Topological Manifolds

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July 25, 2021

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1 Subject Matter

1.1 Paracompactness and Partition of Unity

$$\texttt{LocallyFinite} :: \prod_{X \in \mathsf{TOP}} ?^3 X$$

$$\mathcal{A} : \texttt{LocallyFinite} \iff \forall x \in X \,.\, \exists U \in \mathcal{U}(x) \,.\, \left|\left\{A \in \mathcal{A} : A \cap U \neq \emptyset\right\}\right| < \infty$$

$$\texttt{OpenRefinement} :: \prod X \in \mathsf{TOP} \,.\, \texttt{Cover}(X) \to ?\texttt{OpenCover}(X)$$

$$\mathcal{U} : \texttt{OpenRefinement} \iff \Lambda\mathcal{O} : \texttt{Cover}(X) \,.\, \forall U \in \mathcal{U} \,.\, \exists O \in \mathcal{O} : U \subset O$$

$$\texttt{Paracomapct} :: ?\mathsf{TOP}$$

$$X : \texttt{Paracompact} \iff \forall \mathcal{O} : \texttt{OpenCover}(X) \,.\, \exists \mathcal{U} : \texttt{OpenRefinement}(X) \,\&\, \texttt{LocallyFinite}$$

$$\texttt{Exhaustion} :: \prod_{X \in \mathsf{TOP}} ?\Big(\mathbb{N} \to \texttt{CompactSubset}(X)\Big)$$

$$K : \texttt{Exhaustion} \iff X = \bigcup_{n=1}^{\infty} K_n \,\&\, \forall n \in \mathbb{N} \,.\, K_n \subset \operatorname{int} K_{n+1}$$

$\texttt{ExhaustionExists} :: \forall X : \texttt{T2} \,\&\, \texttt{SecondCountable} \,\&\, \texttt{LocallyCompact} \,.\, \exists \texttt{Exhaustion}(X)$

$\texttt{Proof} \;=$

$\Big(\mathcal{B}, [1]\Big) := \mathcal{C}\texttt{LocallyCompact} : \sum \mathcal{B} : \texttt{Base}(X) \,.\, \forall B \in \mathcal{B} \,.\, \texttt{Precompact}(X, B),$

$[2] := \mathcal{C}\texttt{SecondCountable}(B)\texttt{BaseEquivalence}(X, \mathcal{B}) : |\mathcal{B}| \leq \aleph_0,$

$B := \texttt{enumerate}(B, [2]) : \mathbb{N} \leftrightarrow \mathcal{B},$

$C_1 := \overline{B}_1 : \texttt{Compact}(X),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ C : n \to \texttt{Compact}(X),$

$\texttt{Assume}\ [3] : \forall i \in n \,.\, B_i \subset C_i,$

$\texttt{Assume}\ [4] : \forall i \in [1, n-1]_{\mathbb{N}} \,.\, C_i \subset \operatorname{int} C_i,$

$[5] := \texttt{FiniteCompactUnion} : \texttt{Compact}\,(X)\,,$

$\Big(k, [6]\Big) := \mathcal{C}\texttt{Compact}[5]\mathcal{C}\texttt{Base}(\mathcal{B})\mathcal{D}B : \sum k \in \mathbb{N} \,.\, \bigcup_{i=1}^{n} C_i \subset \bigcup_{i=1}^{k} B_i,$

$k' := \max(k, n+1) \in \mathbb{N},$

$C_{n+1} := \bigcup_{i=1}^{k'} \overline{B}_i : \texttt{Compact}(X),$

$[n.*.1] := \mathcal{D}C_{n+1}\mathcal{C}k' : B_{n+1} \subset C_{n+1},$

$[n.*.2] := \mathcal{D}C_{n+1}\mathcal{C}k'[6] : C_n \subset \operatorname{int} C_{n+1};$

$\rightsquigarrow \Big(C, [3]\Big) := \mathtt{I}\left(\sum\right) : \sum C : \mathbb{N} \to \texttt{Compact}(X) \,.\, \forall n \in \mathbb{N} \,.\, B_n \subset C_n \subset \operatorname{int} C_{n+1},$

$[4] := \mathcal{C}\texttt{Base}(\mathcal{B})\mathcal{D}B[3] : \bigcup_{n=1}^{\infty} C_n = X,$

$[*] := \mathcal{C}^{-1}\texttt{Exhuastiation}[4][3] : \texttt{Exhaustiation}(X, C);$

$\square$

$\texttt{ParacompactnesCondition} :: \forall X : \texttt{T2} \ \& \ \texttt{SecondCountable} \ \& \ \texttt{LocallyCompact} \ . \ \texttt{Paracompact}$

$\texttt{Proof} \ =$

$K := \texttt{ExhaustionExists} : \texttt{Exhaustion}(X),$

$\texttt{Assume} \ \mathcal{O} : \texttt{OpenCover}(X),$

$$\mathcal{U} := \Lambda n \in \mathbb{N} \ . \ \mathcal{O}\texttt{Compact}(K_n)(\mathcal{O}) : \prod_{n=1}^{\infty} \texttt{FiniteSubcover}(X, \mathcal{O}, K_n),$$

$K_0 := \emptyset_X :?X,$

$$\mathcal{O}' := \bigcup_{n=1}^{\infty} \left\{ U \setminus K_{n-1} \middle| U \in \mathcal{U}_n \right\} : \texttt{Refinement}(\mathcal{O}),$$

$\texttt{Assume} \ x \in X,$

$$\Big(n, [1]\Big) := \mathcal{O}\texttt{Exhaustion}(X)(x) : \sum n \in \mathbb{N} \ . \ x \in K_n,$$

$[2] := \mathcal{O}\texttt{Exhaustion}(X)[1] : x \in \operatorname{int} K_{n+1},$

$$[x.*] := \mathcal{D}\mathcal{O}'[2]\texttt{UnionCardinalityBound}(\mathcal{U}_{|n+1})\texttt{FiniteSumIsFinite} : \Big|\{O \in \mathcal{O}' : \operatorname{int} K_{n+1} \cap O \neq \emptyset\}\Big| \leq$$

$$\leq \sum_{i=1}^{n+1} |\mathcal{U}_i| < \infty;$$

$\rightsquigarrow [\mathcal{O}.*] := \mathcal{O}^{-1}\texttt{LocallyFinite} : \texttt{LocallyFinite}(X, \mathcal{O}');$

$\rightsquigarrow [*] := \mathcal{O}^{-1}\texttt{Paracompact} : \texttt{Paracompact}(X);$

$\square$

$\texttt{ParacompactHausdorffIsNormal} :: \forall X : \texttt{T2} \ \& \ {}_{\mathsf{TOP}}\texttt{Paracompact} \ . \ \texttt{T4}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ A, B : \texttt{Closed}(X),$

$\texttt{Assume} \ [1] : A \cap B = \emptyset,$

$\texttt{Assume} \ b \in B,$

$\texttt{Assume} \ a \in A,$

$[2] := \mathcal{O}\texttt{Disjoint}[1](b, a) : b \neq a,$

$$\Big(U_a, V_a, [a.*]\Big) := \mathcal{O}\texttt{T2}\Big(a, b, [2]\Big) : \sum U_b \in \mathcal{U}(b) \ . \ \sum V_b \in \mathcal{U}(a) \ . \ U_n \cap V_b = \emptyset,$$

$$\rightsquigarrow \Big(U, V, [2]\Big) := \texttt{I}\left(\prod\right) : \sum U, V : \prod_{a \in \mathcal{A}} \mathcal{U}(b) \times \mathcal{U}(a) \ . \ \forall a \in A \ . \ U(b) \cap V(a) = \emptyset,$$

$$\mathcal{V} := \{V_a | a \in A\} \cup \left\{A^{\complement}\right\} : \texttt{OpenCover}(X),$$

$\mathcal{V}' := \mathcal{O}\texttt{Paracompact}(X)(\mathcal{V}) : \texttt{Refinement}(X, \mathcal{V}) \ \& \ {}_{??X}\texttt{LocallyFinite}(X),$

$\mathcal{V}'' := \{v \in \mathcal{V}' : \exists a \in A : v \subset V_a\} : \texttt{OpenCover}(X, A) \ \& \ {}_{??X}\texttt{LocallyFinite}(X),$

$\texttt{Assume} \ v : \mathcal{V}'',$

$$\Big(a, [3]\Big) := \mathcal{D}\mathcal{V}'' : \sum a \in A \ . \ v \subset V_a,$$

$[4] := [2][3] : v \cap U_a = \emptyset,$

$[v.*] := \texttt{ClosureAltDef}[4] : b \notin \overline{v};$

$\rightsquigarrow [3] := \texttt{I}(\forall) : \forall v \in \mathcal{V}'' \ . \ b \notin \overline{v},$

$$K_b := \operatorname*{cl}_X \bigcup_{v \in \mathcal{V}''} v : \texttt{Closed}(X),$$

$$[4] := \mathcal{O}^{-1}\texttt{UnionLocallyFiniteUnionClusure} : b \notin \bigcup_{v \in \mathcal{V}''} \overline{v} = K_b,$$

$[b.*] := \mathcal{D}K_b[2] : A \subset \operatorname{int} K_b;$

$$\rightsquigarrow \Big(K, [2]\Big) := \texttt{I}\left(\sum\right) : \sum K : B \to \texttt{Closed}(X) \ . \ \forall b \in B \ . \ A \subset \operatorname{int} K_b \ \& \ b \notin K_b,$$

$\mathcal{U} := \left\{K_b^{\complement} | b \in B\right\} \cup \{B^{\mathcal{C}}\} : \mathtt{OpenCover}(X),$

$\mathcal{U}' := \mathcal{I}\mathtt{Paracompact}(X)(\mathcal{U}) : \mathtt{Refinement}(X, \mathcal{U})\ \&\ {}_{??X}\mathtt{LocallyFinite}(X),$

$\mathcal{U}'' := \{u \in \mathcal{U}' : \exists b \in B : u \subset K_b^{\mathcal{C}}\} : \mathtt{OpenCover}(X, B)\ \&\ {}_{??X}\mathtt{LocallyFinite}(X),$

$[3] := \mathtt{DualLocallyFiniteIntersection}\mathcal{D}\mathcal{U}'' : \bigcap_{u \in \mathcal{U}''} \operatorname{int} u^{\mathcal{C}} \in \mathcal{U}(A),$

$\Big[(A, B).*\Big] := \mathcal{D}\mathcal{U}'' : \bigcap_{u \in \mathcal{U}''} \operatorname{int} u^{\mathcal{C}} \cap \bigcup_{u \in \mathcal{U}''} = \emptyset;$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\mathtt{T4} : \mathtt{T4}(X),$

$\square$

$\mathtt{PartitionOfUnity} :: \prod_{X \in \mathsf{TOP}} \prod \mathcal{O} : \mathtt{OpenCover}(X) \;.\; \mathcal{O} \to X \xrightarrow{\mathsf{TOP}} [0, 1]$

$f : \mathtt{PartitionOfUnity} \iff \forall O \in \mathcal{O} \;.\; f_O\left(O^{\complement}\right) = \{0\}\ \&\ \mathtt{LocallyFinite}(X, \operatorname{supp} f)\ \&\ \sum_{O \in \mathcal{O}} f_O = 1$

$\mathtt{IndexedRefinement} :: \prod_{X \in \mathsf{TOP}} \prod \mathcal{I} : \mathtt{OpenCover}(X) \;.\; ?(\mathcal{O} \to \mathcal{T}(X))$

$\mathcal{U} : \mathtt{IndexedRefinement} \iff \mathtt{OpenCover}(X, \operatorname{Im} \mathcal{U})\ \&\ \forall O \in \mathcal{O} \;.\; \mathcal{U}_O \subset O$

$\mathtt{ParacompactOpenCoverRefiment} :: \forall X : \mathtt{T2}\ \&\ \mathtt{Paracompact} \;.\; \forall \mathcal{O} : \mathtt{OpenCover}(X) \;.$

$\quad .\; \exists \mathcal{V} : \mathtt{IndexedRefinement}(X, \mathcal{V}) : \mathtt{LocallyFinite}(X, \operatorname{Im} \mathcal{V})\ \&\ \forall O \in \mathcal{O} \;.\; \overline{\mathcal{V}}_O \subset O$

$\mathtt{Proof} =$

$\mathtt{Assume}\ x \in X,$

$\Big(O, [2]\Big) := \mathcal{I}\mathtt{OpenCover}(X)(x) : \sum O \in \mathcal{O} \;.\; \sum x \in O,$

$\rightsquigarrow \Big(U_x, [x.*]\Big) := \mathtt{HausdorffAltDef}[2] : \sum U_x \in \mathcal{U}(x) : \overline{U}_x \subset O,$

$\Big(U, [1]\Big) := \mathtt{I}\left(\sum\right) \mathtt{I}\left(\prod\right) : \sum \prod_{x \in X} U_x \in \mathcal{U}(x) \;.\; \exists O \in \mathcal{O} \;.\; \overline{U}_x \subset O;$

$\mathcal{U} := \mathcal{I}\mathtt{Paracompact}(X) : \mathtt{Refinement}(X, \operatorname{Im} \mathcal{U})\ \&\ \mathtt{LocallyFinite}(X),$

$[2] := \mathcal{D}\mathcal{U}[1] : \forall u \in \mathcal{U} \;.\; \exists O \in \mathcal{O} \;.\; \overline{u} \subset O,$

$\mathtt{Assume}\ O \in \mathcal{O},$

$\mathcal{V}_O := \bigcup \left\{u \in \mathcal{U} \middle| \overline{u} \subset O\right\} : \mathtt{Open}(X),$

$[O.*] := \mathtt{LocallyFiniteClosureUnion}[3] : \overline{\mathcal{V}}_O \subset O;$

$\rightsquigarrow \Big(\mathcal{V}, [3]\Big) := \mathtt{I}(\sum) : \sum \mathcal{V} : \mathcal{O} \to \mathcal{T}(X) \;.\; \forall O \in \mathcal{O} \;.\; \overline{\mathcal{V}}_O,$

$[4] := \mathcal{D}\mathcal{V} : \mathtt{IndexedRefinement}(X, \mathcal{O}, \mathcal{V}),$

$[*] := \mathcal{I}\mathcal{U}\mathcal{D}\mathcal{V} : \mathtt{LocallyFinite}\Big(X, \operatorname{Im} \mathcal{V}\Big);$

$\square$

$\texttt{PartitionOfUnityExists} :: \forall X : \texttt{Paracompact} \,\&_{\textsf{TOP}} \texttt{T2} \,.\, \forall \mathcal{O} : \texttt{OpenCover}(X) \,.\, \exists \texttt{PartitionOfUnity}(X, \mathcal{O})$

$\texttt{Proof} \ =$

$[1] := \texttt{ParacompactIsHausdorff}(X) : \texttt{T4}(X),$

$\Big(\mathcal{V}, [2]\Big) := \texttt{ParacompactOpenCoverRefinement}(X, \mathcal{O}) : \sum \mathcal{V} : \texttt{IndexedRefinement}(X, \mathcal{O}) \,\&$

$\quad \& \, \texttt{LocallyFinite}(X) \,.\, \forall O \in \mathcal{O} \,.\, \overline{\mathcal{V}}_O \subset O,$

$\Big(\mathcal{W}', [3]\Big) := \texttt{ParacompactOpenCoverRefinement}(X, \operatorname{Im} \mathcal{V}) : \sum \mathcal{W}' : \texttt{IndexedRefinement}(X, \operatorname{Im} \mathcal{V}) \,\&$

$\quad \& \, \texttt{LocallyFinite}(X) \,.\, \forall V \in \operatorname{Im} \mathcal{V} \,.\, \overline{\mathcal{W}'}_V \subset V,$

$\mathcal{W} := \mathcal{W}'_{\mathcal{V}} : \mathcal{O} \to \operatorname{Im} \mathcal{W}',$

$\Big(f, [4]\Big) := \texttt{NormalAltDef}(X, \overline{\mathcal{W}}, \mathcal{V}) : \sum f : \mathcal{O} \to X \xrightarrow{\textsf{TOP}} [0,1] \,.\, \forall O \in \mathcal{O} \,.\, f\Big(\overline{\mathcal{W}}_O\Big) = \{1\} \,\&\, f\Big(\mathcal{V}^{\complement}_O\Big) = \{0\},$

$F := \sum_{O \in \mathcal{O}} f_O : X \xrightarrow{\textsf{TOP}} [0,1],$

$[5] := \mathcal{I}\,\texttt{OpenCover}(X, \operatorname{Im} \mathcal{W}) \mathcal{D} F : \forall x \in X \,.\, F(x) \neq 0,$

$\phi := \Lambda O \in \mathcal{O} \,.\, \dfrac{f_O}{F} : \mathcal{O} \to X \xrightarrow{\textsf{TOP}} [0,1],$

$[7] := \mathcal{D}\phi\mathcal{D}^{-1} F \mathcal{I}\,\texttt{Inverse}(X \to \mathbb{R})(F) : \sum_{O \in \mathcal{O}} \phi_O = \sum_{O \in \mathcal{O}} \dfrac{f_O}{F} = \dfrac{F}{F} = 1,$

$[8] := \mathcal{D}\phi[4]\mathcal{D}\mathcal{W}\mathcal{I}^{-1} \operatorname{supp} : \forall O \in \mathcal{O} \,.\, \operatorname{supp} \phi_O \subset O,$

$[*] := \mathcal{I}^{-1}\texttt{PartitionOfUnity}[7][8] : \texttt{PartitionOfUnity}(X, \mathcal{O}, \phi);$

$\square$

$\texttt{ParacompactByPartitionOfUnity} :: \forall X : \texttt{T2} \,.\, \forall [0] : \forall \mathcal{O} : \texttt{OpenCover}(X) \,.$

$\quad .\, \exists \texttt{PartitionOfUnity}(X, \mathcal{O}) \,.\, \texttt{Paracompact}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \mathcal{O} : \texttt{OpenCover}(X),$

$f := [0](\mathcal{O}) : \texttt{PartitionOfUnity}(X, \mathcal{O}),$

$[1] := \mathcal{I}_3\texttt{PartitionOfUnity}(X, \mathcal{O}, f) : \sum_{O \in \mathcal{O}} f,$

$\mathcal{V} := \left\{ f_O^{-1}(0,1] \middle| O \in \mathcal{O} \right\} :?\mathcal{T}(X),$

$[2] := \mathcal{D}\mathcal{V}[1]\mathcal{I}\texttt{preimage} : \texttt{OpenCover}(X, \mathcal{V}),$

$[3] := \mathcal{I}_2\texttt{PartitionOfUnity}(X, \mathcal{O}, f) : \texttt{LocallyFinite}(X, \operatorname{supp} f),$

$[4] := \mathcal{D}\mathcal{V}[3]\texttt{ClosureIsSuper} : \texttt{LocallyFinite}(X, \mathcal{V}),$

$[5] := \mathcal{D}\mathcal{V}\texttt{PartitionOfUnity}(X, \mathcal{O}, f) : \forall O \in \mathcal{O} \,.\, \operatorname{supp} f_O \subset O,$

$[6] := \mathcal{D}\mathcal{V}[5]\texttt{ClosureIsSuper} : \forall V \in \mathcal{V} \,.\, \exists O \in \mathcal{O} : V \subset O,$

$[\mathcal{O}] := \mathcal{I}^{-1}\texttt{Refinement}[6][2] : \texttt{Refinemnt}(X, \mathcal{O}, \mathcal{V});$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Paracompact} : \texttt{Paracompact}(X);$

$\square$

$\texttt{CompactPartitionOfUnityIsFinite} :: \forall X : \texttt{Compact} \,.\, \forall \mathcal{O} : \texttt{OpenCover}(X) \,.$

$\quad .\, \forall f : \texttt{PartitionOfUnity}(X, \mathcal{O}) \,.\, \left|\left\{O \in \mathcal{O} | f_O \neq 0\right\}\right| < \infty$

$\texttt{Proof} \;=$

$[1] := \mathit{Cl}\texttt{PartitionOfUnity}(X, \mathcal{O}, f) : \texttt{LocallyFinitr}(X, \operatorname{supp} f),$

$\left(\mathcal{V}, [2]\right) := \mathit{Cl}\texttt{LocallyFinite}[1] : \sum \mathcal{V} : \texttt{OpenCover}(X) \,.\, \forall V \in \mathcal{V} \,.\, \left|\left\{O \in \mathcal{O} : V \cap \operatorname{supp} f_O\right\}\right| < \infty,$

$\mathcal{V}' := \mathit{Cl}\texttt{Compact}(X)(\mathcal{V}) : \texttt{FiniteSubcover}(X, \mathcal{V}),$

$[*] := \mathit{Cl}\texttt{Finite}\left(\mathcal{V}'\right)[2] : \left|\{O \in \mathcal{O} | f_0 \neq 0\}\right| < \infty;$

$\square$

1.2 Proper Maps

$$\mathtt{ProperMap} :: \prod_{X,Y \in \mathsf{TOP}} f : X \to Y$$

$$f : \mathtt{ProperMap} \iff \forall K : \mathtt{CompactSubset}(Y) \,.\, \mathtt{CompactSubset}\Big(X, f^{-1}(K)\Big)$$

$$\mathtt{DivergesToInfinity} :: \prod_{X \in \mathsf{TOP}} ?(\mathbb{N} \to X)$$

$$x : \mathtt{DivergesToInfinity} \iff \lim_{n\to\infty} x_n = \infty \iff \mathtt{ProperMap}(\mathbb{N}, X, x)$$

$\mathtt{DivergenceToInfinityCriterion} :: \forall X : \mathtt{T2} \,\&\, \mathtt{firstConountable} \,.\, \forall x : \mathbb{N} \to X \,.\, \lim_{n\to\infty} x_n \iff \forall n : \mathbb{N} \uparrow \mathbb{N} \,.$

$\mathtt{Proof} \; =$

$\mathtt{Assume}\ [1] : \lim_{n\to\infty} x_n,$

$\mathtt{Assume}\ n : \mathbb{N} \uparrow \mathbb{N},$

$\mathtt{Assume}\ [2] : \mathtt{Convergent}(X, x_n),$

$p := \lim_{n\to\infty} x_n \in X,$

$K := \operatorname{Im} x \cup \{p\} \in ?X,$

$\mathtt{Assume}\ \mathcal{O} : \mathtt{OpenCover}(K),$

$\Big(O, [3]\Big) := \mathtt{EOpenCover}(K, \mathcal{O}) : \sum O \in \mathcal{O} \,.\, p \in O,$

$\Big(M, [4]\Big) := \mathtt{ELimit}(X, x_n, p)(O) : \sum M \in \mathbb{N} \,.\, \forall m \in \mathbb{N} \,.\, m \geq M \Rightarrow x_{n_m} \in O,$

$\Big(\mathcal{O}', [5]\Big) := \mathtt{ELimit}(X, x_n, p)(x_{n_{[1,\ldots,M-1]}}) : \sum \mathcal{O}' : \mathtt{Finite} \,.\, \forall i \in [1, \ldots, M-1] \,.\, \exists O' \in \mathcal{O}' \,.\, x_{n_i} \in O',$

$\leadsto [\mathcal{O}.*] := \mathtt{IFiniteSubcober}[3][5] : \sum \mathtt{FiniteSubcover}(X, \mathcal{O}, \mathcal{O}');$

$\leadsto [4] := \mathtt{ICompactSubset} : \mathtt{CompactSubset}(X, K),$

$[5] := \mathtt{E}K[4] : x^{-1}(K) = \operatorname{Im} m,$

$[6] := \mathtt{E}m[5] : \Big|x^{-1}(K)\Big| = \infty,$

$[7] := \mathtt{E}\mathbb{N}[6] : !\, \mathtt{CompactSubset}\Big(\mathbb{N}, x^{-1}(K)\Big),$

$[1.*] := \mathtt{I}\bot\mathtt{EDevergentToInfinity}[7] : \bot;$

$\leadsto [1] := \mathtt{I}(\Rightarrow)\mathtt{I}(\forall)\mathtt{E}(\bot) : \lim_{n\to\infty} x_n \Rightarrow \forall n : \mathbb{N} \uparrow \mathbb{N} \,.\, !\, \mathtt{Convergent}(X, x_n),$

$\mathtt{Assume}\ [2] : \forall n : \mathbb{N} \uparrow \mathbb{N} \,.\, !\, \mathtt{Convergent}(X, x_n),$

$\mathtt{Assume}\ K : \mathtt{CompactSubset}(X, x_n),$

$\mathtt{Assume}\ [3] : |K \cap \operatorname{Im} x| = \infty,$

$[4] := \mathtt{T2CompactIsSequentiallyCompact}(K)\mathtt{ESequentiallyCompact}[3] : \exists n : \mathbb{N} \uparrow \mathbb{N} \,.\, \mathtt{Convergent}(X, x_n),$

$[3.*] := \mathtt{I}(\bot)[3][4] : \bot;$

$\leadsto [4] := \mathtt{E}(\bot) : |K \cap \operatorname{Im} x| < \infty,$

$[2.*] := \mathtt{IDivergesToInfinity} : \lim_{n\to\infty} x_n = \infty;$

$\leadsto [2] := \mathtt{I}(\Rightarrow) : \forall n : \mathbb{N} \uparrow \mathbb{N} \,.\, !\, \mathtt{Convergent}(X, x_n) \Rightarrow \lim_{n\to\infty} x_n,$

$[*] := \mathtt{E}(\iff) : \lim_{n\to\infty} x_n \iff \forall n : \mathbb{N} \uparrow \mathbb{N} \,.\, !\, \mathtt{Convergent}(X, x_n);$

$\square$

$\texttt{CompositionOfProperMapsIsProper} :: \forall X, Y, Z \in \mathsf{TOP} \;.\; \forall f : \texttt{ProperMap}(X, Y) \;.\; \forall g : \texttt{ProperMap}(Y, Z) \;.$
$\;.\; \texttt{ProperMap}(X, Z, fg)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ProperMapsPreservesDivergenceToInfinity} ::$
$:: \forall X, Y \in \mathsf{TOP} \;.\; \forall f : \texttt{ProperMap}(X, Y) \;.$
$\;.\; \forall x : \texttt{DivergesToInfinity}(X) \;.\; \lim_{n \to \infty} f(x_n)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ProperByCompactDomain} :: \forall X : \texttt{Compact} \;.\; \forall Y : \texttt{T2} \;.\; \forall X \xrightarrow{f} Y : \mathsf{TOP} \;.\; \texttt{ProperMap}(X, Y, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ K : \texttt{CompactSubset}(X),$

$[1] := \texttt{T2CompactIsClosed}(Y, K) : \texttt{Closed}(Y, K),$

$[2] := \mathsf{ETOP}(f)(K)[1] : \texttt{Closed}\Big(X, f^{-1}(K)\Big),$

$[*.K] := \texttt{ClosedCompactSubset}[2] : \texttt{CompactSubset}\Big(X, f^{-1}(K)\Big);$

$\square$

$$\texttt{TotallyUnbounded} :: \prod_{X, Y \in \mathsf{TOP}} ?\mathsf{TOP}(X, Y)$$

$f : \texttt{TotallyUnbounded} \iff \forall x : \texttt{DivergingToInfinity}(X) \;.\; \texttt{DivergingToInfinity}\big(f(x)\big)$

$\texttt{TotallyUnboundedIsProperByDomain} :: \forall X : \texttt{T2}\ \&\ \texttt{SecondCountable} \;.\; \forall Y \in \mathsf{TOP} \;.\; \forall f : \texttt{TotallyUnbounded} \;.$
$\texttt{ProperMap}(X, Y, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ K : \texttt{CompactSubset}(Y),$

$\texttt{Assume}\ [1] : \texttt{IsNotCompactSubset}\Big(X, f^{-1}(K)\Big),$

$[2] := \texttt{SecondCountableCompactIffSequentillyCompact}(X)[1] : \texttt{IsNotSequentiallyCompact}\Big(X, f^{-1}(K)\Big),$

$\Big(x, [3]\Big) := \mathsf{E}[2]\texttt{SequentiallyCompactDivergenceToInfinityCriterion} : \texttt{DivergesToInfinity}\Big(f^{-1}(K)\Big),$

$[4] := \texttt{ETotallyUnbounded}[3]\texttt{Epreimage} : \texttt{DivergesToInfinity}\Big(K, f(x)\Big),$

$[5] := \texttt{IPreimage}\big(f(x), K\big) : \big(f(x)\big)^{-1}(K) = \mathbb{N},$

$[6] := \texttt{EProperMap}(\mathbb{N}, K)[5] : |\mathbb{N}| < \infty,$

$[K.*] := \texttt{I}\bot\texttt{InfiniteNaturalNumbers}[6] : \bot;$

$\rightsquigarrow [*] := \mathsf{E}\bot\mathsf{I}\forall\texttt{IProperMap} : \texttt{ProperMap}(X, Y, f);$

$\square$

$\texttt{ProperByCompactFibers} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : \texttt{ClosedMap}(X, Y) \,.$

$\quad . \forall [0] : \forall y \in Y \,.\, \texttt{CompactSubset}\Big(X, f^{-1}(y)\Big) \,.\, \texttt{ProperMap}(X, Y, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ K : \texttt{CompactSubset}(X),$

$\texttt{Assume}\ \mathcal{O} : \texttt{OpenCover}\Big(X, f^{-1}(K)\Big),$

$\texttt{Assume}\ y \in K,$

$[1] := [0](y) : \texttt{ComapactSubset}\Big(X, f^{-1}(y)\Big),$

$[2] := \texttt{MonotonicPreimage}\Big(X, Y, f, K, \{y\}\Big) : f^{-1}(y) \subset f^{-1}(K(,$

$[3] := \texttt{EOpenCover}[2]\texttt{IOpenCover} : \texttt{OpenCover}\Big(X, f^{-1}(y), /O\Big),$

$\Big(\mathcal{O}'\Big) := \texttt{ECompactSubset}[1][3] : \texttt{FiniteSubCover}\Big(X, f^{-1}(y), \mathcal{O}),$

$C := X \setminus \bigcup_{O \in \mathcal{O}'} O : \texttt{Closed}(X),$

$[4] := \texttt{E}C\texttt{EsetminusE}\mathcal{O}' : f^{-1}(y) \cap C = \emptyset,$

$[5] := \texttt{EClosedMap}(X, Y, f)(K) : \texttt{Closed}(Y, f),$

$[6] := \texttt{Eimage}\Big(f, f^{-1}(y)\Big)\texttt{EDisjoint}[4]\texttt{Icomplement} : y \in f^{\complement}(C),$

$U_y := f^{\complement}(C) : \mathcal{U}(y);$

$\rightsquigarrow \Big(U, [1]\Big) := \texttt{I}\left(\prod\right) : \prod_{y \in K} \mathcal{U}(y) \,.\, \texttt{OpenCover}(Y, K, \operatorname{Im} U),$

$\mathcal{V} := \texttt{ECompactSubset}(Y, K)(\operatorname{Im} U) : \texttt{FiniteSubcover}(Y, K, \operatorname{Im} U),$

$\Big(\mathcal{O}', [2]\Big) := \texttt{E}U\texttt{E}\mathcal{V} : \sum \mathcal{O}' : \mathcal{V} \to \texttt{Finite}(\mathcal{O}) \,.\, \forall V \in \mathcal{V} \,.\, f^{-1}(V) = \bigcup_{O \in \mathcal{O}'} O,$

$\mathcal{O}'' := \bigcup_{V \in \mathcal{V}} \mathcal{O}' :?\mathcal{O},$

$[3] := \texttt{VeryFiniteUnion}(\mathcal{V}, \mathcal{O}')\texttt{I}\mathcal{O}'' : \texttt{FiniteSubset}\Big(\mathcal{O}, \mathcal{O}''\Big),$

$[K.*] := \texttt{EFiniteSubcover}(Y, K, \operatorname{Im} U, \mathcal{V})[1][2][3]\texttt{IFiniteSubcover}\Big(X, f^{-1}(K), \mathcal{O}\Big) :$

$\quad :: \texttt{FiniteSubcover}\Big(X, f^{-1}(K), \mathcal{O}, \mathcal{O}''\Big);$

$\rightsquigarrow [*] := \texttt{I}\forall\texttt{ICompactSubsetI}\forall\texttt{IProperMap} : \texttt{ProperMap}(X, Y, f);$

$\square$

$\texttt{ProperEmbedding} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall f : \texttt{TopologicalEmbedding}(X, Y, f) \,.\, \forall [0] : \texttt{Closed}\Big(Y, f(X)\Big) \,.$

$\quad . \texttt{ProperMap}(X, Y, f)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ProperByLeftInverse} :: \forall X \in \mathsf{TOP} . \forall Y : \texttt{T2} . \forall X \xrightarrow{f} Y : \mathsf{TOP} . \forall g : \texttt{LeftInverse}\mathsf{TOP}, X, Y, f . \texttt{ProperMap}($

$\texttt{Proof} =$

$[1] := \texttt{ELeftInverse}(\mathsf{TOP}, X, Y, f, g) : fg = \mathrm{id},$

$\texttt{Assume}\ K : \texttt{CompactSubset}(Y),$

$\texttt{Assume}\ x : f^{-1}(K),$

$[3] := \texttt{Epreimage} : f(x) \in K,$

$[4] := \texttt{E}(=)\Big([1], fg(x)\Big)\texttt{E}\,\mathrm{id} : fg(x) = x,$

$[x.*] := \texttt{Iimage} : x \in g(K);$

$\rightsquigarrow [2] := \texttt{ISubset} : f^{-1}(K) \subset g(K),$

$[3] := \texttt{CompactImage}(K, g) : \texttt{CompactSubset}\Big(X, g(K)\Big),$

$[4] := \texttt{ProperByCompactDomain}[3] : \texttt{ProperMap}(g(K), Y, f_{g(K)}),$

$[5] := \texttt{EProperMap}[4][2] : \texttt{CompactSubset}\Big(g(K), f^{-1}(K)\Big),$

$\rightsquigarrow [K.*] := \texttt{ComapactSubsetTower}[2][5] : \texttt{CompactSubset}\Big(X, f^{-1}(K)\Big),$

$\rightsquigarrow [*] := \texttt{IProperMap} : \texttt{ProperMap}\Big(X, f^{-1}(K)\Big);$

$\square$

$\texttt{ProperMapRestriction} :: \forall X, Y \in \mathsf{TOP} . \forall f : \texttt{ProperMap}(X, Y) . \forall A : \texttt{Saturated}(X, Y, f) .$

$. \texttt{ProperMap}\Big(f_{|A}, A, f(A)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompactlyGenerated} :: ?\mathsf{TOP}$

$X : \texttt{CompactlyGenerated} \iff \forall A \subset X . \Big(\forall K : \texttt{CompactSubset}(X, A) . \forall \texttt{Closed}(K, A \cap K)\Big) \Rightarrow$

$\Rightarrow \texttt{Closed}(X, A)$

$\texttt{categoryOfCompactlyGenerated} :: \mathsf{CAT}$

$\texttt{categoryOfCompactlyGenerated}\,() = \mathsf{CG} := (\texttt{CompactlyGenerated}, \mathsf{TOP}\ \&\ \texttt{ProperMap}, \circ, \mathrm{id})$

$\texttt{FirstCountableIsCG} :: \forall X : \texttt{FirstCountable} \, . \, X \in \mathsf{CG}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A \in X,$

$\texttt{Assume}\ [1] : \forall K : \texttt{CompactSubset}(X, A) \, . \, \texttt{Closed}(X, A \cap K),$

$\texttt{Assume}\ x \in \overline{A},$

$\Big(a, [2]\Big) := \texttt{AltClosureDefinition}(a) : \sum a : \mathbb{N} \to A \, . \, \lim_{n\to\infty} a_n = x,$

$K := \operatorname{Im} a \cup \{x\} : \texttt{CompactSubset}(X),$

$[3] := [1](A) : \texttt{Closed}(K, A \cap K),$

$[x.*] := \texttt{ClosedByLimits}[3][2]\texttt{Eintersect} : x \in A \cap K \subset A;$

$\leadsto [2] := \texttt{ISubset} : \bar{A} \subset A;$

$[3] := \texttt{Eclosure}(A)[2]\texttt{ISetEq} : A = \bar{A},$

$[*] := \texttt{E}(=)\Big([3], \texttt{Eclosure}\Big) : \texttt{Closed}(X, A);$

$\leadsto [1] := \texttt{ICG} : X \in \mathsf{CG};$

$\square$

$\texttt{LocallyCompactIsCG} :: \forall X : \texttt{LocallyCompact} \, . \, X \in \mathsf{CG}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A \in X,$

$\texttt{Assume}\ [1] : \forall K : \texttt{CompactSubset}(X, A) \, . \, \texttt{Closed}(X, A \cap K),$

$\texttt{Assume}\ x \in \overline{A},$

$\Big(U, K, [2]\Big) := \texttt{ELocallyCompact}(a) : \sum U \in \mathcal{U}(x) \, . \, \sum K : \texttt{CompactSubset}(X, K) \, . \, U \subset K,$

$[3] := [1](K) : \texttt{Closed}(K, K \cap A),$

$\texttt{Assume}\ V : \mathcal{U}(x),$

$[4] := \texttt{ClosureAltDef}(A, x) : U \;\&\; V \cap U \cap A \neq \emptyset,$

$[*.5] := [4][2] : K \cap V \cap A \neq \emptyset;$

$\leadsto [4] := \texttt{ClosureAltDef}[2][4][5] : x \in A \cap K,$

$[5] := \texttt{Eintersect}[1] : x \in A;$

$[3] := \texttt{Eclosure}(A)[2]\texttt{ISetEq} : A = \bar{A},$

$[*] := \texttt{E}(=)\Big([3], \texttt{Eclosure}\Big) : \texttt{Closed}(X, A);$

$\leadsto [1] := \texttt{ICG} : X \in \mathsf{CG};$

$\square$

$\texttt{ClosedMapLemma} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,\&\, \texttt{T2} \,.\, \forall X \xrightarrow{f} Y : \mathsf{TOP} \,.\, \texttt{ProperMap}(X,Y,f) \Rightarrow \texttt{ClosedMap}(X,Y,f)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$\texttt{Assume}\ K : \texttt{CompactSubset}(Y),$

$[1] := \texttt{T2CompactIsClosed} : \texttt{Closed}(Y,K),$

$[2] := \texttt{EProperMap}(X,Y,f)(K) : \texttt{Comcpact}\Big(X, f^{-1}(K)\Big),$

$[2] := \mathsf{ETOP}(X,Y,f)(K) : \texttt{Closed}\Big(X, f^{-1}(K)\Big),$

$[4] := \mathsf{ETOP}(X)\Big(f^{-1}(K), A\Big) : \texttt{Closed}\Big(X, f^{-1}(K)\cap A\Big),$

$[5] := \texttt{ClosedSubset}\Big(X, f^{-1}(K)\Big)[4] : \texttt{Closed}\Big(f^{-1}(K), f^{-1}(K)\cap A\Big),$

$[6] := \texttt{CompactClosedSubset}[5] : \texttt{CompactSubset}\Big(f^{-1}(K), f^{-1}(K)\cap A\Big),$

$[7] := \texttt{ContinuousMapPreservesCompacts}\Big(f^{-1}(K), K, f_{|f^{-1}(K)}\Big)\Big(f^{-1}(K)\cap A\Big) :$

$\quad : \texttt{CompactSubset}\Big(K, K\cap f(A)\Big),$

$[K,*] := \texttt{T2CompactIsClosed}[7] : \texttt{Closed}\Big(K, K\cap f(A)\Big);$

$\rightsquigarrow [A.*] := \mathsf{ECG}(Y)[8] : \texttt{Closed}\Big(Y, f(A)\Big);$

$\rightsquigarrow [*] := \texttt{IClosedMap} : \texttt{ClosedMap}(X,Y,f);$

$\square$

$\texttt{EmbeddingProperIffClosed} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,\&\, \texttt{T2} \,.\, \forall f : \texttt{TopologicalEmbedding}(X,Y) \,.$

$\quad .\, \texttt{ProperMap}(X,Y,f) \iff \texttt{Closed}\Big(Y, f(X)\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SurjectiveProperIsQuotientMap} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,\&\, \texttt{T2} \,.\, \forall X \xrightarrow{f} Y : \mathsf{TOP} \,.$

$\quad .\, \texttt{ProperMap}(X,Y,f) \,\&\, \texttt{Surjective}(X,Y,f) \iff \texttt{QuotientMap}\Big(X,Y,f\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InjectiveProperIsEmbedding} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,\&\, \texttt{T2} \,.\, \forall X \xrightarrow{f} Y : \mathsf{TOP} \,.$

$\quad .\, \texttt{ProperMap}(X,Y,f) \,\&\, \texttt{Injective}(X,Y,f) \iff \texttt{TopologicalEmbedding}\Big(X,Y,f\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{BijectiveProperIsHomeo} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{CG} \,\&\, \texttt{T2} \,.\, \forall X \xrightarrow{f} Y : \mathsf{TOP} \,.$

$\quad .\, \texttt{ProperMap}(X,Y,f) \,\&\, \texttt{Bijective}(X,Y,f) \iff \texttt{Homeomorphism}\Big(X,Y,f\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.3 Topological Manifold

$\texttt{TopologicalManifold} :: \mathbb{Z}_+ \times \texttt{T2} \,\&\, \texttt{SecondCountable}$

$(n, M) : \texttt{TopologicalManifold} \iff (n, M) \in \mathsf{TOPM} \iff \forall x \in M \,.\, \exists U \in \mathcal{U}(x) : U \cong_{\mathsf{TOP}} \mathbb{R}^n$

$\texttt{dimension} :: \mathsf{TOPM} \to \mathbb{Z}_+$

$\texttt{dimension}\,(n, M) = \dim(n, M) := n$

$\texttt{manifold} :: \mathsf{TOPM} \to \mathsf{TOP}$

$\texttt{manifold}\,(n, M) = (n, M) := M$

$\texttt{CoordinateCharts} = \mathcal{CC}_{(n,X)}(x) := \prod n \in \mathbb{Z}_+ \,.\, \prod X \in \mathsf{TOP} \,.\, \prod x \in X \,.\, \sum U \in \mathcal{U}(x) \,.$

$\,.\, U \xleftrightarrow{\mathsf{TOP}} \mathbb{R}^n : \texttt{Type};$

$\texttt{CoordinateChartsExists} :: \forall M \in \mathsf{TOPM} \,.\, \forall x \in M \,.\, \mathcal{CC}_M(x) \neq \emptyset$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SubmanifoldAsOpenSubsets} :: \forall M \in \mathsf{TOPM} \,.\, \forall U \in \mathcal{T}(M) \,.\, U \in \mathsf{TOPM}(\dim M)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{TopologicalManifoldWithBoundary} :: \mathbb{Z}_+ \times \texttt{T2} \,\&\, \texttt{SecondCountable}$

$(n, M) : \texttt{TopologicalManifoldWithBoundary} \iff (n, M) \in \mathsf{TOPM}_\partial \iff$

$|\forall x \in M \,.\, \left(\exists U \in \mathcal{U}(x) : U \cong_{\mathsf{TOP}} \mathbb{R}^n\right)\Big|\left(\exists U \in \mathcal{U}(x) : U \cong_{\mathsf{TOP}} \mathbb{R}^n_+\right)$

$\texttt{dimension} :: \mathsf{TOPM}_\partial \to \mathbb{Z}_+$

$\texttt{dimension}\,(n, M) = \dim(n, M) := n$

$\texttt{manifold} :: \mathsf{TOPM}_\partial \to \mathsf{TOP}$

$\texttt{manifold}\,(n, M) = (n, M) := M$

$\texttt{boundary} :: \prod M \in \mathsf{TOPM}_\partial \,.\, \texttt{Closed}(M)$

$\texttt{boundary}\,() = \partial M := \{m \in M : \mathcal{CC}_M(m) = \emptyset\}$

$\texttt{interior} :: \prod M \in \mathsf{TOPM}_\partial \,.\, \mathsf{TOPM}$

$\texttt{interior}\,() = \operatorname{int} M := M \setminus \partial M$

TopologicalManifoldsLocallyCompact :: $\forall M \in \mathsf{TOPM} \cap \mathsf{TOPM}_\partial \,.\, \texttt{LocallyCompact}(M)$

Proof =

...

$\square$

TopologicalManifoldsParacompact :: $\forall M \in \mathsf{TOPM} \cap \mathsf{TOPM}_\partial \,.\, \texttt{Paracompact}(M)$

Proof =

...

$\square$

ProductOfTopologicalManifolds :: $\forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.\, \forall M : \prod_{i=1}^{n} \mathsf{TOPM}(m_i) \,.\, \prod_{i=1}^{n} M_i \in \mathsf{TOPM}\left(\sum_{i=1}^{n} m_i\right)$

Proof =

...

$\square$

ProductOfTopologicalManifoldsWithBoundary :: $\forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.\, \forall M : \prod_{i=1}^{n} \mathsf{TOPM}_\partial(m_i) \,.$

$\,.\, \prod_{i=1}^{n} M_i \in \mathsf{TOPM}_\partial\left(\sum_{i=1}^{n} m_i\right)$

Proof =

...

$\square$

SumOfTopologicalManifolds :: $\forall n, m \in \mathbb{N} \,.\, \forall M : \prod_{i=1}^{n} \mathsf{TOPM}(m) \,.\, \bigsqcup_{i=1}^{n} M_i \in \mathsf{TOPM}\,(m)$

Proof =

...

$\square$

ProductOfTopologicalManifoldsWithBoundary :: $\forall n \in \mathbb{N} \,.\, \forall m : n \to \mathbb{N} \,.\, \forall M : \prod_{i=1}^{n} \mathsf{TOPM}_\partial(m_i) \,.$

$\,.\, \prod_{i=1}^{n} M_i \in \mathsf{TOPM}_\partial\left(\sum_{i=1}^{n} m_i\right)$

Proof =

...

$\square$

$\texttt{CompactManifoldCoordinateEmbedding} :: \forall M \in \mathsf{TOPM} \;\&\; \mathsf{HC} \;.\; \exists n \in \mathbb{N} : \exists \texttt{HomeomorphicEmbedding}(M, \mathbb{R}^n)$

$\texttt{Proof} \;=$

$d := \dim M \in \mathbb{Z}_+,$

$\Big(\mathcal{O}, [1]\Big) := \mathcal{A}\mathsf{TOPM}(d, M) : \sum \mathcal{O} : \texttt{OpenCover}(X) \;.\; \forall O \in \mathcal{O} \;.\; O \cong \mathbb{R}^d,$

$\mathcal{V} := \mathcal{A}\texttt{Compact}(X)(\mathcal{O}) : \texttt{FiniteSubcover}(X, \mathcal{O}, \mathcal{V}),$

$n := d\Big(|\mathcal{V}| + 1\Big) \in \mathbb{N},$

$f := \texttt{PartitionOfUnityExist}(X, \mathcal{V}) : \texttt{PartitionOfUnity}(X, \mathcal{V}, f),$

$\varphi := \mathcal{A}\texttt{Isomprphic}(\mathsf{TOP}, \mathcal{V}, \mathbb{R}^d) : \prod_{V \in \mathcal{V}} V \xleftrightarrow{\mathsf{TOP}} \mathbb{R}^d,$

$\mathbf{x} := \bigoplus_{U \in \mathcal{U}} f_U(\varphi_U \oplus 1) : X \xrightarrow{\mathsf{TOP}} \mathbb{R}^n,$

$\texttt{Assume}\; p, q \in X,$

$\texttt{Assume}\; [2] : \mathbf{x}(p) = \mathbf{x}(q),$

$\Big(V, [3]\Big) := \mathcal{A}\texttt{PartitionOfUnity}(X, \mathcal{V}, f)(p) : \sum V \in \mathcal{V} \;.\; f_V(p) \neq 0,$

$[4] := \mathcal{D}\mathbf{x}[3][2] : f_V(q) \neq 0,$

$[5] := \mathcal{A}\texttt{PartionOfUnity}(X, \mathcal{V}, f)[3][4] : p, q \in V,$

$[6] := \mathcal{D}\mathbf{x}[2][5] : \varphi_V(p) = \varphi_V(q),$

$\Big[(p, q) . *\Big] := \varphi_V^{-1}[6] : p = q;$

$\rightsquigarrow [2] := \mathcal{A}^{-1}\texttt{Injective} : \mathbf{x} : X \hookrightarrow \mathbb{R}^n,$

$[*] := \texttt{CompactInjectionTHM}[2] : \texttt{HomeomorphicEmbedding}(X, \mathbb{R}^n, \mathbf{x});$

$\square$

$\texttt{FunctionByAZeroSet} :: \forall M \in \mathsf{TOPM} \;.\; \forall A : \texttt{Closed}(M) \;.\; \exists F : M \xrightarrow{\mathsf{TOP}} \mathbb{R}_+ : A = F^{-1}\{0\}$

$\texttt{Proof} \;=$

$d := \dim M : \mathbb{Z}_+,$

$\Big(\mathcal{O}, [1]\Big) := \mathcal{A}\mathsf{TOPM}(d, M) : \sum \mathcal{O} : \texttt{OpenCover}(X) \;.\; \forall O \in \mathcal{O} \;.\; O \cong \mathbb{R}^d,$

$\varphi := \mathcal{A}\texttt{Isomprphic}(\mathsf{TOP}, \mathcal{O}, \mathbb{R}^d) : \prod_{O \in \mathcal{O}} O \xleftrightarrow{\mathsf{TOP}} \mathbb{R}^d,$

$f := \texttt{PartitionOfUnityExist}(X, \mathcal{O}) : \texttt{PartitionOfUnity}(X, \mathcal{O}, f),$

$\Delta := \Lambda O \in \mathcal{O} \;.\; \Lambda x \in X \;.\; \texttt{if}\; x \in O \;\texttt{then}\; \mathrm{dist}\Big(\varphi_O(x), \varphi_O(A \cap O)\Big) \;\texttt{else}\; 0 : \mathcal{O} \to X \to \mathbb{R}_+,$

$F := \sum_{O \in \mathcal{O}} f_O \Delta_O : X \xrightarrow{\mathsf{TOP}} \mathbb{R}_+,$

$[*] := \mathcal{D}F : F^{-1}\{0\} = 0;$

$\square$

$\texttt{ManifoldIsPerfectlyNormal} :: \forall M \in \mathsf{TOPM} . \forall A, B : \texttt{Closed}(M) . \forall [0] : A \cap B = \emptyset . \exists F : X \xrightarrow{\mathsf{TOP}} [0,1] .$
$. A = F^{-1}(0) \ \& \ B = F^{-1}(1)$

Proof $=$

$\Big(f, [1]\Big) := \texttt{FunctionByZeroSet}(X, A) : \sum f : X \xrightarrow{\mathsf{TOP}} \mathbb{R}_+ . A = f^{-1}(0),$

$\Big(g, [2]\Big) := \texttt{FunctionByZeroSet}(X, B) : \sum g : X \xrightarrow{\mathsf{TOP}} \mathbb{R}_+ . B = g^{-1}(0),$

$F := \frac{f}{f+g} : X \xrightarrow{\mathsf{TOP}} [0,1],$

$[*] := \mathcal{O}F[0] : F^{-1}(0) = A \ \& \ F^{-1}(1) = B;$

$\square$

$\texttt{ExhaustionFunction} :: \prod_{X \in \mathsf{TOP}} X \xrightarrow{\mathsf{TOP}} \mathbb{R}$

$f : \texttt{ExhaustionFunction} \iff \forall t \in \mathbb{R} . \texttt{Compact}(X, f^{-1}(-\infty, t))$

$\texttt{TopologicalManifoldIsCompactlyGenerated} :: \mathsf{TOPM} \subset \mathsf{CG}$

Proof $=$

$\dots$

$\square$

$\texttt{TopologicalManifoldWithBounaryIsCompactlyGenerated} :: \mathsf{TOPM}_\partial \subset \mathsf{CG}$

Proof $=$

$\dots$

$\square$

$\texttt{ExhastionFunctionExists} :: \forall M \in \mathsf{TOPM} \, . \, \exists f : \texttt{ExhastionFunction}(X) : f > 0$

$\texttt{Proof} \; =$

$\Big(\mathcal{O}, [1]\Big) := \mathcal{I}\texttt{LocallyCompact}(M) : \sum \mathcal{O} : \texttt{OpenCover}(M) \, . \, \forall O \in \mathcal{O} \, . \, \texttt{Precompact}(X, O),$

$\phi' := \texttt{PartitionOfUnityExist}(X, \mathcal{O}) : \texttt{PartitionOfUnity}(X, \mathcal{O}),$

$[2] := \mathcal{I}\texttt{SecondCountable}(X)\texttt{BaseEquivalence}(X, \mathcal{O}) : |\mathcal{O}| \leq \aleph_0,$

$O := \texttt{Functor}\,(enumerate, ()\, \mathcal{O}) : \mathbb{N} \leftrightarrow \mathcal{O},$

$\phi := \phi'_O : \mathbb{N} \to X \xrightarrow{\mathsf{TOP}} [0, 1],$

$f := \sum_{n=1}^{\infty} n\phi_n : X \xrightarrow{\mathsf{TOP}} \mathbb{R}_{++},$

$[3] := \mathcal{D} f \mathcal{I}\texttt{PartitionOfUnity}(X, \mathcal{O}, \varphi) : f \geq 1,$

$\texttt{Assume } n \in \mathbb{N},$

$\texttt{Assume } x \in X,$

$\texttt{Assume } [4] : f(x) \leq n,$

$\texttt{Assume } [5] : \forall k \in n \, . \, f(x) \notin O_k,$

$[6] := \mathcal{D}\phi\mathcal{I}^2\texttt{PartitionOfUinity}(X, \mathcal{O}, \phi)[5] : 1 = \sum_{k=1}^{n} \phi_k(x) = \sum_{k=n+1}^{n} \phi_k(x),$

$[7] := \mathcal{D} f \mathcal{D}\phi\mathcal{I}\texttt{PartitionOfUinity}(X, \mathcal{O}, \phi)[5]\texttt{PosMultIneq}(\mathbb{R})\texttt{SumIneq}(\mathbb{R})\mathcal{I}\mathsf{RING}(\mathbb{R})[6] : f(x) = \sum_{k=1}^{\infty} k\phi_k(x) =$

$= \sum_{k=n+1}^{\infty} k\phi_k(x) \geq \sum_{k=n+1}^{\infty} (n+1)\phi_k(x) = (n+1) \sum_{k=n+1}^{\infty} \phi_k(x) = (n+1),$

$[5.*] := [4][7] : \bot;$

$\rightsquigarrow [6] := \texttt{E}(\bot)\mathcal{I}^{-1}\texttt{Union} : x \in \bigcup_{i=1}^{n} U_i,$

$[n.*] := \texttt{ClosureIsSuper} : x \in \bigcup_{i=1}^{n} \overline{O}_i;$

$\rightsquigarrow [4] := \texttt{I}(\forall)\mathcal{I}^{-1}\texttt{preimage}(f)\texttt{I}(\forall)\texttt{I}(\Rightarrow) : \forall n \in \mathbb{N} \, . \, f^{-1}(0, n] \subset \bigcup_{i=1}^{n} \overline{O}_i,$

$\texttt{Assume } t : \mathbb{R}_+,$

$\Big(n, [5]\Big) := \mathcal{I}\texttt{Archimedean}(\mathbb{R}, n) : \sum n \in \mathbb{N} \, . \, n \geq t,$

$[6] := \texttt{monotonicPreimage}[4](t) : f^{-1}(0, t] \subset f^{-1}(0, n] \subset \bigcup_{i=1}^{n} \overline{O}_i,$

$[t.*] := \texttt{ClosedSubsetIsCompact}[6] : \texttt{Compact}\Big(X, f^{-1}(0, t]\Big);$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{ExhuastionFunction} : \texttt{ExhuastionFunction}(X, f);$

$\square$

2 Cell Complexes

2.1 Cell Structure

$\texttt{Cell} :: \mathbb{Z}_+ \to ?\mathsf{TOP}$

$B : \texttt{Cell} \iff \Lambda n \in \mathbb{Z}_+ \,.\, B \cong_{\mathsf{TOP}} \mathbb{B}^n$

$\texttt{ClosedCell} :: \mathbb{Z}_+ \to ?\mathsf{TOP}$

$B : \texttt{ClosedCell} \iff \Lambda n \in \mathbb{Z}_+ \,.\, B \cong_{\mathsf{TOP}} \mathbb{D}^n$

$\texttt{CompactConvexBodyIsClosedCell} :: \forall n \in \mathbb{N} \,.\, \forall C : \texttt{Compact} \,\&\, \texttt{ConvexBody}(\mathbb{R}^n) \,.\, \texttt{ClosedCell}(C)$

$\texttt{Proof} \;=$

$[1] := \text{ꓷ}\texttt{ConvexBody}(\mathbb{R}^n, C) : \operatorname{int} C \neq \emptyset,$

$O := \text{ꓷ}\texttt{Nonempty}[1] \in \operatorname{int} C,$

$[2] := \texttt{HeineBorelTHM}(\mathbb{R}^n, C) : \texttt{Bounded} \,\&\, \texttt{Closed}(\mathbb{R}^n, C),$

$[2'] := \texttt{ConvexBodyInteriorIsCore}(\mathbb{R}^n, C)(c) : c \in \operatorname{core} C,$

$\texttt{Assume}\ c \in \partial C,$

$v := c - O \in \mathbb{R}^n,$

$[3] := \text{Ↄ}v\text{ꓷ}O\text{ꓷ}c\text{ꓷ}\,\partial C : v \neq 0,$

$R := \{O + tv | t \in \mathbb{R}_+\} :?\mathbb{R}^n,$

$[4] := \text{Ↄ}R : \texttt{LinearlyConnected}(\mathbb{R}^n, R),$

$[5] := \text{ꓷ}\texttt{Bounded}(\mathbb{R}^n, C)[2]\text{Ↄ}R : R \cap C^{\complement} \neq \emptyset,$

$[6] := \texttt{ClosedConectedIntersection}(\mathbb{R}^n, C, R)[3, 5, 6] : R \cap \partial C \neq \emptyset,$

$\texttt{Assume}\ x, y \in R \cap \partial C,$

$\texttt{Assume}\ [7] : x \neq y,$

$[8] := \text{Ↄ}R[7][3] : x \in (c, y) | y \in (c, x),$

$[9] := \text{ꓷ}\texttt{Closed} \,\&\, \texttt{Convex}(\mathbb{R}^n, C)\text{ꓷ}(x, y) : x, y \in \operatorname{lin} C,$

$[10] := \texttt{ConvexInteriorInCore}(\mathbb{R}^n, C)[2', 9][8] : x \in \operatorname{core} C | y \in \operatorname{core} C,$

$[11] := \texttt{ConvexBodyInteriorIsCore}[10] : x \in \operatorname{int} C | y \in \operatorname{int} C,$

$\Big[(x, y). *\Big] := \text{ꓷ}\,\partial C\text{ꓷ}(x, y)[11] : \perp;$

$[7] := \texttt{E}(\perp)[6] : \texttt{Singleton}(R \cap \partial C),$

$[8] := \text{ꓷ}\texttt{Singleton}[7]\text{Ↄ}R\text{ꓷ}c : R \cap \partial C = \{c\},$

$f(c) := \dfrac{v}{\|v\|} : \mathbb{D}^n;$

$\leadsto f := \texttt{I}(\to) : \partial C \xrightarrow{\mathsf{TOP}} \mathbb{D}^n,$

$[2] := \text{Ↄ}f\text{ꓷ}O : \Big(f : \partial C \leftrightarrow \partial \mathbb{D}^n\Big),$

$[3] := \texttt{CompactOpenMappingTHM}[2] : \Big(f : \partial C \xleftrightarrow{\mathsf{TOP}} \partial \mathbb{D}^n\Big),$

$\varphi := \Lambda v \in \mathbb{D}^n \,.\, \texttt{if}\ v == 0\ \texttt{then}\ O\ \texttt{else}\ O + \|v\| f^{-1}\left(\dfrac{v}{\|v\|}\right) : \mathbb{D}^n \xrightarrow{\mathsf{TOP}} C,$

$[4] := \text{Ↄ}\varphi\text{ꓷ}O : \Big(\varphi : C \leftrightarrow \mathbb{D}^n\Big),$

$[5] := \texttt{CompactOpenMappingTHM}[4] : \Big(\varphi : C \xleftrightarrow{\mathsf{TOP}} \mathbb{D}^n\Big),$

$[*] := \text{ꓷ}^{-1}\texttt{Isomorphic}(\mathsf{TOP})[5] : C \cong_{\mathsf{TOP}} \mathbb{D}^n;$

$\square$

$\texttt{CellDecomposition} :: \prod X : \mathsf{TOP} \,.\, \sum \mathcal{E} : \prod_{n=0}^{\infty} ?\big(?X \,\&\, \texttt{Cell}(n)\big) \,.\, \prod_{n=1}^{\infty} \prod_{E \in \mathcal{E}_n} \mathbb{D}^n \xrightarrow{\mathsf{TOP}} X$

$$(\mathcal{E}, \varphi) : \texttt{CellDecomposition} \iff X = \bigsqcup_{n=0}^{\infty} \bigcup_{E \in \mathcal{E}} E \,\&$$

$$\& \, \forall n \in \mathbb{N} \,.\, \forall E \in \mathcal{E}_n \,.\, \varphi_{n,E|\mathbb{B}^n} : \mathbb{B}^n \overset{\mathsf{TOP}}{\longleftrightarrow} E \,\&$$

$$\& \, \forall n \in \mathbb{N} \,.\, \forall E \in \mathcal{E}_n \,.\, \exists \mathcal{C} : \prod_{i=1}^{n-1} ?\mathcal{E}_i : \partial C = \bigcup_{i=0}^{n-1} \bigcup_{C \in \mathcal{C}_i} C$$

$\texttt{CellComplex} := \sum X : \texttt{T2} \,.\, \texttt{CellDecomposition}(X) : \texttt{Type};$

$\texttt{FiniteCellComplex} :: ?\texttt{CellComplex}$

$$(X, \mathcal{E}, \varphi) : \texttt{FiniteCellComplex} \iff \left| \bigsqcup_{n=0}^{\infty} \mathcal{E}_n \right| < \infty$$

$\texttt{LocallyFiniteCellComplex} :: ?\texttt{CellComplex}$

$$(X, \mathcal{E}, \varphi) : \texttt{FiniteCellComplex} \iff \texttt{LocallyFinite}\left(\bigsqcup_{n=0}^{\infty} \mathcal{E}_n \right)$$

$\texttt{Coherent} :: \prod_{X \in \mathsf{TOP}} ?\texttt{Cover}(X)$

$$\mathcal{C} : \texttt{Coherent} \iff \forall A \subset X \,.\, \Big(\forall C \in \mathcal{C} \,.\, A \cap C \in \mathcal{T}(C) \Big) \Rightarrow A \in \mathcal{T}(X)$$

$\texttt{CoherentContinuity} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall \mathcal{C} : \texttt{Coherent}(X) \,.\, \forall f : X \to Y \,.\, f \in C(X, Y) \iff$

$$\iff \forall D \in \mathcal{C} \,.\, f_{|D} \in C(D, Y)$$

Proof $=$

...

$\square$

$\texttt{CoherentQuotient} :: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{C} : \texttt{Coherent}(X) \,.\, \texttt{QuotientMap}\left(\bigsqcup_{C \in \mathcal{C}} \iota_C \right)$

Proof $=$

...

$\square$

$\texttt{CWComplex} :: ?\texttt{CellComplex}$

$$(X, \mathcal{E}, \varphi) : \texttt{CWComplex} \iff \texttt{Coherent}(\overline{\mathcal{E}}) \,\&$$

$$\forall n \in \mathbb{Z}_+ \,.\, \forall E \in \mathcal{E}_n \,.\, \exists F : \texttt{Finite}\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n \right) : \overline{E} \subset \bigcup_{f \in F} f$$

$\texttt{cellSet} :: \texttt{CellComplex} \to \mathsf{SET}$

$$\texttt{cellSet}\,((X,\mathcal{E},\varphi)) = \mathcal{E} := \bigsqcup_{n=0}^{\infty} \mathcal{E}_n$$

$\texttt{cellDimension} :: \prod (X,\mathcal{E},\phi) : \texttt{CellComplex} \,.\, \mathcal{E} \to \mathbb{Z}_+$

$\texttt{cellDimension}\,(E) = \dim E := \reflectbox{D}\texttt{Singleton}\{n \in \mathbb{N} : E \in \mathcal{E}_n\}$

$\texttt{LocallyFiniteIsCW} :: \forall (X,\mathcal{E},\varphi) : \texttt{LocallyFiniteCellComplex} \,.\, \texttt{CWComplex}(X,\mathcal{E},\varphi)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ A \in ?X,$

$\Big([U],[1]\Big) := \reflectbox{D}\texttt{LocallyFiniteCellComplex}(X,\mathcal{E},\varphi)(A) :$

$$: \sum U : \prod_{a \in A} \mathcal{U}(a) \,.\, \forall a \in A \,.\, \Big|\{e \in \mathcal{E} : \overline{e} \cap U_a \neq \emptyset\}\Big| < \infty,$$

$\texttt{Assume}\ [2] : \forall n \in \mathbb{N} \,.\, \forall e \in E \,.\, \overline{e} \cap A \in \mathcal{T}(\overline{e}),$

$[3] := \reflectbox{D}U\reflectbox{D}\texttt{subsetTopology} : \forall a \in A \,.\, \forall e \in \mathcal{E} \,.\, U_a \cap \overline{e} \in \mathcal{T}(\overline{e}),$

$\texttt{Assume}\ a \in A,$

$\Big(n,e,[4]\Big) := \texttt{enumerate}\reflectbox{D}\texttt{Finite}[2]U_a : \sum n \in \mathbb{N} \,.\, \sum e : n \to \mathcal{E} \,.\, e_n = \{e \in \mathcal{E} : \overline{e} \cap U_a \neq \emptyset\},$

$[5] := \reflectbox{D}^{-1}\texttt{Closed}[3]\texttt{ClosedSubetLemma}(X) : \forall i \in n \,.\, \texttt{Closed}(X, \overline{e}_i \cap A^{\mathcal{C}}),$

$$[6] := \reflectbox{D}\texttt{Intersection}\reflectbox{D}^{-1}\texttt{SetMinus}[4] : A \cap U_a = U_a \setminus \bigcup_{i=1}^{n} \overline{e_i} \cap A^{\mathcal{C}},$$

$[a.*] := \texttt{ClosedFiniteUnion}\ \&\ \texttt{OpenClosedDiff}[6] : A \cap U_a \in \mathcal{T}(X);$

$\rightsquigarrow [A.*] := \texttt{OpenCoverLemma} : A \in \mathcal{T}(X);$

$\rightsquigarrow [1] := \reflectbox{D}^{-1}\texttt{Coherent} : \texttt{Coherent}X, \overline{E},$

$\texttt{Assume}\ E \in \mathcal{E},$

$\Big([U],[2]\Big) := \reflectbox{D}\texttt{LocallyFiniteCellComplex}(X,\mathcal{E},\varphi)(\overline{E}) :$

$$: \sum U : \prod_{a \in A} \mathcal{U}(a) \,.\, \forall a \in \overline{E} \,.\, \Big|\{e \in \mathcal{E} : \overline{e} \cap U_a \neq \emptyset\}\Big| < \infty,$$

$n := \dim E \in \mathbb{Z}_+,$

$[3] := \texttt{CompactMappingTHM}(\mathbb{D}^n, X, \varphi_{n,E})\reflectbox{D}\texttt{CellComplex}(X,\mathcal{E},\varphi_n) : \texttt{Compact}(X,\overline{E}),$

$$\Big(m,a,[4]\Big) := \reflectbox{D}\texttt{Compact}(X,\overline{E})(U) : \sum m \in \mathbb{N} \,.\, \sum a : m \to \overline{E} \,.\, \overline{E} :\subset \bigcup_{i=1}^{m} U_{a_i},$$

$$[E.*] := \texttt{FiniteSumOfFinite}[2][4] : \exists \mathcal{F} : \texttt{Finite}(\mathcal{E}) : \overline{E} \subset \bigcup_{f \in F} f;$$

$\rightsquigarrow [*] := [1]\reflectbox{D}^{-1}\texttt{CWComplex} : \texttt{CWComplex}(X,\mathcal{E},\varphi);$

$\square$

$\texttt{complexDimension} :: \texttt{CellComplex} \to \aleph_1$

$\texttt{complexDimension}\,(X,\mathcal{E},\varphi) = \dim \mathcal{E} := \sup\{\dim e | e \in \mathcal{E}\}$

$\texttt{OpenCellTHM} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \;.\; \forall n \in \mathbb{N} \;.\; \forall [0] : \dim \mathcal{E} = n \;.\; \forall e \in \mathcal{E}_n \;.\; e \in \mathcal{T}(X)$

$\texttt{Proof} \;=$

$[1] := \Game\texttt{CWComplex}(X, \mathcal{E}, \varphi)\Game\texttt{CoherentIQuotientMap}(\varphi_{n,e}) : \texttt{QuotientMap}(\mathbb{D}^n, \overline{e}, \varphi_{n,e}),$

$[2] := \Game\texttt{QuotientMapIsOpen}(\mathbb{D}^n, \overline{e}, \varphi_{n,e})(\mathbb{B}^n)\Game\texttt{CellComplex}(X, \mathcal{E}, \varphi) : \varphi_{e,n}(\mathbb{B}^n) = e \in \mathcal{T}(\overline{e}),$

$\texttt{Assume}\; f \in \mathcal{E},$

$\texttt{Assume}\; [3] : e \neq f,$

$[4] := \Game\texttt{CellComplex}(X, \mathcal{E}, \varphi)\Game\texttt{Partition}(X, \mathcal{E})(e, f)[3] : e \cap f = \emptyset,$

$[5] := \texttt{MonotonicClosure}[4]\Game^{-1}\texttt{boundary} : e \cap \overline{f} \subset \partial_{\overline{f}} f,$

$$\Big(\mathcal{F}, [6]\Big) := \Game\texttt{CWComplex}(X, \mathcal{E}, \varphi)(f) : \sum \mathcal{F} : \prod_{i=1}^{(\dim f)-1} \texttt{Finite}(\mathcal{E}_i) \;.\; \partial_{\overline{f}} f = \bigsqcup_{i=1}^{(\dim f)-1} \bigsqcup_{F \in \mathcal{F}_i} F,$$

$[7] := \texttt{NaturalNegation}\Game \dim \mathcal{E}[0] : (\dim f) - 1 < \dim f \leq n,$

$[8] := \Game\texttt{CellComplex}[7] : \forall F \in \mathcal{F} \;.\; e \cap F = \emptyset,$

$[f.*] := [8][5]\Game\texttt{Topology}\Big(\mathcal{T}\Big(\overline{f}\Big)\Big) : e \cap \overline{f} = \emptyset \in \mathcal{T}(\overline{f});$

$\rightsquigarrow [*] := \Game\texttt{CWComplex}(X, \mathcal{E}, \varphi)\Game\texttt{Coherent}(X, \overline{\mathcal{E}}) : e \in \mathcal{T}(X);$

$\square$

$\texttt{Subcomplex} :: \texttt{CellComplex} \to ?\texttt{CellComlex}$

$(Y, \mathcal{F}, \psi) : \texttt{Subcomplex} \iff \Lambda(X, \mathcal{E}, \varphi) : \texttt{CellComplex} \;.\; (Y, \mathcal{F}, \psi) \subset (X, \mathcal{E}, \varphi) \iff$

$\iff Y \subset X \;\&\; \mathcal{F} \subset \mathcal{E} \;\&\; \forall n \in \mathbb{Z}_+ \;.\; \forall e \in \mathcal{F}_n \;.\; \psi_{n,e} = \varphi_{n,e}$

$\texttt{nSkeleton} :: \prod (X, \mathcal{E}, \varphi) : \texttt{CellComplex} \;.\; \mathbb{Z}_+ \to \texttt{Subcomplex}(X, \mathcal{E}, \varphi)$

$$\texttt{nSkeleton}\,(n) = (X^{\skull n}, \mathcal{E}^{\skull n}, \varphi^{\skull n}) := \left(\bigsqcup_{i=1}^{n} \bigsqcup_{e \in \mathcal{E}_n} e, \mathcal{E}_{|\{0,\ldots,n\}}, \varphi_{|\{0,\ldots,n\}}\right)$$

$\texttt{ClosedSubcomplex} :: \forall (X, \mathcal{E}, \varphi) : \texttt{Subcomplex} \;.\; \forall (Y, \mathcal{F}, \psi) \subset (X, \mathcal{E}, \varphi) \;.\; \texttt{Closed}(Y, X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; e \in \mathcal{E},$

$\texttt{Assume}\; [1] : e \in \mathcal{F},$

$[2] := \Game\texttt{Subcomplex}\,((X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi))\,[1] : \overline{e} \cap Y = \overline{e},$

$[1.*] := \Game\texttt{Closed}(\overline{e}) : \texttt{Closed}(\overline{e}, \overline{e} \cap Y);$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : e \in \mathcal{F} \Rightarrow \texttt{Closed}(\overline{e}, \overline{e} \cap Y),$

$\texttt{Assume}\; [2] : e \notin \mathcal{F},$

$[3] := \Game\texttt{Subcomplex}\,((X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi)) : \overline{e} \cap Y \subset \partial_{\overline{e}} e,$

$$\Big(\mathcal{E}', [4]\Big) := \Game\texttt{CWComplex}(X, \mathcal{E}, \varphi)(e) : \sum \mathcal{E}' : \prod_{i=1}^{(\dim e)-1} \texttt{Finite}(\mathcal{E}_i) \;.\; \partial_{\overline{e}} e = \bigsqcup_{i=1}^{(\dim e)-1} \bigsqcup_{E \in \mathcal{E}'_i} E,$$

$$\mathcal{F}' := \mathcal{F} \cap \mathcal{E} : \prod_{i=1}^{(\dim e)-1} \texttt{Finite}(\mathcal{F}_i),$$

$$[5] := [4][3]\Game\texttt{CellComplex}(Y, \mathcal{F}, \psi) : \overline{e} \cap Y = \bigsqcup_{i=1}^{(\dim e)-1} \bigcup_{F \in \mathcal{F}'_i} \overline{F},$$

$[2.*] := \texttt{FiniteClosedUnion} : \texttt{Closed}\Big(\overline{e}, \overline{e} \cap Y\Big);$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : e \notin \mathcal{F} \Rightarrow \texttt{Closed}(\overline{e}, \overline{e} \cap Y),$

$[e.*] := \mathtt{E}(|)(LEM)(e \in \mathcal{F})[2] : \mathtt{Closed}(\overline{e}, \overline{e \cap Y});$

$\rightsquigarrow [1] := \Game\,\mathtt{CWComplex}(X, \mathcal{E}, \varphi) : \mathtt{Closed}(X, Y);$

$\square$

$\mathtt{CWSubcomplex} :: \forall (X, \mathcal{E}, \varphi) : \mathtt{Subcomplex} \;.\; \forall (Y, \mathcal{F}, \psi) \subset (X, \mathcal{E}, \varphi) \;.\; \mathtt{CWComplex}(Y, \mathcal{F}, \psi)$

$\mathtt{Proof} \;=$

$\mathtt{SkeletonCoherent} :: \forall (X, \mathcal{E}, \varphi) : \mathtt{CWComplex} \;.\; \mathtt{Coherent}\left(X, \left\{X^{\skull n} | n \in \mathbb{Z}_+\right\}\right)$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ A : ?X,$

$\mathtt{Assume}\ [1] : \forall n \in \mathbb{Z}_+ \;.\; X^{\skull n} \cap A \in \mathcal{T}\left(X^{\skull n}\right),$

$\mathtt{Assume}\ e : \mathcal{E},$

$n := \dim e \in \mathbb{Z}_+,$

$[2] := [1](n) : X^{\skull n} \cap A \in \mathcal{T}\left(X^{\skull n}\right),$

$[3] := \Game X^{\skull n} \Game\,\mathtt{SubComplex}\Big((X, \mathcal{E}, \varphi), \left(X^{\skull n}, \mathcal{E}^{\skull n}, \varphi^{\skull n}\right)\Big) : \overline{e} \cap X^{\skull n} \cap A = \overline{e} \cap A,$

$[e.*] := \Game\,\mathtt{SubsetTopology}[3] : \overline{e} \cap A \in \mathcal{T}(\overline{e});$

$\rightsquigarrow [A.*] := \Game\,\mathtt{CWComplex}(X, \mathcal{E}, \varphi) : A \in \mathcal{T}(X);$

$* := \Game^{-1}\mathtt{Coherent} : \mathtt{This};$

$\square$

$\mathtt{RegularCell} :: \prod (X, \mathcal{E}, \varphi) : \mathtt{CellComplex} \;.\; ?\mathcal{E}$

$e : \mathtt{RegularCell} \iff \left(\varphi_{n,e} : \mathbb{D}^n \xleftrightarrow{\mathsf{TOP}} \overline{e}\right) \quad \mathtt{where} \quad n = \dim e$

$\mathtt{RegularComplex} :: ?\mathtt{CellComplex}$

$(X, \mathcal{E}, \varphi) : \mathtt{RegularComplex} \iff \forall e \in \mathcal{E} \;.\; \mathtt{RegularCell}(X, \mathcal{E}, \varphi, e)$

$\mathtt{FiniteDimensionalComplex} :: ?\mathtt{CellComplex}$

$(X, \mathcal{E}, \varphi) : \mathtt{FiniteDimensionalComplex} \iff \dim \mathcal{E} < \infty$

2.2 Topological Properties

$\texttt{ConnectedHasConnectedSkeleton} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \ . \ \texttt{Connected}(X) \Rightarrow \texttt{Connected}(X^{\skull})$

$\texttt{Proof} \ =$

$[1] := \texttt{ConnectedSpheres} : \forall n \in \mathbb{N} \, \texttt{Connected}\left(\mathbb{S}^n\right),$

$[2] := \texttt{ConnectedImage}[2] : \forall n \in \mathbb{N} \ . \ n > 1 \Rightarrow \forall e \in \mathcal{E}_n \ . \ \texttt{Connected}(\partial \, e),$

$\texttt{Assume} \ n \in \mathbb{N},$

$\texttt{Assume} \ [3] :! \texttt{Connected}(X^{\skull n}),$

$\left(A, B, [3.1]\right) := \daleth \texttt{Connected}[3] \daleth^{-1} \texttt{ConnectedComponents} : \sum A, B \in \mathrm{CC}(X^{\skull n}) \ . \ A \neq B,$

$\mathcal{A} := \bigcup \{ e \in E : \overline{e} \cap A \neq \emptyset \ \& \ \dim e \leq n + 1 \} :?X,$

$\mathcal{B} := \bigcup \{ e \in E : \overline{e} \cap B \neq \emptyset \ \& \ \dim e \leq n + 1 \} :?X,$

$[4] := \daleth \mathrm{CC}(X^{\skull n})[3.1][2] : \mathcal{A} \neq \mathcal{B},$

$\texttt{Assume} \ e \in \mathcal{E},$

$[5] := \text{Ɔ}\mathcal{A} : \overline{e} \cap \mathcal{A} = \emptyset \Big| \overline{e} \cap \mathcal{A} = \overline{e},$

$[6] := \text{Ɔ}\mathcal{B} : \overline{e} \cap \mathcal{B} = \emptyset \Big| \overline{e} \cap \mathcal{B} = \overline{e},$

$[e.*] := \daleth \texttt{Topology}[5][6] : \texttt{Clopen}(\overline{e}, \mathcal{A} \cap \overline{e} \ \& \ \mathcal{B} \cap \overline{e});$

$\rightsquigarrow [5] := \daleth \texttt{CWComplex}(X, \mathcal{E}, \varphi) : \texttt{Clopen}(X, \mathcal{A} \ \& \ \mathcal{B}),$

$[3.*] := \daleth \texttt{Connected}[5] : ! \ \texttt{Connected}(X^{\skull (n+1)});$

$\rightsquigarrow [3] := \mathtt{E}(\bot) \mathrm{CC}(X^{\skull}) : \forall n \in \mathbb{N} \ . \ ! \ \texttt{Connected}(X^{\skull n}) \Rightarrow ! \ \texttt{Connected}(X^{\skull (n+1)}),$

$\texttt{Assume} \ [4] : ! \ \texttt{Connected}(X^{\skull}),$

$[5] := \daleth \mathbb{N}[4][3] : \forall n \in \mathbb{N} \ . \ ! \ \texttt{Connected}(X^{\skull n}),$

$\Big((A, \mathcal{A}, \psi), (B, \mathcal{B}, \psi'), [6]\Big) := \daleth \texttt{CellComplex}(X, \mathcal{E}, \varphi)[5] : \sum (A, \mathcal{A}, \psi), (B, \mathcal{B}, \psi') : \texttt{Subcomplex}(X, \mathcal{E}, \varphi) \ .$

$\quad . \ A \sqcup B = X \ \& \ \forall n \in \mathbb{N} \ . \ \texttt{Clopen}\left(X^{\skull n}, A^{\skull n} \ \& \ B^{\skull n}\right),$

$[7] := \daleth^{-1} \texttt{Clopen}[6] : \texttt{Clopen}(X, \mathcal{A} \ \& \ \mathcal{B}),$

$[4.*] := \daleth \texttt{Connected}(X)[7] : \bot;$

$\rightsquigarrow [*] := \mathtt{E}(\bot) : \texttt{Connected}(X^{\skull});$

$\square$

$\texttt{PathConnectedComponentIsClopen} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CellComplex} \ . \ \forall A \in \mathrm{PCC}(X) \ . \ \texttt{Clopen}(X, A)$

$\texttt{Proof} \ =$

$[0] := \texttt{PathConnectedImage} : \forall e \in \mathcal{E} \ . \ \texttt{PathConnected}(\overline{e}),$

$\texttt{Assume} \ e : \mathcal{E},$

$[1] := [0](e) \daleth \mathrm{PCC}(X) : \overline{e} \cap A = \overline{e} \Big| \overline{e} \cap A = \emptyset,$

$[e.*] := \daleth \texttt{Topology}(\overline{e}) : \texttt{Clopen} \overline{e}, \overline{e} \cap A;$

$\rightsquigarrow [*] := \daleth^{-1} \texttt{CWComplex}(X, \mathcal{E}, \varphi) : \texttt{Clopen}(X, A);$

$\square$

$\texttt{ConnectedIffPathConnected} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \; . \; \texttt{Connected}(X) \iff \texttt{PathConnected}(X)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ConnectedSkeletonImplyPathConnected} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \; . \; \forall n \in \mathbb{Z}_+ \; .$

$\quad . \; \texttt{Connected}(X^{\Skull n}) \Rightarrow \texttt{PathConnected}(X)$

$\texttt{Proof} \; =$

$[1] := \texttt{ConnectedHasConnectedSkeleton}(X^{\Skull n}, \mathcal{E}^{\Skull n}, \varphi^{\Skull n}) : \texttt{Connected}(X^{\Skull}),$

$[2] := \texttt{ConnectedIffPathConnected}(X^{\Skull}, \mathcal{E}^{\Skull}, \varphi^{\Skull}) : \texttt{PathConnected}(X^{\Skull}),$

$\texttt{Assume} \; k : \mathbb{N},$

$\texttt{Assume} \; [3] : \texttt{PathConnected}(X^{\Skull k}),$

$\texttt{Assume} \; p, q \in X^{\Skull (k+1)},$

$\Big(e, f, [4]\Big) := \text{Ɑ}\texttt{CellComplex}(X^{\Skull (k+1)}, \mathcal{E}^{\Skull (K+1)}, \varphi^{\Skull (k+1)}) : \sum e, f \in \mathcal{E}^{\Skull (k+1)} \; . \; p \in e \; \& \; q \in f,$

$\texttt{Assume} \; [5] : \dim e \leq k \; \& \; \dim f \leq k,$

$[6] := \text{Ɑ}\texttt{nskeleton}[4] : p, q \in X^{\Skull k},$

$[5.*] := \text{Ɑ}\texttt{PathConnected}\Big(X^{\Skull k}\Big) : \exists \texttt{Path}\Big(X^{\Skull k}, p, q\Big);$

$\rightsquigarrow [5] :\Longrightarrow: (\dim e \leq k \; \& \; \dim f \leq k) \Rightarrow \exists \texttt{Path}\Big(X^{\Skull (k+1)}, p, q\Big),$

$\texttt{Assume} \; [6] : \dim f = (k+1) \Big| \dim e = (k+1),$

$\Big(f', e', [7]\Big) := \text{Ɑ}\texttt{CellComplex}\Big(X^{\Skull (k+1)}, \mathcal{E}^{\Skull (k+1)}, \varphi^{\Skull (k+1)}\Big) : \sum f', e' \in \mathcal{E}^{\Skull k} \; . \; f' \subset \partial f \; \& \; e' \subset \partial e',$

$p' := \text{Ɑ}\texttt{NonEmpty}(e') \in e',$

$q' := \text{Ɑ}\texttt{NonEmpty}(f') \in f',$

$\Big(\alpha, [8]\Big) := \texttt{DiskIsPathConnected}(\overline{e})(p, p') : \sum \alpha \in C\Big([0,1], \overline{e}\Big) \; . \; \alpha(0) = p \; \& \; \alpha(1) = p',$

$\Big(\beta, [9]\Big) := \texttt{DiskIsPathConnected}(\overline{f})(q', q) : \sum \beta \in C\Big([0,1], \overline{f}\Big) \; . \; \alpha(0) = q' \; \& \; \alpha(1) = q,$

$\Big(\omega, [10]\Big) := \text{Ɑ}\texttt{PathConnected}(X^{\Skull k})(p', q') : \sum \omega \in C\Big([0,1], X^{\Skull k}\Big) \; . \; \omega(0) = p' \; \& \; \omega(1) = q',$

$[6.*] := \text{Ɑ}^{-1}\texttt{Path}[8, 9, 10] : \texttt{Path}\Big(X^{\Skull (k+1)}, p, q, \alpha \oplus \omega \oplus \beta\Big);$

$\rightsquigarrow [6] :\Longrightarrow: (\dim e = (k+1) \Big| \dim f \leq k) \Rightarrow \exists \texttt{Path}\Big(X^{\Skull (k+1)}, p, q\Big),$

$\Big[(p, q). *\Big] := \texttt{E}(|)\texttt{LEM}(\ldots)[5][6] : \exists \texttt{Path}\Big(X^{\Skull (k+1)}, p, q, \Big);$

$\rightsquigarrow [3.*] := \text{Ɑ}^{-1}\texttt{PathConnected} : \texttt{PathConnected}\Big(X^{\Skull (k+1)}\Big);$

$\rightsquigarrow [3] := \text{Ɑ}\mathbb{N}[2] : \forall k \in \mathbb{N} \; . \; \texttt{PathConnected}(X^{\Skull k}),$

$\texttt{Assume} \; p, q \in X,$

$\Big(e, f, [4]\Big) := \text{Ɑ}\texttt{CellComplex}(X, \mathcal{E}, \varphi) : \sum e, f \in \mathcal{E} \; . \; p \in e \; \& \; q \in f,$

$m := \max(\dim e, \dim f, 1) \in \mathbb{N},$

$\Big[(p, q). *\Big] := [3](m)\text{Ɑ}\texttt{PathConnected}(p, q) : \exists \texttt{Path}(X^{\Skull m}, p, q);$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{PathConnected} : \texttt{PathConnected}(X);$

$\square$

$\texttt{FiniteSubcomplexLemma} :: \forall (X,\mathcal{E},\varphi) : \texttt{CWComplex} \;.\; \forall e \in \mathcal{E} \;.\; \exists (Y,\mathcal{F},\psi) \subset (X,\mathcal{E},\varphi) \;.$
$\;.\; \texttt{FiniteCellComplex}(Y,\mathcal{F},\psi) \;\&\; \overline{e} \subset Y$

$\texttt{Proof} \;=$

$\Upsilon := \Lambda n \in \mathbb{Z}_+ \;.\; \forall e \in \mathcal{E} \;.\; (\dim e \le n) \Rightarrow \exists (Y,\mathcal{F},\psi) \subset (X,\mathcal{E},\varphi) \;.\; \texttt{FiniteCellComplex}(Y,\mathcal{F},\psi) \;\&\; \overline{e} \subset Y :$
$: \mathbb{Z}_+ \to \texttt{Type},$

$\texttt{Assume}\; e \in \mathcal{E}_0,$

$\Big(x,[2]\Big) := \mathcal{O}\texttt{CellComplex}(X,\mathcal{E},\varphi) : \sum_{x\in X} : e \subset \{x\},$

$[0] := \mathcal{O}^{-1}\texttt{FiniteCellComplex}[2] : \texttt{FiniteCellComplex}\Big(e, 0 \mapsto \{e\}, 0 \mapsto \varphi_{0,e}\Big);$

$\rightsquigarrow [1] := \mathcal{D}^{-1}\Upsilon : \Upsilon(0),$

$\texttt{Assume}\; n : \mathbb{Z}_+,$

$\texttt{Assume}\; [2] : \Upsilon(n),$

$\texttt{Assume}\; e \in \mathcal{E}_{n+1},$

$\Big(\mathcal{F},[3]\Big) := \mathcal{O}\texttt{CWComplex}(X,\mathcal{E},\varphi)(e) : \sum \mathcal{F} : \prod_{i=1}^{n} \texttt{Finite}(\mathcal{E}_i) \;.\; \partial\, e \subset \coprod_{i=1}^{n} \coprod_{f\in\mathcal{F}} f,$

$[n.*] := \mathcal{D}\Upsilon[2](\mathcal{F})[3] : \texttt{FiniteCellComplex}(\overline{e}, e \sqcup \mathcal{F}, \varphi_{(n+1),e} \sqcup \varphi_{\mathcal{F}});$

$\rightsquigarrow [*] := \texttt{CompleteInduction}(\mathbb{Z}_+)[1]\mathcal{D}\Upsilon : \texttt{This};$

$\square$

$\texttt{DiscreteSubsetLemma} :: \forall (X,\mathcal{E},\varphi) : \texttt{CWComplex} \;.\; \forall A \subset X \;.\; \texttt{Closed} \;\&\; \texttt{Discrete}(A) \iff \forall e \in \mathcal{E} \;.\; |e \cap A| < \infty$

$\texttt{Proof} \;=$

$\texttt{Assume}\; [1] : \texttt{Discrete}(A),$

$\texttt{Assume}\; e \in \mathcal{E},$

$n := \dim e : \mathbb{Z}_+,$

$[2] := \mathcal{O}\texttt{CellComlex}(X,\mathcal{E},\varphi)(A)[1] : \texttt{Type}\mathit{Discrete}\Big(\varphi_{n,e}^{-1}(A \cap e)\Big) \;\&\; \varphi_{n,e}^{-1}(A \cap e) \subset \mathbb{B}^n,$

$\texttt{Assume}\; [3] : |A \cap e| = \infty,$

$[4] := \texttt{SequenceCompactDisc}[2]\mathcal{O}\texttt{Discrete}[3] : \overline{\varphi_{n,e}^{-1}(A \cap e) \cap \mathbb{S}^{n-1} \ne \emptyset},$

$[5] := \texttt{ContinuousImagePreservesConvergence}[4] : \overline{e \cap A} \cap \partial e \ne \emptyset,$

$[3.*] := \mathcal{O}\texttt{Discrete}(A)[5] : \perp;$

$\rightsquigarrow [1.*] := \texttt{E}(\perp) : |A \cap e| < \infty;$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow)\texttt{I}(\forall) : \texttt{Discrete}(A) \Rightarrow \forall e \in \mathcal{E} \;.\; |e \cap A| < \infty,$

$\texttt{Assume}\; [2] : \forall e \in \mathcal{E} \;.\; |e \cap A| < \infty,$

$\texttt{Assume}\; B :?A,$

$[3] := \mathcal{O}\texttt{CWComplex}(X,\mathcal{E},\varphi)[2] : \forall e \in \mathcal{E} \;.\; |\overline{e} \cap B| < \infty,$

$[4] := \texttt{FiniteIsClosed}[3] : \forall e \in \mathcal{E} \;.\; \texttt{Closed}\Big(\overline{e}, \overline{e} \cap B\Big),$

$[B.*] := \mathcal{O}\texttt{CWComplex}(X,\mathcal{E},\varphi)[4] : \texttt{Closed}(X,B);$

$\rightsquigarrow [2.*] := \mathcal{O}^{-1}\texttt{Discrete} : \texttt{Discrete}(A);$

$\rightsquigarrow [*] := \texttt{I}(\iff)[1] : \texttt{This};$

$\square$

$\texttt{CompactSubsetOfCWComplex} :: \forall (X, \mathcal{E}, \varphi) . \forall A : \texttt{Closed}(X) . \texttt{Compact}(X, A) \iff$
$\iff \exists (Y, \mathcal{F}, \psi) \subset (X, \mathcal{E}, \varphi) . A \subset Y \ \& \ \texttt{FiniteComplex}(X, \mathcal{E}, \varphi)$

$\texttt{Proof} \ =$

$\mathcal{A} := \{e \in \mathcal{E} : e \cap A \neq \emptyset\} :?\mathcal{E},$

$\texttt{Assume} \ [1] : \texttt{Compact}(X, A),$

$\texttt{Assume} \ [2] : |\mathcal{A}| > \infty,$

$D := \{\varphi_{n,e}(0) | e \in \mathcal{A}, n = \dim e\} :?X,$

$[3] := \mathcal{D} e \mathcal{C} \texttt{CellComplex}(X, \mathcal{E}, \varphi) : \forall e \in \mathcal{E} . |D \cap e| \leq 1,$

$[4] := \mathcal{D} ep[3] \mathcal{C} \texttt{CWComplex}(X, \mathcal{E}, \varphi) : \forall e \in\in \mathcal{E} . |D \cap \overline{e}| < \infty,$

$[5] := \texttt{DiscreteSubsetLemma}[4] : \texttt{Closed} \ \& \ \texttt{Discrete}(X, A),$

$[6] := [2] \mathcal{D} D : |D| = \infty,$

$[2.*] := \texttt{CompactDiscreteSubset}[5][6] : \perp;$

$\leadsto [1.*] := \texttt{E}(\perp) : |\mathcal{A}| < \infty;$

$\leadsto [1] := \texttt{I}(\Rightarrow) : \texttt{Compact}(A) \Rightarrow |\mathcal{A}| < \infty,$

$\texttt{Assume} \ [2] : |\mathcal{A}| < \infty,$

$[3] := \texttt{CompactMappingTHM}(\varphi) : \forall e \in \mathcal{E} . \texttt{Compact}(\overline{e}),$

$[4] := \texttt{CompactFiniteUnion}[2][3] : \texttt{Compact}\left(X, \bigcup_{e \in \mathcal{A}} \overline{e}\right),$

$[5] := \mathcal{D} \mathcal{A} \mathcal{C}^{-1} \texttt{Subset} : A \subset \bigcup_{e \in \mathcal{A}} \overline{e},$

$[2.*] := \texttt{ClosedCompactSubset}[5][4] : \texttt{Compact}(X, A);$

$\leadsto [*] := I(\iff)[1] : \texttt{Compact}(X, \mathcal{A}) \iff |\mathcal{A}| < \infty;$

$\square$

$\texttt{CWComplexFiniteIffCompact} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} . \texttt{Compact}(X) \iff \texttt{FiniteComplex}(X, \mathcal{E}, \varphi)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CWComplexLocallyFiniteIffLocallyCompact} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} . \texttt{LocallyCompact}(X) \iff$
$\iff \texttt{LocallyFiniteComplex}(X, \mathcal{E}, \varphi)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

2.3 Inductive Construction

$\texttt{ByAttachingNCells} :: ?(\texttt{T2}^2 \times \mathbb{N})$

$(X, Y, n) : \texttt{ByAttachingNCells} \iff \sum I \in \mathsf{SET} : \sum \varphi : I \to \mathbb{S}^{n-1} \xrightarrow{\mathsf{TOP}} Y : X = Y \sqcup_{\bigsqcup_{i \in \mathcal{I}} \varphi_i} \bigsqcup_{i \in \mathcal{I}} \mathbb{D}^n$

$\texttt{CharacteristicMapsCoproductIsQuotient} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \, .$

$$. \, \texttt{QuotintMap}\left(\bigsqcup_{e \in \mathcal{E}} \mathbb{D}^{\dim e}, X, \bigsqcup_{e \in \mathcal{E}} \varphi_{\dim e, e} \right)$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{SkeletonByAttachingNCells} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \, . \, \forall n \in \mathbb{N} \, . \, \texttt{ByAttachingNCells}(X^{\text{☠} n}, X^{\text{☠} n-1}, n)$

$\texttt{Proof} \ =$

$\mathcal{I} := \mathcal{E}_n \in \mathsf{SET},$

$\psi := \Lambda e \in \mathcal{E}_n \varphi_{n,e|\mathbb{S}^{n-1}} : \mathcal{I} \to \mathbb{S}^{n-1} \xrightarrow{\mathsf{TOP}} X,$

$[1] := \text{Ↄ}\psi \text{Ɑ}\texttt{CellComplex} : \forall i \in \mathcal{I} \, . \, \operatorname{Im} \psi_i \subset X^{\text{☠} n},$

$\texttt{Assume} \ A : \texttt{SaturatedClosed}(\psi),$

$$[2] := \text{Ɑ}\texttt{SaturatedClosed}(\psi, A) : \texttt{Closed}\left(A \sqcup \bigsqcup_{e \in \mathcal{E}_n} \mathbb{D}^n, A \cap X^{\text{☠}(n-1)} \right) \ \&$$

$$\& \ \forall e \in \mathcal{E}_n \, . \, \texttt{Closed}\left(A \sqcup \bigsqcup_{e \in \mathcal{E}_n} \mathbb{D}^n, A \cap \mathbb{D}^n_e \cap \right),$$

$[3] := \text{Ɑ}\texttt{CWComplex}(X^{\text{☠}(n-1)}, \mathcal{E}, \varphi)[2] : \forall k \in (n-1) \, . \, \forall e \mathcal{E}_k \, . \, \texttt{Closed}(\overline{e} \cap A),$

$[3] := \text{Ɑ}\texttt{CWComplex}(X^{\text{☠}(n-1)}, \mathcal{E}, \varphi)[2] : \forall k \in (n-1) \, . \, \forall e \mathcal{E}_k \, . \, \texttt{Closed}(\overline{e}, \overline{e} \cap A),$

$[4] := \text{Ↄ}\psi[3] : \forall k \in [0, \ldots, n-1]_{\mathbb{Z}_+} \, . \, \forall e \mathcal{E}_k \, . \, \texttt{Closed}(\overline{e}, \overline{e} \cap \widehat{\psi}(A)),$

$[5] := \texttt{ClosedMappingTheorem}[2] : \forall e \in \mathcal{E}_n \, . \, \texttt{Closed}\Big(\overline{e}, \widehat{(\psi)}(A \cap \mathbb{D}^n_e)\Big),$

$[6] := \text{Ↄ}\psi[5] : \forall e \in \mathcal{E}_n \, . \, \texttt{Closed}\Big(\overline{e}, \widehat{(\psi)}(A) \cap e\Big),$

$[A.*] := \text{Ɑ}\texttt{CWComplex}(X, \mathcal{E}, \varphi)[4][6] : \texttt{Closed}(X, \widehat{\psi}(A));$

$$\rightsquigarrow [2] := \text{Ɑ}^{-1}\texttt{QuotientMap} : \texttt{QuotientMap}\left(X^{\text{☠}(n-1)} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \mathbb{D}_n, X^{\text{☠} n}, \widehat{\psi} \right),$$

$[*] := \text{Ɑ}^{-1}\texttt{ByAttachingNCells} : \texttt{ByAttachingNCells}\Big(X^{\text{☠}(n-1)}, X^{\text{☠} n}, n\Big);$

$\square$

$\texttt{CellExtensionTHM} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CellComplex} \ . \ \forall n \in \mathbb{N} \ . \ \forall f : X^{\skull n-1} \xrightarrow{\mathsf{TOP}} [0,1] \ .$

$\quad . \ \exists \bar{f} : X^{\skull n} \xrightarrow{\mathsf{TOP}} [0,1] : \hat{f}_{|X^{\skull(n-1)}} = f \ \& \ \forall e \in \mathcal{E}_n \hat{f}(e) < 1$

$\texttt{Proof} \ =$

$\texttt{Assume}\ e \in \mathcal{E}_n,$

$\texttt{Assume}\ x \in e,$

$p := \varphi^{-1}_{n,e}(x) \in \mathbb{B}^n,$

$\tilde{f}(x) := \texttt{if}\ p == 0\ \texttt{then}\ 0\ \texttt{else}\ \|p\| \left(\frac{p}{\|p\|} \right) \varphi_{e,n} f : \texttt{In}[0,1);$

$\rightsquigarrow \tilde{f} := \texttt{I}(\iff) : \bigcup \mathcal{E}_n \xrightarrow{\mathsf{TOP}} [0,1),$

$\bar{f} := \Lambda x \in X^{\skull n} \texttt{if}\ x \in \bigcup \mathcal{E}_n\ \texttt{then}\ \tilde{f}(x)\ \texttt{else}\ f(x) : X \to [0,1],$

$[1] := \reflectbox{$\mathcal{C}$} \bar{f} : \forall e \in \mathcal{E}_n \ . \ \forall x : \mathbb{N} \to e \ . \ \lim_{n \to \infty} x_n \in \partial\, e_n \Rightarrow$

$\quad \Rightarrow \lim_{n \to \infty} \bar{f}(x_n) = \lim_{n \to \infty} \left\| \varphi^{-1}_{n,e}(x_n) \right\| \left(\frac{\varphi^{-1}_{n,e}(x_n)}{\|\varphi^{-1}_{n,e}(x_n)\|} \right) \varphi_{n,e} f = f\Big(\lim_{n \to \infty} x_n \Big) = \bar{f}\Big(\lim_{n \to \infty} x_n \Big),$

$[*] := \texttt{ContinuousByLimits}[1] : \bar{f} : X^{\skull n} \xrightarrow{\mathsf{TOP}} [0,1];$

$\square$

$\texttt{InductiveConstructionTHM} :: \forall X \in \mathsf{SET} \,.\, \forall Y : \texttt{Increasing}\Big(\mathbb{Z}_+, \texttt{Subset}(X)\Big) \,.$

$$. \forall[0] : \bigcup_{n=0}^{\infty} Y_n = X \,.\, \forall \tau : \prod_{n=0}^{\infty} \texttt{Topology}(Y_n) \,.\, \forall[00] : \texttt{Discrete}(Y_0, \tau_0) \,.$$

$$. \forall[000] : \forall n \in \mathbb{N} \,.\, \texttt{ByAttachingNCells}\Big((Y_n, \tau_n), (Y_{n-1}, \tau_{n-1}), n\Big) \,.\, \exists! \mathcal{T} : \texttt{Topology}(X) :$$

$$: \exists!(\mathcal{E}, \varphi) : \texttt{CellComplex}(X, \mathcal{T}) : \forall n \in \mathbb{Z}_+ \,.\, X^{☠ n} = Y_n$$

$\texttt{Proof} \ =$

$\mathcal{E}_0 := \Big\{\{y\}\Big|y \in Y_0\Big\} :??X,$

$\texttt{Assume}\ e : \mathcal{E}_0,$

$\Big(y, [1]\Big) := \mathcal{D}\mathcal{E}_0(e) : \sum y \in Y_0 \,.\, e = \{y_0\},$

$\varphi_{0,e} := y : \texttt{In}(Y_0);$

$\rightsquigarrow \Big(\varphi_0, [1]\Big) := \mathtt{I}\left(\prod\right) : \prod_{e \in \mathcal{E}_0} \varphi_{0,e} : \{0\} \to Y_0 \,.\, e = \{\varphi_{0,e}(0)\},$

$\texttt{Assume}\ n \in \mathbb{N},$

$[2] := [000](n) : \texttt{ByAttachingNCells}(Y_n, Y_{n-1}, n),$

$\Big(\mathcal{I}, \psi, [3]\Big) := \mathcal{C}\texttt{ByAttachingNCells}[2] : \sum \mathcal{I} \in \mathsf{SET} \,.\, \sum \psi : \mathcal{I} \to \mathbb{S}^{n-1} \xrightarrow{\mathsf{TOP}} Y_{n-1} \,.$

$$. Y_n = Y_{n-1} \sqcup_{\mathrm{id} \sqcup \bigsqcup_{i \in \mathcal{I}} \psi_i} \bigsqcup_{i \in \mathcal{I}} \mathbb{D}^n_i,$$

$\mathcal{E}_n := \Big\{\widehat{\psi}(i, \mathbb{B}^n)\Big|i \in \mathcal{I}\Big\} :??X,$

$\texttt{Assume}\ e \in \mathcal{E}_n,$

$\Big(i, [4]\Big) := \mathcal{D}\mathcal{E}_n(e) : \sum i \in \mathcal{I} \,.\, e = \widehat{\psi}\Big(i, \mathbb{B}^n\Big),$

$\varphi_{n,e} := \Lambda x \in \mathbb{D}^n \,.\, \widehat{\psi}(i, x) : \mathbb{D}^n \to Y_n,$

$[e.*] := \mathcal{D}\psi\mathcal{D}\varphi_{n,e}\mathcal{D}\mathcal{E}_n : \varphi_{n,e}\Big(\mathbb{S}^{n-1}\Big) \subset Y_{n-1} \ \&\ \varphi_{n,e|\mathbb{B}^n} : \mathbb{B}^n \overset{\mathsf{SET}}{\longleftrightarrow} e;$

$\rightsquigarrow \Big(\varphi_n, [4]\Big) := \mathtt{I}\left(\prod\right) : \prod_{e \in \mathcal{E}_n} \varphi_{n,e} : \mathbb{D}^n \xrightarrow{\mathsf{TOP}} Y_n \,.\, \varphi_{n,e}(\mathbb{S}^{n-1}) \subset Y_{n-1} \ \&\ \varphi_{n,e|\mathbb{B}^n} : \mathbb{B}^n \overset{\mathsf{SET}}{\longleftrightarrow} e;$

$\rightsquigarrow \Big(\mathcal{E}, \varphi, [2]\Big) := \mathtt{I}\left(\prod\right) : \prod_{n=1}^{\infty} E_n :??X \,.\, \prod_{e \in \mathcal{E}_n} \varphi_{n,e} : \mathbb{D}^n \xrightarrow{\mathsf{TOP}} Y_n \,.\, \varphi_{n,e}(\mathbb{S}^{n-1}) \subset Y_{n-1} \ \&\ \varphi_{n,e|\mathbb{B}^n} : \mathbb{B}^n \overset{\mathsf{SET}}{\longleftrightarrow} e,$

$\mathcal{T} := \{U \subset X : \forall n \in \mathbb{Z}_+ \,.\, U \cap Y_n \in \tau_n\} :??X,$

$[3] := \mathcal{C}\texttt{ByAttachingNCells}[000] : \forall n \in \mathbb{N} \,.\, \texttt{Closed}\Big((Y_n, \tau_n), (Y_{n-1}, \tau_{n-1})\Big),$

$[4] := \mathcal{D}\mathcal{T}\texttt{FinalTopologyByInclusions}[3] : \texttt{Topology}\Big(X, \mathcal{T}\Big),$

$[5] := \mathcal{D}\mathcal{E}\mathcal{C}\texttt{ByAttachingNCells}[000] : X = \bigsqcup_{n=0}^{\infty} \mathcal{E}_n,$

$[6] := \mathcal{D}\mathcal{E}[2] : \forall n \in \mathbb{N} \,.\, \forall e \in \mathcal{E}_n \,.\, \partial e \subset Y_{n-1},$

$[7] := \mathcal{C}^{-1}\texttt{CellComplex}[6][5] : \texttt{CellComplex}(X, \mathcal{E}, \varphi),$

$[8] := \mathcal{D}\varphi[7][2] : \forall n \in \mathbb{N} \,.\, X^{☠ n} = Y_n,$

$\texttt{Assume}\ x : X,$

$\Big(e, [9]\Big) := [5](x) : \sum e \in \mathcal{E} \mathrel{.} x \in e,$

$n := \dim e \in \mathbb{Z}_+,$

$\texttt{Assume}\ [10] : n = 0,$

$f := \Lambda y \in Y_0 \mathrel{.} x == y : Y_0 \xrightarrow{\mathsf{TOP}} [0,1],$

$[10.*] := \reflectbox{$\mathcal{C}$} f : f^{-1}(1) = \{x\};$

$\rightsquigarrow [10] := \texttt{I}(\Rightarrow)\texttt{I}\left(\sum\right) : n = 0 \Rightarrow \sum f : Y_0 \xrightarrow{\mathsf{TOP}} [0,1] \mathrel{.} f^{-1}(1) = \{x\},$

$\texttt{Assume}\ [11] : n \neq 0,$

$p := \varphi_{n,e}^{-1}(x) \in \mathbb{B}^n,$

$\Big(g, [12]\Big) := \texttt{RegularFunctionalProperty}(\mathbb{D}^n, p, \mathbb{S}^{n-1})[11] : \sum g : \mathbb{D}^n \xrightarrow{\mathsf{TOP}} [0,1] \mathrel{.} g\Big(\mathbb{S}^{n-1}\Big) = 0 \;\&\; g^{-1}(1) = p,$

$f := \Lambda y \in \mathbb{D}^n \mathrel{.} \texttt{if}\ y \in e\ \texttt{then}\ y\ \varphi_{n,e}^{-1}\ g\ \texttt{else}\ 0 : Y_n \xrightarrow{\mathsf{TOP}} [0,1],$

$[11.*] := \reflectbox{$\mathcal{C}$} f[12] : f^{-1}(1) = x;$

$\rightsquigarrow [11] := \texttt{I}(\Rightarrow)\texttt{I}\left(\sum\right) : n \neq 0 \Rightarrow \sum f : Y_n \xrightarrow{\mathsf{TOP}} [0,1] \mathrel{.} f^{-1}(1) = \{x\},$

$\Big(f, [12]\Big) := \texttt{E}(|)\texttt{LEM}(n = 0)[10][11] : \sum f : Y_n \xrightarrow{\mathsf{TOP}} [0,1] \mathrel{.} f^{-1}(1) = \{x\},$

$\Big(\bar{f}, [*]\Big) := \reflectbox{$\mathcal{D}$}\mathbb{N}\reflectbox{$\mathcal{C}$}\mathcal{T}\texttt{CellExtensionTHM}\big(f, [12]\big) : \sum \bar{f} : X \xrightarrow{\mathsf{TOP}} [0,1] \mathrel{.} \bar{f}^{-1}(1) = \{x\};$

$\rightsquigarrow [10] := \texttt{HausdorffByMappings} : \texttt{T2}(X),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [11] : \texttt{CWComplex}(X^{\skull(n-1)}, \mathcal{E}^{\skull(n-1)}, \varphi^{\skull(n-1)}),$

$\texttt{Assume}\ e : \mathcal{E}_n,$

$[12] := \reflectbox{$\mathcal{D}$}\texttt{CellComplex}(X, \mathcal{E}, \varphi) : \partial\, e \subset X^{\skull(n-1)},$

$[13] := \texttt{CompactMappingTHM}(\mathbb{S}^{n-1}, \varphi_n) : \texttt{Compact}(X^{\skull(n-1)}, \partial\, e),$

$[e.*] := \texttt{FiniteComplexIffCompact}[13] : \exists \mathcal{F} : \texttt{Finite}\mathcal{E} \mathrel{.} \partial e \subset \bigcup \mathcal{F};$

$\rightsquigarrow [12] := \texttt{I}(\forall) : \forall e \in \mathcal{E}_n \mathrel{.} \exists \mathcal{F} : \texttt{Finite}\mathcal{E} \mathrel{.} \partial e \subset \bigcup \mathcal{F},$

$\texttt{Assume}\ A :?X^{\skull n},$

$\texttt{Assume}\ [13] : \forall e \in \mathcal{E}^{\skull n} \mathrel{.} \texttt{Closed}(\overline{e}, \overline{e} \cap A),$

$[14] := \reflectbox{$\mathcal{D}$}\texttt{CWComplex}(X^{\skull(n-1)}, \mathcal{E}^{\skull(n-1)}, \varphi^{\skull(n-1)})[13] : \texttt{Closed}\Big(X^{\skull(n-1)}, A \cap X^{\skull(n-1)}\Big),$

$[A.*] := \reflectbox{$\mathcal{D}$}\texttt{QuotientMap}\reflectbox{$\mathcal{C}$}\varphi[14][13] : \texttt{Closed}(X^{\skull n}, A);$

$\rightsquigarrow [13] := \reflectbox{$\mathcal{D}$}^{-1}\texttt{CWComplex} : \texttt{CWComplex}(X^{\skull n}, \mathcal{E}^{\skull n}, \varphi^{\skull n});$

$\rightsquigarrow [11] := \reflectbox{$\mathcal{D}$}\mathbb{N}[00] : \forall n \in \mathbb{N} \mathrel{.} \texttt{CWComplex}(X^{\skull n}, \mathcal{E}^{\skull n}, \varphi^{\skull n}),$

$[*] := \reflectbox{$\mathcal{C}$}\mathcal{T}[11] : \texttt{CWComplex}(X, \mathcal{E}, \varphi);$

$\square$

$\texttt{CWComplexIsParacompact} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \ . \ \texttt{Paracompact}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ \mathcal{O} : \texttt{OpenCover}(X),$

$$\mathcal{O}^{☠} := \Lambda n \in \mathbb{Z}^+ \ . \ \left\{ O \cap X^{☠ n} | O \in \mathcal{O} \right\} : \prod_{n=0}^{\infty} \texttt{OpenCover}\left(X^{☠ n} \right),$$

$$[1] := \text{Ɑ}\texttt{Cover}\left(X^{☠ 0}, \mathcal{O}^{☠ 0} \right) : \forall x \in X^{☠ 0} \ . \ \left\{ O \in \mathcal{O}^{☠ 0} \right\} \neq \emptyset,$$

$$\left(O, [2] \right) := \texttt{Choice}[1] : \prod_{x \in X^{☠ 0}} \sum_{O_x \in \mathcal{O}^{☠ 0}} x \in O_x,$$

$$\phi^0 := \Lambda U \in \mathcal{O}^{☠ 0} \ . \ \Lambda x \in X^{☠ 0} \ . \ \delta_{O_x, U} : \mathcal{O}^{☠ 0} \to X^{☠ 0} \xrightarrow{\mathsf{TOP}} [0, 1],$$

$$[3] := \text{Ɑ}^{-1}\texttt{PartitionOfUnity}\left(X^{☠ 0}, \mathcal{O}^{☠ 0} \right) \text{Ɔ} \phi^0 [2] : \texttt{PartitionOfUnity}\left(X^{☠ n}, \mathcal{O}^{☠ n}, \phi^0 \right),$$

$\texttt{Assume} \ n : \mathbb{Z}_+,$

$$\texttt{Assume} \ \phi^n : \texttt{PartitionOfUnity}\left(X^{☠ n}, \mathcal{O}^{☠ n} \right),$$

$$\texttt{Assume} \ [4] : \forall k \in [0, n)_{\mathbb{Z}_+} \ . \ \phi^{n-1}_{|X^{☠ k}} = \phi^k,$$

$$\texttt{Assume} \ [5] : \forall k \in [0, n)_{\mathbb{Z}_+} \ . \ \forall U \in \mathcal{T}\left(X^{☠ k} \right) \ . \ \phi^k(U) = \{0\} \Rightarrow \exists V \in \mathcal{T}\left(X^{☠ n-1} \right) : U \subset V \ \& \ \phi^{n-1}(V) = \{0\},$$

$$[6] := \texttt{SkeletonbyAttachingNCells}(X, \mathcal{E}, \varphi, n) : \texttt{ByAttachingNCells}\left(X^{☠ n}, X^{☠ (n+1)}, n+1 \right),$$

$$I := \Lambda A \subset \mathbb{S}^n \ . \ \Lambda t \in \mathbb{R}_{++} \ . \ \left\{ x \in \mathbb{D}^{n+1} : |x| > t \ \& \ \frac{x}{\|x\|} \in A \right\} : ?\mathbb{S}^n \to \mathbb{R}_{++} \to \mathbb{D}^{n+1},$$

$$\left(\mathcal{I}, \psi, [7] \right) := \text{Ɑ}\texttt{ByAttachingNCells}[7] : \sum \mathcal{I} \in \mathsf{SET} \ . \ \sum \psi : \mathcal{I} \to \mathbb{S}^n \to X^{☠ n} \ . \ X^{☠ n+1} = X^{☠ n} \sqcup_{\psi} \bigsqcup \mathbb{D}^n,$$

$\texttt{Assume} \ i : \mathcal{I},$

$$\mathcal{V} := \left\{ \psi_i^{-1}(O) \middle| O \in \mathcal{O}^{☠ n} \right\} : \texttt{OpenCover}(\mathbb{S}^n),$$

$$f^i := \Lambda V \in \mathcal{V} \ . \ \Lambda s \in \mathbb{S}^n \ . \ \phi_{\psi_i(V)}\left(\widehat{\psi}(i, s) \right) : \texttt{PartitionOfUnity}(\mathbb{S}^n, \mathcal{V} \cap \mathbb{S}^n),$$

$$[8] := \text{Ɑ}\texttt{CompactPartitionOfUnity}(\mathbb{S}^n, f^i) : \left| \left\{ V \in \mathcal{V} : f^i_O \neq 0 \right\} \right| < \infty,$$

$$m := \left| \left\{ V \in \mathcal{V} : f^i_0 \neq 0 \right\} \right| \in \mathbb{N},$$

$$V := \texttt{enumerate}\left\{ V \in \mathcal{V} : f^i_0 \neq 0 \right\} : m \leftrightarrow \left\{ V \in \mathcal{V} : f^i_0 \neq 0 \right\},$$

$$g := f_V : m \to \mathbb{S}^n \xrightarrow{\mathsf{TOP}} [0, 1],$$

$\texttt{Assume} \ j \in [1, \dots, m]_{\mathbb{N}},$

$$[9] := \text{Ɔ} g_j : \texttt{Compact}(V_j \cap \mathbb{S}^n, \operatorname{supp} g_j),$$

$$\left(t_j, [j.*] \right) := \text{Ɑ}\texttt{Compact}[9]\text{Ɔ}^{-1} I : \sum_{t \in (0,1)} I(\operatorname{supp} g_j, t) \subset V_j;$$

$$\rightsquigarrow \left(t, [9] \right) := \mathtt{I}\left(\prod \right) : \prod_{j=1}^{m} \sum_{t_j \in (0,1)} I(\operatorname{supp} g_j, t) \subset V_j,$$

$$s := \max t \in (0, 1),$$

$$s' := 1 - \frac{(1 - s)}{2} \in (0, 1),$$

$$\sigma := \texttt{bump}\left(\mathbb{D}^{n+1}, \mathbb{D}^{n+1} \setminus I\left(\mathbb{S}^n, s \right), I\left(\mathbb{S}^n \right) \right) : \mathbb{D}^{n+1} \xrightarrow{\mathsf{TOP}} [0, 1],$$

$$\mathcal{W} := \left\{ \psi_i^{-1}(O) \middle| O \in \mathcal{O}^{☠ n+1} \right\} : \texttt{OpenCover}(\mathbb{D}^{n+1}),$$

$$[10] := \text{Ɔ} \mathcal{V} \mathcal{W} : \mathcal{W} \cap \mathbb{S}^n = \mathcal{V},$$

$$h := \texttt{PartitionOfUnityExists}(\mathbb{D}^{n+1}, \mathcal{W}) : \texttt{PartitionOfUnity}\left(\mathbb{D}^{n+1}, \mathcal{W} \right),$$

$$F^i := \Lambda W \in \mathcal{W} \,.\, \Lambda x \in \mathbb{D}^{n+1} \,.\, \sigma(x) h_W(x) \Big(1-\sigma(x)\Big) f_{W\cap\mathbb{S}^n}\left(\frac{x}{\|x\|}\right) : \mathcal{W} \to \mathbb{D}^{n+1} \xrightarrow{\mathsf{TOP}} [0,1],$$

$$[11] := \mathcal{D}F^i : \forall x \in \mathbb{D}^{n+1} \sum_{W\in\mathcal{W}} F^i_W(x) = \sigma(x) \sum_{W\in\mathcal{W}} h_W(x) + \Big(1-\sigma(x)\Big) \sum_{W\in\mathcal{W}} f_{W\cap\mathbb{S}^n}\left(\frac{x}{\|x\|}\right) \sigma(x) + 1 - \sigma(x) = 1,$$

$$[i.*] := \mathcal{A}^{-1}\texttt{PartitionOfUnity}[11]\mathcal{A}\texttt{Compact}(\mathcal{D}^n) : \texttt{PartitionOfUnity}\Big(\mathbb{D}^n, \mathcal{W}, F\Big);$$

$$\rightsquigarrow \Big(F, [8]\Big) := \mathtt{I}\left(\prod\right) : \prod_{i\in\mathcal{I}} \sum F^i : \texttt{PartitionOfUnity}\Big(\mathbb{D}^{n+1}, \widehat{\psi}^{-1}\mathcal{O}^{\skull n+1}\Big) \,.$$

$$.\, \forall O \in \mathcal{O}^{\skull n+1} \,.\, F^i_{\widehat{\psi}^{-1}O|\mathbb{S}^n} = \psi_i \phi^n_{O\cap X^{\skull n}},$$

$$\phi^{n+1} := \Lambda O \in \mathcal{O}^{\skull n+1} \,.\, \Lambda x \in X^{\skull n+1} \,.\, \texttt{if } x \in X^{\skull} \texttt{ then } \phi^n_{O\cap X^{\skull n}}(x) \texttt{ else } F^i_{\widehat{\psi}^{-1}O} \circ \widehat{\psi}_2^{-1}(x)$$

$$\texttt{where} \quad x \in \widehat{\psi}\Big(i, \mathbb{D}^n\Big) : \mathcal{O}^{\skull n+1} \to X^{\skull n+1} \to [0,1],$$

$$[n.*.1] := \mathcal{D}\phi^{n+1} : \forall O \in \mathcal{O}^{\skull n+1} \,.\, \phi^{n+1}_{O|X^{\skull n}} = \phi^n_{O\cap X^{\skull n}},$$

$$[n.*.2] := \mathcal{D}F\mathcal{D}\phi^{n+1}\mathcal{A}\texttt{QuotientMap}(\hat{\psi}) : \forall U \in \mathcal{T}\Big(X^{\skull n}\Big) \,.\, \forall O \in \mathcal{O} \,.\, \phi^n(U)_{O\cap X^{\skull n}} = \{0\} \Rightarrow$$

$$\Rightarrow \exists V \in \mathcal{T}\Big(X^{\skull n+1}\Big) : U \subset V \;\&\; \phi^{n+1}(V) = \{0\},$$

$$[n.*.3] := \mathcal{D}F\mathcal{D}\phi^{n+1}\mathcal{A}\texttt{PartitionOfUnity}\Big(X^{\skull n}, \mathcal{O}^{\skull n}, \phi^n\Big)[5] : \texttt{PartitionOfUnity}\Big(X^{\skull n+1}, \mathcal{O}^{\skull n+1}, \phi^{n+1}\Big);$$

$$\rightsquigarrow \Big(\phi, [4]\Big) := \texttt{PrimitiveRecursion}\mathcal{D}\phi^0 : \prod_{n=0}^{\infty} \sum \phi^n : \texttt{PartitionOfUnity}\Big(X^{\skull n}, \mathcal{O}^{\skull n}\Big) \,.$$

$$.\, \forall n \in \mathbb{N} \,.\, \forall O \in \mathcal{O}^{\skull n+1} \,.\, \phi^{n+1}_{O|X^{\skull n}} = \phi^n_{O\cap X^{\skull n}} \;\&$$

$$\& \; \forall U \in \mathcal{T}\Big(X^{\skull n}\Big) \,.\, \forall O \in \mathcal{O} \,.\, \phi^n(U)_{O\cap X^{\skull n}} = \{0\} \Rightarrow \exists V \in \mathcal{T}\Big(X^{\skull n+1}\Big) : U \subset V \;\&\; \phi^{n+1}(V) = \{0\},$$

$$f := \Lambda O \in \mathcal{O} \,.\, \varinjlim_n \phi^n_{X^{\skull n}\cap O} : \mathcal{O} \to X \xrightarrow{\mathsf{TOP}} [0,1],$$

$$[\mathcal{O}.*] := \mathcal{D}f[4]\mathcal{A}\texttt{CWComplex}(X, \mathcal{E}, \varphi) : \texttt{PartitionOfUnity}(X, \mathcal{O}, f);$$

$$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{ParacompactByPartitionOfUnity} : \texttt{Paracompact}(X);$$

$\square$

$\texttt{CWComplexIsNormal} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.\, \texttt{T4}(X)$

$\texttt{Proof} =$

...

$\square$

$\texttt{countableLocallyEucleadeanCWComplexIsManifold} ::$

$:: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.\, \forall n \in \mathbb{N} \,.\, |\mathcal{E}| \le \aleph_0 \;\&\; \texttt{LocallyEuclidean}(X, n) \Rightarrow X \in \mathsf{TOPM}(n)$

$\texttt{Proof} =$

...

$\square$

$\texttt{DimensionAgrees} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.\, \forall n \in \mathbb{N} \,.\, X \in \mathsf{TOPM}(n) \Rightarrow \dim \mathcal{E} = n$

$\texttt{Proof} =$

...

$\square$

2.4 Embedding Theorems

$$\texttt{EuclideanCWComplex} :: ?\Big(\texttt{CWComplax} \,\&\, \texttt{LocallyFiniteComplex} \,\&$$
$$\& \,\texttt{FiniteDimensionalComplex}\Big)$$
$$(X, \mathcal{E}, \varphi) : \texttt{EuclideanCWComplex} \iff |\mathcal{E}| \leq \aleph_0$$

$$\texttt{QuasiEuclideanCWComplex} :: ?\Big(\texttt{CWComplax} \,\&\, \texttt{LocallyFiniteComplex}\Big)$$
$$(X, \mathcal{E}, \varphi) : \texttt{EuclideanCWComplex} \iff |\mathcal{E}| \leq \aleph_0$$

$$\texttt{CWComplexMetrizationTHM} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.\, \texttt{LocallyFinite}(X, \mathcal{E}, \varphi) \iff \texttt{Metrizable}(X)$$

Proof $=$

...

□

$$\texttt{ComplexEuclideanEmbeddingTheorem} :: \forall (X, \mathcal{E}, \varphi) : \texttt{EucleadinCWComplex} \,.$$
$$.\, \exists \texttt{HomeomorphicEmbedding}\Big(X, \mathbb{R}^{1+2\dim \mathcal{E}}\Big)$$

Proof $=$

...

□

$$\texttt{ComplexQuasiEuclideanEmbeddingTheorem} :: \forall (X, \mathcal{E}, \varphi) : \texttt{QuasiEucleadinCWComplex} \,.$$
$$.\, \exists \texttt{HomeomorphicEmbedding}\Big(X, \mathbb{R}^{\oplus \mathbb{N}}\Big)$$

Proof $=$

...

□

$$\texttt{ComplexQuasiEuclideanEmbeddingTheorem} :: \forall (X, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.$$
$$.\, \exists V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \exists \texttt{HomeomorphicEmbedding}\Big(X, V\Big)$$

Proof $=$

...

□

2.5 Classification of 1D manifolds

$\texttt{ManifoldDimIsTopDim} :: \forall M \in \mathsf{TOPM} \,.\, \dim M = \mathrm{top}\ \dim M$

$\texttt{Proof} \;=$

$n := \dim M \in \mathbb{Z}_+,$

$\texttt{Assume}\ A : \texttt{Closed}(M, A),$

$\texttt{Assume}\ \Psi : A \xrightarrow{\mathsf{TOP}} \mathbb{S}^n,$

$\Big(\mathcal{O}, [1]\Big) := \mathcal{A}\mathsf{TOPM} : \sum \mathcal{O} : \texttt{OpenCover}(M) \,.\, \forall O \in \mathcal{O} \,.\, O \cong \mathbb{B}^n,$

$[2] := \texttt{TopologicalDimInvariant}[1]\texttt{BallDim} : \forall O \in \mathcal{O} \,.\, \mathrm{top}\ \dim O = n,$

$\Big(\psi, [3]\Big) := \texttt{NormalTopologicalDim}[2] : \prod O \in \mathcal{O} \,.\, \sum \psi_O : O \xrightarrow{\mathsf{TOP}} \mathbb{S}^n \,.\, \psi_{O|O\cap A} = \Psi_{|O\cap A},$

$f := \texttt{PartitionOfUnityexists}(M, \mathcal{O}) : \texttt{PartitionOfUnity}(M, \mathcal{O}),$

$\Psi' := \sum_{O\in\mathcal{O}} f_O \psi_O : M \xrightarrow{\mathsf{TOP}} \mathbb{S}^n,$

$[A.*] := \mathcal{A}\texttt{PartitionOfUnity}[3] : \Psi'_{|A} = \Psi;$

$\rightsquigarrow [*] := \texttt{NormalTopologicalDim} : \mathrm{top}\ \dim M = n;$

$\square$

$\texttt{OneManifoldAdmitsRegularCWStruct} :: \forall M \in \mathsf{TOPM}(1) \,.\, \exists (X, \mathcal{E}, \varphi) : \texttt{RegularCWComplex} \,.\, M = X$

$\texttt{Proof} \;=$

$\Big(\mathcal{V}, [1]\Big) := \texttt{ManifoldDimIsTopDim}(M)\mathcal{A}\mathrm{top}\ \dim :$

$: \sum \mathcal{V} : \texttt{OpenCover}(M) \,.\, \forall V \in \mathcal{V} \,.\, V \cong_{\mathsf{TOP}} (0,1)\ \&\ \left|\Big\{U \in \mathcal{V} : U \neq V\ \&\ U \cap B \neq \emptyset\Big\}\right| \leq 2,$

$[2] := \mathcal{A}\texttt{LocallyCompact}(X)\texttt{BaseEq}(\mathcal{V}) : \Big|\mathcal{V}\Big| \leq \aleph_0,$

$V := \texttt{emumerate}(\mathcal{V}) : \mathbb{N} \leftrightarrow \mathcal{V},$

$N := \Lambda n \in \mathbb{N} \,.\, \bigcup_{i=1}^{n} \overline{V}_i : \mathbb{N} \to \texttt{Closed}(M),$

$[3] := \mathcal{A}\texttt{OpenCover}(\mathcal{V})\mathcal{D}N : M = \bigcup_{n=1}^{\infty} N_n,$

$\mathcal{E}^1 := \Big(0 \to \partial V_1, 1 \to V_1\Big) : \{0,1\} \to ?M,$

$\Big(\varphi^1, [4]\Big) := \mathcal{D}\mathcal{E}^1 : \sum \varphi^1 \prod_{i=0}^{1} \prod_{e\in\mathcal{E}_i^1} \mathbb{D}^i \to e \,.\, \texttt{CWComplex}(N_1, \mathcal{E}^1, \varphi^1),$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ (X, \mathcal{E}^n, \varphi^n) : \texttt{RegularCWComplex},$

$\texttt{Assume}\ [5] : X = N_n,$

$\mathcal{E}_0^{n+1} := E_0^n \cup \partial V_{n+1} :?M,$

$\mathcal{E}_1^{n+1} := \Big\{U \cap V_{n-1} | U \in \mathcal{E}_1^n\Big\} \cup \Big\{U \setminus V_{n-1} | U \in \mathcal{E}_1^n\Big\} \cup \Big\{V_{n-1} \setminus U | U \in \mathcal{E}_1^n\Big\} :?\mathcal{T}(M),$

$\Big(\varphi^{n+1}, [n.*]\Big) := [5]\mathcal{D}\mathcal{E}^{n+1} : \sum \varphi^{n+1} \prod_{i=0}^{1} \prod_{e\in\mathcal{E}_i^{n+1}} \mathbb{D}^i \to e \,.\, \texttt{CWComplex}(N_{n+1}, \mathcal{E}^{n+1}, \varphi^{n+1});$

$\rightsquigarrow \Big(\mathcal{E}, [5]\Big) := \mathtt{I}\Big(\sum\Big)\,\mathtt{I}\Big(\prod\Big) : \sum \mathcal{E} : \prod_{n=1}^{\infty} \{0,1\} \to ?N_n \,.\, \forall n \in \mathbb{N} \,.\, \exists (X, \mathcal{E}, \varphi) : \texttt{RegularCWComplex} : X = N_n,$

$\mathcal{E}' := \Lambda_{i=0}^{1} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathcal{E}_i^n : \{0,1\} \to ?M,$

$[6] := \partial\mathcal{E}'[1]\partial\mathcal{E} : \forall i \in \{0,1\} \; . \; \mathcal{E}'_i \neq \emptyset,$

$\Big(\varphi, [7]\Big) := [6][5] : \prod_{i=0}^{1} \prod_{e \in \mathcal{E}'_i} \sum \varphi_{i,e} : \texttt{TopologicalEmbedding}\Big(\mathbb{D}^i, M\Big) \; . \; \operatorname{Im} \varphi_{i,e} = \overline{e},$

$[8] := [3]\partial\mathcal{E}' : \bigcup_{i=1}^{1} \bigcup_{e \in \mathcal{E}_i} e = M,$

$[9] := \partial\mathcal{E}'[7] : \forall e \in \mathcal{E}_1 \; . \; \Big|\texttt{singleton}\, \partial\, e \cap \mathcal{E}_0\Big| = 2,$

$\texttt{Assume}\ A : ?M,$

$\texttt{Assume}\ [10] : \forall e \in \mathcal{E}'_1 \; . \; \texttt{Closed}\Big(\overline{e}, \overline{e} \cap A\Big),$

$[11] := \text{Ɑ}\texttt{CWComplex}[5] : \forall n \in \mathbb{N} \; . \; \texttt{Closed}\Big(N_n \cap A\Big),$

$[12] := \texttt{ClosedSubsetTHM}[11] : \forall n \in \mathbb{N} \; . \; \texttt{Closed}\Big(M, N_n \cap A\Big),$

$[13] := [3]\texttt{UnionDistrivutibity} : \bigcup_{m=1}^{\infty} N_n \cap A = 1,$

$[14] := [1]\partial N : \forall n \in \mathbb{N} \; . \; \texttt{LocallyFinite}(X, N_n),$

$[A.*] := [13][14]\texttt{LocalyFiniteClosureUnion} : \texttt{Closed}(M, A);$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{CWComplex}[7] : \texttt{RegularCwComplex}\Big(M, \mathcal{E}', \varphi\Big);$

$\square$

$\texttt{RegularGraphManifoldEdge} :: \forall (M, \mathcal{E}, \varphi) : \texttt{RegularCWComplex} \; . \; \forall [0] : M \in \mathsf{TOPM} \; .$

$\quad . \; \forall [00] : \dim M = 1 \; . \; \forall e \in \mathcal{E}_1 \; . \; \Big|\partial\, e\Big| = 2$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{RegularGraphManifoldVertex} :: \forall (M, \mathcal{E}, \varphi) : \texttt{RegularCWComplex} \; . \; \forall [0] : M \in \mathsf{TOPM} \; .$

$\quad . \; \forall [00] : \dim M = 1 \; . \; \forall v \in \mathcal{E}_0 \; . \; \Big|\Big\{e \in \mathcal{E}_2 : v \in \overline{e}\Big\}\Big| = 2$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{1DManifoldClassificationTHM} :: \forall M \in \mathsf{TOPM}\ \&\ \texttt{Connected} \; . \; \forall [0] : \dim M = 1 \; . \; M \cong_{\mathsf{TOP}} \mathbb{S}^1 \Big| M \cong_{\mathsf{TOP}} \mathbb{R}$

$\texttt{Proof}\ =$

$\Big((X, \mathcal{E}, \varphi), [1]\Big) := \texttt{OneManifoldAdmitsRegularCWStructure}(M)[1] : \sum (X, \mathcal{E}, \varphi) : \texttt{RegularCWComplex} \; .$

$\quad . \; M = X,$

$\texttt{Assume}\ [2] : \texttt{Compact}(X),$

$[3] := \texttt{CompactIffFinite}[1][2] : |\mathcal{E}| < \infty,$

$n := |\mathcal{E}_0| : \mathbb{N},$

$\Big(v, [3.5]\Big) := \texttt{cyclicEnumerate}(\mathcal{E}_0, \ldots) : \sum v : \mathbb{Z}_n \leftrightarrow \mathcal{E}_0 \; . \; \forall i \in \mathbb{Z}_+ \; . \; \exists e \in \mathcal{E}_1 \; .$

$\quad . \; \lim_{t \to 0} \varphi_{1,e}^{-1}(t) = v_i\ \&\ \lim_{t \to 1} \varphi_{1,e}^{-1}(t) = v_{i+1},$

$\texttt{Assume}\ x \in M,$

$\Big(e, [4]\Big) := \sqcap\texttt{CellComplex}(M, \mathcal{E}, \varphi)(x) : \sum_{e \in \mathcal{E}} x \in e,$

$\texttt{Assume}\ [5] : \dim e = 0,$

$\Big(k, [6]\Big) := \sqcap\texttt{Bijection}(v, e) : \sum_{k=0}^{n-1} e = v_k,$

$f(x) := \exp\left(\frac{2k\mathbf{i}\pi}{n}\right) : \mathbb{S}^1;$

$\rightsquigarrow [6] := \texttt{I}(\Rightarrow) : \dim e = 0 \Rightarrow f(x) \in \mathbb{S}^1,$

$\texttt{Assume}\ [7] : \dim e = 1,$

$\Big(t, [9]\Big) := \sqcap\texttt{Bijection}(v)[3.5] : \sum_{t,l=0}^{n-1} t \neq l \ \&\ \partial e = \{v_t, v_{t+1}\},$

$f(x) := \exp\left(\frac{2\mathbf{i}\pi\Big(t + \varphi_{1,e}^{-1}(x)\Big)}{n}\right) : \mathbb{S}^1;$

$\rightsquigarrow [10] := \texttt{I}(\Rightarrow) : e \in \mathcal{E}_1 \Rightarrow f(x) \in \mathbb{S}^1,$

$[2.*] := \sqcap^{-1}\texttt{CellComplex}(M, \mathcal{E}, \varphi)\sqcup f[3.5] : f : M \xleftrightarrow{\mathsf{TOP}} \mathbb{S}^1;$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : \texttt{Compact}(M) \Rightarrow M \cong_{\mathsf{TOP}} \mathbb{S}^1,$

$\texttt{Assume}\ [3] : !\ \texttt{Compact}(M),$

$[4] := \texttt{CompactIffFinite} : |\mathcal{E}| = \infty,$

$[5] := \texttt{RegularManifoldVertex}(M)[0][1][4] : |\mathcal{E}_0| = |\mathcal{E}_1|,$

$[6] := \sqcap\mathsf{TOPM}(M) : \texttt{SeconCountable}(M),$

$[7] := \texttt{OpenCellTHM}(M)[0] : \forall e \in \mathcal{E}_1 \ .\ e \in \mathcal{T}(X),$

$[8] := \sqcap\texttt{SecondCountable}[7][6] : |\mathcal{E}_0| = \aleph_0,$

$\Big(v, [8.5]\Big) := \sqcap\texttt{Cardinality}(\mathbb{Z}, \mathcal{E}_0) : \sum v : \mathbb{Z} \leftrightarrow \mathcal{E}_0 \ .\ \forall i \in \mathbb{Z} \ .\ \exists e \in \mathcal{E}_1 \ .\ \lim_{t \to 0} \varphi_{1,e}(t) = v_i \ \&\ \lim_{t \to 1} \varphi_{1,e}(t) = v_{i+1},$

$\texttt{Assume}\ x : M,$

$\Big(e, [4]\Big) := \sqcap\texttt{CellComplex}(M, \mathcal{E}, \varphi)(x) : \sum_{e \in \mathcal{E}} x \in e,$

$\texttt{Assume}\ [5] : \dim e = 0,$

$\Big(k, [6]\Big) := \sqcap\texttt{Bijection}(v, e) : \sum_{k \in \mathbb{Z}_+} e = v_k,$

$f(x) := k : \mathbb{R};$

$\rightsquigarrow [6] := \texttt{I}(\Rightarrow) : \dim e = 0 \Rightarrow f(x) \in \mathbb{R},$

$\texttt{Assume}\ [7] : \dim e = 1,$

$\Big(t, [9]\Big) := \sqcap\texttt{Bijection}(v)[3.5] : \sum_{t \in \mathbb{Z}_+} t \neq l \ \&\ \partial e = \{v_t, v_{t+1}\},$

$f(x) := t + \varphi_{1,e}^{-1}(x) : \mathbb{R};$

$\rightsquigarrow [10] := \texttt{I}(\Rightarrow) : e \in \mathcal{E}_1 \Rightarrow f(x) \in \mathbb{R},$

$[3.*] := \sqcap^{-1}\texttt{CellComplex}(M, \mathcal{E}, \varphi)\sqcup f[3.5] : f : M \xleftrightarrow{\mathsf{TOP}} \mathbb{R};$

$\rightsquigarrow [*] := [2]\texttt{I}(\Rightarrow)\texttt{E}(|) : M \cong \mathbb{S}^1 \Big| M \cong \mathbb{R};$

$\square$

$\texttt{1DManifoldWithBoundaryClassification} :: \forall M \in \mathsf{TOPM}_\partial \ \&\ \texttt{Connected}\ .$

$\quad . \forall [0] : \dim M = 1\ .\ \forall [00] : \partial M \neq \emptyset\ .\ M \cong_{\mathsf{TOP}} [0,1] \Big| M \cong_{\mathsf{TOP}} [0,+\infty)$

$\texttt{Proof}\ =$

$N := M \setminus \partial N \in \mathsf{TOPM}(1),$

$[1] := \mathcal{O} N \mathcal{C} \texttt{Compact}[00] : !\ \texttt{Compact}(N),$

$[2] := \texttt{1DManifoldClassification}[1] : N \cong_{\mathsf{TOP}} \mathbb{R},$

$[*] := [00][2] \mathcal{C} \mathsf{TOPM}_\partial(M) : M \cong_{\mathsf{TOP}} [0,1] \Big| M \cong_{\mathsf{TOP}} [0,+\infty);$

$\square$

2.6 Category

$\texttt{CellularMap} :: \prod (X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi) : \texttt{CellComplex} \;.\; ?(X \xrightarrow{\mathsf{TOP}} Y)$

$f : \texttt{CellularMap} \iff \forall n \in \mathbb{Z}_+ \;.\; f\left(X^{\skull n}\right) \subset Y^{\skull n}$

$\texttt{RegularCellMap} :: \prod (X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi) : \texttt{CellComplex} \;.\; \texttt{CellularMap}\Big((X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi)\Big)$

$f : \texttt{RegularCellMap} \iff \forall e \in \mathcal{E} \;.\; f(e) \in \mathcal{F}$

$\texttt{CWCategory} :: \mathsf{CAT}$

$\texttt{CWCategory}\,() = \mathsf{CW} := \left(\texttt{CWComplex}, C, \circ, \mathrm{id}\right)$

$\texttt{CWCellularCategory} :: \mathsf{CAT}$

$\texttt{CWCelluralCategory}\,() = \mathsf{CWC} := \left(\texttt{CWComplex}, \texttt{CellularMap}, \circ, \mathrm{id}\right)$

$\texttt{CWRegularCategory} :: \mathsf{CAT}$

$\texttt{CWRegularCategory}\,() = \mathsf{CWR} := \left(\texttt{CWComplex}, \texttt{RegularMap}, \circ, \mathrm{id}\right)$

$\texttt{RegularImageOfClosedCell} :: \forall (X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi) \in \mathsf{CW} \;.\; \forall r : (X, \mathcal{E}, \varphi) \xrightarrow{\mathsf{CWR}} (Y, \mathcal{F}, \psi) \;.\; \forall e \in \mathcal{E} \;.$

$\quad .\; \exists f \in \mathcal{F} : r\left(\bar{e}\right) = \bar{f}$

$\texttt{Proof} \;=$

$f := r(e) \in \mathcal{F},$

$[1] := \texttt{LimitImage}(r, e)\text{Ↄ}^{-1} f : r\left(\bar{e}\right) \subset \bar{f},$

$[2] := \texttt{CompactMappingTHM}(r, \bar{e}) : \texttt{Closed}\left(Y, r\left(\bar{e}\right)\right),$

$[3] := \texttt{MontonicImage}(e, \bar{e}, r)\texttt{ClosureIsSuper}(e)\text{Ↄ}^{-1} f : f \subset r(\bar{e}),$

$[4] := \text{Ɑ}\,\texttt{closure}[2][3] : \bar{f} \subset r(\bar{e}),$

$[5] := \text{Ɑ}^{-1}\texttt{SetEq} : \bar{f} = r(\bar{e});$

$\square$

$\texttt{RegularImageIsSubcomplex} :: \forall (X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi) \in \mathsf{CW} \;.\; \forall r : (X, \mathcal{E}, \varphi) \xrightarrow{\mathsf{CWR}} (Y, \mathcal{F}, \psi) \;.\; \left(r(X), r(\mathcal{E}), \varphi_{\bullet, r}\right) \subset$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RegularIsQuotientMap} :: \forall (X, \mathcal{E}, \varphi), (Y, \mathcal{F}, \psi) \in \mathsf{CW} \;.\; \forall r : (X, \mathcal{E}, \varphi) \xrightarrow{\mathsf{CWR}} (Y, \mathcal{F}, \psi) \;.\; \texttt{QuotientMap}\Big(X, r(X), r$

$\texttt{Proof} \;=$

$\texttt{Assume}\; A :? r(X),$

$\texttt{Assume}\; [1] : \texttt{Closed}\Big(X, r^{-1}(A)\Big),$

$[2] := \mathcal{A}^{-1}\texttt{subsetTopology}(X, \mathcal{E})[1] : \forall e \in \mathcal{E} \;.\; \texttt{Closed}(\overline{e}, \overline{e} \cap r^{-1}(A)),$

$[3] := \texttt{CompactMapTheorem} : \forall e \in \mathcal{E} \;.\; \texttt{Closed}(\overline{r(e)}, \overline{r}(e) \cap A),$

$[*.A] := \texttt{RegularImageIsSubcomplex}(r)\mathcal{A}\,\mathsf{CW}\Big(r(X), r(\mathcal{E}), \psi_{\bullet,r}\Big) : \texttt{Closed}\Big(r(X), A\Big);$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{QuotientMap} \;.\; \texttt{QuotientMap}\Big(X, r(X), r\Big) :$

;

$\texttt{CWcomplexHasCoproducts} :: \texttt{WithCoproducts}(\mathsf{CWR})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CWcomplexHasFiniteProducts} :: \texttt{withFiniteProducts}(\mathsf{CWR})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3 Simplicial Complexes

3.1 Simplices

$$\texttt{KSimplex} :: \prod_{n=0}^{\infty} \prod_{k=-1}^{n} ?\Big([0,\dots,k]_{\mathbb{Z}} \to \mathbb{R}^n\Big)$$

$$v : \texttt{KSimplex} \iff \dim \operatorname{Aff} v = k$$

$$\texttt{body} :: \prod_{n=1}^{\infty} \prod_{k=-1}^{n} \texttt{KSimplex}(n,k) \to \texttt{Convex}(\mathbb{R}^n)$$

$$\texttt{body}\,(v) = v := \operatorname{conv} v$$

$$\texttt{KFace} :: \prod_{n=0}^{\infty} \prod_{k=-1}^{n} \texttt{KSimplex}(n,k) \to \prod_{t=-1}^{k} ?\texttt{KSimplex}(n,t)$$

$$f : \texttt{KFace} \iff \Lambda v : \texttt{KSimplex}(n,k) \ . \ \Lambda t \in [-1,\dots,k]_{\mathbb{Z}} \ . \ f \in \operatorname{face}(v) \iff$$
$$\iff \Lambda v : \texttt{KSimplex}(n,k) \ . \ \Lambda t \in [-1,\dots,k]_{\mathbb{Z}} \ . \ \exists i : t \hookrightarrow k : f = v_i$$

$$\texttt{simplexBoundary} :: \prod_{n=0}^{\infty} \prod_{k=-1}^{n} \texttt{KSimplex}(n,k) \to \texttt{Compact}(\mathbb{R}^n)$$

$$\texttt{simplexBoundary}\,(v) = \partial v := \bigcup \operatorname{face}(v, k-1)$$

$$\texttt{simplexInterior} :: \prod_{n=0}^{\infty} \prod_{k=-1}^{n} \texttt{KSimplex}(n,k) \to ?\mathbb{R}^n$$

$$\texttt{simplexInterior}\,(v) = \operatorname{int} v := v \setminus \partial v$$

$$\texttt{SimplexIsACell} :: \forall n \in \mathbb{N} \ . \ \forall k \in [n,1]_{\mathbb{N}} \ . \ \texttt{ClosedCell}(k,v)$$

$\texttt{Proof} =$

$\texttt{Assume}\ x \in v,$

$$\Big(t,[1]\Big) := Ɑ v(x) Ɑ^{-1} \operatorname{conv} : \sum t : [0,\dots,k] \to \mathbb{R}_+ \ . \ 1 = \sum_{i=0}^{k} t_i \ \& \ x = \sum_{i=0}^{k} t_i v_i,$$

$$\varphi(x) := (t_1,\dots,t_k) : [0,1]^k;$$

$$\rightsquigarrow \varphi := \mathtt{I}(\to) : v \to [0,1]^k,$$

$$[1] := Ɑ^{-1}\texttt{KSimplex}Ɑ^{-1}\texttt{Simplex} : \Big(\varphi : v \xleftrightarrow{\mathsf{TOP}} [0,1]^k\Big),$$

$$[*] := Ɑ^{-1}\texttt{ClosedXell} : \texttt{ClosedCell}(k,c);$$

$\square$

$$\texttt{SimplicialRetract} :: \prod v : \texttt{KSimplex}(n,k) \ . \ \prod u \in \operatorname{face}(v,t) \ . \ ?\mathbf{AFF}(\mathbb{R}^n,\mathbb{R}^n)$$

$$R : \texttt{SimplicialRetract} \iff R(\operatorname{Im} v) = \operatorname{Im} u \ \& \ \forall i \in t \ . \ R(u_i) = u_i$$

$\texttt{SimplicialRetractTheorem} :: \forall v : \texttt{KSimplex}(n,k) \;.\; \forall u \in \operatorname{face}(v,t) \;.$

$\quad .\; \forall w \in \operatorname{face}(u,s) \;.\; \forall R : \texttt{SimplicialRetract}(u,w) \;.\; \forall[0] : s < t < k \;.\; \exists f : w \sqcup_R v \xleftrightarrow{\mathsf{TOP}} v : f_{|w} = \mathrm{id}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SimplexDiameterTHM} :: \forall v : \texttt{Kimplex}(n,k) \;.\; \operatorname{diam} v = \max_{0 \le i,j \le k} \|v_i - v_j\|$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y \in v,$

$$\Big(t, [1]\Big) := \text{ᗡ} x \text{ᗡ} v \text{ᗡ}\, \texttt{convexCombination} : \sum t : [0,\dots,k] \to [0,1] \;.\; x = \sum_{i=0}^{k} t_i v_i \;\&\; 1 = \sum_{i=0}^{k} t_i,$$

$$\Big(s, [2]\Big) := \text{ᗡ} y \text{ᗡ} v \text{ᗡ}\, \texttt{convexCombination} : \sum s : [0,\dots,k] \to [0,1] \;.\; y = \sum_{i=0}^{k} s_i v_i \;\&\; 1 = \sum_{i=0}^{k} s_i,$$

$$\Big[(x,y).\,*\Big] := [1][2]\texttt{EucleadeanNormConvexity}^{k+1}(n)\text{ᗡ}^{-1}\texttt{max}\|v_\bullet - v_\bullet\|[1][2] :$$

$$: \|x - y\| = \left\| \sum_{i=1}^{k} t_i v_i - \sum_{i=1}^{k} s_i v_i \right\| \le \sum_{i=0}^{k} t_i \left\| v_i - \sum_{j=1}^{k} s_i v_j \right\| \le \sum_{i,j=0}^{k} t_i s_j \|v_i - v_j\| \le \sum_{i,j=0}^{k} t_i s_j \max_{0 \le l,m \le n} \|v_l - v_m\| =$$

$$= \max_{0 \le l,m, \le n} \|v_l - v_m\|;$$

$$\rightsquigarrow [1] := \text{ᗡ}^{-1} \operatorname{diam} v : \operatorname{diam} v \le \max_{0 \le i,j \le k} \|v_i - v_j\|,$$

$$[2] := \max \text{ᗡ} \operatorname{diam} v \text{ᗡ} v : \max_{0 \le i,j \le k} \|v_i - v_j\| \le \operatorname{diam} v,$$

$$[*] := \texttt{DoubleIneq}[1][2] : \operatorname{diam} v = \max_{0 \le i,j \le k} \|v_i - v_j\|;$$

$\square$

$\texttt{barycentre} :: \texttt{KSimplex}(n,k) \to \mathbb{R}^n$

$$\texttt{barycentre}\,(v) = \bar{v} := \sum_{i=0}^{k} \frac{v_i}{k+1}$$

$\texttt{BarycentreDistanceIneq} :: \forall u : \texttt{KSimplex}(n,t) \;.\; \forall w \in \operatorname{face}(u,s) \;.\; \|\bar{u} - \bar{w}\| \le \dfrac{t-s}{t} \operatorname{diam} v$

$\texttt{Proof} \;=$

$$\Big(f, [1]\Big) := \text{ᗡ} \operatorname{face}(u,s)(w) : \sum f \in \operatorname{face}(u, t-s) \;.\; u = f \sqcup w,$$

$$[2] := \text{ᗡ} \bar{u}[1]\text{ᗡ}^{-1}\bar{w}\text{ᗡ}^{-1}\bar{f} : \bar{u} = \frac{s}{t}\bar{w} + \frac{t-s}{t}\bar{f},$$

$$[3] := [2]\texttt{NormHomogen}\text{ᗡ} \operatorname{diam} v\texttt{FractionalDiffIneq} :$$

$$: \|\bar{u} - \bar{w}\| = \left\| \frac{t-s}{t}\bar{f} - \frac{t-s}{t}\bar{w} \right\| = \frac{t-s}{t}\left\|\bar{f} - \bar{w}\right\| \le \frac{t-s}{t} \operatorname{diam} w,$$

$\square$

$\texttt{SimplexIntersectionTHM} :: \forall n \in \mathbb{N} \;.\; \forall (k, v) : \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} \texttt{KSimplex}(n, k) \;.\; \forall [0] : \forall n \in \mathbb{N} \;.\; v^{n+1} \subset v^n \;.$

$\quad . \; \exists t \in \mathbb{Z}_+ : \exists u : \texttt{KSimplex}(n, t) \;.\; u = \bigcap_{n=1}^{\infty} v^n$

$\texttt{Proof} \;=$

$[1] := \texttt{AffineDimSUbse} : \texttt{Decreasing}(\mathbb{N}, \mathbb{Z}_+, k),$

$[2] := \texttt{NonNegInegersBoundedFromBelow}[1] : \texttt{Stabilizes}(\mathbb{N}, \mathbb{Z}_+, k),$

$k' := \lim_{i \to \infty} k_i : \mathbb{Z}_+,$

$i := \texttt{enumerate}\{i \in \mathbb{N} : k_i = k'\} : \mathbb{N} \uparrow \mathbb{N},$

$v := v^i : \mathbb{N} \downarrow \texttt{KSimplex}(n, k'),$

$\Big([3], j\Big) := \texttt{BolzanoWeierstrassTHM}(v) : \sum j : \mathbb{N} \uparrow j \;.\; \texttt{Convergent}(\mathbb{N}, \mathbb{R}^{k'n}, v_j),$

$v := v^j : \texttt{Convergent}(\mathbb{N}, \mathbb{R}^{k' \times n}),$

$w := \lim_{i \to \infty} v^i : \mathbb{R}^{k' \times n},$

$\Big(k'', w', [4]\Big) := \texttt{ConvexlyIndependenExists} : \sum k'' \in [0, \ldots, k'] \;.\; \sum w' \subset w \;.$

$\quad . \; \texttt{ConvexlyIndependent}(k'', \mathbb{R}^n, w'),$

$\Big(f, \sigma, [5]\Big) := \texttt{NaturalSimplexIsoExists}(n, k) : \sum f : \prod_{i=1}^{\infty} \mathbb{R}^n \xrightarrow{\mathbf{AFF}\mathbb{R}} \mathbb{R}^n \;.\; \sum \sigma : \mathbb{N} \to S_{k'} \;.$

$\quad . \; \forall i \in \mathbb{N} \;.\; \forall j \in k' \;.\; f(v^i_j) = \triangle_{\sigma(j)},$

$\ldots$

$\square$

3.2 Euclidean Simplicial Complexes

$$\texttt{SimplicialComplex} :: \prod_{n=0}^{\infty} ? \prod_{k=-1}^{n} ?\texttt{KSimplex}(n,k)$$

$$\triangle : \texttt{SimplicialComplex} \iff \forall k \in [-1,\dots,n] \,.\, \forall s \in \triangle_k \,.\, \forall t \in [-1,\dots,k] \,.\, \forall f \in \operatorname{face}(s,t) \,.\, f \in \triangle_t \;\&$$

$$\& \;\forall k,t \in [-1,\dots,n] \,.\, \forall s \in \triangle_k \,.\, \forall s' \in \triangle_t \,.\, \exists m \in \Big[-1,\dots,\min(k,t)\Big] : s \cap s' \in \operatorname{face}(s,m) \cap \operatorname{face}(s',m) \;\&$$

$$\& \;\texttt{LocallyFinite}\left(\mathbb{R}^n, \bigcap_{k=0}^{\infty} \triangle_k\right)$$

$$\texttt{simplexSet} :: \texttt{SimplicialComplex}(n) \to ??\mathbb{R}^n$$

$$\texttt{simplexSet}\,(\triangle) = \triangle := \bigcup_{k=0}^{n} \triangle_k$$

$$\texttt{polyhedronOf} :: \texttt{SimplicialComplex}(n) \to ?\mathbb{R}^n$$

$$[\texttt{olyhedronOf}\,(\triangle) = \langle\triangle\rangle := \bigcup_{k=0}^{n} \bigcup \triangle_k$$

$$\texttt{FiniteSimplicialComplex} :: ?\texttt{SimplicialComplex}(n)$$
$$\triangle : \texttt{FiniteSimplicialComplex} \iff \triangle < \infty \iff |\triangle| < \infty$$

$$\texttt{simplitialComplexDimension} :: \texttt{SimplicialComplex}(n) \to [0,\dots,n]$$

$$\texttt{simplicialComplexDimension}\,(\triangle) = \dim \triangle := \max\Big\{k \in [0,\dots,n] \Big| \triangle_k \neq \emptyset\Big\}$$

$$\texttt{SimplitialSubcomplex} :: \texttt{SimplicialComplex}(n) \to ?\texttt{SimplicialComplex}(n)$$
$$\triangle' : \texttt{SimplitialSubcomplex} \iff \Lambda\triangle : \texttt{SimplicialComplex}(n) \,.\, \triangle' \subset \triangle \iff \forall k \in [-1,\dots,n] \,.\, \triangle'_k \subset \triangle_k$$

$$\texttt{kSkeletn} :: \prod \triangle \in \texttt{SimplicialComplex}(n) \,.\, [-1,\dots,n] \to \texttt{SimplicialSubcomplex}(n)$$

$$\texttt{kSkeleton}\,(k) = \triangle^{☠k} := \Lambda t \in [-1,\dots,n] \,.\, \texttt{if } t < k \texttt{ then } \emptyset \texttt{ else } \triangle_t$$

$$\texttt{asCellComplex} :: \texttt{SimplicialComplex}(n) \to \texttt{CellComplex}$$

$$\texttt{asCellComplex}\,(\triangle) = \mathcal{CC}(\triangle) := \Big(\langle\triangle\rangle, \Lambda k \in [-1,\dots,\infty] \,.\, \texttt{if } k > n \texttt{ then } \emptyset \texttt{ else } \operatorname{int}\triangle_k,$$

$$, k \mapsto \operatorname{int} t \mapsto \texttt{SimplexIsCell}(n,t)_{|\mathbb{B}^k}\Big)$$

$$\texttt{Triangulation} := \prod_{X \in \mathsf{TOP}} \prod \triangle : \texttt{SimplicialComplex}(n) \,.\, X \overset{\mathsf{TOP}}{\longleftrightarrow} \langle\triangle\rangle :$$

$$: \prod_{n=-1}^{\infty} \mathsf{TOP} \to \texttt{SimplicialComplex}(n) \to \texttt{Type};$$

$$\texttt{Triangulable} :: ?\mathsf{TOP}$$
$$X : \texttt{Triangulable} \iff \exists n \in [-1,\dots,+\infty) \,.\, \exists \triangle \in \texttt{SimplicialComplex}(n) \,.\, \exists\texttt{Triangulation}(X,\triangle)$$

$\texttt{2ManifoldIsTriangulabe} :: \forall M \in \mathsf{TOPM}(2) \, . \, \texttt{Triangulable}(M)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{3ManifoldIsTriangulabe} :: \forall M \in \mathsf{TOPM}(3) \, . \, \texttt{Triangulable}(M)$

$\texttt{Proof} =$

$\dots$

$\square$

3.3 Simplicial Maps

$$\texttt{vertices} :: \prod_{n=0}^{\infty}\prod_{k=-1}^{n} \texttt{KSimplex}(n,k) \to \texttt{Finite}(\mathbb{R}^n)$$

$$\texttt{vertices}\,(s) = \mathrm{vert}(s) := \mathrm{face}(s,0)$$

$$\texttt{VertexMap} := \prod s : \texttt{KSimplex}(n,m)\ .\ \prod t : \texttt{KSimplex}(n,k)\ .\ \mathrm{vert}(s) \to \mathrm{vert}(t) :$$

$$: \prod_{n=-1}^{\infty}\prod_{m,k=-1}^{n} \texttt{KSimplex}(n,m) \times \texttt{KSimplex}(n,k) \to \texttt{Type};$$

$$\texttt{VertexMapExtension} :: \forall n,m \in [0,+\infty)\ .\ \forall l,k \in [-1,n)\ .\ \forall s \in \texttt{KSimplex}(n,l)\ .\ \forall t \in \texttt{KSimplex}(m,k)\ .$$

$$.\ \forall \sigma : \texttt{VertexMap}(s,t)\ .\ \exists f : \mathbb{R}^n \xrightarrow{\mathbb{R}\textbf{-AFF}} \mathbb{R}^m\ .\ f_{|\mathrm{vert}(s)} = \sigma$$

$\texttt{Proof} \ =$

...

$\square$

$$\texttt{SimplicialMap} :: \prod_{n,m=0}^{\infty}\prod \triangle : \texttt{SimplicialComplex}(n)\ .\ \prod \triangle' : \texttt{SimplicialComplex}(n)\ .\ ?(\langle\triangle\rangle \xrightarrow{\mathsf{TOP}} \langle\triangle'\rangle)$$

$$\sigma : \texttt{SimplicialMap} \iff \forall k \in [-1,\dots,n]\ .\ \forall s \in \triangle_k\ .\ \sigma(s) \in \triangle'\ \&\ \exists f : \mathbb{R}^n \xrightarrow{\mathbb{R}\textbf{-AFF}} \mathbb{R}^m\ .\ f(s) = \sigma(s)$$

$$\texttt{vertexMap} :: \texttt{SimplicialMap}(n,m,\triangle,\triangle') \to \triangle^{\skull 0} \to \triangle'^{\skull 0}$$

$$\texttt{vertexMap}\,(\sigma) = \mathrm{vert}(\sigma) := \sigma_{|\triangle^{\skull 0}}$$

$$\texttt{SimplicialMapExtensionTHM} :: \forall n,m \in \mathbb{Z}_+\ .\ \forall \triangle : \texttt{SimplicialComplex}(n)\ .\ \forall \triangle' : \texttt{SimplicialComplex}(m)\ .$$

$$.\ \forall f : \triangle^{\skull 0} \to \triangle'^{\skull 0}\ .\ \exists! \sigma : \texttt{SimplicialMap}(\triangle,\triangle') : \mathrm{vert}(\sigma) = f$$

$\texttt{Proof} \ =$

...

$\square$

3.4 Abstract Simplicial Complexes

$$\texttt{AbstractSimplicialComplex} :: \prod T : \texttt{Type} \,.\, ??\texttt{Finite}(T)$$
$$C : \texttt{AbstractSimplicialComplex} \iff \forall A \in C \,.\, \forall B \subset A \,.\, B \in C$$

$$\texttt{FiniteAbstractComplex} :: ?\texttt{AbstractSimplicialComplex}(T)$$
$$C : \texttt{FiniteAbstractComplex} \iff |C| < \infty$$

$$\texttt{LocallyFiniteAbstractComplex} :: ?\texttt{AbstractSimplicialComplex}(T)$$
$$C : \texttt{LocallyFiniteAbstractComplex} \iff \forall A \in C \,.\, \forall a \in A \,.\, \Big|\{B \in C : a \in A\}\Big|$$

$$\texttt{abstractSimplexDimension} :: \prod C : \texttt{AbstractSimplicialComplex}(T) \,.\, C \to [-1, \ldots, +\infty)$$
$$\texttt{abstractSimplexDimension}\,(A) = \dim_C A := |A| - 1$$

$$\texttt{abstractSimplecialDimension} :: \texttt{AbstractSimplicialComplex}(T) \to [-\infty, \ldots, +\infty]$$
$$\texttt{abstractSimplecialDimension}\,(A) = \dim C := \sup_{A \in C} \dim A$$

$$\texttt{FiniteDimensionalAbstractComplex} :: ?\texttt{AbstractSimplicialComplex}(T)$$
$$C : \texttt{FiniteDimensionalAbstractComplex} \iff |\dim C| < \infty$$

$$\texttt{vertexSet} :: \texttt{AbstractSimplicialComplex}(T) \to ?T$$
$$\texttt{vertexSet}\,(C) = \langle C \rangle := \bigcup_{A \in C} A$$

$$\texttt{AbstractSimplicialMap} :: \prod T, S : \texttt{Type} \,.\, \prod C : \texttt{AbstractSimplicialComplex}(T) \,.$$
$$.\, \prod C' : \texttt{AbstractSimplicialComplex}(S) \,.\, ?(C \to C')$$
$$F : \texttt{AbstractSimplicialMap} \iff \exists f : \langle C \rangle \to \langle C' \rangle : \forall A \in C \,.\, F(A) = f(A)$$

$$\texttt{abstractVertexMap} :: \prod T, S : \texttt{Type} \,.\, \prod C : \texttt{AbstractSimplicialComplex}(T) \,.$$
$$.\, \prod C' : \texttt{AbstractSimplicialComplex}(S) \,.\, \texttt{AbstractSimplicialMap}(C, C') \to \langle C \rangle \to \langle C' \rangle$$
$$\texttt{AbstractVertexMap}\,(F) = \langle F \rangle := \text{Ⅎ}\,\texttt{AbstractSimplicialMap}(C, C', F)$$

$$\texttt{vertexSchema} :: \prod_{n=0}^{\infty} \texttt{SimplicialComplex}(n) \to \texttt{AbstractSimplicialComplex}(\mathbb{R}^n)$$
$$\texttt{vertexSchema}\,(\triangle) = \mathcal{VS}(\triangle) := \left\{ \operatorname{Im} s \,\middle|\, k \in [0, \ldots, n], s \in \triangle_s \right\}$$

$\texttt{AbstractSimplicialIsomorphism} :: \prod T, S : \texttt{Type} \ . \ \prod C : \texttt{AbstractSimplicialComplex}(T) \ .$
$\quad . \ \prod C' : \texttt{AbstractSimplicialComplex}(S) \ . \ ?\texttt{AbstractSimplicalMap}$
$F : \texttt{AbstractSimpliciaIsomorphism} \iff \texttt{Bijection}(C, C', F)$

$\texttt{IsomorphicASC} :: \prod T, S : \texttt{Type} \ . \ ?\texttt{AbstractSimplicialComplex}(T) \times \texttt{AbstractSimplicialComplex}(S)$
$C, C' : \texttt{IsomorphicASC} \iff C \cong C' \iff \exists \texttt{AbstractSimplicialIsomorphism}(C, C)$

$\texttt{GeometricRealization} :: \texttt{AbstractSimplicialComplex}(T) \to ?\sum_{n=0}^{\infty} \texttt{SimplicialComplex}(n)$
$(n, \triangle) : \texttt{GeometricRealization} \iff \Lambda C \in \texttt{AbstractSimplicialComplex}(T) \ . \ \mathcal{VS}(\triangle) \cong C$

$\texttt{GeometricRealizationTHM} :: \forall C \in \texttt{FiniteAbstractComplex}(T) \ . \ \exists \texttt{GeometricRealization}(C)$
$\texttt{Proof} \ =$
$\ldots$
$\square$

4 Compact Surfaces

4.1 Polygones

$\texttt{CompactSurface} := \mathsf{TOPM}(2)\ \&\ \texttt{Compact}\ \&\ \texttt{Connected}\ \&\ \texttt{NonEmpty} : \texttt{Type};$

$\texttt{Polygon} :: ??\mathbb{R}^2$

$P : \texttt{Polygon} \iff P \cong_{\mathsf{TOP}} \mathbb{S}^1\ \&\ \exists \triangle : \texttt{SimplicialComplex}(2) : P = \langle \triangle \rangle$

$\texttt{vertices} :: \texttt{Polygon} \to \texttt{Finite}(\mathbb{R}^2)$

$\texttt{vertices}\,(P) = \mathbf{V}(P) := \triangle_0$

$\texttt{edges} :: \texttt{Polygon} \to \texttt{Finite}(\texttt{KSimplex}(2,1))$

$\texttt{edges}\,(P) = \mathbf{E}(P) := \triangle_1$

$\texttt{PolygonalRegion} :: ?\texttt{Compact}(\mathbb{R}^2)$

$R : \texttt{PolygonalRegion} \iff \texttt{Polygon}(\partial\, R)$

$\texttt{PolygonalComplex} :: ?\mathsf{TOP}$

$$X : \texttt{PolygonalComplex} \iff \exists n \in \mathbb{N} : \exists P : [1,\dots,n] \to \texttt{PolygonalRegion} :$$
$$: \exists E :?(\bigsqcup_{i,j=1}^{n} \mathbf{E}(P_i) \times \bigsqcup_{i=1}^{n} \mathbf{E}(P_i)) : \exists A : \prod (e,f) \in E\ .\ \texttt{SymplecticMap}(e,f) : X = \frac{\bigsqcup_{i=1}^{n} P_i}{A}\ \&$$
$$\&\ \texttt{DiogonalFree}(E)$$

$\texttt{PolygonalComplexIsCW} :: \forall X : \texttt{PolygonalComplex}\ .\ \exists (Y,\mathcal{E},\varphi) : \texttt{FiniteCWComplex} : X = Y$

$\texttt{Proof} =$

$$\Big(n,P,E,A,[1]\Big) := \text{Ɑ}\texttt{PolygonalComplex}(X) : \sum_{n=1}^{\infty} \sum P : [1,\dots,n] \to \texttt{PolygonalRegion}\ .$$
$$.\ \sum E \subset \bigsqcup_{i,j=1}^{n} \mathbf{E}(P_i) \times \mathbf{E}(P_j)\ .\ \sum A : \prod (e,f) \in E\ .\ \texttt{SymplecticMap}(e,f)\ .\ X = \frac{\bigsqcup_{i=1}^{n} P_i}{A},$$

$$\mathcal{E}_2 := \left\{ P_i \setminus \bigcup \mathbf{E}(P_i) \middle| i \in [1,\dots,n] \right\} :?\texttt{Cell}(2),$$

$$\mathcal{E}_1 := \frac{\bigsqcup_{i=1}^{n} \operatorname{int} \mathbf{E}(P_i)}{\operatorname{int} E} :?\texttt{Cell}(1),$$

$$\mathcal{E}_0 := \frac{\bigcup_{i=1}^{n} \mathbf{V}(P_i)}{A_|} :?\texttt{Cell}(0),$$

$\texttt{Assume}\ i \in \{0,1,2\},$

$\texttt{Assume}\ e \in \mathcal{E}_i,$

$$\Big(j,p,[2]\Big) := \text{Ɔ}\mathcal{E}_i : \sum j \in [1,\dots,n]\ .\ \sum p \subset P_j\ .\ e = [p],$$

$[3] := \text{Ɑ}\texttt{Pushout}[1]\text{Ɔ}\mathcal{E}_i(e) : \forall Q,Q' \in [p]\ .\ \forall q \in \bar{Q}\ .\ \forall q' \in \bar{Q}'\ .\ \pi_e(q) = \pi_e(q) \Rightarrow \pi_A(q) = \pi_A(q),$

$\varphi_{i,e} := \pi_{A,[3]} : \bar{e} \to X;$

$$\rightsquigarrow \varphi := \mathtt{I}\left(\prod\right) : \prod_{i=0}^{3} \prod_{e \in \mathcal{E}_n} \bar{e} \to X,$$

$[*] := \text{Ɑ}^{-1}\texttt{FiniteCWComplex}() : \texttt{FiniteCWComplex}(X,\mathcal{E},\varphi);$

$\square$

$\texttt{polygonalDegree} :: \texttt{PolygonalComplex} \to \mathbb{N}$

$\texttt{polygonalDegree}\,(X) = \deg X := n \quad \texttt{where} \quad (n, P, E, A) = \mathit{\Pi}\texttt{PolygonalComplex}(X)$

$\texttt{polygons} :: \prod X : \texttt{PolygonalComplex} \,.\, \Big[1, \ldots, \deg X\Big] \to \texttt{Polygone}$

$\texttt{polygonals}\,() = \mathbf{P}(X) := P \quad \texttt{where} \quad (n, P, E, A) = \texttt{PolygonalComplex}(X)$

$\texttt{edgeEquivalence} :: \prod X : \texttt{PolygonalComplex} \,.\, ?\left(\bigsqcup_{i=1} \mathbf{E}\,\mathbf{P}_i(X) \times \bigsqcup_{i=1} \mathbf{E}\,\mathbf{P}_i(X)\right)$

$\texttt{edgeEquivalence}\,() = \mathbf{E}(X) := E \quad \texttt{where} \quad (n, P, E, A) = \texttt{PolygonalComplex}(X)$

$\texttt{PolygonalComplexIsCompactSurfaceCondition} :: \forall X : \texttt{PolygonalComplex} \,.$

$\quad .\, \texttt{Bijection} \Rightarrow \texttt{CompactSurface}(X)$

$\texttt{Proof} \ =$

$\Big(\mathcal{E}, \varphi, [1]\Big) := \texttt{PolygonalComplexIsCWComplex}(X) : \ldots \texttt{FiniteCWComplex}(X, \mathcal{E}, \varphi),$

$[2] := \texttt{FiniteCWComplexIsCompact}(X, \ldots) : \texttt{Compact}(X),$

$\texttt{Assume}\ x \in X,$

$\Big(i, e, [3]\Big) := \mathit{\Pi}\texttt{Partition}(X, \mathcal{E}) : \sum i \in \{0, 1, 2\} \,.\, \sum e \in \mathcal{E}_i \,.\, x \in e,$

$[x.*] := \texttt{FanTransform}\mathit{\Pi}\texttt{PolygonalComplex}(X) : \exists U \in \mathcal{U}(x) \,.\, \texttt{Cell}(2, X);$

$\rightsquigarrow [3] := \mathit{\Pi}^{-1}\texttt{LocallyEuclidean} : \texttt{LocallyEuclidean}(X),$

$[*] := \mathit{\Pi}^{-1}\texttt{CompactSurface}[3] : \texttt{CompactSurface}(X);$

$\square$

$\texttt{covariantEdgeAssociation} :: \prod a, b : \texttt{KSimplex}(2, 1) \,.\, \texttt{SimplecticMap}(a, b)$

$\texttt{covariantEdgeAssociation}\,() = a \uparrow b := \texttt{AffineMapDetermination}(a_0, b_0) \texttt{where} \det a \uparrow b > 0$

$\texttt{contravariantEdgeAssociation} :: \prod a, b : \texttt{KSimplex}(2, 1) \,.\, \texttt{SimplecticMap}(a, b)$

$\texttt{contravariantEdgeAssociation}\,() = a \downarrow b := \texttt{AffineMapDetermination}(a_0, b_0) \texttt{where} \det a \downarrow b < 0$

$\texttt{standardSquare} :: \texttt{PolygonalRegion}$

$\texttt{standardSquare}\,() = I^2 := [0, 1]^2$

$A := (0, 0) :?\Big(\partial I^2\Big)_0;$

$B := (0, 1) :?\Big(\partial I^2\Big)_0;$

$C := (1, 1) :?\Big(\partial I^2\Big)_0;$

$D := (1, 0) :?\Big(\partial I^2\Big)_0;$

$\texttt{InjectivePair} :: \prod_{X, Y, Z \in \mathsf{SET}} \,.\, ?\Big((X \to Z) \times (X \to Z)\Big)$

$(f, g) : \texttt{InjectivePair} \iff \texttt{Injective}\Big(X \times Y, Z^2, f \times g\Big)$

simplePolygonalComplex :: $\left(\text{InjectivePair}\Big(\{1,2\},\{1,2\},\mathbf{V}(I^2)\Big)\right)\times\Big(\{1,2\}\to\{\uparrow,\downarrow\Big)\to$ CompactSurface

simplePolygonalComplex $(a,b,|) = \mathrm{spc}(a,b,|) := \dfrac{I^2}{a|b}$

SphereAsPolygonalComplex :: $\mathrm{spc}(AB\uparrow BC, AD\uparrow DC) \cong_{\mathsf{TOP}} \mathbb{S}^2$

Proof $=$

...

□

SphereAsPolygonalComplex :: $\mathrm{spc}(AB\uparrow BC, AD\uparrow DC) \cong_{\mathsf{TOP}} \mathbb{S}^2$

Proof $=$

...

□

TorusAsPolygonalComplex :: $\mathrm{spc}(AB\uparrow CD, BC\uparrow AD) \cong_{\mathsf{TOP}} \mathbb{S}^1\times\mathbb{S}^1$

Proof $=$

...

□

ProjectiveSpaceAsPolygonalComplex :: $\mathrm{spc}(AB\downarrow CD, BC\downarrow AD) \cong_{\mathsf{TOP}} \mathbb{RP}^2$

Proof $=$

...

□

ProjectiveSpaceAsPolygonalComplex :: $\mathrm{spc}(AB\downarrow CD, BC\downarrow AD) \cong_{\mathsf{TOP}} \mathbb{RP}^2$

Proof $=$

...

□

bottelOfKlein :: PolygonalComplex

bottelOfKlein $() = \mathbf{K} := \mathrm{spc}(AB\downarrow CD, BC\uparrow AD)$

4.2 Connected Sums

$$\texttt{CircledRegion} :: \prod M \in \mathsf{TOPM}(n) \,.\, ??M$$

$$A : \texttt{CircledRegion} \iff \texttt{Cell}(n, A) \;\&\; \partial A \cong_{\mathsf{TOP}} \mathbb{S}^{n-1}$$

$$\texttt{CicledRegionExists} :: \forall M \in \mathsf{TOPM}(n) \,.\, \exists \texttt{CircledRegion}(M)$$

Proof $=$

...

$\square$

$$\texttt{ConnectedSumsIsWellDefined} :: \forall n \in \mathbb{N} \,.\, \forall M, N \in \mathsf{TOPM}(n) \;\&\; \texttt{Connected} \,.$$

$$.\, \forall U, U' : \texttt{CircledRegion}(M) \,.\, \forall V, V' : \texttt{CircledRegion}(N) \,.$$

$$.\, \frac{(M \setminus U) \bigsqcup (N \setminus V)}{\psi^{-1} \circ \varphi} \cong_{\mathsf{TOP}} \frac{(M \setminus U') \bigsqcup (N \setminus V')}{\psi'^{-1} \circ \varphi'}$$

where $\varphi, \varphi' = \text{Ɑ}\texttt{CircledRegion}(M, U \;\&\; U'); \psi, \psi' = \text{Ɑ}\texttt{CirculedRegion}(N, V \;\&\; V')$

Proof $=$

...

$\square$

$$\texttt{connectedSum} :: \mathsf{TOPM}n \times \mathsf{TOPM}(n) \to \mathsf{TOPM}(n)$$

$$\texttt{connectedSum}\,(M, N) = A\#B := \frac{(M \setminus U) \bigsqcup (N \setminus V)}{\psi^{-1} \circ \varphi}$$

where $U, V = \texttt{CircledRegionExists}(M \;\&\; N)$;

$\varphi = \text{Ɑ}\texttt{CircledRegion}(M, U)$;

$\psi = \text{Ɑ}\texttt{CircledRegion}(N, V)$;

4.3 Polygonal Presentation

$$\texttt{GeneralPolygonalPresentation} :: \prod T : \texttt{Type} \,.\, \prod P : \texttt{Finite}(T) ? \sum_{n=0}^{\infty} \sum k : [1, \ldots, n] \to \mathbb{N} \,.$$

$$.\ \prod_{i=1}^{n} [1, \ldots, k_i] \to \Big(P \times \{1\}\Big) \sqcup \Big(P \times \{-1\}\Big)$$

$$(n, k, w) : \texttt{GeneralPolygonalPresentation} \iff (n, k, w) = \langle P | w_1, \ldots, w_n \rangle \iff$$

$$\iff \forall i \in [1, \ldots, n] \,.\, k_i \geq 3 \;\&\; \forall p \in P \,.\, \exists i \in [1, \ldots, n] : \exists j \in [1, \ldots, k_i] : w_{i,j} = (p, 1) | w_{i,j} = (p, -1)$$

$$\texttt{SpecialPolygonalPresentation} :: \prod T : \texttt{Type} \,.\, \prod p : T \,.\, ?\Big(\{1, 2\} \to \Big\{(p, 1), (p, -1)\Big\}\Big)$$

$$w : \texttt{GeneralPolygonalPresentation} \iff w = \langle p | w_1 w_2 \rangle \iff \top$$

$$\texttt{PolygonalPresentation} := \prod T : \texttt{Type} \,.\, \texttt{GeneralPolygonalPresentation}\Big|$$

$$\Big|\texttt{MaybeIf}\Big(\texttt{Singleton}, \texttt{SpecialPolygonalPresentation}\Big) : \prod T : \texttt{Type} \,.\, \texttt{Finite}(T) \to \mathsf{SET};$$

$$\texttt{PolygonalWord} :: \prod_{X \in \mathsf{SET}} .\ ?\mathrm{Lang}(X)$$

$$w : \texttt{PolygonalWord} \iff |w| \geq 3$$

$$\texttt{wordPolygon} :: \prod T : \texttt{Type} \,.\, \prod P : \texttt{Finite}(T) \,.$$

$$\texttt{NonEmptyWord}\Big(P \times \{1\}\Big) \sqcup \Big(P \times \{-1\}\Big) \to \texttt{Polygon}$$

$$\texttt{wordPolygon}\,((k, w)) = \mathbf{P}(w) := \texttt{RegularPolyonDeterminationByCenterAndRay}\Big(2, \mathrm{len}(w), \{0\} \times \mathbb{R}_+\Big)$$

$$\texttt{wordPresentation} :: \prod T : \texttt{Type} \,.\, \prod P : \texttt{Finite}(T) \,.$$

$$.\ \prod w : \texttt{NonEmptyWord}\Big(P \times \{1\}\Big) \sqcup \Big(P \times \{-1\}\Big) \,.\, \Big[1, \ldots, \mathrm{len}(w)\Big] \to \mathbf{E}\,\mathbf{P}(w)$$

$$\texttt{woedPresentation}\,(i) = \mathbf{E}_i(w) := \texttt{enumerateCounterClockwiseFromRay}\Big(\mathbf{E}\,\mathbf{P}(w), \{0\} \times \mathbb{R}_+\Big)(i)$$

$$\texttt{polygonalRealization} :: \texttt{PolygonalRepresentation} \to \texttt{PolygonalComplex}$$

$$\texttt{polygonalRealization}\,(\langle x | xx \rangle) = \mathrm{real}\langle x | xx \rangle := \mathbb{RP}^2$$

$$\texttt{polygonalRealization}\,\big(\langle x | x^{-1} x \rangle\big) = \mathrm{real}\langle x | x^{-1} x \rangle := \mathbb{S}^2$$

$$\texttt{polygonalRealization}\,\big(\langle x | x x^{-1} \rangle\big) = \mathrm{real}\langle x | x x^{-1} \rangle := \mathbb{S}^2$$

$$\texttt{polygonalRealization}\,\big(\langle x | x^{-1} x^{-1} \rangle\big) = \mathrm{real}\langle x | x^{-1} x^{-1} \rangle := \mathbb{RP}^2$$

$$\texttt{polygonalRealization}\,(\langle X | w_1, \ldots, w_n \rangle) = \mathrm{real}\langle X | w_1, \ldots, w_n \rangle := \frac{\bigsqcup_{i=1}^{n} \mathbf{P}(w_i)}{A}$$

$$\texttt{where} \quad A = \left\{ \mathbf{E}_j(w_i) \downarrow \mathbf{E}_l(w_k) \middle| i, k \in [1, \ldots, n], j \in \Big[1, \ldots, |w_i|\Big], l \in \Big[1, \ldots, |w_k|\Big], w_{i,j} = w_{k,l} \right\} \sqcup$$

$$\sqcup \left\{ \mathbf{E}_j(w_i) \uparrow \mathbf{E}_l(w_k) \middle| i, k \in [1, \ldots, n], j \in \Big[1, \ldots, |w_i|\Big], l \in \Big[1, \ldots, |w_k|\Big], w_{i,j,1} = w_{k,l,1} \;\&\; w_{i,j,2} \neq w_{k,l,2} \right\}$$

$\texttt{SurfacePresentation} :: ?\texttt{PolygonalPresentation}$

$$\langle X \big| w_1, \dots, w_n \rangle : \texttt{SurfacePresentation} \iff \forall x \in X \,.\, \left| \left\{ (i,j) \middle| i \in [1, \dots, n], j \in \Big[1, \dots, |w_i|\Big], w_{i,j,1} = x \right\} \right|$$

$\texttt{SurfacePresentationProperty} :: \forall X : \texttt{SurfacePresentation} \,.\, \texttt{CompactSurface}\Big(\mathrm{real}(X)\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{relabling} :: \prod \left\langle X \middle| w_1, \dots, w_n \right\rangle : \texttt{PolygonalPresentation} \,.\, X \to X^{\complement} \to \texttt{PolygonalPresentation}$$

$$\texttt{relabling}\,(a,b) := \left\langle \Big(X \setminus \{a\}\Big) \sqcup \{b\} \middle| w'_1, \dots w'_n \right\rangle$$

$$\quad \texttt{where} \quad w' = \Lambda i \in [1, \dots, n] \,.\, \Lambda j \in \Big[1, \dots, |w_i|\Big] \,.\, \texttt{if } w_{i,j,1} == a \texttt{ then } (b, w_{i,j,2}) \texttt{ else } w_{i,j}$$

$\texttt{RelablingPreservesRealization} :: \forall \Big\langle X | w_1, \dots, w_n \Big\rangle : \texttt{PolygonalPresentation} \,.$

$\quad . \forall a \in X \,.\, \forall b \in X^{\mathcal{C}} \,.\, \mathrm{real}\ \texttt{relabling}\Big(\langle X | w_1, \dots, w_n \rangle, a, b\Big) \cong_{\mathsf{TOP}} \mathrm{real}\ \langle X | w_1, \dots, w_n \rangle$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{subdividing} :: \prod \left\langle X \middle| w_1, \dots, w_n \right\rangle : \texttt{PolygonalPresentation} \,.\, X \to X^{\complement} \to \texttt{PolygonalPresentation}$$

$$\texttt{subdividing}\,(a,b) := \left\langle X \sqcup \{b\} \middle| w'_1, \dots w'_n \right\rangle$$

$$\quad \texttt{where} \quad w' = \Lambda i \in [1, \dots, n] \,.\, \Big(\Lambda(a,1) \,.\, \Lambda(a,-1) \,.\, w_i\Big)(ab, b^{-1}a^{-1})$$

$\texttt{SubdivisionPreservesRealization} :: \forall \Big\langle X | w_1, \dots, w_n \Big\rangle : \texttt{PolygonalPresentation} \,.$

$\quad . \forall a \in X \,.\, \forall b \in X^{\mathcal{C}} \,.\, \mathrm{real}\ \texttt{subdivision}\Big(\langle X | w_1, \dots, w_n \rangle, a, b\Big) \cong_{\mathsf{TOP}} \mathrm{real}\ \langle X | w_1, \dots, w_n \rangle$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{ConsalidatablePair} :: \prod \langle X | w_1, \dots, w_n \rangle \to ?(X \times X)$$

$$(a,b) : \texttt{ConsalidatablePair} \iff \forall i \in [1, \dots, n] \,.\, \forall j \in \Big[1, \dots, |w_i|\Big] \,.$$

$$\quad . \Big(w_{i,j} = (a,1) \Rightarrow w_{i,j+1} = (b,1)\Big) \,\&\, \Big(w_{i,j} = (a,-1) \Rightarrow w_{i,j-1} = (b,-1)\Big)$$

$$\texttt{consalidation} :: \prod \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.\, \texttt{ConsalidatablePair}\langle X | w_1, \dots, w_n \rangle \to \texttt{Poly}$$

$$\texttt{consalidating}\,(a,b) := \left\langle X \setminus \{b\} \middle| w'_1, \dots w'_n \right\rangle$$

$$\quad \texttt{where} \quad w' = \Lambda i \in [1, \dots, n] \,.\, \Big(\Lambda ab \,.\, \Lambda b^{-1}a^{-1} \,.\, w_i\Big)(a, a^{-1})$$

$\texttt{ConsalidatingPreservesRealization} :: \forall \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.$

$. \forall (a,b) : \texttt{ConsalidatablePair}\text{real } \texttt{consalidating}\Big(\langle X | w_1, \dots, w_n \rangle, a, b \Big) \cong_{\mathsf{TOP}} \text{real } \langle X | w_1, \dots, w_n \rangle$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{reflection} :: \texttt{PolygonalPresentation} \to \texttt{PolygonalPresentation}$

$\texttt{reflection}\,(\langle X | w_1, \dots, w_n \rangle) := \langle X | w_1^{-1}, \dots, w_n^{-1} \rangle$

$\texttt{ReflectionPreservesRealization} :: \forall \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.$

$. \text{ real } \texttt{reflection}\langle X | w_1, \dots, w_n \rangle \cong_{\mathsf{TOP}} \text{real } \langle X | w_1, \dots, w_n \rangle$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{rotation} :: \prod \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.\, [1, \dots, n] \to \texttt{PolygonalPresentation}$

$\texttt{rotation}\,(k) := \langle X | w'_1, \dots, w'_n \rangle$

$\texttt{where} w' = \Lambda i \in [1, \dots, n] \texttt{if } i == k \texttt{ then } w_{i,|w|} w_{i,1} \dots w_{i,|w|-1} \texttt{ else } w_i$

$\texttt{RotationPreservesRealization} :: \forall \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.$

$. \forall i \in [1, \dots, n] \,.\, \text{real } \texttt{rotation}\Big(\langle X | w_1, \dots, w_n \rangle, i \Big) \cong_{\mathsf{TOP}} \text{real } \langle X | w_1, \dots, w_n \rangle$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{cutting} :: \prod \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.$

$. \prod_{k=1}^{n} [1, \dots, |w_k| - 1] \to X^{\complement} \to [1, \dots, n] \to \texttt{PolygonalPresentation}$

$\texttt{cutting}\,(j, z) := \langle X | w'_1, \dots, w'_{n+1} \rangle$

$\texttt{where} \quad w' = \Lambda i \in [1, \dots, n+1] \texttt{if } i < k \texttt{ then } w_i \texttt{ else if}$

$\texttt{else if } i == k \texttt{ then } w_{i,1} \dots w_{i,j} z \texttt{ else if } i == k+1 \texttt{ then } z w_{i,j+1} \dots w_{i,|w_i|} \texttt{ else } w_{i+1}$

$\texttt{CuttingPreservesRealization} :: \forall \langle X | w_1, \dots, w_n \rangle : \texttt{PolygonalPresentation} \,.$

$. \forall i \in [1, \dots, n] \,.\, \forall j \in \Big[1, \dots, |w_i|\Big] \,.\, \forall z \in X^{\complement} \,.\, \text{real } \texttt{cutting}\Big(\langle X | w_1, \dots, w_n \rangle, i, j, z \Big) \cong_{\mathsf{TOP}} \text{real } \langle X | w_1, \dots, w_n$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{PastableIndex} :: \prod \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentation} \,.\, ?[1,\dots,n-1]$

$i : \texttt{PastableIndex} \iff \exists z \in X : w_{i,-1,1} = z = w_{i+1,1,1}$ &

$$\& \left|\left\{(i,j)\middle| i \in [1,\dots,n], j \in \left[1,\dots,|w_i|\right], w_{i,j,1} = z\right\}\right| = 2$$

$\texttt{pasting} :: \prod \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentation} \,.$

$.\, \texttt{PastableIndex} \to \texttt{PolygonalPresentation}$

$\texttt{pasting}\,(i) := \left\langle X \setminus \{z\} \middle| w'_1,\dots,w'_{n-1}\right\rangle$

$\quad \texttt{where} \quad w' = \Lambda i \in [1,\dots,n+1] \texttt{if } i < k \texttt{ then } w_i \texttt{ else if}$

$\quad \texttt{else if } i == k \texttt{ then } \widehat{w_i w_{i+1}}_{\{|w_i|,|w_i|+1\}} \texttt{ else } w_{i-1}$

$\quad \texttt{where} \quad z = \textit{Ɑ}\texttt{PastableIndex}\Big(\langle X|w_1,\dots,w_n\rangle, i\Big)$

$\texttt{PastingPreservesRealization} :: \forall \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentatio} \,.$

$.\, \forall i : \texttt{PastableIndex}\langle X|w_1,\dots,w_m \,.\, \text{real } \texttt{pasting}\Big(\langle X|w_1,\dots,w_n\rangle, i, j, z\Big) \cong_{\mathsf{TOP}} \text{real } \langle X|w_1,\dots,w_n\rangle$

$\texttt{Proof} \;=$

...

□

$\texttt{FoldableIndex} :: \prod \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentation} \,.\, ?[1,\dots,n-1]$

$i : \texttt{FoldableIndex} \iff \exists z \in X \sqcup X^{-1} : \exists u : \texttt{PolygonalWord}(X \sqcup X^{-1}) w_i = uzz^{-1}$ &

$$\& \left|\left\{(i,j)\middle| i \in [1,\dots,n], j \in \left[1,\dots,|w_i|\right], w_{i,j,1} = z\right\}\right| = 2$$

$\texttt{folding} :: \prod \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentation} \,.$

$.\, \texttt{FoldableIndex} \to \texttt{PolygonalPresentation}$

$\texttt{folding}\,(i) := \left\langle X \setminus \{z\} \middle| w'_1,\dots,w'_{n-1}\right\rangle$

$\quad \texttt{where} \quad w' = \Lambda i \in [1,\dots,n+1] \texttt{if } i < k \texttt{ then } w_i \texttt{ else if}$

$\quad \texttt{else if } i == k \texttt{ then } u \texttt{ else } w_i$

$\quad \texttt{where} \quad (z,u) = \textit{Ɑ}\texttt{FoldableIndex}\Big(\langle X|w_1,\dots,w_n\rangle, i\Big)$

$\texttt{FoldingPreservesRealization} :: \forall \langle X|w_1,\dots,w_n\rangle : \texttt{PolygonalPresentatio} \,.$

$.\, \forall i : \texttt{PastableIndex}\langle X|w_1,\dots,w_m \,.\, \text{real } \texttt{folding}\Big(\langle X|w_1,\dots,w_n\rangle, i\Big) \cong_{\mathsf{TOP}} \text{real } \langle X|w_1,\dots,w_n\rangle$

$\texttt{Proof} \;=$

...

□

$\texttt{unfolding} :: \prod \langle X|w_1,\ldots,w_n\rangle : \texttt{PolygonalPresentation}\ .$

$.\ [1,\ldots,n] \to X^{\complement} \to \texttt{PolygonalPresentation}$

$\texttt{unfolding}\,(z,i) := \left\langle X \setminus \{z\} \middle| w_1',\ldots,w_{n-1}' \right\rangle$

$\texttt{where} \quad w' = \Lambda i \in [1,\ldots,n+1]\texttt{if } i < k \texttt{ then } w_i \texttt{ else if}$

$\texttt{else if } i == k \texttt{ then } w_i z z^{-1} \texttt{ else } w_i$

$\texttt{UnfoldingPreservesRealization} :: \forall \langle X|w_1,\ldots,w_n\rangle : \texttt{PolygonalPresentatio}\ .$

$.\ \forall i : [1,\ldots,n]\ .\ \forall z \in X^{\complement}\ .\ \mathrm{real}\ \texttt{unfolding}\Big(\langle X|w_1,\ldots,w_n\rangle, i, z\Big) \cong_{\mathsf{TOP}} \mathrm{real}\ \langle X|w_1,\ldots,w_n\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{ConnectedSumRealization} :: \forall \langle X|w_1\rangle, \langle Y|v_1\rangle : \texttt{SurfacePresentation}\ .$

$.\ \mathrm{real}\langle X \sqcup Y|w_1 v_1\rangle = \mathrm{real}\langle X|w_1\rangle \# \mathrm{real}\langle Y|w_2\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{SpherePresentation} :: \mathbb{S}^2 \cong \mathrm{real}\langle a,b|abb^{-1}a^{-1}\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{TorusPresentation} :: \mathbb{S}^1 \times \mathbb{S}^1 \cong \langle a,b|ab^{-1}ba^{-1}\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{TorusPresentation} :: \mathbb{S}^1 \times \mathbb{S}^1 \cong_{\mathsf{TOP}} \mathrm{real}\langle a,b|ab^{-1}a^{-1}b\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{ProjectiveSpacePresentation} :: \mathbb{RP}^2 \cong_{\mathsf{TOP}} \mathrm{real}\langle a,b|abab\rangle$

$\texttt{Proof} \ =$

...

□

$\texttt{KleinPresentation} :: \mathbf{K} \cong_{\mathsf{TOP}} \mathrm{real}\langle a,b|ab^{-1}ab\rangle$

$\texttt{Proof} \ =$

...

□

4.4 Classification Theorem

EveryCompactSurfaceAdmitsPresentation :: $\forall M$: CompactSurface .

. $\exists P$: SurfacePresentation : $M \cong \operatorname{real} P$

Proof =

...

$\square$

torus :: CompactSurface

torus $() = \mathbb{R} := \mathbb{S} \times \mathbb{S}$

ClassificationOfCompactSurfafacesI :: $\forall M$: CompactSurface .

. $M \cong \mathbb{S}^2 \Big| \exists n \in \mathbb{N} : M \cong \#_{i=1}^{n} \mathbb{T} | M \cong \#_{i=1}^{n} \mathbb{RP}^2$

Proof =

KleinBottelAsSum :: $\mathbf{k} = \mathbb{RP}^2 \# \mathbb{RP}^2$

Proof =

...

$\square$

SumProjectivization :: $\mathbb{T} \# \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Proof =

...

$\square$

4.5 Euler Characteristic

$$\texttt{characteristicOfEuler} :: \texttt{FiniteCWComplex} \to \mathbb{Z}$$

$$\texttt{characteristicOfEuler}\,((X, \mathcal{E}, \varphi)) = \chi(X, \mathcal{E}, \varphi) := \sum_{n=0}^{\infty} (-1)^n |\mathcal{E}_i|$$

$$\texttt{presentationEulerCharacteristic} :: \texttt{PolygonalPresentation} \to \mathbb{Z}$$
$$\texttt{characteristicOfEuler}\,(P) = \chi(P) := \chi(C)$$
$$\quad \texttt{where} \quad C = \texttt{SimplecticComplexIsCW}\ \texttt{PolygonalComplexHasSimplecticStructure}(\text{real } P)$$

$$\texttt{compactSurfacesEulerCharacteristic} :: \texttt{CompactSurface} \to \mathbb{Z}$$
$$\texttt{characteristicOfEuler}\,(M) = \chi(M) := \chi(P)$$
$$\quad \texttt{where} \quad P = \texttt{EveryCompactSurfaceAdmitsPresentation}(M)$$

$$\texttt{SpheresEullerCharacteristic} :: \chi\left(\mathbb{S}^2\right) = 2$$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{ConectedSumOfToriEullerCharacteristic} :: \forall n \in \mathbb{N} \,.\, \chi\left(\#_{i=1}^{n} \mathbb{T}\right) = 2 - 2n$$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{ConectedSumOfToriEullerCharacteristic} :: \forall n \in \mathbb{N} \,.\, \chi\left(\#_{i=1}^{n} \mathbb{RP}^2\right) = 2 - n$$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{EulerCharacteristicIsPreservedByElementaryTransformation} :: \ldots$$

$\texttt{Proof} =$

$\ldots$

$\square$

4.6 Orientability

$\texttt{bandOfMobius} :: \partial\,\mathsf{TOPM}(2)$

$\texttt{bandOfMobius}\,() = \mathbf{MB} := \langle a, b, c | abcb \rangle$

$\texttt{Oriented} :: ?\texttt{PolygonalPresentation}$

$\langle X | w_1, \dots, w_n \rangle : \texttt{Oriented} \iff$

$$\iff \left\{ \Big((i,j), (k,l) \Big) \Big| i, k \in [1, \dots, n]; j \in \Big[1, \dots, |w_i|\Big], l \in \Big[1, \dots, |w_l|\Big], (i,j) \neq (k,l), w_{i,j} = w_{k,l} \right\} = \emptyset$$

$\texttt{Orientable} :: ?\texttt{CompactSurface}$

$M : \texttt{Orientable} \iff \exists P : \texttt{Oriented} \,.\, M \cong_{\mathsf{TOP}} \mathrm{real}(P)$

$\texttt{OrientableSurfacesClassification} :: \forall M : \texttt{Orientable} \,.\, M \cong_{\mathsf{TOPM}} \mathbb{S}^2 \Big| \exists n \in \mathbb{N} : M \cong_{\mathsf{TOPM}} \#_{i=1}^{n} \mathbb{T}$

$\texttt{Proof} =$

$\dots$

$\square$

5 Basic Homotopy

5.1 Homotopy of Maps

$$\texttt{Homotopy} :: \prod X, Y \in \mathsf{TOP} \;.\; (X \xrightarrow{\mathsf{TOP}} Y)^2 \to ?\Big((I \times X) \xrightarrow{\mathsf{TOP}} Y\Big)$$

$$H : \texttt{Homotopy} \iff \Lambda f, g : X \xrightarrow{\mathsf{TOP}} Y \;.\; H(0, \bullet) = f \;\&\; H(1, \bullet) = g$$

$$\texttt{Homotopic} :: \prod X, Y \in \mathsf{TOP} \;.\; ?(X \xrightarrow{\mathsf{TOP}} Y)^2$$

$$(f, g) : \texttt{Homotopic} \iff f \sim g \iff \exists \texttt{Homotopy}(X, Y, f, g)$$

$$\texttt{NullHomotopic} :: \prod X, Y \in \mathsf{TOP} \;.\; ?(X \xrightarrow{\mathsf{TOP}} Y)$$

$$f : \texttt{NullHomotopic} \iff \exists y \in Y \;.\; \exists \texttt{Homotopy}(X, Y, f, y)$$

$$\texttt{HomotopicIsEquivallence} :: \forall X, Y \in \mathsf{TOP} \;.\; \texttt{Equivalence}\Big(C(X, Y), \texttt{Homotopic}(X, Y)\Big)$$

$\texttt{Proof} =$

$\texttt{Assume}\; f \in C(X, Y),$

$H := \Lambda t \in I \;.\; \Lambda x \in X \;.\; f(x) : (I \times X) \xrightarrow{\mathsf{TOP}} Y,$

$[1] := \Game^{-1}\texttt{Homotopy}\reflectbox{$\mathcal{C}$}H : \texttt{Homotopy}(X, Y, f, f, H),$

$[*] := \Game^{-1}\texttt{Homotopic}[1] : f \sim f;$

$\leadsto [1] := \mathtt{I}(\forall) : \forall f \in C(X, Y) \;.\; f \sim f,$

$\texttt{Assume}\; f, g \in C(X, Y),$

$\texttt{Assume}\; [2] : f \sim g,$

$H := \Game\texttt{Homotopic}[2] : \texttt{Homotopy}(X, Y, f, g),$

$H' := \Lambda t \in [0, 1] \;.\; H(1 - t) : \texttt{Homotopy}(X, Y, g, f),$

$[*] := \Game^{-1}\texttt{Homotopic}(H') : g \sim f;$

$\leadsto [2] := \mathtt{I}(\forall)\mathtt{I}(\Rightarrow) : \forall f, g \in C(X, Y) \;.\; f \sim g \Rightarrow g \sim f,$

$\texttt{Assume}\; f, g, h \in C(X, Y),$

$\texttt{Assume}\; [3] : f \sim g,$

$\texttt{Assume}\; [4] : g \sim h,$

$H := \Game\texttt{Homotopic}[3] : \texttt{Homotopy}(X, Y, f, g),$

$H' := \Game\texttt{Homotopic}[4] : \texttt{Homotopy}(X, Y, g, h),$

$H'' := \Lambda t \in I \;.\; \texttt{if}\; y \leq \dfrac{1}{2}\; \texttt{then}\; H(2t)\; \texttt{else}\; H'(2t - 1) : \texttt{Homotopy}(X, Y, f, h),$

$[*] := \Game^{-1}\texttt{Homotopic}(H'') : f \sim h;$

$\leadsto [3] := \mathtt{I}(\forall)\mathtt{I}(\Rightarrow) : \forall f, g, h \in C(X, Y) \;.\; f \sim g \forall g \sim h \Rightarrow f \sim h,$

$[*] := \Game^{-1}\texttt{Equivalence}[1, 2, 3] : \texttt{Equivalence}\Big(C(X, Y), \texttt{Homotopic}(X, Y)\Big);$

$\square$

$\texttt{HomotopicComposition} :: \forall X, Y, Z \in \mathsf{TOP} \;.\; \forall f, g : X \xrightarrow{\mathsf{TOP}} Y \;.\; \forall f', g' : X \xrightarrow{\mathsf{TOP}} Y \;.\; f \sim g \;\&\; f' \sim g' \Rightarrow$
$\quad \Rightarrow f' \circ f \sim g' \circ g$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LineHomotopy} :: \forall X \in \mathsf{TOP} \;.\; \forall C : \texttt{Convex} \;.\; \forall f, g : X \xrightarrow{\mathsf{TOP}} C \;.\; f \sim g$

$\texttt{Proof} \;=$

$H := \Lambda t \in [0,1] \;.\; tf + (1-t)g : \texttt{Homotopy}(X, C),$

$[1] := \partial H\Big(H(0)\Big) : H(0) = f,$

$[2] := \partial H\Big(H(1)\Big) : H(1) = g,$

$[3] := \mathcal{I}^{-1}\texttt{Homotopy} : \texttt{Homotopy}(X, C, f, g, H),$

$[*] := \mathcal{I}^{-1}\texttt{Homotopic}(f, g) : f \sim g;$

$\square$

5.2 Fundamental Group

$\texttt{Stationary} :: \prod X, Y \in \mathsf{TOP} \,.\, \prod f, g :?(X \xrightarrow{\mathsf{TOP}} Y)^2 \,.\, ?X \to ?\texttt{Homotopy}(X, Y, f, g)$

$H : \texttt{Stationary} \iff \prod A \subset X \,.\, \forall t \in [0,1] \,.\, \forall a \in A \,.\, H(t,a) = f(a)$

$\texttt{RelativelyHomotopic} :: \prod X, Y \in \mathsf{TOP} \,.\, ?X \to ?(X \xrightarrow{\mathsf{TOP}} Y)^2$

$(f,g) : \texttt{Homotopic} \iff \Lambda A \subset X \,.\, f \sim_A g \iff \Lambda A \subset X \,.\, \exists \texttt{Stationary}(X, Y, f, g, A)$

$\texttt{FreelyHomotopic} :: \prod X, Y \in \mathsf{TOP} \,.\, ?\texttt{Homotopic}(X, Y)$

$(f,g) : \texttt{FreelyHomotopic} \iff f \sim_! g \iff \forall A \subset X \,.\, (f,g) \,!\, \texttt{RelativelyHomotopic}(X, Y, A)$

$\texttt{PathHomotopic} :: \prod X \in \mathsf{TOP} \,.\, ?\texttt{Homotopic}(I, X)$

$(\alpha, \beta) : \texttt{PathHomotopic} \iff f \approx g \iff \texttt{RelativelyHomotopic}\Big(I, X, \alpha, \beta, \{0,1\}\Big)$

$\texttt{HomotopicIsEquivallence} :: \forall X \in \mathsf{TOP} \,.\, \forall x, y \in X \,.\, \texttt{Equivalence}\Big(\Omega(x,y), \texttt{PathHomotopic}(X) \cap \Omega^2(x,y)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \gamma \in \Omega(x,y),$

$H := \Lambda t \in I \,.\, \Lambda x \in X \,.\, \gamma(x) : I^2 \xrightarrow{\mathsf{TOP}} X,$

$[1] := \mathcal{Q}^{-1}\texttt{Homotopy}\mathcal{O}H : \texttt{PathHomotopy}(X, I, \gamma, \gamma, H),$

$[*] := \mathcal{Q}^{-1}\texttt{PathHomotopic}[1] : \gamma \approx \gamma;$

$\rightsquigarrow [1] := \mathtt{I}(\forall) : \forall \gamma \in \Omega(x,y) \,.\, \gamma \approx \gamma,$

$\texttt{Assume}\ f, g \in \Omega(x,y),$

$\texttt{Assume}\ [2] : f \sim g,$

$H := \mathcal{Q}\texttt{PathHomotopic}[2] : \texttt{Homotopy}(X, Y, \alpha, \beta),$

$H' := \Lambda t \in [0,1] \,.\, H(1-t) : \texttt{Homotopy}(X, I, \alpha, \beta),$

$[*] := \mathcal{Q}^{-1}\texttt{PathHomotopic}(H') : \alpha \approx \beta;$

$\rightsquigarrow [2] := \mathtt{I}(\forall)\mathtt{I}(\Rightarrow) : \forall \alpha, \beta \in \Omega(x,y) \,.\, \alpha \approx \beta \Rightarrow \beta \approx \alpha,$

$\texttt{Assume}\ \alpha, \beta, \gamma \in \Omega(x,y),$

$\texttt{Assume}\ [3] : \alpha \approx \beta,$

$\texttt{Assume}\ [4] : \beta \approx \gamma,$

$H := \mathcal{Q}\texttt{PathHomotopic}[3] : \texttt{Homotopy}(X, I, \alpha, \beta),$

$H' := \mathcal{Q}\texttt{PathHomotopic}[4] : \texttt{Homotopy}(X, I, \beta, \gamma),$

$H'' := \Lambda t \in I \,.\, \texttt{if}\ y \le \frac{1}{2}\ \texttt{then}\ H(2t)\ \texttt{else}\ H'(2t-1) : \texttt{Homotopy}(X, I, \alpha, \gamma),$

$[*] := \mathcal{Q}^{-1}\texttt{PathHomotopic}(H'') : \alpha \approx \gamma;$

$\rightsquigarrow [3] := \mathtt{I}(\forall)\mathtt{I}(\Rightarrow) : \forall \alpha, \beta, \gamma \in C(X,Y) \,.\, \alpha \approx \gamma \,\&\, \beta \approx \gamma \Rightarrow \alpha \approx \gamma,$

$[*] := \mathcal{Q}^{-1}\texttt{Equivalence}[1,2,3] : \texttt{Equivalence}\Big(\Omega(x,y), \texttt{PathHomotopic}(X)\Big);$

$\square$

$\texttt{NullHomotopicPath} :: \prod X \in \mathsf{TOP} \,.\, \prod x \in X \,.\, ?\Omega(x,x)$

$\gamma : \texttt{NullHomotopticPath} \iff \gamma \approx x$

$\texttt{Reparametrization} :: \prod X \in \mathsf{TOP} \,.\, (I \xrightarrow{\mathsf{TOP}} X) \to ?(I \xrightarrow{\mathsf{TOP}} X)$

$\omega : \texttt{Reparametrization} \iff \Lambda\gamma : I \xrightarrow{\mathsf{TOP}} X \,.\, \exists \varphi : I \xrightarrow{\mathsf{TOP}} I : \varphi(0) = 0 \,\&\, \varphi(1) = 1 \,\&\, \omega = \gamma \circ \varphi$

$\texttt{PathHomotopicReparametrization} :: \forall X \in \mathsf{TOP} \,.\, \forall \alpha : I \xrightarrow{\mathsf{TOP}} X \,.\, \forall \beta : \texttt{Reparametrization}(X, \alpha) \,.\, \alpha \approx \beta$

$\texttt{Proof} \ =$

$\Big(\varphi, [1], [2], [3]\Big) := \mathcal{I}\texttt{Reparametrization}(X, \alpha, \beta) : \sum \varphi : (I \xrightarrow{\mathsf{TOP}} I) \,.\, \beta = \alpha \circ \varphi \,\&\, \varphi(0) = 0 \,\&\, \varphi(1) = 1,$

$[4] := \texttt{LineHomotopy}(I, I, \varphi, \mathrm{id}) : \varphi \sim \mathrm{id},$

$[5] := \texttt{HomotopicComposition}(I, I, X, \mathrm{id}, \varphi, \alpha, \alpha)[1, 4] : \alpha \sim \beta,$

$[*] := \mathcal{I}^{-1}\texttt{PathHomotopic}[1, 2, 3, 5] : \alpha \approx \beta;$

$\square$

$\texttt{basedFundamentalGroup} :: \prod X \in \mathsf{TOP} \,.\, X \to \mathsf{SET}$

$\texttt{basedFundamentalGroup}\,(x) = \pi(x) := \dfrac{\Omega(x)}{\texttt{PathHomotopic}}$

$\texttt{joinPaths} :: \prod X \in \mathsf{TOP} \,.\, \prod x, y, z \in X \,.\, \Omega(x, y) \times \Omega(y, z) \to \Omega(x, z)$

$\texttt{joinPaths}\,(\alpha, \beta) = \alpha \circ \beta := \Lambda t \in [0, 1] \,.\, \texttt{if } t \le \dfrac{1}{2} \texttt{ then } \alpha(2t) \texttt{ else } \beta(2t - 1)$

$\texttt{HomotopicLoopJoinIsWellDefine} :: \forall X \in \mathsf{TOP} \,.\, \forall x \in \mathsf{TOP} \,.\, \forall \alpha, \beta \in \pi(x) \,.\, \forall a, a' \in \alpha \,.\, \forall b, b' \in \beta \,.\, [a \circ a'] = [b$

$\texttt{Proof} \ =$

$\Big(H, [1]\Big) := \mathcal{I}\pi(x)(\alpha)(a, a') : \sum H : \texttt{Homotopy}(I, X, a, a') \,.\, \forall t \in I \,.\, H(t, 0) = H(t, 1) = x,$

$\Big(H', [2]\Big) := \mathcal{I}\pi(x)(\alpha)(b, b') : \sum H' : \texttt{Homotopy}(I, X, b, b') \,.\, \forall t \in I \,.\, H'(t, 0) = H'(t, 1) = x,$

$H'' := \Lambda t \in I \,.\, H(t) \circ H'(t) : \texttt{Homotopy}(I, X, ab, a'b'),$

$[*] := \mathcal{I}\pi(x)(H'') : [ab] = [a'b'];$

$\square$

$\texttt{fundamentalGroupOperataion} :: \prod X \in \mathsf{TOP} \,.\, \prod x \in X \,.\, \Omega(x) \times \Omega(y, z) \to \Omega(x, z)$

$\texttt{fundamentalGroupOperation}\,([a], [b]) = [a][b] := [ab]$

$\texttt{FundamentalGroupIsAGroup} :: \forall X \in \mathsf{TOP} \,.\, \forall x \in X \,.\, \Big(\pi(x), (\cdot)\Big) \in \mathsf{GRP}$

$\texttt{Proof} \; =$

$[1] := \texttt{PathHomotopocReparametrizaton}Ɑ\Big(\pi(x), (\cdot)\Big) : \forall \alpha, \beta, \gamma \in \pi(x) \,.\, (\alpha\beta)\gamma = \alpha(\beta\gamma),$

$[2] := Ɑ\Big(\pi(x), (\cdot)\Big) : \forall \alpha \in \pi(x) \,.\, \alpha[x] = [x]\alpha = \alpha,$

$\texttt{Assume}\ [a] \in \pi(x),$

$b := \Lambda s \in I \,.\, a(1-s) \in \Omega(x),$

$H := \Lambda t \in I \,.\, \Lambda s \in I \,.\, \texttt{if}\ s < \frac{1-t}{2}\ \texttt{then}\ a(2s)\ \texttt{else if}\ s \le \frac{1+t}{2}\ \texttt{then}\ a\left(\frac{1-t}{2}\right)\ \texttt{else}\ b\,(2t-1) :$

$\quad : \texttt{Homotopy}(I, X, \alpha\beta, x),$

$[3.*] := Ɑ\pi(x)(H) : [ab] = [x],$

$H' := \Lambda t \in I \,.\, \Lambda s \in I \,.\, \texttt{if}\ s < \frac{1-t}{2}\ \texttt{then}\ b(2s)\ \texttt{else if}\ s \le \frac{1+t}{2}\ \texttt{then}\ b\left(\frac{1-t}{2}\right)\ \texttt{else}\ a\,(2t-1) :$

$\quad : \texttt{Homotopy}(I, X, \alpha\beta, x),$

$[4.*] := Ɑ\pi(x)(H) : [ba] = [x];$

$\rightsquigarrow [3] := \texttt{I}(\forall)\texttt{I}(\exists) : \forall \alpha \in \pi(x) \,.\, \exists \beta \in \pi(x) : \alpha\beta = \beta\alpha = [x],$

$[*] := Ɑ^{-1}\mathsf{GRP} : \Big(\pi(x), (\cdot)\Big) \in \mathsf{GRP};$

$\square$

$\texttt{ChangeOfBasePoint} :: \forall X \in \mathsf{TOP} \,.\, \forall x, y \in X \,.\, \forall [a] \in \pi(x) \,.\, \forall \gamma \in \Omega(y, x) \,.\, \left[\gamma a \gamma^{-1}\right] \in \pi(y)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{IsomorphicFundamentalGroup} :: \forall X \in \mathsf{TOP} \,.\, \forall C \in \mathrm{PCC}(X) \,.\, \forall x, y \in C \,.\, \pi(x) \cong_{\mathsf{GRP}} \pi(y)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{FundamentalGroupsOfConnected} :: \forall X : \texttt{PathConnected} \,.\, \forall x, y \in X \,.\, \pi(x) \cong_{\mathsf{GRP}} \pi(y)$

$\texttt{Proof} \; =$

$\texttt{generalFundamentalGroup} :: \texttt{PathConnected}\ \&\ \texttt{NonEmpty} \to \mathsf{GRP}$

$\texttt{generalFundamentalGroup}\,(X) = \pi(X) := \pi(x) \quad \texttt{where} \quad x = Ɑ\texttt{NonEmpty}(X)$

$\texttt{SimplyConnected} :: ?(\texttt{PathConnected}\ \&\ \texttt{NonEmpty})$

$X : \texttt{SimplyConnected} \iff \Big|\pi(X)\Big| = 1$

$\texttt{ConvexIsSimplyConnected} :: \forall C : \texttt{Convex} \,.\, \texttt{SimplyConnected}(C)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{RealVectorSpaceIsSimplyConnected} :: \forall V : \mathbb{R}\text{-TOPVS} \,.\, \texttt{SimplyConnected}(V)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{circleRepresentative} :: \prod X \in \mathsf{TOP} \,.\, \prod x \in X \,.\, \Omega(x) \to (\mathbb{S}^1 \xrightarrow{\mathsf{TOP}} X)$

$\texttt{circleRepresentative}\,(\gamma) = \tilde{\gamma} := \dfrac{\gamma}{\{0,1\}}$

$\texttt{CircleRepresentativeOfNullHomotopic} :: \forall X \in \mathsf{TOP} \,.\, \forall x \in X \,.\, \forall \gamma : \texttt{NullHomotopic}(x, X) \,.\, \tilde{\gamma} \sim x$

$\texttt{Proof} \ =$

$\Big(H, [1]\Big) := \text{Ɑ}\texttt{NullHomotopic}(X, x, \omega) : \sum H : \texttt{Homotopy}(I, X, \gamma, x) \,.\, \forall t \in I \,.\, H(t,1) = H(t,0) = x,$

$H' := \Lambda t \in I \,.\, \widetilde{H(t)} : \texttt{Homotopy}(\mathbb{S}^1, X, \tilde{\gamma}, x),$

$[*] := \text{Ɑ}^{-1}\texttt{Homotopic} : \gamma \sim x;$

$\square$

$\texttt{NullHomotopicExtension} :: \forall X \in \mathsf{TOP} \,.\, \forall x \in X \,.\, \forall \gamma : \mathbb{S}^1 \xrightarrow{\mathsf{TOP}} X \,.\, \gamma \sim x \Rightarrow$

$\quad \Rightarrow \exists \Gamma : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} X \,.\, \Gamma_{|\mathbb{S}^1} = \gamma$

$\texttt{Proof} \ =$

$H := \text{Ɑ}\texttt{Homotopic}(x, \gamma) : \texttt{Homotopy}(\mathbb{S}^1, X, x, \gamma),$

$\Gamma := \Lambda v \in \mathbf{C}^2 \ .\ \texttt{if}\ v == 0\ \texttt{then}\ x\ \texttt{else}\ H\left(\|v\|, \dfrac{v}{\|v\|}\right) : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} X,$

$[*] := \text{Ɑ}\texttt{Homotopy}(H)\text{Ɔ}\Gamma : \Gamma_{|\mathbb{S}^1} = \gamma;$

$\square$

$\texttt{ExtensionImplyNullHomotopic} :: \forall X \in \mathsf{TOP} \,.\, \forall x \in X \,.\, \forall \gamma : \Omega(x) \,.\, \forall \Gamma : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} X \,.\, \Gamma_{|\mathbb{S}^1} = \tilde{\gamma} \Rightarrow$

$\quad \Rightarrow \texttt{NullHomotopix}(\gamma)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{SquareLemma} :: \forall X \in \mathsf{TOP} \,.\, \forall F : I^2 \xrightarrow{\mathsf{TOP}} X \,.\, fg = hk$

$\quad \texttt{where}$

$\quad f = \Lambda t \in [0,1] \,.\, F(t, 0)$

$\quad g = \Lambda t \in [0,1] \,.\, F(1, t)$

$\quad h = \Lambda t \in [0,1] \,.\, F(0, t)$

$\quad k = \Lambda t \in [0,1] \,.\, F(t, 1)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LebesgueNumber} :: \prod X \in \mathsf{MS} \ . \ \texttt{OpenCover}(X) \to ?\mathbb{R}_{++}$

$\lambda : \texttt{LebesgueNumber} \iff \Lambda\mathcal{O} : \texttt{OpenCover}(X) \ . \ \forall U \subset X \ . \ \operatorname{diam} U < \lambda \Rightarrow \exists O \in \mathcal{O} : U \subset O$

$\texttt{LebesgueNumberLemma} :: \forall X \in \mathsf{MS} \ \& \ \texttt{Compact} \ . \ \forall \mathcal{O} : \texttt{OpenCover}(X) \ . \ \exists \texttt{LebesgueNumber}(\mathcal{O})$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{LoopTacklingTHM} :: \forall M \in \mathsf{TOPM} \ . \ \forall [0] : \dim M \geq 2 \ . \ \forall p, p' \in X \ . \ \forall \gamma \in \Omega(p, p') \ . \ \forall q \in X \setminus \{p, p'\} \ .$
$\quad . \ \exists \gamma' \in \Omega(p, p') : \gamma' \sim \gamma \ \& \ q \notin \operatorname{Im} \gamma'$

$\texttt{Proof} \ =$

$U := \mathcal{J}\texttt{NonEmpty}\ \mathcal{CC}(q) \in \mathcal{CC}(q),$

$V := M \setminus \{q\} \in \mathcal{T}(X),$

$\mathcal{O} := \left\{\gamma^{-1}(U), \gamma^{-1}(V)\right\} : \texttt{OpenCover}(I),$

$\lambda := \texttt{LebesgueNumberLemma}(I, \mathcal{O}) : \texttt{LebesgueNumber}(I, \mathcal{O}),$

$\Big(m, [1]\Big) := \texttt{ReductioInfima}(\mathbb{R}, \lambda) : \sum_{m=1}^{\infty} \frac{1}{m} < \lambda,$

$\texttt{Assume}\ k \in [1, \dots, m-1],$

$\texttt{Assume}\ [2] : \gamma\left(\frac{k}{m}\right) = q,$

$[*] := \mathcal{J}\texttt{LebesgueNumber}[1, 2] : \gamma\left[\frac{(k-1)}{m}, \frac{k}{m}\right], \gamma\left[\frac{k}{m}, \frac{k+1}{m}\right] \subset U;$

$\rightsquigarrow [2] := \texttt{I}(\forall)\texttt{I}(\Rightarrow) : \forall k \in [1, \dots, m-1] \ . \ \gamma\left(\frac{k}{m}\right) = q \Rightarrow \gamma\left[\frac{(k-1)}{m}, \frac{k}{m}\right], \gamma\left[\frac{k}{m}, \frac{k+1}{m}\right] \subset U,$

$A := \left\{k \in [0, \dots, n] : \gamma\left(\frac{k}{m}\right) \neq q\right\} :?[0, \dots, n],$

$l := |A| \in \mathbb{N},$

$a := \frac{\texttt{sort}(A)}{m} : \texttt{increasing}\Big([1, \dots, l], I\Big),$

$[3] := \mathcal{J}\gamma\mathcal{D}a : a_1 = 0 \ \& \ a_l = 1,$

$[4] := \texttt{CellIsPathConnected}(\dim M) : \texttt{PathConnected}\Big(U \setminus \{q\}\Big),$

$[*] := [2][4] : \exists \gamma' \in \Omega(p, p') \ . \ \gamma' \sim \gamma \ \& \ q \notin \operatorname{Im} \gamma,$

$\square$

$\texttt{HigherSphereIsSimplyConnected} :: \forall n \in \mathbb{N} \ . \ n \geq 2 \Rightarrow \texttt{SimplyConnected}\Big(\mathbb{S}^n\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FundamentalGroupIsCountable} :: \forall M \in \mathsf{TOPM} \,.\, \forall p \in M \,.\, \left|\pi(p)\right| \le \aleph_0$

$\texttt{Proof} =$

$\left(\mathcal{O}, [1]\right) := \texttt{ManifoldHasCoverByCoordinateCharts}(M) : \sum \mathcal{O} : \texttt{OpenCover}(M) \,.$
$\quad . \left|\mathcal{O}\right| < \aleph_0 \;\&\; \forall O \in \mathcal{O} \,.\, O \in \mathcal{CC}(M),$

$\texttt{Assume } O, O' \in \mathcal{O},$

$[2] := \mathcal{I}\texttt{LocallyEuclidean}(M, O \;\&\; O')\mathcal{I}\texttt{SecondCountable}(M) : \left|\mathrm{PCC}\left(O \cap O'\right)\right| \le \aleph_0,$

$x := \texttt{Choice} : \prod_{C \in \mathrm{PCC}\left(O \cap O'\right)} C,$

$X_{O,O'} := \operatorname{Im} x : \texttt{Countable}(M);$

$\rightsquigarrow \left(X, [1]\right) := \mathtt{I}\left(\prod\right) : \prod_{O,O' \subset \mathcal{O}} \sum X_{O,O'} : \texttt{Countable}(M) \,.\, \forall C \in \mathrm{PCC}\left(O \cap O'\right) \,.\, X_{O,O'} \cap C \ne \emptyset,$

$\mathcal{X} := \{p\} \cup \bigcup_{O,O' \subset \mathcal{O}} X_{O,O'} :?M;$

$\texttt{Assume } O \in \mathcal{O},$
$\texttt{Assume } x, y \in \mathcal{O} \cap \mathcal{X},$
$\gamma := \mathcal{I}\texttt{LocallyEuCleadean}(O)\mathcal{I}\texttt{PathConnected}(O) : \Omega_O(x, y);$

$\rightsquigarrow \gamma := \mathtt{I}\left(\prod\right) : \prod_{O \in \mathcal{O}} \prod_{x,y \in O \cap \mathcal{X}} \Omega_O(x, y),$

$\Sigma := \left\{ \sigma \in \Omega(p) : \exists n \in \mathbb{N} : \exists O : n \to \mathcal{O} : \exists \prod_{i=1}^{n+1} \sum_{x_i \in \mathcal{X}} x_i, x_{i+1} \in O_i : \sigma = \prod_{i=1}^{n} \gamma_{x_i, x_j}^{O_i} \right\} :?\Omega(x),$

$[2] := \texttt{CountableUnionOfFiniteProdCardBound}(X)\mathcal{I}\Sigma : |\Sigma| \le \aleph_0,$

$\texttt{Assume } \alpha \in \Omega(p),$

$\left(n, O, [3]\right) := \texttt{LebesgueNumberLemma}(I, \alpha^{-1}\mathcal{O}) : \sum_{n=1}^{\infty} \sum_{O : n \to \mathcal{O}} \forall k \in [1, \dots n] \,.\, \alpha\left[\frac{k-1}{n}, \frac{k}{n}\right] \subset O_k,$

$\beta := \Lambda k \in [1, \dots, n] \,.\, \Lambda t \in [0,1] \alpha_{|[(k-1)/n, k/n]}(nt) : [1, \dots, n] \to I \xrightarrow{\mathsf{TOP}} M,$

$[4] := \mathcal{I}\beta : \alpha = \prod_{k=1}^{n} \beta_k,$

$\texttt{Assume } k \in [1, \dots, n-1],$

$[5] := \texttt{OpenHasNoBoundary}[3] : \alpha\left(\frac{k}{n}\right) \in O_k \cap O_{k+1},$

$\left(x, C, [6]\right) := \mathcal{I}\mathcal{X}[5] : \sum x \in \mathcal{X} \,.\, \sum C \in \mathrm{PCC}(O_k \cap O_{k+1}) \,.\, x, \alpha\left(\frac{k}{n}\right) \in C,$

$\delta_k := \mathcal{I}\mathrm{PCC}(C)\mathcal{I}\texttt{PathConnected}(C)[6] : \Omega_C\left(x, \alpha\left(\frac{k}{n}\right)\right);$

$\rightsquigarrow \left(x, \delta\right) := \mathtt{I}\left(\prod\right) : \prod_{k=1}^{n-1} \sum_{x_k \in \mathcal{X}} \Omega_{O_k \cap O_{k+1}}\left(x_k, \alpha\left(\frac{k}{n}\right)\right),$

$\delta_0 := t \mapsto p \in \Omega(p),$
$\delta_n := t \mapsto p \in \Omega(p),$

$\beta' := \Lambda k \in [1, \dots, n] \,.\, \delta_{k-1}\beta_k\delta_k^{-1} : [1, \dots, n] \to I \xrightarrow{\mathsf{TOP}} M,$

$[5] := \mathcal{I}\beta'[4] : \alpha \sim \prod_{k=1}^{n} \beta'_k,$

$[*] := \mathcal{D}\Sigma[5]\mathcal{A}\texttt{SimplyConnected}(O) : \exists\sigma \in \Sigma\ .\ \sigma \approx \alpha;$

$\rightsquigarrow [3] := \texttt{I}(\forall) : \forall\alpha \in \Omega(p)\ .\ \exists\sigma \in \Sigma : \alpha \approx \sigma,$

$[*] := \mathcal{A}\pi(p)[2][3] : \left|\pi(p)\right| \leq \aleph_0;$

$\square$

$\texttt{fundamentalGroupoid} :: \textsf{TOP} \to \textsf{GROUPOID}$

$$\texttt{fundamentalGroupoid}\,(X) = \Pi(X) := \left(X, \frac{omega}{\texttt{Homotopic}}, \circ, x \mapsto (t \mapsto x)\right)$$

$\texttt{FundamentalGroupoidIsGroupoid} :: \forall X \in \textsf{TOP}\ .\ \texttt{Groupoid}\Big(\Pi(X)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{manifolfdsFundamentalGroupoidHasCountableMorphisms} :: \forall X \in \textsf{TOPM}\ .\ \forall x, y \in X\ .\ \left|\Omega(x, y)\right| \leq \aleph_0$

$\texttt{Proof} =$

...

$\square$

5.3 Induced Functors

$\texttt{PathHomotopyPreservedByC} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall f \in \mathsf{TOP}(X,Y) \,.\, \forall \alpha, \beta \in I \to X \,.\, \alpha \sim \beta \Rightarrow \alpha f \sim \beta f$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{inducedFunctor} :: \prod X, Y \in \mathsf{TOP} \,.\, \Pi(X) \xrightarrow{\mathsf{CAT}} \Pi(Y)$

$\texttt{inducedFunctor}\,(f) = f_* := \Big(f, \Lambda\gamma \in \Pi(X)(p,q) \,.\, f \circ \gamma\Big)$

$\texttt{FundamentalGroupoidIsFunctor} :: \texttt{Covariant}(\mathsf{TOP}, \mathsf{GROUPOID}, \Pi)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FundamentalGroupIsomorphism} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall \varphi : X \xleftrightarrow{\mathsf{TOP}} Y \,.\, \forall x \in X \,.\, \pi(x) \cong_{\mathsf{GRP}} \pi\big(\varphi(x)\big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Retraction} :: \prod X \in \mathsf{TOP} \,.\, \prod R \subset X \,.\, ?(X \xrightarrow{\mathsf{TOP}} R)$

$f : \texttt{Retraction} \iff \iota_R f = \mathrm{id}_R$

$\texttt{Retract} :: \prod X \in \mathsf{TOP} \,.\, \prod R \subset X \,.\, ??X$

$R : \texttt{Retraction} \iff \exists \texttt{Retract}(X, R)$

$\texttt{RetractOfCompactSpaceIsCompact} :: \forall X : \texttt{Compact} \,.\, \forall R : \texttt{Retract}(X) \,.\, \texttt{Compact}(R)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{RetractOfRetractIsRetract} :: \forall X : \texttt{Compact} \,.\, \forall R : \texttt{Retract}(X) \,.\, \forall S : \texttt{Retract}(R) \,.\, \texttt{Retract}(x, S)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{RetractOfConnectedSpaceIsConnected} :: \forall X : \texttt{Connected} \,.\, \forall R : \texttt{Retract}(X) \,.\, \texttt{Connected}(R)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{InjectiveRetractFunctorProperty} :: \forall X \in \mathsf{TOP} \,.\, \forall R : \texttt{Retract}(X) \,.\, \forall p, q \in R \,.$

$\quad . \,\texttt{Injective}\Big(\Pi(R)(p,q), \Pi(X)(p,q), \iota_{R*}\Big)$

$\texttt{Proof} \ =$

$r := \mathcal{A}\texttt{Retract}(X) : \texttt{Retraction}(X, R),$

$\texttt{Assume}\ \alpha, \beta \in \Pi(R)(p,q),$

$\texttt{Assume}\ [1] : \iota_{R*}(\alpha) = \iota_{R*}(\beta),$

$[2] := \mathcal{A}\texttt{Retraction}(X, R, r)\mathcal{A}\iota_R[1] : \iota_{R*}r_*(\alpha) = \alpha \ \&\ \iota_{R*}r^*(\beta) = \beta,$

$\Big[(\alpha, \beta).*\Big] := \texttt{PathHomotopyPreservedByC} : \alpha = \beta;$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{Injective} : \texttt{Injective}(\iota_*);$

$\square$

$\texttt{SurjectiveRetractFunctorProperty} :: \forall X \in \mathsf{TOP} \,.\, \forall R : \texttt{Retract}(X) \,.$

$\quad . \,\forall r : \forall p, q \in R \,.\, \texttt{Surjective}\Big(\Pi(R)(p,q), \Pi(X)(p,q), r_*\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \alpha : \Pi(R)(p,q),$

$[\alpha.*] := \mathcal{A}\texttt{Retraction}(X, R, r) : \iota_{R*}r_*(\alpha) = \alpha;$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{Surjectrve} : \texttt{Surjective}\Big(\Pi(R)(p,q), \Pi(X)(p,1), r_*\Big);$

$\square$

$\texttt{RetractOfSimplyConnectedIsSimplyConnected} :: \forall X : \texttt{SimplyConnected} \,.\, \forall R : \texttt{Retract}(X) \,.$

$\quad . \,\texttt{SimpltConnected}(R)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FundamentalGroupoidPreservesProducts} :: \forall X, Y \in \mathsf{TOP} \,.\, \Pi(X \times Y) = \Pi(X) \times \Pi(Y)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

5.4 Homotopy Equivalence

$\texttt{HomotopyCategory} :: \mathsf{CAT}$

$\texttt{HomotopyCategory}() = \mathsf{HTOP} := \left(\mathsf{TOP}, \frac{\mathsf{TOP}}{\texttt{Homotopic}}, [\circ], [\mathrm{id}]\right)$

$\texttt{HomotopyEquivalence} :: \forall X, Y \in \mathsf{TOP} \,.\, X \cong_{\mathsf{HTOP}} Y \,.\iff$

$\iff \exists \phi : X \xrightarrow{\mathsf{TOP}} Y : \exists \psi : Y \xrightarrow{\mathsf{TOP}} X \phi\psi \sim \underset{X}{\mathrm{id}} \;\&\; \psi\phi \sim \underset{Y}{\mathrm{id}}$

$\texttt{Proof} =$

...

$\square$

$\texttt{DeformationRetraction} :: \prod X \in \mathsf{TOP} \,.\, \prod R \subset X \,.\, ?\texttt{Retraction}$

$r : \texttt{DeformationRetraction} \iff [\iota_R] =_{\mathsf{HTOP}} [r]^{-1}$

$\texttt{DeformationRetract} :: \prod X \in \mathsf{TOP} \,.\, ?\texttt{Subset}(X)$

$R : \texttt{DeformationRetract} \iff \exists\texttt{DeformationRetraction}(X, R)$

$\texttt{StrongDeformationRetraction} :: \prod X \in \mathsf{TOP} \,.\, \prod R \subset X \,.\, ?\texttt{Retraction}$

$r : \texttt{StrongDeformationRetraction} \iff r\iota_R \sim_R \mathrm{id}$

$\texttt{StrongDeformationRetract} :: \prod X \in \mathsf{TOP} \,.\, ?\texttt{Subset}(X)$

$R : \texttt{StrongDeformationRetract} \iff \exists\texttt{StrongDeformationRetraction}(X, R)$

$\texttt{Contractible} :: ?\mathsf{TOP}$

$X : \texttt{Contractible} \iff \forall x \in X \,.\, \left[\underset{X}{\mathrm{id}}\right] =_{\mathsf{HTOP}} [x]$

$\texttt{StarShapedIsContractible} :: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall A : \texttt{RelativelyStarshaped}(V) \,.\, \texttt{Contractible}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ContractibilityCondition} :: \forall X \in \mathsf{TOP} \,.\, \texttt{Contractible}(X) \iff X \cong_{\mathsf{HTOP}} \mathbf{pt}$

$\texttt{Proof} =$

...

$\square$

$\texttt{PathTranslationLemma} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall \varphi, \psi : X \xrightarrow{\mathsf{TOP}} Y \,.\, \forall H : \texttt{Homotopy}(X, Y, \varphi, \psi) \,.\, \varphi_* \Phi_h = \psi_*$

$\quad \texttt{where}$

$\quad h = \Lambda p \in X \,.\, \Lambda t \in [0,1] \,.\, H(p,t),$

$\quad \Phi_h = \Lambda p, q \in X \,.\, \Lambda \gamma \in \Pi(Y)\Big(\varphi(p), \varphi(q)\Big) \,.\, h_p^{-1} \gamma h_q$

$\texttt{Proof} \ =$

$\texttt{Assume}\ p, q \in X,$

$\texttt{Assume}\ \gamma \in \Pi(X)(p,q),$

$[1] := \mathit{\Pi} \varphi_* \mathit{\Pi} \Phi_h : \varphi_* \Phi_h(\gamma) = \Phi_h\Big(\gamma\varphi\Big) = h_p^{-1}(\gamma\varphi) h_q,$

$H' := \Lambda t \in [0,1] \,.\, h^{-1}_{p|[0,1-t]}(\gamma H(1-t, p)) h_{p|[0,1-t]} : \texttt{Homotopy}(I, Y, \varphi_* \Phi_h(\gamma), \psi_*(\gamma)),$

$\Big[(p,q).*\Big] := \mathit{\Pi}\texttt{Homotopic}[1] : \varphi_* \Phi_h(\gamma) = \psi_*(\gamma);$

$\rightsquigarrow [*] := \texttt{I}(=, \rightarrow) : \varphi_* \Phi_h = \psi_*;$

$\square$

$\texttt{HomotopyInvariance} :: \forall X, Y \in \mathsf{TOP} \,.\, \forall [\varphi] : X \xleftrightarrow{\mathsf{HTOP}} Y \,.\, \varphi_* : \Pi(X) \xleftrightarrow{\mathsf{SGRPD}} \Pi(Y)$

$\texttt{Proof} \ =$

$(\psi, [1]) := \texttt{HomotopyEquivalence}(X, Y, \varphi) : \sum \psi : Y \xrightarrow{\mathsf{TOP}} X \,.\, \underset{X}{\mathrm{id}} \sim \varphi\psi \ \&\ \underset{Y}{\mathrm{id}} \sim \psi\varphi,$

$H := \mathit{\Pi}\texttt{Homotopic}[1_1] : \texttt{Homotopy}(X, X, \varphi\psi, \mathrm{id}_X),$

$h := \Lambda p \in X \,.\, \Lambda t \in I \,.\, H(t,p) : X \to I \xrightarrow{\mathsf{TOP}} X,$

$\Phi := \Lambda p, q \in X \,.\, \Lambda \gamma \in \Omega(p,q) \,.\, h_p^{-1} \gamma h_q : X^2 \to \Omega(p,q) \to \Omega(p,q),$

$[2] := \texttt{PathTranslationLemma}(X, X, \varphi\psi, \mathrm{id}_X, H) : \varphi_* \psi_* = \Phi,$

$[3] := \texttt{IsoAsComposition}[2] : \texttt{Injective}(\varphi_*) \ \&\ \texttt{Surjectve}(\psi_*),$

$H' := \mathit{\Pi}\texttt{Homotopic}[1_2] : \texttt{Homotopy}(Y, Y, \psi\varphi, \mathrm{id}_Y),$

$h' := \Lambda p \in Y \,.\, \Lambda t \in I \,.\, H'(t,p) : Y \to I \xrightarrow{\mathsf{TOP}} Y,$

$\Phi' := \Lambda p, q \in Y \,.\, \Lambda \gamma \in \Omega(p,q) \,.\, h'^{-1}_p \gamma h'_q : Y^2 \to \Omega(p,q) \to \Omega(p,q),$

$[4] := \texttt{PathTranslationLemma}(Y, Y, \psi\varphi, \mathrm{id}_X, H) : \psi_* \varphi_* = \Phi',$

$[5] := \texttt{IsoAsComposition}[2] : \texttt{Injective}(\psi_*) \ \&\ \texttt{Surjectve}(\varphi_*),$

$[*] := \texttt{GrpIsomorphism}[3][5] : \varphi_* : \Pi(X) \xleftrightarrow{\mathsf{SGRPD}} \Pi(Y);$

$\square$

$\texttt{MappingCyllinder} :: \prod_{X, Y \in \mathsf{TOP}} (X \xrightarrow{\mathsf{TOP}} Y) \to \mathsf{TOP}$

$\texttt{MappingCyllinder}\,(f) = Z_f := (X \times I) \sqcup_\varphi Y$

$\quad \texttt{where}$

$\quad \varphi = \Lambda(x, 0) \in X \times \{0\} \,.\, f(x)$

$\texttt{MappingCyllinderTHM} :: \forall X, Y \in \mathsf{HTOP} \,.\, \forall [f] : X \xleftrightarrow{\mathsf{TOP}} Y \,.\, \forall \pi_X(Z_f), \pi_Y(Z_f) : \texttt{DeformationRetract}(Z_f)$

$\texttt{Proof} \;=$

$q := \Lambda(p,t) \times \in X \times I \,.\, \iota_{X\times I}(p,t) : X \times I \xrightarrow{\mathsf{TOP}} Z_f,$

$q' := \Lambda y \in Y \,.\, \iota_Y(y) : Y \xrightarrow{\mathsf{TOP}} Z_f,$

$\texttt{Assume}\ z : Z_f,$

$[1] := \mathcal{A} Z_f(z) : \exists x \in X : \exists t \in I : z = [x,t] \Big| \exists y \in Y : z = [y],$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ t \in I,$

$\texttt{Assume}\ [2] : z = [x,t],$

$H(z) := \Lambda s \in [0,1] \,.\, \Big[x, t(1-s)\Big] : I \to Z_f,$

$A(z) := [x,0] : Z_f(z);$

$\leadsto [2] := \texttt{I}(\forall) : \forall x \in X \,.\, \forall t \in I \,.\, z = [x,t] \Rightarrow A(z) \in Z_f, H : I \to Z_f,$

$\texttt{Assume}\ y \in Y,$

$\texttt{Assume}\ [3] : z = [y],$

$H(z) := \Lambda s \in [0,1] \,.\, [y] : I \to Z_f,$

$A(z) := [y] : Z_f;$

$\leadsto [3] := \texttt{I}(\forall) : \forall y \in Y \,.\, z = [y] \Rightarrow A(z) \in Z_f, H : I \to Z_f,$

$A(z) := \texttt{E}(|)[1,2,3]\mathcal{A} Z_f : Z_f,$

$H := \texttt{E}(|)[1,2,3]\mathcal{A} Z_f : I \to Z_f;$

$\leadsto A := \texttt{I}(\to) : Z_f \xrightarrow{\mathsf{TOP}} Z_f,$

$\leadsto H := \texttt{I}(\to) : I \xrightarrow{\mathsf{TOP}} Z_f \xrightarrow{\mathsf{TOP}} Z_f,$

$[1] := \mathcal{D} H : \texttt{Homotopy}(Z_f, Z_f, \mathrm{id}, A, H),$

$[2] := \mathcal{A}^{-1}\texttt{StrongDeformationRetract}[1] : \texttt{StrongDeformationRetraction}\Big(Z_f, q'(Y)\Big),$

$\Big(g, [3]\Big) := \texttt{HomotopyEquivalence}(X, Y, f) : \sum g : Y \xrightarrow{\mathsf{TOP}} X \,.\, fg \sim \mathrm{id}_X \;\&\; gf \sim \mathrm{id}_Y,$

$F := \mathcal{A}\mathsf{HTOP}[3_1] : \texttt{Homotopy}(X, X, fg, \mathrm{id}_X),$

$G := \mathcal{A}\mathsf{HTOP}[3_2] : \texttt{Homotopy}(Y, Y, gf, \mathrm{id}_Y),$

$\texttt{Assume}\ z : Z_f,$

$[1] := \mathcal{A} Z_f(z) : \exists x \in X : \exists t \in I : z = [x,t] \Big| \exists y \in Y : z = [y],$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ t \in I,$

$\texttt{Assume}\ [2] : z = [x,t],$

$H'(z) := \Lambda s \in [0,1] \,.\, \Big[F\big(f(x), 1-t\big)\Big] : I \to Z_f,$

$H''(z) := \Lambda s \in [0,1] \,.\, \Big[G(x, st), t\Big] : I \to Z_f;$

$\leadsto [2] := \texttt{I}(\forall) : \forall x \in X \,.\, \forall t \in I \,.\, z = [x,t] \Rightarrow H', H'' : I \to Z_f,$

$\texttt{Assume}\ y \in Y,$

$\texttt{Assume}\ [3] : z = [y],$

$H'(z) := \Lambda s \in [0,1] \,.\, \Big[F(y, 1-t)\Big] : I \to Z_f,$

$H''(z) := \Lambda s \in [0,1] \,.\, \Big(g(y), t\Big) : I \to Z_f;$

$\leadsto [3] := \texttt{I}(\forall) : \forall y \in Y \,.\, z = [y] \Rightarrow H', H'' : I \to Z_f,$

$H' := \texttt{E}(|)[1,2,3]\mathcal{A} Z_f : Z_f,$

$$
\begin{aligned}
&H'' := \mathtt{E}(|)[1,2,3]\text{Ɑ} Z_f : I \to Z_f;\\
&\rightsquigarrow H' := \mathtt{I}(\to) : I \xrightarrow{\textsf{TOP}} Z_f \xrightarrow{\textsf{TOP}} Z_f,\\
&\rightsquigarrow H'' := \mathtt{I}(\to) : I \xrightarrow{\textsf{TOP}} Z_f \xrightarrow{\textsf{TOP}} Z_f,\\
&B := H'(0) : Z_f \xrightarrow{\textsf{TOP}} Z_f,\\
&C := H'(1) : Z_f \xrightarrow{\textsf{TOP}} Z_f,\\
&[4] := \text{Ɔ}B : \mathtt{Homotopty}(Z_f, Z_f, \mathrm{id}_{Z_f}, B),\\
&[5] := \text{Ɔ}C : \mathtt{Homotopty}(Z_f, Z_f, \mathrm{id}_{Z_f}, C),\\
&[*] := \text{Ɑ}^{-1}\mathtt{StrongDeformationRetract}[1] :: \mathtt{StrongDeformationRetraction}\Big(Z_f, q(X), C\Big),
\end{aligned}
$$

$\square$

5.5 The Circle

$\texttt{circleParametrization} :: \mathbb{R} \to \mathbb{S}^1$
$\texttt{circleParametrisation}\,(t) = s(t) := \exp(it)$

$$\texttt{Lift} :: \prod_{X \in \mathsf{TOP}} (X \xrightarrow{\mathsf{TOP}} \mathbb{S}^1) \to ?(X \xrightarrow{\mathsf{TOP}} \mathbb{R})$$

$g : \texttt{Lift} \iff \Lambda f : X \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \,.\, gs = f$

$$\texttt{Liftable} :: \prod_{X \in \mathsf{TOP}} ?(X \xrightarrow{\mathsf{TOP}} \mathbb{S}^1)$$

$f : \texttt{Liftable} \iff \exists \texttt{Lift}(X, f)$

$$\texttt{SpirallingAtlas} :: \forall z \in \mathbb{S}^1 \,.\, \exists U \in \mathcal{U}(z) : \exists U' : \mathbb{N} \to \texttt{OpenInterval}(\mathbb{R}) : s^{-1}(U) = \bigsqcup_{n=1}^{\infty} U'_n \;\&$$

$$\& \; \forall n \in \mathbb{N} \,.\, s_{|U'_n} : U'_n \xleftrightarrow{\mathsf{TOP}} U$$

$\texttt{Proof} \;=$

$U := \mathbb{S}^1 \setminus \{-z\} \in \mathcal{U}(z),$

$[1] := \text{Ɑ}\mathsf{TOP}(\mathbb{R}, \mathbb{S}^1, s) : s^{-1}(U) \in \mathcal{T}(\mathbb{R}),$

$$\Big(N, U', [3]\Big) := \texttt{RealOpenSubsetRepresentation} : \sum N \in \aleph_0 \,.\, \sum U' : N \to \texttt{OpenInterval}(\mathbb{R}) \,.$$

$$.\; s^{-1}(U) = \bigsqcup_{n=1}^{N} U'_n,$$

$$\Big(t, [2]\Big) := \text{Ɑ}s\text{Ɔ}U : \sum t \in \mathbb{R} : s^{-1\complement}(U) = \bigsqcup_{n=-\infty}^{\infty} \{t + 2\pi n\},$$

$[4] := [3][2] : N = \mathbb{N},$

$[5] := \text{Ɑ}s[3][2] : \forall n \in \mathbb{N} \,.\, |U'_n| < 2\pi,$

$[*] := \text{Ɑ}s[5] : \forall n \in \mathbb{N} \,.\, S_{|U'_n} : U'_n \xleftrightarrow{\mathsf{TOP}} U;$

$\square$

$\texttt{EvenlyCovered} :: ?\mathcal{T}(\mathbb{S}^1)$

$$U : \texttt{EvenlyCovered} \iff \exists U' : \mathbb{N} \to \texttt{OpenInterval}(\mathbb{R}) : s^{-1}(U) = \bigsqcup_{n=1}^{\infty} U'_n \;\&$$

$$\& \; \forall n \in \mathbb{N} \,.\, s_{|U'_n} : U'_n \xleftrightarrow{\mathsf{TOP}} U$$

$\texttt{CircleSectionLemma} :: \forall U : \texttt{EvenlyCovered} \,.\, \forall z \in U \,.\, \forall r \in s^{-1}(z) \exists \sigma : \texttt{LocalSection}(U, \mathbb{R}, s) \,.\, \sigma(z) = r$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\mathtt{UniqueLiftingProperty} :: \forall X : \mathtt{Connected} \;.\; \forall f : X \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \;.\; \forall g, g' : \mathtt{Lift}(X, f) \;.\; \forall x \in X \;.$

$\quad .\; \forall [0] : g(x) = g'(x) \;.\; g = g'$

$\mathtt{Proof} \;=$

$\mathcal{X} := \{x \mathbf{i} x : g(x) = g'(x)\} :?X,$

$[1] := [0]Ɔ\mathcal{X} : \mathcal{X} \neq \emptyset,$

$[2] := ᗡ\mathtt{Lift}(g) : gs = f,$

$[3] := ᗡ\mathtt{Lift}(g') : g's = f,$

$\mathtt{Assume}\; p : \mathcal{X},$

$\Big(U, [4]\Big) := \mathtt{SpirallingAtlas}\big(f(x)\big) : U : \mathtt{EvenlyCovered} \;.\; f(p) \in U,$

$U', [5] := ᗡ\mathtt{EvenlyCovered}(U) : \sum \mathbb{N} \to \mathcal{T}(\mathbb{R}) \;.\; \bigsqcup_{n=1}^{\infty} s^{-1}(U) = \bigsqcup_{n=1}^{\infty} U' \;\&\; \forall n \in \mathbb{N} \;.\; s_{|U'_n} U'_n \xleftrightarrow{\mathsf{TOP}} U,$

$n, [6] := ᗡ\mathtt{Preimage}[5_1][2]Ɔ\mathcal{X} : \sum n \in \mathbb{N} \;.\; g(p) \in U',$

$V := g^{-1}(U'_n) \cap g'^{-1}(U'_n) : \mathcal{T}(X),$

$[7] := [6]ƆV : p \in V,$

$[p.*] := [2][3]ƆV[5_2][6] : V \subset \mathcal{X};$

$\rightsquigarrow [4] := \mathtt{OpenByOpenCover} : \mathcal{X} \in \mathcal{T}(\mathcal{X}),$

$[5] := ᗡ\mathtt{Continuous}(g - g')Ɔ\mathcal{X} : \mathtt{Closed}(X, \mathcal{X}),$

$[6] := ᗡ\mathtt{Connected}(X)[1, 4, 5] : \mathcal{X} = X,$

$[*] := [6]Ɔ\mathcal{X} : g = g';$

$\square$

$\mathtt{HomotopyLifitingProperty} :: \forall X : \mathtt{LocallyConnected} \;.\; \forall f, f' : X \xrightarrow{\mathsf{TOP}} \mathbb{S}_1 \;.\; \forall H : \mathtt{Homotopy}(X, \mathbb{S}^1, f, f') \;.$

$\quad .\; \forall g : \mathtt{Lift}(X, f) \;.\; \exists! g' : \mathtt{Lift}(X, f') : \exists! \widetilde{H} : \mathtt{Homotopy}(X, \mathbb{R}, g, g')$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\; x \in X,$

$\mathtt{Assume}\; t \in I,$

$\Big(U, [1]\Big) := \mathtt{SpirallingAtlas}\big(H(t, x)\big) : U : \mathtt{EvenlyCovered} \;.\; H(t, x) \in U,$

$W' := H^{-1}(U) : \mathcal{U}(t, x),$

$\Big(V_t, J_t, [2]\Big) := ᗡ\mathtt{ProductTopology} : \sum J \in \mathcal{U}(t) \;.\; \sum V \in \mathcal{U}(x) \;.\; J \times V \subset W';$

$\rightsquigarrow (V, J, [1]) := \mathtt{I}\left(\prod\right) : \prod_{t \in I} \sum V_t \in \mathcal{U}(x) \;.\; \sum J_t \in \mathcal{U}(t) \;.\; \exists U : \mathtt{UniformlyCovered} : V_t \times J_t \subset H^{-1}(U),$

$[2] := ᗡ^{-1}\mathtt{OpencCover}ᗡ(V, J) : \mathtt{OpenCover}\Big(I \times \{x\}, J \times V\Big),$

$\Big(n, t, [3]\Big) := ᗡ\mathtt{Compact}(I \times \{x\}, J \times V) : \sum_{n=1}^{\infty} \sum t : [1, \ldots, n] \to I \;.\; \mathtt{OpenCover}\Big(I \times \{x\}, J_t \times V_t\Big),$

$\Big(W, [4]\Big) := ᗡ\mathtt{LocallyConncected}\left(\bigcap_{i=1}^{n} V_i, x\right) : \sum W \in \mathcal{U}(x) \;\&\; \mathtt{Connected} \;.\; W \subset \bigcup_{i=1}^{n} V_i,$

$\lambda := \mathtt{LebesgueNumberExists}(J_t) : \mathtt{LebesgueNumber}(J_t),$

$\Big(m, [5]\Big) := \mathtt{ReductionInfima}(\lambda) : \sum_{m=1}^{\infty} \frac{1}{m} < \lambda,$

$[6] := ᗡ\mathtt{LebesgueNumber}[5][1] : \forall j \in [1, \ldots, m] \;.\; \exists U : \mathtt{UniformlyCovered} \;.\; H\left(\left[\frac{j-1}{m}, \frac{j}{m}\right] \times W\right) \subset U,$

$\texttt{Assume}\ j \in [1, \dots, m],$

$\left(U, [7]\right) := [6](j) : \sum U : \texttt{UniformlyCovered}\ .\ H\left(\left[\frac{j-1}{m}, \frac{j}{m}\right] \times W\right) \subset U,$

$\left(\sigma, [8]\right) := \texttt{CircleSectionLemma}\left(U, f(x), g(x)\right) : \sum \sigma : \texttt{LocalSection}(U, s)\ .\ f\sigma(x) = g(x),$

$\texttt{Assume}\ [9] : j = 1,$

$\texttt{Assume}\ (t, x) : \left[0, \frac{1}{m}\right] \times W,$

$\widetilde{H}(t, x) := H\sigma_1 : \mathbb{R};$

$\rightsquigarrow \widetilde{H} := \mathtt{I}(\to) : \left[0, \frac{1}{m}\right] \times W \xrightarrow{\mathsf{TOP}} \mathbb{R},$

$[10] := \text{Ɑ}\texttt{LocalSection}(\sigma)\text{Ɔ}\widetilde{H} : \forall x \in W\ .\ \widetilde{H}s(x, 0) = s\sigma H(x, 0) = H(x, 0) = f(x),$

$[9.*] := \texttt{UniqueLiftingProperty}[10] : \forall x \in W\ .\ \widetilde{H}(x, 0) = g(x);$

$\rightsquigarrow (\widetilde{H}, [9]) := \mathtt{I}\left(\sum\right) : \sum \widetilde{H} : \left[0, \frac{1}{m}\right] \times W \to \mathbb{R}\ .\ \forall x \in W\ .\ \widetilde{H}(x, 0) = g(x),$

$\texttt{Assume}\ [10] : j > 1,$

$\texttt{Assume}\ \widetilde{H} : \left[0, \frac{j-1}{m}\right] \times W \to \mathbb{R},$

$\texttt{Assume}\ [11] : \forall x \in W\ .\ \widetilde{H}(0, x) = g(x),$

$\texttt{Assume}\ (t, x) : \left[\frac{k-1}{m}, \frac{1}{m}\right] \times W,$

$\widetilde{H}(t, x) := H\sigma_1 : \mathbb{R};$

$\rightsquigarrow \widetilde{H} := \mathtt{I}(\to)\texttt{UniqueLiftProperty}() : \left[0, \frac{k}{m}\right] \times W \xrightarrow{\mathsf{TOP}} \mathbb{R},$

$[12] := \text{Ɑ}\texttt{LocalSection}(\sigma)\text{Ɔ}\widetilde{H} : \forall x \in W\ .\ \widetilde{H}s(x, 0) = s\sigma H(x, 0) = H(x, 0) = f(x),$

$[10.*] := \texttt{UniqueLiftingProperty}[10] : \forall x \in W\ .\ \widetilde{H}(x, 0) = g(x);$

$\rightsquigarrow (\widetilde{H}, [9]) := \text{Ɑ}[1, \dots, m] : \sum \widetilde{H} : I \times W \to \mathbb{R}\ .\ \forall x \in W\ .\ \widetilde{H}(x, 0) = g(x);$

$\rightsquigarrow (\widetilde{H}, [1]) := \texttt{UniqueLiftProperty} : \sum \widetilde{H} : I \times X \to \mathbb{R}\ .\ \forall x \in W\ .\ \widetilde{H}(x, 0) = g(x),$

$g' := \Lambda x \in X\ .\ \widetilde{H}(x, 1) : X \xrightarrow{\mathsf{TOP}} \mathbb{R},$

$[*] := \text{Ɑ}^{-1}\texttt{Homotopy} : \texttt{Homotopy}(X, \mathbb{R}, g, g', \widetilde{H});$

$\square$

$\texttt{PathLifitingProperty1} :: \forall \gamma : I \xrightarrow{\mathsf{TOP}} \mathbb{S}^1\ .\ \exists \texttt{Lift}(I, \gamma)$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{PathLifitingProperty2} :: \forall \gamma : I \xrightarrow{\mathsf{TOP}} \mathbb{S}^1\ .\ \forall \alpha, \beta : \texttt{Lift}(I, \gamma)\ .\ \exists n \in \mathbb{N}\ .\ \alpha - \beta = 2\pi n$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{PathHomotopyCriterion} :: \forall f, f' : I \to \mathbb{S}^1 \,.\, \forall g : \texttt{Lift}(I, f) \,.\, \forall g' : \texttt{Lift}(I, f') \,.\, \forall [0] : f(0) = f'(0) \,.$
$\quad . \forall [00] : f(1) = f'(1) \,.\, \forall [000] : g(0) = g'(0) \,.\, g(1) = g'(1) \iff f \approx f'$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{windingNumber} :: \prod z \in \mathbb{S}_1 \,.\, \Omega(z) \to \mathbb{Z}$
$$\texttt{windingNumber}\,(\gamma) = w(\gamma) := \frac{\alpha(1) - \alpha(0)}{2\pi}$$
$\quad \texttt{where}$
$\quad \alpha = \texttt{PathLifitingProperty1}(\gamma)$

$\texttt{WindingNumberRotationInvariance} :: \forall z, u \in \mathbb{S}_1 \,.\, \forall \gamma \in \Omega(z) \,.\, w(u\gamma) = w(\gamma)$
$\texttt{Proof} =$
$\Big(x, [1]\Big) := \texttt{PolarRepresentation}(u) : \sum x \in [0, 2\pi) \,.\, u = e^{\mathrm{i}x},$
$\alpha := \texttt{PathLifitingProperty}(\gamma) : \texttt{Lift}(I, \gamma),$
$\beta := \alpha + x : I \to \mathbb{R},$
$[2] := \Lambda t \in I \,.\, [1] \mathcal{I}\texttt{Lift}(I, \gamma, \alpha)(t) \mathcal{I} \mathsf{RING}(\mathbb{C}) \texttt{ExponentProduct}(\mathrm{i}t, \mathrm{i}\alpha(t)) \mathcal{O}^{-1}\beta :$
$\quad : \forall t \in I \,.\, u\gamma(t) = e^{\mathrm{i}x} e^{\mathrm{i}\alpha(t)} = \exp\Big(\mathrm{i}\big(\alpha(t) + x\big)\Big) = e^{\mathrm{i}\beta(t)},$
$[3] := \mathcal{I}^{-1}\texttt{Lift}[2] : \texttt{Lift}(I, u\gamma, \beta),$
$[*] := \mathcal{I} w(u\gamma)[3] \mathcal{O} \beta \mathcal{I}^{-1} w(\gamma) : w(u\gamma) = w(\gamma);$
$\square$

$\texttt{WindingNumberLoopClassification} :: \forall z \in \mathbb{S}^1 \,.\, \forall \gamma, \gamma' \in \Omega(z) \,.\, w(\gamma) = w(\gamma') \iff \gamma \approx \gamma'$
$\texttt{Proof} =$
$\alpha := \texttt{PathLifitingProperty}(\gamma) : \texttt{Lift}(I, \gamma),$
$\alpha' := \texttt{PathLifitingProperty}(\gamma') : \texttt{Lift}(I, \gamma'),$
$\beta := \alpha' - \alpha'(0) + \alpha(0) : I \to \mathbb{R},$
$[0] := \mathcal{I}\Omega(z) \mathcal{O} \beta \texttt{PathLifitingProperty} : \texttt{Lift}(I, \omega', \beta),$
$\texttt{Assume}\ [1] : w(\gamma) = w(\gamma'),$
$[2] := [1] \mathcal{I} w \mathcal{I} \alpha \mathcal{I} \alpha' : \alpha(1) - \alpha(0) = \alpha'(1) - \alpha'(0),$
$[3] := \mathcal{O}\beta[2] : \beta = \alpha' - \alpha'(1) + \alpha'(1),$
$[4] := [3](1) : \beta(1) = \alpha(1),$
$[5] := \mathcal{O}\beta(0) : \beta(0) = \alpha(0),$
$[1.*] := \texttt{PathHomotopyCriterion}[0][4][5] : \gamma \approx \gamma';$
$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : w(\gamma) = w(\gamma') \Rightarrow \gamma \approx \gamma',$
$\texttt{Assume}\ [2] : \gamma \approx \gamma',$
$[3] := \mathcal{O}\beta(0) : \beta(0) = \alpha(0),$
$[4] := \texttt{PathHomotopyCriterion}(\gamma, \gamma', \alpha, \beta)[3] : \alpha(1) = \beta(1),$
$[*] := \mathcal{I}2[3][4] : w(\gamma) = w(\gamma');$
$\square$

$\texttt{FundamentlGroupOfThCircle} :: \pi(\mathbb{S}^1) \cong_{\mathsf{GRP}} \mathbb{Z}$

$\texttt{Proof} \ =$

$F := \Lambda n \in \mathbb{Z} \,.\, [s]^{\circ n} : \mathbb{Z} \to \pi(\mathbb{S}^1),$

$G := \Lambda[\gamma] \in \pi(\mathbb{S}^1) \,.\, w(\gamma) : \pi(\mathbb{S}^1) \to \mathbb{Z},$

$[1] := \Lambda n \in \mathbb{Z} \,.\, \mathcal{O}F\mathcal{O}G\mathcal{I}s\mathcal{I}w : \forall n \in \mathbb{Z} \,.\, FG(n) = G[s]^{\circ n} = w\Big(s^n\Big) = n,$

$[2] := \Lambda[\gamma] \in \pi(\mathbb{S}^1) \,.\, \mathcal{O}G\mathcal{O}w\texttt{WindingNumberLoopClassification}\Big(\gamma, s^{w(\gamma)}\Big) :$

$\quad : \forall[\gamma] \in \pi(\mathbb{S}^1) \,.\, GF[\gamma] = F\big(w(\gamma)\big) = [s]^{\circ w(\gamma)} = [\gamma],$

$[3] := \mathcal{I}^{-1}\texttt{Bijection}[1][2] : F : \mathbb{Z} \xleftrightarrow{\mathsf{SET}} \pi(\mathbb{S}^1),$

$[4] := \Lambda n, m \in \mathbb{Z} \,.\, \mathcal{O}F(n+m)\texttt{GroupExponentiation}\Big(\pi\big(\mathbb{S}^1\big)\Big)\mathcal{O}^{-1}F :$

$\quad : \forall n, m \in \mathbb{Z} \,.\, F(n+m) = [s]^{\circ(n+m)} = [s]^{\circ n}[s]^{\circ m} = F(n)F(m),$

$[*] := \mathcal{I}\mathsf{GRP}[4][3] : F : \mathbb{Z} \xleftrightarrow{\mathsf{GRP}} \pi(\mathbb{S}^1);$

$\square$

$\texttt{FundamentalGroupOfThePuncturedPlane} :: \pi\Big(\mathbb{C} \setminus \{0\}\Big) \cong_{\mathsf{GRP}} \mathbb{Z}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FundamentalGroupOfTheNTorus} :: \forall n \in \mathbb{N} \,.\, \pi\Big(\mathbb{T}^n\Big) \cong_{\mathsf{GRP}} \mathbb{Z}^n$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{degreeOfTheMap} :: \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}_1) \to \mathbb{Z}$

$\texttt{degreeOfTheMap}\,(f) = \deg f := w(sf)$

$\texttt{DegreeCharacterisation} :: \forall f : \mathrm{End}_{\mathsf{TOP}}\big(\mathbb{S}_1\big) \,.\, \deg f = \deg\left(\frac{f}{f(0)}\right)_*$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{HomotopicMapsHaveSameDegree} :: \forall f, f' : \mathrm{End}_{\mathsf{TOP}}\big(\mathbb{S}_1\big) \,.\, f \sim f' \Rightarrow \deg f = \deg f'$

$\texttt{Proof} \ =$

$[1] := \texttt{RotatioIsHomotopicToId}\left(\frac{f}{f(0)}, \frac{f'}{f'(0)}\right) : \frac{f}{f(0)} \sim \frac{f'}{f'(0)},$

$\Big(h, [2]\Big) := \texttt{PathTranslationLemma}[1] : \sum h \in \Omega(1) \,.\, \left(\frac{f}{f(0)}\right)_* \Phi_h = \left(\frac{f'}{f'(0)}\right)_*,$

$[3] := \mathcal{I}\mathsf{ABEL}(\mathbb{Z})\texttt{FundamentalGroupOfTheCircle}\mathcal{I}\Phi_h : \Phi_h = \mathrm{id},$

$[4] := [2][3] : \left(\frac{f}{f(0)}\right)_* = \left(\frac{f'}{f'(0)}\right)_*,$

$[*] := \texttt{DegreeCharacterisation}^2(f)(f')[4] : \deg f = \deg f';$

$\square$

$\texttt{DegreeIsHomo} :: \forall f,g \in \mathrm{End}_{\mathsf{TOP}}\left(\mathbb{S}_1\right) \;.\; \deg fg = \deg f \deg g$

$\texttt{Proof} \;=$

$[*] := \texttt{DegreeCharacterisation}(fg) \daleth \texttt{Field}(\mathbb{C})\texttt{HomoDegIsHomo}(\ldots)\texttt{DegreeCharacterisation}^2(f)(\mu_{f(0)}g)$
$\quad\texttt{HomotopicMapsHaveSameDegrees}(g, \mu_{f(0)}g) :$

$$: \deg fg = \deg\left(\frac{fg}{fg(1)}\right)_* = \deg\left(\frac{1}{fg(1)}g\left(\frac{f(1)f}{f(1)}\right)\right)_* = \deg\left(\frac{\mu_{f(1)}g}{fg(1)}\right)_* \deg\left(\frac{f}{f(1)}\right)_* = \deg g \deg f;$$

$\square$

$\texttt{DegreeClassificationOfCircleEnd} :: \forall f,f' \in \mathrm{End}_{\mathsf{TOP}}\left(\mathbb{S}^1\right) \;.\; \deg f = \deg f' \Rightarrow f \sim f'$

$\texttt{Proof} \;=$

$[1] := \daleth\, \deg : w(sf) = w(sf'),$

$[2] := \texttt{WindingNumberLoopClassification} : sf \approx sf',$

$\texttt{Assume}\ [3] : f(1) = f'(1),$

$H := \daleth\texttt{Homotopic} : \texttt{Homotopy}(I, \mathbb{S}^1, sf, sf'),$

$\widetilde{H} := \dfrac{H}{\mathrm{id}\times s} : \texttt{Homotopy}\left(\mathbb{S}^1, \mathbb{S}^1, f, f'\right),$

$[3.*] := \daleth^{-1}\texttt{Homotopic}(\widetilde{H}) : f \sim f';$

$\rightsquigarrow [*] := \texttt{RotationIsHomotopicToId} : f \sim f';$

$\square$

$\texttt{SurjectiveByDegree} :: \forall f \in \mathrm{End}_{\mathsf{TOP}}\left(\mathbb{S}^1\right) \;.\; \deg f \neq 0 \Rightarrow \texttt{Surjective}(\mathbb{S}^1, \mathbb{S}^1, f)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{HasFixedPointByDegree} :: \forall f \in \mathrm{End}_{\mathsf{TOP}}\left(\mathbb{S}^1\right) \;.\; \deg f \neq 1 \Rightarrow \mathrm{Fix}(f) \neq \emptyset$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \mathrm{Fix}(f) = \emptyset,$

$H := \Lambda t \in I \;.\; \Lambda z \in \mathbb{S}^1 \;.\; \dfrac{(1-t)f(z) - tz}{\left\|(1-t)f(z) - tz\right\|} : \texttt{Homotopy}\left(\mathbb{S}^1, \mathbb{S}^1, f, \mathrm{inv}(\mathbb{C}, +)\right),$

$[2] := \texttt{HomtopicMapsHaveSameDegree}(H) : \deg f = \deg \mathrm{inv}(\mathbb{C}, +) = 1,$

$[1.*] := [2][0] : \perp;$

$\rightsquigarrow [*] := \texttt{E}(\perp) : \mathrm{Fix}(f) \neq \emptyset;$

$\square$

$\texttt{DegreeZeroByExtensionToTheCell} :: \forall f : \mathbb{S}^1 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \;.\; \forall F : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \;.\; F_{|\mathbb{S}^1} = f \Rightarrow \deg f = 0$

$\texttt{Proof} \;=$

$[1] := \daleth\texttt{CircleRepresentative} : \widetilde{sf} = f,$

$[2] := \texttt{ExtensionImplyNullHomotopic}[0][1] : \texttt{NullHomotopic}(sf),$

$[3] := \daleth\texttt{NullHomotopic}\daleth^{-1}w : w(sf) = 0,$

$[*] := \daleth\, \deg : \deg f = 0;$

$\square$

$\texttt{MainTheoremOfAlgebra} :: \forall p \in \mathbb{C}[x] \ . \ \deg p > 1 \Rightarrow \rho(p) \neq \emptyset$

$\texttt{Proof} \ =$

$n := \deg p \in \mathbb{N},$

$\Big(a, [1]\Big) := \text{ᗡ}p\text{Ɔ}n : \sum a : [1, \dots, n] \to \mathbb{C} \ . \ p(x) \sim x^n + \sum_{i=1}^{n} a_i x^{i-1},$

$\texttt{Assume} \ [2] : \rho(p) \neq \emptyset,$

$f := \Lambda z \in \mathbb{D}^2 \ . \ \dfrac{p(z)}{\|p(z)\|} : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1,$

$[3] := \texttt{DegreeZeroByExtensionToTheCell} : \deg f_{|\mathbb{S}^1} = 0,$

$H := \Lambda t \in I \ . \ \Lambda z \in \mathbb{S}^1 \ . \ \dfrac{t^n p\left(\frac{z}{t}\right)}{\left\|t^n p\left(\frac{z}{t}\right)\right\|} : \texttt{Homotopy}\Big(p, z^n\Big),$

$[4] := \texttt{HomotopicMapsHaveSameDegree}(H) : \deg p = n,$

$[2.*] := [4][3] : \bot;$

$\rightsquigarrow [*] := \texttt{E}(\bot) : \rho(p) \neq \emptyset;$

$\square$

$\texttt{BrauwerFixedPointTHM} :: \forall f : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} \mathbb{D}^2 \ . \ \mathrm{Fix}(f) \neq \emptyset$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : \mathrm{Fix}(f) = \emptyset,$

$F := \dfrac{x - f(x)}{\Big\|x - f(x)\Big\|} : \mathbb{D}^2 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1,$

$[2] := \texttt{DegreeZeroByExtensionToTheCell} : \deg F_{|\mathbb{S}^1} = 0,$

$H := \Lambda t \in I \ . \ \Lambda z \in \mathbb{S}^1 \ . \ \dfrac{z - tf(z)}{\Big\|z - tf(z)\Big\|} : \texttt{Homotopy}\Big(\mathbb{S}^1, \mathbb{S}^1, F_{|\mathbb{S}^1}, \mathrm{id}\Big),$

$[3] := \texttt{HomotopyPreservesDegree}(H) : \deg F_{|\mathbb{S}^1} = 1,$

$[1.*] := [2][3] : \bot;$

$\rightsquigarrow [*] := \texttt{E}(\bot) : \mathrm{Fix}(f) \neq \emptyset;$

$\square$

$\texttt{InjectiveDegree} :: \forall f : \mathbb{S}^1 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \ . \ \texttt{Injective}(\mathbb{S}_1, \mathbb{S}_1, f) \Rightarrow |\deg f| = 1$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DeifferentDegreesImplyAntipodalValuesExists} ::$

$:: \forall f, g : \mathbb{S}^1 \xrightarrow{\mathsf{TOP}} \mathbb{S}^1 \ . \ \deg f \neq \deg g \Rightarrow \exists z \in \mathbb{S}^1 : f(z) = -g(z)$

$\texttt{Proof} \ =$

$\dots$

$\square$

5.6 Index for Plane Vector Fields

$$\mathtt{FlatVectroField} = \mathfrak{X} := \prod_{n=0}^{\infty} \mathbb{R}^n \xrightarrow{\mathsf{TOP}} \mathbb{R}^n : \mathbb{Z}_+ \to \mathbb{R}\text{-}\mathsf{VS};$$

$$\mathtt{SingularPoint} :: \prod_{n=0}^{\infty} \mathfrak{X}(n) \to ?\mathbb{R}^n$$

$$s : \mathtt{SingularPoint} \iff \Lambda V \in \mathfrak{X}(n) \;.\; s \in \mathcal{S}_V \iff V(s) = 0$$

$$\mathtt{RegularPoint} :: \prod_{n=0}^{\infty} \mathfrak{X}(n) \to ?\mathbb{R}^n$$

$$r : \mathtt{SingularPoint} \iff \Lambda V \in \mathfrak{X}(n) \;.\; r \in \mathcal{R}_V \iff V(r) \neq 0$$

$$\mathtt{IsolatedSingularPoint} :: \prod_{n=0}^{\infty} \prod V \in \mathfrak{X}(n) \;.\; ?\mathcal{S}_V$$

$$s : \mathtt{IsolatedSingularPoint} \iff \exists U \in \mathcal{U}(s) \;.\; U \setminus \{s\} \subset \mathcal{R}_V$$

$$\mathtt{RegularLoop} :: \prod_{V \in \mathfrak{X}(2)} ?\left(I \xrightarrow{\mathsf{TOP}} \mathcal{R}_V\right)$$

$$\gamma : \mathtt{RegularLoop} \iff \gamma(0) = \gamma(1)$$

$$\mathtt{windingNumberInThePuncturedSpace} :: \left(I \xrightarrow{\mathsf{TOP}} \mathbb{R}^2 \setminus \{0\}\right) \to \mathbb{Z}$$

$$\mathtt{windingNumberInThePuncturedSpace}(\gamma) = w(\gamma) := w\left(\frac{\gamma}{\|\gamma\|}\right)$$

$$\mathtt{windingNumberRelativeToAVectorField} :: \prod_{V \in \mathfrak{X}(2)} \mathtt{RegularLoop}(V) \to \mathbb{Z}$$

$$\mathtt{windingNumberRelativeToAVectorField}(\gamma) = w_V(\gamma) := w(\gamma V)$$

$\mathtt{HomotopyPreservesVectorFieldWindingNumber} :: \forall V \in \mathfrak{X}(2) \;.\; \forall \gamma, \gamma' \;.\; \gamma \sim \gamma' \Rightarrow w_V(\gamma) = w_V(\gamma')$

$\mathtt{Proof}\ =$

$H := \mathcal{I}\mathtt{Homotopic}(\gamma, \gamma') : \mathtt{Homotopy}(I, \mathcal{R}_V, \gamma, \gamma),$

$[1] := \mathtt{HomotopyPreservedByC}(H) : \mathtt{Homotopy}(I, \mathbb{R}^2 \setminus \{0\}, \gamma V, \gamma' V, HV),$

$[2] := \mathtt{HomotopyPreservesWindingNumber}[1] : w(\gamma V) = w(\gamma' V),$

$[*] := \mathcal{I}^{-1} w_V[2] : w_V(\gamma) = w_V(\gamma');$

$\square$

$\texttt{IndexIsWellDefined} :: \forall V \in \mathfrak{X}(2) \,.\, \forall p : \texttt{IsolatedSingularPoint} \,.\, \exists \varepsilon \in \mathbb{R}_{++} : \exists n \in \mathbb{Z} \,.$
$\quad .\, \forall t \in (0, \varepsilon] \,.\, w_V(p + ts) = n$

$\texttt{Proof} =$

$\Big(U, [1]\Big) := \mathcal{O}\texttt{SingularPoint}(2, V, p) : \sum U \in \mathcal{U}(p) \,.\, U \setminus \{p\} \subset \mathcal{R}_V,$

$\Big(\varepsilon, [2]\Big) := \texttt{OpenInMetricSpace}(U, p) : \sum \varepsilon \in \mathbb{R}_{++} \,.\, \mathbb{B}^2(p, \varepsilon) \subset U,$

$[*] := \texttt{HomotopyPreservesVectorFieldsWindingNumber}[1][2] : \forall t \in (0, \varepsilon] \,.\, w_V(p + ts) = w_V(p + \varepsilon s);$

$\square$

$$\texttt{indexOfASingularPoint} :: \prod_{V \in \mathfrak{X}(2)} \texttt{IsolatedSingularPoint}(V) \to \mathbb{Z}$$

$\texttt{indexOfASingularPoint}\,(p) = \operatorname{ind}_V p := w_V(p + \varepsilon s) \quad \texttt{where} \quad \varepsilon = \texttt{IndexIsWellDefined}(V, p)$

$\texttt{IndexOfManyPoints} :: \forall V \in \mathfrak{X}(2) \,.\, \forall \gamma : \texttt{RegularLoop}(V) \,.\, \forall U \in \mathcal{T}(\mathbb{R}^2) \,.\, \forall [0] : \partial U = \operatorname{Im} \gamma \,.$

$$.\, \forall n \in \mathbb{N} \,.\, \forall p : [1, \dots, n] \twoheadrightarrow \mathcal{S}_n \cap U \,.\, w_V(\gamma) = \sum_{i=1}^{n} \operatorname{ind}_V p$$

$\texttt{Proof} =$

Note, that γ is hamotopic to a flower with n petels, each containg one singular point.

Use some some complex analysis and compute the winding number as complex path integral.

$\square$

5.7 Degree Theory Of The Torus

torusDegree :: $\left(\mathbb{T}^2 \xrightarrow{\mathsf{TOP}} \mathbb{T}^2\right) \to \mathbb{Z}^{2\times 2}$

f : torusDegree $\iff D(f) \iff \Lambda i, j \in \{1, 2\}\ .\ \deg \iota_i f \pi_j$

HomotopyPreservesTorusDegree :: $\forall f, f' : \left(\mathbb{T}^2 \xrightarrow{\mathsf{TOP}} \mathbb{T}^2\right) .\ f \sim f' \iff D(f) = D(f')$

Proof =

($\Rightarrow$) If H is a homotopy of tori maps it evently constraints to the homotopy of circle maps $\iota_i f \pi_j$ and $\iota_i f' \pi_j$
By degree theory of the circle $D(f) = D(f')$.

($\Leftarrow$) Use some lifting theory of $\mathbb{T}^2 = \dfrac{\mathbb{R}^2}{\mathbb{Z}^2}$

Note that $D(f) = D(f')$ imply that $f_* = f'_*$.

Assume that $f[0] = f'[0]$ and define $g = f - f'$ with $D(g) = 0$

There is a lift $\tilde{g} : \mathbb{T} \to \mathbb{R}^2$ with $g = \tilde{g}\pi$

Then, there is a homotopy $H(t, x) = \pi(t\tilde{g}(x))$ between g and 0

Hence $f \sim f'$ by topological groop theory

□

ToricDegreeComposition :: $\forall f, g : \mathrm{End}_{\mathsf{TOP}}(\mathbb{T}^2) D(fg) = D(g)D(f)$

Proof =

use properties of homomrphisms f_*, g_*

□

ToricDegreeisSurjective :: Surjective$(\mathbb{T}^2, \mathbb{Z}^{2\times 2} D)$

Proof =

try $f(u, v) = \begin{pmatrix} u^n v^m \\ u^k v^l \end{pmatrix}$

□

ToricHomeeIfInvertibleDegee :: $\forall f \in \mathrm{End}_{\mathsf{TOP}}(\mathbb{T}^2)[f] \in \mathrm{Aut}_{\mathsf{HTOP}}(\mathbb{T}^2) \iff$ Invertibe$(\mathbb{Z}^{2\times 2}, D(f))$

Proof =

use properties of homomrphisms f_*, g_*

□

5.8 Seifert-van-Kampen Theorem

$$\texttt{SeifertVanKampenDecomposition} ::? \left(\sum X : \texttt{PathConnected}\ \&\ .\ \Big(\mathcal{T}(X)\ \&\ \texttt{PathConnected}\Big)^2\right)$$

$$(X,U,V) : \texttt{SeifertVanKampenDecomposition} \iff X = U \cup V\ \&\ \texttt{PathConnected}\ \&\ \texttt{NonEmpty}(U \cap V)$$

$$\texttt{mapOfSeifertVanKampen} :: \prod (X,U,V) : \texttt{SeifertVanKampenDecomposition}\ .\ \pi(U) \sqcup_{\mathsf{GRP}} \pi(V) \xrightarrow{\mathsf{GRP}} \pi(X)$$

$$\texttt{mapOfSeifertVanKampen}\left(\prod_{i=1}^{n}[\alpha_i]_U[\beta_i]_V\right) = \Phi\left(\prod_{i=1}^{n}[\alpha_i]_U[\beta_i]_V\right) := \left[\prod_{i=1}^{n}\alpha_i\beta_i\right]_X$$

$$\texttt{subgroupOfSeifertVanKampen} :: \prod (X,U,V) : \texttt{SeifertVanKampenDecomposition}\ .$$
$$.\ \texttt{Subgroup}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)\Big)$$

$$\texttt{subgroupOfSeifertVanKampen}\,() = \bar{C}(X,U,V) := N\left(\left\langle [\gamma]_U[\gamma]_V^{-1} \middle| [\gamma]_{U\cap V} \in \pi(U \cap V)\right\rangle\right)$$

$$\texttt{SeifertVanKampenLemma1} :: \forall (X,U,V) : \texttt{SeifertVanKampenDecomposition}\ .$$
$$.\ \texttt{Surjective}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V), \pi(X), \Phi_{X,U,V}\Big)$$

$\texttt{Proof}\ =$

$p := \mathcal{A}\texttt{NonEmpty}(U \cap V) \in U \cap V,$

$\texttt{Assume}\ \gamma \in \Omega_X(p),$

$\lambda := \texttt{LebesgueNumberLemma}\Big(I, \big(\gamma^{-1}(U), \gamma^{-1}(V)\big)\Big)\mathcal{A}\texttt{SeifertVanKampenDecomposition}(X,U,V) :$
$: \texttt{LebesgueNumber}\Big(I, \big(\gamma^{-1}(U), \gamma^{-1}(V)\big)\Big),$

$\Big(n, [1]\Big) := \mathcal{A}\texttt{ReductioInfima}(\mathbb{R}, \lambda) : \sum_{n=1}^{\infty}\frac{1}{n} < \lambda,$

$\alpha := \Lambda k \in [1,\dots,n]\ .\ \Lambda t \in I\ .\ \gamma\left(\frac{k-1}{n} + \frac{t}{n}\right) : [1,\dots,n] \to (I \xrightarrow{\mathsf{TOP}} X),$

$[2] := \mathcal{D}\alpha\mathcal{A}\texttt{LebesgueNumber}\Big(I, \big(\gamma^{-1}(U), \gamma^{-1}(V)\big)\Big) : \forall k \in [1,\dots,n]\ .\ \operatorname{Im}\alpha_k \subset U | \operatorname{Im}\alpha_k \subset V,$

$W := \Lambda k \in [1,\dots,n]\ .\ \texttt{if}\ \operatorname{Im}\alpha_k \subset U \cap V\ \texttt{then}\ U \cap V\ \texttt{else if}\ \operatorname{Im}\alpha_k \subset U\ \texttt{then}\ U\ \texttt{else}\ V :$
$: [1,\dots,n] \to \{U, V, U \cap V\},$

$\texttt{Assume}\ k \in [1,\dots,n],$

$[3] := \mathcal{D}W[2](k) : \operatorname{Im}\alpha_k \subset W_k,$

$h_k := \mathcal{A}\texttt{SeifertVanKampenDecomposition}(X,U,V)\mathcal{A}\texttt{PathConnected}(W_k, p, \alpha_i(1))[3] : \Omega_{W_k}(p, \alpha_i(1));$

$\rightsquigarrow h := \mathtt{I}\left(\prod\right) : \prod_{k=1}^{n}\Omega_{W_k}(p, \alpha_k(1)),$

$h_0 := \Lambda t \in I\ .\ p \in \Omega_{U\cap V}(p),$

$\beta := \Lambda i \in [1,\dots,n]\ .\ h_{k-1}\alpha_k h_k^{-1} : [1,\dots,n] \to \Omega_X(p),$

$[3] := \mathcal{D}\beta\mathcal{A}\Pi(X) : [\gamma]_X = \prod_{i=1}^{n}[\beta_i]_X,$

$W' := \Lambda k \in [1,\dots,n]\ .\ \texttt{if}\ \operatorname{Im}\alpha_k \subset U\ \texttt{then}\ U\ \texttt{else}\ V :: [1,\dots,n] \to \{U, V\},$

$\omega := \prod_{i=1}^{n}[\beta_i]_{W'_i} : \pi(U) \sqcup_{\mathsf{GRP}} \pi(V),$

$[\gamma.*] := ⅁\Phi[3] : \Phi(\omega) = [\gamma]_X;$

$\rightsquigarrow [*] := ⅁^{-1}\texttt{Surjective} : \texttt{Surjective}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V), \pi(X)\Big);$

$\square$

$\texttt{SeifertVanKampenLemma2} :: \forall(X, U, V) : \texttt{SeifertVanKampenDecomposition} \ . \ \bar{C} \subset \ker\Phi$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [\gamma]_{U\cap V} \in \pi(U \cap V),$

$[\gamma.*] := ⅁\Phi⅁\texttt{Inverse} : \Phi\left([\gamma]_U[\gamma]_V^{-1}\right) = [\gamma]_X[\gamma]_X^{-1} = e;$

$\rightsquigarrow [*] := ⅁\bar{C} : \bar{C} \subset \ker\Phi;$

$\square$

$\texttt{SeifertVanKampenLemma3} :: \forall(X, U, V) : \texttt{SeifertVanKampenDecomposition} \ .$

$\quad . \ \ker\Phi \subset \bar{C}$

$\texttt{Proof} \ =$

$p := ⅁\texttt{NonEmpty}(U \cap V) \in U \cap V,$

$\texttt{Assume} \ x \in \ker\Phi,$

$\Big(n, \alpha, \beta, [1]\Big) := ⅁\pi(U) \sqcup_{\mathsf{GRP}} \pi(V) : \sum_{n=1}^{\infty}\sum \alpha : n \to \Omega_U(p) \ . \ \sum \beta : n \to \Omega_V(p) \ . \ x = \prod_{i=1}^{n}[\alpha_i]_V[\beta_i]_U,$

$[2] := ⅁ \ker\Phi[1] : \prod_{i=1}^{n}\alpha_i\beta_i \approx_{X,p} p,$

$H := ⅁\texttt{RelativeHomotopic}[2] : \texttt{RelativeHomotopy}\left(I, X, p, \prod_{i=1}^{n}\alpha_i\beta_i, p\right),$

$\lambda := \texttt{LebesgueNumberLemma}\Big(I \times I, \left(H^{-1}(U), H^{-1}(V)\right)\Big)⅁\texttt{SeifertVanKampenDecomposition}(X, U, V) :$

$\quad : \texttt{LebesgueNumber}\Big(I \times I, \left(H^{-1}(U), \mathrm{H}^{-1}(V)\right)\Big),$

$\Big(m, [1]\Big) := ⅁\texttt{ReductioInfima}(\mathbb{R}, \lambda) : \sum_{m=1}^{\infty}\frac{1}{m} < \lambda,$

$S := \Lambda i, j \in [1, \dots, m] \ . \ \left[\frac{i-1}{m}, \frac{i}{m}\right] \times \left[\frac{j-1}{m}, \frac{j}{m}\right] : [1, \dots, m]^2 \to ?[0,1]^2,$

$[3] := \Game S⅁\texttt{LebesgueNumber}\Big(I \times I, \left(H^{-1}(U), H^{-1}(V), \lambda\right) : \forall i, j \in [1, \dots, m] \ . \ H(S_{i,j}) \subset U | H(S_{i,j}) \subset V,$

$v := \Lambda i, j \in [1, \dots, m] \ . \ H\left(\frac{i}{m}, \frac{j}{m}\right) : [1, \dots, m]^2 \to X,$

$\xi := \Lambda i, j \in [1, \dots, m] \ . \ \Lambda t \in I \ . \ H\left(\frac{i-1}{m} + \frac{t}{m}, \frac{j}{m}\right) : [1, \dots, m]^2 \to I \to X,$

$\zeta := \Lambda i, j \in [1, \dots, m] \ . \ \Lambda t \in [1, \dots, m] \ . \ H\left(\frac{i}{m}, \frac{j-1}{m} + \frac{t}{m}\right) : [1, \dots, m]^2 \to I \to X,$

$W := \Lambda i, j \in [1, \dots, m] \ . \ \texttt{if} \ H(S_{i,j}) \subset U \cap W \ \texttt{then} \ U \cap W \ \texttt{else if} \ H(S_{i,j}) \subset U \ \texttt{then} \ U \ \texttt{else} \ V :$

$\quad : [1, \dots, m]^2 \to \{U, V, U \cap V\},$

$[4] := ⅁H\Game\xi : \prod_{i=1}^{n}\alpha_i\beta_i = \prod_{i=1}^{m}\xi_{1,i},$

$\texttt{Assume}\ i, j \in [1, \dots, n],$

$[5] := \mathcal{E} W[3](i, j) : H(S_{i,j}) \subset W_{i,j},$

$h_k := \mathcal{I}\,\texttt{SeifertVanKampenDecomposition}(X, U, V)\mathcal{I}\,\texttt{PathConnected}(W_{i,j}, p, v_{i,j})[5] : \Omega_{W_{i,j}}(p, v_{i,j});$

$$\rightsquigarrow h := \texttt{I}\left(\prod\right) : \prod_{i,j=1}^{m} \Omega_{W_{i,j}}(p, v_{i,j}(1)),$$

$h_0 := \Lambda j \in [0, \dots, m]\ .\ \Lambda t \in I\ .\ p \in [0, \dots, m]\Omega_{U \cap V}(p),$

$\mu := \Lambda i \in [1, \dots, n]\ .\ h_{i-1,j}\xi_{i,j}h_{i,j}^{-1} : [1, \dots, m]^2 \to \Omega_X(p),$

$\nu := \Lambda i \in [1, \dots, n]\ .\ h_{i-1,j}\zeta_{i,j}h_{i,j}^{-1} : [1, \dots, m]^2 \to \Omega_X(p),$

$W' := \Lambda i, j \in [1, \dots, m]^2\ .\ \texttt{if}\ H(S_{i,j}) \subset U\ \texttt{then}\ U\ \texttt{else}\ V :: [1, \dots, m]^2 \to \{U, V\},$

$$[5] := [1][4]\mathcal{E}\mu : x = \prod_{i=1}^{m} [\mu_{1,i}]_{W'_{1,i}},$$

$\texttt{Assume}\ \gamma : \Omega_{U \cap V}(p),$

$[\gamma.*] := \mathcal{I}\,\texttt{Inverse}(\pi(U))\mathcal{I}\,\mathsf{GRP}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)\Big)\mathcal{I}\bar{C} :$

$$: \Big[[\gamma]_V\Big]_{\bar{C}} = \left[\left([\gamma]_U[\gamma]_U^{-1}\right)[\gamma]_V\right]_{\bar{C}} = \left[[\gamma]_U\left([\gamma]_U^{-1}\right)[\gamma]_V\right)\Big]_{\bar{C}} = \Big[[\gamma]_U\Big]_{\bar{C}};$$

$$\rightsquigarrow [6] := \texttt{I}(\forall) : \forall \gamma \in \Omega_{U \cap V}(p)\ .\ \Big[[\gamma]_U\Big]_{\bar{C}} = \Big[[\gamma]_V\Big]_{\bar{C}},$$

$\texttt{Assume}\ k \in [1, \dots, m-1],$

$$\texttt{Assume}\ [7] : [x]_{\bar{C}} = \left[\prod_{i=1}^{n} [\mu_{k,i}]_{W'_{k,i}}\right]_{\bar{C}},$$

$\texttt{Assume}\ i \in [1, \dots, m-1],$

$[8] := \texttt{SquareLemma}\mathcal{E}\xi\mathcal{E}\zeta : \xi_{k,i+1} \approx_{W_{i,j}} \zeta_{k+1,i}\xi_{k+1,i+1}\zeta_{k+1,i+1}^{-1},$

$[i.*] := [8]\mathcal{E}\mu\mathcal{E}\nu : \mu_{k,i+1} \approx_{W_{i,j}} \nu_{k+1,i}\mu_{k,i+1}\nu_{k+1,i+1}^{-1};$

$[8] := \texttt{I}(\forall) : \forall i \in [0, \dots, i]\ .\ \mu_{k,i+1} \approx_{W_{i,j}} \nu_{k+1,i}\mu_{k+1,i+1}\nu_{k+1,i+1}^{-1};$

$[k.*] := [7][8][6]\mathcal{I}\,\texttt{Inverse} :$

$$: [x]_{\bar{C}} = \left[\prod_{i=1}^{n} [\mu_{k,i}]_{W'_{k,i}}\right]_{\bar{C}} = \left[\prod_{i=0}^{n} [\mu_{k,i+1}]_{W'_{k,i+1}}\right]_{\bar{C}} = \left[\prod_{i=0}^{n} [\nu_{k+1,i}]_{W'_{k+1,i}}[\mu_{k+1,i+1}]_{W'_{k+1,i+1}}[\nu_{k+1,i+1}]_{W'_{k+1,i+1}}\right]_{\bar{C}} =$$

$$= \left[\prod_{i=1}^{n} [\mu_{k+1,i}]_{W'_{k+1,i}}\right]_{\bar{C}};$$

$$\rightsquigarrow [7] := \mathcal{I}\,\texttt{Primitive}[1, \dots, m] : \forall k \in [1, \dots, m]\ .\ [x]_{\bar{C}} = \left[\prod_{i=1}^{n} [\mu_{k,i}]_{W'_{k,i}}\right]_{\bar{C}};$$

$$[8] := [7](m) : [x]_{\bar{C}} = \left[\prod_{i=1}^{n} [\mu_{m,i}]_{W'_{m,i}}\right]_{\bar{C}},$$

$[9] := [8]\mathcal{E}\mu\mathcal{I}H : [x]_{\bar{C}} = e,$

$[x.*] := \mathcal{I}\,\texttt{Coset}[9] : x \in \bar{C};$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Subset} : \ker \Phi \subset \bar{C};$

$\square$

$\texttt{SeifertVanKampenTheorem} :: \forall(X,U,V) : \texttt{SeifertVanKampenDecomposition} \;.\; \pi(X) \cong_{\mathsf{GRP}} \dfrac{\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)}{\bar{C}}$

$\texttt{Proof} \;=$

$[1] := \texttt{SeifertVanKampenLemma2} : \hat{C} \subset \ker \Phi,$

$[2] := \texttt{SeifertVanKampenLemma3} : \ker \Phi \subset \hat{C},$

$[3] := \reflectbox{$D$}\texttt{SetEq}[1][2] : \ker \Phi \subset \hat{C},$

$[4] := \texttt{SeifertVanKampenLemma4} : \texttt{Surjective}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V), \pi(X), \Phi\Big),$

$[*] := \texttt{IsomorphismTHM}[3][4] : \pi(X) \cong_{\mathsf{GRP}} \dfrac{\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)}{\bar{C}};$

$\square$

$\texttt{SeifertVanKampenTheorem} :: \forall(X,U,V) : \texttt{SeifertVanKampenDecomposition} \;.\; \pi(X) \cong_{\mathsf{GRP}} \dfrac{\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)}{\bar{C}}$

$\texttt{Proof} \;=$

$[1] := \texttt{SeifertVanKampenLemma2} : \hat{C} \subset \ker \Phi,$

$[2] := \texttt{SeifertVanKampenLemma3} : \ker \Phi \subset \hat{C},$

$[3] := \reflectbox{$D$}\texttt{SetEq}[1][2] : \ker \Phi \subset \hat{C},$

$[4] := \texttt{SeifertVanKampenLemma4} : \texttt{Surjective}\Big(\pi(U) \sqcup_{\mathsf{GRP}} \pi(V), \pi(X), \Phi\Big),$

$[*] := \texttt{IsomorphismTHM}[3][4] : \pi(X) \cong_{\mathsf{GRP}} \dfrac{\pi(U) \sqcup_{\mathsf{GRP}} \pi(V)}{\bar{C}};$

$\square$

$\texttt{SpecialSeifertVanKampenTheorem1} :: \forall(X,U,V) : \texttt{SeifertVanKampenDecomposition} \;.$

$\;.\; \texttt{SimplyConnecte}(U \cap V) \Rightarrow \pi(X) \cong_{\mathsf{GRP}} \pi(U) \sqcup_{\mathsf{GRP}} \pi(V)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SpecialSeifertVanKampenTheorem2} :: \forall(X,U,V) : \texttt{SeifertVanKampenDecomposition} \;.$

$\;.\; \texttt{SimplyConnecte}(V) \Rightarrow \pi(X) \cong_{\mathsf{GRP}} \dfrac{\pi(U)}{N\Big(\iota_{U*}\pi(U \cap V)\Big)}$

$\texttt{Proof} \;=$

$\dots$

$\square$

5.9 Applications to Geometric Topology

$$\texttt{wedgeSum} :: \prod_{\mathcal{I} \in \mathsf{SET}} (I \to \mathsf{TOP}_*) \to \mathsf{TOP}_*$$

$$\texttt{wedgeSum}\,(X) = \bigvee_{i \in \emptyset} X_i := \{p\}$$

$$\texttt{wedgeSum}\,(X) = \bigvee_{i \in \mathcal{I}} X_i := \left(\frac{\bigsqcup_{i \in \mathcal{I}} X_i}{\left\{ (i, \mathrm{pt}\ X_i) \middle| i \in \mathcal{I} \right\}}, [\mathrm{pt}\ X] \right)$$

$$\texttt{NondegenarateBasepoint} :: ?\mathsf{TOP}^*$$

$$(X, p) : \texttt{NondegenerateBasepoint} \iff \exists U \in \mathcal{U}(p)\ .\ \texttt{StrongDeformationRetract}\Big(U, \{p\}\Big)$$

$$\texttt{WedgeProductOpenUnion} :: \forall \mathcal{I} \in \texttt{Set}\ .\ \forall X : \mathcal{I} \to \mathsf{TOP}^*\ .$$

$$.\ \forall U : \prod_{i \in \mathcal{I}} \mathcal{U}\Big(\mathrm{pt}(X_i)\Big)\ .\ \bigcup_{i \in \mathcal{I}} \iota_i q(U_i) \in \mathcal{T}\left(\bigvee_{i \in I} X_i \right)$$

$$\texttt{Proof} =$$

$$q := \mathcal{C}\texttt{wedgeSum}(X)\mathcal{C}\texttt{quotientSpace} : \texttt{QuotienMap}\left(\bigsqcup_{i \in \mathcal{I}} X_i, \bigvee_{i \in \mathcal{I}} X_i \right),$$

$$\mathcal{J} := \Lambda i' in \mathcal{I}\ .\ \mathcal{I} \setminus \{i\} : \mathcal{I} \to ?\mathcal{I},$$

$$[1] := \mathcal{C}\texttt{wedgeSum}(U)\mathcal{C}U\mathcal{O}^{-1}q : \forall i \in \mathcal{I}\ .\ q^{-1}(\iota_i q U_i) = \iota_i U_i \cup \bigcup_{j \in \mathcal{J}_i} \Big\{ \Big(j, \mathrm{pt}(X_j)\Big) \Big\},$$

$$[2] := \texttt{PreimageUnion}(q, U)[1]\mathcal{C}\texttt{Union} :=$$

$$= q^{-1}\left(\bigcup_{i \in \mathcal{I}} \iota_i q U_i \right) = \bigcup_{i \in \mathcal{I}} q^{-1}(\iota_i q U_i) = \bigcup_{i \in \mathcal{I}} \left(\iota_i U_i \cup \bigcup_{j \in \mathcal{J}_i} \Big\{ \Big(j, \mathrm{pt}(X_j)\Big) \Big\} \right) = \bigcup_{i \in \mathcal{I}} \iota_i U_i,$$

$$[*] := \mathcal{C}\texttt{QuotientMap}(q)\mathcal{C}\mathsf{TOP}\left(\bigsqcup_{i \in \mathcal{I}} X_i \right)[2] : \bigcup_{i \in \mathcal{I}} \iota_i q U_i \in \mathcal{T}\left(\bigwedge_{i \in \mathcal{I}} X_i \right);$$

$\square$

$\texttt{NondegenerateWedgeSum} :: \forall \mathcal{I} \in \texttt{Set} . \forall X : \mathcal{I} \to \texttt{NondegenerateBasepoint} .$

$$. \texttt{NondegenerateBasePoint} \left(\bigvee_{i \in \mathcal{I}} X_i \right)$$

$\texttt{Proof} \; =$

$$\Big(U, [1]\Big) := \mathcal{C}\!\!\!I\texttt{NondegenerateBasepoint}(X) : \prod_{i \in \mathcal{I}} \sum U_i \in \mathcal{U}\Big(\mathrm{pt}(X_i)\Big) .$$

$$. \texttt{StrongDeformationRetract}\left(U_i, \Big\{\mathrm{pt}(X_i)\Big\}\right),$$

$$\star := \mathrm{pt}\left(\bigvee_{i \in \mathcal{I}} X_i\right) \in \bigvee_{i \in \mathcal{I}} X_i,$$

$$V := \bigvee_{i \in \mathcal{I}} \Big(U_i, \mathrm{pt}(X_i)\Big) \in \mathcal{U}(\star),$$

$\texttt{Assume} \; v \in V,$

$$[2] := \mathcal{D}V : v = \star | \exists i \in \mathcal{I} . v \in U_i \setminus \Big\{\mathrm{pt}(X_i)\Big\},$$

$\texttt{Assume} \; [3] : v = \star,$

$$G(\bullet, v) := \Lambda t \in I . \star : I \xrightarrow{\mathsf{TOP}} V;$$

$$\rightsquigarrow [3] := \texttt{I}(\Rightarrow) : v = \star \Rightarrow (I \xrightarrow{\mathsf{TOP}} V);$$

$\texttt{Assume} \; i \in \mathcal{I},$

$$\texttt{Assume} \; [4] : v \in U_i \setminus \Big\{\mathrm{pt}(X_i)\Big\},$$

$$H := \mathcal{C}\!\!\!I\texttt{StrongDeformationRetract}\left(U_i, \Big\{\mathrm{pt}(X_i)\Big\}\right) : \texttt{RelativeHomotopy}\left(U_i, \Big\{\mathrm{pt}(X_i)\Big\}, \mathrm{id}, \mathrm{pt}(X_i)\right),$$

$$G(\bullet, v) := H(\bullet, v)\iota_i : I \xrightarrow{\mathsf{TOP}} V;$$

$$\rightsquigarrow [4] := \texttt{I}(\Rightarrow) : v \neq \star \Rightarrow (I \xrightarrow{\mathsf{TOP}} V);$$

$$H(\bullet, v) := \texttt{E}(|)[2,3,4] : I \xrightarrow{\mathsf{TOP}} V;$$

$$\rightsquigarrow H := \texttt{I}(\to) : I \xrightarrow{\mathsf{TOP}} V \xrightarrow{\mathsf{TOP}} V,$$

$$[2] := \mathcal{C}\!\!\!I\texttt{QuotientSpace}\mathcal{D}H : \texttt{RelatriveHomotopy}\Big(V, \{\star\}, \mathrm{id}, \star\Big),$$

$$[*] := \mathcal{C}\!\!\!I^{-1}\texttt{NondegenerateBasePoint} : \texttt{NondegenerateBasePoint}\left(\bigvee_{i \in \mathcal{I}} X_i\right);$$

$\square$

$\texttt{FundamentalGroupOfWedgeSum} :: \forall \mathcal{I} : \texttt{Finite} . \forall X : \mathcal{I} \to \texttt{NondegenerateBasepoint} .$

$$. \pi\left(\bigvee_{i\in\mathcal{I}} X_i\right) = \bigsqcup_{i\in\mathcal{I}} \pi(X_i)$$

$\texttt{Proof} =$

$\texttt{Assume } X, Y : \texttt{NondegenerateBasepoint},$

$\Big(U, [1]\Big) := \mathcal{O}\texttt{NondegenerateBasepoint}(X) : \sum U \in \mathcal{U}\Big(\mathrm{pt}(X)\Big) .$

$\quad . \texttt{StrongDeformationRetract}\Big(U, \Big\{\mathrm{pt}(X)\Big\}\Big),$

$\Big(V, [2]\Big) := \mathcal{O}\texttt{NondegenerateBasepoint}(Y) : \prod_{i\in\mathcal{I}} \sum U_i \in \mathcal{U}\Big(\mathrm{pt}(X_i)\Big) .$

$\quad . \texttt{StrongDeformationRetract}\Big(U_i, \Big\{\mathrm{pt}(X_i)\Big\}\Big),$

$U' := Y \cup U \in \mathcal{T}(X \wedge Y) \ \& \ \texttt{Connected},$

$V' := Y \cup U \in \mathcal{T}(X \wedge Y) \ \& \ \texttt{Connected},$

$[3] := \mathcal{O}^{-1}\texttt{union}\mathcal{D}U'\mathcal{D}V' : U' \cup V' = X,$

$[4] := \mathcal{O}^{-1}\texttt{intersection}\mathcal{D}U'\mathcal{D}V' : U' \cap V' = U \cup V,$

$[5] := [1][2][4] : \texttt{Connected}(U' \cap V'),$

$[6] := \mathcal{O}^{-1}\texttt{SeifertVanKampenDecomposition}[5][3] : \texttt{SeifertVanKampenDecomposition}(X \vee Y, U', V'),$

$[7] := \mathcal{D}U'[1] : \pi(U') \cong_{\mathsf{GRP}} \pi(Y),$

$[8] := \mathcal{D}V'[2] : \pi(V') \cong_{\mathsf{GRP}} \pi(X),$

$[9] := [1][2][4]\mathcal{O}^{-1}\mathcal{O}^{-1}\texttt{SimplyConnected} : \texttt{SimplyConnected}(U' \cap V'),$

$\big[(X, Y).*\big] := \texttt{SpecialSeifertVanKampenTHM1}[6, 7, 8, 9] : \pi(X \vee Y) = \pi(X) \sqcup \pi(Y);$

$\rightsquigarrow [*] := \mathcal{O}\mathbb{N}\texttt{NonDegenerateWedgeSum} :$

$$: \forall \mathcal{I} : \texttt{Finite} . \forall X : \mathcal{I} \to \texttt{NondegenerateBasepoint} . \pi\left(\bigvee_{i\in\mathcal{I}} X_i\right) = \bigsqcup_{i\in\mathcal{I}} \pi(X_i);$$

□

$\texttt{FundamentalGroupOfBuquetOfCircles} :: \forall n \in \mathbb{N} . \pi\Big(\mathbb{S}^{1(\vee n)}\Big) = F_{\mathsf{GRP}}[1, \dots, n]$

$\texttt{Proof} =$

$\dots$

□

$\texttt{CWGraph} :: ?\texttt{FiniteCommplex} \ \& \ \texttt{Connected}$

$C : \texttt{CWGraph} \iff \dim C = 1$

$\texttt{cwGraphReprsentation} :: \texttt{CWgraph} \leftrightarrow \sum X \in \mathsf{TOP} . ?X \times \texttt{Multiset}(X \times X)$

$\texttt{cwGraphRepresentation}\left((X, \mathcal{E}, \varphi)\right) = (X, V, E) := (X, \mathcal{E}_0, \varphi_1(\mathcal{E}_1))$

$\texttt{SelfLoop} :: \prod (X, V, E) : \texttt{CWGraph} . ?E$

$(x, y) : \texttt{SelfLoop} \iff x = y$

$\texttt{MultipleEdge} :: \prod (X, V, E) : \texttt{CWGraph} . ?E$

$(e : \texttt{MultipleEdge} \iff |e|_E > 1$

$$\texttt{SimpleGraph} :: ?\texttt{CWGraph}$$
$$G : \texttt{SimpleGraph} \iff \texttt{SelfLoop}(G) = \texttt{MultipleEdge}(G) = \emptyset$$

$$\texttt{EdgePath} :: \prod G : \texttt{CWGraph}\ ?\sum_{n=1}^{\infty}[1,\ldots,n] \to E_G$$
$$p : \texttt{EdgePath} \iff \forall i \in [1,\ldots,n-1]\ .\ p_{i,2} = p_{i+1}, 1$$

$$\texttt{EdgePath} :: \prod G : \texttt{CWGraph}\ .\ \sum_{n=0}^{\infty}\Big([1,\ldots,n+1] \to V_G) \times \Big([1,\ldots,n] \to E_G\Big)$$
$$(n,v,e) : \texttt{EdgePath} \iff \forall i \in [1,\ldots,n]\ .\ e_{i,1} = v_i \ \&\ e_{i,2} = v_{i+1}$$

$$\texttt{length} :: \prod G : \texttt{CWGraph}\ .\ \texttt{EdgePath}(G) \to \mathbb{Z}_+$$
$$\texttt{length}\,(n,v,e) = \big|(n,v,e)\big| := n$$

$$\texttt{verticePath} :: \prod G : \texttt{CWGraph}\ .\ \prod \gamma : \texttt{EdgePath}(G)\ .\ \Big[1,\ldots,|\gamma|+1\Big] \to V_G$$
$$\texttt{verticePath}\,(i) = v_\gamma^i := \gamma_{2,i}$$

$$\texttt{edgePath} :: \prod G : \texttt{CWGraph}\ .\ \prod \gamma : \texttt{EdgePath}(G)\ .\ \Big[1,\ldots,|\gamma|\Big] \to E_G$$
$$\texttt{edgePath}\,(i) = e_\gamma^i := \gamma_{3,i}$$

$$\texttt{initialVertex} :: \prod G : \texttt{CWGraph}\ .\ \texttt{EdgePath}(G) \to V_G$$
$$\texttt{initialVertex}\,(\gamma) = \operatorname{init}\gamma := v_\gamma^1$$

$$\texttt{terminalVertex} :: \prod G : \texttt{CWGraph}\ .\ \texttt{EdgePath}(G) \to V_G$$
$$\texttt{terminalVertex}\,(\gamma) = \operatorname{init}\gamma := v_\gamma^{|\gamma|+1}$$

$$\texttt{Closed} :: \prod G : \texttt{CWGraph}\ .\ ?\texttt{EdgePath}(G)$$
$$\gamma : \texttt{Closed} \iff \operatorname{init}\gamma = \operatorname{tetm}\gamma$$

$$\texttt{otherIndices} :: \prod G : \texttt{CWGraph}\ .\ \prod \gamma : \texttt{EdgePath}(G)\ .\ [1,\ldots|\gamma|] \to ??[1,\ldots,|\gamma|+1]$$
$$\texttt{otherIndices}\,(1) = \hat{I}_\gamma^1 := \big[2,\ldots,|\gamma|\big]$$
$$\texttt{otherIndices}\,(i) = \hat{I}_\gamma^i := \big[1,\ldots,|\gamma|+1\big] \setminus \{i\}$$

$$\texttt{Simple} :: \prod G : \texttt{CWGraph}\ .\ ?G$$
$$\gamma : \texttt{Simple} \iff \forall i \in \big[1,\ldots,|\gamma|\big]\ .\ \forall j \in \hat{I}_\gamma^i\ .\ v_\gamma^i \neq v_\gamma^j \ \&\ \forall i,j \in \big[1,\ldots,|\gamma|\big]\ .\ i \neq j \Rightarrow e_\gamma^i \neq e_\gamma^j$$

$$\texttt{Cycle} :: \prod G : \texttt{CWGraph}\ .\ \Big(\texttt{Closed}\ \&\ \texttt{Simple}\Big)(G)$$
$$\gamma : \texttt{Cycle} \iff |\gamma| \geq 1$$

$$\texttt{CWTree} :: ?\texttt{CWGraph}$$
$$T : \texttt{CWTree} \iff \texttt{Cycle}(T) = \emptyset$$

$\texttt{CWTreeIsSimplyConnected} :: \forall (X,V,E) : \texttt{CWTree} \;.\; \texttt{SimplyConnected}(X)$

$\texttt{Proof} \;=$

$[1] := \mathit{\Pi}^{-1}\texttt{SimplyConnected} : \texttt{SimplyConnected}(\star),$

$\texttt{Assume}\; n \in \mathbb{N},$

$\texttt{Assume}\; [2] : \forall m \in [1,\dots,n] \;.\; \forall \Gamma : \texttt{CWTree} \;.\; |E_\Gamma| = m \Rightarrow \texttt{SimplyConnected}(\Gamma),$

$\texttt{Assume}\; (X,V,E) : \texttt{CWTree} \;\&\; \texttt{FiniteComplex} \;\&\; \texttt{Connected},$

$\texttt{Assume}\; [3] : |V| = n+1,$

$\texttt{Assume}\; [4] : \forall v \in V \;.\; \deg v > 1,$

$[5] := \mathit{\Pi}\deg[4] : \forall n \in \mathbb{N} \;.\; \exists \gamma : \texttt{EdgePath}(V,E) : |\gamma| = n,$

$[6] := [3][5] : \exists \texttt{Cycle}(G),$

$[4.*] := \mathit{\Pi}\texttt{CWTree}(X,V,E)[6] : \bot;$

$\rightsquigarrow \Big(v,[4]\Big) := \texttt{E}(\bot) : \sum v \in V \;.\; \deg v = 1,$

$\Big(e,[5]\Big) := \mathit{\Pi}\deg v[4] : \sum e \in E \;.\; e_2 = v,$

$[6] := \mathit{\Pi}\deg v[4][5] : \forall f \in E \;.\; f_1 = v | f_2 = v \Rightarrow f = e,$

$w := e_2 \in V,$

$\Gamma := \Big(X \setminus \{v\}, V \setminus \{v\}, E \setminus \{e\}\Big) : \texttt{CWTree},$

$[7] := \mathcal{I}\Gamma[2][6] : \texttt{SimpleConncected}(\Gamma),$

$[n.*] := \texttt{FundamentalGroupOfWedgeSum}\Big((\Gamma,w),(I,w)\Big) : \texttt{SimplyConnected}(X);$

$\rightsquigarrow [*] := \mathit{\Pi}\mathbb{N}[1] : \texttt{SimplyConnected}(X);$

$\square$

$\texttt{SpanningTree} :: \prod (X,V,E) : \texttt{CWGraph} \;.\; ?\texttt{CWTree}$

$(Y,V',E') : \texttt{SpanningTree} \iff Y \subset X \;\&\; V = V' \;\&\; E' \subset E$

$\texttt{SpanningTreeExists} :: \forall(X,V,E) : \texttt{CWGraph} \;.\; \exists\texttt{SpanningTree}(X,V,E)$

$\texttt{Proof} \;=$

$[1] := \text{Ɑ}^{-1}\texttt{SpanningTree} : \texttt{SpanningTree}\Big((\star,\star,\emptyset),(\star,\star,\emptyset)\Big),$

$\texttt{Assume}\; n \in \mathbb{N},$

$\texttt{Assume}\;[2] : \forall(X,V,E) : \texttt{CWGraph} \;.\; |E| < n \Rightarrow \exists\texttt{SpanningTree}(X,V,E),$

$\texttt{Assume}\;(X,V,E) : \texttt{CWGraph},$

$\texttt{Assume}\;[3] : |E| = n,$

$\texttt{Assume}\;[4] : \texttt{CWTree}(X,V,E),$

$[4.*] := \text{Ɑ}^{-1}\texttt{SpanningTree}\text{Ɑ}\texttt{CWTree} : \texttt{SpanningTree}\Big((X,V,E),(X,V,E)\Big);$

$\rightsquigarrow [4] := \texttt{I}(\Rightarrow) : \texttt{CWTree}(X,V,E) \Rightarrow \exists\texttt{SpanningTree}(X,V,E),$

$\texttt{Assume}\;[5] : !\,\texttt{CWTree}(X,V,E),$

$\gamma := \text{Ɑ}\texttt{CWTree}[5] : \texttt{Cycle}(X,V,E),$

$[6] := \text{Ɑ}\texttt{Cycle}(\gamma) : |\gamma| \geq 1,$

$\Gamma := \big(X,V,E \setminus \{e^1_\gamma\}\big) : \texttt{CWGraph},$

$[7] := \text{Ɔ}\Gamma : |E_\Gamma| = n-1,$

$T := [2][7] : \texttt{SpanningTree}(\Gamma),$

$[5.*] := \text{Ɔ}T\text{Ɔ}\Gamma\text{Ɑ}^{-1}\texttt{SpanningTree} : \texttt{SpanningTree}\Big((X,V,E),T\Big);$

$\rightsquigarrow [5] := \texttt{I}(\Rightarrow : !\,\texttt{CWTree}(X,V,E) \Rightarrow \exists\texttt{SpanningTree}(X,V,E),$

$[n.*] := \texttt{E}(|)\texttt{LEM}[4][5] : \exists\texttt{SpanningTree}(X,V,E);$

$\rightsquigarrow [*] := \text{Ɑ}\mathbb{N} : \exists\texttt{SpanningTree}(X,V,E);$

$\square$

$\texttt{FundamentalGroupOfAGraph} :: \forall(X,V,E) : \texttt{CWGraph} \;.\; \pi(X) = F_{\mathsf{GRP}}[1,\ldots,n]$

$\quad \texttt{where} \quad (X',V,E') = \texttt{SpanningTreeExists}(X,V,E), n = |E \setminus E'|$

$\texttt{Proof} \;=$

$[1] := \texttt{CWTreeIsSimplyConnected}(X',V,E') : \texttt{SimplyConnected}(X'),$

$[2] := \text{Ɑ}\texttt{SimplyConnected}[1] : X' \cong_{\mathsf{HTOP}} \star,$

$[3] := [2]\text{Ɑ}X' : X \cong_{\mathsf{HTOP}} \mathbb{S}^{1(\vee n)},$

$[*] := \texttt{FundamentalGroupOfBuquetOfCircles}[3] : \pi(X) = F_{\mathsf{GRP}}[1,\ldots,n];$

$\square$

FundamentalGroupByAttachingADisk $:: \forall X : \texttt{Connected} \, . \, \forall (X', \varphi) : \texttt{ByAttachingNCell}(X, 2) \, .$

$\quad . \, \pi(X') \cong_{\mathsf{GRP}} \dfrac{\pi(X)}{N(\tau)} \quad \texttt{where} \quad \tau = [s\varphi]_X$

Proof $=$

$q := \texttt{quotientMap}(X', \varphi) : \texttt{QuotientMap}\Big(X \sqcup \mathbb{D}^2, X'\Big),$

$U := q\Big(\big(2, \mathbb{D}^2 \setminus \{0\}\big) \sqcup (1, X)\Big) : \mathcal{T}(X'),$

$[1] := \texttt{ConnectedImage}^2(q, \dots)\texttt{ConnectedByItersection} : \texttt{Connected}(U),$

$V := q(2, \mathbb{B}^2)) : \mathcal{T}(X'),$

$[2] := \texttt{ConnectedImage}(q, \mathbb{B}^2) : \texttt{Connected}(V),$

$[3] := \text{Ↄ}U\text{Ↄ}V\text{ɑ}^{-1}\texttt{Union} : U \cup V = X',$

$[4] := \text{Ↄ}U\text{Ↄ}V\text{ɑ}^{-1}\texttt{Intersect} : U \cap V = q(2, \mathbb{B} \setminus \{0\}),$

$[5] := \texttt{ConnectedImage}\big(q, \mathbb{B}^2 \setminus \{0\}\big)[4] : \texttt{Connected}(U \cap V),$

$[6] := \texttt{CPreservesHomotopy}[4] : U \cap V \cong_{\mathsf{HTOP}} \varphi(\mathbb{S}^1),$

$[7] := \text{ɑ}\texttt{ByAttachingNCell}(X, 2, X', \varphi)\texttt{FundamentalGroupIsomorphism} : \varphi\pi\mathbb{S}^1 = \Big\langle [s\varphi] \Big\rangle,$

$[8] := \text{ɑ}\texttt{BuAttacjongNCell}(X, 2, X', \varphi)\text{Ↄ}V : \texttt{SimplyConnected}(V),$

$[9] := \text{ɑ}^{-1}\texttt{SeifertVanKampenDecomposition}[1][2][3][5] : \texttt{SeifertVanKampenDecomposition}(X', U, V),$

$[*] := \texttt{SpecialSeifertVanKampenTHM2}(X', U, V)[7][8] : \pi(X') = \dfrac{\pi(X)}{N(\tau)};$

$\square$

FundamentalGroupByAttachingHigherCell $:: \forall n \in \mathbb{N} \, . \, \forall [0] : n > 2 \, . \, \forall X : \texttt{Connected} \, .$

$\quad . \, \forall (X', \varphi) : \texttt{ByAttachingNCell}(X, n) \, .$

$\quad . \, \pi(X') \cong_{\mathsf{GRP}} \pi(X)$

Proof $=$

...

$\square$

FundamentalGroupOfCWComplex $:: \forall (X, \mathcal{E}, \varphi) : \texttt{Connected} \,\&\, \texttt{FiniteCWComplex} \, . \, \pi(X) = \dfrac{\pi(X^{☠})}{N\Big\{[s\varphi_{2,e}]_X \Big| e \in \mathcal{E}_2\Big\}}$

Proof $=$

...

$\square$

FundamentalGroupByPolygonalPresentation $:: \forall X : \texttt{CompactSurface} \, . \, \forall [0] : X = \mathrm{real}\langle a_1, \dots, a_n | w \rangle \, .$

$\quad . \, \pi(X) = \langle a_1, \dots, a_n | w \rangle_{\mathsf{GRP}}$

Proof $=$

...

$\square$

ClassificationOfCompacttSurfaces2 $:: \forall n, m \in \mathbb{N} \, . \, \mathbb{S}^2 \not\cong_{\mathsf{TOP}} \#_{i=1}^{m} \mathbb{T}^2 \not\cong_{\mathsf{TOP}} \#_{i=1}^{n} \mathbb{RP}^2$

Proof $=$

...

$\square$

$\texttt{EulerCharacteristicIsTopologicalInvariant} :: \forall X, Y : \texttt{CompactSurface} \,.\, \forall [0] : X \cong_{\mathsf{TOP}} Y \,.\, \chi(X) = \chi(Y)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{OrientabilityIsTopologicalInvariant} :: \forall X, Y : \texttt{CompactSurface} \,.\, \forall [0] : X \cong_{\mathsf{TOP}} Y \,.\, \texttt{Orientable}(X) \iff$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$$\texttt{genus} :: \texttt{CompactSurface} \to \mathbb{Z}_+$$

$$\texttt{genus}\left(\left[\mathbb{S}^2\right]\right) = \operatorname{gen} \mathbb{S}^2 := 0$$

$$\texttt{genus}\left(\left[\#_{i=1}^{n} \mathbb{T}^2\right]\right) = \operatorname{gen}\left[\#_{i=1}^{n} \mathbb{T}^2\right] := n$$

$$\texttt{genus}\left(\left[\#_{i=1}^{n} \mathbb{RP}^2\right]\right) = \operatorname{gen}\left[\#_{i=1}^{n} \mathbb{RP}^2\right] := n$$

6 Covering Theory

6.1 Covering Map

$\texttt{EvenlyCovered} :: \prod X, Y \in \mathsf{TOP} \, . \, (X \xrightarrow{\mathsf{TOP}} Y) \to ?\mathcal{T}(Y)$

$U : \texttt{EvenlyCovered} \iff \Lambda f : X \xrightarrow{\mathsf{TOP}} Y \, . \, \exists n \in \mathbb{N} : \exists V : [1, \dots, n] \to \mathcal{T}(X) \, \& \, \texttt{Connected} : f^{-1}(U) = \bigsqcup_{i=1}^{n} V_i \, \&$

$\& \, \forall i \in [1, \dots, n] \, . \, \texttt{Homeo}\Big(f_{|V_i}, U\Big)$

$\texttt{EvenlyCoveredIsConnected} :: \forall X, Y \in \mathsf{TOP} \, . \, \forall f : X \xrightarrow{\mathsf{TOP}} Y \, . \, \forall U : \texttt{EvenlyCovered}(X, Y, f) \, . \, \texttt{Connected}(U)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{CoveringMap} :: \prod X : \texttt{Connected} \, \& \, \texttt{Locally PathConnected} \, . \, \prod B \in \mathsf{TOP} \, .$

$. \, ?\Big(\texttt{Surjective} \, \& \, \texttt{Continuous}\Big)(X, B)$

$f : \texttt{CoveringMap} \iff \forall p \in B \, . \, \exists U \in \mathcal{U}(p) : \texttt{EvenlyConvered}(X, Y, f, U)$

$\texttt{CoveringMapisLocalHomeo} :: \forall f : \texttt{CoveringMap}(X, B) \, . \, f : \texttt{Local Homeo}(X, B)$

$\texttt{Proof} \; =$

$\texttt{Assume} \; x \in X,$

$\Big(U, [1]\Big) := \text{Ɑ}\texttt{CoveringMap}(X, B, f)\big(f(U)\big) : \sum U : \texttt{EvenlyCovered}(X, B, f) \, . \, f(x) \in U,$

$\Big(V, [x.*]\Big) := \text{Ɑ}\texttt{EvenlyCovered}(X, B, f) : \sum V \in \mathcal{U}(x) \, . \, \texttt{Homeomorphism}(V, U, f_{|V});$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{Local Homeomorphism} : \texttt{Local Homeomorphism}(X, B, f),$

$\square$

$\texttt{CoveringMapProduct} :: \forall n \in \mathbb{N} \,.\, \forall X : [1,\dots,n] \to \texttt{Connected} \,\&\, \texttt{Locally PathConnected} \,.$

$\quad . \forall B : [1,\dots,n] \to X \,.\, \forall f : [1,\dots,n] \to \texttt{CoveringMap}(X_i, B_i) \,.\, \prod_{i=1}^n f_i : \texttt{CoveringMap}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n B_i\right)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; p : \prod_{i=1}^n B_i,$

$U := \lambda i \in [1,\dots,n] \,.\, \mathcal{O}\texttt{CoveringMap}(f)(p_i) : \prod_{i=1}^n \texttt{EvenlyCovered}(X_i, B_i, f_i),$

$U' := \prod_{i=1}^n U' \in \mathcal{T}(U'),$

$\Big(\mathcal{I}, V, [1], [2]\Big) := \mathcal{O}\texttt{EvenlyCovered}(X, B, f, U) : \prod_{i=1}^n \sum_{\mathcal{I}_i \in \mathsf{SET}} \sum V_{i,j} : \texttt{Connected} \,\&\, \mathcal{T}(X_i) \,.$

$\quad . \; f_i^{-1}(U_i) = \bigsqcup_{j \in \mathcal{I}_i} V_{i,j} \,\&\, \forall j \in \mathcal{I}_i \,.\, \texttt{Homeomorphism}\Big(V_{i,j}, U_i, f_{i|V_{i,j}}^{|U_i}\Big),$

$V' := \Lambda j : \prod_{i=1}^n \mathcal{I}_i \,.\, \prod_{i=1}^n V_{i,j_i} : \prod_{i=1}^n \mathcal{I}_i \to \mathcal{T}\left(\prod_{i=1}^n X_i\right),$

$[3] := \mathcal{D}U'\texttt{ProductPreImage}[1]\mathcal{D}^{-1}V' : \left(\prod_{i=1}^n f_i\right)^{-1}(U') = \left(\prod_{i=1}^n f_i\right)^{-1}\left(\prod_{i=1}^n U_i\right) = \prod_{i=1}^n f_i^{-1}(U_i) = \prod_{i=1}^n \bigsqcup_{j \in \mathcal{I}_i} V_{i,j} =$

$\quad = \bigsqcup j \in \prod_{i=1}^n \mathcal{I}_j \,.\, V'_j,$

$[4] := \texttt{HomeoProduct}[1] : \forall j \in \prod_{i=1}^n \mathcal{I}_i \texttt{Homeomorphism}\left(V'_j, U', \left(\prod_{i=1}^n f_i\right)_{|V'_j}^{|U'}\right),$

$[p.*] := \mathcal{O}^{-1}\texttt{EvenlyCovered}[3][4] : \texttt{EvenlyCovered}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n B_i, \prod_{i=1}^n f_i, U'\right);$

$\rightsquigarrow [*] := \mathcal{O}^{-1}\texttt{CoveringMap} : \texttt{CoveringMap}\left(\prod_{i=1}^n X_i, \prod_{i=1}^n B_i, \prod_{i=1}^n f_i\right);$

$\square$

$\texttt{CoveringMapHasLocalSection} :: \forall f : \texttt{CoveringMap}(X, B) \,.\, \forall U : \texttt{EvenlyCovered}(X, B, f) \,.$

$\quad . \exists \sigma : \texttt{LocalSection}(U, X, f)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CoveringMapHasNumber} :: \forall f : \texttt{CoveringMap}(X,B) \,.\, \exists n \in \mathsf{CARD} \,.\, \forall p \in B \,.\, \left|f^{-1}(p)\right| = n$

$\texttt{Proof} \ =$

$[1] := \mathcal{C}\texttt{Surjective}(f)\texttt{ConnectedImage} : \texttt{Connected}(B),$

$\texttt{Assume}\ n \in \mathsf{CARD},$

$\texttt{Assume}\ p \in B,$

$\texttt{Assume}\ [2] : \left|f^{-1}(p)\right| = n,$

$U := \mathcal{C}\texttt{CoveringMap}(f)(p) : \texttt{EvenlyCovered}(X,B,f),$

$[p.*] := \mathcal{C}\texttt{EvenlyCovered}(X,B,f,U)[2]\mathcal{C}^{-1}\texttt{card}[2] : \forall u \in U \,.\, \left|f^{-1}(u)\right| = n;$

$\leadsto [n.*] := \texttt{OpenByOpenCover} : \forall n \in \mathsf{CARD} \,.\, \left\{q \in B : \left|f^{-1}(q)\right| = 1\right\} \in \mathcal{T}(B);$

$\leadsto [*] := \mathcal{C}\texttt{Connected}[1] : \exists n \in \mathsf{CARD} \,.\, \forall p \in B \,.\, \left|f^{-1}(p)\right| = n;$

$\square$

$\texttt{coveringNumber} :: \texttt{Covering}(X,B) \to \mathsf{CARD}$

$\texttt{coveringNumber}\,(f) = \mathrm{num}\ f := \texttt{CoveringMapHasNumber}$

$\texttt{HausdorffByCovering} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, \texttt{Hausdorff}(B) \Rightarrow \texttt{Hausdorff}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x, y \in X,$

$\texttt{Assume}\ [1] : x \neq y,$

$\Big(U, [2]\Big) := \texttt{ECoveringMap}\big(X,B,c,c(x)\big) : \sum U : \texttt{EvenlyCovered}(X,B,c) \,.\, c(x) \in U,$

$\Big(V, [3]\Big) := \texttt{ECoveringMap}\big(X,B,c,c(y)\big) : \sum V : \texttt{EvenlyCovered}(X,B,c) \,.\, c(y) \in V,$

$\Big(U', [4]\Big) := \texttt{EEvenlyCovered}\big(X,B,c,x\big)[2] : \sum U' \in \mathcal{U}(x)\ \&\ \texttt{StronglyConnected} \,.\, U \cong_{\mathsf{TOP}} U',$

$\Big(V', [5]\Big) := \texttt{EEvenlyCovered}\big(X,B,c,y\big)[3] : \sum V' \in \mathcal{U}(y)\ \&\ \texttt{StronglyConnected} \,.\, V \cong_{\mathsf{TOP}} V',$

$\texttt{Assume}\ [6] : c(y) = c(x),$

$[6.*] := \texttt{E}U'\texttt{E}V'\texttt{EEvenlyCovered} : U' \cap V' \neq \emptyset;$

$\leadsto [6] := \texttt{I} \Rightarrow : c(y) = c(x) \Rightarrow \exists u \in \mathcal{U}(x) : \exists v \in \mathcal{U}(y) : u \cap v = \emptyset,$

$\texttt{Assume}\ [7] : c(x) \neq c(y),$

$(U'', V'', [8]) := \texttt{EHausdorff}(U \cap V)\big(c(x), c(y)\big) : \sum U'' \in \mathcal{U}(x) \,.\, \sum V'' \in \mathcal{U}(y) \,.$

$\quad U'' \subset U\ \&\ V'' \subset V\ \&\ U'' \cap V'' = \emptyset,$

$[*] := \texttt{DisjointPreimage}\Big(c, U'', V'', [8]\Big) : c^{-1}U'' \cap c^{-1}V'' = \emptyset;$

$\leadsto [7] := \texttt{I} \Rightarrow : c(y) \neq c(x) \Rightarrow \exists u \in \mathcal{U}(x) : \exists v \in \mathcal{U}(y) : u \cap v = \emptyset,$

$\Big[(x,y).*\Big] := \texttt{E}(|)\texttt{LEM}\Big(c(x) = c(y)\Big) : \exists u \in \mathcal{U}(x) : \exists v \in \mathcal{U}(y) : u \cap v = \emptyset;$

$\leadsto * := \texttt{IHausdorff} : \texttt{Hausdorff}(X);$

$\square$

$\texttt{ManifoldByCoveringBase} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, B \in \mathsf{TOPM} \Rightarrow X \in \mathsf{TOPM}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ManifoldByCoveringSpace} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \texttt{Haussdorff}(B) \,\&\, X \in \mathsf{TOPM} \Rightarrow B \in \mathsf{TOPM}$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CoveringRestricion} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall A \subset B \,.\, \texttt{Locally PathConnected}(A) \Rightarrow$

$$\Rightarrow \forall C : \mathrm{PCC}\Big(c^{-1}(A)\Big) \,.\, ?\texttt{CoveringMap}\Big(C, A, c_{|C}\Big)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CoveringInducesCWStructure} :: \forall (B, \mathcal{E}, \varphi) : \texttt{CWComplex} \,.\, \forall c : \texttt{CoveringMap}(X, B) \,.$

$$.\, \exists (Y, \mathcal{F}, \psi) : \texttt{CWComplex} : Y = X \,\&\, (X, \mathcal{F}, \psi) \xrightarrow{\mathsf{CWR}} (B, \mathcal{E}, \varphi)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CoveringByRegularity} :: \forall X, Y : \texttt{StronglyConnected} \ \& \ \mathsf{CG} \ \& \ \texttt{T2} \ . \ \forall X \xrightarrow{f} Y : \mathsf{CG} \ .$

$\quad . \ \texttt{Local Homeomorphism}(X, Y, f) \Rightarrow \texttt{Covering}(X, Y, f)$

$\texttt{Proof} \ =$

$[1] := \texttt{ClosedMapLemma}(X, Y, f)\texttt{EClosedMap}(X, Y, f)(X) : \texttt{Closed}\Big(Y, f(X)\Big),$

$[2] := \texttt{LocalHomeoIsOpen}(X, Y, f)\texttt{EOpenMap}(X, Y, f)(X) : \texttt{Open}\Big(Y, f(X)\Big),$

$[3] := \texttt{EConnected}(X)\texttt{NonEmptyImage} : f(X) \neq \emptyset,$

$[4] := \texttt{EConnected}(Y)[1, 2, 3] : f(X) = Y,$

$\texttt{Assume} \ y \in Y,$

$[5] := \texttt{EProperMap}(X, y, f)\{y\} : \texttt{CompactSubset}\Big(X, f^{-1}(y)\Big),$

$[6] := \texttt{Eimage}[4] : f^{-1}(y) \neq \emptyset,$

$\Big(U', [7]\Big) := \texttt{ELocalHomeo}(X, Y, f)\Big(f^{-1}(y)\Big) :$

$\quad : \sum U' : \prod_{x \in f^{-1}(y)} \mathcal{U}(x) \ . \ \forall x \in f^{-1}(y) \ . \ f(U'_x) \in \mathcal{T}(Y) \ \& \ f_{|u'_x} : \texttt{Homeomorphism}\Big(U', f(U')\Big),$

$\texttt{Assume} \ [8] : \Big|f^{-1}(y)\Big| = \infty,$

$\Big(\mathcal{X}, [9]\Big) := \texttt{ECompactSubset}\Big(f^{-1}(y)\Big)(U') : \sum \mathcal{X} : \texttt{Finite}\Big(f^{-1}(y)\Big) \ . \ \texttt{OpenCover}\Big(f^{-1}(y), U'_{\mathcal{X}}\Big),$

$\Big(x, x', [10]\Big) := \texttt{DirichletPrinciple}[8][9] : \sum x \in \mathcal{X} \ . \ \sum x' \in f^{-1}(y) : x \neq x' \ \& \ x \in U'_x,$

$[11] := \texttt{EpreimageE}x\texttt{E}x' : f(x) = y = f(x'),$

$[12] := [7](x)\texttt{EHemeomorphisn}[10] : f(x) \neq f(x'),$

$[8.*] := \texttt{I}(\bot)[11][12] : \bot;$

$[8] := \texttt{E}(\bot) : \Big|f^{-1}(y)\Big| < \infty,$

$\Big(U, [9]\Big) := \texttt{EStronglyConnected}(X)(U' : \sum U : \prod_{x \in f^{-1}(y)} \mathcal{U}(x) \ \& \ \texttt{StronglyConnected} \ . \ \forall x \in f^{-1}(y) \ . \ U_x \subset U'_x,$

$V := \bigcap_{x \in f^{-1}(y)} f(U_x) \in \mathcal{U}(y),$

$[y.*] := \texttt{E}V\texttt{IEvenlyCovered} : \texttt{EvenlyCovered}(X, Y, f, V);$

$\rightsquigarrow [*] := \texttt{ICoveringMap} : \texttt{CoveringMap}(X, Y, f);$

$\square$

$\texttt{CoveringIsProperIffFinite} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \texttt{ProperMap}(X, B, c) \iff \text{num}\ c < \infty$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{ProperMap}(X, B, c),$

$\texttt{Assume}\ p \in B,$

$[2] := \texttt{EProperMap}(X, B, c)\{p\} : \texttt{CompactSubset}\Big(X, f^{-1}(p)\Big),$

$[3] := \texttt{ECoveringMap}\Big(X, B, c)\{p\} : \texttt{DiscreteSubset}\Big(X, c^{-1}(p)\Big),$

$[1.*] := \texttt{DiscreteCompactIsFinite}[2][3] : \Big|c^{-1}(p)\Big| < \infty;$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : \texttt{Proper}(X, B, c) \Rightarrow \text{num}\ c < \infty,$

$\texttt{Assume}\ [2] : \text{num}\ c < \infty,$

$n := \text{num}\ c \in \mathbb{N},$

$[3] := \texttt{DiscreteCompactIsFiniteE}(\text{num}\ c)[2] : \forall p \in B \,.\, \texttt{CompactSubset}\Big(X, c^{-1}(p)\Big),$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$\Big(U, [4]\Big) := \texttt{E}(\text{num}\ c)[2] : \sum U : [1, \ldots, n] \to \mathcal{T}(X) \,.\, A \subset \bigcup_{i=1}^{n} \ \&\ \forall i \in [1, \ldots, n] \,.\, \texttt{Homeomorphism}(U_i, c(U_i), c_{|U_i}),$

$[5] := \texttt{ClosedSubset}(X, U, A)\texttt{EHomeomorphism}[4] : \forall i \in [1, \ldots, n] \,.\, \texttt{Closed}\Big(f(U_i), c(U_i \cap A)\Big),$

$[6] := [4]\texttt{ImageUnion} : c(A) = c\left(\bigcup_{i=1}^{n} A \cap U_i\right) = \bigcup_{i=1}^{n} c(A \cap U_i),$

$[A.*] := \texttt{ClosedUnion}[5][6] : \texttt{Closed}\Big(B, c(A)\Big);$

$\rightsquigarrow [4] := \texttt{IClosedMap} : \texttt{ClosedMap}\Big(X, c(A)\Big),$

$[2.*] := \texttt{ProperByCompactFibers}[3][4] : \texttt{ProperMap}(X, B, c);$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : \text{num}\ c < \infty \Rightarrow \texttt{Proper}(X, B, c),$

$\rightsquigarrow [*] := \texttt{I}(\iff)[1][2] : \texttt{Proper}(X, B, c) \iff \text{num}\ c < \infty,$

$\square$

$\texttt{ProperMapCompact} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \texttt{Compact}(X) \iff \text{num}\ c < \infty\ \&\ \texttt{Compact}(B)$

$\texttt{Proof} =$

$\ldots$

$\square$

6.2 Lifting

$\texttt{Lift} :: \prod c : \texttt{CoveringMap}(X,B) \,.\, \prod Y \in \mathsf{TOP} \,.\, (Y \xrightarrow{\mathsf{TOP}} B) \to ?(Y \xrightarrow{\mathsf{TOP}} X)$

$g : \texttt{Lift} \iff \Lambda f : Y \xrightarrow{\mathsf{TOP}} B \,.\, f = gc$

$\texttt{UniqueLiftingProperty} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, \forall Y : \texttt{Connected} \,.\, \forall f : Y \xrightarrow{\mathsf{TOP}} X \,.\, \forall g, g' : \texttt{Lift}(c,f) \,.$
$\quad . \forall y \in Y \,.\, g(y) = g'(y) \Rightarrow g = g'$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{HomotopyLiftingProperty} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, \forall Y : \texttt{Locally Connected} \,.\, \forall f, f' : Y \xrightarrow{\mathsf{TOP}} B \,.$
$\quad . \forall H : \texttt{Homotopy}(Y,B,f,f') \,.\, \forall g : \texttt{Lift}(c,f) \,.\, \exists! g' : \texttt{Lift}(c,g') : \exists! G : \texttt{Homotopy}(Y,X,g,g') : Gc = H$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{PathLifitngProperty} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, \forall \gamma : I \xrightarrow{\mathsf{TOP}} B \,.\, \forall x \in c^{-1}\Big(f(0)\Big) \,.$
$\quad . \exists! \xi : \texttt{Lift}(c,\gamma) : \xi(0) = x$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{pathLift} :: \prod c : \texttt{CoveringMap}(X,B) \,.\, \prod I \xrightarrow{\gamma} B : \mathsf{TOP} \,.\, c^{-1}\Big(\gamma(0)\Big) \to \texttt{Lift}(c,\gamma)$

$\texttt{pathLift}\,(x) = \tilde{\gamma}_x := \texttt{PathLifitingProperty}(c,\gamma,x)$

$\texttt{MonodromyTHM1} :: \forall c : \texttt{CoveringMap}(X,B) \,.\, \forall p, q \in B \,.\, \forall \alpha, \beta \in \Omega(p,q) \,.\, \forall x \in c^{-1}(p) \,.\, \tilde{\alpha}_x \sim \tilde{\beta}_x \iff \alpha \sim \beta$

$\texttt{Proof} =$

$[1] := \mathit{Ɑ}\texttt{Lift}(c,\alpha)\mathit{Ɑ}\tilde{\alpha}_x : \tilde{\alpha}_x c = \alpha,$

$[2] := \mathit{Ɑ}\texttt{Lift}(c,\beta)\mathit{Ɑ}\tilde{\beta}_x : \tilde{\beta}_x c = \beta,$

$\texttt{Assume}\ [3] : \tilde{\alpha}_x \sim \tilde{\beta}_x,$

$[1.*] := \texttt{CPreservesHomotopy}[1,2,3] : \alpha \sim \beta;$

$\rightsquigarrow [3] := \texttt{I}(\Rightarrow) : \tilde{\alpha}_x \sim \tilde{\beta}_x \Rightarrow \alpha \sim \beta,$

$\texttt{Assume}\ [4] : \alpha \sim \beta,$

$H := \mathit{Ɑ}\texttt{Homotopic}[4] : \texttt{Homotopy}(I,B,\alpha,\beta),$

$\Big(\gamma, G, [5]\Big) := \texttt{HomotopyLiftingProperty}(c,H,\tilde{\alpha}_x) : \sum \gamma : \texttt{Lift}(c,\gamma) \,.\, \sum G : \texttt{Homotopy}(I,X,\tilde{\alpha}_x,\gamma) \,.$
$\quad . H = Gc,$

$[6] := \mathit{Ɑ}\texttt{CoveringMap}[5] : \gamma(0) = \tilde{\beta}_x(0),$

$[7] := \texttt{UniqueLifitingProperty}[6] : \gamma = \tilde{\beta}_x,$

$4.* := \mathit{Ɑ}^{-1}\texttt{Homotopic}(G)[7] : \tilde{\alpha}_x \sim \tilde{\beta}_x;$

$\rightsquigarrow [4] := \texttt{I}(\Rightarrow) : \tilde{\alpha}_x \sim \tilde{\beta}_x \Rightarrow \alpha \sim \beta,$

$\rightsquigarrow [5] := \texttt{I}(\iff)[4][5] : \tilde{\alpha}_x \sim \tilde{\beta}_x \Rightarrow \alpha \sim \beta,$

$\square$

$\texttt{MonodromyTHM2} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall p, q \in B \,.\, \forall \alpha, \beta \in \Omega(p, q) \,.\, \forall x \in c^{-1}(p) \,.\, \alpha \sim \beta \Rightarrow \tilde{\alpha}_x(1) = \tilde{\beta}_x(1)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{InjectivityTheorem} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall x \in X \,.\, \texttt{Injective}\Big(\pi(X, x), \pi\big(B, c(x)\big), c_*\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; \alpha, \beta \in \Omega(x),$

$[1] := \mathit{\Pi}\texttt{Lift}\big(c, c_*(\alpha)\big)\mathit{\Pi}\widetilde{c(\alpha)}_x : \widetilde{c_*(\alpha)}_x c = c_*(\alpha),$

$[2] := \mathit{\Pi}\texttt{Lift}\big(c, c_*\big(\beta\big)\big)\mathit{\Pi}\widetilde{c(\beta)}_x : \widetilde{c_*\beta}_x c = c_*(\beta),$

$[3] := \mathit{\Pi} c_* \mathit{\Pi}\texttt{pathLift}(\alpha)\texttt{UniqueLiftingProperty}[1] : \widetilde{c_*\alpha}_x = \alpha,$

$[4] := \mathit{\Pi} c_* \mathit{\Pi}\texttt{pathLift}(\beta)\texttt{UniqueLiftingProperty}[2] : \widetilde{c_*\beta}_x = \beta,$

$\texttt{Assume}\; [5] : c_*\alpha \sim c_*\beta,$

$[6] := \texttt{MonodromyTHM1}(c, \alpha, \beta, x)[3] : \widetilde{c_*\alpha}_x \sim \widetilde{c_*\beta}_x,$

$[5.*] := \texttt{E}^2(=)\Big([3], [4]\Big)[6] : \alpha \sim;$

$\rightsquigarrow [*] := \mathit{\Pi}^{-1}\texttt{Injective} : \texttt{Injective}\Big(\pi(X, x), \pi\big(B, c(x)\big), c_*\Big);$

$\square$

$\texttt{coveringInducedSubgroup} :: \prod c : \texttt{CoveringMap}(X, B) \,.\, \texttt{Subgroup}\Big(\pi(B)\Big)$

$\texttt{coveringInducedSubgroup}\,() = \pi(c) := c_*\pi(X)$

$\texttt{LiftingCriterion} :: \forall c : \texttt{CoverinngMap}(X, B) \,.\, \forall Y : \texttt{StronglyConnected} \,.\, \forall Y \xrightarrow{f} B : \mathsf{TOP} \,.\, \forall y \in Y \,.$

$\quad .\, \forall x \in X \,.\, \forall c(x) = f(y) \,.\, \Big(\exists f' : \texttt{Lift}(c, f) : f'(y) = x\Big) \iff f_*\pi(Y, y) \subset \pi(c)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; f' : \texttt{Lift}(c, f),$

$\texttt{Assume}\; [1] : f'(y) = y,$

$[2] := \mathit{\Pi}\texttt{Lift}(c, f, f') : f = f'c,$

$[3] := \mathit{\Pi}\texttt{Covariant}(\pi)[2] : f_* = f'_* c_*,$

$[f'.*] := \texttt{ImageComposition}[3]\mathit{\Pi}^{-1}\pi(c) : f_*\pi(Y, y) \subset \pi(c);$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : \Big(\exists f' : \texttt{Lift}(c, f) : f'(y) = x\Big) \Rightarrow f_*\pi(Y, y) \subset \pi(c),$

$\texttt{Assume}\; [2] : f_*\pi(Y, y) \subset \pi(c),$

$\texttt{Assume}\; u \in Y,$

$\gamma := \mathit{\Pi}\texttt{PathConnected}(Y)(y, u) \in \Omega(y, u),$

$f'(y) := \widetilde{f_*\gamma}_x(1) \in X,$

$\texttt{Assume}\; \delta \in \Omega(y, u),$

$[3] := [2](\delta\gamma^{-1}) : f_*\big[\delta\gamma^{-1}\big] \in \pi(c),$

$\Big(\xi, [4]\Big) := \mathit{\Pi}\pi(c)[3] : \sum \xi \in \Omega(x) \,.\, f_*\Big[(\delta\gamma^{-1})\Big] = c_*\Big[\xi\Big],$

$[5] := \mathit{\Pi}\mathsf{GRP}(\pi(Y) \xrightarrow{f_*} \pi(B))\mathit{\Pi}\pi : f(\delta)f^{-1}(\gamma) \cong_x c(\xi),$

$[6] := [5]f(\gamma) : c(\xi)f(\gamma) = f(\delta),$

$[u.*] := \texttt{MonodromyTHM2}[6] \mathcal{A}\texttt{pathComposition} : \widetilde{f_* \delta}_x(1) = (\widetilde{c_* \xi)(f_* \gamma})_x(1) = \widetilde{f_* \gamma}_x(1);$

$\rightsquigarrow Y \xrightarrow{f'} X := \texttt{WellDefine} : \mathsf{SET},$

$\texttt{Assume}\ V : \texttt{PathConnsectedSubset}(Y),$

$\texttt{Assume}\ u, v \in V,$

$\gamma := \mathcal{A}\texttt{PathConnected}(Y)(y, u) \in \Omega(y, u),$

$\delta := \mathcal{A}\texttt{PathConnected}(V)(u, v) \in \Omega(u, v),$

$[4] := \mathcal{O} f : f'_*(\gamma\delta) = (\widetilde{f_* \gamma)(f_*}\delta)_x,$

$[V.*] := \mathcal{A}\texttt{pathComposition}\mathcal{A}\texttt{mapping} : f'_* \delta \in \Omega(f'(u), f'(v));$

$\rightsquigarrow [4] := \texttt{I}(\forall) : \forall V : \texttt{PathConnectedSubset}(Y) \ . \ \texttt{PathConnectedSubset}\Big(X, f(V')\Big),$

$\texttt{Assume}\ u \in Y,$

$\Big(V, [5]\Big) := \mathcal{A}\texttt{CoveringMap}(X, B, c)\Big(f(u)\Big) : \sum V : \texttt{EvenlyCovered}\big(X, B, c\big) \ . \ f(u) \in V,$

$\Big(W, \sigma, [6]\Big) := \texttt{EvenlyCoveredHasLoclaSection} : \sum (W, \sigma) : \texttt{LocalSection}(c) \ . \ f'(u) \in U,$

$\Big(U, [7]\Big) := \mathcal{A}\texttt{LocallyPathConnected}\Big(Y, f^{-1}(V)\Big)(u) : \sum U \in \mathcal{U}(u)\ \&\ \texttt{PathConnected}(U) \ . \ U \subset f^V,$

$[8] := [4](U) : \texttt{PathConnected}\Big(f'(U)\Big),$

$[9] := [4](U)\mathcal{O} f' : f'(U) \subset c^{-1}(W),$

$[10] := [9]\mathcal{A} U : f'(U) \subset W,$

$[11] := \mathcal{A}\texttt{LoclalSection}(\sigma)[10] : \forall v \in U \ . \ f's(u) = f'(u) = f\sigma s(u),$

$[12] := \mathcal{A}\texttt{InjectionRightInversion}[11] : \forall v \in U \ . \ f'(u) = f\sigma(s),$

$[u.*] := \mathcal{A}\mathsf{CAT}(\mathsf{TOP})(f, \sigma)\texttt{E}(=)[12] : f'_{|U} \in \mathsf{TOP}(U, X);$

$\rightsquigarrow [2.*] := \texttt{LocallyContinuousIsContinuous} : f' \in \mathsf{TOP}(Y, X);$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : f_* \pi(Y, y) \subset \pi(c) \Rightarrow \Big(\exists f' : \texttt{Lift}(c, f) : f'(y) = x\Big),$

$[*] := \texttt{I}(\iff)[2] : \Big(\exists f' : \texttt{Lift}(c, f) : f'(y) = x\Big) \iff f_* \pi(Y, y) \subset \pi(c);$

$\square$

$\texttt{SimplyLifting} :: \forall c : \texttt{CoverinngMap}(X, B) \ . \ \forall Y : \texttt{StronglyConnected}\ \&\ \texttt{SimplyConnected} \ .$

$\quad . \ \forall Y \xrightarrow{f} B : \mathsf{TOP} \ . \ \forall y \in Y \ . \ \forall x \in X \ . \ \forall c(x) = f(y) \ . \ \exists f' : \texttt{Lift}(c, f) : f'(y) = x$

$\texttt{Proof}\ =$

$\ldots$

$\square$

6.3 Transitive Group Action

$\texttt{Transitive} :: \prod G \in \mathsf{GRP} \,.\, \prod X \in \mathsf{SET} \,.\, ?(X \curvearrowleft G)$

$\alpha : \texttt{Transitive} \iff \forall x, y \in X \,.\, \exists g \in G \,.\, \alpha(x, g) =$

$\texttt{TransitiveGSetStabilizerAction} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall \alpha : X \curvearrowleft G \,.$
$\quad \forall x \in X \,.\, \forall g \in G \,.\, \mathrm{Stab}(xg) = g^{-1}\mathrm{Stab}(x)g$

$\texttt{Proof} =$

$\texttt{Assume}\ h : \mathrm{Stab}(xg),$

$[1] := \mathcal{C}\mathrm{Stab}(xg)h : xgh = xg,$

$[2] := [1]g^{-1} : xghg^{-1} = x,$

$[3] := \mathcal{C}\mathrm{Stab}(x)[2] : ghg^{-1} \in \mathrm{Stab}(x),$

$[4] := g^{-1}[3]g : h \in g^{-1}\mathrm{Stab}(x)g;$

$\leadsto [1] := \mathcal{C}^{-1}\texttt{Subset} : \mathrm{Stab}(xg) \subset g^{-1}\mathrm{Stab}(x)g,$

$\texttt{Assume}\ h : g^{-1}\mathrm{Stab}(x)g,$

$\big(f, [2]\big) := \mathcal{C}\texttt{Coset}\mathcal{C}h : \sum f \in \mathrm{Stab}(x) \,.\, h = g^{-1}fg,$

$[3] := \texttt{E}(=)\big(xgh, [2]\big)\mathcal{C}\texttt{Inverse}(G, d)\mathcal{C}\mathrm{Stab}(x)(f) : xgh = xgg^{-1}fg = xfg = xg,$

$[4] := \mathcal{C}\mathrm{Stab}(xg)[3] : g \in \mathrm{Stab}(xg);$

$\leadsto [2] := \texttt{I} \subset : g^{-1}\mathrm{Stab}(x)g \subset \mathrm{Stab}(xg),$

$[*] := \mathcal{C}^{-1}\texttt{SetEq} : \mathrm{Stab}(xg) = g^{-1}\mathrm{Stab}(x)g;$

$\square$

$\texttt{StabIsAGMap} :: \forall X : G\text{-}\mathsf{SET} \,.\, X \xrightarrow{\mathrm{Stab}_X} \Gamma_G : G\text{-}\mathsf{SET}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{StabAreOrbit} :: \forall \alpha : \texttt{Transitive}(G, X) \,.\, \left\{\mathrm{Stab}_\alpha(x) \middle| x \in X\right\} \in O_{\Gamma_G}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{isotropyType} :: \texttt{Transitive}(G, X) \to O_{\Gamma_G}$

$\texttt{isotropyType}\,(\alpha) = \mathrm{type}\ \alpha := \left\{\mathrm{Stab}_\alpha(x) \middle| x \in X\right\}$

GMapsBetweenTransitiveAreDeterminedByOnePoint ::

$:: \forall G \in \mathsf{GRP} \,.\, \forall X : \mathtt{Transitive}(G) \,.\, \forall Y : G\text{-}\mathsf{SET} \,.\, \forall X \xrightarrow{\alpha,\beta} Y : G\text{-}\mathsf{SET} \,.\, \forall x \in X \,.$

$. \; \alpha(x) = \beta(x) \Rightarrow f = g$

Proof $=$

Assume $v \in X$,

$\Big(g, [1]\Big) := \mathcal{A}\mathtt{Transitive}(G, X)(x, v) : \sum g \in G \,.\, xg = v,$

$[v.*] := \mathtt{E}(=)\Big([1], \alpha(v)\Big)\mathcal{A} X \xrightarrow{\alpha} Y : G\text{-}\mathsf{SET}[0]\mathcal{A} X \xrightarrow{\beta} Y : G\text{-}\mathsf{SET}\mathtt{E}(=)([1]) :$

$= \alpha(v) = \alpha(xg) = \alpha(x)g = \beta(x)g = \beta(xg) = \beta(v);$

$\rightsquigarrow [*] := \mathtt{I}(=, \rightarrow) : \alpha = \beta;$

$\square$

GMapsBetweenTransitiveAreSujective ::

$:: \forall G \in \mathsf{GRP} \,.\, \forall X \in G\text{-}\mathsf{SET} \;\&\; \mathtt{Nonempty} \,.\, \forall \beta : \mathtt{Transitive}(G, Y) \,.\, \forall X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} \,.\, \mathtt{Surjective}(X, Y, f)$

Proof $=$

$y := f(x) \in Y,$

Assume $u \in Y$,

$\Big(g, [1]\Big) := \mathcal{A}\mathtt{Transitive}(G, Y)(y, u) : \sum g \in G \,.\, yg = u,$

$[u.*] := [1]\mathcal{O}y\mathcal{A} X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} : u = yg = f(x)g = f(xg);$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\mathtt{Surjective} : \mathtt{Surjective}(X, Y, f);$

$\square$

$\texttt{ExistanceOfTransitiveGMap} :: \forall G \in \mathsf{GRP} \;.\; \forall X : \texttt{Transitive}(G) \;.\; \forall Y : \texttt{Transitive}(G) \;.$

$\;.\; \forall x \in X \;.\; \forall y \in Y \;.\; \left(\exists X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} : f(x) = y\right) \iff \mathrm{Stab}(x) \subset \mathrm{Stab}(y)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; f \in G\text{-}\mathsf{SET}(X, Y),$

$\texttt{Assume}\; [1] : f(x) = y,$

$\texttt{Assume}\; g \in \mathrm{Stab}(x),$

$[2] := \mathcal{O}\mathrm{Stab}(x)(g) : xg = x,$

$[3] := [1]\texttt{E}(=)[2]\mathcal{O}G\text{-}\mathsf{SET}(X, Y)\texttt{E}(=)[1] : y = f(x) = f(xg) = f(x)g = yg,$

$[g.*] := \mathcal{O}\mathrm{Stab}(y)[3] : g \in \mathrm{Stab}(y);$

$\rightsquigarrow [f.*] := \mathcal{O}^{-1}\texttt{Subset} : \mathrm{Stab}(x) \subset \mathrm{Stab}(y);$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : \left(\exists X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} : f(x) = y\right) \Rightarrow \mathrm{Stab}(x) \subset \mathrm{Stab}(y),$

$\texttt{Assume}\; [2] : \mathrm{Stab}(x) \subset \mathrm{Stab}(y),$

$\texttt{Assume}\; v \in X,$

$\left(g, [3]\right) := \mathcal{O}\texttt{Transitive}(G, X)(x, v) : \sum g \in G \;.\; xg = v,$

$f(v) := yg \in Y,$

$\texttt{Assume}\; h \in G,$

$\texttt{Assume}\; [4] : xh = v,$

$[5] := [3]g^{-1}\texttt{E}(=)[4] : x = vg^{-1} = xhg^{-1},$

$[6] := \mathcal{O}\texttt{Stab}(x)[5] : hg^{-1} \in \mathrm{Stab}(x),$

$[7] := \mathcal{O}\texttt{Subset}([2])[6] : hg^{-1} \in \mathrm{Stab}(y),$

$[x.*] := \texttt{E}(=)\left(\mathcal{O}\mathrm{Stab}(y)[7], yg\right)\mathcal{O}\texttt{Inverse}(G)(g) : yg = yhg^{-1}g = yh;$

$\rightsquigarrow f := \texttt{WellDefined} : X \to Y,$

$\texttt{Assume}\; v \in X,$

$\texttt{Assume}\; g \in G,$

$\left(h, [3]\right) := \mathcal{O}\texttt{Transitive}(G, X)(x, vg) : \sum h \in G \;.\; xh = vg,$

$[4] := [3]g^{-1} : v = xhg^{-1},$

$[5] := \mathcal{D}f : f(v) = yhg^{-1},$

$[v.*] := \mathcal{D}f(vg)[3]\mathcal{O}\texttt{Inverse}(G)(g)\mathcal{O}\texttt{Identity}(G)\texttt{E}(=)[5] : f(vg) = yh = yhg^{-1}g = f(v)g;$

$\rightsquigarrow [3] := \mathcal{O}G\text{-}\mathsf{SET} : f \in G\text{-}\mathsf{SET}(X, Y),$

$[2.*] := \mathcal{D}f(x) : f(x) = y;$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow : \mathrm{Stab}(x) \subset \mathrm{Stab}(y) \Rightarrow \left(\exists X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} : f(x) = y\right),$

$[*] := \texttt{I} \iff [1][2] : \mathrm{Stab}(x) \subset \mathrm{Stab}(y) \iff \left(\exists X \xrightarrow{f} \beta : G\text{-}\mathsf{SET} : f(x) = y\right);$

$\square$

$\texttt{GSetInversion} :: \forall G \in \mathsf{GRP} \,.\, \forall X \xrightarrow{f} Y : G\text{-}\texttt{Set} \,.\, \forall [0] : \texttt{Bijiection}(X, Y, f) \,.\, Y \xrightarrow{f^{-1}} X : G\text{-}\texttt{Set}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ y \in Y,$

$\Big(x, [1]\Big) := \mathcal{I}\texttt{Bijection} : \sum x \in X \,.\, y = f(x),$

$\texttt{Assume}\ g \in G,$

$[y.*] := \texttt{E}(=)\Big([1], f^{-1}(yg)\Big)\mathcal{I}\texttt{Inverse}(f) : f^{-1}(yg) = f^{-1}\Big(f(x)g\Big) = ff^{-1}(xg) = xg;$

$\rightsquigarrow [*] := \mathcal{I} G\text{-}\mathsf{SET} : f^{-1} \in G\text{-}\mathsf{SET}(Y, X);$

$\square$

$\texttt{GSetIsomorphismExistance} :: \forall G \in \mathsf{GRP} \,.\, \forall X, Y : \texttt{Transitive}(G) \,.\, \forall x \in X \,.\, \forall y \in Y \,.$

$\quad . \Big(\exists X \xleftrightarrow{f} Y : G\text{-}\mathsf{SET} : f(x) = y\Big) \iff \mathrm{Stab}(x) = \mathrm{Stab}(y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \texttt{Isomorphism}(G\text{-}\mathsf{SET}, X, Y),$

$\texttt{Assume}\ [1] : f(x) = y,$

$[2] := \texttt{ExistanceOfTransitiveGMap}\Big(f, [1]\Big) : \mathrm{Stab}(x) \subset \mathrm{Stab}(y),$

$[3] := \texttt{ExistanceOfTransitiveGMap}\Big(f^{-1}, [1]\Big) : \mathrm{Stab}(y) \subset \mathrm{Stab}(x),$

$[f.*] := \mathcal{I}^{-1}\texttt{SetEq} : \mathrm{Stab}(x) = \mathrm{Stab}(y);$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : \Big(\exists X \xleftrightarrow{f} Y : G\text{-}\mathsf{SET} : f(x) = y\Big) \Rightarrow \mathrm{Stab}(x) = \mathrm{Stab}(y),$

$\texttt{Assume}\ [2] : \mathrm{Stab}(x) = \mathrm{Stab}(y),$

$\Big(f, [3]\Big) := \texttt{ExistanceOfTransitiveMap}[2] : \sum f \in G\text{-}\mathsf{SET}(X, Y) \,.\, f(x) = y,$

$\Big(f', [4]\Big) := \texttt{ExistanceOfTransitiveMap}[2] : \sum f' \in G\text{-}\mathsf{SET}(Y, X) \,.\, f'(y) = x,$

$[5] := [3][4] : ff'(x) = x,$

$[6] := \texttt{GMapsBetweenTransitiveAreDeterminedByOnePoint}[5] : ff' = \mathrm{id},$

$[7] := [3][4] : f'f(y) = y,$

$[8] := \texttt{GMapsBetweenTransitiveAreDeterminedByOnePoint}[7] : f'f = \mathrm{id},$

$[2.*] := \texttt{GSetInversion}[6][8] : \texttt{Isomorphism}(G\text{-}\mathsf{SET}, X, Y, f);$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow : \mathrm{Stab}(x) = \mathrm{Stab}(y) \Rightarrow \Big(\exists X \xleftrightarrow{f} Y : G\text{-}\mathsf{SET} : f(x) = y\Big),$

$[*] := \texttt{I} \iff [1][2] : \Big(\exists X \xleftrightarrow{f} Y : G\text{-}\mathsf{SET} : f(x) = y\Big) \iff \mathrm{Stab}(x) = \mathrm{Stab}(y);$

$\square$

$\texttt{TransitiveIsomorphismCriterion} :: \forall G \in \mathsf{GRP} \,.\, \forall X, Y : \texttt{Transitive}(G) \,.\, X \cong_{G\text{-}\mathsf{SET}} Y \iff$

$\quad \iff \mathrm{type}(X) = \mathrm{type}(Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TransitiveAutomrphismExists} ::$

$$:: \forall G \in \mathsf{GRP} \ . \ \forall X : \texttt{Transitive}(G) \ . \ \forall x \in X \ . \ \forall g \in N\big(\mathrm{Stab}(x)\big) \ . \ \exists! f \in \mathrm{Aut}_{G\text{-}\mathsf{SET}}(X) \ . \ f(x) = f(xg)$$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{structutalAutomorphism} ::$

$$:: \prod G \in \mathsf{GRP} \ . \ \prod X : \texttt{Transitive}(G) \ . \ \prod x \in X \ . \ N\big(\mathrm{Stab}(x)\big) \xrightarrow{\mathsf{GRP}} \mathrm{Aut}_{G\text{-}\mathsf{SET}}(X)$$

$\texttt{structuralAutomorphism}\,(g) = \varphi_{x,g} := \texttt{TransitiveutomorphismExists}(G, X, x, g)$

$\texttt{StrucuralAutomorphismIsSurjective} ::$

$$:: \forall G \in \mathsf{GRP} \ . \ \forall X : \texttt{Transitive}(G) \ . \ \forall x \in X \ . \ \texttt{Surjective}\Big(N\big(\mathrm{Stab}(x)\big), \mathrm{Aut}_{G\text{-}\mathsf{SET}}(X), \varphi_x\Big)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathrm{Aut}_{G\text{-}\mathsf{SET}}(X),$

$y := f(x) \in X,$

$\Big(g, [1]\Big) := \Lbag\texttt{Transitive}(G, X)(x, y) : \sum g \in G \ . \ y = xg,$

$\texttt{Assume}\ h \in \mathrm{Stab}(x),$

$[2] := \texttt{E}(=)\Big([1], xghg^{-1}\Big)\Rbag y \Lbag G\text{-}\mathsf{SET}(X, X)(f)(x, h)\Lbag \mathrm{Stab}(x)(h)\Rbag^{-1} y\texttt{E}(=)[1]\Lbag\texttt{Inverse}(G, g) :$

$\quad : xghg^{-1} = yhg^{-1} = f(x)hg^{-1} = f(xh)g^{-1} = f(x)g^{-1} = yg^{-1} = xgg^{-1} = y,$

$[h.*] := \Lbag\mathrm{Stab}(x)[2] : ghg^{-1} \in \mathrm{Stab}(x);$

$\rightsquigarrow [2] := \Lbag N\Big(\mathrm{Stab}(x)\Big) : g \in N\Big(\mathrm{Stab}(x)\Big),$

$[*] := \texttt{GMapsBetweenTransitiveAreDeterminedByOnePoint}\Lbag\varphi_{x,g}\Rbag y[1] : \varphi_{x,g} = f,$

$\square$

$\texttt{TransitiveAutomorphismStructure} ::$

$$:: \forall G \in \mathsf{GRP} \ . \ \forall X : \texttt{Transitive}(G, X) \ . \ \forall x \in X \ . \ \mathrm{End}_{G\text{-}\mathsf{SET}}(X) \cong_{\mathsf{GRP}} \frac{N\big(\mathrm{Stab}(x)\big)}{\mathrm{Stab}(x)}$$

$\texttt{Proof} \ =$

...

$\square$

6.4 Monodromy Action

$\texttt{monodromyAction} :: \prod c : \texttt{CoveringMap}(X, B) \,.\, \prod p \in B \,.\, c^{-1}(p) \curvearrowleft \pi(B)$

$\texttt{monodromyAction}\,(x, [\gamma]) = x \curvearrowleft_{c,p} [\gamma] := \widetilde{\gamma}_x(1)$

$\texttt{MonodromyActionIsTransitive} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall p \in B \,.\, \texttt{Transitive}(\curvearrowleft_{c,p})$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{StabilizerOfMonodromyGroup} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall p \in B \,.\, \forall x \in c^{-1}(p) \,.\, \mathrm{Stab}_{\curvearrowleft_{c,p}}(x) = \pi(c)$

$\texttt{Proof} =$

$\texttt{Assume}\ g \in \pi(c),$

$\Big(\gamma, [1]\Big) := \text{Д}\pi(c)(x) : \sum \gamma \in \Omega(x) = g = \big[c_*\gamma\big],$

$[g.*] := \texttt{E}(\curvearrowleft_{c,p})(xg)[1]\texttt{UniqueLifitingProperty}(c)\texttt{E}\Omega(x) : xg = \widetilde{[c_*\gamma]}_x(1) = \gamma(1) = x;$

$\rightsquigarrow [1] := \texttt{I}\Subset : \pi(c) \subset \mathrm{Stab}_{\curvearrowleft_{c,p}}(x),$

$\texttt{Assume}\ g \in \mathrm{Stab}_{\curvearrowleft_{c,p}}(x),$

$[2] := \texttt{E}\mathrm{Stab}_{\curvearrowleft_{c,p}}(x)(g) : x = xg = \widetilde{g}_x(1),$

$\Big(\gamma, [3]\Big) := \texttt{E}\widetilde{g}_x(1)[2] : \sum \gamma \in \Omega(x) \,.\, g = [c_*\gamma],$

$[2.*] := \texttt{E}\pi(c)[3] : g \in \pi(c);$

$\rightsquigarrow [2] := \texttt{I}\Subset : \mathrm{Stab}_{\curvearrowleft_{c,p}}(x) \subset \pi(c),$

$[*] := \texttt{ISubsetEq}[1][2] : \mathrm{Stab}_{\curvearrowleft_{c,p}}(x) = \pi(c);$

$\square$

$\texttt{FreeIsSimplyConnected} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall p \in B \,.\, \texttt{Free}(\curvearrowleft_{c,p}) \iff \texttt{SimplyConnected}(X)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SimplyConnectedCoveringStructure} :: \forall c : \texttt{CoveringMap}(X, B) \,.\, \texttt{SimplyConnected}(X) \Rightarrow \forall p \in B \,.$
$\quad .\, \Big|c^{-1}(p)\Big| = \Big|\pi(B)\Big|$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SimplyConnectedCoveringBase} :: ::$
$\quad \forall c : \texttt{CoveringMap}(X, B) \,.\, \texttt{SimplyConnected}(B) \Rightarrow \texttt{Homeomorphism}(X, B, c)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{MonodromyConjugacyTHM} ::$

$$:: \forall c : \texttt{CoveringMap}(X, B) \,.\, \forall p \in B \,.\, \forall x' \in X \,.\, \texttt{Orbit}\Big(\Gamma_{\pi(B)}, \Big\{c_*\pi(X, x) \Big| x \in c^{-1}(p)\Big\}\Big)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{NormalCovering} :: ?\texttt{CoveringMap}(X, B)$

$c : \texttt{NormalCovering} \iff \forall x \in X \,.\, c_*\pi(X, x) \lhd \pi\Big(B, c(x)\Big)$

$\texttt{NormalCoveringCharacterization} :: \forall c : \texttt{Covering}(X) \,.\, \forall x \in X \,.\, \forall [0] : c^*\pi(X, x) \lhd \pi(B, c(x)) \,.\, \texttt{NormalCove}$

$\texttt{Proof} \;=$

$p := c(x) \in B,$

$\texttt{Assume}\; y \in X,$

$q := c(y) \in B,$

$\xi := \mathcal{A}\texttt{PathConnected}(X)(x, y) \in \Omega(x, y),$

$\beta := c_*\xi \in \Omega(p, q),$

$[1] := \texttt{ChangeOfBasePoint}(\beta) : \texttt{Isomorphism}(\mathsf{GRP}, \pi(X, x), \pi(X, y), \gamma_{[\xi]}),$

$[2] := \texttt{ChangeOfBasePoint}(\xi) : \texttt{Isomorphism}(\mathsf{GRP}, \pi(B, p), \pi(B, q), \gamma_{[\beta]}),$

$[3] := \texttt{E}(\gamma)\mathcal{A}\mathsf{GRP}\Big(\pi(X), \pi(B)\Big)(c_*)\texttt{I}(\beta)\texttt{I}(\gamma) : c_*\gamma_{[\xi]} = \gamma_{[\beta]}c_*,$

$[4] := \texttt{Imapping}[3] : \gamma_\beta\Big(c_*\pi(X, x)\Big) = c_*\pi(X, y),$

$[y.*] := \texttt{IsomorphismPreservesNormal} : c_*\pi(X, y) \lhd \pi\Big(B, c(y)\Big);$

$\rightsquigarrow [*] := \texttt{INormalCove} : \texttt{NormalCover}(X, B, c),$

$\square$

6.5 Category of Coverings

$\texttt{CoveringMorphism} :: \prod B \in \mathsf{TOP}\ .$

$\quad .\ \prod a : \texttt{CoveringMapping}(X, B)\ .\ \prod b : \texttt{CoveringMapping}(Y, B)\ .\ ?\mathsf{TOP}(X, Y)$

$f : \texttt{CoveringMorphism} \iff fb = a$

$\texttt{coveringCategory} :: \mathsf{TOP} \to \mathsf{CAT}$

$\texttt{coveringCategory}\,(B) = \mathsf{COV}(B) := \left(\sum X \in \mathsf{TOP}\ .\ \texttt{CoveringMap}(X, B), \texttt{CoveringMorphism}(B), \circ, \mathrm{id}\right)$

$\texttt{coveringMorphismUniqueness} :: \forall (X, a), (Y, b) \in \mathsf{COV}(B)\ .\ \forall f, g \in \mathsf{COV}(a, b)\ .\ \forall x \in X\ .\ f(x) = g(x) \Rightarrow f = g$

$\texttt{Proof}\ =$

$[1] := \texttt{ECovariant}(a, b)\texttt{ILift} : \texttt{Lift}(a, b, f\ \&\ g),$

$[*] := \texttt{UniqueLiftingTHM}[0][1] : f = g;$

$\square$

$\texttt{CoveringMorphismGSetInduction} :: \forall (X, a), (y, b) \in \mathsf{COV}(B)\ .\ \forall f \in \mathsf{COV}(a, b)\ .\ \forall p \in B\ .$

$\quad .\ f_{|a^{-1}(p)} \in \pi(B, p)\text{-}\mathsf{SET}\left(a^{-1}(p), b^{-1}(p)\right)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ x \in a^{-1}(p),$

$\texttt{Assume}\ g \in \pi(B, p),$

$\left(\xi, [1]\right) := \mathcal{C}\texttt{PathLiftingProperty}(a, g) : \sum \xi : I \to X\ .\ g = [a_* \xi],$

$[2] := [1]\texttt{E}\mathsf{COV}(a, b)(f)\texttt{ECovariant}(\mathsf{TOP}_*, \mathsf{GRP}, \pi)\texttt{E}(f_*) : g = [a_* \xi] = \left[(f b_*)(\xi)\right] = [f_* b_* \xi] = \left[b_* f(\xi)\right],$

$[g.*] := \texttt{E} \curvearrowleft_{a,p} \texttt{IcompositonI} \curvearrowleft_{b,p}: f(xg) = f\left(\gamma(1)\right) = \gamma f(1) = f(x)g;$

$\rightsquigarrow [*] := \texttt{I}\pi(B, p)\text{-}\mathsf{SET}\left(a^{-1}(p), b^{-1}(p)\right) : f_{|a^{-1}(p)} \in \pi(B, p)\text{-}\mathsf{SET}\left(a^{-1}(p), b^{-1}(p)\right);$

$\square$

$\texttt{CoveringMorphismIsCoveringMap} :: \forall (a,X),(b,Y) \in \mathsf{COV}(B) \,.\, \forall f \in \mathsf{COV}(B)\Big((a,X),(b,Y)\Big) \,.\, f : \texttt{CoveringM}$

$\texttt{Proof} =$

$[1] := \texttt{ECoveringMorphism}(f) : fb = a,$

$\texttt{Assume}\ y \in Y,$

$p := b(y) \in B,$

$[2] := \texttt{ESurjective}(a,p) : a^{-1}(p) \neq \emptyset,$

$[3] := \texttt{CoveringMorphismGSetInduction}(a,b,f) : f_{|a^{-1}(p)} \in \pi(B,p)\text{-}\mathsf{SET}\Big(a^{-1}(p), b^{-1}(p)\Big),$

$[4] := \texttt{TransitiveGMApIsSurjective}[3] : \texttt{Surjective}(a^{-1}(p), b^{-1}(p), f_{|a^{-1}(p)}),$

$[y.*] := \texttt{EspecificationESurjection}[4] : y \in \operatorname{Im} f_{|a^{-1}(p)} \subset f_{|b^{-1}(p)};$

$\leadsto [2] := \texttt{ISurjective} : \texttt{Surjective}(X,Y,a),$

$\texttt{Assume}\ y \in Y,$

$p := b(y) \in B,$

$\Big(U', [2]\Big) := \texttt{ECoveringMap}(X,B,a)(p) : \sum U : \texttt{EvenlyCovered}(X,B,a) \,.\, p \in U,$

$\Big(U'', [3]\Big) := \texttt{ECoveringMap}(Y,B,b)(p) : \sum U : \texttt{EvenlyCovered}(Y,B,b) \,.\, p \in U,$

$U''' := U' \cap U'' \in \mathcal{U}(p),$

$U := \texttt{ELocallyPathConnected}(B)\texttt{PathConnectedIsConnected} \in \mathrm{PCC}(U''') \cap \mathcal{U}(p),$

$\Big(V, [4]\Big) := \texttt{EEvenlyCovered}(X,B,b,U)\texttt{EdisjointUnion}[1]\texttt{E}p : \sum V \in \mathcal{U}(y)\ \&\ \texttt{StronglyConnected} \,.$

$\quad . \texttt{Homeomprphism}(b_{|V}, V, U),$

$[5] := \mathsf{ETOP}(X,Y)(f) : \texttt{Clopen}\Big(b^{-1}(U), f^{-1}(V)\Big),$

$[6] := \texttt{CompositionPreimage}[1][5] : \texttt{Clopen}\Big(a^{-1}(U), f^{-1}(V)\Big),$

$[\mathcal{I}, W, [y.*]] := \texttt{EEvenlyCloded}(U'')\texttt{E}U[6][1] :$

$$: \sum \mathcal{I} \in \mathsf{SET} \,.\, \sum W : \mathcal{I} \to \mathcal{T}(X)\ \&\ \texttt{StronglyConnected} \,.\, f^{-1}(V) = \bigsqcup_{i \in \mathcal{I}} W_i \forall i \in \mathcal{I} \,.$$

$\quad . \texttt{Homeomorphis}(W_i, V, f_{|W_i});$

$\leadsto [[*] := \texttt{ICoveringMap} : \texttt{CoveringMap}(X,Y,f);$

$\square$

$\texttt{CoveringMorphisimCriterion} :: \forall (a,X),(b,Y) \in \mathsf{COV}(B) \,.\, \forall x \in X \,.\, \forall y \in Y \,.\, \forall [0] : a(x) = b(y) \,.$

$\quad . \exists f \in \mathsf{COV}(B)(a,b) : f(x) = f(y) \iff a_*\pi(X,x) \subset b_*\pi(Y,y)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CoveringIsomorphisCriterion} :: \forall (a,X),(b,Y) \in \mathsf{COV}(B) \,.\, \forall x \in X \,.\, \forall y \in Y \,.\, \forall [0] : a(x) = b(y) \,.$

$\quad . \exists f : \texttt{Isomorphism}\Big(\mathsf{COV}(B), a, b\Big) : f(x) = f(y) \iff a_*\pi(X,x) = b_*\pi(Y,y)$

$\texttt{Proof} =$

...

$\square$

6.6 The Universal Covering Space

$\texttt{UniversalCover} :: \prod B \in \mathsf{TOP} \ . \ ?\mathsf{COV}(B)$

$(Z,z) : \texttt{UniversalCover} \iff \forall (X,c) \in \mathsf{COV}(B) \ . \ \exists (Z,z) \xrightarrow{f} (X,c) : \mathsf{COV}(B)$

$\texttt{SimplyConnectedIsUniversalCover} :: \forall B \in \mathsf{TOP} \ . \ \forall (Z,z) \in \mathsf{COV}(B) \ .$
$\quad . \ \texttt{SimplyConnected}(Z) \Rightarrow \texttt{UniversalCover}(z,Z)$

$\texttt{Proof} \ =$

$[1] := \texttt{ESimplyConnectedI}\pi(Z) : \pi(Z) = \star,$

$[2] := \texttt{Eimage}(z_*)[1]\texttt{I}\pi(z) : \pi(z) = z_*\pi(Z) = z_*\star = \star,$

$\texttt{Assume} \ (X,c) \in \mathsf{COV}(B),$

$[3] := \texttt{TrivialSubgroup}\Big(\pi(c)\Big) : \star \subset \pi(c),$

$\Big[(X,c).*\Big] := \texttt{CoveringMorphismCriterion}\Big((Z,z),(X,c)\Big)[2][3] : \exists f : (z,Z) \xrightarrow{f} (X,c) : \mathsf{COV}(B);$

$\rightsquigarrow [*] := \texttt{IUniversalCover} : \texttt{UniversalCover}\Big(B,(Z,z)\Big);$

$\texttt{LocallyConnectedCoversAreIsomorphic} :: \forall B \in \mathsf{TOP} \ . \ \forall (X,x),(Y,y) \in \mathsf{COV}(B) \ .$
$\quad . \ \texttt{SimplyConnected}(X \ \& \ Y) \Rightarrow (X,x) \cong_{\mathsf{COV}(B)} (Y,y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{Reasonable} := \texttt{Connected} \ \& \ \texttt{Locally SimplyConnected} : \texttt{Type};$

$\texttt{UniversalCoverExists} :: \forall B : \texttt{Reasonable} \ . \ \exists (Z,z) \in \mathsf{COV}(B) : \texttt{SimplyConnected}(Z)$

$\texttt{Proof} \ =$

$p := \texttt{ENonEmpty} \in B,$

$Z := \Big\{\texttt{pathClass}(\gamma) \Big| I \xrightarrow{\gamma} B : \mathsf{TOP}, \gamma(0) = x\Big\} \in \mathsf{SET},$

$z := \Lambda[\gamma] \in Z \ . \ \gamma(1) : Z \to B,$

$V := \Lambda[\gamma] \in Z \ . \ \Lambda U \in \mathcal{U}(z) \ . \ \Big\{[\gamma \circ \omega] \Big| \mathcal{I} \xrightarrow{\gamma} B : \mathsf{TOP}, \omega(0) = \gamma(1)\Big\} : \prod_{[\gamma] \in Z} \mathcal{U}(\gamma(1)) \to ?Z,$

$\mathcal{B} := \Big\{V_{[\gamma],U} \Big| [\gamma] \in Z, U \in \mathcal{U}\big(\gamma(1)\big)\Big\} : ??Z,$

$[1] := \texttt{E}\mathcal{B}\texttt{EPathCompositionICover} : \texttt{Cover}(Z,\mathcal{B}),$

$\texttt{Assume} \ v,v' \in \mathcal{B},$

$\texttt{Assume} \ [2] : v \cap v' \neq \emptyset,$

$[\alpha],[\beta],U,U',[3] := \texttt{E}\mathcal{B}(v,v') : \sum [\alpha],[\beta] \in Z \ . \ \sum U \in \mathcal{U}\big(\alpha(0)\big) \ . \ \sum U' \in \mathcal{U}\big(\beta(0)\big) \ . \ v = V_{[\alpha],U} \ \& \ v' = V_{[\beta],U'},$

$[4] := \mathcal{D}V[2][3] : [\alpha] = [\beta],$

$\Big[(v,v').*\Big] := \texttt{Eintersecion}[4]\texttt{I}V\texttt{E}\mathcal{B} : v \cap v' = V_{[\alpha],U\cap U'} \in \mathcal{B};$

$\rightsquigarrow [2] := \texttt{IBase}(Z) : \texttt{Base}(Z,\mathcal{B}),$

$Z := (Z, \texttt{topology}(\mathcal{B})) \in \mathsf{TOP},$

$[3] := \texttt{ESimplyConnected}(B)\texttt{E}V\texttt{ISimplyConnected}(Z) : \forall [\gamma] \in Z \ . \ \forall U \in \mathcal{U}\big(\gamma(0)\big) \ \& \ \texttt{SimplyConnected} \ .$
$\quad . \ \texttt{SimplyConnected}(V_{[\gamma],U}),$

$[4] := \texttt{EReasonable}(B)[2] : \forall [\gamma] \in Z \ . \ \exists V \in \mathcal{U}\big([\gamma]\big) : \texttt{SimplyConnected}(V),$

$[5] := \texttt{ELocally SimplyConnected}[4]\texttt{ILocally PathConnected} : \texttt{Locally PathConnected}(Z),$

$[6] := \texttt{EConnected}(B)\mathcal{I}Z : \forall[\gamma] \in Z \,.\, \Omega\Big([\gamma], p\Big) \neq \emptyset,$

$[7] := \texttt{IPathConnected}[6] : \texttt{PathConnected}(Z),$

$[8] := \texttt{PathConnectedIsConnected}[7] : \texttt{Connected}(Z),$

$\texttt{Assume}\ U : \mathcal{T}(B),$

$E := \Big\{[\gamma] \in Z : \gamma(1) \in U\Big\} :?Z,$

$[U.*] := \texttt{E}z\texttt{I}V_{e,U}\texttt{E}\mathcal{T}(Z) : z^{-1}(U) = \bigcup_{e \in E} V_{e,U} \in \mathcal{T}(Z);$

$\rightsquigarrow [9] := \texttt{I}\mathsf{TOP} : Z \xrightarrow{z} B : \mathsf{TOP},$

$\texttt{Assume}\ U : \mathcal{T}(B)\ \&\ \texttt{SimplyConnected},$

$E := \Big\{[\gamma] \in Z : \gamma(1) \in U\Big\} :?Z,$

$[10] := \texttt{E}z\texttt{I}V_{e,U} : z^{-1}(U) = \bigcup_{e \in E} V_{e,U},$

$[11] := \texttt{E}V_{e,U}[11]\texttt{IDisjointUnion} : z^{-1}(U) = \bigsqcup_{e \in E} V_{e,U},$

$\texttt{Assume}\ e \in E,$

$[12] := \texttt{E}(z)\texttt{IBijective} : \texttt{Surjective}(V_{e,U}, U, z_{|V_{e,U}}),$

$\texttt{Assume}\ [\gamma], [\gamma'] \in V_{e,U},$

$\texttt{Assume}\ [13] : z[\gamma] = z[\gamma'],$

$[14] := \texttt{E}z[13] : \gamma(1) = \gamma'(1),$

$\Big[\alpha, [15]\Big] := \texttt{E}V_{e,U}[\gamma] : \sum \alpha \in \Omega\big(e(1), \gamma(1)\big) \,.\, \gamma \sim e \circ \alpha,$

$\Big[\beta, [16]\Big] := \texttt{E}V_{e,U}[\gamma] : \sum \beta \in \Omega\big(e(1), \gamma(1)\big) \,.\, \gamma \sim e \circ \beta,$

$\Big[\big([\gamma], [\gamma']\big).*\Big] := \mathcal{I}\texttt{SimplyConnected}[15][16] : [\gamma] = [\gamma'];$

$\rightsquigarrow [12] := \texttt{IInjective} : \texttt{Injective}(V_{e,U}, U, z_{|V_{e,U}}),$

$\texttt{Assume}\ W : \mathcal{T}(V_{e,U}),$

$(\mathcal{O}, [13]) := \texttt{E}W : \sum \mathcal{O} \in \prod_{f \in V_{e,U}} ?\mathcal{U}(f(1)) \,.\, W = \bigcup_{f \in W} \bigcup_{O \in \mathcal{O}_f} V_{f,O_f},$

$[W.*] := \texttt{E}z[13] : z(W) = \bigcup_{f \in W} \bigcup_{O \in \mathcal{O}_f} O;$

$\rightsquigarrow [14] := \texttt{IOpenMap} : \texttt{OpenMap}\Big(V_{e,U}, U, z_{V_{e,U}}\Big),$

$[U.*] := \texttt{IHomeomophism}[12][13][14] : \texttt{Homeomorphism}\Big(V_{e,U}, U, z_{V_{e,U}}\Big);$

$\rightsquigarrow [10] := \texttt{EReasonable}(B)\texttt{I}\mathsf{COV}(B) : (Z, z) \in \mathsf{COV}(B),$

$\texttt{Assume}\ \Gamma \in \Omega_Z\big([p]\big),$

$\gamma := \Gamma z : \Omega_B(p),$

$H := \Lambda t \in I \,.\, \Lambda s \in \mathcal{I} \,.\, \gamma(st) : \mathcal{I}^2 \to B,$

$[11] := \texttt{E}z\texttt{E}H : \forall t \in I \,.\, [H_t]z = H_t(1) = \gamma(t),$

$[12] := \texttt{ILiftE}\gamma[11] : \texttt{Lift}(I, B, \gamma, \tilde{\Gamma}_x\ \&\ H),$

$[13] := \texttt{UniqueLiftProperty}[12] : \Gamma = [H],$

$[\Gamma.*] := \mathcal{I}\Omega_Z\Big([x]\Big) : \Gamma = [p];$

$\rightsquigarrow [*] := \mathcal{C}^{-1}\texttt{SimplyConnected} : \texttt{SimplyConnected}(Z);$

$\square$

6.7 Borsuk-Ulam Theory

$\texttt{OddFunction} :: \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1)?$
$f : \texttt{OddFunction} \iff \forall s \in \mathbb{S}_1 \,.\, f(-s) = -f(s)$

$\texttt{EvenFunction} :: \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1)?$
$f : \texttt{EvenFunction} \iff \forall s \in \mathbb{S}_1 \,.\, f(-s) = f(s)$

$\texttt{OddSquareCommuter} :: \forall f : \texttt{OddFunction} \,.\, \exists g : \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1) \,.\, \deg f = \deg g \,.\, f^2 = \Lambda z \in \mathbb{S}^1 \,.\, g(z^2)$
$\texttt{Proof} =$
$\Big(r, [1]\Big) := \texttt{ComplexRootCovers} : \sum r : \mathbb{R} \xrightarrow{\mathsf{TOP}} \mathbb{R} \,.\, \forall t \in \mathbb{R} \,.\, rs^2(t) = s(t),$
$g := \Lambda z \in \mathbb{S}_1 \,.\, f^2\bigg(\exp\Big(\mathrm{i}r\Big(\mathrm{Arg}\; z\Big)\Big)\bigg) : \mathbb{S}^1 \to ?\mathbb{S}^1,$
$\texttt{Assume}\; z : \mathbb{S}^1,$
$\Big(t, [2]\Big) := \mathtt{E}\mathrm{Arg}(z) : \sum t \in [0, 2\pi) \,.\, \mathrm{Arg}\; z = \{t + 2\pi n | n \in \mathbb{Z}\},$
$[3] := \mathtt{E}g[2]\mathtt{E}r\texttt{ExpHomo}(\mathbb{C})\texttt{EOddFunction}(f)\texttt{SignSquare} : g(z) = f^2\bigg(\exp\Big(\mathrm{i}r\Big(\mathrm{Arg}\; z\Big)\Big)\bigg) =$
$= \left\{f^2\left(\exp\left(\frac{\mathrm{i}t}{2} + n\pi\right)\right) | n \in \mathbb{Z}\right\} = \left\{f^2\left(\pm\exp\left(\frac{\mathrm{i}t}{2}\right)\right)\right\} = \left\{(\pm)^2 f^2\left(\exp\left(\frac{\mathrm{i}t}{2}\right)\right)\right\} = \left\{f^2\left(\exp\left(\frac{\mathrm{i}t}{t}\right)\right)\right\},$
$(z.*) := \texttt{ISingleton}[3] : \texttt{Singleton}\Big(g(z)\Big);$
$\rightsquigarrow [2] := \texttt{WellDefine} : g \in \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1),$
$[3] := \mathtt{E}g\mathtt{E}r : \forall z \in \mathbb{S}^1 \,.\, f^2(z) = g(z^2),$
$[4] := \texttt{DegreeCompostition}[3]\texttt{DegreeComposition} : 2\deg f = \deg f^2 = \deg g(\bullet^2) = 2\deg g,$
$[5] := \mathtt{E}\,!\,\texttt{ZeroDivisor}(\mathbb{Z}, 2) : \deg f = \deg g;$
$\square$

$\texttt{SquareCommuter} :: \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1) \to ?\mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1)$
$g : \texttt{SquareCommuter} \iff \Lambda f \in \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1) \,.\, \deg f = \deg g \;\&\; f^2 = g(\bullet^2)$

$\texttt{oddSquareCommuter} :: \prod f : \texttt{OddFunction} \,.\, \texttt{SquareCommutor}(f)$
$\texttt{oddSquareCommuter}\,() = g_f := \texttt{OddSquareCommuter}(f)$

$\texttt{EvenDegreeLifting} :: \forall f : \texttt{OddFunction} \,.\, \forall [0] : \texttt{Even}(\deg f) \,.\, \exists \tilde{g} : \texttt{Lift}(\bullet^2, g_f)$
$\texttt{Proof} =$
$[1] := \texttt{EEvenE}\deg f[0] : f_*\pi(\mathbb{S}_1) \subset 2\mathbb{Z},$
$[2] := \texttt{PowerDegree}(2)[1] : f_*^2\pi(\mathbb{S}_1) \subset 4\mathbb{Z},$
$[3] := \texttt{ESquaeCommuter}(f, g_f)[2] : (\bullet^2 g_f)_*\pi(\mathbb{S}_1) \subset 4\mathbb{Z},$
$[4] := \texttt{PowerDegree}(2)[3] : g_{f*}\pi(\mathbb{S}_1) \subset 2\mathbb{Z},$
$[*] := \texttt{LiftingCriterion}(\mathbb{S}^1, \mathbb{S}^1, \bullet^2, g_f)[4] : \exists \tilde{g} : \texttt{Lift}(\bullet^2, g_f);$
$\square$

$\texttt{OddFunctionsHasOddDegree} :: \forall f : \texttt{OddFunction} \;.\; \texttt{Odd}(\deg f)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; [1] : \texttt{Even}(\deg f),$

$\tilde{g} := \texttt{EvenDegreeLifting} : \texttt{Lift}(\bullet^2, g_f),$

$[2] := \texttt{ELift}(\bullet^2, g_f) : (\tilde{g})^2 = g_f,$

$[3] := \texttt{E}(=)\Big(\bullet^2\, g_f, [2]\Big) : (\bullet^2 \tilde{g})^2 = \bullet^2 g_f,$

$[4] := \texttt{ILift}[3] : \texttt{Lift}(\bullet^2, \bullet^2\tilde{g}, \bullet^2 g_f),$

$[5] := \texttt{ESquareCommetor}(f, g_f) : f^2 = \bullet^2 g_f,$

$[6] := \texttt{ILift}[3] : \texttt{Lift}(\bullet^2, f, \bullet^2 g_f),$

$[7] := \texttt{EOddFunction}(f)[5][3] : f(-1) = \tilde{g}(1) | f(1) = \tilde{g}(1),$

$[8] := \texttt{UniqueLiftingProperty}[4][6][7] : f = \bullet^2 \tilde{g},$

$[9] := \texttt{IEvenFunctionE}\; \bullet^2\; [8] : \texttt{EvenFunction}(f),$

$[1.*] := \texttt{EOddFuncyion}(f)\texttt{EEvenFunction}(f) : \bot;$

$\leadsto [*] := \texttt{E}(\bot)\texttt{E}(|)\texttt{OddOrEven} : \texttt{Even}(\deg f);$

$\square$

$\texttt{EvenFunctionsHaveEvenDegrees} :: \forall f : \texttt{EvenFunction} \;.\; \texttt{Even}(f)$

$\texttt{Proof} \; =$

$\Big(r, [1]\Big) := \texttt{ComplexRootCovers} : \sum r : \mathbb{R} \xrightarrow{\mathsf{TOP}} \mathbb{R} \;.\; \forall t \in \mathbb{R} \;.\; rs^2(t) = s(t),$

$g := \Lambda z \in \mathbb{S}_1 \;.\; f\Bigg(\exp\bigg(\mathrm{i}r\Big(\operatorname{Arg} z\Big)\bigg)\Bigg) : \mathbb{S}^1 \to ?\mathbb{S}^1,$

$\texttt{Assume}\; z : \mathbb{S}^1,$

$\Big(t, [2]\Big) := \mathtt{E}\operatorname{Arg}(z) : \sum t \in [0, 2\pi) \;.\; \operatorname{Arg} z = \{t + 2\pi n | n \in \mathbb{Z}\},$

$$[3] := \texttt{E}g[2]\texttt{E}r\texttt{ExpHomo}(\mathbb{C})\texttt{EOddFunction}(f)\texttt{SignSquare} : g(z) = f\Bigg(\exp\bigg(\mathrm{i}r\Big(\operatorname{Arg} z\Big)\bigg)\Bigg) =$$

$$= \left\{ f\left(\exp\left(\frac{\mathrm{i}t}{2} + n\pi\right)\right) | n \in \mathbb{Z} \right\} = \left\{ f\left(\pm\exp\left(\frac{\mathrm{i}t}{2}\right)\right)\right\} = \left\{ f\left(\exp\left(\frac{\mathrm{i}t}{t}\right)\right)\right\},$$

$(z.*) := \texttt{ISingleton}[3] : \texttt{Singleton}\Big(g(z)\Big);$

$\leadsto [2] := \texttt{WellDefine} : g \in \operatorname{End}_{\mathsf{TOP}}(\mathbb{S}^1),$

$[3] := \texttt{E}g\texttt{E}r : \forall z \in \mathbb{S}^1 \;.\; f(z) = g(z^2),$

$[4] := \texttt{DegreeCompostition}[3]\texttt{DegreeComposition} : \deg f = \deg g(\bullet^2) = 2 \deg g,$

$[*] := \texttt{IEven}[4] : \texttt{Even}(\deg f);$

$\square$

$\texttt{BorsukUlamTheorem} :: \forall \mathbb{S}^2 \xrightarrow{F} \mathbb{R}^2 \;.\; \exists v \in \mathbb{S}^2 \;.\; F(v) = F(-v)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \forall v \in \mathbb{S}^2 \;.\; F(v) \neq F(-v),$

$f' := \Lambda v \in \mathbb{S}^2 \;.\; \dfrac{F(v) - F(-v)}{\|F(v) - F(-v)\|} : \mathsf{TOP}(\mathbb{S}^2, \mathbb{S}^1),$

$\texttt{Assume}\ L : \texttt{VectorPlane}(\mathbb{R}^3),$

$f_L := f'_{|L \cap \mathbb{S}^2} : \mathrm{End}_{\mathsf{TOP}}(\mathbb{S}^1),$

$f_L := \texttt{E}f\texttt{IOddFunction} : \texttt{OddFunction}(f),$

$[2] := \texttt{OddFunctionHasOddDegree}(f) : \texttt{Odd}(\deg f),$

$H := \texttt{HigherSphereIsSimplyConnected} : \texttt{Homotopy}(L \cap \mathbb{S}^2, \star),$

$[3] := \texttt{CPreservesHomotopy}(f, H) : \texttt{Homotopy}(f, f(\star), Hf),$

$[4] := \texttt{HomotopyPreserrvesDegree}[3] : \deg f = 0,$

$[1.*] := \texttt{E}(\deg)[4][2]\texttt{I}(\bot) : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot : \exists v \in \mathbb{S}^2 \;.\; F(v) = F(-v);$

$\square$

$\texttt{HamSandwichTHM} :: \forall U_1, U_2, U_3 : \texttt{Open}\ \&\ \texttt{ConnectedSubset}(\mathbb{R}_3) \;.\; \exists H : \texttt{Hyperplane}(\mathbb{R}^3) : \forall i \in \{1,2,3\} \;.$
$\quad \mathrm{Vol}(U_i \cap H_-) = \mathrm{Vol}(U_i \cap H_+)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ p \in \mathbb{S}_2,$

$\texttt{Assume}\ H : \texttt{HyperplenThrough}(p),$

$[1] := \texttt{E}\mathrm{Vol} : \mathrm{Vol}(U_3) = \mathrm{Vol}(U_3 \cap H_-) + \mathrm{Vol}(U_2 \cap H_+),$

$[2] := \texttt{EreverseHyperplane} : \mathrm{Vol}(U_3 \cap H_-) = \mathrm{Vol}(U_3 \cap -H_+),$

$[H.*] := [1][2]\texttt{I}\exists :$

$$\mathrm{Vol}(U_3 \cap H_-) \leq \frac{1}{2}\mathrm{Vol}(U_3) \Rightarrow \exists H' : \texttt{HyperplaneThrough}(p) : \mathrm{Vol}(U_3 \cap H'_-) \geq \frac{1}{2}\mathrm{Vol}(U_3)\ \&$$
$$\&\ \mathrm{Vol}(U_3 \cap H_-) \geq \frac{1}{2}\mathrm{Vol}(U_3) \Rightarrow \exists H' : \texttt{HyperplaneThrough}(p) : \mathrm{Vol}(U_3 \cap H'_-) \leq \frac{1}{2}\mathrm{Vol}(U_3);$$

$\rightsquigarrow \Big(H_p, [1]\Big) := \texttt{IntermidiateVlaueTheirem} :$

$$\sum H^p : \texttt{HyperplaneThrough}(p) \;.\; \mathrm{Vol}(U_3 \cap H^p_-) = \mathrm{Vol}(U_3 \cap H^p_+);$$

$\rightsquigarrow H := \texttt{I}\Big(\prod\Big) : \prod p \in \mathbb{S}^2 \;.\; \sum H^p \texttt{HyperplaneThrough}(p) \;.\; \mathrm{Vol}(U_3 \cap H^p_-) = \mathrm{Vol}(U_3 \cap H^p_+),$

$F := \Lambda p \in \mathbb{S}^2 \;.\; (\mathrm{Vol}(U_1 \cap H^p_+), \mathrm{Vol}(U_3 \cap H^p_+)) : \mathsf{TOP}(\mathbb{S}^2, \mathbb{R}^2),$

$\Big(p, [1]\Big) := \texttt{BorsukUlamTHM} : \sum p \in \mathbb{S}^2 \;.\; F(p) = F(-p),$

$[*] := \texttt{E}F\texttt{E}H\texttt{E}\mathrm{Vol} : \forall i \in \{1,2,3\} \;.\; \mathrm{Vol}(U_i \cap H^p_+) = \mathrm{Vol}(U_i \cap H^p_+);$

$\square$

6.8 Galois Covering Theory

deckTransformationGroup :: $\prod B \in \mathsf{TOP} \,.\, \mathsf{COV}(B) \to \mathsf{GRP}$

deckTransforamtionGroup $\left(X \xrightarrow{c} B\right) = \mathrm{Gal}(X \xrightarrow{c} B) := \mathrm{Aut}_{\mathsf{COV}(B)}(X, B, c)$

DeckTransformationMonodromy :: $\forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.\, \forall f, g \in \mathrm{Gal}(X \xrightarrow{c} B) \,.\, \forall x \in X \,.$
$\quad . \, f(x) = g(x) \Rightarrow f = g$

Proof =

...

□

DeckTransformationIsGSetIso :: $\forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.\, \forall f \in \mathrm{Gal}(X \xrightarrow{c} B) \,.\, \forall p \in B \,.$
$\quad . \, f_{|c^{-1}(x)} \in \mathrm{Aut}_{\pi(B,p)\text{-}\mathsf{SET}}\left(c^{-1}(x)\right)$

Proof =

...

□

DeckTransformationActsFreely :: $\forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.$ Free & HomeoAction$\left(X, \mathrm{Gal}(X \xrightarrow{c} B)\right)$

Proof =

...

□

DeckTransformationOrbitCriterion :: $\forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.\, \forall p \in B \,.\, \forall x, y \in f^{-1}(p) \,.$
$\quad . \, \exists f : \mathrm{Gal}(X \xrightarrow{c} B) : f(x) = y \iff c_*(X, x) = c_*(X, y)$

Proof =

...

□

NormalCoveringHasTransitiveGal ::
$\quad :: \forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.\, \left(\forall p \in B \,.\, \text{Transitive}\left(c^{-1}(p), \mathrm{Gal}(X \xrightarrow{c} B)\right)\right) \iff$ NormalCovering$(X \xrightarrow{c} B)$

Proof =

...

□

$\texttt{DeckTransformationIso} :: \forall (X \xrightarrow{c} B) \in \mathsf{COV}(B) \,.\, \forall p \in B \,.\; \mathrm{Gal}(X \xrightarrow{c} B) \cong_{\mathsf{GRP}} \mathrm{Aut}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p)$

$\texttt{Proof} =$

$\varphi := \Lambda f \in \mathrm{Gal}(X \xrightarrow{c} B) \,.\, f_{|f^{-1}(p)} \in \mathrm{Gal}(X \xrightarrow{c} B) \to \mathrm{Aut}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p),$

$[1] := \mathtt{E}\,\mathrm{Gal}(X \xrightarrow{c} B)\mathtt{E}\mathrm{Aut}_{\pi(B,p)\text{-}\mathsf{SET}}\mathtt{E}\varphi\texttt{IHomo} : \mathsf{GRP}\Big(\mathrm{Gal}(X \xrightarrow{c} B), \mathrm{End}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p), \varphi\Big),$

$[2] := \texttt{DeckTransformationIso}(X \xrightarrow{c} B)\mathtt{E}\varphi\texttt{IInjective} : \texttt{Injective}\Big(\mathrm{Gal}(X \xrightarrow{c} B), \mathrm{End}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p), \varphi\Big),$

$\texttt{Assume}\ \sigma \in \mathrm{End}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p),$

$[4] := \texttt{GSetIsomorphismExistance}(\sigma) : \forall x \in c^{-1}(p) \,.\, \mathrm{Stab}(x) = \mathrm{Stab}\,\sigma(x);$

$[5] := \texttt{StabilizerOfMonodromyGroup}[4] : \forall x \in c^{-1}(p) \,.\, c_*\pi(X,x) = c_*\pi\Big(X, \sigma(x)\Big),$

$\texttt{Assume}\ x \in X,$

$\Big(f, [6]\Big) := \texttt{DeckTransformationOrbitCriterion}[5](x) : \sum f \in \mathrm{Gal}(X \xrightarrow{c} B) \,.\, f(x) = \sigma(x),$

$[7] := \mathtt{E}(\varphi)[6] : \varphi(f)(x) = \sigma(x),$

$[x.*] := \texttt{MonodromyActionIsTransitive}(X \xrightarrow{c} B)$

$\quad \texttt{GMapsBetweenTransitiveAreDeterminedByOnePoint}(\pi(B,p) : \varphi(f) = \sigma;$

$\leadsto [\sigma.*] := \texttt{ENonEmpty}(X) : \varphi(f) = \sigma;$

$[3] := \texttt{ISurjective} : \texttt{Surjective}\Big(\mathrm{Gal}(X \xrightarrow{c} B), \mathrm{End}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p), \varphi\Big),$

$[*] := \texttt{IIsomorphism}[1,2,3] : \texttt{Isomorphism}\Big(\mathsf{GRP}, \mathrm{Gal}(X \xrightarrow{c} B), \mathrm{End}_{\pi(B,p)\text{-}\mathsf{SET}} f^{-1}(p), \varphi\Big);$

$\square$

$\texttt{DeckStructuralMorphismExists} :: \forall (X \xrightarrow{c} B) : \mathsf{COV}(B) \,.\, \forall p \in B \,.\, \forall x \in c^{-1}(x) \,.\, \forall \gamma \in N(c_*\pi(X,x)) \,.\, \exists! f \in \mathrm{G}$

$\texttt{Proof} =$

$[1] := \mathtt{E} \curvearrowleft_{c,p} (\gamma)\mathtt{E}N\mathtt{I}\pi : c_*\pi(X,x) = c_*\pi(X, x\gamma),$

$[*] := \texttt{EDeckTransformationOrbitCriterion}[3] : \exists f \in \mathrm{Gal}(X \xrightarrow{c} B) : f(x) = x\gamma;$

$\square$

$\texttt{deckStructuralMorphism} :: \prod (X \xrightarrow{c} B) : \mathsf{COV}(B) \,.\, \prod_{p \in B} \prod_{x \in f^{-1}(x)} N(c_*\pi(X,x)) \xrightarrow{\mathsf{GRP}} \mathrm{Gal}(X \xrightarrow{c} B)$

$\texttt{deckStructuralMorphism}\,(\gamma) = \Delta_\gamma^{c,p,x} := \texttt{DeckStrucuralMorphismExists}$

$\texttt{DeckStructuralMorphismIsSurjective} :: \forall (X \xrightarrow{c} B) : \mathsf{COV}(B) \,.\, \forall p \in B \,.\, \forall x \in c^{-1}(x) \,.$

$\quad .\, \Delta^{c,p,x} : \texttt{Surjective}\Big(N(c_*\pi(X,x), \mathrm{Gal}(X \xrightarrow{c} B)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{DeckGroupStructure} :: \forall (X \xrightarrow{c} B) : \mathsf{COV}(B) \,.\, \forall p \in B \,.\, \forall x \in c^{-1}(x) \,.\; \mathrm{Gal}(X \xrightarrow{c} B) \cong_{\mathsf{GRP}} \dfrac{N\Big(c_*\pi(X,x)\Big)}{c_*\pi(X,x)}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{NormalDeckGroupStructure} :: \forall (X \xrightarrow{c} B) : \texttt{NormalCover} \,.\, \forall p \in B \,.\, \forall x \in c^{-1}(x) \,.$

$\,.\, \mathrm{Gal}(X \xrightarrow{c} B) \cong_{\mathsf{GRP}} \dfrac{\pi(B,p)}{c_*\pi(X,x)}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SimplyConnectedGroupStructure} :: \forall (X \xrightarrow{c} B) : \mathsf{COV}(B) \,.\, \forall p \in B \,.$

$\,.\, \texttt{SimplyConnected}(X) \Rightarrow \mathrm{Gal}(X \xrightarrow{c} B) \cong_{\mathsf{GRP}} \pi(B,p)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CoveringAction} :: \prod X \in \mathsf{TOP} \,.\, \prod G \in \mathsf{GRP} \,.\, ?(X \curvearrowleft_{\mathsf{TOP}} G)$

$(\cdot) : \texttt{CoveringAction} \iff \forall x \in X \,.\, \exists U \in \mathcal{U}(x) : \forall g \in G \,.\, g \neq e \Rightarrow U \cap Ug = \emptyset$

$\texttt{CoveringActionCovering} :: \forall X : \texttt{StronglyConnected} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall \alpha : \texttt{CoveringAction}(X,G) \,.$

$\,.\, X \xrightarrow{\pi} \dfrac{X}{\alpha} : \mathsf{COV}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [x] \in \dfrac{X}{\alpha},$

$\Big(U', [1]\Big) := \texttt{ECoveringAction}(X,G,\alpha)(x) : \sum U' \in \mathcal{U}(x) \,.\, \forall g \in G \,.\, g \neq e \Rightarrow U' \cap U'g = \emptyset,$

$\Big(U'', [2]\Big) := \texttt{EStronglyConnected}(X)(x, U') : \sum U'' \in \mathcal{U}(x) \,.\, \texttt{StronglyConnected}(X) \,\&\, U'' \subset U',$

$U := \pi U'' :? \dfrac{X}{\alpha},$

$[2] := \texttt{E}U\texttt{I}x : [x] \in U,$

$[3] := \texttt{E}U\texttt{EquotientByGroupAction}(X,G,\alpha) : \pi^{-1}(U) = \bigcup_{g \in G} U''g,$

$[5] := [1][3] : \pi^{-1}(U) = \bigsqcup_{g \in G} U''g,$

$[6] := \texttt{EHomeAction}(X,G,\alpha)[3]\texttt{I}\mathcal{T}(X) : \pi^{-1}(U) \in \mathcal{T}(X),$

$[7] := [2]\texttt{EquotientTopology}[6] : U \in \mathcal{U}[x],$

$[8] := \texttt{EHomeoAction}(X,G,\alpha)\texttt{CPreservesStronglyConnect}(U'') : \forall g \in G \,.\, \texttt{StronglyConnected}(U''x),$

$\Big[[x].*\Big] := \texttt{IEvenlyCovered}[6,7,8] : \texttt{EvenlyCovered}\left(X, \dfrac{X}{\alpha}, \pi, U\right);$

$\rightsquigarrow [*] := \texttt{ICoveringMap} : \texttt{CoveringMap}\left(X, \dfrac{X}{\alpha}, \pi, U\right);$

$\square$

$\texttt{ActionCoveringIsNormal} :: \forall X : \texttt{StronglyConnected} . \forall G \in \mathsf{GRP} . \forall \alpha : \texttt{CoveringAction}(X, G) .$

$. \texttt{NormalCovering}\left(X, \dfrac{X}{\alpha}, \pi\right)$

$\texttt{Proof} =$

$\texttt{Assume } g \in G,$

$\texttt{Assume } x \in X,$

$[x.*] := \texttt{E}\dfrac{X}{\alpha}(x) : \pi(xg) = \pi(x);$

$\leadsto [g.*] := \texttt{E}\,\mathrm{Gal}\left(X \xrightarrow{\pi} \dfrac{X}{\alpha}\right) : g \in \mathrm{Gal}\left(X \xrightarrow{\pi} \dfrac{X}{\alpha}\right);$

$\leadsto [1] := \texttt{ISubset} : G \subset \mathrm{Gal}\left(X \xrightarrow{\pi} \dfrac{X}{\alpha}\right),$

$[2] := \texttt{EOrbit}(\alpha)[1]\texttt{ITransitive} : \texttt{Transitive}\left(G, \mathrm{Gal}\left(X \xrightarrow{\pi} \dfrac{X}{\alpha}\right)\right),$

$[*] := \texttt{NormalCoveringHasTransitiveGal}[2] : \texttt{NormalCovering}\left(G, \dfrac{G}{\alpha}, \pi\right);$

$\square$

$\texttt{ActionCoveringDeckTransformationGroup} :: \forall X : \texttt{StronglyConnected} . \forall G \in \mathsf{GRP} .$

$. \forall \alpha : \texttt{CoveringAction}(X, G) . \ \mathrm{Gal}\left(X \xrightarrow{\pi} \dfrac{X}{\alpha}\right) = G$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ActionCoveringByDiscreteSubgroup} ::$

$:: \forall G : \texttt{StronglyConnected} \ \& \ \texttt{TopologicaGroup} . \forall H : \texttt{DiscreteSubgroup}(G) . \texttt{CoveringAction}(G, H, \cdot)$

$\texttt{Proof} =$

$\left(W, [1]\right) := \texttt{EDiscreteSubgroup}(G, H)(e)\texttt{E}\mathsf{TOPGRP}(G) :$

$: \sum W \in \mathcal{U}(e) . W \cap H = \{e\} \ \& \ \texttt{Balanced}\left(G, (-1, 1)W\right),$

$\texttt{Assume } g \in G,$

$U := gW \in \mathcal{U}(g),$

$\texttt{Assume } h : H,$

$\texttt{Assume } [2] \in h \neq e,$

$\texttt{Assume } [3] \in U \cap Uh \neq \emptyset,$

$\left([4]\right) := \texttt{E}U : gW = gWh,$

$\left(a, b, [5]\right) := [1][4] : \sum a, b \in W \ . \ ga = gbh,$

$[6] := b^{-1}g^{-1}[5] : b^{-1}a = h,$

$[7] := \texttt{EBalanced}(G, W)(a) : h \in W,$

$[3.*] := [1][2][7]\texttt{I}(\perp) : \perp;$

$\leadsto [h.*] := \texttt{E}(\perp) : U \cap Uh = \emptyset;$

$\leadsto [*] := \texttt{ICoveringAction} : \texttt{CoveringAction}(G, H, \cdot);$

$\square$

$\texttt{CoveringOfGRP} ::$

$:: \forall G, H : \texttt{StronglyConnected} \ \& \ \texttt{TopologicaGroup} \ . \ \forall G \xrightarrow{\varphi} H : \mathsf{TOPGRP} \ .$

$. \ \texttt{Discrete}\Big(\ker \varphi\Big) \ \& \ \texttt{Closed}(G, H, \varphi) \ \& \ \texttt{Open}(G, H, \varphi) \Rightarrow \texttt{CoveringMap}(G, H, \varphi)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CoveringClassification} ::$

$:: \forall B : \texttt{Reasonable} \ .$

$. \ \texttt{Bijection}\left(\texttt{Isoclass}\ \big(\mathsf{COV}(B)\big), \dfrac{\texttt{Subgroup}\ \pi(X)}{\Gamma}, \Big(\Lambda\big[X \xrightarrow{c} B\big] : \texttt{Isoclass}\ \mathsf{COV}(B) \ . \ \big[\pi(c)\big]_\Gamma\Big)\right)$

$\texttt{Proof} \ =$

$F := \Lambda X \xrightarrow{c} B : \mathsf{COV}(B) \ . \ \big[\pi(c)\big]_\Gamma : \dfrac{\texttt{Subgroup}\ \pi(X)}{\Gamma},$

$\Big(\hat{F}, [1]\Big) := \texttt{E}F\texttt{CoveringIsomorphismCriterion} : \sum \hat{F} : \texttt{Isoclass}\ \Big(\mathsf{COV}(B)\Big) \hookrightarrow \dfrac{\texttt{Subgroup}\ \pi(X)}{\Gamma} \ .$

$: \forall (X, c) \in \mathsf{COV}(B) \ . \ \hat{F}[X, c] = F(X, c),$

$(Z, z) := \texttt{UniversalCoverExists}(B) : \texttt{UniversalCover}(B),$

$[2] := \texttt{EUniversalCover}(B, Z, z)\texttt{SimplyConnectedGroupStructure}(B, Z, z) : \mathrm{Gal}(Z \xrightarrow{z} B) \cong_{\mathsf{GRP}} \pi(B),$

$\varphi := \texttt{EIsomorphic}[2] : \texttt{Isomorphism}\Big(\mathsf{GRP}, \mathrm{Gal}(Z \xrightarrow{z} B), \pi(B)\Big),$

$\texttt{Assume}\ [H] : \dfrac{\texttt{Subgroup}\ \pi(X)}{\Gamma},$

$H' := \varphi(H) : \texttt{Subgroup}\ \mathrm{Gal}(Z \xrightarrow{z} B),$

$[3] := \texttt{ICoveringAction} : \texttt{CoveringAction}\Big(Z, H', \texttt{application}\Big),$

$Q := \dfrac{Z}{H'} \in \mathsf{TOP},$

$[4] := \texttt{ActionCoveringIsNormal}(Z, H') : \texttt{NormalCovering}\ (Z, Q, \pi_Q)\ ,$

$[5] := \mathrm{E}H'\mathrm{E}\ \mathrm{Gal}(Z \xrightarrow{z} B) : \forall q \in Q \ . \ \Big|z\big(\pi_Q^{-1}(q)\big)\Big| = 1,$

$\Big(\hat{z}, [6]\Big) := [5]\texttt{FiberMap} : \sum_{\hat{z} \in \mathsf{TOP}(Q, B)} \forall p \in Z \ . \ z(p) = \hat{z}[p],$

$\texttt{Assume}\ p \in B,$

$\Big(U, [6]\Big) := \texttt{ECoveringMap}(Z, B, z) : \sum_{U \in \mathcal{U}(p)} \texttt{EvenlyCovered}(Z, B, z, U),$

$[7] := \texttt{Epreimage}[6] : \pi_Q^{-1}\hat{z}^{-1}(U) = z^{-1}(U),$

$\Big(W, [7.1]\Big) := \texttt{EEvenlyCoveres}(Z, V, z, U) : \sum W : \mathcal{T}(Z) \ \& \ \texttt{StronglyConnected} \ .$

$. \ z^{-1}(U) = \bigsqcup_{g \in \mathrm{Gal}(Z \xrightarrow{z} B)} Wg \ . \ \forall g \in \mathrm{Gal}(Z \xrightarrow{z} B) \ . \ \texttt{Homeomorphism}(Wg, U, z_{|Wg}),$

$\texttt{Assume}\ V \in \mathrm{PCC}(\hat{z}^{-1}(U)),$

$[8] := \texttt{ELocallyPathConnected}(Q)\mathrm{EPCC} : \texttt{Clopen}\Big(V, \hat{z}^{-1}(U)\Big),$

$[9] := \mathrm{E}\mathsf{TOP}(\pi_Q)[7][8] : \texttt{Clopen}\Big(\pi^{-1}(V), z^{-1}(U)\Big),$

$\Big(g, [10]\Big) := \texttt{EClopen}[9][7.1] : \sum g \in \mathrm{Gal}(Z \xrightarrow{z} B) \ . \ \pi^{-1}(V) = \bigsqcup_{h \in H'} Wgh,$

$[p.*] := [10][7.2][5] : \texttt{Homeomorphism}(V, U, \hat{z}_{|V});$

$\rightsquigarrow [6] := \texttt{ICoveringMap} : \texttt{CoverigMap}(Q, B, \hat{z});$

$\texttt{Assume}\ p \in B,$

$\texttt{Assume}\ q \in \hat{z}^{-1}(p),$

$[7] := \texttt{StabilizerOfMonodromyAction}(\hat{z}, p, q) : \mathrm{Stab}_{\curvearrowleft_{\hat{z},p}}(q) = \pi(\hat{z}),$

$[8] := \mathtt{E}\hat{z}[7] : H \subset \pi(\hat{z}),$

$\Big(u, [9]\Big) := \mathtt{E}Q(q) : \sum u \in Z \ .\ q = [u],$

$\texttt{Assume}\ \gamma \in \pi(\hat{z}),$

$[10] := [8](\gamma)[9]\mathtt{E}\mathsf{COV}(B)(Z, Q)(\pi)\mathtt{I}\varphi : q = q\gamma = [u]\gamma = [u\gamma] = \Big[\varphi(\gamma)(u)\Big],$

$[\gamma.*] := \mathtt{E}Q[10] : \gamma \in H;$

$\rightsquigarrow [10] := \texttt{ISubset} : H \subset \pi(\hat{z}),$

$[p.*] := \texttt{ISubsetEq}[8][10] : H = \pi(\hat{z});$

$\rightsquigarrow [7] := \texttt{ENonEmpty}(B) : H = \pi(\hat{z}),$

$[H.*] := \mathtt{I}\hat{F}[7] : H = \hat{F}(\hat{z});$

$\rightsquigarrow [3] := \texttt{ISurjective} : \texttt{Surjective}(\hat{F}),$

$[*] := \texttt{IBijective}[1][3] : \texttt{Bijective}(\hat{F});$

$\square$

$\texttt{HausdorffActionQuotientCriterion} :: \forall X \in \mathsf{TOP}\ .\ \forall G \in \mathsf{GRP}\ .\ \forall \alpha : X \curvearrowleft_{\mathsf{TOP}} G\ .\ \mathtt{T2}\left(\frac{X}{\alpha}\right) \iff$

$\iff \left(\forall x, y \in X\ .\ y \notin O_\alpha(x) \Rightarrow \Big(\exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : \forall g \in G U \cap V g = \emptyset\Big)\right)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : \mathtt{T2}\left(\frac{X}{\alpha}\right),$

$\texttt{Assume}\ x, y \in X,$

$\texttt{Assume}\ [2] : y \notin O_\alpha(x),$

$[3] := \mathtt{E}\frac{X}{\alpha}[2] : [x]_\alpha \neq [y]_\alpha,$

$\Big(U, V, [4]\Big) := \mathtt{ET3}\left(\frac{X}{\alpha}\right)\Big([x], [y]\Big) : \sum U \in \mathcal{U}[x]\ .\ \sum V \in \mathcal{U}[x]\ .\ V \cap U = \emptyset,$

$U' := \pi_\alpha^{-1}(U) : \mathcal{U}(x),$

$V' := \pi_\alpha^{-1}(V) : \mathcal{U}(x),$

$[5] := \mathtt{E}U'\mathtt{E}V'\texttt{DisjointPreimage}[4] : U' \cap V' = \emptyset,$

$[6] := \mathtt{E}V'\mathtt{E}\pi_\alpha : \forall g \in G\ .\ V'g = V',$

$[1.*] := \forall g \in G\ .\ \mathtt{E}\Big(=, [6](g), [5]\Big) : \forall g \in G\ .\ U' \cap V'g = \emptyset;$

$\rightsquigarrow [1] := \mathtt{I}(\Rightarrow) : \mathtt{T2}\left(\frac{X}{\alpha}\right) \Rightarrow \left(\forall x, y \in X\ .\ y \notin O_\alpha(x) \Rightarrow \Big(\exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : \forall g \in G U \cap V g = \emptyset\Big)\right),$

$\texttt{Assume}\ [2] : \forall x, y \in X\ .\ y \notin O_\alpha(x) \Rightarrow \Big(\exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : \forall g \in G U \cap V g = \emptyset\Big),$

$\texttt{Assume}\ [x], [y] : \dfrac{X}{\alpha},$

$\texttt{Assume}\ [3] : \forall [x] \neq [y],$

$[4] := \texttt{E}\pi_\alpha[3]\texttt{I}O_\alpha : y \notin O_\alpha(x),$

$$\Big(U, V, [5]\Big) := [2]\Big(x, y, [4]\Big) : \sum_{U \in \mathcal{U}(x)} \sum_{V \in \mathcal{U}(y)} U \cap V g = \emptyset,$$

$$U' := \bigcap_{g \in G} U g \in \mathcal{U}(x),$$

$$V' := \bigcap_{g \in G} V g \in \mathcal{U}(y),$$

$[6] := \texttt{E}U'\texttt{E}\mathsf{GRP}(G) : \forall g \in G\ .\ U' g = U',$

$[7] := \texttt{E}V'\texttt{E}\mathsf{GRP}(G) : \forall g \in G\ .\ V' g = V',$

$[8] := \texttt{E}U'\texttt{E}V'[5] : U' \cap V',$

$[9] := \texttt{E}\pi_\alpha[6] : \pi^{-1}\pi(U') = U',$

$[10] := \texttt{EQuotinetMap}[9] : \pi(U') \in \mathcal{U}[x],$

$[11] := \texttt{E}\pi_\alpha[7] : \pi^{-1}\pi(V') = V',$

$[12] := \texttt{EQuotinetMap}[9] : \pi(V') \in \mathcal{U}[y],$

$$\Big[\Big([x], [y]\Big) . *\Big] := \texttt{E}\pi_\alpha[6][7][8]\texttt{I}\pi_\alpha : \pi(V') \cap \pi(U') = \emptyset;$$

$$\leadsto [2.*] := \texttt{IT2} : \texttt{T2}\left(\frac{X}{\alpha}\right);$$

$$\leadsto [2] := \texttt{I}(\Rightarrow) : \left(\forall x, y \in X\ .\ y \notin O_\alpha(x) \Rightarrow \Big(\exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : \forall g \in G U \cap V g = \emptyset\Big)\right) \Rightarrow \texttt{T2}\left(\frac{X}{\alpha}\right),$$

$$[3] := \texttt{I}(\iff)[1][2] : \left(\forall x, y \in X\ .\ y \notin O_\alpha(x) \Rightarrow \Big(\exists U \in \mathcal{U}(x) : \exists V \in \mathcal{U}(y) : \forall g \in G U \cap V g = \emptyset\Big)\right) \iff$$
$$\iff \texttt{T2}\left(\frac{X}{\alpha}\right);$$

$\square$

$\texttt{ProperAction} :: \prod X \in \mathsf{TOP}\ .\ \prod G \in \mathsf{TOPGRP}\ .\ ?(X \curvearrowleft_{\mathsf{TOP}} G)$

$\alpha : \texttt{ProperAction} \iff \texttt{ProperMap}(X \times G, X^2, \Lambda(x, g) \in X \times G\ .\ (x, xg))$

$\texttt{ProperActionCriterion} :: \forall X : \texttt{T2} \,.\, \forall G \in \mathsf{TOPGR} \,\&\, \texttt{Compact} \,.\, \forall \alpha : X \curvearrowleft_{\mathsf{TOP}} G \texttt{ProperAction}(X, G, \alpha)$

$\texttt{Proof} \ =$

$\theta := \Lambda(g, x) \in G \times X \,.\, (xg, x) : \mathsf{TOP}(G \times X, X^2),$

$\texttt{Assume}\ K : \texttt{CompactSubset}(X \times X),$

$[1] := \texttt{CompactImage}(K, \pi_2) : \texttt{CompactSubset}(X, \pi_2 K),$

$[2] := \texttt{T2CompactIsClosed}(X^2, K) : \texttt{Closed}(X^2, K),$

$[3] := \texttt{E}\mathsf{TOP}(G \times X, X^2)[2] : \texttt{Closed}\Big(G \times X, \theta^{-1}(K)\Big),$

$[4] := \texttt{E}\theta\Big(\theta^{-1}(K)\Big) : \theta^{-1}(K) \subset G \times K,$

$[5] := \texttt{TychonoffTHM}(G, K) : \texttt{Compact}(G \times K),$

$[6] := \texttt{ClosedSubset}\big(G \times X, G \times K, [3]\big) : \texttt{Closed}\Big(G \times K, \theta^{-1}(K)\Big),$

$[7] := \texttt{ClosedCompactSubset}[6] : \texttt{CompactSubset}\Big(G \times K, \theta^{-1}(K)\Big),$

$[K.*] := \texttt{ComapcCompactSubset}[5][7] : \texttt{CompactSubset}\Big(G \times X, \theta^{-1}(K)\Big);$

$\leadsto [*] := \texttt{IProperAction} : \texttt{ProperAction}\big(G, X\big);$

$\square$

$\texttt{ProperActionbyCompactOrbit} :: \forall X : \texttt{T2} . \forall G \in \mathsf{TOPGR} . \forall \alpha : X \curvearrowleft_{\mathsf{TOP}} G . \texttt{ProperAction}(X, G, \alpha) \iff$

$$\iff \left(\forall K : \texttt{CompactSubset}(X) . \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big)\right)$$

$\texttt{Proof} =$

$\theta := \Lambda(g, x) \in G \times X . (xg, x) : \mathsf{TOP}(G \times X, X^2),$

$\texttt{Assume } [1] : \texttt{ProperAction}(X, G, \alpha),$

$\texttt{Assume } K : \texttt{CompactSubset}(X),$

$[2] := \texttt{TychonoffTHM}(K, K) : \texttt{CompactSubset}(X^2, K^2),$

$[3] := \texttt{E}\theta\texttt{IsetBuilder} : \theta^{-1}(K^2) = \Big\{(g, x) \in G \times K : gx \in K\Big\},$

$[4] := \texttt{EProperAction}(X, G, \alpha) : \texttt{ComapactSubset}\Big(G \times X, \theta^{-1}(K^2)\Big),$

$[1.*] := \texttt{CompactImage}[4][3] : \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big);$

$\rightsquigarrow [1] := \texttt{I}\forall\texttt{I} \Rightarrow : \texttt{ProperAction}(X, G, \alpha) \Rightarrow$

$$\Rightarrow \left(\forall K : \texttt{CompactSubset}(X) . \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big)\right),$$

$\texttt{Assume } [2] : \forall K : \texttt{CompactSubset}(X) . \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big),$

$\texttt{Assume } K : \texttt{CompactSubset}(X \times X),$

$[3] := \texttt{CompactImage}(K, \pi_2) : \texttt{CompactSubset}(X, \pi_2 K),$

$[4] := \texttt{CompactImage}(K, \pi_1) : \texttt{CompactSubset}(X, \pi_1 K),$

$L := \pi_1 K \cap \pi_2 K : \texttt{CompactSubset}(X),$

$H := \{g \in G : K \cap gK = \emptyset\} : \texttt{CompacSubset}(G),$

$[5] := \texttt{PreimageSubsetE}\theta\texttt{I}H : \theta^{-1}(K) \subset \theta^{-1}(L \times L) = \Big\{(g, x) \in G \times X . gx \in L\Big\} \subset H \times L,$

$[6] := \texttt{TychonoffTHM}(H, L) : \texttt{CompactSubset}(H \times L, G \times X),$

$[7] := \texttt{CompactClosedSubset}[5] : \texttt{CompactSubset}\Big(H \times L, \theta^{-1}(K)\Big),$

$[K.*] := \texttt{CompactCompactSubset}[7][6] : \texttt{CompactSubset}\Big(G \times X, \theta^{-1}(K)\Big);$

$\rightsquigarrow [2.*] := \texttt{IProperAction} : \texttt{ProperAction}(X, G, \alpha);$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow : \left(\forall K : \texttt{CompactSubset}(X) . \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big)\right) \Rightarrow$

$$\Rightarrow \texttt{ProperAction}(X, G, \alpha),$$

$[*] := \texttt{I} \iff [1][2] : \texttt{ProperAction}(X, G, \alpha) \iff$

$$\iff \left(\forall K : \texttt{CompactSubset}(X) . \texttt{CompactSubset}\Big(G, \{g \in G : K \cap gK \neq \emptyset\}\Big)\right);$$

$\texttt{HausdorffByProperAction} :: \forall X : \texttt{T2} \ \& \ \texttt{LocallyCompact} \ . \ \forall G \in \mathsf{TOPGR} \ . \ \forall \alpha : \texttt{ProperAction}(X, G) \ .$

$$\texttt{T2}\left(\frac{X}{\alpha}\right)$$

$\texttt{Proof} \ =$

$\theta := \Lambda(g, x) \in G \times X \ . \ (xg, x) : \texttt{Type}ProperMap(G \times X, X^2),$

$[1] := \texttt{HausdorffProduct}(X, X) \ \& \ \texttt{LocallyCompactProduct}(X, X) : \texttt{T2} \ \& \ \texttt{LocallyCompact}(X^2),$

$[2] := \texttt{EmbeddingProperIffClosed}(\theta)[1] : \texttt{ClosedMap}(G \times X, X^2, \theta),$

$[3] := \texttt{EClosedMap}(G \times X, X^2, \theta)(G \times X) : \texttt{Closed}\Big(X^2, \theta(G \times X)\Big),$

$[*] := \texttt{T2ByClosedOrbitRelation}[3] : \texttt{T2}\left(\frac{X}{\alpha}\right);$

$\square$

ProperByDiscreteAction ::

$$:: \forall X \in \mathsf{TOP} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall \alpha : \texttt{CoveringAction} \,.\, \texttt{T2}\left(\frac{X}{G}\right) \Rightarrow \texttt{ProperAction}(X, G, \alpha)$$

Proof $=$

$Q := \dfrac{X}{G} \in \mathsf{TOP}$,

$[1] := \texttt{ActionCoveringIsNormal}(X, G, \alpha) : \texttt{NormalCovering}(Q, X, \pi_Q)$,

$\mathcal{O} := \{(x, xg) | x \in X, g \in G\} :?(X \times X)$,

$[2] := \texttt{HausdorffIfRelationIsClosed}(X, G) : \texttt{Closed}(X, G)$,

$[3] := \texttt{HausdorffByCovering}[1] : \texttt{T2}(X)$,

Assume $K : \texttt{CompactSubset}(X \times X)$,

$H := \{g \in G : K \cap gK \neq \emptyset\} :?G$,

Assume $[4] : !\, \texttt{CompactSubset}\Big(G, H\Big)$,

$[5] := \texttt{EDiscreteGroup}(G)[4] : |H| = \infty$,

Assume $g \in H$,

$\Big(x_g, [6]\Big) := \mathtt{E}H(g) : \sum x \in K \,.\, xg \in K$,

$F(g) := (x_g g, x_g) : K \times K$;

$\rightsquigarrow F := \mathtt{I}(\rightarrow) : H \to K \times K$,

$[6] := \texttt{EFree}(X, G, \alpha)\mathtt{I}F : \texttt{Injective}(H, K \times K, F)$,

$[7] := [5][6] : \Big|F(H)\Big| = \infty$,

$(x, y) := \texttt{LimitCompact}(K \times K)[7] : \texttt{LimitPoint}\Big(F(H)\Big)$,

$[8] := \mathtt{E}F(H)\mathtt{E}\mathcal{O}\mathtt{ISubsetI}F(H)\mathtt{I}\mathcal{O} : F(X) \subset \mathcal{O}$,

$[9] := \texttt{ClosedLimit}[8](x, y) : (x, y) \in \mathcal{O}$,

$\Big(g, [10]\Big) := \mathtt{E}\mathcal{O}[9] : \sum g \in G \,.\, x = yg$,

$\Big(U, [11]\Big) := \texttt{HausdorffByGroupActionQuotientCriterion} \in \sum_{U \in \mathcal{U}(y)} \forall g \in G \,.\, gU \cap U = \emptyset$,

$V := Vg \in \mathcal{U}(x)$,

$[12] := \texttt{ELimitPoint}(x, y)(U \times V) : \Big|U \times V \cap F(H)\Big| = \infty$,

Assume $h \in H$,

Assume $[13] : F(h) \in V \times U$,

$p := \pi_2 F(h) \in U$,

$[14] := (\mathtt{E}U\mathtt{E}F)(p) : pg = ph$,

$[15] := \texttt{EFree}\alpha[14] : g = h$,

$\rightsquigarrow [13] := \mathtt{I}\mathsf{CARD} : |H| = 1$;

$[14] := \mathtt{I}\bot[13][5] : \bot$;

$\rightsquigarrow [4] := \mathtt{E}(\bot) : \texttt{CompactSubset}(G, H)$;

$[*] := \texttt{ProperActioByCompactOrbit}[4] : \texttt{ProperAction}(X, G, \alpha)$;

$\square$

$\texttt{RegularCoveringAction} :: \forall X : \texttt{StronglyConnected} \ \& \ \texttt{LocallyCompact} \ \& \ \texttt{T2} \ . \ \forall G : \texttt{DiscreteGroup} \ .$

$. \ \forall \alpha : \texttt{ProperAction} \ \& \ \texttt{Free}(X, G) \ . \ \texttt{CoveringAction}(X, G, \alpha) \ \& \ \texttt{T2}\left(\frac{X}{G}\right) \ \&$

$\& \ \texttt{NormalCovering}\left(X, \frac{X}{G}, \pi\right)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ p \in X,$

$\Big(V, [1]\Big) := \texttt{ELocallyComapct}(X, p) : \sum V \in \mathcal{U}(p) \ . \ \texttt{CompactSubset}(X, \overline{V}),$

$K := \overline{V} : \texttt{CompactSubset}(X),$

$H := \{g \in G : K \cap gK \neq \emptyset\} : \texttt{CompactSubset}(G),$

$[2] := \texttt{EDiscreteGroup}(G)(H) : |H| < \infty,$

$m := |H| \in \mathbb{N},$

$h := \texttt{enumerate}(H) : [1, \ldots, m] \leftrightarrow H,$

$[3] := \texttt{EFree}(X, G, \alpha)(p) : \forall g \in G \ . \ pg = p \iff g = e,$

$\Big(W, W', [4]\Big) := \texttt{ET2}[3](p, ph) : \prod_{i=1}^{m} \sum_{W_i \in \mathcal{U}(p)} \sum_{W'_i \in \mathcal{U}(ph_i)} W_i \cap W'_i = \emptyset,$

$U := V \cap \bigcap_{i=1}^{m} W_i \cap W'_i h_i^{-1} \in \mathcal{U}(p),$

$\texttt{Assume} \ i \in [1, \ldots, m],$

$\texttt{Assume} \ u \in U,$

$[5] := \texttt{E}U(u)\texttt{I}W'_i h_i^{-1} : u \in W'_i h_i^{-1},$

$[6] := [5]h_i : uh_i \in W'_i,$

$[i.*] := [4][6] : uh_i \notin U;$

$\rightsquigarrow [4] := \texttt{IDisjointI}\forall : \forall i \in [1, \ldots, m] \ . \ Uh_i \cap U = \emptyset,$

$\texttt{Assume} \ g : H^{\complement},$

$\texttt{Assume} \ [5] : g \neq e,$

$\texttt{Assume} \ u \in U,$

$[6] := \texttt{E}U(u)[1] : ug \in Kg,$

$[7] := \texttt{E}H^{\complement}(g) : Kg \cap K = \emptyset,$

$[g.*] := \texttt{I}U[6][7] : ug \notin U;$

$\rightsquigarrow [5] := \texttt{IDisjointI}(\Rightarrow)\texttt{I}(\forall) : \forall g \in H^{\complement} \ . \ g \neq e \Rightarrow Ug \cap U = \emptyset,$

$[p.*] := [4][5] : \forall G \in G^{\complement} \ . \ g \neq e \Rightarrow Ug \cap U = \emptyset;$

$\rightsquigarrow [1] := \texttt{HausdorffByGroupQuotientCriterion} : \texttt{T2}\left(\frac{X}{\alpha}\right),$

$[*] := \texttt{ProperByDiscreteAction}[1] : \texttt{ProperAction}(X, G, \alpha);$

$\square$

$\texttt{ManifoldByProperAction} :: \forall X \in \mathsf{TOPM} \ . \ \forall G : \texttt{DiscreteGroup} \ .$

$. \ \forall \alpha : \texttt{ProperAction} \ \& \ \texttt{Free}(X, G) \ . \ \frac{X}{G} \in \mathsf{TOPM}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

6.9 Applications to Geometic Topology

$\texttt{SphereCoversProjectiveSpace} :: \forall n \in \mathbb{N} \,.\, (\mathbb{S}^n, \pi) \in \mathsf{COV}(\mathbb{RP}^n)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ p \in \mathbb{RP}^n,$

$\Big(e, [1]\Big) := \texttt{ProjectiveCoordinatesExists}(n, p) : \sum e : \texttt{ProjectiveCooedinates}(\mathbb{R}, n) \,.\, p_e = [1, 0, \dots, 0],$

$U := \left\{ q \in \mathbb{RP}^n \middle| p^1_e \neq 0 \right\} \in \mathcal{U}(p),$

$V_+ := \left\{ x \in \mathbb{S}^n \middle| x^1_e > 0 \right\} : \mathcal{T}(\mathbb{S}^1) \ \&\ \texttt{StronglyConnected},$

$V_- := \left\{ x \in \mathbb{S}^n \middle| x^1_e < 0 \right\} : \mathcal{T}(\mathbb{S}^1) \ \&\ \texttt{StronglyConnected},$

$[2] := \mathtt{E}U\mathtt{I}V_+\mathtt{I}V_- : \pi^{-1}(U) = V_- \sqcup V_+,$

$[*.1] := \mathtt{E}\pi\mathtt{I}V_+\mathtt{I}U : \texttt{Homeomorphism}\Big(U, V_+, \pi_{|V_+}\Big),$

$[*.2] := \mathtt{E}\pi\mathtt{I}V_-\mathtt{I}U : \texttt{Homeomorphism}\Big(U, V_-, \pi_{|V_-}\Big);$

$\rightsquigarrow [*] := \texttt{ICoveringMap} : \texttt{CoveringMap}(\mathbb{S}^n, \mathbb{RP}^n, \pi);$

$\square$

$\texttt{complexSquareRootSpace} :: ?\mathbb{C}^2$

$\texttt{complexSquareRootSpace}\,() = \sqrt{\mathbb{C}} := \left\{ (z, w) \in \mathbb{C}^2 \middle| z \neq 0, z = w^2 \right\}$

$\texttt{ComplexSquareRootCovers} :: \ (\sqrt{\mathbb{C}}, \pi_1) \in \mathsf{COV}(\mathbb{C} \setminus \{0\})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ p : \mathbb{C} \setminus \{0\},$

$A := \texttt{if}\ p \in \mathbb{R}_{--}\ \texttt{then}\ \Im\ \texttt{else}\ \Re \in \mathbb{C} \to \mathbb{R},$

$U := \texttt{if}\ p \in \mathbb{R}_{--}\ \texttt{then}\ \{z \in \mathbb{C} \setminus \{0\} : z \notin \mathbb{R}_{++}\}\ \texttt{else}\ \{z \in \mathbb{C} \setminus \{0\} : z \notin \mathbb{R}_{--}\} \in \mathcal{U}\Big(\mathbb{C} \setminus \{0\}\Big),$

$[1] := \mathtt{E}U\mathtt{E}A : \forall (u, v) \in \pi_1^{-1}(U) \,.\, A(v) \neq 0,$

$V_+ := \left\{ (u, v) \in \sqrt{\mathbb{C}} \middle| A(v) > 0 \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \&\ \texttt{StronglyConnected},$

$V_- := \left\{ x \in \sqrt{\mathbb{C}} \middle| A(v) < 0 \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \&\ \texttt{StronglyConnected},$

$[2] := \mathtt{E}U\mathtt{E}V_+\mathtt{E}V_-\mathtt{E}A : \pi_1^{-1}U = V_+ \cup V_-,$

$[*.1] := \mathtt{E}\pi\mathtt{I}V_+\mathtt{I}U : \texttt{Homeomorphism}\Big(U, V_+, \pi_{|V_+}\Big),$

$[*.2] := \mathtt{E}\pi\mathtt{I}V_-\mathtt{I}U : \texttt{Homeomorphism}\Big(U, V_-, \pi_{|V_-}\Big);$

$\rightsquigarrow [*] := \texttt{ICoveringMap} : \texttt{CoveringMap}\Big(\sqrt{\mathbb{C}}, \mathbb{C} \setminus \{0\}, \pi_1\Big);$

$\square$

$\texttt{TorusCoversKleinBottel} :: \exists c : \texttt{CoveringMap}(\mathbb{T}^2, \mathbf{KB}) : \text{num } c = 2$

$\texttt{Proof} \ =$

$[1] := \texttt{TorusAsGroup} : \mathbb{T}^2 = \dfrac{\mathbb{R}^2}{\mathbb{Z}^2},$

$[2] := [1]\texttt{I}\mathbf{KB} : \dfrac{\mathbb{T}^2}{[s,t] \sim [s+1/2, 1-t]} \cong_{\mathsf{TOP}} \mathbf{KB},$

$\texttt{Assume } p \in \mathbf{KB}^2,$

$\texttt{Assume } [3] : \forall t \in I \ . \ p \neq [0,t],$

$U := (0, 1/2) \times I \in \mathcal{U}(p),$

$V_+ := \left\{ [a,b] \in \mathbb{T} \middle| a > \dfrac{1}{2} \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \& \ \texttt{StronglyConnected},$

$V_- := \left\{ [a,b] \in \mathbb{T} \middle| a < \dfrac{1}{2} \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \& \ \texttt{StronglyConnected},$

$[3.*.1] := \texttt{E}\pi\texttt{I}V_+\texttt{I}U : \texttt{Homeomorphism}\Big(U, V_+, \pi_{|V_+}\Big),$

$[3.*.2] := \texttt{E}\pi\texttt{I}V_-\texttt{I}U : \texttt{Homeomorphism}\Big(U, V_-, \pi_{|V_-}\Big);$

$\rightsquigarrow [3] := \texttt{I}(\Rightarrow) : \Big(\forall t \in I \ . \ p \neq [0,t]\Big) \Rightarrow \exists\texttt{EvenlyCovered}(\mathbb{T}^2, \mathbf{KB}, \pi),$

$\texttt{Assume } t \in I,$

$\texttt{Assume } [4] : p = [0,t],$

$U := \Big([0, 1/4) \sqcup (1/4, 1/2)\Big) \times I \in \mathcal{U}(p),$

$V_+ := \left\{ [a,b] \in \mathbb{T} \middle| \dfrac{1}{4} < a < \dfrac{3}{4} \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \& \ \texttt{StronglyConnected},$

$V_- := \left\{ [a,b] \in \mathbb{T} \middle| a < \dfrac{1}{4} \vee a > \dfrac{3}{4} \right\} : \mathcal{T}\sqrt{\mathbb{C}} \ \& \ \texttt{StronglyConnected},$

$[4.*.1] := \texttt{E}\pi\texttt{I}V_+\texttt{I}U : \texttt{Homeomorphism}\Big(U, V_+, \pi_{|V_+}\Big),$

$[4.*.2] := \texttt{E}\pi\texttt{I}V_-\texttt{I}U : \texttt{Homeomorphism}\Big(U, V_-, \pi_{|V_-}\Big);$

$\rightsquigarrow [4] := \texttt{I}(\Rightarrow) : \Big(\forall t \in I \ . \ p \neq [0,t]\Big) \Rightarrow \exists\texttt{EvenlyCovered}\Big(\mathbb{T}^2, \mathbf{KB}, \pi\Big),$

$[p.*] := \texttt{E}(|)\texttt{LEM}[3][4] : \exists\texttt{EvenlyCovered}\Big(\mathbb{T}^2, \mathbf{KB}, \pi\Big);$

$\rightsquigarrow [3] := \texttt{ICoveringMap} : \texttt{CoveringMap}\Big(\mathbb{T}^2, \mathbf{KB}, \pi\Big);$

$[4] := \texttt{I} \ \text{num}[2][3] : \text{num } \pi = 2;$

$\square$

$\texttt{CoveringOfConnectedSum} :: \forall B, D \in \mathsf{TOPM}(n) \ \& \ \texttt{Connected} \ . \ \forall c : \texttt{CoveringMap}(X, B) \ . \ \forall k \in \mathbb{N} \ .$

$. \ \forall [0] : \text{numa} = k \ . \ \exists c' : \texttt{Covering}\left(X \# \mathop{\#}_{i=1}^{k} D, B \# D\right) \ . \ \text{num } c' = k$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{OriantableCoversNonoriantable} :: \forall M : \texttt{Nonoriantable} \ . \ \exists N : \texttt{Oriantable} :$

$: \forall c : \texttt{CoveringMap}(N, M) : \text{num } c = 2 \ \& \ \text{gen } M = 1 + \text{gen } N$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{toriParametricCovering} :: \left(\mathbb{Z}^2 \setminus \{0\}\right)^2 \to \texttt{NormalCovering}(\mathbb{T}^2, \mathbb{T}^2)$

$\texttt{toriParametricCovering}\,(a, b) = \tau_{a,b} := \Lambda(u, v) \in \mathbb{T}^2 \;.\; \left(u^{a_1} v^{a_2}, u^{b_1} v^{b_2}\right)$

$\texttt{ClassificationOfToriCoverings} :: \forall (X, c) \in \mathsf{COV}(\mathbb{T}^2) \;.\; X = \mathbb{R}^2 \;\&\; c = s \times s \Big|$

$\Big| X = \mathbb{R} \times \mathbb{S}^1 \;\&\; \exists a, b \in \mathbb{Z}^2 \;.\; c = \tau_{a,b} \circ (s \times \mathrm{id}) \Big|$

$\Big| X = \mathbb{T}^2 \;\&\; \exists a, b \in \mathbb{Z}^2 \;.\; c = \tau_{a,b}$

$\texttt{Proof} =$

...

$\square$

$\texttt{spaceOfLens} :: \texttt{Coprime} \to \mathsf{TOPM}(3)$

$\texttt{spaceOfLence}\,(n, m) = \mathbb{L}_{n,m} := \dfrac{\mathbb{S}^3}{\alpha}$ **where**

$$\alpha = \Lambda(z_1, z_2) \in \mathbb{S}^3 \;.\; \Lambda k \in \frac{\mathbb{Z}}{n\mathbb{Z}} \;.\; \left(\exp\left(\frac{2\pi \mathrm{i} k}{n}\right) z_1, \exp\left(\frac{2\pi \mathrm{i} k m}{n}\right) z_2\right)$$

6.10 Hyperbolic Cells and Presentation