Measure.Know

Uncultured Tramp

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$$\int_A d\lambda$$

Contents

1 Basic measure theory

1.1 Limits of Sets

$\texttt{IncreasingSetSeq} ::?\mathbb{N} \to \texttt{Set}$
$A : \texttt{IncreasingSetSeq} \iff A\uparrow \iff \forall n \in \mathbb{N} \,.\, A_n \subset A_{n+1}$

$\texttt{IncreasingTo} ::?\texttt{IncreasingSetSeq} \times \texttt{Set}$

$$(A, \alpha) : \texttt{IncreasingTo} \iff A \uparrow \alpha \iff \alpha = \bigcup_{i=1}^{\infty} A_i$$

$\texttt{DecreasingSetSeq} ::?\mathbb{N} \to \texttt{Set}$
$A : \texttt{DecreasingSetSeq} \iff A\downarrow \iff \forall n \in \mathbb{N} \,.\, A_{n+1} \subset A_n$

$\texttt{IncreasingTo} ::?\texttt{IncreasingSetSeq} \times \texttt{Set}$

$$(A, \alpha) : \texttt{DecreasingTo} \iff A \downarrow \alpha \iff \alpha = \bigcap_{i=1}^{\infty} A_i$$

$\texttt{ComplimentLimit1} :: \forall A \uparrow \alpha \,.\, A^c \downarrow \alpha^c$

$\texttt{ComplimentLimit2} :: \forall A \downarrow \alpha \,.\, A^c \uparrow \alpha^c$

$$\texttt{ToDisjoint} :: \forall A : \mathbb{N} \to ?\Omega \,.\, \exists A' : \texttt{Disjoint}(\Omega, \mathbb{N}) : \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$$

$$\texttt{ToDisjoint}(A) = \Lambda n \in \mathbb{N} \,.\, \bigcap_{i=1}^{n-1} A_i^{\complement} \cap A_n$$

$$\texttt{toDisjoint} :: \forall A :\uparrow_{\Omega} \,.\, \exists A' : \texttt{Disjoint}(\Omega, \mathbb{N}) : \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$$

$$\texttt{toDisjoint}(A) = \Lambda n \in \mathbb{N} \,.\, \texttt{if } n = 1 \texttt{ then } A_1 \texttt{ else } (A_n \setminus A_{n-1})$$

$$\limsup :: (\mathbb{N} \to ?\Omega) \to ?\Omega$$

$$\limsup A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$\liminf :: (\mathbb{N} \to ?\Omega) \to ?\Omega$$

$$\liminf A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

LimSupComplement $:: \forall A : \mathbb{N} \to ?\Omega \ . \ (\limsup A)^{\complement} = \liminf A^{\complement}$

Scatch :

$$(\limsup A)^{\complement} = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right)^{\complement} = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)^{\complement} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^{\complement} =$$
$$= \liminf A^{\complement}$$

LimInfComplement $:: \forall A : \mathbb{N} \to ?\Omega \ . \ (\liminf A)^{\complement} = \limsup A^{\complement}$

Scatch :

$$(\liminf A)^{\complement} = \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right)^{\complement} = \bigcap_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)^{\complement} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^{\complement} =$$
$$= \limsup A^{\complement}$$

InfSubsetSup $:: \forall A : \mathbb{N} \to ?\Omega \ . \ \liminf A \subset \limsup A$

Proof $=$

Assume $A : \mathbb{N} \to ?\Omega$,

Assume $n \in \mathbb{N}$,

Assume $m \in \mathbb{N}$,

Assume $a \in \bigcap_{k=m}^{\infty} A_k$,

$$a \in \bigcap_{k=m}^{\infty} A_k \rightsquigarrow a \in A_{n+m} \rightsquigarrow a \in \bigcup_{k=n}^{\infty} A_k;$$

$$\bigcap_{k=m}^{\infty} A_k \subset \bigcup_{k=n}^{\infty} A_k;$$

$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} A_k \subset \bigcup_{k=n}^{\infty} A_k \rightsquigarrow \liminf A \subset set \bigcup_{k=n}^{\infty} A_k;$$

$$\liminf A \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \rightsquigarrow \liminf A \subset \limsup A; \square$$

$\texttt{IncreasingLimit} :: \forall A \uparrow \alpha \ . \ \liminf A = \limsup A = \alpha$

$\texttt{Proof} =$

$\texttt{Assume}\ A \uparrow \alpha,$

$\texttt{Assume}\ a \in \alpha,$

$(A \uparrow \alpha, a \in \alpha) \rightsquigarrow \exists N \in \mathbb{N} : \forall n > N \ . \ a \in A_n\ \texttt{Extract},$

$$\forall n > N \ . \ a \in A_n \rightsquigarrow \forall n > N \ . \ a \in \bigcap_{k=n}^{\infty} A_n \rightsquigarrow a \in \liminf A,$$

$$\forall n > N \ . \ a \in A_n \rightsquigarrow \forall n \in \mathbb{N} \ . \ a \in \bigcup_{k=n}^{\infty} A_n \rightsquigarrow a \in \limsup A;$$

$\alpha \subset \liminf A, \alpha \subset \limsup A\ \texttt{as}\ (1),$

$$A \uparrow \alpha \iff \alpha = \bigcup_{n=1}^{\infty} A_n \rightsquigarrow \liminf A, \limsup A \subset \alpha\ \texttt{as}\ (2),$$

$(1,2) \rightsquigarrow \liminf A = \limsup A = \alpha; \square$

$\texttt{DecreasingLimit} :: \forall A \downarrow \alpha \ . \ \liminf A = \limsup A = \alpha$

$\texttt{Proof} =$

$\texttt{Assume}\ A \downarrow \alpha,$

$\texttt{Assume}\ a \in \limsup A,$

$$a \in \limsup A \to \forall n \in \mathbb{N} \ . \ a \in \bigcup_{k=n}^{\infty} A_k\ \texttt{as}\ (2)$$

$\texttt{Assume}\ n \in \mathbb{N},$

$$(2)(n) \rightsquigarrow a \in \bigcup_{k=n}^{\infty} A_k \rightsquigarrow \exists m \in \mathbb{N} : m \geq n : a \in A_m\ \texttt{Extract},$$

$A \downarrow .a \in A_m \rightsquigarrow \forall k \in \mathbb{N} : k \leq m \ . \ a \in A_k \rightsquigarrow_n a \in A_n;$

$\forall n \in \mathbb{N} \ . \ a \in A_n \rightsquigarrow a \in \alpha;$

$\limsup A \subset \alpha\ \texttt{as}\ (2),$

$\alpha \subset \liminf A, (2) \rightsquigarrow \liminf A = \limsup A = \alpha; \square$

$\texttt{SetLimit} :: (\mathbb{N} \to ?\Omega) \to ??\Omega$

$\alpha : \texttt{SetLimit}(A) \iff \liminf A = \limsup A = \alpha$

$\texttt{Example}$ 1.

$A := \Lambda n \in \mathbb{N} \,.\, \texttt{if } n : \texttt{Odd then } (-1/n, 1] \texttt{ else } (-1, 1/n)$

$\limsup A = (-1, 1],$

$\liminf A = \{0\},$

$\texttt{Example}$ 2.

$A := \Lambda n \in \mathbb{N} \,.\, \mathbb{B}^2 \big(((-1)^n/n, 0), 1\big)$

$\limsup A = \overline{\mathbb{B}}^2 (0, 1) \setminus \{(0, 1), (0, -1)\}$

$\liminf A = \mathbb{B}^2 (0, 1)$

$\texttt{Example}$ 3.

$\limsup x = X$

$A := \Lambda n \in \mathbb{N} \,.\, (-\infty, x_n]$

$(-\infty, X) \subset \limsup A \subset (-\infty, X]$

$\liminf y = Y$

$B := \Lambda n \,.\, (-\infty, y_n]$

$(-\infty, Y) \subset \liminf A \subset (-\infty, Y]$

$\texttt{Example}$ 4.

$a < b < c < d$

$A := \Lambda n \in \mathbb{N} \,.\, \texttt{if } n : \texttt{Odd then } (a, b) \texttt{ else } (c, d)$

$\liminf A = \emptyset$

$\limsup A = (a, b) \cup (c, d)$

1.2 Fields and Measures

$$\texttt{Algebra} :: \prod \Omega : \texttt{Set} \,.\, ???\Omega$$

$$\mathcal{F} : \texttt{Algebra} \iff \Omega \in \mathcal{F} \wedge \forall A, B \in \mathcal{F} \,.\, A^{\complement} \in \mathcal{F} \wedge A \cup B \in \mathcal{F}$$

$$\sigma\texttt{-Algebra} :: ?\texttt{Algebra}(\Omega)$$

$$\mathcal{F} : \sigma\texttt{-Algebra} \iff \forall A : \mathbb{N} \to \mathcal{F} \,.\, \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

$$\texttt{GenSigmaAlgebra} :: ??\Omega \to \sigma\texttt{-Algebra}\,(\Omega)$$

$$\texttt{GenSigmaAlgebra}(A) = \sigma_\Omega(A) = \bigcap\{\mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) : A \subset \mathcal{F}\}$$

$\texttt{AlgebraContraction} :: \forall \mathcal{A} :??\Omega \,.\, \forall A \in ?\Omega \,.\, [\sigma_\Omega(\mathcal{A})] \cap A = \sigma_A([\mathcal{A}] \cap A)$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{A} :??\Omega,$

$\texttt{Assume}\ A :?\Omega,$

$[\sigma_\Omega(\mathcal{A})] \cap A : \sigma\texttt{-Algebra}\,(A)\,, [\mathcal{A}] \cap A \subset [\sigma_\Omega(\mathcal{A})] \cap A \rightsquigarrow$

$\quad \rightsquigarrow \sigma_A([\mathcal{A}] \cap A) \subset [\sigma_\Omega(\mathcal{A})] \cap A\ \texttt{as}\ (1),$

$G := \{B \in \sigma_\Omega(\mathcal{A}) : B \cap A \in \sigma_A(\mathcal{A} \cap A)\}$

$G : \sigma\texttt{-Algebra}\,(\Omega) \rightsquigarrow G = \sigma_\Omega(\mathcal{A}) \rightsquigarrow [\sigma_\Omega(\mathcal{A})] \cap A \subset \sigma_A([\mathcal{A}] \cap A)\ \texttt{as}\ (2),$

$(1, 2) \rightsquigarrow [\mathcal{A}] \cap A = [\sigma_\Omega(\mathcal{A})] \cap A; ; \square$

$$\texttt{Measure} :: \prod \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \,.\, ?\mathcal{F} \to \overset{\infty}{\mathbb{R}}_+$$

$$\mu : \texttt{Measure} \iff \forall A : \texttt{Disjoint}(\Omega, \mathbb{N}) \& \mathcal{F} \,.\, \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=1}^{n} \mu(A_i)$$

$$\texttt{Probability} :: ?\texttt{Measure}(\Omega, \mathcal{F})$$

$$\mathbb{P} : \texttt{Probability} \iff \mathbb{P}(\Omega) = 1$$

$$\texttt{MeasureSpace} := \sum \Omega : \texttt{Set} \,.\, \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \,.\, \texttt{Measure}(\Omega, \mathcal{F})$$

$$\texttt{ProbbilitySpace} := \sum \Omega : \texttt{Set} \,.\, \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \,.\, \texttt{Probability}(\Omega, \mathcal{F})$$

$$\texttt{SetFunction} :: \prod \mathcal{F} : \texttt{Algebra}(\Omega) \,.\, ?\mathcal{F} \to \overset{\infty}{\mathbb{R}}$$

$$f : \texttt{SetFunction} \iff \{-\infty, \infty\} \not\subset \operatorname{Im} f \wedge \exists A \in \mathcal{F} : f(A) \in \mathbb{R}$$

$\texttt{Charge} :: ?\texttt{SetFunction}(\Omega, \mathcal{F})$

$f : \texttt{Charge} \iff \forall (A, B) : \texttt{DisjointPair}(\Omega) \;.\; f(A \cup B) = f(A) + f(B)$

$\texttt{CountablyAdditive} :: \prod \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \;.\; ?\texttt{SetFunction}(\Omega, \mathcal{F})$

$$f : \texttt{CountablyAdditive} \iff \forall A : \texttt{Disjoint}(\Omega, \mathbb{N})\&\mathcal{F} \;.\; \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=1}^{n} \mu(A_i)$$

$\texttt{ConcentratedOn} :: \prod \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \;.\; \mathcal{F} \to ?\texttt{Measure}(\Omega, \mathcal{F})$

$\mu : \texttt{ConcentratedOn}(A) \iff \mu\left(A^{\complement}\right) = 0$

$\texttt{EmptyIsZero} :: \forall f : \texttt{Charge}(\Omega, \mathcal{F}) \;.\; f(\emptyset) = 0$

$\texttt{Scatch}:$

$f : \texttt{SetFunction}(\Omega.\mathcal{F}) \rightsquigarrow \exists A \in \mathcal{F} : f(A) \in \mathbb{R}\ \texttt{Extract}$

$f(A) = f(A) + f(\emptyset) \rightsquigarrow f(\emptyset) = 0\square$

$\texttt{UnionDecomposition} :: \forall f : \texttt{Charge}(\Omega, \mathcal{F}) \;.\; \forall A, B \in \mathcal{F} \;.$

$\quad .\; f(A \cup B) = f(A) + f(B) - f(A \cap B)$

$\texttt{Scatch}:$

$$\begin{aligned} f(A \cup B) &= f\left(A \cap B^{\complement}\right) + f(A \cap B) + f\left(B \cap A^{\complement}\right) = \\ &= \left(f\left(A \cap B^{\complement}\right) + f(A \cap B)\right) + \left(f\left(B \cap A^{\complement}\right) + f(A \cup B)\right) - f(A \cap B) = \\ &= f(A) + f(B) - f(A \cap B)\square \end{aligned}$$

$$\texttt{MeasureInequality} :: \forall \mu : \texttt{Measure}(\Omega.\mathcal{F}) \;.\; \forall A : \mathbb{N} \to \mathcal{F} \;.\; \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

$\texttt{Finite} :: ?\texttt{SetFunction}(\Omega, \mathcal{F})$

$f : \texttt{Finite} \iff \operatorname{Im} f \subset M$

$\sigma\texttt{-Finite} : ?\texttt{SetFunction}(\Omega, \mathcal{F})$

$$f : \sigma\texttt{-Finite} \iff \exists A : \mathbb{N} \to \mathcal{F} : \bigcup_{n=1}^{\infty} A_n = \Omega \;\wedge\; \forall n \in \mathbb{N} \;.\; f(A_n) \in \mathbb{R}$$

$\texttt{MeasureUpperContinuity} :: \forall \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\ .\ \forall \mu : \texttt{CountablyAdditive}(\mathcal{F})\ .$
$\quad .\ \forall A \uparrow \alpha\ .\ \lim_{n\to\infty} \mu(A_n) = \mu(\alpha)$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\,,$

$\texttt{Assume}\ \mu : \texttt{CountablyAdditive}(\mathcal{F}),$

$\texttt{Assume}\ A \uparrow \alpha,$

$A' := \texttt{toDisjoint}(A),$

$$\mu(\alpha) = \mu\left(\bigcup_{n=1}^{\infty} A'_n\right) = \sum_{n=1}^{\infty} \mu(A'_n)$$

$\texttt{Assume}\ n \in \mathbb{N},$

$$\sum_{n=1}^{n} \mu(A'_k) = \mu(A_1) + \sum_{k=2}^{n} \mu(A_k \setminus A_{k-1}) = \mu(A_k);$$

$$\forall n \in \mathbb{N}\ .\ \sum_{n=1}^{n} \mu(A'_k) = \mu(A_k)\ \texttt{as}\ (1)$$

$\texttt{FromDef Seria}\ (1) \leadsto \lim_{n\to\infty} \mu(A_n) = \mu(\alpha);;;\square$

$\texttt{MeasureLowerContinuity} :: \forall \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\ .\ \forall \mu : \texttt{CountablyAdditive}(\mathcal{F})\ .$
$\quad .\ \forall A \downarrow \alpha\ .\ \lim_{n\to\infty} \mu(A_n) = \mu(\alpha)$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\,,$

$\texttt{Assume}\ \mu : \texttt{CountablyAdditive}(\mathcal{F}),$

$\texttt{Assume}\ A \downarrow \alpha,$

$B := A_1 \setminus A,$

$B \uparrow A_1 \setminus \alpha \leadsto \lim_{n\to\infty} \mu(A_1 \setminus A_n) = \mu(A_1 \setminus \alpha) = \mu(A_1) - \mu(\alpha)\ \texttt{as}\ (1),$

$\lim_{n\to\infty} \mu(A_1 \setminus A_n) = \lim_{n\to\infty} \mu(A_1) - \mu(A_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)\ \texttt{as}\ (2),$

$(1,2) \to \mu(\alpha) = \lim_{n\to\infty} \mu(A_n);;;\square$

$\texttt{UpperContinuous} :: \prod \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\ .\ ?\texttt{Charge}(\mathcal{F})$

$\mu : \texttt{UpperContinuous} \iff \forall A \uparrow \alpha\ .\ \lim_{n\to\infty} \mu\,(A_n) = \mu(\alpha)$

$\texttt{LowerContinuous} :: \prod \mathcal{F} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\ .\ ?\texttt{Charge}(\mathcal{F})$

$\mu : \texttt{LowerContinuous} \iff \forall A \downarrow \alpha\ .\ \lim_{n\to\infty} \mu\,(A_n) = \mu(\alpha)$

$\texttt{CountablyAdditivityMark1} :: \forall \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega)\ .\ \forall \mu : \texttt{UpperContinuous}(\mathcal{F})\ .$
$\quad .\ \mu : \texttt{CountablyAdditive}(\mathcal{F})$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega)\,,$

$\texttt{Assume}\ \mu : \texttt{UpperContinuous}(\mathcal{F}),$

$\texttt{Assume}\ A : \texttt{Disjoint}(\Omega, \mathbb{N}) : \mathrm{Im}(A) \subset \mathcal{F},$

$$B := \Lambda n \in \mathbb{N}\ .\ \bigcup_{k=1}^{n} A_k,$$

$$B \uparrow \bigcup_{n=1}^{\infty} A_n,$$

$$\mu : \texttt{UpperContinuous}(\mathcal{F}) \rightsquigarrow \lim_{n\to\infty} \mu(B_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)\ \texttt{as}\ (1),$$

$\texttt{Assume}\ n \in \mathbb{N},$

$\mu : \texttt{UpperContinuous}(\mathcal{F}) \rightsquigarrow \mu : \texttt{Charge}(\mathcal{F}),$

$$\texttt{Charge}(\mathcal{F})(\mu)(A_{1,\dots,n}) \rightsquigarrow \mu(B_n) = \sum_{k=1}^{n} \mu(A_n);$$

$$\forall n \in \mathbb{N}\ .\ \mu(B_n) = \sum_{k=1}^{n} \mu(A_n)\ \texttt{as}\ (2),$$

$$\texttt{Seria}(2) \rightsquigarrow \lim_{n\to\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(A_n) \rightsquigarrow_{(1)} \sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right);$$

$\mu : \texttt{CountablyAdditive}(\mathcal{F});;\square$

$\texttt{CountablyAdditivityMark1} :: \forall \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega)\ .\ \forall \mu : \texttt{LowerContinuous}(\mathcal{F})\ .$
$\quad .\ \mu : \texttt{CountablyAdditive}(\mathcal{F})$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega)\,,$

$\texttt{Assume}\ \mu : \texttt{UpperContinuous}(\mathcal{F}),$

$\texttt{Assume}\ A : \texttt{Disjoint}(\Omega, \mathbb{N}) : \mathrm{Im}(A) \subset \mathcal{F},$

$$\alpha := \bigcup_{n=1}^{\infty} A_n,$$

$$B := \Lambda n \in \mathbb{N}\ .\ \alpha \setminus \bigcup_{k=1}^{n} A_k,$$

$$B \downarrow \emptyset \rightsquigarrow \lim_{n\to\infty} \mu(B_n) = 0,$$

$\texttt{Assume}\ n \in \mathbb{N},$

$$\mu(\alpha) = \mu(B_n) + \sum_{k=1}^{n} \mu(A_n) \rightsquigarrow \sum_{k=1}^{n} \mu(A_n) = \mu(\alpha) - \mu(B_n);$$

$$\sum_{n=1}^{\infty} \mu(A_n) = \lim_{n\to\infty} \mu(\alpha) - \mu(B_n) = \mu(\alpha) - \lim_{n\to\infty} \mu(B_n) = \mu(\alpha);$$

$\mu : \texttt{CountablyAdditive}(\mathcal{F});;\square$

$\texttt{Example}$ 1
$\texttt{Assume}\ \Omega : \texttt{Infinite\&Countable},$
$\mathcal{F} := 2^{\Omega} : \sigma\text{-}\texttt{Algebra}\,(\Omega)\,,$
$\mu := \Lambda A \in \mathcal{F}\ .\ \texttt{if}\ A : \texttt{Finite then}\ 0\ \texttt{else}\ \infty : \texttt{Measure}(A),$
$\texttt{Assume}\ n \in \mathbb{N},$
$\texttt{Assume}\ A : \texttt{DisjointElems}(\mathcal{F}, n),$
$\texttt{Assume}\ \exists k \in n : A_k : \texttt{Infinite},$

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \infty = \sum_{k=1}^{n} \mu(A_1);$$

$\texttt{Assume}\ \forall k \in n\ .\ A_k : \texttt{Finite}$

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = 0 = \sum_{k=1}^{n} \mu(A_1);;$$

$\mu : \texttt{Charge}(\mathcal{F}),$
$\Omega : \texttt{Infinite\&Countable} \rightsquigarrow \exists \omega : \mathbb{N} \leftrightarrow \Omega\ \texttt{Extract},$

$$\mu(\Omega) = \infty \neq 0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} \mu\big(\{\omega_n\}\big) \rightsquigarrow \mu\ !\ \texttt{CountablyAdditive}\,(\mathcal{F})\,.$$

$\texttt{Example}$ 2
$\texttt{Assume}\ \Omega : \texttt{Infinite},$
$\mu := \Lambda A \in 2^{\Omega}\ .\ \#A$
$\Omega : \texttt{Infinite} \rightsquigarrow \exists Z :? \Omega : Z : \texttt{Infinite\&Countable}\ \texttt{Extract},$
$Z : \texttt{Infinite\&Countable} \rightsquigarrow \exists z : \mathbb{N} \leftrightarrow \Omega\ \texttt{Extract},$

$$A := \Lambda n \in \mathbb{N}\ .\ Z \setminus \bigcup_{k=1}^{n} \{z_n\},$$

$A \downarrow \emptyset,$
$\texttt{Assume}\ n \in \mathbb{N},$
$\mu(A_n) = \infty;$
$\forall n \in \mathbb{N}\ .\ \mu(A_n) = \infty,$
$\lim_{n\to\infty} \mu(A_n) = \infty,$
$\mu\ !\ \texttt{CountablyAdditive}\,(\mathcal{F})\,.$

$\texttt{Example}$ 3

$\texttt{Assume}\ \Omega : \texttt{Infinite\&Countable},$

$\mathcal{F} := \left\{ A :?\Omega : \#A < \infty \vee \#A^{\complement} < \infty \right\} : \texttt{Algebra}(\Omega),$

$\mu := \Lambda A \in \mathcal{F}\ .\ \texttt{if}\ A : \texttt{Finite}\ \texttt{then}\ 0\ \texttt{else}\ 1 : \mathcal{F} \to \overset{\infty}{\mathbb{R}},$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ A : \texttt{DisjointElems}(\mathcal{F}, n),$

$\texttt{Assume Alternative}\ \ \exists k \in n : A_k : \texttt{Infinite Extract},$

$\texttt{Assume}\ i \in n : i \neq k,$

$A : \texttt{DisjointElems}(\mathcal{F}, n), i \neq k \rightsquigarrow A_i \subset A_k^{\complement} \rightsquigarrow_{\jmath(\mathcal{F})}$

$\quad \rightsquigarrow_{\jmath(\mathcal{F})} A_i : \texttt{Finite};$

$$\forall i \in n : i \neq k\ .\ A_i : \texttt{Finite} \rightsquigarrow_{\jmath(\mu)} \sum_{i=1}^{n} \mu(A_i) = 1 = \mu\left(\bigcup_{i=1}^{n} A_i\right);$$

$\texttt{Close Alternative}\ \forall k \in n\ .\ A_k : \texttt{Finite},$

$$\sum_{i=1}^{n} \mu(A_i) = 0 = \mu\left(\bigcup_{i=1}^{n} A_i\right);;$$

$\mu : \texttt{Charge}\,(\mathcal{F})\,,$

$\ldots$ (as in Ex. 1)

$\mu\ !\ \texttt{CountablyAdditive}\,(\mathcal{F})\,.$

$\texttt{Example}$ 4

$$\mathcal{F} := \left\{ \bigcup_{k=1}^{n} I_i \middle| n \in \mathbb{N}, I : \texttt{DisjointElem}\big(\texttt{RightSemiclosed}(\mathbb{R}), n\big) \right\}$$

$\texttt{def}\ \mu : \mathcal{F} \to \overset{\infty}{\mathbb{R}}$

$\mu(-\infty, a] = a,$

$\mu(a, b] = a - b,$

$\mu(b.\infty) = -b,$

$\mu(\mathbb{R}) = 0,$

$$\mu\left(\bigcup_{i=1}^{n} I_I)\right) = \sum_{i=1}^{n} \mu(I_i)\ \texttt{Having}\ I : \texttt{DisjointElem}\big(\texttt{RightSemiclosed}(\mathbb{R}), n\big)$$

Check $(-\infty, n)$.

$\texttt{UT1} :: \forall \mu : \texttt{Charge}\,(\mathcal{F}) : \mu \geq 0 \,.\, \forall A : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}) : \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \,.$

$$. \, \mu \left(\bigcup_{n=1}^{\infty} A \right) \geq \sum_{n=1}^{\infty} \mu(A)$$

$\texttt{Proof} =$

$\texttt{Assume}\ \mu : \texttt{Charge}\,(\mathcal{F})\,; \mu \geq 0,$

$\texttt{Assume}\ A : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}) : \bigcup_{n=1}^{\infty} A_n \in \mathcal{F},$

$\alpha := \bigcup_{n=1}^{\infty} A_n \in \mathcal{F},$

$\texttt{Assume}\ n \in \mathbb{N},$

$a_n := \mu \left(\alpha \setminus \bigcup_{i=1}^{n} A_n \right),$

$\mu(\alpha) = a_n + \sum_{k=1}^{n} \mu(A_n) \geq \sum_{k=1}^{n} \mu(A_n);$

$\mu(\alpha) \geq \sum_{k=1}^{\infty} \mu(A_n) \square$

$\texttt{UT2} :: \forall f : \Omega \to \Omega' \,.\, \forall \mathcal{B} :??\Omega' \,.\, \sigma(f^{-1}(\mathcal{B})) = f^{-1}\sigma(\mathcal{B})$

$\texttt{Proof} =$

$f : \Omega \to \Omega',$

$\mathcal{B} :??\Omega',$

$\texttt{Assume}\ A \in f^{-1}\sigma(\mathcal{B}),$

$A \in f^{-1}\sigma(\mathcal{B}) \rightsquigarrow \exists B \in \sigma(\mathcal{B}) : A = f^{-1}B\ \texttt{Extract},$

$\sigma(B) : \sigma\texttt{-Algebra}\,(\Omega') \rightsquigarrow B^{\complement} \in \sigma(B) \rightsquigarrow f^{-1}\left(B^{\complement}\right) = A^{\complement} \in f^{-1}\sigma(\mathcal{B});$

$(1) : \forall A \in f^{-1}\sigma(\mathcal{B}) \,.\, A^{\complement} \in f^{-1}\sigma(\mathcal{B}),$

$\texttt{Assume}\ A : \mathbb{N} \to f^{-1}\sigma(\mathcal{B}),$

$B := f(A) : \mathbb{N} \to \sigma(\mathcal{B}),$

$\sigma(B) : \sigma\texttt{-Algebra}\,(\Omega') \rightsquigarrow \bigcup_{n=1}^{\infty} B_n \in \sigma(\mathcal{B}) \rightsquigarrow f^{-1} \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \in f^{-1}\sigma(\mathcal{B});$

$(2) : \forall A : \mathbb{N} \to f^{-1}\sigma(\mathcal{B}) \,.\, \bigcup_{n=1}^{\infty} A_n, \in f^{-1}\sigma(\mathcal{B})$

$(1,2) \rightsquigarrow f^{-1}\sigma(\mathcal{B}) : \sigma\texttt{-Algebra}\,(\Omega)\ \texttt{as}\ (3),$

$\eth\sigma \rightsquigarrow \mathcal{B} \subset \sigma(\mathcal{B}) \rightsquigarrow (4) : f^{-1}\mathcal{B} \subset f^{-1}\sigma(\mathcal{B}),$

$\eth\sigma(3,4) \rightsquigarrow (5) : \sigma(f^{-1}\mathcal{B}) \subset f^{-1}\sigma(\mathcal{B}),$

$G := \big\{ B \in \sigma(\mathcal{B}) : f^{-1}(B) \in \sigma(f^{-1}\mathcal{B}) \big\} \ : ??\Omega,$

$\texttt{Assume}\ B \in \mathcal{B},$

$f^{-1}(B) \in f^{-1}(\mathcal{B}) \leadsto_{\eth\sigma} f^{-1}(B) \in \sigma(f^{-1}(\mathcal{B})) \leadsto_{\jmath G} B \in G;$

$(6) : \mathcal{B} \subset G,$

$\texttt{Assume}\ B \in G,$

$B \in G \leadsto_{\jmath G} f^{-1}(B) \in \sigma(f^{-1}(\mathcal{B})),$

$\eth\sigma \leadsto \sigma\big(f^{-1}(\mathcal{B})\big) : \sigma\text{-}\texttt{Algebra}\,(\Omega) \leadsto$

$\leadsto \big(f^{-1}(B)\big)^{\complement} = f^{-1}\Big(B^{\complement}\Big) \in \sigma\big(f^{-1}(\mathcal{B})\big) \leadsto B^{\complement} \in f^{-1}(\mathcal{B});$

$(7) : \forall B \in G \ .\ B^{\complement} \in G,$

$\texttt{Assume}\ B : \mathbb{N} \to G,$

$\eth G \leadsto f^{-1}(B) : \mathbb{N} \to \sigma\big(f^{-1}(\mathcal{B})\big),$

$\eth\sigma \leadsto \sigma\big(f^{-1}(\mathcal{B})\big) : \sigma\text{-}\texttt{Algebra}\,(\Omega) \leadsto$

$$\leadsto \bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \in \sigma\big(f^{-1}(\mathcal{B})\big) \leadsto \bigcup_{n=1}^{\infty} B_n \in G;$$

$$(8) : \forall B : \mathbb{N} \to G \ .\ \bigcup_{n=1}^{\infty} B_n \in G,$$

$(7, 8) \leadsto G : \sigma\text{-}\texttt{Algebra}\,(\Omega')\ \texttt{as}\ (9),$

$\eth\sigma(6, 9) \leadsto (10) \leadsto \sigma(\mathcal{B}) \subset G,$

$\eth G \leadsto (11) : G \subset \sigma(\mathcal{B}),$

$(10, 11) \leadsto G = \sigma(\mathcal{B}) \leadsto (12) : f^{-1}\ \sigma\ \mathcal{B} \subset \sigma\ f^{-1}\ \mathcal{B},$

$(5, 12) \leadsto f^{-1}\ \sigma\ \mathcal{B} = \sigma\ f^{-1}\ \mathcal{B};;\square$

$\texttt{UT3} :: \forall \mu : \texttt{FiniteMeasure}(\Omega, \mathcal{F}) \ .\ \forall A : \texttt{DisjountElems}(\mathcal{F}, X) : \forall x \in X \ .\ A_x > 0 \ .\ \#X \le \aleph_0$

$\texttt{Proof}\ =$

$\texttt{Assume}\ \mu : \texttt{FiniteMeasure}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ A : \texttt{DisjountElems}(\mathcal{F}, X) : \forall x \in X \ .\ A_x > 0,$

$b := \Lambda n \in \mathbb{N} \ .\ 1/n,$

$\texttt{Assume}\ n \in \mathbb{N},$

$Y_n := \{x \in X : \mu(A_x) \ge b_n\},$

$\texttt{Assume}\ a : \#Y_n \ge \aleph_0,$

$a \leadsto \exists y : \mathbb{N} \hookrightarrow Y_n\ \texttt{Extract},$

$$(1) : \mu\left(\bigcup_{i=1}^{\infty} A_{y_i}\right) = \sum_{i=1}^{\infty} \mu\big(A_{y_i}\big) \ge \sum_{i=1}^{\infty} b_n = \infty,$$

$$\mu : \texttt{FiniteMeasure}(\Omega, \mathcal{F}) \leadsto \mu\left(\bigcup_{i=1}^{\infty} A_{y_i}\right) < \infty \leadsto_{(1)} \perp;$$

$(1) : \#Y_n \leq \aleph_0;$
$Y : \mathbb{N} \to ?X,$
$(1) : \forall n \in \mathbb{N} \,.\, \#Y_n < \aleph_0,$

$$\eth(A, Y) \rightsquigarrow Y \uparrow X \rightsquigarrow (2) : X = \bigcup_{n=1}^{\infty} Y_n,$$

$(1, 2, \#\mathbb{N} = \aleph_0) \rightsquigarrow \#X \leq \aleph_0; ; \square$

1.3 Measure Extension

$\texttt{PreBorel} :: \texttt{Algebra}$

$$\texttt{PreBorel} := \left\{ \bigcup_{i=1}^{n} A_i \middle| n \in \mathbb{N}, A : n \to \texttt{Semiclosed}(\mathbb{R}) \right\}$$

$\texttt{PreBorelPreMeasure} :: ?\texttt{CountablyAdditive}\,(\texttt{PreBorel})$

$\mu : \texttt{PreBorelPreMeasure} \iff \exists F : \texttt{Increasig\&RightContinuous}(\mathbb{R}) :$

$\quad : \mu(a, b] = F(b) - F(a)$

$\texttt{Extentible} :: ?\texttt{CountablyAdditive}(\Omega, \mathcal{F})\&\mathcal{F} \to \overset{\infty}{\mathbb{R}}_+$

$$\mu : \texttt{Extentible} \iff \forall A : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}) : \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \ .\ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

$\texttt{FExtentible} :: ?\texttt{Extendible}(\Omega, \mathcal{F})$

$\mu : \texttt{FExtentible} \iff \forall A \in \mathcal{F} \ .\ \mu(F) < \infty$

$\texttt{MonotonicityI} :: \forall P : \texttt{FExtendible}(\mathcal{F}, \Omega)\ .$

$\quad .\ \forall A \uparrow \alpha, B \uparrow \beta : \operatorname{Im} A \cup \operatorname{Im} B \subset \mathcal{F} : \alpha \subset \beta\ .\ \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(B_n)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ P : \texttt{FinitlyAdditive}(\mathcal{F}, \Omega) : \operatorname{Im} P \subset [0, 1] : P(\Omega) = 1,$

$\texttt{Assume}\ \forall A \uparrow \alpha, B \uparrow \beta : \operatorname{Im} A \cup \operatorname{Im} B \subset \mathcal{F} : \alpha \subset \beta\ .\ \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P(B_n),$

$\texttt{Assume}\ n, m \in \mathbb{N} : n < m,$

$A_n \subset A_m, P \geq 0 \leadsto P(A_n) \leq P(A_m),$

$B_n \subset B_m, P \geq 0 \leadsto P(B_n) \leq P(B_m);$

$P(A), P(B) : \texttt{Nondecreasing}\big(\mathbb{N}, [0, 1]\big),$

$P \leq 1 \leadsto P(A), P(B) : \texttt{Bounded}\big(\mathbb{N}, [0, 1]\big),$

$\texttt{BoundedConvergence} \leadsto P(A), P(B) : \texttt{Convergent},$

$\texttt{Assume}\ n \in \mathbb{N},$

$C := A_n \cap A',$

$C \uparrow A_n,$

$A_n \in \mathcal{F}, A : \texttt{Extendible}(\Omega, \mathcal{F}) \leadsto \lim_{m\to\infty} \mu(C_n) = \mu(A_n),$

$\texttt{Assume}\ m \in \mathbb{N},$

$C_n \subset A'_m \leadsto \mu(C_m) \leq \mu(A'_m);$

$\lim_{m\to\infty} C_m \leq \lim_{m\to\infty} A'_m \leadsto \mu(A_n) \leq \lim_{m\to\infty} A'_m;$

$\lim_{m\to\infty} A_m \leq \lim_{m\to\infty} A'_m;\ ;\square$

$\texttt{SetOfUnions} :: ??\Omega \to ??\Omega$

$$\texttt{SetOfUnions}(S) = U(S) = \left\{ \bigcup_{n=1}^{\infty} A_n \middle| A : \mathbb{N} \to S \right\}$$

$\texttt{Premeasure} :: \prod \mathcal{F} : \texttt{Algebra}(\Omega) \,.\, \prod c \in \mathbb{R}_{++} \,.\, ?U(\mathcal{F}) \to [0, c]$

$P : \texttt{Premeasure} \iff P_{|\mathcal{F}} : \texttt{Extendible}(\mathcal{F}, \Omega) : P(\Omega) = c,$

$\quad \forall A, B \in U(\mathcal{F}) \,.\, P(A \cup B) = P(A) + P(B) - P(A \cap B),$

$\quad \forall A, B \in U(\mathcal{F}) : A \subset B \,.\, P(A) \le P(B),$

$\quad \forall A : \mathbb{N} \to U(\mathcal{F}) : A \uparrow \alpha \,.\, \alpha \in S(\mathcal{F}) \wedge \lim_{n\to\infty} P(A_n) = P(\alpha).$

$\texttt{ExtensionI} :: \forall \mu : \texttt{FExtendible}(\Omega, \mathcal{F}) : \mu(\Omega) = 1 \,.\, \exists P : \texttt{Premeasure}(\Omega, \mathcal{F}) : P_{|\mathcal{F}} = \mu$

$\texttt{Proof} \;\; =$

$\texttt{Assume}\ \mu : \texttt{Extendible}(\Omega, \mathcal{F}),$

$$P := \Lambda A \in U(\mathcal{F}) \,.\, \texttt{if}\ A \in \mathcal{F}\ \texttt{then}\ \mu(A)\ \texttt{else}\ \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} \eth U(A)_i \right),$$

$\mu(\Omega) = 1 \rightsquigarrow (1) : P(\Omega) = 1,$

$\texttt{Assume}\ A, B \in U(\mathcal{F}),$

$$a := \Lambda n \in \mathbb{N} \,.\, \bigcup_{i=1}^{n} \eth U(A)_i : \mathbb{N} \to \mathcal{F} : a \uparrow A,$$

$$b := \Lambda n \in \mathbb{N} \,.\, \bigcup_{i=1}^{n} \eth U(B)_i : \mathbb{N} \to \mathcal{F} : b \uparrow B,$$

$a \cap b \uparrow A \cap B,$

$a \cup b \uparrow A \cup B,$

$\texttt{Assume}\ n \in \mathbb{N},$

$(2) : P(a_n \cup b_n) = P(a_n) + P(b_n) - P(a_n \cap b_n);$

$(2) : P(A \cup B) = P(A) + P(B) - P(A);$

$(2) : \forall A, B \in U(\mathcal{F}) \,.\, P(A \cup B) = P(A) + P(B) - P(A),$

$\texttt{MonotonicityI}(\mu) = (3) : \forall A, B \in U(\mathcal{F}) : A \subset B \,.\, P(A) \le P(B),$

$\texttt{Assume}\ A : \mathbb{N} \to U(\mathcal{F}) : A \uparrow \ \alpha,$

$\texttt{Assume}\ n \in \mathbb{N},$

$$a^n := \Lambda m \in \mathbb{N} \,.\, \bigcup_{i=1}^{m} \eth U(A_n)_i : \mathbb{N} \to \mathcal{F} : a^n \uparrow A_n;$$

$a : \mathbb{N} \to \mathbb{N} \to \mathcal{F},$

$$a' := \Lambda n \in \mathbb{N} \,.\, \bigcup_{i=1}^{n} a_n^i : \mathbb{N} \to \mathcal{F},$$

$a' \uparrow \alpha \rightsquigarrow \alpha \in U(\mathcal{F}),$

$\texttt{Assume}\ n, m \in \mathbb{N} : n \leq m,$
$a^n_m \subset a'_m \rightsquigarrow P(a^n_m) \leq P(a'_m),$
$a'_m \subset A_m \rightsquigarrow P(a'_m) \leq P(A_m) \rightsquigarrow P(a^n_m) \leq P(A_m);$
$(4) : \forall n \in \mathbb{N} : \lim_{m\to\infty} P(a^n_m) \leq \lim_{m\to\infty} P(a'_m) \leq \lim_{m\to\infty} P(A_m),$
$(4) \rightsquigarrow \forall n \in \mathbb{N} \,.\, P(A_n) \leq \lim_{m\to\infty} P(a'_n) \leq \lim_{m\to\infty} P(A_m) \rightsquigarrow$
$\rightsquigarrow \lim_{n\to\infty} P(A_n) \leq \lim_{m\to\infty} P(a'_n) \leq \lim_{m\to\infty} P(A_m) \rightsquigarrow$
$\rightsquigarrow (4) : \lim_{m\to\infty} P(A_m) = \lim_{n\to\infty} P(a'_n),$
$a' \uparrow \alpha, a' : \mathbb{N} \to \mathcal{F} \rightsquigarrow \lim_{n\to\infty} P(a'_n) = P(\alpha) \rightsquigarrow \lim_{n\to\infty} P(A_n) = P(\alpha);$
$(4) : \forall A : \mathbb{N} \to U(\mathcal{F}) : A \uparrow \alpha \,.\, \alpha \in S(\mathcal{F}) \wedge \lim_{n\to\infty} P(A_n) = P(\alpha),$
$(1, 2, 3, 4) \rightsquigarrow P : \texttt{Premeasure}(\Omega, \mathcal{F}); \square$

$\texttt{OuterMeasure}\ :: ?\left(?\Omega \to \overset{\infty}{\mathbb{R}}_+\right)$
$\mu : \texttt{OuterMeasure} \iff \mu(\emptyset) = 0,$
$\forall A, B \in ?\Omega : A \subset B \,.\, \mu(A) \leq \mu(B),$
$\forall A : \mathbb{N} \to ?\Omega \,.\, \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$

$\texttt{OuterExtension} :: \prod \mu : \texttt{Premeasure}(\Omega, \mathcal{F}) \,.\, ?(?\Omega \to \mathbb{R}^{\infty})$
$P : \texttt{OuterExtension} \iff P_{|U(\mathcal{F})} = \mu,$
$\forall A, B \in ?\Omega \,.\, P(A \cup B) + P(A \cap B) \leq P(A) + P(B),$
$\forall A, B \in ?\Omega : A \subset B \,.\, P(A) \leq P(B),$
$\forall A \uparrow_{\Omega} \alpha : \lim_{n\to\infty} P(A_n) = P(\alpha),$

$\texttt{ExtensionII} :: \forall \mu : \texttt{Premeasure}(\Omega, \mathcal{F}) \; . \; \exists P : \texttt{OuterExtension}(\mu)$

$\texttt{Proof} \;\; =$

$\texttt{Assume}\ \mu : \texttt{Premeasure}(\Omega, \mathcal{F}),$

$P := \Lambda A \in ?\Omega \; . \; \inf\{\mu(S) | S \in U(\mathcal{F}) : A \subset S\},$

$\texttt{Assume}\ A, B \in 2^\Omega,$

$\eth P \rightsquigarrow \exists a : \mathbb{N} \to U(\mathcal{F}) : \forall n \in \mathbb{N} \; . \; A \subset a_n : \mu(a_n) \downarrow P(A)\ \texttt{Extract},$

$\eth P \rightsquigarrow \exists b : \mathbb{N} \to U(\mathcal{F}) : \forall n \in \mathbb{N} \; . \; B \subset b_n : \mu(b_n) \downarrow P(B)\ \texttt{Extract},$

$\texttt{Assume}\ n \in \mathbb{N},$

$A \subset a_n, B \subset b_n \rightsquigarrow A \cap B \subset a_n \cap b_n, A \cup B \subset a_n \cup b_n,$

$\mu(a_n) + \mu(b_n) = \mu(a_n \cup b_n) + \mu(a_n \cap b_n);$

$$P(A \cap B) + P(A \cup B) \le \lim_{n\to\infty} \mu(a_n \cup b_n) + \mu(a_n \cap b_n) = \lim_{n\to\infty} \mu(a_n) + \mu(b_n) =$$
$$= P(A) + P(B);$$

$(1) : \forall A, B \in ?\Omega \; . \; P(A \cup B) + P(A \cap B) \le P(A) + P(B),$

$\texttt{Assume}\ A : A \uparrow_\Omega \alpha,$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$

$\texttt{Assume}\ n \in \mathbb{N},$

$\eth P \rightsquigarrow \exists B \in U(\mathcal{F}) : \mu(B) \le P(A_n) + \epsilon/n!\ \texttt{Extract as}\ B_n;$

$$\beta := \bigcup_{n=1}^{\infty} B_n \in U(\mathcal{F}),$$

$\texttt{Assume}\ n \in \mathbb{N},$

$$\texttt{Assume}\ P\left(\bigcup_{k=1}^{n} B_k\right) \le P(A_n) + \sum_{k=1}^{n} \epsilon/k!,$$

$$P\left(\bigcup_{k=1}^{n+1} B_k\right) = P\left(\bigcup_{k=1}^{n} B_k \cup B_{k+1}\right) = P\left(\bigcup_{k=1}^{n} B_k\right) + P(B_{n+1}) - P\left(\bigcup_{k=1}^{n} B_k \cap B_{k+1}\right)$$

$$A_k \subset \bigcup_{k=1}^{n} B_k \cap B_{k+1} \rightsquigarrow P\left(\bigcup_{k=1}^{n} B_k \cap B_{k+1}\right) \ge P(A_n),$$

$$P\left(\bigcup_{k=1}^{n+1} B_k\right) \le P(A_n) + P(A_{n+1}) + \sum_{k=1}^{n+1} \epsilon/n! - P(A_n) = P(A_{n+1}) + \sum_{k=1}^{n+1} \epsilon/n!;;$$

$\texttt{Induction} \rightsquigarrow P(\alpha) \le P(\beta) \le \lim_{n\to\infty} P(A_{n+1}) + \epsilon \mathrm{e};$

$\lim_{n\to\infty} P(A_n) \le P(\alpha) \le \lim_{n\to\infty} P(A_n) \rightsquigarrow P(\alpha) = \lim_{n\to\infty} P(A_n);$

$(2) : \forall A \uparrow_\Omega \alpha : \lim_{n\to\infty} P(A_n) = P(\alpha),$

$(1, 2) \rightsquigarrow P : \texttt{OuterExtension}(\mu); \square$

$\texttt{OuterTHM} :: \forall \mu : \texttt{Premeasure}(\Omega, \mathcal{F})\ .\ \forall P : \texttt{OuterExtension}(\mu)\ .\ P : \texttt{OuterMeasure}(\Omega)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \mu : \texttt{Premeasure}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ P : \texttt{OuterExtension}(\mu),$

$\texttt{Assume}\ A : \mathbb{N} \to ?\Omega,$

$\texttt{Assume}\ n \in \mathbb{N},$

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq P\left(\bigcup_{k=1}^{n} A_k\right) + \sum_{k=1}^{n} P\left(A_k \cap \bigcup_{i=k+1}^{n} A_i\right) \leq \sum_{k=1}^{n} P(A_k);$$

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right) \leq \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{k=1}^{\infty} P(A_k);$$

$P : \texttt{OuterMeasure}(\Omega); \square$

$\texttt{relativeAlgebra} :: \texttt{OuterMeasure}(\Omega) \to ??\Omega$

$$\texttt{relativeAlgebra}(P) = H(P) := \left\{A \in \Omega : P(A) + P\left(A^{\complement}\right) = P(\Omega)\right\}$$

$\texttt{extension} :: \texttt{FExtendible}(\Omega, \mathcal{F}) \to \texttt{OuterMeasure}(\Omega) \& \texttt{OuterExtension}(\Omega, \mathcal{F})$

$\texttt{extension}(\mu) = P_\mu := \texttt{OuterTHM}\ \texttt{ExtensionII}\ \texttt{ExtensionI}\ \mu$

$\texttt{ExtensionIII} :: \forall \mu : \texttt{FExtendible}(\Omega, \mathcal{F})\ .\ H(P_\mu) : \sigma\text{-}\texttt{Algebra}\,(\Omega) : \mathcal{F} \subset H(P_\mu)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \mu : \texttt{Premeasure}(\Omega, \mathcal{F}),$

$P := P_\mu,$

$\texttt{Assume}\ A \in U(\mathcal{F}),$

$a := \eth(U)(\mathcal{F}, A) : \mathbb{N} \to \mathcal{F} : a \uparrow A,$

$\texttt{Assume}\ n \in \mathbb{N},$

$$a_n \subset A \rightsquigarrow A^{\complement} \subset a_n^{\complement} \rightsquigarrow P\left(A^{\complement}\right) \leq P\left(a_n^{\complement}\right)$$

$$P(a_n) + P\left(A^{\complement}\right) \leq P(a_n) + P\left(a_n^{\complement}\right) = P(\Omega);$$

$$(1) : P(A) + P\left(A^{\complement}\right) \leq \lim_{n\to\infty} P(a_n) + P\left(A^{\complement}\right) \leq P(\Omega),$$

$$(2) : P(A) + P\left(A^{\complement}\right) \geq P\left(A \cup A^{\complement}\right) + P\left(A \cap A^{\complement}\right) = P(\Omega) + P(\emptyset) = P(\Omega),$$

$$(1,2) \rightsquigarrow P(A) + P\left(A^{\complement}\right) = P(\Omega) \rightsquigarrow A \in H(P);$$

$(1) : U(\mathcal{F}) \subset H(P);$

$\texttt{Assume}\ A, B \in H(P)$,

$\texttt{ExtensionII} \rightsquigarrow (2) : P(A \cap B) + P(A \cup B) \leq P(A) + P(B)$,

$\texttt{ExtensionII} \rightsquigarrow (3) : P\Big((A \cap B)^\complement\Big) + P\Big((A \cup B)\Big)^\complement) \leq P\Big(A^\complement\Big) + P\Big(B^\complement\Big)$,

$(2,3) \rightsquigarrow (4) : P\Big((A \cap B)^\complement\Big) + P\Big((A \cup B)^\complement\Big) + P(A \cap B) + P(A \cup B)$

$\quad \leq P(A) + P(B) + P\Big(A^\complement\Big) + P\Big(B^\complement\Big) = 2P(\Omega)$,

$(5) : P(\Omega) \leq P\Big((A \cup B)^\complement\Big) + P(A \cup B)$,

$(6) : P(\Omega) \leq P\Big((A \cap B)^\complement\Big) + P(A \cap B)$,

$(4,5,6) \rightsquigarrow P(\Omega) = P\Big((A \cup B)^\complement\Big) + P(A \cup B), P(\Omega) = P\Big((A \cap B)^\complement\Big) + P(A \cap B) \rightsquigarrow$

$\quad \rightsquigarrow A \cap B, A \cup B \in H(P)$;

$H(P) : (\texttt{Algebra})(\Omega)$,

$\texttt{Assume}\ A : \mathbb{N} \to H(P) : A \uparrow \alpha$,

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++}$,

$\texttt{ExtensionII} \rightsquigarrow \lim_{n \to \infty} P(A_n) = P(\alpha) \rightsquigarrow \exists n \in \mathbb{N} \,.\, P(\alpha) \leq P(A_n) + \epsilon\ \texttt{Extract}$,

$A_n \subset \alpha \rightsquigarrow \alpha^\complement \subset A_n^\complement \rightsquigarrow P\Big(\alpha^\complement\Big) \leq P\Big(A_n^\complement\Big)$,

$P(\Omega) \leq P\Big(\alpha^\complement\Big) + P(\alpha) \leq P\Big(A_n^\complement\Big) + P(A_n) + \epsilon = P(\Omega) + \epsilon$;

$P\Big(\alpha^\complement\Big) + P(\alpha) = P(\Omega) \rightsquigarrow P(\alpha) \in H(P)$;

$H(P) : \sigma\texttt{-Algebra}\,(()\,\Omega); \square$

$\texttt{extensionToSpace} : \texttt{FExtendible}(\Omega, \mathcal{F}) \to \texttt{MeasureSpace}$

$\texttt{extensionToSpace}(\mu) = \mu^{\bigcirc} := (\Omega, \sigma(\mathcal{F}), P_{\mu|\sigma(\mathcal{F})})$

$\texttt{Complete} :: ?\texttt{MeasureSpace}$

$(\Omega, \mathcal{F}, \mu) : \texttt{Complete} \iff \forall A \in \mathcal{F} : \mu(\mathcal{F}) = 0 \,.\, \forall S \subset A \,.\, S \in \mathcal{F}$

$\texttt{algCompletion} :: \texttt{Measure}(\Omega, \mathcal{F}) \to \sigma\texttt{-Algebra}\,(\Omega)$

$\texttt{algCompletion}(\mu) = C(\mu) = [\mathcal{F}] \cup [\{A \subset \Omega | \exists F \in \mathcal{F} : \mu(F) = 0 : A \subset F\}]$

$\texttt{completion} :: \texttt{MeasureSpace} \to \texttt{Complete}$

$\texttt{completion}(\Omega, \mathcal{F}, \mu) = \widehat{(\Omega, \mathcal{F}, \mu)} := (\Omega, C(\mu), \hat{\mu} := \Lambda A \cup N \in_{\eth} C(\mu) \,.\, \mu(A)$

$\texttt{ComplementaryCompletion} :: \forall \mu : \texttt{FExtendible}(\Omega, \mathcal{F}) \;.\; \widehat{\mu^{\bigcirc}} = (\Omega, H(P_\mu), P_\mu)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mu : \texttt{FExtendible}(\Omega, \mathcal{F}),$

$P := P_\mu,$

$\texttt{Assume}\ A \in H(P),$

$\eth H(P)(A) \rightsquigarrow A^{\complement} \in H(P),$

$a := \eth P(A) : \mathbb{N} \to U(\mathcal{F}) : P(a) \downarrow P(A),$

$b := \eth P\left(A^{\complement}\right) : \mathbb{N} \to U(\mathcal{F}) : P(b) \downarrow P\left(A^{\complement}\right),$

$a' := b^{\complement} : \mathbb{N} \to \sigma(\mathcal{F})$

$\eth H(P)(A) \rightsquigarrow (1) : P(\Omega) = P(A) + P\left(A^{\complement}\right) = \lim_{n\to\infty} P(a_n) + \lim_{n\to\infty} P(b_n);$

$(2) : P(\Omega) = \lim_{n\to\infty} P(b_n) + P(a'_n);$

$(1,2) \rightsquigarrow \lim_{n\to\infty} P(a'_n) = \lim_{n\to\infty} P(a_n) = P(A);$

$\alpha := \bigcap_{n=1}^{\infty} a_n \in \sigma(\mathcal{F}),$

$\alpha' := \bigcup_{n=1}^{\infty} a'_n \in \sigma(\mathcal{F}),$

$\beta := \alpha \cap \alpha'^{\complement} \in \sigma(\mathcal{F}),$

$P(A) = P(\alpha) = P(\alpha') + P(\beta) = P(A) + P(\beta) \rightsquigarrow P(\beta) = 0,$

$\alpha' \subset A \subset \alpha \rightsquigarrow A \cap \alpha'^{\complement} \subset \beta \rightsquigarrow A = \alpha' \cup \left(A \cap \alpha'^{\complement}\right) \in C(P, \sigma(\mathcal{F}));$

$(1) : H(P) \subset C(P, \sigma(\mathcal{F})),$

$\texttt{Assume}\ A \in C(P, \sigma(\mathcal{F})),$

$(B, N) := \eth C(P, \sigma(\mathcal{F})) : \sigma(\mathcal{F}) \times \left\{A \in 2^{\Omega} : \exists N \in \sigma(\mathcal{F}) : P(N) = 0 : A \subset P\right\} : A = B \cup N,$

$B \in \sigma(\mathcal{F}) \rightsquigarrow B \in H(P),$

$M = \eth(N) \in \sigma(\mathcal{F}) : P(M) = 0 : N \subset M,$

$M \in \sigma(\mathcal{F}) \rightsquigarrow M \in H(P),$

$\eth\texttt{Complete}(\Omega, H(P), P)(N.M) \rightsquigarrow N \in H(P),$

$B, N \in H(P) \rightsquigarrow A = B \cup H \in H(P);$

$(2) : C(P, \sigma(F)) \subset H(P),$

$(1,2) \rightsquigarrow C(P, \sigma(F) = H(P); \square$

$\texttt{MonotoneClass} :: ???\Omega$

$M : \texttt{MonotoneClass} \iff \forall A : \mathbb{N} \to M : A \downarrow \alpha \,.\, \alpha \in M,$

$\forall A : \mathbb{N} \to M : A \uparrow \alpha \,.\, \alpha \in M,$

$\texttt{MonotoneClassTHM} :: \forall M : \texttt{MonotoneClass}(\Omega) \,.\, \forall \mathcal{F} : \texttt{Algebra}(\Omega) : \mathcal{F} \subset M \,.\, \sigma(\mathcal{F}) \subset M$

$\texttt{Proof} =$

$\texttt{Assume}\ M : \texttt{MonotoneClass}(\Omega)$

$\texttt{Assume}\ \mathcal{F} : \texttt{Algebra}(\Omega) : \mathcal{F} \subset M,$

$N := \min\{N : \texttt{MonotoneClass}(\Omega) : \mathcal{F} \subset N\},$

$\eth N(M) \rightsquigarrow N \subset M$

$\texttt{Assume}\ A \in \mathcal{F},$

$N' = \{B \in N : A \cap B \in N \wedge A^{\complement} \cap B \in N \wedge A \cap B^{\complement} \in N\} \subset N,$

$\eth\texttt{Algebra}(\mathcal{F}) \rightsquigarrow \mathcal{F} \subset N',$

$\eth(N'), \eth\bigcap, \eth\bigcup \rightsquigarrow N' : \texttt{MonotoneClass}(\Omega) \rightsquigarrow N = N';$

$(1) : \forall A \in \mathcal{F} \,.\, \forall B \in N \,.\, A \cap B \in N \wedge A^{\complement} \cap B \in N \wedge A \cap B^{\complement} \in N,$

$\texttt{Assume}\ A \in N,$

$N' = \{B \in N : A \cap B \in N \wedge A^{\complement} \cap B \in N \wedge A \cap B^{\complement} \in N\} \subset N,$

$(1) \rightsquigarrow \mathcal{F} \subset N',$

$\eth(N'), \eth\bigcap, \eth\bigcup \rightsquigarrow N' : \texttt{MonotoneClass}(\Omega) \rightsquigarrow N = N';$

$N : \texttt{Algebra}(\Omega),$

$N : \texttt{MonotoneClass}(\Omega) \& \texttt{Algebra}(\Omega) \rightsquigarrow N : \sigma\text{-}\texttt{Algebra}\,(\Omega) \rightsquigarrow \sigma(F) \subset N \rightsquigarrow \sigma(F) \subset M; ; \square$

$\sigma\text{-}\texttt{Finite} ::? \sum \mathcal{F} : \texttt{Algebra}(\Omega) \,.\, \mathcal{F} \to \overset{\infty}{\mathbb{R}}_+$

$(\mathcal{F}, \mu) : \sigma\text{-}\texttt{Finite}\,(\Omega) \iff \exists A : \mathbb{N} \to \mathcal{F} : \forall n \in \mathbb{N} \,.\, \mu(A_n) < \infty : A \uparrow \Omega$

$\texttt{CaratheodoryExtension} :: \forall(\mathcal{F},\mu) : \sigma\texttt{-Finite}\,(\Omega) : \mu : \texttt{Extendible}(\mathcal{F},\Omega)\ .$

$\quad .\ \exists!\lambda : \texttt{Measure}(\Omega,\sigma(\mathcal{F})) : \lambda_{|\mathcal{F}} = \mu$

$\texttt{Proof}\ =$

$\texttt{Assume}\ (\mathcal{F},\mu) : \sigma\texttt{-Finite}\,(\Omega) : \mu : \texttt{Extendible}(\mathcal{F},\Omega),$

$\omega := \texttt{toDisjoint}\ \eth\sigma\texttt{-Finite}\,(\Omega)\,(\mu,\mathcal{F}),$

$\texttt{Assume}\ n \in \mathbb{N},$

$p := \Lambda A \in \mathcal{F}\ .\ \mu(\omega_n \cap A) : \texttt{Extendible}(\Omega,\mathcal{F}),$

$\mu(\omega_n) < \infty \rightsquigarrow p : \texttt{FExtendible}(\Omega,\mathcal{F}),$

$P_n := \texttt{extension}(p) : \texttt{FiniteMeasure}(\Omega,\sigma(\mathcal{F}));$

$$\lambda := \Lambda A \in \sigma(\mathcal{F})\ .\ \sum_{n=1}^{\infty} P_n(A) : \sigma(\mathcal{F}) \to \overset{\infty}{\mathbb{R}}_+;$$

$\texttt{Assume}\ \lambda' : \texttt{Measure}(\Omega,\sigma(\mathcal{F})) : \lambda'_{|\mathcal{F}} = \mu,$

$\eth\lambda \rightsquigarrow (1) : \lambda'_{|\mathcal{F}} = \lambda_{|\mathcal{F}},$

$\texttt{Assume}\ m \in \mathbb{N},$

$P'_n := \Lambda A \in \sigma(\mathcal{F})\ .\ \lambda'(A \cap \omega_n) : \texttt{FiniteMeasure}(\Omega,\sigma(\mathcal{F})),$

$M := \{A \in \sigma(\mathcal{F}) : P(A) = P'(A')\},$

$P, P' : \texttt{Measure}(\Omega,\mathcal{F}) \rightsquigarrow M : \texttt{MonotomeClass}(\Omega),$

$(1) \rightsquigarrow \mathcal{F} \subset M$

$,\texttt{MonotoneClassTHM} \rightsquigarrow \sigma(\mathcal{F}) \subset M \rightsquigarrow P_n = P'_n;$

$(2) : \forall n \in \mathbb{N}\ .\ P_n = P'_m,$

$$(\eth(\lambda',P',\omega),2) \rightsquigarrow \lambda' = \sum_{n=1}^{\infty} P'_n = \sum_{n=1}^{\infty} P_n = \lambda; \square$$

$\texttt{ApproximationI} :: \forall\mathcal{F} : \texttt{Algebra}\ .\ \forall P : \texttt{FiniteMeasure}(\Omega,\sigma(\mathcal{F}))\ .$

$\quad .\ \forall A \in \sigma(\mathcal{F})\ .\ \forall\epsilon \in \mathbb{R}_{++}\ .\ \exists B \in \mathcal{F} : P(A \triangle B) \le \epsilon$

$\texttt{Assume}\ F : \texttt{Algebra},$

$\texttt{Assume}\ P : \texttt{FiniteMeasure}(\Omega,\sigma(\mathcal{F})),$

$\texttt{Assume}\ A \in \sigma(\mathcal{F}),$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$

$B := \texttt{ExtensionII}(A) : \mathbb{N} \to U(\mathcal{F}) : \forall n \in \mathbb{N}\ .\ A \subset B_n\ .\ P(B) \downarrow P(A),$

$P(B) \downarrow P(A) \rightsquigarrow \exists n \in \mathbb{N} : P(B_n) \le P(A) + \epsilon/2,\ \texttt{Extract}$

$C = \eth U(\mathcal{F})(B_n) : \mathbb{N} \to \mathcal{F} : C \uparrow B_n,$

$P : \texttt{UpperContinous}(\sigma(\mathcal{F})) \rightsquigarrow \lim\limits_{m\to\infty} P(C_m) = P(B_n) \rightsquigarrow$

$\quad \rightsquigarrow \exists n \in \mathbb{N} : P(B_n) \le P(C_m) + \epsilon/2,$

$P(A \triangle C_m) \le P(A \triangle B_n \cup B_n \triangle C_m) =$

$\quad = P(A \triangle B_n) + P(B_n \triangle C_m) - P(A \triangle B_n \cap B_n \triangle C_m) \le$

$\leq P(A \triangle B_n) + P(B_n \triangle C_m) = P(B_n) - P(A) + P(B_n) - p(C_m) = \epsilon;;;;\square$

$\texttt{ApproximationII} :: \forall \mathcal{F} : \texttt{Algebra}(\Omega) \,.\, \forall \mu : \texttt{Measure}(\Omega, \sigma(\mathcal{F})) : (\mu, \mathcal{F}) : \sigma\texttt{-Finite}\,(\Omega) \,.$
$\quad .\, \forall A \in \sigma(\mathcal{F}) \,.\, \forall \epsilon \in \mathbb{R}_{++} \,.\, \exists B \in \mathcal{F} : \mu(A \triangle B) \leq \epsilon$

$\texttt{Assume}\ F : \texttt{Algebra},$

$\texttt{Assume}\ \mu : \texttt{Measure}(\Omega, \sigma(\mathcal{F})) : (\mu, \mathcal{F}) : \sigma\texttt{-Finite}\,(\Omega)\,,$

$\texttt{Assume}\ A \in \sigma(\mathcal{F}),$

$\texttt{Assume}\ e \in \mathbb{R}_{++},$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$

$$\omega := \texttt{toDisjoint}\ \eth\sigma\texttt{-Finite}\,(\Omega)\,(\mathcal{F}, \mu) : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}) : \forall n \in \mathbb{N} \,.\, \mu(\omega_n) < \infty : \bigcup_{n=1}^{\infty} \omega_n = \Omega,$$

$\texttt{Assume}\ n \in \mathbb{N},$

$P_n := \Lambda A \in \sigma(\mathcal{F}) \,.\, \mu(A \cup \omega_n) : \texttt{FiniteMeasure}(\Omega, \mathcal{F}),$

$B_n := \texttt{ApproximationI}(\mathcal{F})(P_n)(A)(\epsilon/\mathrm{e}(n-1)!) \in F : P(B_n \triangle A) \leq \epsilon/\mathrm{e}(n-1)!,$

$P_n(A \triangle B_n) = \mu((A \triangle B_n) \cap \omega_n) = \mu((A \triangle (B_n \cap \omega_n)) \cap \omega_n) = P_n(A \triangle (B_n \cap \omega_n));$

$$C := \bigcup_{n=1}^{\infty} B_n,$$

$\texttt{Assume}\ n \in \mathbb{N};$

$P_n(A \triangle C) = P_n(A \triangle B_n);$

$$\mu(A \triangle C) = \sum_{n=1}^{\infty} P_n(A \triangle C) = \sum_{n=1}^{\infty} (A \triangle B_n) \leq \epsilon;$$

$$0 = \sum_{n=1}^{\infty} P_n(A \triangle B_n) = \sum_{n=1}^{\infty} P_n(A \setminus B_n \cup B_n \setminus A) = \sum_{n=1}^{\infty} P_n(A \setminus B_n) + P_n(B_n \setminus A) =$$

$$= \sum_{n=1}^{\infty} P_n(A \setminus B_n) + \sum_{n}^{\infty} P_n(B_n \setminus A) = \lim_{n \to \infty} \mu\left(A \setminus \bigcup_{i=1}^{n} B_i\right) + \lim_{n \to \infty} \mu\left(\bigcup_{i=1}^{n} B_i \setminus A\right) \rightsquigarrow$$

$$\rightsquigarrow \exists n \in \mathbb{N} : \mu\left(\bigcup_{i=1}^{n} B_i \setminus A\right) \leq e/2 \wedge \mu\left(A \setminus \bigcup_{i=1}^{n} B_i\right) \leq e/2\texttt{Extract},$$

$$\beta := \bigcup_{i=1}^{n} B_i \in \mathcal{F},$$

$\mu\,(A \triangle \beta) = \mu(A \setminus \beta) + \mu(\beta \setminus A) \leq e;;;;\square$

1.4 Lebesgue-Steltjes Measures and Distributions on the Real Line

$\texttt{borelSets} :: \prod X : \texttt{TopologicalSpace} \, . \, \sigma\texttt{-Algebra}\,(X)$

$\texttt{borelSets} = \mathcal{B}(X) := \sigma(\mathcal{T}_X)$

$\texttt{Lebesgue-Stieltjes}(\mathbb{R}) :: ?\texttt{Measure}(\mathbb{R}, \mathcal{B}(\mathbb{R})),$

$\mu : \texttt{Lebesgue-Stieltjes} \iff \forall I \in \mathcal{B}(\mathbb{R}) : \texttt{Bounded} \, . \, \mu(I) < \infty$

$\texttt{DistributionFunction}(\mathbb{R}) \;:: ?\texttt{RightContinuous\&Increasing}\left(\overset{\infty}{\mathbb{R}}, \overset{\infty}{\mathbb{R}}\right)$

$F : \texttt{DistributionFunction}(\mathbb{R}) \iff \lim_{x\to\infty} F(x) = F(\infty)$

$\texttt{MeasureAsDistribution} :: \forall \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}) \, . \, \forall x, c \in \mathbb{R} \, .$

$\quad . \, \exists F : \texttt{DistributionFunction}(\mathbb{R}) : F(x) = c : \forall (a, b] \in \texttt{SemiClosed}(\mathbb{R}) \, .$

$\quad . \, \mu(a, b] = F(b) - F(a)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}),$

$\texttt{Assume}\ x, c \in \mathbb{R},$

$F :: \mathbb{R} \to \mathbb{R}$

$F(x) = c$

$F(a) =$

$|a < x = -\mu(a, x] + F(x)$

$|a > x = \mu(x, a] - F(x),$

$\texttt{Assume}\ a, b \in \mathbb{R} : b > a,$

$\eth F \leadsto F(b) - F(a) = \mu(a, b] \geq 0 \leadsto F(b) \geq F(a);$

$F : \texttt{Increasing}(\mathbb{R}, \mathbb{R}),$

$\texttt{Assume}\ a : \mathbb{N} \to \mathbb{R} : a \downarrow A,$

$\texttt{Assume}\ n \in \mathbb{N},$

$(A, a] \downarrow \emptyset \leadsto \lim_{n\to\infty} F(a_n) - F(A) = \lim_{n\to\infty} \mu(A, a_n] = 0 \leadsto \lim_{n\to\infty} F(a_n) = F(A);$

$F : \texttt{RightContinuous}(\mathbb{R}, \mathbb{R}),$

$F : \texttt{DistributionFunction}(\mathbb{R});; \square$

$\texttt{toDistribution} :: \texttt{Lebesgue-Stieltjes}(\overset{\infty}{\mathbb{R}}) \to \texttt{DistributionFunction}(\overset{\infty}{\mathbb{R}})$

$\texttt{toDistribution}(\mu) = F_\mu := \texttt{MeasureAsDistribution}(\mu, 0, 0)$

$\texttt{DistributionAsMeasure} :: \forall F : \texttt{DistributionFunction}(\mathbb{R}) \,.\, \exists! \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}) :$

$: \texttt{MeasureAsDistribution}(\mu, 0, F(0)) = F$

$\texttt{Proof} \ =$

$\texttt{Assume}\ F : \texttt{DistributionFunction}(\mathbb{R})$

$\mu :: \texttt{Preborel} \to \overset{\infty}{\mathbb{R}}_+$

$\mu(\emptyset) = 0$

$\mu(\mathbb{R}) = \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} F(x)$

$\mu(a, \infty] = \lim_{x\to\infty} F(x) - F(a)$

$\mu(\infty, a] = F(a) - \lim_{x\to-\infty} F(x)$

$\mu(a, b] = F(b) - F(a)$

$$\mu\left(\bigsqcup_{i=1}^{n} I_n\right) = \sum_{i=1}^{n} \mu(I_n)$$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ I : \texttt{DisjointElems}(\texttt{Preborel}, A),$

$$\eth\mu \rightsquigarrow \mu\left(\bigsqcup_{i=1}^{n} I_n\right) = \sum_{i=1}^{n} \mu(I_n);$$

$\mu : \texttt{CountablyAdditive}(\mathbb{R}, \texttt{Preborel}),$

$\texttt{Assume}\ A : \mathbb{N} \to \texttt{Preborel} : A \uparrow \alpha : \alpha \in \texttt{Preborel}$

$$\bigsqcup_{i=1}^{n} I_n := \alpha$$

$$\mu\left(\bigsqcup_{i=1}^{n} I_n\right) = \sum_{k=1}^{n} \mu(I_k);$$

$\texttt{Assume}\ k \in n,$

$B^k := A \cap I_k,$

$a^k := \inf B^k,$

$b^k := \sup B^k,$

$C^k := (a^k, b^k) \setminus B_k,$

$\lim_{m\to\infty} \mu(B^k_m) = \lim_{m\to\infty} \mu(a^k_m, b^k_m] - \mu(C^k_m) = F(\lim_{m\to\infty} b^k_m) - \lim_{m\to\infty} F(a^k_m),$

$F : \texttt{RightContinouos} \rightsquigarrow \lim_{m\to\infty} \mu(B^k_m) = F(\lim_{m\to\infty} b^k_m) - F(\lim_{m\to\infty} a^k_m) = \mu(I_k);$

$$\lim_{m\to\infty} \mu(A_m) = \sum_{k=1}^{n} \lim_{m\to\infty} \mu(B^k_m) = \sum_{k=1}^{n} \mu(I_n) = \mu(\alpha);$$

$\mu : \texttt{Extendible}(\mathbb{R}, \texttt{Preborel}),$

$\lim_{n\to\infty} (-n, n] = \mathbb{R} \rightsquigarrow (\mu, \texttt{Preborel}) : \sigma\text{-}\texttt{Finite}\,(\mathbb{R})\,,$

$\lambda := \texttt{CarathedoryExtension}(\mu)$,
$\texttt{Assume}\ I : \mathcal{B}(\mathbb{R}) : I : \texttt{Bounded}(\mathbb{R})$,
$\eth\texttt{Bounded}(\mathbb{R}) \rightsquigarrow \exists a, b \in \mathbb{R} : a < b : I \subset (a, b]\ \texttt{Extract}$,
$\lambda(I) \leq \lambda(a, b] = F(b) - F(a) < \infty$;
$\lambda : \texttt{Lebesgue-Stieltjes}(\mathbb{R})$,
$\jmath\lambda \rightsquigarrow \texttt{MeasureAsDistribution}(\lambda, 0, F(0)) = F; \square$

$\texttt{toMeasure} :: \texttt{DistributionFunction}(\mathbb{R}) \to \texttt{Lebesgue-Stieltjes}(\mathbb{R})$
$\texttt{toMeasure}(F) = \mu_F := \texttt{DistributionAsMeasure}(F)$

$\texttt{LebesgueMeasure} :: \texttt{Lebesgue-Stieltjes}(\mathbb{R})$
$\texttt{LebesgueMeasure} = \lambda := \texttt{ToMeasure}(\mathrm{id})$

$\texttt{LebesgueMesurable} :: ??\mathbb{R}$
$\texttt{LebesgueMesurable} = \overline{\mathcal{B}}(\mathbb{R}) := \mathcal{B}(\mathbb{R}) \cup \{A \subset \mathbb{R} : \exists B \in \mathcal{B}(\mathbb{R}) : A \subset B : \lambda(B) = 0\}$

$\texttt{Lebesgue-Stieltjes}(\mathbb{R}^n) :: ?\texttt{Measure}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
$\mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n) \iff \forall a, b \in \mathbb{R}^n :: a \prec b \ .\ \mu(a, b] < \infty$

$\texttt{DistributionFunction}(\mathbb{R}^n) :: \mathbb{R}^n \to \mathbb{R}$
$F : \texttt{DistributionFunction}(\mathbb{R}^n) :: \forall m \in \mathbb{N} \ .\ \forall a \in \mathbb{R}^{n-1}\ .$
$\quad .\ \Lambda x \in \mathbb{R} \ .\ F(a_{1\ldots(m-1)} \oplus x \oplus a_{m\ldots n+1}) : \texttt{DistributionFunction}(\mathbb{R})$

$\texttt{Difference} :: (\mathbb{R}^n \to \mathbb{R}) \to \mathbb{R} \to \mathbb{R} \to n \to \mathbb{R}^{n-1} \to \mathbb{R}$
$\texttt{Difference}(F, a, b, m) = \Delta^m_{b,a} F :=$
$\quad := \Lambda x \in \mathbb{R} \ .\ F(x_{1\ldots(m-1)} \oplus b \oplus a_{m\ldots n+1}) - F(x_{1\ldots(m-1)} \oplus a \oplus a_{m\ldots n+1})$

$\texttt{toMeasure} :: \texttt{DistributionFunction}(\mathbb{R}^n) \to \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n)$
$\texttt{toMeasure}(F) = \mu_F := \texttt{CaratheodoryExtension}(\mu)$
$\quad \texttt{where}$
$\qquad \mu(a, b] = \left(\bigcirc_{i=1}^n \Delta^i_{a_i, b_i}\right) F$
$\ldots$

$\texttt{CompactApproximationI} :: \forall \mu : \texttt{FiniteMeasure}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \mathrel{.} \forall A \in \mathcal{B}(\mathbb{R}^n) \mathrel{.}$

$\quad . \ \mu(A) = \sup \left\{ \mu(K) \middle| K : \texttt{Compact}(\mathbb{R}^n) : K \subset A \right\}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ \big(\mu, \mathcal{B}(\mathbb{R}^n)\big) : \sigma\texttt{-Finite}(\mathbb{R}^n) : \mu : \texttt{FiniteMeasure},$

$G := \Big\{ A \in \mathcal{B}(\mathbb{R}^n) : \mu(A) = \sup \left\{ \mu(K) \middle| K : \texttt{Compact}(\mathbb{R}^n) : K \subset A \right\} \Big\},$

$\texttt{Assume} \ A \uparrow_G \alpha,$

$\texttt{Assume} \ \epsilon \in \mathbb{R}_{++},$

$\texttt{Assume} \ n \in \mathbb{N},$

$\eth A(n) \rightsquigarrow A_n \in G,$

$A_n \in G, \eth \sup(A_n, \epsilon) \rightsquigarrow \exists K \in \texttt{Compact} : K \subset B : \mu(A_n) \leq \mu(K) + \epsilon \ \texttt{Extract as} \ K_n,$

$A_n \subset \alpha, K_n \subset \alpha \rightsquigarrow K_n \subset \alpha,$

$K_n \subset \alpha \rightsquigarrow \mu(K_n) \leq \mu(\alpha);$

$\lim_{n\to\infty} \mu(K_n) \leq \mu(\alpha) = \lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \mu(K_n) + \epsilon;$

$\mu(K) \uparrow \mu(\alpha) \rightsquigarrow \alpha \in G;$

$\texttt{Assume} \ A \downarrow_G \alpha,$

$\texttt{Assume} \ \epsilon \in \mathbb{R}_{++},$

$\texttt{Assume} \ n \in \mathbb{N},$

$\eth A(n) \rightsquigarrow A_n \in G,$

$A_n \in G, \eth \sup(A_n, \epsilon 2^{-n}) \rightsquigarrow \exists K \in \texttt{Compact} : K \subset B : \mu(A_n) \leq \mu(K) + \epsilon 2^{-n} \ \texttt{Extract as} \ K_n;$

$$C := \bigcap_{n=1}^{\infty} K_n,$$

$\eth C \rightsquigarrow \forall n \in \mathbb{N} \mathrel{.} C \subset A_n \rightsquigarrow C \subset A,$

$$\mu(\alpha) - \mu(C) = \mu(\alpha \setminus C) \leq \mu \left(\bigcup_{n=1}^{\infty} (B_n \setminus K_n) \right) \leq \sum_{n=1}^{\infty} \mu(B_n \setminus K_n) \leq \epsilon;$$

$\mu(\alpha) = \sup \left\{ \mu(K) \middle| K : \texttt{Compact}(\mathbb{R}^n) : K \subset A \right\} \rightsquigarrow \alpha \in G;$

$G : \texttt{MonotoneClass},$

$\texttt{Assume} \ A \in \texttt{Preborel}(\mathbb{R}^n),$

$$\eth\texttt{Preborel}(A) \rightsquigarrow \exists n \in \mathbb{N} : \exists (a,b] : n \to \texttt{Halfinterval}(\mathbb{R}^n) : A = \bigsqcup_{k=1}^{n} (a_k, b_k] \ \texttt{Extract},$$

$\texttt{Assume} \ k \in n,$

$x := \Lambda m \in \mathbb{N} \mathrel{.} a_k + (b_k - a_k)/(2n),$

$\forall m \in \mathbb{N} \mathrel{.} [x_m, b_k] : \texttt{Compact}(\mathbb{R}^n),$

$[x, b_k] \uparrow (a_k, b_k] \rightsquigarrow (a_k, b_k] \in G;$

$G : \texttt{MonotoneClass}(\mathbb{R}^n) \rightsquigarrow A \in G;$

$\texttt{Preborel} \subset G,$

$G : \texttt{MonotoneClass}(\mathbb{R}^n) \rightsquigarrow \mathcal{B}(\mathbb{R}^n) \subset G; \square$

$\texttt{CompactApproximationII} :: \forall (\mathcal{B}(\mathbb{R}^n), \mu) : \sigma\texttt{-Finite}(\mathbb{R}^n) \;.\; \forall A \in \mathcal{B}(\mathbb{R}^n) \;.$

$\quad . \; \mu(A) = \sup \left\{ \mu(K) \middle| K : \texttt{Compact}(\mathbb{R}^n) : K \subset A \right\}$

$\texttt{Proof} \;=$

$\texttt{Assume} \; (\mathcal{B}(\mathbb{R}^n), \mu) : \sigma\texttt{-Finite}(\mathbb{R}^n),$

$$A := \eth\sigma\texttt{-Finite}(\mathbb{R}^n)\,(\mathcal{B}(\mathbb{R}^n), \mu) : \mathbb{N} \to \mathcal{B}(\mathbb{R}^n) : \mathbb{R}^n = \bigcup_{n=1}^{\infty} A_n : \forall n \in \mathbb{N} \;.\; \mu(A_n) < \infty,$$

$\texttt{Assume} \; B \in \mathcal{B}(\mathbb{R}^n),$

$\texttt{Assume} \; n \in \mathbb{N},$

$$\beta_n := B \cap \bigcup_{k=1}^{n} A_k$$

$M_n := \Lambda X \in \mathcal{B}(\mathbb{R}^n) \;.\; \mu(X \cap \beta_n)$

$K_n := \texttt{CompactApproximationI}(M_n, B, 1/n);$

$$\lim_{n\to\infty} \mu(K_n) \le \mu(B) = \lim_{n\to\infty} M_n(B) \le \lim_{n\to\infty} M_n(K_n) + 1/n = \lim_{n\to\infty} M_n(K_n) \le \lim_{n\to\infty} \mu(K_n) \rightsquigarrow$$

$$\quad \rightsquigarrow \mu(K) \uparrow \mu(B);\,;\square$$

$\texttt{OpenApproximationI} :: \forall \mu : \texttt{FiniteMeasure}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \;.\; \forall A \in \mathcal{B}(\mathbb{R}^n) \;.$

$\quad . \; \mu(A) = \sup \left\{ \mu(K) \middle| K : \texttt{Compact}(\mathbb{R}^n) : K \subset A \right\}$

$\texttt{Proof} \;=$

$\texttt{Assume} \; \mu : \texttt{FiniteMeasure}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)),$

$\texttt{Assume} \; A \in \mathcal{B}(\mathbb{R}^n),$

$\eth\sigma\texttt{-Algebra}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\,(A) \to A^{\complement} \in \mathcal{B}(\mathbb{R}^n),$

$$K := \texttt{CompactApproximationI}(\mu)\left(A^{\complement}\right) : \mathbb{N} \to \texttt{Compact}(\mathbb{R}^n) : \mu(K) \uparrow \mu\left(A^{\complement}\right),$$

$$\lim_{n\to\infty} \mu\left(K_n^{\complement}\right) = \lim_{n\to\infty} \mu(\Omega) - \mu(K_n) = \mu(\Omega) - \lim_{n\to\infty} K_n = \mu(\Omega) - \mu\left(A^{\complement}\right) = \mu(A) \rightsquigarrow$$

$$\quad \rightsquigarrow \mu\left(K^{\complement}\right) \downarrow \mu(A),$$

$\eth\texttt{Closed}(\mathbb{R}^n)(\eth\texttt{Compact}(\mathbb{R}^n)(K)) \rightsquigarrow K^{\complement} : \mathbb{N} \to \texttt{Open}(\mathbb{R}^n)$

$\texttt{OpenApproximationII} :: \forall(\mathcal{B}(\mathbb{R}^n), \mu) : \sigma\texttt{-Finite}(\mathbb{R}^n) : \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n) \,.\, \forall A \in \mathcal{B}(\mathbb{R}^n) \,.$

$\quad . \, \mu(A) = \inf \left\{ \mu(U) \middle| U : \texttt{Open}(\mathbb{R}^n) : A \subset U \right\}$

$\texttt{Proof} \; =$

$\texttt{Assume} \; (\mathcal{B}(\mathbb{R}^n), \mu) : \sigma\texttt{-Finite}(\mathbb{R}^n) : \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n)$

$\texttt{Assume} \; A \in \sigma\texttt{-Algebra}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

$$B := \eth\sigma\texttt{-Finite}(\mathbb{R}^n)\,(\mathcal{B}(\mathbb{R}^n), \mu) : \mathbb{N} \to \mathcal{B}(\mathbb{R}^n) : \mathbb{R}^n = \bigsqcup_{n=1}^{\infty} B_n :$$

$\quad : \forall n \in \mathbb{N} \,.\, \mu(B_n) < \infty : B_n : \texttt{Bounded}(\mathbb{R}^n),$

$C := \eth\texttt{Lebesgue-Stieltjes}(\mathbb{R}^n)(B) : \prod n \in \mathbb{N} \,.\, \mathcal{U}(B_n) : \forall n \in \mathbb{N} \,.\, \mu(C_n) < \infty,$

$\texttt{Assume} \; \epsilon \in \mathbb{R}_{++},$

$\texttt{Assume} \; n \in \mathbb{N},$

$M_n := \Lambda S \in \mathcal{B}(\mathbb{R}^n) \,.\, \mu(S \cap C_n) : \texttt{FiniteMeasure}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

$U_n := C_n \cap \texttt{OpenApproximationI}(M_n, A \cap B_n, \epsilon/2^n) : \texttt{Open}(\mathbb{R}^n) : M_n(U_n) - M_n(A \cap B_n) \leq \epsilon/2^n : A \subset U_n;$

$$O := \bigcup_{n=1}^{\infty} U_n,$$

$$\mu(A) \leq \; \mu(O) \leq \sum_{n=1}^{\infty} \mu(U_n) = \sum_{n=1}^{\infty} M_n(U_n) \leq$$

$$\leq \sum_{n=1}^{\infty} M_n(A \cap B_n) + \epsilon = \sum_{n=1}^{\infty} \mu(A \cap B_n) + \epsilon = \mu(A) + \epsilon;$$

$\mu(A) = \inf \left\{ \mu(U) \middle| U : \texttt{Open}(\mathbb{R}^n) : A \subset U \right\};; \square$

$\texttt{Example}$ 1.3.1

$$F(x) := \begin{cases} 0 & x < -1 \\ 1 + x & -1 \leq x < 0 \\ 2 + x^2 & 0 \leq x < 2 \\ 9 & x \geq 2 \end{cases} : \texttt{DistributionFunction}(\mathbb{R}, \mathcal{B}\mathbb{R})$$

$\mu := \texttt{ToMeasure} : \texttt{Measure}(\mathbb{R}, \mathcal{B}\mathbb{R})$

$\mu\{2\} = 3,$

$\mu[-1/2, 3) = 9 - 1/2 = 8.5,$

$\mu(-1, 0] \cup (1, 2) = 2 + (6 - 3) = 5$

$\mu[0, 1/2) \cup (1, 2] = (2.25 - 2) + (9 - 3) = 6.25$

$\mu(-\infty, -1/2) \cup (1/2, \infty) = 1/2 + (9 - 2.25) = 7.25$

$\texttt{Example}$ 1.3.2

$\texttt{Assume}\ \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}) : F_\mu \in \mathcal{M}_{\mathsf{TOP}}(\mathbb{R}, \mathbb{R}),$

$\texttt{Assume}\ N : \texttt{Countable}(\mathbb{R}),$

$$\mu(N) = \sum_{n \in N} \mu\{n\} = \sum_{n \in N} F(n) - \lim_{x \to n-0} F(x) = 0;$$

$\texttt{Assume}\ \mu = U[0,1],$

$\mu([0,1] \cap \mathbb{Q}) = 1,$

$\mu\left([0,1]^{\complement}\right) = 0,$

$\texttt{Assume}\ A : \mathcal{B}\mathbb{R}\ !\ \texttt{Dense}(\mathbb{R}),$

$I := \eth A : \texttt{Open}(\mathbb{R}) : I \subset A^{\complement},$

$\lambda\left(A^{\complement}\right) \geq \lambda(I) \geq 0;;$

$\texttt{BorelTranslationInvariance} :: \forall A \in \mathcal{B}\mathbb{R}^n\ .\ \forall a \in \mathbb{R}^n\ .\ a + A \in \mathcal{B}\mathbb{R}^n \wedge -A \in \mathcal{B}\mathbb{R}^n$

$\texttt{Proof}\ =$

$G := \{A \in \mathcal{B}\mathbb{R}^n : \forall a \in \mathbb{R}^n : a + A \in \mathcal{B}\mathbb{R}^n \wedge -A \in \mathcal{B}\mathbb{R}^n\} :?\mathcal{B}\mathbb{R}^n,$

$\texttt{Assume}\ (a, b] \in \texttt{Halfinterval}(\mathbb{R}^n)$

$\texttt{Assume}\ r \in \mathbb{R}^n,$

$r + (a, b] = (a + r, b + r] : \texttt{Halfinterval}(\mathbb{R}^n) \rightsquigarrow r + (a, b] : \mathcal{B}\mathbb{R}^n\ \texttt{as}\ (1),$

$-(a, b] = [b, a) \in \mathcal{B}\mathbb{R}^n\ \texttt{as}\ (2),$

$(1, 2) \rightsquigarrow (a, b] \in G,$

$\texttt{HalfInterval}(\mathbb{R}^n) \subset G,$

$\texttt{Assume}\ X : \mathbb{N} \to G,$

$\texttt{Assume}\ a \in \mathbb{R}^n,$

$f := \Lambda v \in \mathbb{R}^n\ .\ v + a : \texttt{ISO}_{\mathsf{SET}}(\mathbb{R}^n, \mathbb{R}^n),$

$g := \Lambda v \in \mathbb{R}^n\ .\ -v : \texttt{ISO}_{\mathsf{SET}}(\mathbb{R}^n, \mathbb{R}^n),$

$-A^{\complement} = g\left(A^{\complement}\right) = g(A^{\complement}) = (-A)^{\complement}\ \texttt{as}\ (1),$

$\eth(G)(X) \rightsquigarrow -A \in \mathcal{B}\mathbb{R}^n \rightsquigarrow_{(1)} -A^{\complement} \in \mathcal{B}\mathbb{R}^n,$

$a + A^{\complement} = f\left(A^{\complement}\right) = f(A)^{\complement} = (a + A)^{\complement}\ \texttt{as}\ (2),$

$\eth(G)(A) \rightsquigarrow a + A \in \mathcal{B}\mathbb{R}^n \rightsquigarrow_{(2)} a + A^{\complement} \in \mathcal{B}\mathbb{R}^n,$

$$-\bigcup_{n=1}^{\infty} X_n = g\left(\bigcup_{n=1}^{\infty} X_n\right) = \bigcup_{n=1}^{\infty} g(X_n) = \bigcup_{n=1}^{\infty} -X_n\ \texttt{as}\ (3)$$

$$\eth(G)(X) \rightsquigarrow \forall n \in \mathbb{N}\ .\ -X_n \in \mathcal{B}\mathbb{R}^n \rightsquigarrow_{(3)} -\bigcup_{n=1}^{\infty} X_n \in \mathcal{B}\mathbb{R}^n,$$

$$a + \bigcup_{n=1}^{\infty} X_n = f\left(\bigcup_{n=1}^{\infty} X_n\right) = \bigcup_{n=1}^{\infty} f(X_n) = \bigcup_{n=1}^{\infty} a + X_n \ \texttt{as}\ (4)$$

$$\eth(G)(X) \rightsquigarrow \forall n \in \mathbb{N} \ . \ a + X_n \in \mathcal{B}\mathbb{R}^n \rightsquigarrow_{(4)} a + \bigcup_{n=1}^{\infty} X_n \in \mathcal{B}\mathbb{R}^n;$$

$A^{\complement} \in G,$

$$\bigcup_{n=1}^{\infty} X_n \in G;$$

$G : \sigma\texttt{-Algebra}\,(\mathbb{R}^n) \rightsquigarrow \mathcal{B}\mathbb{R}^n = \sigma\{\texttt{Halfinterval}\} \subset G,$

$\eth G \rightsquigarrow G \subset \mathcal{B}\mathbb{R} \rightsquigarrow G = \mathcal{B}\mathbb{R}^d \square$

$\texttt{LebesgueTranslationInvariance} :: \forall A \in \mathcal{B}\mathbb{R}n \ . \ \forall a \in \mathbb{R}^n \ . \ \lambda(a + A) = \lambda(A)$

$\texttt{Proof} \ =$

$G := \{A \in \mathcal{B}\overset{\infty n}{\mathbb{R}} : \forall a \in \mathbb{R}^n \ . \ \lambda(A) + \lambda(a + A)\}$

$\texttt{Assume}\ (a, b] : \texttt{Halfinterval},$

$\texttt{Assume}\ r \in \mathbb{R}^n,$

$\lambda(r + (a, b]) = \lambda(a + r, b + r] = b + r - a - r = b - a = \lambda(a, b];;$

$\texttt{Halfinterval} \subset G \ \texttt{as}\ (1),$

$\texttt{Assume}\ A \uparrow_G \alpha,$

$\texttt{Assume}\ B \downarrow_G \beta,$

$\texttt{Assume}\ a \in \mathbb{R}^n,$

$\lambda(a + \alpha) = \lim_{n\to\infty} \lambda(a + A_n) = \lim_{n\to\infty} \lambda(A_n) = \lambda(\alpha),$

$\lambda(a + \beta) = \lim_{n\to\infty} \lambda(a + B_n) = \lim_{n\to\infty} \lambda(B_n) = \lambda(\beta);$

$G : \texttt{MonotoneClass}(\mathbb{R}^n) \rightsquigarrow_{(1)} G = \mathcal{B}\mathbb{R}^n \square$

$\texttt{InvariantIsLebesgue} :: \forall \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n) \ .$

$\quad \texttt{if}\ \forall A \in \mathcal{B}\mathbb{R}^n \ . \ \forall a \in \mathbb{R}^n \ . \ \mu(A + a) = \mu(A) \ \texttt{then}\ \exists c \in \mathbb{R}_+ : \mu = c\lambda,$

$\texttt{Assume}\ \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^n),$

$\texttt{Assume}\ (*) : A \in \mathcal{B}\mathbb{R}^n \ . \ \forall a \in \mathbb{R}^n \ . \ \mu(A + a) = \mu(A),$

$\texttt{Assume}\ I, J : \texttt{Halfcube} : \lambda(I) = \lambda(J) : \mu(I) \neq \mu(J),$

$\eth I, J \rightsquigarrow \exists v \in \mathbb{R}^n \ . \ I = J + a \ \texttt{Extract},$

$\mu(I) = \mu(J + a) =_* \mu(J) \rightsquigarrow \bot;$

$(1) : \forall I, J : \texttt{Halfcube} \ . \ \lambda(I) = \lambda(J) \Rightarrow \mu(I) = \mu(J),$

$c := \mu(0, 1] : \mathbb{R}_+,$

$\eth(\texttt{Measure})(\mu) \rightsquigarrow (1) : \mu_{|\texttt{Halfcube}} = c\lambda_{\texttt{Halfcube}},$

$\mathcal{B}(\mathbb{R}^n) = \sigma(\texttt{Halfcube}(\mathbb{R}^n)) \rightsquigarrow_{(1)} \mu = c\lambda;; \square$

$\texttt{Example}$ 1.3.3
$R := \{(x,y) \in \mathbb{R}^2 : x - y \in \mathbb{Q}\} : \texttt{Eq}(\mathbb{R})$,
$B := \texttt{eqclasses}(R)$,
$A := \texttt{choice}(B, [0,1])$,
$\texttt{Assume}\ r, q \in \mathbb{Q} : r \neq q$,
$\texttt{Assume}\ a \in A + r \cap A + q$,
$\eth(a) \rightsquigarrow \exists x, y \in A : x + r = a = y + q$,
$x - y = q - r \in \mathbb{Q} \rightsquigarrow x = y \rightsquigarrow q = r \rightsquigarrow \perp$,
$\forall r, q \in \mathbb{Q} \,.\, A + r \cap A + q = \emptyset$,
$\texttt{Assume}\ A : \texttt{Measurable}(\lambda)$,
$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} q + A \subset [0,2] \rightsquigarrow 2 \geq \lambda \left(\bigcup_{q \in \mathbb{Q} \cap [0,1]} q + A \right) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda(q + A) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda(A) \rightsquigarrow \lambda(A) = 0,$$
$\lambda(\mathbb{R}) = \bigcup_{q \in \mathbb{Q}} \lambda(A + q) = \bigcup_{q \in \mathbb{Q}} \lambda(A) = 0 \rightsquigarrow \perp$,
$A \,!\, \texttt{LebesgueMeasurable}(\mathbb{R}^n)\Box$

$\texttt{Example}$ 1.3.4
$F := \Lambda(x,y) \in \mathbb{R}^2 \,.\, \texttt{if}\ x + y > 1\ \texttt{then}\ \ln(x+y)\ \texttt{else}\ 0$,
$\mu_F(0.5, 2.5] = F(0.5, 0.5) - F(0.5, 2.5) - F(2.5, 0.5) + F(2.5, 2.5) = \ln(5) - 2\ln(3) =$
$\quad = \ln(5) - \ln(9) < 0 \rightsquigarrow F \,!\, \texttt{DistributionFunction}(\mathbb{R}^2)$,

$\texttt{Example}$ 1.3.5
$F := \Lambda(x,y) \in \mathbb{R}^2 \,.\, \texttt{if}\ x < 0 | y < 0\ \texttt{then}\ 0\ \texttt{else}\ xy + 1 : \mathbb{R}^2 \to \mathbb{R}$,
$\texttt{discont}(F) = [(0,0), (\infty, 0)] \cup [(0,0), (\infty] \,!\, \texttt{Countable}$

1.5 Lebesgue-Steltjes Measures in Multivariable Context [!]

1.6 Categorical Viewpoint: Boolean Algebras [!]

$\texttt{Boolean} :: ?\texttt{Commutative}$

$B : \texttt{Boolean} \iff \forall b \in B \,.\, b^2 = b$

$\mathcal{F} : \sigma\texttt{-Algebra}\,(()\,\Omega) \Rightarrow (\mathcal{F}, \cap, \ \triangle\ ) : \texttt{Boolean}$

$\sigma\texttt{-Ideal} :: \prod \mathcal{F} : \sigma\texttt{-Algebra}\,(\Omega) \ \,.\, ?\texttt{Ideal}(\mathcal{F})$

$N : \sigma\texttt{-Ideal} \iff \forall A : \mathbb{N} \to N \,.\, \bigcup_{n=1}^{\infty} A_n \in N$

$\texttt{ZeroSpace} := \sum \Omega : \texttt{Set} \,.\, \mathcal{F} : \sum \sigma\texttt{-Algebra}\,(\Omega) \ \,.\, \sigma\texttt{-Ideal}\,(\mathcal{F})$

$\texttt{Localizable} :: ?\texttt{ZeroSpace}$

$(\Omega, \mathcal{F}, N) : \texttt{Localizable} \iff \forall A : ?\frac{\mathcal{F}}{N} \,.\, \sup A \in \frac{\mathcal{F}}{N}$

$\texttt{IdealMeasure} :: \prod (\Omega, \mathcal{F}, N) : \texttt{ZeroSpace} \,.\, ?\texttt{Measure}(\Omega, \mathcal{F})$

$\mu : \texttt{IdealMeasure} \iff \forall A \in \mathcal{F} \,.\, \mu(A) = 0 \iff A \in N$

2 Lebesgue Integration

2.1 Measurable Functions

$$\texttt{Measurable} :: \prod (\Omega, \mathcal{F}), (\Omega', \mathcal{F}') \ . \ ?\Omega \to \Omega'$$
$$f : \texttt{Measurable} \iff \forall A \in \mathcal{F}' \ . \ f^{-1}(A) \in \mathcal{F}$$

$\texttt{BorelMeasurableCriterion} :: \forall f : (\Omega, \mathcal{F}) \to \mathbb{R} : \forall c \in \mathbb{R} \ . \ f^{-1}(-\infty, c) \in \mathcal{F} \ . \ f : \texttt{Measurable}(\Omega, \mathcal{F})$
$\texttt{Proof} \ =$
$\texttt{Assume} \ f : (\Omega, \mathcal{F}) \to \mathbb{R} : \forall c \in \mathbb{R} \ . \ f^{-1}(-\infty, c) \in \mathcal{F},$
$\texttt{Assume} \ A \in \mathcal{B}\mathbb{R},$
$\mathcal{B}\mathbb{R} = \sigma(\{(-\infty, c] | c \in \mathbb{R}\}),$
$(c, T) := \jmath\sigma(\{(-\infty, c] | c \in \mathbb{R}\})(A) : \sum \mathbb{N} \to \mathbb{R} \ . \ \texttt{SetAlgTransform} : T((-\infty, c]) = A,$
$\texttt{Assume} \ n \in \mathbb{N},$
$(1) := \eth(f)((-\infty, c_n]) : f^{-1}(-\infty, c_n] \in \mathcal{F},$
$(1) : \forall n \in \mathbb{N} \ . \ f^{-1}(-\infty, c_n] \in \mathcal{F},$
$f^{-1}(A) = f^{-1}(T(-\infty, c]) = T(f^{-1}[(-\infty, c_n)]_{n=1}^{\infty}) \in_{(1)} \mathcal{F};$
$f : \texttt{Measurable}(\Omega, \mathcal{F}); \square$

$\texttt{MeasurableMax} :: \forall f, g : \texttt{Mesurable}(\Omega, \mathcal{F}) \ . \ \max(g, f) : \texttt{Measurable}(\Omega, \mathcal{F})$
$\texttt{Proof} \ =$
$\texttt{Assume} \ f, g : \texttt{Mesurable}(\Omega, \mathcal{F}),$
$\texttt{Assume} \ c \in \mathbb{R},$
$\max(f, g)^{-1}(-\infty, c] = f^{-1}(-\infty, c] \cap g^{-1}(-\infty, c] \in F;$
$\texttt{BorelMeasurableCriterion} \rightsquigarrow \max(g, f) : \texttt{Measurable}(\Omega, \mathcal{F}); \square$

$\texttt{MeasurableMin} :: \forall f, g : \texttt{Mesurable}(\Omega, \mathcal{F}) \ . \ \min(g, f) : \texttt{Measurable}(\Omega, \mathcal{F})$
$\texttt{Proof} \ =$
$\texttt{Assume} \ f, g : \texttt{Mesurable}(\Omega, \mathcal{F}),$
$\texttt{Assume} \ c \in \mathbb{R},$
$\min(f, g)^{-1}(-\infty, c] = f^{-1}(-\infty, c] \cup g^{-1}(-\infty, c] \in F;$
$\texttt{BorelMeasurableCriterion} \rightsquigarrow \min(g, f) : \texttt{Measurable}(\Omega, \mathcal{F}); \square$

$\texttt{IndicatorMeasurable} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \forall A \in \mathcal{F} \ . \ I_A f : \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Proof} \ =$

$\texttt{Assume} \ f : \texttt{Measurable}(\Omega, \mathcal{F})$

$\texttt{Assume} \ A \in \mathcal{F}$

$\texttt{Assume} \ B \in \mathcal{B}\mathbb{R},$

$(1) := \texttt{LawOfExcludeMiddle}(B, 0) : 0 \in B | 0 \notin B$

$\texttt{Assume} \ (2) : 0 \in B,$

$I_A f^{-1}(B) = (A \cap f^{-1}(B)) \cup A^{\complement} \in \mathcal{F}$

$\texttt{Assume} \ (2) : 0 \notin B$

$I_A f^{-1}(B) = (A \cap f^{-1}(B)) \in F;$

$(1) \rightsquigarrow f^{-1}(B) \in \mathcal{F};$

$I_A f : \texttt{Measurable}(\Omega, \mathcal{F}) \square$

$\texttt{MeasurableConvergance} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, F) \ . \ \forall \phi : f_n \xrightarrow{p} \phi \ . \ \phi : \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Proof} \ =$

$\texttt{Assume} \ f : \mathbb{N} \to \texttt{Measurable}(\Omega, F),$

$\texttt{Assume} \ \phi : f_n \xrightarrow{p} \phi,$

$\texttt{Assume} \ c \in \mathbb{R},$

$$\phi^{-1}(-\infty, c) = \{\omega \in \Omega : \phi(\omega) \leq c\} = \{\omega \in \Omega : \lim_{n \to \infty} f_n(\omega) \leq c\} =$$

$$= \bigcup_{n \in \mathbb{N}} \in \mathbb{N} \{\omega \in \Omega : \exists K : \texttt{Infinite}(\mathbb{N}) : \forall k \in K \ . \ f_k(\omega) \leq c - 1/n\} =$$

$$= \bigcup_{n \in \mathbb{N}} \liminf_{n \in \mathbb{N}} \{\omega \in \Omega : f_n(\omega) \leq c - 1/n\} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} f_m^{-1}(-\infty, c + 1/n] \in \mathcal{F};$$

$\texttt{BorelMeasurableCriterion} \rightsquigarrow \phi : \texttt{Measurable}(\Omega, \mathcal{F});; \square$

$$\texttt{Simple} \ :: ?\texttt{Mesurable}(\Omega, \mathcal{F}) \left(\overset{\infty}{\mathbb{R}}, \mathcal{B}\left(\overset{\infty}{\mathbb{R}} \right) \right)$$

$$f : \texttt{Simple} \iff \exists r \in \mathbb{N} : \exists A : r \to \mathcal{F} : \exists a : r \to \mathcal{F} : f = \sum_{k=1}^{r} a_k I_{A_k}$$

$\texttt{SimpleApproximationI} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) : f > 0 \ .$

$\quad \exists S : \mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F}) : \forall n \in \mathbb{N} \ . \ S_n > 0 : S \xrightarrow{p} f$

$\texttt{SimpleApproximationII} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ .$

$\quad \exists S : \mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F}) : \forall n \in \mathbb{N} \ . \ |S_n| \leq |f| : S \xrightarrow{p} f$

$\texttt{MeasurableAlgebra} :: \forall f, g : \texttt{Measurable}(\Omega, \mathcal{F}) \;.\; f+g, fg, f-g, f/g : \texttt{Measurable}(\Omega, \mathcal{F}) \;.\; f+g, fg, f-g, f/$

$\texttt{MeasurableComposition} :: \forall f : \texttt{Measurable}\ A\ B \forall g : \texttt{Measurable}\ B\ C \;.\; f \circ g : \texttt{Measurable}\ A\ B$

$\texttt{LebesgueMeasurableMap} ::?(\mathbb{R}^m, \mathcal{B}\mathbb{R}^m) \to (\mathbb{R}^n, \mathcal{B}\mathbb{R}^m)$
$f : \texttt{LebesgueMeasurableMap} \iff \forall A \in \mathcal{B}\mathbb{R}^m \;.\; f^{-1}A : \texttt{LebesgueMeasurable}(\mathbb{R}^n)$

$\texttt{MultivariateMeasurability} :: \forall f : \Omega \to \mathbb{R}^n \;.$
$\quad f : \texttt{Measurable}(\Omega, \mathcal{F})(\mathbb{R}^n, \mathcal{B}\mathbb{R}^n) \iff \forall i \in n \;.\; f^i : \texttt{Measurable}(\Omega, \mathcal{F})$

$\texttt{Example}$ 2.2.1
$\texttt{Assume}\ f : \texttt{Measurable}(\Omega, F) : f > 0,$
$S : \mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F})$

$$S_n := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \left[f^{-1} \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right) \right]_\in + nf^{-1}[n, \infty)$$

$\texttt{Assume}\ f : \texttt{Bounded} \rightsquigarrow \exists b : \forall \omega \in \Omega \;.\; f(\omega) \leq \Omega\ \texttt{Extract},$
$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$
$N = \lceil \max(b, \log_2 \epsilon^{-1}) \rceil,$
$\texttt{Assume}\ n \in \mathbb{N} : n \geq N,$
$\texttt{Assume}\ \omega \in \Omega,$

$$f(\omega) - S_n(\omega) \leq \frac{1}{2^n} \leq \frac{1}{2^N} \leq \epsilon;;;;$$

$f : \texttt{Bounded} \Rightarrow S \rightrightarrows f$
$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$
$\texttt{Assume}\ \omega \in \Omega,$
$N = \lceil \max(f(\omega), \log_2 \epsilon^{-1}) \rceil,$
$\texttt{Assume}\ n \in \mathbb{N} : n \geq N,$

$$f(\omega) - S_n(\omega) \leq \frac{1}{2^n} \leq \frac{1}{2^N} \leq \epsilon;;;$$

$S \xrightarrow{p} f; \square$

$\texttt{ConditionalMeasurable} :: \forall f, g : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B}) \;.\; \forall A \in \mathcal{F} \;.$
$\quad \Lambda \omega \in \Omega \;.\; \texttt{if}\ \omega \in A\ \texttt{then}\ f(\omega)\ \texttt{else}\ g(\omega) : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B})$
$\texttt{Proof}\ =$
$\texttt{Assume}\ f, g : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B}),$
$\texttt{Assume}\ A \in \mathcal{F},$
$\texttt{Assume}\ h := \Lambda \omega \in \Omega \;.\; \texttt{if}\ \omega \in A\ \texttt{then}\ f(\omega)\ \texttt{else}\ g(\omega) : \Omega \to R$
$\texttt{Assume}\ B \in \mathcal{B},$

$$h^{-1}(B) = (f^{-1}(B) \cap A) \cup \left(f^{-1}(B) \cap A^\complement \right) \in \mathcal{F};$$

$\texttt{MeasurableInf} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}) \ . \ \inf_{n\in\mathbb{N}} f_n, \sup_{n\in\mathbb{N}} f_n : \texttt{Measurable}(\Omega, \mathcal{F})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F})$

$\texttt{Assume}\ c \in \mathbb{R},$

$\left(\inf_{n\in\mathbb{N}} f_n\right)^{-1}(-\infty, c] = \bigcup_{n=1}^{\infty} f_n^{-1}(-\infty, c] \in \mathcal{F},$

$\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}(-\infty, c] = \bigcap_{n=1}^{\infty} f_n^{-1}(-\infty, c] \in \mathcal{F};$

$\texttt{BorelMeasurableCriterion} \leadsto \inf_{n\in\mathbb{N}} f_n, \sup_{n\in\mathbb{N}} f_n : \texttt{Measurable}(\Omega, \mathcal{F}); \square$

$\texttt{MeasurableAlmost} :: \forall (\Omega, \mathcal{F}, \mu) : \texttt{CompleteMeasure} \ . \ \forall f : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B}) \ .$
$\quad . \ \forall A \in \mathcal{F} : \mu(A) = 0 \ . \ \forall g : \Omega \to R : f_{|A^\complement} = g_{|A^\complement} \ . \ g : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (\Omega, \mathcal{F}, \mu) : \texttt{CompleteMS},$

$\texttt{Assume}\ f : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B},$

$\texttt{Assume}\ A \in \mathcal{F} : \mu(A) = 0,$

$\texttt{Assume}\ g : \Omega \to R : f_{|A^\complement} = g_{|A^\complement},$

$\texttt{Assume}\ B \in \mathcal{B},$

$g^{-1}(B) \cap A \subset A \leadsto \ \texttt{as}\ (\Omega, \mathcal{F}, \mu) : \texttt{CompleteMeasure} \leadsto g^{-1}(B) \cap A \in \mathcal{F}$

$g^{-1}(B) = \left(g^{-1}(B) \cap A\right) \cup \left(g^{-1}(B) \cap A^\complement\right) = \left(g^{-1}(B) \cap A\right) \cup \left(f^{-1}(B) \cap A^\complement\right) \in \mathcal{F};$

$g : \texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B});;;;\square$

$\texttt{ClosedUnion} :: \prod X : \texttt{TopologicalSpace} \ . \ ??X$

$A : \texttt{ClosedUnion} \iff A : F_\sigma \iff \exists K : \mathbb{N} \to \texttt{Closed}(X) \ . \ A = \bigcup_{n=1}^{\infty} K_n$

$\texttt{DiscontTHM} \forall f : \mathbb{R}^n \to \mathbb{R}^m : \texttt{discont}(f) : F_\sigma$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{R}^n \to \mathbb{R}^m$

$s :: \mathbb{R}^n \to \overset{\infty}{\mathbb{R}}$

$s(p) = \sup\{ \lim_{n,m\to\infty} d(f(x_n), f(y_m)) | x, y : \mathbb{N} \to \mathbb{R}_n : \lim_{n\to\infty} x_n = a = \lim_{m\to\infty} Y_m : f(x), f(y) : \texttt{Convergent}\}$

$\texttt{Assume}\ p \in \{p \in \mathbb{R}^n : s(p) < 1/n\},$

$\texttt{Assume}\ (1) : \forall U \in \mathcal{U}(p) \ . \ \exists u \in U : s(u) \geq 1/n,$

$(1) \leadsto \exists x \in \mathbb{N} \to \mathbb{R}^n : \forall n \in \mathbb{N} \ . \ s(x_n) \geq 1/n$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++}$,

$\eth(x) \rightsquigarrow \exists a, b \in\in \mathbb{R}^n : d(a,p) < \epsilon : d(b,p) \leq \epsilon : d(f(a), f(b)) > 1/n + \epsilon$;

$s(p) \geq 1/n \rightsquigarrow \bot$,

$\exists U \in \mathcal{U}(p) : \forall u \in U \,.\, s(u) < 1/n$ $\texttt{Extract as}$ U_p;

$\{p \in \mathbb{R}^n : s(p) < 1/n\} = \bigcup_{p \in \mathbb{R}^n : s(p) < 1/n} U_p \rightsquigarrow \{p \in \mathbb{R}^n : s(p) < 1/n\} : \texttt{Open}(\mathbb{R}^n) \rightsquigarrow$

$\{p \in \mathbb{R}^n : s(p) \geq 1/n\} : \texttt{Closed}(\mathbb{R}^n)$;

$\texttt{discont}(f) : F_\sigma$ □

$\texttt{NoIrrationalDisconts} : \forall f : \mathbb{R} \to \mathbb{R} \,.\, \texttt{discont}(f) \neq \mathbb{Q}^{\complement}$

$\texttt{Assume}\ f : \mathbb{R} \to \mathbb{R} : \texttt{discont}(f) = \mathbb{Q}^{\complement}$,

$\texttt{DiscontTHM} \rightsquigarrow \mathbb{Q}^{\complement} : F_\sigma \rightsquigarrow \exists K : \mathbb{N} \to \texttt{Closed}(\mathbb{R}) : \mathbb{Q}^{\complement} = \bigcup_{n=1}^{\infty} K_n$ $\texttt{Extract}$,

$\mathbb{Q} = \bigcap_{n=1}^{\infty} K_n^{\complement} \rightsquigarrow \forall n \in \mathbb{N} \,.\, \mathbb{Q} \subset K^{\complement}$,

$\texttt{Assume}\ n \in \mathbb{N}$,

$\mathbb{Q} \subset K_n^{\complement}$,

$K_n : \texttt{Closed}(\mathbb{R}) \rightsquigarrow K_n^{\complement} : \texttt{Open}(\mathbb{R})$,

$\mathbb{Q} : \texttt{Dense}(\mathbb{R}) \rightsquigarrow K_n^{\complement} = \mathbb{R}$;

$\mathbb{Q} = \mathbb{R} \rightsquigarrow \bot$; □

$\texttt{SteinhausLemma} :: \forall a : \mathbb{N} \to \mathbb{N} \to \mathbb{R} : \exists c \in \mathbb{R}_{++} : \forall n \in \mathbb{N} \,.\, \sum_{k=1}^{\infty} a_{n,k} = 1 \wedge$

$\wedge \sum_{k=1}^{\infty} |a_{n,k}| < c \wedge : \lim_{k \to \infty} x_{k,n} = 0 \,.\, \exists x : \mathbb{N} \to \{0,1\} :$

$: \Lambda n \in \mathbb{N} : \sum_{k=1}^{\infty} x_k a_{n,k}\ !\ \texttt{Convergent}\left(\overset{\infty}{\mathbb{R}}\right)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ a : \mathbb{N} \to \mathbb{N} \to \mathbb{R} : \exists c \in \mathbb{R}_{++} : \forall n \in \mathbb{N} \,.\, \sum_{k=1}^{\infty} a_{n,k} = 1 \wedge \sum_{k=1}^{\infty} |a_{n,k}| < c \wedge : \lim_{k \to \infty} x_{k,n} = 0$,

$\texttt{Iterate}\ n, k \in \mathbb{N}\ \texttt{Over}\ i \in \mathbb{N}\ \texttt{With}\ n_1 = 1, k_1 = 1$,

$\texttt{Assume}\ j \in k_i$,

$\eth(a)(3) \rightsquigarrow \exists N \in \mathbb{N} : \forall m \in N \,.\, |a_{m,k_i}| \leq \frac{1}{8k_i}$ $\texttt{Extract as}$ N_j;

$n_{i+1} := \max\{n_i + 1\} \cup N[k_i]$,

$$\eth(n_{i+1}) \rightsquigarrow \sum_{j=1}^{k_i} |a_{n_{i+1},j}| \leq \sum_{j=1}^{k_i} \frac{1}{8k_i} = 1/8,$$

$$\eth(a)(2) \rightsquigarrow \exists K \in \mathbb{N} \;.\; \sum_{j=K+1}^{\infty} |a_{n_{i+1},j}| < 1/8 \; \texttt{Extract},$$

$$k_{i+1} := \max k_i + 1, K;$$

$$n : \texttt{Subseqer} : \forall i \in \mathbb{N} \;.\; \sum_{j=1}^{k_i} |a_{n_{i+1},j}| \leq 1/8,$$

$$k : \texttt{Subseqer} : \forall i \in \mathbb{N} \;.\; \sum_{j=k_{i+1}+1}^{\infty} |a_{n_{i+1},j}| < 1/8,$$

$x : \mathbb{N} \to \{0, 1\}$

$x(m)$

$|\exists s \in \mathbb{N} \;.\; k_{2s-1} < m < k_{2s} = 0$

$|\texttt{otherwise} = 1$

$$\tau := \Lambda n \in \mathbb{N} : \sum_{k=1}^{\infty} x_k a_{n,k},$$

$\texttt{Assume}\; m \in \mathbb{N},$

$\texttt{Assume}\; m : \texttt{Odd},$

$$\eth(t, x) \rightsquigarrow |\tau_{n_{m+1}}| \leq \sum_{i=1}^{\infty} x_i |a_{n_{m+1},i}| \leq \sum_{i=1}^{k_n} |a_{n_{m+1},i}| + \sum_{i=k_{n+1}+1}^{\infty} |a_{n_{m+1},i}| \leq 1/4,$$

$\tau_{n_{m+1}} \leq 1/4;$

$|\lim_{m\to\infty} \tau_m| \neq \infty,$

$z\texttt{Assume}\; m : \texttt{Even},$

$$\sum_{i=k_n+1}^{k_{n+1}} a_{n_{m+1},i} \geq 3/4$$

$$\tau_{n_{m+1}} \geq 3/4 - \left| \sum_{i=1}^{k_n} |a_{n_{m+1},i}| + \sum_{i=k_{n+1}+1}^{\infty} |a_{n_{m+1},i}| \right| \geq 1/2;;$$

$$\tau \;!\; \texttt{Convergent} \left(\overset{\infty}{\mathbb{R}}\right); \square$$

$\texttt{VitaliHahnSaksI} :: \forall P : \mathbb{N} \to \texttt{Probability}(\Omega, \mathcal{F}) \,.\, \forall \mathbb{P} : \mathcal{F} \to \mathbb{R} :$

$: \forall A \in \mathcal{F} \,.\, \lim_{n\to\infty} P_n(A) = \mathbb{P}(A) \,.\, P : \texttt{Probability}(\Omega, \mathcal{F}),$

$\texttt{Proof} \ =$

$\texttt{Assume}\ P : \mathbb{N} \to \texttt{Probability}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \mathbb{P} : \mathcal{F} \to \mathbb{R} : \forall A \in \mathcal{F} \,.\, \lim_{n\to\infty} P_n(A) = \mathbb{P}(A),$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ A : \texttt{DisjointElem}(\mathcal{F}, n),$

$$\alpha := \bigcup_{k=1}^{n} A_k \in \mathcal{F},$$

$$1 = \lim_{n\to\infty} P_n(\Omega) = \lim_{n\to\infty} P_n(\alpha) + P_n\left(\alpha^{\complement}\right) = \lim_{n\to\infty} P_n(\alpha) + \lim_{n\to\infty} P_n\left(\alpha^{\complement}\right) = \mathbb{P}(\alpha) + \mathbb{P}\left(\alpha^{\complement}\right) \leadsto$$

$$\leadsto (1) : 1 - \mathbb{P}\left(\alpha^{\complement}\right) = \mathbb{P}(\alpha)$$

$$1 = \lim_{n\to\infty} P_n(\alpha) + \lim_{n\to\infty} P_n\left(\alpha^{\complement}\right) = \lim_{m\to\infty} \sum_{k=1}^{n} P_m(A_k) + \mathbb{P}\left(\alpha^{\complement}\right) = \sum_{k=1}^{n} \lim_{m\to\infty} P_m(A_k) + \mathbb{P}\left(\alpha^{\complement}\right) =$$

$$= \sum_{k=1}^{n} \mathbb{P}(A_k) + \mathbb{P}\left(\alpha^{\complement}\right) \leadsto (2) : 1 - \mathbb{P}\left(\alpha^{\complement}\right) = \sum_{k=1}^{n} \mathbb{P}(A_k),$$

$$(1, 2) \leadsto \sum_{k=1}^{n} \mathbb{P}(A_k) = \mathbb{P}(\alpha);$$

$\mathbb{P} : \texttt{Charge}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ A : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}),$

$$c := \sum_{n=1}^{\infty} \mathbb{P}(A_n) \in \mathbb{R}_+ : c \leq 1,$$

$$\alpha := \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$$

$\texttt{Assume}\ c = 1$

$$\mathbb{P}(\Omega) = \sum_{n=1}^{\infty} \mathbb{P}(A_n \sum_{n=1}^{\infty} \mathbb{P}(A_n)),$$

$\texttt{Assume}\ \mathbb{P}(\alpha) < 1,$

$\texttt{Assume}\ m \in \mathbb{N},$

$$\mathbb{P}(\alpha) = \lim_{n\to\infty} P_n(\alpha) \geq \lim_{n\to\infty} P_n\left(\bigcup_{k=1}^{m} A_k\right) == \sum_{k}^{m} = 1 \lim_{n\to\infty} P_n(A_k) = \sum_{k}^{m} = 1\mathbb{P}(A_k);$$

$$\sum_{k=1}^{m} \mathbb{P}(A_k) < 1 \leadsto \bot;$$

$$\mathbb{P}(\alpha) = 1 \leadsto \mathbb{P}(\alpha) = \sum_{n=1}^{\infty} \mathbb{P}(A_n);$$

$\texttt{Assume}\ c < 1,$

$\alpha = \Omega,\ \texttt{Use}\ \mathbb{P} : \texttt{Charge}(\Omega, \mathcal{F}),$

$a :: \mathbb{N} \to \mathbb{N} \to \mathbb{R}$

$a(n,k) := (1-c)^{-1}(P_n(A_k) - \mathbb{P}(A_k)),$

$\texttt{Assume}\ n \in \mathbb{N},$

$$\sum_{k=1}^{\infty} a_{n,k} = (1-c)^{-1}\left(\sum_{k=1}^{\infty} P_n(A_k) - \sum_{k=1}^{\infty} \mathbb{P}(A_k)\right) = (1-c)^{-1}\left(P_n(\alpha) - \sum_{k=1}^{\infty} \mathbb{P}(A_k)\right) = 1;$$

$$\sum_{k=1}^{\infty} |a_{n,k}| \leq (1-c)^{-1}\left(\sum_{k=1}^{\infty} P_n(A_k) + \sum_{k=1}^{\infty} \mathbb{P}(A_k)\right) = (1-c)^{-1}\left(P_n(\alpha) + \sum_{k=1}^{\infty} \mathbb{P}(A_k)\right) = \frac{1+c}{1-c},$$

$$\lim_{k\to\infty} a_{k,n} = (1-c)^{-1}(P_k(A_n) - \mathbb{P}(A_n)) = (1-c)^{-1}(\mathbb{P}(A_n) - \mathbb{P}(A_n)) = 0;$$

$$\texttt{SteinhausLemma} \rightsquigarrow \exists x : \mathbb{N} \to \{0,1\} : \Lambda n \in \mathbb{N} : \sum_{k=1}^{\infty} x_k a_{n,k}\ !\ \texttt{Convergent}\left(\overset{\infty}{\mathbb{R}}\right)\ \texttt{Extract},$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} x_k a_{n,k} = (1-c)^{-1}\left(\lim_{n\to\infty} P_n\left(\bigcup_{k\in\mathbb{N}:x_k=1} A_k\right)\right) - (1-c)^{-1}\sum_{k=1}^{\infty} x_k \mathbb{P}\left(A_k\right) =$$

$$= (1-c)^{-1}\mathbb{P}\left(\bigcup_{k\in\mathbb{N}:x_k=1} A_k\right) - (1-c)^{-1}\sum_{k=1}^{\infty} x_k \mathbb{P}\left(A_k\right) \rightsquigarrow$$

$$\rightsquigarrow \sum_{k=1}^{\infty} x_k a_{n,k} : \texttt{Convergrent}(\mathbb{R}) \rightsquigarrow \perp;$$

$$\mathbb{P}(\alpha) = \sum_{n=1}^{\infty} \mathbb{P}(A_n);$$

$\mathbb{P} : \texttt{Probability}(\Omega, \mathcal{F});; \square$

2.2 Integration: Definition and Basic Results

$$\texttt{integrate} :: \texttt{Simple}(\Omega, \mathcal{F}) \to \texttt{Measure}(\Omega, \mathcal{F}) \to \overset{\infty}{\mathbb{R}}$$

$$\texttt{integrate}\left(\sum_{n=1}^{r} a_n I_{A_n}\right)(\mu) = \int_\Omega \sum_{n=1}^{r} a_n I_{A_n} d\mu = \sum_{n=1}^{r} a_n \mu(A_n)$$

$$\texttt{Extend integrate} :: \texttt{Measurable}(\Omega, \mathcal{F}) \& \texttt{Positive} \to \texttt{Measure}(\Omega, \mathcal{F}) \to \overset{\infty}{\mathbb{R}}$$

$$\int_\Omega f d\mu = \sup \left\{ \int_\Omega S d\mu | S : \texttt{Simple}(\Omega, \mathcal{F}) : 0 < S \le f \right\}$$

$$\texttt{IntegralExist} :: \prod \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ ?\texttt{Measurable}(\Omega, \mathcal{F})$$

$$f : \texttt{IntegralExist} \iff \int_O f^+ d\mu < \infty | \int_O f^- d\mu < \infty$$

$$\texttt{Extend integrate} :: \prod \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ \texttt{IntegralExist} \to \overset{\infty}{\mathbb{R}}$$

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

$$\texttt{integrateOver} :: \mathcal{F} \to \texttt{Measurable}(\Omega, \mathcal{F}) \to \texttt{Measure}(\Omega, \mathcal{F}) \to \overset{\infty}{\mathbb{R}}$$

$$\texttt{integrateOver}(\texttt{Assume}, f, \mu) = \int_A f d\mu = \int_\Omega f I_A d\mu$$

$$\texttt{IntegralHomogenity} :: \forall A \in \mathcal{F} \ .$$
$$\forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ . \ \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ \forall c \in \mathbb{R} \ . \ \int_A c f d\mu = c \int_A f d\mu$$

$$\texttt{IntegralInequality} : \forall A \in \mathcal{F} \ .$$
$$\forall f, g : \texttt{Measurable}(\Omega, \mathcal{F}) : f \ge g \ . \ \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ \int_A f d\mu \ge \int_A g d\mu$$

$$\texttt{IntegralModuleInequality} : \forall A \in \mathcal{F} \ .$$
$$\forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ . \ \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ \int_A |f| d\mu \ge \left| \int_A f d\mu \right|$$

$$\texttt{Integrable} :: \prod \mu : \texttt{Measure}(\Omega, \mathcal{F}) \ . \ ?\texttt{IntegralExists}(\Omega, \mathcal{F})$$

$$f : \texttt{Integrable}(\Omega, \mathcal{F}) \iff \int_\Omega f d\mu < \infty$$

$\texttt{FunToMeasureI} :: \forall f : \texttt{Simple}(\Omega, \mathcal{F}) \, . \, \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \, .$

$$\Lambda A \in \mathcal{F} \, . \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F})$$

$\texttt{Proof} \ =$

$\texttt{Assume } f : \texttt{Simple}(\Omega, \mathcal{F}),$

$\texttt{Assume } \mu : \texttt{Measure}(\Omega, \mathcal{F}),$

$$f : \texttt{Simple}(\Omega, \mathcal{F}) \rightsquigarrow \exists n \in \mathbb{N} \, . \, \exists x : n \to \mathbb{R} \, . \, \exists A : n \to \mathcal{F} : f = \sum_{k=1}^{n} x_k I_{A_k} \ \texttt{Extract},$$

$\texttt{Assume } B : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}),$

$$\beta := \bigcup_{m=1}^{\infty} B_m \in \mathcal{F},$$

$$\int_\beta f d\mu = \sum_{k=1}^{n} x_k \mu(A_k \cap \beta) = \sum_{k=1}^{n} x_k \sum_{m=1}^{\infty} \mu(A_k \cap B_m) = \sum_{m=1}^{\infty} \sum_{k=1}^{n} x_k \mu(A_k \cap B_m) = \sum_{m=1}^{\infty} \int_{B_m} f d\mu;$$

$$\Lambda A \in \mathcal{F} \, . \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F});; \square$$

$\texttt{FunToMeasureII} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) : f > 0 \, . \, \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \, .$

$$\Lambda A \in \mathcal{F} \, . \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F})$$

$\texttt{Proof} \ =$

$\texttt{Assume } f : \texttt{Simple}(\Omega, \mathcal{F}),$

$\texttt{Assume } \mu : \texttt{Measure}(\Omega, \mathcal{F}),$

$\texttt{Assume } B : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N}),$

$$\beta := \bigcup_{m=1}^{\infty} B_m \in \mathcal{F},$$

$$\eth(\texttt{integrate}) \rightsquigarrow \int_\beta f d\mu \leq \sum_{n=1}^{\infty} \int_{B_n} f d\mu$$

$\texttt{Assume } n \in \mathbb{N},$

$$\eth(\texttt{integrate}) \exists S : \mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F}) : \int_{B_n} S d\mu \uparrow \int_{B_n} d\mu \ \texttt{Extract as } S^n;$$

$R : \mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F})$

$R_n(x)$

$|\exists k \in n : x \in B^k = S_n^k(x)$

$|\texttt{otherwise} = 0$

$$\eth(R) \rightsquigarrow \lim_{n\to\infty} \int R_n d\mu = \sum_{n=1}^{\infty} \int_{B_n} f d\mu,$$

$$\forall_{n\in\mathbb{N}} R_n \in \{S : \texttt{Simple}(\Omega, \mathcal{F}) : 0 < S \le f\} \rightsquigarrow \int_\beta f d\mu \ge \sum_{n=1}^{\infty} \int_{B_n} fd \rightsquigarrow \int_\beta f d\mu = \sum_{n=1}^{\infty} \int_{B_n} fd$$

$$\Lambda A \in \mathcal{F} \ . \ \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F});;\square$$

$\texttt{FunToMeasureIII} :: \forall f : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) \ . \ \Lambda A \in \mathcal{F} \ . \ \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F})$

$\texttt{Proof} \ =$

$\texttt{Assume} \ f : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu)$

$\texttt{Assume} \ B : \texttt{DisjointElems}(\mathcal{F}, \mathbb{N})$

$$\beta := \bigcup_{m=1}^{\infty} B_m \in \mathcal{F},$$

$$\int_\beta f d\mu = \int_\beta f^+ d\mu - \int_\beta f^- d\mu = \sum_{n=1}^{\infty} \int_{B_n} f^+ d\mu - \sum_{n=1}^{\infty} \int_{B_n} f^- d\mu = \sum_{n=1}^{\infty} \int_{B_n} f^+ d\mu - \int_{B_n} f^- d\mu =$$

$$= \sum_{n=1}^{\infty} \int_{B_n} f d\mu;$$

$$\Lambda A \in \mathcal{F} \ . \ \int_A f d\mu : \texttt{CountablyAdditive}(\Omega, \mathcal{F});\square$$

$\texttt{IntegrableCriterionI} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ .$
$\quad f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \iff |f| : \texttt{Integrable}(\Omega, \mathcal{F}, \mu)$

$\texttt{IntegrableCriterionII} :: \forall f : \texttt{Measurable}(\Omega, F) \ . \ \forall g : \texttt{Integrable}(\Omega, \mathcal{F}) \ .$
$\quad . \ |f| < g \Rightarrow f : \texttt{Integrable}$

$\texttt{almostEverywhere} :: \prod(\Omega, \mathcal{F}, \mu) : \texttt{MeasureSpace} \ . \ (?\Omega \to \texttt{Type}) \to \texttt{Type}$

$A : \texttt{almostEverywhere}(\Omega, \mathcal{F}, \mu)(T) \iff A : \mathrm{a\ .\ e\ .}\ [\mu] T \iff \exists Z \in \mathcal{F} : \mu(Z) = 0 : A : T(\Omega \setminus Z)$

$\texttt{ZeroAlmostEverywhere} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ . \ \forall P : \mathrm{a\ .\ e\ .}\ [\mu]\ f = 0 \ . \ \int_\Omega f d\mu = 0$

$\texttt{EquallAlmostEverywhere} :: \forall f : \texttt{Measurable}(\Omega, \mathcal{F}) \ . \ \forall g : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) \ .$
$\quad . \ \forall P : \mathrm{a\ .\ e\ .}\ [\mu]\ f = g \Rightarrow f : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) \wedge \int_\Omega f d\mu = \int_\Omega g d\mu$

$\texttt{IntegrableAEFinite} :: \forall f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \ . \ \mathrm{a\ .\ e\ .}\ [\mu]\ f < \infty$

$\texttt{ZeroAEZero} :: \forall f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) : f \ge 0 \ . \ \forall P : \int_\Omega f d\mu = 0 \ . \ \mathrm{a\ .\ e\ .}\ [\mu]\ f = 0$

$\texttt{IntegralAdditive} :: \forall f,g : \texttt{Measurable}(\Omega,\mathcal{F}) : f+g : \texttt{IntegralExists}(\Omega,\mathcal{F},\mu) :$

$$: \int_\Omega f d\mu + \int_\Omega f d\mu \in \overset{\infty}{\mathbb{R}} \ . \ \int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega f d\mu$$

$\texttt{SumIntegral} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega,\mathcal{F}) : \forall n \in \mathbb{N}\, f_n \geq 0 \ . \ \sum_{n=1}^{\infty} \int_\Omega f_n d\mu = \int_\Omega \sum_{n=1}^{\infty} f_n d\mu$

$\texttt{AEInequelity} :: \forall \mu : \texttt{FiniteMeasure}(\Omega,\mathcal{F}) \ . \ \forall f,g : \texttt{IntegralExists}(\Omega,\mathcal{F},\mu) :$

$$\forall A \in \mathcal{F} \ . \ \int_A f d\mu \leq \int_A g d\mu \ . \ \mathrm{a\,.\,e\,.}\ [\mu]\, f \leq g$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f,g : \texttt{IntegralExists}(\Omega,\mathcal{F},\mu) : \forall A \in \mathcal{F} \ . \ \int_A f d\mu \leq \int_A g d\mu$

$A :: \mathbb{N} \to \mathcal{F}$

$A_n = \{\omega \in \Omega : f(\omega) \geq g(\omega) + 1/n : |g(\omega)| \leq n\},$

$\eth(A) \rightsquigarrow A \uparrow \{\omega \in \Omega : f(\omega) > g(\omega) : g(\omega) > -\infty\}$

$\texttt{Assume}\ n \in \mathbb{N},$

$$\int_{A_n} g_n d\mu \geq \int_{A_n} f_n d\mu \geq \int_{A_n} g_n d\mu + \frac{1}{n}\mu(A_n),$$

$$\left|\int_{A_n} g d\mu\right| \leq \int_{A_n} |g| d\mu \leq n\mu(A_n) < \infty \rightsquigarrow \mu(A_n) = 0;$$

$$\mu\{\omega \in \Omega : f(\omega) > g(\omega) : g(\omega) > -\infty\} = \sum_{n=1}^{\infty} \mu(A_n) = 0,$$

$C :: \mathbb{N} \to \mathcal{F}$

$C_n = \{\omega \in \Omega : g(\omega) = -\infty : f(\omega) > -n\},$

$C \uparrow \{\omega \in \Omega : f(\omega) > g(\omega) : g(\omega) = -\infty\},$

$\texttt{Assume}\ n \in \mathbb{N},$

$$-\infty\mu(C_n) = \int_{C_n} g d\mu \geq \int_{C_n} f d\mu \geq -n\mu(C_n) \rightsquigarrow \mu(C_n) = 0;$$

$$\mu\{\omega \in \Omega : f(\omega) > g(\omega) : g(\omega) = -\infty\} = \sum_{n=1}^{\infty} \mu(C_n) = 0 \rightsquigarrow \mathrm{a\,.\,e\,.}\ [\mu]\, f \leq g;; \ \square$$

2.3 Convergence of Integrals

MonotoneConvergence $:: \forall h : \mathbb{N} \to$ Measurable$(\Omega, \mathcal{F}) : \forall n \in \mathbb{N} \, . \, h_n \geq 0 \, . \, \forall h \uparrow_p H \, . \, \int h d\mu \uparrow \int H d\mu$

Proof $=$

Assume $h : \mathbb{N} \to$ Measurable$(\Omega, \mathcal{F}) : \forall n \in \mathbb{N} \, . \, h_n > 0$,

Assume $h \uparrow_p H \rightsquigarrow \lim_{n\to\infty} \int_\Omega h_n d\mu \leq \int_\Omega H d\mu$

Assume $S :$ Simple$(\Omega, \mathcal{F}) : 0 \leq S \leq H$,

Assume $c \in (0, 1)$,

$h \uparrow_p H \rightsquigarrow \lim_{n\to\infty} \int f_n d\mu > c \int_\Omega S d\mu;$

$\lim_{n\to\infty} \int f_n d\mu \geq \sup_c \sup_S \lim_{n\to\infty} c \int S d\mu = \int_O H d\mu \rightsquigarrow \int h d\mu \uparrow \int H d\mu; ; \square$

ExtendedMonotoneConvergenceUp $:: \forall h :$ IntegralExists$(\Omega, \mathcal{F}, \mu) : \int_\Omega h d\mu > -\infty \, .$

$. \, \forall f : \mathbb{N} \to$ IntegralExists$(\Omega, \mathcal{F}, \mu) : \forall n \in \mathbb{N} \, . \, f_n \geq h \, . \, \forall f \uparrow \phi \, . \, \int_\Omega f d\mu \uparrow \int_\Omega \phi d\mu$

Proof $=$

Assume $h :$ IntegralExists$(\Omega, \mathcal{F}, \mu) : \int_\Omega h d\mu > -\infty$,

Assume $f : \mathbb{N} \to$ IntegralExists$(\Omega, \mathcal{F}, \mu) : \forall n \in \mathbb{N} \, . \, f_n \geq h$,

Assume $f \uparrow \phi$,

Alternative $\int_\Omega h d\mu = \infty | \int_\Omega h d\mu < \infty$,

Assume Alternative $\int_\Omega h d\mu = \infty$,

Assume $n \in \mathbb{N}$,

$f_n \geq h, \int_\Omega h d\mu = \infty \rightsquigarrow \int_\Omega f_n d\mu \geq \int_\Omega h d\mu = \infty \rightsquigarrow \int_\Omega f_n d\mu = \infty;$

$\int_\Omega \phi d\mu = \infty \rightsquigarrow \int_\Omega f d\mu \uparrow \int_\Omega \phi d\mu;$

$\texttt{Close Alternative} \int_\Omega h d\mu < \infty,$

$\texttt{IntegrableAEFinite}(h) \rightsquigarrow (1) : \mathrm{a.e.}\ [\mu]\, h < \infty,$

$h' := \Lambda\omega \in \Omega\ .\ \texttt{if}\ h(\omega) < \infty\ \texttt{then}\ h(\omega)\ \texttt{else}\ 0,$

$\texttt{EquallAlmostEverywhere}(h, h', (1)) \rightsquigarrow (2) : \int_\Omega h' d\mu = \int_\Omega h d\mu,$

$\texttt{Assume}\ n \in \mathbb{N},$

$f_n \geq h \geq h' \rightsquigarrow (3) : f_n \geq h',$

$(3) \rightsquigarrow f_n - h' \geq 0;$

$(3) : f - h' \geq 0,$

$\texttt{MonotoneConvergence}(f - h', (3), f - h' \uparrow \phi - h') \rightsquigarrow (4) : \int_\Omega (f - h') d\mu \uparrow \int_\Omega (\phi - h') d\mu,$

$(4) \rightsquigarrow \int_\Omega f d\mu + \int_\Omega h' d\mu = \int_\Omega (f + h') d\mu \uparrow \int_\Omega (\phi + h') d\mu = \int_\Omega \phi d\mu + \int_\Omega h' d\mu \rightsquigarrow \int_\Omega f d\mu \uparrow \int_\Omega \phi d\mu;;;; \square$

$\texttt{ExtendedMonotoneConvergenceDown} :: \forall h : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : \int_\Omega h d\mu < \infty\ .$

$.\ \forall f : \mathbb{N} \to \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : \forall n \in \mathbb{N}\ .\ f_n \leq h\ .\ \forall f \downarrow \phi\ .\ \int_\Omega f d\mu \downarrow \int_\Omega \phi d\mu$

$\texttt{Proof}\ =$

$\texttt{Assume}\ h : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : \int_\Omega h d\mu < \infty,$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : \forall n \in \mathbb{N}\ .\ f_n \leq h,$

$\texttt{Assume}\ int_\Omega f d\mu \downarrow \int_\Omega \phi d\mu,$

$\texttt{ExtendedMonotoneConvergenceUp}(-h, -f, -f \uparrow -\phi) \rightsquigarrow$

$\rightsquigarrow \int_\Omega -f d\mu \uparrow \int_\Omega -\phi d\mu \rightsquigarrow \int_\Omega f d\mu \downarrow \int_\Omega \phi d\mu;;; \square$

$\texttt{limInf} :: \left(\mathbb{N} \to \Omega \to \overset{\infty}{\mathbb{R}}\right) \to \Omega \to \overset{\infty}{\mathbb{R}}$

$\texttt{limInf}(f, \omega) = \left(\liminf_{n\to\infty} f_n\right)(\omega) = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k(\omega)$

$\texttt{limSup} :: \left(\mathbb{N} \to \Omega \to \overset{\infty}{\mathbb{R}}\right) \to \Omega \to \overset{\infty}{\mathbb{R}}$

$\texttt{limSup}(f, \omega) = \left(\liminf_{n\to\infty} f_n\right)(\omega) = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k(\omega)$

$\texttt{FatouLemma} :: \forall f : \mathbb{N} \to \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) .$

$$. \forall g : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : f \geq g : \int_\Omega g d\mu > -\infty \ . \ \liminf_{n\to\infty} \int_\Omega f_n d\mu \geq \int_\Omega \liminf_{n\to\infty} f_n d\mu$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ g : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) : f \geq g : \int_\Omega g d\mu > -\infty,$

$h :: \mathbb{N} \to \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu)$

$h_n = \inf_{k>n} f_k,$

$H := \liminf_{n\to\infty} f_n,$

$\eth(\liminf) \rightsquigarrow h \uparrow H$

$\texttt{Assume}\ n \in \mathbb{N},$

$\eth(g) \rightsquigarrow h_n \geq g,$

$\eth(h) \rightsquigarrow f_n \geq h;$

$(1) : h \geq g,$

$(*) : f \geq h$

$$\texttt{ExtendedMonotoneConvergenceUp}(h, g, (1), h \uparrow H) \rightsquigarrow (2) : \int_\Omega H d\mu = \lim_{n\to\infty} \int \int_\Omega h_n d\mu,$$

$$(*) \rightsquigarrow \lim_{n\to\infty} \int_\Omega h_n d = \liminf_{n\to\infty} \int_\Omega h_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu \rightsquigarrow \int_\Omega H d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu; ; \square$$

$\texttt{DominatedConvergance} :: \forall D : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \ . \ \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) :$

$$: \forall n \in \mathbb{N} \ . \ |f_n| \leq D \ . \ \forall P : \mathrm{a\ .\ e\ .}\ [\mu]\, f \xrightarrow{p} \phi \ . \ \phi : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \land \lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega \phi d\mu \qquad \texttt{Proof} \ =$$

$\texttt{Assume}\ D : \texttt{Integrable}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) :$

$: \forall n \in \mathbb{N} \ . \ |f_n| \leq D,$

$\texttt{Assume}\ P : \mathrm{a\ .\ e\ .}\ [\mu]\, f \xrightarrow{p} \phi,$

$P, \eth(f) \rightsquigarrow (1) : |\phi| < D$

$\texttt{IntegrableCriterionII}(\phi, D, 1) \rightsquigarrow \phi : \texttt{Integrable}(\Omega, \mathcal{F}, \mu),$

$$\int_\Omega \phi d\mu = \int_\Omega \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu \leq \lim_{n\to\infty} \int_\Omega f_n d\mu \leq$$

$$\leq \limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n d\mu = \int_\Omega \phi d\mu \rightsquigarrow \lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega \phi d\mu; ; \square$$

$\texttt{PDominatedConvergance} :: \forall p \in \mathbb{R}_{++} \ .$

$$. \forall D : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) : |D|^p : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \ . \ \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) :$$

$$: \forall n \in \mathbb{N} \ . \ |f_n| \leq D \ . \ \forall P : \mathrm{a\ .\ e\ .}\ [\mu]\, f \xrightarrow{p} \phi \ . \ |\phi|^p : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \land \lim_{n\to\infty} \int_\Omega |f_n - \phi| d\mu = 0$$

$$\texttt{LimIntegral} :: \forall f : (a,b) \times (d,c) \to \mathbb{R} : \left(\exists g : \texttt{Measurable}((d,c), \mathcal{B}(d,c)) : g > |f| : \int_a^b g d\mu < \infty\right)$$

$$: \forall x \in (a,b) \;.\; \Lambda y \in (d,c) \;.\; f(x,t) : \texttt{Measurable}((d,c), \mathcal{B}(d,c)) \;.\; \forall x' \in (a,b) \;.$$

$$. \; \forall P : \forall y \in (c,d) \;.\; \lim_{x \to x'} f(x,t) \in \mathbb{R} \;.\; \int_a^b f(x,y) d\mu(y) = \int_a^b \lim_{x \to x'} f(x,y) d\mu(y)$$

Proof

$$\texttt{Assume } \forall f : (a,b) \times (d,c) \to \mathbb{R} : \left(\exists g : \texttt{Measurable}((d,c), \mathcal{B}(d,c)) : g > |f| : \int_a^b g d\mu < \infty\right)$$

$$: \forall x \in (a,b) \;.\; \Lambda y \in (d,c) \;.\; f(x,t) : \texttt{Measurable}((d,c), \mathcal{B}(d,c)),$$

Assume $x' \in (a,b)$,

Assume $P : \forall y \in (c,d) \;.\; \lim_{x \to x'} f(x,t) \in \mathbb{R}$,

$P \rightsquigarrow \exists X : \mathbb{N} \to (a,b) : \lim_{n \to \infty} X_n = x' : \forall y \in (a,b] \;.\; \lim_{n \to \infty} f(X_n, y) = \lim_{n \to \infty} f(x', y)$ Extract,

$\texttt{DominatedConvergence}(f(X_n, \cdot), g, \lim_{n \to \infty} f(X_n, \cdot) = \lim_{x \to \infty x'} f(x, \cdot)) :$

$$: \int_c^d \lim_{x \to x'} f(x,y) d\mu(y) = \lim_{n \to \infty} \int_c^d f(X_n, y) d\mu(y) = \lim_{x \to x'} \int_c^d f(x,y) d\mu(y);;; \square$$

2.4 Lebesgue VS Riemann [!]

2.5 Categorical Viewpoint: Integral As a Functor [!]

$\texttt{Transport} :: \texttt{Measure}(\Omega, \mathcal{F}) \to \texttt{Measure}(R, \mathcal{B}) \to ?\texttt{Measurable}(\Omega, \mathcal{F})(R, \mathcal{B})$

$T : \texttt{Transport}(\alpha, \beta) \iff \forall B \in \mathcal{B} \; . \; \beta(B) = \alpha(T^{-1}B)$

$\texttt{TransportIntegral} :: \forall T : \texttt{Transport}(\Omega, \mathcal{F})(\Omega', \mathcal{F}')(\alpha, \beta) \; . \; \forall f : \texttt{IntegralExists}(\Omega', \mathcal{F}', \beta) \; .$

$\quad . \; \forall A \in \mathcal{F}' \; . \; \displaystyle\int_A f d\beta = \int_{T^{-1}A} f \circ T d\alpha$

$\texttt{Proof} \; =$

$\texttt{Assume } T : \texttt{Transport}(\Omega, \mathcal{F})(\Omega', \mathcal{F}')(\alpha, \beta),$

$\texttt{Assume } f : \texttt{IntegralExists}(\Omega', \mathcal{F}', \beta),$

$\texttt{Assume } A \in \mathcal{F}',$

$\texttt{Assume } B \in \mathcal{F}',$

$\displaystyle\int_A I_B d\beta = \beta(A \cap B) = \alpha(T^{-1}(A \cap B)) = \alpha(T^{-1}A \cap T^{-1}B) = \int_{T^{-1}A} I_{T^{-1}B} d\alpha = \int_{T^{-1}A} I_B \circ T d\alpha,$

$(1) : \forall B \in \mathcal{F}' \; . \; \displaystyle\int_A I_B d\beta = \int_{T^{-1}A} I_B \circ T d\alpha,$

$\texttt{Assume } S : \texttt{Simple}(\Omega, \mathcal{F}),$

$\eth\texttt{Simple}(S) \rightsquigarrow: \exists n \in \mathbb{N} \; . \; \exists x : n \to \mathbb{R} \; . \; \exists B : \texttt{DisjointElems}(\mathcal{F}', n) : S = \displaystyle\sum_{k=1}^{n} x_k I_{B_k},$

$\displaystyle\int_A S d\beta = \int_A \sum_{k=1}^{n} x_k I_{B_k} d\beta = \sum_{k=1}^{n} x_k \int_A I_{B_k} d\beta = \sum_{k=1}^{n} x_k \int_{T^{-1}A} I_B \circ T d\alpha = \int_{T^{-1}A} S \circ T d\alpha;$

$(2) : \forall S : \texttt{Simple}(\Omega, \mathcal{F}) \; . \; \displaystyle\int_A S d\beta = \int_{T^{-1}A} S \circ T d\alpha,$

$\texttt{Assume } g : \texttt{IntegralExists}(\Omega, \mathcal{F}, \beta) : g > 0 \; .$

$S := \texttt{SimpleApproximationI}(g) : \, :\mathbb{N} \to \texttt{Simple}(\Omega, \mathcal{F}) : 0 \le S \le g : S \uparrow_p g,$

$\texttt{MonotoneConvergence}(S, g, 0) \rightsquigarrow \displaystyle\int_A g d\beta = \lim_{n\to\infty} \int_A S_n d\beta = \lim_{n\to\infty} \int_{T^{-1}A} S_n \circ T d\alpha =$

$\quad = \displaystyle\int_{T^{-1}A} g \circ T d\alpha;$

$(3) : \forall f : \texttt{IntegralExists}(\Omega, \mathcal{F}, \mu) \; . \; \displaystyle\int_A g d\beta = \int_{T^{-1}A} g \circ T d\alpha,$

$\displaystyle\int_A f d\beta = \int_A f^+ d\beta - \int_A f^- d\beta = \int_{T^{-1}A} f^+ d \circ T d\alpha - \int_{T^{-1}A} f^- d \circ T d\alpha = \int_{T^{-1}A} f \circ T d\alpha; ; \square$

$\mathsf{MEAS} :: \texttt{Category}$
$\mathcal{O}_{\mathsf{MEAS}} := \texttt{MeasureSpace}$
$\mathcal{M}_{\mathsf{MEAS}}(A, B) := \texttt{Transport}(A, B)$
$\cdot_{\mathsf{MEAS}} := \circ$

$\mathsf{I} :: \texttt{Functor}(\mathsf{MEAS}, \mathsf{VS}(\mathbb{R}))$
$\mathsf{I}(A) = (\texttt{Integrable}(A))^*$
$\mathsf{I}(T) = \Lambda v \in \mathsf{I}(A) \, . \, \Lambda f \in \texttt{Integrable}(B) \, . \, \langle v, f \circ T \rangle$

$\mathsf{BOR} :: \texttt{Category}$
$\mathcal{O}_{\mathsf{BOR}} := \texttt{MeasurableSpace}$
$\mathcal{M}_{\mathsf{BOR}}(A, B) := \texttt{Measurable}(A, B)$
$\cdot_{\mathsf{BOR}} := \circ$

$\mathsf{BOR}_0 :: \texttt{Category}$
$\mathcal{O}_{\mathsf{BOR}_0} := \texttt{Localizeble}$

$$\mathcal{M}_{\mathsf{BOR}_0}((\Omega, \mathcal{F}, N), (\Omega', \mathcal{F}', M)) := \frac{\{f : \texttt{Measurable}(\Omega, \mathcal{F})(\Omega', \mathcal{F}') : \forall Z \in M \, . \, f^{-1}(Z) \in N\}}{\big\{(f, g) \in \texttt{Measurable}(\Omega, \mathcal{F})(\Omega', \mathcal{F}')^2 : \{\omega \in \Omega : f(\omega) \neq g(\omega)\big\} \in N\}}$$

$\cdot_{\mathsf{BOR}_0} := \circ$

3 Radon-Nikodym Theory

3.1 Jordan-Hahn Decomposition

$\texttt{Diffusion} ::?\texttt{Lebesgue-Stieltjes}(\mathbb{R}^n)$

$\mu : \texttt{Diffusion} \iff F_\mu \in \mathcal{M}_{\mathsf{TOP}}(\mathbb{R}^n, \mathbb{R})$

$\texttt{AbsolutelyContinuous} :: \texttt{SignedMeasure}(\Omega, \mathcal{F}) \to ?\texttt{SignedMeasure}(\Omega, \mathcal{F})$

$\alpha : \texttt{AbsolutelyContinuous}(\beta) \iff \alpha \ll \beta \iff \forall A \in \mathcal{F} : \beta(A) = 0 \; . \; \alpha(A) = 0$

$\texttt{SupIsAttainable} :: \forall H : \texttt{CountablyAdditive}(\Omega, \mathcal{F}) \; . \; \exists A \in \mathcal{F} : H(A) = \sup_{S \in \mathcal{F}} H(S)$

$\texttt{Proof} \; =$

$\texttt{Assume} \; H : \texttt{CountablyAdditive}(\Omega, \mathcal{F}),$

$s := \sup_{S \in \mathcal{F}} H(S) \in \overset{\infty}{\mathbb{R}},$

$A := \eth(\sup)(\sup_{S \in \mathcal{F}} H(S)) : \mathbb{N} \to \mathcal{F} : \lim_{n \to \infty} H(A_n) = s$

$$\alpha = \bigcup_{n=1}^{\infty} A_n,$$

$\texttt{Assume} \; n \in \mathbb{N},$

$\texttt{Assume} \; b \in \mathbb{B}^n,$

$$a_{n,b} := \bigcap_{n=1}^{\infty} \texttt{bToC}(A_n, b_n, \alpha);$$

$$B_n = \bigcup \{a_{n,b} | b \in \mathbb{B} : a_{n,b} > 0\}$$

$\texttt{Assume} \; r \in \mathbb{N} : r > n,$

$$H(A_n) \le H(B_n) \le H\left(\bigcup_{k=n}^{r} B_k\right),$$

$$H(A_n) \le H\left(\bigcup_{k=n}^{\infty} B_k\right),$$

$c := \liminf B \in \mathcal{F}$

$$\bigcup_{k=n}^{\infty} B_k \downarrow_n C,$$

$$\eth(B) \rightsquigarrow \lim_{n \to \infty} H\left(\bigcup_{k=n}^{\infty} B_k\right) = H(C),$$

$$s = \lim_{n \to \infty} H(A_n) \le lim_{n \to \infty} H\left(\bigcup_{k=n}^{\infty} B_k\right) = H(C) \le s \rightsquigarrow H(C) = s; \; \square$$

$\texttt{upperVariation} :: \texttt{CountablyAdditive}(\Omega, \mathcal{F}) \to \texttt{Measure}(\Omega, \mathcal{F})$
$\texttt{upperVariation}(H) = H^+ := \Lambda A \in H \, . \, \sup\{H(B) | B \in F : B \subset A\}$

$\texttt{lowerVariation} :: \texttt{CountablyAdditive}(\Omega, \mathcal{F}) \to \texttt{Measure}(\Omega, \mathcal{F})$
$\texttt{lowerVariation}(H) = H^- := \Lambda A \in H \, . \, -\inf\{H(B) | B \in F : B \subset A\}$

$\texttt{absVariation} :: \texttt{CountablyAdditive}(\Omega, \mathcal{F}) \to \texttt{Measure}(\Omega, \mathcal{F})$
$\texttt{absVariation}(H) = |H| := H^+ + H^-$

$\texttt{JordanHahnDecomposition} :: \forall H : \texttt{CountablyAdditive}(\Omega, \mathcal{F}) \, . \, H = H^+ - H^-$
$\texttt{Proof} \ =$
$\texttt{Assume } H : \texttt{CountablyAdditive}(\Omega, \mathcal{F}),$
$\texttt{Assume } P : H > -\infty,$
$D := \texttt{SupIsAttainable}(H) \in \mathcal{F} : H(D) = \inf_{A \in \mathcal{F}} H(A),$
$\texttt{Assume } A \in \mathcal{F},$
$\texttt{Assume } R : -H^-(A) < H(A \cap D),$
$R \rightsquigarrow \exists C \in F : C \subset B : H(A \cap D) > H(C) \ \texttt{Extract},$
$H(A \cap D) > H(C) > H(D) \rightsquigarrow H(D) > H(D) - H(A \cap D) + H(C) = H((D \setminus (A \cap D)) \cup C),$
$(D \setminus (A \cap D)) \cup C \in \mathcal{F} \rightsquigarrow \bot;$
$-H^-(A) \geq H(A \cap D) \rightsquigarrow_{\eth H^-} -H^-(A) = H(A \cap D),$
$-H^-(A) \leq H(\emptyset) = 0;$
$H^- = \Lambda A \in \mathcal{F} \, . \, -H(A \cap D)$
$H^- \geq 0 \rightsquigarrow H^- : \texttt{Measure}(\Omega, \mathcal{F})$
$\texttt{Assume } A \in \mathcal{F},$
$H(A) = H(A \cap D) + H\left(A \cap D^\complement\right) = -H^- + H\left(A \cap D^\complement\right),$
$\texttt{Assume } R : H\left(A \cap D^\complement\right) < 0,$
$H^-(A) = H(A \cap D) < H(A \cap D) + H\left(A \cap D^\complement\right) \rightsquigarrow \bot;$
$H\left(A \cap D^\complement\right) \geq 0;$
$(1) : \forall A \in \mathcal{F} \, . \, H\left(A \cap D^\complement\right) \geq 0,$
$\texttt{Assume } B \in F : B \subset A,$
$H(B) = H(B \cap D) + H\left(B \cap D^\complement\right) \leq H\left(B \cap D^\complement\right) \leq_1 H\left(A \cap D^\complement\right),$
$H^+(A) \leq H\left(A \cap D^\complement\right) \rightsquigarrow H^+(A) = H\left(A \cap D^\complement\right);$
$H = H^+ - H^-; \ \square$

$$
\begin{aligned}
&|\lambda_1+\lambda_2| = (\lambda_1+\lambda_2)^+ + (\lambda_1+\lambda_2)^- = \\
&\quad = (\lambda_1^+ + \lambda_2^+ - \lambda_1^- - \lambda_2^-)^+ + (\lambda_1^+ + \lambda_2^+ - \lambda_1^- - \lambda_2^-)^- \leq \lambda_1^+ + \lambda_2^+ + \lambda_1^- + \lambda_2^- = |\lambda_1| + |\lambda_2|
\end{aligned}
$$

3.2 Radon-Nikodym Theorem

$\texttt{RadonNykodymI} :: \forall P : \texttt{FiniteMeasure}(\Omega, \mathcal{F}) \,.\, \forall \mu : \texttt{FiniteMeasure} : \mu \ll P \,.$

$\quad \exists f : \texttt{Measurable}(\Omega, \mathcal{F}) : \forall A \in f : \mu(A) = \int_A f dP$

$\texttt{Proof} \ =$

$\texttt{Assume}\ P : \texttt{FiniteMeasure}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \mu : \texttt{FiniteMeasure} : \mu \ll P,$

$F := \{f : \texttt{Intregrable}(\Omega, \mathcal{F}, P) : \forall A \in \mathcal{F} \,.\, \int_A f dP \leq \mu(A)))\}$

$\texttt{Assume}\ f : \mathbb{N} \to F,$

$\phi := \sup_{n \in \mathbb{N}} f : \texttt{Integrable}(\Omega, \mathcal{F}, P),$

$\texttt{Assume}\ n \in \mathbb{N},$

$g_n :=:= \Lambda \omega \in \Omega \,.\, \max\{f_k(\omega) : k\} \in F;$

$M = \eth g : g \uparrow \phi,$

$B = \eth(F, \Im g \subset F) : \forall n \in \mathbb{N} \,.\, \forall A \in \mathcal{F} \,.\, \int_A g_n dP \leq \mu(A),$

$MC = \texttt{MonotoneConvergence}(g, M) : \int_\Omega g dP \uparrow \int_\Omega \phi dP$

$\texttt{Assume}\ A \in \mathcal{F},$

$B \leadsto \mu(A) \geq \int_A g dP = \int_\Omega I_A g dP \uparrow \int_\Omega I_A \phi dP = \int_A \phi dP;$

$\phi \in F;$

$0 \in F \leadsto F \neq \emptyset \leadsto \sup F = \max F \neq \emptyset,$

$\texttt{Assume}\ f \in \max F,$

$\lambda := \Lambda A \in \mathcal{F} \,.\, \mu(A) - \int_A \phi dP : \texttt{Measure}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ A \in \mathcal{F} : P(A) = 0$

$\mu \ll P \leadsto (1) : \mu(A) = 0,$

$(1) \leadsto \lambda(A) = \mu(A) - \int_A \phi dP = 0;$

$(1) : \lambda \ll P,$

$\texttt{Assume}\ A1 : \lambda \neq 0$

$A1, f \in F \leadsto \lambda(\Omega) > 0 \leadsto \exists k \in \mathbb{R}_{++} : P(\Omega) - k\lambda(\Omega) < 0\ \texttt{Extract},$

$H := P - k\lambda : \texttt{SignedMeasure}(\Omega, \mathcal{F})$

$D := \texttt{SupIsAttainable}(H) \in \mathcal{F} : H(D) = \inf_{A \in \mathcal{F}} H(A)$

$D1 : \eth(h, H) D \neq \emptyset \wedge H(D) < 0$

$\texttt{Assume}\ Z : P(D) = 0,$

$H(D) = P(D) - \lambda(D) =_{Z,1} 0 \leadsto_{D1} \perp$

$(2) : P(D) > 0,$

$h := (1/k) I_D : \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ A \in \mathcal{F}$,

$$\int_A hdP = 1/kP(A \cap D) \leq \lambda(A \cap D) \leq \lambda(A) = \mu(A) - \int_A fdP,$$

$$\int_A (h+g)dP \leq \mu(A);$$

$h + f \in F$,

$h + f > f \rightsquigarrow f \notin \max F \rightsquigarrow \bot$;

$$H = 0 \rightsquigarrow \forall A \in f : \mu(A) = \int_A fdP;\ ;\ \square$$

$\texttt{RadonNykodym} :: \forall \mu : \sigma\texttt{-Finite}\,(\Omega, \mathcal{F})\ .\ \forall H : \texttt{SignedMeasure}(\Omega, \mathcal{F}) : H \ll \mu\ .$

$$.\ \exists f \in \texttt{Measurable}(\Omega, \mathcal{F}) : \forall A \in \mathcal{F}\ .\ \int_A f d\mu = H(A)$$

$\texttt{Density} :: \texttt{Measure}(\Omega, \mathcal{F}) \to \texttt{SignedMeasure}(\Omega, \mathcal{F}) \to ?\texttt{Measureable}(\Omega, \mathcal{F})$

$$f : \texttt{Density}(\mu, \lambda) \iff forall A \in \mathcal{F}\ .\ \int_A f d\mu = \lambda(A)$$

$\texttt{density} :: \prod \mu : \texttt{Measure}(\Omega, \mathcal{F})\ .\ \prod \lambda : \texttt{SignedMeasure} : \lambda \ll \mu\ .\ \texttt{Density}(\mu, \lambda)$

$\texttt{density} = f_\lambda = \texttt{RadonNykodym}(\mu, \lambda)$

$\texttt{Singular} :: \texttt{Measure}(\Omega, \mathcal{F}) \to ?\texttt{Measure}(\Omega, \mathcal{F})$

$$\lambda : \texttt{Singular}(\mu) \iff \lambda \bot \mu \iff \exists A \in \mathcal{F} : \lambda(A) = 0 \wedge \mu\left(A^{\complement}\right) = 0$$

$\texttt{MutSingular} :: \texttt{SignedMeasure}(\Omega, \mathcal{F}) \to ?\texttt{SigneMeasure}(\Omega, \mathcal{F})$

$\lambda : \texttt{Singular}(\mu) \iff \lambda \bot \mu \iff |\mu| \bot |\lambda|$

$$\texttt{BorelCantelli} :: \forall \mu : \texttt{Measure}(\Omega, \mathcal{F})\ .\ \forall A : \mathbb{N} \to \mathcal{F}\ .\ \forall S : \sum_{n=1}^{\infty} \mu(A_n) < \infty\ .\ \mu(\limsup A) = 0$$

$\texttt{Proof} =$

$\texttt{Assume}\ \mu : \texttt{Measure}(\Omega, \mathcal{F})$,

$\texttt{Assume}\ A : \mathbb{N} \to \mathcal{F}$,

$$\texttt{Assume}\ S : \sum_{n=1}^{\infty} \mu(A_n) < \infty,$$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++}$,

$$k = \texttt{SumConverge}(S, \epsilon) : \mathbb{N} : \sum_{n=1}^{\infty} \mu(A_n) < \epsilon,$$

$$D1 : \epsilon > \sum_{n=1}^{\infty} \mu(A_n) \geq \mu\left(\bigcup_{n=k}^{\infty} A_n\right) \geq \mu\left(\bigcap_{m=k}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) \geq \mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = \mu(\limsup A);$$

$\mu(\limsup A) = 0; ; ; \square$

$\texttt{RNProperties} :: \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \;.\; \forall \lambda_1, \lambda_2 : \texttt{SignedMeasure}(\Omega, \mathcal{F})$

$\forall P1 : \lambda_1 \perp \mu \& \lambda_2 \perp \mu \;.\; \mu = \lambda_1 + \lambda_2,$

$\forall P2 : \lambda_1 \ll \mu \;.\; |\lambda_1| \ll \mu,$

$\forall P3 : \lambda_1 \ll \mu \;\&\; \lambda_2 \perp \mu \;.\; \lambda_1 \perp \lambda_2,$

$\forall P4 : \lambda_1 \ll \mu \;\&\; \lambda_1 \perp \mu \;.\; \lambda_1 = 0,$

$\forall P5 : |\lambda_1| < 0 \;.\; \lambda_1 \ll \mu \iff \lim_{\mu(A) \to 0} \lambda_1(A) = 0;$

$\texttt{LebesgueDecomposition} :: \forall \mu : \texttt{Measure}(\Omega, \mathcal{F}) \;.\; \forall H : \texttt{SignedMeasure}(\Omega, \mathcal{F}) :$

$\quad : (|H| : \sigma\texttt{-Finite}\,(\Omega, \mathcal{F})) \;.\; \exists \alpha, \beta : \texttt{SignedMeasure}(\Omega, \mathcal{F}) : \alpha \ll \mu : \beta \perp \mu : H = \alpha + \beta$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mu : \texttt{Measure}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ P : \texttt{FiniteMeasure}(\Omega, \mathcal{F}),$

$\mathcal{B} := \{A \in \mathcal{F} : \mu(A) = 0\} :?\mathcal{F},$

$s := \sup\{P(b) | b \in \mathcal{B}\} \in \overset{\infty}{\mathbb{R}}_+,$

$b := \eth(\sup)(s) : \mathbb{N} \to B : P(b) \uparrow s,$

$B := \bigcup_{n=1}^{\infty} b_n \in \mathcal{F},$

$MUC1 := \texttt{MeasureUpperContinuity}(\mu, b) : \mu(B) = \lim_{n \to \infty} \mu(b_n) = 0 \rightsquigarrow B \in \mathcal{B},$

$MUC2 := \texttt{MeasureUpperContinuity}(P, b) : P(B) = \lim_{n \to \infty} P(b_n) = s,$

$D1 : s =_{MUC2} P(B) \leq P(\Omega) < \infty,$

$D1 \rightsquigarrow s \in \mathbb{R}_{++},$

$\alpha = \Lambda A \in \mathcal{F} := P\left(A \cap B^{\complement}\right) : \texttt{Measure}(\Omega, \mathcal{F}),$

$\beta = \Lambda A \in \mathcal{F} := P\,(A \cap B) : \texttt{Measure}(\Omega, \mathcal{F}),$

$S1 := \eth(\perp)(\beta, (B, MUC1)) : \beta \perp \mu,$

$\texttt{Assume}\ A \in \mathcal{F} : \mu(A) = 0,$

$\texttt{Assume}\ C1 : \alpha(A) > 0,$

$\eth(\mathcal{B})(\eth A) \rightsquigarrow A \in \mathcal{B} \rightsquigarrow B \cup A \in \mathcal{B},$

$P(A \cup B) = P(B) + P(A \cap B^{\complement}) > P(B) \rightsquigarrow B \neq s \rightsquigarrow \perp,$

$\alpha(A) = 0;$

$\alpha \ll \mu,$

$P = \alpha + \beta,$

$A1 : \forall P : \texttt{FiniteMeasure}(\Omega, \mathcal{F}) \;.\; \exists \alpha, \beta : \texttt{SignedMeasure}(\Omega, \mathcal{F}) : \alpha \ll \mu : \beta \perp \mu : P = \alpha + \beta,$

$\texttt{Assume}\ \lambda : \sigma\texttt{-Finite}\,(\Omega, \mathcal{F})\,,$

$$a := \eth\sigma\texttt{-Finite}\,(\Omega, \mathcal{F}) : \mathbb{N} \to \mathcal{F} : \bigcup_{n=1}^{\infty} a_n = \Omega : \forall n \in \mathbb{N}\ .\ \lambda(a) < \infty,$$

$\texttt{Assume}\ n \in \mathbb{N},$

$P := \Lambda A \in F\ .\ \lambda(a_n \cap A) : \texttt{FiniteMeaure}(\Omega, \mathcal{F}),$

$(\alpha_n, \beta_n) := A1(P) : \texttt{SignedMeasure}^2(\Omega, \mathcal{F}) : \alpha_n \ll \mu : \beta_n \perp \mu : P = \alpha_n + \beta_n;$

$$\alpha := \sum_{n=1}^{\infty} \alpha_n,$$

$$\beta := \sum_{n=1}^{\infty} \beta_n,$$

$\lambda = \alpha + \beta;$

$A2 : \forall \lambda : \sigma\texttt{-Finite}\,(\Omega, \mathcal{F})\ \ .\ \exists \alpha, \beta : \texttt{SignedMeasure}(\Omega, \mathcal{F}) : \alpha \ll \mu : \beta \perp \mu : \lambda = \alpha + \beta,$

$\texttt{Assume}\ H : \texttt{SignedMeasure}(\Omega, \mathcal{F}) : (|H| : \sigma\texttt{-Finite}\,(()\,\Omega, F)),$

$|H| : \sigma\texttt{-Finite}\,(\Omega, F) \rightsquigarrow H^+, H^- : \sigma\texttt{-Finite}\,(\Omega, \mathcal{F})\,,$

$(\alpha^+, \beta^+) := A2\left(H^+\right),$

$(\alpha^-, \beta^-) := A2\left(H^-\right),$

$\alpha := \alpha^+ - \alpha^-,$

$\beta := \beta^+ - \beta^-,$

$H = H^+ - H^- := \alpha^+ + \beta^+ - \alpha^- - \beta^- := \alpha - \beta;;\square$

$\texttt{ChainRule} :: \forall \mu, \lambda : \texttt{Measure}(\Omega, \mathcal{F})\ .\ \forall f : \texttt{Density}(\mu, \lambda)\ .\ \forall g : \texttt{Measurable}(\Omega, \mathcal{F})\ .$

$$.\ \int_\Omega g d\lambda = \int_\Omega g f d\mu$$

$\texttt{Proof}\ =$

$g \geq 0,$

$$\int_\Omega g d\lambda = \lim_{n\to\infty} \sum_{k=1}^{m_n} a_k^n \lambda(A_k^n) = \lim_{n\to\infty} \sum_{k=1}^{m_n} a_k^n \int_{A_k^n} f d\mu = \lim_{n\to\infty} \int_\Omega \sum_{k=1}^{m_n} a_k^n I_{A_k^n} f d\mu =$$

$$= \lim_{n\to\infty} \int_\Omega S_n f d\mu = \int_\Omega g f d\mu$$

$-$

$$\int g d\lambda = \int_\Omega g^+ d\lambda - \int_\Omega g^- d\lambda = \int_\Omega g^+ f d\mu - \int_\Omega g^- f d\mu = \int_\Omega g f d\mu$$

3.3 Absolutely Continuous Functions

$$\texttt{AbsCont} :: \prod [a,b] : \texttt{CInterval} \;.\; ?([a,b] \to \mathbb{R})$$

$$f : \texttt{AbsCont} \iff \forall \epsilon \in \mathbb{R}_{++} \;.\; \exists \delta \in \mathbb{R} \;.\; \forall n \in \mathbb{N} \;.$$

$$.\; \forall [c,d] : \mathbb{N} \to \texttt{CInterval} : \sum_{k=1}^{n} d_k - c_k \leq \delta \;.\; \sum_{k=1}^{n} |f(c_k) - f(d_k)| \leq \epsilon$$

$\texttt{AbsContDistr} :: \forall \alpha, \beta : \texttt{Measure}([a,b], \mathcal{B}[a,b]) \;.\; F_\alpha - F_\beta : \texttt{AbsCont}[a,b] \iff \alpha - \beta \ll \lambda$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \alpha, \beta : \texttt{Measure}([a,b], \mathcal{B}[a,b]),$

$H := \alpha - \beta : \texttt{SignedMeasure}\big([a,b], \mathcal{B}[a,b]\big),$

$\texttt{Assume}\ [c,d] : \texttt{SubInterval}[a,b],$

$H[c,d] = F_\alpha(d) - F_\alpha(c) - F_\beta(d) + F_\beta(c) = G(d) - G(c);$

$R1 : G = F_H,$

$\texttt{Assume}\ P1 : F_\alpha - F_\beta : \texttt{AbsCont}[a,b],$

$G := F_\alpha - F_\beta : \texttt{AbsCont}[a,b],$

$\texttt{Assume}\ A \in \mathcal{F} : \lambda(A) = 0,$

$V := \texttt{OuterRegularity}(A) : \mathbb{N} \to \mathcal{T}_{[a,b]} \;.\; \lambda(V) \downarrow 0 : \alpha(V) \downarrow \alpha(A) : \beta(V) \downarrow \beta(A)$

$W := \texttt{toDeacrising}(V),$

$\texttt{Assume}\ \epsilon \in \mathbb{R},$

$\delta := \texttt{AbsCont}[a,b](G) \in \mathbb{R}_{++} : \forall n \in \mathbb{N} \;.\; \forall (c,d) : \mathbb{N} \to \texttt{CInterval} :$

$$: \forall k \in n \;.\; d_k - c_k \leq \delta \;.\; \sum_{k=1}^{n} |G(c_k) - G(d_k)| \leq \epsilon,$$

$$N := \eth(w, \lambda(W) \downarrow 0) \in \mathbb{N} : \forall n \geq N \;.\; \exists m \in \mathbb{N} : \exists (c,d) : m \to \texttt{Subinterval}[a,b] : W_n = \bigsqcup_{k=1}^{m} (c_k, d_k) :$$

$$: \sum_{k=1}^{m} \lambda(c_k, d_k) \leq \delta,$$

$$(m, (c,d)) := \eth(N) : \sum m \in \mathbb{N} \to \mathbb{N} \;.\; m \to \texttt{Subinterval}[a,b] : W_n \subset \bigsqcup_{k=1}^{m} (c_k, d_k) : \sum_{k=1}^{m} \lambda(c,d) \leq \delta,$$

$$\eth(\delta, N) \rightsquigarrow \forall n \geq N \;.\; |H(W_n)| \leq \left| \sum_{k=1}^{m_n} H(c_k, d_k) \right| \leq \sum_{k=1}^{m_n} |H(c_k, d_k| = \sum_{k=1}^{m_n} |G(d_k) - G(c_k)| \leq \epsilon \rightsquigarrow$$

$$\lim_n \to \infty |H(W_n)| \leq \epsilon,$$

$$H(A) = \lim_{n\to\infty} H(A_n) = 0;$$

$R2 : \forall P : F_\alpha - F_\beta : \texttt{AbsCont}[a,b] \;.\; H \ll \lambda,$

$\texttt{Assume}\ P : H \ll \lambda,$

$R3 := \eth(\ll)(P) \rightsquigarrow |H| \ll \lambda,$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$

$\delta := \texttt{RNProperties}(|H|, R3)(\epsilon) : \forall A \in \mathcal{B}[a,b] : \lambda(A) \leq \delta \,.\, |H|(A) \leq \epsilon,$

$n \in \mathbb{N}$

$$\texttt{Assume}\ [c,d] : n \to \texttt{SubInterval}[c,d] : \sum_{k=1}^{n} d_k - c_k \leq \delta,$$

$$R4 \leq \texttt{Subaditivity}(\lambda) : \lambda\left(\bigcup_{k=1}^{n}[c_k,d_k]\right) \leq \sum_{k=1}^{n} d_k - c_k \leq \delta$$

$$R5 : \eth(\delta)\left(\bigcup_{k=1}^{n}[c_k,d_k], R4\right) : |H|\left(\bigcup_{k=1}^{n}[c_k,d_k]\right) \leq \epsilon,$$

$$R5 \rightsquigarrow \epsilon \geq |H|\left(\bigcup_{k=1}^{n}[c_k,d_k]\right) = \sum_{k=1}^{n}|H|[c_k,d_k] \geq \sum_{k=1}^{n}|H[c_k,d_k]| = \sum_{k=1}^{n}|G(d_k) - G(c_k)|;;;$$

$G : \texttt{AbsCont}[a,b];$

$R3 : \forall P : H \ll \lambda \,.\, F_\alpha - F_\beta : \texttt{AbsCont}[a,b],$

$\texttt{IFFI}(R2, R3) : F_\alpha - F_\beta : \texttt{AbsCont}[a,b] \iff \alpha - \beta \ll \lambda;\ \square$

$$\texttt{variation} : ([a,b] \to \mathbb{R}) \overset{\infty}{\to} \mathbb{R}_+$$

$$\texttt{variation}(f) = V_f := \sup\left\{\sum_{k=1}^{\texttt{size}(P)-1} |f(P_k) - f(P_{k+1})| \Big| P : \texttt{Partition}[a,b]\right\}$$

$\texttt{BoundedVariation} ::? \big([a,b] \to \mathbb{R}\big)$

$f : \texttt{BoundedVariation} \iff V_f < \infty$

$\texttt{ACIsBV} :: \forall f : \texttt{AbsCont}[a,b] \,.\, f : \texttt{Bounded}Variation$

$\texttt{Proof}\ =$

$\texttt{Assume}\ f : \texttt{AbsCont}[a,b],$

$\texttt{Assume}\ [a,b] : \texttt{Compact},$

$$(y, y') := \texttt{ExtremeValue}(f, [a,b]) : \mathbb{R}^2 : y = \max_{x\in[a,b]} f(x) : y' = \min_{x\in[a,b]} f(x),$$

$c := y - y' \in \mathbb{R}_+,$

$$\delta := \eth\texttt{AbsCont}(f)(c) \in \mathbb{R}_{++} : \forall[c,d] : \forall n \in \mathbb{N} \,.\, \forall[c,d] : n \to \texttt{Subinterval}[a,b]$$

$$:: \sum_{k=1}^{n} d_k - c_k \geq \delta \,.\, \sum_{k=1}^{n}|f(c_k) - d(d_k| \leq c,$$

$$n := \left\lceil \frac{b-a}{\delta} \right\rceil \in \mathbb{N},$$

$\mathtt{Assume}\ P : \mathtt{Partition}[a,b]$,

$m := \mathtt{size}(P) \in \mathbb{N}$,

$R1 := \eth\mathtt{Partition}(P) : \sum_{k=1}^{m-1} P_{k+1} - P_k = b - a$,

$K := \{i \in m-1 : P_{i+1} - P_i > \delta\} :?(m-1)$,

$k := \#K \in \mathbb{Z}_+$,

$R1 \rightsquigarrow k \le n$,

$L := K^{\complement} :?(m-1)$,

$(l, A) := \mathtt{JollyPartitioningLemma}(\delta, L) : \sum l \in \mathbb{N} : l + k \le n \;.\; l \to \mathcal{B}[a,b] : \forall i \in l \;.\; \lambda(A_i) \le \delta :$

$\quad : \forall i \in L \;.\; \exists j \in l : [P_i, P_{i+1}] \subset A_j$,

$\mathtt{Assume}\ j \in l$

$I_j := \{i \in m-1 : i \in A_j\} :?(m-1)$

$\lambda(A_j) \le \delta \rightsquigarrow \sum_{i \in I_j} P_{i+1} - P_i \le \delta \rightsquigarrow \sum_{i \in I_j} |f(P_{i+1}) - f(P_i)| \le c$;

$$\sum_{i=1}^{m-1} \big|f(P_i) - f(P_{i+1})\big| = \sum_{i \in K} \big|f(P_i) - f(P_{i+1})\big| + \sum_{i \in L} \big|f(P_i) - f(P_{i+1})\big| \le$$

$$\le kc + \sum_{j=1}^{l} \sum_{i \in I_j} \big|f(P_i) - f(P_{i+1})\big| \le kc + lc \le nc < \infty;$$

$f : \mathtt{BoundedVariation}$; $\square$

$\mathtt{BVDecomposition} :: \forall f : \mathtt{BoundedVariation}[a,b] \;.\; \exists F, G : \mathtt{Increasing}[a,b] : f = F - G$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ f : \mathtt{BoundedVariation}[a,b]$,

$F := \Lambda x \in [a,b] \;.\; V_{f_{|[a,x]}} : \mathtt{Increasing}[a,b]$,

$G := F - f : [a,b] \to \mathbb{R}$,

$\mathtt{Assume}\ x, y \in [a,b] : y > x$,

$G(y) - G(x) = V_{f_{|[a,y]}} - V_{f_{|[a,x]}} + f(x) - f(y) \le V_{f_{|[x,y]}} - |f(x) - f(y)| \ge 0$;

$G : \mathtt{Increasing}[a,b]$,

$f = F - G$; $\square$

$\mathtt{ACDecomposition} :: \forall f : \mathtt{AbsCont}[a,b] \;.\; \exists F, G : \mathtt{Increasing}\ \&\ \mathtt{AbsCont}[a,b] : f = F - G$

$\texttt{ACIsIntegral} :: \forall F : \texttt{AbsCont}[a,b] \ . \ \exists f : \texttt{Integrable}([a,b], \mathcal{B}[a,b], \lambda) : \forall x \in [a,b] \ .$

$$. \ F(x) - F(a) = \int_a^x f d\lambda$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ F : \texttt{AbsCont}[a,b],$

$V, W := \texttt{ACDecomposition}(F) :: \texttt{Increasing} \ \& \ \texttt{AbsCont}[a,b] : F = V - W$

$V, W : \texttt{AbsCont}[a,b] \rightsquigarrow V, W : \mathcal{M}_{\mathsf{TOP}}\big([a,b], \mathbb{R}\big),$

$V, W : \texttt{Increasing}\&\mathcal{M}_{\mathsf{TOP}}\big([a,b], \mathbb{R}\big) \rightsquigarrow V, W : \texttt{Distribution}([a,b], \mathcal{B}[a,b]),$

$V, W : \texttt{AbsCont}[a,b] \rightsquigarrow V - W : \texttt{AbsCont}[a,b],$

$R1 := \texttt{AbsContDistr}(\mu_V, \mu_W) : \mu_V - \mu_W \ll \lambda,$

$f := \texttt{RadonNikodym}(\mu_V - \mu_W, \lambda, R1) : \texttt{Integrable}\big([a,b], \mathcal{B}[a,b], \lambda\big) : \forall A \in \mathcal{B}[a,b] \ .$

$$. \ (\mu_V - \mu_W)(A) = \int_A f d\lambda$$

$,\texttt{Assume} \ x \in [a,b],$

$[a,x] \in \mathcal{B}[a,b],$

$$\int_a^x f d\lambda = \mu_V[a,x] - \mu_W[a,x] = V(x) - V(a) - W(x) + W(a) = F(x) - F(a); ; \square$$

3.4 Differntiation of Measures

$$\texttt{RadonCharge} ::?\texttt{Charge}(\mathbb{R}^d, \mathcal{B}\mathbb{R}^d)$$
$$H : \texttt{RadonCharge} \iff \forall A : \texttt{Bounded}\,\mathbb{R}^d \,.\, |H(A)| < \infty$$

$$\texttt{UpperRadonDifferential} :: \texttt{RadonCharge}(d) \to \mathbb{R}^d \to \mathbb{R}$$
$$\texttt{UpperRadonDifferential}(H)(x) = (\overline{D}\,\mu)(x) = \lim_{r\to 0} \sup_{C:\texttt{Cube}(d,r)} \frac{H(C)}{\lambda(C)}$$

$$\texttt{UpperRadonDifferential} :: \texttt{RadonCharge}(d) \to \mathbb{R}^d \to \mathbb{R}$$
$$\texttt{UpperRadonDifferential}(H)(x) = (\overline{D}\,\mu)(x) = \lim_{r\to 0} \sup_{C:\texttt{Cube}(d,r)} \frac{H(C)}{\lambda(C)}$$

$$\texttt{LowerRadonDifferential} :: \texttt{RadonCharge}(d) \to \mathbb{R}^d \to \mathbb{R}$$
$$\texttt{LowerRadonDifferential}(H)(x) = (\underline{D}\,\mu)(x) = \lim_{r\to 0} \inf_{C:\texttt{Cube}(d,r)} \frac{H(C)}{\lambda(C)}$$

$$\texttt{DifferentiableChargeAt} :: \mathbb{R}^d \to ?\texttt{RadonCharge}(d)$$
$$H : \texttt{DifferentiableChargeAt}(x) \iff (\underline{D}H)(x) = (\overline{D}H)(x)$$

$$\texttt{DifferentiableCharge} ::?\texttt{RadonCharge}(d)$$
$$H : \texttt{DifferentiableCharge} \iff \forall x \in \mathbb{R}^d \,.\, H : \texttt{DifferentiableChargeAt}(x)$$

$$\texttt{RadonDifferentiate} :: \texttt{DifferentiableCharge}(d) \to \mathbb{R}^d \to \mathbb{R}$$
$$\texttt{RadonDifferentiate}(H)(x) = D\,H\,x := \overline{D}\,H\,x$$

$\texttt{DisjointCubesLemma} :: \forall n \in \mathbb{N} \,.\, \forall C : n \to \texttt{Cube}(d)\,.$

$$.\, \exists s \in n \,.\, \exists i : s \hookrightarrow n : \lambda\left(\bigcup_{j=1}^{n} C_j\right) \leq 3^d \sum_{k=1}^{s} \lambda(C_{i_k})$$

$\texttt{Proof} \;=$

$\texttt{Assume} \;\in \mathbb{N}$

$\texttt{Assume}\; C : n \to \texttt{Cube}(d),$

$C' := \texttt{sort}(C, <, \mathrm{diam}),$

$\texttt{Iterate over}\; k \in n \;\texttt{with}\; \mathbf{C}^1 = C', A_1 = \emptyset$

$i_k := \texttt{index}(\mathbf{C}_1^k)(C),$

$A_k := \bigcup_{j=1}^{k} C_{i_j},$

$C^{k+1} := [\mathbf{C}_j^k : \mathbf{C}_j^k \cap A_k = \emptyset];$

$\texttt{until}\; \mathbf{C}^k = [\,];$

$s := \mathrm{Dom}\, i,$

$\texttt{Assume}\; j \in N,$

$\texttt{Assume}\; c : \forall k \in s \,.\, C_j \cap C_{i_k} = \emptyset,$

$c \rightsquigarrow C^{s+1} \neq [\,] \rightsquigarrow \bot;;$

$(1) : \forall j \in n \,.\, \exists k \in s \,.\, C_j \cap C_{i_k} \neq \emptyset,$

$\texttt{Assume}\; k \in s,$

$B_k := \texttt{cube}(\mathrm{center}\, C_{i_k}\, 3\mathrm{diam}\, C_{i_k}),$

$\texttt{Assume}\; \ell \in n : \lambda(C_\ell) \leq \lambda(C_{i_k}) : C_\ell \cap C_{i_k} \neq \emptyset,$

$C_\ell \subset B_k;$

$$\eth(B_k), \eth(C), (1) \rightsquigarrow \lambda\left(\bigcup_{j=1}^{n} C_j\right) \leq \lambda\left(\bigcup_{j=1}^{s} B_j\right) \leq \sum_{j=1}^{s} \lambda(B_j) = 3^d \sum_{j=1}^{s} \lambda(C_{i_j});; \;\square$$

$\texttt{ZeroDifferentialLemma} :: \forall \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^d, \mathcal{B}\mathbb{R}^d) \,.\, \forall A \in \mathcal{B}\mathbb{R}^d \,.$

$\quad . \forall a : \mu(A) = 0 \,.\, D\mu_{|A} = 0 \text{ a . e . } [\lambda]$

$\texttt{Proof} \ =$

$\texttt{Assume } \mu : \texttt{Lebesgue-Stieltjes}(\mathbb{R}^d, \mathcal{B}\mathbb{R}^d),$

$\texttt{Assume } \forall A \in \mathcal{B}\mathbb{R}^d,$

$\texttt{Assume } a : \mu(A) = 0$

$\texttt{Assume } t \in \mathbb{R}_{++},$

$B := \{x \in A : D\,\mu\,x > t\},$

$\texttt{Assume } K : \texttt{Compact}(B),$

$\texttt{Assume } r \in \mathbb{R}_{++},$

$\texttt{Assume } x \in K,$

$x \in K \rightsquigarrow x \in B \rightsquigarrow D\,\mu\,x > t \rightsquigarrow: \exists q : \mathbb{N} \to \mathbb{R}_{++} : q \downarrow 0 : \forall n \in \mathbb{N} \,.$

$$\frac{\mu\,\texttt{cube}(x, q_n)}{\lambda\,\texttt{cube}(x, q_n)} > t \texttt{ Extract},$$

$a_n \downarrow 0 \rightsquigarrow \exists n \in \mathbb{N} \,.\, q_n < r \texttt{ Extract},$

$C_x := \texttt{cube}(x, q_n);$

$C : K \to \texttt{Cube}(\mathbb{R}^d) : \forall x \in K \,.\, x \in C_x,$

$C : \texttt{OCover}(K), K : \texttt{Compact}(B) \rightsquigarrow \exists O : \texttt{FSubOCover}(K, C) \texttt{ Extract},$

$n := \#O \in \mathbb{N},$

$k := \texttt{DisjointCubesLemma}(O),$

$$\lambda(K) \le \lambda\left(\bigcup_{i=1}^n O_i\right) \le 3^d \sum_{i=1}^n \lambda(O_{k_i}) \le \frac{3^k}{t} \sum_{i=1}^n \mu(O_{k_i}) = \frac{3^k}{t}\mu\left(\bigcup_{i=1}^n C_{k_i}\right) \le \frac{3^k}{t}\mu(\mathbb{B}(K, r));$$

$$\lambda(K) \le \frac{3^k}{t}\mu(K) \le \frac{3^k}{t}\mu(A) = 0;$$

$\lambda(B) = 0;$

$D\mu_{|A} = 0 \text{ a . e . } [\lambda];$

$\texttt{LebesgueDecompasitionDerivatives} :: \forall H : \texttt{RadonCharge}(d)\ .$

$\quad .\ \forall \alpha, \beta : \texttt{RadonCharge}(d) : (\alpha, \beta) = \texttt{LebesgueDecomposition}\, H\ .$

$\quad .\ H : \texttt{RadonDifferentiableAt}\ \&\ DH = Df_\alpha \quad \mathrm{a.e.}\ [\lambda]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ H : \texttt{RadonCharge}(d),$

$\texttt{Assume}\ \alpha, \beta : \texttt{Measure}(\mathbb{R}^d, \mathcal{B}\mathbb{R}^d) : (\alpha, \beta) = \texttt{LebesgueDecomposition}\, H,$

$\texttt{Assume}\ a \in \mathbb{R}_{++},$

$A := \{x \in \mathbb{R}^d : f_\alpha(x) < a\} : \mathcal{B}\mathbb{R}^d,$

$B := A^{\complement} : \mathcal{B}\mathbb{R}^d$

$\mu := \Lambda E \in \mathcal{B}\mathbb{R}^d\ .\ \int_{E\cap B} (f_\alpha - a)\mathrm{d}\lambda : \texttt{Measure}(\mathbb{R}^d, \mathcal{B}\mathbb{R}^d),$

$\texttt{Assume}\ r \in \mathbb{R}_{++},$

$\texttt{Assume}\ C : \texttt{Cube}(D) : \operatorname{diam} C < r,$

$R1 :: \alpha(C) - a\lambda(C) = \int_C (f_\alpha - a)\mathrm{d}\lambda \le \int_{C\cap B} (f_\alpha - a)\mathrm{d}\lambda,$

$\eth\mu \rightsquigarrow R_2 : \mu(A) = 0,$

$R_3 := \texttt{ZeroDifferentialLemma}(R_2) : D\mu_{|A} = 0 \quad \mathrm{a.e.}\ [\lambda],$

$R_1 \rightsquigarrow R_4 : \dfrac{\alpha(C)}{\lambda(C)} \le a + \dfrac{\mu(C)}{\lambda(C)};$

$R_5 : \overline{D}\alpha_{|A} \le a \quad \mathrm{a.e.}\ \left[\lambda_{|\mathcal{B}A}\right],$

$E_a := \{x \in \mathbb{R}^d : f_\alpha(x) < a < \overline{D}\alpha(x)\} \in \mathcal{B}\mathbb{R}^d,$

$R_5 \rightsquigarrow \lambda(E_a) = 0,$

$R_6 :: \{\overline{D}\alpha > f_\alpha\} \subset \bigcup_{q\in\mathbb{Q}_{++}} E_q,$

$R_6 \rightsquigarrow R_7 : \lambda\{\overline{D}\alpha > f_\alpha\} \le \lambda\left(\bigcup_{q\in\mathbb{Q}_{++}} E_q\right) \le \sum_{q\in\mathbb{Q}_{++}} \lambda(E_q) = 0,$

$\dots,$

$D\alpha = f_\alpha \quad \mathrm{a.e.}\ [\lambda],$

$\dots$

$D\beta = 0 \quad \mathrm{a.e.}\ [\lambda],$

$DH = D\alpha + D\beta = f_\alpha \quad \mathrm{a.e.}\ [\lambda] \square$

$\texttt{MonotoneIsAlmostDiffrerentiable} :: \forall f : \texttt{Increasing}[a,b] \,.\, f : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right]$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \texttt{Increasing}[a,b],$

$R_1 := \texttt{MonotoneDisconts}(f) : \#\texttt{Discont}(f) \le \aleph_0,$

$R_1 \rightsquigarrow \exists G : \texttt{DistributionFunction}[a,b] : f = G \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right] \texttt{ Extract},$

$R_2 := \texttt{LebesgueDecompasitionDerivatives}(\mu_G) : \left(\mu_G : \texttt{RadonDifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right]\right)$

$\texttt{Assume}\ x \in [a,b] : (\mu_G : \texttt{RadonDifferentiableAt}(x)),$

$\mu_G : \texttt{RadonDifferentiableAt}(x) \rightsquigarrow G : \texttt{ContAt}(x),$

$$\lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0}\lim_{k\to 0} \frac{G(x+h) - G(x-k)}{h+k} = D\mu_G(x);$$

$G : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right] \rightsquigarrow f : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right] \;\square$

$\texttt{ACHasNewtonProperty} :: \forall F : [a,b] \to \mathbb{R} \,.$

$$F : \texttt{AbsCont}[a,b] \iff \forall x \in [a,b] \,.\, F(x) - F(a) = \int_a^x F'(t)\mathrm{d}t$$

$\texttt{Proof} \;=$ $\texttt{Assume}$

$\texttt{Assume}\ F : \texttt{AbsCont}[a,b],$

$(A,B) := \texttt{ACDecomposition} : \texttt{Increasing} \,\&\, \texttt{AbsCont}[a,b] : F = A - B,$

$dA := \texttt{MonotoneIsAlmostDiffrerentiable}(A) : \left(A : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right]\right),$

$dB := \texttt{MonotoneIsAlmostDiffrerentiable}(B) : \left(B : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right]\right),$

$dA, dB, \eth(A,B) \rightsquigarrow dF : \left(F : \texttt{DifferentiableAt} \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right]\right),$

$f := \texttt{ACIsIntegral}(F) : \texttt{Integrable}([a,b], \mathcal{B}[a,b],) : F(x) - F(a) = \int_a^x f(t)\mathrm{d}t,$

$H := \Lambda A \in \mathcal{B}[a,b] \,.\, \int_A f \mathrm{d}\lambda : \texttt{RadonCharge}(1),$

$E_1 := \texttt{LebesgueDecompasitionDerivatives}(H) :: f = DH \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right],$

$\texttt{Assume}\ x \in [a,b] : (F : \texttt{DifferentiableAt}(x)),$

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\lim_{k\to 0} \frac{F(x+h) - F(x-k)}{h+k} = DH;$$

$E_1 \rightsquigarrow E_2 : f = F' \quad \text{a . e .} \, \left[\lambda_{|\mathcal{B}[a,b]}\right],$

$E_2 \rightsquigarrow \forall x \in [a,b] \,.\, F(x) - F(a) = \int_a^x F'(t)\mathrm{d}t;$

$I : F : \texttt{AbsCont}[a,b] \Rightarrow \forall x \in [a,b] \,.\, F(x) - F(a) = \int_a^x F'(t)\mathrm{d}t$

$\texttt{Assume}\ a : forall x \in [a,b] \,.\, F(x) - F(a) = \int_a^x F'(t)\mathrm{d}t,$

$\texttt{ACIsIntegral}(a) \rightsquigarrow F : \texttt{AbsCont}[a,b];$

$F : \texttt{AbsCont}[a,b] \iff \forall x \in [a,b] \,.\, F(x) - F(a) = \int_a^x F'(t)\mathrm{d}t\square$

$\texttt{VectorChangeOfVariable} :: \forall V, W : \texttt{Open}(\mathbb{R}^d) \;.\; \forall f : \texttt{Integrable}(W, \mathcal{B}W, \lambda) \;.\; \forall T : \texttt{Iso}_{\texttt{DIFF}}(V, W) \;.$

$$\int_W f d\lambda = \int_V |\det \nabla T|(f \circ T) \mathrm{d}\lambda$$

3.5 Categorical Viewpoint:Developing Borel-Null [!]

4 Convergence in Measure

4.1 Measure Topology

$\texttt{measurePseudoMetrics} :: \prod(\Omega, \mathcal{F}, \mu) \in \mathsf{MEAS} \,.\, \texttt{PseudoDistance}(\texttt{Measurable}(\Omega, \mathcal{F}))$

$\texttt{measurePseudoMetrics}(f, g) = d_\mu(f, g) := \inf_{\delta > 0} \mu\{\omega \in \Omega : |f(\omega) - g(\omega)| > \delta\} + \delta$

$\texttt{implicit} :: \texttt{Measure}(\Omega, \mathcal{F}) \to \texttt{Topology}(\texttt{Measurable}(\Omega, \mathcal{F}))$

$\mu := \texttt{pseudometricTopology}(d_\mu)$

$\texttt{ConvergenceInMeasure} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}) \,.\, \forall \phi : \texttt{Measurable}(\Omega, \mathcal{F}) \,.$

$\quad f \to_\mu \phi \iff \forall \epsilon \in \mathbb{R}_{++} \,.\, \lim_{n \to \infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon\} = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \phi : \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ C : f \to_\mu \phi,$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++};$

$\texttt{Assume}\ a \in \mathbb{R}_{++} : a < \epsilon,$

$\texttt{Assume}\ L : \lim_{n \to \infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon\} > a,$

$\texttt{Assume}\ \delta \in \mathbb{R}_{++},$

$\texttt{Assume Alternative}\ \delta \geq a,$

$\forall n \in \mathbb{N} \,.\, \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \delta\} + \delta \geq \delta \geq a;$

$\texttt{Close Alternative}\ \delta < a,$

$\exists N : \texttt{Infinite}(\mathbb{N}). \forall n \in N \,.\, \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \delta\} + \delta \geq \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \delta\} \geq a;$

$\exists N : \texttt{Infinite}(\mathbb{N}) \,.\, \forall n \in N \,.\, \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \delta|\} + \delta \geq a;$

$\exists N : \texttt{Infinite}(\mathbb{N}) \,.\, \forall n \in N \,.\, d_\mu(f_n, \phi) \geq a \leadsto \lim_{n \to \infty} f_n \neq \phi \leadsto \bot;;;$

$\forall \epsilon \in \mathbb{R}_{++} \,.\, \lim_{n \to \infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon\} = 0;$

$\ldots$

$\texttt{Assume}\ A : \forall \epsilon \in \mathbb{R}_{++} \,.\, \lim_{n \to \infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon\} = 0,$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_n,$

$L := A(\epsilon/2) : \lim_{n \to \infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon/2\} = 0,$

$N := L(\epsilon/2) \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq N \,.\, \mu\{\omega \in \Omega : |f_n(\omega) - \phi| > \epsilon/2\} < \epsilon/2,$

$\texttt{Assume}\ n \in \mathbb{N} : n \geq N,$

$d_\mu(f_n, \phi) = \inf_{\delta > 0} \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \delta\} + \delta \leq \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \epsilon/2\} + \epsilon/2 < \epsilon;;;$

$\square$

4.2 Comparisson with other modes of convergence

$\texttt{LpImplyMeasure} :: \forall f : \mathbb{N} \to L^p(\Omega, \mathcal{F}, \mu) \,.\, \forall \phi : L^p(\Omega, \mathcal{F}, \mu) \,.\, f \to_{L^p} \phi \Rightarrow f \to_\mu \phi$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \mathbb{N} \to L^p(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ \phi : L^p(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ C : f \to_{L^p} \phi,$

$\texttt{Assume}\ \epsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ n : \mathbb{N},$

$$I := \texttt{ChebishevIneq}(|f_n - \phi|, \epsilon) : \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| < \epsilon\} \le \frac{\|f_n(\omega) - \phi(\omega)\|_p}{\epsilon^p};$$

$$\rightsquigarrow R := \texttt{LimIneq}(\eth(\to_{L^p})(C), \texttt{IneqLim}(\cdot)) : \lim_{n\to\infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| < \epsilon\} \le$$

$$\le \lim_{n\to\infty} \frac{\|f_n - \phi\|_p}{\epsilon^p} = 0;$$

$\rightsquigarrow R := \texttt{ConvergenceInMeasure}(f, \phi, \cdot) : f \to_\mu \phi;;;$

$\square$

$\texttt{UniformAEConvergence} :: \prod(\Omega, \mathcal{F}, \mu) : \mathsf{MEAS} \,.$

$\quad .\, ?\Big(\big(\mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu)\big) \times \texttt{Measurable}(\Omega, \mathcal{F}, \mu)\Big)$

$(f, \phi) : \texttt{UniformAEConvergence} \iff f \rightrightarrows_\mu \phi \iff \forall \epsilon : \mathbb{R}_{++} \,.$

$\quad .\, \exists A \in \mathcal{F} : \mu(A) < \epsilon : f_{|A^\complement} \rightrightarrows \phi_{|A^\complement}$

$\texttt{AEUnifImplyMeasure} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \,.\, \forall \phi : \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \,.$

$\quad .\, f \rightrightarrows_\mu \phi \Rightarrow f \to_\mu \phi$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ \phi : \texttt{Measurable}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ C : f \rightrightarrows_\mu \phi,$

$\texttt{Assume}\ \epsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ a : \mathbb{R}_{++},$

$A := C(a) : \mathcal{F} : \mu(A) < a : f_{|A^\complement} \rightrightarrows \phi_{|A^\complement},$

$N := \eth(A)(\epsilon) : \forall n \in \mathbb{N} : n \ge N \,.\, \{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \epsilon\} \subset A,$

$\texttt{Assume}\ n : \mathbb{N} : n \ge N,$

$S := \eth(N)(n) : \{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \epsilon\} \subset A,$

$I := \texttt{MeasureMonotonicity}(\mu, S)\eth(A) : \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \epsilon\} \le \mu(A) < a;;$

$$\rightsquigarrow L := \eth[\texttt{Limit}](\cdot) : \lim_{n\to\infty} \mu\{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| > \epsilon\} = 0;$$

$\rightsquigarrow R := \texttt{ConvergenceInMeasure}(f, \phi, \cdot) : f \to_\mu \phi;;;$

$\square$

$\texttt{AEUnifImplyAE} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \,.\, \forall \phi : \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \,.$

$\quad .\, f \rightrightarrows_\mu \phi \Rightarrow f \to \phi \quad \mathrm{a\,.\,e\,.}\ [\mu]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ \phi : \texttt{Measurable}(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume}\ C : f \rightrightarrows_\mu \phi,$

$\texttt{Assume}\ n : \mathbb{N},$

$A_n := C(1/n) : \mathcal{F} : \mu(A_n) < \dfrac{1}{n} : f_{|A_n^\complement} \rightrightarrows \phi_{|A_n^\complement},$

$L := \texttt{UnifImplyPointwise}(\eth_2 A_n) : f_{|A_n^\complement} \to \phi_{|A_n^\complement};$

$\leadsto A := (\cdot) : \prod n : \mathbb{N} \,.\, \mathcal{F} : \mu(A_n) < \dfrac{1}{n} : f_{|A_n^\complement} \to \phi_{|A_n^\complement},$

$B := \bigcup_{n=1}^{\infty} A_n^\complement : \mathcal{F},$

$L := \eth(B.\eth(A)) : f_{|B} \to \phi_{|B},$

$Z := \texttt{IneqLim}(\texttt{LimIneq}\,\Lambda n \in \mathbb{N} \,.\, \texttt{MeasurMonotonicity}\left(A_n, \bigcap_{k=1}^{\infty} A_k\right) \texttt{UnionCompliment}(B),$

$\quad , \texttt{LimEq}(\texttt{Lim}(\lambda n \in \mathbb{N} \,.\, 1/n), \mu(A), \eth_1 A)) : \mu\left(B^\complement\right) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \leq \lim_{n\to\infty} \mu(A_k) = 0;$

$\leadsto R := \eth \mathrm{a\,.\,e\,.}\ [\mu]\,(\Lambda \omega \in \Omega \,.\, f(\omega) \to \phi(\omega))(\cdot) : f \to \phi \quad \mathrm{a\,.\,e\,.}\ [\mu]\,;\,;\,,$

$\square$

$\texttt{AEUnifSubseq} :: \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \,.\, \forall \phi : \texttt{Measurable}(\Omega, \mathcal{F}) : f \to_\mu \phi \,.$

$\quad .\, \exists n : \texttt{Subseqer} : f_n \rightrightarrows_\mu \phi$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \phi : \texttt{Measurable}(\Omega, \mathcal{F}, \mu) : f \to_\mu \phi,$

$(T) := \eth(\mu)\ \texttt{as Topology} : (\mu : \texttt{Semimetrizable}),$

$(C) := \texttt{ConvergingIsCauchy}(T, f, \eth(\phi)) : (f : \texttt{Cauchy}),$

$\texttt{Assume}\ k : \mathbb{N},$

$M := \eth\texttt{Cauchy}(f)(2^{-k}) : \mathbb{N} : \forall n, m \in \mathbb{N} : n \geq M : m \geq M \,.\, d_\mu(f_n, f_m) < 2^{-k},$

$N_k := \max(M, N_{k-1} + 1) : \mathbb{N};$

$\leadsto N := (\cdot) : \texttt{Subseqer} : \forall k : \mathbb{N} \,.\, \forall n, m : \mathbb{N} : n \geq N_k : m \geq N_k \,.\, d_\mu(f_n, f_m) < 2^{-k},$

$g := f_N : \mathbb{N} \to \texttt{Measurable} : \forall k : \mathbb{N} \,.\, d_\mu(g_k, g_{k+1}) < 2^{-k},$

$\texttt{Assume}\ k : \mathbb{N},$

$A_k := \{\omega \in \Omega : |g_k(\omega) - g_{k+1}(\omega)| > 2^{-k}\} : \mathcal{F},$

$(I) := \eth(g, A) : 2^{-k} \geq d_\mu(g_k, g_{k+1}) + \epsilon =$

$= [x < 2^{-k}] = \{\omega \in \Omega : |g_k(\omega) - g_{k+1}(\omega)| > x\} + x \geq \mu(A_k);$

$\rightsquigarrow A := (\cdot) : \mathbb{N} \to \mathcal{F} : \forall k : \mathbb{N} \; . \; \mu(A_k) \leq 2^{-k},$

$\alpha := \limsup A : \mathcal{F},$

$Z := \texttt{BorellCanteli}(\alpha, \eth(A)) : \mu(\alpha) = 0,$

$\texttt{Assume}\ \omega : \alpha^{\complement},$

$FF := \eth(\limsup)(\alpha, A, \omega) : \#\{k \in \mathbb{N} : \omega \in A_k\} < \aleph_0,$

$L := \texttt{CauchyCriterion}(f(\omega), A, FF) : (f(\omega) : \texttt{Cauchy}),$

$R := \eth\texttt{Complete}(\mathbb{R}, f(\omega)) : (f(\omega) : \texttt{Converge}(\mathbb{R}));$

$\rightsquigarrow AC := \eth(\mathrm{a.e.}\ [\mu])(\cdot, Z) : (f : \texttt{Converge}(\mathbb{R})) \quad \mathrm{a.e.}\ [\mu],$

$\gamma := [AC] \lim_{n \to \infty} g_n : \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ k : \mathbb{N},$

$$B_k := \bigcup_{n=k}^{\infty} A_k : \mathcal{F};$$

$\rightsquigarrow B := (\cdot) : \mathbb{N} \to \mathcal{F},$

$L := \eth(B, A) : \lim_{n \to \infty} \mu(B_n) = 0,$

$W := \texttt{WeierstrassMTest}(\eth(B), L) : g \rightrightarrows_\mu \gamma,$

$LM := \texttt{AEUnifImplyMeasure}(g, \gamma, W) : g \to_\mu \gamma,$

$LL := \texttt{ConvergingSubseqAgrees}(f, g, \eth f, LM) : g \to_\mu \phi,$

$E := \texttt{TopoEquelInMeasure}(LM, LL) : \phi = \gamma \quad \mathrm{a.e.}\ [\mu],$

$R := \texttt{TopoEquelAEUniform}(W, E) : g \rightrightarrows_\mu \phi,$

□

$\texttt{AEProbabilityLemma} :: \forall P : \texttt{Probability}(\Omega, \mathcal{F}) \; . \; \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}) \; .$

$. \; \forall \phi : \texttt{Measurable}(\Omega, \mathcal{F}) \; . \; f \to \phi \quad \mathrm{a.e.}\ [P] \iff$

$$\iff \forall \delta : \mathbb{R}_{++} \; . \; \lim_{n \to \infty} P\left(\bigcup_{k=n}^{\infty} \{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| \geq \delta\} = 0\right)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ P : \texttt{Probability}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \phi : \texttt{Measuravle}(\Omega, F),$

$\texttt{Assume}\ \delta : \mathbb{R}_{++},$

$\texttt{Assume}\ n : \mathbb{N},$

$B_n^\delta := \{\omega \in \Omega : |f_n(\omega) - \phi(\omega)| \geq \delta\} : \mathcal{F};$

$\rightsquigarrow B^\delta := (\cdot) : \mathbb{N} \to \mathcal{F},$

$\beta^\delta := \limsup B : \mathcal{F},$

$$CD := \eth(\limsup)(B^\delta)(\beta^\delta) : \bigcup_{k=n}^{\infty} B_k^\delta \downarrow_n \beta^\delta,$$

$$MC := \texttt{MeasureLowerContinuity}(P, CD) : \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} B_k^\delta\right) = P(\beta^\delta);$$

$$\rightsquigarrow E := \ldots : \{\omega \in \Omega : f(\omega) \not\to \phi(\omega)\} = \bigcup_{\delta\in\mathbb{R}_{++}} \beta_\delta,$$

$\texttt{Assume}\ C : f \to \phi \quad \text{a.e.}\ [P],$

$\texttt{Assume}\ \delta : \mathbb{R},$

$$R := \ldots : 0 = P(\{\omega \in \Omega : f(\omega) \not\to \phi(\omega)\}) = P\left(\bigcup_{x\in\mathbb{R}_{++}} \beta_x\right) \geq P(\beta_\delta) \rightsquigarrow$$

$$\rightsquigarrow P(\beta_\delta) = 0 \rightsquigarrow \lim_{n\to\infty}\left(\bigcup_{k=n}^{\infty} B_k^\delta\right) = 0;;$$

$\texttt{Assume}\ A : (*),$

$$R := \ldots : P(\{\omega \in \Omega : f(\omega) \not\to \phi(\omega)\}) = P\left(\bigcup_{x\in\mathbb{R}_{++}} \beta^\delta\right) =$$

$$= P\left(\bigcup_{x\in\mathbb{Q}_{++}} \beta^\delta\right) \leq \sum_{x\in\mathbb{Q}_{++}} P(\beta^\delta) = \sum_{x\in\mathbb{Q}_{++}} \lim_{n\to\infty}\left(\bigcup_{k=n}^{\infty} B_k^x\right) = 0 \rightsquigarrow$$

$$\rightsquigarrow f \to \phi \quad \text{a.e.}\ [P];;$$

$\square$

$\texttt{Egoroff} :: \forall P : \texttt{Probability}(\Omega, \mathcal{F}) . \forall f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}) .$
$\forall \phi : \texttt{Measurable}\Omega, \mathcal{F} : f \to \phi \quad \text{a.e.}\ [P]\ . f \rightrightarrows_P \phi$

$\texttt{Proof} =$

$\texttt{Assume}\ P : \texttt{Probability}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ f : \mathbb{N} \to \texttt{Measurable}(\Omega, \mathcal{F}),$

$\texttt{Assume}\ \phi : \texttt{Measurable}\Omega, \mathcal{F} : f \to \phi \quad \text{a.e.}\ [P],$

$\texttt{Assume}\ \epsilon : \mathbb{R}_{++},$

$\texttt{Assume}\ i : \mathbb{N},$

$$A_j := \bigcup_{k=n}^{\infty} \{\omega \in \Omega : |f_k(\omega) - \phi(\omega)| \geq i^{-1}\} : \mathcal{F},$$

$$N_j := \texttt{AEProbabilityLemma}(P, f, \phi)(\epsilon 2^{-i}) : \mathbb{N} : \forall n \in \mathbb{N} : n \geq N_j\ . P(A_j) < \epsilon 2^{-i};$$

$$\rightsquigarrow N := (\cdot) : \mathbb{N} \to \mathbb{N},$$

$$\alpha := \bigcup A : \mathcal{F},$$

$$I := \eth\texttt{Measure}(P)(\eth(A, \alpha)) : P(\alpha) \leq \sum_{i=1}^{\infty} P(A_i) < \epsilon,$$

$$U := \eth A : \forall \omega \in A^{\complement}\ . f(\omega) \rightrightarrows \phi(\omega);$$

$$\rightsquigarrow R := \eth[\rightrightarrows_P](\cdot) : f \rightrightarrows_P \phi;$$

$\square$

5 Products over Measure spaces

5.1 Product Measure Theorem

$\texttt{SigmaAlgebraProduct} :: \sigma\texttt{-Algebra}(A) \to \sigma\texttt{-Algebra}(B) \to \texttt{Set}(A \times B)$

$\texttt{SigmaAlgebraProduct}(\mathcal{A}, \mathcal{B}) = \mathcal{A} \times \mathcal{B} := \{a \times b | a \in \mathcal{A}, b \in \mathcal{B}\}$

$\texttt{BorProduct} :: \mathsf{BOR} \to \mathsf{BOR} \to \mathsf{BOR}$

$\texttt{BorProduct}((A, \mathcal{A}), (B, \mathcal{B})) = (A, \mathcal{A}) \times (B, \mathcal{B}) := (A \times B, \sigma(\mathcal{A} \times B))$

$\texttt{Uniformly } \sigma\texttt{-Finite} :: ?(X \to \texttt{Measure}(Y))$

$$\mu : \texttt{Uniformly } \sigma\texttt{-Finite} \iff \exists b : \mathbb{N} \to \mathcal{F}_Y : \exists k : \mathbb{N} \to \mathbb{R}_+ : \bigcup_{n=1}^{\infty} b_n = Y :$$

$$: \forall x \in X \,.\, \forall n \in \mathbb{N} \,.\, \mu(x, b_n) \le k_n$$

$\texttt{SlicingMeasure} :: \prod X \in \mathsf{MEAS} \,.\, \prod Y \in \mathsf{BOR} \,.\, X \to \texttt{Measure}(Y)$

$\mu : \texttt{SlicingMeasure} \iff \forall b \in \mathcal{F}_Y \,.\, \Lambda x \in X \,.\, \mu(x, b) : \texttt{Measurable}(F_{\mathsf{BOR}} X)$

$\texttt{RectangularAlgrebraTHM} :: \forall X, Y : \mathsf{BOR} \,.\, \forall G : \texttt{MonotoneClass}(X \times Y) : \mathcal{F}_X \times \mathcal{F}_Y \subset G \,.\, \sigma(\mathcal{F}_X \times \mathcal{F}_Y) \subset G$

$\texttt{Proof} =$

$\texttt{Assume } x \times y : \mathcal{F}_X \times \mathcal{F}_Y,$

$(1) := \texttt{ProductComplement}(x \times y) : (x \times y) = x^{\complement} \times y \cap x \times y^{\complement} \cap x^{\complement} \times y^{\complement},$

$(2) := \eth\texttt{MonotoneClass}(1, \eth(G)) : (x \times y)^{\complement};$

$\leadsto (\mathcal{F}_X \times \mathcal{F}_Y, 1) := (\mathcal{F}_X \times \mathcal{F}_Y, \eth\texttt{ComplementClosed}(\cdot) : \texttt{ComplementClosed}(G),$

$(2) := \texttt{MonotoneClassTHM}(1) : \sigma(\mathcal{F}_X \times F_Y) \subset G;$

$\square$

$\texttt{MeasurableSection} :: \forall X, Y : \mathsf{BOR} \,.\, \forall A : \mathcal{F}_{X \times Y} \,.\, \forall x : X \,.\, \mathrm{section}(A, x) \in \mathcal{F}_Y$

$\texttt{Proof} =$

$B := \{A \in \mathcal{F}_{X \times Y} : \mathrm{section}(A, x) \in \mathcal{F}_Y\} : \sigma\texttt{-Algebra}(X \times Y),$

$(I) := \eth B : \{a \times b | a \in F_X, b \in F_Y\} \subset B,$

$(II) := \eth(\mathcal{F}_X \times F_Y)(\eth(\sigma)(I)) : \mathcal{F}_{X \times Y} \subset B \leadsto \mathcal{F}_{X \times Y} = B;;;$

$\square$

$\texttt{MeasurableSlicing} :: \forall S : \texttt{SlicingMeasure}(X, U) \,.\, \forall A : \mathcal{F}_{X\times Y} \,.\, .$
$\quad \Lambda x \in X \,.\, S(x, \mathrm{section}(A, x)) : \texttt{Measurable}(F_{\mathsf{BOR}} X)$

$\texttt{Proof} \ =$

$B := \{A \in \mathcal{F}_{X\times Y} : \Lambda x \in X \,.\, S(x, \mathrm{section}(A, x)) : \texttt{Measurable}(F_{\mathsf{BOR}} X)\} :$
$\quad : \texttt{Set}(F_{\mathsf{BOR}} X \times Y),$

$\texttt{Assume}\ b : \mathbb{N} \to B,$

$\texttt{Assume}\ \beta : \mathcal{F}_{X\times Y} : b \uparrow \beta,$

$(1) := \texttt{SectionIsMonotonic}(b, \beta) : \forall x : X \,.\, \mathrm{section}(x, b_n) \uparrow \mathrm{section}(x, \beta),$

$(2) := \texttt{MeasureUpperContinuity}(\Lambda x \in X \,.\, S(x, b), (1)) : \Lambda x \in X \,.\, S(x, b_n) \uparrow \Lambda x \in X \,.\, S(x, \beta),$

$(3) := \texttt{MonotoneConvergenceTHM}(2) : (x \in X \,.\, S(x, \beta) : \texttt{Measurable}(F_{\mathsf{BOR}} X)),$

$(4) := \eth(B)(3) : \beta \in B;;$

$\rightsquigarrow (1\star) := \texttt{UniversalIntroduction}(\cdot) : \forall b : \mathbb{N} \to B \,.\, \forall \beta : \mathcal{F}_{X\times Y} : b \uparrow \beta \,.\, \beta \in B,$

$\texttt{Assume}\ b : \mathbb{N} \to B,$

$\texttt{Assume}\ \beta : \mathcal{F}_{X\times Y} : b \downarrow \beta,$

$(1) := \texttt{SectionIsMonotonic}(b, \beta) : \forall x : X \,.\, \mathrm{section}(x, b_n) \downarrow \mathrm{section}(x, \beta),$

$(2) := \texttt{MeasureLowerContinuity}(\Lambda x \in X \,.\, S(x, b), (1)) : \Lambda x \in X \,.\, S(x, b_n) \downarrow \Lambda x \in X \,.\, S(x, \beta),$

$(3) := \texttt{MonotoneConvergenceTHM}(2) : (x \in X \,.\, S(x, \beta) : \texttt{Measurable}(F_{\mathsf{BOR}} X)),$

$(4) := \eth(B)(3) : \beta \in B;;$

$\rightsquigarrow (2\star) := \texttt{UniversalIntroduction}(\cdot) : \forall b : \mathbb{N} \to B \,.\, \forall \beta : \mathcal{F}_{X\times Y} : b \downarrow \beta \,.\, \beta \in B,$

$(1) := \eth\texttt{MonotoneClass}(1\star, 2\star) : B : \texttt{MonotoneClass}(X \times Y),$

$\texttt{Assume}\ a : \mathcal{F}_X,$

$\texttt{Assume}\ b : \mathcal{F}_Y,$

$(2) := \eth\mathrm{section}(a \times b) : \Lambda x \in X \,.\, S(x, \mathrm{section}(x, a \times b)) = \Lambda x \in X \,.\, S(x, b),$

$(3) := \eth\texttt{SlicingMeasure}(S)(b)(2) : (\Lambda x \in X \,.\, S(x, \mathrm{section}(x, a \times b)) : \texttt{Measurable}(F_{\mathsf{BOR}} X)),$

$(4) := \eth B(3) : a \times b \in B;$

$\rightsquigarrow (2) := \eth\mathcal{F}_X \times \mathcal{F}_Y(\cdot) : \mathcal{F}_X \times \mathcal{F}_Y \subset B,$

$(3) := \texttt{RectangularAlgebraTHM}(X, Y, B)(2) : \mathrm{Alg}(F_X \times \mathcal{F}_Y) \subset B,$

$(4) := \texttt{MonotoneClassTHM}(1, 3) : \sigma(\mathcal{F}_X \times \mathcal{F}_Y) \subset B,$

$(5) := \texttt{SetEqIntroduction}(4, \eth B) : \mathcal{F}_{X\times Y} = B;;$

$\square$

$\texttt{ProductMeasureTheorem} :: \forall X : \mathsf{MEAS} \,.\, \forall Y : \mathsf{BOR} \,.\, \forall S : \texttt{SlicingMeasure}(X, Y) \,.$

$\quad .\, \exists! \gamma : \texttt{Measure}(F_{\mathsf{BOR}} X \times Y) : \forall A : \mathcal{F}_{F_{\mathsf{BOR}} X \times Y} \,.\, \gamma(A) = \int_X S(x, \mathrm{section}(A, x)) \mathrm{d}\mu_X$

$\texttt{Proof} \ =$

$\gamma := \Lambda A \in \mathcal{F}_{F_{\mathsf{BOR}} X \times Y} \,.\, \int_X S(x, \mathrm{section}(A, x))\, \mathrm{d}\mu_X(x) : \mathcal{F}_{F_{\mathsf{BOR}} X \times Y} \overset{\infty}{\to} \mathbb{R}_+,$

$\texttt{Assume}\ A : \texttt{Disjoint}(\mathbb{N}, \mathcal{F}_{F_{\mathsf{BOR}} X \times Y}),$

$(1) := \eth\texttt{Measure}(S(x, \cdot)) : \int_X S\left(x, \mathrm{section}\left(\bigcap_{n=1}^{\infty} A_n, x\right)\right) \mathrm{d}\mu_X(x) = \int_X \sum_{i=1}^{n} S(x, \mathrm{section}(A_n, x))\, \mathrm{d}\mu_X(x),$

$(2) := \texttt{IntegralSum}(2) : \int_X S\left(x, \mathrm{section}\left(\bigcap_{n=1}^{\infty} A_n, x\right)\right) \mathrm{d}\mu_X(x) = \sum_{n=1}^{\infty} \int_X S(x, \mathrm{section}(A_n, x))\, \mathrm{d}\mu_X(x),$

$(3) := \eth\gamma(2) : \gamma\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \gamma(A_n);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Measure}(\cdot) : (\gamma : \texttt{Measure}(\mathcal{F}_{F_{\mathsf{BOR}} X \times Y}));$

$\square$

$\texttt{productMeasure} :: \texttt{SlicingMeasure}(X, Y) \to \texttt{Measure}(\mathcal{F}_{X \times Y})$

$\texttt{productMeasure}\,(S) := \texttt{ProductMeasureTHM}(S)$

$\texttt{ProductProbabilityTheorem} :: \forall X : \texttt{ProbabilitySpace} \,.\, \forall Y : \mathsf{BOR} \,.\, \forall P : \texttt{SlicingMeasure} :$

$\quad \forall x : X \,.\, S(x, Y) = 1 \,.\, \texttt{productMeasure}(P) : \texttt{Probability}(X \times Y)$

$\texttt{Proof} \ =$

$\mathbb{P} := \texttt{productMeasure(P)} : \texttt{Measure}(\mathcal{F}_{X \times Y}),$

$(1) := \texttt{EqE}(\eth\texttt{section}(\eth\texttt{SlicingMeasure}(P), X \times Y)) :$

$\quad : \int_X P(x | \texttt{section}(X \times Y, x))\, \mathrm{d}\mu_X(x) = \int_X P(x|Y)\, \mathrm{d}\mu_X(x),$

$(2) := (1)\texttt{EqE}(\eth(P)) : \int_X P(x | \texttt{section}(X \times Y, x))\, \mathrm{d}\mu_X(x) = \int_X \mathrm{d}\mu_X(x),$

$(3) := (2)\eth\texttt{Probability}(\mu_X) : \int_X P(x | \texttt{section}(X \times Y, x))\, \mathrm{d}\mu_X(x) = 1,$

$(4) := \eth\mathbb{P}(X \times Y(3) : \mathbb{P}(X \times Y) = 1,$

$(*) := \eth^{-1}\texttt{Probability} : (\mathbb{P} : \texttt{Probability}(X \times Y));$

$\square$

$\texttt{ProductSFTHM} :: \forall S : \texttt{SlicingMeasure} \ \& \ \texttt{Uniformly}\,\sigma\texttt{-Finite}\,(X, Y) : (\mu_X : \sigma\texttt{-Finite}\,(X))\ .$

$\quad . \ \texttt{productMeasure}(S) : \sigma\texttt{-Finite}\,(X \times Y)$

$\texttt{Proof} \ =$

$$(B, b) := \eth(\texttt{Uniformly}\ \sigma\texttt{-Finite}\,(X \times Y))(S) : \sum B : \mathbb{N} \to \mathcal{F}_Y : \bigcup_{n=1}^{\infty} B_n = Y\ .$$

$$\quad . \ \mathbb{N} \to \mathbb{R}_+ : \forall x : X\ .\ \forall n : \mathbb{N}\ .\ S(x, B_n) \le b_n,$$

$$A := \eth\sigma\texttt{-Finite}\,(X)\,(\mu_X) : \mathbb{N} \to \mathcal{F}_X : \bigcup_{n=1}^{\infty} A_n = X : \mu_X(A) < \infty,$$

$$(1) := \texttt{ProductPartition}(A, B) : \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \times B_m = X \times Y,$$

$\gamma := \texttt{productMeasure}(S) : \texttt{Measure}(X \times Y),$

$\texttt{Assume}\ n, m : \mathbb{N},$

$$(2) := \eth\gamma(\texttt{Assume}\ {}_n \times B_n)\texttt{IntIneq}(\eth b_m)\texttt{MeasureAsIntegral}(\mu_X, A_n)\eth(A_n) :$$

$$\quad : \gamma(A_n \times B_m) = \int_{A_n} S(x, B_m)\,\mathrm{d}\mu_X(x) \le \int_{A_n} b_m \mathrm{d}\mu_X = b_m \mu_X(A) < \infty;$$

$\rightsquigarrow (2) := \texttt{UI} : \forall n, m : \mathbb{N}\ .\ \gamma(A_n \times B_m) < \infty,$

$(*) := \eth^{-1}\sigma\texttt{-Finite}\,(X \times Y)\,(\gamma, A \times B, 1, 2) : (\gamma : \sigma\texttt{-Finite}\,(X \times U));$

$\square$

$\texttt{productOfMeasures} :: \mathsf{MEAS} \to \mathsf{MEAS} \to \mathsf{MEAS}$

$$\texttt{productOfMeasures}\,((X, \mathcal{F}, \mu), (Y, \mathcal{G}, \nu)) = \mu \times \nu := \left(X \times Y, \sigma(\mathcal{F} \times \mathcal{G}), A \mapsto \int_X \nu(\mathrm{section}(A, x))\,\mathrm{d}\mu(x)\right)$$

$\texttt{ClassicalPMTHM} :: \forall X, Y : \mathsf{MEAS}\ .\ \forall A \times B : F_X \times F_Y\ .\ \mu_X \times \mu_Y(A \times B) = \mu_X(A)\mu_Y(B)$

$\texttt{Proof} \ =$

$$(*) := \eth\texttt{productOfMeasure}(X, Y)\texttt{ProductSection}(A, B)\texttt{IntegralHomogenity}(\mu_Y(B)$$

$$\quad \texttt{MeasureAsIntegral}(\mu_X, A) : \mu_X \times \mu_Y(A \times B) = \int_X \mu_Y(\texttt{section}(A \times B), x)\,\mathrm{d}\mu_X(x) =$$

$$\quad = \int_X \mu_Y(B) I_A\,\mathrm{d}\mu_X = \mu_Y(B) \int_X I_A \mathrm{d}\mu_X = \mu_Y(B)\mu_X(A);$$

$\square$

$\texttt{MeasureProductCommute} :: \forall X, Y : \mathsf{MEAS}\ .\ \mu_X \times \mu_Y = \mu_Y \times \mu_X \circ \texttt{swap}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A \times B : \mathcal{F}_X \times \mathcal{F}_Y,$

$(1) := \texttt{ClassicalPMTHM}(X, Y)(A \times B) : \mu_X \times \mu_Y(A \times B) = \mu_X(A)\mu_Y(B),$

$(2) := \texttt{ClassicalPMTHM}(Y, X)(B \times A) : \mu_Y \times \mu_X(B \times A) = \mu_Y(B)\mu_X(A),$

$(3) := (1)(2) : \mu_X \times \mu_Y(A \times B) = \mu_Y \times \mu_X(B \times A);$

$\rightsquigarrow (*) := \texttt{SwapIntro}(\cdot) : \mu_X \times \mu_Y = \mu_Y \times \mu_X \circ \texttt{swap};$

$\square$

5.2 Fubbini Theorem

$\texttt{MeasrableOnProduct} :: \forall X, Y : \mathsf{BOR} \,.\, \forall f : \texttt{Masurable}(X \times Y) \,.\, \forall x : X \,.\, \Lambda y : Y \,.\, f(x,y) : \texttt{Measurable}(Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : \mathcal{B}\overset{\infty}{\mathbb{R}},$

$(1) := \texttt{InversePointProduct}(f, x, A) : f^{-1}(x, \cdot)(A) = \texttt{section}(f^{-1}(A), x),$

$(2) := \eth\texttt{Measurable}(X \times Y)(f)(A) : f^{-1}(A) : F_{X \times Y},$

$(3) := (1)\texttt{MeasurableSection}(x, f^{-1}(A)) : f^{-1}(x, \cdot);$

$\leadsto (*) := \eth^{-1}\texttt{Measurable}(X)(\cdot) : \Lambda y : Y \,.\, f(x,y) : \texttt{Measurable}(Y);$

$\square$

$Y : \mathsf{BOR}$

$X : \mathsf{MEAS}$

$S : \texttt{SlicingMeasure}(X, Y)\ \&\ \texttt{Uniformly}\sigma\texttt{-Finite}\,(X, Y)$

$\nu = \texttt{productMeasure}(S)$

$\texttt{FubiniI} :: \forall f : \texttt{Measurable}(X \times Y) : f > 0 \,.\, \forall A : \mathcal{F}_{X \times Y} \,.$

$$. \ \Lambda x : X \,.\, \int_{\texttt{section}(A,x)} f(x,y)\,\mathrm{d}S(x,y) : \texttt{Measurable}(X)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ B : \mathcal{F}_Y,$

$\texttt{Assume}\ \phi : \texttt{Simple}(X \times Y),$

$$(n, b, c) := \eth\texttt{Simple}(X \times Y) : \mathbb{N} \times n \to \mathcal{F}_{X \times Y} \times n \to \mathbb{R}_{++} : \phi = \sum_{i=1}^{n} c_i I_{b_i},$$

$$(1) := \eth(n, b, c) \to \int_B \phi\,\mathrm{d}S : \int_B \phi\,\mathrm{d}S = \sum_{i=1}^{n} c_i S(x, \texttt{section}(X \times B \cup b_i, x)),$$

$$(2) := (1)\texttt{MeasrableSlicing}(S, X \times B \cup b) : \int \phi\,\mathrm{d}S : \texttt{Measurable}(X);$$

$$\leadsto (1) := UI(\cdot) : \forall \phi : \texttt{Simple}(X \times Y) \,.\, \int_B f\,\mathrm{d}S : \texttt{Measurable}(X),$$

$\phi := \texttt{SimpleApprox}(f) : \mathbb{N} \to \texttt{Simple}(X \times Y) : \phi_n \uparrow f,$

$$(2) := \texttt{MonotoneConvergence}\left(\int_B \phi\,\mathrm{d}S, \int_B f\,\mathrm{d}S\right) : \int_B \phi\,\mathrm{d}S : \texttt{Measurable}(X);$$

$\leadsto (1) := \eth^{-1}\texttt{SlicingMeasure}(\cdot) : fS : \texttt{SlicingMeasure}(X \times Y),$

$$(2) := \texttt{MeasurableSlicing}(fS) : \Lambda x \in X \,.\, \int_{A_x} f(x,y)\,\mathrm{d}S(x,y) : \texttt{Measurable}(X);$$

$\square$

$\texttt{FubiniII} :: \forall f : \texttt{Measurable}(X \times Y) : f \geq 0 \,.\, \forall A : \mathcal{F}_{X \times Y} \,.\, \int_X \int_{A_x} f(x,y)\,\mathrm{d}S(x,y)\,\mathrm{d}\mu(x) = \int_A f\,\mathrm{d}\nu(S)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ B : \mathcal{F}_Y,$

$(1) := \eth\texttt{Indicator}(B)\eth\texttt{productMeasure}\eth\texttt{Indicator}(B) =$

$: \int_X \int_{A_x} I_B\,\mathrm{d}S\,\mathrm{d}\mu = \int_X \int_{A_x \cap B_x} \mathrm{d}S\,\mathrm{d}\mu = \nu(A \cap B) = \int_A I_B\,\mathrm{d}\nu;$

$\rightsquigarrow (1) := \texttt{UI}(\cdot) : \forall B : \mathcal{F}_{X \times Y} \,.\, \int_X \int_{A_x} I_B\,\mathrm{d}S\,\mathrm{d}\mu = \int_A I_B\,\mathrm{d}\nu,$

$\texttt{Assume}\ \phi : \texttt{Simple}(X \times Y),$

$(n, b, c) := \eth\texttt{Simple}(X \times Y) : \mathbb{N} \times n \to \mathcal{F}_{X \times Y} \times n \to \mathbb{R}_{++} : \phi = \sum_{i=1}^{n} c_i I_{b_i},$

$(2) := \ldots : \int_X \int_{A_x} \phi\,\mathrm{d}S\mathrm{d}\mu = \int_X \int_{A_x} \sum_{i=1}^{n} c_i I_{b_i}\,\mathrm{d}S\mathrm{d}\mu = \sum_{i=1}^{n} c_i \int_X \int_{A_x} I_{b_i}\,\mathrm{d}S\mathrm{d}\mu =$

$= \sum_{i=1}^{n} c_i \int_A I_{b_i}\,\mathrm{d}\nu = \int_A \sum_{i=1}^{n} I_{b_i}\,\mathrm{d}\nu = \int_A \phi\mathrm{d}\nu;$

$\rightsquigarrow (2) := \texttt{UI}(\cdot : \forall \phi : \texttt{Simple}(X \times Y) \,.\, \int_X \int_{A_x} \phi\,\mathrm{d}S\,\mathrm{d}\mu = \int_A \phi\mathrm{d}\nu,$

$\phi := \texttt{SimpleApproximation}(f) : \mathbb{N} \to \texttt{Simple}(X \times Y) : \phi \uparrow f,$

$(3) := \ldots : \int_X \int_{A_x} f\,\mathrm{d}S\,\mathrm{d}\mu = \int_X \int_{A_x} \lim_{n \to \infty} \phi_n\,\mathrm{d}S\,\mathrm{d}\mu = \lim_{n \to \infty} \int_X \int_{A_x} \phi_n\,\mathrm{d}S\,\mathrm{d}\mu = \lim_{n \to \infty} \int_A \phi_n\,\mathrm{d}\nu =$

$= \int_A \lim_{n \to \infty} \phi_n\,\mathrm{d}\nu = \int_A f\,\mathrm{d}\nu;$

$\square$

$\texttt{ToneliI} :: \forall f : \texttt{IntegralExists}(X \times Y, \nu) \,.\, \Lambda x \in X \,.\, \Lambda y \in Y \,.\, f(x,y) : \texttt{IntegralExists}(Y, S(x))\mathrm{a.e.}\ [\mu]$

$\texttt{Proof} \ =$

$(1) := \texttt{FubiniII}(f_+, X \times Y) : \int_X \int_Y f_+\,\mathrm{d}S\,\mathrm{d}\mu = \int_{X \times Y} f_+\mathrm{d}\nu,$

$(2) := \texttt{FuniniII}(f_-, X \times Y) : \int_X \int_Y f_-\,\mathrm{d}S\,\mathrm{d}\mu = \int_{X \times Y} f_-\mathrm{d}\nu,$

$(3) := \eth\texttt{IntegralExists}(\mu)\eth\texttt{Integrate}(f, \nu)((1), (2))\eth\texttt{Integrate}(f, S(x)) :$

$: \texttt{Error} \neq \int_{X \times Y} f\,\mathrm{d}\nu = \int_{X \times Y} f_+\,\mathrm{d}\nu - \int_{X \times Y} f_-\,\mathrm{d}\nu = \int_X \int_Y f_+\,\mathrm{d}S\,\mathrm{d}\mu - \int_X \int_Y f_-\,\mathrm{d}S\,\mathrm{d}\mu = \int_X \int_Y f\,\mathrm{d}S\,\mathrm{d}\mu,$

$(4) := \texttt{IntegralEq}\left(\mu, \int_Y f\,\mathrm{d}S, \texttt{Error}\right) : \int_Y f\mathrm{d}\,S \neq \texttt{Error}\,\mathrm{a.e.}\ [\mu],$

$(*) := \eth^{-1}\texttt{IntegralExists}(4) : (f : \texttt{IntegralExists}(Y, S)\,\mathrm{a.e.}\ [\mu]);$

$\square$

$\texttt{ToneliII} :: \forall f : \texttt{Integrable}(X \times Y, \nu) \,.\, \Lambda x \in X \,.\, \Lambda y \in Y \,.\, f(x,y) : \texttt{Integrable}(Y, S(x))\text{a . e . } [\mu]$

$\texttt{Proof} \ =$

$(1) := \texttt{FubiniII}(f_+, X \times Y) : \displaystyle\int_X \int_Y f_+ \, \mathrm{d}S \, \mathrm{d}\mu = \int_{X \times Y} f_+ \mathrm{d}\nu,$

$(2) := \texttt{FuniniII}(f_-, X \times Y) : \displaystyle\int_X \int_Y f_- \, \mathrm{d}S \, \mathrm{d}\mu = \int_{X \times Y} f_- \mathrm{d}\nu,$

$(3) := \eth\texttt{IntegralExists}(\mu)\eth\texttt{Integrate}(f, \nu)((1), (2))\eth\texttt{Integrate}(f, S(x)) :$

$: \infty > \displaystyle\int_{X \times Y} |f| \, \mathrm{d}\nu = \int_{X \times Y} f_+ \, \mathrm{d}\nu + \int_{X \times Y} f_- \, \mathrm{d}\nu = \int_X \int_Y f_+ \, \mathrm{d}S \, \mathrm{d}\mu + \int_X \int_Y f_- \, \mathrm{d}S \, \mathrm{d}\mu = \int_X \int_Y f \, \mathrm{d}S \, \mathrm{d}\mu,$

$(4) := \texttt{IntegralIneq}\left(\mu, \displaystyle\int_Y f \, \mathrm{d}S, \infty\right) : \displaystyle\int_Y |f| \, \mathrm{d}S < \infty \,\text{a . e . } [\mu] \,,$

$(*) := \eth^{-1}\texttt{Integrable}(4) : (f : \texttt{Integrable}(Y, S) \text{ a . e . } [\mu]);$

$\square$

$\texttt{Toneli0} :: \forall f : \texttt{IntegralExists}(X \times Y, \nu) \,.$

$. \, \exists \phi : \texttt{IntegralExists}(X \times Y, \nu) : \displaystyle\int_Y \phi \, \mathrm{d}S : \texttt{Measurable}(X) : \phi =_\mu f$

$\texttt{Proof} \ =$

$(1) := \texttt{ToneliI}(f) : f : \texttt{Integrable}(Y, S) \text{ a . e . } [\mu] \,,$

$\phi := \Lambda(a, b) \in X \times Y \,.\, \texttt{if} \displaystyle\int_Y f(a, y) \, \mathrm{d}S(x, y) = \texttt{Error then } 0 \texttt{ else } f(a, b) : \texttt{Integralexists},$

$(2) := \texttt{FubiniI}(\phi_+) : \displaystyle\int_Y f_+ \, \mathrm{d}S : \texttt{Measurable}(X),$

$(3) := \texttt{FubiniI}(\phi_-) : \displaystyle\int_Y f_- \, \mathrm{d}S : \texttt{Measurable}(X),$

$(4) := \texttt{AdditiveIntegral}(\phi_+, -\phi_-) : \displaystyle\int_Y \phi_+ \, \mathrm{d}S - \int_Y \phi_- \, \mathrm{d}S = \int_Y \phi \mathrm{d}S,$

$(*) := \texttt{ContinousPreserveMeasureable}(2, 3, 4) : \displaystyle\int_Y \phi \, \mathrm{d}S : \texttt{Measurable}(X),$

$\square$

$\texttt{FubiniToneli} :: \forall f : \texttt{Measurable}(X \times Y) : \displaystyle\int_{X \times Y} |f| \, \mathrm{d}\nu < \infty$

$. \displaystyle\int_{X \times Y} f \, \mathrm{d}\nu = \int_X \int_Y f \mathrm{d}S \, \mathrm{d}\mu$

$\texttt{Proof} \ =$

$\texttt{ClassicalFubini} :: \forall \nu : \texttt{Measure}(Y) \,.\, \forall f : \texttt{IntegralExists}(X \times Y, \mu \times \nu) \,.$

$. \displaystyle\int_{X \times Y} f \, \mathrm{d}\mu \times \nu = \int_X \int_Y f \, \mathrm{d}\mu \, \mathrm{d}\nu = \int_Y \int_X f \, \mathrm{d}\nu \, \mathrm{d}\mu$

$\texttt{Proof} \ =$

5.3 Infinite Products

$$\texttt{Cylinder} :: \prod X : \mathbb{N} \to \texttt{Set} \,.\, \prod n \in \mathbb{N} \,.\, ?\left(\prod_{i=1}^{n} X_i\right) \to ? \prod_{i=1}^{\infty} X_i$$

$$C : \texttt{Cylinder(A)} \iff \pi_{1,\ldots,n} C = A$$

$$\texttt{MeasurableCylinder} :: \prod X : \mathbb{N} \to \mathsf{BOR} \,.\, \prod n \in \mathbb{N} \,.\, \mathcal{F}_{\prod_{i=1}^{n} X_i} \to ? \prod_{i=1}^{\infty} X_i$$

$$C : \texttt{MeasurableCylinder}(A) : \texttt{C : Cylinder(A)} \iff$$

$$\texttt{infiniteBorProduct} :: (\mathbb{N} \to \mathsf{BOR}) \to \mathsf{BOR}$$

$$\texttt{InfiniteBorProduct}\,(X_i, \mathcal{F}_i) = \prod_{i=1}^{n} (X_i, \mathcal{F}_i) := \left(\prod_{i=1}^{\infty} X_i, \sigma(\texttt{MeasurableCylinder}(X))\right)$$

$$\texttt{cylinder} :: \prod X : \mathbb{N} \to \texttt{Set} \,.\, \prod n \in \mathbb{N} \,.\, \prod A \subset \prod_{i=1}^{n} X_i \to \texttt{Cylinder}(X, n, A)$$

$$\texttt{cylinder}\,(A) := A \times \prod_{i=n+1}^{\infty} X_i$$

$$\texttt{DiscreteRandomProcess} :: \prod X : \mathbb{N} \to \mathsf{BOR} \,.\, ?\left(\prod n : \mathbb{N} \,.\, \prod_{i=1}^{n} X_i \to \texttt{Measure} X_{n+1}\right)$$

$$\texttt{P} :: \texttt{DiscreteRandomProcess}$$

$$\iff \forall n \in \mathbb{N} \,.\, \forall A \in \mathcal{F}_{\prod_{n \in N} X_n} \,.\, \Lambda x \in \prod_{n \in N^{\complement}} \,.\, P(x, A) : \texttt{Measureble} \prod_{n \in N^{\complement}} X_n \qquad head$$

5.4 Categorical Viewpoint: Products in Borel-Null [!]

6 Convergence of Measures [!]