

# Topological Groups

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March 6, 2022

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# 1 Uniform Spaces

Uniform spaces generalizes many concept of metric topology beyond real numbers.

## 1.1 Connector Uniformities

One tool defining Uniformities are connectors or entourages.

### 1.1.1 Connectors

Connectors can be thought as large binary matrices with ones on diagonals.

$$\text{Connector} :: \prod_{X \in \text{SET}} ?(X \rightarrow ?X)$$

$$U : \text{Connector} \iff \forall x \in X . x \in U(x)$$

$$\text{SemimetricConnector} :: \prod X \in \text{SMS} . \mathbb{R}_{++} \rightarrow \text{Connector}(X)$$

$$\text{SemimetricConnector}(\varepsilon) = \mathbb{B}_\varepsilon := \Lambda x \in X . \mathbb{B}(x, \varepsilon)$$

$$\text{connectorAsSubset} :: \prod_{X \in \text{SET}} \text{Connector}(X) \rightarrow ?X^2$$

$$\text{connectorAsSubset}(U) = U := \left\{ (x, y) \in X^2 \mid y \in U(x) \right\}$$

$$\text{ConnectorContainsDiagonal} :: \forall X \in \text{SET} . \forall U : \text{Connector}(X) . \Delta(X) \subset U$$

**Proof** =

...

□

$$\text{SubsetWithDiagonal} :: \prod_{X \in \text{SET}} ??X^2$$

$$V : \text{SubsetWithDiagonal} \iff \Delta(X) \subset U$$

$$\text{connectorFromDiagonalSubset} :: \prod_{X \in \text{SET}} \text{SubsetWithDiagonal}(X) \rightarrow \text{Connector}(X)$$

$$\text{connectorFromDiagonalSubset}(V) = U := \Lambda x \in X . V_x$$

$$\text{transpose} :: \prod_{X \in \text{SET}} \text{Connector}(X) \rightarrow \text{Connector}(X)$$

$$\text{transpose}(U) = U^\top := \left\{ (y, x) \mid (x, y) \in U \right\}$$

$$\text{IdempotentTransposition} :: \forall X \in \text{SET} . \forall U : \text{Connector}(X) . (U^\top)^\top = U$$

**Proof** =

...

□

**ConnectorContainment** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . U \subset V \Rightarrow \forall x \in X . U(x) \subset V(x)$

**Proof** =

...

□

**ConnectorTransposeContainment** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . U \subset V \Rightarrow \forall x \in X . U^\top \subset V^\top$

**Proof** =

...

□

**ConnectorSetAlgebra** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . \text{Connector}(X, U \cap V \ \& \ X \cup V)$

**Proof** =

...

□

**TransposeIntersection** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . (U \cap V)^\top = U^\top \cap V^\top$

**Proof** =

...

□

**TransposeComposition** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . (U \circ V)^\top = V^\top \circ U^\top$

**Proof** =

...

□

**SemimetricConnectorComposition** ::  $\forall X \in \text{SMS} . \forall t, s \in \mathbb{R}_{++} . \mathbb{B}_t \circ \mathbb{B}_s \subset \mathbb{B}_{t+s}$

**Proof** =

**Assume**  $(x, z) \in \mathbb{B}_t \circ \mathbb{B}_s,$

$(y, [1]) := \text{Ecomposition}(\mathbb{B}_t, \mathbb{B}_s) : \sum_{y \in X} (x, y) \in \mathbb{B}_t \ \& \ (y, z) \in \mathbb{B}_s,$

$[2] := \text{EB}_t[1.1] : d(x, y) < t,$

$[3] := \text{EB}_s[1.2] : d(y, z) < s,$

$[4] := \text{TriangleIneq}(X)[2][3] : d(x, z) \leq d(x, y) + d(y, z) < t + s,$

$[(x, z).*] := \text{IB}_{t+s} : (x, z) \in \mathbb{B}_{t+s};$

$\leadsto [*] := \text{I} \subset \mathbb{B}_t \circ \mathbb{B}_s \subset \mathbb{B}_{s+t};$

□

**CompositionContainment** ::  $\forall X \in \text{SET} . \forall U, V : \text{Connector}(X) . U \subset U \circ V \ \& \ U \subset V \circ U$

**Proof** =

$[1] := \text{ConnectorContainsDiagonal}(X, V) : \Delta(X) \subset V,$

$[*] := \text{Ecomposition}[1] : U \subset U \circ V \ \& \ U \subset V \circ U;$

□

**SemimetricConnectorIsMonotonic** ::  $\forall X \in \text{SMS} . \mathbb{R}_{++} \xrightarrow{\mathbb{B}} ?X^2 : \text{POSET}$

**Proof** =

**SemimetricConnectorIntersection** ::  $\forall X \in \mathbf{SMS} . \forall t, s \in \mathbb{R}_{++} . \mathbb{B}_t \cap \mathbb{B}_s = \mathbb{B}_{s \wedge t}$

**Proof** =

[1] := **TosetMeet**( $\mathbb{R}_{++}, t, s$ ) :  $s \wedge t = t \mid s \wedge t = s$ ,

[2] := **SemimetricConnectorIsMonotonic**( $X$ )**NestedIntersection**( $X$ )**E** = $_{\mathbb{R}}$ :

:  $s \wedge t = t \Rightarrow \mathbb{B}_t \cap \mathbb{B}_s = \mathbb{B}_t = \mathbb{B}_{t \wedge s}$  &  $s \wedge t = s \Rightarrow \mathbb{B}_s \cap \mathbb{B}_t = \mathbb{B}_s = \mathbb{B}_{t \wedge s}$ ,

[\*] := **E**[1][2] :  $\mathbb{B}_s \cap \mathbb{B}_t = \mathbb{B}_{t \wedge s}$ ;

□

**ConnectorProductContainment** ::  $\forall X \in \mathbf{SET} . \forall U, V : \mathbf{Connector}(X) . \forall x \in X . U(x) \times V(x) \subset V \circ U^\top$

**Proof** =

**Assume**  $(a, b) \in U(x) \times V(x)$ ,

[1] := **EConnector**( $X$ )**E**( $a, b$ ) :  $(x, a) \in U$  &  $(x, b) \in V$ ,

$\left[ (a, b).* \right] := \mathbf{E} \circ [1] : (a, b) \in V \circ U^\top$ ;

$\leadsto [*] := \mathbf{I} \subset U(x) \times V(x) \subset V \circ U^\top$ ;

□

**ConnectorProductContainment2** ::

:  $\forall X \in \mathbf{SET} . \forall V : \mathbf{SymmetricConnector}(X) . \forall x \in X . V(x) \times V(x) \subset V \circ V$

**Proof** =

...

□

### 1.1.2 Uniform Topology

Unifirmities generalize concept of cells or balls from metric topology. They are filters of connectors with nice properties. Also, they define topology similarly to a distance metric.

$$\text{Uniformity} :: \prod_{X \in \text{SET}} ?\text{Filter}(X^2)$$

$$\mathcal{U} : \text{Uniformity} \iff \forall U \in \mathcal{U} . \text{WithDiagonal}(X, U) \ \& \ U^\top \in \mathcal{U} \ \& \ \exists V \in \mathcal{U} . V \circ V \subset U$$

$$\text{UniformityBase} :: \prod_{X \in \text{SET}} ?\text{Filterbase}(X^2)$$

$$\mathcal{B} : \text{UniformityBase} \iff \forall U \in \mathcal{B} . \text{WithDiagonal}(X, U) \ \& \ \exists V \in \mathcal{B} . V \subset U^\top \ \& \ \exists V \in \mathcal{B} . V \circ V \subset U$$

$$\text{UniformSpace} := \sum_{X \in \text{SET}} \text{Uniformity}(X) : \text{Type};$$

$$\text{UniformBaseGeneratesUnifomity} :: \forall X \in \text{SET} . \forall \mathcal{B} : \text{UniformBase}(X) . \text{Uniformity}(X, \langle \mathcal{B} \rangle)$$

Proof =

...

□

$$\text{BasesOfUniformityAreUniformityBases} :: \forall X \in \text{SET} . \forall \mathcal{U} : \text{Uniformity}(X) . \forall \mathcal{B} : \text{FilterBase}(X) . \\ . (\mathcal{B} \subset \mathcal{U} \ \& \ \forall U \in \mathcal{U} . \exists B \in \mathcal{B} . B \subset U) \Rightarrow U = \langle \mathcal{B} \rangle \ \& \ \text{UniformBase}(X, \mathcal{B})$$

Proof =

$$\text{SemimetricFilterbase} :: \forall X \in \text{SMS} . \exists \mathcal{B} \Rightarrow \text{UniformBase}(X, \{\mathbb{B}_\varepsilon | \varepsilon \in \mathbb{R}_{++}\})$$

Proof =

$$\mathcal{B} := \{\mathbb{B}_\varepsilon | \varepsilon \in \mathbb{R}_{++}\} : ??X^2,$$

$$[1] := \text{SemimetricConnectorInersection}(X) \text{IB} : \forall U, V \in \mathcal{B} . U \cap V \in \mathcal{B},$$

$$[2] := \text{EB}[0] \text{IB} : \exists \mathcal{B},$$

$$[3] := \text{IFilterbase}[1][2] : \text{Filtrerbase}(X, \mathcal{B}),$$

$$[4] := \text{EBESymmetric}(X, \mathbb{R}, d) \text{IB} : \forall U \in \mathcal{B} . U^\top \in \mathcal{B},$$

Assume  $U \in \mathcal{B}$ ,

$$(t, [5]) := \text{EB}(U) : \sum_{t \in \mathbb{R}_{++}} (U = \mathbb{B}_t),$$

$$[U.*] := \text{SemimetricConnectorComposition} \left( X, \frac{t}{2}, \frac{t}{2} \right) : \mathbb{B}_{\frac{t}{2}} \circ \mathbb{B}_{\frac{t}{2}} \subset \mathbb{B}_t;$$

$$\leadsto [5] := \text{EBI} \forall : \forall U \in \mathcal{B} . \exists V \in \mathcal{B} . V \circ V \subset U,$$

$$[*] := \text{IUniformBase}[3] \text{EBE} \text{IB}[4][5] : \text{UniformBase}(X, \mathcal{B});$$

□

$$\text{uniformTopology} :: \prod_{X \in \text{SET}} \text{Uniformity}(X) \rightarrow \text{Topology}(X)$$

$$\text{uniformTopology}(\mathcal{U}) = \mathcal{T}_{\mathcal{U}} := \left\{ O \subset X \mid \forall x \in O . \exists U \in \mathcal{U} . U(x) \subset O \right\}$$

$$\mathcal{T} := \text{uniformTopology}(\mathcal{U}) : ??X,$$

$$[1] := \text{ETExNihilo}(X) \text{I} \mathcal{T} : \emptyset \in X,$$

$$[2] := \text{E}_1 \text{Filter}(X, \mathcal{U}) : \mathcal{U} \neq \emptyset,$$

$$[3] := \text{ET}[2] \text{UniversumContainsAll}(X) : X \in \mathcal{T},$$

$$[4] := \text{ET}[1] \text{UnionContainsSubsets}(X) : \forall \mathcal{O} \subset \mathcal{T} . \bigcup \mathcal{O} \in \mathcal{T},$$

$$\text{Assume } n \in \mathbb{N},$$

$$\text{Assume } V : \{1, \dots, n\} \rightarrow \mathcal{T},$$

$$\text{Assume } x \in \bigcap_{k=1}^n V_k,$$

$$(U, [5]) := \text{ET}(V) : \sum U : \{1, \dots, n\} \rightarrow \mathcal{U} . \forall k \in n . U_k(x) \subset V_k,$$

$$W := \bigcap_{k=1}^n U_k \in \mathcal{U},$$

$$[n.*] := \text{EWSubsetIntersectionI} W : W(x) \subset \bigcap_{k=1}^n V_k;$$

$$\leadsto [5] := \text{ET} : \forall n \in \mathbb{N} . \forall V : \{1, \dots, n\} \rightarrow \mathcal{T} . \bigcap_{k=1}^n V_k \in \mathcal{T},$$

$$[*] := \text{ITopology}[1][3][4][5] : \text{Topology}(X);$$

□

$$\text{uniformity} :: \prod (X, \mathcal{U}) : \text{UniformSpace} . \text{Uniformity}(X)$$

$$\text{Uniformity}() = \mathcal{U}_X := \mathcal{U}$$

$$\text{UniAsTop} :: \text{UniformSpace} \rightarrow \text{TOP}$$

$$\text{UniAsTop}(X) = X := (X, \text{uniformTopology}(\mathcal{U}_X))$$

$$\text{Stronger} :: \prod_{X \in \text{SET}} ?\text{Uniformity}^2(X)$$

$$(\mathcal{U}, \mathcal{V}) : \text{Stronger} \iff \mathcal{U} \geq \mathcal{V} \iff \mathcal{T}_{\mathcal{V}} \subset \mathcal{T}_{\mathcal{U}}$$

$$\text{EquivalentUniformities} :: \prod_{X \in \text{SET}} ?\text{Uniformity}^2(X)$$

$$(\mathcal{U}, \mathcal{V}) : \text{EquivalenUniformities} \iff \mathcal{U} \cong \mathcal{V} \iff \mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\mathcal{U}}$$

$$\text{BaseOfUniformity} :: \prod X : \text{UniformSpace} . ?\text{UniformBase}(X)$$

$$\mathcal{B} : \text{BaseOfUniformity} \iff \langle \mathcal{B} \rangle = \mathcal{U}_X$$



$\text{discreteUniformity} :: \prod_{X \in \text{SET}} \text{Uniformity}(X)$

$\text{discreteUniformity}() := \left\{ U \in X^2 : \Delta(X) \subset U \right\}$

$\text{codiscreteUniformity} :: \prod_{X \in \text{SET}} \text{Uniformity}(X)$

$\text{codiscreteUniformity}() := \{X \times X\}$

$\text{DiscreteUniformityTopology} :: \forall X \in \text{SET} . \mathcal{T}(X, \text{discreteUniformity}(X)) = 2^X$

Proof =

...

□

$\text{CodiscreteUniformityTopology} :: \forall X \in \text{SET} . \mathcal{T}(X, \text{codiscreteUniformity}(X)) = \{\emptyset, X\}$

Proof =

...

□

$\text{relativeUniformity} :: \prod X : \text{UniformSpace} . 2^X \rightarrow \text{UniformSpace}$

$\text{relativeUniformity}(A) = A := \left( A, \{U \cap A^2 \mid U \in \mathcal{U}_X\} \right)$

### 1.1.3 Symmetric Connectors

Symmetric connectors are particularly nice. Every uniformity has a base of symmetric connectors.

$$\text{SymmetricConnector} :: \prod_{X \in \text{SET}} ?\text{Connector}(X)$$

$$U : \text{SymmetricConnector} \iff U^\top = U$$

$$\text{SymmetricBase} :: \prod_{X \in \text{SET}} ?\text{UniformBase}(X)$$

$$\mathcal{B} : \text{SymmetricBase} \iff \forall U \in \mathcal{B} . \text{SymmetricConnector}(X, U)$$

$$\text{SymmetricBaseExists} :: \forall X : \text{UniformSpace} . \exists \mathcal{B} : \text{SymmetricBase}(X) . \mathcal{U}_X = \langle \mathcal{B} \rangle$$

**Proof** =

$$\mathcal{B} := \{U \in \mathcal{U}_X : \text{SymmetricConnector}(X, U)\} : ?\mathcal{U}_X,$$

$$[1] := \text{EFilter}(X, \mathcal{U}_X) : \exists \mathcal{U}_X,$$

$$[2] := \text{E}^\top \text{ISymmetricConnector} : \forall U \in \mathcal{U}_X . \text{SymmetricConnector}(X, U \cap U^\top),$$

$$[3] := [1][2] : \exists \mathcal{B},$$

$$[4] := \text{EBESymmetricConnector}(X) \text{E} \cap \text{ISymmetricConnector}(X) \text{IB} : \forall U, V \in \mathcal{B} . U \cap V \in \mathcal{B},$$

$$[5] := \text{IFilterbase}[3][4] : \text{Filterbase}(X, \mathcal{B}),$$

$$[6] := \text{EBESymmetricConnector}(X) \text{IB} : \forall U \in \mathcal{B} . U^\top \in \mathcal{B},$$

**Assume**  $U \in \mathcal{B}$ ,

$$(V, [7]) := \text{E}_3 \text{Uniformity}(X, \mathcal{U}, U) : \sum V \in \mathcal{U} . V \circ V \subset U,$$

$$[8] := \text{CompositionContainment}(X, V, V)[7] : V \subset V \circ V \subset U,$$

$$[9] := \text{ESymmetricConnector}(X, U)[8] : V^\top \subset U,$$

$$W := V \cap V^\top : \text{SymmetricConnector}(X),$$

$$[10] := \text{EWEVI}W : W \in \mathcal{U},$$

$$[11] := \text{EW}[8][9] \text{IW} : W \subset U,$$

$$[U.*] := [11][7] \text{EWIntersectionSubsetIW} : W \circ W \subset U;$$

$$\leadsto [7] := \text{ISymmetricBase}[5][6] : \text{SymmetricBase}(X, \mathcal{B}),$$

$$[*] := \text{EB}[2] \text{GeneratingFilterbaseIW} : \mathcal{U}_X = \langle \mathcal{B} \rangle;$$

□

### 1.1.4 Neighborhoods

Every base of uniformities has a corresponding base of neighborhoods.

$$\text{uniformityAssociatedBase} :: \prod X : \text{UniformSpace} . \prod_{x \in X} \text{BaseOfUniformity}(X) \rightarrow \text{NeighborhoodBase}(X, a)$$

$$\text{uniformityAssociatedBase}(\mathcal{B}) = \tilde{\mathcal{B}}_x := \left\{ \{y \in X : \exists V \in \mathcal{U}_X : V(y) \subset U(x)\} \mid U \in \mathcal{B} \right\}$$

Assume  $U \in \mathcal{U}_X$ ,

$$G := \{y \in X : \exists V \in \mathcal{U}_X : V(y) \subset U(x)\} : ?X,$$

$$[1] := \text{SelfSubset}(X, U(x)) : U(x) \subset U(x),$$

$$[2] := \text{EG}[1] \text{IG} : x \in G,$$

$$[3] := \text{EConnectorEG} : G \subset U(x),$$

Assume  $g \in G$ ,

$$(V, [3]) := \text{EG}(g) : \sum V : \text{Connector} . V(g) \subset U(x),$$

$$(W, [4]) := \text{EUniformity}(X, U) : \sum W \in \mathcal{U}_X . W \circ W \subset V,$$

Assume  $y \in W(g)$ ,

Assume  $z \in W(y)$ ,

$$[5] := \text{EConnector}(W) \text{Ey} : (g, y) \in W,$$

$$[6] := \text{EConnector}(W) \text{Ez} : (y, z) \in W,$$

$$[7] := \text{I}(W \circ W)[5][6] : (g, z) \in W \circ W,$$

$$[z.*] := [7][4][3] : z \in V(g) \subset U(x);$$

$$\sim [5] := \text{I} \subset : W(y) \subset U(x),$$

$$[y.*] := \text{EG}[5] : y \in G;$$

$$\sim [U.*] := \text{EuniformTopology} : G \in \mathcal{T}(X);$$

$$\sim [1] := \text{I}\forall \text{I}\exists : \forall U \in \mathcal{B} . \exists G \in \mathcal{T}(X) . x \in G \subset U,$$

Assume  $N \in \mathcal{U}(x)$ ,

$$(U, [2]) := \text{EuniformTopology}(X) \text{EN} : \sum U \in \mathcal{U}_X . x \in U(x) \subset N,$$

$$(B, [3]) := \text{EUniformBase}(\mathcal{U}_X, \mathcal{B}) : \sum B \in \mathcal{B} . x \in B(x) \subset U(x),$$

$$(G, [N.*]) := [1][2][3] : \sum G \in \tilde{\mathcal{B}}_x . G \subset N;$$

$$\sim [*] := \text{INeighborhoodBase} : \text{NeighborhoodBase}(X, \tilde{\mathcal{B}}_x);$$

□

$$\text{UniformNeighborhood} :: \prod X : \text{UniformSpace} . ?X \rightarrow ??X$$

$$B : \text{UniformNeighborhood} \iff \Lambda A \subset X . \exists U \in \mathcal{U}_X . U(A) \subset B$$

**UniformNeighborhoodIsANeighborhood** ::  $\forall X : \text{UniformSpace} . \forall A \subset X .$

$. \forall N : \text{UniformNeighborhood}(X, A) . \text{Neighborhood}(X, A, N)$

**Proof** =

$(U, [1]) := \text{EUniformNeighborhood}(X, A, N) : \sum U \in \mathcal{U}_X . U(A) \subset B,$

$[2] := \text{EU} : \forall a \in A . \exists O \in \mathcal{U}(a) . O \subset U(a) \subset U(A) \subset B,$

$O := \bigcup_{a \in A} [2](a) : \mathcal{T}(X),$

$[3] := \text{EO} : A \subset O \subset B,$

$[*] := \text{INeighborhood}[3] : \text{Neighborhood}(X, A, N);$

□

**EveryCompactNeighborhoodIsUniform** ::  $\forall X : \text{UniformSpace} . \forall A : \text{CompactSubset}(X) .$

$. \forall N : \text{Neighborhood}(X, A) . \text{UniformNeighborhood}(X, A, N)$

**Proof** =

$(O, [1]) := \text{ENeighborhood}(X, A, N) : \sum O \in \mathcal{T}(X) . A \subset O \subset N,$

$(U, [2]) := \text{EuniformTopology}[1] : \prod U' : A \rightarrow \mathcal{U}_X . \forall a \in A . U'_a(a) \subset O,$

$(U, [22]) := \text{EUniformity}[2] : \prod U : A \rightarrow \mathcal{U}_X . \forall a \in A . U_a \circ U_a(a) \subset O,$

$(V, [3]) := \text{EB}[22] : \prod V : \prod_{a \in A} \mathcal{U}(a) . \forall a \in A . V_a \subset U_a(a),$

$[4] := \text{EV} : \text{OpenCover}(X, A, \text{Im } V),$

$(n, a, [5]) := \text{ECompactSubset}(X, A, [4]) : \sum_{n=1}^{\infty} \sum \{1, \dots, n\} \rightarrow A . \text{OpenCover}(X, A, \text{Im } V_a),$

$[6] := [5][3] : \text{Cover}(X, A, U_a(a)),$

$W := \bigcap_{k=1}^n U_{a_k} \in \mathcal{U}_X,$

**Assume**  $w \in W(A),$

$(b, [7]) := \text{E}w : \sum b \in A . w \in W(b),$

$(k, [8]) := \text{ECover}[6](b) : \sum_{k=1}^n b \in U_{a_k}(a_k),$

$[9] := \text{EW}[7](k) : (b, w) \in U_{a_k},$

$[w.*] := \text{EConnector}(X, U_{a_k})[7][9] : w \in U_{a_k} \circ U_{a_k}(a_k);$

$\leadsto [7] := \text{I} \subset [22][2][1] : W(A) \subset \bigcup_{a \in A} U_a \circ U_a(a) \subset N,$

$[*] := \text{IUniformNeighborhood}[7] : \text{UniformNeighborhood}(X, A, N);$

□

**ConentorAsNeighborhoodOfDiagonal** ::  $\forall X : \text{UniformSpace} . \forall U \in \mathcal{U}_X . \text{Neighborhood}(X^2, \Delta(X), U)$

**Proof** =

...

□

### 1.1.5 Closures and regularity

There is a special way to compute closures in uniform spaces. Also, any separated uniform space is regular

$$\text{ClosureFormula} :: \forall X : \text{UniformSpace} . \forall \mathcal{B} : \text{BaseOfUniformity}(X) . \forall A \subset X . \overline{A} = \bigcap_{V \in \mathcal{B}} \bigcup_{a \in A} V(a)$$

Proof =

Assume  $x \in \overline{A}$ ,

Assume  $V \in \mathcal{B}$ ,

[1] :=  $\text{SymmetrocBaseExists}(x) : \sum W : \text{SymmetricConnector}(X) . W \subset V$ ,

[2] :=  $\text{ClosureAltDef}(X, A) \text{EB}_{\tilde{x}} : \exists A \cap W(x)$ ,

$y := \text{E}\exists[2] \in W(x)$ ,

[3] :=  $\text{ESymmetricConnector}(X, W) \text{E}y : x \in W(y)$ ,

[4] := [1][3] :  $x \in V(x)$ ,

$[x.*] := \text{UnionSubset}[4] : x \in \bigcup_{a \in A} V(a)$ ;

$\leadsto [1] := \text{I} \subset : \overline{A} \subset \bigcup_{a \in A} V(a)$ ,

Assume  $x \in \overline{A}^c$ ,

$(U, [2]) := \text{SymmetrocBaseExists}(X) \text{ClosureAltDef}(X, A) \text{EB}_{\tilde{x}} :$   
 $: \sum U : \text{SymmetricConnector}(X) . U(x) \cap A = \emptyset$ ,

$(V, [3]) := \text{EUniformBase}(X, \mathcal{B}, U) : \sum V \in \mathcal{B} . V \subset U$ ,

Assume [4] :  $x \in \bigcup_{a \in A} V(a)$ ,

$(a, [5]) := \text{EUnionEx} : \sum_{a \in A} x \in V(a)$ ,

[6] := [3][5] :  $x \in U(a)$ ,

[7] :=  $\text{ESymmetricConnector}(X, U)[6] : a \in U(x)$ ,

[8] := [2][7] :  $\perp$ ;

$\leadsto [x.*] := \text{E}\perp : x \notin \bigcup_{a \in A} V(a)$ ;

$\leadsto [*] := [1] \text{ISetEq} : \overline{A} = \bigcap_{V \in \mathcal{B}} \bigcup_{a \in A} V(a)$ ;

□

**EveryUniformSpaceIsRegular** ::  $\forall X : \text{UniformSpace} . \text{T0}(X) \Rightarrow \text{T3}(X)$

**Proof** =

**Assume**  $x \in X$ ,

**Assume**  $N \in \mathcal{U}(x)$ ,

$(U, [1]) := \text{EuniformTopology}(X) : \sum U \in \mathcal{U}_X . U \subset N$ ,

$(W, [2]) := \text{EUniformSpace}(X, U) : \sum W \in \mathcal{U}_X . W \circ W \subset U$ ,

$[x.*] := \text{ClosureFormula}(X, V(x))[2][1] : \overline{V(x)} \subset \bigcup_{a \in V(x)} V(a) \subset U(x) \subset N$ ;

$\leadsto [1] := \text{I}\forall : \forall x \in X . \forall N \in \mathcal{U}(x) . \exists V \in \mathcal{U}(x) . \overline{V} \subset N$ ,

$[*] := \text{RegularT0IsT3}[1] : \text{T3}(X)$ ;

□

**UniformT3SpaceIntersectsToDiagonal** ::  $\forall X : \text{UniformSpace} . \text{T3}(X) \iff \bigcap \mathcal{U}_X = \Delta(X)$

**Proof** =

...

□

**CloppenByConnector** ::  $\forall X : \text{UniformSpace} . \forall A \subset X . \forall U : \text{Connector}(X) .$

$. \bigcup_{a \in A} U(a) \subset A \Rightarrow \text{Cloppen}(X, A)$

**Proof** =

$[1] := \Lambda a \in A . [0] \text{UnionSubset } A, U(a) : \forall a \in A . U(a) \subset A$ ,

$[2] := \text{EuniformTopology}[1] : A \in \mathcal{T}(X)$ ,

$[3] := \text{ClosureFormula}(X, A)[0] : \overline{A} \subset A$ ,

$[4] := \text{Eclosure}(X, A)[3] : \overline{A} = A$ ,

$[*] := \text{ICloppen}[2][4] : \text{Cloppen}(X, A)$ ;

□

**ConnectorsCloppenAggregation** ::  $\forall X : \text{UniformSpace} . \forall U \in \mathcal{U}_X . \forall A \subset X . \text{Cloppen} \left( X, \bigcup_{n=1}^{\infty} \bigcup_{a \in A} U^{on}(a) \right)$

**Proof** =

$B := \bigcup_{n=1}^{\infty} \bigcup_{a \in A} U^{on}(a) : ?X$ ,

$[1] := \text{EBEConnector}(X, U) \text{CompositionContainment}(X) \text{IB} :$

$: U(B) = U \bigcup_{n=1}^{\infty} U^{on}(A) = \bigcup_{n=2}^{\infty} U^{on}(A) = \bigcup_{n=1}^{\infty} U^{on}(A) = B$ ,

$[*] := \text{CloppenByConnector}[1] : \text{Cloppen}(X, B)$ ;

□

### 1.1.6 Closed Connectors

There are always a base of closed connectors

$$\text{ConnectorClosureFormula} :: \forall X : \text{UniformSpace} . \forall \mathcal{B} : \text{BaseOfUniformity}(X) . \forall U \in \mathcal{U}_X . \\ . \overline{U} = \bigcap \{V \circ U \circ V : V \in \mathcal{B}\}$$

**Proof** =

$$\text{Assume } (a, b) \in \overline{U}^c,$$

$$(V, [1]) := \text{ClosureAltDef}(X^2, U) \mathbf{E}(a, b) : \sum V \in \mathcal{U}(a, b) . V \cap U = \emptyset,$$

$$(W, [2]) := \text{EproductTopology}(X, V) \text{EuniformTopology}(X) \widetilde{\text{EBSymmetricBaseExists}}(X) : \\ : \sum W : \text{SymmetricConnector}(X) . W(a) \times W(b) \subset V,$$

$$(O, [3]) := \text{EUniformBase}(X, \mathcal{B}, W) : \sum O \in \mathcal{B} . O \subset W,$$

$$\text{Assume } [4] : O \circ U \circ O(a, b),$$

$$(x, y, [5]) := \text{EConnector}(X, O \& U)[4] : \sum x, y \in X . x \in O(a) \& y \in U(x) \& b \in O(y),$$

$$[6] := [5.1][3] : W(a, x),$$

$$[7] := [5.2][3] \text{ESymmetricConnector}(X, W) : W(b, y),$$

$$[8] := [5.3] : U(x, y),$$

$$[9] := \mathbf{I} \times [6][7][2] : (x, y) \subset W(a) \times W(b) \subset V,$$

$$[4.*] := [9][1] : \perp;$$

$$\leadsto [4] := \mathbf{E}\perp : \neg O \circ U \circ O(a, b),$$

$$\left[ (a, b). * \right] := \text{CompositionContainment}[4] : \neg O(a, b);$$

$$\leadsto [1] := \mathbf{I}\exists \mathbf{I}\forall : \forall (a, b) \in \overline{V}^c . \exists B \in \mathcal{B} . \neg B \circ U \circ B(a, b),$$

$$\text{Assume } (a, b) \in \overline{U},$$

$$\text{Assume } B \in \mathcal{B},$$

$$(W, [2]) := \text{SymmetricBaseExists}(X, B) : \sum W : \text{SymmetricConnector}(X) . W \subset B,$$

$$[3] := \text{EproductTopology}(X, V) \text{EuniformTopology}(X) \widetilde{\text{EB}}[2] : \exists (W(a) \times W(b)) \cap U,$$

$$(x, y) := \mathbf{E}\exists [3] \in (W(a) \times W(b)) \cap U,$$

$$[*] := \text{EConnector}(X, U \& W) \text{ESymmetricConnector}(X, W) : (a, b) \in W \circ U \circ W \subset B \circ U \circ B;$$

$$\leadsto [*] := [1] : \overline{U} = \bigcap \{V \circ U \circ V : V \in \mathcal{B}\};$$

□

$$\text{UniformityTrisection} :: \forall X : \text{UniformSpace} . \forall U \in \mathcal{U}_X . \exists V : \text{SymmetricConnector}(X) . V \circ V \circ V \subset U$$

**Proof** =

$$(V, [1]) := \text{EUniformity}(X, U) : \sum V \in \mathcal{U}_X . V \circ V \subset U,$$

$$(W, [2]) := \text{EUniformity}(X, V) : \sum W \in \mathcal{U}_X . W \circ W \subset V,$$

$$[3] := \text{MonotonicContainMent}(X)[1][2] : W \circ W \circ W \subset W \circ W \circ W \circ W \subset V \circ V \subset U,$$

$$(O, [4]) := \text{SymmetricBaseExists}(X, W) : \sum O : \text{SymmetricConnector}(X) . O \subset W,$$

$$[5] := [3][4] : O \circ O \circ O \subset U;$$

□

$\text{ClosedConnector} :: \prod X : \text{UniformSpace} . ?\mathcal{U}_X$

$U : \text{ClosedConnector} \iff \text{Closed}(X^2, U)$

$\text{ClosedConnectorsBaseExists} :: \forall X : \text{UniformSpace} . \exists \mathcal{B} : \text{BaseOfUniformity}(X) .$

$. \forall U \in \mathcal{B} . \text{ClosedConnector} \ \& \ \text{SymmetricConnector}(X, U)$

**Proof** =

$\mathcal{B} := \left\{ U \in \mathcal{U}_X : \text{ClosedConnector} \ \& \ \text{SymmetricConnector}(X, U) \right\} : ?\mathcal{U}_X,$

$S := \left\{ U \in \mathcal{U}_X : \text{SymmetricConnector}(X, U) \right\} : \text{BaseOfUniformity}(X),$

**Assume**  $U \in \mathcal{U}_X,$

$(V, [1]) := \text{UniformityTrisection}(X, U) : \sum V : \text{SymmetricConnector}(X) . V \circ V \circ V \subset U,$

$[2] := \text{ConnectorClosureFormula}(X, V)[1] : \overline{V} \subset V \circ V \circ V \subset U,$

$[4] := \text{CinconnectorClosureFormula}(X, V, S) : \overline{V} = \bigcap \{W \circ V \circ W \mid W \in S\},$

$[U.*] := \text{ESymmetricConnector}(X, V)[4] \text{ISymmetricConnector} : \text{SymmetricConnector}(X, \overline{V});$

$\leadsto [*] := \text{IBIBaseOfUniformity} : \text{BaseOfUniformity}(X, \mathcal{B});$

□

$\text{UniformityTrisection2} :: \forall X : \text{UniformSpace} . \forall U \in \mathcal{U}_X .$

$. \exists V : \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(X) . V \circ V \circ V \subset U$

**Proof** =

...

□



### 1.1.7 Uniform Convergence

Uniform convergence of continuous functions preserves continuity.

$$\begin{aligned} \text{UniformlyConvergent} &:: \prod X \in \text{SET} . \prod Y : \text{UniformSpace} . ? \left( \text{Net}(X \rightarrow Y) \times (X \rightarrow Y) \right) \\ &\left( (\Delta, f), g \right) : \text{UniformlyConvergent} \iff f_{\delta \in \Delta} \rightrightarrows g \iff \\ &\iff \forall U \in \mathcal{U}_Y . \exists \delta_0 \in \Delta . \forall \delta \geq \delta_0 . \forall x \in X . \left( f_\delta(x), g(x) \right) \in U \end{aligned}$$

$$\begin{aligned} \text{UniformConvergencePreservesContinuity} &:: \forall X \in \text{TOP} . \forall Y : \text{UniformSpace} . \forall g : X \rightarrow Y . \\ & . \forall (\Delta, f) : \text{Net}(\text{TOP}(X, Y)) . f_{\delta \in \Delta} \rightrightarrows g \Rightarrow g \in \text{TOP}(X, Y) \end{aligned}$$

**Proof** =

**Assume**  $x \in X$ ,

**Assume**  $O \in \mathcal{U}(g(x))$ ,

$$(U, [1]) := \text{EuniformTopology}(Y, O, g(x)) : \sum U \in \mathcal{U}_Y . U(g(x)) \subset O,$$

$$(V, [2]) := \text{UniformityTrisection2}(X, U) :$$

$$: \sum V : \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(Y) . V \circ V \circ V \subset U,$$

$$(\delta, [3]) := \text{E}[0](V) : \sum_{\delta \in \mathcal{D}} \forall t \geq \delta . \forall u \in X . \left( f_t(u), g(u) \right) \in V,$$

$$(O', [4]) := \text{EB}(V, g(x)) : \sum O' \in \mathcal{T}(Y) . g(x) \in O' \subset V(g(x)),$$

$$[5] := \text{ETOP}(X, Y, f_\delta, O') : f_\delta^{-1}(O') \in \mathcal{T}(X),$$

**Assume**  $u \in f_\delta^{-1}(O')$ ,

$$[6] := \text{EuEpreimage} : f_\delta(u) \in O' \subset V(g(x)),$$

$$[7] := [3](\delta, u) : \left( f_\delta(u), g(u) \right) \in V,$$

$$[x.*] := \text{ESymmetricConnector}(Y, V)[6][7][2][1] : \left( g(u), g(x) \right) \in O;$$

$$\rightsquigarrow [*] := \text{ContinuityIsLocal}(X, Y) : g \in \text{TOP}(X, Y);$$

□

### **1.1.8 Pseudo-Uniformities [ $\infty$ ]**

There are more general concepts than uniformities. Will be written on demand.

### **1.1.9 $v$ -Closure [ $\infty$ ]**

This is about columns in connectors. Will be written on demand.

### **1.1.10 Transitive Uniformities [ $\infty$ ]**

Transitive uniformities are very special. Will be written on demand.

## 1.2 Covering Uniformities $[\infty]$

One can also define uniformities as families of covers. Will be written on demand.

## 1.3 Uniform Continuity

### 1.3.1 Uniform Maps

The notion of uniform map nicely generalizes from metric spaces.

$\text{UniformlyContinuous} :: \prod X, Y : \text{UniformSpace} . ?(X \rightarrow Y)$

$\varphi : \text{UniformlyContinuous} \iff \forall U \in \mathcal{U}_Y . \exists V \in \mathcal{U}_X . (\varphi \times \varphi)(V) \subset U$

$\text{MetricUniformContinuity} :: \forall X, Y \in \text{SMS} . \forall \varphi : X \rightarrow Y . \text{UniformlyContinuous}(X, Y, \varphi) \iff$   
 $\iff \forall \varepsilon \in \mathbb{R}_{++} . \exists \delta \in \mathbb{R}_{++} . \forall a, b \in X . d(a, b) < \delta \Rightarrow d(\varphi(a), \varphi(b)) < \varepsilon$

**Proof** =

...

□

$\text{UniformContinuousIsContinuous} :: \forall X, Y : \text{UniformSpace} . \forall \varphi : \text{UniformlyContinuous}(X, Y) .$   
 $\varphi \in \text{TOP}(X, Y)$

**Proof** =

**Assume**  $x \in X$ ,

**Assume**  $O \in \mathcal{U}(\varphi(x))$ ,

$(U, [1]) := \text{EuniformTopology}(Y, \varphi(x)) : \sum U \in \mathcal{U}_Y . U(\varphi(x)) \subset O$ ,

$(V, [2]) := \text{EUniformlyContinuous}(X, Y, \varphi, U) : \sum V \in \mathcal{U}_X . (\varphi \times \varphi)(V) \subset U$ ,

$(W, [3]) := \text{EB}_{\tilde{x}}(V) : \sum W \in \mathcal{U}(x) . W \subset V(x)$ ,

$[x.*] := [1][2][3] : \varphi(W) \subset O$ ;

$\leadsto [*] := \text{ContinuityIsLocal} : \varphi \in \text{TOP}(X, Y)$ ;

□

$\text{UniformityInclusionByIdentityContinuity} ::$

$: \forall X \in \text{SET} . \mathcal{U}, \mathcal{V} : \text{Uniformity}(X) . \mathcal{U} \leq \mathcal{V} \iff \text{UniformlyContinuous}((X, \mathcal{V}), (X, \mathcal{U}), \text{id})$

**Proof** =

...

□

$\text{Unimorphism} :: \prod X, Y : \text{UniformSpace} . ?\text{Homeomorphis}(X, Y)$

$\varphi : \text{Unimorphism} \iff \text{UniformlyContinuous}(X, Y, \varphi) \ \& \ \text{UniformlyContinuous}(Y, X, \varphi^{-1})$

**UniformConvergencePreservesContinuity** ::  $\forall X, Y \in \mathbf{UNI} . \forall g : X \rightarrow Y .$

$. \forall (\Delta, f) : \mathbf{Net}(\mathbf{UNI}(X, Y)) . f_{\delta \in \Delta} \rightrightarrows g \Rightarrow g \in \mathbf{UNI}(X, Y)$

**Proof** =

**Assume**  $V \in \mathcal{U}_Y,$

$(W, [1]) := \mathbf{UniformityTrisection}(Y, V) : \sum W : \mathbf{SymmetricConnector}(Y) . W \circ W \circ W \subset V,$

$(\delta, [2]) := \mathbf{EUniformConvergence}[0](W) : \sum_{\delta \in \Delta} \forall x \in X . \forall t > \delta . (f_t(x), g(x)) \in VW,$

$(U, [3]) := \mathbf{EUniformlyContinuous}(X, Y, f_\delta, W) : \sum U \in \mathcal{U}_X . f_\delta \times f_\delta(U) \subset W,$

**Assume**  $(a, b) \in U,$

$[4] := [3](a, b) : (f_\delta(a), f_\delta(b)) \in W,$

$[5] := [2](a, \delta) : (f_\delta(a), g(a)) \in W,$

$[6] := [3](b, \delta) : (f_\delta(b), g(b)) \in W,$

$[(a, b). *] := [4][5][6]\mathbf{EConnector}(Y, W)[1] : (g(a), g(b)) \in W \circ W \circ W \subset V;$

$\leadsto [V.*] := \mathbf{I} \subset (g \times g)(U) \subset V;$

$\leadsto [*] := \mathbf{IUniformlyContinuous} : g \in \mathbf{UNI}(X, Y);$

□

### 1.3.2 Category of Uniform Spaces, Initial and Final Uniformity

With uniform maps as morphism we have a bicomplete category. The notion of final and initial uniformity is very useful for constructing limits. Note, that embedding in TOP reflects limits.

**UniformCategory** :: CAT

**UniformCategory** () = UNI := ( **UniformSpace**, **UniformlyContinuous**, o, id )

**UniformSeparetadCategory** :: CAT

**UniformSeparatedCategory** () = UNIS := ( **UniformSpace** & T3, **UniformlyContinuous**, o, id )

**initialUniformity** ::  $\prod_{X, I \in \text{SET}} \prod_{Y: I \rightarrow \text{UNI}} \left( \prod_{i \in I} X \rightarrow Y_i \right) \rightarrow \text{Uniformity}(X)$

**initialUniformity** ( $\phi$ ) =  $\mathcal{I}_X(I, Y, \phi) := \min \left\{ \mathcal{U} : \text{Uniformity}(X) : \forall i \in I . \phi_i \in \text{UNI}((X, \mathcal{U}), Y) \right\}$

**InitialUniformityExpression** ::  $\forall X, I \in \text{SET} . \forall Y : I \rightarrow \text{UNI} . \forall \phi : \prod_{i \in I} X \rightarrow Y_i .$

$\cdot \mathcal{I}_X(I, Y, \phi) = \left\langle \left\{ (\phi_i \times \phi_i)^{-1}(V) \mid i \in I, V \in \mathcal{U}_{Y_i} \right\} \right\rangle_{\mathcal{F}}$

**Proof** =

...

□

**productUniformSpace** ::  $\prod_{I \in \text{SET}} (I \rightarrow \text{UNI}) \rightarrow \text{UNI}$

**productUniformSpace** ( $X$ ) =  $\prod_{i \in I} X_i := \left( \prod_{i \in I} X_i, \mathcal{I}(I, X, \pi) \right)$

**UniformCategoryIsComplete** :: **Complete**(UNI)

**Proof** =

...

□

**LimitsOfUniAgreeWithTop** ::  $\forall (I, X, \phi) : \text{Diagram}(\text{UNI}) . \lim_{\text{UNI}}(I, X, \phi) \cong_{\text{TOP}} \lim_{\text{TOP}}(I, X, \phi)$

**Proof** =

...

□

**supremumUniformity** ::  $\prod_{X, I \in \text{SET}} (I \rightarrow \text{Uniformity}(X)) \rightarrow \text{Uniformity}(X)$

**supremumUniformity** ( $\mathcal{U}$ ) =  $\bigvee_{i \in I} \mathcal{U}_i := \mathcal{I}_X(I, (X, \mathcal{U}), \text{id})$

**InitialUniformityUniversalProperty** ::

$$\begin{aligned} &:: \forall I, X \in \text{SET} . \forall Y : I \rightarrow \text{UNI} . \forall \phi : \left( \prod_{i \in I} X \rightarrow Y_i \right) . \forall Q \in \text{UNI} . \forall \psi : Q \rightarrow X . \\ & . \psi \in \text{UNI} \left( Q, \left( X, \mathcal{I}_X(I, Y, \phi) \right) \right) \iff \forall i \in I . \psi \phi_i \in \text{UNI}(Q, Y_i) \end{aligned}$$

**Proof** =

...

□

$$\text{finalUniformity} :: \prod_{Y, I \in \text{SET}} \prod_{X : I \rightarrow \text{UNI}} \left( \prod_{i \in I} X_i \rightarrow Y \right) \rightarrow \text{Uniformity}(X)$$

$$\text{finalUniformity}(\phi) = \mathcal{F}_Y(I, X, \phi) := \max \left\{ \mathcal{V} : \text{Uniformity}(Y) : \forall i \in I . \phi_i \in \text{UNI} \left( X_i, (Y, \mathcal{V}) \right) \right\}$$

$$\mathfrak{V} := \left\{ \mathcal{V} : \text{Uniformity}(Y) : \forall i \in I . \phi_i \in \text{UNI} \left( X_i, (Y, \mathcal{V}) \right) \right\} : ?\text{Uniformity}(Y),$$

$$\mathcal{U} := \bigvee_{V \in \mathfrak{V}} V : \text{Uniformity}(X),$$

**Assume**  $i \in I$ ,

$$[1] := \text{E}\mathfrak{V}(i) : \forall \mathcal{V} \in \mathfrak{V} . \phi_i \in \text{UNI} \left( X_i, (Y, \mathcal{V}) \right),$$

$$[2] := \text{E}\mathcal{U} : \mathcal{U} = \mathcal{I}_Y \left( \mathfrak{V}, \Lambda \mathcal{V} \in \mathfrak{V} . (Y, \mathcal{V}), \text{id} \right),$$

$$[i.*] := \text{InitialUniformityUniversalProperty}[1][2] : \phi_i \in \text{UNI} \left( X_i, (Y, \mathcal{U}) \right);$$

$$\leadsto [1] := \text{I}\mathfrak{V} : \mathcal{U} \in \mathfrak{V},$$

$$[*] := \text{E}\mathcal{U}[1] \text{I}\mathcal{F}_Y(I, X, \phi) : \mathcal{U} = \mathcal{F}_Y(I, X, \phi);$$

□

**FinalUniformityUniversalProperty** ::

$$\begin{aligned} &:: \forall I, Y \in \text{SET} . \forall X : I \rightarrow \text{UNI} . \forall \phi : \left( \prod_{i \in I} X_i \rightarrow Y \right) . \forall Z \in \text{UNI} . \forall \psi : Y \rightarrow Z . \\ & . \psi \in \text{UNI} \left( \left( Y, \mathcal{F}_Y(I, X, \phi) \right), Z \right) \iff \forall i \in I . \phi_i \psi \in \text{UNI}(X_i, Z) \end{aligned}$$

**Proof** =

...

□

**UniformCategoryIsBicomplete** :: **Bicomplete**(UNI)

**Proof** =

...

□

$$\text{quotientUniformity} :: \prod_{X \in \text{UNI}} \text{Equivalence}(X) \rightarrow \text{UNI}$$

$$\text{quotientUniformity}(\sim) = \frac{X}{\sim} := \left( \frac{X}{\sim}, \mathcal{F}(\star, \star \mapsto X, \pi_{\sim}) \right)$$

$\text{infimumUniformity} :: \prod_{I, X \in \text{SET}} \left( I \rightarrow \text{Uniformity}(X) \right) \rightarrow \text{Uniformity}(X)$

$\text{infimumUniformity}(\mathcal{U}) = \bigwedge_{i \in \mathcal{I}} \mathcal{U}_i := \left( X, \mathcal{F}_X \left( I, (X, \mathcal{U}), \text{id} \right) \right)$

$\text{UniformSeparatedCategoryIsBicomplete} :: \text{Bicomplete}(\text{UNIS})$

$\text{Proof} =$

...

□

$\text{MinUniformityBase} ::$

$:: \forall X \in \text{SET} . \forall \mathcal{U}, \mathcal{V} : \text{Uniformity}(X) . \text{BaseOfUniformity} \left( X, \mathcal{U} \wedge \mathcal{V}, \{U \circ V \mid U \in \mathcal{U}, V \in \mathcal{V}\} \right)$

$\text{Proof} =$

...

□



### 1.3.3 Uniform Covers

Notion of uniform covers is useful for identifying uniform maps.

$$\text{UniformCover} :: \prod_{X \in \text{UNI}} ?\text{Cover}(X)$$

$$\mathcal{C} : \text{UniformCover} \iff \exists U \in \mathcal{U}_X . \forall x \in X . \exists C \in \mathcal{C} . U(x) \subset C$$

$$\text{UniformContinuityByPreimages} :: \forall X, Y \in \text{UNI} . \forall \varphi : X \rightarrow Y .$$

$$. \varphi \in \text{UNI}(X, Y) \iff \forall V \in \mathcal{U}_Y . \text{UniformCover} \left( X, \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} \right)$$

Proof =

Assume [1] :  $\varphi \in \text{UNI}(X, Y)$ ,

Assume  $V \in \mathcal{U}_Y$ ,

$\mathcal{C} := \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} : ??X$ ,

$(U, [2]) := \text{EUniformlyContinuous}(X, Y, \varphi)[1] : \sum U \in \mathcal{U}_X . (\varphi \times \varphi)(U) \subset V$ ,

Assume  $x \in X$ ,

$[3] := [2](x) : \varphi(U(x)) \subset V(\varphi(x))$ ,

$[x, *] := \varphi^{-1}[3] : U(x) \subset \varphi^{-1}(V(\varphi(x)))$ ;

$\leadsto [1.*] := \text{IUniformCover} : \text{UniformCover}(X, \mathcal{C})$ ;

$\leadsto [1] := \text{I} \Rightarrow : \varphi \in \text{UNI}(X, Y) \Rightarrow \forall V \in \mathcal{U}_Y . \text{UniformCover} \left( X, \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} \right)$ ,

Assume [2] :  $\forall V \in \mathcal{U}_Y . \text{UniformCover} \left( X, \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} \right)$ ,

Assume  $V \in \mathcal{U}_Y$ ,

$\mathcal{C} := \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} : \text{UniformCover}(X)$ ,

$(U, [3]) := \text{EUniformCover}(X) : \forall x \in X . \exists C \in \mathcal{C} . U(x) \subset C$ ,

$[V.*] := \varphi(\text{EC}[3])\text{I} \times : (\varphi \times \varphi)(U) \subset V$ ;

$\leadsto [2.*] := \text{IUniformlyContinuous} : \varphi \in \text{UNI}(X, Y)$ ;

$\leadsto [*] := \text{I}(\iff)[1] : \varphi \in \text{UNI}(X, Y) \iff \forall V \in \mathcal{U}_Y . \text{UniformCover} \left( X, \left\{ \varphi^{-1}(V(y)) \mid y \in Y \right\} \right)$ ;

□

### 1.3.4 Compact Uniform Spaces

All continuous maps with compact uniform domains are uniformly continuous.

**EveryCoverOfCompactSpaceIsUniform** ::

::  $\forall X \in \text{UNI} . \text{Compact}(X) \Rightarrow \forall \mathcal{O} : \text{OpenCover}(X) . \text{UniformCover}(X, \mathcal{O})$

**Proof** =

$(\mathcal{O}', [1]) := \text{ECompact}(X, \mathcal{O}) : \sum \mathcal{O}' : \text{FiniteSubset}(\mathcal{O}) . \text{Cover}(X, \mathcal{O}'),$   
 $(V, [2]) := \text{EuniformTopology}[1] : \sum V : \prod_{O \in \mathcal{O}'} \prod_{x \in O} \mathcal{U}_X . \forall O \in \mathcal{O}' . \forall x \in O . V_{O,x}(x) \subset O,$   
 $(W, [21]) := \text{SymmetricBaseExists}(X, V) :$   
 $: \sum W : \prod_{O \in \mathcal{O}'} \prod_{x \in O} \text{SymmetricConnector}(X) . \forall O \in \mathcal{O}' . \forall x \in O . W_{O,x} \circ W_{O,x} \subset U_{O,x},$   
 $(\mathcal{O}', [3]) := \text{EB}(W)[2] : \sum \mathcal{O}' : \prod_{O \in \mathcal{O}'} \prod_{x \in O} x \in O'_{O,x} \subset W_{O,x}(x),$   
 $[4] := \text{ECover}(X, \mathcal{O}')[1][3] : \text{OpenCover}(X, \text{Im } \mathcal{O}'),$   
 $(n, x, \theta, [5]) := \text{ECompact}(X, \text{Im } \mathcal{O}') : \sum_{n=1}^{\infty} \sum x : \{1, \dots, n\} \rightarrow X .$   
 $. \sum \theta : \{1, \dots, n\} \rightarrow \mathcal{O}' . \left( \forall k \in \{1, \dots, n\} . x_k \in \theta_k \right) \& \text{OpenCover}(X, \text{Im } \mathcal{O}'_{\theta,x}),$   
 $[6] := [5][3] : \text{Cover}(X, \text{Im } W_{\theta,x}(x)),$   
 $U := \bigcap_{k=1}^n W_{\theta_k, x_k} \in \mathcal{U}_X,$

**Assume**  $y \in X,$

$(k, [7]) := \text{ECover}[6] : \sum_{k=1}^n y \in W_{\theta_k, x_k}(x_k),$

**Assume**  $u \in U(y),$

$[8] := \text{EuEU}(k) : u \in W_{\theta_k, x_k}(y),$

$[y.*] := \text{ESymmetricConnector}(X, W_{\theta_k, x_k})[21][2] : u \in W_{\theta_k, x_k} \circ W_{\theta, x_k}(x_k) \subset V_{\theta_k, x_k} \subset \theta_k;$

$\leadsto [*] := \text{IUniformCover} : \text{UniformCover}(X, \mathcal{O});$

□

**CompactUniformContinuity** ::  $\forall X, Y \in \text{UNI} . \text{Compact}(X) \Rightarrow \text{TOP}(X, Y) = \text{UNI}(X, Y)$

**Proof** =

...

□

**CompactUniformityEquivalence** ::  $\forall X, Y \in \text{UNI} . \text{Compact}(X \& Y) \Rightarrow (X \cong_{\text{TOP}} Y \Rightarrow X \cong_{\text{UNI}} Y)$

**Proof** =

...

□

## 1.4 Completeness

As metric spaces, the uniform spaces can be complete.

### 1.4.1 Cauchy Filterbase

Cauchy Filterbases are natural generalizations of Cauchy sequences.

$$\text{CauchyFilterbase} :: \prod_{X \in \text{UNI}} ?\text{Filterbase}(X)$$

$$\mathcal{F} : \text{CauchyFilterbase} \iff \forall U \in \mathcal{U}_X . \exists F \in \mathcal{F} . F \times F \subset U$$

$$\text{EveryConvergentFilterbaseIsCauchy} ::$$

$$:: \forall X \in \text{UNI} . \forall \mathcal{F} : \text{ConvergentFilterbase}(X) . \text{CauchyFilterbase}(X, \mathcal{F})$$

$$\text{Proof} =$$

$$(x, [1]) := \text{EConvergentFilterbase}(X, \mathcal{F}) : \sum_{x \in X} \lim \mathcal{F} = x,$$

$$\text{Assume } U \in \mathcal{U}_X,$$

$$(V, [3]) := \text{SymmetricBaseExists}(X) : \sum V : \text{SymmetricConnector}(X) . V \circ V \subset X,$$

$$(O, [4]) := \text{EB}_{\tilde{x}}(V) : \sum O \in \mathcal{U}(x) . O \subset V(x),$$

$$(F, [5]) := \text{E} \lim[1](O) : \sum F \in \mathcal{F} . F \subset O,$$

$$[U.*] := [4][5]\text{ProductSubset}(X) : F \times F \subset V \circ V \subset U;$$

$$\leadsto [*] := \text{ICauchyFilterbase} : \text{CauchyFilterbase}(X, \mathcal{F});$$

□

$$\text{UniformMapsPreserveCauchyFilters} ::$$

$$:: \forall X, Y \in \text{UNI} . \forall \varphi \in \text{UNI}(X, Y) . \forall \mathcal{F} : \text{CauchyFilterbase}(X) . \text{CauchyFilterbase}(Y, \varphi(\mathcal{F}))$$

$$\text{Proof} =$$

$$\text{Assume } V \in \mathcal{U}_Y,$$

$$(U, [1]) := \text{EUniformlyContinuous}(X, Y, f, V) : \sum U \in \mathcal{U}_X . \varphi \times \varphi(U) \subset V,$$

$$(F, [2]) := \text{ECauchyFilterbase}(X, \mathcal{F}, U) : \sum F \in \mathcal{F} . F \times F \subset U,$$

$$[V.*] := [1][2] : \varphi(F) \times \varphi(F) \subset V;$$

$$\leadsto [*] := \text{ICauchyFilterbase} : \text{CauchyFilterbase}(Y, \varphi(\mathcal{F}));$$

□

**CauchyClustersAreLimits** ::  $\forall X \in \mathbf{UNI} . \forall \mathcal{F} : \mathbf{CauchyFilterbase}(X) . \forall x : \mathbf{Cluster}(X, \mathcal{F}) . x \in \lim \mathcal{F}$

**Proof** =

**Assume**  $O \in \mathcal{U}(x)$ ,

$(U, [1]) := \mathbf{EuniformTopology}(U, O) : \sum U \in \mathcal{U}_X . U(x) \subset O$ ,

$(V, [2]) := \mathbf{ClosedBaseExists}(X, U) : \sum V : \mathbf{ClosedConnector}(X) . V \subset U$ ,

$(F, [3]) := \mathbf{ECauchyFilterbase}(X, \mathcal{F}, V) : \sum F \in \mathcal{F} . F \times F \subset V$ ,

**Assume**  $f \in F$ ,

$[5] := \mathbf{ECluster}(X, \mathcal{F}, x, F) \mathbf{EClosedConnector}(X, V)[3] : (x, f) \in \overline{F} \times F \subset V$ ,

$[f.*] := [2] \mathbf{EConnector}(X, U)[1] : f \in O$ ;

$\leadsto [O.*] := \mathbf{I} \subset : F \subset O$ ;

$\leadsto [*] := \mathbf{I} \lim : x \in \lim \mathcal{F}$ ;

□

**SupUniformityCauchyFilterbase** ::

$:: \forall X, I \in \mathbf{SET} . \forall \mathcal{U} : I \rightarrow \mathbf{Uniformity}(X) . \forall \mathcal{F} : \mathbf{Filterbase}(X) .$

$. \mathbf{CauchyFilterbase} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), \mathcal{F} \right) \iff \forall i \in I . \mathbf{CauchyFilterbase} \left( (X, \mathcal{U}_i), \mathcal{F} \right)$

**Proof** =

**Assume**  $[1] : \mathbf{CauchyFilterbase} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), \mathcal{F} \right)$ ,

$[2] := \mathbf{E} \bigvee_{i \in I} \mathcal{U}_i : \forall i \in I . \text{id} \in \mathbf{UNI} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), (X, \mathcal{U}_i) \right)$ ,

$[1.*] := \mathbf{UniformMapsPreserveCauchyFilters}[1][2] : \forall i \in I . \mathbf{CauchyFilterbase} \left( (X, \mathcal{U}_i), \mathcal{F} \right)$ ;

$\leadsto [1] := \mathbf{I} \Rightarrow : \mathbf{CauchyFilterbase} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), \mathcal{F} \right) \Rightarrow \forall i \in I . \mathbf{CauchyFilterbase} \left( (X, \mathcal{U}_i), \mathcal{F} \right)$ ,

**Assume**  $[1] : \forall i \in I . \mathbf{CauchyFilterbase} \left( (X, \mathcal{U}_i), \mathcal{F} \right)$ ,

**Assume**  $U \in \bigvee_{i \in I} \mathcal{U}_i$ ,

$(n, i, V, [2]) := \mathbf{E} \bigvee_{i \in I} \mathcal{U}_i(U) \mathbf{InitialUnifomityExpression} : \sum_{n=0}^{\infty} \sum i : \{1, \dots, n\} \rightarrow I . \sum V : \prod_{k=1}^{\infty} \mathcal{U}_{i_k} . \bigcap_{k=1}^n V_k \subset U$

$(F, [3]) := \Lambda k \in \{1, \dots, n\} . \mathbf{ECauchyFilterbase}(X, \mathcal{U}_{i_k}, V_k) : \sum F : \{1, \dots, n\} \rightarrow \mathcal{F} . \forall k \in \{1, \dots, n\} . F_k \times F_k \subset V_k$

$(G, [4]) := \mathbf{EFilterbase}(X, F) \in \sum G \in \mathcal{F} . G \subset \bigcap_{k=1}^n F_k$ ,

$[U.*] := [4][3] \mathbf{SubsetIntersect}[2] : G \times G \subset U$ ;

$\leadsto [2.*] := \mathbf{ICauchyFilterbase} : \mathbf{CauchyFilterbase} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), \mathcal{F} \right)$ ;

$\leadsto [*] := \mathbf{I} \iff [*] : \mathbf{CauchyFilterbase} \left( \left( X, \bigvee_{i \in I} \mathcal{U}_i \right), \mathcal{F} \right) \iff \forall i \in I . \mathbf{CauchyFilterbase} \left( (X, \mathcal{U}_i), \mathcal{F} \right)$ ;

□

### 1.4.2 Complete Uniform Spaces

Now complete spaces are those, where all Cauchy filters are converging

`CompleteUniformSpace` :: ?UNI

$X : \text{CompleteUniformSpace} \iff \forall \mathcal{F} : \text{CauchyFilterbase}(X) . \text{ConvergentFilterbase}(X, \mathcal{F})$

`CauchySequence` ::  $\prod_{X \in \text{UNI}} \mathbb{N} \rightarrow X$

$x : \text{CauchySequence} \iff \text{CauchyFilterbase}\left(X, \{\{x_n, n \geq m\} \mid m \in \mathbb{N}\}\right)$

`SequenceCompleteUniformSpace` :: ?UNI

$X : \text{SequenceCompleteUniformSpace} \iff \forall x : \text{CauchySequence}(X) . \text{Convergent}(X, x)$

`IsomorphicCompleteness` ::  $\forall X, Y \in \text{UNI} . X \cong_{\text{UNI}} Y \Rightarrow$

$\Rightarrow \left( \text{CompleteUniformSpace}(X) \iff \text{CompleteUniformSpace}(Y) \right)$

`Proof` =

...

□

`IsomorphicSequenceCompleteness` ::  $\forall X, Y \in \text{UNI} . X \cong_{\text{UNI}} Y \Rightarrow$

$\Rightarrow \left( \text{SequenceCompleteUniformSpace}(X) \iff \text{SequenceCompleteUniformSpace}(Y) \right)$

`Proof` =

...

□

`LargerEqUniformityIsComplete` ::  $\forall X : \text{CompleteUniformSpace} . \forall \mathcal{V} \geq \mathcal{U}_X .$

$. \mathcal{V} \cong \mathcal{U}_X \Rightarrow \text{CompleteUniformSpace}(X, \mathcal{V})$

`Proof` =

...

□

`LargerEqUniformityIsComplete` ::  $\forall X : \text{CompleteUniformSpace} . \forall \mathcal{V} \geq \mathcal{U}_X .$

$. \mathcal{V} \cong \mathcal{U}_X \Rightarrow \text{CompleteUniformSpace}(X, \mathcal{V})$

`Proof` =

...

□

`LargerEqUniformityIsSeqComplete` ::  $\forall X : \text{SequenceCompleteUniformSpace} . \forall \mathcal{V} \geq \mathcal{U}_X .$

$. \mathcal{V} \cong \mathcal{U}_X \Rightarrow \text{SequenceCompleteUniformSpace}(X, \mathcal{V})$

`Proof` =

...

□

**ClosedOfCompleteIsComplete** ::

$:: \forall X : \text{CompleteUniformSpace} . \forall A : \text{Closed}(X) . \text{CompleteUniformSpace}(A)$

**Proof** =

**Assume**  $\mathcal{F} : \text{CauchyFilterbase}(A)$ ,

$\mathcal{F}' := \{F \cup B \mid F \in \mathcal{F}, B \subset X\} : \text{Filter}(X)$ ,

**Assume**  $U \in \mathcal{U}_X$ ,

$U' := U \cap A \times A \in \mathcal{U}_A$ ,

$(F, [1]) := \text{ECauchyFilterbase}(A, \mathcal{F}) \text{EU}' \text{IntersectionIsSubset}(X, U', U, A \times A) : \sum_{F \in \mathcal{F}} F \times F \subset U' \subset U$ ,

$[U.*] := \text{EFUnionWithEmpty}(X) \text{IF}' : F \in \mathcal{F}'$ ;

$\leadsto [1] := \text{ICauchyFilterbase} : \text{CauchyFilterbase}(X, \mathcal{F}')$ ,

$x := \text{ECompleteUniformSpace}(X, \mathcal{F}') \in \lim \mathcal{F}'$ ,

$[2] := \text{EClosed}(X, A) \text{EF}' \text{ClosedFilterLimit} : x \in A$ ,

$[\mathcal{F}.*] := \text{EF}' \text{Ex}[2] : x \in \lim \mathcal{F}$ ;

$\leadsto [*] := \text{ICompleteUniformSpace} : \text{CompleteUniformSpace}(A)$ ;

□

**CompleteProductTHM** ::  $\forall I \in \text{SET} . \forall X : I \rightarrow \text{UNI} . (\forall i \in I . \exists X_i) \Rightarrow$

$\Rightarrow \text{CompleteUniformSpace} \left( \prod_{i \in I} X_i \right) \iff \forall i \in I . \text{CompleteUniformSpace}(X_i)$

**Proof** =

**Assume**  $[1] : \text{CompleteUniformSpace} \left( \prod_{i \in I} X_i \right)$ ,

**Assume**  $i \in I$ ,

$J := I \setminus \{i\} : ?I$ ,

$x := \text{Choice} \left( J, X, [0] \right) : \prod_{j \in J} X_j$ ,

**Assume**  $\mathcal{F} : \text{CauchyFilterbase}(X_i)$ ,

$\mathcal{F}' := \left\{ F \times_i \prod_{j \in J} \{x_j\} \mid F \in \mathcal{F} \right\} : \text{Filterbase} \left( \prod_{i \in I} X_i \right)$ ,

$\phi := \Lambda u \in X_i . \Lambda k \in I . \text{if } i == k \text{ then } u \text{ else } x_k : X_i \rightarrow \prod_{i \in I} X_i$ ,

$[1] := \text{E}\phi \text{E}\mathcal{F}' : \mathcal{F}' = \phi_*(\mathcal{F})$ ,

$[2] := \text{E}\phi \text{E}\pi_i \text{Id} \text{ECAT}(\text{UNI}) : \phi \pi_i = \text{id} \in \text{UNI}(X_i, X_i)$ ,

$[3] := \Lambda j \in J . \text{E}\phi \text{E}\pi_j \text{EUniformity}(\mathcal{U}_{X_j}) : \forall j \in J . \phi \pi_j = x_j \in \text{UNI}(X_i, X_j)$ ,

$[4] := \text{E} \prod_{i \in I} X_i \text{InitialUniformityUniversalProperty}[2][3] : \phi \in \text{UNI} \left( X_i, \prod_{i \in I} X_i \right)$ ,

$[5] := \text{UniformMapsPreserveCauchyFilters}[1][4] : \text{CauchyFilterbase} \left( \prod_{i \in I} X_i, \mathcal{F}' \right)$ ,

$f := \lim \mathcal{F}' \in \prod_{i \in I} X_i$ ,

Assume  $O \in \mathcal{U}(f_i)$ ,

$$O' := O \times_i \prod_{j \in J} X_j \in \mathcal{U}(f),$$

$$(F', [6]) := \mathbf{E} \lim[5](O') : \sum F' \in \mathcal{F}' . F' \subset O',$$

$$F := \pi_i(F') : ?X,$$

$$[7] := \mathbf{E} F \mathbf{E} \mathcal{F}' : F \in \mathcal{F},$$

$$[O.*] := \mathbf{E} F \mathbf{E} O' [6] : F \subset O;$$

$$\leadsto [1.*] := \mathbf{I} \lim : f_i \in \lim \mathcal{F};$$

$$\leadsto [1] := \mathbf{I} \Rightarrow : \mathbf{CompleteUniformSpace} \left( \prod_{i \in I} X_i \right) \Rightarrow \forall i \in I . \mathbf{CompleteUniformSpace}(X_i),$$

$$\text{Assume } [2] : \forall i \in I . \mathbf{CompleteUniformSpace}(X_i),$$

$$\text{Assume } \mathcal{F} : \mathbf{CauchyFilterbase} \left( \prod_{i \in I} X_i \right),$$

$$[3] := \Lambda i \in I . \mathbf{UniformMapsPreserveCauchyFilters} \left( \prod_{i \in I} X_i, X_i, \pi_i, \mathcal{F} \right) :$$

$$: \forall i \in \mathcal{I} . \mathbf{CauchyFilterbase} \left( X_i, \pi_i(\mathcal{F}) \right),$$

$$f := \Lambda i \in I . \lim \pi_i(\mathcal{F}) \in \prod_{i \in I} X_i,$$

$$\text{Assume } O : \mathcal{U}(f),$$

$$(J, E, [4]) := \mathbf{EproductTopology}(I, X, O) : \sum J : \mathbf{Finite}(I) . \sum E : \prod_{j \in J} \mathcal{T}(X_j) . f \in \prod_{j \in J} E_j \times \prod_{j \in J^c} X_j \subset O,$$

$$(F, [5]) := \mathbf{E} f(E) : \sum F \in \prod_{j \in J} \pi_j(\mathcal{F}) . F_j \subset E_j,$$

$$[6] := \Lambda j \in J . \mathbf{E} F \mathbf{E} \pi_j \mathbf{E} \mathbf{Filter} \left( \prod_{i \in I} X_i, \mathcal{F} \right) : \forall j \in J . F_j \times_j \prod_{i \in \{j\}^c} X_i \in \mathcal{F},$$

$$[7] := \mathbf{E} \mathbf{Filter} \left( \prod_{i \in I} X_i, \mathcal{F} \right) [6] : \prod_{j \in J} F_j \times \prod_{j \in J^c} X_j \in \mathcal{F},$$

$$[O.*] := [7][5][4] : \prod_{j \in J} F_j \times \prod_{j \in J^c} X_j \subset O;$$

$$\leadsto [2.*] := \mathbf{I} \lim : f \in \lim \mathcal{F};$$

$$\leadsto [*] := \mathbf{I} \iff [1] : \mathbf{CompleteUniformSpace} \left( \prod_{i \in I} X_i \right) \iff \forall i \in I . \mathbf{CompleteUniformSpace}(X_i);$$

□

There is also an extension theorem.

### 1.4.3 Extension Theorem

**UCExtensionTheorem** ::  $\forall X \in \text{UNI} . \forall Y : \text{CompleteUniformSpace} . \forall D : \text{Dense}(X) . \forall \phi \in \text{UNI}(D, Y) .$   
 $. \exists \Phi \in \text{UNI}(X, Y) . \Phi|_D = \phi$

**Proof** =

**Assume**  $x \in X$ ,

$\mathcal{F}_x := \left\{ U(x) \cap D \mid U \in \mathcal{U}_X \right\} :??D$ ,

[1] :=  $\text{E}\mathcal{F}_x\text{EUniformity}(X) : \text{Filterbase}(X, \mathcal{F}_x)$ ,

[2] :=  $\text{E}\mathcal{F}_x\text{EuniformTopology}(X)\text{EDense}(X, D) : \lim \mathcal{F}_x = x$ ,

[3] :=  $\text{ConvergenrFilterbaseIsCauchy}[2] : \text{CauchyFilterbase}(X, \mathcal{F}_x)$ ,

[4] :=  $\text{UniformBasePreservesCauchyFilters}(X, Y, \phi)[3] : \text{CauchyFilterbase}(Y, \phi(\mathcal{F}_x))$ ,

$\Phi(x) := \text{ECompleteUniformSpace}(Y, \phi(\mathcal{F}_x))\text{EConvergentFilterbase} \in \lim \phi(\mathcal{F}_x)$ ;

$\leadsto \Phi := \text{I}(\rightarrow) : X \rightarrow Y$ ,

[1] :=  $\text{E}\Phi\text{ContinuousPreserveLimits} : \Phi|_D = \phi$ ,

**Assume**  $V \in \mathcal{U}_Y$ ,

$(W, [2]) := \text{UniformityTrisection2}(Y, V) : \sum W \in \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(Y) .$   
 $. W \circ W \circ W \subset V$ ,

$(U, [3]) := \text{EUNI}(D, X, \phi, W) : \sum U \in \mathcal{U}_D . (\phi \times \phi)(U) \subset W$ ,

$(U', [4]) := \text{EsubsetUniformity}(X, D, U) : \sum U' \in \mathcal{U}_X . U = U' \cap (D \times D)$ ,

$(O, [5]) := \text{UniformityTrisection2}(X, U') : \sum O : \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(X) .$   
 $. O \circ O \circ O \subset U'$ ,

**Assume**  $(a, b) \in O$ ,

$r := \text{EDense}(X, D)\text{E}\tilde{\mathcal{B}}_a(O) \in O(a) \cap D$ ,

$s := \text{EDense}(X, D)\text{E}\tilde{\mathcal{B}}_b(O) \in O(b) \cap D$ ,

[6] :=  $\text{ESymmetricConnector}(Y, O)[5] : (r, s) \in U'$ ,

[7] :=  $[6][4][4] : (\phi(r), \phi(s)) \in U'$ ,

**Assume**  $N : \mathcal{U}(\Phi(a))$ ,

$(F, [8]) := \text{E}\Phi(N) : \sum F \in \mathcal{F}_a \ \phi(F) \subset N$ ,

$(I, [9]) := \text{E}\mathcal{F}_a(F) : \sum I \in \mathcal{U}_X . F = I(a) \cap D$ ,

$d := \text{EUniformity}(\mathcal{U}_X)\text{EDense}(X, D) \in I(a) \cap O(a) \cap D$ ,

[10] :=  $\text{Ed}[9][8] : \phi(d) \in N$ ,

[11] :=  $\text{EdErESymmetricConnector}(X, O)[5] : (d, r) \in O \circ O \subset U'$ ,

[12] :=  $[11][3][4] : (\phi(d), \phi(r)) \in W$ ,

[13] :=  $\text{I}\exists[12][10] : \exists N \cap W(\phi(r))$ ;

$\leadsto [8] := \text{EClosedConnector}(X, O) : \Phi(a) \in W(\phi(r))$ ,



Assume  $N : \mathcal{U}(\Phi(b))$ ,

$(F, [9]) := \mathbf{E}\Phi(N) : \sum F \in \mathcal{F}_b \phi(F) \subset N$ ,

$(I, [10]) := \mathbf{E}\mathcal{F}_b(F) : \sum I \in \mathcal{U}_X . F = I(b) \cap D$ ,

$d := \mathbf{E}\mathbf{Uniformity}(\mathcal{U}_X)\mathbf{EDense}(X, D) \in I(b) \cap O(b) \cap D$ ,

$[11] := \mathbf{E}d[10][9] : \phi(d) \in N$ ,

$[12] := \mathbf{E}d\mathbf{Es}\mathbf{ESymmetricConnector}(X, O)[5] : (d, s) \in O \circ O \subset U'$ ,

$[13] := [12][3][4] : (\phi(d), \phi(s)) \in W$ ,

$[14] := \mathbf{I}\exists[13][10] : \exists N \cap W(\phi(s))$ ;

$\leadsto [9] := \mathbf{E}\mathbf{ClosedConnector}(X, O) : \Phi(b) \in W(\phi(s))$ ,

$\left[ (x, y). * \right] := \mathbf{E}\mathbf{SymmetricConnector}(Y, W)[7][8][9] : (\Phi(a), \Phi(b)) \in W \circ W \circ W \subset V$ ;

$\leadsto [*] := \mathbf{I}\mathbf{UNI} : \Phi \in \mathbf{UNI}(X, Y)$ ;

□

**UCUniqueExtensionTheorem** ::

::  $\forall X \in \mathbf{UNI} . \forall Y : \mathbf{CompleteUniformSpace} \ \& \ \mathbf{T3} . \forall D : \mathbf{Dense}(X) . \forall \phi \in \mathbf{UNI}(D, Y) .$

.  $\exists ! \Phi \in \mathbf{UNI}(X, Y) . \Phi|_D = \phi$

**Proof** =

...

□

**UCbyUCRestricton** ::  $\forall X, Y \in \mathbf{UNI} . \forall \varphi \in \mathbf{TOP}(X, Y) . \forall D : \mathbf{Dense}(X) . \varphi|_D \in \mathbf{UNI}(D, Y) \Rightarrow \varphi \in \mathbf{UNI}(X, Y)$

**Proof** =

...

□

**UnimorphismExtension** ::  $\forall X, Y : \mathbf{CompleteUniformSpace} \ \& \ \mathbf{T3} . \forall A : \mathbf{Dense}(X) . \forall B : \mathbf{Dense}(Y) . A \cong_{\mathbf{UNI}} B$

**Proof** =

...

□

There is also a notions of total boundednes. Subsets of uniform space are compacts iff they complete and totally bounded.

#### 1.4.4 Total Boundednes

$$\text{Small} :: \prod_{X \in \text{UNI}} \mathcal{U}_X \rightarrow ??X$$

$$A : \text{Small} \iff \Lambda U \in \mathcal{U}_X . A \times A \subset U$$

$$\text{SmallSymmetricConnector} ::$$

$$:: \forall X \in \text{UNI} . \forall U \in \mathcal{U}_X . \forall V : \text{SymmetricConnector}(X) . \forall x \in X . V \circ V \subset U \Rightarrow \text{Small}(X, U, V(x))$$

$$\text{Proof} =$$

$$[1] := \text{ConnectorProductContainment}(X, V, x) : V(x) \times V(x) \subset V \circ V,$$

$$[2] := [1][0] : V(x) \times V(x) \subset U,$$

$$[*] := \text{ISmall}[2] : \text{Small}(X, U, V(x));$$

□

$$\text{TotallyBounded} :: \prod_{X \in \text{UNI}} ??X$$

$$A : \text{TotallyBounded} \iff \forall U \in \mathcal{U}_X . \exists n \in \mathbb{N} . \exists C : \{1, \dots, n\} \rightarrow \text{Small}(X, U) . A \subset \bigcap_{k=1}^n C_k$$

$$\text{InTBEveryUltrafilterIsCauchy} :: \forall X : \text{UNI} . \text{TotallyBounded}(X, X) \Rightarrow$$

$$\Rightarrow \forall \mathcal{F} : \text{Ultrafilter}(X) . \text{CauchyFilterbase}(X, \mathcal{F})$$

$$\text{Proof} =$$

$$\text{Assume } U \in \mathcal{U}_X,$$

$$(n, C, [1]) := \text{ETotallyBounded}(X, X, U) : \sum_{n=1}^{\infty} \sum \{1, \dots, n\} \rightarrow \text{Small}(X, U) . X = \bigcup_{k=1}^n C_k,$$

$$(k, [2]) := \text{UltrafilterUnion}[1] : \sum_{k=1}^n C_k \in \mathcal{F},$$

$$[U.*] := \text{ESmall}(X, U, C_k) : C_k \times C_k \subset U;$$

$$\leadsto [*] := \text{ICauchyFilterbase} : \text{CauchyFilterbase}(X, \mathcal{F});$$

□

**TotallyBoundedClosure** ::  $\forall X \in \text{UNI} . \forall A : \text{TotallyBounded}(X) . \text{TotallyBounded}(X, \overline{A})$

**Proof** =

**Assume**  $U \in \mathcal{U}_X$ ,

$(V, [1]) := \text{ClosedConnectorsBaseExists} : \sum V : \text{ClosedConnector}(X) . V \subset U$ ,

$(n, C, [2]) := \text{ETotallyBounded}(X, A, V) : \sum_{n=1}^{\infty} \sum C : \{1, \dots, n\} \rightarrow \text{Small}(X, V) . A \subset \bigcup_{k=1}^n C_k$ ,

$[3] := \text{ESmall}(C) : \forall k \in \{1, \dots, n\} . C_k \times C_k \subset V$ ,

$[4] := \text{EClosedConnector}(X, V)[3][1] : \forall k \in \{1, \dots, n\} . \overline{C}_k \times \overline{C}_k \subset V \subset U$ ,

$[5] := \text{ISmall}[4] : \forall k \in \{1, \dots, n\} . \text{Small}(X, U, \overline{C}_k)$ ,

$[U.*] := \text{FiniteClosureUnion}[4] : \overline{A} \subset \bigcup_{k=1}^n \overline{C}_k$ ;

$\leadsto [*] := \text{ITotallyBounded} : \text{TotallyBounded}(X, \overline{A})$ ;

□

**UCPreservesTB** ::  $\forall X, Y \in \text{UNI} . \forall \varphi \in \text{UNI}(X, Y) . \forall A : \text{TotallyBounded}(X) . \text{TotallyBounded}(Y, f(A))$

**Proof** =

**Assume**  $V \in \mathcal{U}_X$ ,

$(U, [1]) := \text{EUniformlyContinuous}(X, Y, \varphi, V) : \sum_{U \in \mathcal{U}_X} (\varphi \times \varphi)(U) \subset V$ ,

$(n, C, [2]) := \text{ETotallyBounded}(X, A, U) : \sum_{n=1}^{\infty} \sum C : \{1, \dots, n\} \rightarrow \text{Small}(X, U) . A \subset \bigcap_{k=1}^n C_k$ ,

$[3] := \text{ESmall}(C) : \forall k \in \{1, \dots, n\} . C_k \times C_k \subset U$ ,

$[4] := \varphi[3][1] : \forall k \in \{1, \dots, n\} . \varphi(C_k) \times \varphi(C_k) \subset V$ ,

$[5] := \text{ISmall}[4] : \forall k \in \{1, \dots, n\} . \text{Small}(Y, V, \varphi(C_k))$ ,

$[V.*] := \text{UnionMap}(X, Y, \varphi)[2] : \varphi(A) \subset \bigcap_{k=1}^n \varphi(C_k)$ ;

$\leadsto [*] := \text{ITotallyBounded} : \text{TotallyBounded}(Y, \varphi(A))$ ;

□

**TBByUltrafilters** ::  $\forall X : \text{UNI} . \left( \forall \mathcal{F} : \text{Ultrafilter}(X) . \text{CauchyFilterbase}(X, \mathcal{F}) \Rightarrow \right.$

$\Rightarrow \text{TotallyBounded}(X, X)$

**Proof** =

**Assume** [1] :  $\neg \text{TotallyBounded}(X, X)$ ,

$(U, [2]) := \text{ETotallyBounded}[1] : \sum U : \text{SymmetricConnector}(X) . \forall A : \text{Finite}(X) . X \neq U(A),$

$(x, [3]) := \text{EU}[2] : x : \mathbb{N} \rightarrow X . \forall n, m \in \mathbb{N} . n \neq m \Rightarrow (x_n, x_m) \notin U,$

$A := \Lambda n \in \mathbb{N} . \{x_m | m \geq n\} : \mathbb{N} \rightarrow ?X,$

**Assume**  $\mathcal{F} : \text{Filter}(X)$ ,

**Assume** [4] :  $\text{Im } A \subset \mathcal{F}$ ,

**Assume** [5] :  $\text{CauchyFilterbase}(X, \mathcal{F})$ ,

$(F, [6]) := \text{ECauchyFilterbase}(X, \mathcal{F}, U) : \sum_{F \in \mathcal{F}} F \times F \subset U,$

$(N, [7]) := \text{EFilter}(X, \mathcal{F}, F)[4] \text{EA} : \sum N : \text{Infinite}(\mathbb{N}) . \forall n \in N . x_n \in F,$

$[\mathcal{F}.*] := [7][3] : \perp;$

$\sim [4] := \text{IV} : \forall \mathcal{F} : \text{Filter}(X) . \text{Im } A \subset \mathcal{F} \Rightarrow \neg \text{CauchyFilterbase}(X, \mathcal{F}),$

$(\mathcal{F}, [5]) := \text{UltrafilterTHM}(X, \text{Im } A) : \sum \mathcal{F} : \text{Ultrafilter}(X) . \text{Im } A \subset \mathcal{F},$

[6] := [4][5] :  $\neg \text{CauchyFilterbase}(X, \mathcal{F})$ ,

[7] := [0]( $\mathcal{F}$ ) :  $\text{CauchyFilterbase}(X, \mathcal{F})$ ,

[1.\*] := [6][7] :  $\perp;$

$\sim [*] := \text{E}\perp : \text{TotallyBounded}(X, X);$

□

**UniCompactnessTHM** ::  $\forall X \in \text{UNI} . \text{Compact}(X) \iff \text{TotallyBounded} \ \& \ \text{CompleteUniformSpace}(X)$

**Proof** =

**Assume** [1] :  $\text{Compact}(X)$ ,

[2] := **CompactnessByUltrafilters** :  $\forall \mathcal{F} : \text{Ultrafilter}(X) . \text{ConvergentFilterbase}(X, \mathcal{F})$ ,

[3] := **EveryConvergenFilterbaseIsCauchy**[2] :  $\forall \mathcal{F} : \text{Ultrafilter}(X) . \text{CauchyFilterbase}(X, \mathcal{F})$ ,

[1.\*.1] := **TBByUltrafilters**[3] :  $\text{TotallyBounded}(X, X)$ ,

**Assume**  $\mathcal{F} : \text{CauchyFilterbase}(X)$ ,

$(x, [4]) := \text{inCompactFilterHasCluster} : \sum_{x \in X} \text{Cluster}(X, \mathcal{F}, x),$

$[\mathcal{F}.*] := \text{CuychyClustersAreLimits} : x \in \lim \mathcal{F};$

$\sim [4] := \text{IConvergentFilterbaseIV} : \forall \mathcal{F} : \text{CauchyFilterbase}(X) . \text{ConvergentFilterbase}(X, \mathcal{F})$ ,

[1.\*.2] := **ICompleteUniformSpace**[4] :  $\text{CompleteUniformSpace}(X, \mathcal{F})$ ;

$\sim [1] := \text{I} \Rightarrow \text{Compact}(X) \Rightarrow \text{TotallyBounded} \ \& \ \text{CompleteUniformSpace}(X),$

**Assume** [2] :  $\text{TotallyBounded} \ \& \ \text{CompleteUniformSpace}(X)$ ,

[3] := **InTBEveryUltrafilterIsCauchy**( $X$ ) :  $\forall \mathcal{F} : \text{Ultrafilter}(X) . \text{CauchyFilterbase}(X, \mathcal{F})$ ,

[4] := **ECompleteUniformSpace**( $X$ )[2] :  $\forall \mathcal{F} : \text{Ultrafilter}(X) . \text{ConvergentFilterbase}(X, \mathcal{F})$ ,

[2.\*] := **CompactByUltrafilters**[4] :  $\text{Compact}(X)$ ;

$\sim [*] := \text{I} \Rightarrow \text{I} \iff [1] : \text{Compact}(X) \iff \text{TotallyBounded} \ \& \ \text{CompleteUniformSpace}(X);$

□

1.4.5 Bounded Sets [∞]

There is also boundedness, but this property is wierd. This chapter will be written then there is a demand for it.

$$\begin{aligned} \text{Bounded} &:: \prod_{X \in \text{UNI}} ??X \\ A : \text{Bounded} &\iff \forall U \in \mathcal{U}_X . \exists n \in \mathbb{N} . \exists F : \text{Finite}(X) . A \subset U^{on}(F) \end{aligned}$$

## 1.5 Special Constructions

Many operation which can be used to construct metric spaces from given topological data, can be also extended for uniformities.

### 1.5.1 Uniformization

As analogy of metrization there is an uniformization.

$$\text{ringUniformity} :: \prod_{X \in \text{TOP}} \text{Uniformity}(X)$$

$$\text{ringUniformity} () = \mathcal{C}_X := \mathcal{I}_X(C(X), \mathbb{R}, \text{id}_{C(X)})$$

$$\text{CompletelyRegularUniformization} :: \forall X : \text{CompletelyRegular} . (X, \mathcal{C}_X) \cong_{\text{TOP}} X$$

Proof =

...

□

$$\text{EveryMetricSpaceAdmitsCompleteStruct} :: \forall X \in \text{MS} . \exists \mathcal{U} : \text{Uniformity}(X) .$$

$$. \text{CompleteUniformSpace}(X, \mathcal{U}) \ \& \ X \cong_{\text{TOP}} (X, \mathcal{U})$$

Proof =

$$\mathcal{U} := \mathbb{B}_X \vee \mathcal{C}_X : \text{Uniformity}(X),$$

$$\text{Assume } \mathcal{F} : \text{CauchyFilterbase}(X, \mathcal{U}),$$

$$[1] := \text{EUSupUniformityCauchyFilterbase}(X, \mathcal{U}, \mathcal{F}) : \text{CauchyFilterbase}(X, \mathcal{B}_X),$$

$$(f, [2]) := \text{ECompletion}(X, \widehat{X}, \mathcal{F}) : \sum f \in \widehat{X} . f = \lim \mathcal{F},$$

$$\phi := \lambda x \in X . d(x, f) \in C(X),$$

$$\text{Assume } [3] : f \notin X,$$

$$[4] := \text{EUE}\phi\text{continuousInverse} : \frac{1}{\phi} \in \text{UNI}(X, \mathbb{R}),$$

$$[5] := \text{Ef} : \lim_{\phi(\mathcal{F})} \frac{1}{\phi} = \infty,$$

$$[6] := \text{UniformMapsPreserveFilters}[5][4]\text{ECAuchyFilterbase} : \perp;$$

$$\leadsto [\mathcal{F}.*] := \text{E}\perp : f \in X;$$

$$\leadsto [*] := \text{ICompleteUniformSpace} : \text{CompleteUniformSpace}(X, \mathcal{U});$$

□

$$\text{CompactUniformityIsExists} :: \forall X \in \text{HC} . \exists ! \mathcal{U} : \text{Uniformity}(X) . (X, \mathcal{U}) \cong_{\text{TOP}} X$$

Proof =

...

□

$$\text{betaUniformity} :: \prod X : \text{Tychonoff} . \text{Uniformity}(X)$$

$$\text{betaUniformity} () = \mathcal{B}_X := \mathcal{U}_{\beta X} \cap (X \times X)$$

$$\text{alphaUniformity} :: \prod X : \text{LocallyComapct} \ \& \ \text{T2} . \text{Uniformity}(X)$$

$$\text{alphaUniformity} () = \mathcal{A}_X := \mathcal{U}_{\omega X} \cap (X \times X)$$

$\text{NormalBetaUniformity} :: \forall X : \mathbf{T4} \ \& \ \neg \text{Compact} .$   
 $\quad . \neg \text{CompleteUniformSpace}(X, \mathcal{B}_X) \ \& \ \text{SequenceCompleteUniformSpace}(X, \mathcal{B}_X)$   
 Proof =  
 ...  
 $\square$

$\text{MetrizableViewSpace} :: ?\mathbf{UNI}$   
 $X : \text{MetrizableViewSpace} \iff \exists d : \text{Metric}(X) . \mathcal{U}_X = \mathbb{B}_{(X,d)}$

$\text{SemimetrizableUniformSpace} :: ?\mathbf{UNI}$   
 $X : \text{SemimetrizableUniformSpace} \iff \exists d : \text{Semimetric}(X) . \mathcal{U}_X = \mathbb{B}_{(X,d)}$

$\text{NormalBetaIsNotMetrizable} :: \forall X : \mathbf{T4} \ \& \ \neg \text{Compact} . \neg \text{MetrizableViewSpace}(X, \mathcal{B}_X)$   
 Proof =  
 ...  
 $\square$

$\text{TychonoffBetaIsNotMetrizable} :: \forall X : \mathbf{Tychonoff} \ \& \ \neg \text{Compact} . \neg \text{MetrizableViewSpace}(X, \mathcal{B}_X)$   
 Proof =  
 ...  
 $\square$

$\text{RingUniformityCompleteIffRealCompact} :: \forall X : \mathbf{Tychonoff} . \text{CompleteUniformSpace}(X, \mathcal{C}_X) \iff \text{Realcompact}$   
 Proof =  
 ...  
 $\square$

$\text{Uniformizable} :: ?\mathbf{TOP}$   
 $X : \text{Uniformizable} \iff \exists \mathcal{U} \in \text{Uniformity}(X) . (X, \mathcal{U}) \cong_{\mathbf{TOP}} X$

Gages is a different kind of generalizations of metric spaces. It uses a set of metrics to define its topology. Gage spaces are completely regular, but not neccesarly normal.

### 1.5.2 Gages

**GageSubbase** ::  $\forall X \in \text{SET} . \forall \mathfrak{R} : ?\text{Semimetric}(X) . \exists \mathfrak{R} \Rightarrow \text{Subbase} \left( \{ \mathbb{B}_\rho(x, \varepsilon) \mid \rho \in \mathfrak{R}, x \in X, \varepsilon \in \mathbb{R}_{++} \} \right)$

**Proof** =

...

□

**gageTopology** ::  $\prod X \in \text{SET} . \text{NonEmpty Semimetric}(X) \rightarrow \text{Topology}(X)$

**gageTopology** ( $\mathfrak{R}$ ) =  $\mathcal{T}_{\mathfrak{R}} := \left\langle \{ \mathbb{B}_\rho(x, \varepsilon) \mid \rho \in \mathfrak{R}, x \in X, \varepsilon \in \mathbb{R}_{++} \} \right\rangle$

**gageUniformity** ::  $\prod X \in \text{SET} . \text{NonEmpty Semimetric}(X) \rightarrow \text{Uniformity}(X)$

**gageUniformity** ( $\mathfrak{R}$ ) =  $\mathcal{U}_{\mathfrak{R}} := \bigvee_{\rho \in \mathfrak{R}} \mathbb{B}_{(X, \rho)}$

**GageTopology** ::  $\forall X \in \text{SET} . \forall \mathfrak{R} : ?\text{Semimetric}(X) . (X, \mathcal{T}_{\mathfrak{R}}) \cong_{\text{TOP}} (X, \mathcal{U}_{\mathfrak{R}})$

**Proof** =

...

□

**GageSpace** :: ?TOP

$X : \text{GageSpace} \iff \exists \mathfrak{R} : \text{NonEmpty Semimetric}(X) . X \cong_{\text{TOP}} (X, \mathcal{T}_{\mathfrak{R}})$

**EveryGageSpaceIsUniformizable** ::  $\forall X : \text{GageSpace} . \text{Uniformizable}(X)$

**Proof** =

...

□

**EveryGageSpaceIsCompletelyRegular** ::  $\forall X : \text{GageSpace} . \text{CompletelyRegular}(X)$

**Proof** =

...

□



### 1.5.3 Metrization

Uniform spaces can be metrized if they have countable base of uniformity. It turns out that gage topologies and uniform topologies are the same thing.

$$\text{triangulization} :: \prod_{X \in \text{SET}} (X \times X \rightarrow \mathbb{R}_+) \rightarrow \text{TriangleIneq}(X)$$

$$\text{triangulization}(f) = d_f := \Lambda x, y \in X . \inf \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \mid n \in \mathbb{N}, u : \{1, \dots, n\} \rightarrow X, u_1 = x, u_n = y \right\}$$

Assume  $x, y, z \in X$ ,

$$A := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \mid n \in \mathbb{N}, u : \{1, \dots, n\} \rightarrow X, u_1 = x, u_n = y \right\} : ?\mathbb{R}_+,$$

$$B := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \mid n \in \mathbb{N}, u : \{1, \dots, n\} \rightarrow X, u_1 = y, u_n = z \right\} : ?\mathbb{R}_+,$$

$$C := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \mid n \in \mathbb{N}, u : \{1, \dots, n\} \rightarrow X, u_1 = x, u_n = z \right\} : ?\mathbb{R}_+,$$

Assume  $a \in A$ ,

Assume  $b \in B$ ,

$$(n, u, [1]) := \text{EA}(a) : \sum_{n=1}^{\infty} \sum u : \{1, \dots, n\} \rightarrow X . u_1 = x \ \& \ u_n = y \ \& \ a = \sum_{i=1}^{n-1} f(u_i, u_{i+1}),$$

$$(m, v, [2]) := \text{EB}(b) : \sum_{m=1}^{\infty} \sum u : \{1, \dots, m\} \rightarrow X . v_1 = y \ \& \ v_m = z \ \& \ b = \sum_{i=1}^{m-1} f(v_i, v_{i+1}),$$

$$[(a, b). *] := [1][2]\text{EC} : a + b = \sum_{i=1}^{n-1} f(u_i, u_{i+1}) + \sum_{i=1}^{m-1} f(v_i, v_{i+1}) \in C;$$

$$\leadsto [*] := \text{Id}_f : d_f(x, z) \leq d_f(x, y) + d_f(y, z);$$

□

`CountableUniformSpace` :: ?UNI

$$X : \text{CountableUniformSpace} \iff \exists \mathcal{B} : \text{BaseOfUniformity}(X) . |\mathcal{B}| \leq \aleph_0$$

**SemimetrizationOfUniformSpace** ::  $\forall X : \text{CountableUniformSpace} . \text{SemimetrizableUniformSpace}(X)$

**Proof** =

$(\mathcal{B}, [1]) := \text{ECountableUniformSpace}(X) : \sum \mathcal{B} : \text{BaseOfUniformity}(X) . |\mathcal{B}| \leq \aleph_0,$

$B := \text{enumerate}(\mathcal{B}) : \mathbb{N} \rightarrow \mathcal{B},$

$(V, [2]) := \text{recursion}(\mathbb{Z}_+, X \times X, \Lambda V \in \mathcal{U}_X . \Lambda n \in \mathbb{N} . \text{UniformityTrisection}(X, V \cap B_n)) :$

$: \sum V : \mathbb{N} \rightarrow \text{SymmetricConnector}(X) . \forall n \in \mathbb{Z}_+ . V_{n+1} \circ V_{n+1} \circ V_{n+1} \subset V_n \cap B_{n+1},$

$\lambda := \Lambda x, y \in X . \inf \{2^{-n} | n \in \mathbb{Z}_+, (x, y) \in V_n\} : X \times X \rightarrow \mathbb{R}_{++},$

$\rho := d_\lambda : \text{TriangleIneq}(X),$

$[3] := \text{E}\lambda \text{E}\rho : \text{Semimetric}(X, \rho),$

**Assume**  $x, y, z, w \in X,$

$r := \max \left( \lambda(w, x), \lambda(x, y), \lambda(y, z) \right) : \mathbb{R}_{++},$

$(n, [4]) := \text{ErE}\lambda : \sum n \in \mathbb{Z}_+ \cup \{\infty\} . 2^{-n} = r,$

$[5] := [4] \text{Er} : \max \left( \lambda(w, x), \lambda(x, y), \lambda(y, z) \right) \leq 2^{-n},$

$[6] := \text{E}\lambda [5] : (w, x), (x, y), (y, z) \in V_n,$

$[7] := \text{ESymmetricConnector}(X, V_n)[2][6] : (w, z) \in V_{n-1},$

$[8] := \text{I}\lambda [7] : \lambda(w, z) \leq 2^{-n+1},$

$\left[ (x, y, z, w).* \right] := [8][4] \text{Er} : \lambda(w, z) \leq 2 \max \left( \lambda(w, x), \lambda(x, y), \lambda(y, z) \right);$

$\leadsto [4] := \text{I}\forall : \forall x, y, z, w \in X . \lambda(w, z) \leq 2 \max \left( \lambda(w, x), \lambda(x, y), \lambda(y, z) \right),$

$[5] := \text{E} \max [4] : \forall n \in \mathbb{N} . \forall x : \{1, \dots, n\} \rightarrow X . \lambda(x_1, x_n) \leq 2 \sum_{i=1}^{n-1} \lambda(x_i, x_{i+1}),$

$[6] := \text{E}\rho [5] : \rho \leq \lambda \leq 2\rho,$

**Assume**  $U \in \mathcal{U}_X,$

$(n, [7]) := \text{EBaseOfUniformity}(X, \mathcal{B})[2] : \sum_{n=1}^{\infty} V_n \subset B_n \subset U,$

$[U.*] := [6][7] : \mathbb{B}_\rho(2^{-n}) \subset U;$

$\leadsto [8] := \text{EUniformity}(X, \mathbb{B}_\rho) : \mathcal{U}_X \subset \mathbb{B}_\rho,$

**Assume**  $\varepsilon : \mathbb{R}_{++},$

$(n, [8]) := \text{ExponentialLimit}(2, \varepsilon) : \sum n \in \mathbb{N} . 2^{-n} < \varepsilon,$

$[\varepsilon.*] := \text{E}\rho [8] : V_n \subset \mathbb{B}_\rho(\varepsilon);$

$\leadsto [9] := \text{EUniformity}(X, \mathcal{U}_X : \mathbb{B}_\rho \subset \mathcal{U}_X,$

$[*] := \text{ISetEq}[8][9] : \mathbb{B}_\rho = \mathcal{U}_X;$

□

**UniformGageSpace** :: ?UNI

$X : \text{UniformGageSpace} \iff \exists \mathfrak{X} : \text{NonEmpty} \text{Semimetric} X . \mathcal{U}_X = \mathcal{U}_{\mathfrak{X}}$

**EveryUniformSpaceIsAGage** ::  $\forall X \in \text{UNI} . \text{GageSpace}(X)$

**Proof** =

$\mathfrak{U} := \{\mathcal{V} \in \text{Uniformity}(X) : \text{CountableUniformSpace}(X, \mathcal{V}) \ \& \ \mathcal{V} \subset \mathcal{U}_X\} : ?\text{Uniformity}(X),$   
 $(\rho, [1]) := \text{SemimetrizationOfUniformSpace} : \rho : \mathfrak{U} \rightarrow \text{Semimetric}(X) . \forall \mathcal{U} \in \mathfrak{U} . \mathbb{B}_{\rho_{\mathcal{U}}} = \mathcal{U},$   
 $\mathfrak{R} := \text{Im } \rho : \text{NonEmpty Semimetric}(X),$   
 $[2] := \text{EI} \sup[1] \text{IU}_{\mathfrak{R}} : \mathcal{U}_X = \bigvee_{\rho \in \mathfrak{R}} \mathfrak{U} = \bigvee_{\rho \in \mathfrak{R}} \mathbb{B}_{\rho} = \mathcal{U}_{\mathfrak{R}},$   
 $[*] := \text{IGageUniformSpace}[2] : \text{GageUniformSpace}(X);$

□

**UniformizableIffCompletelyRegular** ::  $\forall X \in \text{TOP} . \text{CompletelyRegular}(X) \iff \text{Uniformizable}(X)$

**Proof** =

...

□

**UnimorphicEmbedding** ::  $\prod X, Y \in \text{UNI} . ?\text{UNI}(X, Y)$

$\varphi : \text{UnimorphicEmbedding} \iff \text{Unimorphism}\left(X, \varphi(X), \varphi|_{\varphi(X)}\right)$

**UnimorphicEmbeddingToAProduct** ::

$:: \forall X \in \text{UNI} . \exists I \in \text{SET} . \exists Y : I \rightarrow \text{SMS} . \exists \text{UnimorphicEmbedding} \left( X, \prod_{i \in I} Y_i \right)$

**Proof** =

$(\mathfrak{R}, [1]) := \text{EveryUniformSpaceIsAGage}(X) : \sum \mathfrak{R} : \text{NonEmpty Semimetric}(X) . \mathcal{U}_X = \mathcal{U}_{\mathfrak{R}},$

$\varphi := \lambda x \in X . \lambda \rho \in \mathfrak{R} . x : X \rightarrow X^{\mathfrak{R}},$

$[2] := \text{E}\varphi : \forall \rho \in \mathfrak{R} . \varphi \pi_{\rho} = \text{id}_X \in \text{UNI}\left(X, (X, \rho)\right),$

$[3] := \text{InitialUniformityUniversalProperty}[1] : \varphi \in \text{UNI} \left( X, \prod_{\rho \in \mathfrak{R}} (X, \rho) \right),$

$[4] := \text{E}\varphi : \varphi(X) = \Delta^{\mathfrak{R}}(X),$

**Assume**  $n \in \mathbb{N},$

**Assume**  $\rho : \{1, \dots, n\} \rightarrow \mathfrak{R},$

**Assume**  $\varepsilon : \{1, \dots, n\} \rightarrow \mathfrak{R},$

$R := \text{Im } \rho : \text{Finite}(\mathfrak{R}),$

$[5] := \text{EproductUniformity} : \left( \Delta^{\mathfrak{R}}(X) \times \Delta^{\mathfrak{R}}(X) \right) \cap \prod_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i) \times X^{R^{\mathbb{G}}} \in \mathcal{U}(\dots),$

$[n.*] := \text{E}\varphi : (\varphi^{-1} \times \varphi^{-1}) \left( \Delta^{\mathfrak{R}}(X) \times \Delta^{\mathfrak{R}}(X) \right) \cap \prod_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i) \times X^{R^{\mathbb{G}}} \subset \bigcap_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i);$

$\sim [*] := \text{EgageUniformity}[1] \text{IUnimorphicEmbedding} : \text{UnimorphicEmbedding} \left( X, \prod_{\rho \in \mathfrak{R}} (X, \rho), \varphi \right);$

□

Also there is a completion.

### 1.5.4 Completion

$$\text{Completion} :: \prod_{X \in \text{UNI}} ? \sum Y : \text{CompleteUniformSpace} . \text{UnimorphicEmbedding}(X, Y)$$

$$\iota : \text{Completion} \iff \text{Dense}(Y, \iota(X))$$

$$\text{SeparableCompletion} :: \prod_{X \in \text{UNI}} ? \text{Completion}(X)$$

$$(Y, \iota) : \text{SeparableCompletion} \iff Y \in \text{UNIS}$$

$$\text{EveryUniformSpaceHasACompltion} :: \forall X \in \text{UNI} . \exists \text{Completion}(X)$$

**Proof** =

$$\begin{aligned} (I, Y, \varphi) &:= \text{UnimorphicEmbeddingToAProduct} : \sum_{I \in \text{SET}} \sum_{Y: I \rightarrow \text{SMS}} \sum \varphi : \text{UnimorphicEmbedding} \left( X, \prod_{i \in I} Y_i \right), \\ (\hat{Y}, \iota, [1]) &:= \Lambda i \in I . \text{SemimetricCompletionExists}(Y_i) : \prod_{i \in I} \text{Completion}(Y_i), \end{aligned}$$

$$Z := \text{cl}_{(\prod_{i \in I} \hat{Y}_i)} \varphi(X) : \text{Closed} \left( \prod_{i \in I} \hat{Y}_i \right),$$

$$[2] := \text{CompleteProductTHM}(I, \hat{Y}_i) \text{ClosedOfCompleteIsComplete} : \text{CompleteUniformSpace}(Z),$$

$$\psi := \varphi \prod_{i \in I} \iota_i : \text{HomeomorphicEmbedding}(X, Z),$$

$$[3] := \text{EZIDense} : \text{Dense}(Z, \psi(X)),$$

$$[*] := \text{ICompletion} : \text{Completion}(X, Z, \psi);$$

□

$$\text{UnimorphicEmbeddingToAMetricProduct} ::$$

$$:: \forall X \in \text{UNIS} . \exists I \in \text{SET} . \exists Y : I \rightarrow \text{MS} . \exists \text{UnimorphicEmbedding} \left( X, \prod_{i \in I} Y_i \right)$$

**Proof** =

$$\begin{aligned} (I, Y, \varphi) &:= \text{UnimorphicEmbeddingToAProduct} : \sum_{I \in \text{SET}} \sum_{Y: I \rightarrow \text{SMS}} \sum \varphi : \text{UnimorphicEmbedding} \left( X, \prod_{i \in I} Y_i \right), \\ (Z, \phi) &:= \Lambda i \in I . \text{MetricQuotient} : \prod_{i \in I} \sum_{Z_i \in \text{MS}} \text{Isometry}(Y_i, Z_i), \end{aligned}$$

$$\psi := \varphi \prod_{i \in I} \psi_i \in \text{UNI} \left( X, \prod_{i \in I} Z_i \right),$$

$$[1] := \text{E}\psi \text{EUNIS}(X) \text{EUnimorphicEmbedding}(\varphi) : \text{Injective} \left( X, \prod_{i \in I} Z_i, \psi \right),$$

$$[2] := [1] \text{ElimIsometry}(\phi) : \text{IsometricEmbedding} \left( \varphi(X), \prod_{i \in I} Z_i \left( \prod_{i \in I} \phi_i \right)_{|\varphi(X)} \right),$$

$$[*] := \text{E}\psi [2] : \text{UnimorphicEmbedding} \left( X, \prod_{i \in I} Z_i, \psi \right);$$

□

**SeparatedSpaceHasSeparatedCompletion** ::  $\forall X \in \mathbf{UNIS} . \exists \text{SeparableCompletion}(X)$

**Proof** =

...

□

**SeparableCompletionAreUnique** ::  $\forall X \in \mathbf{UNIS} . \forall (Y, \iota), (Y', \iota') : \text{SeparableCompletion}(X) . Y \cong_{\mathbf{UNI}} Y'$

**Proof** =

...

□

**separableCompletion** ::  $\prod X \in \mathbf{UNIS} . \text{SeparableCompletion}(X)$

**separableCompletion** () =  $(\gamma X, \iota_{\gamma X}) := \text{SeparatedSpaceHasSeparatedCompletion}(X)$

**UniformlyContinuousByDenseSubset** ::

$:: \forall X \in \mathbf{UNI} . \forall Y \in \mathbf{UNIS} . \forall \varphi \in \mathbf{TOP}(X, Y) . \forall D : \text{Dense}(X) . \varphi|_D \in \mathbf{UNI}(D, Y) \Rightarrow \varphi \in \mathbf{UNI}(X, Y)$

**Proof** =

...

□

**UniformityEqualityTHM** ::  $\forall Y \in \mathbf{SET} . \forall X \subset X . \forall \mathcal{U}, \mathcal{V} \in \text{Uniformity}(X) .$

$(Y, \mathcal{U}), (Y, \mathcal{V}) \in \mathbf{UNIS} \ \& \ \text{Dense}\left((Y, \mathcal{U}) \ \& \ (Y, \mathcal{V}), X\right) \ \& \ \mathcal{U} \cap X \times X = \mathcal{V} \cap X \times X \Rightarrow \mathcal{U} = \mathcal{V}$

**Proof** =

...

□

### 1.5.5 Uniformly Continuous Metric [\*]

Will be written on demand.

## 1.6 Function Spaces

Many sets of functions have natural uniformities.



### 1.6.1 Pointwise Uniformity

Functions with their values in an uniform space can be given an uniformity which corresponds to a pointwise convergence. Turns out it corresponds to the product uniformity.

$$\text{evaluation} :: \prod_{X, Y \in \text{SET}} X \rightarrow (X \rightarrow Y) \rightarrow Y$$

$$\text{evaluation}(x, f) = \epsilon_x(f) := f(x)$$

$$\text{pointwiseUniformSpace} :: \text{SET} \times \text{UNI} \rightarrow \text{UNI}$$

$$\text{pointwiseUniformSpace}(X, Y) = X \rightarrow_{\text{pt}} Y := (X \rightarrow Y, \mathcal{I}(X, Y, \epsilon))$$

$$\text{PointwiseUniformSpaceIsAProduct} :: \forall X \in \text{SET} . \forall Y \in \text{UNI} . (X \rightarrow_{\text{pt}} Y) \cong_{\text{UNI}} Y^X$$

Proof =

...

□

$$\text{PointwisePreservesSeparation} :: \forall X \in \text{SET} . \forall Y \in \text{UNIS} . (X \rightarrow_{\text{pt}} Y) \in \text{UNIS}$$

Proof =

...

□

$$\text{PointwisePreservesCompleteness} :: \forall X \in \text{SET} . \forall Y : \text{CompleteUniformSpace} . \\ . \text{CompleteUniformSpace}(X \rightarrow_{\text{pt}} Y)$$

Proof =

...

□

$$\text{PointwisePreservesSeparatedCompleteness} :: \forall X \in \text{SET} . \forall Y : \text{SequenceCompleteUniformSpace} . \\ . \text{SequenceCompleteUniformSpace}(X \rightarrow_{\text{pt}} Y)$$

Proof =

...

□

$$\text{PointwiseFilterConvergence} :: \forall X \in \text{SET} . \forall Y \in \text{UNI} . \forall f : X \rightarrow Y . \forall \mathcal{F} : \text{Filter}(X \rightarrow_{\text{pt}} Y) . \\ . f \in \lim \mathcal{F} \iff \forall x \in X . f(x) \in \lim \epsilon_x(\mathcal{F})$$

Proof =

...

□

$$\text{PointwiseCauchyFilters} :: \forall X \in \text{SET} . \forall Y \in \text{UNI} . \forall f : X \rightarrow Y . \forall \mathcal{F} : \text{Filter}(X \rightarrow_{\text{pt}} Y) . \\ . \text{CauchyFilterbase}(X \rightarrow_{\text{pt}} Y, \mathcal{F}) \iff \forall x \in X . \text{CauchyFilterbase}(X \rightarrow_{\text{pt}} Y, \epsilon_x(\mathcal{F}))$$

Proof =

...

□

**PointwiseCompactness**  $:: \forall X \in \mathbf{SET} . \forall Y \in \mathbf{UNI} . \forall K \subset (X \rightarrow_{\text{pt}} Y) .$

$. \text{CompactSubset}(X \rightarrow_{\text{pt}} Y, K) \iff \text{Closed}(X \rightarrow_{\text{pt}} Y, K) \ \& \ \forall x \in X . \text{CompactSubset}\left(Y, \overline{\epsilon_x(K)}\right)$

**Proof** =

...

□

**PointwiseCompleteness**  $:: \forall X \in \mathbf{SET} . \forall Y \in \mathbf{UNI} . \forall K \subset (X \rightarrow_{\text{pt}} Y) .$

$. \text{CompleteUniformSpace}(K) \iff \text{Closed}(X \rightarrow_{\text{pt}} Y, K) \ \& \ \forall x \in X . \text{CompleteUniformSpace}\left(\overline{\epsilon_x(K)}\right)$

**Proof** =

...

□

## 1.6.2 Uniformity of Uniform Convergence

It is also possible to define uniformity of uniform convergence.

$$\text{uniformUniformity} :: \prod_{X \in \text{SET}} \prod_{Y \in \text{UNI}} \text{Uniformity}(X \rightarrow Y)$$

$$\text{uniformUniformity}() = \mathcal{UU}(X, Y) := \left\{ \left\{ (f, g) \in (X \rightarrow Y)^2 \mid \forall x \in X . (f(x), g(x)) \in U \right\} \mid U \in \mathcal{U}(Y) \right\}$$

$$\text{uniformConvergenceSpace} :: \text{SET} \rightarrow \text{UNI} \rightarrow \text{UNI}$$

$$\text{uniformConvergenceSpace}(X, Y) = (X \rightarrow_{\mathcal{U}} Y) := (X \rightarrow Y, \mathcal{UU}(X, Y))$$

$$\begin{aligned} \text{PointwiseFilterConvergence} &:: \forall X \in \text{SET} . \forall Y \in \text{UNI} . \forall f : X \rightarrow Y . \forall \mathcal{F} : \text{Filter}(X \rightarrow_{\mathcal{U}} Y) . \\ &. f \in \lim_{\mathcal{U}} \mathcal{F} \iff \text{CauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}) \ \& \ f \in \lim_{\text{pt}} \mathcal{F} \end{aligned}$$

**Proof** =

$$\text{Assume } [1] : f \in \lim_{\mathcal{U}} \mathcal{F},$$

$$[2] := \text{ConvergentIsCauchy}[1] : \text{CauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}),$$

$$\text{Assume } x \in X,$$

$$\text{Assume } O : \mathcal{U}(f(x)),$$

$$[3] := \text{EuniformTopologyEUU}(X, Y) : O^X \in \mathcal{T}(X \rightarrow_{\mathcal{U}} Y),$$

$$(F, [4]) := [1][3] : \sum F \in \mathcal{F} . F \subset O^X,$$

$$[x.*] := \epsilon_x[4] : \epsilon_x(F) \subset O;$$

$$\leadsto [3] := \text{I}\forall : \forall x \in X . f(x) \in \lim \epsilon_x \mathcal{F},$$

$$[1.*] := \text{PointwiseFilterConvergence}[3] : f \in \lim_{\text{pt}} \mathcal{F};$$

$$\leadsto [1] := \text{I} \Rightarrow : f \in \lim_{\mathcal{U}} \mathcal{F} \Rightarrow \text{CauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}) \ \& \ f \in \lim_{\text{pt}} \mathcal{F},$$

$$\text{Assume } [2] : \text{CauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}),$$

$$\text{Assume } [3] : f \in \lim_{\text{pt}} \mathcal{F},$$

$$\text{Assume } O \in \mathcal{U}_{X \rightarrow_{\mathcal{U}} Y}(f),$$

$$(U, [4]) := \text{EuniformTopologu}(O) : \sum U \in \mathcal{UU}(X, Y) . U(f) \subset O,$$

$$(W, [5]) := \text{SymmetricConnectorBaseExists}(Y, V) \text{ClosedConnectorBaseExists}(Y, V) :$$

$$: \sum W : \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(X \rightarrow_{\mathcal{U}} Y) . W \subset U,$$

$$(V, [6]) := \text{EUU}(X, Y, U) : \sum V \in \text{SymmetricConnector} \ \& \ \text{ClosedConnector}(Y) .$$

$$. W = \left\{ (f, g) \in (X \rightarrow Y)^2 \mid \forall x \in X . (f(x), g(x)) \in V \right\},$$

$$(F, [7]) := \text{ECauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}, U) : \sum F \in \mathcal{F} . F \times F \subset W,$$

$$\text{Assume } g \in F,$$

$$[9] := [7][8] \text{ESymmetricConnector}(Y, V) : \forall x \in X . F(x) \subset V(g(x)),$$

$$\text{Assume } x \in X,$$

$$[10] := [3](x) : \lim \mathcal{F}(x) = f(x),$$

$$\text{Assume } E \in \mathcal{U}_Y,$$

$$(G, [11]) := \text{EConvergent}[10] : \sum G \in \mathcal{F} . G(x) \subset E(f(x)),$$

$$\begin{aligned}
[E.*] &:= [11][9](x) : F(x) \cap G(x) \subset V\Big(g(x)\Big) \cap E\Big(f(x)\Big); \\
\leadsto [g.*] &:= \textcolor{blue}{\mathcal{U}}_Y \textcolor{blue}{EClosedConnector}(Y,V) : f(x) \in V\Big(g(x)\Big); \\
\leadsto [O.*] &:= [4][5][6] : G \subset O; \\
\leadsto [2.*] &:= \textcolor{blue}{I} \lim : \lim_{\mathcal{U}} \mathcal{F} = f; \\
\leadsto [*] &:= \textcolor{blue}{I} \iff : \lim_{\mathcal{U}} \mathcal{F} = f \iff \textcolor{blue}{CauchyFilterbase}(X \rightarrow_{\mathcal{U}} Y, \mathcal{F}) \ \& \ f \in \lim_{\text{pt}} \mathcal{F}; \\
&\square
\end{aligned}$$

### 1.6.3 Uniform Convergence over S

For any family  $\mathcal{S}$  of subset of the domain, there is possible to define a uniformity for uniform convergence over  $\mathcal{S}$ . It turns out that both pointwise and uniform convergence are special cases of this.

$\mathbf{fUniformity} :: \prod X \in \mathbf{SET} . \prod Y \in \mathbf{UNI} . ??X \rightarrow \mathbf{Uniformity}(X \rightarrow Y)$

$\mathbf{fUniformity}(\mathcal{S}) = \mathcal{F}(X, Y, \mathcal{S}) := \mathcal{I}\left(\mathcal{S}, \Lambda S \in \mathcal{S} . \left(S \rightarrow Y, \mathcal{U}\mathcal{U}(S, Y)\right), \Lambda S \in \mathcal{S} . \Lambda f : X \rightarrow Y . f|_S\right)$

$\mathbf{FiniteFUniformityIsPointwise} :: \forall X \in \mathbf{SET} . \forall Y \in \mathbf{UNI} . \left(X \rightarrow Y, \mathcal{F}(X, Y, \mathbf{Finite}(Y))\right) \cong_{\mathbf{UNI}} X \rightarrow_{\mathbf{pt}} Y$

**Proof** =

...

□

$\mathbf{GlobaFUniformityIsUniform} :: \forall X \in \mathbf{SET} . \forall Y \in \mathbf{UNI} . \left(X \rightarrow Y, \mathcal{F}(X, Y, \{Y\})\right) \cong_{\mathbf{UNI}} X \rightarrow_{\mathcal{U}} Y$

**Proof** =

$\mathbf{compactConvergence} :: \mathbf{TOP} \rightarrow \mathbf{UNI} \rightarrow \mathbf{UNI}$

$\mathbf{compactConvergence}(X, Y) = X \rightarrow_{\mathbb{K}} Y := \left(X \rightarrow Y, \mathcal{F}(X, Y, \mathbf{CompactSubset}(X))\right)$

$\mathbf{precompactConvergence} :: \mathbf{UNI} \rightarrow \mathbf{UNI} \rightarrow \mathbf{UNI}$

$\mathbf{precompactConvergence}(X, Y) = X \rightarrow_{\lambda} Y := \left(X \rightarrow Y, \mathcal{F}(X, Y, \mathbf{TotallyBounded}(X))\right)$

$\mathbf{ClosedContinousCriterion} :: \forall X \in \mathbf{TOP} . \forall Y \in \mathbf{UNI} . \forall \mathcal{S} : ??X .$

$. \forall x \in X . \exists S \in \mathcal{S} . x \in \text{int } S \Rightarrow \mathbf{Closed}\left(\left(X \rightarrow Y, \mathcal{F}(X, Y, \mathcal{S})\right), C(X, Y)\right)$

**Proof** =

**Assume**  $f \in \overline{C(X, Y)}$ ,

**Assume**  $x \in X$ ,

$\left(S, [2]\right) := [0](x) : \sum S \in \mathcal{S} . x \in \text{int } S,$

$[x.*] := \mathbf{EF}(X, Y, \mathcal{S})\mathbf{UniformLimitIsContinuous}(f) : f|_S \in C(S, Y);$

$\leadsto [f.*] := \mathbf{ContinuityIsLocal} : f \in C(S, Y);$

$\leadsto [*] := \mathbf{ClosedByLimits} : \mathbf{Closed}\left(\left(X \rightarrow Y, \mathcal{F}(X, Y, \mathcal{S})\right), C(X, Y)\right);$

□

$\mathbf{FunctionalTopologyCompleteness} :: \forall X \in \mathbf{SET} . \forall Y \in \mathbf{UNI} . \forall \mathcal{S} : ??X .$

$. \mathbf{CompleteUniformSpace}(Y) \ \& \ \mathbf{Cover}(X, \mathcal{S}) \iff \mathbf{CompleteUniformSpace}\left(X \rightarrow Y, \mathcal{F}(X, Y, \mathcal{S})\right)$

**Proof** =

...

□

### 1.6.4 Equicontinuity and Uniform Equicontinuity[!]

Notions of equicontinuity and Arzello-Ascoli theorem also generalizes nicely. There is no proofs in this chapter. They may be provided on demand.

**EquicontinuousAtAPoint** ::  $\prod X \in \text{TOP} . \prod Y \in \text{UNI} . X \rightarrow ??(X \rightarrow Y)$

$F : \text{EquicontinuousAtAPoint} \iff \Lambda x \in X . \forall U \in \mathcal{U}_Y . \exists O \in \mathcal{U}(x) . \forall f \in F . f(O) \subset V(f(x))$

**UniformlyEquicontinuousAtAPoint** ::  $\prod X, Y \in \text{UNI} . X \rightarrow ??(X \rightarrow Y)$

$F : \text{UniformlyEquicontinuousAtAPoint} \iff \Lambda x \in X . \forall U \in \mathcal{U}_Y . \exists V \in \mathcal{U}_X . \forall f \in F . f(V(x)) \subset V(f(x))$

**Equicontinuous** ::  $\prod X \in \text{TOP} . \prod Y \in \text{UNI} . ??(X \rightarrow Y)$

$F : \text{Equicontinuous} \iff \forall x \in X . \text{EquicontinuousAtAPoint}(X, Y, F, x)$

**UniformlyEquicontinuous** ::  $\prod X, Y \in \text{UNI} . ??(X \rightarrow Y)$

$F : \text{UniformlyEquicontinuous} \iff \forall x \in X . \text{UniformlyEquicontinuousAtAPoint}(X, Y, F, x)$

**UniformlyEquicontinuousAltDef** ::  $\forall X, Y \in \text{UNI} . \forall F : ??(X \rightarrow Y) .$

$. \text{UniformlyEquicontinuous}(X, Y, F) \iff \forall V \in \mathcal{U}_Y . \exists U \in \mathcal{U}_X . \forall f \in F . (f \times f)(U) \subset V$

**Proof** =

...

□

**BourbakiJointEquicontinuityTheorem** ::  $\forall T \in \text{SET} . \forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall \mathcal{S} : ??T .$

$. \forall f : T \times X \rightarrow Y . \Lambda x \in X . \Lambda t \in T . f(t, x) \in \text{TOP}\left(X, \left(T \rightarrow Y, \mathcal{F}(T, Y, \mathcal{S})\right)\right) \iff$

$\iff \forall S \in \mathcal{S} . \text{Equicontinuous}\left(X, Y\{\Lambda x \in X . f(t, x) | t \in S\}\right)$

**Proof** =

...

□

**BourbakiJointUniformEquicontinuityTheorem** ::  $\forall T \in \text{SET} . \forall X, Y \in \text{UNI} . \forall \mathcal{S} : ??T .$

$. \forall f : T \times X \rightarrow Y . \Lambda x \in X . \Lambda t \in T . f(t, x) \in \text{UNI}\left(X, \left(T \rightarrow Y, \mathcal{F}(T, Y, \mathcal{S})\right)\right) \iff$

$\iff \forall S \in \mathcal{S} . \text{UniformlyEquicontinuous}\left(X, Y\{\Lambda x \in X . f(t, x) | t \in S\}\right)$

**Proof** =

...

□

**EquicontinuityClosureTHM** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall F : ?(X \rightarrow Y) .$

. **Equicontinuous** $(X, Y, F) \iff \text{Equicontinuous}(X, Y, \text{cl}_{\text{pt}} F)$

**Proof** =

...

□

**UniformEquicontinuityClosureTHM** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall F : ?(X \rightarrow Y) .$

. **UnifomlyEquicontinuous** $(X, Y, F) \iff \text{UnifomlyEquicontinuous}(X, Y, \text{cl}_{\text{pt}} F)$

**Proof** =

...

□

**EquicontinuitySClosureTHM** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall \mathcal{S} : \text{Cover}(X) . \forall F : ?(X \rightarrow Y) .$

. **Equicontinuous** $(X, Y, F) \iff \text{Equicontinuous}(X, Y, \text{cl}_{\mathcal{S}} F)$

**Proof** =

...

□

**UniformEquicontinuitySClosureTHM** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall \mathcal{S} : \text{Cover}(X) . \forall F : ?(X \rightarrow Y) .$

. **UnifomlyEquicontinuous** $(X, Y, F) \iff \text{UnifomlyEquicontinuous}(X, Y, \text{cl}_{\mathcal{S}} F)$

**Proof** =

...

□

**PointwiseConvergenceIsCompact** ::

::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall F : \text{Equicontinuous}(X, Y) . (F, \text{pt}) \cong_{\text{UNI}} (F, \mathbb{K})$

**Proof** =

...

□

**OnCompactPpointwiseConvergenceIsUniform** ::

::  $\forall X : \text{Compact} . \forall Y \in \text{UNI} . \forall F : \text{Equicontinuous}(X, Y) . \left( F, \mathcal{U}(X, Y) \right) \cong_{\text{UNI}} (F, \text{pt})$

**Proof** =

...

□

**PointwiseAndCompactClosureAgree** ::

::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall F : \text{Equicontinuous}(X, Y) . \text{cl}_{\text{pt}} F = \text{cl}_{\mathbb{K}} F$

**Proof** =

...

□

**EquicontinuousSquareTHM** ::

$$:: \forall X : \text{Compact} . \forall Y \in \text{UNI} . \forall F : \text{Equicontinuous}(X, Y) . \left( F^2, \mathcal{U}^2(X, Y) \right) \cong_{\text{UNI}} (F^2, \text{pt}^2)$$

**Proof** =

...

□

**SConvergenceTotalBoundnessImpleq** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall \mathcal{S} : \text{Cover}(X) .$

$$. \forall F : \text{TotallyBounded}(X \rightarrow Y, \mathcal{F}(X, Y, \mathcal{S})) . \left( \forall S \in \mathcal{S} . F|_S \subset \text{TOP}(S, Y) \right) \Rightarrow \\ \Rightarrow \forall S \in \mathcal{S} . \text{Equicontinuous}(S, Y, F|_S)$$

**Proof** =

...

□

**SConvergenceTotalBoundnessImpleTB** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNI} . \forall \mathcal{S} : \text{Cover}(X) .$

$$. \forall F : \text{TotallyBounded}(X \rightarrow Y, \mathcal{F}(X, Y, \mathcal{S})) . \left( \forall S \in \mathcal{S} . F|_S \subset \text{TOP}(S, Y) \right) \Rightarrow \\ \Rightarrow \forall x \in X . \text{TotallyBounded}(Y, F(x))$$

**Proof** =

...

□

**ArzeloAscolliTHM** ::  $\forall X \in \text{TOP} . \forall Y \in \text{UNIS} . \forall F \subset C(X, Y) .$

$$. \text{CompactSubset}(X \rightarrow_{\mathbb{K}} Y, F) \iff \text{Closed}(X \rightarrow_{\mathbb{K}} Y, F) \ \& \ \forall x \in X . \text{CompactSubset}(Y, \overline{F(x)}) \ \& \\ \& \ \forall K : \text{CompactSubset}(X) . \text{Equicontinuous}(X, Y, F|_K)$$

**Proof** =

...

□



## 2 Topological Groups Basics

Groups can be equipid with a topology in such a way that their alebraic and topological structrure interplay.

### 2.1 Group Topology

There are different ways to equip a group with a group topology, but all of them have some necessary properties.

#### 2.1.1 Category of Topological Groups

Topological groups form a complete category.

$$\begin{aligned} \text{TopologicalGroup} &:: ? \sum G \in \text{GRP} . \text{Topology}(G) \\ (G, \mathcal{T}) &: \text{TopologicalGroup} \iff \\ &\iff \circ_G \in \text{TOP}\left((G, \mathcal{T}) \times (G, \mathcal{T}), (G, \mathcal{T})\right) \ \& \ \Lambda g \in G . g^{-1} \in \text{TOP}\left((G, \mathcal{T}), (G, \mathcal{T})\right) \end{aligned}$$

$$\begin{aligned} \text{topologicalGroupAsGroup} &:: \text{TopologicalGroup} \rightarrow \text{GRP} \\ \text{topologicalGroupAsGroup} (G, \mathcal{T}) &= (G, \mathcal{T}) := G \end{aligned}$$

$$\begin{aligned} \text{topologicalGroupAsTopologicalSpace} &:: \text{TopologicalGroup} \rightarrow \text{TOP} \\ \text{topologicalGroupAsTopologicalSpace} (G, \mathcal{T}) &= (G, \mathcal{T}) := (G, \mathcal{T}) \end{aligned}$$

$$\begin{aligned} \text{categoryOfTopologicalGroups} &:: \text{CAT} \\ \text{categoryOfTopologicalGroups} () &= \text{TGRP} := (\text{TopologicalGroup}, \text{TOP} \ \& \ \text{GRP}, \circ, \text{id}) \end{aligned}$$

**HomomorphismWeakTopologyIsGroupTopology** ::

$$:: \forall I \in \text{SET} . \forall G \in \text{GRP} . \forall H : I \rightarrow \text{TGRP} . \forall \phi : \prod_{i \in I} \text{GRP}(G, H_i) . \left( G, \mathcal{W}(I, H, \phi) \right) \in \text{TGRP}$$

**Proof** =

$$[1] := \Lambda i \in I . \text{E}_2 \text{GRP}(G, H_i, \phi_i) \text{E} \mathcal{W}_G(I, H, \phi) \text{E} \text{TGRP}(H_i) \text{E} \text{CAT}(\text{TOP}) :$$

$$: \forall i \in I . \text{inv}_G \phi_i = \phi_i \text{inv}_{H_i} \in \text{TOP}\left(\left(G, \mathcal{W}(I, H, \phi)\right), H_i\right),$$

$$[2] := \text{WeakTopologyUniversalProperty}[1] : \text{inv}_G \in \text{Aut}_{\text{TOP}}\left(G, \mathcal{W}(I, H, \phi)\right),$$

$$[3] := \Lambda i \in I . \text{E}_1 \text{GRP}(G, H_i, \phi_i) \text{E} \mathcal{W}_G(I, H, \phi) \text{E} \text{TGRP}(H_i) \text{E} \text{CAT}(\text{TOP}) :$$

$$: \forall i \in I . (\cdot_G)(\phi_i \times \phi_i) = (\phi_i \times \phi_i)(\cdot_{H_i}) \in \text{TOP}\left(\left(G, \mathcal{W}(I, H, \phi)\right)^2, H_i\right),$$

$$[4] := \text{WeakTopologyUniversalProperty}[3] : \cdot_G \in \text{TOP}\left(\left(G, \mathcal{W}(I, H, \phi)\right)^2, \left(G, \mathcal{W}(I, H, \phi)\right)\right),$$

$$[3] := \text{ITGRP}[2][4] : \left(G, \mathcal{W}(I, H, \phi)\right) \in \text{TGRP},$$

□

$$\text{SupTopologyIsGroup} :: \forall G \in \text{GRP} . \forall I \in \text{SET} . \forall \mathcal{T} : \mathcal{T} \rightarrow \text{TGRP}(G) . \left( G, \bigvee_{i \in I} \mathcal{T}_i \right) \in \text{TGRP}$$

**Proof** =

...

□

`TopologicalGroupsAreComplete :: Complete(TGRP)`

`Proof =`

`...`

`□`

### 2.1.2 Absolute Values and Invariant Metrics

Group topology can be determined by an absolute value function, or by an invariant metric. Absolute value functions and invariant metrics are the same.

$$\text{AbsoluteValue} :: \prod_{G \in \text{GRP}} (G \rightarrow \mathbb{R}_+)$$

$$\alpha : \text{AbsoluteValue} \iff \alpha(e) = 0 \ \& \ \forall g \in G . \alpha(g) = \alpha(g^{-1}) \ \& \ \forall g, h \in G . \alpha(gh) \leq \alpha(g) + \alpha(h) \ \& \\ \& \ \forall x : \mathbb{N} \rightarrow G . \forall g \in G . \lim_{n \rightarrow \infty} \alpha(x_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} \alpha(gx_n g^{-1}) = 0$$

$$\text{absoluteValueAsSemimetric} :: \prod_{G \in \text{GRP}} \text{AbsoluteValue}(G) \rightarrow \text{Semimetric}(G)$$

$$\text{absoluteValueAsSemimetric}(\alpha) = d_\alpha := \Lambda a, b \in G . \alpha(ab^{-1})$$

$$[1] := \Lambda g \in G . \text{Ed}_\alpha(g, g) \text{InverseMeaning}(G) \text{E}_1 \text{AbsoluteValue}(G, \alpha) : \\ : \forall g \in G . d_\alpha(g, g) = \alpha(gg^{-1}) = \alpha(e) = 0,$$

$$[2] := \Lambda g, h \in G . \text{Ed}_\alpha(g, h) \text{ProductInverse}(G) \text{E}_2 \text{AbsoluteValue}(G, \alpha, g^{-1}) \text{Id}_\alpha(h, g) : \\ : \forall g, h \in G . d_\alpha(g, h) = \alpha(gh^{-1}) = \alpha((hg^{-1})^{-1}) = \alpha(hg^{-1}) = d_\alpha(h, g),$$

$$[3] := \Lambda f, g, h \in G . \text{Ed}_\alpha(g, h) \text{InverseMeaning}(G, g) \text{E}_3 \text{AbsoluteValue}(G, \alpha, g^{-1}, fg^{-1}, gh^{-1}) \text{Id}_\alpha(f, g) \text{Id}_\alpha(g, h) : \\ : \forall f, g, h \in G . d_\alpha(f, h) = \alpha(fh^{-1}) = \alpha(fg^{-1}gh^{-1}) \leq \alpha(fg^{-1}) + \alpha(gh^{-1}) = d_\alpha(f, g) + d_\alpha(g, h),$$

$$[*] := \text{ISemimetric}(G)[1][2][3] : \text{Semimetric}(G, d_\alpha);$$

□

$$\text{LeftInvariantMetric} :: \prod_{G \in \text{GRP}} ?\text{Semimetric}(X)$$

$$\rho : \text{LeftInvariantMetric} \iff \forall a, b, g \in G . d(ag, bg) = d(a, b)$$

$$\text{RightInvariantMetric} :: \prod_{G \in \text{GRP}} ?\text{Semimetric}(X)$$

$$\rho : \text{RightInvariantMetric} \iff \forall a, b, g \in G . d(ga, gb) = d(a, b)$$

$$\text{TwosidedInvariantMetric} := \prod_{G \in \text{GRP}} \text{LeftInvariantMetric} \ \& \ \text{RightInvariantMetric} : \text{GRP} \rightarrow \text{Type};$$

$$\text{AbsoluteValueMetricIsRightInvariant} ::$$

$$:: \forall G \in \text{GRP} . \forall \alpha : \text{AbsoluteValue}(G) . \text{RightInvariantMetric}(G, d_\alpha)$$

$$\text{Proof} =$$

$$[1] := \Lambda a, b, g \in G . \text{Ed}_\alpha(ag, bg) \text{InverseProduct}(G) \text{InverseMeaning}(G, g) \text{Id}_\alpha(ag, bg) : \\ : \forall a, b, g \in G . d_\alpha(ag, bg) = \alpha(agg^{-1}b^{-1}) = \alpha(ab^{-1}) = d_\alpha(a, b),$$

$$[*] := \text{IRightInvariantMetric}[1] : \text{RightInvariantMetric}(G, d_\alpha);$$

□

**AbsoluteValueMetrizesATopologicalGroup** ::  $\forall G \in \text{TGRP} . \forall \alpha : \text{AbsoluteValue}(G) . (G, d_\alpha) \in \text{TGRP}$

**Proof** =

**Assume**  $g \in G$ ,

**Assume**  $x : \mathbb{N} \rightarrow G$ ,

**Assume** [1] :  $\lim_{n \rightarrow \infty} x_n = g$ ,

[2] := **MetricLimit**[1] :  $\lim_{n=1} d_\alpha(x_n, g) = 0$ ,

[3] :=  $E d_\alpha : 0 = \lim_{n \rightarrow \infty} \alpha(x_n g^{-1})$ ,

[4] :=  $E_4 \text{AbsoluteValue}(G, \alpha, g^{-1})[4] \text{InverseProperty}(G, \alpha) :$   
 $: 0 = \lim_{n \rightarrow \infty} \alpha(g^{-1} x_n g^{-1} g) = \lim_{n \rightarrow \infty} \alpha(g^{-1} x_n) = \lim_{n \rightarrow \infty} d_\alpha(g^{-1}, x_n^{-1})$ ,

$[g.*] := \text{MetricLimit}[4] : \lim_{n=1} x_n^{-1} = g^{-1}$ ;

$\leadsto [1] := \text{ContinuousByLimits} : \Lambda g \in G . g^{-1} \in \text{Aut}_{\text{TOP}}(G, d_\alpha)$ ,

**Assume**  $g, h \in G$ ,

**Assume**  $x, y : \mathbb{N} \rightarrow G$ ,

**Assume** [2] :  $\lim_{n \rightarrow \infty} x_n = g$ ,

**Assume** [3] :  $\lim_{n \rightarrow \infty} y_n = h$ ,

[4] := **MetricLimit**[2]  $E d_\alpha : \lim_{n=1} d_\alpha(x_n, g) = \lim_{n=1} \alpha(x_n g^{-1}) = 0$ ,

[5] := **MetricLimit**[3]  $E d_\alpha : \lim_{n=1} d_\alpha(y_n, h) = \lim_{n=1} \alpha(y_n h^{-1}) 0$ ,

[6] :=  $\Lambda n \in \mathbb{N} . E d_\alpha E_2 \text{AbsoluteValue}(G, \alpha, x_n^{-1}) \text{InverseMeaning} E_3 \text{AbsoluteValue}(G, \alpha, y_n h^{-1}, g^{-1} x_n)$   
 $E \text{Symmetric}(G, G, d_\alpha) \text{LimitSum}[4][5] : \lim_{n=1} d_\alpha(x_n y_n, gh) = \lim_{n=1} \alpha(x_n y_n h^{-1} g^{-1}) = \lim_{n=1} \alpha(x_n^{-1} x_n y_n h^{-1} g^{-1} x_n) =$   
 $= \lim_{n=1} \alpha(y_n h^{-1} g^{-1} x_n) \leq \lim_{n=1} \alpha(y_n h^{-1}) + \alpha(g^{-1} x_n) = \lim_{n=1} d_\alpha(y_n, h) + \lim_{n=1} d_\alpha(x_n, g) = 0$ ;

[7] := **NonNegtiveZeroBound**[8] :  $\lim_{n=1} d_\alpha(x_n y_n, gh) = 0$ ,

$[8.*] := \text{MetricLimit}[7] : \lim_{n \rightarrow \infty} x_n y_n = gh$ ;

$\leadsto [2] := \text{ContinuousByLimits} : \circ \in \text{TOP}((X, d_\alpha)^2, (X, d_\alpha))$ ,

$[*] := \text{ITGRP}[1][2] : (X, d_\alpha) \in \text{TGRP}$ ;

□

$\text{absoluteValueFromRIM} :: \prod_{G \in \text{TGRP}} \text{RightInvariantMetric}(G) \rightarrow \text{AbsoluteValue}(F)$

$\text{absoluteValueFromRIM}(\rho) = \alpha_\rho := \Lambda g \in G . \rho(g, e)$

$[1] := \text{E}\alpha_\rho(e) \text{E}_1 \text{Semimetric}(\rho) : \alpha_\rho(e) = \rho(e, e) = 0,$

$[2] := \Lambda g \in G . \text{E}\alpha_\rho(g^{-1}) \text{ERightInvariantMetric}(G, \rho) \text{ESymmetric}(G, \rho) \text{I}\alpha_\rho(g^{-1}) :$   
 $: \forall g \in G . \alpha_\rho(g^{-1}) = \rho(g^{-1}, e) = \rho(e, g) = \rho(g, e) = \alpha_\rho(g),$

$[3] := \Lambda g, h \in G . \text{E}\alpha_\rho(gh) \text{ERightInvariantMetric}(G, \rho, gh, e, h^{-1}) \text{ETriangleIneq}(G, \rho, g, e, h^{-1})$   
 $\text{ERightInvariantMetric}(G, \rho, e, h^{-1}, h) \text{I}\alpha_\rho : \forall g, h \in G . \alpha_\rho(gh) = \rho(gh, e) = \rho(g, h^{-1}) \leq$   
 $\leq \rho(g, e) + \rho(e, h^{-1}) = \rho(g, e) + \rho(h, e) = \alpha_\rho(g) + \alpha_\rho(h),$

**Assume**  $x : \mathbb{N} \rightarrow G,$

**Assume**  $[4] : \lim_{n \rightarrow \infty} \alpha_\rho(x_n) = 0,$

**Assume**  $g \in G,$

$[5] := [4] \text{E}\alpha_\rho \text{MetricLimit}(G, \rho) : \lim_{n \rightarrow \infty} x_n = e,$

$[6] := \text{ETGRP}(F) g[5] g^{-1} : \lim_{n \rightarrow \infty} g x_n g^{-1} = e,$

$[x.*] := [6] \text{MetricLimit}(G, \rho) \text{I}\alpha_\rho : \lim_{n \rightarrow \infty} \alpha_\rho(g x_n g^{-1}) = 0;$

$\leadsto [*] := \text{IAbsoluteValue}[1, 2, 3] : \text{AbsoluteValue}(G, \alpha_\rho);$

□

**AbsoluteValueGeneratingMetricCondition** ::

$:: \forall G \in \text{GRP} . \forall \alpha : \text{AbsoluteValue}(G) . \text{Metric}(G, d_\alpha) \iff \forall g \in G . g \neq e \Rightarrow \alpha(g) > 0$

**Proof** =

**Assume**  $a, b \in G,$

**Assume**  $[1] : a \neq b,$

$[2] := \text{EGRP}(G)[1] : ab^{-1} \neq e,$

$\left[ (a, b). * \right] := \text{E} : d_\alpha(a, b) = \alpha(ab^{-1}) > 0;$

$\leadsto [*] := \text{IMetric} : \text{Metric}(G, d),$

□

**LEMBYIsometry** ::  $\forall G \in \text{GRP} . \forall \text{Semimetric}(G, \rho) .$

$. \text{LeftInvariantMetric}(G, \rho) \iff \forall a \in A . \text{Isometry}(G, G, \lambda_a)$

**Proof** =

...

□

$\text{metricInversion} :: \prod_{G \in \text{GRP}} \text{Semimetric}(G) \rightarrow \text{Semimetric}(G)$

$\text{metricInversion}(\rho) = \rho^{-1} := \Lambda g, h \in G . \rho(g^{-1}, h^{-1})$

**InvariantInversion** ::  $\forall G \in \text{GRP} . \forall \rho : \text{LeftInvariantMetric}(G) . \text{RightInvariantMetric}(G, \rho^{-1})$

**Proof** =

...

□

### 2.1.3 Neighbourhoods of Unity

Topology of a topological group is fully determined by the neighborhood system of its unity.

$$\text{SymmetricSet} :: \prod_{G \in \text{GRP}} ?G$$

$$S : \text{SymmetricSet} \iff \text{inv}(S) = S$$

$$\text{UnityHasSymmetricHoodBase} :: \forall G \in \text{TGRP} . \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) : V \subset U \ \& \ \text{SymmetricSet}(G, V)$$

**Proof** =

$$V := U \cap \text{inv}(U) : ?G,$$

$$[1] := \text{unityInverse}(G) \text{EV} : e \in V,$$

$$[2] := \text{ETGRP}(G) \text{EV} : V \in \mathcal{T}(G),$$

$$[*] := \text{EV} : \text{inv}(V) = V;$$

□

$$\text{InverseContinuityAtUnityCriterion} ::$$

$$: \forall G \in \text{TOP} \ \& \ \text{GRP} . \text{inv}_G \in C_e(G, G) \iff \forall U \in \mathcal{U}(e) . \text{inv}(U) \in \mathcal{U}(e)$$

**Proof** =

...

□

$$\text{MultContinuityAtUnityCriterion} ::$$

$$: \forall G \in \text{TOP} \ \& \ \text{GRP} . (\cdot_G) \in C_{(e,e)}(G^2, G) \iff \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) : VV \subset U$$

**Proof** =

$$\text{Assume } [1] : (\cdot_G) \in C_{(e,e)}(G^2, G),$$

$$\text{Assume } U \in \mathcal{U}(e),$$

$$W := (\cdot_G)^{-1}(U) \in \mathcal{U}(e, e),$$

$$(A, B, [2]) := \text{ProductTopologyBae}(G, G, W) : \sum A, B \in \mathcal{U}(e) . A \times B \subset W,$$

$$V := A \cap B \in \mathcal{U}(e),$$

$$[1*] := \text{EV}[2] : VV \subset U;$$

$$\leadsto [1] := \text{I} \Rightarrow : (\cdot_G) \in C_{(e,e)}(G^2, G) \Rightarrow \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) : VV \subset U,$$

$$\text{Assume } [2] : \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) : VV \subset U,$$

$$\text{Assume } U \in \mathcal{U}(e),$$

$$(V, [3]) := [2](U) : \sum V \in \mathcal{U}(e) . VV \subset U,$$

$$[U.*] := \text{ProductTopologyBase}(G, G, V) : V \times V \in \mathcal{U}(e, e);$$

$$\leadsto [2.*] := \text{IC}_{(e,e)} : (\cdot_G) \in C_{(e,e)}(G^2, G);$$

$$\leadsto [*] := \text{I} \iff [1] : (\cdot_G) \in C_{(e,e)}(G^2, G) \iff \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) : VV \subset U;$$

□

**TopologicalGroupAltDef** ::  $\forall G \in \text{GRP} . \forall \mathcal{T} : \text{Topology}(G) . (G, \mathcal{T}) \in \text{TGRP} \iff$   
 $\iff \left( \forall g, h \in G . \forall U \in \mathcal{U}(h) . gU \in \mathcal{U}(gh) \right) \& \left( \forall U \in \mathcal{U}(e) . \forall \text{inv } U \in \mathcal{U}(e) . \right)$   
 $\& \& \left( \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) . VV \subset U \right) \& \left( \forall U \in \mathcal{U}(e) . \forall g \in G . \exists V \in \mathcal{U}(e) . aVa^{-1} \subset U \right)$

**Proof** =

[1] := **MultContinuityAtUnityCriterion**[0.3] :  $(\cdot_G) \in C_{(e,e)}(G^2, G)$ ,

**Assume**  $(\Delta, g) : \text{Net}(G)$ ,

**Assume** [2] :  $e \in \lim_{\delta \in \Delta} g_\delta$ ,

**Assume**  $a \in G$ ,

**Assume**  $U \in \mathcal{U}(e)$ ,

$(V, [3]) := [0.4](U) : \sum V \in \mathcal{U}(e) . aVa^{-1} \subset U$ ,

$(\delta, [4]) := \mathbf{E}[3](V) : \sum \delta \in \Delta . \forall \sigma \geq \delta . g_\sigma \in V$ ,

$[(\Delta, g).3] := [3][4] : \forall \sigma \geq \delta . ag_\sigma a^{-1} \subset U$ ;

$\leadsto [2] := \mathbf{I}C_e : \forall a \in G . (\Lambda g \in G . aga^{-1}) \in C_e(G, G)$ ,

**Assume**  $(\Delta, x), (\Delta, y) : \text{Net}(G)$ ,

**Assume**  $g, h \in G$ ,

**Assume** [3] :  $g \in \lim_{\delta \in \Delta} x_\delta$ ,

**Assume** [4] :  $h \in \lim_{\delta \in \Delta} y_\delta$ ,

[5] := [0.1][3] :  $e \in \lim_{\delta \in \Delta} g^{-1}x_\delta$ ,

[6] := [0.1][4] :  $e \in \lim_{\delta \in \Delta} y_\delta h^{-1}$ ,

[7] := [1][5][6] :  $e \in \lim_{\delta \in \Delta} g^{-1}x_\delta y_\delta h^{-1}$ ,

[8] := [2](g)[7] :  $e \in \lim_{\delta \in \Delta} x_\delta y_\delta h^{-1}h^{-1}g^{-1}$ ,

$[\dots *] := [0.1][8] : hg \in \lim_{\delta \in \Delta} x_\delta y_\delta$ ;

$\leadsto [3] := \text{ContinuityByNets} : (\cdot_G) \in \text{TOP}(G^2, G)$ ,

**Assume**  $g \in G$ ,

**Assume**  $U \in \mathcal{U}(g)$ ,

[4] := [0.1](U,  $g^{-1}$ ) :  $g^{-1}(U) \in \mathcal{U}(e)$ ,

[5] := [0.2][4] :  $\text{inv}(g^{-1}(U)) = \text{inv}(U)g \in \mathcal{U}(e)$ ,

[6] := [0.1][5] :  $g^{-1}\text{inv}(U)g \in \mathcal{U}(g^{-1})$ ,

$[g.*] := [3][6] : \text{inv}(U) \in \mathcal{U}(g^{-1})$ ;

$\leadsto [4] := \mathbf{I}\text{TOP} : \text{inv} \in \text{TOP}(G, G)$ ,

[\*] := **ITGRP**[3][4] :  $(G, \mathcal{T}) \in \text{TGRP}$ ;

□

**ConjugationIsAutomorphism** ::  $\forall G \in \text{TGRP} . \forall g \in G . \gamma_g \in \text{Aut}_{\text{TGRP}}(G)$

**Proof** =

...

□

**TopologicalGroupAltDef2** ::  $\forall G \in \text{GRP} . \forall \mathcal{T} : \text{Topology} . G \in \text{TGRP} \iff$

$$\iff \left( \Lambda g, h \in G . gh^{-1} \right) \in \text{TOP} \left( (G, \mathcal{T})^2, (G, \mathcal{T}) \right)$$

**Proof** =

$$\phi := \Lambda g, h \in G . gh^{-1} : \text{TOP} \left( (G, \mathcal{T})^2, (G, \mathcal{T}) \right),$$

$$[1] := \text{E}\phi\text{I}\text{inv} : \text{inv} = \Lambda g \in G . \phi(e, g),$$

$$[2] := \text{ITOP}[1] : \text{inv} \in \text{TOP} \left( (G, \mathcal{T}), (G, \mathcal{T}) \right),$$

$$[3] := \text{E}\phi\text{I}\cdot : (\cdot_G) = (\text{id} \times \text{inv})\phi,$$

$$[4] := \text{ITOP}[3] : (\cdot_G) \in \text{TOP} \left( (G, \mathcal{T})^2, (G, \mathcal{T}) \right),$$

$$[*] := \text{ITGRP}[2][4] : (G, \mathcal{T}) \in \text{TGRP};$$

□



### 2.1.4 Uniformity and Regularity

Topological groups are uniform spaces and, hence completely regular.

$$\text{leftGroupConnector} :: \prod_{G \in \text{TGRP}} \mathcal{U}(e) \rightarrow \text{Connector}(G)$$

$$\text{leftGroupConnector}(U) = U_L := \{(a, b) \in G^2 \mid a^{-1}b \in U\}$$

$$\text{LeftConnectorsIntersection} :: \forall G \in \text{TGRP} . \forall U, V \in \mathcal{U}(e) . (U \cap V)_L = U_L \cap V_L$$

**Proof** =

...

□

$$\text{LeftConnectorTranspose} :: \forall G \in \text{TGRP} . \forall U \in \mathcal{U}(e) . (U_L)^\top = (\text{inv}(U))_L$$

**Proof** =

...

□

$$\text{LeftConnectorsCompose} :: \forall G \in \text{TGRP} . \forall U, V \in \mathcal{U}(e) . U_L \circ V_L = (VU)_L$$

**Proof** =

$$\text{Assume } (a, c) \in U_L \circ V_L,$$

$$(b, [2]) := \mathbf{E}(\circ)(U_L \circ V_L)(a, c) : \sum b \in G . (a, b) \in U_L \ \& \ (b, c) \in V_L,$$

$$[3] := \mathbf{E}U_L[2] : a^{-1}b \in U \ \& \ b^{-1}c \in V,$$

$$[4] := [3.1][3.2]\text{InverseMeaning}(G, G) : a^{-1}c = a^{-1}bb^{-1}c = a^{-1}c \in UV,$$

$$\left[ (a, c). * \right] := \mathbf{I}(VU)_L[4] : (a, c) \in (VU)_L;$$

$$\leadsto [1] := \mathbf{I} \subset : U_L \circ V_L \subset (VU)_L,$$

$$\text{Assume } (a, c) \in (VU)_L,$$

$$[2] := \mathbf{E}(VU)_L(a, c) : a^{-1}c \in VU,$$

$$(v, u, [3]) := \mathbf{E}VU[2] : \sum v \in V . \sum u \in U . a^{-1}c = vu,$$

$$[4] := [3]u^{-1} : a^{-1}cu^{-1} = v \in V,$$

$$[5] := \mathbf{I}V_L[4] : (a, cu^{-1}) \in V_L,$$

$$[6] := v^{-1}[3] : v^{-1}a^{-1}c = u \in U,$$

$$[7] := \mathbf{I}U_L[6] : (av, c) \in U_L,$$

$$[8] := [3]\text{InverseMeaning}(G, u) : cu^{-1} = avuu^{-1} = av,$$

$$\left[ (a, c). * \right] := \mathbf{I}U_L \circ V_L[5][7][8] : (a, c) \in U_L \circ V_L;$$

$$\leadsto [*] := \mathbf{ISetEq}[1] : U_L \circ V_L = (VU)_L;$$

□

$$\text{leftGroupUniformity} :: \prod_{G \in \text{TGRP}} \text{Uniformity}(G)$$

$$\text{leftGroupUniformity}() = \mathcal{L}_G := \left\langle \{U_L \mid U \in \mathcal{U}_G(e)\} \right\rangle_{\mathcal{F}}$$

**TopologicalGroupIsUniformizableByLeftUniformities** ::  $\forall G \in \mathbf{TGRP} . G \cong_{\mathbf{TOP}} (G, \mathcal{L}_G)$

**Proof** =

**Assume**  $g \in G$ ,

**Assume**  $U \in \mathcal{U}(g)$ ,

**Assume**  $u \in U$ ,

$\varphi := \lambda x \in G . u^{-1}x : \mathbf{Aut}_{\mathbf{TOP}}(G)$ ,

$V := \varphi(U) : \mathcal{U}(e)$ ,

$[1] := \mathbf{EGRP}(G)\mathbf{EVI}V_L : U = V_L(u)$ ,

$[u.*] := \mathbf{EqIsSubset}[1] : V_L(u) \subset U$ ;

$\leadsto [U.*] := \mathbf{EuniformTopology} : U \in \mathcal{U}_{\mathcal{L}_G}(g)$ ;

$\leadsto [1] := \mathbf{I} \subset \mathcal{U}(g) \subset \mathcal{U}_{\mathcal{L}_G}(g)$ ,

**Assume**  $U : \mathcal{U}_{\mathcal{L}_G}(g)$ ,

$(V, [2]) := \mathbf{EL}_G : \sum V \in \mathcal{U}(e) . U = V_L(g)$ ,

$[3] := \mathbf{EV}_L : U = gV$ ,

$[U.*] := \mathbf{ETGRP}(G)[3] : U \in \mathcal{U}(g)$ ;

$\leadsto [x.*] := \mathbf{ISetEq}[1] : \mathcal{U}(g) = \mathcal{U}_{\mathcal{L}_G}(g)$ ;

$\leadsto [*] := \mathbf{TopologyEqByHoods} : G \cong_{\mathbf{TOP}} (G, \mathcal{L}_G)$ ;

□

**rightGroupConnector** ::  $\prod_{G \in \mathbf{TGRP}} \mathcal{U}(e) \rightarrow \mathbf{Connector}(G)$

**rightGroupConnector**  $(U) = U_R := \{(a, b) \in G^2 . ab^{-1} \in U\}$

**rightConnectorsIntersection** ::  $\forall G \in \mathbf{TGRP} . \forall U, V \in \mathcal{U}(e) . (U \cap V)_R = U_R \cap V_R$

**Proof** =

...

□

**RightConnectorTranspose** ::  $\forall G \in \mathbf{TGRP} . \forall U \in \mathcal{U}(e) . (U_R)^\top = (\mathbf{inv}(U))_R$

**Proof** =

...

□

**RightConnectorsCompose** ::  $\forall G \in \mathbf{TGRP} . \forall U, V \in \mathcal{U}(e) . U_R \circ V_R = (VU)_R$

**Proof** =

...

□

**rightGroupUniformity** ::  $\prod_{G \in \mathbf{TGRP}} \mathbf{Uniformity}(G)$

**rightGroupUniformity**  $() = \mathcal{R}_G := \left\langle \{U_R | U \in \mathcal{U}_G(e)\} \right\rangle_{\mathcal{F}}$

**TopologicalGroupIsUniformizableByRightUniformities** ::  $\forall G \in \text{TGRP} . G \cong_{\text{TOP}} (G, \mathcal{R}_G)$

**Proof** =

...

□

**TopologicalGroupsAreCompletelyRegular** ::  $\forall G \in \text{TGRP} . \text{CompletelyRegular}(G)$

**Proof** =

...

□

**SeparatedTopologicalGroupsAreTychonoff** ::  $\forall G \in \text{TGRP} . \text{T0}(G) \iff \text{Tychonoff}(G)$

**Proof** =

...

□

**upperTwoSidedUniformity** ::  $\prod_{G \in \text{TGRP}} \text{Uniformity}(G)$

**upperTwoSidedUniformity** () =  $\mathcal{S}_G^\vee := \mathcal{L}_G \vee \mathcal{R}_G$

**TopologicalGroupIsUniformizableByTwoSidedUniformities** ::  $\forall G \in \text{TGRP} . G \cong_{\text{TOP}} (G, \mathcal{S}_G^\vee)$

**Proof** =

...

□

**supConnector** ::  $\prod G \in \text{TGRP} . \mathcal{U}(e) \rightarrow \text{Connector}(G)$

**supGroupConnector** (U) =  $U_\vee := U_L \cap U_R$

**TwoSidedUniformityBase** ::  $\forall G \in \text{TGRP} . \text{BaseOfUniformity}\left(G, \mathcal{S}_G^\vee, \{U_\vee | U : \text{SymmetricSet}(G) \ \& \ \mathcal{U}(e)\}\right)$

**Proof** =

**Assume**  $U \in \mathcal{S}_G^\vee$ ,

$(O, [1]) := \text{ES}^v \text{ee}_G \text{EUE} \mathcal{L}_G : \sum O \in \mathcal{U}(e) . O_L \subset U,$

$E := O \cap \text{inv} O' \in \text{SymmetricSet}(G) \ \& \ \mathcal{U}(e),$

$[U.*] := \text{EE}_\vee \text{IntersectionIsSubset}(G \times G, E_L, E_R) \text{EELeftConnectorIntersection}(G) \text{IntersectionIsSubs}$

$\leadsto [*] := \text{IBaseOfUniformity} : \text{BaseOfUniformity}\left(G, \mathcal{S}_G^\vee, \{U_\vee | U : \text{SymmetricSet}(G) \ \& \ \mathcal{U}(e)\}\right),$

□

**ClosureInTopologicalGroup** ::  $\forall G \in \text{TGRP} . \forall A \subset G . \overline{A} = \bigcap \{AU \mid U \in \mathcal{U}(e)\}$

**Proof** =

...

□

**ClosureInTopologicalGroup1** ::  $\forall G \in \mathbf{TGRP} . \forall A \subset G . \overline{A} = \bigcap \left\{ UA \mid U \in \mathcal{U}(e) \right\}$

**Proof** =

...

□

**ClosureInversion** ::  $\forall G \in \mathbf{TGRP} . \forall A \subset G . \overline{A^{-1}} = \overline{A}^{-1}$

**Proof** =

$[*] := \mathbf{ClosureInTopologicalGroup}(G, A) \mathbf{EAut}_{\mathbf{TOP}}(G, \text{inv}) \mathbf{EGRP}(G)$

$\mathbf{Iimage}(\text{inv}) \mathbf{ClosureInTopologicalGroup2}(G, A) :$

$$\begin{aligned} : \overline{A^{-1}} &= \bigcap \left\{ A^{-1}U \mid U \in \mathcal{U}(e) \right\} = \bigcap \left\{ A^{-1}U^{-1} \mid U \in \mathcal{U}(e) \right\} = \bigcap \left\{ (UA)^{-1} \mid U \in \mathcal{U}(e) \right\} = \\ &= \left( \bigcap \left\{ AU \mid U \in \mathcal{U}(e) \right\} \right)^{-1} = \overline{A}^{-1}; \end{aligned}$$

□

**ConjugationClosure** ::  $\forall G \in \mathbf{TGRP} . \forall A \subset G . \forall g \in G . g\overline{A}g^{-1} = \overline{gAg^{-1}}$

**Proof** =

...

□

**ClosureMult** ::  $\forall G \in \mathbf{TGRP} . \forall A, B \subset G . (\overline{A})(\overline{B}) \subset \overline{AB}$

**Proof** =

...

□

**OpenProduct** ::  $\forall G \in \mathbf{TGRP} . \forall U \in \mathcal{T}(G) . \forall A \subset G . UA, UA \in \mathcal{T}(G)$

**Proof** =

...

□

**ClosedProduct** ::  $\forall G \in \mathbf{TGRP} . \forall x \in G . \forall A : \mathbf{Closed}(G) . \mathbf{Closed}(G, xA \ \& \ Ax)$

**Proof** =

...

□

**ClosureOfSubgroup** ::  $\forall G \in \text{TGRP} . \forall H \subset_{\text{GRP}} G . \overline{H} \subset_{\text{GRP}} G$

**Proof** =

[1] := **SetMultSubset** $\left(G, \overline{H}, (\overline{H})^{-1}\right)$  **ClosureInversion** $\left(G, H\right)$  **ClosureNult** $(G, H, H^{-1})$  **ESubgroup** $(H)$  :  

$$: \overline{H} \subset \overline{H}(\overline{H})^{-1} = \overline{H}(\overline{H^{-1}}) \subset \overline{HH^{-1}} = \overline{H},$$
  
[2] := **DoubleIneqLemma** $\left(?G, \overline{H}, \overline{H}(\overline{H})^{-1}\right)$  [1] :  $\overline{H} = \overline{H}(\overline{H})^{-1},$   
[\*] := **SubgroupAltDef**[2] :  $\overline{H} \subset_{\text{GRP}} G;$   
□

**ClosureOfNormalSubgroup** ::  $\forall G \in \text{TGRP} . \forall H \triangleleft G . \overline{H} \triangleleft G$

**Proof** =

[1] :=  $\Lambda g \in G .$  **ConjugationClosure** $(G, H, g)$  **ENormalSubgroup** $(G, H)$  :  $\forall g \in G . g\overline{H}g^{-1} = \overline{gHg^{-1}} = \overline{H},$   
[\*] := **I**  $\triangleleft$  [1] :  $\overline{H} \triangleleft G;$   
□

**AbelianClosureIsAbelian** ::  $\forall G \in \text{TGRP} . \forall H \subset_{\text{GRP}} G . \text{T2}(G) \ \& \ H \in \text{ABEL} \Rightarrow \overline{H} \in \text{ABEL}$

**Proof** =

$\varphi := \Lambda g, h \in G . ghg^{-1}h^{-1} \in \text{TOP}(G^2, G),$   
[1] := **T2HasClosedPoints** $(G)$  **ClosedPreimage** $(G^2, G, \varphi, \{e\})$  : **Closed** $\left(G \times G, \varphi^{-1}\{e\}\right),$   
[2] := **AbelianHasTrivialCommutator** $(G, H)$  **E** $\varphi : H \times H \subset \varphi^{-1}\{e\},$   
[3] := [1][2] **EClosure** $(G \times G)$  **ClosureProduct** $(G)$  :  $\overline{H} \times \overline{H} \subset \overline{H \times H} \subset \varphi^{-1}\{e\}$  **E** $\varphi,$   
[\*] := **AbelianByTrivialCommutot**[3] :  $\overline{H} \in \text{ABEL};$   
□

**OpenGroupsAreClopen** ::  $\forall G \in \text{TGRP} . \forall H \subset_{\text{GRP}} G . H \in \mathcal{T}(G) \Rightarrow \text{Clopen}(G, H)$

**Proof** =

[1] := **Outproduct** $(G, H)$  **EsetProduct** $(G, H^{\complement}, H)$  **ETOP** $(G)[0]$  :  $H^{\complement} = H^{\complement}H = \bigcap_{x \in H^{\complement}} xH \in \mathcal{T}(G),$   
[2] := **IClosed**[1] : **Closed** $(G, H),$   
[\*] := **IClopen**[2][0] : **Clopen** $(G, H);$   
□

**OpenGroupProduceClopenSets** ::  $\forall G \in \text{TGRP} . \forall H \subset_{\text{GRP}} G . \forall A \subset G . H \in \mathcal{T}(G) \Rightarrow \text{Clopen}(G, AH \ \& \ HA)$

**Proof** =

**Assume**  $x \in (AH)^{\complement},$   
[1] := **E** $x : \forall a \in A . \forall h \in H . ah \neq x,$   
**Assume**  $h \in H,$   
**Assume** [2] :  $xh \in AH,$   
[3] := [2] $h^{-1}$  **ESubgroup** $(G, H)$  :  $x \in AHh^{-1} = AH,$   
[1.\*] := [1][3] :  $\perp;$   
 $\leadsto$  [1] := **E** $\perp$  **OpenProduct** $(G, H, (AH)^{\complement})$  :  $(AH)^{\complement} = (AH)^{\complement}H \in \mathcal{T}(G),$   
[2] := **IClosed**[1] : **Closed** $(G, AH),$   
[3] := **OpenProduct** $(G, H, A)$  :  $AH \in \mathcal{T}(G),$   
[\*] := **IClopen**[2][3] : **Clopen** $(G, AH);$   
□

**OpenGroupIntersection** ::  $\forall G \in \text{TGRP} . \bigcap \text{Subgroup} \ \& \ \mathcal{T}(G) \triangleleft G$

**Proof** =

$Z := \bigcap \text{Subgroup} \ \& \ \mathcal{T}(G) : \text{Subgroup}(G),$

**Assume**  $g \in G,$

**Assume**  $z \in Z,$

**Assume**  $H : \text{Subgroup} \ \& \ \text{Open}(G),$

$[1] := \text{ETGRP} : \text{Subgroup} \ \& \ \text{Open}(G, g^{-1}Hg),$

$[2] := \text{EZEx}[1] : z \in g^{-1}Hg,$

$[H.*] := g[2]g^{-1} : gzg^{-1} \in H;$

$\leadsto [g.*] := \text{I}z : gzg^{-1} \in Z;$

$[*] := \text{INormalSubgroup} : Z \triangleleft G;$

□

**ClosedGroupIntersection** ::  $\forall G \in \text{TGRP} . \bigcap \left\{ H \subset_{\text{GRP}} G : \text{Closed}(G, H) \ \& \ H \neq \{e\} \right\} \triangleleft G$

**Proof** =

$\mathcal{A} := \left\{ H \subset_{\text{GRP}} H : \text{Closed}(G, H) \ \& \ H \neq \{e\} \right\} : ?\text{Subgroup}(G),$

$Z := \bigcap \mathcal{A} : \text{Subgroup}(G),$

**Assume**  $g \in G,$

**Assume**  $z \in Z,$

**Assume**  $H \in \mathcal{A},$

$[1] := \text{ETGRPBijectionPreservesCardinality}(G)\text{E}\mathcal{A} : g^{-1}Hg \in \mathcal{A},$

$[2] := \text{EZEx}[1] : z \in g^{-1}Hg,$

$[H.*] := g[2]g^{-1} : gzg^{-1} \in H;$

$\leadsto [g.*] := \text{I}z : gzg^{-1} \in Z;$

$[*] := \text{INormalSubgroup} : Z \triangleleft G;$

□

**ClosureAnnihilation** ::  $\forall G \in \text{TGRP} . \forall A, B \subset G . \forall U \in \mathcal{T}(G) . \overline{AU\overline{B}} = AU\overline{B}$

**Proof** =

...

□

**DoubleClosureExpression** ::  $\forall X \in \text{TOP} . \forall A \subset X . \forall B : \text{CompactSubset}(X) . \overline{\overline{A}B} = \bigcap_{U \in \mathcal{U}(e)} AU\overline{B}$

**Proof** =

...

□

### 2.1.5 Metrics for Twosided Uniformities

There is a special form of metrics for twosided uniformities.

$$\mathbf{V}\text{-Semimetric} :: \prod_{G \in \mathbf{GRP}} ?\mathbf{Semimetric}(G)$$

$$\rho : \mathbf{V}\text{-Semimetric} \iff \mathbf{V}\text{-Semimetric} \iff \exists \sigma : \mathbf{LeftInvariantMetric}(G) . \forall g, h \in G . \\ \rho(g, h) = \sigma(g, h) + \sigma(g^{-1}, h^{-1})$$

**VeeMetricConstructionIsUnique** ::

$$:: \forall G \in \mathbf{GRP} . \forall \rho : \mathbf{V}\text{-Semimetric}(G) . \exists ! \sigma : \mathbf{LeftInvariantMetric}(G) . \forall g, h \in G . \\ . \rho(g, h) = \sigma(g, h) + \sigma(g^{-1}, h^{-1})$$

**Proof** =

$$\left( \sigma, [1] \right) := \mathbf{EV}\text{-Semimetric}(G, \rho) :$$

$$: \sum \sigma : \mathbf{LeftInvariantMetric}(G) . \forall g, h \in G . \rho(g, h) = \sigma(g, h) + \sigma(g^{-1}, h^{-1}),$$

$$[2] := q \Lambda g \in G . \mathbf{E} \alpha_\rho(g) [1] \mathbf{EL} \mathbf{LeftInvariantMetric}(G, \sigma) \mathbf{I} \alpha_\sigma :$$

$$: \alpha_\rho(g) = \rho(g, e) = \sigma(g, e) + \sigma(g^{-1}, e) = 2\sigma(g, e) = 2\alpha_\sigma(g),$$

$$[3] := \mathbf{I}(=, \rightarrow) [2] : \alpha_\rho = 2\alpha_\sigma,$$

$$[*] := d \left( \frac{[3]}{2} \right) : \sigma = d_{\frac{\alpha_\rho}{2}};$$

□

**VeeMetricMetrisesUpperTwoSidedUniformity** ::  $\forall G \in \mathbf{GRP} . \forall \rho : \mathbf{V}\text{-Semimetric}(G) . \mathcal{S}_{(G, \rho)}^\vee = \mathbb{B}_\rho$

**Proof** =

...

□

### 2.1.6 Ellis Theorem

If group has a locally compact Hausdorff topology then it is only enough to have continuous multiplication to show that the topology is a group topology!

$$\text{EllisTopology} :: \prod_{G \in \text{GRP}} \text{?Topology}(G)$$

$$\mathcal{T} : \text{EllisTopology} \iff \text{T2}(G, \mathcal{T}) \ \& \ \text{LocallyCompact}(G, \mathcal{T}) \ \& \ \cdot_G \in \text{TOP}(G^2, G)$$

$$\text{EllisCompactInversionLemma} ::$$

$$:: \forall G \in \text{GRP} . \forall \mathcal{T} : \text{EllisTopology}(G) . \forall K : \text{CompactSubset}(G, \mathcal{T}) . \text{Closed}\big((G, \mathcal{T}), K^{-1}\big)$$

$$\text{Proof} =$$

$$\text{Assume } b \in \overline{K^{-1}},$$

$$(\mathcal{F}, [1]) := \text{ClosureByLimits}(G, K^{-1}, b) : \sum \mathcal{F} : \text{Filter}(K^{-1}) . b = \lim \mathcal{F},$$

$$a := \text{FilterCompact}(K, \text{inv}_* \mathcal{F}) : \text{Cluster}\big(K, \text{inv}_* \mathcal{F}\big),$$

$$(\mathcal{G}, [2]) := \text{ClusterConvergingFilter} : \sum \mathcal{G} : \text{Filter}(K) . a = \lim \mathcal{G} \ \& \ \text{inv}_* \mathcal{F} \subset \mathcal{G},$$

$$[3] := [2.1][1] : b = \lim \text{inv}_* \mathcal{G},$$

$$\mathcal{L} := \mathcal{G} \times \text{inv}_* \mathcal{G} : \text{Filterbase}(K \times K^{-1}),$$

$$[4] := \text{EL}[1][3] : \lim \mathcal{L} = (a, b),$$

$$[5] := \text{E}_3 \text{EllisTopology}(G, \mathcal{T})[4] : \lim(\cdot_G)(\mathcal{L}) = ab,$$

$$[6] := \text{ELFilterLimit} : \text{Cluster}\big(G, (\cdot_G)(\mathcal{L}), e\big),$$

$$[7] := \text{E}_1 \text{EllisTopology}(G, \mathcal{T}) \text{T2HasUniqueClusters} : ab = e,$$

$$[b.*] := a^{-1}[7] : b \in K^{-1};$$

$$\leadsto [1] := \text{I} \subset \overline{K^{-1}} \subset K^{-1},$$

$$[*] := \text{EClosure}[1] : \overline{K^{-1}} = K^{-1},$$

□



**EllisCountableInversionLemma** ::

$$:: \forall G \in \text{GRP} . \forall \mathcal{T} : \text{EllisTopology}(G) . \forall A : \text{Countable}(G) . \forall b \in \overline{A} . b^{-1} \in \overline{A^{-1}}$$

**Proof** =

$$\left( \mathcal{F}, [1] \right) := \text{ClosureByLimits}(G, K^{-1}, b) : \sum \mathcal{F} : \text{Filter}(K^{-1}) . b = \lim \mathcal{F},$$

$$H := \langle A \cup \{b\} \rangle_{\text{GRP}} : \text{Subgroup}(G),$$

$$[2] := \text{GeneratedByConutableIsCountable}(G) \text{EH} : |H| \leq \aleph_0,$$

$$\text{Assume } K \in \mathcal{K}(e),$$

$$\text{Assume } y \in \overline{H},$$

$$[3] := \text{E}_3 \text{EllisTopology}(G, \mathcal{T}) : yK \in \mathcal{K}(y),$$

$$[4] := \text{ClosureAltDef}(G, H) \text{EK}[3] : \exists yK \cap H,$$

$$\left( k, [5] \right) := \text{E}\exists[4] : \sum k \in K . yk \in H,$$

$$[y.*] := [5]k^{-1} : y \in HK^{-1};$$

$$\leadsto [3] := \text{I} \subset: \overline{H} \subset HK^{-1},$$

$$[4] := \Lambda g \in H . \text{E}_3 \text{EllisTopology}(G, \mathcal{T}) \text{HomeoClusureEq} \left( (G, \mathcal{T}), H, \lambda_g \right) \text{ESubgroup}(G, H) : \\ : \forall g \in G . g\overline{H} = \overline{gH} = \overline{H},$$

$$[5] := [3] \Lambda x \in H . \text{InverseMeaning}(G, x)[4] :$$

$$: \overline{H} = \bigcup_{x \in H} xK^{-1} \cap \overline{H} = \bigcup_{x \in H} x(K^{-1} \cap x^{-1}\overline{H}) = \bigcup_{x \in H} x(K^{-1} \cap \overline{H}),$$

$$[6] := \text{EllisCompactInversionLemma}(G, \mathcal{T}, K) : \text{Closed} \left( (G, \mathcal{T}), K^{-1} \right),$$

$$[7] := \text{E}_{1,2} \text{EllisTopology}(G, \mathcal{T}) \text{BaireCategoryTHM} : \text{Baire}(\overline{H}, \mathcal{T} \cap \overline{H}),$$

$$\left( h, [8] \right) := \text{EBaire}(\overline{H}, \mathcal{T} \cap \overline{H})[5] : \sum h \in H . \exists^* x \in \overline{H} . x \in h(K^{-1} \cap \overline{H}),$$

$$[9] := \text{E}\exists^*[8] : \neg \text{Dense} \left( \overline{H}, \overline{H} \setminus h(K^{-1} \cap \overline{H}) \right),$$

$$\left( U, [10] \right) := \text{EDense}[9] : \sum U \in \mathcal{T} . U \subset h(K^{-1} \cap \overline{H}) \ \& \ \exists U \cap \overline{H},$$

$$u := \text{ClosureAltDef} \left( (G, \mathcal{T}), H, U \right) [10] \in H \cap U,$$

$$[11] := [4] \text{Eu} : bu^{-1}(U \cap \overline{H}) = bu^{-1}U \cap \overline{H} \in \mathcal{U}_{\overline{H}}(b),$$

$$\left( F, [12] \right) := \text{EFilterConvergece}[1][11][10][4] \text{IntersectionIsSubset} : \\ : F \subset bu^{-1}(U \cap \overline{H}) \subset bu^{-1}h(K^{-1} \cap \overline{H}) \subset bu^{-1}hK^{-1},$$

$$\left( F, [12] \right) := \text{EFilterConvergece}[1][11][10][4] \text{IntersectionIsSubset} : \\ : F \subset bu^{-1}(U \cap \overline{H}) \subset bu^{-1}h(K^{-1} \cap \overline{H}) \subset bu^{-1}hK^{-1},$$

$$[13] := [12]^{-1} : F^{-1} \subset Kb^{-1}uh^{-1},$$

$$a := \text{CompactHasCluseter} \left( (A, \mathcal{T} \cap A), Kb^{-1}uh^{-1}, \mathcal{F}^{-1} \right) : \text{Cluster} \left( (A, \mathcal{T} \cap A), Kb^{-1}uh^{-1}, \mathcal{F}^{-1} \right),$$

$$[14] := [13] \text{Ea} : a \in Kb^{-1}uh^{-1} \cap \overline{A^{-1}},$$

$$[*] := \text{ByAnalogy} \left( \text{EllisCompactInversionLemma} \right) [14] : b^{-1} = a \in \overline{A^{-1}};$$

□

**EllisInversCompactnessLemma** ::

::  $\forall G \in \text{GRP} . \forall \mathcal{T} : \text{EllisTopology} . \forall K : \text{CompactSubset}(G, \mathcal{T}) . \text{CompactSubset}\left((G, \mathcal{T}), K^{-1}\right)$

**Proof** =

[1] := **EllisCompactInversionLemma** $(G, \mathcal{T}, K) : \text{Closed}\left((G, \mathcal{T}), K^{-1}\right),$

$U := \text{E}_2\text{EllisTopology}(G, \mathcal{T})\text{ELocallyCompact}(G, e) \in \mathcal{K}(e),$

**Assume** [2] :  $\forall A : \text{Finite}(G) . K^{-1} \not\subset AU,$

$(k, [3]) := \text{EN}[2] : \sum k : \mathbb{N} \rightarrow K^{-1} . \forall n \in \mathbb{N} . k_{n+1} \notin \bigcup_{i=1}^n k_i U,$

$(b, [4]) := \text{CompactHasClusetr}\left((G, \mathcal{T}), K, k^{-1}\right) : \sum b \in K . \text{Cluster}\left((G, \mathcal{T}), k^{-1}, b\right),$

$(V, [5]) := \text{E}_1\text{EllisTopology}(G, \mathcal{T})\text{ProductTopologyBase}(\mathcal{T}, \mathcal{T}, U) : \sum V \in \mathcal{U}(e) . V^2 \subset U,$

$(n, [6]) := \text{ECluster}\left((G, \mathcal{T}), k^{-1}, b, Vb\right) : \sum_{n=1}^{\infty} k_n^{-1} \in Vb,$

[7] :=  $k_n[6]b^{-1} : b^{-1} \in k_n V,$

$A := \{k_m^{-1} | m > n\} : ?K,$

[8] := **ECluster** $\left((G, \mathcal{T}), k^{-1}, b\right)\text{EA} : b \in \overline{A},$

[9] := **EllisCountableInversionLemma**[8] :  $b^{-1} \in \overline{A^{-1}},$

$(m, [10]) := \text{EA}[9]\text{ClosureAltdef}(b^{-1}V) : \sum m > n . k_m \in b^{-1}V,$

[11] := [10][7][5] :  $k_m \in k_n V^2 \subset k_n U,$

[2.\*] := [2][11] :  $\perp;$

$\leadsto (A, [2]) := \text{E}\perp : \exists A : \text{Finite}(G) . K^{-1} \subset AU,$

[3] := **E**<sub>1</sub>**EllisTopology** $(G, \mathcal{T})\text{FiniteCompactUnion} : \text{CompactSubset}\left((G, \mathcal{T}), AU\right),$

[\*] := **ClosedSubsetOfCompactIsCompact**[2][3] :  $\text{CompactSubset}\left((G, \mathcal{T}), K^{-1}\right),$

□

**EllisTheorem** ::  $\forall G \in \text{GRP} . \forall \mathcal{T} : \text{EllisTopology}(G) . (G, \mathcal{T}) \in \text{TGRP}$

**Proof** =

**Assume**  $U \in \mathcal{U}(e)$ ,

**Assume** [1] :  $\forall K \in \mathcal{K}(e) . K^{-1} \not\subset U$ ,

$\mathcal{F} := \left\{ K^{-1} \cap U^c \mid K \in \mathcal{K}(e) \right\} : ?\text{CompactSubset}(K, \mathcal{T})$ ,

[2] := [1]**E** $\mathcal{F} : \emptyset \notin \mathcal{F}$ ,

[3] := **E**<sub>2</sub>**EllisTopologyELocallyCompact**( $G, \mathcal{T}$ )**E** $\mathcal{F} : \mathcal{F} \neq \emptyset$ ,

[4] := **E** $\mathcal{K}(e)$ **E** $\text{Aut}_{\text{SET}}(G, \text{inv})$ **E** $\mathcal{F} : \forall A, B \in \mathcal{F} . A \cap B \in \mathcal{F}$ ,

[5] := **I****Filterbase**[2 – 4] : **Filterbase** $\left((G, \mathcal{T}), \mathcal{F}\right)$ ,

[6] := **CantorIntersectionTHM**[5] :  $\exists \bigcap \mathcal{F}$ ,

[7] := **E**<sub>1,2</sub>**EllisTopology**( $G, \mathcal{T}$ )**RegularNbhdBaseIntersection** :  $\bigcap \mathcal{K}(e) = \{e\}$ ,

[8] := **BijectionOfIntersection**( $G, G, \text{inv}$ )[7] :  $\bigcap \text{inv}_* \mathcal{K}(e) = \{e\}$ ,

[9] := [8]**E** $\mathcal{F} : \bigcap \mathcal{F} = \emptyset$ ,

[1.\*] := [9][6] :  $\perp$ ;

$\leadsto \left(K, [U.*]\right) := \text{E}\perp : \sum K \in \mathcal{K}(e) . K^{-1} \subset U$ ;

$\leadsto [1] := \text{I}C_e : \text{inv} \in C_e \left((G, \mathcal{T}), (G, \mathcal{T})\right)$ ,

[2] := [1]**E**<sub>3</sub>**EllisTopology**( $G, \mathcal{T}$ ) :  $\text{inv} \in \text{Aut}_{\text{TOP}}(G, \mathcal{T})$ ,

[\*] := **I****TGRP**[2]**E**<sub>3</sub>**EllisTopology**( $G, \mathcal{T}$ ) :  $(G, \mathcal{T}) \in \text{TGRP}$ ;

□

### 2.1.7 Topological Groups with Ultrametrics

Balls around the unity produced by ultrametric are subgroups, and hence clopen.

$$\text{Ultravalue} :: \prod_{G \in \text{GRP}} ?\text{AbsoluteValue}(G)$$

$$\alpha : \text{Ultravalue} \iff \forall a, b \in A . \alpha(ab) \leq \max(\alpha(a), \alpha(b))$$

$$\text{UltravalueProduceUltrametric} :: \forall A \in \text{ABEL} . \forall \alpha : \text{Ultravalue}(A) . \text{Ultrametric}(A, d_\alpha)$$

**Proof** =

...

□

$$\text{UltrametricCellsAreSubgroups} :: \forall A \in \text{ABEL} . \forall \alpha : \text{Ultravalue}(A) . \forall r \in \mathbb{R}_{++} \mathbb{B}(0, r) \triangleleft A$$

**Proof** =

**Assume**  $a, b \in \mathbb{B}(0, r)$ ,

$$[1] := \text{Ed}_\alpha \text{EUltravalue}(A) \text{Id}_\alpha \text{E}a, b \in \mathbb{B}(0, r) :$$

$$: d_\alpha(0, a + b) = \alpha(a + b) \leq \max(\alpha(a), \alpha(b)) = \max(d_\alpha(0, a), d_\alpha(0, b)) < r,$$

$$[(a, b). *] := \text{EB}(0, r)[1] : a + b \in \mathbb{B}(0, r);$$

$$\leadsto [1] := \text{I}\forall : \forall a, b \in \mathbb{B}(0, r) . a + b \in \mathbb{B}(0, r),$$

$$[2] := \text{EAbsoluteValue}(A, \alpha) : \forall a \in \mathbb{B}(0, r) . a^{-1} \in \mathbb{B}(0, r),$$

$$[*] := \text{ISubgroup} : \mathbb{B}(0, r) \subset_{\text{GRP}} G;$$

□

$$\text{UltrametrizableGroupHasBaseOfSubgroups} ::$$

$$:: \forall A \in \text{ABEL} . \forall \alpha : \text{Ultravalue}(A) . \exists \mathcal{N} : \text{NeighborhoodBase}(A, \rho) . \forall N \in \mathcal{N} . N \triangleleft A$$

**Proof** =

...

□

$$\text{UltrametrizableGroupHasClopenBalls} ::$$

$$:: \forall G \in \text{TGRP} . \forall \alpha : \text{Ultravalue}(G) . \forall r \in \mathbb{R} . \text{Clopen}(G, \mathbb{B}_\alpha(e, r))$$

**Proof** =

...

□

$$\text{UltrametrizableGroupsAreZeroDim} ::$$

$$:: \forall G \in \text{GRP} . \forall \alpha : \text{Ultravalue}(G) . \dim_{\text{TOP}}(G, d_\alpha) = 0$$

**Proof** =

...

□

### 2.1.8 Some Interesting Examples

Sometimes our basic expectations fail.

**SumOfIntegersIsNotClosed** ::  $\alpha \in \mathbb{R} \setminus \mathbb{Q} . \neg \text{Closed}(\mathbb{R}, \mathbb{Z} + \alpha\mathbb{Z})$

**Proof** =

$A := \mathbb{Z} + \alpha\mathbb{Z} : ?\mathbb{R}$ ,

$[1] := \text{IrrationalGenDense}(\alpha) \text{I} A : \text{Dense}(\mathbb{R}, A)$ ,

**Assume**  $[2] : \text{Closed}(\mathbb{R}, A)$ ,

$[3] := \text{EDense}(\mathbb{R}, A)[2] : A = \mathbb{R}$ ,

$(n, m, [4]) := \text{EA}[3] \left( \frac{\alpha}{2} \right) : \sum n, m \in \mathbb{Z} . \frac{\alpha}{2} = n\alpha + m$ ,

$[5] := [4] - n\alpha : \frac{1 - 2n}{2}\alpha = m$ ,

$[6] := \frac{2}{1 - 2n}[5] \text{I} \mathbb{Q} : \alpha = \frac{2m}{1 - 2n} \in \mathbb{Q}$ ,

$[2.*] := \text{E}\alpha[6] : \perp$ ;

$\leadsto [*] := \text{E}\perp : \neg \text{Closed}(\mathbb{R}, A)$ ,

□

**PositiveRaysTopology** ::  $\text{Topology}(\mathbb{Z})$

**PositiveRaysTopology** () =  $\mathcal{T}_{+\infty} := \left\{ [n, \dots, +\infty) \mid n \in \mathbb{Z} \right\} \cup \{\emptyset, \mathbb{Z}\}$

**PositiveRayTopologyHasContinuousAddition** ::  $(+\mathbb{Z}) \in \text{TOP}\left((\mathbb{Z}, \mathcal{T}_{+\infty})^2, (\mathbb{Z}, \mathcal{T}_{+\infty})\right)$

**Proof** =

**Assume**  $U \in \mathcal{T}_{+\infty}$ ,

$(n, [1]) := \text{E}\mathcal{T}_{+\infty} : \sum n \in \mathbb{Z} . U = [n, +\infty)$ ,

$[2] := \text{E}(+\mathbb{Z})[1] : (+\mathbb{Z})^{-1}U = \left\{ (k, l) \in \mathbb{Z}^2 \mid k + l \geq n \right\}$ ,

$[3] := \Lambda(k, l) \in (+\mathbb{Z})^{-1}U[2] \text{SumIneq}[2] : \forall (k, l) \in (+\mathbb{Z})^{-1}U . [k, \dots, +\infty) \times (l, \dots, +\infty) \subset (+\mathbb{Z})^{-1}U$ ,

$[U.*] := \text{ProductTopologyBase}[3] : (+\mathbb{Z})^{-1}U \in \mathcal{T}\left((\mathbb{Z}, \mathcal{T}_{+\infty})^2\right)$ ;

$\leadsto [*] := \text{ETOP} : (+\mathbb{Z}) \in \text{TOP}\left((\mathbb{Z}, \mathcal{T}_{+\infty})^2, (\mathbb{Z}, \mathcal{T}_{+\infty})\right)$ ;

□

**PositiveRaysIsNotGroupTopology** ::  $(\mathbb{Z}, \mathcal{T}_{+\infty}) \notin \text{TGRP}$

**Proof** =

$[1] := \text{E}\mathcal{T}_{+\infty}(0) : [0, \dots, +\infty) \in \mathcal{T}_{+\infty}$ ,

$[2] := \text{Einv}_{\mathbb{Z}} : \text{inv}_{\mathbb{Z}}[0, \dots, +\infty) = (-\infty, \dots, 0] \notin \mathcal{T}_{+\infty}$ ,

$[3] := [1][2] : \text{inv}_{\mathbb{Z}} \notin \text{Aut}_{\text{TOP}}(\mathbb{Z}, \mathcal{T}_{+\infty})$ ,

$[*] := \text{ETGRP}[3] : (\mathbb{Z}, \mathcal{T}_{+\infty}) \notin \text{TGRP}$ ;

□

## 2.2 Further Topological Properties

Topological group structure simplifies work with some topological and metric concepts

### 2.2.1 Continuous Homomorphism

For topological groups it is especially easy to prove that morphism is continuous.

**PointContinuityImpliesContinuity** ::  
:  $\forall G, H \in \text{TGRP} . \forall g \in G . \forall \varphi \in \text{GRP} \ \& \ C_g(G, H) . \varphi \in \text{TGRP}(G, H)$   
**Proof** =  
**Assume**  $h \in \text{Im } \varphi$ ,  
 $(p, [1]) := \text{EIm } \varphi h : \sum p \in G . h = \varphi(p)$ ,  
**Assume**  $U \in \mathcal{U}(h)$ ,  
 $[2] := \text{ETGRP}(H) : \varphi(g)h^{-1}U \in \mathcal{U}(\varphi(g))$ ,  
 $[3] := \text{EC}_h(G, H)[2] : \varphi^{-1}(\varphi(g)h^{-1}U) \in \mathcal{U}(g)$ ,  
 $[h.*] := \text{ETGRP}(G)\text{EGRP}(G, H, \varphi)[1][3] : \varphi^{-1}(U) = pg^{-1}\varphi^{-1}(\varphi(g)h^{-1}U) \in U(p)$ ;  
 $\leadsto [*] := \text{ContinuityIsLocal} : \varphi \in \text{TGRP}(G, H)$ ;  
 $\square$

**IdentityOpenessImpliesOpeness** ::  $\forall G, H \in \text{TGRP} . \forall \varphi \in \text{GRP}(G, H) .$   
:  $(\forall U \in \mathcal{U}_G(e) . \varphi(U) \in \mathcal{U}_H(e)) \Rightarrow \text{OpenMap}(G, H, \varphi)$   
**Proof** =  
**Assume**  $g \in G$ ,  
**Assume**  $U \in \mathcal{U}_G(g)$ ,  
 $[1] := \text{ETGRP} : g^{-1}U \in \mathcal{U}_G(e)$ ,  
 $[2] := [0][1] : \varphi(g^{-1}U) \in \mathcal{U}_H(e)$ ,  
 $[g.*] := \text{EGRP}(G, H, \varphi)\text{ETGRP}(H) : \varphi(U) = \varphi(g)(g^{-1}U) \in \mathcal{U}_H(\varphi(g))$ ;  
 $\leadsto [*] := \text{OpennessIsLocal} : \text{OpenMap}(G, H, \varphi)$ ;  
 $\square$

**LeftUniformityUCCriterion** ::  $\forall G, H \in \mathbf{GRP} . \forall \varphi : G \rightarrow H . \varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right) \iff$

$$\iff \forall V \in \mathcal{U}_H(e) . \exists U \in \mathcal{U}_G(e) . \forall g \in G . \varphi(gU) \subset \varphi(g)V$$

**Proof** =

**Assume** [1] :  $\varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right),$

**Assume**  $B : \mathcal{U}_H(e),$

[2] :=  $\mathbf{EL}_H(V) : V_L \in \mathcal{L}_H,$

$(U, [3]) := \mathbf{EUNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right)(\varphi) : \sum U \in \mathcal{L}_G . (\varphi \times \varphi)U \subset V_L,$

$(W, [4]) := \mathbf{EBaseOfUniformity}\left(G, \mathcal{L}_G, (\mathcal{U}_G(e))_L, U\right) : \sum W \in \mathcal{U}_G(e) . W_L \subset U,$

**Assume**  $g \in G,$

[5] :=  $\mathbf{EW}_L(g) : gW = W_L(g),$

[1.\*] :=  $\mathbf{E}(=)\left([5], \varphi(gW)\right) \mathbf{EConnector}(G, W_L \& U) \mathbf{MonotonicImage}\left(G, H, W_L(g), U(g)[4]\right) [3] \mathbf{EV}_L(\varphi(g)) :$

$$\varphi(gW) = \varphi(W_L(g)) \subset \varphi(U(g)) \subset V_L(\varphi(g)) = \varphi(g)V;$$

$\leadsto [1] := \mathbf{I}(\Rightarrow) : \varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right) \Rightarrow \forall V \in \mathcal{U}_H(e) . \exists U \in \mathcal{U}_G(e) . \forall g \in G . \varphi(gU) \subset \varphi(g)V,$

**Assume** [2] :  $\forall V \in \mathcal{U}_H(e) . \exists U \in \mathcal{U}_G(e) . \forall g \in G . \varphi(gU) \subset \varphi(g)V,$

**Assume**  $V \in \mathcal{L}_H,$

$(W, [3]) := \mathbf{EBaseOfUniformity}\left(H, \mathcal{L}_H, (\mathcal{U}_H(e))_L, V\right) : \sum W \in \mathcal{U}_H(e) . W_L \subset V,$

$(U, [4]) := [2](W) : \sum U \in \mathcal{U}_G(e) . \forall g \in G . \varphi(gU) \subset \varphi(g)W,$

[V.\*] :=  $[4][3] : (\varphi \times \varphi)U_L \subset W_L \subset V;$

$\leadsto [2.*] := \mathbf{IUNI} : \varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right);$

$\leadsto [*] := \mathbf{I}(\iff)[1] :$

$$: \varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right) \iff \forall V \in \mathcal{U}_H(e) . \exists U \in \mathcal{U}_G(e) . \forall g \in G . \varphi(gU) \subset \varphi(g)V;$$

□

**TopologicalHomomorphismsAreUniformlyContinuous** ::

$$: \forall G, H \in \mathbf{TGRP} . \forall \varphi \in \mathbf{TGRP}(G, H) . \varphi \in \mathbf{UNI}\left((G, \mathcal{L}_G), (H, \mathcal{L}_H)\right)$$

**Proof** =

...

□

### 2.2.2 Metrization

Topological groups with a countable base of neighborhood can be metrized in a way compatible with their algebraic structure.

**TGRPTrisection** ::

::  $\forall G \in \text{TGRP} . \forall U \in \mathcal{U}(e) . \exists V \in \mathcal{U}(e) \ \& \ \text{SymmetricSet}(G) . VVV \subset U$

**Proof** =

...

□

**LeftGroupMetrization** ::  $\forall G \in \text{TGRP} . \forall \mathcal{N} : \text{NeighborhoodBase}(G, e) .$

.  $|\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \text{LeftInvariantMetric}(G) . (G, \rho) \cong_{\text{TOP}} G$

**Proof** =

$N := \text{enumerate}(\mathcal{N}) : \mathbb{N} \rightarrow \mathcal{N},$

$(V, [2],) := \text{rec2}\left(G, \Lambda n \in \mathbb{N} \Lambda U \in \mathcal{U}(e) . \text{TGRPTrisection}(G, U \cap N_n)\right) :$

$: \sum \mathbb{Z}_+ \rightarrow \mathcal{U}(e) \ \& \ \text{SymmetricSet}(G) . \forall n \in \mathbb{N} . V_n V_n V_n \subset U_{n-1},$

$\alpha := \Lambda g \in G . \inf\{2^{-n} | n \in \mathbb{Z}_+, g \in V_n\} : G \rightarrow \mathbb{R}_+,$

$[3] := \text{E}\alpha \text{E}\Lambda n \in \mathbb{Z}_+ . \text{ESymmetricSet}(G, V_n) : \forall g \in G . \alpha(g) = \alpha(g^{-1}),$

$[4] := \text{E}\alpha \text{E}\Lambda n \in \mathbb{Z}_+ . \text{E}V_n \mathcal{U}_G(e) : \alpha(e) = 0,$

$[5] := \text{E}\alpha[2] : \forall a, b, c \in G . \alpha(a, b, c) \leq 2 \max \alpha(\alpha(a), \alpha(b), \alpha(c)),$

$\beta := \Lambda g \in G . \inf \left\{ \sum_{i=1}^n \alpha(h_i h_{i-1}^{-1}) \mid n \in \mathbb{N}, h : \{0, \dots, n\} \rightarrow G, h_0 = e, h_n = g \right\} : G \rightarrow \mathbb{R}_{++},$

**Assume**  $h, g : G,$

**Assume**  $\varepsilon : \mathbb{R}_{++},$

$(n, a, [6]) := \text{E}\beta\left(g, \frac{\varepsilon}{2}\right) : \sum_{n=1}^{\infty} \sum_{\{0, \dots, n\} \rightarrow G} a_0 = e \ \& \ a_n = g \ \& \ \frac{\varepsilon}{2} + \sum_{i=1}^n \alpha(a_i a_{i-1}^{-1}) = \beta(g),$

$(m, b, [7]) := \text{E}\beta\left(h, \frac{\varepsilon}{2}\right) : \sum_{m=1}^{\infty} \sum_{\{0, \dots, m\} \rightarrow G} b_0 = e \ \& \ b_m = h \ \& \ \frac{\varepsilon}{2} + \sum_{i=1}^m \alpha(b_i b_{i-1}^{-1}) = \beta(h),$

$c := \text{concat}\left(b, a_{\{1, \dots, n\}} h\right) : \{0, \dots, n+m\} \rightarrow G,$

$\left[(h, g) . *\right] := \text{E}\beta(gh) \text{Ec}[6][7] : \beta(gh) \leq \sum_{i=1}^{n+m} \alpha(c_i c_{i+1}^{-1}) = \sum_{i=1}^n \alpha(a_i a_{i+1}^{-1}) + \sum_{i=1}^m \alpha(b_i b_{i+1}^{-1}) \leq \beta(g) + \beta(h) + \varepsilon;$

$\leadsto [6] := \text{I}\forall : \forall h, g \in G . \beta(gh) \leq \beta(g) + \beta(h),$

$[7] := \text{E}\beta \text{EBaseOfUniformity}(G, \mathcal{L}_G, \mathcal{N}) : \forall \mathcal{F} : \text{Filter}(G) . e \in \lim \mathcal{F} \iff 0 = \lim \beta(\mathcal{F}),$

$[*] := \text{TopologyByFilters} : (G, d_\beta) \cong_{\text{TOP}} G;$

□

**RightGroupMetrization** ::  $\forall G \in \text{TGRP} . \forall \mathcal{N} : \text{NeighborhoodBase}(G, e) .$

.  $|\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \text{RightInvariantMetric}(G) . (G, \rho) \cong_{\text{TOP}} G$

**Proof** =

...

□



**VeeGroupMetrization** ::  $\forall G \in \mathbf{TGRP} . \forall \mathcal{N} : \mathbf{NeighborhoodBase}(G, e) .$   
 $. |\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \mathbf{V-Semimetric}(G) . (G, \rho) \cong_{\mathbf{TOP}} G$

**Proof** =

...

□

### 2.2.3 Completeness

Completeness showcases some symmetry between key uniformities

**CauchyFilterInversion** ::  $\forall G \in \text{TGRP} . \forall \mathcal{F} : \text{FilterBase} .$   
 $. \forall \text{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}) \iff \text{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}^{-1})$   
**Proof** =  
**Assume** [1] :  $\text{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F})$ ,  
**Assume**  $U \in \mathcal{R}$ ,  
 $(V, [2]) := \text{ER}(U) : \sum V \in \mathcal{U}(e) . V_R \subset U$ ,  
 $(F, [3]) := \text{ECauchyFilterbase}(G, \mathcal{L}, \mathcal{F}, V_L^\top) : \sum F \in \mathcal{F} . F \times F \subset V_L^\top$ ,  
**Assume**  $(x, y) : V_L^\top$ ,  
 $[4] := \text{EV}_L : y^{-1}x \in V$ ,  
 $[(x, y). *] := \text{IV}_R[4] : (x^{-1}, y^{-1}) \in V_R$ ;  
 $\leadsto [4] := \text{I} \subset (\text{inv} \times \text{inv})(V_L^\top) \subset V_R$ ,  
 $[U.*] := [3][4][2] : F^{-1} \times F^{-1} \subset U$ ;  
 $\leadsto [*] := \text{ICauchyFilterbase} : \text{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}^{-1})$ ;  
 $\square$

**TwoSidedCauchyFilters** ::  $\forall G \in \text{TGRP} . \forall \mathcal{F} : \text{FilterBase} .$   
 $. \text{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}) \iff \text{CauchyFilterbase}(G, \mathcal{R} \& \mathcal{R}, \mathcal{F})$   
**Proof** =  
 $[*] := \text{ES}^\vee \text{SupUniformityCauchyFilterbase}(G, (\mathcal{L}, \mathcal{R})) :$   
 $: \text{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}) \iff \text{CauchyFilterbase}(G, \mathcal{R} \& \mathcal{R}, \mathcal{F})$ ;  
 $\square$

**TwoSidedCauchyFilterInversion** ::  
 $:: \forall G \in \text{TGRP} . \forall \mathcal{F} \in \text{CauchyFilterbase}(G, \mathcal{S}^\vee) . \text{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}^{-1})$   
**Proof** =  
 Combain two previous results.  
 $\square$

**LeftRightMutualCompleteness** ::  
 $:: \forall G \in \text{TGRP} . \text{CompleteUniformSpace}(G, \mathcal{L}) \iff \text{CompleteUniformSpace}(G, \mathcal{R})$   
**Proof** =  
 $\dots$   
 $\square$

**LeftOrRightCompletenessImPLYTwoSided** ::  
 $:: \forall G \in \text{TGRP} . \text{CompleteUniformSpace}(G, \mathcal{L}|\mathcal{R}) \Rightarrow \text{CompleteUniformSpace}(G, \mathcal{S}^\vee)$   
**Proof** =  
 $\dots$   
 $\square$

**CompleteByNbhd** ::

$$:: \forall G \in \text{TGRP} . \text{CompleteUniformSpace}(G, \mathcal{L}_G) \iff \exists N \in \mathcal{N}(e_G) . \text{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2)$$

**Proof** =

$$[1] := \Lambda T : \text{CompleteUniformSpace}(G, \mathcal{L}_G) . T :$$

$$: \text{CompleteUniformSpace}(G, \mathcal{L}_G) \Rightarrow \exists N \in \mathcal{N}(e_G) . \text{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2),$$

$$\text{Assume } N \in \mathcal{N}_G(e_G),$$

$$(U, [0]) := \text{EN}_G(e_G) : \sum U \in \mathcal{U}_G(e_G) . U \subset N,$$

$$\text{Assume } [2] : \text{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2),$$

$$\text{Assume } \mathcal{F} : \text{CauchyFilterbase}(G, \mathcal{L}_G),$$

$$(F, [3]) := \text{ECauchyFilterbase}(G, \mathcal{L}_G, \mathcal{F}, U_L) : \sum F \in \mathcal{F} . F \times F \subset U_L,$$

$$f := \text{EFilterbase}(G, \mathcal{F}, F) \in F,$$

$$[4] := \text{EU}_L[3] : f^{-1}F \subset U,$$

$$[5] := \text{EFilterbase}(G, \mathcal{F})[4] : \forall F' \in \mathcal{F} . f^{-1}F' \cap N \neq \emptyset,$$

$$[6] := \text{ETGRP}(G) \text{ICauchyFilterbase} : \text{CauchyFilterbase}(N, \mathcal{L}_G \cap N^2, f^{-1}\mathcal{F} \cap N),$$

$$(g, [7]) := \text{ECompleteUniformSpace}(N, \mathcal{L}_G \cap N^2, f^{-1}\mathcal{F} \cap N) : \sum g \in N . g \in \lim f^{-1}\mathcal{F} \cap N,$$

$$[\mathcal{F}.*] := \text{ETGRP}(G)[7] : fg \in \lim \mathcal{F};$$

$$\leadsto [N.*] := \text{ICompleteUniformSpace} : \text{CompleteUniformSpace}(G, \mathcal{L}_G);$$

$$\leadsto [*] := \text{I} \iff [1] :$$

$$: \text{CompleteUniformSpace}(G, \mathcal{L}_G) \iff \exists N \in \mathcal{N}(e_G) . \text{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2);$$

□

**LocallyCompactGroupIsComplete** ::  $\forall G \in \text{TGRP} \ \& \ \text{LocallyCompact} . \text{CompleteUniformSpace}(G, \mathcal{L})$

**Proof** =

...

□

**LocallyCompactGroupRegularity** ::  $\forall G \in \text{TGRP} \ \& \ \text{LocallyCompact} \ \& \ \text{T0} . \text{T4}(G)$

**Proof** =

$$\text{Assume } E : \text{Closed}(G),$$

$$\text{Assume } \phi \in \text{TOP}(E, \mathbb{R}),$$

$$(\hat{\text{id}}_E, [1]) := \text{OnePointCompactificationExtensition}(E, \text{id}) : \sum \hat{\text{id}}_E \in \text{UNI}(E^{\text{pt}}, E) . \hat{\text{id}}_{E|E} = \text{id}_E,$$

$$\Phi := \hat{\text{id}}\phi \in \text{TOP}(E^{\text{pt}}, \mathbb{R}),$$

$$(\hat{\Phi}, [2]) := \text{TietzeExtensionTheorem}(E^{\text{pt}}, \mathbb{R}, G^{\text{pt}}, \Phi) : \sum \hat{\Phi} \in \text{UNI}(G^{\text{pt}}, \mathbb{R}) . \hat{\Phi}|_{E^{\text{pt}}} = \Phi,$$

$$(\hat{\text{id}}_G, [4]) := \text{OnePointCompactificationExtensition}(E, \text{id}) : \sum \hat{\text{id}}_G \in \text{UNI}(G^{\text{pt}}, G) . \hat{\text{id}}_{G|G} = \text{id}_G,$$

$$\varphi := \hat{\text{id}}_G \hat{\Phi} : \text{UNI}(G, \mathbb{R}),$$

$$[\phi.*] := \text{E}\varphi : \varphi|_E = \phi;$$

$$\leadsto [*] := \text{TietzeExtensionTHM} : \text{T4}(G);$$

□

## 2.2.4 Completion

One can use symmetry properties of Cauchy filters mentioned above to show that separable completion of a separable group in its two-sided uniformity is a topological group with a two-sided uniformities again.

**LeftCauchyLemma** ::  $\forall G : \text{TGRP} . \forall \mathcal{F} : \text{CauchyFilterbase}(G, \mathcal{L}) . \forall U \in \mathcal{U}(e) . \exists F \in \mathcal{F} . F^{-1}F \subset U$

**Proof** =

...

□

**LeftCauchyFilterProduct** ::

$:: \forall G \in \text{TGRP} . \forall \mathcal{F}, \mathcal{F}' : \text{CauchyFilterbase}(G, \mathcal{L}) . \text{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}\mathcal{F}')$

**Proof** =

**Assume**  $U' \in \mathcal{L}$ ,

$(U, [1]) := \text{EL}(U') : \sum U \in \mathcal{U}(e) . U_L \subset U'$ ,

$(V, [2]) := \text{TGRPTrisection}(G, U) : V : \text{SymmetricSet}(G) . V \in \mathcal{U}(e) \ \& \ VVV \subset U$ ,

$(F', [3]) := \text{LeftCauchyLemma}(G, \mathcal{F}', V) : \sum F' \in \mathcal{F}' . F'^{-1}F' \subset V$ ,

$f := \text{EFilterbase}(G, \mathcal{F}', F') \text{E} \exists \in F'$ ,

$(W, [4]) := \text{TopologicalGroupAltDef}_4(G, V, f^{-1}) : \sum W \in \mathcal{U}(e) . f^{-1}Wf \subset V$ ,

$(F, [5]) := \text{LeftCauchyLemma}(G, \mathcal{F}, W) : \sum F \in \mathcal{F} . F^{-1}F \subset W$ ,

$[U' . * . 1] := \text{EsetProduct}(F, F') : FF' \in \mathcal{F}\mathcal{F}'$ ,

$[U' . * . 2] := \text{ProductInverse}(G) \text{InverseMeaning}^2(G, f)[5][4][3][2][1] :$

$: (FF')^{-1}FF' = F'^{-1}F^{-1}FF'^{-1} = F'^{-1}ff^{-1}F^{-1}Fff^{-1}F'^{-1} \subset F'^{-1}ff^{-1}Wff^{-1}F'^{-1} \subset F'^{-1}fVf^{-1}F'^{-1} \subset VVV \subset U \subset U'(e);$

$\leadsto [*] := \text{ELICauchyFilterbase} : \text{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}\mathcal{F}')$ ;

□

**RightCauchyFilterProduct** ::

$:: \forall G \in \text{TGRP} . \forall \mathcal{F}, \mathcal{F}' : \text{CauchyFilterbase}(G, \mathcal{R}) . \text{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}\mathcal{F}')$

**Proof** =

...

□

**UpperTwoSidedCauchyFilterProduct** ::

$:: \forall G \in \text{TGRP} . \forall \mathcal{F}, \mathcal{F}' : \text{CauchyFilterbase}(G, \mathcal{S}^\vee) . \text{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}\mathcal{F}')$

**Proof** =

...

□

**TopologicalGroupCompletion** ::  $\prod_{G \in \text{TGRP}} \sum_{H \in \text{TGRP}} \text{TGRP}(G, H)$

$\iota : \text{TopologicalGroupCompletion} \iff \text{Completion}\left((G, \mathcal{S}_G^\vee), (H, \mathcal{S}_H^\vee), \iota\right)$

**ContinuityByDenseSetAndPoints** ::  $\forall X \in \text{TOP} . \forall Y : \text{Regular} . \forall \phi : X \rightarrow Y . \forall D : \text{Dense}(X) .$   
 $\left( \forall x \in X . \phi|_{D \cup \{x\}} \in \text{TOP}(D \cup \{x\}, Y) \right) \Rightarrow \phi \in \text{TOP}(X, Y)$

**Proof** =

**Assume**  $x \in X$ ,

**Assume**  $V \in \mathcal{U}(x \ \phi)$ ,

$C := \bar{V} : \text{Closed}(Y)$ ,

$U := \phi^{-1}(V) \cap (D \cup \{x\}) : \mathcal{U}_{D \cup \{x\}}(x)$ ,

$(W, [2]) := \text{SubspaceTopology}(X, U) : \sum W \in \mathcal{U}(x) . U = W \cap (D \cup \{x\})$ ,

$[3] := \text{EWEC} : \phi(W \cap D) \subset V \subset C$ ,

**Assume**  $w \in W$ ,

**Assume**  $O \in \mathcal{U}(w \ \phi)$ ,

$I := \phi^{-1}(O) \cap (D \cup \{x\}) : \mathcal{U}_{D \cup \{w\}}(w)$ ,

$(E, [4]) := \text{SubspaceTopology}(X, U) : \sum E \in \mathcal{U}(x) . I = E \cap (D \cup \{w\})$ ,

$[5] := \text{EE} : \phi(E \cap D) \subset O$ ,

$[6] := \text{EEEWEDense}(X, D) : \exists (E \cap W \cap D)$ ,

$[7] := \text{ImageIntersection}[3][5] : \phi(W \cap E \cap D) \subset O \cap V$ ,

$[w.*] := [4][7]\text{IE} : \exists O \cap V$ ;

$\leadsto [x.*] := \text{IimageECClosureAltDef} : \phi(W) \subset C$ ,

$\leadsto [*] := \text{ContinuityIsLocalBase}(X, Y, \phi) \text{RegularHasClosedNbhdBase}(Y) : \phi \in \text{TOP}(X, Y)$ ;

□

**SeparatedTopologicalGroupHasComplition** ::  $\forall G \in \text{TGRP} \ \& \ \text{T0} . \exists \text{TopologicalGroupCompletion}(G)$

**Proof** =

$(H, \iota) := \text{SeparatedHasSeparatedCompletion}(G, \mathcal{S}_G^\vee) : \text{SeparatedCompletion}(G, \mathcal{S}_G^\vee)$ ,

**Assume**  $f, f' \in H$ ,

$(\mathcal{F}, F', [1]) := \text{ESeparatedCompletion}(G, H, \iota) :$

$: \sum \mathcal{F}, \mathcal{F}' : \text{CauchyFilterbase}(G, \mathcal{S}_G^\vee) . f = \lim \mathcal{F} \ \& \ f' = \lim \mathcal{F}'$ ,

$[2] := \text{UpperTwoSidedCauchyFilterProduct}(G, \mathcal{F}, \mathcal{F}') : \text{CauchyFilterbase}\left((G, \mathcal{S}_G^\vee), \mathcal{F}\mathcal{F}'\right)$ ,

$f \cdot_H f' := \lim \mathcal{F}\mathcal{F}' \in H$ ;

$\leadsto \cdot_H := \text{ETGRP}(G) \text{UNI}(G, H, \iota) : H \times H \rightarrow H$ ,

**Assume**  $f \in H$ ,

$(\mathcal{F}, [1]) := \text{ESeparatedCompletion}(G, H, \iota) : \sum \mathcal{F} : \text{CauchyFilterbase}\left((G, \mathcal{S}_G^\vee)\right) . f = \lim \mathcal{F}$ ,

$[2] := \text{TwoSidedCauchyFilterInversion}(G, \mathcal{F}) : \text{CauchyFilterbase}\left((G, \mathcal{S}_G^\vee), \mathcal{F}^{-1}\right)$ ,

$\text{inv}_H f := \lim \mathcal{F}^{-1} \in H$ ;

$\leadsto \text{inv}_H := \text{ETGRP}(G) \text{UNI}(G, H, \iota) : H \rightarrow H$ ,

$[1] := \text{ContinuityByDenseSetAndPoints}(H \times H, H) \text{E} \cdot_H : \cdot_H \in \text{TOP}(H \times H, H)$ ,

$[2] := \text{ContinuityByDenseSetAndPoints}(H, H) \text{E} \text{inv}_H : \text{inv}_H \in \text{TOP}(H, H)$ ,

$[*.1] := \text{ETGRP}(G) \text{ContinuityByDenseSetAndPoints}(\dots) \text{ITGRP} : H \in \text{TGRP}$ ,

$[*.2] := \text{E} \cdot_H \text{E} \text{inv}_H : \iota \in \text{TGRP}(G, H)$ ,

$[*.3] := \text{E} \cdot_H \text{E} \text{inv}_H : \mathcal{U}_H = \mathcal{S}_H^\vee$ ;

□

## 2.2.5 Baire's Category

Being Baire and topological group structure interplay nicely.

**ClopenSubgroupByNonemptyInterior** ::  $\forall G : \text{TGRP} . \forall H \subset_{\text{GRP}} G . \exists \text{int } H \Rightarrow \text{Clopen}(G, H)$

**Proof** =

$u := \mathbf{E}\exists[0] \in \text{int } H,$

$(U, [1]) := \mathbf{E} \text{int } H(u) : \sum U \in \mathcal{U}(u) . U \subset H,$

$[2] := \Lambda h \in H . \mathbf{E} \text{Aut}_{\text{TOP}}(G, \lambda_{hu^{-1}}, U) : \forall h \in H . hu^{-1}U \in \mathcal{U}(h),$

$[3] := \Lambda h \in H . \mathbf{E} \text{Subgroup}(G, H, hu^{-1}, U)[1] : \forall h \in H . hu^{-1}U \subset H,$

$[4] := \text{OpenByCover}[2][3] : \text{Open}(G, H),$

$[5] := \text{OpenSubgroupsAreClopen}[5] : \text{Open}(G, H);$

□

**ClopenSubgroupGeneration** ::  $\forall G \in \text{TGRP} . \forall U : \text{SymmetricSet} \ \& \ \text{Open}(G) . \text{Clopen}(G, \langle U \rangle_{\text{GRP}})$

**Proof** =

$H := \langle U \rangle_{\text{GRP}} : \text{Subgroup}(G),$

$[1] := \text{GeneratedSubgroupIsSuper}(G, U) \mathbf{I} H : U \subset H,$

$[2] := \mathbf{I} \text{interior}[1] : \exists \text{int } H,$

$[*] := \text{ClopenSubgroupByNonemptyInterior}[2] : \text{Clopen}(G, H);$

□

**FirstCategoryByClopenSets** ::  $\forall X \in \text{TOP} . \forall \mathcal{C} : \text{Disjoint}(\text{Clopen} \ \& \ \text{Meager}(X)) . X = \bigcup \mathcal{C} \Rightarrow \neg \exists^* X$

**Proof** =

$(N, [1]) := \mathbf{E} ? \text{Meager}(X) : \sum N : \mathcal{C} \rightarrow \mathbb{N} \rightarrow \text{NowhereDense}(X) . \forall C \in \mathcal{C} . C = \bigcup_{n=1}^{\infty} N_{C,n},$

$M := \Lambda n \in \mathbb{N} . \bigcup_{C \in \mathcal{C}} N_{C,n} : \mathbb{N} \rightarrow ?X,$

$[2] := \Lambda n \in \mathbb{N} . \mathbf{E} M_n \mathbf{E} \text{cl } \mathbf{E} \text{int } \Lambda C \in \mathcal{C} . \mathbf{E} \text{Clopen}(X, C) \mathbf{E} \text{NowhereDense}(X, N_{C,n}) \mathbf{E} \text{EmptySum}(X) :$   
 $: \forall n \in \mathbb{N} . \text{int cl } M_n = \text{int cl } \bigcap_{C \in \mathcal{C}} N_{C,n} = \bigcap_{C \in \mathcal{C}} \text{cl int } N_{C,n} = \bigcap_{C \in \mathcal{C}} \emptyset = \emptyset,$

$[3] := \mathbf{I} \text{NowhereDense}[2] : \forall n \in \mathbb{N} . \text{NowhereDense}(X, M_n),$

$[4] := [0][1] \text{UnionCommutativity}(X) \mathbf{I} M_n : X = \bigcup_{C \in \mathcal{C}} C = \bigcup_{C \in \mathcal{C}} \bigcap_{n=1}^{\infty} N_{C,n} = \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{C}} N_{C,n} = \bigcap_{n=1}^{\infty} M_n,$

$[*] := \mathbf{I} \text{Meager}[3][4] : \text{Meager}(X, X);$

□

**TGRP**BaireCondition ::  $\forall G \in \text{TGRP} . \text{BaireSpace}(X) \iff \exists^* X$

**Proof** =

[1] :=  $\Lambda T : \text{BaireSpace}(X) . \text{EBaireSpace}(X, X) \text{I} \exists^* : \text{BaireSpace}(X) \Rightarrow \exists^* X$ ,

**Assume** [2] :  $\exists^* X$ ,

**Assume** [3] :  $\neg \text{BaireSpace}(X)$ ,

$(U, [3]) := \text{EBaireSpace}(X) : \sum U \in \mathcal{T}(X) . \exists U \ \& \ \neg \exists^* U$ ,

$u := \text{E}\exists[3.1] \in U$ ,

$V := u^{-1}U \in \mathcal{U}(e)$ ,

$W := V \cap V^{-1} : \text{SymmetricSet}(X)$ ,

[4] :=  $\text{EW} : W \in \mathcal{U}(e) \ \& \ \text{Meager}(X, W)$ ,

$H := \langle W \rangle_{\text{GRP}} : \text{Subgroup}(G, H)$ ,

[5] :=  $\text{ClopenSubgroupGeneration}(G, W) \text{I} H : \text{Clopen}(G, H)$ ,

[6] :=  $\Lambda n \in \mathbb{N} . \text{MeagerOpenImage}\left(G^n, G, \prod, W^{\times n}\right) : \forall n \in \mathbb{N} . \text{Meager}(G, W^n)$ ,

[7] :=  $\text{EHMeagerCountableUnion}(G)[6] : \text{Meager}(G, H)$ ,

$\mathcal{C} := \{gH \mid g \in G\} : ?\left(\text{Clopen} \ \& \ \text{Meager}(G)\right)$ ,

[8] :=  $\text{DisjointCosets}(G) \text{EC} : \text{Disjoint}\left(\text{Clopen} \ \& \ \text{Meager}(G), \mathcal{C}\right)$ ,

[9] :=  $\text{EGRP}(G) \text{EC} : G = \bigcap_{C \in \mathcal{C}} C$ ,

[10] :=  $\text{FirstCategoryByClopenSets}(G, \mathcal{C}) : \neg \exists^* G$ ,

[3.\*] := [2][10] :  $\perp$ ;

$\leadsto$  [2.\*] :=  $\text{E}\perp : \text{BaireSpace}(G, H)$ ;

$\leadsto$  [\*] :=  $\text{I} \iff [1] : \text{BaireSpace}(G, H) \iff \exists^* G$ ;

□

**BaireGroup** := TGRP & BaireSpace : Type;

**DenseGdeltaSugGroupsAreUnique** ::  $\forall G : \text{BaireGroup} . \forall H \subset_{\text{GRP}} G . \text{Dense} \ \& \ G_\delta(G, H) . H = G$

**Proof** =

[1] :=  $\text{EDense} \ \& \ G_\delta(G, H) \text{I} \text{Meager}(G) : \text{Meager}\left(G, H^c\right)$ ,

**Assume** [2] :  $G \neq H$ ,

[3] :=  $\text{I}\lambda_G[2] : |\lambda_G G| > 1$ ,

[4] :=  $\text{ETGRPMeagerSubset}(G)[1][3] : \text{Meager}(G, H)$ ,

[5] :=  $\text{MeagerUnion}[1][4] : \text{Meager}(G, G)$ ,

[2] :=  $\text{NPGRBairConditionEBaireGroup}(G)[5] : \perp$ ;

$\leadsto$  [\*] :=  $\text{E}\perp : G = H$ ;

□

**GDeltaSubgroupIsClosed** ::  $\forall G : \text{BaireGroup} \ \& \ \text{CompleteMetricSpace} . \forall H \subset_{\text{GRP}} G .$   
 $. G_\delta(H) \Rightarrow \text{Closed}(G, H)$

**Proof** =

[1] := **ClosureOfSubgroup**( $G, H$ ) :  $\overline{H} \subset G$ ,  
[2] := **ClosedSubsetsAreComplete**( $G, \overline{H}$ ) :  $\text{CompleteMetricSpace}(\overline{H})$ ,  
[3] := **BairCategoryTHM2**[2] :  $\text{BaireSpace}(\overline{H})$ ,  
[4] := **DenseGdeltaSubgroupsAreUnique**( $\overline{H}, H$ ) :  $H = \overline{H}$ ,  
[5] := **ClosedByClosure**[4] :  $\text{Closed}(G, H)$ ;  
□

**DiscontiniousRealEndomorphismsHaveDenseGraphs** ::

$:: \forall \phi \in \text{End}_{\text{GRP}}(\mathbb{R}, +) . \phi \notin \text{End}_{\text{TOP}}(\mathbb{R}) \Rightarrow \text{Dense}(\mathbb{R}^2, G(\phi))$

**Proof** =

...  
□

**ContinuityByGraph** ::  $\forall \phi \in \text{End}_{\text{GRP}}(\mathbb{R}, +) . \phi \in \text{End}_{\text{TOP}}(\mathbb{R}) \iff G_\delta(\mathbb{R}^2, G(\phi))$

**Proof** =

...  
□



## 2.2.6 Connectedness

Connected groups can be generated by any neighborhood of unity. There is no proofs in this chapter, results are pretty trivial.

**ConnectedGroupGeneration** ::  $\forall G : \text{TGRP} \ \& \ \text{Connected} . \forall N \in \mathcal{N}(e) . G = \langle N \rangle_{\text{GRP}}$

**Proof** =

...

□

**ConnectedGroupCardinality** ::  $\forall G : \text{TGRP} \ \& \ \text{Connected} \ \& \ \text{T0} . G \neq \star \Rightarrow |G| > \aleph_0$

**Proof** =

...

□

**ConnectedSubgroupSeparabilityCondition** ::  $\forall G \in \text{TGRP} \ \& \ \text{Connected} . \forall U \in \mathcal{U}(e) .$   
     $\text{Separable}(U) \Rightarrow \text{Separable}(G)$

**Proof** =

...

□

**ConnectedGroupCountability** ::  $\forall G : \text{TGRP} \ \& \ \text{Connected} \ \& \ \text{FirstCountable} \ \& \ \text{LocallyCompact} .$   
     $\text{SecondCountable}(G)$

**Proof** =

...

□



## 2.2.7 Group of an Interval's Homeomorphisms

Group of an Interval's Homeomorphisms is an interesting exmple. it is complete in thir upper two-sided uniformity. But not in the on-sided ones. Theis shows that they don't have corresponding group completions.

**homeoAbsoluteValue** :: **AbsoluteValue**  $\left(\text{Aut}_{\text{TOP}}[0, 1]\right)$

**homeoAbsoluteValue**  $(f) = v(f) := \sup_{t \in [0, 1]} |t - f(t)|$

[1] := **Ev**(id)  $\Lambda t \in [0, 1]$  . **InverseMeaning**  $(\mathbb{R}, t)$  **E** | **E** **sup** :  $v(\text{id}) = \sup_{t \in [0, 1]} |t - t| = \sup_{t \in [0, 1]} 0 = 0$ ,

[2] :=  $\Lambda f \in \text{Aut}_{\text{TOP}}[0, 1]$  . **Ev**  $(f^{-1})$  **Substitution**  $([0, 1], f, y \mapsto f(x))$  **E** **AbsValue**  $(\mathbb{R}, | \bullet |)$  **I**  $v(f)$  :  
 $: v(f^{-1}) = \sup_{y \in [0, 1]} |y - f^{-1}(y)| = \sup_{x \in [0, 1]} |f(x) - x| = \sup_{x \in [0, 1]} |x - f(x)| = v(f)$ ,

[3] :=  $\Lambda f, g \in \text{Aut}_{\text{TOP}}[0, 1]$  **Ev**  $(fg)$  **Substitution**  $([0, 1], f^{-1}, x \mapsto f^{-1}(y))$  **TriangleIneq**  $(\mathbb{R}, | \bullet |, f^{-1}(y), y, g(y))$   
**SupSumIneq**  $(\mathbb{R})$  **I**  $v(f^{-1})$  **I**  $v(g)$  [2] :  $v(fg) = \sup_{x \in [0, 1]} |x - fg(x)| = \sup_{y \in [0, 1]} |f^{-1}(y) - g(y)| \leq$   
 $\leq \sup_{y \in [0, 1]} |f^{-1}(y) - y| + |y - g(y)| \leq \sup_{y \in [0, 1]} |y - f^{-1}(y)| + \sup_{y \in [0, 1]} |y - g(y)| \leq v(f^{-1}) + v(g) \leq v(f) + v(g)$ ,

**Assume**  $f : \mathbb{N} \rightarrow \text{Aut}_{\text{TOP}}[0, 1]$ ,

**Assume** [4] :  $\lim_{n \rightarrow \infty} v(f_n) = 0$ ,

**Assume**  $g \in \text{Aut}_{\text{TOP}}[0, 1]$ ,

$[f.*] := \lim_{n \rightarrow \infty} \text{Ev}(g^{-1} f_n g) \text{Substitution}([0, 1], g, x \mapsto g(y)) \text{E} \text{Aut}_{\text{UNI}}([0, 1], g) [4]$  :  
 $: \lim_{n \rightarrow \infty} v(g^{-1} f_n g) = \lim_{n \rightarrow \infty} \sup_{x \in [0, 1]} |x - g^{-1} f_n g(x)| = \lim_{n \rightarrow \infty} \sup_{x \in [0, 1]} |g(y) - f_n g(y)| = 0$ ;

$\leadsto [*] := \text{IAbsoluteValue} : \text{AbsoluteValue}(\text{Aut}_{\text{UNI}}[0, 1], v)$ ;

□

**HomeoAbsValueProducesUniformMetric** ::  $\forall f, g \in \text{Aut}_{\text{TOP}}[0, 1] . d_v(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$

**Proof** =

$[*] := \text{Ed}_v \text{EvSubstitution}([0, 1], f, f(x) \mapsto x)$  :  
 $: d_v(f, g) = v(g^{-1} f) = \sup_{x \in [0, 1]} |x - g^{-1} f(x)| = \sup_{y \in [0, 1]} |g(y) - f(y)|$ ;

□

**HomeoAreNotLeftComplete** ::  $\neg \text{CompleteUniformSpace}(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{L})$

**Proof** =

**Assume** [1] : **CompleteUniformSpace**  $(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{L})$ ,

[2] := **HomeoAbsValueProducesUniformMetric** **E** **CompleteUniformSpace**  $(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{L})$  **IClosed** :  
 $: \text{Closed}(\text{End}_{\text{TOP}}[0, 1], \text{Aut}_{\text{TOP}}[0, 1])$ ,

$f := \Lambda n \in \mathbb{N} . \Lambda t \in [0, 1] . 2 \sqrt[n]{\frac{1}{2} t} \left[ t < \frac{1}{2} \right] + \sqrt[n]{t} \left[ t \geq \frac{1}{2} \right] : \mathbb{N} \rightarrow \text{Aut}_{\text{TOP}}[0, 1]$ ,

[3] := **E**  $\lim_{n \rightarrow \infty} f_n : \lim_{n \rightarrow \infty} f_n = 2t \left[ t < \frac{1}{2} \right] + \left[ t \geq \frac{1}{2} \right] \notin \text{Aut}_{\text{TOP}}[0, 1]$ ,

$[1.*] := \text{ClosedHasLimits}(\text{Aut}_{\text{TOP}}[0, 1]) [2] [3] : \perp$ ;

$\leadsto [*] := \text{E} \perp : \neg \text{CompleteUniformSpace}(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{L})$ ,

□

**HomeoIsUpperComplete** :: **CompleteUniformSpace** $\left(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{S}^\vee\right)$

**Proof** =

**Assume**  $f : \text{CauchySeq}\left(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{S}^\vee\right),$

$[1] := \text{TwoSidedCauchyFilters}(\text{Aut}_{\text{TOP}}[0, 1], v, f) :$

$: \text{CauchySeq}\left(\text{Aut}_{\text{TOP}}[0, 1], \|\bullet\|_\infty, f\right) \ \& \ \text{CauchySeq}\left(\text{Aut}_{\text{TOP}}[0, 1], \|\bullet\|_\infty, f^{-1}\right),$

$(g, [2]) := \text{ECompleteUniformSpace}\left(\text{End}_{\text{TOP}}[0, 1], \|\bullet\|_\infty, f\right) : \sum g \in \text{End}_{\text{TOP}}[0, 1] \cdot \lim_{n \rightarrow \infty} f_n = g,$

$(h, [3]) := \text{ECompleteUniformSpace}\left(\text{End}_{\text{TOP}}[0, 1], \|\bullet\|_\infty, f^{-1}\right) : \sum h \in \text{End}_{\text{TOP}}[0, 1] \cdot \lim_{n \rightarrow \infty} f_n^{-1} = h,$

**Assume**  $x : [0, 1],$

**Assume**  $\varepsilon : \mathbb{R}_{++},$

$(N, [4]) := \text{Ev}[3](\varepsilon) : \sum N \in \mathbb{N} \cdot \forall n \geq N \cdot \left|gh(x) - gf_n^{-1}(x)\right| < \frac{\varepsilon}{2},$

$(M, [5]) := \text{Ev}[2](\varepsilon) : \sum M \in \mathbb{N} \cdot \forall m \geq M \cdot \left|gf_m^{-1} - x\right| < \frac{\varepsilon}{2},$

$n := \max(N, M) \in \mathbb{N},$

$[x.*] := \text{TriangleIneq}[4][5] : \left|gh(x) - x\right| \leq \left|gh(x) - gf_n^{-1}(x)\right| + \left|gf_n^{-1} - x\right| < \varepsilon;$

$\leadsto [4] := \text{Iinv} : gh = \text{id},$

$[5] := \text{ByAnalogy}[4] : hg = \text{id},$

$[f.*] := \text{EAut}_{\text{TOP}}[0, 1][4][5] : g \in \text{Aut}_{\text{TOP}}[0, 1];$

$\leadsto [*] := \text{ICompleteUniformSpace} : \text{CompleteUniformSpace}\left(\text{Aut}_{\text{TOP}}[0, 1], v, \mathcal{S}^\vee\right);$

□

**HomeoHasDistinctUniformities** ::  $\mathcal{L}_v \neq \mathcal{S}_v^\vee \ \& \ \mathcal{L}_v \neq \mathcal{R}_v$

**Proof** =

...

□

**HomeoHasNoLeftGroupCompletion** ::  $\neg \exists \text{TopologicalGroupCompletion}\left(\text{End}_{\text{TOP}}[0, 1], v, \mathcal{L}\right)$

**Proof** =

...

□

## 2.3 Further Group Properties[0]

This chapter will include topics such initial and final construcures, quotients, group action, free groups. It will be written on demand in a first return.

## 2.4 Some Analytic Properties[1]

Many famous theorems of functional analysis can be proved for topological groups. This chapter will be written on demand in a first return. Prerequisite: Further Group Properties

## 2.5 Representation[5]

Results about topological groups can be applied to their representations. Prerequisite: Further Group Properties, Some Analytic Properties, Haar measure, group representations

### 2.5.1 Continuous Characters[5]

Characters of topological groups are Also Continuous

$$\text{dualGroup} :: \text{TGRP} \rightarrow \text{TGRP}$$

$$\text{dualGroup}(G) = G^* := \text{TGRP}(G, \mathbb{S}^1)$$

### 2.5.2 Keller's Theorem[6]

Keller's theorem states that every compact convex body in a Hilbert space is homeomorphic to a Hilbert cube. In 1993 Agaev published a proof, which uses representation of topological groups as a main tool. Prerequisite: Spectral theory

## 2.6 Almost Metrizable Groups[ $\infty$ ]

Almost metrizable spaces are those where every compact admits a countable system of neighborhood. Almost metrizable groups are of some esoteric interest.

## 3 Polish Groups

### 3.1 Basisc

#### 3.1.1 Definition and Examples

`PolishGroup := TGRP & Polish : Type;`

`cli :: ?PolishGroup`

$G : \text{cli} \iff \exists \rho : \text{LeftInvariantMetric}(G) . \text{Complete}(G, \rho) \ \& \ (G, \rho) \cong_{\text{TOP}} G$

`DiscreteIsCli ::  $\forall G \in \text{TGRP} . |G| \leq \aleph_0 \ \& \ \text{Discrete}(G) \Rightarrow \text{cli}(G)$`

`Proof =`

...

□

`NiceIsCli ::  $\forall G \in \text{TGRP} . \text{FirstCountable} \ \& \ \text{LocallyCompact} \ \& \ \text{T0} . \text{cli}(G)$`

`Proof =`

First countable topological groups are LIM metrizable.

First countable topological groups are second countable.

Second countable and locally compact metrizable spaces are polish.

Locally compact topological groups are complete in their left uniformity, and hence LIM-complete. □

`RealsAndComplexAreCli :: cli( $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$ ,  $(\mathbb{R} \setminus \{0\}, *)$ ,  $(\mathbb{C} \setminus \{0\}, *)$ )`

`Proof =`

Result for additive groups is well known fact of Reals Analysis

For multiplicative groups nice is cli.

□

`ProductOfPolishGroupsIsPolish ::  $\forall n \in \sigma(\omega) . \forall G : n \rightarrow \text{PolishGroup} . \text{PolishGroup} \left( \prod_{i=1}^n G \right)$`

`Proof =`

product of topological groups is a topological group.

Countable product of polish spaces is polish.

□



**UnitaryOperatorsAreCli** ::  $\forall V \in \mathbb{C}\text{-HIL} \ \& \ \text{Separable} . \text{cli}(\mathbf{U}(V))$

**Proof** =

[1] :=  $\Lambda A, B, C \in \mathbf{U}(V) . \Lambda x \in V . \text{ERING}(\mathbf{B}(V)) \text{EIsometry}(V, C) :$   
 $: \forall A, B, C \in \mathbf{U}(V) . \forall x \in V . \|(CA - CB)x\| = \|C(A - B)x\| = \|(A - B)x\|,$   
[2] :=  $\text{IoperatorNorm}(V)[1] : \forall A, B, C \in \mathbf{U}(V) . \|C(A - B)\| = \|A - B\|,$   
 $\sim [3] := \text{ILeftInvariantMetric}[2] : \text{LeftInvariantMetric}(\mathbf{U}(V), \Lambda A, B \in \mathbf{U}(V) . \|A - B\|),$   
[4] :=  $\text{MultiplicationContinuousOnBoundedSets}(V, \mathbf{U}(V)) \text{AdjoiningIsContinuous}(V) \text{ITGRP} :$   
 $. \mathbf{U}(V) \in \text{TGRP},$   
[5] :=  $\text{EU}(V) : \mathbf{U}(V) = \left( \Lambda A \in \mathbb{S}(\mathcal{B}(V)) . A^*A \right)^{-1}(\text{id}) \cap \left( \Lambda A \in \mathbb{S}(\mathcal{B}(V)) . A^*A \right)^{-1}(\text{id}),$   
[6] :=  $\text{MultiplicationContinuousOnBoundedSets}(V, \mathbf{U}(V)) \text{IClosed} : \text{Closed}(\mathcal{B}(V), \mathbf{U}(V)),$   
[7] :=  $\text{ClosedSetsAreGDelta}[5] : G_\delta(\mathcal{B}(V), \mathbf{U}(V)),$   
[8] :=  $\text{CompleteSubsetIsGDelta} : \text{Complete}(\mathbf{U}(V)),$   
[\*] :=  $\text{Icli}[7][4][3] : \text{cli}(\mathbf{U}(V));$

□

**UnitaryOperatorsAreNotLocallyCompact** ::  $\forall V : \mathbb{C}\text{-HIL} \ \& \ \text{Separable} . \neg \text{LocallyCompact}(\mathbf{U}(V))$

**Proof** =

Fix any  $v \in V, \|v\| = 1$ , then the evaluation map  $\epsilon_v$  is a bounded, surjective operator on  $\mathcal{B}(V)$ .  
Then by open mapping theorem,  $\epsilon_v$  is an open mapping.  
As  $\epsilon_v$  also surjective it preserves local compactness.  
 $\epsilon_v$  maps  $\mathbf{U}(V)$  onto  $\mathbb{S}(0, 1)$ , but Spheres are not locally Compact in  $V$

□

**CompactMetrizizableHomeoArePolish** ::  $\forall X : \text{Compact} \ \& \ \text{Metrizable}(X) . \text{PolishGroup}(\text{Aut}_{\text{TOP}}(X))$

**Proof** =

...

□

**HomeoOfIntervalAreNotCli** ::  $\neg \text{cli}(\text{Aut}_{\text{TOP}}[0, 1])$

**Proof** =

□

**IsometryGroupIsCLI** ::

$:: \forall X \in \text{MS} . \text{Complete} \ \& \ \text{Separable}(X) \Rightarrow \text{cli}(\text{Aut}_{\text{MS} \rightarrow \cdot}(X), \mathcal{W}(X, X, \epsilon))$

**Proof** =

...

□

**InfinitePermutationsArePolishGroup** :: **PolishGroup**( $S_\infty$ )

**Proof** =

...

□

**IsometryGroupIsCompact** ::  $\forall X \in \mathbf{MS} . \mathbf{Compact}(X) \Rightarrow \mathbf{CompactSubset}(\mathbf{Aut}_{\mathbf{TOP}}(X), \mathbf{Aut}_{\mathbf{MS}_{\circ \rightarrow}}(X))$

**Proof** =

Assume  $f$  is a sequence of isometries.

Let  $(E_n)_{n=1}^\infty$  be a sequence of finite  $\frac{1}{n}$ -nets, such that  $E_{n+1} \subset E_n$ .

Then, it is possible to select a collection  $(g^n)_{n=1}^\infty$  of subsequences of  $f$

Let  $g^n$  be converging on each  $x \in X_n$  to some  $L(x)$  and  $\forall m \in \mathbb{N} . d(L(x), g_m^n) \leq \frac{1}{m}$ .

This is possible as  $X$  is compact and  $E_n$  is finite.

Set  $h_n = g_n^n$ , Then  $h$  is converging  $D = \bigcup_{n=1}^\infty E_n$  to an isometry  $L$ .

But  $D$  is dense in  $X$  and  $L$  is uniformly continuous, so there is an extension  $\widehat{L}$  over whole  $X$ .

$\widehat{L}$  is an isometry as metric is continuous.

We show that  $h$  converges to  $L$  pointwise.

Assume  $x \in X$ , assume  $\varepsilon \in \mathbb{R}_{++}$ .

Then there is an  $y \in D$  such that  $d(x, y) < \varepsilon$ .

Also there is an  $N \in \mathbb{N}$ , such that  $d(L(y), h_n(y)) < \varepsilon$  for all  $n \geq N$ . Let  $n$  be such.

So,  $d(\widehat{L}(x), h_n(x)) \leq d(\widehat{L}(x), \widehat{L}(y)) + d(\widehat{L}(y), h_n(y)) + d(h_n(y), h_n(x)) = 2d(y, x) + d(L(y), h_n(y)) < 3\varepsilon$ .

Hence,  $f$  has a subsequence which converges to  $L$  pointwise.

As  $f$  was arbitraty, it means that group of isometries is compact.

□

**NatIdealIsGDelta** ::  $\forall I : \mathbf{Ideal}(\mathbb{N}) . G_\delta(\mathcal{C}, I) \Rightarrow \mathbf{Closed}(\mathcal{C}, I)$

**Proof** =

Cantor's space seen as  $\mathcal{C} = {}^\omega\mathbb{N} = \prod_{n=1}^\infty \frac{\mathbb{Z}}{2\mathbb{Z}}$  is a polish group, hence Baire.

Note, that ideals are subgroups for this structure.

So, from the theorem in chapter 2.2.5 of this treatise the result follows.

□

**FrechetIdealOfNatIsOnlyFSigma** ::  $F_\sigma(\mathcal{C}, \mathbf{Finite}(\mathbb{N})) \& \neg G_\delta(\mathcal{C}, \mathbf{Finite}(\mathbb{N}))$

**Proof** =

Set of finite subsets is countable and  $\mathcal{C}$  is separated, so it is  $F_\sigma$ .

By previous result, if Frechet Ideal was  $G_\delta$  it would be Closed.

But Finite subsets are dense in  $\mathcal{C}$  so this contradicts the definition of Frechet's ideal.

□



### 3.1.2 Baire Groups Redux

Polish groups are bair, so bair properties will be useful.

**PettisBPTTheorem** ::  $\forall G \in \text{BaireGroup} . \forall A \in \mathbf{BP}(G) . \exists^* A \Rightarrow \exists U \in \mathcal{U}(e) . U \subset A^{-1}A$

**Proof** =

$(U, E, [1]) := \mathbf{E}\exists^* A : \sum U \in \mathcal{T}(G) . \sum E : \text{Meager}(G) . A = U \triangle E \ \& \ \exists U,$

$([2]) := \mathbf{E}\exists^* A : \sum U \in \mathcal{T}(G) . \sum E : \text{Meager}(G) . A = U \triangle E \ \& \ \exists^* U,$

$g := \mathbf{E}\exists^* U \in U \cap A,$

$(V, [3]) := \text{TopologicalGroupAltDef}(G, g^{-1}U) : \sum V \in \mathcal{U}(e) . VV^{-1} \subset g^{-1}U,$

$[4] := \text{ESetProduct}[3] : \forall v \in V . gV \subset U \cap Uv,$

**Assume**  $v \in V,$

$[6] := \text{CheckingBooleanTables} : (U \cap Uv) \triangle (A \cap Av) \subset (U \triangle A) \cup (U \triangle A)v = E \cup Ev,$

$[7] := \text{MeagerUnion}(G)\text{MeagerSubset}(G)[6] : \text{Meager}(G, (U \cap Uv) \triangle (A \cap Av)),$

$[v.*] := \Lambda T : A \cap Av = \emptyset . T[7]\text{EBaireSpace}(G)\mathbf{E}\perp : A \cap Av \neq \emptyset;$

$\leadsto [5] := \mathbf{I}\forall : \forall v \in V . A \cap Av \neq \emptyset,$

$[*] := \text{ISetPtooduct} : V \subset A^{-1}A;$

□

**BaireGroupMeasurableIsContinuous** ::

$: \forall G : \text{BaireGroup} . \forall H \in \text{TGRP} \ \& \ \text{Separable} . \forall \phi \in \text{GRP} \ \& \ \text{BairMeasurable}(G, H) . \phi \in \text{TOP}(G, H)$

**Proof** =

$(h, [1]) := \text{ESeparable}(H) : \sum \mathbb{N} \rightarrow h . \text{Dense}(H, \text{Im } h),$

**Assume**  $U : \mathcal{U}_H(e),$

$(V, [2]) := \text{TopologicalGroupAltDef}(H, U) : \sum V \in \mathcal{U}_H(e) . V^{-1}V \subset U,$

$[3] := \text{EDense}[1](V) : H = \bigcup_{n=1}^{\infty} h_n V,$

$[4] := \text{UniversalPreimage}(G, H, \phi)[3]\text{PreimageUnion}(G, H, \phi) :$

$: G = \phi^{-1}(H) = \phi^{-1}\left(\bigcup_{n=1}^{\infty} h_n V\right) = \bigcup_{n=1}^{\infty} \phi^{-1}(h_n V),$

$(n, [5]) := \text{EBaireSpace}(G)[4] : \sum_{n=1}^{\infty} \exists^* \phi^{-1}(h_n V),$

$(W, [6]) := \text{PettisBPTTheorem}(G, \phi^{-1}(h_n V)) : \sum W \in \mathcal{U}_G(e) . W \subset \left(\phi^{-1}(h_n V)\right)^{-1} \phi^{-1}(h_n V),$

$[U.*] := \phi([6])[2] : \phi(W) \subset V^{-1}V \subset [U];$

$\leadsto [2] := \mathbf{I}C_e : \phi \in C_e(G, H),$

$[*] := \text{PointContinuityImplyContinuity} : \phi \in \text{TOP}(G, H);$

□

**NonMeagerBPSubgroupIsClopen** ::  $\forall G : \text{BaireGroup} . \forall H \subset_{\text{GRP}} G . \mathbf{BP}(H) \ \& \ \exists^* H \Rightarrow \text{Clopen}(H)$

**Proof** =

By Pettis Theorem there is an open  $U \subset HH^{-1} = H$  .

But, then  $H = UH$  is open, and hence clopen .

□

**RealSetWithoutBPExists** ::  $\mathbf{BP}^c(\mathbb{R}) \neq \emptyset$

**Proof** =

Let  $h$  be a Hamel basis for  $\mathbb{R}$  taken as  $\mathbb{Q}$ -vector space. .

Define  $A = \{a \in \mathbb{R} | a_1 = 0\}$  .

Then  $\mathbb{R}$  is a countable union of translates of  $A$ , so  $A$  can't be meager .

Then, if  $A$  has Baire property it must be clopen by the previous remark .

But  $\mathbb{R}$  are connected, producing a contradiction .

□

**ContinuousActionTheorem** ::

$\forall G \in \mathbf{GRP} . \forall \mathcal{T} : \mathbf{Topology}(G) . \forall X : \mathbf{Metrizable} . \forall \alpha \in \mathbf{GRP} \ \& \ \mathbf{TOP}\big((G, \mathcal{T}), \mathbf{Aut}_{\mathbf{TOP}}(X)\big) .$

$. \mathbf{BaireSpace} \ \& \ \mathbf{Metrizable}(X, \mathcal{T}) \ \& \ \left( \forall g \in G . \lambda_g \in \mathbf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big) \right) \Rightarrow \alpha \in \mathbf{TOP}\big((G, \mathcal{T}) \times X, X\big)$

**Proof** =

Use jpoint continuity theorem from descriptive set theory..

□

**TopologicalGroupBySeparateContinuity** ::

$\forall G \in \mathbf{GRP} . \forall \mathcal{T} : \mathbf{Topology}(G) .$

$. \mathbf{BaireSpace} \ \& \ \mathbf{Metrizable}(X, \mathcal{T}) \ \& \ \mathbf{inv}_G \in \mathbf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big) \ \&$

$\ \& \ \left( \forall g \in G . \lambda_g, \rho_g \in \mathbf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big) \right) \Rightarrow (G, \mathcal{T}) \in \mathbf{TGRP}$

**Proof** =

...

□

**MillerStabilizerTHM** ::  $\forall G : \text{BaireGroup} . \forall X : \text{T1} \ \& \ \text{SecondCountable} . \forall \alpha \in \text{GRP}(G, \text{Aut}_{\text{SET}}(X)) .$

$. \left( \forall H \subset_{\text{GRP}} G . \text{Closed}(G, H) \Rightarrow \text{BaireSpace}(H) \right) \ \&$   
 $\ \& \ \left( \forall x \in X . \forall H \subset_{\text{GRP}} G . \text{Closed}(G, H) \Rightarrow \text{BairMeasurable}(H, X, \Lambda h \in H . \alpha(h)(x)) \right) \Rightarrow$   
 $\Rightarrow \forall x \in X . \text{Closed}(G, \text{Stab}(\alpha, x))$

**Proof** =

$H := \overline{\text{Stab}(\alpha, x)} : \text{Closed}(G),$

$[1] := \text{SubgroupClosure}(H) : \text{Subgroup}(G, H),$

$[2] := [0.1][1] : \text{BaireGroup}(H),$

$[4] := \text{DenseInAClosure}(G, \text{Stab}(\alpha, x)) : \text{Dense}(H, \text{Stab}(\alpha, x)),$

**Assume**  $[5] : \exists^* \text{Stab}(\alpha, x),$

$[6] := \text{NonMeagerBPSubgroupIsClopen}[5] : \text{Clopen}(H, \text{Stab}(\alpha, x)),$

$[5.*] := [4][6] : H = \text{Stab}(\alpha, x);$

$\leadsto [5] := \text{I}(\rightarrow) : \exists^* \text{Stab}(\alpha, x) \rightarrow H = \text{Stab}(\alpha, x),$

**Assume**  $[6] : \neg \exists^* \text{Stab}(\alpha, x),$

$(V, [7]) := \text{ESecondCountable}(X) : \sum V : \mathbb{N} \rightarrow \mathcal{T}(X) . \text{BaseOfTopology}(X, \text{Im } V),$

$[8] := \text{BaseSeparation}(X, V) : \forall x, y \in X . x \neq y \rightarrow \exists n, m \in \mathbb{N} . x \in V_n \ \& \ y \in V_m \ \& \ V_m \cap V_n = \emptyset,$

$\phi := \Lambda h \in H . \alpha(h)(x) : G \rightarrow X,$

$A := \phi^{-1}(V) : \mathbb{N} \rightarrow \mathbf{BP}(H),$

$[9] := \text{EStabEA} : \forall h \in H . \left( \forall n \in \mathbb{N} . A_n h = A_n \right) \iff h \in \text{Stab}(\alpha, x),$

$[10] := \text{EA} : \forall h \in H . h \text{Stab}(\alpha, x) = \bigcap \{A_n | n \in \mathbb{N}, g \in A_n\},$

$[11] := \text{FirstTopological01Law}[9] : \forall n \in \mathbb{N} . \forall^* A_n \Big| \neg \exists^* A_n,$

$[12] := \text{EBaireGroup}(H)[6][11][10] : \forall h \in H . \exists n \in \mathbb{N} . h \in A_n \ \& \ \neg \exists^* A_n,$

$[13] := \text{MeagerUnion} : \neg \exists H,$

$[6.*] := \text{EBaireGroup}(H)[12] : \perp;$

$\leadsto [*] := \text{E}\perp\text{E}[5] : H = \text{Stab}(\alpha, x);$

□

3.1.3 Universal Polish Group

The proof of this result is rather convoluted and we may need Keller’s theorm for it.

UspenskiTHM ::  $\forall G : \text{PolishGroup} . \exists H \subset_{\text{GRP}} \text{Aut}_{\text{TOP}}[0,1]^{\mathbb{N}} . H \cong_{\text{TGRP}} G$   
Proof =  
...  
 $\square$

$$G^* := \text{Aut}_{\text{TOP}}[0,1]^{\mathbb{N}} : \text{TGRP};$$

### 3.1.4 Selectors and Transversal

**Selector** ::  $\prod_{X \in \text{SET}} \text{Equivalence}(X) \rightarrow ?(X \rightarrow X)$

$\sigma : \text{Selector} \iff \Lambda E : \text{Equivalence}(X) . \forall C : \text{EquivalenceClass}(X, E) . \forall x, y \in C . \sigma(x) = \sigma(y) \ \& \ \sigma(x) \in$

**Transversal** ::  $\prod_{X \in \text{SET}} \text{Equivalence}(X) \rightarrow ??X$

$T : \text{Transversal} \iff \Lambda E : \text{Equivalence}(X) . \forall C : \text{EquivalenceClass}(X, E) . |C \cap T| = 1$

**selectorAsTransversal** ::  $\prod_{X \in \text{SET}} \prod E : \text{Equivalence}(X) . \text{Selector}(X, E) \rightarrow \text{Transversal}(X, E)$

**selectorAsTransversal**  $(\sigma) = \sigma := \text{Im } \sigma$

**transversalAsSelector** ::  $\prod_{X \in \text{SET}} \prod E : \text{Equivalence}(X) . \text{Transversal}(X, T) \rightarrow \text{Transversal}(X, T)$

**transversalAsSelector**  $(T) = T := \Lambda x \in X . \text{ESingleton}(X) \text{ETransversal}(T, [x])$

**saturation** ::  $\prod_{X \in \text{SET}} \text{Equivalence}(X) \rightarrow ?X \rightarrow ?X$

**saturation**  $(E, A) = [A]_E := \{x \in X : \exists a \in A : xEa\}$

**BorelSelectors** ::  $\forall X : \text{Polish} . \forall E : \text{Equivalence} .$

$. \forall [0.1] : \forall C : \text{EquivalenceClass}(X, E) . \text{Closed}(X, C) .$

$. \forall [0.2] : \forall U \in \mathcal{T}(X) . [U]_E \in \mathcal{B}(X) .$

$. \exists \sigma : \text{Selector}(X) . \sigma \in \text{End}_{\text{BOR}}(X)$

**Proof** =

$\varphi := \Lambda x \in X . [x]_E : X \rightarrow \text{EFF}(X),$

$[1] := \Lambda U \in \mathcal{T}(X) . \text{E}\varphi\text{I}[U]_E[0.2] : \forall U \in \mathcal{T}(X) . \varphi^{-1}\{A \in \text{EFF}(X) : \exists A \cap U\} = [U]_E \in \mathcal{B}(X),$

$[2] := \text{IBOR}[1] : \varphi \in \text{End}_{\text{BOR}}(X);$

$(\delta, [3]) := \text{SelectionTHM}(X) : \sum \delta : \mathbb{N} \rightarrow \text{BOR}(X, \text{EFF}(X)) . \forall A \in \text{EFF}(X) . \text{Dense}(A, \delta_{\mathbb{N}}(A)),$

$\sigma := \varphi\delta_1 : \text{BOR}(X, X),$

$[*] := \text{E}\sigma[3] : \text{Selector}(X, E, \sigma);$

□

**SubgroupSelector** ::  $\forall G : \text{PolishGroup} . \forall H : \text{Closed} \ \& \ \text{Subgroup}(G) . \exists \sigma \in \text{Aut}_{\text{BOR}}(G) .$

$. \text{Selector}(G, \text{Coset}(G, H), \sigma)$

**Proof** =

$[1] := \text{ETGRP}(G) : \forall g \in G . \text{Closed}(G, gH),$

$[2] := \Lambda U \in \mathcal{T}(G) \text{ETGRP}(G) \text{ETOP}(G) : \forall U \in \mathcal{T}(G) . [U]_H = \bigcup_{g \in G} gU \in \mathcal{T}(G),$

$[3] := \text{BorelSelector}(G, \sim_H)[1][2] : \exists \sigma \in \text{Aut}_{\text{BOR}}(G) . \text{Selector}(G, \text{Coset}(G, H), \sigma);$

□



### 3.1.5 Borel Space of Polish Groups

**ClosedSubgroupsAreBorel** :: **Closed** & **Subgroup**( $G^\star$ )  $\in \mathcal{B}(\text{EFF}(G^\star))$

**Proof** =

$S := \text{Closed} \ \& \ \text{Subgroup}(G^\star) : ??G^\star,$

$(\delta, [1]) := \text{SelectionTHM}(G^\star) : \sum \delta : \mathbb{N} \rightarrow \text{BOR}(G^\star, \text{EFF}(G^\star)) . \forall A \in \text{EFF}(G^\star) . \text{Dense}(A, \delta_{\mathbb{N}}(A)),$

$[*] :=: S = \left\{ A \in \text{EFF}(G^\star) \middle| e \in A \right\} \cap \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (\delta_n \delta_m^{-1})^{-1} \{ (x, A) \in G^\star \times \text{EFF}(G^\star) \middle| x \in A \};$

□

### 3.1.6 Standard Borel Groups

$\text{BorelGroup} :: ? \sum G \in \text{GRP} . \sigma\text{-Algebra}(G)$

$(G, \mathcal{A}) : \text{BorelGroup} \iff$

$\iff \circ_G \in \text{BOR}\left((G, \mathcal{T}) \times (G, \mathcal{T}), (G, \mathcal{T})\right) \ \& \ \Lambda g \in G . g^{-1} \in \text{BOR}\left((G, \mathcal{T}), (G, \mathcal{T})\right)$

$\text{groupOfBorelAsGroup} :: \text{BorelGroup} \rightarrow \text{GRP}$

$\text{groupOfBorelAsGroup}(G, \mathcal{A}) = (G, \mathcal{A}) := G$

$\text{groupOfBorelAsMeasurableSpace} :: \text{BorelGroup} \rightarrow \text{BOR}$

$\text{groupOfBorelAsMeasurableSpace}(G, \mathcal{A}) = (G, \mathcal{A}) := (G, \mathcal{A})$

$\text{StandardBorelGroup} := \text{BorelGroup} \ \& \ \text{StandardBorelSpace} : \text{Type};$

$\text{StandardGroupTopologyUniqueness} :: \forall G : \text{StandardBorelGroup} . \forall \mathcal{A}, \mathcal{B} : \text{Topology}(G) .$

$. \forall [0.1] : \text{PolishGroup}\left((G, \mathcal{A}) \ \& \ (G, \mathcal{B})\right) . \forall [0.2] : \mathcal{A}(G) = \sigma(\mathcal{A}) = \sigma(\mathcal{B}) . \mathcal{A} = \mathcal{B}$

$\text{Proof} =$

$[1] := \text{BorelIsBairMeasurable}[0.2] :$

$: \text{BairMeasurable}\left((G, \mathcal{A}), (G, \mathcal{B}), \text{id}_G\right) \ \& \ \text{BairMeasurable}\left((G, \mathcal{B}), (G, \mathcal{A}), \text{id}_G\right),$

$[2] := \text{BairMeasurableIsContinuous}[1][0.1] : \text{TOP}\left((G, \mathcal{A}), (G, \mathcal{B}), \text{id}_G\right) \ \& \ \text{TOP}\left((G, \mathcal{B}), (G, \mathcal{A}), \text{id}_G\right),$

$[3] := \text{Eid}[2] : \mathcal{A} = \mathcal{B};$

□

$\text{Polishable} :: ?\text{StandardBorelGroup}$

$G : \text{Polishable} \iff \exists \mathcal{T} : \text{Topology}(G) . \text{PolishGroup}(G, \mathcal{T}) \ \& \ \mathcal{A}(G) = \sigma(\mathcal{A})$

$\text{eventuallyOneGroup} :: \text{TGRP}$

$\text{eventuallyOneGroup}() = \mathbb{T}_1^{\mathbb{N}} := \left\{s \in \mathbb{T}^{\mathbb{N}} : \left|\{n \in \mathbb{N} . s_n \neq 1\}\right| < \infty\right\}$

$\text{EventuallyOneGroupIsBorel} :: \mathbb{T}_1^{\mathbb{N}} \in \mathcal{B}(\mathbb{T}^{\mathbb{N}})$

$\text{Proof} =$

$\mathbb{T}_1^{\mathbb{N}} = \bigcup_{n=1}^{\infty} \mathbb{T}^n \times \{1\}^{\mathbb{N}}$

□

$\text{EventuallyOneGroupIsStandard} :: \text{StandardBorelGroup}(\mathbb{T}_1^{\mathbb{N}})$

$\text{Proof} =$

The space  $\bigsqcup_{n=1}^{\infty} \mathbb{T}^n$  is standard Borel.

Factorizing by closed sets should produce  $\mathbb{T}_1^{\mathbb{N}}$  with polish topology.

□

**EventuallyOneGroupIsNotPolishable** ::  $\neg \text{Polishable}(\mathbb{T}_1^{\mathbb{N}})$

**Proof** =

**Assume**  $[1] : \text{Polishable}(\mathbb{T}_1^{\mathbb{N}})$ ,

$(\mathcal{T}, [2]) := \mathbf{E}[1] : \sum \mathcal{T} : \text{Topology}(X) . \text{PolishGroup}(\mathbb{T}_1^{\mathbb{N}}, \mathcal{T}) \ \& \ \mathbf{A}(\mathbb{T}_1^{\mathbb{N}}) = \sigma(\mathcal{T})$ ,

$[3] := \text{PolishIsGDelta}[2.1] : G_{\delta}(\mathbb{T}^{\mathbb{N}}, \mathbb{T}_1^{\mathbb{N}})$ ,

$[4] := \text{GDeltaSubgroupIsClosed}[3] : \text{Closed}(\mathbb{T}^{\mathbb{N}}, \mathbb{T}_1^{\mathbb{N}})$ ,

$x := \Lambda n \in \mathbb{N} . \Lambda m \in \mathbb{N} . \exp\left(\frac{i}{n}\right) : \mathbb{N} \rightarrow \mathbb{T}^{\mathbb{N}}$ ,

$[6] := \mathbf{ExE}\mathbb{T}_1^{\mathbb{N}} : \forall n \in \mathbb{N} . x_n \notin \mathbb{T}_1^{\mathbb{N}}$ ,

$[7] := \mathbf{ExE}\mathbb{T}^{\mathbb{N}}\mathbf{E}\mathbb{T}_1^{\mathbb{N}} : \lim_{n \rightarrow \infty} x_n = 1 \in \mathbb{T}_1^{\mathbb{N}}$ ,

$[1.*] := \text{ClosedBySequences}[4][7] : \perp$ ;

$\leadsto [*] := \mathbf{E}\perp : \neg \text{Polishable}(\mathbb{T}^{\mathbb{N}})$ ;

□

**L2SequencesArePolishable** ::  $\text{Polishable}(l_2)$

**Proof** =

$l_2$  is polish as a separable Hilbert space.

□

## 3.2 Borel Action

### 3.2.1 E Separation

**EInvariant** ::  $\prod X \in \mathbf{Set} . \mathbf{Equivalence}(X) \rightarrow ??X$

$A : \mathbf{EInvariant} \iff \forall x \in A . \forall y \in [x]_E . y \in A$

**ESeparation** ::  $\forall X : \mathbf{StandardBorelSpace} . \forall E : \mathbf{Equivalence}(X) . \forall [0.1] : E \in \Sigma_1^1(X^2) .$   
 $. \forall A, B \in \Sigma_1^1(X) \ \& \ \mathbf{EInvariant}(X, E) . \forall [0.2] : \mathbf{DisjointPair}(X, A, B) .$   
 $. \exists C \in \mathcal{B}(X) \ \& \ \mathbf{EInvariant}(X, E) . A \subset C \ \& \ B \cap C = \emptyset$

**Proof** =

Assume  $X$  is Polish without loss of generality..

It is possible to give strong topology to the set  $\frac{X}{E}$ , so the projection  $\pi_E$  is continuous.

With this structure  $\pi_E(A), \pi_E(B)$  are analytic in  $\frac{X}{E}$ .

They also disjoint as they were E-Invariant.

As  $E$  is analytic set itself, we can realize by a pair of continuous maps  $\phi_1, \phi_2 : \mathcal{B} \rightarrow X$  .

Then,  $x \sim_E y$  iff there is a  $b \in \mathcal{B}$  such that  $\phi_1(b) = x$  and  $\phi_2(b) = y$ .

Thus,  $\frac{X}{E}$  is equivalent to pushout  $X \sqcup_{\mathcal{B}, \phi} X$ .

So,  $\frac{X}{E}$  must be Polish .

Now apply Suslin separation theorem in  $\frac{X}{E}$  to separate  $A$  and  $B$  by some  $C$  .

Then  $\pi_E^{-1}(C)$  is Borel and  $E$ -invariant, it also separates  $A$  and  $B$ .

...

□

**PolishTopologicalGroupCondition** ::  $\forall G \in \mathbf{GRP} . \forall \mathcal{T} : \mathbf{PolishTopology}(G) .$   
 $. \forall [0] : \forall g \in G . \lambda_g, \rho_g \in \mathbf{Aut}_{\mathbf{TOP}}(G, \mathcal{T}) . (G, \mathcal{T}) \in \mathbf{TGRP}$

**Proof** =

...

□

BlackwellTHM ::

$:: \forall X : \text{StandardBorelSpace} . \forall A : \mathbb{N} \rightarrow \mathcal{S}_X . \forall S \subset X . \text{EInvariant}(X, E, S) \ \& \ \mathcal{B}(X) \iff S \in \sigma(\text{Im } A)$

where  $E = \left\{ (x, y) \in X^2 : \forall n \in \mathbb{N} . x \in A_n \iff y \in A_n \right\}$

Proof =

By properties of logical  $\iff$  it is obvious that  $E$  is equivalence.

Firstly, we show that each  $A_n$  is  $E$ -invariant.

consider  $x \in A_n$  and  $y \in [x]_E$ , then by definition of  $E$  we also have  $y \in A_n$ .

It is clear that union of  $E$ -invariant sets is  $E$ -invariant.

Now assume that  $B$  is  $E$ -invariant.

Assume  $x \in B^c$  and  $y \in [x]_E$ .

If  $y$  was in  $B$  then by symmetry  $x$  would also be in  $B$ , so  $y \in B^c$ .

So, invariant subsets form a  $\sigma$ -algebra containing all  $A_n$ .

Thus,  $\sigma(\text{Im } A)$  is all  $E$ -invariant.

Now assume  $B$  is Borel and  $E$ -invariant.

Note, that equivalence class of  $E$  are Borel and correspond to elements of  $\mathcal{C}$ .

For  $c \in \mathcal{C}$  the set  $\alpha_c = \bigcap_{c_n=1} A_n \cap \bigcap_{c_n=0} A_n^c$  is equivalence class of  $E$ .

And every equivalence class of  $E$  can be expressed as some  $\alpha_c$ .

Thus, every equivalence class belongs to  $\sigma(\text{Im } A)$ .

Nevertheless, we always can express  $B = \bigcup_{c \in C} \alpha_c$  for some  $C \subset \mathcal{C}$ .

Consider a mapping  $\psi : \frac{X}{E} \rightarrow \mathcal{C}$  defined by  $\psi(\alpha_c) = c$ .

This mapping is a measurable (see argument about prebase next) injection.

So  $C = \psi[B]_E$  must be measurable in  $\mathcal{C}$ .

but topology on  $\mathcal{C}$  can be generated by prebase of sets of form  $\{c \in \mathcal{C} | c_n = j\}$ , where  $j = 1, 0$  and  $n \in \mathbb{N}$ .

And such sets also generate the Borel algebra of  $\mathcal{C}$ .

Now sets  $C = \{c \in \mathcal{C} | c_n = 1\}$  corresponds directly to  $A_n$ .

So,  $B$  must belong to  $\sigma(\text{Im } A)$ .

□

### 3.2.2 Subject

**BorelAction** :=  $\Lambda X : \text{StandardBorelSpace} . \Lambda G : \text{StandardBorelGroup} . G \curvearrowright_{\text{BOR}} X =$   
 $= \Lambda X : \text{StandardBorelSpace} . \Lambda G : \text{StandardBorelGroup} . \text{GRP}\left(G, \text{Aut}_{\text{BOR}}(X)\right) \& \text{BOR}(G \times X, X) :$   
 $: \text{StandardBorelSpace} \rightarrow \text{StandardBorelGroup} \rightarrow \text{Type};$

**BorelOrbitRelationIsAnalytic** ::

$:: \forall X : \text{StandardBorelSpace} . \forall G : \text{StandardBorelGroup} . \forall \alpha : G \curvearrowright_{\text{BOR}} X . E_\alpha \in \Sigma_1^1(X^2)$

**Proof** =

Enrich topology on  $X \times G$  so  $\alpha$  is continuous.

Then, mapping  $\beta : (x, g) \mapsto (x, \alpha(x, g))$  is also continuous.

But its image is  $E_\alpha$ .

□

**FreeBorelOrbitIsBorel** ::

$:: \forall X : \text{StandardBorelSpace} . \forall G : \text{StandardBorelGroup} . \forall \alpha : G \curvearrowright_{\text{BOR}} X . \text{Free}(G, X, \alpha) \Rightarrow E_\alpha \in \mathcal{B}(X^2)$

**Proof** =

In this case  $\beta$  will be injective.

So  $E_\alpha$  is Borel by Injective Image Theorem.

□

**LocallyCompactContinuousOrbitIsFSigma** ::

$:: \forall X : \text{Polish} . \forall G : \text{PolishGroup} \& \text{LocallyCompact} . \forall \alpha : G \curvearrowright_{\text{TOP}} X . E_\alpha \in F_\sigma(X^2)$

**Proof** =

...

□

**MillerBorelOrbitTHM** ::

$:: \forall G : \text{PolishGroup} . \forall X : \text{StandardBorelSpace} . \forall \alpha : G \curvearrowright_{\text{BOR}} X . \forall x \in X . O_\alpha(x) \in \mathcal{S}_X$

**Proof** =

[1] := **MillerStabilizerTHM**( $G, \alpha$ ) :  $\forall x \in X . \text{Closed}\left(G, \text{Stab}(\alpha, x)\right),$

$\left(T, [2]\right) := \text{SubgroupSelector}[1](x) : T \in \mathcal{B}(G) . \forall g \in G . \left|T \cap g\text{Stab}(\alpha, x)\right| = 1,$

[3] := **EGroupAction**( $G, X, \alpha$ ) :  $\forall g, h \in G . gx = hx \iff \exists f \in G . g, h \in f\text{Stab}(\alpha, x),$

$\varphi := \Lambda g \in G . gx \in \text{BOR}(G, X),$

[4] :=  $\text{E}\varphi[2][3] : \text{Injective}\left(T, X, \varphi|_T\right),$

[5] :=  $\text{E}\varphi\text{IO}_\alpha : \varphi(T) = O_\alpha(x),$

[6] := **InjectiveImageTHM** :  $O_\alpha(x) \in \mathcal{S}_X;$

□

**BorelHomo** ::  $\forall G : \text{PolishGroup} . \forall H : \text{StandardBorelGroup} . \forall \varphi \in \text{BOR} \cap \text{GRP}(G, H) . \varphi(G) \in \mathcal{S}_H$

**Proof** =

Define Borel action  $\alpha : G \curvearrowright H$  by  $\alpha(g, h) = \varphi(g)h$ .

Then  $\varphi(G) = O_\alpha(e)$ .

By Miller's Theorem it must be measurable.

□

### 3.2.3 Vaught Transform

$\text{actionSaturation} :: \prod G \in \text{GRP} . \prod X \in \text{SET} . \prod \alpha : G \curvearrowright X . ?X \rightarrow ?X$

$\text{actionSaturation}(A) = [A]_\alpha := \{x \in X : \exists g \in G . gx \in A\}$

$\text{actionHull} :: \prod G \in \text{GRP} . \prod X \in \text{SET} . \prod \alpha : G \curvearrowright X . ?X \rightarrow ?X$

$\text{actionHull}(A) = (A)_\alpha := \{x \in X : \forall g \in G . gx \in A\}$

$\text{SaturationAndHullRelation} :: \forall G \in \text{GRP} . \forall X \in \text{SET} . \forall \alpha : G \curvearrowright X . \forall A \subset X . (A)_\alpha \subset A \subset [A]_\alpha$

**Proof** =

This is obvious.

□

$\text{AnalyticSaturation} ::$

$:: \forall G : \text{StandardBorelGroup} . \forall X : \text{StandardBorelSpace} . \forall \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \in \mathcal{S}_X . [A]_\alpha \in \Sigma_1^1(X)$

**Proof** =

View  $[A]_\alpha$  as an image of  $G \times A$  under  $\alpha$ .

□

$\text{CoanalyticHull} ::$

$:: \forall G : \text{StandardBorelGroup} . \forall X : \text{StandardBorelSpace} . \forall \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \in \mathcal{S}_X . (A)_\alpha \in \Pi_1^1(X)$

**Proof** =

View  $(A)_\alpha^c$  as an image of  $G \times A^c$  under  $\alpha$ .

□

$\text{nonmeagerVaughtTransform} :: \prod G \in \text{StandardBorelGroup} . \prod X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . ?X \rightarrow ?X$

$\text{nonmeagerVaughtTransform}(A) = A_\alpha^* := \{x \in X : \exists^* g \in G . gx \in A\}$

$\text{comeagerVaughtTransform} :: \prod G \in \text{StandardBorelGroup} . \prod X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . ?X \rightarrow ?X$

$\text{comeagerVaughtTransform}(A) = A_\alpha^\Delta := \{x \in X : \forall^* g \in G . gx \in A\}$

$\text{nonmeagerLocalVaughtTransform} :: \prod G \in \text{StandardBorelGroup} . \prod X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \mathcal{T}(G) \rightarrow ?X \rightarrow ?X$

$\text{nonmeagerLocalVaughtTransform}(A, U) = A_\alpha^{*U} := \{x \in X : \exists^* g \in U . gx \in A\}$

$\text{comeagerLocalVaughtTransform} :: \prod G \in \text{StandardBorelGroup} . \prod X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \mathcal{T}(G) \rightarrow ?X \rightarrow ?X$

$\text{comeagerLocalVaughtTransform}(A, U) = A_\alpha^{\Delta U} := \{x \in X : \forall^* g \in U . gx \in A\}$



**VaughtTransformsRelation** ::

$:: \forall G \in \text{StandardBorelGroup} . \forall X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \subset X . (A)_\alpha \subset A_\alpha^\Delta \subset A_\alpha^* \subset A \subset [A]_\alpha$

**Proof** =

Obvious.

□

**VaughtTransformsInvariant** ::  $\forall G \in \text{StandardBorelGroup} . \forall X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \subset X . \text{Invariant}(G, X, \alpha, A_\alpha^\Delta \text{ \& } A_\alpha^*)$

**Proof** =

...

□

**VaughtTransformInvarianceCriterion** ::  $\forall G \in \text{StandardBorelGroup} . \forall X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \subset X . \text{Invariant}(G, X, \alpha, A) \iff A_\alpha^\Delta = A_\alpha^*$

**Proof** =

...

□

**LocalVaughtTransformIsBorel** ::  $\forall G \in \text{StandardBorelGroup} . \forall X \in \text{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\text{BOR}} X . \forall A \subset \mathcal{S}_X . \forall U \in \mathcal{T}(G) . A_\alpha^{\Delta U}, A_\alpha^{*U} \in \mathcal{S}_X$

**Proof** =

Follows from Novikov-Montgomery theorem.

□

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