Multilinear Algebra

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1 Vector Spaces and Modules

<u>`def`</u> R-`Module` $:: \forall R : \texttt{Ring} \,.\, \exists M : \texttt{Abelean} \,.\, \mathcal{H}_{\mathsf{RING}}(R, \texttt{End}(M))$

<u>`def`</u> $\mathcal{L}(A, B) :: \forall A, B : R\text{-}\texttt{Module} \,.\, ?\mathcal{H}_{\mathsf{GRP}}(A, B)$

$T : \mathcal{L}(A, B) \iff \forall r \in R \,.\, \forall a \in A \,.\, Tra = rTa$

<u>`def`</u> $\texttt{Submodule}(M) :: \forall M : R\text{-}\texttt{Module} \,.\, R\text{-}\texttt{Module} \wedge ?M$

$(S, \psi) : \texttt{Submodule}((X, \phi)) \iff \phi_{|S} = \psi$

<u>`def`</u> $\ker :: \mathcal{L}((X, \phi), B) \to \texttt{Submodule}(X, \phi)$

$\ker T = \big(S \leftarrow \{x \in X : Tx = 0\}, \phi_{|S}\big)$

<u>`def`</u> $\operatorname{Im} :: \mathcal{L}(A, (Y, \phi)) \to \texttt{Submodule}(Y, \phi)$

$\operatorname{Im} T = \big(S \leftarrow TA, \phi_{|S}\big)$

<u>`def`</u> $\operatorname{Coker} :: \mathcal{L}(A, B) \to R\text{-}\texttt{Module}$

$$\operatorname{Coker} T = \frac{B}{\operatorname{Im} T}$$

<u>`def`</u> $\operatorname{coim} :: \mathcal{L}(A, B) \to R\text{-}\texttt{Module}$

$$\operatorname{coim} T = \frac{A}{\ker T}$$

$\underline{\texttt{def}}$ $\cdot\text{-}\mathsf{MOD} :: \texttt{Ring} \to \texttt{Category}$

$\mathcal{O}(R\text{-}\mathsf{MOD}) = R\text{-}\texttt{Module}$

$\mathcal{H}_{R\text{-}\mathsf{MOD}}(A, B) = \mathcal{L}(A, B)$

$\cdot_{R\text{-}\mathsf{MOD}} = \circ$

$\underline{\texttt{def}}$ $K\text{-}\texttt{VectorSpace} :: \forall R : \texttt{Field} \,.\, \exists M : \texttt{Abelean} \,.\, \mathcal{H}_{\mathsf{RING}}(K, \texttt{End}(M))$

$\underline{\texttt{def}}$ $\mathcal{L}(A, B) :: \forall A, B : K\text{-}\texttt{VectorSpace} \,.\, ?\mathcal{H}_{\mathsf{GRP}}(A, B)$

$T : \mathcal{L}(A, B) \iff \forall r \in R \,.\, \forall a \in A \,.\, Tra = rTa$

$\underline{\texttt{def}}$ $\texttt{Subspace}(V) :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, K\text{-}\texttt{VectorSpace} \wedge ?V$

$(S, \psi) : \texttt{Submodule}((X, \phi)) \iff \phi_{|S} = \psi$

$\underline{\texttt{def}}$ $\ker :: \mathcal{L}((X, \phi), B) \to \texttt{Subspace}(X, \phi)$

$$\ker T = \big(S \leftarrow \{x \in X : Tx = 0\}, \phi_{|S}\big)$$

$\underline{\texttt{def}}$ $\operatorname{Im} :: \mathcal{L}(A, (Y, \phi)) \to \texttt{Subspace}(Y, \phi)$

$$\operatorname{Im} T = \big(S \leftarrow TA, \phi_{|S}\big)$$

$\underline{\texttt{def}}$ $\operatorname{Coker} :: \mathcal{L}(A, B) \to K\text{-}\texttt{VectorSpace}$

$$\operatorname{Coker} T = \frac{B}{\operatorname{Im} T}$$

$\underline{\texttt{def}}$ $\operatorname{coim} :: \mathcal{L}(A, B) \to$

$$\operatorname{coim} T = \frac{A}{\ker T}$$

$\underline{\texttt{def}}$ $\cdot\text{-}\mathsf{VS} :: \texttt{Field} \to \texttt{Category}$

$\mathcal{O}(K\text{-}\mathsf{VS}) = K\text{-}\texttt{VectorSpace}$

$\mathcal{H}_{K\text{-}\mathsf{VS}}(A, B) = \mathcal{L}(A, B)$

$\cdot_{K\text{-}\mathsf{VS}} = \circ$

1.1 Basis of a Module

<u>def</u> $\texttt{Free} :: \forall R : \texttt{Ring} \,.\, \texttt{Set} \to R\text{-}\texttt{Module}$

$F^R(S) = \Big(\big\{f : S \to R : \{x \in S : f(x) \neq 0\} : \texttt{Finite}\big\}, \lambda r \,.\, \lambda v \,.\, (\lambda x \in S \,.\, r)v\Big)$

<u>def</u> $\texttt{IndexedSet} :: \forall M : R\text{-}\texttt{Module} \,.\, \exists I : \texttt{Set} \,.\, I \to M$

<u>def</u> $\texttt{orthant} :: \forall I : \texttt{Set} \,.\, I \to F^R(I)$

$e_j = \lambda y \,.\, \delta_{x,y}$

<u>def</u> $\texttt{LinearCombination} :: \forall I : \texttt{Set} \,.\, \forall M : R\text{-}\texttt{Module} \,.\, (I \to M) \to F^R(I) \to M$

$L(v)(\alpha) = \sum_{i \in I} \alpha_j v_j$

<u>def</u> $\texttt{LinearlyIndependent} :: \forall M : R\text{-}\texttt{Module} \,.\, ?\texttt{IndexedSet}(M)$

$(I, v) : \texttt{LinearlyIndependent} \iff L_I(v) : \texttt{Injective}$

<u>def</u> $\texttt{Generates} :: \forall M : R\text{-}\texttt{Module} \,.\, ?\texttt{IndexedSet}(M)$

$(I, v) : \texttt{Generates} \iff L_I(v) : \texttt{Surjective}$

<u>def</u> $\texttt{Basis} :: \texttt{LinearlyIndependent} \wedge \texttt{Generates}$

<u>def</u> $\text{rank} :: \forall R : \texttt{IntegralDomain} \,.\, R\text{-}\texttt{Module} \to \texttt{Cardinal}$

$\operatorname{rank} M = \max_{(I,v) : \texttt{LinearlyIndependent}(M)} |I|$

<u>def</u> $\dim :: K\text{-}\texttt{VectorSpace} \to \texttt{Cardinal}$

$\dim V = \operatorname{rank} V$

<u>def</u> $\texttt{MaximalLI} :: \forall M : R\text{-}\texttt{Module} \,.\, \texttt{Maximal}\{w : \texttt{LinearlyIndependent}(V) : v \subset w\}$

<u>def</u> $\texttt{MinimalGenerator} :: \forall M : R\text{-}\texttt{Module} \,.\, \texttt{Minimal}\{w : \texttt{Generates}(V)\}$

<u>thm</u> $\texttt{maxLInd} :: \forall M : R\text{-}\texttt{Module} \,.\, \forall v : \texttt{LinearlyIndependent}(M) \,.\, \exists w : \texttt{MaximalLI}(M) : v \subset w$

<u>thm</u> $\texttt{freeBasis} :: \forall M : R\text{-}\texttt{Module} \,.\, (\exists S : \texttt{Set} : M = F^R(S)) \iff \texttt{Basis}(M)$

<u>thm</u> $\texttt{basisFree} :: \forall b : \texttt{IndexedSet}(M) \,.\, b : \texttt{Basis}(M) \iff M \cong F^R(b)$

<u>thm</u> $\texttt{maxIsBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \text{id} : \texttt{MaximalLI}(V) \to \texttt{Basis}(V)$

<u>thm</u> $\texttt{completeBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \forall v : \texttt{LinearlyIndependent}(V) \,.$

$.\, \exists b : \texttt{Basis}(V) : (v \subset b)$

<u>thm</u> $\texttt{minIsBasis} :: \forall V : K\text{-}\texttt{VectorSpace} \,.\, \text{id} : \texttt{MinimalGenerator}(V) \to \texttt{Basis}(V)$

<u>thm</u> $\texttt{lIndBound} :: \forall R : \texttt{IntegralDomain} \,.\, \forall M : R\text{-}\texttt{Module} \,.\, \forall b : \texttt{Basis}(M) \,.$

$.\, \forall v : \texttt{LinearlyIndependent}(M) \,.\, |v| \leq |b|$

$\underline{\texttt{def}}$ $\texttt{IBN} :: ?\texttt{Ring}$

$\texttt{IBN}(R) = \forall n, m \in \mathbb{N} \;.\; R^n \cong R^m \iff m = n$

$\underline{\texttt{thm}}$ $\texttt{commIsIBN} :: \forall R : \texttt{Commutative} \;.\; \texttt{IBN}(R)$

$\texttt{commIsIBN}(R) =$

$(R, |n, m \in R \vdash$

$(\Rightarrow) = |A : R^n \cong R^m \vdash\rightarrow \exists T : \texttt{Iso}_{R\text{-MOD}}(R^n, R^m) \rightarrow (1)$

$V.3.5 \rightarrow \exists \mathfrak{m} : \texttt{Maximal}(R) \rightarrow (2)$

$(2) \rightarrow \dfrac{R}{\mathfrak{m}} : \texttt{Field}$

$\underline{\texttt{def}}$ $T' : \left(\dfrac{R}{\mathfrak{m}}\right)^n \rightarrow \left(\dfrac{R}{\mathfrak{m}}\right)^m$

$T'(\bar{v}) = \overline{(Tv)}$

$(T', |\bar{v} \in \left(\dfrac{R}{\mathfrak{m}}\right)^m \vdash$

$(1) \rightarrow \exists w \in R^n : v = Tw \rightarrow (3)$

$T'(\bar{w}) = \overline{(Tw)} = \bar{v}|) : \texttt{Surjective}\left(\left(\dfrac{R}{\mathfrak{m}}\right)^n, \left(\dfrac{R}{\mathfrak{m}}\right)^m\right)$

$(T', |\bar{v} \in \left(\dfrac{R}{\mathfrak{m}}\right)^n : \bar{v} \neq 0 \vdash$

$\Gamma_0 : \bar{v} \neq 0 \rightarrow v \notin \mathfrak{m}R^n \rightarrow \exists i \in \mathbb{I}_n : v_i \notin \mathfrak{m} - (1) \rightarrow$

$\rightarrow Tv \notin \mathfrak{m}R^m \rightarrow T'(\bar{v}) \neq 0|) : \texttt{Injective}\left(\left(\dfrac{R}{\mathfrak{m}}\right)^n, \left(\dfrac{R}{\mathfrak{m}}\right)^m\right) \rightarrow$

$\rightarrow T' : \texttt{Iso}_{R\text{-VS}}\left(\left(\dfrac{R}{\mathfrak{m}}\right)^n, \left(\dfrac{R}{\mathfrak{m}}\right)^m\right) \rightarrow \left(\dfrac{R}{\mathfrak{m}}\right)^n \cong \left(\dfrac{R}{\mathfrak{m}}\right)^m -$

$- \texttt{IBN}\left(\dfrac{R}{\mathfrak{m}}\right) \rightarrow n = m| : R^n \cong R^m \Rightarrow m = n$

$(\Leftarrow) = |A : n = m \vdash$

$\texttt{freeBasis}(R)(\mathrm{id}(R^n)) \rightarrow \exists(\mathbb{I}_n, e) : \texttt{Basis}(R^n)$

$\texttt{freeBasis}(R)(\mathrm{id}(R^m)) \rightarrow \exists(\mathbb{I}_m, f) : \texttt{Basis}(R^m)$

$\underline{\texttt{def}}$ $T : R^n \rightarrow R^m$

$$T \sum_{i=1}^{n} v_i e_i = \sum_{i=1}^{n} v_i f_i$$

$A : n = m \to T : \texttt{Iso}_{R\text{-}\mathsf{MOD}}\,(R^n, R^m) \to R^n \cong R^m| :$
$| : n = m \Rightarrow R^n \cong R^m$
$(\Rightarrow, \Leftarrow) \to R^n \cong R^m \iff m = n| : \texttt{IBN}(R))\square$

1.2 Composition serias of modules

<u>def</u> $\texttt{Simple} :: ?R\text{-}\texttt{Module}$

$\texttt{Simple}(M) = \forall S : \texttt{Submodule}(M) \; . \; S = (0) | S = M$

<u>def</u> $\texttt{CompositionSeria} :: R\text{-}\texttt{Module} \rightarrow ?\texttt{List}(R\text{-}\texttt{Module})$

$$\texttt{CompositionSeria}(M)(S) = (S_0 = M) \wedge \forall i \in \mathbb{I}_S \; . \; (S_{i+1} : \texttt{Submodule}(S_i) \wedge \frac{S_i}{S_{i+1}} : \texttt{Simple}) \wedge S_{|\mathbb{I}_S|} = 0$$

<u>def</u> $\mathrm{len} :: \texttt{CompositionSeria}(M) \rightarrow \mathbb{N}^\infty$

$\mathrm{len}(S) = |\mathbb{I}_S| - 1$

$\underline{\text{thm}}$ $\texttt{JordanHolder} :: \forall M : R\text{-}\texttt{Module} \,.\, \forall A, B : \texttt{CompositionSeria}(M) \,.$

$. \operatorname{len} A = \operatorname{len} B$

$\texttt{JordanHolder}(M, A, B) =$

$\underline{\text{def}} \quad P :: \mathbb{Z}_+ \to \mathcal{T}$

$P(n) = \forall k \in \mathbb{I}_n \,.\, \forall M : R\text{-}\texttt{Module} \,.\, \forall A, B : \texttt{CompositionSeria}(M) \,.$

$. \operatorname{len} A = k \Rightarrow \operatorname{len} A = \operatorname{len} B$

$(0) = |M : R\text{-}\texttt{Module} \vdash$

$|A, B : \texttt{CompositionSeria}(M) \vdash$

$|Q : \operatorname{len} A = 0 \vdash$

$Q : \operatorname{len} A = 0 \to M = 0 \to A = B||| : P(0)$

$h = |n \in \mathbb{N} \vdash$

$|p : P(n-1) \vdash$

$|M : R\text{-}\texttt{Module} \vdash$

$|A, B : \texttt{CompositionSeria}(M) \vdash$

$|Q : \operatorname{len} A = n \vdash$

$\beta = |Z : A_1 = B_1 \vdash$

$Z : A_1 = B_1 \to \operatorname{tail} A, \operatorname{tail} B : \texttt{CompositionSeria}(A_1)$

$Q : \operatorname{len} A = n \to (1) : \operatorname{len} \operatorname{tail} A = n - 1$

$p(A_1, \operatorname{tail} A, \operatorname{tail} B, (1)) : \operatorname{len} \operatorname{tail} B = \operatorname{len} \operatorname{tail} A \to$

$\to \operatorname{len} A = \operatorname{len} B| : A_1 = B_1 \Rightarrow \operatorname{len} A = \operatorname{len} B$

$\alpha = |Z : A_1 \neq B_1 \vdash$

$A_1, B_1 : \texttt{Submodule}(M) \to A_1 + B_1 : \texttt{Submodule}(M)$

$$A_1 \subset A_1 + B_1 - \left(\frac{M}{A_1} : \texttt{Simple}\right) \to (1) : A_1 + B_1 = M$$

$\underline{\text{def}} \quad K = A_1 \cap B_1$

$$\frac{A_1}{K} =_K \frac{A_1}{A_1 \cup B_1} \cong_{\dots} \frac{A_1 + B_1}{B_1} =_{(1)} \frac{M}{B_1} : \texttt{Simple}$$

$$\frac{A_1}{K} =_K \frac{A_1}{A_1 \cup B_1} \cong_{\dots} \frac{A_1 + B_1}{A_1} =_{(1)} \frac{M}{A_1} : \texttt{Simple}$$

$(\dots) \to \exists X :: \texttt{CompositionSeria}(K) \to (2)$

$\underline{\text{def}} \quad A' = [M, A_1] \oplus X; B' = [M, B_1] \oplus X$

$$
\begin{aligned}
&\qquad\qquad \texttt{constr}(A', B') \to (3) : \operatorname{len} A' = \operatorname{len} B' \\
&\qquad\qquad \underline{\texttt{def}} \quad m = \operatorname{len} B \to (4) : \operatorname{len}\operatorname{tail} B = m - 1 \\
&\qquad\qquad Q : \operatorname{len} A = n \to (5) : \operatorname{len}\operatorname{tail} A = m - 1 \\
&\qquad\qquad (6) = p(M, \operatorname{tail} A, \operatorname{tail} A', (5)) : \operatorname{len}\operatorname{tail} A = \operatorname{len}\operatorname{tail} A' \\
&\qquad\qquad (7) = p(M, \operatorname{tail} B, \operatorname{tail} B', (6)) : \operatorname{len}\operatorname{tail} B = \operatorname{len}\operatorname{tail} B' \\
&\qquad\qquad (3, 6, 7) \to \operatorname{len}\operatorname{tail} A = \operatorname{len}\operatorname{tail} B \to \operatorname{len} A = \operatorname{len} B| : \\
&\qquad\quad | : A_1 \neq B_1 \Rightarrow \operatorname{len} A = \operatorname{len} B \\
&\qquad\quad \texttt{EM}(\alpha, \beta) : \operatorname{len} A = \operatorname{len} B| : \forall n \in \mathbb{N} \ . \ P(n-1) \Rightarrow P(n) \\
&\texttt{Ind}(P, 0, h) : \forall n \in \mathbb{Z}_+ \ . \ P(n)
\end{aligned}
$$

$$
\begin{aligned}
&\underline{\texttt{def}} \quad (*) :: \overset{\infty}{\mathbb{Z}_+} \to \mathcal{T} \\
&\quad (*)(c) = \forall M \in R\text{-}\texttt{Module} \ . \ \forall A, B : \texttt{CompositionSeria}(M) \ . \\
&\qquad\qquad \operatorname{len} A = c \Rightarrow \operatorname{len} A = \operatorname{len} B
\end{aligned}
$$

$$
\begin{aligned}
&|c \in \overset{\infty}{\mathbb{Z}_+} \vdash \\
&\quad \alpha = |c \in \mathbb{Z}_+ \vdash \\
&\qquad \texttt{Ind}(P, 0, h)(c) : (*)(c)| : c \in \mathbb{Z}_+ \to (*)(c) \\
&\quad \beta = |Z : c = \infty \vdash \\
&\qquad \ldots \\
&\qquad\quad |\operatorname{len} B < \infty \vdash \\
&\qquad\qquad \underline{\texttt{def}} \quad m = \operatorname{len} B \\
&\qquad\qquad (1) = \texttt{Ind}(P, 0, h)(m)(M, B, A, \texttt{constr}(m)) : \operatorname{len} A = m \\
&\qquad\qquad (Z, 1) \to \bot| : \operatorname{len} B = \infty \to \operatorname{len} A = \operatorname{len} B| : c \notin \mathbb{Z}_+ \to (*)(c) \\
&\quad \texttt{EM}(\alpha, \beta) : (*)(c)| : \ldots \square
\end{aligned}
$$

$$\underline{\texttt{thm}} \quad \texttt{maxInGen} :: \forall R : \texttt{IntegralDomain} \, . \, \forall S : \texttt{Set} \, . \, \exists w : \texttt{MaximalLI}(F^R(S)) : w \subset S$$

1.3 Lie Algebras

<u>`thm`</u> $\texttt{ex1} :: \mathbb{R} \cong_{\mathbb{Q}\text{-}\mathsf{vs}} \mathbb{C}$

<u>`def`</u> $\mathcal{M}_n(R) :: \texttt{Ring} \to \mathbb{N} \to \texttt{Set}$

$\mathcal{M}_n(R) = \mathbb{I}_n^2 \to R$

<u>`def`</u> $\mathfrak{gl}_n(R) :: \texttt{Ring} \to \mathbb{N} \to \texttt{Ring}$

$$\mathfrak{gl}_n(R) = \big(\mathcal{M}_n(R), +, \lambda(A, B)\ .\ \lambda(i, j)\ .\ \sum_{k=1}^{n} A_{i,k} B_{k,j}\big)$$

<u>`def`</u> $\operatorname{trace} :: \mathcal{M}_n(R) \to R$

$$\operatorname{trace} M = \sum_{i=1}^{n} M_{i,i}$$

<u>`def`</u> $(\cdot)^\top :: \mathcal{M}_n(R) \to \mathcal{M}_n(R)$

$M^\top = \lambda(i, j)\ .\ M_{j,i}$

<u>`def`</u> $(\cdot)^\dagger :: \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$

$(M)^\dagger = \lambda(i, j)\ .\ \overline{M_{j,i}}$

<u>`def`</u> $\mathfrak{sl}_n(R) :: \texttt{Ring} \to \mathbb{N} \to \texttt{Ring}$

$\mathfrak{sl}_n(R) = \{M \in \mathfrak{gl}_n(R) : \operatorname{trace} M = 0\}$

<u>`def`</u> $\mathfrak{so}_n(R) :: \texttt{Ring} \to \mathbb{N} \to \texttt{Ring}$

$\mathfrak{so}_n(R) = \{M \in \mathfrak{sl}_n(R) : M + M^\top = 0\}$

<u>`def`</u> $\mathcal{L}_n(M) :: \forall n : \texttt{Set}\ .\ \forall M :: R\text{-}\texttt{Module}\ .\ \forall ?(M^n \to M)$

$$T : \mathcal{L}_n(M) \iff \forall i \in n\ .\ \forall v \in M^{n-\{i\}}\ .\ \lambda m \in M\ .\ T(v \otimes \bigotimes_i m) \in \mathcal{L}(M)$$

<u>`def`</u> $\texttt{LieBracket} :: \forall V : K\text{-}\texttt{VectorSpace}\ .\ \mathcal{L}_2(V)$

$[\cdot, \cdot] : \texttt{LieBracket} \iff \forall u, v, w \in V\ .\ [v, v] = 0\ \wedge$

$\wedge\ [[u, v], w] + [[v, w], u] + [[w, u], v] = 0$

<u>`def`</u> $\texttt{LieAlgebra} :: \exists V : K\text{-}\texttt{VectorSpace}\ .\ \texttt{LieBracket}(V)$

2 Linear Maps

$\underline{\texttt{def}}$ $\texttt{FDim} :: ?R\texttt{-Module}$

$\texttt{FDim(M)} = \operatorname{rank} M \in \mathbb{N}$

$\underline{\texttt{def}}$ $\texttt{matrix} :: \forall n, m : \mathbb{N} \;.\; \mathcal{L}(F^R(n), F^R(m)) \to$

$\to \texttt{Basis}(F^R(n)) \to \texttt{Basis}(F^R(n)) \to \mathcal{M}_{m,n}$

$T_{e,e'}(i,j) = (Te_i)_{e_j}$

$\underline{\texttt{def}}$ $\texttt{linear} :: \forall n, m : \mathbb{N} \;.\; \mathcal{M}_{m,n}(R) \to \texttt{Basis}(F^R(n)) \to \texttt{Basis}(F^R(n))$

$\to \mathcal{L}(F^R(n), F^R(m))$

$$\texttt{linear}(M, e, e')(v) = \sum_{i=1}^{n} \sum_{j=1}^{m} v_{e_j} M_{(j,i)} e'_i$$

$\underline{\texttt{def}}$ $\texttt{Row} :: \forall n, m : \mathbb{N} \;.\; \mathcal{M}_{m,n} \to \mathbb{I}_m \to R^n$

$\mathcal{R}_i(M)(j) = M_{(i,j)}$

$\underline{\texttt{def}}$ $\texttt{Column} :: \forall n, m : \mathbb{N} \;.\; \mathcal{M}_{m,n} \to \mathbb{I}_n \to R^m$

$\mathcal{C}_i(M)(j) = M_{(j,i)}$

$\underline{\texttt{def}}$ $\texttt{Invertible} :: ?\mathcal{M}_n$

$\texttt{Invertible}(A) = \exists B \in \mathcal{M}_n \;.\; AB = I$

2.1 Row-Echellon form

2.2 Introduction to Grassmanian

2.3 Determinants

2.4 Nakayma Lemma

2.5 Grothendieck group

<u>def</u> $\texttt{Complex} :: \forall \mathcal{C} : \texttt{AbelianCategory} \;.\; ?\texttt{List}\left(\sum A, B : \mathcal{O}(\mathcal{C}) \;.\; \mathcal{H}_{\mathcal{C}}(A, B)\right)$

$$\texttt{Complex}(A, B, \phi)_\bullet \iff \forall i \in \mathbb{I}^{(+1)}_{(A,B,\phi)_\bullet} \;.\; A_i = B_{i-1} \wedge \operatorname{Im} \phi_{i-1} \subset \ker \phi_i \wedge$$
$$\wedge \text{ first } A = 0 \wedge \text{last } B = 0$$

<u>def</u> $\texttt{Exact} :: ?\texttt{Complex}(\mathcal{C})$

$$\texttt{Exact}(A, B, \phi)_\bullet = \mathbb{I}^{(+1)}_{(A,B,\phi)_\bullet} \;.\; \operatorname{Im} \phi_{i-1} \cong \ker \phi_i$$

<u>def</u> $\texttt{Homology} :: \forall (A, B, \phi)_\bullet : \texttt{Complex}(\mathcal{C}) \;.\; \mathbb{I}^{(+1)}_{(A,B,\phi)_\bullet} \to \mathcal{O}(\mathcal{C})$

$$H_i(A, B, \phi)_\bullet = \frac{\ker \phi_i}{\operatorname{Im} \phi_{i-1}}$$

<u>def</u> $\texttt{Short} :: ?\texttt{Exact}(\mathcal{C})$

$$\texttt{Short}(X_\bullet) \iff \exists A, B, C \in \mathcal{O}(\mathcal{C}) \;.\; \exists \phi \in \mathcal{H}_{\mathcal{C}}(A, B) \;.\; \exists \psi \in \mathcal{H}_{\mathcal{C}}(B, C) \;.$$
$$X_\bullet = \big[(0, A, 0), (A, B, \phi), (B, C, \psi), (C, 0, 0)\big]$$

<u>def</u> $\texttt{EullerCharacteristic} :: \texttt{Complex}(K\text{-}\mathsf{VS}^f) \to \mathbb{Z}$

$$\chi(V, \_, \_)_\bullet = \sum_{i=1}^{\operatorname{len} V - 1} (-1)^i \dim V_i$$

<u>thm</u> $\texttt{shortClaim} :: \forall (U, V, W, \_, \_) : \texttt{Short}(K\text{-}\mathsf{VS}^f) \;.$

$$.\;\dim V = \dim U + \dim W$$

<u>thm</u> $\texttt{EullerHomology} :: \forall V_\bullet : \texttt{Cpmplex}(K\text{-}\mathsf{VS}^f) \;.$

$$.\;\chi(V_\bullet) = \sum_{i=1}^{\operatorname{len} V_\bullet - 1} (-1)^i \dim H_i(V_\bullet)$$

$\underline{\texttt{def}}$ $\quad E :: \forall \mathcal{C} : \texttt{AbelianCategory\&Small} \;.\; \texttt{Subgroup}\Big(F_{\mathsf{AB}}\big(\texttt{IsoClass}(\mathcal{C})\big)\Big)$

$$E(\mathcal{C}) = \Big(\big\{[A] - [B] - [C] \mid (A, B, C, \_, \_) : \texttt{Short}(\mathcal{C})\big\}\Big)$$

$\underline{\texttt{def}}$ $\quad \texttt{GrothendieckGroup} :: \texttt{AbelianCategory\&Small} \to \texttt{Abelean}$

$$K(\mathcal{C}) = \frac{F_{\mathsf{AB}}\big(\texttt{IsoClass}(\mathcal{C})\big)}{E(\mathcal{C})}$$

$\underline{\texttt{def}}$ $\quad \texttt{GrothendieckProjection} :: \forall \mathcal{C} \in \texttt{AbelianCategory} \;.\; \mathcal{O}(\mathcal{C}) \to K(\mathcal{C})$

$$[A]_K = \pi_{K(\mathcal{C})}[A]$$

$\underline{\texttt{def}}$ $\quad \texttt{GrothendieckCharacteristic} :: \forall \mathcal{C} \in \texttt{AbelianCategory} \;.$

$$. \; \texttt{Complex}(\mathcal{C}) \to K(\mathcal{C})$$

$$\chi_K(V, \_, \_)_\bullet = \sum_{i=1}^{\operatorname{len} V - 1} (-1)^i [V_i]_K$$

$\underline{\texttt{thm}}$ $\quad \texttt{GtorhendiekHomology} :: \forall \mathcal{C} \in \texttt{AbelianCategory} \;.$

$$V_\bullet : \texttt{Complex}(\mathcal{C}) \;.\; \chi_K(V_\bullet) = \sum_{i=1}^{\operatorname{len} V_\bullet - 1} (-1)^i [H_i(V_\bullet)]_K$$

$\underline{\texttt{thm}}$ $\quad \texttt{St1} :: K(k\text{-}\mathsf{VS}^f) \cong \mathbb{Z}$

$\underline{\texttt{thm}}$ $\quad \texttt{St2} :: K(\mathsf{ABEL}^{fg}) \cong \mathbb{Z}$

$\underline{\texttt{thm}}$ $\quad \texttt{St3} :: K(k\text{-}\mathsf{VS}^{fg}) \cong 0$

$\underline{\texttt{thm}}$ $\quad \texttt{St4} :: K(\mathsf{ABEL}^f) \succeq (\mathbb{Q}, \cdot)$

3 Presentation and Resolution

3.1 Presentation

def $\texttt{torsion} :: \forall M : R\text{-}\texttt{Module} \,.\, \texttt{Set}(M)$

$\mathrm{Tor}_R\, M = \{m \in M : \exists r \in R : r \neq 0 : rm = 0\}$

def $\texttt{TorsionFree} :: ?R\text{-}\texttt{Module}$

$\texttt{TorsionFree}(M) = \big(\mathrm{Tor}_R\, (M) = \{0\}\big)$

def $\texttt{Torsion} :: ?R\text{-}\texttt{Module}$

$\texttt{Torsion}(M) = (\mathrm{Tor}_R\, (M) = M)$

def $\texttt{Ciclic} :: ?R\text{-}\texttt{Module}$

$\texttt{Ciclic}(M) = \exists m \in M : M = (m)$

def $\texttt{Annihilator} :: R\text{-}\texttt{Module} \to \texttt{Ideal}(R)$

$\mathrm{Ann}\, M = \{r \in R : \forall m \in M \,.\, rm = 0\}$

def $\texttt{FPresented} :: ?R\text{-}\texttt{Module}$

$\mathrm{FPresented}(M) = \exists n, m \in \mathbb{N} : \exists \phi : \mathcal{L}(R^n, M) : \exists \psi \in \mathcal{L}(R^m, R^n) :$
$[(0, R^m, 0), (R^m, R^n, \psi), (R^n, M, \phi), (M, 0, 0)] : \texttt{Short}$

def $\texttt{presentation} :: \texttt{FPresented} \to \texttt{Short}$

$\texttt{presentation}(M) =$
$(\underline{\texttt{def}}\quad \texttt{FPresrnted}) \to [(0, R^m, 0), (R^m, R^n, \psi), (R^n, M, \phi), (M, 0, 0)]$

def $\texttt{Resolution} :: \forall M : R\text{-}\texttt{Module} \,.\, ?\texttt{Exact}(R\text{-}\mathsf{MOD})$

$\texttt{Resolution}(M)(V, W, \phi)_\bullet = \exists n : \texttt{Nonproductive} : \mathrm{len}(V, W, \phi) = n \wedge$
$\wedge\, V_n = M \wedge \exists m \in \mathbb{I}_{n-1} \to \mathbb{N} : \forall i \in \mathbb{I}_{n-1} \,.\, V_i = R^{m_i}$

$\underline{\texttt{thm}}$ $\texttt{subTorsionFree} :: \forall M : \texttt{TorsionFree} \; . \; \forall S : \texttt{Submodule}(M) \; . \; S : \texttt{TorsionFree}$

$\texttt{Proof}(M, S) =$

If S has a torsion element then it also belongs to M. But M is a torsion free, hence a contradiction. $\square$

$\underline{\texttt{thm}}$ $\texttt{sumTorsionFree} :: \forall I : \texttt{Set} \; . \; \forall T : I \to \texttt{TorsionFree}(R) \; . \; \bigoplus_{i \in I} T_i : \texttt{TorsionFree}(R)$

$\texttt{Proof}(I, T) =$

Assume that $\bigoplus_{i \in I} T_i$ is not torsion free. Then there $\exists v \in \bigoplus_{i \in I} T_i : \exists r \in R :$ $: v, r \neq 0 \wedge rv = 0$. As $v \neq 0$ there $\exists i \in I : v_i \neq 0$. But as $rv = 0$ we know $rv_i = 0$ but this means that T_i is not torsion free, a contradiction. $\square$

$\underline{\texttt{thm}}$ $\texttt{sumTorsionFree} :: \forall I : \texttt{Set} \; . \; \forall R : \texttt{IntegralDomain} \; . \; R^I : \texttt{TorsionFree}(R)$

$\texttt{Proof}(I, R) =$

By definition of $\texttt{IntegralDomain}$ we have $R : \texttt{TorsionFree}(R)$. Then by $\texttt{sumTorsionFree}$ theorem $R^I : \texttt{TorsionFree}(R)$ $\square$

$\underline{\texttt{thm}}$ $\texttt{CMField} :: \forall R : \texttt{IntegralDomain} \; . \; (\forall M : \texttt{Cyclic}(R) \; . \; M : \texttt{TorsionFree}(R))$

$\Rightarrow R : \texttt{Field}$

$\texttt{Proof}(R, P) =$

Assume that $c \in R : c \neq 0$. Then $\dfrac{R}{(c)}$ is generated by $\bar{1}$ so it is ciclic as R-module. However, $\bar{1} \in \mathrm{Tor}_R \dfrac{R}{(c)}$ as $c\bar{1} = \bar{0}$ by definition of quatioent ring. This fact and hipothesis P provides $\dfrac{R}{(c)} \cong 1$, which in its own turn means that $R = (c)$. Hence c is invertible in R. Here we deduce that R is a field. $\square$

$\underline{\texttt{thm}}$ $\texttt{NoetherianPresentation} :: \forall R : \texttt{Noetherian} \; . \; \forall M : \texttt{FG}(R) \; . \; M : \texttt{FP}(R)$

$\texttt{Proof}(R, M) =$

As M is finetly generated where is $g : \mathbb{I}_n \to M$, a finite collection of generators of M.

3.2 Associated primes

3.3 Kozsul complex