semicontinuous Functions

**UpperSemicontinuous** ::  $\prod X : \text{TOP} . ?(X \rightarrow \mathbb{R})$

$f : \text{UpperSemicontinuous} \iff f \in C_{1/2}(X) \iff \forall x \in X . \forall r \in \mathbb{R} . f(x) > r \Rightarrow$   
 $\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U . f(u) > r$

**LowerSemicontinuous** ::  $\prod X : \text{TOP} . ?(X \rightarrow \mathbb{R})$

$f : \text{LowerSemicontinuous} \iff f \in C^{1/2}(X) \iff \forall x \in X . \forall r \in \mathbb{R} . f(x) < r \Rightarrow$   
 $\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U . f(u) < r$

**UpperSemicontinuous** ::  $\prod X : \text{TOP} . \prod R : \text{Poset} . ?(X \rightarrow R)$

$f : \text{UpperSemicontinuous} \iff f \in C^{1/2}(X, R) \iff \forall x \in X . \forall r \in R . f(x) > r \Rightarrow$   
 $\Rightarrow \exists U \in \mathcal{U}(x) : \forall u \in U . f(u) > r$

**EquivalentUpperSemicontinuous** ::  $\forall X \in \text{TOP} . \forall f : C \rightarrow \mathbb{R} . f \in C^{1/2}(X) \iff \forall r \in \mathbb{R} .$   
 $. \left\{ x \in X : f(x) \leq r \right\} : \text{Closed}(X)$

**Proof** =

$A := \left\{ x \in X : f(x) \leq r \right\} : ?X,$

**Assume**  $x : X,$

**Assume**  $[1] : f(x) > r,$

$\left( U_x, [x.*] \right) := \partial C^{1/2}(X)[1] : \sum U_x \in \mathcal{U}(x) . \forall u \in U_x . f(u) > r;$

$\leadsto [1] := \text{OpenByInnerCover} : A^c \in \mathcal{T}(X),$

$[*] := \partial \text{Closed}[1] : \left( A : \text{Closed}(X) \right);$

□

**EquivalentLowerSemicontinuous** ::  $\forall X \in \text{TOP} . \forall f : C \rightarrow \mathbb{R} . \forall f \in C_{1/2}(X) \iff \forall r \in \mathbb{R} .$   
 $. \left\{ x \in X : f(x) \geq r \right\} : \text{Closed}(X)$

**Proof** =

...

□

**ContinuousByLoweAndUpperSemicontinuity** ::  $\forall X \in \text{TOP} . C^{1/2}(X) \cap C_{1/2}(X) = C(X)$

**Proof** =

...

□

**SemicontinuousReversion** ::  $\forall X \in \text{TOP} . C^{1/2}(X) = -C_{1/2}(X)$

**Proof** =

...

□

**UpperSemicontinuousAlgebra** ::  $\forall X \in \mathbf{TOP} . \forall f, g \in C^{1/2}(X) . f + g, \max(f, g), \min(f, g) \in C^{1/2}(X)$

**Proof** =

...

□

**UpperSemicontinuousAlgebra2** ::  $\forall X \in \mathbf{TOP} . \forall f, g \in C^{1/2}(X) . f, g > 0 \Rightarrow f * g \in C^{1/2}(X)$

**Proof** =

...

□

**LoweSemicontinuousInfimum** ::  $\forall X \in \mathbf{TOP} . \forall I \in \mathbf{SET} . \forall f : I \rightarrow C_{1/2}(X) . \forall b : X \rightarrow \mathbb{R} .$   
 $. \forall [0] : \forall i \in I . f_i \geq b . \inf_{i \in I} f \in C_{1/2}(X)$

**Proof** =

...

□

**UpperSemicontinuousSupremum** ::  $\forall X \in \mathbf{TOP} . \forall I \in \mathbf{SET} . \forall f : I \rightarrow C^{1/2}(X) . \forall b : X \rightarrow \mathbb{R} .$   
 $. \forall [0] : \forall i \in I . f_i \geq b . \sup_{i \in I} f \in C^{1/2}(X)$

**Proof** =

...

□

**ForteTheorem** ::  $\forall X \in \mathbf{TOP} . \forall f \in C^{1/2}(X) . \exists U : \mathbb{N} \rightarrow \mathcal{T} \ \& \ \mathbf{Dense}(X) . \forall x \in \bigcap_{n=1}^{\infty} U_n . f : \mathbf{ContinuousAt}(x)$

**Proof** =

...

□

**TychonoffBySemicontinuousApproximation** ::  $\forall X : \mathbf{T1} . X : \mathbf{Tychonoff} \iff \forall f \in C^{1/2}(X) .$   
 $. \exists I \in \mathbf{SET} : \exists g : I \rightarrow C(X) : f = \sup_{i \in I} g_i$

**Proof** =

...

□

**NormalBySemicontinuousMidpoint** ::  $\forall X : \mathbf{T1} . X : \mathbf{T4} \iff \forall f \in C_{1/2}(X) . \forall g \in C^{1/2}(X) .$   
 $. f \leq g \Rightarrow \exists h \in C(X) : f \leq h \leq g$

**Proof** =

...

□

**PerfectlyNormalBySemicontinuousApproximation** ::  $\forall X : \mathbf{T1} . X : \mathbf{PerfectlyNormal} \iff \forall f \in C^{1/2} .$   
 $. \exists g : \mathbb{N} \rightarrow C(X) . g \uparrow f$

**Proof** =

...

□

**PerfectlyNormalBySemicontinuousMidpoint** ::  $\forall X : \mathbf{T1} . X : \mathbf{PerfectlyNormal} \iff \forall f \in C_{1/2} . \forall g \in C^{1/2} .$   
 $. f \leq g \Rightarrow \exists h \in C(X) . f \leq g \leq h \ \& \ \forall x \in X . f(x) < g(x) \Rightarrow f(x) < h(x) < g(x)$

**Proof** =

...

□

**LowerSemicontinuousSubspaceValued** ::  $\prod X, Y \in \mathbf{TOP} . ?(Y \rightarrow \mathbf{Closed}(X))$

$F : \mathbf{LowerSemicontinuousSubspaceValued} \iff F \in \mathcal{C}^{1/2}(X, Y) \iff$   
 $\iff \forall U \in \mathcal{T}(X) . \{y \in Y : F(y) \cap U \neq \emptyset\} \in \mathcal{T}(Y)$

**UpperSemicontinuousSubspaceValued** ::  $\prod X, Y \in \mathbf{TOP} . ?(Y \rightarrow \mathbf{Closed}(X))$

$F : \mathbf{UpperSemicontinuousSubspaceValued} \iff F \in \mathcal{C}_{1/2}(X, Y) \iff$   
 $\iff \forall U \in \mathcal{T}(X) . \{y \in Y : F(y) \subset U\} \in \mathcal{T}(Y)$

**ContinuousBySubspaceSemicontinuity** ::  $\forall X : \mathbf{T1} . \forall Y \in \mathbf{TOP} . \forall f : Y \rightarrow X . f : Y \xrightarrow{\mathbf{TOP}} X \iff$   
 $\iff \Lambda y \in Y . \{f(y)\} \in \mathcal{C}^{1/2} \cap \mathcal{C}_{1/2}(X, Y)$

**Proof** =

...

□

**OpenBySubspaceSemicontinuity** ::  $\forall X \in \mathbf{TOP} . \forall Y : \mathbf{T1} . \forall f : X \rightarrow Y . f : \mathbf{Open} \iff$   
 $\iff \Lambda y \in Y . f^{-1}\{y\} \in \mathcal{C}^{1/2}(X, Y)$

**Proof** =

...

□

**ClosedBySubspaceSemicontinuity** ::  $\forall X \in \mathbf{TOP} . \forall Y : \mathbf{T1} . \forall f : X \rightarrow Y . f : \mathbf{Closed} \iff$   
 $\iff \Lambda y \in Y . f^{-1}\{y\} \in \mathcal{C}_{1/2}(X, Y)$

**Proof** =

...

□

**OpenBySubspaceSemicontinuity** ::  $\forall Y \in \mathbf{TOP} . \forall f : Y \rightarrow \mathbb{R} . f \in C^{1/2}(Y) \iff$   
 $\iff \Lambda y \in Y . (-\infty, f(y)] \in \mathcal{C}^{1/2}(\mathbb{R}, Y)$

**Proof** =

...

□

**ClosedBySubspaceSemicontinuity** ::  $\forall Y \in \mathbf{TOP} . \forall f : Y \rightarrow \mathbb{R} . f \in C_{1/2}(Y) \iff$   
 $\iff \Lambda y \in Y . (-\infty, f(y)] \in \mathcal{C}_{1/2}(\mathbb{R}, Y)$

**Proof** =

...

□

**LowerSemicontinuousUnion** ::  $\forall X, Y \in \mathbf{TOP} . \forall I \in \mathbf{SET} . \forall F : I \rightarrow \mathcal{C}^{1/2}(X, Y) .$   
 $. \left( \Lambda y \in Y . \overline{\bigcup_{i \in I} F_i(y)} \right) \in \mathcal{C}^{1/2}(X, Y)$

**Proof** =

...

□

**UpperSemicontinuousUnion** ::  $\forall X, Y \in \mathbf{TOP} . \forall F, G \in \mathcal{C}_{1/2}(X, Y) . . \left( \Lambda y \in Y . F(y) \cup G(y) \right) \in \mathcal{C}_{1/2}(X, Y)$

**Proof** =

...

□

**UpperSemicontinuousIntersect** ::  $\forall X : \mathbf{T4} . \forall Y \in \mathbf{TOP} . \forall F, G \mathcal{C}_{1/2}(X, Y) .$   
 $. \left( \Lambda y \in Y . F(y) \cap G(y) \right) \in \mathcal{C}_{1/2}(X, Y)$

**Proof** =

...

□



## 1.5 Properties Preserved by Continuous Transformations

### 1.5.1 Compact Sets

$\text{Compact} :: \prod X \in \text{TOP} . ??X$

$K : \text{Compact} \iff \forall \mathcal{O} : \text{OpenCover}(X, K) . \exists \mathcal{O}' \subset \mathcal{O} : \mathcal{O}' : \text{Finite} \ \& \ \text{OpenCover}(X, K)$

$\text{FiniteIntersectionProperty} :: \prod X : ??\text{SET} . ?X$

$A : \text{FiniteIntersectionProperty} \iff \forall B : \text{Finite}(A) . \bigcap_{b \in B} b \neq \emptyset$

$\text{CompactByFiniteIntersection} :: \forall X \in \text{TOP} . X : \text{Compact}(X) \iff$   
 $\iff \forall A : \text{FiniteIntersectionProperty Closed}(X) . \bigcup_{a \in A} a \neq \emptyset$

Proof =

...

□

$\text{CompactAsSubset} :: \forall X \in \text{TOP} . \forall A \subset X . A : \text{Compact}(A) \Rightarrow A : \text{Compact}(X)$

Proof =

...

□

$\text{CompactSubset} :: \forall X \in \text{TOP} . \forall [0] : (X : \text{Compact}(X)) . \forall A : \text{Closed}(X) . A : \text{Compact}(X)$

Proof =

...

□

$\text{CompactAsSubspace} :: \forall X \in \text{TOP} . \forall A : \text{Compact}(X) . A : \text{Compact}(A)$

Proof =

...

□

$\text{CompactaUnion} :: \forall X \in \text{TOP} . \forall n \in \mathbb{N} . \forall A : n \rightarrow \text{Compact}(X) . \bigcup_{i=1}^n A_i : \text{Compact}(X)$

Proof =

...

□

$\text{CompactaIntersection} :: \forall X \in \text{TOP} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{Compact}(X) . \bigcap_{i \in I} A_i : \text{Compact}(X)$

Proof =

...

□

**FiniteCompactIntersection** ::  $\forall X \in \mathbf{TOP} . \forall U \in \mathcal{T}(X) . \forall I \in \mathbf{SET} . \forall A : I \rightarrow \mathbf{Closed}(X) . \forall i \in I .$

$$. \forall [0] : \left( A_i : \mathbf{Compact} \right) . \forall [00] : \bigcap_{i \in I} \subset U . \exists F : \mathbf{Finite}(I) : \bigcup_{i \in F} A_i \subset U$$

**Proof** =

$$J := I \setminus \{i\} : \mathbf{Subset}(J),$$

$$V := \Lambda j \in J . A_j^c : \mathbf{Subset}(I),$$

$$[1] := jV[00] : \left( \text{Im } V : \mathbf{TypeOpenCover}(A_i \cap U^c) \right),$$

$$\left( V', [2] \right) := \mathfrak{d}\mathbf{Compact}(X) : \sum V' : \mathbf{Finite}(\text{Im } F) . V' : \mathbf{OpenCover}(A_i \cap U^c),$$

$$\left( K, [3] \right) := \mathfrak{d}\mathbf{image}(V') : \sum K : \mathbf{Finite}(J) . V' = V(K),$$

$$F := K \cup \{i\} : \mathbf{Finite}(I),$$

$$[4] := \mathfrak{d}\mathbf{OpenCover}[2]jF : \bigcup_{i \in F} A_i \subset U;$$

□

**CompactSeparation1** ::  $\forall X : \mathbf{T3} . \forall A : \mathbf{Compact}(X) . \forall B : \mathbf{Closed}(A) .$

$$. \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$$

**Proof** =

...

□

**CompactSeparation2** ::  $\forall X : \mathbf{T2} . \forall A : \mathbf{Compact}(X) . \forall B : \mathbf{Compact}(A) .$

$$. \exists U \in \mathcal{U}(A) : \exists V \in \mathcal{U}(B) : U \cap V = \emptyset$$

**Proof** =

...

□

**CompactIsNormal** ::  $\forall X : \mathbf{T2} \ \& \ \mathbf{Compact} . X : \mathbf{T4}$

**Proof** =

...

□

**TychonoffComapctSeparation** ::  $\forall X : \mathbf{Tychonoff} . \forall A : \mathbf{Compact}(X) . \forall B : \mathbf{Closed}(X) .$

$$. \exists f \in C(X) : f(A) = \{0\} \ \& \ f(B) = \{1\}$$

**Proof** =

...

□

**FiniteIsCompact** ::  $\forall X \in \mathbf{TOP} . \forall F : \mathbf{Finite}$

**Proof** =

**CompactIsClosed** ::  $\forall X : \mathbf{T2} . \forall A : \mathbf{Compact}(X) . A : \mathbf{Closed}(X)$

**Proof** =

**Assume**  $x : A^c$ ,

$[1] := \mathbf{FiniteIsCompact}(X)(\{x\}) : (\{x\} : \mathbf{Compact}(X))$ ,

$(U, V, [2]) := \mathbf{CompactSeparation2}(X, A, \{x\})[1] : \sum U \in \mathcal{U}(A) . \sum V \in \mathcal{U}(x) . V \cap U = \emptyset$ ,

$[3] := \mathbf{ClosureEquivalent}(A)[2] : x \notin \overline{A}$ ;

$\sim [1] := \mathfrak{d}^{-1} \mathbf{SetEq} : A = \overline{A}$ ,

$[*] := \mathbf{ClosedByClosure}[1] : (A : \mathbf{Closed}(X))$ ;

□

**CompactImage** ::  $\forall X : \mathbf{Compact} . \forall Y \in \mathbf{TOP} . \forall f \in C \ \& \ \mathbf{Surjective}(X, Y) . Y : \mathbf{Compact}$

**Proof** =

...

□

**CompactImageClosure** ::  $\forall X : \mathbf{Compact} . \forall Y : \mathbf{T2} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . \forall A \subset X . f(\overline{A}) = \overline{f(A)}$

**Proof** =

...

□

**CompactClosedMap** ::  $\forall X : \mathbf{Compact} . \forall Y : \mathbf{T2} . \forall f : X \xrightarrow{\mathbf{TOP}} Y . f : \mathbf{Closed}(X, Y)$

**Proof** =

...

□

**CompactHomeo** ::  $\forall X : \mathbf{Compact} . \forall Y : \mathbf{T2} . \forall f \in C \ \& \ \mathbf{Bijective}(X, Y) . f : X \xleftrightarrow{\mathbf{TOP}} Y$

**Proof** =

...

□

**KuratowskiLemma** ::  $\forall X, Y \in \mathbf{TOP} . \forall A : \mathbf{Compact}(X) . \forall y \in Y . \forall W \in \mathcal{U}(A \times \{y\}) .$   
 $\quad . \exists U \in \mathcal{U}(A) . \exists V \in \mathcal{U}(y) . U \times V \subset W$

**Proof** =

...

□

**KuratowskiTHM1** ::  $\forall X : \text{Compact} \ \& \ \text{T2} . \forall Y \in \text{TOP} . \pi_2 : \text{Closed}(X \times Y, Y)$

**Proof** =

**Assume**  $A : \text{Closed}(X \times Y),$

$U := A^c : \text{Open}(X \times Y),$

$B := \pi_1(A) : ?X,$

$[1] := \text{CompactClosedMap} : (B : \text{Closed}(X)),$

**Assume**  $y : (\pi_2(A))^c,$

$[2] := \text{compliment} \pi_2 B : B \times y \subset U,$

$(W, V, [3]) := \text{KuratowskiLemma}[2] : \sum W \in \mathcal{U}(B) . \sum V \in \mathcal{U}(y) . V \times W \subset U,$

$[4] := [3]_J U \pi_2 : U \cap A = \emptyset;$

$\leadsto [2] := \text{OpenByInnerCover} : (\pi_2(A))^c \in \mathcal{T}(Y),$

$[3] := \text{compliment}^{-1} \text{Closed}[2] : (\pi_2 A : \text{Closed}(X));$

$\leadsto [*] := \text{compliment}^{-1} \text{Closed} : (\pi_2 : \text{Closed}(X \times Y, Y));$

□

**KuratowskiProperty** ::  $? \text{TOP}$

$X : \text{KuratowskiProperty} \iff \forall Y : \text{T4} . \pi_2 : \text{Closed}(X, X \times Y)$

**KuratowskiTHM2** ::  $\forall X : \text{KuratowskiProperty} . X : \text{Compact} \ \& \ \text{T2}$

**Proof** =

...

□

**CompactGraphTheorem** ::  $\forall X \in \text{TOP} . \forall Y : \text{Compact} \ \& \ \text{T2} . \forall f : X \rightarrow Y . f \in C(X, Y) \iff G(f) : \text{Closed}(X,$

**Proof** =

$[1] := \text{ClosedGraphThm}(X, Y)(f) : \text{Left} \Rightarrow \text{Right},$

**Assume**  $[2] : (G(f) : \text{Closed}(X, Y)),$

**Assume**  $A : \text{Closed}(Y),$

$B := X \times A : \text{Closed}(X \times Y),$

$C := B \cap G(f) : \text{Closed}(X \times Y),$

$[3] := \text{compliment} G(f)_J C \text{compliment}^{-1} \pi . f^{-1}(A) = \pi_1(C),$

$[A.*] := \text{KuratowskiTHM}[3] : (f^{-1}(A) : \text{Closed}(X));$

$\leadsto [2.*] := \text{compliment}^{-1} \text{Continuous} : f \in C(X, Y);$

$\leadsto [*] := I(\iff) I(\Rightarrow)[1] : f \in C(X, Y) \iff G(f) : \text{Closed}(X, U);$

□

**CompactLimitTheorem** ::  $\forall X \in \text{TOP} . X : \text{Compact} \iff \forall D : \text{DirectedSet} . \forall x : \text{Net}(X, D) . \exists \text{Cluster}(x)$

**Proof** =

**Assume** [1] :  $X : \text{Compact}$ ,

**Assume**  $D : \text{DirectedSet}$ ,

**Assume**  $x : \text{Net}(X, D)$ ,

$A := \Lambda n \in D . \overline{\{x_i | i \geq n\}} : D \rightarrow \text{Closed}(X)$ ,

[2] :=  $\text{DirectedSet}_J A \text{FiniteIntersectionProperty} : (A : \text{FiniteIntersectionProperty Closed}(X))$ ,

[3] :=  $\text{CompactByFiniteIntersection}([2]) : \bigcap_{n \in D} A_n \neq \emptyset$ ,

$B := \bigcap_{n \in D} A_n : \text{Closed}(X)$ ,

$(c, [5]) := \text{Nonempty} : \sum c \in X . c \in B$ ,

**Assume**  $U : \mathcal{U}(c)$ ,

**Assume**  $n : D$ ,

[6] :=  $J B [5] \text{intersection} : c \in A_n$ ,

$[U.*] := J A_n \text{AlternativeClosure} : \exists m \in D : m \geq n \ \& \ x_m \in U$ ;

$\leadsto [1.*] := \text{Cluster}(x) : (c : \text{Cluster})$ ;

$\leadsto [1] := I(\Rightarrow) : \text{LEFT} \Rightarrow \text{RIGHT}$ ,

**Assume** [2] : **Right**,

**Assume**  $A : \text{FiniteIntersectionProperty}(X)$ ,

$D := \left\{ \bigcap F \middle| F : \text{Finite}(A) \right\} : ??X$ ,

[3] :=  $\text{FiniteIntersectionProperty}_J D : \forall d \in D . d \neq \emptyset$ ,

$(x, [4]) := \text{Nonempty}[3] : \sum x : D \rightarrow X . \forall d \in D . x \in d$ ,

$c := [2](x) : \text{Cluster}(x)$ ,

[5] :=  $\text{Cluster}(x)[4] : c \in \bigcap A$ ,

$[A.*] := \text{Nonempty}[5] : \bigcap A \neq \emptyset$ ;

$\leadsto [2.*] := \text{CompactByFiniteIntersection} : (X : \text{Compact})$ ;

$\leadsto [*] := I(\Rightarrow)I(\iff)[1] : \text{THIS}$ ;

□

**CompactQuotientMap** ::

$:: \forall X : \text{Compact} . \forall Y : \text{T2} . \forall f \in \text{TOP} \ \& \ \text{Surjective}(X, Y) . \text{QuotientMap}(X, Y, f)$

**Proof** =

Mapping  $f$  must be closed, so it is a quotient map.

□

## 1.5.2 Connected Spaces

**Connected** :: ?TOP

$X : \text{Connected} \iff \text{Clopen}(X) = \{\emptyset, X\}$

**ConnectedProduct** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{Connected} . \prod_{i \in I} X_i$

**Proof** =

**Assume**  $U : \text{Clopen}(X)$ ,

**Assume**  $i : I$ ,

**Assume**  $p : \prod_{j \in \{i\}^c} X_j$ ,

$[1] := \text{ProjectionHomeo}(I, X, i, p) : \{p\} \times X_i \cong_{\text{TOP}} X_i$ ,

$[2] := \mathfrak{D} : U \cap \{p\} \times X_i : \text{Clopen}(X)$ ,

$[i.*] := U \cap \{p\} \times X_i = \emptyset \mid U \cap \{p\} \times X_i = \{p\} \times X_i$ ;

$\leadsto [1] := I(\forall) : \forall i \in I . \exists E \subset \prod_{j \in \{i\}^c} X_j . U = X_i \times E$ ,

$[U.*] := \text{Choice}(X) : U = \prod_{i \in I} X_i \mid U = \emptyset$ ;

$\leadsto [*] := \mathfrak{D}^{-1} \text{Connected} : \text{Connected} \left( \prod_{i \in I} X_i \right)$ ;

□

**MainTheoremOfConnectedSpace** ::  $\forall X : \text{Connected} . \forall Y \in \text{TOP} . \forall f \in C(X, Y) . \text{Connected}(Y)$

**Proof** =

...

□

**ConnectedAltDef** ::  $\forall X \in \text{TOP} . \text{Connected}(X) \iff \forall f \in C(X, 2) . \text{Constant}(X, 2, f)$

**Proof** =

...

□

**ConnectedSubset** ::  $\prod X : \text{TOP} . ??X$

$A : \text{ConnectedSubset} \iff \text{Connected}(X, A) \iff \text{Connected}(A)$

**ConnectedUnion1** ::  $\forall X \in \text{TOP} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{Connected}(X) .$

$\text{PairwiseIntersecting}(X, I, A) \Rightarrow \text{Connected} \left( X, \bigcup_{i \in I} A_i \right)$

**Proof** =

...

□

**ConnectedUnion2** ::  $\forall X \in \text{TOP} . \forall I \in \text{SET} . \forall A : I \rightarrow \text{Connected}(X) .$

$$(\exists i \in I . \forall j \in I . A_i \cap A_j \neq \emptyset) \Rightarrow \text{Connected} \left( X, \bigcup_{i \in I} A_i \right)$$

**Proof** =

...

□

**IntermediateValueTheorem** ::  $\forall X : \text{Connected} . \forall f \in C(X) . a, b \in X . f(a) < 0 \ \& \ f(b) > 0 \Rightarrow$   
 $\Rightarrow \exists c \in X : f(c) = 0$

**Proof** =

...

□

**ClosureOfConnectedIsConnected** ::  $\forall X \in \text{TOP} . \forall A : \text{Connected}(X) . \text{cl } A : \text{Connected}$

**Proof** =

**Assume**  $f : C(\text{cl } A, 2),$

[1] := **ContinuousRestriction**  $\left( f, A, \text{ClosureIsSuper}(A) \right) : f|_A \in C(A, 2),$

[2] := **AltConnectedDef**  $(A, f) : \text{Constant}(X, 2, f|_A),$

[3] :=  $\text{DualConstant}(X, 2, f) \text{ClosureContinuation} : \text{Constant}(\text{cl } A, 2);$

$\leadsto [*] := \text{AltConnectedDef} : \text{Connected}(X, \text{cl } A);$

...

□

**connectedComponents** ::  $\prod X \in \text{TOP} . ??X$

**connectedComponents**  $() = CC(X) := \sup \left\{ A \subset X : \text{Connected}(X) \right\}$

**ConnectedComponentsAreClosed** ::  $\forall X \in \text{TOP} . \forall A \in CC(X) . \text{Closed}(X, A)$

**Proof** =

...

□

**ConnectedComponentsDisjointCover** ::  $\forall X \in \text{TOP} . \bigsqcup_{A \in CC(X)} A = X$

**Proof** =

...

□

**connectedComponentsOf** ::  $\prod X \in \text{TOP} . x \rightarrow CC(X)$

**connectedComponentsOf**  $(x) = CC(x) := \text{ConnectedComponentsDisjointCover}(X, A$

$\text{LocallyConnectededConnctedComponentsAreClopen} :: \forall X : \text{Locally Connected} . \forall A \in CC(X) .$   
 $\text{. Clopen}(X, A)$

$\text{Proof} =$

$\dots$

$\square$



### 1.5.3 Path-Connected Spaces

$$\text{pathSpace} :: \prod X \in \text{TOP} . X^2 \rightarrow ?C([0, 1], X)$$

$$\text{pathSpace}(x, y) = \Omega(x, y) := \left\{ \gamma \in C([0, 1], X) \right\}$$

$$\text{joinPaths} :: \prod X \in \text{TOP} . \prod x, y, z \in X . \Omega(x, y) \times \Omega(y, z) \rightarrow \Omega(x, z)$$

$$\text{joinPaths}(\alpha, \beta) = \alpha\beta := \Lambda t \in [0, 1] . \text{if } t \leq \frac{1}{2} \text{ then } \alpha(2t) \text{ else } \beta(2t - 1)$$

$$\text{pathCategory} :: \text{TOP} \rightarrow \text{SCAT}$$

$$\text{pathCategory}(X) = \omega(X, X) := \left( X, \Omega, \text{joinPaths}, \text{constant}([0, 1], X) \right)$$

$$\text{reversePath} :: \prod X \in \text{TOP} . \prod x, y \in X . \uparrow \Omega(x, y) \rightarrow \uparrow \Omega(y, x)$$

$$\text{reversePath}(\gamma) = \gamma^\frown := \Lambda t \in [0, 1] . \gamma(1 - t)$$

$$\text{Subpath} :: \prod X \in \text{TOP} . \uparrow \Omega(X, X) \rightarrow ? \uparrow \Omega(X, X)$$

$$\alpha : \text{Subpath} \iff \Lambda \gamma \in \Omega(X, X) . \alpha \subset \gamma \iff \Lambda \gamma \in \Omega(X, X) . \exists \phi : \text{Nondeacrizing}([0, 1], [0, 1]) . \alpha = \phi\gamma$$

$$\text{pathMesh} :: \prod X \in \text{TOP} . \uparrow \Omega(X, X) \rightarrow \text{SET}$$

$$\text{pathMesh}(\gamma) = M(\gamma) := \left\{ (n, \alpha) : \text{Chain } \Omega(X, X) : \prod_{i=1}^n \alpha_i = \gamma \right\}$$

$$\text{PathMeshLess} :: \prod X \in \text{TOP} . \prod \gamma \in \uparrow \Omega(X, X) . ?(M(\omega) \times M(\omega))$$

$$(n, \alpha), (m, \beta) : \text{PathMeshLess} \iff (n, \alpha) \leq (m, \beta) \iff \forall i \in n . \exists j \in m : \alpha_i \subset \beta_j$$

**PathMeshIsDirected** ::  $\forall X \in \mathbf{TOP} . \forall \gamma \in \uparrow \Omega(X, X) . M(\gamma) : \mathbf{DirectedSet}$

**Proof** =

**Assume**  $(n, \alpha), (m, \beta) : M(\gamma),$

$[1] := \mathfrak{d}M(\gamma)(n, \alpha) : \forall i \in n . \alpha_i \subset \gamma,$

$[2] := \mathfrak{d}M(\gamma)(m, \beta) : \forall i \in m . \beta_i \subset \gamma,$

$(\phi, [3]) := \mathfrak{d}\mathbf{Subpath}[1] : \prod_{i=1}^n \sum_{\phi:[0,1] \uparrow [0,1]} . \alpha = \phi_i \gamma,$

$(\psi, [4]) := \mathfrak{d}\mathbf{Subpath}[2] : \prod_{i=1}^m \sum_{\phi:[0,1] \uparrow [0,1]} . \beta = \psi_i \gamma,$

$T := \phi(n)(0) \cup \phi(n)(1) \cup \psi(m)(0) \cup \psi(m)(1) : ?[0, 1],$

$(N, t) := \mathbf{sort}(T) : \sum_{N=2}^{\infty} t : \mathbf{Increasing} \ \& \ \mathbf{Bijection}(N, T),$

$M := N - 1 : \mathbb{N},$

$\omega := \Lambda i \in M . \Lambda \lambda \in [0, 1] . \gamma \Big( (1 - \lambda)t_i + \lambda t_{i+1} \Big) : M(\omega),$

$[\dots *] := j\omega : (M, \omega) \leq (n, \alpha) \ \& \ (M, \omega) \leq (m, \beta);$

$\leadsto [*] := \mathfrak{d}^{-1} \mathbf{DirectedSet} : \mathbf{DirectedSet} \Big( M(\omega), \mathbf{PathMehLess}(\omega) \Big);$

□

**PathConnected** ::  $? \mathbf{TOP}$

$X : \mathbf{PathConnected} \iff \forall x, y \in X . \Omega(x, Y) \neq \emptyset$

**PathConnectedIsConnected** ::  $\forall X : \mathbf{PathConnected} . \mathbf{Connected}(X)$

**Proof** =

...

□

**PathConnectedSubset** ::  $\prod X \in \mathbf{TOP} . ? \mathbf{Connected}(X)$

$A : \mathbf{PathConnectedSubset} \iff A : \mathbf{PathConnected}(X) \iff \Big( A, \mathbf{subsetTopology}(X, A) \Big) : \mathbf{PathConected}$

**MainPathConnectedSpaceTHM** ::  $\forall X : \mathbf{PathConnected} . \forall Y \in \mathbf{TOP} . \forall f \in C(X, Y) . \mathbf{PathConnected}(Y, f(X))$

**Proof** =

...

□

**PathConnectedPair** ::  $\prod X \in \mathbf{TOP} . ?(X \times X)$

$(x, y) : \mathbf{PathConnectedPair} \iff \Omega(x, y) \neq \emptyset$

**PathConnectedPairIsEquivalence** ::  $\forall X \in \mathbf{TOP} . \mathbf{PathConnectedPairIsEquivalence}(X)$

**Proof** =

...

□

`pathConnectedComponents` ::  $\prod X \in \text{TOP} . ??X$

`pathConnectedComponents` () =  $\text{PCC}(X) := \text{classes PathConnectedPair}(X)$

`LocallyPathConnectedProperty` ::  $\forall X : \text{LocallyPathConnected} . \forall U \in \text{PCC}(X) . U : \text{Clopen}(X)$

`Proof` =

...

□

### 1.5.4 Totally Disconnected Spaces

$\text{TotallyDisconnected} :: ?\text{TOP}$

$X : \text{TotallyDisconnectedSpace} \iff \forall A \in \text{CC}(X) . A : \text{Singleton}$

$\text{TotallyDisconectesByBase} :: \forall X \in \text{TOP} . \langle \text{Clopen}(X) \rangle_{\text{TOP}} = X \iff \text{TotallyBounded}(X)$

$\text{Proof} =$

### 1.5.5 Sequential Spaces

$\text{limitOfSequences} :: \prod X : \text{TOP} . (X \rightarrow \mathbb{N}) \rightarrow ?X$

$\text{limitOfSequences}(x) = \lim_{n \rightarrow \infty} x_n := \lim_{n \in \mathbb{N}} x_n$

$\text{SequentialSpace} :: ?\text{TOP}$

$X : \text{SequentialSpace} \iff \forall A \subset X . A : \text{Closed} \iff \forall x : \text{Net}(\mathbb{N}, X) . x_{\mathbb{N}} \subset A \Rightarrow \bar{x} \subset A$

$\text{FrechetSpace} :: ?\text{TOP}$

$X : \text{FrechetSpace} \iff \forall A \subset X . \forall p \in \bar{A} . \exists x : \text{Net}(\mathbb{N}, X) : x_{\mathbb{N}} \subset A : p = \lim_{n \rightarrow \infty} x_n$

$\text{FirstCountableIsFrechetSpace} :: \forall X : \text{FirstCounable} . X : \text{FrechetSpace}$

Proof =

...

□

$\text{FrechetSpaceIsSequential} :: \forall X : \text{FrechetSpace} . X : \text{SequentialSpace}$

Proof =

...

□

$\text{ContinuousByLimits} :: \forall X : \text{SequentialSpace} . \forall Y \in \text{TOP} . \forall f : X \rightarrow Y . f \in C(X, Y) \iff$   
 $\iff \forall x : \mathbb{N} \rightarrow X . \lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right)$

Proof =

...

□

$\text{T1ByLimitNumber} :: \forall X \in \text{TOP} . \forall [0] : \forall x : \mathbb{N} \rightarrow X . \left| \lim_{n \rightarrow \infty} x_n \right| \leq 1 . X : \text{T1}$

Proof =

...

□

$\text{T2ByLimitNumber} :: \forall X : \text{FirstCountable} . X : \text{T2} \iff \forall x : \mathbb{N} \rightarrow X . \left| \lim_{n \rightarrow \infty} x_n \right| \leq 1$

Proof =

...

□

## 2 Compacta

In the previous chapter I introduced compactness on the elementary rather. The chapter is fully devoted to developing these chapters to the level required by advanced abstract analysis. Three sections of this chapter are independent on conceptual level, but latter fragments may reference earlier ones occasionally.

### 2.1 Genera

As well as the separation axioms, the compactness may be introduced as a form of regularity, a property which makes space easy to handle. As it was shown during our previous encounter with compacts, they, indeed, may have nice properties. However, being compact is too restrictive, as many common natural spaces like real numbers  $\mathbb{R}$  are not compact. In this section I investigate some other predicates which in many ways similar to compactness, in the sense of making spaces into handy structures of comprehensible size, on the other hand still far more common than the compactness itself.

#### 2.1.1 Lindelöf Spaces

First of all I investigate Lindelöf spaces. Which are spaces which are more general than compacts as only countable subcovers may be extracted from the arbitrary covers. It is obvious that compact spaces are Lindelöf. Also as with compacts I show that this property is inherited by all closed sets. In difference with compactness which allows elevating T2 to T4 separation axioms, the Lindelöf property allows only elevating T3 to T4.

**Lindelöf** :: ?TOP

$X : \text{Lindelöf} \iff \forall \mathcal{O} : \text{OpenCover}(X) . \exists \mathcal{O}' \subset \mathcal{O} . |\mathcal{O}'| \leq \aleph_0 \ \& \ \mathcal{O}' : \text{OpenCover}(X)$

**CompactIsLindelöf** ::  $\forall X : \text{Compact} . \text{Lindelöf}(X)$

**Proof** =

Obvious.

□

**SecondCountableIsLindelöf** ::  $\forall X : \text{SecondCountable} . \text{Lindelöf}(X)$

**Proof** =

Let  $\mathcal{B}$  be a countable base for  $X$ .

As  $\mathcal{O}$  is a cover of  $X$  it is possible to construct a function  $O : X \rightarrow \mathcal{O}$  such that  $x \in O_x$ .

By definition of the base it must be possible to construct a function  $B : X \rightarrow \mathcal{B}$  such that  $x \in B_x \subset O_x$ .

Then  $\text{Im } B \subset \mathcal{B}$  must be countable.

Then for each  $U \in \text{Im } B$  select  $O'_U$  with  $U \subset O'_U$ , which is possible for by construction of  $B$ .

Then  $\text{Im } \mathcal{O}'$  must be countable as an image of countable set.

$x \in U_x$  for any  $x \in X$ , so  $x \in O'_{U_x}$ .

Thus,  $\text{Im } \mathcal{O}'$  forms a countable subcover of  $\mathcal{O}$ .

□

**ClosedLindelöfSubspaceIsLindelöf** ::  $\forall X : \text{Lindelöf} . \forall F : \text{Closed}(X) . \text{Lindelöf}(F)$

**Proof** =

Assume  $\mathcal{O}$  is an open cover of  $F$ .

Then  $\mathcal{O}' = \left\{ O \cup F^c \mid O \in \mathcal{O} \right\}$  is an open cover of  $X$ .

Select a countable subcover  $\mathcal{C}$  of  $\mathcal{O}'$  by Lindelöf property.

Then  $\mathcal{C}' = \{ O \cap F \mid O \in \mathcal{C} \}$  is a countable subcover of  $\mathcal{O}$ .

□

**CoverSeparationLemma** ::

$$\begin{aligned} &:: \forall X \in \mathbf{TOP} . \forall A, B \subset X . \forall V, W : \mathbb{N} \rightarrow \mathcal{T}(X) . \forall \aleph : A \subset \bigcup_{n=1}^{\infty} V_n . \forall \beth : \forall n \in \mathbb{N} . \overline{V} \cap B = \emptyset . \\ & . \forall \beth : B \subset \bigcup_{n=1}^{\infty} W_n . \forall \beth : \forall n \in \mathbb{N} . \overline{W} \cap A = \emptyset . \text{Separated}(X, A, B) \end{aligned}$$

**Proof** =

Without loss of generality assume that  $V$  and  $W$  are increasing.

Otherwise construct  $V'_n = \bigcup_{k=1}^n V_k$  and  $W'_n = \bigcup_{k=1}^n W_k$ .

Clearly, these constructions inherit properties  $\aleph', \beth', \beth', \beth'$ .

Also without loss of generality assume that  $V_n \cap \overline{W}_n = \emptyset$  and  $W_n \cap \overline{V}_n = \emptyset$ .

Otherwise construct  $V'_n = V_n \setminus \overline{W}_n$  and  $W'_n = W_n \setminus \overline{V}_n$ .

Clearly, these constructions inherit properties  $\aleph', \beth', \beth', \beth'$ .

Define open sets  $G = \bigcap_{n=1}^{\infty} V_n$  and  $H = \bigcap_{n=1}^{\infty} W_n$ .

Then  $A \subset G$  and  $B \subset H$  by  $\aleph$  and  $\beth$ .

Take any  $x \in G \cap H$ .

Then there exists some  $n, m \in \mathbb{N}$  such that  $x \in V_n \cap W_m$ .

Take  $k = \max(m, n)$ .

Then  $x \in V_k \cap W_k$  as both  $V$  and  $W$  are increasing.

But  $V_k \cap \overline{W}_k = \emptyset$ , so this is a contradiction!

Thus,  $H \cap G = \emptyset$  and  $H$  and  $G$  provide the separation of  $A$  and  $B$ .

□

**LindelöfRegularity** ::  $\forall X : \text{Lindelöf} \ \& \ \mathbf{T3} . \mathbf{T4} \ \& \ \text{Tychonoff}(X)$

**Proof** =

Assume  $A$  and  $B$  are both closed sets in  $X$ .

As  $X$  is **T3** for every  $a \in A$  it must be possible to select open  $U_a$  such that  $a \in U_a$  and  $\overline{U}_a \cap B = \emptyset$ .

As  $A$  itself is Lindelöf it must be possible to select a countable subcover  $\mathcal{U}$  from  $U_A$ .

By the similar process we may select a countable cover  $\mathcal{V}$  for  $B$ .

But this leads us to open separation lemma, so  $A$  and  $B$  can be separated.

Thus  $X$  is **T4** and hence **T3.5** or Tychonoff.

□

**LindelöfFSigmaIsLindelöf** ::  $\forall X : \text{Compact} . \forall A \in F_{\sigma}(X) . \text{Lindelöf}(A)$

**Proof** =

Represent  $A = \bigcup_{n=1}^{\infty} C_n$ , where each  $C_n$  is closed.

Then each  $C_n$  is Lindelöf, and if  $\mathcal{O}$  is an open cover for  $A$ , then  $\mathcal{O}$  is open cover for each  $C_n$ .

Hence open countable subcovers  $\mathcal{O}'_n$  can be selected for each  $C_n$ .

Then  $\mathcal{O}'' = \bigcup_{n=1}^{\infty} \mathcal{O}'_n$  is still countable and is a cover of  $A$ .

□

### 2.1.2 Locally Compact Spaces

Another intuitive approach on how the niceness of compactness can be extended to the broader class of spaces is the Locall Compactness. So, not the whole space is comprehandable anymore, but every point has a comprehandable neighborhood. In this case a T2 separation axiom can only be lifted to the separation axiom T3.5.

**LocallyCompact** :: ?TOP

$X : \text{LocallyCompact} \iff \forall x \in X . \exists U \in \mathcal{U}(x) : \overline{U} : \text{Compact}(X)$

**LocallyCompactIsTychonoff** ::  $\forall X : \text{LocallyCompact} \ \& \ \text{T2} . X : \text{Tychonoff}$

**Proof** =

...

□

**LocallyCompactSeparation** ::  $\forall X : \text{LocallyCompact} . \forall A : \text{Compact}(X) . \forall U \in \mathcal{U}(A) .$   
 $: \exists V \in \mathcal{U}(A) : \overline{V} : \text{Compact}(X) \ \& \ \overline{V} \subset U$

**Proof** =

...

□

**LocallyCompactSubset1** ::  $\forall X : \text{LocallyCompact} . \forall A \subset X . \forall U : \text{Open}(X) . \forall K : \text{Closed}(X) .$   
 $. A = U \cap K \Rightarrow A : \text{LocallyCompact}$

**Proof** =

...

□

**LocallyCompactSubset2** ::  $\forall X : \text{T2} \ \& \ \text{LocallyCompact} . \forall A \subset X .$   
 $. A : \text{LocallyCompact} \Rightarrow \exists U : \text{Open}(X) : \exists K : \text{Closed}(X) :$

**Proof** =

...

□

**LocallyCompactRepresentation** ::  $\forall X \in \text{TOP} . X : \text{LocallyCompact} \iff \exists K : \text{Compact} : \exists U \in \mathcal{T}(K) .$   
 $. U \cong_{\text{TOP}} X$

**Proof** =

...

□

**LocallyCompactSum** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \prod_{i \in I} X_i : \text{LocallyCompact} \iff$   
 $\iff \forall i \in I . X_i : \text{LocallyCompact}$

**Proof** =

...

□



**LocallyCompactProduct** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{TOP} . \prod_{i \in I} X_i : \text{LocallyCompact} \iff$   
 $\iff \left( \forall i \in I . X_i : \text{LocallyCompact} \right) \ \& \ \left| \{ i \in I : X_i ! \text{Compact} \} \right| < \infty$

**Proof** =

...  
□

**LocallyCompactByMapping** ::  $\forall X : \text{LocallyCompact} . \forall Y : \text{T2} . \forall f : \text{Surjection} \ \& \ C(X, Y) .$   
 $. Y : \text{LocallyCompact}$

**Proof** =

...  
□

**WhiteheadQuotientTheorem** ::  $\forall X : \text{LocallyCompact} . \forall Y, Z \in \text{TOP} . \forall \pi : \text{QuotientMapping}(\text{TOP}; Y, Z) .$   
 $. \text{id}_X \times \pi : \text{QuotientMapping}(\text{TOP}; X \times Y, X \times Z)$

**Proof** =

...  
□

### 2.1.3 Countably Compact Spaces

The predicate dual to being Lindelöf is countable compactness. This duality is present in the sense that both combined form true compactness. The duality suggest that countable compactness allows elevating **T2** separation axiom to **T3** if the space is first countable.

**CountablyCompact** :: ?TOP

$X : \text{CountablyCompact} \iff \forall \mathcal{O} : \text{OpenCover}(X) . |\mathcal{O}| \leq \aleph_0 \Rightarrow \exists \mathcal{O}' \subset \mathcal{O} : |\mathcal{O}'| < \infty . \ \& \ \mathcal{O}' : \text{OpenCover}(X)$

**CompactnessDecomposition** ::  $\forall X \in \text{TOP} . \forall \text{CountableCompact} \ \& \ \text{Lindelöf}(X) \iff \text{Compact}(X)$

**Proof** =

This is obvious.

□

**CondensationPoint** ::  $\prod_{X \in \text{TOP}} ?X \rightarrow ?X$

$x : \text{CondensationPoint} \iff \nexists A \subset X . \forall U \in \mathcal{U}(x) . |A \cap U| = \infty$

**CountableCompactAltDef** ::

$:: \forall X \in \text{TOP} . \text{CountableCompact}(X) \iff$

$\iff \forall x : \mathbb{N} \rightarrow X . \exists \text{Cluster}(X, x) \iff$

$\iff \forall A : \text{Infinite}(X) . \exists \text{CondensationPoint}(X, A)$

**Proof** =

#### 2.1.4 Sequentially Compact Spaces

As it is now evident that countably compact need to be first countable to lift **T2** to **T3**. The first countable countably compact space is in fact sequentially compact. So this concept seems to be more wholesome as convergence is an such important part of analysis.

**SequentiallyCompact** :: ?TOP

$X : \text{SequentiallyCompact} \iff \forall x : \mathbb{N} \rightarrow X . \exists y \subset x : y : \text{Convergent}(X)$

### 2.1.5 Pseudocompact Spaces

An important property of compacts is that every continuous function is bounded. Spaces which share this property are called pseudocompact.

$$\mathbf{boundedFunctions} :: \mathbf{TOP} \rightarrow \mathbf{SET}$$

$$\mathbf{boundedFunctions}(X) = C_b(X) := \left\{ f \in C(X) : \exists a, b \in \mathbb{R} : f(X) \subset [a, b] \right\}$$

$$\mathbf{Pseudocompact} :: ?\mathbf{TOP}$$

$$X : \mathbf{Pseudocompact} \iff C(X) = C_b(X)$$

## 2.2 Category of Compact Spaces

Compactness can be studied in two distinct ways. Firstly, it is possible to look at compact subsets of general topological sets. On the other hand, one may investigate how compact sets interact with each other. The later topic leads to the language of category theory naturally.

### 2.2.1 Filters in categories

One way to apply this language in the smart way is the use of the filter monad. Deep study of this subject is leading to the area known as monoidal topology. However, here I only scratch the surface with the basic results and definitions. Note, that there some complications in the interpretation of  $F(\mathcal{C})(\mathcal{F}, \mathfrak{F})$

`filterFunctor` :: `Covariant`(`SET`, `SET`)

`filterFunctor` ( $X$ ) =  $F(X) := \text{Filter}(X)$

`filterFunctor` ( $X, Y, f$ ) =  $F_{X,Y}(f) := \Lambda \mathcal{F} \in \text{Filter}(X) . \{f(A) | A \in \mathcal{F}\}$

Obviosly, as  $\mathcal{F} \neq \emptyset$  then  $F_{X,Y}(f)(\mathcal{F}) \neq \emptyset$ .

Also, as  $\emptyset \notin \mathcal{F}$  then  $f \notin F_{X,Y}(f)(\mathcal{F})$ .

All this is true as an image non-emptyset can't be empty.

Now assume  $f(A), f(B) \in F_{X,Y}(f)(\mathcal{F})$  .

Then there is  $\emptyset \neq C \in \mathcal{F}$  such that  $C \subset A \cap B$ .

Thus  $f(C) \subset f(A \cap B) \subset f(A) \cap f(B)$ .

And  $f(C) \in F_{X,Y}(f)(\mathcal{F})$ , so  $F_{X,Y}(f)(\mathcal{F})$  is a filterbase.

□

`pointFilter` ::  $\prod_{X \in \text{SET}} X \rightarrow F(X)$

`pointFilter` ( $x$ ) =  $\dot{x} := \{A \subset X : A(x)\}$

`sumOfKowalsky` ::  $\prod_{X \in \text{SET}} F^2(X) \rightarrow F(X)$

`sumOfKovalsky` ( $\mathfrak{F}$ ) =  $m_X(\mathfrak{F}) := \left\{ A \subset X : \{\mathcal{F} \in F(X) | A \in \mathcal{F}\} \in \langle \mathfrak{F} \rangle \right\}$

As  $\emptyset \notin \mathcal{F}$  for any  $\mathcal{F} \in F(X)$  it follows that  $\emptyset \notin m_X(\mathfrak{F})$ .

Also note that  $\{\mathcal{F} \in F(X) | X \in \mathcal{F}\} = F(X)$  .

And  $F(X) \in \mathfrak{F}$ , so  $X \in m_X(\mathfrak{F})$ .

Whence  $m_X(\mathfrak{F}) \neq \emptyset$ .

If  $A, B \in m_X(\mathfrak{F})$  then  $\{\mathcal{F} \in F(X) | A \in \mathcal{F}\}, \{\mathcal{F} \in F(X) | B \in \mathcal{F}\} \in \mathfrak{F}$ .

But  $\{\mathcal{F} \in F(X) | A \in \mathcal{F}\} \cap \{\mathcal{F} \in F(X) | B \in \mathcal{F}\} = \{\mathcal{F} \in F(X) | A \in \mathcal{F}, B \in \mathcal{F}\} = \{\mathcal{F} \in F(X) | A \cap B \in \mathcal{F}\} \in \mathfrak{F}$ .

Hence  $A \cap B \in m_X(\mathfrak{F})$ .

Now assume that  $A \in m_X(\mathfrak{F})$  and  $A \subset B$  .

Then  $\{\mathcal{F} \in F(X) | A \in \mathcal{F}\} \in \mathfrak{F}$  .

Also by property of filters being upward closed  $\{\mathcal{F} \in F(X) | A \in \mathcal{F}\} \subset \{\mathcal{F} \in F(X) | B \in \mathcal{F}\}$  .

But as  $\mathfrak{F}$  is also filter, then  $\{\mathcal{F} \in F(X) | B \in \mathcal{F}\} \in \mathfrak{F}$  and hence  $B \in \mathfrak{F}$ .

And hence  $m_X(\mathfrak{F})$  is a filter.

□

**FilterConvergence** ::  $\prod_{X \in \mathbf{SET}} ?(X \times \mathbf{F}(X))$

$\mathcal{C} : \mathbf{FilterConvergence} \iff \forall x \in X . \mathcal{C}(x, \dot{x}) \ \&$

$\& \forall \mathfrak{F} \in \mathbf{F}^2(X) . \exists \mathcal{F} \in \mathbf{F}(X) . \mathbf{F}(\mathcal{C})(\mathcal{F}, \mathfrak{F}) \ \& \ \mathcal{C}(x, \mathcal{F}) \Rightarrow \mathcal{C}(x, m_X(\mathfrak{F}))$

**FilterConvergenceIsTopology** ::

$:: \forall X \in \mathbf{SET} . \forall \mathcal{C} : \mathbf{FilterConvergence}(X) . \exists ! \tau : \mathbf{Topology}(X) . \forall (x, \mathcal{F}) \in X \times \mathbf{F}(X) .$   
 $. (x, \mathcal{F}) \in \mathcal{C} \iff x \in \lim_{(X, \tau)} \mathcal{F}$

**Proof** =

Let  $\mathcal{C}$  be a filter convergence in the sense defined above.

Define  $\tau = \left\{ U \subset X : \forall (x, \mathcal{F}) \in \mathcal{C} . U(x) \Rightarrow \langle \mathcal{F} \rangle(U) \right\}$ .

Then  $\emptyset$  is trivially in  $\tau$ .

If  $(x, \mathcal{F}) \in \mathcal{C}$ , then  $\mathcal{F} \neq \emptyset$  so there are  $A \subset \mathcal{F}$ .

This means that  $A \subset X \in \mathcal{F}$  as  $\langle \mathcal{F} \rangle$  must be upward closed.

Assume  $U : I \rightarrow \tau$  is the collection of sets, and  $(x, \mathcal{F}) \in \mathcal{C}$  is such that  $\bigcup_{i \in I} U_i(x)$ .

Then there is  $i \in I$  such that  $U_i(x)$ , so  $\langle \mathcal{F} \rangle(U_i)$ .

But as  $\langle \mathcal{F} \rangle$  is upwards closed  $\langle \mathcal{F} \rangle \left( \bigcup_{i \in I} U_i \right)$ .

Thus  $\bigcup_{i \in I} U_i \in \tau$ .

Now take  $n \in \mathbb{N}$  and  $U : \{1, \dots, n\} \rightarrow \tau$  and also  $(x, \mathcal{F}) \in \mathcal{C}$  such that  $\bigcap_{i=1}^n U_i(x)$ .

Then  $U_i(x)$  for any  $i \in \{1, \dots, n\}$ , so  $\langle \mathcal{F} \rangle(U_i)$ .

But  $\langle \mathcal{F} \rangle$  is intersection closed, so  $\langle \mathcal{F} \rangle \left( \bigcap_{i=1}^n U_i \right)$ .

Thus  $\bigcup_{i \in I} U_i \in \tau$  and we proved that  $\tau$  is topology.

Then by construction  $(x, \mathcal{F}) \in \mathcal{C} \Rightarrow x \in \lim_{(X, \tau)} \mathcal{F}$ .

Now take  $(x, \mathcal{F}) \in X \times \mathbf{F}(X)$ , such that  $x \in \lim_{(X, \tau)} \mathcal{F}$ .

This means that  $\mathcal{F}(U)$  for all  $U \in \mathcal{U}(x)$ .

The idea is to show that  $\mathcal{F} = m_X(\mathfrak{F})$  for some filter of filters  $\mathfrak{F}$  and  $\mathbf{F}(\mathcal{C})(\dot{x}, \mathfrak{F})$ .

So define  $\mathfrak{F} = \left\{ \{ \langle \mathcal{G} \rangle \mid \mathcal{G} \in \mathbf{F}(X), A \in \mathcal{G} \} \mid A \in \mathcal{F} \right\}$ .

$\mathfrak{F}$  is a filterbase: it is nonempty as  $\mathcal{F}$  is and any element of  $\mathfrak{F}$  contains  $\mathcal{F}$ .

It is also closed by intersections as

$\{ \langle \mathcal{G} \rangle \mid \mathcal{G} \in \mathbf{F}(X), A \in \mathcal{G} \} \cap \{ \langle \mathcal{G} \rangle \mid \mathcal{G} \in \mathbf{F}(X), B \in \mathcal{G} \} = \{ \langle \mathcal{G} \rangle \mid \mathcal{G} \in \mathbf{F}(X), A \cap B \in \mathcal{G} \}$

as filters are upward and intersection closed.

Then  $\mathcal{F} \subset m_x(\mathfrak{F})$ .

Assume  $A \in m_x(\mathfrak{F})$  .

Then there is  $B \in \mathcal{F}$  such that  $\{\langle \mathcal{G} \rangle \mid \mathcal{G} \in F(X), B \in \mathcal{G}\} \subset \{\langle \mathcal{G} \rangle \mid \mathcal{G} \in F(X), A \in \mathcal{G}\}$  .

But this means that  $A \in \langle \mathcal{F} \rangle$  .

So we may say that  $m_x(\mathfrak{F}) = \langle \mathcal{F} \rangle$  as filter.

By construction  $\dot{x} \in \{\langle \mathcal{G} \rangle \mid \mathcal{G} \in F(X), N \in \mathcal{G}\} \in \mathfrak{F}$  for every  $N \in \mathcal{N}_\tau(x)$ .

This must be enough to shaw that  $F(\mathcal{C})(\dot{x}, \mathfrak{F})$ .

And as  $\mathcal{C}(x, \dot{x})$  by definition, it follows that  $\mathcal{C}(x, \mathcal{F})$  .

If  $\sigma$  and  $\tau$  are two topologies with the desired property, then  $\mathcal{N}_\tau(x) = \mathcal{N}_\sigma(x)$  for all  $x \in X$ .

So  $\tau = \sigma$ .

□

`convergenceAsTopology` ::  $\prod_{X \in \text{SET}} \text{FilterConvergnce}(X) \rightarrow \text{Topology}(X)$

`convergenceAsTopology` ( $\mathcal{C}$ ) =  $\tau_{\mathcal{C}} := \text{FilterConvergenceIsTopology}$

□

`CompactTopologyByConvergne` ::

$:: \forall X \in \text{SET} . \forall \mathcal{C} : \text{FilterConvergence}(X) . \text{Compact}(X, \tau_{\mathcal{C}}) \iff$   
 $\iff \forall \mathcal{F} : \text{Ultrafilter}(X) . \exists x \in X . (x, \mathcal{F}) \in \mathcal{C}$

`Proof` =

( $\Rightarrow$ ) Firstly, assume that  $(X, \tau_{\mathcal{C}})$  is compact.

Take some  $\mathcal{F} \in F(X)$ .

If  $\mathcal{F}$  has no limits, then for any  $x \in X$  we may select  $U \in \mathcal{U}(x)$  such that  $U \notin \mathcal{F}$  .

Then there exists a finite subcover  $\mathcal{O}$  of  $\text{Im } U$ .

There is a set  $A \in \mathcal{F}$  as  $\mathcal{F}$  is a filter.

But as  $\mathcal{O}$  is a cover there must be  $O \in \mathcal{O}$  such that  $O \cap A \in \mathcal{F}$  .

Furthermore, as  $\mathcal{F}$  is upward closed,  $O \in \mathcal{F}$ , which produces a contradiction.

( $\Leftarrow$ ) Now assume that condion on  $\mathcal{C}$  holds.

Then every ultrafilter has a limit.

But this also means that every net has a convergent subnet .

But by `CompactLimitTHM` this means that  $(X, \tau_{\mathcal{C}})$  is compact .

□

`HausdorffTopologyByConvergne` ::

$:: \forall X \in \text{SET} . \forall \mathcal{C} : \text{FilterConvergence}(X) . \text{Hausdorff}(X, \tau_{\mathcal{C}}) \iff$   
 $\iff \forall \mathcal{F} : \text{Ultrafilter}(X) . \left| \left\{ \exists x \in X . (x, \mathcal{F}) \in \mathcal{C} \right\} \right| \leq 1$

`Proof` =

( $\Rightarrow$ ) Assume  $x, y \in X$  are two distinct limits of the ultrafilter  $\mathcal{F}$ .

Then there are disjoint  $U \in \mathcal{U}(x)$  and  $V \in \mathcal{U}(y)$ .

But then  $U, V \in \mathcal{F}$  but  $U \cap V = \emptyset \in \mathcal{F}$ , which is a contradiction.

( $\Leftarrow$ ) For  $x, y \in X$  there are filters  $\mathcal{N}(x), \mathcal{N}(y) \in F(X)$  such that  $x = \lim \mathcal{N}(x)$  and  $y = \lim \mathcal{N}(y)$ .

There are ultrafilters  $\mathcal{F}_x, \mathcal{F}_y$  such that  $\mathcal{N}(x) \subset \mathcal{F}_x$  and  $\mathcal{N}(y) \subset \mathcal{F}_y$ .

If for any pair of open neighborhoods  $(U, V) \in \mathcal{U}(x) \times \mathcal{U}(y)$ , then  $U \in \mathcal{N}(x)$  and  $V \in \mathcal{N}(y)$ .

But this means that  $x, y \in \lim \mathcal{F}_x$  and  $x, y \in \lim \mathcal{F}_y$ , which contradicts the assumption.

□

## 2.2.2 Compact Hausdorff Spaces

In Last section it was shown that the [FilterConvergence](#) relation  $\mathcal{C}$  becomes a function  $\lim : \mathbf{UF}(X) \rightarrow X$  if and only if the space  $(X, \tau_{\mathcal{C}})$  is compact and Hausdorff. Thus, it makes compact Hausdorff spaces of a particular interst as a category. In particular in this category any invertible continuous bijection will have a continuous inverse. Here I aslo prove the Tychonoff theorem, which is a very important result such that any product of compacts is compact.

[CategoryOfCompacta](#) :: CAT

[CategoryOfCompacta](#) () = HC :=  $\left( \text{Compact} \ \& \ \mathbf{T2}, C, \circ, \text{id} \right)$

[CompactExtensionCriterion](#) ::  $\forall X \in \mathbf{TOP} . \forall Y \in \mathbf{HC} . \forall D : \text{Dense}(X) . \forall f : D \xrightarrow{\text{TOP}} X .$

$. \left( \exists F : X \xrightarrow{\text{TOP}} Y : F|_D = f \right) \iff \forall A, B : \text{Closed}(Y) . A \cap B = \emptyset \Rightarrow \left( \text{cl}_x f^{-1}(A) \right) \cap \left( \text{cl}_y f^{-1}(B) \right) = \emptyset$

[Proof](#) =

[Assume](#)  $F : X \xrightarrow{\text{TOP}} Y,$

[Assume](#) [1] :  $F|_D = f,$

[Assume](#)  $A, B : \text{Closed}(Y),$

[Assume](#) [2] :  $A \cap B = \emptyset,$

[3] := [1][ClosedContainsLimits](#) :  $\text{cl}_X f^{-1}(A) = F^{-1}(A),$

[4] := [2][ClosedContainsLimits](#)(B) :  $\text{cl}_X f^{-1}(B) = F^{-1}(B),$

[F.\*] := [DisjointPreimage](#)[2][3][4] :  $\text{cl}_X f^{-1}(A) \cap \text{cl}_X f^{-1}(B) = \emptyset,$

$\leadsto$  [1] :=  $I(\Rightarrow) : \mathbf{LEFT} \Rightarrow \mathbf{RIGHT},$



Assume [2] : Right,

Assume  $x : X$ ,

Assume  $a, b : \text{Net}(\mathcal{U}_X(x), D)$ ,

Assume [3] :  $\forall U \in \mathcal{U}_X(x) . a_U, b_U \in U$ ,

Assume  $A : \text{Cluster}(f(a))$ ,

Assume  $B : \text{Cluster}(f(b))$ ,

Assume [4] :  $A \neq B$ ,

[5] := CompactIsNormal( $Y$ ) :  $(Y : \text{T4})$ ,

$(U, V, [6]) := \text{Urysohn}(Y)(A, B)[4][5] : \sum U \in \mathcal{U}(A) . \sum V \in \mathcal{U}(B) . \overline{U} \cap \overline{B} = \emptyset$ ,

[7] := [2][6] :  $\text{cl}_X f^{-1} \overline{U} \cap \text{cl}_X f^{-1} \overline{V} = \emptyset$ ,

[8] := MonotonicClosure :  $\text{cl}_X f^{-1}(U) \cap \text{cl}_X f^{-1}(V) = \emptyset$ ,

[9] := [8]( $x$ ) :  $x \notin \text{cl}_X f^{-1}(U) \Big| x \notin \text{cl}_X f^{-1}(V)$ ,

[10] :=  $\text{Cluster}(f(a))(A)(U)[3]\text{ClosureEqual}(X) : x \in \text{cl}_X f^{-1}(U)$ ,

[11] :=  $\text{Cluster}(f(b))(B)(V)[3]\text{ClosureEqual}(X) : x \in \text{cl}_X f^{-1}(V)$ ,

$[(a, b).*] := [9][10][11] : \perp$ ;

$\leadsto [3] := I(\forall)I(\Rightarrow)I(\forall)E(\perp) : \forall a, b : \text{Net}(\mathcal{U}_X(x), D) .$

$. (\forall U \in \mathcal{U}_X(x) . a_U, b_U \in \text{Im } U) \Rightarrow \forall A : \text{Cluster}(f(a)) . \forall B : \text{Cluster}(f(b)) . A = B$ ;

$(a, [4]) := \text{Dense}(X)(D) : \sum a : \text{Net}(\mathcal{U}_X(x), D) . \forall U \in \mathcal{U}(x) . a_U \in U$ ,

$F(x) := \lim_{U \in \mathcal{U}(x)} f(a_U) : Y$ ;

$\leadsto F := I(\rightarrow) : X \rightarrow Y$ ,

[3] :=  $jF : F|_D = f$ ,

Assume  $A : \text{Closed}(Y)$ ,

[4] :=  $jF(A) : F^{-1}(A) = \text{cl}_X f^{-1}(A)$ ,

$[*.A] := \text{cl}_X [4] : F^{-1}(A) : \text{Closed}(X)$ ;

$\leadsto [2.*] := \text{C}^{-1} : F \in C(X, Y)$ ;

$[*] := I(\iff)[1]I(\Rightarrow) : \text{This}$ ;

□

**CompactCoproduct** ::  $\forall I \in \mathbf{SET} . \forall X : I \rightarrow \mathbf{TOP} . \prod_{i \in I} X_i \in \mathbf{HC} \iff \text{Im } X \subset \mathbf{HC} \ \& \ I : \mathbf{Finite}$

**Proof** =

...

□

**TychonoffTheorem** ::  $\forall I \in \mathbf{SET} . \forall X : I \rightarrow \mathbf{TOP} . \prod_{i \in I} X_i \in \mathbf{HC} \iff \text{Im } X \subset \mathbf{HC}$

**Proof** =

...

□

**TychonoffUniversality** ::  $\forall \kappa : \mathbf{InfiniteCardinal} . [0, 1]^\kappa : \mathbf{Universal} \left\{ X \in \mathbf{HC} : w(X) = \kappa \right\}$

**Proof** =

...

□

**TychonoffCriterion** ::  $\forall X \in \mathbf{TOP} . X : \mathbf{Tychonoff} \iff \exists K \in \mathbf{HC} : \exists \mathbf{HomeomorphicEmbedding}(X, K)$

**Proof** =

...

□

**WallaceTheorem** ::  $\forall I \in \mathbf{SET} . \forall X : I \rightarrow \mathbf{TOP} . \forall K : \prod_{i \in I} \mathbf{HC} \ \& \ \mathbf{Subspace}(X_i) . \forall W : \mathbf{Open} \prod_{i \in I} X_i .$   
 $\forall [0] : \prod_{i \in I} K_i \subset W . \exists U : \prod_{i \in I} \mathbf{Open}(X_i) : \prod_{i \in I} K_i \subset \prod_{i \in I} U_i \subset W \ \& \ \left| \{i \in I : U_i \neq X_i\} \right| < \infty$

**Proof** =

...

□

**AlexandroffTheorem** ::  $\forall X \in \mathbf{HC} . \forall E : \mathbf{Equivalence}(X) . E : \mathbf{Closed} \iff \exists Y : \mathbf{T2} : \exists f : X \xrightarrow{\mathbf{TOP}} Y :$   
 $\frac{X}{E} \cong_{\mathbf{TOP}} \frac{X}{f}$

**Proof** =

...

□

### 2.2.3 Compactly Generated Spaces

One issue with compact Hausdorff spaces is that it do not have all categorical limits and colimits. This problem may be overcome by introducing notion of compactly generated spaces.

`CompactlyGenerated` :: ?T2

$X : \text{CompactlyGenerated} \iff \exists Q : \text{LocallyCompact} : \exists \pi : \text{QuotientMapping}(\text{TOP}, Q, X)$

`compactlyGenerated` :: CAT

$\text{compactlyGenerated}() = \text{CG} := (\text{CompactlyGenerated}, C, \circ, \text{id})$

`LocallyCompactIsCompactlyGenerated` ::  $\forall X : \text{LocallyCompact} . \forall X \in \text{CG}$

`Proof` =

...

□

`CompactlyGeneratedAlternativeDefinition` ::  $\forall X : \text{T2} . X \in \text{CG} \iff \forall A \subset X .$

$. A : \text{Closed}(X) \iff \forall K : \text{Compact}(X) . K \cap A : \text{Closed}(K)$

`Proof` =

...

□

`DualCompactlyGeneratedAlternativeDefinition` ::  $\forall X : \text{T2} . X \in \text{CG} \iff \forall A \subset X .$

$. A \in \mathcal{T}(X) \iff \forall K : \text{Compact}(X) . K \cap A \in \mathcal{T}(K)$

`Proof` =

...

□

`SequentialHausdorffIsCompactlyGenerated` ::  $\forall X : \text{T2} \ \& \ \text{Sequential} . X \in \text{CG}$

`Proof` =

...

□

`CompactlyGeneratedContinuousMapping` ::  $\forall X \in \text{CG} . \forall Y \in \text{TOP} . \forall f : X \rightarrow Y . f \in C(X, Y) \iff \forall K : \text{Compact}(Y) . f|_{f^{-1}K} : \text{Closed}(f^{-1}K, K)$

`Proof` =

...

□

`CompactlyGeneratedClosedMapping` ::  $\forall X \in \text{TOP} . \forall Y \in \text{CG} . \forall f : X \rightarrow Y .$

$. f : \text{Closed} \iff \forall K : \text{Compact}(Y) . f|_{f^{-1}K} : \text{Closed}(f^{-1}(K), K)$

`Proof` =

...

□

**CompactlyGeneratedOpentMapping** ::  $\forall X \in \text{TOP} . \forall Y \in \text{CG} . \forall f : X \rightarrow Y .$

$. f : \text{Open} \iff \forall K : \text{Compact}(Y) . f|_{f^{-1}K} : \text{Open}(f^{-1}(K), K)$

**Proof** =

...

□

**CompactlyGeneratedQuotientMapping** ::  $\forall X \in \text{TOP} . \forall Y \in \text{CG} . \forall f : X \rightarrow Y .$

$. f : \text{QuotientMapping} \iff \forall K : \text{Compact}(Y) . f|_{f^{-1}K} : \text{QuotientMapping}(f^{-1}(K), K)$

**Proof** =

...

□

**CompactlyGeneratedTransition** ::  $\forall X \in \text{CG} . \forall Y : \text{T2} . \forall \pi : \text{QuotientMapping}(X, Y) . Y \in \text{CG}$

**Proof** =

...

□

**CompactlyGeneratedSpacesHaveFiniteProducts** ::  $\text{CG} : \text{HasFiniteProducts}$

**Proof** =

...

□

**spaceKaonization** ::  $\text{TOP} \rightarrow \text{CG}$

**spaceKaonization**  $(X) = kX := \left( X, \{U \subset X : \forall K : \text{Compact}(X) \ \& \ \text{T2} . K \cap U \in \mathcal{T}(K)\} \right)$

**kaonizationFunctor** ::  $\text{TOP} \xrightarrow{\text{CAT}} \text{CG}$

**kaonizationFunctor**  $() = k := (\text{spaceKaonization}, \text{id})$

## 2.2.4 Compact-Open Topology

Here we try to set some notion of exponential objects for topological spaces. The problem comes from the fact that in case of pointwise topology, the currying operation is not continuous. The compact-open topology is acceptable in the sense, that currying is continuous. It turns out that compact-open topology is a generalization of uniform convergence. So many familiar results hold here, including Ascoli-theorem.

**DomainImageSet** ::  $\prod X, Y \in \mathbf{TOP} . ?X \rightarrow ?Y \rightarrow ?C(X, Y)$

**DomainImageSet**  $(A, B) = M(A, B) := \left\{ f \in C(X, Y) : f(A) \subset B \right\}$

**compactOpenTopology** ::  $\mathbf{TOP} \rightarrow \mathbf{TOP} \rightarrow \mathbf{TOP}$

**compactOpenTopology**  $(X, Y) = \mathcal{C}(X, Y) := \left\langle \left\langle \left\{ M(K, U) \mid K : \mathbf{Compact}(K), U \in \mathcal{T}(Y) \right\} \right\rangle \right\rangle_{\mathbf{TOP}}$

**RightCompositionIsContinuous** ::  $\forall X, Y, Z \in \mathbf{TOP} . \forall g \in C(Y, Z) . \rho_g \in C(\mathcal{C}(X, Y), \mathcal{C}(Y, Z))$

**Proof** =

...

□

**LetCompositionIsContinuous** ::  $\forall X : \mathbf{T2} . \forall g \in C(Y, Z) . \rho_g \in C(\mathcal{C}(X, Y), \mathcal{C}(Y, Z))$

**Proof** =

...

□

**CompactOpenTopologyIsProper** ::  $\forall X, Y \in \mathbf{TOP} . \mathcal{TC}(X, Y) : \mathbf{Proper}(X, Y)$

**Proof** =

...

□

**CompositionIsContinuous** ::  $\forall X, Z \in \mathbf{TOP} . \forall Y : \mathbf{LocallyCompact} . \circ_{X, Y, Z} \in C(\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z), \mathcal{C}(X, Z))$

**Proof** =

...

□

**CompactOpenTopologyIsAcceptable** ::  $\forall X : \mathbf{LocallyCompact} . \forall Y \in \mathbf{TOP} . \mathcal{TC}(X, Y) : \mathbf{Acceptable}$

**Proof** =

...

□

**CurryHomeomorphicEmbedding** ::  $\forall Y \in \mathbf{TOP} . \forall X, Z : \mathbf{T2} .$

$\Lambda : \mathbf{HomeomorphicEmbedding}(\mathcal{C}(X \times Z, Y), \mathcal{C}(X, \mathcal{C}(Z, Y)))$

**Proof** =

...

□

**LocallyCompactCurryHomeo** ::  $\forall Y \in \text{TOP} . \forall Z : \text{T2} . \forall X : \text{LocallyCompact} .$

$. \Lambda : \text{Homeo}(\mathcal{C}(X \times Z, Y), \mathcal{C}(X, \mathcal{C}(Z, Y)))$

**Proof** =

...

□

**CompactlyGeneratedCurryHomeo** ::  $\forall X, Y, Z \in \text{TOP} . \forall [0] : X \times Z \in \text{CG} .$

$. \Lambda : \text{Homeo}(\mathcal{C}(X \times Z, Y), \mathcal{C}(X, \mathcal{C}(Z, Y)))$

**Proof** =

...

□

**ExponentsOfCompactlyGeneratedSpace** ::  $\mathcal{C} : \text{Exponent}(\text{CG})$

**Proof** =

...

□

**CompactOpenTopologyPreservesRegularity** ::  $\forall i \in \{1, 2, 3, 3.5\} . \forall X \in \text{TOP} . \forall Y \in T(i) . \mathcal{C}(X, Y) \in T(i)$

**Proof** =

...

□

**ContinuousSupremum** ::  $\forall X \in \text{TOP} . \forall K : \text{Compact}(X) . \Lambda f \in C(X, \mathbb{R}) \sup_{x \in K} f(x) \in C(\mathcal{C}(X, \mathbb{R}))$

**Proof** =

...

□

**CompactOpenTopologyPreservesWeight** ::  $\forall X, Y \in \text{TOP} . \forall \kappa : \text{InfiniteCardinal} . [1] : w(X), w(Y) \leq \kappa .$   
 $. w\mathcal{C}(X, Y) \leq \kappa$

**Proof** =

...

□

**EvenlyContinuous** ::  $\forall X, Y \in \text{TOP} . ??\mathcal{C}(X, Y)$

$F : \text{EvenlyContinuous} \iff \forall x \in X . \forall y \in Y . \forall V \in \mathcal{U}(y) . \exists U \in \mathcal{U}(x) : \exists W \in \mathcal{U}(y) : (F \cap M(x, W)) \subset V$

**EvenlyContinuousClosure** ::  $\forall X \in \text{TOP} . \forall Y : \text{T3} . \forall F : \text{EvenlyContinuous}(X, Y) .$

$. \text{cl}_{\mathcal{C}(X, Y)} F : \text{EvenlyContinuous}(X, Y)$

**Proof** =

...

□

**AscoliTheorem** ::  $\forall X \in \mathbf{CG} . \forall Y : \mathbf{T3} . \forall F \subset \mathcal{C}(X, Y)$

$$. F : \mathbf{EvenlyContinuous} \ \& \ \forall [0] : \forall x \in X . \overline{F(x)} : \mathbf{Compact}(Y) \iff F : \mathbf{Compact}(\mathcal{C}(X, Y))$$

**Proof** =

...

□

**LocalAscoliTheorem** ::  $\forall X \in \mathbf{CG} . \forall Y : \mathbf{T3} . \forall F \subset \mathcal{C}(X, Y)$

$$. \forall K : \mathbf{Compact}(X) . F|_K : \mathbf{EvenlyContinuous} K, Y \ \& \ \forall [0] : \forall x \in X . \overline{F(x)} : \mathbf{Compact}(Y) \iff F : \mathbf{Compact}$$

**Proof** =

...

□

**DiniTheorem** ::  $\forall X : \mathbf{Compact} . \forall f : \mathbf{Monotonic}(C(X)) . \forall F : X \rightarrow \mathbb{R} .$

$$\left( \forall x \in X . \lim_{n \rightarrow \infty} f_i(x) = F(x) . \right) \Rightarrow f \Rightarrow F$$

**Proof** =

...

□

**StoneWeierstassTheorem** ::  $\forall X : \mathbf{Compact} . \forall P : \mathbf{Subalgebra}(\mathbb{R}, C(X)) .$

$$. \mathbf{SeparatesPoints}(X)(P) \Rightarrow P : \mathbf{Dense}(C(X), \mathbf{uniformTopology}(X, \mathbb{R}))$$

**Proof** =

...

□

## 2.3 Compactifications

### 2.3.1 Subject

$\text{Compactification} :: \prod X : \text{TOP} . ? \sum K \in \text{HC} . \text{HomeomorphicEmbedding}(X, K)$   
 $(K, \iota) : \text{Compactification} \iff \text{cl}_K \iota(X) = K$

$\text{CompactificationIfTychonoff} :: \forall X \in \text{TOP} . X : \text{Tychonoff} \iff \exists \text{Compactification}(X)$

$\text{Proof} =$

...

□

$\text{CompactificationWeight} :: \forall X : \text{Tychonoff} . \exists (K, \iota) : \text{Compactification} : w(K) = w(X)$

$\text{Proof} =$

...

□

$\text{compactificationCategory} :: \text{TOP} \rightarrow \text{CAT}$

$\text{compactificationCategory}(X) = \mathcal{C}(X) :=$

$= \left( \text{Compactification}, (A, \alpha), (B, \beta) \mapsto \{f : A \xrightarrow{\text{TOP}} B . \alpha f = \beta\}, \circ, \text{id} \right)$

$\text{CompactificationCardinalityBound} :: \forall X \in \text{TOP} . \forall (K, \iota) \in \mathcal{C}(X) . |K| \leq \exp \exp d(X)$

$\text{Proof} =$

...

□

$\text{CompactificationWeightBound} :: \forall X \in \text{TOP} . \forall (K, \iota) \in \mathcal{C}(X) . w(K) \leq \exp d(X)$

$\text{Proof} =$

...

□

$\text{CompactificationCategoryIsPoset} :: \forall X \in \text{TOP} . \mathcal{C}(X) : \text{Poset}$

$\text{Proof} =$

...

□

$\text{EquivalentCompactificationCriterion} :: \forall X \in \text{TOP} . \forall (A, \alpha), (B, \beta) \in \mathcal{C}(X) . (A, \alpha) \cong_{\mathcal{C}(X)} (B, \beta) \iff$   
 $\iff \forall x, y : \text{Closed}(X) . \overline{\alpha x} \cap \overline{\alpha y} \iff \overline{\beta x} \cap \overline{\beta y} .$

$\text{Proof} =$

...

□



`reminder` ::  $\prod X \in \text{TOP} . \prod (K, \iota) : \text{Compactification}(X) . ?K$

`reminder` () =  $\text{rem } \iota := K \setminus \iota(X)$

`CompactificationReminderTheorem` ::  $\forall X \in \text{TOP} . \forall (A, \alpha), (B, \beta) \in \mathcal{C}(X) . \forall f : (A, \alpha) \xrightarrow{\mathcal{C}(X)} (B, \beta) .$   
. $f(\text{rem } \alpha) = \text{rem } \beta$

`Proof` =

...

□

`LocallyCompactCompactification1` ::  $\forall X : \text{Tychonoff} . X : \text{LocallyComapct} \iff \forall (K, \iota) \in \mathcal{C}(X) .$   
. $\text{rem } \iota : \text{Closed}(K)$

`Proof` =

...

□

`LocallyCompactCompactification2` ::  $\forall X : \text{Tychonoff} . X : \text{LocallyComapct} \iff \exists (K, \iota) \in \mathcal{C}(X) :$   
. $\text{rem } \iota : \text{Closed}(K)$

`Proof` =

...

□

`CompactificationLeastUpperBoundProperty` ::  $\forall X \in \text{TOP} . \mathcal{C}(X) : \text{LUBProperty}$

`Proof` =

...

□

`compactificationOfStoneAndChech` ::  $\prod X : \text{Tychonoff} . \mathcal{C}(X)$

`compatificationOfStoneAndChech` () =  $\beta X := \sup \mathcal{C}(X)$

`AlexandroffCompactificationTHM` ::  $\forall X : \text{LocallyCompact} . \exists (K, \iota) \in \mathcal{C}(X) . |\text{rem } \iota| = 1$

`Proof` =

...

□

`onePointCompactification` ::  $\prod X : \text{LocallyCompact} . \mathcal{C}(X)$

`onePointCompactification` () =  $\omega X := \text{AlexandroffCompactificationTheorem}(X)$

`OnePointIsInf` ::  $\forall X : \text{LocallyComapct} ! \text{Compact} \omega X = \inf \mathcal{C}(X)$

`Proof` =

...

□

**LocallyCompactByOneMinimalCompactification** ::  $\forall X : \text{Tychonoff} . \forall (K, \iota) \in \mathcal{C}(X) .$   
 $(K, \iota) = \inf \mathcal{C}(X) \Rightarrow X : \text{LocallyCompact}$

**Proof** =

...  
 □

**ReconstructionByReminder** ::  $\forall Y \in \text{HC} . \forall X : \text{LocallyCompact} . \forall (K, \iota) \in \mathcal{C}(X) . \forall f : \text{rem } \iota \xrightarrow{\text{TOP}} Y .$   
 $f(\text{rem } \iota) = Y \Rightarrow \exists (Y', \gamma) : \text{rem } \gamma = Y$

**Proof** =

...  
 □

### 2.3.2 Stone-Čech Functor

$$\mathbf{StoneCechSpace} :: \prod X \in \mathbf{TOP} . \sum \Omega : \mathbf{HC} . X \xrightarrow{\mathbf{TOP}} \Omega$$

$$(\Omega, \varphi) : \mathbf{StoneCechSpace} \iff \forall K \in \mathbf{HC} . \forall f : X \xrightarrow{\mathbf{TOP}} K . \exists g : \Omega \xrightarrow{\mathbf{HC}} K : f = \varphi g$$

$$\mathbf{StoneCechSpaceExists} :: \forall X \in \mathbf{TOP} . \exists \mathbf{StoneCechSpace}(X)$$

**Proof** =

...

□

$$\mathbf{StoneCechSpaceHomeo} :: \forall X \in \mathbf{TOP} . \forall (A, \varphi), (B, \psi) : \mathbf{StoneCechSpace}(X) . A \cong_{\mathbf{HC}} B$$

**Proof** =

...

□

$$\mathbf{StoneCechFunctor} :: \mathbf{TOP} \xrightarrow{\mathbf{CAT}} \mathbf{HC}$$

$$\mathbf{StoneCechFunctor}(X) = \beta X := \mathbf{StoneCechSpaceExists} \ \& \ \mathbf{StoneCechSpaceHome}(X)$$

$$\mathbf{StoneCechFunctor}(X, Y, f) = \beta f := \mathbf{StoneCechSpace}(\omega X)(f \varphi_Y)$$

$$\mathbf{StoneCechConsistancy} :: \forall X : \mathbf{Tychonoff} . \mathbf{CompactificationOfStoneAndCech}(X) : \mathbf{StoneCechSpace}(X)$$

**Proof** =

...

□

$$\mathbf{StoneCechAdjoint} :: \beta : \mathbf{LeftAdjoint}(U_{\mathbf{HC}, \mathbf{TOP}})$$

**Proof** =

...

□

$$\mathbf{CompleteSeparationInStoneCech} :: \forall X \in \mathbf{TOP} . \forall (A, B) : \mathbf{CompletelySeparated}(X) .$$

$$. (\overline{\varphi_X A}, \overline{\varphi_X B}) : \mathbf{CompletelySeparated}(\beta X)$$

**Proof** =

...

□

$$\mathbf{StoneCechByCompleteSeparation} :: \forall X \in \mathbf{TOP} . \forall (K, \iota) \in \mathcal{C}(X) .$$

$$. \forall [1] : \forall (A, B) : \mathbf{CompletelySeparated}(X) . (\overline{\iota A}, \overline{\iota B}) : \mathbf{CompletelySeparated}(K) . (K, \iota) \cong_{\mathcal{C}(X)} (\beta X, \varphi_X)$$

**Proof** =

...

□

$$\mathbf{StoneCechClopenSubset} :: \forall X : \mathbf{Tychonoff} . \forall A : \mathbf{Clopen}(X) . \overline{\varphi_X A} : \mathbf{Clopen}(\beta X)$$

**Proof** =

...

□

**StoneCechSubspaceCompactification** ::  $\forall X : \mathbf{Tychonoff} . \forall A \subset X .$

$$. \forall [0] : \forall f : A \xrightarrow{\text{TOP}} [0, 1] . \exists F : X \xrightarrow{\text{TOP}} [0, 1] : F|_A = f . (\text{cl}_{\beta X} A, \varphi_x) \in \mathcal{C}(A)$$

**Proof** =

...

□

**NormalStoneCechSubspaceCompactification** ::  $\forall X : \mathbf{T4} . \forall A \subset X \text{cl}_{\beta X} A \cong_{\text{TOP}} \beta A$

**Proof** =

...

□

**StoneCechSuperspaceCompactification** ::  $\forall X : \mathbf{Tychonoff} . \forall A \subset \beta X . X \subset A \Rightarrow \beta A = \beta X$

**Proof** =

...

□

**DiscreteStoneCechCardinality** ::  $\forall X \in \mathbf{SET} . |\beta D X| = \exp \exp |X|$

**Proof** =

...

□

**DiscreteStoneCechWeight** ::  $\forall X \in \mathbf{SET} . w(\beta D X) = \exp |X|$

**Proof** =

**ClopenSubsetInDiscreteStoneCech** ::  $\forall X \in \mathbf{SET} . \forall x \in \beta D X . \forall U \in \mathcal{U}_{\beta D X}(x) .$   
 $. \exists V : \mathbf{Clopen}(\beta D X) : V \subset U$

**Proof** =

$$(W, [1]) := \mathbf{AltT4}(\beta D X, U) : \sum W : \mathbf{Open}(\beta D X) . \overline{W} \subset U,$$

$$A := \varphi_{DX} \varphi_{DX}^{-1}(W) : ?\beta D X,$$

$$[2] := \mathfrak{d}\mathbf{StoneCechClopenSubset}(X, A) : (\overline{A} : \mathbf{Clopen}(\varphi_X X)),$$

$$[3] := jA\mathfrak{d}\mathbf{preimage} : A \subset W,$$

$$[4] := \mathbf{MonotonicClosure}(\beta D X)[3][1] : \overline{A} \subset \overline{W} \subset U;$$

□

**NaturalNumbersStoneCechSelfsimilarity** ::  $\forall A : \mathbf{Closed} \ \& \ \mathbf{Infinite}(\beta \mathbb{N}) . \exists B \subset A : B \cong_{\text{TOP}} \beta \mathbb{N}$

**Proof** =

...

□

**NaturalStoneCechConvergentSequences** ::  $\forall x : \mathbf{Convergent}(\mathbb{N}, \beta \mathbb{N}) . x : \mathbf{FinallyConstant}$

**Proof** =

...

□

### 2.3.3 Wallman Extension

**FilterDisownesEmptySet** ::  $\forall X \in \text{SET} . \forall \mathcal{X} \in ??X . \forall F : \text{Filter}(\mathcal{X}) . \emptyset \notin F$

**Proof** =

...

□

**FilterIntersectionClosed** ::  $\forall X \in \text{SET} . \forall \mathcal{X} \in ??X . \forall F : \text{Filter}(\mathcal{X}) . \forall A, B \in F . A \cap B \in F$

**Proof** =

...

□

**GreedyUltrafilters** ::  $\forall X \in \text{SET} . \forall \mathcal{X} \in ??X . \forall F : \text{Ultrafilter}(\mathcal{X}) . \forall A \in \mathcal{X} . \forall [1] : \forall B \in F . A \cap B \neq \emptyset .$

**Proof** =

...

□

**UltrafilterSupercomplete** ::  $\forall X \in \text{SET} . \forall \mathcal{X} \in ??X . \forall F : \text{Ultrafilter}(\mathcal{X}) . \forall A \in F . \forall B \in \mathcal{X} . A \subset B \Rightarrow B \in F$

**Proof** =

...

□

**DifferentUltrafilters** ::  $\forall X \in \text{SET} . \forall \mathcal{X} \in ??X . \forall F, G : \text{Ultrafilter}(\mathcal{X}) . F \neq G \iff \exists A \in F : \exists B \in G : A \cap B = \emptyset$

**Proof** =

**PrincipleUltrafilter** ::  $\prod X \in \text{SET} . \prod \mathcal{X} \in ??X . ?\text{Ultrafilter}(\mathcal{X})$

$F : \text{PrincipleUltrafilter} \iff \exists x \in X : \bigcap_{A \in F} A = \{x\}$

**NonPrincipleUltrafilter** ::  $\prod X \in \text{SET} . \prod \mathcal{X} \in ??X . ?\text{Ultrafilter}(\mathcal{X})$

$F : \text{NonPrincipleUltrafilter} \iff \bigcap_{A \in F} A = \emptyset$

**T1UltrafilterClassification** ::  $\forall X : \text{T1} . \forall F : \text{Ultrafilter Closed}(X) .$

$. F : \text{PrincipleUltrafilter Closed}(X) \Big| F : \text{NonPrincipleUltrafilter Closed}(X)$

**Proof** =

...

□

WallmanExtension :: T1 → T1 & Compact

WallmanExtension (X) = W(X) :=  $\left\langle \left\{ \{F : \text{Ultrafilter Closed}(X) : \exists A \in F : A \subset U\} \mid U \in \mathcal{T}(X) \right\} \right\rangle_{\text{TOP}}$

WallmanEmbedding ::  $\prod X \in \text{T1} . X \xrightarrow{\text{TOP}} W(X)$

WallmanEmbedding (X) =  $w_X := \left\{ A : \text{Closed}(X) : x \in X \right\}$

WallmanExtensionTheorem ::  $\forall X : \text{T1} . \overline{w(X)} = W(X)$

Proof =

...

□

WallmanExtensionUniversality ::  $\forall X : \text{T1} . \forall Z : \text{Compact} . \forall f : X \xrightarrow{\text{TOP}} Z . \exists g : W(X) \xrightarrow{\text{TOP}} Z : f = wg$

Proof =

...

□

WallmanExtensionRegularity ::  $\forall X : \text{T1} . W(X) : \text{T2} \iff X : \text{T4}$

Proof =

...

□

WallmanStoneCechEquivalence ::  $\forall X : \text{T4} . W(X) \cong_{\text{TOP}} \beta X$

Proof =

...

□

### 2.3.4 Perfect Mappings

**Perfect** ::  $\prod X : \mathbf{T2} . \prod Y \in \mathbf{TOP} . ?\mathbf{Closed}(X, Y)$

$f : \mathbf{Perfect} \iff \forall y \in Y . f^{-1}(y) : \mathbf{Compact}$

**PerfectInjection** ::  $\forall X : \mathbf{T2} . \forall Y \in \mathbf{TOP} . \forall f : \mathbf{Injection}(X, Y) .$

$. f : \mathbf{Perfect}(X, Y) \iff f : \mathbf{Closed}(X, Y)$

**Proof** =

...

□

**PerfectInclusion** ::  $\forall X : \mathbf{T2} . \forall A \subset X . \iota_A : \mathbf{Perfect}(A, X) \iff A : \mathbf{Closed}(X)$

**Proof** =

...

□

**PerfectProjection** ::  $\forall X \in \mathbf{HC} . \forall Y : \mathbf{T2} . \pi_Y : \mathbf{Perfect}(X \times Y, Y)$

**Proof** =

...

□

**CompactPerfectPreimage** ::  $\forall X : \mathbf{T2} . \forall Y \in \mathbf{TOP} . \forall f : \mathbf{Perfect}(X, Y) . \forall K \subset Y . \forall [0] : K \in \mathbf{HC} .$   
 $. f^{-1}(K) \in \mathbf{HC}$

**Proof** =

[1] := **HausdorffSubset**( $X, f^{-1}(K)$ ) :  $(f^{-1}(K) : \mathbf{T2})$ ,

**Assume**  $A : \mathbf{Filter} \mathbf{Closed}(f^{-1}(K))$ ,

[2] :=  $\mathfrak{d}\mathbf{Closed}(X, Y)(f) : f(A) \in ?\mathbf{Closed}(Y)$ ,

[3] := **ImageIntersection**( $f$ ) :  $(f(A) : \mathbf{FiniteIntersectionProperty}(K))$ ,

[4] := **CompactByFiniteIntersection**[3] :  $\bigcap_{a \in A} f(a) \neq \emptyset$ ,

$y$ ) :=  $\mathfrak{d}\mathbf{NonEmpty}$ [4] :  $\bigcap_{a \in A} f(a)$ ,

[5] :=  $\mathfrak{d}\mathbf{Perfect}(X, Y)(f) : (f^{-1}(y) : \mathbf{Compact})$ ,

$B := A \cap f^{-1}(y) : ?\mathbf{Closed}(f^{-1}(x))$ ,

[6] :=  $\jmath B \jmath y : \forall b \in B . b \neq \emptyset$ ,

[7] := **FilterRestriction**[6]( $A, B$ ) :  $(B : \mathbf{Filter}(\mathbf{Closed}(f^{-1}y)))$ ,

[8] :=  $\mathfrak{d}\mathbf{Filter}(B)\mathfrak{d}^{-1}\mathbf{FiniteIntersection} : (B : \mathbf{FiniteIntersectionProperty}(\mathbf{Closed}(f^{-1}y)))$ ,

[9] := **CompactByFiniteIntersection**( $B$ ) :  $\bigcap B \neq \emptyset$ ,

[\*] := **SubsetIntersectionNonEmpty**( $A, B$ )[9] :  $\bigcap A \neq \emptyset$ ;

$\leadsto [*] := \mathbf{CompactByFilterPrincipality} : (f^{-1}K : \mathbf{Compact})$ ;

□

**PerfectComposition** ::  $\forall X, Y : \mathbf{T2} . \forall Z \in \mathbf{TOP} . \forall f : \mathbf{Perfect}(X, Y) . \forall g : \mathbf{Perfect}(Y, X) .$   
 $\quad . fg : \mathbf{Perfect}(X, Y)$

**Proof** =

**Assume**  $z : Z$ ,

$[1] := \mathfrak{d}\mathbf{Perfect}(g)(z) : (g^{-1}(z) : \mathbf{Compact}(Y)),$

$[2] := \mathbf{CompactPerfectPreimage}(f)[1] : (f^{-1}g^{-1}(z) : \mathbf{Compact}(X)),$

$[z.*] := \mathbf{PreimageComposition}(f, g, z)[2] : ((fg)^{-1}(z) : \mathbf{Compact}(X));$

$\leadsto [*] := \mathfrak{d}^{-1}\mathbf{Perfect}(X, Z) : (fg : \mathbf{Perfect});$

□

**PerfectRestriction** ::  $\forall X : \mathbf{T2} . \forall Z \in \mathbf{TOP} . \forall f : \mathbf{Perfect}(X, Y) .$   
 $\quad . \forall A : \mathbf{Closed}(X) . f|_A : \mathbf{Perfect}$

**Proof** =

...

□

**PerfectProduct** ::  $\forall I \in \mathbf{SET} . \forall X : I \rightarrow \mathbf{T2} . \forall Y : I \rightarrow \mathbf{TOP} .$

$\quad . \forall f : \prod_{i \in I} X_i \xrightarrow{\mathbf{TOP}} Y_i .$

$\quad . \prod_{i \in I} f_i : \mathbf{Perfect} \left( \prod_{i \in I} X_i, \prod_{i \in I} Y_i \right) \iff \forall i \in I . f_i : \mathbf{Perfect}(X_i, Y_i)$

**Proof** =

...

□



### 2.3.5 Abstract Baire Theory

$\text{CechCompleteSpace} :: ?\text{Tychonoff}$

$X : \text{CechCompleteSpace} \iff \forall (K, \gamma) \in \mathcal{K}(X) . \text{rem } \gamma : F_\sigma(K)$

$\text{BaireSpace} :: ?\text{TOP}$

$X : \text{BaireSpace} \iff \forall A : \mathbb{N} \rightarrow \text{NowhereDense}(X) . \bigcup_{i=1}^{\infty} A_i : \text{Codense}(X)$

$\text{BaireCategoryTheorem} :: \forall X : \text{CechCompleteSpace} . X : \text{BaireSpace}$

$\text{Proof} =$

...

□

$\text{DualBaireProperty} :: \forall X : \text{BaireSpace} . \forall U : \mathbb{N} \rightarrow \mathcal{T} \ \& \ \text{Dense}(X) . \bigcap_{i=1}^{\infty} U_i : \text{Dense}(X)$

$\text{Proof} =$

...

□

### 2.3.6 Realcompact Spaces

$\text{Realcompact} :: ?\text{TOP}$

$X : \text{Realcompact} \iff \exists \kappa \in \text{CARD} : \exists A : \text{Closed}(\mathbb{R}^\kappa) : \exists \varphi : \text{HomeomorphicEmbedding}(X, A)$

## 3 Cardinal Functions

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