TangentSpace.Know

Uncultured Tramp

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1 Tangent Structure

1.1 Tangent Spaces

$$\texttt{TangentSpace} :: \prod M : \texttt{SmoothManifold} \,.\, M \to \texttt{VectorSpace}\,(\mathbb{R})$$
$$T_pM := \{p\} \times \mathbb{R}^{\dim M}$$

$$\texttt{Germ} :: \prod M : \texttt{SmoothManifold} \,.\, M \to \texttt{EqClass}\,(C^\infty\,(M))$$
$$\mathcal{F}_p\,(M) := \big\{\{g : C^\infty\,(M) : D_pg = D_pf : g(p) = f(p)\} | f : C^\infty\,(M)\,\big\}$$

$$\texttt{DerivationAtPoint} :: \prod M : \texttt{SmoothManifold} \,.\, \prod p \in M \,.\, ?\big(\mathcal{F}_p\,(M) \to \mathbb{R}\big)$$
$$\mathcal{D} : \mathrm{Dir}\,(M,p) \iff \forall f, g \in \mathcal{F}_p\,(M)\ .\ \mathcal{D}(fg) = \mathcal{D}(f)g(p) + f(p)\mathcal{D}(g)$$

$$\begin{aligned}&\mathrm{alg} :: \prod M : \texttt{SmoothManifold} \,.\, \prod p \in M \,.\\ &\quad \mathrm{chartCentredAt}(M,P) \to T_pM \to \mathrm{Dir}\,(M,p)\\ &\mathrm{alg}(U,x)(p,v)f := \sum_{i=1}^{|x|} v_i \frac{\delta f}{\delta x^i}\Big|_p\end{aligned}$$

$$\begin{aligned}&\mathrm{fromAlg} :: \prod M : \texttt{SmoothManifold} \,.\, \prod p \in M \,.\\ &\quad \mathrm{chartCentredAt}(M,P) \to \mathrm{Dir}\,(M,p) \to T_pM\\ &\mathrm{fromAlg}(U,x)\mathcal{D} = \Big(p, \big(\mathcal{D}x^i\big)_{i=1}^{|x|}\Big)\end{aligned}$$

$$\begin{aligned}&\texttt{PhysicalView} :: \prod M : \texttt{SmoothManifold} \,.\, \prod p \in M \,.\\ &\quad \texttt{EqClass}\,\big(\{p\} \times \mathbb{R}^{\dim M} \times \mathrm{chartCentredAt}(M,p)\big)\\ &\texttt{PhysicalView}(M,p) = \big\{\{(p, w, (V,y)) \in \{p\} \times \mathbb{R}^{\dim M} \times \mathrm{chartCentredAt}(M,p)\\ &\quad : w = D(x^{-1}y)|_{x(p)}v\}(p, v.(U,x)) \in \{p\} \times \mathbb{R}^{\dim M} \times \mathrm{chartCentredAt}(M,p)\big\}\end{aligned}$$

$$\text{phys} :: \prod M : \texttt{SmoothManifold} \; . \; \prod p \in M \; .$$
$$\quad \text{chartCentredAt}(M, P) \to T_p M \to \texttt{PhysicalView}(M, p)$$
$$\text{phys}(U, x)(p, v) := (p, v, (U, x))$$

$$\text{fromPhys} :: \prod M : \texttt{SmoothManifold} \; . \; \prod p \in M \; .$$
$$\quad \text{chartCentredAt}(M, P) \to \texttt{PhysicalView}(M, p) \to T_p M$$
$$\text{fromPhys}(U, x)(p, w, (V, y)) = \left(p, D(y^{-1}x)|_{y(p)} w\right)$$

$$\texttt{MovementDirection} :: \prod M : \texttt{SmoothManifold} \; . \; \prod p \in M \; .$$
$$\quad \texttt{EqClass}\left(\texttt{CentredCurve}\left(M, p\right)\right)$$
$$\texttt{MovementDirection}(M, p) = \big\{\{\beta \in \texttt{CentredCurve}\left(M, p\right) : \beta'(0) = \alpha'(0)\}|$$
$$\quad |\alpha : \texttt{CentredCurve}\left(M, p\right)\}$$

$$\text{kin} :: \prod M : \texttt{SmoothManifold} \; . \; \prod p \in M \; .$$
$$\quad \text{chartCentredAt}(M, P) \to T_p M \to \texttt{MovementDirection}(M, p)$$
$$\text{kin}(U, x)(p, v) := \Lambda t \in (-1, 1) \; . \; x^{-1}(tv)$$

$$\text{fromKin} :: \prod M : \texttt{SmoothManifold} \; . \; \prod p \in M \; .$$
$$\quad \text{chartCentredAt}(M, P) \to \texttt{MovementDirection}(M, p) \to T_p M$$
$$\text{fromKin}(U, x)\gamma = \left(p, (\gamma x)'(0)\right)$$

1.2 Tangent Maps

$\texttt{TangentMap} :: \prod M, N : \texttt{SmoothManifold} \ . \ \prod p \in M \ .$

$\quad . \ \prod f : C^{\infty}(M, N) \ . \ \mathcal{L}\left(T_p M, T_{f(p)} N\right)$

$T_p f(p, v) = (f(p), Df|_p v)$

$\texttt{DerivationTransfer} :: \prod M, N : \texttt{SmoothManifold} \ . \ \prod p \in M \ . \ \prod f : C^{\infty}(M, N) \ .$

$\quad . \ \mathcal{L}\left(\texttt{DerivationAtPoint}(M, p), \texttt{DerivationAtPoint}(N, f(p))\right)$

$T_p f(\mathcal{D})\phi = \mathcal{D}(\phi \circ f)$

$\texttt{PhysicalShift} :: \prod M, N : \texttt{SmoothManifold} \ . \ \prod p \in M \ . \ \prod f : C^{\infty}(M, N) \ .$

$\quad . \ \mathcal{L}\left(\texttt{PhysicalView}(M, p), \texttt{PhysicalView}(N, f(p))\right)$

$T_p f(p, v, (U, x)) = \big(f(p), D(y \circ f \circ x)v, (V, y)\big)$

  where

    $(V, y) : \text{chartCentredAt}(N, f(p))$

$\texttt{DirectionTransfer} :: \prod M, N : \texttt{SmoothManifold} \ . \ \prod p \in M \ . \ \prod f : C^{\infty}(M, N) \ .$

$\quad . \ \mathcal{L}\left(\texttt{MovementDirection}(M, p), \texttt{MovementDirection}(N, f(p))\right)$

$T_p f \gamma = f \circ \gamma$

$\texttt{Differential} :: \prod M : \texttt{SmoothManifold} \ . \ \prod p \in M \ . \ C^{\infty}(M) \to T_p^* M$

$df(p)(p, v) = [\text{alg}(\cdot)(p, v)(f)]$

1.3 Categorical Viewpoint on Tangent Spaces

$\texttt{PointedManifolds} :: \texttt{Category}$

$\mathcal{O}(\mathsf{PM}) = \sum M : \texttt{SmoothManifold} \,.\, M$

$\mathcal{M}((M,p),(N,q)) = \{f : C^\infty(M) : f(p) = q\}$

$f \cdot g = g \circ f$

$\texttt{TangentFunctor} :: \texttt{Functor}(\mathsf{PM}, \mathsf{VS}(\mathbb{R}))$

$T(M,p) = T_p M$

$Tf = T_p f$

$\texttt{CotangentFunctor} :: \texttt{ContrFunctor}(\mathsf{PM}, \mathsf{VS}(\mathbb{R}))$

$T(M,p) = T_p^* M$

$Tf = (T_p f)^*$

1.4 Critical Points and Value

$$\texttt{RegularPoint} :: \prod M, N : \texttt{SmoothManifold}.C^\infty(M,N) \to ?M$$

$$p : \texttt{RegularPoint}(f) \iff T_p f : \texttt{Surjective}\left(T_p M, T_{f(p)} N\right)$$

$$\texttt{CriticalPoint} :: \prod M, N : \texttt{SmoothManifold} \,.\, C^\infty(M,N) \to ?M$$

$$p : \texttt{CriticalPoint}(f) \iff p\,!\,\texttt{RegularPoint}(f)$$

$$\texttt{RegularValue} :: \prod M, N : \texttt{SmoothManifold} \,.\, C^\infty(M,N) \to ?N$$

$$q : \texttt{RegularValue}(f) \iff \forall p \in f^{-1}q \,.\, v : \texttt{RegularPoint}(f)$$

$$\texttt{CriticalValue} :: \prod M, N : \texttt{SmoothManifold} \,.\, C^\infty(M,N) \to ?N$$

$$q : \texttt{CriticalValue}(f) \iff q\,!\,\texttt{RegularValue}(f)$$

$$\texttt{Zero} :: \prod M : \texttt{SmoothManifold} \,.\, ??M$$

$$A : \texttt{Zero}(M) \iff \forall (x, U) \in \texttt{Admissible}(M) \,.\, x(U \cap A) : \texttt{Zero}$$

$$\texttt{Sard} :: \forall M, N : \texttt{SmoothManifold} \,.\, \forall f : C^\infty(M,N) \,.\, \texttt{CriticalValue}(f) : \texttt{Zero}$$

$$\texttt{NonDegenerate} :: \prod M, N : \texttt{SmoothManifold} \prod f : C^\infty(M,N) \,.\, ?\texttt{CriticalPoint}(f)$$

$$p : \texttt{NonDegenerate}(f) \iff \operatorname{rank} H(f,p) \neq \{0\}$$

$$\texttt{MorseLemma} :: \forall M : \texttt{SmoothManifold} \,.\, \forall f : C^\infty(M) \,.\, \forall p : \texttt{NonDegenerate}(f) \,.$$
$$.\ \exists (U, x) : \text{chartCentredAt}(M,p) : \exists b \in \{s, 1\}^{|x|} \forall u \in U \,.$$
$$.\ f(u) = f(p) + \sum_{i=1}^{|x|} s_i (x^i(u))^2$$

1.5 Rank and Level Set

$\texttt{ConstantRank} :: \prod M, N : \texttt{SmoothManifold} \ . \ ?C^\infty (M, N) \times \mathbb{Z}_+$

$(f, k) : \texttt{ConstantRank} \iff \forall p \in M \ . \ \operatorname{rank} T_p M = k$

$\texttt{LevelSubmanifold} :: \prod M, N : \texttt{SmoothManifold}. \prod f : C^\infty (N, M) \ .$
$\ . \ \forall q \in N \ . \ \forall k \in \mathbb{Z}_+ \ . \ \texttt{if} \ \forall p \in f^{-1} q \ . \ \exists U \in \mathcal{U}(p) : (f_{|U}, k) : \texttt{ConstantRank}$
$\texttt{then} \ f^{-1} q : \texttt{SubManifold}(M) \wedge \operatorname{codim} f^{-1} q = k$

$\texttt{Transverse} :: \prod M, N : \texttt{SmoothManifold} \ . \ ?C^\infty(M, N) \times \texttt{SubManifold}(N)$

$f \pitchfork S \iff \forall p \in f^{-1}(S) \ . \ T_{f(p)} N = T_p f T_p M + T_{f(p)} S$

$\texttt{Transversality} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall f : C^\infty(M, N) \ .$
$\ . \ \forall S : \texttt{SubManifold}(N) \ . \ \texttt{if} \ f \pitchfork S \ \texttt{then} \ f^{-1}(S) : \texttt{SubManifold}(M) \wedge$
$\wedge \operatorname{codim} f^{-1}(S) = \operatorname{codim} S$

$\texttt{Transverse} :: \prod M, M', N : \texttt{SmoothManifold} \ . \ ?C^\infty(M, N) \times C^\infty (M', N)$

$f \pitchfork g \iff \forall q \in fM \cap gM' \ . \ \forall p \in f^{-1}(q) \ . \ \forall p' \in g^{-1}(q) \ . \ T_{f(p)} N = T_p f T_p M + T_{p'} g T_{p'} M'$

$\texttt{TransversalPullbacks} :: \forall M, M', N : \texttt{SmoothManifold} \ . \ \forall f : C^\infty(M, N) \ .$
$\ . \ \forall f' C^\infty (M', N) \ \texttt{if} \ f \pitchfork f' \texttt{then} \ (f, f')^{-1}(\Delta(N)) : \texttt{SubManifold}(M \times M')$

1.6 Tangent Bundles

$\texttt{TangentBundle} :: \texttt{SmoothManifold} \to \texttt{SmoothManifold}$

$$TM = \left(\bigcup_{p \in M} T_p M, A_M, \left\{ (U_\alpha \times \mathbb{R}^{\dim M}, x_\alpha \oplus \mathrm{id}) : \alpha \in A \right\} \right)$$

$\texttt{TangentMap} :: \prod M, N : \texttt{SmoothManifold} \,.\, C^\infty(M, N) \to C^\infty(TM, TN)$

$Tf(p, v) = (f(p), T_p f(v))$

$\texttt{Differential} :: \prod M, N : \texttt{SmoothManifold} \,.\, C^\infty(M, N) \to C^\infty\left(TM, \mathbb{R}^{\dim N}\right)$

$df(p, v) = T_p f(v)$

$\texttt{TBProjection} :: \prod M : \texttt{SmoothManifold} \,.\, C^\infty\left(TM, \mathbb{R}^{\dim M}\right)$

$\pi(p, v) = v$

$\texttt{Trivialization} :: \prod M : \texttt{SmoothManifold} \,.\, ?D^\infty\left(TM, M \times \mathbb{R}^{\dim M}\right)$

$F : \texttt{Trivialization} \iff \forall p \in M \,.\, \lambda v \in \mathbb{R}^{\dim M} \,.\, F(p, c) : \mathcal{L}\mathbb{R}^{\dim M}$

$\texttt{LocalTrivialization} :: \prod M : \texttt{SmoothManifold} \,.\, ?D^\infty\left(TM, M \times \mathbb{R}^{\dim M}\right) \times \texttt{Chart}(M)$

$(F, U) : \texttt{LocalTrivialization} \iff F|_U : \texttt{Trivialization}(M)$

1.7 Cotangent Bundles

$$\texttt{CotangentBundle} :: \texttt{SmoothManifold} \to \texttt{SmoothManifold}$$
$$T^*M = \left(\bigcup_{p \in M} T_p^* M, A_M, \{(U_\alpha \times \mathbb{R}^{\dim M}, x_\alpha \oplus \mathrm{id}) : \alpha \in A\} \right)$$

1.8 Categorical View on Tangent Bundles

$$\texttt{TangentFunctor} :: \texttt{Functor}\left(\mathsf{SM}, \mathsf{VS}\left(\mathbb{R}\right)\right)$$
$$T(M) = TM$$
$$T(f) = Tf$$

1.9 Vector Fields

$$\texttt{Section} :: \prod M, N : \texttt{SmoothManifold} \, . \, ?(M \to N)$$
$$f : \texttt{Section} \iff \exists \sigma : C^\infty(N, M) : \sigma \circ \pi = \mathrm{id}$$

$$\texttt{VectorField} :: \prod M : \texttt{SmoothManifold} \, . \, ?C^\infty(M, TM)$$
$$X \in \mathfrak{X}(M) \iff X : \texttt{Section}(\pi)$$

$$\texttt{CoordinateFrame} :: \prod M : \texttt{SmoothManifold} \, . \, \prod (U, x) : \texttt{Chart}(M) \, . \, \texttt{List}(\mathfrak{X}(U))$$
$$\texttt{CoordinateFrame}(M, (U, x)) = \left(\frac{\delta}{\delta x^i} \right)_{i=1}^{\dim M}$$

$$(\mathfrak{X}(M), +, \cdot) : \texttt{VectorSpace}\left(\mathbb{R}\right)$$

$$(\mathfrak{X}(M), +, \cdot) : \texttt{Module}(C^\infty(M))$$

$\texttt{VectorFieldAlong} :: \prod M, N : \texttt{SmoothManifold} \; . \; C^\infty(M, N) \to ?C^\infty(M, TN)$

$X \in \mathfrak{X}_f(M, N) \iff \pi \circ X = f$

$\texttt{Derivation} :: \prod M : \texttt{SmoothManifold} \; . \; ?\mathcal{L}C^\infty(M)$

$\mathcal{D} : \texttt{Derivation} \iff \forall f, g \in C^\infty(M) \; . \; \mathcal{D}(fg) = f\mathcal{D}(g) + \mathcal{D}(f)g$

$\texttt{LieDerivative} :: \prod M : \texttt{SmoothManifold} \; . \; \mathfrak{X}(M) \to C^\infty(M) \to C^\infty(M)$

$\mathcal{L}_X f(p) = df(X_p)$

$\texttt{FeildsAsDerivations} :: \forall M : \texttt{SmoothManifold} \; . \; \forall X \in \mathfrak{X}(M) \; . \; \mathcal{L}_X : \texttt{Derivation}(M)$

$\texttt{Proof}(M, X) =$

$(\mathcal{L}_X, | f, g \in C^\infty(M)$

$|c \in \mathbb{R}$

$\mathcal{L}_X(f + g) = d(f + g)(X) = df(X) + dg(X) = \mathcal{L}_X(f) + \mathcal{L}_X(g);$

$\mathcal{L}_X(cf) = d(cf)(X) = cdf(X) = c\mathcal{L}_X(f)$

$||) : \mathcal{L}C^\infty(M)$

$(\mathcal{L}_X, | f, g \in C^\infty(M)$

$\mathcal{L}_X(fg) = d(fg)(X) = fd(g)(X) + d(f)g(X) = f\mathcal{L}_X(g) + \mathcal{L}_X(f)g;$

$|) : \texttt{Derivation}(M) \; \square$

$\texttt{DerivationsAsFields} :: \forall M : \texttt{SmoothManifold} \; . \; \forall \mathcal{D} \in \texttt{Derivation}(M) \; .$

$. \; \exists! X \in \mathfrak{X}(M) : \mathcal{D} = \mathcal{L}_X$

$\texttt{DerivationsBracket} :: \texttt{Derivation}^2(M) \to \texttt{Derivation}(M)$

$[\mathcal{D}, \mathcal{D}'] = \mathcal{D} \circ \mathcal{D}' - \mathcal{D}' \circ \mathcal{D}$

$\texttt{LieBracket} :: \mathfrak{X}^2(M) \to \mathfrak{X}(M)$

$[X, Y] = \texttt{DerivationsAsFields}([\mathcal{L}_X, \mathcal{L}_Y])$

$(\mathfrak{X}(M), [\cdot, \cdot]) : \texttt{LieAlgebra}$

$\texttt{PullBack} :: \prod M, N : \texttt{SmoothManifold} \ . \ D^\infty(M, N) \to \mathfrak{X}(N) \to \mathfrak{X}(M)$
$f^*Y = Tf^{-1} \circ Y \circ f$

$\texttt{PushForward} :: \prod M, N : \texttt{SmoothManifold} \ . \ D^\infty(M, N) \to \mathfrak{X}(M) \to \mathfrak{X}(N)$
$f_*X = Tf \circ X \circ f^{-1}$

$\texttt{LiePullBack} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall \phi : D^\infty(M, N) \ . \ \forall g : C^\infty(N) \ .$
$\quad . \ \forall Y : \mathfrak{X}(N) \ . \ \mathcal{L}_{\phi^* Y} \phi^* g = \phi^* \mathcal{L}_Y g$

$\texttt{LiePushForward} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall \phi : D^\infty(M, N) \ . \ \forall f : C^\infty(M) \ .$
$\quad . \ \forall X : \mathfrak{X}(M) \ . \ \mathcal{L}_{\phi_* X} \phi_* f = \phi_* \mathcal{L}_X f$

$\texttt{Related} :: \prod M, N : \texttt{SmoothManifold} \ . \ C^\infty(M, N) \to ?(\mathfrak{X}(M) \times \mathfrak{X}(N))$
$(X, Y) : \texttt{Related}(f) \iff Tf \circ X = Y \circ f$

$\texttt{RelatedMark} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall f : C^\infty(M, N) \ . \ \forall X : \mathfrak{X}(M) \ . \ \forall Y : \mathfrak{X}(N) \ .$
$\quad . \ (X, Y) : \texttt{Related}(f) \iff \forall g : C^\infty(N) \ . \ X \cdot (g \circ f) = (Y \cdot g) \circ f$

$\texttt{BracketRelated} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall f : C^\infty(M, N) \ .$
$\quad . \ \forall (X, Y), (X', Y') : \texttt{Related}(f) \ . \ \big([X, X'], [Y, Y']\big) : \texttt{Related}(f)$

$\texttt{BracketPullBack} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall \phi : D^\infty(M, N) \ .$
$\quad . \ \forall X, X' : \mathfrak{X}(N) \ . \ [\phi^* X_1, \phi^* X_2] = \phi^* [X_1, X_2]$

$\texttt{LieRelated} :: \forall M, N : \texttt{SmoothManifold} \ . \ \forall f : C^\infty(M, N) \ . \ \forall g \in C^\infty(N) \ .$
$\quad . \ \forall X \in \mathfrak{X}(M) \ . \ \forall Y \in \mathfrak{X}(N) \ . \ \mathcal{L}_X (f^* g) = f^* \mathcal{L}_Y g$

1.10 Integral Curves and Flows

$\texttt{IntegralCurve} :: \prod M : \texttt{SmoothManifold} \,.\, ?\texttt{Curve}(M)$

$\gamma : \texttt{IntegralCurve} \iff \exists X \in \mathfrak{X}(M) : \dot{\gamma} = X \circ \gamma$

$\texttt{Flow} :: \prod M : \texttt{SmoothManifold} \,.\, \mathbb{R} \to D^\infty(M, M)$

$\Phi : \texttt{Flow} \iff \forall t, s \in \mathbb{R} \,.\, \Phi_t \circ \Phi_s = \Phi_{t+s} \wedge \Phi_t^{-1} = \Phi_{-t}$

$\texttt{flowField} :: \texttt{Flow}(M) \to \texttt{Field}(M)$

$$X^\Phi = \Lambda p \in M \,.\, \frac{\mathrm{d}}{\mathrm{d}t} \Phi(0, p)$$

$\texttt{ODESolution} :: \prod X : \texttt{Field}(M) \,.\, ?\texttt{IntegralCurve}(M)$

$c : \texttt{ODESolution} \iff \dot{c} = X(c)$

$\texttt{SolutionUniquenes} :: \forall X \in \texttt{Field}(M) \,.\, \forall a, b \in \texttt{ODESolution}(X) : a(0) = b(0) \,.$

$\quad . \forall p \in \operatorname{Dom} a \cap \operatorname{Dom} b \,.\, a(p) = b(p)$

$\texttt{FlowBox} :: \prod X : \texttt{Field}(M) \,.\, \prod p \in M \,.\, ?\sum U : \mathcal{U}(p) \,.\, \sum p \in \mathbb{R}_{++}^\infty \,.\, C^\infty\big((-a, a) \times U, M\big)$

$(U, a, \phi) : \texttt{FlowBox} \iff \forall q \in U \,.\, t \mapsto \phi(t, q) : \texttt{ODESolution}(X) \wedge \phi(0, q) = q$

$\quad \forall t \in (-a, a) \,.\, q \mapsto \phi(t, p) : D^\infty(U, \phi(t, U))$

$\texttt{Complete} :: ?\texttt{Field}(M)$

$X : \texttt{Complete} \iff \exists (U, a, \phi) : \texttt{FlowBox}(X) : U = M \wedge a = \infty$

$\texttt{maximalIntegralCurve} :: \texttt{Field}(M) \to M \to \texttt{IntegralCurve}(M)$

$$\texttt{maximalIntegralCurve}(X, p) = J_p^X := \bigcup_{J \in \mathcal{J}} J$$

$\quad \texttt{where } \mathcal{J} = \big\{J : \texttt{ODESolution}(X) : J(0) = p\big\}$

$\texttt{maximalFlow} :: \texttt{Field}(M) \to \texttt{Flow}(M)$

$$\texttt{maximalFlow}(X) = \mathcal{D}_X := \bigcup_{p \in M} J_p^X \times \{\}$$

$\texttt{support} :: \texttt{Field}(M) \to \texttt{Closed}(M)$

$\texttt{support}(X) = \operatorname{supp} X = \operatorname{cl} \{p \in M : X(p) \neq 0\}$

1.11 Lie Derivative of a Vector Field

$$\texttt{LieDerivative} :: \mathfrak{X}(M) \to \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$\texttt{LieDerivative}(X) = \mathcal{L}_X := Y \mapsto [X, Y]$$

$$\texttt{lieDerivativeOfAFunction} :: \mathfrak{X}(M) \to C^\infty(M) \to M \to \mathbb{R}$$
$$\texttt{lieDerivativeOfAFunction}(X, f, p) = \mathcal{L}_X f(p) := \frac{\mathrm{d}}{\mathrm{d}t}(t, p)\phi^X f$$

$$\texttt{FlowTaylor} :: \forall X : \mathfrak{X}M \;.\; \forall p \in U \;.\; \forall U : \mathcal{U}(p) \;.\; \forall f \in C^\infty(U) \;.\; \exists \delta \in \mathbb{R}_{++} : \exists V : \mathcal{U}(p) :$$
$$: \phi^X([-\delta, \delta] \times V) \subset U : \exists g \in C^\infty([-\delta, \delta] \times V) : \forall t \in [-\delta, \delta] \;.$$
$$. \forall q \in V f(\phi^X(t, q)) = f(q) + tg(t, q) \wedge g(0, q) = X_q f$$

...

1.12 1-Forms

$$\texttt{1 - Form} :: \prod M : \texttt{SManifold} \;.\; C^\infty(M, T^*M)$$
$$\alpha : \texttt{1 - Form} \iff \alpha \in \Omega^1(M) \iff \alpha : \texttt{Injective}$$

$$\texttt{differential} :: C^\infty(M) \to \Omega^1(M)$$
$$\texttt{differential}(f) = df := p \mapsto h \mapsto \lim_{t \to \infty} \frac{f(h(t)) - f(h(0))}{t}$$

$$\texttt{applyFormToVector} :: \Omega^1(M) \to TM \to \mathbb{R}$$
$$\texttt{applyFormToVector}(\alpha, (p, v)) = \alpha(p, v) := \alpha(p)(v)$$

$$\texttt{applyFormToField} :: \Omega^1(M) \to \mathfrak{X}(M) \to C^\infty(M)$$
$$\texttt{applyFormToField}(\alpha, X) = \alpha(X) := p \mapsto \langle \alpha(p), X(p) \rangle$$

$$\texttt{holonomicCoordinateCoframe} :: \prod (U, x) : \texttt{Chart}(M) \;.\; \dim M \to \Omega^1(U)$$
$$\texttt{holonomicCoordinateCoframe}(i) = \mathrm{d}x^i := \mathrm{d}x^i$$

$$\texttt{Exact} :: ?\Omega^1(M)$$
$$\alpha : \texttt{Exact} \iff \exists f \in C^\infty(M) : \alpha = \mathrm{d}f$$

$$\texttt{pullBack} :: \Omega^1(N) \to C^\infty(M, N) \to \Omega^1(M)$$
$$\texttt{pullBack}(\alpha, \phi) = \phi^* \alpha := (p, v) \mapsto \big\langle \alpha \; \phi \; p, (T_p \phi) v \big\rangle$$

$\texttt{pushForward} :: \Omega^1(M) \to D^\infty(M, N) \to \Omega^1(N)$

$\texttt{pushForward}(\alpha, \phi) = \phi_* \alpha := (p, v) \mapsto \left\langle \alpha\ \phi^{-1}\ p, (T_p\phi)^{-1} v \right\rangle$

$\texttt{DifferentialNaturalPullBack} :: \forall f : C^\infty(N)\ .\ \forall \phi : C^\infty(M, N)\ .\ \phi^* df = d\phi^* f$

$\texttt{Proof} =$

$\texttt{Assume}\ f : C^\infty(N),$

$\texttt{Assume}\ \phi : C^\infty(M, N),$

$\texttt{Assume}\ (p, v) \in TM,$

$q := \phi(p),$

$\phi^* \mathrm{d} f(p, v) = \big\langle \mathrm{d} f|_q, (T_p\phi) v \big\rangle = \big\langle (T_p\phi)^* \mathrm{d} f|_{\phi(p)}, v \big\rangle = \langle \mathrm{d}(f \circ \phi)|_p, v \rangle =$

$\quad = \langle \mathrm{d}(\phi^* f)|_p, v \rangle = \mathrm{d}\phi^* f(p, v);$

$\forall (p, v) \in TM\ .\ \phi^* df(p, v) = d\phi^* f(p, v) \Rightarrow \phi^* df = d\phi^* f;;$

$\forall f : C^\infty(N)\ .\ \forall \phi : C^\infty(M, N)\ .\ \phi^* df = d\phi^* f\ \square$

$\texttt{canonicalCotangentForm} :: \forall M : \texttt{SManifold}\ .\ \Omega^1(T^* M)$

$\theta = ((p, v), u) \mapsto \langle v, T_{(p,v)} \pi u \rangle$

$$\left(T_{(p,v)} \pi \frac{\delta}{\delta p^i} = \lim_{t \to 0} \frac{\pi \dfrac{\delta(p, v)}{\delta p^i}(t) - \pi \dfrac{\delta(p, v)}{\delta p^i}(0)}{t} = \lim_{t \to 0} \frac{v - v}{t} = 0 \right)$$

$$\theta \left((p, v), \frac{\delta}{\delta p^i} \right) = \left\langle v, T_{(p,v)} \pi \frac{\delta}{\delta p^i} \right\rangle = \langle v, 0 \rangle = 0$$

$$\theta \left((p, v), \frac{\delta}{\delta q^i} \right) = \left\langle v, T_{(p,v)} \pi \frac{\delta}{\delta q^i} \right\rangle = \left\langle v, \frac{\delta}{\delta x^i} \right\rangle = v^i$$

$$\theta(p, v) = \sum_{i=1}^{n} v^i \mathrm{d} q^i$$

1.13 Line Integrals

$\texttt{lineIntegral} :: \texttt{SmoothPath}(M) \to \Omega^1(M) \to \mathbb{R}$

$\texttt{lineIntegral}(\gamma, \omega) = \int_\gamma \omega := \int_0^1 \gamma^* \omega(t) dt$

$\texttt{PSPath} :: ?\texttt{Path}(M)$

$\gamma : \texttt{PSPath} \iff \exists t : \texttt{IntervalPartition}[0,1] : \forall i \in \mathbb{I}(t) \;.\; \gamma|_{[t_i, t_{i+1}]} : \texttt{SmoothPath}$

$\texttt{lineIntegral} :: \texttt{PSPath}(M) \to \Omega^1(M) \to \mathbb{R}$

$\texttt{lineIntegral}(\gamma, \omega) = \int_\gamma \omega := \sum_{i \in \mathbb{I}} \int_{t_i}^{t_{i+1}} \gamma^* \omega(t) dt$

$\quad \texttt{where}\ t = \texttt{PSPath}(M)(\gamma)$

$\texttt{NewtonOnLine} :: \forall M : \texttt{Smanifold} \;.\; \forall p, q \in M \;.\; \forall \gamma : \texttt{PSPath}(M)(p,q) \;.\; \forall f : C^\infty(M) \;.$

$\quad . \int_\gamma \mathrm{d}f = f(q) - f(p)$

$\texttt{Scatch} :$

$\int_\gamma \mathrm{d}f = \int_0^1 \gamma^* \mathrm{d}f(t) dt = \int_0^1 \mathrm{d}\gamma^* f dt = \gamma^* f(1) - \gamma^* f(0) = f(q) - f(p)$

$\texttt{Conservative} :: ?\Omega(M)$

$\omega : \texttt{Conservative} \iff \forall \gamma : \texttt{PSPath} \,\&\, \texttt{Loop}(M) \;.\; \int_\gamma \omega = 0$

$\texttt{DifferentiabilityMark} :: \forall M : \texttt{Smanifold} \;.\; \forall f : M \to \mathbb{R} \;.\; \texttt{if}$

$\quad . \forall p \in M \;.\; \forall v \in V(M) \;.\; \forall c : \texttt{SCurve} : \dot{c}(0) = v : c(0) = p \;.$

$\quad . \frac{\mathrm{d}fc}{\mathrm{d}t}|_0 = \alpha(v) \;.\; f : C^\infty(M)$

$\texttt{Scatch} :$

$\mathrm{d}f = \alpha$

1.14 Moving Frames