# Ordered Fields

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# Contents

# 1 Totally Ordered Fields

## 1.1 Basic Definitions and Inequelity Algebra

$$\texttt{OrderedField} :: \sum S : \texttt{Set} \,.\, (S \times S \to S)^2 \times \texttt{TotalOrder}(S)$$

$$(k, +, \cdot, \geq) : \texttt{OrderedField} \iff (k, +, \cdot) : \texttt{Field} \,\&\, \forall x, y, z \in k \,.$$
$$x \leq y \Rightarrow z + x \geq z + y$$
$$x \geq 0 \,\&\, y \geq 0 \Rightarrow xy \geq 0$$

$$\texttt{addToIneq} :: \prod k : \texttt{OrderedField} \,.\, \prod a, b, c \in k \,.\, a \geq b \to a + c \geq b + c$$
$$\texttt{addToIneq}\,(x) = x + c := \eth\texttt{OrderedField}(k)(a, b, c)$$

$$\texttt{implicit} :: \texttt{OrderedField} \to \texttt{Field}$$
$$\texttt{implicit}\,(S, +, \cdot, \geq) := (S, +, \cdot)$$

$$\texttt{implicit} :: \texttt{OrderedField} \to \texttt{Poset}$$
$$\texttt{implicit}\,(S, +, \cdot, \geq) := (S, \geq)$$

$\texttt{NegateIneq} :: \forall k : \texttt{OrderedField} \,.\, \forall a \in k \,.\, -a \geq 0 \geq a | a \geq 0 \geq -a$

$\texttt{Proof} =$

$\texttt{Assume}\ (1) : a \geq 0,$

$\texttt{Assume}\ (2) : -a \geq 0,$

$(3) := \eth\texttt{OrderedField}(k)(-a, 0, a) : 0 \geq a,$

$(4) := \eth\texttt{Order}(1)(3) : 0 = a,$

$() := \texttt{ZeroInverse}(4) : 0 \geq -a;$

$\rightsquigarrow (2) := I(\Rightarrow) : -a \geq 0 \Rightarrow 0 \geq -a,$

$(3) := \texttt{SelfImplication}(0 \geq -a) : 0 \geq -a \Rightarrow 0 \geq -a,$

$(4) := \eth\texttt{Total}(\geq)(0, -a) : -a \geq 0 | 0 \geq -a,$

$(5) := E(|)(2, 3, 4) : 0 \geq -a,$

$() := I(I)(5)(-a \geq 0 \geq a) : -a \geq 0 \geq a | a \geq 0 \geq -a;$

$\rightsquigarrow (1) := I(\Rightarrow) : a \geq 0 \Rightarrow -a \geq 0 \geq a | a \geq 0 \geq -a,$

$\texttt{Assume}\ (2) : 0 \geq a,$

$\texttt{Assume}\ (3) : 0 \geq -a,$

$(4) := \eth\texttt{OrderedField}(k)(0, -a, a) : a \geq 0,$

$(5) := \eth\texttt{Order}(1)(3) : 0 = a,$

$() := \texttt{ZeroInverse}(4) : 0 \geq -a;$

$\rightsquigarrow (3) := I(\Rightarrow) : 0 \geq -a \Rightarrow -a \geq 0,$

$(4) := \texttt{SelfImplication}(-a \geq 0) : -a \geq 0 \Rightarrow -a \geq 0,$

$(5) := \eth\texttt{Total}(\geq)(0, -a) : -a \geq 0 | 0 \geq -a,$

$(6) := E(|)(2, 3, 4) : -a \geq -0,$

$() := I(I)(5)(-a \geq 0 \geq a) : -a \geq 0 \geq a | a \geq 0 \geq -a;$

$\rightsquigarrow (2) := I(\Rightarrow) : 0 \geq a \Rightarrow -a \geq 0 \geq a | a \geq 0 \geq -a,$

$(3) := \eth\texttt{Total}(\geq)(2) : 0 \geq a | a \geq 0,$

$(*) := E(|)(3, 2, 1) : -a \geq 0 \geq a | a \geq 0 \geq -a,$

$\square$

$\texttt{neqIneq} :: \prod k : \texttt{OrderedField} \,.\, \prod a.b \in k \,.\, a \le b \to -b \le -a$

$\texttt{neqIneq}\,(x) = -x := \texttt{NegateIneq}(x)(b-a)$

$\texttt{AddIneq} :: \forall k : \texttt{OrderedField} \,.\, \forall a, b, c, d \in k \,.\, a \le b \,\&\, c \le d \Rightarrow a + c \le b + d$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{OrderedField}(k)(a, b, c) : a + c \le b + c,$

$(2) := \eth\texttt{OrderedField}(k)(c, d, b) : b + c \le c + d,$

$(*) := \eth\texttt{Transitive}(k)(1)(2) : a + c \le b + d;$

$\square$

$\texttt{addIneq} :: \prod k : \texttt{OrderedField} \,.\, \prod a, b, c, d \in k \,.\, a \ge b \,\&\, c \ge d \to a + c \ge b + d$

$\texttt{addIneq}\,(x, y) = x + y := \texttt{AddIneq}(b, a, d, c)(x, y)$

$\texttt{UnityIsGreaterThenZero} :: \forall k : \texttt{OrderedField} \,.\, 1_k > 0_k$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : 0_k \ge 1_k,$

$(2) := \texttt{NegateIneq}(1) : -1_k \ge 0_k,$

$(3) := \eth\texttt{OrderedField}(k)(-1_k, -1_k) : 1_k \ge 0_K,$

$(4) := \eth\texttt{Antisymmetric}(\ge)(1, 3) : 0_k = 1_k,$

$(5) := \texttt{FieldContradiction}(4) : \perp;$

$\rightsquigarrow (*) := E(\perp) : 1_k \ge 0_k;$

$\square$

$\texttt{positivePart} :: \prod k : \texttt{OrderedField} \,.\, ?k$

$\texttt{positivePart}\,() = k_{++} := \{x \in k : x > 0\}$

$\texttt{nonNegativePart} :: \prod k : \texttt{OrderedField} \,.\, ?k$

$\texttt{positivePart}\,() = k_{+} := \{x \in k : x \ge 0\}$

$\texttt{MultIneq} :: \forall k : \texttt{OrderedField} \,.\, \forall a, b, c, d \in k_+ \,.\, a \ge b \,\&\, c \ge d \Rightarrow ac \ge bd$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{OrderedField}(k)(c, d, -d) : c - d \ge 0,$

$(2) := \eth\texttt{OrderedField}(k)(a, b, -b) : a - b \ge 0,$

$(3) := \eth\texttt{Distributive}(k, +, \cdot)(c, d, a)\eth\texttt{OrderedField}(k)(c - d, a)(1) : a(c - d) = ac - ad \ge 0,$

$(4) := \eth\texttt{Distributive}(k, +, \cdot)(a, b, d)\eth\texttt{OrderedField}(k)(a - b, d)(2) : (a - b)d = ad - bd \ge 0,$

$(5) := \eth\texttt{OrderedField}(k)(ac - ad, 0, ad)(3) : ac \ge ad,$

$(6) := \eth\texttt{OrderedField}(k)(ad - bd, 0, bd)(4) : ad \ge bd,$

$(*) := \eth\texttt{Transitive}(\ge)(5, 6) : ac \ge bd,$

$\square$

$\texttt{multIneq} :: \prod k : \texttt{OrderedField} \,.\, \forall a, b, c, d \in k_+ \,.\, a \ge b \,\&\, c \ge d \to ac \ge bd$

$\texttt{multIneq}\,(x, y) = x \cdot y := \texttt{MulIneq}(k)(a, b, c, d)(x, y)$

$\texttt{InverseOfPositiveIsPositive} :: \forall k : \texttt{OrderedField} \,.\, \forall a \in k_{++} \,.\, a^{-1} \in k_{++}$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{inverse}(a)\texttt{UnityIsGreaterThenZero} : aa^{-1} = 1 > 0,$

$\texttt{Assume}\ (2) : 0 > a^{-1},$

$(3) := -(2) : -a^{-1} > 0,$

$(4)) := (1) \cdot (3) : -a > 0,$

$(5) := \texttt{StrictOrderContradiction}(\eth k_{++}(\eth a))(4) : \perp;$

$\rightsquigarrow (*) := E(\perp) : a^{-1} > 0;$

$\texttt{InverseIneq} :: \forall k : \texttt{OrderedField} \,.\, \forall a, b \in k_{++} \,.\, \forall (0) : a \geq b \,.\, b^{-1} \geq a^{-1}$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Reflexive}(\geq)(b^{-1}) : b^{-1} \geq b^{-1},$

$(2) := \eth\texttt{Reflexive}(\geq)(a^{-1}) : a^{-1} \geq a^{-1},$

$(3) := \texttt{InversOfPositiveIsPositive}(a) : a^{-1} > 0,$

$(4) := \texttt{InverseOfPositiveIsPositive}(b) : b^{-1} > 0,$

$(*) := (0) \cdot (1) \cdot (2) : b^{-1} \geq a^{-1},$

$\square$

$\texttt{InverseIneq} :: \prod k : \texttt{OrderedField} \,.\, \forall a, b \in k_{++} \,.\, a \geq b \to b^{-1} \geq a^{-1}$

$\texttt{InverseIneq}\,(x) = x^{-1} := \texttt{InverseIneq}(x)$

$\texttt{SquareIsNonNeg} :: \forall k : \texttt{OrderedField} \,.\, \forall a \in k \,.\, a^2 \in k_{+}$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Antisimmetric}(\geq)(a, 0) : a \geq 0 | 0 \geq a,$

$\texttt{Assume}\ (2) : a \geq 0,$

$() := (2)^2 : a^2 \geq 0;$

$\rightsquigarrow (2) := I(\Rightarrow) : a \geq 0 \Rightarrow a^2 \geq 0,$

$\texttt{Assume}\ (3) : 0 \geq a,$

$(4) := -(3) : -a \geq 0,$

$(5) := (4)^2 : a^2 \geq 0;$

$\rightsquigarrow (3) := I(\Rightarrow) : 0 \geq a \Rightarrow a^2 \geq 0,$

$(*) := E(|)(1, 2, 3) : a^2 \geq 0;$

$\square$

## 1.2 Roots of Inequelities

$\texttt{RootEq} :: \forall R : \texttt{OrderedField} \,.\, \forall x, y \in R_+ \,.\, x^2 = y^2 \iff x = y$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : x^2 = y^2,$

$(2) := \eth x^2 : \Big(x : \texttt{Root}(\Lambda u \in R \,.\, x^2 - u^2)\Big),$

$(3) := \eth y^2(1) : \Big(y : \texttt{Root}(\Lambda u \in R \,.\, x^2 - u^2)\Big),$

$(4) := \texttt{SimpleQuadaraticRoots}(x) : \texttt{Root}(\Lambda u \in R \,.\, x^2 - u^2) = \{-x, x\},$

$(5) := (3)(4) : y = x | y = -x,$

$(6) := \texttt{NegValue}(\eth R_+)(\eth x) : -x \leq 0,$

$() := \texttt{LEM}(6)(5)(\eth R_+)(\eth y) : x = y;$

$\rightsquigarrow (1) := I(\Rightarrow) : x^2 = y^2 \Rightarrow x = y,$

$\texttt{Assume}\ (2) : x = y,$

$() := E(\rightarrow, =)(\texttt{square})(2) : x^2 = y^2;$

$\rightsquigarrow () := I(\Rightarrow) : x = y \Rightarrow x^2 = y^2,$

$(*) := I(\iff)(1)(2) : x^2 = y^2 \iff x = y;$

$\square$

$\texttt{RootIneq} :: \forall R : \texttt{OrderedField} \,.\, \forall x, y \in R_+ \,.\, x^2 \geq y^2 \iff x \geq y$

$\texttt{Proof} \ =$

$\texttt{Assume}\ 1 : x^2 \geq y^2,$

$\texttt{Assume}\ 2 : x < y,$

$(3) := (2)^2 : x^2 < y^2,$

$() := \eth\texttt{Antisymmetric}(\texttt{order})(1, 3) : \bot;$

$\rightsquigarrow (2) := E(\bot) : x \geq y;$

$\rightsquigarrow (1) := I(\Rightarrow) : x^2 \geq y^2 \Rightarrow x \geq y,$

$\texttt{Assume}\ (2) : x \geq y,$

$() := (2)^2 : x^2 \geq y^2;$

$\rightsquigarrow (3) := I(\Rightarrow) : x \geq y \Rightarrow x^2 \geq y^2,$

$(*) := I(\iff) : x^2 \geq y^2 \iff x \geq y;$

$\square$

$\texttt{rootEq} :: \prod R : \texttt{OrderedField} \,.\, \prod a, b \in R_+ \,.\, a^2 = b^2 \rightarrow a = b$

$\texttt{RootEq}\,(x) = \sqrt{x} := \texttt{RootEq}(x)$

$\texttt{rootIneq} :: \prod R : \texttt{OrderedField} \,.\, \prod a, b \in R_+ \,.\, a^2 \geq b^2 \rightarrow a \geq b$

$\texttt{rootIneq}\,(x) = \sqrt{x} := \texttt{RootIneq}(x)$

## 1.3 Archimedean Property

$\texttt{OrderedFieldHasCharZero} :: \forall k : \texttt{OrderedField} \, . \, \mathrm{char}\, k = 0$

$\texttt{Proof} \; =$

$(0) := \texttt{UnityIsGreaterThenZero}(k) : 1_k > 0_k,$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; (1) : \mathrm{char}\, k = n,$

$(2) := \eth\mathrm{char}(2) : n_k = 0,$

$\texttt{Assume}\; m : \mathbb{N},$

$\texttt{Assume}\; (3) : m_k > 0_k,$

$(4) := ((3) + 1_k)(0) : m_k + 1_k \geq 1_k > 0_k;$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall m \in \mathbb{N} \, . \, m_k > 0_k \Rightarrow m_k + 1 > 0_k,$

$(3) := E(\mathbb{N})(0)(2) : \forall m \in \mathbb{N} \, . \, m_k > 0,$

$(4) := (3)(n) : n_k > 0,$

$() := \texttt{StrictOrderContradiction}(1)(4) : \perp;$

$\rightsquigarrow (*) := \eth^1\mathrm{char} I(\forall)E(\perp) : \mathrm{char}\, k = 0;$

$\square$

$\texttt{Archimedean} :: ?\texttt{OrderedField}$

$k : \texttt{Archimedean} \iff \forall a \in k \, . \, \exists n \in \mathbb{N} \, . \, n_k \geq a$

$\texttt{InverseArchimedean} :: \forall k : \texttt{Archimedean} \, . \, \forall x \in k^+ \, . \, \exists n \in \mathbb{N} \, . \, \frac{1}{n} \leq x$

$\texttt{Proof} \; =$

$(1) := \texttt{InverseOfPosistiveIsPositive}(k)(x) : x^{-1} \in k^+ +,$

$(n, 2) := \eth\texttt{Archimedean}(x^{-1}) : \sum n \in \mathbb{N} \, . \, n > x^{-1},$

$(*) := (2)^{-1} : n^{-1} < x;$

## 1.4 Rational Numbers as Example

$\texttt{RationalNumbersAreOrderedField} :: \mathbb{Q} : \texttt{OrderedField}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \frac{a}{b}, \frac{c}{d}, \frac{x}{y} : \mathbb{Q},$

$\texttt{Assume}\ (1) : \frac{a}{b} \geq \frac{c}{d},$

$(2) := \eth(\geq_{\mathbb{Q}})(2) : ad \geq cb,$

$(3) := y \cdot_{\mathbb{Z}} (2) +_{\mathbb{Z}} bdx : ady + bdx \geq bcy + bdx,$

$(4) := \left(\frac{1}{bdy}\right) \cdot_{\mathbb{Q}} (3) : \frac{ady + bdx}{bdy} \geq \frac{bcy + bdx}{bdy},$

$() := \eth^{-1} +_{\mathbb{Q}} (4) : \frac{a}{b} + \frac{x}{y} \leq \frac{c}{d} + \frac{x}{y};$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall a, b, c \in \mathbb{Q} \,.\, a \geq b \Rightarrow a + c \geq b + c,$

$\texttt{Assume}\ \frac{a}{b}, \frac{c}{d} : \mathbb{Q},$

$\texttt{Assume}\ (2) : \frac{a}{b} \geq 0 \ \&\ \frac{c}{d} \geq 0,$

$(3) := \eth(\geq_{\mathbb{Q}})(2) : a \geq 0 \ \&\ c \geq 0,$

$(4) := (3)_1 \cdot_{\mathbb{Z}} (3)_2 : ac \geq 0,$

$(5) := (4) \cdot_{\mathbb{Q}} \frac{1}{bf} : \frac{ac}{bd} \geq 0;$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall x, y \in \mathbb{Q} \,.\, x > 0 \ \&\ y > 0 \Rightarrow xy > 0,$

$(*) := \eth^{-1}\texttt{OrderedField}(\mathbb{Q})(1)(2) : \Big(\mathbb{Q} : \texttt{OrderedField}\Big);$

$\square$

$\texttt{RationalNumbersAreArchimedean} :: \mathbb{Q} : \texttt{Archimedean}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \frac{a}{b} : \mathbb{Q},$

$(1) := \texttt{IntegerIneq}(a) : a \leq |a| + 1,$

$(2) := \texttt{NaturalMultIneq}(2, b) : a \leq b|a| + b,$

$() := \eth(\leq_{\mathbb{Q}}) : \frac{a}{b} \leq b|a| + b;$

$\rightsquigarrow (1) := I(\forall)I(\exists)(b|a| + b) : \forall q \in \mathbb{Q} \,.\, \exists n \in \mathbb{N} \,.\, q \leq n,$

$(2) := \eth^{-1}\texttt{Archimedean}(\mathbb{Q})(1) : \Big(\mathbb{Q} : \texttt{Archimedean}\Big);$

$\square$

# 2 Value Fields

## 2.1 Absolute Values

$\texttt{AbsoluteValue} :: \prod K : \texttt{Field} \, . \prod R : \texttt{OrderedField} \, . \, ?(K \to R_+)$

$a : \texttt{AbsoluteValue} \iff \forall x \in K \, . \, a(x) = 0 \iff x = 0 \;\&$

$\& \; \forall x, y \in K \, . \, a(xy) = a(x)a(y) \;\&\; \forall x, y \in K \, . \, a(x+y) \leq a(x) + a(y)$

$\texttt{AbsoluteValueField} = \prod R : \texttt{OrderedField} \, . \sum K : \texttt{Field} \, . \, \texttt{AbsoluteValueField}(K, R)$

$\texttt{synecdoche} :: \texttt{AbsoluteValueField} \to \texttt{Field}$

$\texttt{synecdoche}\,(K, a) := K$

$\texttt{absoluteValue} :: \prod (K, a) : \texttt{AbsoluteValueField}(R) \, . \, \texttt{AbsoluteValue}(K, R)$

$\texttt{absoluteVlaue}\,() = |\cdot|_{(K,a)} := a$

$\texttt{IdValue} :: \forall K : \texttt{AbsoluteValueField}(R) \, . \, |1_K| = 1$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Field}(K) : 1 \neq 0,$

$(2) := \eth_2\texttt{AbsoluteValue}(|\cdot|_K)(1,1) : |1|^2 = |1|,$

$(3) := |1|^{-1}((2) - |1|) : |1|(|1| - 1) = 0,$

$(4) := \eth_1\texttt{AbsoluteValue}(1) : |1| \neq 0,$

$(5) := \texttt{PolynomialRoots}(1)(4) : |1| = 1;$

$\square$

$\texttt{NegIdValue} :: \forall K : \texttt{AbsoluteValueField}(R) \, . \, |-1_K| = 1$

$\texttt{Proof} \;=$

$(1) := \texttt{IdValueNegSquere}\eth_2\texttt{AbsoluteValue}(|\cdot|_K)(-1,-1) : 1 = |1| = \left|(-1)^2\right| = |-1|^2,$

$(2) := \eth\texttt{AbsoluteValue}(-1) : |-1| > 0,$

$(*) := \texttt{PolinomialRoots}(1)(2) : |-1| = 1;$

$\square$

$\texttt{NegValue} :: \forall K : \texttt{AbsoluteValueField}(R) \, . \, \forall x \in K \, . \, |-x| = |x|$

$\texttt{Proof} \;=$

$(*) := \eth\texttt{negate}(-x, |-x|)\eth_2\texttt{AbsoluteValue}(-1, x)\texttt{NegIdValue} : |-x| = |-1 \cdot x| = |-1||x| = |x|;$

$\square$

$\texttt{positiveVersion} :: \prod R : \texttt{OrderedField} \, . \, R \to R_+$

$\texttt{positiveVersion}\,(x) = |x| := \texttt{if } x \geq 0 \texttt{ then } x \texttt{ else } -x$

$\texttt{PositiveIsNotLess} :: \forall R : \texttt{OrderedField} \,.\, \forall x \in R \,.\, x \leq |x|$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : x \geq 0,$

$(2) := \eth\texttt{positiveVersion} : |x| = x,$

$() := \eth\texttt{Reflexive}\Big(\texttt{order}(R)\Big)(2) : x \leq |x|;$

$\rightsquigarrow (1) := I(\Rightarrow) : x \geq 0 \Rightarrow x \leq |x|,$

$\texttt{Assume}\ (2) : x \leq 0,$

$(3) := \eth\texttt{positiveVersion}(x)\eth(R_+) : |x| \geq 0,$

$() := (2)(3) : x \leq |x|;$

$\rightsquigarrow (2) := I(\Rightarrow) : x < 0 \Rightarrow x \leq |x|,$

$(3) := \eth\texttt{Total}\Big(\texttt{order}(R)\Big)(x, 0) : x \geq 0 | x \leq 0,$

$(*) := E(|)(1, 2, 3) : x \leq |x|;$

$\square$

$\texttt{PositiveSquering} :: \forall R : \texttt{OrderedField} \,.\, \forall x \in R \,.\, x^2 = |x|^2$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : x \geq 0,$

$(2) := \eth\texttt{positiveVersion} : x = |x|,$

$(4) := \eth x^2(2) : x^2 = x \cdot x = |x||x| = |x|^2;$

$\rightsquigarrow (1) := I(\Rightarrow) : x \geq 0 \Rightarrow x^2 = |x|^2,$

$\texttt{Assume}\ (2) : x \leq 0,$

$(3) := \eth\texttt{positiveVersion} : |x| = -x,$

$() := \eth x^2(3) : x^2 = (-x) \cdot (-x) = |x||x| = |x|^2;$

$\rightsquigarrow (2) := I(\Rightarrow) : x \leq 0 \Rightarrow x^2 = |x|^2,$

$(3) := \eth\texttt{Total}\Big(\texttt{order}(R)\Big)(x, 0) : x \geq 0 | x \leq 0,$

$(*) := E(|)(1, 2, 3) : x = |x|^2;$

$\square$

$\texttt{PositiveVersionIsAbsoluteValue} :: \forall R : \texttt{OrderedField} \,.\, \texttt{positiveVersion}(R) : \texttt{AbsoluteValue}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \texttt{In}(R),$

$\texttt{Assume}\ (1) : x \neq 0,$

$(2) := \eth\texttt{positiveVersion} : |x| = x||x| = -x,$

$\texttt{Assume}\ (3) : |x| = x,$

$() := (2)(3) : |x| \neq 0;$

$\rightsquigarrow (3) := I(\Rightarrow) : |x| = x \Rightarrow |x| \neq 0,$

$\texttt{Assume}\ (4) : |x| = -x,$

$() := (1)\eth\texttt{neg}(4) : |x| \neq 0;$

$\rightsquigarrow (4) := I(\Rightarrow) : |x| = -x \Rightarrow |x| \neq 0,$

$(5) := E(|)(2,3,4) : |x| \neq 0;$

$\rightsquigarrow (1) := I(\Rightarrow) : x \neq 0 \Rightarrow |x| \neq 0,$

$\texttt{Assume}\ (2) : |x| \neq 0,$

$(3) := \eth\texttt{positiveVersion}(0) : |0| = 0,$

$() := (2)(3) : x \neq 0;$

$\rightsquigarrow (1) := I(\iff)(1)I(\Rightarrow) : x \neq 0 \iff |x| \neq 0,$

$\texttt{Assume}\ x, y : \texttt{In}(R),$

$(2) := \texttt{PositiveSquere}(R)(xy) : (xy)^2 \geq 0,$

$(3) := \texttt{PositiveSquering}(R)(xy) : |xy|^2 = \left|(xy)^2\right| = (xy)^2 = x^2y^2 = \left|x^2\right|\left|y^2\right| = |x|^2|y|^2,$

$() := \sqrt{(3)} : |xy| = |x||y|;$

$\rightsquigarrow (2) := I(\forall) : x, y \in R \,.\, |xy| = |x||y|,$

$\texttt{Assume}\ x, y : \texttt{In}(R),$

$(3) := \texttt{PositiveIsNotLess}(2xy)(2)(2, |x|, |y|) : 2xy \leq |2xy| = 2|x||y|,$

$(4) := \texttt{PositiveIsSquare}\eth\texttt{positiveVersion}\Big(\texttt{PositiveSquering}(x+y)\Big)\texttt{Binom}(2)\ (2)\texttt{Binom}^{-1}(2) :$

$\quad : |x+y|^2 = |(x+y)^2| = (x+y)^2 \leq x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2 \leq \Big(|x| + |y|\Big)^2,$

$() := \sqrt{(4)} : |x+y| \leq |x| + |y|;$

$\rightsquigarrow (3) := I(\forall) : \forall x, y \in R \,.\, |x+y| \leq |x| + |y|,$

$(*) := \eth^{-1}\texttt{AbsoluteValue}(1,2,3) : |x+y| \leq |x| + |y|;$

$\square$

$\texttt{InverseAbsValue} :: \forall K : \texttt{AbsoluteValueField}(R) \,.\, \forall x \in K \,.\, \forall (0) : x \neq 0 \,.\, \left|x^{-1}\right| = |x|^{-1}$

$\texttt{Proof} \;=$

$(1) := \texttt{IdValue}(K)\eth^{-1}\texttt{inverse}(x)\eth_2\texttt{absoluteValue}(K)(x, x^{-1}) : 1_R = |1_K| = |xx^{-1}| = |x|\left|x^{-1}\right|,$

$(*) := |x|^{-1} \cdot (1) : |x|^{-1} = \left|x^{-1}\right|;$

$\square$

$\texttt{InverseTriangleIneq} :: \forall K : \texttt{AbsoluteValueField}(R) \,.\, \forall x, y \in K \,.\, \big||x| - |y|\big| \leq |x - y|$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : |x| - |y| \geq 0,$

$(2) := \texttt{AddSubstract}(x, y)(|x|)\eth_3\texttt{AbsoluteValue}(R) : |x| = |x - y + y| \leq |x - y| + |y|,$

$() := (\eth\texttt{positiveVersion}(1)\Big((2) - |y|\Big) : \big||x| - |y|\big| = |x| - |y| \leq |x - y|;$

$\rightsquigarrow (1) := I(\Rightarrow) : |x| - |y| \geq 0 \Rightarrow ||x| - |y|| \leq |x - y|,$

$\texttt{Assume}\ (2) : |x| - |y| \leq 0,$

$(2) := \texttt{AddSubstract}(x, y)(|x|)\eth_3\texttt{AbsoluteValue}(R) : |y| = |x - x + y| \leq |x - y| + |x|,$

$() := (\texttt{NegValue}\eth\texttt{positiveVersion}(1)\Big((2) - |y|\Big) : \big||x| - |y|\big| = |y| - |x| \leq |x - y|;$

$\rightsquigarrow (2) := I(\Rightarrow) : |x| - |y| \leq 0 \Rightarrow ||x| - |y|| \leq |x - y|,$

$(3) := \eth\texttt{Total}\Big(\texttt{order}(R)\Big)(|x| - |y|, 0) : |x| - |y| \geq 0 \ \Big| \ |x| - |y| \leq 0,$

$(*) := E(|)(1, 2, 3) : \Big||x| - |y|\Big| \leq |x - y|;$

$\square$

$\texttt{IteratedTriangleIneq} :: \forall K : \texttt{AbsoluteValueField}(R) \,.\, \forall n \in \mathbb{N} \,.\, \forall x : n \to K \,.\, \left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : n = 1,$

$() := \eth\texttt{Reflexive}(\texttt{order}(R))(|x_1|) : |x_1| \leq |x_1|;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{IteratedTriangleIneq} : \texttt{TriangleTriangleIneq}(1),$

$\texttt{Assume}\ n - 1 : \mathbb{N},$

$\texttt{Assume}\ (2) : \texttt{IteratedTriangleIneq}(n - 1),$

$() := \eth_3\texttt{AbsoluteValue}(\texttt{absoluteValue}(K))(2) : \left|\sum_{i=1}^{n} x_i\right| \leq |x_n| + \left|\sum_{i=1}^{n-1} x_i\right| \leq \sum_{i=1}^{n} |x_i|;$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)\eth^{-1}\texttt{IteratedTriangleIneq}(n) :$

$\quad \forall n - 1 \in \mathbb{N} \,.\, \texttt{IteratedTriangleIneq}(n - 1) \Rightarrow \texttt{IteratedTrangleIneq}(n),$

$(*) := E(\mathbb{N})(1)(2) : \texttt{IteratedTriangleIneq};$

$\square$

$\texttt{IteratedAbsHomogen} :: \forall K : \texttt{AbsoluteValueField}(R) \,.\, \forall n \in \mathbb{N} \,.\, \forall x : n \to K \,.\, \left|\prod_{i=1}^{n} x_i\right| = \prod_{i=1}^{n} |x_i|$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : n = 1,$

$() := I(=)\big(|x_1|\big) : |x_1| = |x_1|;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{IteratedAbsHomogen} : \texttt{IteratedAbsHomogen}(1),$

$\texttt{Assume}\ (2) : n - 1 \in \mathbb{N},$

$\texttt{Assume}\ (3) : \texttt{IteratedAbsHomogen}(n - 1),$

$(4) := \eth_2\texttt{AbsoluteValue}\big(\texttt{absoluteValue}(K)\big)\left(x_n, \prod_{i=1}^{n-1} x_i\right)(2)(x_{|n-1}) : \left|\prod_{i=1}^{n} x_i\right| = |x_n|\left|\prod_{i=1}^{n-1} x_i\right| \leq \prod_{i=1}^{n} |x_i|;$

$\rightsquigarrow (2) := I(\forall)I(\Rightarrow)\eth^{-1}\texttt{IteratedAbsHomogen}(n) :$

$\quad \forall n - 1 \in \mathbb{N} \,.\, \texttt{IteratedAbsHomogen}(n - 1) \Rightarrow \texttt{IteratedAbsHomogen}(n),$

$(*) := E(\mathbb{N})(1)(2) : \texttt{IteratedAbsHomogen};$

$\square$

## 2.2 Conjugation

$$\texttt{ConjugationMap} :: \prod K : \texttt{AbsoluteValueField}(R) \,.\, \prod (1) : R \subset_{\mathsf{RING}} K \,.\, ?(K \to K)$$

$$\zeta : \texttt{ConjugationMap} \iff \forall x, y \in K \,.\, x\zeta(x) = |x|^2 \;\&\; \zeta(xy) = \zeta(x)\zeta(y) \;\&$$
$$\& \; \zeta(x+y) = \zeta(x) + \zeta(y) \;\&\; \zeta\big(\zeta(x)\big) = x$$

$$\texttt{ConjugationField} = \sum K : \texttt{AbsoluteValueField}(R) \,.\, \sum (1) : R \subset_{\mathsf{RING}} K \,.\, \texttt{ConjugationMap}(k)$$

$$\texttt{synecdoche} :: \texttt{ConjugationField}(R) \to \texttt{AbsoluteValueField}(R)$$
$$\texttt{synecdoche}\,((K, \dots, \zeta)) := K$$

$$\texttt{conjugationMap} :: \prod (K, (1), \zeta) : \texttt{ConjugationField}(R) \,.\, \texttt{ConjugationMap}(K, (1))$$
$$\texttt{conjugationMap}\,() = \bar{\cdot} := \zeta$$

$$\texttt{RealStrucure} :: \prod K : \texttt{ConjugationField}(R) \,.\, ?K$$
$$x : \texttt{RealStructure} \iff x \in \mathbb{R}(K) \iff \overline{x} = x$$