

# Topological Vector Spaces 2

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August 14, 2022

# Contents

<b>1</b>	<b>Abstract Topological Vector Spaces</b>	<b>3</b>
1.1	Minkowski's Theory . . . . .	3
1.1.1	Intro and Definition . . . . .	3
1.1.2	Absorbent and Balanced Sets . . . . .	4
1.1.3	Topology and Convexity . . . . .	7
1.1.4	Semimetrization . . . . .	11
1.1.5	Completion . . . . .	12
1.1.6	Continuous Decompositions . . . . .	14
1.1.7	Finite Dimension Conditions . . . . .	16
1.1.8	Case of Ultravalued Field . . . . .	19
1.1.9	Some Interesting Examples . . . . .	27
1.1.10	Seminorms . . . . .	29
1.1.11	Topology of Locally Convex Space . . . . .	33
1.1.12	Spaces of Continuous Functions . . . . .	39
1.1.13	Constructions . . . . .	43
1.1.14	Non-Archimedean Spaces . . . . .	45
1.2	Towards Bornology . . . . .	49
1.2.1	Bounded Sets . . . . .	49
1.2.2	Stability under Operations . . . . .	51
1.2.3	Locally Bounded Maps . . . . .	53
1.2.4	Liouville's Theorem . . . . .	55
1.2.5	p-convexity . . . . .	56
1.2.6	Bornology . . . . .	57
1.2.7	Interesting Examples and Facts . . . . .	62
1.3	Hahn-Banach Theory . . . . .	62
1.4	Barelled and Bornological Spaces . . . . .	62
1.5	Towards Approximation Theory . . . . .	62
<b>2</b>	<b>Spaces of Distributions</b>	<b>62</b>
<b>3</b>	<b>Ordered Topological Vector Spaces</b>	<b>63</b>
3.1	Reisz Spaces and Banach Lattices . . . . .	63
3.1.1	Order Unit Norm . . . . .	63
3.1.2	Topological Vector Lattices . . . . .	64
3.1.3	Lattice of Continuous Functions . . . . .	65

# 1 Abstract Topological Vector Spaces

## 1.1 Minkowski's Theory

### 1.1.1 Intro and Definition

$\text{TopologicalVectorSpace} :: \prod k : \text{TopologicalField} . ? \sum_{V \in k\text{-VS}} \text{Topology}(V)$

$(V, \tau) : \text{TopologicalVectorSpace} \iff \cdot_V \in \text{TOP}(k \times (V, \tau), (V, \tau)) \ \& \ +_V \in \text{TOP}((V, \tau) \times (V, \tau), (V, \tau))$

$k :: \text{TopologicalField};$

$\text{VectorTopology} := \lambda V \in k\text{-VS} . \text{TopologicalVectorSpace}(V) : \prod_{V \in k\text{-VS}} V . ? \text{Topology}(V);$

$\text{categoryOfTopologicalVectorSpaces} :: \text{TopologicalField} \rightarrow \text{CAT}$

$\text{categoryOfTopologicalVectorSpaces}(k) = k\text{-TVS} :=$   
 $:= (\text{TopologicalVectorSpace}(k), k\text{-VS} \cap \text{TOP}, \circ, \text{id})$

$\text{categoryOfTopologicalVectorSpaces} :: \text{TopologicalField} \rightarrow \text{CAT}$

$\text{categoryOfHausdorffTopologicalVectorSpaces}(k) = k\text{-HTVS} :=$   
 $:= (\text{TopologicalVectorSpace}(k) \ \& \ \text{T2}, k\text{-VS} \cap \text{TOP}, \circ, \text{id})$

$\text{asTopologicalGroup} :: k\text{-TVS} \rightarrow \text{TGRP}$

$\text{asTopologicalGroup}(V) = V := V$

$\text{asVectorSpace} :: k\text{-TVS} \rightarrow k\text{-VS}$

$\text{asVectorSpace}(V) = V := V$

### 1.1.2 Absorbent and Balanced Sets

$k :: \text{AbsoluteValueField}(\mathbb{R});$

$\text{Balanced} :: \prod_{V:k\text{-TVS}} ??V$

$B : \text{Balanced} \iff \mathbb{D}_k(0,1)B \subset B$

$\text{Absorbent} :: \prod_{k:\text{AbsoluteValueField}(\mathbb{R})} \prod_{V:k\text{-TVS}} ??V$

$A : \text{Absorbent} \iff \forall v \in V . \exists \rho \in \mathbb{R}_{++} . \forall \alpha \in \mathbb{D}_k(0, \rho) . \alpha v \in A$

$\text{VectorSubspaceIsBalanced} :: \forall V \in k\text{-TVS} . \forall U \subset_{k\text{-VS}} V . \text{Balanced}(V, U)$

**Proof** =

Obvious.

□

$\text{AbsorbentVectorSubspaceIswhole} :: \forall V \in k\text{-TVS} . \forall U \subset_{k\text{-VS}} V . \text{Absorbent}(V, U) \Rightarrow V$

**Proof** =

Take  $v \in V$ .

By definition of absorbent there is  $\alpha \in k_*$  such that  $\alpha v \in U$ .

But then  $v = \alpha^{-1} \alpha v \in U$ .

So,  $U = V$ .

□

$\text{BalancedSetsAreDedekindComplete} :: \forall V \in k\text{-TVS} . \text{OrderDedekindComplete}(\text{Balanced}(V))$

**Proof** =

Assume  $\beta$  is a set of balanced sets in  $V$ .

If  $v \in \bigcup \beta$ , then there is a  $B \in \beta$  such that  $v \in B$ .

And by definition of balanced  $\alpha v \in B \subset \bigcup \beta$  for any  $\alpha \in \mathbb{B}_k(0,1)$ .

So  $\bigcup \beta$  is Balanced.

if  $v \in \bigcap \beta$ , then  $v \in B$  for any  $B \in \beta$ .

And by definition of balanced  $\alpha v \in B \subset \bigcap \beta$  for any  $\alpha \in \mathbb{B}_k(0,1)$  and for all  $B \in \beta$ .

So  $\bigcap \beta$  is Balanced.

□

$\text{AbsorbentAreClosedUnderUnions} :: \forall V \in k\text{-TVS} . \forall \alpha : ?\text{Absorbent}(V) . \text{Absorbent}(V, \bigcup \alpha)$

**Proof** =

This is obvious.

□

**AbsorbentAreClosedUnderFiniteIntersections** ::

$$:: \forall V \in k\text{-TVS} . \forall \alpha : \text{Finite}(\text{Absorbent}(V)) . \text{Absorbent}\left(V, \bigcap \alpha\right)$$

**Proof** =

Say  $n = |\alpha|$ .

if  $n = 0$ , then  $\bigcap \alpha = V$  which is always absorbent.

otherwisr represent  $\alpha = \{A_1, \dots, A_n\}$  and assume  $v \in V$ .

Select a finite sequence  $\rho : \{1, \dots, n\} \rightarrow \mathbb{R}_{++}$ , with  $\rho_i$  absorbing  $v$  for  $A_i$ .

Let  $\sigma = \min\{\rho_1, \dots, \rho_n\}$ .

Then  $\sigma$  is absorbing for every  $A_i$ , so it is absorbing for  $\bigcap \alpha$ .

□

In case of infinite intersiction the minimum may not exit.

$$\text{balancedHull} :: \prod_{V:k\text{-TVS}} 2^V \rightarrow \text{Balanced}(V)$$

$$\text{balancedHull}(A) = \text{bal } A := \bigcap \left\{ B : \text{Balanced}(V), A \subset B \right\}$$

$$\text{BalancedHullProductExpression} :: \forall_{V \in k\text{-TVS}} \forall A \subset V . \text{bal } A = \mathbb{B}_k(0, 1)A$$

**Proof** =

Clearly  $\mathbb{B}_k(0, 1)A$  is balanced.

Assume that  $B$  is a balanced set such that  $A \subset B$ .

Then  $\mathbb{B}_k(0, 1)A \subset \mathbb{B}_k(0, 1)B \subset B$  as  $B$  as balanced.

This proves the result as balanced hull of  $A$  may beviewed as the smallest balanced set containing  $A$ .

□

$$\text{balancedCore} :: \prod_{V:k\text{-TVS}} 2^V \rightarrow \text{Balanced}(V)$$

$$\text{balancedCore}(A) = A^{\text{bal}} := \bigcup \left\{ B : \text{Balanced}(V), B \subset A \right\}$$

$$\text{BalancedCoreAsIntersction} :: \forall_{V \in k\text{-TVS}} \forall A \subset V . \text{bal } A = \bigcap_{\alpha \in \mathbb{B}_k^c(0, 1)} \alpha A$$

**Proof** =

Firstly, I show that  $B = \bigcap_{\alpha \in \mathbb{B}_k^c(0, 1)} \alpha A$  is balanced.

Assume  $v \in B$ .

Then,  $v \in \alpha A$  for all  $\alpha \in \mathbb{B}_k^c(0, 1)$ .

Thus  $\mathbb{B}_k(0, 1)v \subset A$ .

By definition  $A^{\text{bal}}$  as a union this means, that  $v \in A^{\text{bal}}$ , so  $B \subset A^{\text{bal}}$ .

Assume now that  $v \in A^{\text{bal}}$ .

Then  $\mathbb{B}_k(0, 1)v \subset \mathbb{B}_k(0, 1)A^{\text{bal}} \subset A^{\text{bal}} \subset A$  As  $A^{\text{bal}}$  is a union of subsets.

But this mean that  $v \in B$ , so  $A = B$ .

□

**ClosedBalancedCoreIsOpen** ::  $\forall V : k\text{-TVS} . \forall F : \text{Closed}(V) . \text{Closed}(V, F^{\text{bal}})$

**Proof** =

Multiplication by non-zero scalar is a homeomorphism.

So result follows from intersection representation as  $\alpha F$  will be closed.

□

**LinearMapsBalancedToBalanced** ::

$:: \forall V, W : k\text{-TVS} . \forall T \in k\text{-VS}(V, W) . \forall B : \text{Balanced}(V) . \text{Balanced}(W, T(B))$

**Proof** =

Assume  $w \in T(B)$  and  $\alpha \in \mathbb{D}_k(0, 1)$ .

Then there is  $v \in B$  such that  $T(v) = w$ .

as  $B$  is balanced  $\alpha v \in B$ .

Thus  $\alpha w = \alpha T(v) = T(\alpha v) \in T(B)$ .

This proves that  $T(B)$  is balanced.

□

**LinearSurjectiveMapsAbsorbentToAbsorbent** ::

$:: \forall V, W : k\text{-TVS} . \forall T \in k\text{-VS} \ \& \ \text{Surjective}(V, W) . \forall A : \text{Absorbent}(V) . \text{Absorbent}(W, T(A))$

**Proof** =

Assume  $w \in W$ .

Then there is  $v \in V$  such that  $T(v) = w$  as  $T$  is surjective.

Then there exists  $\rho \in \mathbb{R}_{++}$  such that  $\mathbb{D}(0, \rho)v \subset A$  as  $A$  is absorbent.

Take  $\alpha \in \mathbb{D}(0, \rho)$ .

Then  $\alpha w = \alpha T(v) = T(\alpha v) \in T(A)$ .

This proves that  $T(A)$  is absorbent.

□

**BalancedPreimageIsBalanced** ::

$:: \forall V, W : k\text{-TVS} . \forall T \in k\text{-VS}(V, W) . \forall B : \text{Balanced}(W) . \text{Balanced}(V, T^{-1}(B))$

**Proof** =

Take  $v \in T^{-1}(B)$  and  $\alpha \in \mathbb{D}_k(0, 1)$ .

Then  $T(v) \in B$ , but also  $T(\alpha v) = \alpha T(v) \in B$  as  $B$  is balanced.

But this means that  $\alpha v \in T^{-1}(B)$ .

□

**BalancedPreimageIsBalanced** ::

$:: \forall V, W : k\text{-TVS} . \forall T \in k\text{-VS}(V, W) . \forall A : \text{Absorbent}(W) . \text{Absorbent}(V, T^{-1}(A))$

**Proof** =

Take  $v \in V$ .

Then there is  $\rho \in \mathbb{R}_{++}$  such that  $T(\alpha v) = \alpha T(v) \in A$  for any  $\alpha \in \mathbb{D}_k(0, \rho)$  as  $A$  is absorbent.

But this means that  $\alpha v \in T^{-1}(A)$ .

□

### 1.1.3 Topology and Convexity

$$\text{Disc} := \Lambda V \in k\text{-TVS} . \text{Convex} \ \& \ \text{Balanced}(V) : \prod_{V \in k\text{-TVS}} ??V;$$

**DiscCharacterization** ::

$$:: \forall V \in k\text{-TVS} . \forall D \subset V . \text{Disc}(V, D) \iff \forall v, w \in D . \forall \alpha, \beta \in k . |\alpha| + |\beta| \leq 1 \Rightarrow \alpha v + \beta w \in D$$

**Proof** =

Firstly, assume that  $D$  is a Disc.

Take  $v, w \in D$  and  $\alpha, \beta \in k$  such that  $|\alpha| + |\beta| \leq 1$ .

$\alpha v, \beta w \in D$  as  $D$  is balanced.

So if  $\alpha = 0$  or  $\beta = 0$  then  $\alpha v + \beta w = \alpha v \in V$  or  $\alpha v + \beta w = \beta w \in V$ .

Otherwise,  $|\alpha| + |\beta| \neq 0$  and  $\frac{|\alpha|}{|\alpha| + |\beta|} + \frac{|\beta|}{|\alpha| + |\beta|} = 1$ .

Also,  $\frac{|\alpha| + |\beta|}{|\alpha|} \alpha v, \frac{|\alpha| + |\beta|}{|\beta|} \beta w \in D$  as  $|\alpha| + |\beta| \leq 1$  and  $D$  is absorbent.

Then  $\alpha v + \beta w = \frac{|\alpha|}{|\alpha| + |\beta|} \frac{|\alpha| + |\beta|}{|\alpha|} \alpha v + \frac{|\beta|}{|\alpha| + |\beta|} \frac{|\alpha| + |\beta|}{|\beta|} \beta w \in D$  as  $D$  is convex.

Now assume that the condition holds.

Then convexity and being balanced is obvious.

□

$$\text{DiskedHull} :: \forall V \in K\text{-TVS} . \forall A \subset V . \bigcap \left\{ D : \text{Disc}(V), A \subset D \right\} = \text{conv bal } A$$

**Proof** =

Firstly we need to show that  $\text{conv bal } A$  is balanced.

Assume  $v \in \text{conv bal } A$  and  $\alpha \in \mathbb{D}_k(0, 1)$ .

If  $\alpha = 0$  then  $\alpha v = 0 \in \text{bal } A \subset \text{conv bal } A$ .

Otherwise, if  $C$  is convex in  $V$ , then  $\frac{\alpha}{|\alpha|} C$  is also convex.

Also if  $\text{bal } A \subset C$  then  $\text{bal } A = \frac{\alpha}{|\alpha|} \text{bal } A \subset \frac{\alpha}{|\alpha|} C$  as  $\text{bal } A$  is balanced.

Thus,  $\frac{\alpha}{|\alpha|} v \in \text{conv bal } A$ .

Also, as it was said  $0 \in \text{bal } A \subset \text{conv bal } A$ .

So  $\alpha v = \frac{|\alpha|}{|\alpha|} \alpha v + (1 - |\alpha|) 0 \in \text{conv bal } A$  as  $\text{conv bal } A$  is convex.

So  $\text{conv bal } A$  is a disk and  $B = \bigcap \left\{ D : \text{Disc}(V), A \subset D \right\} \subset \text{conv bal } A$ .

Now assume that  $D$  is a disk such that  $A \subset D$ .

Then  $\text{bal } A \subset D$  as  $D$  is balanced.

Furthermore,  $\text{conv bal } A \subset D$  as  $D$  is convex.

Thus  $\text{conv bal } A = B$ .

□

**TVSIsConnected** ::  $\forall V \in k\text{-TVS} . \text{Connected}(k) \Rightarrow \text{Connected}(V)$

**Proof** =

Note that  $V = \bigcup_{v \in V} kv$ .

Each  $kv$  is connected as continuous image of connected  $k$ .

Then all lines  $kv$  intersect at 0, so  $V$  is connected.

□

**AbsorbentNeighborhoodsOfZero** ::  $\forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \text{Absorbent}(V, U)$

**Proof** =

Assume  $v \in V$ .

Then  $\lim_{\alpha \rightarrow 0} \alpha v = 0$ .

So, there exists  $\rho \in \mathbb{R}_{++}$  such that  $\mathbb{B}_k(0, \rho)v \subset U$ .

Then  $\mathbb{D}_k\left(0, \frac{\rho}{2}\right)v \subset \mathbb{B}_k(0, \rho)v \subset U$ .

Thus,  $U$  is absorbent.

□

**NeighborhoodsOfZeroScaling** ::  $\forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \forall \alpha \in k_* . \alpha U \in \mathcal{U}_V(0)$

**Proof** =

$\alpha \cdot \bullet$  is a homeomorphism, so  $\alpha U$  is open.

Obviously,  $0 = \alpha 0 \in \alpha U$  as  $0 \in U$ .

Thus,  $U \in \mathcal{U}_V(0)$ .

□

**EachNeighborhoodsOfZeroContainsBalancedNeighborhoods** ::

::  $\forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \exists W \in \mathcal{U}_V(0) . W \subset U \ \& \ \text{Balanced}(V, W)$

**Proof** =

$(\cdot)^{-1}(U)$  is open in  $k \times V$ .

So there exist  $W \in \mathcal{U}_V(0)$  and  $\rho \in \mathbb{R}_{++}$  such that  $\mathbb{B}_k(0, \rho) \times W \subset (\cdot)^{-1}(U)$  as  $0 \in (\cdot)^{-1}(U)$ .

This means that  $\mathbb{B}_k(0, \rho)W \subset U$ .

Also, note that  $\mathbb{B}_k(0, \rho)W = \bigcup_{|\alpha| < \rho} \alpha W \in \mathcal{U}_V(0)$ .

Assume  $v \in \mathbb{B}_k(0, \rho)W$  and  $\alpha \in \mathbb{D}_k(0, 1)$ .

Then there is  $w \in W$  and  $\beta \in \mathbb{B}_k(0, \rho)$  such that  $v = w\beta$ .

But  $\alpha\beta$  is also in  $\mathbb{B}_k(0, \rho)$  and so  $\alpha v = \alpha\beta w \in \mathbb{B}_k(0, \rho)W$ .

Thus,  $\mathbb{B}_k(0, \rho)W$  is balanced.

□

**ClosedAndBalancedNeighborhoodBase** ::

::  $\forall V \in k\text{-TVS} . \exists \mathcal{F} : \text{Filterbase}(V, \mathcal{U}_V(0)) . \forall F \in \mathcal{F} . \text{Closed} \ \& \ \text{Balanced}(V, F)$

**Proof** =

Pretty obvious.

□



$\text{LocallyConvexSpace} :: ?k\text{-TVS}$

$V : \text{LocallyConvexSpace} \iff \exists \mathcal{F} : \text{Filterbase} \left( V, \mathcal{N}_V(0) \right) . \forall F \in \mathcal{F} . \text{Convex}(F, \mathcal{F})$

$\text{categoryOfLocallyConvexSpaces} :: \text{AbsoluteValueField}(\mathbb{R}) \rightarrow \text{CAT}$

$\text{categoryOfLocallyConvexSpaces}(k) = k\text{-LCS} :=$   
 $:= (\text{LocallyConvexSpace}(k), k\text{-VS} \cap \text{TOP}, \circ, \text{id})$

$\text{categoryOfTopologicalVectorSpaces} :: \text{AbsoluteValueField}(\mathbb{R}) \rightarrow \text{CAT}$

$\text{categoryOfHausdorffTopologicalVectorSpaces}(k) = k\text{-LCHS} :=$   
 $:= (\text{LocallyConvexSpace}(k) \ \& \ \text{T2}, k\text{-VS} \cap \text{TOP}, \circ, \text{id})$

$\text{NormedSpaceIsLocallyConvex} :: \text{NORM}(k) \subset k\text{-LCHS}$

**Proof** =

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

□

$\text{NormedSpaceIsLocallyConvex} :: \text{NORM}(k) \subset k\text{-LCHS}$

**Proof** =

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

□

$\text{LCSHasDiscBase} :: \forall V \in k\text{-LCS} . \exists \mathcal{F} : \text{Filterbase} \left( V, \mathcal{N}_V(0), \mathcal{F} \right) . \forall F \in \mathcal{F} . \text{Disc}(V, F)$

**Proof** =

Take  $U \in \mathcal{N}_V(0)$ .

Then there exists a convex neighborhood  $C \in \mathcal{N}_V(0)$  with  $C \subset U$  as  $V$  is locally convex.

Then there is  $B \subset C$  which is a balanced neiborhood which was proved for all topological vector spaces.

Then  $\text{conv } B \subset C$  is convex and still an neighborhood of zero.

But convex hull of the balanced set is balanced,hence  $\text{conv } B$  is a disc .

□

$\text{LCSHasOpenDiscBase} :: \forall V \in k\text{-LCS} . \exists \mathcal{F} : \text{Filterbase} \left( V, \mathcal{N}_V(0), \mathcal{F} \right) . \forall F \in \mathcal{F} . \text{Disc} \ \& \ \text{Open}(V, F)$

**Proof** =

...

□

$\text{LCSHasClosedDiscBase} :: \forall V \in k\text{-LCS} . \exists \mathcal{F} : \text{Filterbase} \left( V, \mathcal{N}_V(0), \mathcal{F} \right) . \forall F \in \mathcal{F} . \text{Disc} \ \& \ \text{Closed}(V, F)$

**Proof** =

...

□

**VectorTopologyByAbsorbentAndBalancedSets** ::

$$:: \forall V \in k\text{-VS} . \forall \mathcal{F} : \text{GroupFilterbase}(V) . \forall \mathfrak{N} : \mathcal{F} \subset \text{Balanced} \ \& \ \text{Absorbent}(V) . \left( V, \langle \mathcal{F} \rangle_{\text{TGRP}} \right) \in k\text{-TVS}$$

**Proof** =

As  $F \in \mathcal{F}$  is balanced, then  $F = -F$ , so  $\langle \mathcal{F} \rangle_{\text{TGRP}}$  is a group topology for  $(V, +)$ .

Now assume  $F \in \mathcal{F}$  and  $\alpha \in k_*$ .

Then there exists balanced  $U \in \langle \mathcal{F} \rangle_{\text{TGRP}}$  such that  $0 \in U$  and  $2U \subset U + U \subset F$ .

Then there exists balanced  $U \in \langle \mathcal{F} \rangle_{\text{TGRP}}$  such that  $0 \in U$  and  $2U \subset U + U \subset F$ .

This can be generalized to the case when  $U \in \langle \mathcal{F} \rangle_{\text{TGRP}}$  and  $2^n U \subset F$ .

So, we can take such  $U$  that  $|\alpha| \leq 2^n$  and  $\alpha U \subset 2^n U \subset F$  for any  $\alpha \in k_*$  and  $F \in \mathcal{F}$ .

Now consider  $\alpha \in k_*$ ,  $v \in V$  and  $F \in \mathcal{F}$ .

There exists  $U \in \mathcal{F}(0)$  such that  $U + U + U \subset F$ .

As  $U$  is absorbent there is  $\rho \in (0, 1)$  such that  $\mathbb{B}(0, \rho)v \subset U \subset F$ .

Thus,  $\text{Cell}(0, \rho)(v + U) = \mathbb{B}(0, \rho)v + \mathbb{B}(0, \rho)U = U + U \subset F$ .

Now, assume  $\alpha \neq 0$ .

There is  $U' \in \mathcal{F}$  such that  $\alpha U' \subset U$ .

Then there is also a  $W \in \mathcal{F}$  such that  $W \subset U' \cap U$ .

Thus,  $\mathbb{B}(\alpha, \rho)(v + W) = \alpha v + \alpha W + \mathbb{B}(0, \rho)(v + W) \subset \alpha v + U + U + U \subset \alpha v + F$ .

This proves that scalar multiplication is continuous.

□

**LocallyConvexTopologyByDiscFilterbase** ::

$$:: \forall V \in k\text{-VS} . \forall \mathcal{F} : \text{Filterbase}(V) . \forall \mathfrak{N} : \mathcal{F} \subset \text{Disc} \ \& \ \text{Absorbent}(V) .$$

$$. \forall \sqsupset : \forall F \in \mathcal{F} . \exists \alpha \in (0, 1/2) . \alpha F \in \mathcal{F} . \left( V, \langle \mathcal{F} \rangle_{\text{TGRP}} \right) \in k\text{-LCS}$$

**Proof** =

We need to show that  $\mathcal{F}$  is a group filterbase.

Assume  $F \in \mathcal{F}$ .

By assumption there are  $\alpha \in (0, 1/2)$  such that  $\alpha F \in \mathcal{F}$ .

Then, as  $\alpha F$  is convex and  $F$  is absorbent  $\alpha F + \alpha F = 2\alpha F \subset F$ .

Thus, by previous theorem  $(V, \langle \mathcal{F} \rangle_{\text{TGRP}})$  is a topological vector space.

And it is locally convex as there is a filterbase consisting of disks by construction.

□

### 1.1.4 Semimetrization

**FSeminorm** ::  $\prod V \in k\text{-VS} . ?(V \rightarrow \mathbb{R}_+)$

$\sigma : \text{FSeminorm} \iff \left( \forall \alpha \in \mathbb{D}_k(0, 1) . \forall v \in V . \sigma(\alpha v) \leq \sigma(v) \right) \&$   
 $\& \left( \forall v \in V . \lim_{n \rightarrow \infty} \sigma\left(\frac{v}{n}\right) \right) \& (\forall v, w \in V . \sigma(v + w) \leq \sigma(v) + \sigma(w))$

**FNorm** ::  $\prod V \in k\text{-VS} . ?\text{FSeminorm}(V)$

$\sigma : \text{FNorm} \iff \forall v \in V . \sigma(v) = 0 \iff v = 0$

**FSeminormSemimetrization** ::  $\forall V \in k\text{-VS} . \forall \sigma : \text{FSeminorm} . \exists \tau : \text{VectorTopology}(V) . \sigma \in C(V, \tau)$

**Proof** =

I will show that  $\sigma$  is a value.

Firstly, note that  $\sigma(-v) \leq \sigma(v)$  and  $\sigma(v) \leq \sigma(-v)$ , so  $\sigma(v) = \sigma(-v)$ .

Also  $\sigma(0) = \sigma\left(\frac{0}{n}\right) \rightarrow 0$ , so  $\sigma(0) = 0$ .

Other properties of value follows trivially by commutativity of  $+_V$ .

Now I show that scalar multiplication is continuous in topology defined by semimetric  $\rho(v, w) = \sigma(v - w)$ .

There are neighborhoods of zero defined by relation  $\sigma(v) < \varepsilon$ .

By first property of F-seminorm these balls are ballanced.

And by second property of F-seminorm these balls are absorbent.

So produced topology of  $\rho$  is a vector space topology.

□

**FNormSemimetrization** ::  $\forall V \in k\text{-VS} . \forall \sigma : \text{FNorm} . \exists \tau : \text{VectorTopology}(V) . \sigma \in C(V, \tau) \& \text{T2}(V, \tau)$

**Proof** =

In this case  $\rho$  is a metric, so resulting topology musy be Hausdorff.

□

**subspaceSeminorm** ::  $\prod V \in k\text{-VS} . \prod U \subset_{k\text{-VS}} V . \text{FSeminorm}(V) \rightarrow \text{FSeminorm}\left(\frac{V}{U}\right)$

$\text{subspaceSeminorm}(\sigma) = [\sigma]_U := \Lambda[v] \in \frac{V}{U} . \inf_{u \in U} \sigma(v + u)$

**SubspaceSemimetrization** ::  $\forall V \in k\text{-TVS} \& \text{Semimetrizable} . \forall U \subset_{k\text{-VS}} V . \text{Semimetrizable}\left(\frac{V}{U}\right)$

**Proof** =

...

□

### 1.1.5 Completion

**Completion** ::  $\prod_{V \in k\text{-TVS}} ? \sum_{W \in k\text{-TVS}} \text{TopologicalEmbedding}(V, W)$

$(W, \iota) : \text{Completion} \iff \text{Complete}(V) \ \& \ \text{Dense}(W, \iota(V))$

**EveryTVSHasACompletion** ::  $\forall V \in k\text{-TVS} . \exists \text{Completion}(V)$

**Proof** =

As with topological Groups.

□

**TopologicalVectorSpaceSubset** ::  $\prod_{V \in k\text{-TVS}} ??V$

$U : \text{TopologicalVectorSpaceSubset} \iff U \subset_{k\text{-TVS}} V \iff U \subset_{k\text{-VS}} V \ \& \ \text{Closed}(V, U)$

**CompletenessQuotient** ::  $\forall V \in k\text{-TVS} . \forall U \subset k\text{-TVS} V . \text{Complete}(V) \Rightarrow \text{Complete}\left(\frac{V}{U}\right)$

**Proof** =

As with topological groups.

□

**BalancedHullOfTotallyBoundedIsTotallyBounded** ::

$:: \forall V \in k\text{-TVS} . \forall B : \text{TotallyBounded}(V) . \text{TotallyBounded}(V, \text{bal } B)$

**Proof** =

Embed  $B$  in a completion of  $\hat{V}$  of  $V$ .

Then  $\text{cl } B$  is a compact in  $\hat{V}$ .

As  $\mathbb{D}_k(0, 1)$  is compact in  $k$ , then  $\mathbb{D}_k(0, 1)\text{cl}_{\hat{V}} B$  is compact is continuous image of compact  $\mathbb{D}_k(0, 1) \times \text{cl}_{\hat{V}} B$ .

Then  $\text{bal } B = \mathbb{D}_k(0, 1)B$  is totally bounded as a subset of compact  $\mathbb{D}_k(0, 1)\text{cl}_{\hat{V}} B$ .

□

**BalancedHullOfCompactIsCompacts** ::

$:: \forall V \in k\text{-TVS} . \forall K : \text{CompactSubset}(V) . \text{CompactSubset}(V, \text{bal } K)$

**Proof** =

$\mathbb{D}_k(0, 1)K$  is compact as an image of compact  $\mathbb{D}_k(0, 1) \times K$ .

□

**ConvexHullofTotallyBoundedAsTotallyBounded** ::

$:: \forall V \in k\text{-LCS} . \forall B : \text{TotallyBounded}(V) . \text{TotallyBounded}(V, \text{conv } B)$

**Proof** =

In order to show that  $\text{conv } B$  is totally bounded we need to show that  $\text{conv } B$  can be covered by finite number of translates  $(U + v_i)_{i=1}^n$  for any  $U \in \mathcal{U}_V(0)$  .

Select disc  $D \in \mathcal{U}_V(0)$  such that  $D + D \subset U$ .

This is possible as  $V$  is locally convex.

As  $K$  totally bounded there are a finite set of translates such that  $K \subset (D + v_i)_{i=1}^n \subset \text{conv}\{v_1, \dots, v_n\} + D$ .

As sum of convex sets is convex  $\text{conv } K \subset \text{conv}\{v_1, \dots, v_n\} + D$  .

As  $\text{conv}\{v_1, \dots, v_n\}$  is compact it is possible to select a finite set of  $m$  translates  $u_i$  of  $D$  such that

$$\text{conv } K \subset \bigcup_{i=1}^m (D + u_i).$$

So  $\text{conv } K$  is totally bounded.

□

**ConvexHullofTotallyBoundedAsTotallyBounded** ::

$:: \forall V \in k\text{-LCSComplete} . \forall K : \text{CompactSubset}(V) . \text{CompactSubset}(V, \text{conv } K)$

**Proof** =

$\text{conv } K$  is closed.

And as it was shown in the previous theorem  $\text{conv } K$  is also totally bounded, hence compact.

□

### 1.1.6 Continuous Decompositions

**TopologicalComplement** ::  $\prod V : k\text{-TVS} . ?\text{LinearComplement}(V)$

$(U, W) : \text{TopologicalComplement} \iff V =_{k\text{-TVS}} U \oplus W \iff$   
 $\iff \text{Homeomorphism}(U \oplus W, V, \Lambda(u, w) \in U \oplus W . u + w)$

**TopologicalComplementsByContinuousProjection** ::

$:: \forall V \in k\text{-TVS} . \forall U, W : \text{LinearComplement}(V) . U \oplus W =_{k\text{-TVS}} V \iff P_{U,W} \in \text{End}_{\text{TOP}}(V)$

**Proof** =

Define  $T : U \oplus W \rightarrow V$  by  $T(u, w) = u + w$ .

$(\Rightarrow)$  : Assume that  $T$  is a homeomorphism.

There is an expression  $P_{U,W} = T^{-1}P_1I_U$ , where  $P_1 : U \oplus W \rightarrow U$  is a projection, and  $I_U : U \rightarrow V$  is a natural embedding.

Thus,  $P_{U,W}$  is continuous as composition of continuous functions.

$(\Leftarrow)$  : Assume  $(\Delta, u_\delta + w_\delta)$  is a net in  $V$  converging to 0 .

Then by continuity  $0 = P_{U,W}(0) = P_{U,W}(\lim_{\delta \in \Delta} u_\delta + w_\delta) = \lim_{\delta \in \Delta} P_{U,W}(u_\delta + w_\delta) = \lim_{\delta \in \Delta} u_\delta$ .

Also  $E - P_{U,W} = P_{W,U}$  is continuous.

So by the argument similar to one above  $\lim_{\delta \in \Delta} w_\delta = 0$ .

Thus,  $\lim_{\delta \in \Delta} (u_\delta, w_\delta) = 0$  and  $T^{-1}$  is continuous meaning that  $T$  is homeomorphism.

□

**TopologicalComplementsByIsomorphicQuotient** ::

$:: \forall V \in k\text{-TVS} . \forall U, W : \text{LinearComplement}(V) . U \oplus W =_{k\text{-TVS}} V \iff \text{Homeomorphism}\left(W, \frac{V}{U}, \pi_{U|W}\right)$

**Proof** =

$\pi_U$  is a quotient map, and hence continuous.

$(\Rightarrow)$  : Assume  $(\Delta, [U + w_\delta])$  is a net in  $\frac{V}{U}$  converging to zero.

But this means that  $\lim_{\delta} w_\delta = 0$  and  $\lim_{\delta} \pi_{U|W}^{-1}[U + w_\delta] = \lim_{\delta} w_\delta = 0$ .

So  $\pi_{U|W}$  is homeomorphism.

$(\Leftarrow)$  : write  $P_{U,W} = \pi_U \pi_{U|W}^{-1} I_W$ .

This is continuous as a composition of continuous functions.

So by the previous theorem  $V = U \oplus_{k\text{-TVS}} W$ .

□

**ComplementedImpliesClosed** ::  $\forall V \in k\text{-TVS} \forall (U, W) : \text{TopologicalComplement}(V) . \text{Closed}(V, U)$

**Proof** =

By previous theorem  $P_{W,U}$  is continuous.

Thus,  $U = \ker P_{W,U}$  is closed.

□

**MaximalSubspace** ::  $\prod_{V \in k\text{-VS}} ?\text{VectorSubspace}(V)$

$U : \text{MaximalSubspace} \iff \forall W \subset_{k\text{-VS}} V . U \subsetneq W \Rightarrow W = V$

**MaximalClosedSubspace** ::

::  $\forall V \in k\text{-TVS} . \forall U \subset_{k\text{-VS}} V .$

.  $\text{MaximalSubspace} \ \& \ \text{Closed}(V, U) \iff \exists f \in \text{TOP}(V, k) . U = \ker f \ \& \ f \neq 0$

**Proof** =

$(\Rightarrow)$  : Assume  $U$  is closed and maximal subspace in  $V$ .

As  $U$  is maximal it should have a codimension 1.

So where exists  $v \in U^c$  such that  $V = U \oplus \langle v \rangle$ .

As  $U$  is closed, where exists a balanced open subset  $O \in \mathcal{U}_V(0)$  such that  $(O + v) \cap U = \emptyset$ .

assume  $u + \alpha v \in O$  is such that  $|\alpha| > 1$  and  $u \in U$ .

Then, as  $O$  is balanced,  $\alpha^{-1}u + v \in O$ .

But, then  $(\alpha^{-1}u + v) - v = \alpha^{-1}u \in (O + v) \cap U$ , which is a contradiction.

Thus,  $u + \alpha v \in \sigma O$  implies that  $|\alpha| < |\sigma|$ .

Define  $f(u + \alpha v) = \alpha : V \rightarrow k$ .

Consider a net  $v_\delta = u_\delta + \alpha_\delta v$  converging to zero with  $u_\delta$  in  $U$ .

But the previous remark shows that  $f(v_\delta) = \alpha_\delta$  converges to zero.

**SchroederBernsteinTHM** ::

::  $\forall V, V' \in k\text{-TVS} . \forall \aleph : V \cong_{k\text{-TVS}} V \oplus V . \forall \beth : V' \cong_{k\text{-TVS}} V' \oplus V' .$

.  $\forall \beth : \text{TopologicalComplement}(V, V') . \forall \beth : \text{TopologicalComplement}(V', V') . V \cong_{k\text{-TVS}} V'$

**Proof** =

Write  $V \cong V' \oplus U = (V' \oplus V') \oplus U \cong V' \oplus (V' \oplus U) \cong V' \oplus V$ .

Symmetrically,  $V' \cong V' \oplus V$ .

Thus,  $V \cong V \oplus V' \cong V'$ .

□

### 1.1.7 Finite Dimension Conditions

**OneDimTVS** ::  $\forall V \in k\text{-HTVS} . \dim V = 1 \iff V \cong_{k\text{-TVS}} k$

**Proof** =

As dimension is invariant for linear isomorphism ( $\Leftarrow$ ) is obvious .

( $\Rightarrow$ ) : As  $\dim V = 1$  there is a  $v \in V$  such that  $v \neq 0$  and  $V = kv$ .

Then the map  $T(\alpha v) = \alpha$  is a linear isomorphism .

fix some  $\rho \in \mathbb{R}_{++}$ .

As  $V$  is Hausdorff there must exist an open set  $U \in \mathcal{U}_V(0)$  such that  $\rho v \notin U$ .

Furthermore,  $U$  must have a balanced subset  $W \in \mathcal{U}_V(0)$ .

As  $W$  is balanced,  $W \subset \mathbb{B}(0, \rho)v$  .

So,  $\alpha_\delta v \rightarrow 0 \iff \alpha_\delta \rightarrow 0$  .

Thus,  $T$  must be a homeomorphism.

□

**FinDimIsomorphism** ::

$\forall V \in k\text{-HTVS} . \forall n \in \mathbb{N} . \dim V = n \iff V \cong_{k\text{-TVS}} (k^n, \|\bullet\|_\infty)$

**Proof** =

I modify the proof of the previous theorem.

By algebraic there must exist a base  $\mathbf{e} = (e_1, \dots, e_n)$  of  $V$  .

fix  $\rho$  in  $\mathbb{R}_{++}$ .

As  $V$  is Hausdorff and each  $e_i \neq 0$  there  $U \subset \mathcal{U}_V(0)$  such  $\rho e_i \notin U$  for any  $i \in \{1, \dots, n\}$ .

So there exists a balanced subset  $W$  of  $U$  such that  $W \subset \mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \rho) \cdot \mathbf{e}$ .

Thus, the mapping  $\alpha \cdot \mathbf{e} \mapsto \alpha$  is continuous.

Also, if  $U \in \mathcal{U}_V(0)$  the set  $U$  must be absorbent,

so there is a sequence  $\rho_1, \dots, \rho_n \in \mathbb{R}_{++}$  such that  $\mathbb{D}_k(0, \rho_i)e_i \subset U$ .

Let  $\sigma = \min(\rho_1, \dots, \rho_n) \in \mathbb{R}_{++}$ .

Then  $\mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \sigma) \cdot \mathbf{e} \subset U$  .

So, the inverse  $\alpha \mapsto \alpha \cdot \mathbf{e}$  is also continuous.

□

**FDimdSubspaceIsClosed** ::  $\forall V \in k\text{-HTVS} . \forall U \subset_{k\text{-VS}} V . \dim U < \infty \Rightarrow \text{Closed}(V, U)$

**Proof** =

$U$  is Hausdorff as a subset of Hausdorff space.

Then  $U$  is isomorphic to  $\ell_{k, \dim U}^\infty$  which is complete.

So,  $U$  can be viewed as a uniform embedding of complete space into  $V$ , and hence must be closed.

□



**ClosedFDimSum** ::  $\forall V \in k\text{-TVS} . \forall U \subset_{k\text{-TVS}} V . \forall W \subset_{k\text{-VS}} V . \dim W < \infty \Rightarrow \text{Closed}(V, U + W)$

**Proof** =

As  $U$  is closed in  $V$  the quotient  $\frac{V}{U}$  must be Hausdorff.

As  $\dim P_U(W) \leq \dim W$  the image  $P_U(W)$  is still finite dimensional.

So by previous theorem  $P_U(W)$  is closed in  $\frac{V}{U}$ .

But then the preimage  $U + W = P_U^{-1}P_U(W)$  is closed as quotient map  $P_U$  is continuous.

□

**FiniteDimensionalDomain** ::  $\forall V, U \in k\text{-HTVS} . \forall T \in k\text{-VS}(V, U) .$

$\dim V < \infty \Rightarrow T \in k\text{-TVS}(V, U)$

**Proof** =

$\dim T(V) \leq \dim V$ , thus  $T(V)$  must be finite dimensional.

Thus both  $V$  and  $T(V)$  are isomorphic to copies of  $l_k^\infty$  with corresponding finite dimensions.

And  $T$  must be continuous as any mapping between such spaces does.

**FiniteDimensionalCodomain** ::  $\forall V, U \in k\text{-HTVS} . \forall T \in k\text{-TVS} \& \text{Surjective}(V, U) .$

$\dim U < \infty \Rightarrow \text{Open}(V, U, T)$

**Proof** =

By isomorphism theorem  $\frac{V}{\ker T} \cong_{k\text{-VS}} T(V) = U$ .

So  $\dim \frac{V}{\ker T} < \infty$ .

Also  $\frac{V}{\ker T}$  is Hausdorff as  $T$  is continuous.

So by previous theorem the isomorphism must  $\frac{V}{\ker T} \cong_{k\text{-VS}} U$  must be continuous.

So  $U$  is also finite dimensional Hausdorff this bijection is homeomorphism and so  $\frac{V}{\ker T} \cong_{k\text{-TVS}} U$ .

Denote this homeomorphism by  $S$ .

Then  $T$  factors as  $P_{\ker T}S$  and both these maps are open.

□

**FDimIffLocallyCompact** ::  $\forall V \in k\text{-HTVS} . \dim V < \infty \iff \text{LocallyCompact}(V)$

**Proof** =

$(\Rightarrow)$  :  $V$  is homeomorphic to  $l_{k, \dim V}^\infty$  and this space is locally compact..

This can be easily shown by considering a base of closed cubes.

So  $V$  is locally compact.

$(\Leftarrow)$  : now consider the case when  $V$  is locally compact.

Then there exists a compact balanced neighborhood of zero, say  $K$ .

Take  $U$  to be any other open neighborhood and choose  $W \in \mathcal{U}_V(0)$  such balanced set that  $W + W \subset U$ .

As  $K$  is compact, it is totally bounded and hence can be covered by a finite set of translates  $K \subset \bigcup_{i=1}^n W + v_i$ .

As  $W$  is absorbent and balanced there is  $\rho \in (1, +\infty)$  such that each  $v_i \in \rho W$ .

Then  $K \subset \bigcup_{i=1}^n W + v_i \subset W + \rho W \subset \rho W + \rho W = \rho(W + W) \subset \rho U$ .

Thus, sets of form  $2^{-n}K$  form base at zero.

As  $K$  is totally bounded it can be covered by a finite set of translates  $K \subset \bigcup_{i=1}^n \frac{1}{2}K + e_i$ .

$F = \text{span } e$  is finite-dimensional and hence closed.

$K \subset \bigcup_{i=1}^n \frac{1}{2}K + e_i \subset \frac{1}{2}K + F$ .

But also  $\alpha F = F$  for any non-zero scalar  $\alpha$ .

So  $\frac{1}{2}K \subset \frac{1}{4}K + F$ .

Iterating this relation and substituting we get the result that  $K \subset \frac{1}{2^n}K + F$  for any  $n \in \mathbb{N}$ .

This can be rewritten as  $K \subset \bigcap_{n=1}^{\infty} \frac{1}{2^n}K + F = F$ .

But  $K$  spans whole  $V$ , and so  $V = F$  which is finite dimensional.

□

**FDimCompactConvexHullIsCompact** ::

$\forall V \in k\text{-TVS} . \forall K : \text{CompactSubset}(V) . \dim V < \infty \Rightarrow \text{CompactSubset}(V, \text{conv } K)$ .

**Proof** =

Let  $n = \dim V$ .

$\text{conv } K$  consists of convex combination of form  $\sum_{i=1}^{2n+1} \lambda_i x_i$  where  $\lambda \geq 0$  and  $\sum_{i=1}^{2n+1} \lambda_i = 1$  and  $x_i \in K$ .

This condition can be express as  $\lambda \in \Delta_{2n+1} \subset k^{2n+1}$ .

But  $\Delta_{2n+1}$  is also compact, and so is  $\Delta_{2n+1} \times K^{2n+1}$  by Tychonoff's theorem.

So  $\text{conv } K = (\cdot)(\Delta_{2n+1} \times K^{2n+1})$  is compact as a continuous image of a compact.

□

### 1.1.8 Case of Ultravalued Field

$k : \text{UltravaluedField};$

$\text{AbsolutelyKConvex} :: \prod_{V:k\text{-TVS}} ??V$

$A : \text{AbsolutelyKConvex} \iff \mathbb{D}_k(0,1)A + \mathbb{D}_k(0,1)A = A$

$\text{KConvex} :: \prod_{V:k\text{-TVS}} ??V$

$V : \text{KConvex} \iff \exists v \in V . \exists A : \text{AbsolutelyKConvex}(V) . C = A + v$

$\text{AbsolutelyKConvexByZeroContaintment} :: \forall V \in k\text{-TVS} . \forall C : \text{KConvex}(V) . 0 \in C \Rightarrow \text{AbsolutelyKConvex}(V)$

**Proof** =

$C$  must be a translate of absolutely K-Convex set, so write  $C = A + v$ .

As  $A$  is absolutely K-Convex, then  $\alpha(x + v) + \beta(y + v) - v \in C$  for any  $x, y \in C$  and  $\alpha, \beta \in \mathbb{D}_k(0,1)$ .

Take  $\alpha = \beta = 1, y = 0$ .

Then the expression above reduces to  $x + v \in C$ .

But this means that  $A \subset C$ .

On the other hand,  $\alpha(x + v) + \beta(y + v) \in A$  for any  $x, y \in C$  and  $\alpha, \beta \in \mathbb{D}_k(0,1)$ .

Taking  $\alpha = 1, \beta = -1, y = 0$ , produces  $x \in A$ .

Thus  $C \subset A$  and  $C = A$  is absolutely K-convex.

□

$\text{TripleCombinationKConvexityCondition} ::$

$:: \forall V \in k\text{-TVS} . \forall C \subset V .$

$. \text{KConvex}(V, C) \iff \forall x, y, z \in C . \forall \alpha, \beta, \gamma \in \mathbb{D}_k(0,1) . \alpha + \beta + \gamma = 1 \Rightarrow \alpha x + \beta y + \gamma z \in C$

**Proof** =

1 ( $\Rightarrow$ ) : assume that  $C$  is K-convex.

1.1  $C$  must be a translate of absolutely K-Convex set, so write  $C = A + v$ .

1.2 Then  $\alpha x + \beta y + \gamma z = \alpha(x - v) + \beta(y - v) + \gamma(z - v) + v \in C$ .

2 ( $\Leftarrow$ ).

2.1 If  $C = \emptyset$  then it is trivially K-convex, so assume the contrary.

2.2 Take  $v \in V$  and let  $A = C - v$ .

2.3  $A$  is absolutely K-convex.

2.3.1 Assume  $x, y \in C$  and  $\alpha, \beta \in \mathbb{D}_k(0,1)$ .

2.3.2  $1 - \alpha - \beta \in \mathbb{D}_k(0,1)$  .

2.3.2.1  $|1 - \alpha - \beta| \leq \max(1, |\alpha|, |\beta|) = 1$  .

2.3.3 Then by the hypothesis  $\alpha x + \beta y + (1 - \alpha - \beta)v \in C$  .

2.3.4 Translating by  $-v$  gives  $\alpha(x - v) + \beta(y - v) = \alpha x + \beta y + (1 - \alpha - \beta)v - v \in A$  .

□

**convexCombinationKConvexityCondition** ::

$:: \forall V \in k\text{-TVS} . \forall \mathbb{K} : \text{res char } k \neq 2 . \forall C \subset V .$

$. \text{KConvex}(V, C) \iff \forall x, y \in C . \forall \alpha \in \mathbb{D}_k(0, 1) . \alpha x + (1 - \alpha)y + \gamma z \in C$

**Proof** =

1 ( $\Rightarrow$ ) This direction is obvious.

1.1 The convex combination is a weaker form of triple combination in the previous result.

2 ( $\Leftarrow$ ) .

2.1 If  $C = \emptyset$  then it is trivially K-convex, so assume the contrary.

2.2 Take  $v \in V$  and let  $A = C - v$ .

2.3  $A$  is absolutely K-convex.

2.3.1 Assume  $x, y \in C$  and  $\alpha, \beta \in \mathbb{D}_k(0, 1)$ .

2.3.2 Rewrite  $\alpha(x - v) + \beta(y - v) + v = \frac{1}{2}(2\alpha x + (1 - 2\alpha)v) + \frac{1}{2}(2\beta y + (1 - 2\beta)v)$ .

2.3.3 Both  $\frac{1}{2}(2\alpha x + (1 - 2\alpha)v)$  and  $\frac{1}{2}(2\beta y + (1 - 2\beta)v)$  in  $C$ .

2.3.3.1 for ultravalue  $|2\alpha| = |\alpha + \alpha| \leq |\alpha| = 1$  .

2.3.3.2 Same holds for  $\beta$ .

2.3.3.3 So the convex combination hypothesis can be applied.

2.3.4 clearly  $\frac{1}{2} + \frac{1}{2} = 1$ , so  $\alpha(x - v) + \beta(y - v) \in A$  .

2.3.4.1  $\left| \frac{1}{2} \right| = 1$  as residual characteristic of the field is not 2.

□

**AbsolutelyKConvexIntersection** ::  $\forall V : k\text{-TVS} . \forall I \in \text{SET} .$

$. \forall A : I \rightarrow \text{AbsolutelyKConvex}(V) . \text{AbsolutelyKConvex} \left( V, \bigcap_{i \in I} A_i \right)$

**Proof** =

Obvious.

□

**KConvexIntersection** ::  $\forall V : k\text{-TVS} . \forall I \in \text{SET} .$

$$. \forall C : I \rightarrow \mathbf{KConvex}(V) . \mathbf{KConvex} \left( V, \bigcap_{i \in I} C_i \right)$$

**Proof** =

1 Assume that  $\bigcap_{i \in I} C_i \neq \emptyset$ .

1.1 Otherwise the condition is trivial.

2 Take any  $v \in \bigcap_{i \in I} C_i$ .

3 Then  $\left( \bigcap_{i \in I} C_i \right) - v$  is absolutely K-convex and  $\bigcap_{i \in I} C_i$  is K-convex.

3.1  $\left( \bigcap_{i \in I} C_i \right) - v = \bigcap_{i \in I} (C_i - v)$  as translation by  $v$  is bijective.

3.2 Then every  $C_i - v$  are K-convex sets, which contain zero, so they are absolutely K-Convex.

3.3 So, the intersection  $\bigcap_{i \in I} (C_i - v)$  is also absolutely K-Convex.

□

**kConvexHull** ::  $\prod_{V : k\text{-TVS}} (?V) \rightarrow \mathbf{KConvex}(V)$

**kConvexHull** ( $X$ ) =  $K\text{-conv } X := \bigcap \left\{ C : \mathbf{KConvex}(V), X \subset C \right\}$

**KConvexHullByLinearCombinations** ::

::  $\forall V \in k\text{-TVS} . \forall X \subset V .$

$$. K\text{-conv } X = \left\{ x_{n+1} + \sum_{i=1}^n \alpha_i (x_i - x_{n+1}) \mid n \in \mathbb{Z}_+, \alpha : \{1, \dots, n\} \rightarrow \mathbb{D}_k(0, 1), x : \{1, \dots, n+1\} \rightarrow X \right\}$$

**Proof** =

1 Let  $B$  denote the set defined above.

2  $B$  is K-Convex.

2.1 Note, that  $x_{n+1}$  in definition can be fixed.

2.2 Then  $B - x_{n+1}$  is obviously absolutely K-convex.

3  $X \subset B$ .

3.1 Just take  $n = 1, \alpha_1 = 1$ .

4 So  $K\text{-conv } X \subset B$ .

5 If  $C$  is K-convex, then  $B \subset C$ .

5.1 Some  $x_{n+1} \in X$  must also be contained in  $C$ .

5.2 So  $C - x_{n+1}$  is absolutely K-convex.

5.3 So by induction  $\sum_{i=1}^n \alpha_i (x_i - x_{n+1}) \in C - x_{n+1}$ .

6 Thus,  $B \subset K\text{-conv } X$ , and so  $B = K\text{-conv } X$ .

□

$\mathbf{kDiskHull} :: \prod_{V:k\text{-TVS}} (?V) \rightarrow \mathbf{AbsolutelyKConvex}(V)$

$\mathbf{kDiscHull}(X) = K\text{-disc } X := \bigcap \left\{ C : \mathbf{AbsolutelyKConvex}(V), X \subset C \right\}$

$\mathbf{AbsolutelyKConvexInterior} :: \forall V : k\text{-TVS} . \forall A : \mathbf{AbsolutelyKConvex}(V) . \text{int } A = \emptyset \mid \text{int } A = A$

**Proof** =

1 assume  $\text{int } A \neq \emptyset$ .

2 Take  $v \in \text{int } A$ .

3 Without loss of generality assume  $v = 0$ .

3.1 Then  $A - v$  is an isomorphic absolutely convex set with  $0 \in \text{int } A$ .

4 Take any  $U \in \mathcal{U}_V(0)$  such that  $U \subset \text{int } A \subset A$ .

5 Now take arbitrary  $v \in A$ .

6 Then  $U + v \subset A$ .

6.1  $U + v$  consists of elements  $u + v$  with  $u \in U \subset A$ .

6.2 As  $v \in A$  also and  $A$  is absolutely K-convex it must be the case that  $u + v \in A$ .

7 As translation is a homeomorphism  $U + v$  is open and so  $v \in \text{int } A$ .

□

$\mathbf{OpenKDiscHull} :: \forall V : k\text{-TVS} . \forall U : \mathbf{Open}(V) . \mathbf{Open}(V, K\text{-disc } U)$

**Proof** =

1  $K\text{-disc } U$  is absolutely K-convex.

2  $U \subset K\text{-disc } U$ , so  $\text{int } K\text{-disc } U \neq \emptyset$ .

3 But this means that  $K\text{-disc } U$  is open.

□

$\mathbf{LocallyKConvexSpace} :: ?k\text{-TVS}$

$V : \mathbf{LocallyKConvexSpace} \iff \exists \mathcal{F} : \mathbf{Filterbase}(V, \mathcal{U}_V(0)) . \forall F \in \mathcal{F} . \mathbf{KConvex}(V, F)$

**NonarchimedeanVSHasZeroTopDim** ::  $\forall V : \text{LocallyKConvexSpace}(k) \ \& \ \text{T2} . \dim_{\text{TOP}} V = 0$

**Proof** =

1  $V$  has a base of closed K-discs.

1.1 Consider  $U \in \mathcal{U}_V(0)$ .

1.2 Then there exists an open K-disc  $D$  such that  $0 \in D \subset \overline{D} \subset U$ .

1.3 Then  $\overline{D}$  is a K-disk.

1.3.1 If  $u, v \in \overline{D}$  it means that every their open neighborhood meet  $D$ .

1.3.2 Assume  $\alpha, \beta \in \mathbb{D}_k(0, 1)$ .

1.3.3 Consider an open neighborhood  $W$  of  $\alpha u + \beta v$ .

1.3.4 Then there is an open neighborhood of zero  $O + O \subset W - \alpha u - \beta v$ .

1.3.5 Consider the case  $\alpha \neq 0 \neq \beta$ .

1.3.6 Then there must be some  $u' \in D \cap \frac{1}{\alpha}(O + \alpha u)$ .

1.3.7 Then there is also  $v' \in D \cap \frac{1}{\beta}(O + \beta v)$ .

1.3.8 Then  $\alpha u' + \beta v' \in D$  as  $D$  is absolutely K-convex.

1.3.9 Also  $\alpha u' + \beta v' \in O + O + \alpha u + \beta v \subset W$ .

1.3.10 As  $W$  was arbitrary this means that  $\alpha u + \beta v \in \overline{D}$ .

1.4  $\overline{D} \subset U$ .

1.4.1 This is true as  $V$  is Hausdorff, and Hence regular.

2 But then every K-disc in this base is clopen.

2.1 To be in base every K-disc  $D$  should contain an element of  $U_V(0)$ .

2.2 Hence  $D$  has non-empty interior.

2.3 But This means that  $D$  is open.

3 Thus  $\dim_{\text{TOP}} V = 0$ .

□

**RelativelyKConvex** ::  $\prod_{V_k\text{-TVS}} \prod_{A \subset V} ??A$

$R : \text{RelativelyKConvex} \iff \exists C : \text{KConvex}(K) . R = C \cap A$

**KConvexFilterbase** ::  $\prod V : k\text{-TVS} . \prod_{A \subset V} ?\text{Filterbase}(A)$

$\mathcal{F} : \text{KConvexFilterbase} \iff \forall F \in \mathcal{F} . \text{RelativelyKConvex}(V, A, F)$

**CCompact** ::  $\prod_{V_k\text{-TVS}} ??V$

$K : \text{CCompact} \iff \forall \mathcal{F} : \text{KConvexFilterbase}(V, K) . \exists \text{AdherencePoint}(V, \mathcal{F})$

$|\cdot| \neq \Lambda \alpha \in k . [\alpha \neq 0]$

**EveryCompactIsCCompact** ::  $\forall V : k\text{-TVS} . \forall K : \text{Compact}(V, K) . \text{CCompact}(V, K)$

**Proof** =

- 1 Assume  $\mathcal{F}$  is a K-Convex filterbase on  $K$ .
  - 2 Then associated ultrafilter must have a limit.
  - 3 This limit is an adherence point of  $\mathcal{F}$ .
- 

**ClosedSubsetOfCCompact** ::  $\forall V : k\text{-HTVS} . \forall K : \text{CCompact}(V) . \forall L : \text{Closed}(K) \ \& \ \text{KConvex}(V) .$   
 $\text{CCompact}(V, L)$

**Proof** =

- 1 Assume  $\mathcal{F}$  is a K-Convex filterbase on  $L$ .
  - 2 Then the  $\mathcal{F}$  is also a K-Convex filterbase for  $K$ .
  - 3 Then, there is an adherence point  $p \in K$  fo  $\mathcal{F}'$ .
  - 4  $p$  is also an adherence point for  $\mathcal{F}$ .
  - 4.1 Take any  $U \in \mathcal{U}_V(p)$  .
  - 4.2 Then  $F \cap K \cap U \neq \emptyset$  for any  $F \in \mathcal{F}$  .
  - 4.3 Bat all these  $F \subset L$ .
  - 4.4 Thus  $p \in \text{cl}_K L = L$ .
- 

**MaximalConvexFilterbase** ::

$\forall V : \text{LocallyKConvexSpace}(k) . \forall C : \text{KConvex}(V) . \forall \mathcal{F} \in \max \text{KConvexFilterbase}(V, C) .$   
 $\text{. } \forall p \in C . \text{AherencePoint}(C, \mathcal{F}, p) \iff \lim \mathcal{F} = p$

**Proof** =

- 1 ( $\Rightarrow$ ) : Assume  $p$  is an adherence point for  $\mathcal{F}$  in  $C$ .
  - 1.1 Then  $\forall F \in \mathcal{F} . \forall U \in \mathcal{U}_V(p) . U \cap F \neq \emptyset$  .
  - 1.2 Assume that  $U \in \mathcal{U}_C(p)$ .
  - 1.3 Then there exist a K-convex  $D$  and open  $W \in \mathcal{U}_C(p)$  such that  $W \subset D \subset V$ .
  - 1.4 Then  $\forall F \in \mathcal{F} . D \cap F \neq \emptyset$ .
  - 1.4.1  $\forall F \in \mathcal{F} . W \cap F \neq \emptyset$ .
  - 1.4.2  $W \subset D$  .
  - 1.5 As  $\mathcal{F}$  is maximal  $D \in \mathcal{F}$ .
  - 1.6 Thus,  $p = \lim \mathcal{F}$ .
  - 2 ( $\Leftarrow$ ) : Now Assume  $p = \lim \mathcal{F}$ .
  - 2.1 Then  $\forall U \in \mathcal{U}_C(p) . \exists F \in \mathcal{F} . F \subset U$ .
  - 2.2 Take arbitrary  $U \in \mathcal{U}_C(p)$  and  $F \in \mathcal{F}$ .
  - 2.3 Then by (2.1) there exists  $G \in \mathcal{F}$  such that  $G \subset U$ .
  - 2.4 As  $\mathcal{F}$  is a filterbase  $G \cap F \neq \emptyset$ .
  - 2.5 Thus  $F \cap U \neq \emptyset$ .
  - 2.6 This proves that  $p$  is and adherence point for  $\mathcal{F}$ .
-



**KConvexAndCcompactIsClosed** ::

$:: \forall V : \text{LocallyKConvexSpace}(k) . \forall K : \text{CCompact} \ \& \ \text{KConvex}(V) . \text{Closed}(V, K)$

**Proof** =

- 1 Assume  $p$  is a Limit point for  $K$ .
  - 2 Then there exists an filter  $\mathcal{F}$  in  $K$  such that  $p = \lim \mathcal{F}$ .
  - 2.1 Take  $\mathcal{N}_V(p) \cap K$  for example.
  - 3 Then  $p$  is an adherence point of  $\mathcal{F}$ .
  - 4 construct a K-convex filterbase  $\mathcal{C}$  from  $\mathcal{F}$ .
  - 4.1 For example, use the fact that  $V$  is locally K-convex.
  - 4.2 Let  $C$  be the intersections of  $K$  and K-convex neighborhoods of  $p$ .
  - 5 Then  $p$  is still a limit point of  $\mathcal{C}$  in  $V$ .
  - 6 There also must exist an adherence point of  $\mathcal{C}$  in  $K$ , say  $q$ .
  - 7 But as  $V$  is Hausdorff and  $\mathcal{C}$  has a limit it must be the case  $q = p$ .
  - 8 Thus  $K$  has all its limit points and must be closed.
- 

**CCompactProduct** ::  $\forall I \in \text{Set} . \forall V : I \rightarrow k\text{-TVS} . \forall C : \prod_{i \in I} \text{CCompact}(V_i) . \text{CCompact} \left( \prod_{i \in I} V_i, \prod_{i \in I} C_i \right)$

**Proof** =

Same proof as Tychonoff's theorem's proof with filters, but with  $k$ -convex sets.

□

**CCompactCombination** ::  $\forall V : \text{LocallyKConvexSpace} k . \forall n \in \mathbb{Z}_+ .$

$. \forall D : \{1, \dots, n\} \rightarrow \text{AbsolutelyKConvex} \ \& \ \text{CCompact}(V) . \text{CCompact} \left( V, K\text{-conv} \bigcup_{i=1}^n D_i \right)$

**Proof** =

- 1 I will give a proof by induction.
  - 2  $K\text{-conv} \bigcup_{i=1}^n D_i = \emptyset$  in case  $n = 0$  and is trivially c-compact.
  - 3  $K\text{-conv} \bigcup_{i=1}^{n+1} D_i = K\text{-conv} \left( D_{n+1} + \bigcup_{i=1}^n D_i \right)$  by the result expressing K-convex hulls by linear combinations.
  - 4 So for the induction step we need to prove case of two c-compacts  $D_1$  and  $D_2$ .
  - 5 assume  $\mathcal{F}$  is a closed k-convex filterbase on  $K\text{-conv} D_1 \cup D_2$ .
  - 6 Let  $\mathcal{F}' = \left\{ \{(x, y) \in D_1 \times D_2 : \exists \alpha, \beta \in \mathbb{D}_k(0, 1) . \alpha x + \beta y \in F\} \mid F \in \mathcal{F} \right\}$ .
  - 7 Then  $\mathcal{F}'$  is a k-convex filterbase on  $D_1 \times D_2$ .
  - 8  $D_1 \times D_2$  is c-compact.
  - 9 So there is an adherence point  $(x, y)$  of  $\mathcal{F}'$ .
  - 10 Let  $C = K\text{-disc}\{x, y\}$ .
  - 11 Then  $C$  is c-compact K-disc.
  - 12 Then  $\overline{F} \cap C \neq \emptyset$  for all  $F \in \mathcal{F}$ .
  - 13 So  $\mathcal{F}'' = \{\overline{F} \cap C \mid F \in \mathcal{F}\}$  is a filterbas on  $C$ .
  - 14 So there exists an adherence point  $P$  of  $\mathcal{F}''$ .
  - 15 But  $p$  is als an adherence point of  $\mathcal{F}$  then.
-

**CCompactIffSphericallyComplete** :: **CCompact**( $k$ )  $\iff$  **SphericallyComplete**( $k$ )

**Proof** =

1 ( $\Rightarrow$ ) : Assume that  $k$  is c-compact.

1.1 Let  $B : \mathbb{N} \rightarrow 2^k$  be a deacrising sequence of closed balls.

1.2 Then  $\mathcal{B} = \{B_i | i \in \mathbb{N}\}$  is a  $k$ -convex filter.

1.3 So there must exist and adherence point  $\beta$  of  $\mathcal{B}$ .

1.4 Then  $\beta \in B_n$  for every  $n \in \mathbb{N}$ .

1.4.1  $B_n \cap U \neq \emptyset$  for every  $U \in \mathcal{U}_k(\beta)$ .

1.4.2 This means that  $\beta \in \overline{B_n}$ .

1.4.3 But  $B_n = \overline{B_n}$  as  $B_n$  is closed.

1.5 Which can be rendered as  $\beta \in \bigcap_{n=1}^{\infty} B_n$ .

2 ( $\Rightarrow$ ) : Assume that  $k$  is spherically complete.

2.1 we claim that every  $k$ -convex set in  $k$  is either  $\emptyset$  or a ball.

2.1.1 Assume  $A$  is an absolutely  $k$ -convex set such that  $\emptyset \neq A \neq k$ .

2.1.2 Take  $\omega \in A^\complement$ .

2.1.3 Then  $\omega \neq 0$ .

2.1.4 Then every  $\omega'$  such that  $|\omega| \leq |\omega'|$  is not in  $A$ .

2.1.4.1 Assume there is some  $\omega' \in A$  such that  $|\omega| \leq |\omega'|$ .

2.1.4.2 Then  $\left| \frac{\omega}{\omega'} \right| \leq 1$ .

2.1.4.3 Thus, as  $A$  is a  $k$ -disc,  $\omega = \frac{\omega}{\omega'} \omega' \in A$ .

2.1.5 So the set  $R = \left\{ |\omega| \mid \omega \in A^\complement \right\}$  is bounded from above.

2.1.6 Let  $r = \sup R$ .

2.1.7 Take  $\alpha \in A$  and  $\beta \in k$  with  $|\beta| \leq |\alpha|$ .

2.1.8 Then  $\beta \in A$ .

2.1.9 so  $A$  is a ball of radius  $r$  open or closed depending on inclusion of  $r$  to  $R$ .

2.2 Also note, that in non-archimedian space any balls are either disjoint or contained in one or another.

2.3 So any  $k$ -convex filterbase  $\mathcal{F}$  in  $k$  can be represented as a decreasing sequence of balls, closed or open.

2.4 Construct sequence of closed balls  $\mathcal{B}$  by taking closures.

2.4.1 radii of balls will form a set  $R$  bounded from below by 0.

2.4.2 let  $\delta = \inf R$ .

2.4.3 Then there exists a decreasing sequence of balls  $B$  with respective radii  $r$  such that  $\lim_{n \rightarrow \infty} r_n = \delta$ .

2.4.3.1 This is true as all elements in the filterbase  $\mathcal{F}$  must have non-empty intersection.

2.5 Then there exists  $\beta \in \bigcap \mathcal{B}$ .

2.4.4 Take  $\mathcal{B} = \{B_n | n \in \mathbb{N}\}$ .

2.6  $\beta$  is an adherence point of  $\mathcal{F}$ .

2.6.1 There is some  $B \in \mathcal{B}$  such  $\beta \in B \subset \overline{F}$  for every element  $F \in \mathcal{F}$ .

2.6.2 Then  $F \cap U \neq \emptyset$  for every  $U \in \mathcal{U}_k(\beta)$ .

□

### 1.1.9 Some Interesting Examples

$k :: \text{AbsoluteValueField}(\mathbb{R})$

$\text{NonLocallyConvexSpace} :: \exists V : k\text{-TVS} . \neg \text{LocallyConvexSpace}(V)$

**Proof** =

1 Let  $V = L^p(\mathbb{R}, \lambda)$  for  $p \in (0, 1)$ .

2 Its topology can be metrized by the metric  $\rho(f, g) = \int |f - g|^p$ .

2.1 we use inequality of form  $\left( \sum_{i=1}^n \alpha_i \right)^p \leq \sum_{i=1}^n \alpha_i$  for  $\alpha_i > 0$ .

3 on the other hand  $\text{conv } \mathbb{B}_V(0, \sigma) \subset \mathbb{B}_V(0, 2^{p-1}\sigma)$ .

3.1 Assume  $f \in \mathbb{B}_V(0, \sigma)$ .

3.2 Define  $F(t) = \int_{-\infty}^t |f|^p$ .

3.3 Then  $F$  is a continuous function on  $[-\infty, +\infty]$  such that  $F(-\infty) = 0$  and  $F(+\infty) = \rho(0, f)$ .

3.4 By intermediate value theorem there exists  $t \in \mathbb{R}$  such that  $F(t) = \frac{\rho(0, f)}{2}$ .

3.5 Let  $g(x) = f(x)\delta_x(-\infty, t)$ ,  $h(x) = f(x)\delta_x(t, +\infty)$ .

3.6 Then  $\rho(g, 0) \leq \frac{\sigma}{2}$  and  $\rho(h, 0) \leq \frac{\sigma}{2}$  and  $f = h + g = \frac{2}{\sigma}g + \frac{2}{\sigma}h$ .

3.7 But  $2g, 2h \in \mathbb{B}_V(0, 2^{p-1}\sigma)$ , so  $f \in \text{conv } \mathbb{B}_V(0, 2^{p-1}\sigma)$ .

4 By iterating one gets  $\text{conv } \mathbb{B}_V(0, \sigma) = V$ .

5 So there are no non-trivial convex neighborhoods of 0.

□

$\text{NonCompactConvexHullOfTheCompact} :: \exists V : k\text{-TVS} . \exists K : \text{CompactSubset}(V) . \neg \text{CompactSubset}(V, \text{conv } K)$

**Proof** =

1 Let  $V = \ell^1$ .

2 Let  $K = \left\{ 0, \delta_1^\bullet, \dots, \frac{1}{n}\delta_n^\bullet, \dots \right\}$ .

3 Define  $\xi_n = \frac{1}{\sum_{i=1}^n 2^{-i}} \sum_{t=1}^n \frac{2^{-t}}{t} \delta_t^\bullet \in \text{conv } K$ .

4 Then  $\zeta = \lim_{n \rightarrow \infty} \xi_n = \sum_{t=1}^{\infty} \frac{2^{-t}}{t} \delta_t^\bullet$ .

5 But then  $\zeta_i \neq 0$  for all  $i \in \mathbb{N}$ , but this means that  $\zeta \notin \text{conv } K$ , so  $K$  is not compact.

□

**NoncomplimentedClosedSubspaceExist** ::  $\exists V : k\text{-TVS} . \exists U \subset_{k\text{-TVS}} V . \neg \text{TopologicalComplement}(V, U)$

**Proof** =

1 Let  $V = \ell^\infty$  .

2 Let  $U = c_0$ .

...

□

$k$  :: **UltravaluedField**

**PathologicalConvexSet** ::

::  $\text{res } k = \mathbb{F}_2 \Rightarrow \exists V : k\text{-TVS} . \exists A : \neg \text{KConvex}(V) . \forall a, b \in A . \forall \lambda \in \mathbb{D}_k(0, 1) . \lambda a + (1 - \lambda)b \in A$

**Proof** =

1 Let  $V = k^3$  and let  $A = \left\{ a \in \mathbb{D}_k(0, 1) : \exists i \in \{1, 2, 3\} . a_i \in \mathbb{B}_k(0, 1) \right\}$ .

2  $A$  has desired property for convex combinations of two elements.

2.1 Assume  $\lambda \in \mathbb{D}_k(0, 1)$  and  $a, b \in A$ .

2.2 Note, either  $|\lambda| = 1$  or  $|1 - \lambda| = 1$ .

2.2.1  $1 = [1] = [1 - \lambda + \lambda] = [1 - \lambda] + [\lambda]$  in a residue1 field  $\mathbb{F}_2$  .

2.3 There exists some  $i, j \in \{1, 2, 3\}$  such that  $|a_i| < 1$  and  $|b_j| < 1$ .

2.4 So  $|\lambda a_i| = |\lambda||a_i| < 1$  and  $|(1 - \lambda)b_j| = |1 - \lambda||b_j| < 1$ .

2.5 so either  $|\lambda a_i + (1 - \lambda)b_i| < 1$  or  $|\lambda a_j + (1 - \lambda)b_j| < 1$ .

3  $A$  is not K-convex.

3.1  $(-1, 1, 1) \notin A$ .

3.1.1  $|-1| = |1| = 1$  .

3.2 on the othe hand  $(-1, 1, 1) = -1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3 \in K\text{-conv } A$  .

□

### 1.1.10 Seminorms

$k :: \text{AbsoluteValueField}(\mathbb{R})$

$\text{Seminorm} :: \prod V : k\text{-VS} . ?(V \rightarrow \mathbb{R}_{++})$

$\nu : \text{Seminorm} \iff \forall v, w \in V . \nu(v + w) \leq \nu(v) + \nu(w) \ \& \ \forall v \in V . \forall \lambda \in k . \nu(\lambda v) = |\lambda| \nu(v)$

$\text{ZeroSeminorm} :: \forall V : k\text{-VS} . \forall \nu : \text{Seminorm}(V) . \nu(0) = 0$

**Proof** =

1  $\nu(0) = \nu(\lambda 0) = |\lambda| \nu(0)$  for any  $\lambda \in k$ .

2 This means that  $\nu(0)$  is not invertible in  $k$ .

3 So  $\nu(0) = 0$  .

□

$\text{SymmetricSeminorm} :: \forall V : k\text{-VS} . \forall \nu : \text{Seminorm}(V) . \forall v \in V . \nu(-v) = \nu(v)$

**Proof** =

1  $\nu(-v) = |-1| \nu(v) = \nu(v)$ .

□

$\text{SumOfSeminorms} :: \forall V : k\text{-VS} . \forall n \in \mathbb{N} . \forall \nu : \{1, \dots, n\} \rightarrow \text{Seminorm}(V) . \text{Seminorm}\left(V, \sum_{i=1}^n \nu_i\right)$

**Proof** =

Obvious.

□

$\text{MaxOfSeminorms} :: \forall V : k\text{-VS} . \forall n \in \mathbb{N} . \forall \nu : \{1, \dots, n\} \rightarrow \text{Seminorm}(V) . \text{Seminorm}\left(v, \max_{1 \leq i \leq n} \nu_i\right)$

**Proof** =

Obvious.

□

Note: this means that seminorms over  $V$  form an ordered tropical semiring with  $0 = -\infty$ .

$\text{seminormsFunctor} :: \text{Contravariant}(k\text{-VS}, \text{TSRING})$

$\text{seminormsFunctor}(V) = \text{SMN}(V) := \text{Seminorm}(V)$

$\text{seminormsFunctor}(V, W, T) = \text{SMN}_{V,W}(T) := T^*$

**seminormCell** ::  $\prod V \in k\text{-VS} . \text{Seminorm}(V) \rightarrow ?V$

**seminormCell** ( $\nu$ ) =  $\mathbb{B}(\nu) := \{v \in V : \nu(v) < 1\}$

**seminormDisc** ::  $\prod V \in k\text{-VS} . \text{Seminorm}(V) \rightarrow ?V$

**seminormDisc** ( $\nu$ ) =  $\mathbb{D}(\nu) := \{v \in V : \nu(v) \leq 1\}$

**SeminormIneq** ::  $\forall V \in k\text{-VS} . \forall \nu, \nu' : \text{Seminorm}(V) . \nu \leq \nu' \iff \mathbb{B}(\nu') \subset \mathbb{B}(\nu)$

**Proof** =

Obvious.

□

Note: This means that  $\mathbb{B}$  is an antitone map or functor  $\text{SMN}(V) \rightarrow 2^V$ .

Moreover, both  $\mathbb{B}$  and  $\mathbb{D}$  are natural transform from **SMN** to the lattice of absorbent discs.

**SeminormScaling** ::  $\forall V \in k\text{-VS} . \forall \nu \in \text{SMN}(V) . \forall \lambda \in \mathbb{R}_{++} . \lambda \mathbb{B}(\nu) = \mathbb{B}(\lambda^{-1}\nu)$

**Proof** =

Obvious.

□

**SeminormCellIsAbsobentDisc** ::  $\forall V \in k\text{-VS} \forall \nu \in \text{SMN}(V) . \text{Absorbent} \ \& \ \text{Disc}(V, \mathbb{B}(\nu))$

**Proof** =

Obvious.

□

**SeminormCellClosureTheorem** ::  $\forall V \in k\text{-TVS} . \forall \nu \in \text{SMN} \ \& \ C(V) . \text{cl}_V \mathbb{B}(\nu) = \mathbb{D}(\nu)$

**Proof** =

1 Assume  $v \in \mathbb{D}(\nu)$ .

2 then the sequence  $u_n = \left(1 - \frac{1}{n}\right) v \in \mathbb{B}(\nu)$  has limit  $v$ .

3 So  $\mathbb{D}(\nu) \subset \text{cl}_V \mathbb{B}(\nu)$ .

4 On the other hand  $\mathbb{D}(\nu) = \nu^{-1}[0, 1]$  is closed.

5 So  $\text{cl}_V \mathbb{B}(\nu) \subset \mathbb{D}(\nu)$  and  $\mathbb{D}(\nu) = \text{cl}_V \mathbb{B}(\nu)$ .

□

**SeminormContinuity** ::  $\forall V : k\text{-TVS} . \forall \nu \in \text{SMN}(V) .$

- (1)  $\nu \in \text{UNI}(V, \mathbb{R}) \iff$
- (2)  $\mathbb{B}(\nu) \in \mathcal{T}(V) \iff$
- (3)  $\mathbb{D}(\nu) \in \mathcal{N}(V) \iff$
- (4)  $\text{ContinuousAt}(V, \mathbb{R}, 0, \nu)$

**Proof** =

1 (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) obvious.

2 (3)  $\Rightarrow$  (4).

2.1 As non-zero scalar multiplication is a homeomorphism  $\lambda \mathbb{D}(\nu) \in \mathcal{N}(V)$  for all  $\lambda \in \mathbb{R}_{++}$ .

2.2 consider a net  $v$  such that  $\lim_{\delta} v_{\delta} = 0$ .

2.3 Eventually  $v_{\delta} \in \lambda \mathbb{D}(\nu)$  for any  $\lambda \in \mathbb{R}_{++}$ .

2.4 This means that  $\lim_{\delta} \nu(v_{\delta}) = 0$ .

3 (4)  $\Rightarrow$  (1).

3.1  $\nu^{-1}[0, \lambda)$  is open for any  $\lambda \in \mathbb{R}_{++}$ .

3.2 As  $V$  is a topological group there is  $U \in \mathcal{U}_V(0)$  such that  $U - U \subset \nu^{-1}[0, \lambda)$ .

3.3 Thus,  $\nu(x - y) < \lambda$  for any  $x, y \in U$ .

3.4 Let  $v \in V$  be arbitraty .

3.5 Take  $u \in v + U$ .

3.6 Then  $\nu(u) = \nu(u + v - v) \leq \nu(u - v) + \nu(v) \leq \nu(v) + \lambda$ .

3.7 On the other hand  $\nu(u) \geq \nu(v) - \nu(u - v) \geq \nu(v) - \lambda$  as  $\nu(v) = \nu(v - u + u) \leq \nu(u) + \nu(u - v)$ .

3.8 So  $|\nu(u) - \nu(v)| \leq \lambda$ .

□

**SeminormContinuityByDomination** ::

$:: \forall V : k\text{-TVS} . \forall \nu \in \text{SMN}(V) . \forall \mu \in \text{SMN} \ \& \ C(V) . \nu \leq \mu \Rightarrow \nu \in \text{UNI}(V, \mathbb{R})$

**Proof** =

By antitonicity  $\mathbb{B}(\mu) \subset \mathbb{B}(\nu) \subset \mathbb{D}(\nu)$  .

But  $\mathbb{B}(\mu)$  is open, so  $\mathbb{D}(\nu) \in \mathcal{N}_V(0)$  .

Thus  $\nu$  is uniformly continuous.

□

**GaugesOfDiscsProduceSeminorms** ::  $\forall V \in k\text{-VS} . \forall D : \text{Disc} \ \& \ \text{Absorbent}(D) . \gamma(\bullet|D) \in \text{SMN}(V)$

**Proof** =

1 Discs are convex, so  $\gamma(\bullet|D)$  is a convex function.

2 Take some  $v \in V$ .

2.1 Let  $I_v = \{\lambda \in \mathbb{R}_{++} : \lambda^{-1}v \in D\}$ .

2.2 As  $D$  is absorbent,  $I_v \neq \emptyset$ .

2.3 As  $D$  is balanced then if  $\alpha \in I_v$  and  $\beta \geq \alpha$ , then  $\beta \in I$ .

2.4 Thus,  $I_v = \left(\gamma(v|D), +\infty\right)$ .

2.5 Then it is clear that  $I_{\lambda v} = \lambda I_v = \left(\lambda \gamma(v|D), +\infty\right) = \left(\gamma(\lambda v|D), +\infty\right)$ .

3 So  $\gamma(\bullet|D)$  is positively homogeneous.

4  $\gamma(\bullet|D)$  is subadditive.

4.1 Take some  $v, w \in V$ .

4.2 Write  $\gamma(v+w|D) = \gamma\left(\frac{2}{2}v + \frac{2}{2}w|D\right) \leq \frac{1}{2}\gamma(2v|D) + \frac{1}{2}\gamma(2w|D) = \gamma(v|D) + \gamma(w|D)$ .

□

Note: Cells and gauges produce a Functor isomorphism.

This isomorphism is between  $\text{SMN} : k\text{-VS} \rightarrow \text{ORD}$  and some absorbent disc functor, open or closed.

**GaugeContinuity** ::  $\forall V \in k\text{-TVS} . \forall D : \text{Disc} \ \& \ \text{Absorbent}(D) . \gamma(\bullet|D) \in C(V) \iff D \in \mathcal{N}_V(0)$

**Proof** =

1 This follows from seminorm continuity theorem as  $\mathbb{B}(\gamma(\bullet|D)) \subset D \subset \mathbb{D}(\gamma(\bullet|D))$ .

□

**Sublinear** ::  $\prod V : k\text{-VS} . ?(V \rightarrow \mathbb{R})$

$\phi : \text{Sublinear} \iff \phi \in \mathcal{SL}(V) \iff \forall v, w \in V . \phi(v+w) \leq \phi(v) + \phi(w) \ \& \ \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . \phi(\alpha v) = \alpha \phi(v)$

**seminormFromSublinear** ::  $\prod V : k\text{-VS} . \text{Sublinear}(V) \rightarrow \text{SMN}(V)$

**seminormFromSublinear**  $(\phi) = \nu_\phi := \Lambda v \in V . \max\left(\phi(v), \phi(-v)\right)$

1 Either  $\phi(v) \geq 0$  or  $\phi(-v) \geq 0$ .

1.1 From positive homogeneity  $\phi(0) = 0$ .

1.2 Write  $0 = \phi(0) = \phi(v-v) \leq \phi(v) + \phi(-v)$ .

2 So  $\nu_\phi$  has positive range .

3 Minkowsky Inequality holds also.

3.1  $\nu_\phi(v+w) = \max\left(\phi(v+w), \phi(-v-w)\right) \leq \max\left(\phi(v) + \phi(w), \phi(-v) + \phi(-w)\right) \leq \max\left(\phi(v), \phi(-v)\right) + \max\left(\phi(w), \phi(-w)\right) = \nu_\phi(v) + \nu_\phi(w)$ .

□



### 1.1.11 Topology of Locally Convex Space

$$\text{seminormTopology} :: \prod_{V \in k\text{-VS}} ?\text{SMN}(V) \rightarrow \text{VectorTopology}(V)$$

$$\text{seminormTopology}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) := \mathcal{W}_V(\mathcal{N}, \mathbb{R}, \text{id})$$

**HausdorffSeminormTopology** ::

$$:: \forall V \in k\text{-VS} . \forall \mathcal{N} \subset \text{SMN}(V) . \text{T2}\left(V, \mathcal{T}(\mathcal{N})\right) \iff \forall v \in \mathcal{V} . v \neq 0 \Rightarrow \exists \nu \in \mathcal{N} . \nu(v) \neq 0$$

**Proof** =

- 1 If such norm  $\nu$  exists then  $v$  can be sparated from 0 by an open set.
- 2 For topological group  $(V, +)$  this is enough.

□

**SeminormTopologyBase** ::

$$:: \forall V \in k\text{-VS} . \forall \mathcal{N} \subset \text{SMN}(V) . \text{Base}\left(V, \mathcal{T}(\mathcal{N}), \left\{ \lambda \mathbb{B}(\nu) \mid \lambda \in \mathbb{R}_{++}, \nu \in \mathcal{N} \right\}\right)$$

**Proof** =

- 1 Seems obvious by weak topology definition.

□

$$\text{SeminormTopologyIsLC} :: \forall V \in k\text{-VS} . \forall \mathcal{N} \subset \text{SMN}(V) . \left(V, \mathcal{T}(\mathcal{N})\right) \in k\text{-LCS}$$

**Proof** =

- 1 This holds as the base is convex.

□

$$\text{EveryLCSHasSeminormTopology} :: \forall V \in k\text{-LCS} . \exists \mathcal{N} \subset \text{SMN}(V) . \mathcal{T}_V = \mathcal{T}(\mathcal{N})$$

**Proof** =

- 1 As we working with froup topologies it is enough to work with zero equivalence.
- 2 Take  $U \in \mathcal{U}_V(0)$ .
- 3 Then there exists a disc  $D \subset U$ .
- 4  $\gamma(\bullet|D)$  is continuous gauge for  $V$ .
- 5 So  $U \in \mathcal{T}\left(\left\{ \gamma(\bullet|D) \right\}\right)$ .
- 6 Define  $\mathcal{N}$  to be set of all such gauges.
- 7 Then  $\mathcal{T}_V \subset \mathcal{T}(\mathcal{N})$ .
- 8 On the other hand  $\mathcal{T}(\mathcal{N}) \subset \mathcal{T}_V$  as all gauges are continuous.

□

Note: There should exists a  $k\text{-VS} \rightarrow \text{ORD}$  functor equivalence.

Take functors of saturated seminorm cones an locally convex topologies.

**Saturated** ::  $\prod_{V \in k\text{-VS}} ??\text{SMN}(k)$

$\mathcal{N} : \text{Saturated} \iff \forall \nu, \mu \in \mathcal{N} . \max(\nu, \mu) \in \mathcal{N} \iff$

**saturatedSeminormCones** :: **Covariant**( $k\text{-VS}$ , **ORD**)

**saturatedSeminormCones** ( $V$ ) = **SSC**( $V$ ) := **Saturated**( $V$ ) & **ConvexCone**( $\mathcal{SL}(V)$ )

**saturatedSeminormCones** ( $V, W, *$ ) = **SSC** <sub>$V, W$</sub> ( $T$ ) :=  $(T^*)^{-1}$

**SeminormedProductTopolgy** ::

$$\forall I \in \text{SET} . \forall V : I \rightarrow k\text{-TVS} . \forall \mathcal{N} : \prod_{i \in I} ??\text{SMN}(V) . \prod_{i \in I} (V_i, \mathcal{T}(\mathcal{N}_i)) \cong_{\text{TOP}} \left( \prod_{i \in I} V_i, \left\{ \pi_i^* \nu \mid i \in I, \nu \in \mathcal{N}_i \right\} \right)$$

**Proof** =

1 This may be seen as functorial eqiavalence interacting with limits.

2 And weak topologies are limits.

□

**LocallyConvexProduct** ::

$$\forall I \in \text{SET} . \forall V : I \rightarrow k\text{-LCS} . \prod_{i \in I} V_i \in k\text{-LCS}$$

**Proof** =

1 Now this is obvious.

□

**LocallyConvexSemimetrizability** ::

$$:: \forall V \in k\text{-LCS} . \text{Semimetrizable}(V) \iff \exists \nu : \mathbb{N} \uparrow C(V) \ \& \ \text{SMN}(V) . \mathcal{T}_V = \mathcal{T}(\text{Im } \nu)$$

**Proof** =

1( $\Rightarrow$ ) assume  $V$  is semimetrizable.

1.1 Then there exists a decreasing sequence of disked neighborhoods of unity  $D$  which generate the topology.

1.2 Then  $\gamma(\bullet|D_n)$  is clearly a sequence of seminorms we seek.

2( $\Leftarrow$ ) assume  $\nu$  are seminorms of the hypothesis.

$$2.1 \text{ Define } \mu(x) = \sum_{n=1}^{\infty} 2^{1-n} \frac{\nu_n(x)}{1 + \nu_n(x)}.$$

2.2 Then  $\mu$  is an F-seminorm.

2.2.1 Assume  $\alpha \in \mathbb{D}_k(0, 1)$  and  $v \in V$ .

$$2.2.2 \text{ Then } \frac{\nu_n(\alpha v)}{1 + \nu_n(\alpha v)} = \frac{|\alpha| \nu_n(v)}{1 + |\alpha| \nu_n(v)} \leq \frac{\nu_n(v)}{1 + \nu_n(v)} \text{ for any } n \in \mathbb{N}.$$

2.2.2.1 Note, that  $f(x) = \frac{x}{1+x}$  is increasing for  $x > 0$ .

$$2.2.2.1.1 \ f'(x) = \frac{1}{(1+x)^2} > 0.$$

2.2.2.2 And  $|\alpha| \nu_n(v) \leq \nu_n(v)$  for any  $n \in \mathbb{N}$ .

2.2.3 Thus  $\mu(\alpha v) \leq \mu(v)$ .

$$2.2.4 \text{ Also } \lim_{m \rightarrow \infty} \mu\left(\frac{v}{m}\right) = \lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} 2^{1-n} \frac{\nu_n(v/m)}{1 + \nu_n(v/m)} = \sum_{n=1}^{\infty} \lim_{m \rightarrow \infty} \frac{2^{1-n}}{m} \frac{\nu_n(v)}{1 + \nu_n(v/m)} = 0$$

by dominated convergence theorem with dominator  $x_n = 2^{2-n}$ .

2.2.5 The Minkowsky inequality for  $\mu$  is obvious from metric topology

2.3 By construction  $\mu$  is continuous in a topology defined by  $(\nu_n)_{n=1}^{\infty}$  by construction.

2.3.1  $\mu$  is a uniform limit of continuous functions.

$$2.4 \text{ Also F-seminorm } 2^{1-n} \frac{\nu_n}{\nu_n + 1} \leq \mu \text{ for each } n.$$

$$2.5 \text{ so each F-seminorm } 2^{1-n} \frac{\nu_n}{\nu_n + 1} \text{ is continuous in the topology defined by } \mu.$$

2.6 But this means that each  $\nu_n$  is also continuous in this topology .

□

**continuousDual** ::  $\prod k : \text{TopologicalField} . k\text{-TVS} \rightarrow k\text{-VS}$

$$\text{continuousDual}(V) = V' := V^* \cap \text{TOP}(V, k)$$

**DiscontinuousFunctionalExists** ::  $\forall V \in k\text{-LCS} . \forall \aleph : \text{Semimetrizable}(V) . \forall \beth : \dim V = \infty . \exists (V^* \setminus V')$

**Proof** =

1 Let  $\rho$  be a semimetric for  $V$  .

2 Then there exists an infinite linearly independent sequence  $(e_n)_{n=1}^{\infty}$ .

3 Extend  $(e_n)_{n=1}^{\infty}$  to a Hamel basis  $H$ .

4 As  $V$  is semimetrizable it is possible to select a countables decreasing base of absorbent discs  $(D_n)_{n=1}^{\infty}$ .

5 Then it is possible to select  $\lambda_n$  such that  $\lambda_n e_n \in D_n$ .

6 Obviously, then  $\lim_{n \rightarrow \infty} \lambda_n e_n = 0$ .

7 Define linear functional  $f$  by  $f(e_n) = \frac{1}{\lambda_n}$  and  $f(h) = 0$  if  $h$  is linearly independent from all  $e_n$ .

8 Then clearly  $\lim_{n \rightarrow \infty} f(\lambda_n e_n) = 1$ , so  $f$  can't be continuous.

□

**FinitieDimensionByContinuousFunctionals** ::

$$:: \forall V : \text{NormedSpace}(k) . \dim V < \infty \iff V' = V^*$$

**Proof** =

1 As  $V$  is metric and locally convex this follows from the precious result.

□

**FinestLocallyConvexSpaceIsNotMetrizizable** ::

$$:: \forall V \in k\text{-VS} . \forall \aleph : \dim V = \aleph . \neg \text{Metrizizable}(V, \mathcal{W}_V(V^*, k, \text{id}))$$

**Proof** =

1 As  $V$  is locally convex this follows from the precious result.

□

**defininigSeminorms** ::  $\prod V \in k\text{-LCS} . \text{SSC}(V)$

**definingSeminorms** () =  $\text{ssc}(V) := \text{SMN}(V) \cap \text{TOP}(V, \mathbb{R})$

**ConvergenceInLocallyConvexSpace** ::

$$:: \forall V : k\text{-LCS} . \forall (\Delta, x) : \text{Net}(V) . \forall v \in V . \lim_{\delta \in \Delta} x_\delta = v \iff \forall \nu \in \text{ssc}(V) . \lim_{\delta \in \Delta} \nu(x_\delta - v) = 0$$

**Proof** =

1 ( $\Rightarrow$ ) This is obvious as each  $\nu$  is continuous.

2 ( $\Leftarrow$ ) Assume  $D$  is an open disc in  $V$ .

2.1 as  $D$  is open disc then  $\gamma(\bullet|D) \in \text{ssc}(V)$  is continuous.

2.2 But this meand that  $\lim_{\delta \in \Delta} \gamma(x_\delta - v|D) = 0$ .

2.3 So  $x_\delta - v$  is eventually inside  $D$ .

2.4 As  $D$  was arbitraty this means that  $\lim_{\delta \in \Delta} x_\delta = v$ .

□

**CauchyPropertyInLocallyConvexSpace** ::

$$:: \forall V : k\text{-LCS} . \forall (\Delta, x) : \text{Cauchy}(V) . \forall \nu \in \text{ssc}(V) . \text{Cauchy}(V, \Delta, \nu(x))$$

**Proof** =

1 This is true as every  $\nu$  is uniformly continuous.

□

**LocallyConvexContinuityCriterion** ::

$$:: \forall V, W : k\text{-LCS} . \forall T \in k\text{-VS}(V, W) . T \in k\text{-LCS} \iff \forall \nu \in \text{ssc}(W) . \exists \mu \in \text{ssc}(V) . T^*\nu \leq \mu$$

**Proof** =

1 ( $\Rightarrow$ ) True as  $T^*\nu$  is continuous as composition and  $T^*\nu \leq T^*\nu$ .

2 ( $\Leftarrow$ ) As  $T^*\nu \leq \mu$  the seminorm  $T^*\nu$  is continuous by domination.

2.1 Then the result follows by universal property of weak topology.

□

**ContinuousIfBounded** ::

$$:: \forall V, W : \text{NormedSpace}(k) . \forall T \in k\text{-VS}(V, W) . T \in \text{TOP}(V, W) \iff T \in \mathcal{B}(V, W)$$

**Proof** =

1 Now this is obvious specification of the previous result.

□

Note: This is interesting how the fundamental theorem of elementary functional analysis can be seen as application of the universal property of weak topology.

**KernelSeparationLemma** ::  $\forall V : k\text{-VS} . \forall f \in V^* . \forall v \in V . \forall \mathfrak{U} : f(v) = 1 .$   
 $. \forall U : \text{Balanced}(V) . (v + U) \cap \ker f = \emptyset \iff \forall u \in U . |f(u)| < 1$

**Proof** =

1 ( $\Rightarrow$ ) Assume  $x + U \cap \ker f = \emptyset$  .

1.1 Assume there is  $u \in U$  such that  $|f(u)| \geq 1$  .

1.2 As  $U$  is balanced, then  $w = -\frac{u}{f(u)} \in U$ .

1.3 But  $f(v + w) = f(v) + f(w) = 1 - 1 = 0$ , a contradiction !.

2 ( $\Leftarrow$ ) Assume  $\forall u \in U . |f(u)| < 1$  is the case.

2.1  $f(v) \neq -f(u)$  for any  $u \in U$ .

2.2 So  $f(v + u) = f(v) + f(u) \neq 0$ .

□

**ContinuousByClosedKernel** ::  $\forall V \in k\text{-TVS} . \forall f \in V^* . f \in V' \iff \text{Closed}(V, \ker f)$

**Proof** =

1 ( $\Rightarrow$ ) This direction is obvious as  $k$  is Hausdorff.

2 ( $\Leftarrow$ ) Now assume  $\ker f$  is closed.

2.1 If  $f = 0$  then continuity is trivial.

2.2 So assume there is  $x$  such that  $f(x) \neq 0$ .

2.2.1 Without loss of generality assume  $f(x) = 1$ .

2.2.2 Then there is some balanced open  $U$  such that  $U_\gamma + x \cap \ker f = \emptyset$ .

2.2.3 But this means that  $\forall u \in U . |f(u)| < 1$ .

2.2.4 This means that  $\mathbb{D}(|f|) \in \mathcal{N}_V(0)$ .

2.3 So  $f$  is continuous.

□

**ContinuousByRealPart** ::  $\forall V \in \mathbb{C}\text{-TVS} . \forall f \in V^* . f \in V' \iff \Re f \in C(V)$

**Proof** =

1 write  $f(v) = \Re f(v) - i\Re f(iv)$  .

□

**ContinuousFunctionalIsOpen** ::  $\forall V \in k\text{-TVS} . \forall f \in V' . f \neq 0 \Rightarrow \text{Open}(V, k, f)$

**Proof** =

1 As  $f \neq 0$  this must be the case that  $f$  is surjective.

2 So  $f$  is open as it linear, continuous and surjective.

□

ContinuityOfMultilinearMap ::

$$:: \forall n \in \mathbb{N} . \forall V : \{1, \dots, n\} \rightarrow k\text{-LCS} . \forall W \in k\text{-LCS} . \forall A : \bigotimes_{i=1}^n V_i \rightarrow W .$$

$$. A \in k\text{-TVS} \left( \bigotimes_{i=1}^n V_i, W \right) \iff \forall \nu : \prod_{i \in I} \text{ssc}(V_i) . \forall \mu \in \text{ssc}(W) . \exists \lambda \in \mathbb{R}_{++} . A\mu \leq \lambda \prod_{i=1}^n \nu_i$$

**Proof** =

This follows from the theory of norms on tensor spaces.

□

### 1.1.12 Spaces of Continuous Functions

$\text{compactOpenTopology} :: \prod X \in \text{TOP} . \text{Topology}(\text{TOP}(X, k))$

$\text{compactOpenTopology} () = \kappa_X := \mathcal{T}\left(\{ \Lambda f \in \text{TOP}(X, k) . \sup_{x \in K} |f(x)| \mid K \in \mathbf{K}(X) \}\right)$

$\text{SpaceWithCompactOpenTopology} :: \forall X \in \text{TOP} . V = (\text{TOP}(X, k), \kappa_X) \in k\text{-LCHS}$

**Proof** =

- 1 Topology on  $V$  is generated by seminorms, so  $V$  is locally convex.
  - 2 As sets  $\{x\}$  are dcompact, the evaluation seminorm  $\epsilon_x : f \mapsto |f(x)|$  is continuous for  $V$ .
  - 3 If  $f \neq 0$  then there is some  $x \in X$  such that  $f(x) \neq 0$ .
  - 4 So  $\epsilon_x(f) \neq 0$  and this means that  $V$  is Hausdorff.
- 

$\text{Hemicompact} :: ?\text{TOP}$

$X : \text{Hemicompact} \iff \exists \mathcal{C} : \text{Countable}(\mathbf{K}(X)) . \forall K \in \mathbf{K}(X) . \exists F \in \mathcal{C} . K \subset F$

$\text{CompactOpenTopologyMetrization} :: \forall X \in \text{T3.5} . \text{Hemicompact}(X) \iff \text{Metrizible}(\text{TOP}(X, k), \kappa_X)$

**Proof** =

- 1 ( $\Rightarrow$ ) Assume  $X$  is hemicompact.
    - 1.1 Then let  $F$  be an enumeration of the set  $\mathcal{C}$  from the definition of hemicompact.
    - 1.2 Without loss of generality we may assume that  $F$  is increasing.
    - 1.3 Then  $\nu_n(f) = \sup_{x \in F_n} |f(x)|$  is an increasing family of seminorms.
    - 1.4 By hemicompactness  $\nu_n$  defines  $\kappa_X$ .
    - 1.5 So the  $\kappa_X$  is metrizable.
  - 2 ( $\Leftarrow$ ) now assume  $\kappa_X$  is metrizable.
    - 2.1 Then there is a countable base defined by sup-functionals for some compacts  $F_n$ .
    - 2.2 Then for any compact  $K$  its sup-functional is less then a scalar multiple of a sup-functional of some  $F_n$ .
    - 2.3 Assume This is the case, but  $K \not\subset F_n$ .
    - 2.4 Then there is some  $x \in K \setminus F_n$ .
    - 2.5 Also there is some  $f \in \text{TOP}(X, k)$  such that  $f(x) = 1$  and  $f(F_n) = \{0\}$ .
      - 2.5.1 This is true as  $X$  is Tychonoff and Hausdorff.
    - 2.6 Then  $\sup_{x \in K} |f(x)| \geq \sup_{x \in F_n} |f(x)|$  which is a contradiction.
    - 2.7 So  $X$  must be hemicompact.
- 

$\text{KRSpace} :: \text{TOP} \rightarrow ?\text{TOP}$

$X : \text{KRSpace} \iff \Lambda Y \in \text{TOP} \forall f : X \rightarrow Y . \left( \forall K \in \mathbf{K}(X) . f|_K \in \text{TOP}(K, Y) \right) \Rightarrow f \in \text{TOP}(X, Y)$

$$\text{CompactOpenTopologyCompleteness} :: \forall X : \text{T3.5} . \text{KRSpace}(k, X) \iff \text{Complete}(\text{TOP}(X, k), \kappa_X)$$

**Proof** =

- 1 ( $\Rightarrow$ ): Assume  $X$  is a KRSpaces for  $k$ .
  - 1.1 Take  $f$  to be a Cauchy sequence for  $\kappa_X$ .
  - 1.2 Then  $f(x)$  is also Cauchy as  $\{x\}$  is compact for any  $x \in X$ .
  - 1.3 Thus, as  $k$  is complete  $F = \lim_{n \rightarrow \infty} f_n$  exists.
  - 1.4 On every compact  $K$  the convergence of  $f|_K$  towards  $F|_K$  is uniform so  $F|_K$  is continuous.
  - 1.5 But as  $X$  is KRSpace the whole  $F$  must be continuous.
  - 1.6 So  $\kappa_X$  is complete.
  - 2 ( $\Leftarrow$ ): Now assume that  $\kappa_X$  is complete.
  - 2.1 Take some  $f : X \rightarrow k$  such that  $f|_K$  is continuous for any compact  $K$ .
  - 2.2 Then by Tietze extension theorem  $f|_K$  can be extended to a continuous function  $F_K : \beta X \rightarrow k$ .
  - 2.3 By properties of Tietze-Urysohn extension we may assume that  $\sup F_K = \sup f|_K$ .
  - 2.4 Define  $g_K = F_K|_X$ .
  - 2.5 The set  $\mathbf{K}(X)$  is directed.
  - 2.6 Then  $g_K$  is a Cauchy net.
  - 2.6.1 Take  $K$  be a compact in  $X$  and let  $\nu_K(f) = \sup_{x \in K} |f|$ .
  - 2.6.2 Then  $\nu_K(g_L - g_H) = 0$  for any  $L, H \in \mathbf{K}(X)$  such that  $K \subset L$  and  $K \subset H$ .
  - 2.6.3 So  $g_L - g_H \in \mathbb{B}(\nu_K)$  in this case.
  - 2.7 Thus there exists a continuous limit  $G$  for  $\kappa_X$ .
  - 2.8 But  $G = f$ .
  - 2.8.1 If  $x \in X$  then  $g_K(x) = f(x)$  for any  $K \in \mathbf{K}(X)$  such that  $x \in K$ .
  - 2.9 Thus  $f$  is continuous.
- 

$$\text{pointwiseConvergenceTopology} :: \prod X \in \text{TOP} . \text{Topology}(\text{TOP}(X, k))$$

$$\text{pointwiseConvergenceTopology} () = \pi_X := \mathcal{T}(\{\Lambda f \in \text{TOP}(X, k) . |f(x)| \mid x \in X\})$$

$$\text{SpaceWithPointwiseConvergenceTopology} :: \forall X \in \text{TOP} . V = (\text{TOP}(X, k), \kappa_X) \in k\text{-LCHS}$$

**Proof** =

- 1 Topology on  $V$  is generated by seminorms, so  $V$  is locally convex.
  - 2 If  $f \neq 0$  then there is some  $x \in X$  such that  $f(x) \neq 0$ .
  - 3 So  $\epsilon_x(f) \neq 0$  and this means that  $V$  is Hausdorff.
- 

**PointwiseConvergence** ::

$$:: \forall X \in \text{TOP} . \forall (\Delta, f) : \text{Net}(\text{TOP}(X, k)) . \forall g \in \text{TOP}(X, k) . \lim_{\delta \in \Delta} f_\delta =_{\pi_X} g \iff \forall x \in X . \lim_{\delta \in \Delta} f_\delta(x) = g(x)$$

**Proof** =

...

□



**Equicontinuous** ::  $\prod X \in \text{TOP} . \prod G \in \text{TGRP} . ??\text{TOP}(X, G)$

$\mathcal{F} : \text{Equicontinuous} \iff \forall x \in X . \forall V \in \mathcal{U}_G(e) . \exists U \in \mathcal{U}_X(x) . \forall f \in \mathcal{F} . f(U) \subset f(x)V$

**Equibounded** ::  $\prod X \in \text{TOP} . ??\text{TOP}(X, k)$

$\mathcal{F} : \text{Equibounded} \iff \forall x \in X . \exists \beta \in \mathbb{R}_{++} . \forall f \in \mathcal{F} . |f(x)| \leq \beta$

**EquicontinuousTopologyEquality** ::  $\forall X \in \text{TOP} . \forall \mathcal{F} : \text{Equicontinuous}(X, k) . (\mathcal{F}, \kappa_X) = (\mathcal{F}, \pi_X)$

**Proof** =

1 Firstly,  $\kappa_X \subset$ .

1.1 Take  $g \in \mathcal{F}$ .

1.2 Assume  $U \in \kappa_X(g)$  has form  $U = \left\{ f \in \text{TOP}(X, k) : \sup_{x \in K} |f(x) - g(x)| < \alpha \right\}$

for some compact  $K$  and  $\alpha \in \mathbb{R}_{++}$ .

1.3 Then for each  $x \in K$  there is some  $W_x \in \mathcal{U}_X(x)$  such that  $f(W_x) \subset f(x) + \mathbb{B}_k(0, \alpha/4)$  for each  $f \in \mathcal{F}$ .

1.4 As  $K$  is compact and  $W$  is an open cover we can select a finite family of points  $(x_i)_{i=1}^n$

such that  $K \subset \bigcup_{i=1}^n W_{x_i}$ .

1.5 Let  $\epsilon_y$  stand for evaluation seminorm  $\epsilon_y(f) = |f(y)|$ .

1.6 Then  $V = \bigcap_{i=1}^n \frac{\alpha}{2} \mathbb{B}(\epsilon_{x_i}) + g \in \pi_X$  and  $V \subset U$  in  $\mathcal{F}$ .

1.6.1 Take some  $f \in V \cap \mathcal{F}$  and some  $y \in K$ .

1.6.2 Then there is some  $i \in \{1, \dots, n\}$  such that  $y \in W_{x_i}$ .

1.6.3  $|f(y) - g(y)| \leq |f(y) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(y)| < \alpha$ .

1.6.4 So  $\sup_K |f - g| < \alpha$ .

1.7 This means that  $U$  is open in  $\pi_X$ .

2 This is obvious from definition that  $\pi_X \subset \kappa_X$  and  $\pi_X = \kappa_X$ .

□

**PointwiseClosureEquicontinuous** ::

$:: \forall X \in \text{TOP} . \forall \mathcal{F} : \text{Equicontinuous}(X, k) . \text{Equicontinuous}\left(X, k, \text{cl}_{\pi_X} \mathcal{F}\right)$

**Proof** =

1 Take  $x \in X$  and  $V \in \mathcal{U}_k(0)$ .

2 Then by equicontinuity there is  $U \in \mathcal{U}_X(x)$  such that  $f(U) \subset f(x) + V$  for any  $f \in \mathcal{F}$ .

3 Take  $g$  to be a limit point in  $\mathcal{F}$ .

4 Then there is sequence  $f$  such that  $\lim_{n \rightarrow \infty} f_n = g$  pointwise.

5 Take some  $u \in U$ .

6 Then  $g(u) = \lim_{n \rightarrow \infty} f_n(u)$ .

7 Then  $|g(u) - g(x)| \leq |g(u) - f_n(u)| + |f_n(u) - f_n(x)| + |f_n(x) - g(x)| \leq 3\varepsilon$  for suitably choosen  $n$ .

8 So  $\text{cl}_{\pi_X} \mathcal{F}$  is equicontinuous.

□

ArzeloAscoli1 ::

$:: \forall X \in \text{TOP} . \forall \mathcal{F} : \text{Equicontinuous}(X, k) \ \& \ \text{Equibounded}(X) \ \& \ \text{Closed}(\text{TOP}(X, k), \kappa_X, \mathcal{F}) .$   
 $. \text{CompactSubset}(\text{TOP}(X, k), \pi_X, \mathcal{F})$

Proof =

1 Each  $\mathcal{F}(x)$  is a compact subset of  $k$  by Heine-Borel Lemma.

2 So by Tychonoff theorem  $\prod \mathcal{F}(x)$  is compact in the product topology.

3 But  $\mathcal{F}$  is a closed subset of  $\prod \mathcal{F}(x)$  in  $\pi_X$ , so  $\mathcal{F}$  is also compact in  $\pi_X$ .

4 As  $\mathcal{F}$  is equicontinuous  $\pi_X$  is equal to  $\kappa_X$  on  $\mathcal{F}$ , so  $\mathcal{F}$  is also compact in  $\kappa_X$ .

□

ArzeloAscoli2 ::

$:: \forall X : \text{LocallyCompact} . \forall \mathcal{F} : \text{CompactSubset}(\text{TOP}(X, k), \kappa_X, \mathcal{F}) .$   
 $. \text{Equicontinuous}(X, k, \mathcal{F}) \ \& \ \text{Equibounded}(X, \mathcal{F}) \ \& \ \text{Closed}(\text{TOP}(X, k), \pi_X, \mathcal{F})$

Proof =

...

□

### 1.1.13 Constructions

**SubspaceQuotientSeminorm** ::

$$:: \forall V \in k\text{-LCS} . \forall U \subset_{k\text{-VS}} V . \mathcal{T} \left( \frac{V}{U} \right) = \mathcal{T} \left( \left\{ \Lambda[v] \in \frac{V}{U} . \inf_{u \in U} \nu(v+u) \mid \nu \in \text{ssc}(V) \right\} \right)$$

**Proof** =

- 1 Let  $\nu \in \text{ssc}(V)$ .
- 2 define  $\mu = \Lambda[v] \in \frac{V}{U} . \inf_{u \in U} \nu(v+u)$ .
- 3 Then  $\mu$  is a seminorm.
- 3.1  $[v] = 0$  imply  $v \in U$ .
- 3.2 So  $\mu = 0$  as  $\nu(w) \geq 0$  and  $\nu = 0$ .
- 3.3 Take  $[v] \in \frac{V}{U}$  and  $\alpha \in k$ .
- 3.4 Then  $\mu[\alpha v] = \inf_{u \in U} \nu(\alpha v + u) = \inf_{u \in U} \nu(\alpha v + \alpha u) = |\alpha| \inf_{u \in U} \nu(v+u) = |\alpha| \mu[v]$ .
- 3.5 Now take  $v, w \in V$ .
- 3.6 Then  $\mu[v+w] = \inf_{u \in U} \nu(v+w+u) = \inf_{u, o \in U} \nu(v+w+u+o) \leq \inf_{u, o \in U} \nu(v+u) + \nu(w+o) =$   
 $= \inf_{u \in U} \nu(v+u) + \inf_{o \in U} \nu(v+o) \mu[v] + \mu[w]$ .
- 4 Then  $\mathbb{B}(\mu) = \pi_U \mathbb{B}(\nu)$ .
- 5 As open cells as above form a base of topology on  $V$ ,  
 and quotion topology is an image topology, the result follows.

□

**LocallyConvexQuotient** ::  $\forall V \in k\text{-LCS} . \forall U \subset_{k\text{-VS}} V . \forall \frac{V}{U} \in k\text{-LCS}$

**Proof** =

- 1 This is True as topology on  $\frac{V}{U}$  is generated by seminorms.

□

**kernelOfSeminorm** ::  $\prod_{V \in k\text{-VS}} \text{SMN}(V) \rightarrow \text{VectorSubspace}(V)$

**kernelOfSeminorm**( $\nu$ ) =  $\ker \nu := \nu^{-1}\{0\}$

**SeminormedCompletion** ::  $\forall V : \text{SeminormedSpace}(k) . \exists (\hat{V}, \iota) : \text{TVSCompletion}(V) . \text{SMS}(k, \hat{V})$

**Proof** =

- 1 Take  $[v] \in \hat{V}$ .
- 2 Then  $[v]$  can associated with Cauchy sequence  $v$ .
- 3 Define  $\nu_{\hat{V}}[v] = \lim_{n \rightarrow \infty} \nu_V(v_n)$ .
- 3.1 As  $\nu_V$  is uniformly continuous the  $\nu_V(v_n)$  must be again Cauchy, and hence convergent as  $k$  is complete.
- 3.2 Use completion metric argument to see that  $\nu_{\hat{V}}$  is *Uniquelydetermined*.
- 3.2.1 Assume  $x$  and  $y$  are both Cauchy sequences for  $[v]$ .
- 3.2.2 Then  $\lim_{n \rightarrow \infty} |\nu_V(x_n) - \nu_V(y_n)| \leq \lim_{n \rightarrow \infty} \nu_V(x_n - y_n) = \lim_{n \rightarrow \infty} \rho_V(x_n, y_n) = 0$ .

□

**SeminormedSpaceProductEmbedding** ::  $\forall V \in k\text{-LCS} . \exists I \in \text{SET} . \exists W : I \rightarrow \text{SeminormedSpace} .$

$$. \exists U \subset_{k\text{-VS}} \prod_{i \in I} W_i . V \cong_{k\text{-TVS}} W$$

**Proof** =

- 1 For  $\nu \in \text{ssc}(V)$  define  $W = (V, \nu)$ .
  - 2 Then the mapping  $x \mapsto (x)_{\nu \in \text{ssc}(V)}$  is an isomorphism.
- 

**BanachSpaceProductEmbedding** ::  $\forall V \in k\text{-LCHS} . \exists I \in \text{SET} . \exists W : I \rightarrow \text{BAN}(k) .$

$$. \exists U \subset_{k\text{-VS}} \prod_{i \in I} W_i . V \cong_{k\text{-TVS}} W$$

**Proof** =

- 1 For  $\nu \in \text{ssc}(V)$  define  $W = \widehat{\left( \frac{V}{\ker \nu} \right)}$ .
  - 2 Then each  $W_\nu$  is an Banach space.
  - 3 Then the mapping  $\phi : x \mapsto ([x]_{\ker \nu})_{\nu \in \text{ssc}(V)}$  is an isomorphism.
  - 3.1  $\phi$  is one-to-one as  $V$  is hausdorff.
  - 3.1.1 For any  $v \in V$  such that  $v \neq 0$  exists  $\nu \in \text{ssc}(V)$  such that  $\nu(v) \neq 0$ .
  - 3.1.2 So  $[v]_{\ker \nu} \neq 0$ .
- 

**LCSCompletion** ::  $\forall V \in k\text{-LCS} . \exists (\hat{V}, \iota) : \text{TVSCompletion}(V) . \hat{V} \in k\text{-LCS}$

**Proof** =

- 1 Construct product emedding  $\phi : V \hookrightarrow \prod_{\nu \in \text{ssc}(V)} W_\nu$  as in the previous theorem.
  - 3 This embedding can be extended to the embedding into a complete vecor space  $\prod_{\nu \in \text{ssc}(V)} \hat{W}_\nu$ .
  - 3.1 The product of complete spaces is complete.
  - 4 Then  $\text{cl}_{\hat{W}} \phi(V)$  is a closed subset of the complete space.
  - 5 So  $\hat{V} = \text{cl}_{\hat{W}} \phi(V)$  is the sought completion.
- 

**LCHSCompletion** ::  $\forall V \in k\text{-LCHS} . \exists (\hat{V}, \iota) : \text{TVSCompletion}(V) . \hat{V} \in k\text{-LCHS}$

**Proof** =

- 1 Same argument as above.
-

### 1.1.14 Non-Archimedean Spaces

$k : \text{UltravaluedField};$

$\text{Ultraseminorm} :: \prod_{V \in k\text{-VS}} ?\text{SMN}(V)$

$\nu : \text{Ultraseminorm} \iff \forall v, w \in V . \nu(v + w) \leq \max(\nu(v), \nu(w))$

$\text{UltraseminormMaximumPrinciple} ::$

$:: \forall V \in k\text{-VS} . \forall v, w \in V . \forall \nu : \text{Ultraseminorm}(V) . \nu(v) < \nu(w) \Rightarrow \nu(v + w) = \nu(w)$

$\text{Proof} =$

1  $\nu(w + v) \leq \max(\nu(w), \nu(v)) = \nu(w)$  .

2  $\nu(w) = \nu(v - (w + v)) \leq \max(\nu(v), \nu(w + v)) = \nu(w + v)$ .

2.1 This must be the case as  $\nu(v) < \nu(w)$ .

3  $\nu(w) = \nu(w + v)$  .

□

$\text{Ultradisc} ::$

$:: \forall V \in k\text{-VS} . \forall \nu : \text{Ultraseminorm}(V) . \text{AbsolutelyKConvex} \ \& \ \text{Absorbent}(V, \mathbb{B}(\nu))$

$\text{Proof} =$

1 Assume  $v, w \in \mathbb{B}(\nu)$  and  $\alpha, \beta \in \mathbb{D}_k(0, 1)$ .

2 Then  $\nu(\alpha v + \beta w) \leq |\alpha|\nu(v) + |\beta|\nu(w) < 1$  .

3 So  $\mathbb{B}(\nu)$  is K-convex.

4 Take  $v \in V$  such that  $\nu(v) \neq 0$ .

5 Then  $\alpha v \in \mathbb{B}(\nu)$  for any  $\alpha \in k$  such that  $|\alpha| < \nu^{-1}(v)$ .

6 So  $\mathbb{B}(\nu)$  is absorbent.

□

$\text{ultragauge} :: \prod_{V \in k\text{-VS}} \text{AbsolutelyKConvex} \ \& \ \text{Absorbent}(V) \rightarrow \text{Ultraseminorm}(V)$

$\text{ultragauge}(D) = v(\bullet|D) := \lambda v \in V . \inf \left\{ |\alpha| \mid \alpha \in k : v \in \alpha D \right\}$

1 It is obvious that the ultragauge is a seminorm.

2 Now take  $v, w \in V$ .

3 Then as  $D$  is K-convex  $v(v + w|D) \leq \max(v(v|D), v(w|D))$ .

3.1 Take a sequence  $\alpha, \beta : \mathbb{N} \rightarrow k_*$  such that  $\alpha_n v \in D, \beta_n w \in D, \lim_{n \rightarrow \infty} |\alpha_n|^{-1} = v(v|D), \lim_{n \rightarrow \infty} |\beta_n|^{-1} = v(w|D)$ .

3.2 Define  $\gamma_n = \arg \max_{\tau \in \{\alpha_n, \beta_n\}} |\tau|$  .

3.3 Then  $\gamma_n(v + w) \in D$  as  $D$  is K-Convex.

3.4 Then  $v(v + w|D) \leq |\gamma_n| \leq \max(|\alpha_n|, |\beta_n|)$  .

3.5 Taking limits gives  $v(v + w|D) \leq \max(v(v|D), v(w|D))$ .

□

**UltragaugBound** ::

$$:: \forall V \in k\text{-VS} . \forall D : \text{AbsolutelyKConvex} \ \& \ \text{Absorbent}(V) . \mathbb{B}(v(\bullet|D)) \subset D \subset \mathbb{D}(v(\bullet|D))$$

**Proof** =

Pretty obvious.

□

**UltragaugContinuity** ::

$$:: \forall V \in k\text{-TVS} . \forall D : \text{AbsolutelyKConvex} \ \& \ \text{Absorbent}(V) . D \in \mathcal{N}_V \iff v(\bullet|D) \in C(V)$$

**Proof** =

1 ( $\Rightarrow$ ) Assume  $D$  has non-empty interior.

1.1 By previous result this implies that  $D$  is open.

$$1.2 \text{ Then } v^{-1}([0, \rho), D) = \bigcup_{\alpha \in \mathbb{D}(0, \rho)} \alpha D.$$

1.3 But  $\alpha D$  is also open as multiplication by  $\alpha$  is a homeomorphism.

1.4 So the ultragaugage must be continuous.

2 ( $\Leftarrow$ ) Assume that ultragaugage is continuous.

$$2.1 \text{ Then } v^{-1}([0, \rho), D) \subset D.$$

2.2 So  $D$  has non-empty interior.

□

$$\text{topologyOfUltraseminorms} :: \prod_{V \in k\text{-VS}} ?\text{Ultraseminorm}(V) \rightarrow \text{VectorTopology}(V)$$

$$\text{topologyOfUltraseminorms}(\Upsilon) = \mathcal{T}(\Upsilon) := \mathcal{W}_V(\Upsilon, \mathbb{R}, \text{id})$$

**UltraseminormsDefineLocallyKConvexTopology** ::

$$:: \forall V \in k\text{-VS} . \forall \Upsilon : ?\text{Ultraseminorm}(V) . \text{LocallyKConvexSpace}(k, V, \mathcal{T}(\Upsilon))$$

**Proof** =

1 Take  $v \in \Upsilon$ .

2 Then  $\mathbb{B}(v)$  is absolutely K-convex.

2.1 See ultradisc theorem.

□

**LocallyKConvexTopologyIsGeneratedByUltraseminorms** ::

$$:: \forall V : \text{LocallyKConvexSpace}(k) . \exists \Upsilon : ?\text{Ultraseminorm}(V) . \mathcal{T}_V = \mathcal{T}(\Upsilon)$$

**Proof** =

Take ultragauges for the K-discs generating the locally K-convex topology.

□

$$\text{definingUltraseminorms} :: \prod V : \text{LocallyKConvexSpace}(k) . ?\text{Ultraseminorm}(V)$$

$$\text{definingUltraseminorms}(V) = \text{suc} := C(V) \cap \text{Ultraseminorm}(V)$$

**Ultrasemimetrization** ::

$$\begin{aligned} &:: \forall V \in \text{LocallyKConvexSpace}(k) . \text{Ultrasemimetrizable}(V) \iff \\ &\iff \exists v : \mathbb{N} \uparrow \text{Ultraseminorm}(V) . \mathcal{T}_V = \mathcal{T}(\text{Im } v) \end{aligned}$$

**Proof** =

1 This is simmlar to normal semimetrization theorem .

$$2 \text{ Define an F-seminorm } \mu(v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{v_n(v)}{1 + v_n(v)}.$$

3 The only difference is in the proving the ultrametric property.

3.1 Take some  $v, w \in V$ .

$$3.2 \text{ Then } v_n(v + w) \leq \max(v_n(v), v_n(w)).$$

$$3.3 \text{ But as th function } \frac{x}{x+1} \text{ is increasing } \frac{v_n(v+w)}{1+v_n(v+w)} \leq \max\left(\frac{v_n(v)}{1+v_n(v)}, \frac{v_n(w)}{1+v_n(w)}\right).$$

$$4 \text{ Thus } \mu(v+w) \leq \max(\mu(v), \mu(w)) \text{ for any } v, w \in V.$$

5 So  $\mu$  defines an ultrasemimetric.

□

**LocallyCCompact** :: ? $k$ -TVS

$$V : \text{LocallyCCompact} \iff \exists \mathcal{F} : \text{Filterbase}(\mathcal{N}_0(V)) . \forall F \in \mathcal{F} . \text{CCompact} \ \& \ \text{AbsolutelyKConvex}(V, F)$$

$$\text{Ultrnorm} :: \prod_{V \in k\text{-VS}} ?\text{Ultraseminorm}(V)$$

$$v : \text{Ultrnorm} \iff \forall v \in V . v(v) = 0 \iff v = 0$$

$$\text{UltrnormedSpace} :: ? \sum_{V \in k\text{-TVS}} \text{Ultraseminorm}(V)$$

$$(V, v) : \text{UltrnormedSpace} \iff \mathcal{T}_V = \mathcal{T}\{v\}$$

$$|\cdot|_k \neq \Lambda \alpha \in k . [k \neq 0]$$

**LocallyCCompactHasLocllyCCompactField** ::  $\forall V : \text{LocallyCCompact}(k) . \dim V > 0 \Rightarrow \text{LocallyCCompact}(k,$

**Proof** =

1 As  $k$  has non-trivial valuation Every one-dimensional subspace of  $V$  is isomorphic to  $k$ .

2 Let  $L$  be such one-dimensional subspace.

3 And let  $C$  be a C-compact neighborhood of 0 in  $V$ .

4 The  $C \cap L$  is C-compact and and reltively open in  $L$ .

5 So  $L$  is locally compact.

6 And so is  $k$  as it is isomorphic to  $L$ .

□

**LocallyCCompactIsFinDim** ::  $\forall V : \text{Ultrarnormed} \ \& \ \text{LocallyCCompact}(k) . \dim V < \infty$

**Proof** =

- 1 Assume that  $\dim V = \infty$ .
- 2 As  $V$  is Locally C-compact there is a C-compact neighborhood  $C$  of 0 in  $V$ .
- 3 Then there is a ball  $D \subset C$  if radius  $\rho$ .
- 4 Select a topologically linearly independent sequence  $(e_i)_{i=1}^\infty$  such that  $\|e_i\| = \rho$ .
- 5 Define  $F_i = \text{cl}_V K\text{-conv} (e_j)_{j=i}^\infty$ .
- 6 Then  $(F_i)_{i=1}^\infty$  is a closed k-convex filterbase on  $C$ .
- 6.1 The K-convex filterbase property is obvious.
- 6.2 As all points  $e_i, e_j$  are separated, sets  $(e_j)_{j=i}^\infty$  are closed.
- 6.3 And k-convex hull of closed sets must be closed.
- 7 This mean that  $\bigcap_{i=1}^\infty F_i \neq \emptyset$  as  $C$  is C-convex.
- 8 On the other hand, clearly  $\bigcap_{i=1}^\infty F_i = \emptyset$ , a contradiction!
- 8.1 Assume  $v \in \bigcap_{i=1}^\infty F_i = \emptyset$ .
- 8.2 Then  $v \in \text{span}(e_i)_{i=1}^\infty$  by construction.
- 8.3 But as  $v \in F_{i=1}$  it means that its  $e_i$  coefficient must be 0.
- 8.4 So it must be the case that  $v = 0$ .
- 8.5 But 0 do not belong to any  $F_i$ .

□

**LocallyCCompactIsCCompact** ::  $\forall V : \text{LocallyCCompact} \ \& \ \text{LocallyKConvexSpace}(k) . \text{CCompact}(V)$

**Proof** =

- 1  $k$  is C-compact.
- 1.1 Let  $\mathcal{F}$  be a K-convex Filter on  $k$ .
- 1.2 Then  $\mathcal{F}$  can be structured as a monotonic sequence of balls.
- 1.3 If  $\mathcal{F}$  there is a ball  $D$  such tha all small enough elements  $F \in \mathcal{F}$  are in  $D$ .
- 1.4 but all closed discs are isomorphic in  $k$ .
- 1.5 Thus  $D$  is C-compact.
- 1.6 So  $\mathcal{F}$  must have an adherence point in  $D$ .
- 1.7 So it also has an adherence point in  $k$ , and  $k$  is C-compact.
- 2 Then  $V \cong k^n$  as  $V$  must be finite-dimensional.
- 3 And  $k^n$  is C-compact as a product of C-compact sets.

□



## 1.2 Towards Bornology

### 1.2.1 Bounded Sets

$k : \text{AbsoluteValueField}(\mathbb{R}) \Big| \text{UltravaluedField};$

**Bounded** ::  $\prod V \in k\text{-TVS} . ??V$

$B : \text{Bounded} \iff \forall U \in \mathcal{U}_V(0) . \exists \lambda \in \mathbb{R} . \forall \alpha \in k . |\alpha| \geq \lambda \Rightarrow B \subset \alpha U$

**BoundedByBase** ::

$:: \forall V \in k\text{-TVS} . \forall B \subset V . \forall \beta : \text{BalancedBaseBase}(V) . \text{Bounded}(V) \iff \forall U \in \mathcal{B} . \exists \alpha \in k . \alpha B \subset U$

**Proof** =

Obvious.

□

**BoundedBySeminorms** ::

$:: \forall V \in k\text{-LCS} . \forall B \subset V . \forall \beta : \text{BalancedBaseBase}(V) . \text{Bounded}(V) \iff \forall \nu \in \text{ssc} . \text{Bounded}(B, \nu|_B)$

**Proof** =

Obvious.

□

**TotallyBoundedIsBounded** ::  $\forall V \in k\text{-TVS} . \forall B : \text{TotallyBounded}(V) . \text{Bounded}(V, B)$

**Proof** =

1 Assume  $U \in \mathcal{U}_V(0)$ .

2 Then there exists a balanced and absorbing  $W \in \mathcal{U}_V(0)$  such that  $W + W \subset U$ .

3 As  $B$  is totally bounded there is a finite subset  $F \subset V$  such that  $B \subset W + F$ .

4 As  $W$  is absorbing there exists  $\alpha \in k$  such that  $F \subset \alpha W$ .

5 Without loss of generality we may assume that  $|\alpha| > 1$ .

5 So, as  $W$  is balanced  $W \subset \alpha W$ .

6 Thus,  $B \subset \alpha W + \alpha W = \alpha(W + W) \subset \alpha U$ .

□

**KolomogorovsBoundednessCriterion** ::

$:: \forall V \in k\text{-TVS} . \forall B \subset V . \text{Bounded}(V, B) \iff \forall \alpha : \mathbb{N} \rightarrow k . \forall b : \mathbb{N} \rightarrow B . \left( \lim_{n \rightarrow \infty} \alpha_n = 0 \Rightarrow \lim_{n \rightarrow \infty} \alpha_n b_n = 0 \right)$

**Proof** =

1 ( $\Rightarrow$ ) This direction is obvious.

2 ( $\Leftarrow$ ) Assume  $B$  is not bounded.

2.1 Then there is an  $U \in \mathcal{U}_V(0)$  such that for any  $\rho \in \mathbb{R}_{++}$  there is  $\alpha \in k$  such that  $|\alpha| \geq \rho$  and  $U \not\subset \alpha B$ .

2.2 So there exists sequences  $\alpha$  with  $|\alpha_n| \leq \frac{1}{n}$  and  $b : \mathbb{N} \rightarrow B$  such that  $\alpha_n b_n \notin U$ .

2.3  $|\alpha_n| \leq \frac{1}{n}$  imply that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ .

2.4 On the other hand  $\alpha_n b_n \notin U$  imply that  $\lim_{n \rightarrow \infty} \alpha_n b_n \neq 0$ .

2.5 This contradicts an initial assumption.

□

**BoundednesByCountableSubsets** ::  $\forall V \in k\text{-TVS} . \forall B \subset V . \text{Bounded}(V, B) \iff \forall C : \text{CountableSubset}(V, B)$

**Proof** =

This follows from Kolmogorov's criterion.

□

**BoundedMetrizationTHM** ::  $\forall V \in k\text{-TVS} . \forall N \in \mathcal{N}_V(0) . \text{Bounded}(V, N) \Rightarrow \text{Semimetrizable}(V)$

**Proof** =

$\left(\frac{1}{n}N\right)_{n=1}^{\infty}$  is a countable base of vector topology for  $V$ .

□

**BoundedNormizationTHM** ::  $\forall V \in k\text{-TVS} . \forall N \in \mathcal{N}_V(0) . \text{Bounded} \ \& \ \text{Disc}(V, N) \Rightarrow \text{Seminormable}(V)$

**Proof** =

Topology may be determined by  $\gamma(\bullet|N)$ .

□

### 1.2.2 Stability under Operations

**SubsetOfBounded** ::  $\forall V \in k\text{-TVS} . \forall B : \text{Bounded}(V) . \forall C \subset B . \text{Bounded}(V, B)$

**Proof** =

Obvious.

□

**BoundedUnion** ::  $\forall V \in k\text{-TVS} . \forall B, C : \text{Bounded}(V) . \text{Bounded}(V, B \cap C)$

**Proof** =

Select max.

□

**BoundedScale** ::  $\forall V \in k\text{-TVS} . \forall B : \text{Bounded}(V) . \forall \alpha \in k . \text{Bounded}(V, \alpha B)$

**Proof** =

Rescale.

□

**BoundedSum** ::  $\forall V \in k\text{-TVS} . \forall B, C : \text{Bounded}(V) . \forall \alpha \in k . \text{Bounded}(V, B + C)$

**Proof** =

Assume  $U \in \mathcal{U}_V(0)$ .

Select  $V \in \mathcal{U}_V(0)$  such that  $V + V \subset U$ .

Then there are two  $V$ -absorbtion factors  $\rho$  and  $\sigma$  for  $B$  and  $C$  respectively.

If  $\alpha \in k$  is such that  $|\alpha| \geq \max(\rho, \sigma)$ , then  $B + C \subset \alpha V + \alpha V = \alpha(V + V) \subset \alpha U$ .

□

**BoundedQuotient** ::  $\forall V \in k\text{-TVS} . \forall W \subset_{k\text{-VS}} V . \forall B : \text{Bounded}(V) . \forall \text{Bounded} \left( \frac{V}{W}, \pi_W(B) \right)$

**Proof** =

Use the preimage to determine the absorbtion factor.

□

**BoundedProducts** ::  $\forall I \in \text{SET} . \forall V : I \rightarrow k\text{-TVS} . \forall B : \prod_{i \in I} \text{Bounded}(V_i) . \text{Bounded} \left( \prod_{i \in I} V_i, \prod_{i \in I} B_i \right)$

**Proof** =

Assume  $U \in \mathcal{U}_{\prod_{i \in I} V_i}(0)$ .

Then there exists  $W \in \prod_{i \in I} \mathcal{T}(V_i)$  such that that  $W_i \neq V_i$  only for a finite set of indices  $J \subset I$  and  $\prod_{i \in I} W_i \subset U$ .

Then find a  $W_i$ -absorbtion factor  $\rho_i$  for each  $i \in J$ .

Then  $\prod_{i \in I} B_i \subset \alpha \prod_{i \in I} W_i \subset \alpha U$  for any  $\alpha \in k$  with  $|\alpha| \geq \max_{i \in J} \rho_i$ .

□

**BoundedClosure** ::  $\forall V \in k\text{-TVS} . \forall B : \text{Bounded}(V) . \text{Bounded}(V, \overline{B})$

**Proof** =

$\overline{B}$  is bounded for the base of closed neighborhoods of unity.

Thus,  $\overline{B}$  is bounded in a general sence.

□

**BoundedBalancedHull** ::  $\forall V \in k\text{-TVS} . \forall B : \text{Bounded}(V) . \text{Bounded}(V, \text{bal } B)$

**Proof** =

$\text{bal } B$  is bounded for the base of balanced neighborhoods of unity.

Thus,  $\overline{B}$  is bounded in a general sence.

□

**BoundedConvexHull** ::  $\forall V \in k\text{-LCS} . \forall B : \text{Bounded}(V) . \text{Bounded}(V, \text{conv } B)$

**Proof** =

$\text{conv } B$  is bounded for the base of disced neighborhoods of unity.

Thus,  $\overline{B}$  is bounded in a general sence.

□

**BoundedBase** ::  $\prod_{V \in k\text{-TVS}} \text{Bounded}(V)$

$\beta : \text{BoundedBase} \iff \forall B : \text{Bounded}(V) . \exists B' \in \beta . B \subset B'$

**ClosedDiscsAsBoundedBase** ::  $\forall V \in k\text{-LCS} . \text{BoundedBase}(V, \text{Closed} \ \& \ \text{Disc}(V))$

**Proof** =

Assume  $B$  is bounded in  $V$ .

Then the disced hull of  $B$  is also bounded.

□

### 1.2.3 Locally Bounded Maps

$k : \text{AbsoluteValueField}(\mathbb{R});$

$\text{LocallyBounded} :: \prod_{V, W : k\text{-TVS}} ?(V \rightarrow W)$

$f : \text{LocallyBounded} \iff \forall B : \text{Bounded}(V) . \text{Bounded}(W, f(B))$

$\text{Homogeneous} :: \prod_{V, W : k\text{-VS}} ?(V \rightarrow W)$

$f : \text{Homogeneous} \iff \exists \delta \in \mathbb{R}_{++} . \forall v \in V . \forall \rho \in \mathbb{R}_{++} . f(\rho v) = \rho^\delta f(v)$

$\text{ContunuousHomogenuousIsLocallyBounded} ::$

$:: \forall V, W \in k\text{-TVS} . \forall f : \text{TOP} \ \& \ \text{Homogeneous}(V, W) . \text{LocallyBounded}(V, W, f)$

**Proof** =

Pretty obvious if you use basic properties.

□

$\text{BoundedProductsConverse} :: \forall I \in \text{SET} . \forall V : I \rightarrow k\text{-TVS} . \forall B \subset \prod_{i \in I} V_i .$

$. \left( \forall i \in I . \text{Bounded}(V_i, \pi_i(B)) \right) \iff \text{Bounded} \left( \prod_{i \in I} V_i, \prod_{i \in I} B_i \right)$

**Proof** =

1 ( $\Rightarrow$ ) .

1.1 As  $\pi_i(B)$  is bounde, so is  $\prod_{i \in I} \pi_i(B)$ .

1.2 Then  $B$  is bounded as  $B \subset \prod_{i \in I} \pi_i(B)$ .

2 ( $\Leftarrow$ ).

2.1 In product topology each  $\pi_i$  is continuous linear and so locally bounded.

□

$\text{MultilinearIsLocallyBounded} ::$

$:: \forall n \in \mathbb{N} . \forall V : \{1, \dots, n\} \rightarrow k\text{-TVS} . \forall W \in k\text{-TVS} .$

$. \forall T \in \mathcal{L}(V; W) \text{TOP} \left( \prod_{i=1}^n V_i, W \right) . \text{LocallyBounded} \left( \prod_{i=1}^n V_i, W, T \right)$

**Proof** =

Multilinear maps are homogeneous of degree  $n$ .

□

**BoundedSetsInWeakTopology** ::  $\forall V \in k\text{-VS} . \forall I \in \text{SET} . \forall W : I \rightarrow k\text{-TVS} .$

$$. \forall T : \prod_{i \in I} k\text{-VS}(V, W_i) . \forall B \subset V . \text{Bounded}\left((V, \mathcal{W}(I, W, T)), B\right) \iff \forall i \in I . \text{Bounded}\left(W_i, T_i(B)\right)$$

**Proof** =

1 This is simmlar to the case with products.

2 We may assume that topology is determined by one map  $T : V \rightarrow \prod_{i \in I} W_i$  .

3 Then  $\prod_{i \in I} T_i(B)$  is bounded in  $\prod_{i \in I} W_i$  .

4 Assume  $U$  is a neighborhood in the weak topology .

5 Then it must be a preimage of some open  $O \in \prod_{i \in I} W_i$ .

6 So find an  $O$ -absorbing scale  $\rho$  for  $\prod_{i \in I} T_i(B)$  and use it as  $U$ -absorbing scale for  $B$ .

6.1 Take some  $\alpha \in k$  such that  $|\alpha| \geq \rho$ .

6.2 Then  $T(b) \in \alpha O$  for any  $b \in B$  .

6.3 By thaking inverse image  $b \in T^{-1}(\alpha O) = \alpha T^{-1}(O) = \alpha U$ .

□

**ContinuityByBoundedImage** ::

$$:: \forall V, W \in k\text{-TVS} . \forall T \in k\text{-VS}(V, W) . \forall U \in \mathcal{U}_V(0) . \text{Bounded}\left(W, T(U)\right) \Rightarrow T \in k\text{-TVS}(V, W)$$

**Proof** =

Assume  $O \in \mathcal{U}_W(0)$ .

Then there exists an  $\rho \in \mathbb{R}_{++}$  such that  $T(U) \subset rO$ .

But this means that  $T(r^{-1}U) \subset O$ .

Then by topological group theory  $T$  is continuous.

□

**LocallyBoundedWithSemimetrizableDomainIsContinuous** ::

$$:: \forall V, W \in k\text{-TVS} . \forall T \in k\text{-VS} \ \& \ \text{LocallyBounded}(V, W) . \text{Semimetrizable}(V) \Rightarrow T \in k\text{-TVS} .$$

**Proof** =

1 Let  $U$  be a decreasing countable base of neighborhoods of 0 in  $V$ .

2 Assume that  $T$  is discontinuous.

2.1 By group topology  $T$  must be discontinuous at 0.

2.2 Then there is a  $O \in \mathcal{U}_W(0)$  such that  $T^{-1}(O)$  is not a neighborhood of 0 in  $V$ .

2.3 So  $\frac{1}{n}U_n \not\subset T^{-1}(O)$ .

2.4 It must be possible select a sequence  $u$  such that  $u_n \in U_n$  and  $Tu_n \notin O$ .

2.5 As  $U$  is a neighborhood base it follows that  $\lim_{n \rightarrow \infty} nu_n = 0$  .

2.6 This means that  $\{nu_n | n \in \mathbb{N}\}$  is bounded.

2.6.1 Given  $E \in \mathcal{U}_V(0)$  there is only finite amount of numbers  $n$  such that  $nu_n \notin E$ .

2.6.2 So it is posible to find  $E$ -absorbtion scales for this finite number and take max.

2.7 So  $nTu_n$  can be also be viewed as a sequence in a bounded subset of  $W$ .

2.8 So there exist an  $O$ -absorbtion scale  $\alpha$  for  $nTu_n$ .

2.9 That is  $nTu_n \in \alpha O$  for all  $n \in \mathbb{N}$  .

2.10 By archemedian property there exists  $n \in \mathbb{N}$  such that  $n \geq \alpha$ , so  $Tu_n \in O$ , a contradiction!

□

### 1.2.4 Liouville's Theorem

$$\text{Bounded} :: \prod_{X \in \mathbf{Set}} \prod_{V \in k\text{-TVS}} ?(X \rightarrow V)$$

$$f : \text{Bounded} \iff \text{Bounded}(V, f(X))$$

$$\text{Analytic} :: \prod_{U \in \mathcal{T}(\mathbb{C})} \prod_{V \in \mathbb{C}\text{-TVS}} ?(X \rightarrow V)$$

$$f : \text{Analytic} \iff \forall u \in U . \exists v \in V . \lim_{z \rightarrow u} \frac{f(z) - f(u)}{z - u} = v$$

$$\text{Entire} := \lambda V \in \mathbb{C}\text{-TVS} . \text{Analytic}(\mathbb{C}, V) : \mathbb{C}\text{-TVS} \rightarrow \text{Type};$$

$$\text{ContinuousComposition} ::$$

$$:: \forall U \in \mathcal{T}(\mathbb{C}) . \forall V \in \mathbb{C}\text{-TVS} . \forall v : \text{Analytic}(U, V) . \forall f \in V' . \text{Analytic}(U, \mathbb{C}, f(v))$$

$$\text{Proof} =$$

Use the continuity of  $f$  on the limit, which defines derivative.

□

$$\text{Total} :: \prod_{V \in k\text{-VS}} ??V$$

$$A : \text{Total} \iff \forall v \in V . \left( \forall f \in A . f(v) = 0 \right) \Rightarrow v = 0$$

$$\text{LiouvillesTheorem} ::$$

$$:: \forall V \in \mathbb{C}\text{-TVS} . \forall v : \text{Bounded}(\mathbb{C}, V) \ \& \ \text{Entire}(V) . \forall \mathbb{N} : \text{Total}(V, V') . \text{TypeConstant}(\mathbb{C}, V, v)$$

$$\text{Proof} =$$

$f(v)$  is an entire bounded function for every  $f \in V'$ .

So  $f(v)$  must be constant by classical Liouville theorem.

But this means that  $f(v(\alpha) - v(\beta)) = f(v(\alpha)) - f(v(\beta)) = 0$  for every  $\alpha, \beta \in \mathbb{C}$ .

But as  $V'$  is total this means that  $v$  is constant.

□

### 1.2.5 p-convexity

$$\mathbf{PConvex} :: \prod_{V \in \mathbf{R-VS}} \mathbb{R}_{++} \rightarrow ??V$$

$$A : \mathbf{PConvex} \iff \Lambda p \in \mathbb{R}_{++} . \forall \alpha, \beta \in \mathbb{R}_{++} . \forall v, w \in A . \alpha^p + \beta^p = 1 \Rightarrow \alpha v + \beta w \in A$$

$$\mathbf{AbsolutelyPConvex} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \rightarrow ??V$$

$$A : \mathbf{AbsolutelyPConvex} \iff \Lambda p \in \mathbb{R}_{++} . \forall \alpha, \beta \in k . \forall v, w \in A . |\alpha|^p + |\beta|^p \leq 1 \Rightarrow \alpha v + \beta w \in A$$

$$\mathbf{PSeminorm} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \rightarrow ?\mathbf{Sublinear}(V, \mathbb{R})$$

$$\nu : \mathbf{PSeminorm} \iff \Lambda p \in \mathbb{R}_{++} . \forall \alpha \in k . \forall v \in A . \|\alpha v\| = |\alpha|^p \|v\|$$

$$\mathbf{PSeminorm} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \rightarrow ?\mathbf{Sublinear}(V, \mathbb{R})$$

$$\nu : \mathbf{PSeminorm} \iff \Lambda p \in \mathbb{R}_{++} . \forall \alpha \in k . \forall v \in A . \|\alpha v\| = |\alpha|^p \|v\|$$

$$\mathbf{pSeminormedTopology} :: \prod_{V \in k\text{-VS}} \mathbf{PSeminorm}(V, k, p) \rightarrow \mathbf{Topology}(V)$$

$$\mathbf{pSeminormedTopology}(\nu) = \mathcal{T}(\nu) := \left\langle \left\{ \{w \in W : \nu(v - w) < \rho\} \mid v \in V, \rho \in \mathbb{R}_{++} \right\} \right\rangle$$

$$\mathbf{PSeminormable} :: \mathbb{R}_{++} \rightarrow ?k\text{-TVS}$$

$$V : \mathbf{PSeminormable} \iff \Lambda p \in \mathbb{R}_{++} . \exists \nu : \mathbf{PSeminorm}(V) . \mathcal{T}(V) = \mathcal{T}(\nu)$$

$$\mathbf{PSeminormableSpace} ::$$

$$:: \forall V \in k\text{-TVS} . \forall p \in \mathbb{R}_{++} \mathbf{PSeminormable}(V, p) \iff \exists U \in \mathcal{U}_V(0) . \mathbf{Bounded}(V, U) \ \& \ \mathbf{PConvex}(V, p, U)$$

**Proof** =

$\left(\frac{1}{n}U\right)_{n=1}^{\infty}$  is a countable base of vector topology for  $V$ .

The gauges defined by  $U$  are p-seminorms.

□



### 1.2.6 Bornology

$k : \text{AbsoluteValueField}(\mathbb{R}) \mid \text{UltravaluedField};$

$\text{Bornology} := \Lambda X \in \text{SET} . \text{Ideal}(2^X) : \text{SET} \rightarrow \text{Type};$

$\text{BoundedStructure} := \sum_{X \in \text{SET}} \text{Bornology}(X) : \text{Type};$

$\text{asSet} :: \text{BoundedStructure} \rightarrow \text{Set}$

$\text{asSet}(X, \beta) = (X, \beta) := X$

$\text{bornology} :: \prod (X, \beta) : \text{BoundedStructure} . \text{Bornology}(X)$

$\text{bornology}() = \mathcal{B}(X, \beta) := \beta$

$\text{Bounded} :: \prod X : \text{BoundedStructure} . ??X$

$B : \text{Bounded} \iff B \in \mathcal{B}(X)$

$\text{CompactsAreBornology} :: \forall X \in \text{TOP} . \text{Bornology}(X, \text{RelativeCompacts}(X))$

$\text{Proof} =$

This is obvious.

□

$\text{standardBornology} :: k\text{-TVS} \rightarrow \text{BoundedStructure}$

$\text{standardBornology}(V) = V := (V, \text{Bounded}(V))$

$\text{BornologyBase} :: \prod X : \text{BoundedStructure} . ??\mathcal{B}(X)$

$\mathcal{C} : \text{BornologyBase} \iff \forall B \in \mathcal{B}(X) . \exists C \in \mathcal{C} . B \subset C$

$\text{generateBornology} :: \prod_{X \in \text{SET}} ??X \rightarrow \text{Bornology}(X)$

$\text{generateBornology}(\alpha) = \langle \alpha \rangle_{\text{BORN}} := \left\{ A \subset X : \exists n \in \mathbb{N} . \exists C : \{1, \dots, n\} \rightarrow \alpha . A \subset \bigcup_{i=1}^n C_i \right\}$

$\text{bornologicalCategory} :: \text{CAT}$

$\text{bornologicalCategory}() = \text{BORN} :=$

$:= \left( \text{BoundedStructure}, \Lambda X, Y : \text{BoundedStructure} . \{f : X \rightarrow Y . \forall B \in \mathcal{B}(X) . f(B) \in \mathcal{B}(Y)\}, \circ, \text{id} \right)$

$$\text{strongBornology} :: \prod_{X \in \text{SET}} \prod Y : \text{BoundedStructure} . (X \rightarrow Y) \rightarrow \text{Bornology}(X)$$

$$\text{strongBornology}(f) = \mathcal{S}(Y, f) := \langle f^{-1}\mathcal{B}(Y) \rangle_{\text{BORN}}$$

$$\text{weekBornology} :: \prod_{Y \in \text{SET}} \prod X : \text{BoundedStructure} . (X \rightarrow Y) \rightarrow \text{Bornology}(Y)$$

$$\text{weakBornology}(f) = \mathcal{W}(X, f) := \langle f\mathcal{B}(X) \rangle_{\text{BORN}}$$

By use of week and strong notions, we may define subset bornology, quotient bornology or any kind of limit bornologies.

$$\text{supBornology} :: \prod_{X, I \in \text{SET}} \left( I \rightarrow \text{Bornology}(X) \right) \rightarrow \text{Bornology}(X)$$

$$\text{supBornology}(\beta) = \bigvee_{i \in I} \beta_i := \left\langle \bigcup_{i \in I} \beta_i \right\rangle_{\text{BORN}}$$

$$\text{infBornology} :: \prod_{X, I \in \text{SET}} \left( I \rightarrow \text{Bornology}(X) \right) \rightarrow \text{Bornology}(X)$$

$$\text{infBornology}(\beta) = \bigwedge_{i \in I} \beta_i := \left\langle \bigcap_{i \in I} \beta_i \right\rangle_{\text{BORN}}$$

This shows that a set of bornologies forms a complete lattice.

$$\text{VectorBornology} :: \prod V \in k\text{-VS} . ?\text{Bornology}(V)$$

$$\beta : \text{VectorBornology} \iff +_V \in \text{BORN}\left((V, \beta) \times (V, \beta), (V, \beta)\right) \ \& \cdot_V \in \text{BORN}\left(k \times (V, \beta), (V, \beta)\right)$$

$$\text{ConvexBornology} :: \prod V \in k\text{-VS} . ?\text{VectorBornology}(V)$$

$$\beta : \text{ConvexBornology} \iff \exists \gamma : \text{BornologyBase}(V, \beta) . \forall B \in \gamma . \text{Convex}(V, B)$$

**VectorBornologyCharacterisation** ::

$$:: \forall V \in k\text{-VS} . \forall \beta : \text{Bornology}(V) .$$

$$. \text{VectorBornology}(V, \beta) \iff \forall A, B \in \beta . A + B \in \beta \ \& \ \forall A \in \beta . \text{bal } A \in \beta$$

**Proof** =

1 ( $\Rightarrow$ ).

1.1  $A + B \in \beta$  as addition is locally bounded.

1.2  $\text{bal } A = \mathbb{D}_k(0, 1)A$  and scalar multiplication is uniformly bounded.

2 ( $\Leftarrow$ ) .

2.1  $A + B \in \beta$  implies that addition is locally bounded .

2.2  $\mathbb{D}_k(0, r)A \in \beta$ .

2.2.1 By archimedean property of  $\mathbb{R}$  there is  $n \in \mathbb{N}$  such that  $n \geq r$ .

2.2.2 But  $\mathbb{D}_k(0, r)A \subset \mathbb{D}_k(0, n)A \subset \sum_{i=1}^n \mathbb{D}(0, 1)A = \sum_{i=1}^n \text{bal } A \in \beta$  .

2.2.3 As  $\beta$  is ideal  $\mathbb{D}_k(0, r)A$ .

2.3 As  $k$  has bornology base of discs the scalar multiplication must be continuous.

□

**EquicontinuousBornology** ::  $\forall X \in \text{TOP} . \text{VectorBornology}(\text{TOP}(X, k), \text{Equicontinuous}(X, k))$

**Proof** =

- 1 Denote by  $\eta$  the set of equicontinuous subsets of  $\text{TOP}(X, k)$ .
  - 2 It is obvious that  $\eta$  is downwards closed.
  - 3  $\eta$  is also closed under finite unions.
  - 3.1 Assume  $A, B \in \eta$ , also assume  $U \in \mathcal{U}_k(0)$  and  $x \in X$ .
  - 3.2 Then there exists  $V \in \mathcal{U}_X(x)$  such that  $f(V) \subset U + f(x)$  for all  $f \in A$ .
  - 3.3 Also there is  $W \in \mathcal{U}_X(x)$  such that  $f(W) \subset U + f(x)$  for all  $f \in B$ .
  - 3.4 Then taking  $V \cap W$  should for  $A \cup B$ .
  - 4 Also  $\eta$  is closed under addition.
  - 4.1 Assume  $A, B \in \eta$ , also assume  $U \in \mathcal{U}_k(0)$  and  $x \in X$ .
  - 4.2 Then there exists  $O \in \mathcal{U}_k(0)$  such that  $O + O \subset U$ .
  - 4.3 Then there exists  $V \in \mathcal{U}_X(x)$  such that  $f(V) \subset O + f(x)$  for all  $f \in A$ .
  - 4.4 Also there is  $W \in \mathcal{U}_X(x)$  such that  $f(W) \subset O + f(x)$  for all  $f \in B$ .
  - 4.5 A function  $h \in A + B$  can be expressed as  $h = f + g$  for  $f \in A$  and  $g \in B$ .
  - 4.6 So  $h(V \cap W) = f(V \cap W) + g(V \cap W) \subset O + O + f(x) + g(x) \subset U + h(x)$ .
  - 5 Scalar multiplication is locally bounded.
  - 5.1 Assume  $A \in \eta$ , also assume  $U \in \mathcal{U}_k(0)$  and  $x \in X$ .
  - 5.2 Then there exist a balanced  $W \in \mathcal{U}_k(0)$  such that  $W \subset U$ .
  - 5.3 Then there exists  $V \in \mathcal{U}_X(x)$  such that  $f(V) \subset W + f(x)$  for all  $f \in A$ .
  - 5.4 Then for any  $f \in \text{bal } A = \mathbb{D}_k(0, 1)A$  there is  $g \in A$  and  $\alpha \in \mathbb{D}_k(0, 1)$  such that  $f = \alpha g$ .
  - 5.5 Then  $f(V) = \alpha g(V) \subset \alpha W + \alpha g(x) = W + f(x) \subset U + f(x)$ .
- 

**closure** ::  $\prod_{X \in \text{TOP}} \text{Bornology}(X) \rightarrow \text{Bornology}(X)$

**closure**  $(\beta) = \text{cl } \beta := \left\langle \{\text{cl } B \mid B \in \beta\} \right\rangle_{\text{BORN}}$

**interior** ::  $\prod_{X \in \text{TOP}} \text{Bornology}(X) \rightarrow \text{Bornology}(X)$

**interior**  $(\beta) = \text{int } \beta := \left\langle \{\text{int } B \mid B \in \beta\} \right\rangle_{\text{BORN}}$

**InteriorClosureOrder** ::  $\forall X \in \text{TOP} . \forall \beta : \text{Bornology}(X) . \text{int } \beta \subset \beta \subset \text{cl } \beta$

**Proof** =

This follows from the fact that  $\beta$  is closed under taking subsets.

And  $\text{int } A \subset A \subset \text{cl } A$  for any  $A \subset X$ .

□

**MonotonicInterior** ::  $\forall X \in \text{TOP} . \forall \alpha, \beta : \text{Bornology}(X) . \alpha \subset \beta \Rightarrow \text{int } \alpha \subset \text{int } \beta$

**Proof** =

Obvious.

□

**MonotonicClosure** ::  $\forall X \in \text{TOP} . \forall \alpha, \beta : \text{Bornology}(X) . \alpha \subset \beta \Rightarrow \text{cl } \alpha \subset \text{cl } \beta$

**Proof** =

Obvious.

□

**Open** ::  $\forall X \in \text{TOP} . ?\text{Bornology}(X)$

$\beta : \text{Open} \iff \text{int } \beta = \beta$

**Closed** ::  $\forall X \in \text{TOP} . ?\text{Bornology}(X)$

$\beta : \text{Closed} \iff \text{cl } \beta = \beta$

**Proper** := **Closed** & **Open** :  $\prod_{X \in \text{TOP}} ?\text{Bornology}(X);$

**ClodednessAltDef** ::

$\forall X \in \text{TOP} . \forall \beta : \text{Bornology}(X) . \text{Closed}(X, \beta) \iff \text{BornologyBase}(X, \beta, \beta \cap \text{Closed}(X))$

**Proof** =

1 ( $\Rightarrow$ ).

1.1 If  $A \in \beta$ , then  $A \subset \text{cl } A$ .

1.2 Also  $\text{cl } A \in \beta$ .

2 ( $\Leftarrow$ ).

2.1 Assume  $A \in \beta$ .

2.2 Then there is a closed set  $F \in \beta$  such that  $A \subset F$ .

2.3 But  $A \subset \text{cl } A \subset F$ .

2.4 So  $\text{cl } A \in \beta$  as  $\beta$  is closed under taking subsets.

□

**LocallyBounded** ::  $? \text{TOP} \ \& \ \text{BORN}$

$X : \text{LocallyBounded} \iff \forall x \in X . \mathcal{N}_V(x) \cap \beta \neq \emptyset$

**CompactsAreBoundedInLocallyBoundedSpace** ::  $\forall X : \text{LocallyBounded} . \mathcal{K}(X) \subset \mathcal{B}(X)$

**Proof** =

1 Take  $K$  to be compact in  $X$ .

2 Select a bounded Neighborhood  $U_x$  for each point  $x \in K$ .

3 As  $K$  is compact there is a finite subcover  $(x_i)_{i=1}^n$ .

4 Then  $\bigcup_{i=1}^n U_{x_i} \in \mathcal{B}(X)$  as  $\mathcal{B}(X)$  is an ideal.

5 But  $K \subset \bigcup_{i=1}^n U_{x_i} \in \mathcal{B}(X)$ , so  $K \in \mathcal{B}(X)$ , as  $\mathcal{B}(X)$  is an ideal.

□

$\text{semimetricBornology} :: \prod_{X \in \text{SET}} \text{Semimetric}(X) \rightarrow \text{Bornology}(X)$

$\text{semimetricBornology}(\rho) = \mathcal{B}(\rho) := \langle \mathbb{B}_X(X, \mathbb{R}_{++}) \rangle_{\text{BORN}}$

$\text{Semimetrizable} :: ?\text{TOP} \ \& \ \text{BORN}$

$X : \text{Semimetrizable} \iff \exists \rho : \text{Semimetric}(X) . \mathcal{T}(X) = \mathcal{T}(\rho) \ \& \ \mathcal{B}(X) = \mathcal{B}(\rho)$

$\text{SemimetrizationTHM} ::$

$:: \forall (X, \tau, \beta) \in \text{TOP} \ \& \ \text{BORN} .$

$. \text{Semimetrizable}(X, \tau, \beta) \iff$

$\iff \text{Semimetrizable}(X, \tau) \ \& \ \text{LocallyBounded} \ \& \ \text{Proper}(X, \beta) \ \& \ \exists \beta' : \text{BornologyBase}(X) . |\beta'| \leq \aleph_0$

$\text{Proof} =$

...

□

1.2.7 Interesting Examples and Facts

## 1.3 Hahn-Banach Theory

1.3.1 Dominated Extension

1.3.2 Mazur-Orlich Theorem

1.3.3 Minimal Sublinear functionals

## 1.4 Barelled and Bornological Spaces

## 1.5 Towards Approximation Theory

# 2 Spaces of Distributions

## 3 Ordered Topological Vector Spaces

### 3.1 Reisz Spaces and Banach Lattices

#### 3.1.1 Order Unit Norm

**OrderUnitDefinesASublinear** ::

$:: \forall V : \text{OrderedVectorSpace}(\mathbb{R}) . \forall u : \text{OrderUnit}(V) . \text{Sublinear}(V, \Lambda v \in V . \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\})$

**Proof** =

1 Write  $\omega(v) = \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\}$ .

2 Obviously  $\omega$  is positively homogeneous.

3 Now take  $v, w \in V$ .

3.1 Define  $\alpha = \omega(v) + \omega(w)$ .

3.2 Then  $v + w \leq (\omega(v) + \omega(w))u = \alpha u$ .

3.3 So  $\omega(v + w) \leq \alpha = \omega(v) + \omega(w)$ .

□

**orderUnitFunctional** ::  $\prod V : \text{OrderedVectorSpace}(\mathbb{R}) . \text{OrderUnit}(V) \rightarrow \text{Sublinear}(V)$

**orderUnitFunctional**  $(u) = \omega_u := \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\}$

**orderUnitSeminorm** ::  $\prod V : \text{ArchedeanVectorSpace}(\mathbb{R}) . \text{OrderUnit}(V) \rightarrow \text{SMN}(V)$

**orderUnitFunctional**  $(u) = \nu_u := \Lambda v \in V . \max(\omega_u(v), \omega_u(-v))$

**UnitDiscIsAnInterval** ::  $\forall V : \text{ArchedeanVectorSpace}(\mathbb{R}) . \forall u : \text{OrderUnit}(V) . \mathbb{D}(\nu_u) = [-u, u]$

**Proof** =

1 Obvious.

□

### 3.1.2 Topological Vector Lattices

$\text{TopologicalVectorLattice} :: ?\mathbb{R}\text{-TVS} \ \& \ \text{RieszSpace}$

$V : \text{TopologicalVectorLattice} \iff \text{Closed}(V, \mathcal{C}_V) \ \& \ \exists \mathcal{B} : \text{NeighborhoodBase}(V, 0) . \forall B \in \mathcal{B} . \text{OrderConvex}(V, B)$

$\text{BanachLattice} :: ?\text{NormedSpace} \ \& \ \text{RieszSpace}$

$V : \text{BanachLattice} \iff \forall v, w \in V . |v| \leq |w| \Rightarrow \|v\| \leq \|w\|$

$\text{MSpace} :: ?\text{NormedSpace} \ \& \ \text{RieszSpace}$

$V : \text{MSpace} \iff \forall v, w \in V_+ . \|v \vee w\| = \|v\| \vee \|w\|$

$\text{LSpace} :: ?\text{NormedSpace} \ \& \ \text{RieszSpace}$

$V : \text{LSpace} \iff \forall v, w \in V_+ . \|v + w\| = \|v\| + \|w\|$



### 3.1.3 Lattice of Continuous Functions

**ExtremellyDisconnected** ::

$$:: \forall X \in \text{TOP} . \text{ExtremellyDisconnected}(X) \iff \forall U, V \in \mathcal{T}(X) . UV = \emptyset \Rightarrow \text{cl}_X U \cap \text{cl}_X V = \emptyset$$

**Proof** =

**OrderCompletenessOfContinuousFunctions** ::

$$:: \forall X \in \text{ExtremellyDisconnected} . \text{OrderDedekindComplete}(C(X))$$

**Proof** =

...

□

**OrderCompletenessOfContinuousFunctions** ::

$$:: \forall X : \text{T3.5} . \text{OrderDedekindComplete}(C(X)) \Rightarrow \text{ExtremellyDisconnected}(X)$$

**Proof** =

...

□

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