Differential Analysis

Uncultured Tramp

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Contents

1 Differentiable Maps

1.1 Tangent Maps

$$\texttt{localDivergence} :: \prod V, W : \mathsf{BAN}(K) \,.\, \prod U : \texttt{OPEN}(W) \,.\, (U \to W)^2 \to U \to \mathbb{R}_{++} \to \mathbb{R}_+$$
$$\texttt{localDivergence}\,(f, g, p, r) := \sup\{\|f(x) - g(x)\| | x \in \mathbb{B}_V(p, r) \cap U\}$$

$$\texttt{TangentAt} :: \prod V, W : \mathsf{BAN}(K) \,.\, \prod U : \texttt{OPEN}(W) \,.\, U \to ?(U \to W)$$
$$(p, f, g) : \texttt{TangentAt} \iff \lim_{r \to 0} \frac{\texttt{localDivergence}(f, g, p, r)}{r} = 0$$

$$\texttt{TangentAtIsEqRelation} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(W) \,.$$
$$.\, \texttt{TangentAt}(V, W, U)(p) : \texttt{Equavalence}(U \to W)$$
$$\texttt{Proof} =$$

1) For Reflexivity use that $\|f(x) - f(x)\| = 0$ as Constant.
2) For Symmetry use that addition in vector spaces is commutative.
3) For Transitivity use that

$$\sup_{x \in \mathbb{B}(p,r)} \frac{\|f(x) - h(x)\|}{r} \le \sup_{x \in \mathbb{B}_U(p,r)} \frac{\|f(x) - g(x)\| + \|g(x) - h(x)\|}{r} \le$$
$$\le \sup_{x \in \mathbb{B}_U(p,r)} \frac{\|f(x) - g(x)\|}{r} + \sup_{x \in \mathbb{B}_U(p,r)} \frac{\|g(x) - h(x)\|}{r}$$

and that both summands converges to 0 with r. $\square$

1.2 Differential

$$\texttt{Differential} :: \prod V, W : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(V) \;.\; U \to (U \to W) \to ?\mathcal{L}(V, W)$$

$$A : \texttt{Differential}(p, f) \iff \Big(\big(\Lambda x \in U \;.\; f(p) - f(x), \Lambda x \in U \;.\; A(p - x)\big) : \texttt{TangentAt}(V, W, U)(p) \Big)$$

$$\texttt{DifferentiableAt} :: \prod V, W : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(V) \;.\; U \to ?(U \to W)$$

$$f : \texttt{DifferentiableAt}(p) \iff \exists \texttt{Differential}(p, f)$$

$$\texttt{Differentiable} :: \prod V, W : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(V) \;.\; ?(U \to W)$$

$$f : \texttt{Differentiable} \iff \forall p \in U \;.\; f : \texttt{DifferentiableAt}(p)$$

$\texttt{DifferentialUnique} :: \forall f : \texttt{DifferentiableAt}(V, W, U)(p) \;.\; \exists! A : \texttt{Differential}(V, W, U)(p, f)$

$\texttt{Proof} =$

$A := \eth\texttt{DifferentiableAt}(V, W, U)(p)(f) : \texttt{Differential}(V, W, U)(p, f),$

$\texttt{Assume}\ B : \texttt{Differential}(V, W, U)(p, f),$

$(1) := \eth\texttt{Differential}(V, W, U)(p, f)(A) :$

$\quad : \Big(\big(\Lambda x \in U \;.\; f(p) - f(x), \Lambda x \in U \;.\; A(p - x)\big) : \texttt{TangentAt}(V, W, U)(p) \Big),$

$(2) := \eth\texttt{Differential}(V, W, U)(p, f)(B) :$

$\quad : \Big(\big(\Lambda x \in U \;.\; f(p) - f(x), \Lambda x \in U \;.\; B(p - x)\big) : \texttt{TangentAt}(V, W, U)(p) \Big),$

$(3) := \eth\texttt{Transitive}\Big(\texttt{TangentAt}(V, W, U)(p)\Big)(1)(2) : \Big(A(p) - A, B(p) - B\Big) : \texttt{TangentAt}(V, W, U)(p),$

$(4) := \ldots \eth^{-1}\texttt{operatorNorm} : \forall r \in \mathbb{R}_{++} \;.\; \sup_{x \in \mathbb{B}_U(p, r)} \|(A - B)(p - x)\| = r\|A - B\|,$

$(5) := \eth\texttt{TangentAt}(V, W, U)(p)(3)(4) : \|A - B\| = 0,$

$(6) := \eth\texttt{Hypernorm}(5) : A = B;$

$\rightsquigarrow (*) := \forall(I) : \forall B : \texttt{Differential}(V, W, U)(p, f),$

$\square$

$$\texttt{differential} :: \prod V, W : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(K) \;.\; \prod f : \texttt{Differentiable}(V, W, U) \;.$$
$$.\; \prod p \in U \;.\; \texttt{Differential}(V, W, U)(p, f)$$

$$\texttt{differential}\,() = \mathrm{D}f|_p := \texttt{DifferentialUnique}(f, p)$$

$$\texttt{differentialAt} :: \prod V, W \in \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(K) \;.\; \prod p \in U \;.$$
$$.\; \prod f : \texttt{DifferentiableAt}(V, W, U)(p) \;.\; \texttt{Differential}(V, W, U)(p, f)$$

$$\texttt{differential}\,() = \mathrm{D}f|_p := \texttt{DifferentialUnique}(f, p)$$

$$\texttt{ContinuoslyDifferentiable} :: \prod V, W : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(V) \;.\; ?\texttt{Differentiable}(V, W, U)$$

$$f : \texttt{ContinuouslyDifferentiable} \iff f \in C^1(U, W) \iff$$
$$\iff \forall p \in U \;.\; \mathrm{D}f|_p \in \mathcal{B}(V, W)\ \&\ \mathrm{D}f : C(U, \mathcal{B}(V, W))$$

$\texttt{LinearDifferentiation} :: \forall V, W \in \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(W) \,.\, \forall p \in U . \mathrm{D}|_p \in \mathcal{L}\left(\texttt{DifferentiableAt}(V, W, U)(p\right.$

$\texttt{Proof} \ =$

Use the fact that

$$\sup_{x \in \mathbb{B}(p,r)} \frac{\|f(x) + g(x) - f(p) - f(g) - \mathrm{D}f|_p(x-p) - \mathrm{D}g|_p(x-p)\|}{r} \le$$
$$\le \sup_{x \in \mathbb{B}(p,r)} \frac{\|f(x) - f(p) - \mathrm{D}f|_p(x-p)\|}{r} + \sup_{x \in \mathbb{B}(p,r)} \frac{\|g(x) - g(p) - \mathrm{D}g|_p(x-p)\|}{r} \to 0$$

To Prove additivity. Use absolute homogenity of the norm to prove homogenity.

$\square$

$\texttt{DerivativeOfLinearMap} :: \forall V, W \in \mathsf{BAN}(K) \,.\, \forall T \in \mathcal{B}(V, W) \,.\, \mathrm{D}T = T$

$\texttt{Proof} \ =$

Use zero operator norm argument.

$\square$

$\texttt{DerivativeOfMultilinear} :: \forall n \in \mathbb{N} \,.\, \forall V : n \to \mathsf{BAN}(K) \,.\, \forall W \in \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}\Big((V_i)_{i=1}^n ; W\Big) \,.$

$$. \forall p, v \in \prod_{i=1}^n V_i \,.\, \mathrm{D}T|_p v = \sum_{i=1}^n T\Big((p_j)_{j=1}^{i-1} \oplus v_i \oplus (p_j)_{j=i+1}^n\Big)$$

$\texttt{Proof} \ =$

rewrite

$$T(x) - T(p) - \sum_{i=1}^n T\Big((p_j)_{j=1}^{i-1} \oplus v_i \oplus (p_j)_{j=i+1}^n\Big)$$

as

$$\sum_{i=1}^n \Big(T\big((p_j)_{j=1}^{i-1} \oplus v_i \oplus (p_j)_{j=i+1}^n\big) - T(v) - T\big((p_j)_{j=1}^{i-1} \oplus v_i - p_i \oplus (p_j)_{j=i+1}^n\big)\Big) + \phi,$$

where $\phi = O(r^2)$, hence the derivative is defined correctly.

$\square$

$\texttt{DerivativeOfTheInverse} :: \forall V : \texttt{BanachAlgebra}(K) \,.\, \forall u : \texttt{Invertible}(V) \,.\, \forall h \in V \,.$

$$. \mathrm{D}\,\texttt{inv}|_u(h) = -u^{-1} h u^{-1}$$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{inv} : (u+h)^{-1} - u^{-1} = (u-h)^{-1}\big(u - (u-h)\big)u^{-1} = -(u-h)^{-1} h u^{-1},$

$(2) := \eth\texttt{operatorNorm}(\ldots) : \Big\|(u-h)^{-1} h u^{-1} - u^{-1} h u^{-1}\Big\| = \Big\|\big((u-h)^{-1} - u^{-1}\big) h u^{-1}\Big\| \le$

$\quad \le \big\|(u-h)^{-1} - u^{-1}\big\| \|h\| \|u^{-1}\|,$

$(3) := \eth\texttt{Continuous}(\texttt{inv}) : \lim_{h \to 0} \big\|(u-h)^{-1} - u^{-1}\big\| = 0,$

$(*) := \eth\texttt{Differential}(1)(2)(3) : \mathrm{D}\texttt{inv}|_u h = -u^{-1} h u^{-1};$

$\square$

DifferentialOfComposition $:: \forall F, G, H : \mathsf{BAN}(K) . \forall U : \mathtt{Open}(F) . \forall V : \mathtt{Open}(G) .$

$. \forall f : \mathtt{Differentiable}(F, G, U) . \forall g : \mathtt{Differentiable}(G, H, V) . \forall s : f(F) \subset V .$

$. \forall p \in U . \mathrm{D} g \circ f|_p = \mathrm{D} g|_{f(p)} \mathrm{D} f|_p$

Proof $=$

$(\phi, 1) := \eth^{-1} \mathtt{AsymptoticalyBounded} \eth \mathtt{DifferentiableAt}(G, H, V)(p)(f) :$

$: \sum \phi : U \to V . \phi(x) = O_0(x) \ \& \ \forall x \in U . f(x) = f(p) + \mathrm{D} f|_p(x-p) + \phi(x-p),$

$(\psi, 2) := \eth^{-1} \mathtt{AsymptoticalyBounded} \eth \mathtt{DiffereniableAt(G,H,V)}(f(p))(g) :$

$: \sum \psi : V \to H . \psi(x) = O_0(x) \ \& \ \forall x \in V . g(x) = g\big(f(p)\big) + \mathrm{D} g|_{f(p)}\big(x - f(p)\big) + \psi(x - f(p)),$

$(3) := \mathtt{AsymptoticalyBoundedComposition}(\psi, \phi) : \psi \circ \phi = O_0(x),$

$(4) := \mathtt{AsymptoticalyBoundedComposition}(\mathrm{D} g|_{f(p)} \circ \phi) : \mathrm{D} g|_{f(p)} \circ \phi = O_0(x),$

$(5) := (1)(2)(g \circ f) : g \circ f = g\big(f(p)\big) + \mathrm{D} g|_{f(p)} \mathrm{D} f|_x(x-p) + \mathrm{D} g|_{f(p)} \phi(p-x) + \psi\big(\phi(p-x)\big),$

$(*) := \eth^{-1} \mathtt{Differential}(F, H, U)(p)(5) \eth \mathtt{AsymptoticalyBounded}(3)(4) : \mathrm{D} g \circ f|_p = \mathrm{D} g|_{f(p)} \mathrm{D} f_p;$

$\square$

DifferentialOfCompositionAtAPoint $:: \forall F, G, H : \mathsf{BAN}(K) . \forall U : \mathtt{Open}(F) . \forall V : \mathtt{Open}(H) . \forall p \in U .$

$. \forall f : \mathtt{DifferentiableAt}(F, G, U)(p) . \forall s : f(p) \in U \forall g : \mathtt{DifferentiableAt}(G, H, V)(f(p)) .$

$. \mathrm{D} g \circ f|_p = \mathrm{D} g|_{f(p)} \mathrm{D} f|_p$

Proof $=$

...

$\square$

1.3 Partial and Coordinate Derivatives

$$\texttt{coordinateDerivative} :: \prod n \in \mathbb{N} \,.\, \prod V \in \mathsf{BAN}(K) \,.\, \prod W : n \to \mathsf{BAN}(K) \,.\, \prod U : \texttt{Open}(V) \,.$$
$$.\, \prod f : \prod i \in n \,.\, \texttt{Differentiable}(V, W_i, U) \,.\, \prod i \in n \,.\, \prod p \in U \,.\, \texttt{Differential}(V, W_i, U)(p, f_i)$$
$$\texttt{coordinateDerivative}\,() = \mathrm{D} f_i|_p := f_i'(p)$$

$$\texttt{PartiallyDifferentiable} :: \prod n \in \mathbb{N} \,.\, \prod V : n \to \mathsf{BAN}(K) \,.\, \prod W \in \mathsf{BAN}(K) \,.$$
$$.\, \prod U \prod i \in n \,.\, \texttt{Open}(V_i) \,.\, ?\left(\prod_{i=1}^{n} V_i \to W\right)$$
$$f : \texttt{PartiallyDifferentiable} \iff \forall p \in \prod_{i=1}^{n} U_i \,.\, \forall i \in n \,.\, \Lambda v \in U_i \,.\, f\Big((p_j)_{j=1}^{i-1} \oplus w \oplus (p_j)_{j=i+1}^{n}\Big) :$$
$$: \texttt{Differentiable}(V_i, v, U_i)$$

$$\texttt{CoordinatewiseDifferentiability} :: \forall n \in \mathbb{N} \,.\, \forall V \in \mathsf{BAN}(K) \,.\, \forall W : n \to \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(V) \,.$$
$$.\, \forall f : \prod i \in n \,.\, \texttt{Differentiable}(V, W_i, U) \,.$$
$$.\, (f) : \texttt{Differentiable}\left(V, \prod_{i=1}^{n} W_i, U\right) \;\&\; \forall p \in U \,.\, \mathrm{D}(f)|_p = \sum_{i=1}^{n} \iota_W^i f_i'(p)$$

Proof $=$

Use representation

$$f = \sum_{i=1}^{n} \iota_W^i f_i'(p)$$

as ι_W^i is linear by composition and linearity theorems results follow

□

$$\texttt{partialDerivative} :: \forall n \in \mathbb{N} \,.\, \forall V : n \to \mathsf{BAN}(K) \,.\, \forall f : \texttt{PartiallyDifferentiable}(V, W, U) \,.$$
$$.\, \prod p \in \prod_{i=1}^{n} U_i \,.\, \prod i \in n \,.\, \texttt{Differential}(V_i, W, U_i)\Big(p_i, \Lambda w \in U_i \,.\, f\Big((p_j)_{j=1}^{i-1} \oplus v \oplus (p_j)_{j=i+1}^{n}\Big)\Big)$$
$$\texttt{partialDerivative}\,() = \mathrm{D}_i f|_p := \mathrm{D}\Lambda v \in U_i \,.\, f\Big((p_j)_{j=1}^{i-1} \oplus v \oplus (p_j)_{j=i+1}^{n}\Big)|_{p_i}$$

$\texttt{DifferentiableIsAlwaysPartial} :: \forall n \in \mathbb{N} . \forall V : n \to \mathsf{BAN}(K) .$

$$. \forall U : \prod i \in n . \texttt{Open}(V_i) \forall f : \texttt{Differentiable}\left(\prod_{i=1}^n V_i, W, \prod_{i=1}^n U_i\right) . f : \texttt{PartiallyDifferentiable}(V, W, U)$$

Proof =

Partial derivatives are exactly

$\mathrm{D} f \iota_V^{i,p}|_{p_i} = \mathrm{D} f|_p \mathrm{D}\iota_V^i|_{p_i},$

So partial derivatives exist.

□

$\texttt{SmoothnessByPartialDerivatives} :: \forall f : \texttt{Differentiable}\left(\prod_{i=1}^n V_i, W, \prod_{i=1}^n U_i\right) .$

$$. f \in C^1 \iff \forall i \in n . \mathrm{D}_i f : C\Big(U_i, \mathcal{B}(V_i, W)\Big)$$

Proof =

use representation:

$$\mathrm{D} f|p = \mathrm{D} f \sum_{i=1}^n \iota_V^{i,p}|p = \sum_{i=1}^n \mathrm{D}_i f|_{p_i}$$

result follows from the continuoity of sum.

□

1.4 Mean Value Theorem

$$\texttt{RightDifferentiable} :: \prod F : \mathsf{BAN}(K) \ . \ \forall [a,b] : \texttt{Interval}(\mathbb{R}) \ . \ ?[a,b] \to F$$

$$f : \texttt{RightDifferentiable} \iff \forall r \in [a,b) \ . \ \exists v \in F \ . \ \lim_{t \downarrow r} \frac{f(t) - f(r)}{t - r} = v$$

$$\texttt{rightDerivative} :: \texttt{RightDifferentiable}(F, [a,b]) \to [a,b) \to K \to F$$

$$\texttt{rightDerivative}\,(f, r, h) = f'_{\text{right}}(r) := h \lim_{t \downarrow r} \frac{f(t) - f(r)}{t - r}$$

$$\texttt{LeftDifferentiable} :: \prod F : \mathsf{BAN}(K) \ . \ ?[a,b] \to F$$

$$f : \texttt{LeftDifferentiable} \iff \forall r \in (a,b] \ . \ \exists v \in F \ . \ \lim_{t \uparrow r} \frac{f(t) - f(r)}{r - t} = v$$

$$\texttt{leftDerivative} :: \texttt{LeftDifferentiable}(F, [a,b]) \to [a,b) \to K \to F$$

$$\texttt{rightDerivative}\,(f, r, h) = f'_{\text{left}}(r) := \forall r \in (a,b] \ . \ h \lim_{t \uparrow r} \frac{f(t) - f(r)}{r - t}$$

$$\texttt{MeanValueTheorem} :: \forall F : \mathsf{BAN}(\mathbb{R}) \ . \ \forall f : \texttt{RightDifferentiable}(F, [a,b]) \ .$$
$$. \ \forall g : \texttt{RightDifferentiable}(\mathbb{R}, [a,b]) \ . \ \forall I : \forall r \in (a,b) \ . \ \|f'_{\text{right}}(r)\| \le g'_{\text{right}}(r) \ . \ \|f(a) - f(b)\| \le g(a) - g(b)$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ \varepsilon : \mathbb{R}_{++},$

$U := \Big\{ x \in [a,b] : \|f(x) - f(a)\| > g(x) - g(a) + \varepsilon(x - a) + \varepsilon \Big\} : \texttt{Set}\Big([a,b]\Big),$

$\varphi := \Lambda x \in [a,b] . \|f(x) - f(a)\| - g(x) + g(a) + \varepsilon(x - a) : C([a,b], \mathbb{R}),$

$(1) := \eth U \eth^{-1} \varphi : U = \varphi^{-1}(\varepsilon, +\infty),$

$(2) := \eth C([a,b], \mathbb{R})(\varphi)(1) : \Big(U : \texttt{Open}(x)\Big),$

$(3) := \eth U(a) : a \notin U,$

$\texttt{Assume} \ A : U \ne \emptyset,$

$c := \inf U : [a,b],$

$(4) := \eth c(2)(3) : c < U,$

$(5) := \eth \varphi(a) : a \in \varphi^{-1}[0, \varepsilon),$

$(6) := \texttt{OpenByNeighbourhoods}\Big(a, \varphi^{-1}[0, \varepsilon)\Big) \eth U \eth c : a < c,$

$(7) := \eth \texttt{LowerBound}([a,b])(U)(c) : c < b,$

$(8) := I(c)(6,7) \eth \texttt{rightDifferential} : \lim_{t \downarrow c} \dfrac{\|f(t) - f(c)\|}{t - c} \le \lim_{t \downarrow c} \dfrac{g(t) - g(c)}{t - c},$

$(u, 9) := \eth \texttt{LimitIneq}(\varepsilon)(8) \eth c \eth(U) : \sum u \in U \ . \ \|f(u) - f(c)\| \le g(u) - g(c) + \varepsilon(u - c),$

$(10 := \texttt{AddNoneg}(\varepsilon)(9) \eth U : u \notin U,$

$11 := \texttt{NotInAndIn}(\eth u, (10)) : \perp;$

$\rightsquigarrow 4 := \texttt{Contradiction} : U = \emptyset,$

$(5) := \texttt{Antiset}(U)(4) : \forall x \in [a,b] \ . \ \|f(x) - f(a)\| \le g(x) - g(a) + \varepsilon(x - a) + \varepsilon;$

$\rightsquigarrow (1) := I(\forall)(\varepsilon) : \forall \varepsilon \in \mathbb{R}_{++} \ . \ \forall x \in [a,b] \ . \ \|f(x) - f(a)\| \le g(x) - g(a) + \varepsilon(x - a) + \varepsilon,$

$(*) := \lim_{\varepsilon \downarrow 0} \lim_{x \uparrow b} (1)(\varepsilon, x) : \|f(b) - f(a)\| \le g(b) - g(a);$

$\square$

$\texttt{BanachMeanValueTheorem} :: \forall V, W \in \mathsf{BAN}(K) \ . \ \forall U : \texttt{Open}(V) \ . \ \forall f : \texttt{Differentiable}(V, W, U) \ .$
$\ . \ \forall [a, b] : \texttt{Interval}(U) \ . \ \|f(b) - f(a)\| \le \sup_{v \in [a,b]} \|\mathrm{D}f|_v (b - a)\|$

$\texttt{Proof} \ =$
Apply mean value theorem to the contracted function
$\varphi(t) = f((1 - t)a + tb) : [a, b] \to W$
having
$\|\varphi'(t)\| = \left\|\mathrm{D}f|_{tb+(1-t)a}(b - a)\right\| \le \sup_{v \in [a,b]} \|\mathrm{D}f|_v (b - a)\|$
with the last function treated as constant.
This provides
$\|f(1) - f(0)\| = \|\varphi(b) - \varphi(a)\| \le (1 - 0) \sup_{v \in [a,b]} \|\mathrm{D}f|_v (b - a)\|$

$\square$

$\texttt{Lipschitz} :: \prod V, W : \mathsf{BAN}(K) \ . \ \forall U : \texttt{Open}(V) \ . \ \mathbb{R}_+ \to ?(U \to W)$
$f : \texttt{Lipschitz}(k) \iff \forall a, b \in U \ . \ \|f(b) - f(a)\| \le k\|b - a\|$

$\texttt{LipschitzByDerivatives} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall U : \texttt{Open} \ \& \ \texttt{Convex}(V) \ .$
$\ . \ \forall f : \texttt{Differentiable}(V, W, U) \ . \ \forall k \in \mathbb{R}_+ \ . \ \forall \mathbf{I} : \sup_{v \in U} \|\mathrm{D}f|_v\| < k \ . \ f : \texttt{Lipschitz}(k)$

$\texttt{Proof} \ =$
As U is convex for each two distinct points $a, b \in U$ the inerval $[a, b] \subset U$.
Apply previous theorem and $\mathbf{I}$. Result follows
$\square$

$\texttt{ZeroDerivativeConstant} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall U : \texttt{Open} \ \& \ \texttt{Convex}(V) \ .$
$\ . \ \forall f : \texttt{Differentiable}(V, W, U) \ . \ \forall \mathbf{I} : \sup_{v \in U} \|\mathrm{D}f|_v\| = 0 \ . \ f : \texttt{Constant}(U, W)$

$\texttt{Proof} \ =$
By previous theorem function is
Apply previous theorem and $\mathbf{I}$. Result follows
$\square$

$\texttt{ZeroDerivativeConstanII} :: \forall V, W : \mathsf{BAN}(K) \ . \ \forall U : \texttt{Open} \ \& \ \texttt{Connected}(V) \ .$
$\ . \ \forall f : \texttt{Differentiable}(V, W, U) \ . \ \forall \mathbf{I} : \sup_{v \in U} \|\mathrm{D}f|_v\| = 0 \ . \ f : \texttt{Constant}(U, W)$

$\texttt{Proof} \ =$
By building balls around each point and the privious theorem the function is locally constand
And as the set is connected is a constant. $\square$

$\texttt{PolygonalLine} :: \prod V : \mathsf{BAN}(K) \,.\, V \to V \to ?\Big([0,1] \to V\Big)$

$\gamma : \texttt{PolygonalLine}(x,y) \iff \exists n \in \mathbb{N} \,.\, \exists\Big([a,b],(1)\Big) : \sum [a,b] : n \to \texttt{Interval}(V) \,.\, \,.\, \forall i \in n-1 \,.\, b_n = a_{n+1} \,.$

$\quad . \, \gamma = \texttt{join}(n, \Lambda i \in n \,.\, \Lambda t \in [0,1] \,.\, tb_n + (1-t)a_n) \;\&\; x = a_1 \;\&\; y = b_n$

$\texttt{PolygonalLineConnected} :: \prod V : \mathsf{BAN}(K) \,.\, ??V$

$U : \texttt{PolygonalLineConnected} \iff \forall U : \mathsf{BAN}(K) \,.\, \forall x,y \in U \,.\, \exists \gamma : \texttt{PolygonalLine}(x,y) : \Im\gamma \subset U$

$\texttt{PolygonalLineConnected} :: \forall V : \mathsf{BAN}(K) \,.\, \forall U :??V \,.\, U : \texttt{PolygonalLineConnected}(V) \iff$

$\quad \iff U : \texttt{Connected}(V)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{length} :: \prod V : \mathsf{BAN}(K) \,.\, \texttt{PolygonalLine}(\_,\_) \to \mathbb{R}_{++}$

$$\texttt{length}\,(\gamma) = |\gamma| := \sum_{i=1}^{n} \|b_i - a_i\|$$

where

$\quad ([a,b],n) = \eth\texttt{PolygonalLine}(\gamma)$

$\square$

$\texttt{innerDistance} :: \prod V : \mathsf{BAN}(K) \,.\, \prod U : \texttt{Connected}(V) \,.\, \texttt{Distance}(U)$

$\texttt{innerDistance}\,(x,y) = d_U(x,y) := \inf\{|\gamma| | \gamma : \texttt{PolygonalLine}(x,y)\}$

$\texttt{InnerMeanValueTheorem} :: \forall V,W \in \mathsf{BAN}(K) \,.\, \forall U \in \texttt{Open} \;\&\; \texttt{Connected}(V) \,.$

$\quad \forall f : \texttt{Differentiable}(V,W,U) \,.\, \forall k \in \mathbb{R}_{++} \,.\, \forall \mathbf{I} : \sup_{v \in U} \|\mathrm{D}f|_v\| \le k \,.\, \forall x,y \in U \,.\, \|f(x) - f(y)\| \le k d_U(x,y)$

$\texttt{Proof} \;=$

Firstly, by the generalized mean value theorem (see Cartan) for polygonal line γ connecting x and y define

$\varphi(t) = f\Big(\gamma(t)\Big) : [0,1] \to W$

with

$$\|\mathrm{D}\varphi|_t\| = \left\|\mathrm{D}f|_{\gamma(t)} \sum_{i=1}^{n}(b_i - a_i)\right\| \le k \sum_{i=1}^{n} \|b_i - a_i\| = k|\gamma|.$$

taking infimum over all such γ provides

$\|f(x) - f(y)\| \le k d_U(x,y)$

$\square$

$\texttt{ConstantDerivative} :: \forall V,W \in \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(V) \,.\, \forall f : \texttt{Differentiable}(V,W,U) \,.$
$\forall C \in \mathcal{L}(V,W) \,.\, \forall E : \mathrm{D}f = C \,.\, \exists A : \texttt{Affine}(V,W) : f = A_{|U}$

$\texttt{Proof} =$

By E it holds that $\mathrm{D}f - C = 0$, but
$0 = \mathrm{D}f - C = \mathrm{D}(f - C_{|U})$.
Hence, $f - C_{|U} = w$ is a constant, but this means that $f = C_{|U} + w$.
$\square$

$\texttt{ConvexIsRightDifferentiabile} :: \forall [a,b] : \texttt{Interval}(\mathbb{R}) \,.\, \forall f : \texttt{Convex}\Big([a,b]\Big) \,.$
$. \; f : \texttt{RightDifferentiable}\Big([a,b],(a,b),\mathbb{R}\Big)$

$\texttt{Proof} =$

$\texttt{Assume}\ t : \texttt{In}\Big((a,b)\Big),$

$\texttt{Assume}\ y,x : \sum y,x : \texttt{In}\Big([0,b-t) \,.\, y > x,$

$(1) := \texttt{EpigrafTHM}(f)(t,x,h) : \dfrac{f(t+x)-f(t)}{x} \le \dfrac{f(t+y)-f(t)}{y};$

$\leadsto (1) := \eth^{-1}\texttt{NonDecreasing} : \left(\Lambda h \in (0,b-t) \,.\, \dfrac{f(t+h)-f(t)}{h} : \texttt{Monotonic}\Big([0,b-t),\mathbb{R}\Big)\right),$

$(2) := \texttt{MonotoneLimAsInf}(1) : \lim\limits_{h \downarrow 0} \dfrac{f(t+h)-f(t)}{h} = \inf\left\{\dfrac{f(t+h)-f(t)}{h} | h \in (0,b-t)\right\},$

$v := \eth\texttt{OpenInterval}(a,b)(t) : \texttt{In}(a,t),$

$\texttt{Assume}\ h^+ : \texttt{In}\Big([0,b-t)\Big),$

$h^- := v - t : \mathbb{R}_{--},$

$(3) := \texttt{EpigrafTHM}(f)(t,h^-,h^+) : \dfrac{f(t)-f(t+h^-)}{h^-} \le \dfrac{f(t)-f(t+h^+)}{h^+};$

$\leadsto (3) := \eth\texttt{BoundedFromBelow} : \Big(\Lambda h \in (0,b-t) \,.\, f(t+h)-f(t)\Big) : \texttt{BpundedFromBelow}\Big([0,b-t]\Big)\Big),$

$(u,4) := (1)\texttt{BoundedMonotonicConvergence}(2,3) : \sum u \in \mathbb{R} \,.\, \lim\limits_{h \downarrow 0} \dfrac{f(t+h)-f(t)}{t} = 0;$

$\leadsto (5) := \eth\texttt{RightDifferentiable}\Big([a,b],(a,b),\mathbb{R}\Big) :$ $\square$

,

NormDifferentiability $:: \forall W \in \mathsf{BAN}(\mathbb{R}) \ . \ \forall [a,b] \in \mathtt{Interval}(V) \ . \ \forall f : [a,b] \to V \ .$

$\forall g : [a,b] \to \mathbb{R}_+ \ . \ \forall E : g = \|f\| \ . \ \forall (p,1) : \sum p \in (a,b] \ . \ f : \mathtt{RightDifferentiableAt}([a,b],W)(p) \ .$

$g : \mathtt{RightDifferentiableAt}([a,b],W)(p)$

Proof $=$

Assume $t \in [a,b)$, then $G(h) = \|f(t) + \mathrm{D}_r f|_t h\|$ is a convex function on any interval around 0.
Byi the previous theorem G is right-differentiable at 0.
 We know that f admits representatation for $s \in (t,b)$:
$f(s) = f(t) + \mathrm{D}_r f|_t (s-t) + O\big(s-t\big)$,
Thus,as norm is Lipschitz

$O(s-t) = -\Big\|O(s-t)\Big\| \leq g(s) - G(s-t) \leq \Big\|O(s-t)\Big\| = O(s-t)$

$g(s) = G(s-t) + O(s-t)$

where first summand is right-differentiable at t and last summand is neglegible.

□

DifferentialTarget $:: \forall f : \mathtt{RightDifferentiableAt} \ \& \ C\Big([a,b],W\Big) \ . \ \forall K : \mathtt{Closed} \ \& \ \mathtt{Convex}(W) \ .$

$. \ \forall T : \forall t \in (a,b) \ . \ \mathrm{D}_r f|_t \in K \ . \ \dfrac{f(b)-f(a)}{b-a} \in K$

Proof $=$

Assume $(x,y,1) : \sum x,y \in (a,b) \ . \ x < y,$

$g := \Lambda t \in (x,y) \ . \ \dfrac{f(t)-f(x)}{t-x} : C\Big((x,y),W\Big),$

Assume $\varepsilon : \mathbb{R}_{++},$

$U := g^{-1}\Big(\mathtt{inflate}(K,\epsilon)^{\complement} \cap (x,y)\Big) : \mathtt{Open}(x,y),$

Assume $A : U \neq \emptyset,$

$u := \inf U : [x,y),$

$(2) := \eth K(x) : \lim_{t \downarrow x} g(t) \in K,$

$(3) := \eth C\Big((x,y),W\Big)(g)\eth U \eth u(2) : x < u,$

$(4) := \mathtt{OpenLowerBound}(U,u) : u < U,$

$(v,5) := \eth\mathtt{LowerBound}(u) : \sum v : \mathbb{N} \to U \ . \ \lim_{n\to\infty} v_n = u,$

$(6) := \mathtt{ContConvergent}(g)(v,5)(4) : \lim_{n\to\infty} g(v_n) \in \mathtt{inflate}(K,\epsilon),$

$(n,7) := \eth\mathtt{RightDifferential}(f)(u)(\eth v)C(\varepsilon) : \sum n \in \mathbb{N} \ . \ d\left(\dfrac{f(v_n)-f(u)}{v_n-u}, K\right) < \epsilon,$

$(8) := \eth g(v_n)\mathtt{FracSumIntro}(u) : g(v_n) = \dfrac{f(v_n)-f(x)}{v_n-x} == \dfrac{v_n-u}{v_n-x}\dfrac{f(v_n-f(u)}{v_n-u} + \dfrac{u-x}{v_n-x}g(u),$

$(9) := \eth U \eth\mathtt{Convex}(E)\Big(\mathtt{inflate}(K,\epsilon)\Big)(8)(7) : v_n \notin U,$

$(10) := \mathtt{InAndNotIn}(\eth v, 9) : \bot;$

$\rightsquigarrow (2) := \mathtt{Contradiction} : U = \emptyset,$

$(3) := \mathtt{AntiSet}(\eth U)(2) : \forall t \in (x,y) \ . \ d(g(t),K) < \epsilon;$

$\rightsquigarrow (2) := \lim_{\epsilon\to 0} I(\forall)(\epsilon) : \forall x \in (a,b) \, \forall t \in (x,b) \ . \ \dfrac{f(t)-f(x)}{t-x} \in K,$

$(4) := \lim(x \to a) \lim_{t \to b}(2) : \dfrac{f(b)-f(a)}{b-a} \in K;$

□

$\texttt{ConvergenceByDerivatives} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open} \,\&\, \texttt{InnerBounded} \,\&\, \texttt{Connected}(V) \,.$

$\quad . \forall f : \mathbb{N} \to \texttt{Differentiable} V, W, U \,.\, \forall a \in U \,.\, \forall A_1 : \Big(f(a) : \texttt{Convergent}(W)\Big) \,.\, \forall g : U \to \mathcal{B}(V, W) \,.$

$\quad . \forall A_2 : \mathrm{D}f \rightrightarrows g \,.\, \exists \varphi : \texttt{Differentiable}(V, W, U) \,.\, f \rightrightarrows \varphi \,\&\, \mathrm{D}\varphi = g$

$\texttt{Proof} \ =$

$(1) := \texttt{ConvergentIsCauchy}(\mathrm{D}f) : \Big(\mathrm{D}f : \texttt{Cauchy}\big(U \to \mathcal{B}(V, W), \texttt{supmetric}\big)\Big),$

$R := \texttt{diam}(U) : \mathbb{R}_{++},$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$(N, 2) := \eth\texttt{Cauchy}\big(U \to \mathcal{B}(V, W), \texttt{supmetric}\big)(\mathrm{D}f)\left(\frac{\varepsilon}{R}\right) : N \in \mathbb{N} \,.\, \forall (n, m, B) :$

$\quad : \sum n, m \in \mathbb{N} \,.\, n \geq N \,\&\, m \geq n \,.\, \|\mathrm{D}_f - g\| \leq \frac{\varepsilon}{R},$

$\texttt{Assume}\ (n, m, 3) : \sum n, m \in \mathbb{N} \,.\, n \geq N \,\&\, m \geq N,$

$h := \Lambda u \in U \,.\, f_n(u) - f_m(u) : \texttt{Differentiable}(V, W, U),$

$(4) := \sup_{x \in U} \texttt{InnerMeanValueTHM}(h, x, a) : \sup_{x \in U} \|f_n(x) - f_m(x) - f_n(a) + f_m(a)\| \leq$

$\quad \leq : w \sup_{x \in U} \sup_{y \in U} \|\mathrm{D}f_n|_y - \mathrm{D}f_m|_y\| d_U(x, a) == R\|\mathrm{D}f_n - \mathrm{D}f_m\| \leq \varepsilon;$

$\leadsto (2) := \eth^{-1}\texttt{Cauchy}\big(C_\infty(U)\big) : \Big(f - f(a) : \texttt{Cauchy}\big(C_\infty(U, W)\big)\Big),$

$(3) := \eth\texttt{Complete}(C_\infty)(2) - \lim_{n \to \infty} f(a) : \Big(f : \texttt{Convergent}\big(C_\infty(U)\big)\Big),$

$\varphi := \lim_{n \to \infty} f_n : C_\infty,$

$\texttt{Assume}\ u : U,$

$(4) := \texttt{ConverginSum} : f_n - f_n(u) - \mathrm{D}f_n|_u\, \texttt{minus}(u) : \texttt{Convergent}\big(C_\infty(U)\big),$

$(5) := \texttt{SublineaByConvergence}(4) : \varphi(x) - \varphi(u) - g(u)(x - u) = O\Big(\|x - u\|\Big),$

$(6) := \eth^{-1}\texttt{Differential}(6) : \mathrm{D}\varphi|_u = g(u);$

$\leadsto (*) := I(\forall) : \mathrm{D}\varphi = g,$

$\square$

$\texttt{ConvergenceByDerivativeUnbounded} :: \forall V, W : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open} \,\&\, \texttt{Connected}(V) \,.$

$\quad . \forall f : \mathbb{N} \to \texttt{Differentiable} V, W, U \,.\, \forall a \in U \,.\, \forall A_1 : \Big(f(a) : \texttt{Convergent}(W)\Big) \,.\, \forall g : U \to \mathcal{B}(V, W) \,.$

$\quad . \forall A_2 : \mathrm{D}f \rightrightarrows g \,.\, \exists \varphi : \texttt{Differentiable}(V, W, U) \,.\, f\mathrm{D}\varphi = g$

$\texttt{Proof} \ =$

Every point $u \in U$ will have bounded Neighbourhood to which we can apply previous theorem. Gluing provide

$\square$

$\texttt{COnebyPartialDerivatives} :: \forall n \in \mathbb{N} \,.\, \forall W \in \mathsf{BAN}(K) \,.\, \forall V : n \to \mathsf{BAN}(K) \,.$

$. \forall U : \texttt{Open}\left(\prod_{i=1}^{n} V_i\right) . \forall f : \texttt{PartiallyDifferentiable}(V, W, U) \,.$

$. \forall A : \forall i \in n \,.\, \mathrm{D}_i f : C\left(U, \mathcal{B}(V_i, W)\right) \,.\, f : C^1\left(U, W\right)$

$\texttt{Proof} \; =$

$$T := \Lambda p \in U \,.\, \Lambda h \in \prod_{i=1}^{n} \,.\, \sum_{i=1}^{n} \mathrm{D}_i f|_p h_i : \prod_{i=1}^{n} U_i \to \mathcal{B}\left(\prod_{i=1}^{n} V_i, W\right),$$

$\texttt{Assume}\; p : U,$

$$I := \Big\{\{1, \ldots, i\} | i \in \mathbb{N} : i < n\Big\} :??n,$$

$$\eta := \Lambda i \in I \,.\, i \cap \max(i) + 1 : I \to ??n,$$

$$\varphi := \Lambda i \in I \,.\, \Lambda x \in \prod_{i=1}^{n} U \,.\, f\left(\hat{p}_x^{\eta(i)}\right) - f\left(\hat{p}_x^{i}\right) - T(p)(x - p) : I \to U \to V,$$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$\texttt{Assume}\; i : n,$

$$(\delta_i, 1_i) := A(p, \varepsilon) : \sum \delta \in \mathbb{R}_{++} \,.\, \forall v \in \mathbb{B}(p, \delta) \,.\, \|\mathrm{D}_i f|_p - \mathrm{D}_i f|_v\| < \varepsilon;$$

$$\rightsquigarrow (\delta, 1) := I\left(\prod\right) : \prod i \in n \,.\, \sum \delta \in \mathbb{R}_{++} \,.\, \forall v \in \mathbb{B}(p, \delta) \,.\, \|\mathrm{D}_i f|_p - \mathrm{D}_i f|_v\| < \varepsilon,$$

$$\Delta := \max(\delta) : \mathbb{R}_{++},$$

$\texttt{Assume}\; x : \mathbb{B}(p, \Delta),$

$\texttt{Assume}\; i : n,$

$$h := \Lambda \xi \in \pi_i \mathbb{B}(p, \Delta) \,.\, f\left(\hat{p}_{\hat{x}_\xi^{\{i\}}}^{\{1,\ldots,i\}}\right) - \mathrm{D}_i f|_p (x - p) : \pi_i \mathbb{B}(p, \Delta) \to W,$$

$$(2) := \texttt{BanachMeanValueTHM}(h)(p, x) : \frac{\|h(x) - h(p)\|}{\|x - p\|} \le \sup_{v \in [x, p]} \Big\|\left(\mathrm{D}_i f|_v - \mathrm{D}_i f|_p\right)(x - p)|\Big\| \le \|x - p\|\varepsilon,$$

$$\rightsquigarrow (2) := \forall i \in I \,.\, \eth^{-1}\texttt{Convergent}(W) : \forall i \in I \,.\, \lim_{x \to p} \frac{\|\varphi_i(x)\|}{\|x - p\|} = 0,$$

$$(2) := \lim_{x \to p} \eth^{-1} \varphi \omega(f)(2) : \lim_{x \to p} \frac{\|f(x) - f(p) - T(p)(x - p)\|}{\|x - p\|} = 0,$$

$$(1) := \eth^{-1}\texttt{DifferentialAt}\left(\prod_{i=1}^{n} V, W, U\right) : \mathrm{D}f|_p = T(p);$$

$$\rightsquigarrow (1) := \eth^{-1}\texttt{Differentiable}\left(\prod_{i=1}^{n} V_i, U, W\right) : \left(f : \texttt{Differentiable}\left(\prod_{i=1}^{n} V_i, U, W\right)\right),$$

$$(*) := \texttt{SmoothByPartialDerivatives}(f) : f \in C^1(U, W);$$

$\square$

$\texttt{StronglyTangentToZeroAt} :: \prod V, W \in \mathsf{BAN}(K) \ . \ \prod U : \texttt{Open}(V) \ . \ ?(U \times U \to V)$

$(a, f) : \texttt{StronglyTangenToZeroAt} \iff f(a) = 0 \ \& \ \forall \varepsilon \in \mathbb{R}_{++} \ . \ \exists r \in \mathbb{R}_{++} \ . \ f_{|\mathbb{B}(a.r)} : \texttt{Lipschitz}(U, W, \varepsilon)$

$\texttt{StronglyDifferentiable} :: ?\texttt{Differentiable}(V, W, U)$

$f : \texttt{StronglyDifferentiable} \iff \forall p \in U \ . \ \Big(p, f(p) - f - \mathrm{D}f|_p \texttt{minus}(p)\Big) :$

$: \texttt{StronglyTangentToZeroAt}(V, W, U)$

$\texttt{ContinuouslyDifferentiableAreStrong} :: \forall f \in C^1(U, W) \ . \ : \texttt{StronglyDifferentiable}(V, W, U)$

Proof $=$

...

$\square$

1.5 Inverse function theorem

$\texttt{CategoryOfSmoothMaps} :: \texttt{Category}$

$\texttt{CategoryOfSmoothMaps}\,() = \mathsf{DIFF}(1) :=$

$$= \left(\sum H \in \mathsf{BAN}(K) \;.\; \texttt{Open}(H), \big((H,U),(G,V)\big) \mapsto C^1(U,V), \circ\right)$$

$\texttt{DiffeomorphismByInvertibility} :: \forall (H,U),(G,V) \in \mathsf{SMH}(1) \;.\; \forall f : (H,U) \to_{\mathsf{DIFF}} (G,V) \;\&\; U \leftrightarrow_{\mathsf{TOP}} V \;.$

$\quad . \; \forall u \in U \iff \mathrm{D}f|_u : \texttt{Invertible}(H,G) \;.\; f : (H,U) \leftrightarrow (V,U)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; R : \forall u \in U \;.\; \mathrm{D}f|_u : \texttt{Inverible}(H,G),$

$\texttt{Assume}\; y : V,$

$(x,1) := \eth\texttt{Bijective}(f)(y) : \sum x \in U \;.\; f(x) = y,$

$(O,\phi,2) := \eth\texttt{DifferntiableAt}(H,G,V)(f,x) : \sum W \in \mathcal{U}(x) \;.\; \sim \phi : W \to G \;.\; \phi(w) = O(\|w-x\|) \;\&$

$\quad \& \; \forall w \in W \;.\; f(w) = f(x) + \mathrm{D}f|_x(o-x) + \phi(w),$

$(A,3) := \texttt{SmoothIsStronglyDifferentiable}\left(\dfrac{\|(Df|_x)^{-1}\|^{-1}}{2}\right) : \sum A \in \dot{\mathcal{U}}(x) \;.\; \forall a \in A \;.$

$\quad . \; \|\phi(a)\| \le \dfrac{\left\|(\mathrm{D}f|_x)^{-1}\right\|^{-1}\|a-x\|}{2},$

$\texttt{Assume}\; a : A,$

$(4) := \|(\mathrm{D}f|_x)^{-1}(2)(a)\| : \left\|(\mathrm{D}f^{-1}|_x)^{-1}(f(a)-f(x))\right\| \ge \Big|\|a-x\| - \left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\|\Big|,$

$(5) := \eth\texttt{operatorNorm}(\mathrm{D}f|_x)^{-1}(4)(3) : \left\|(\mathrm{D}f|_x)^{-1}\right\|\|f(a)-f(x)\| \ge \|a-x\|\left|1 - \dfrac{\left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\|}{\|a-x\|}\right|,$

$$(6) := (5)\left(\frac{\left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\|}{\|a-x\|\left|1-\frac{\|(\mathrm{D}f|_x)^{-1}\phi(a)\|}{\|a-x\|}\right|}\right) : \left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\| \le \frac{\left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\|\left\|(\mathrm{D}f|_x)^{-1}\right\|\|f(a)-f(x)\|}{\|a-x\|\left|1-\frac{\|(\mathrm{D}f|_x)^{-1}\phi(a)\|}{\|a-x\|}\right|};$$

$\rightsquigarrow (3) := \eth^{-1}\texttt{AsymptoticAtZero}(2) : \left\|(\mathrm{D}f|_x)^{-1}\phi(a)\right\| = O(\|f(a)-f(x)\|),$

$(4) := \texttt{diffirintiate}\Big(f^{-1}, y, (\mathrm{D}f|_x)^{-1}\Big)\eth^{-1}x(2)(3) : \lim\limits_{v\to y}\dfrac{\|f^{-1}v - f^{-1}y - (\mathrm{D}f|_u)^{-1}(y-v)\|}{\|y-v\|} =$

$$= \lim_{v\to y}\frac{\left\|f^{-1}v - x - (\mathrm{D}f|_x)^{-1}\Big(x + \mathrm{D}f|_x\big(f^{-1}(v)-x\big) + \phi\big(f^{-1}(v)\big)\Big)\right\|}{\|y-v\|} = \lim_{v\to y}\frac{\left\|\big(\mathrm{D}f|_x\big)^{-1}\phi\big(f^{-1}(v)\big)\right\|}{\|y-v\|} = 0,$$

$(5) := \eth^{-1}\texttt{Differential}(G,H,V)(f^{-1},y)(4) : \mathrm{D}f^{-1}|_y = (\mathrm{D}f|_x)^{-1};$

$\rightsquigarrow (1) := I(\forall) : \forall y \in V \;.\; \mathrm{D}f|_y = (\mathrm{D}f|_{f^{-1}(y)})^{-1},$

$(2) := \eth^{-1}\texttt{Differntiable}(G,H,V)(1) : \Big(f^{-1} : \texttt{Differentiable}(G,H,V)\Big),$

$(3) := \eth^{-1}C^1(V,U)\texttt{ContinuousComposition}(1) : f^{-1} \in C^1(V,U),$

$(4) := \eth\mathsf{DIFF}(3) : \Big(f : (H,U) \leftrightarrow_{\mathsf{DIFF}} (G,V)\Big);$

$\rightsquigarrow (1) := I(\Rightarrow) : \Big(\forall u \in U \;.\; \mathrm{D}f|_u : \texttt{Invertible}(\mathcal{B}(H,G))\Big) \Rightarrow f : (H,U) \leftrightarrow_{\mathsf{DIFF}} (G,V),$

$\texttt{Assume}\ R : \Big(f : (H,U) \leftrightarrow_{\mathsf{DIFF}} (G,V)\Big),$

$\texttt{Assume}\ u : U,$

$(2) := \eth\texttt{Inverse}(f) : f^{-1}f = I,$

$(3) := \texttt{DerivativeOfLinear}(2) : \mathrm{D}f^{-1}f|_u = I,$

$(4) := \texttt{ChainRule}(f^{-1}, f)(3) : I = \mathrm{D}f^{-1}f|_u = \mathrm{D}f^{-1}|_{f(u)}\mathrm{D}f|_u,$

$(5) := \eth^{-1}\texttt{LeftInverse}(4) : \Big(\mathrm{D}f^{-1}|_u : \texttt{LeftInvers}(\mathrm{D}f|_u)\Big),$

$(6) := \eth\texttt{Inverse}(f^{-1}) : ff^{-1} = I,$

$(7) := \texttt{DerivativeOfLinear}(6) : \mathrm{D}ff^{-1}|_{f(u)} = I,$

$(8) := \texttt{ChainRule}(f^{-1}, f)(7) : I = \mathrm{D}ff^{-1}|_{f(u)} = \mathrm{D}f|_u\mathrm{D}f^{-1}|_{f(u)},$

$(9) := \eth^{-1}\texttt{RightInverse}(8) : \Big(\mathrm{D}f^{-1}|_u : \texttt{RightInverse}(\mathrm{D}f|_u)\Big),$

$(10) := \eth\texttt{Invertible}(H,G)(5,9) : \Big(\mathrm{D}f|_u : \texttt{Invertible}(H,G)\Big);$

$\leadsto (*) := I(\forall)(1)I(\Rightarrow)I(\forall) : \texttt{This};$

$\square$

$\texttt{HomeoByContraction} :: \forall E : \mathsf{BAN}(K)\ .\ \forall(p,r,f,1) : \sum(p,r) : E \times \mathbb{R}_{++}\ .\ \sum f : \mathbb{B}(p,r) \to E\ .$

$\quad \Lambda x \in \mathbb{B}(p,r)\ .\ x - f(x) : \texttt{Contraction}\Big(\mathbb{B}(p,r), E\Big)\ .\ \exists U : \texttt{Open}(E)\ .\ f : \mathbb{B}(p,r) \leftrightarrow_{\mathsf{TOP}} U$

$\texttt{Proof}\ =$

$\varphi := \Lambda x \in \mathbb{B}(p,r)\ .\ x - f(x) : \mathbb{B}(p,r) \to E,$

$(k,2) := \eth\texttt{Contraction}(1) : \sum k \in (0,1)\ .\ \forall x,y \in \mathbb{B}(p,r)\ .\ \|\varphi(x) - \varphi(y)\| \le k\|x-y\|,$

$\texttt{Assume}\ x,y : \mathbb{B}(p,r),$

$(3) := \eth^{-1}\varphi\Big(\|f(x) - f(y)\Big)\eth\texttt{Seminorm}(E)(\varphi(x) - \varphi(y), x-y)(2) :$

$\quad : \|f(x) - f(y)\| \le \|\varphi(x) - \varphi(y)\| + \|x-y\| \le (1+k)\|x-y\|;$

$\leadsto (3) := \eth^{-1}\texttt{Lipschitz} : \Big(f : \texttt{Lipschitz}\big(\mathbb{B}(p,r), E, 1+k\big)\Big),$

$\texttt{Assume}\ x,y : \mathbb{B}(p,r),$

$(4) := \eth^{-1}\varphi\texttt{InverseTriabgleIneq}(x-y, \varphi(x) - \varphi(y))\texttt{AbsValIneq}(2) :$

$\quad : \|f(x) - f(y)\| \ge \big|\|x-y\| - \|\varphi(x) - \varphi(y)\big| \ge (1-k)\|x-y\|;$

$\leadsto (4) := I(\forall) : \forall x,y \in \mathbb{B}(p,r)\ .\ \|f(x) - f(y)\| \ge (1-k)\|x-y\|,$

$\texttt{Assume}\ z : \Im f,$

$\texttt{Assume}\ (x,y,5) : \sum x,y \in \mathbb{B}(p,r)\ .\ f(x) = z\ \&\ f(y) = z,$

$(6) := (4)(x,y) : 0 \ge (1-k)\|x-y\|,$

$(7) := \texttt{NonnegativeNonpositive}(4)\texttt{ZeroNorm}(E) : x = y;$

$\leadsto (5) := \eth\texttt{Injective} : \Big(f : \mathbb{B}(p,r) \hookrightarrow E\Big),$

$\texttt{Assume}\ y : \mathbb{B}\left(f(p), \dfrac{r}{(1-k)}\right),$

$x_0 := p : \mathbb{B}(f(p), r),$

$\texttt{Assume}\ n : \mathbb{N},$

$x_n := y + \varphi(x_{n-1}) : E,$

$\texttt{Assume}\ Q : n = 1,$

$A_1 := \eth x_1 : \|x_1 - x_0\| = \|y + p - f(p) - p\| = \|y - f(p)\|;$

$\leadsto A^1 := I(\Rightarrow) : n = 1 \Rightarrow \|x_n - a\| = \dfrac{1-k^n}{1-k}\|y - f(p)\|,$

$\texttt{Assume}\ Q : n > 1\ \&\ A_{n-1}\ \&\ B_{n-1},$

$A_n := \texttt{TriangleIneq}(x_n, x_{n-1}, p)\eth x(2)A[n]\eth y :$

$$: \|x_n - p\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - p\| \le \|\varphi(x_{n-1}) - \varphi(x_{n-2})\| + \frac{1-k^{n-1}}{1-k}\|y - f(p)\| \le$$

$$\le k^{n-1}\|y - f(p)\| - \frac{1-k^{n-1}}{1-k}\|y - f(p)\| = \frac{1-k^n}{1-k}\|y - f(p)\| < r;$$

$\rightsquigarrow x := I(\rightarrow)\texttt{Induction}(A) : \mathbb{N} \to \mathbb{B}(p, r),$

$(6) := \texttt{SeriaCauchy}(2)(\eth x) : \Big(x : \texttt{Cauchy}\big(\mathbb{B}(p, r)\big)\Big),$

$X := \lim_{n\to\infty} x_n : \mathbb{B}(p, r),$

$(7) := \texttt{ContinuousLimit}(X, \eth x) : X = y + \varphi(X),$

$(8) := \eth\varphi(7) : y = \varphi(x);$

$$\rightsquigarrow (6) := \eth^{-1}\texttt{Bijection}(5) : \left(f : \mathbb{B}(p, r) \leftrightarrow \mathbb{B}\left(f(p), \frac{(1-k)}{r}\right)\right),$$

$$(*) := \eth^{-1}\texttt{Homeomorphism}(6)(2)(3) : \left(f : \mathbb{B}(p, r) \leftrightarrow_{\mathsf{TOP}} \mathbb{B}\left(f(p), \frac{(1-k)}{r}\right)\right);$$

$\square$

$\texttt{LocalInversionTheorem} :: \forall E, F : \mathsf{BAN}(K) . \forall U : \texttt{Open}(E) . \forall f : C^1(U, F) .$

$. \forall(p, 1) : \sum p \in U . \mathrm{D}f|_p : \texttt{Invertible}\Big(\mathcal{B}(E, F)\Big) . \exists V \in \mathcal{U}(p) : \exists W \in \mathcal{U}(f(p)) :$

$: \Big(f : (E, V) \leftrightarrow_{\mathsf{DIFF}(1)} (F, W)\Big)$

$\texttt{Proof} =$

$\varphi := \Lambda x \in U . x - (\mathrm{D}f|_p)^{-1}f(x) : U \to E,$

$(1) := \texttt{SmoothIsStronglyDifferentiable}(\mathrm{D}f|_p)^{-1}f : \Big((\mathrm{D}f|_p)^{-1}f : \texttt{StronglyDifferentiableAt}(E, F, U, p)\Big),$

$(r, 2) := \eth\texttt{StronglyDifferentiable}(E, F, U, p)(1/2)(\mathrm{D}f|_p)^{-1}f : \sum r \in \mathbb{R}_{++} . \varphi_{|\mathbb{B}(p,r)} : \texttt{Contraction},$

$(V', W', 3) := \mathrm{D}f|_p\texttt{HomeoByContraction} : \sum(V', W') \in \mathcal{U}(p) \times \mathcal{U}(f(p)) . f : V' \leftrightarrow_{\mathsf{TOP}} W',$

$(V, 4) := V' \cap \texttt{OpenInvertible}\Big(\mathcal{B}(E, F)\Big) : \sum V \in \mathcal{U}(p) . V \subset V'\ \&\ \forall v \in V . \mathrm{D}f|_v : \texttt{Invertible}\Big(\mathcal{B}(E, F)\Big),$

$W := f(V) : \mathcal{U}(f(p)),$

$(*) := \texttt{DiffeoByInvertibility}(3, 4) : \Big(f : (V, E) \leftrightarrow_{\mathsf{DIFF}(1)} (W, F)\Big);$

$\square$

$\texttt{LocalInversionCollorarly} :: \forall E, F : \mathsf{BAN}(K) . \forall U : \texttt{Open}(E) . \forall f : C^1 U, F .$

$. \forall q : \forall u \in U . \mathrm{D}f|_u : \texttt{Inverible}\Big(\mathcal{B}(E, F)\Big) . \Big(f : (E, U) \leftrightarrow_{\mathsf{DIFF}(1)} (F, f(U))\Big)$

$\texttt{Proof} =$

1.6 ImplicitFunctionTheorem

$\texttt{ImplicitFunctionTHM} :: \forall E,F,G : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(V \oplus F) \,.\, \forall f : C^1(U,G) \,.$

$\quad . \forall(a,b,1) : \sum (a,b) \in U \,.\, f(a,b) = 0 \;\&\; \mathrm{D}_2 f|_{(a,b)} : \texttt{Invertible}\Big(\mathcal{B}(F,G)\Big) \,.$

$\quad . \exists(V,W,g,2) : \sum (V,W) \in \mathcal{U}(a,b) \times \mathcal{U}(a) \,.\, \sum g \in C^1(W,F) \,.$

$\quad . \forall(x,y) \in V \,.\, f(x,y) = 0 \iff x \in W \;\&\; y = g(x)$

$\texttt{Proof} \;=$

$\varphi := \Lambda(x,y) \in U \,.\, \big(x, f(x,y)\big) : C(U, E \oplus G),$

$(2) := \mathrm{D}\eth\varphi|_{(a,b)} : \mathrm{D}\varphi|_{(a,b)} = \begin{bmatrix} I & 0 \\ \mathrm{D}_1 f|_{(a,b)} & \mathrm{D}_2 f|_{(a,b)} \end{bmatrix},$

$(3) := \texttt{LTOperatorInvertible}(2) : \Bigg(\mathrm{D}\varphi|_{(a,b)}; \texttt{Invertible}\Big(\mathcal{B}(E \oplus F, E \oplus G)\Big)\Bigg),$

$(4) := \texttt{LTOperatorInversion}(2,3) : (\mathrm{D}\varphi|_{(a,b)})^{-1} = \begin{bmatrix} I & 0 \\ -(\mathrm{D}_2 f|_{(a,b)})^{-1}\mathrm{D}_1 f|_{(a,b)} & (\mathrm{D}_2 f|_{(a,b)})^{-1} \end{bmatrix},$

$(V, W', 5) := \texttt{LocalInversionTHM}(3) :$

$\quad : \sum (V,W') \in \mathcal{U}(a,b) \times \mathcal{U}(a,0) \,.\, \varphi : (E \oplus F, V) \leftrightarrow_{\mathsf{DIFF}(1)} (E \oplus G, W'),$

$(W,6) := \texttt{horisontalSliceAt}(W',0) : \sum W \in \mathcal{U}(a) \,.\, \forall(w,0) \in W' \,.\, w \in W,$

$g := \Lambda w \in W \,.\, \pi_2 \phi^{-1}(w,0) : C^1(W,F),$

$\texttt{Assume}\ (x,y) : V,$

$\texttt{Assume}\ (7) : f(x,y) = 0,$

$(8) := \eth\varphi(7) : \varphi(x,y) = (x,0),$

$(9) := (5)\eth(x,y) : (x,0) \in W',$

$(10) := (6)(9,8) : x \in W,$

$(11) := \eth g(8) : g(x) = y;$

$\rightsquigarrow (7) := I(\Rightarrow) : f(x,y) = 0 \Rightarrow x \in W \;\&\; g(x) = y,$

$\texttt{Assume}\ (8) : x \in W \;\&\; g(x) = y,$

$(10) := \eth g \eth\varphi(8) : f(x,y) = 0;$

$\rightsquigarrow (*) := I(\forall)I(\iff)(7) : \forall(x,y) \in V \,.\, f(x,y) = 0 \iff x \in W \;\&\; g(x) = y;$

$\square$

$\texttt{LocalImpilcitFunction} :: \prod E,F,G : \mathsf{BAN}(K) \,.\, \prod U : \texttt{Open}(V) \,.\, C^1(U,G) \to U \to$

$\quad \to \sum (V,W) : \texttt{Open}(U) \times \texttt{Open}(E) \,.\, C^1(W,F)$

$(V,W,g) : \texttt{LocalImplicitFunction}\big(f,(a,b)\big) \iff$

$\quad \iff (a,b) \in U \;\&\; a \in W \;\&\; \forall(x,y) \in U \,.\, f(x,y) = f(a,b) \iff x \in W \;\&\; g(x) = y$

$\texttt{ImplicitFunctionUnique} :: \forall E, F, G : \mathsf{BAN}(K) \;.\; \forall U : \texttt{Open}(E \oplus F) \;.\; \forall f : C^1(U, G) \;.\; \forall (a, b) \in U \;.$
$\;.\; \forall (V, W, g), (V', W', h) : \texttt{loclaImplicitFunction}(E, F, G, U)(f, (a, b)) \;.$
$\;.\; \forall (X, 1) : \sum X \in \mathcal{U}(a) \;\&\; \texttt{Connecected}(E) \;.\; X \subset W \cap W' \;.\; h_{|X} = g_{|X}$

$\texttt{Proof} \;=$

$A := \{x \in X : h(x) = g(x)\} : \texttt{Set}(X),$

$\texttt{Assume}\; x : \texttt{Convergent}(A),$

$a := \lim_{n \to \infty} x_n : X,$

$(2) := \texttt{ContinuousLimit}(g, h, x)\eth A : f(a) = h(a),$

$(3) := \eth^{-1} A(2) : a \in A;$

$\rightsquigarrow (2) := \texttt{ClosedByConvergence} : \Big(A : \texttt{Closed}(X)\Big),$

$(3) := \eth X \eth \texttt{LocalImplicitFunction}(E, F, G, U)(f, (a, b))(g, h) : a \in A,$

$(4) := \eth^{-1}\emptyset(3) : A \neq \emptyset,$

$\texttt{Assume}\; a : \texttt{LimitPoint}(A),$

$(x, 5) := \eth A \eth \texttt{LimitPoint}(A)(a) : \sum x : \mathbb{N} \to A^{\complement} \;.\; \lim_{n \to \infty} x_n = a,$

$(6) := \texttt{ContinuousLimit}\eth A(a) : \lim_{n \to \infty} \varphi(x_n, g(x_n)) = \varphi(a, g(a)) = \varphi(a, h(a)) = \lim_{n \to \infty} \varphi(x_n, h(x_n)),$

$(n, 7) := (6)(V) : \sum n \in \mathbb{N}\,(x_n, h(x_n)) \in V,$

$(8) := \eth \texttt{LocalImplicitFunction}(E, F, G, U)(f, (a, b))(g, h) : (x_n, g_n(x_n)) = \varphi^{-1}(x_n, f(x_n, g(x_n))) = \varphi^{-1}(x_n, f($

$(9) := E(=, \times)(8) : h(x_n) = g(x_n),$

$(10) := \eth x(9) : \perp;$

$\rightsquigarrow (5) := \texttt{LimitlessIsOpen} : \Big(A : \texttt{Open}(X)\Big),$

$(6) := \eth \texttt{Connected}(X)(5)(4) : A = X,$

$(*) := \eth A(6) : g_X = h_{|X};$

$\square$

1.7 Higher Order Derivative

$$\texttt{NDifferentiable} :: \prod E, F : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(E) \;.\; \mathbb{N} \to ?(U \to F)$$

$$f : \texttt{NDifferentiable}(1) \iff f : \texttt{Differentiable}(E, F, U)$$

$$f : \texttt{NDifferentiable}(n) \iff f : \texttt{Differentiable}(E, F, U) \;\&$$

$$\& \; \mathrm{D}f : \texttt{NDiffernetiable}\Big(E, \mathcal{B}(E, F), U\Big)(n-1)$$

$$\texttt{nDerivative} :: \prod E, F : \mathsf{BAN}(K) \;.\; \prod U : \texttt{Open}(E) \;.\; \prod n \in \mathbb{N} \;.$$

$$.\; U \to \texttt{NDifferentiable}(E, F, U) \to \mathcal{B}\Big((E)_{i=1}^{n}; G\Big)$$

$$\texttt{nDerivatve}\,(u, f, h) = \mathrm{D}^n f|_u h := \mathrm{D}\mathrm{D}^{n-1} f|_u|_u h$$

$$\texttt{NPartiallyDifferentiable} :: \prod m \in \mathbb{N} \;.\; \prod E : m \to \mathsf{BAN}(K) \;.\; \prod F \in \mathsf{BAN}(K) \;.$$

$$.\; \prod U : \texttt{Open}\left(\prod_{i=1}^{n} E\right) \;.\; \mathbb{N} \to ?(U \to F)$$

$$f : \texttt{NPartiallyDifferentiable}(1) \iff f : \texttt{PartiallyDifferentiable}(E, F, U)$$

$$f : \texttt{NPartiallyDifferentiable}(n) \iff f : \texttt{PartiallyDifferentiable}(E, F, U) \;\&$$

$$\& \; \forall i \in m \;.\; \mathrm{D}_i f : \texttt{NPartialyDifferentiable}(E, F, U)(n-1)$$

$$\texttt{nPartialDerivative} :: \prod m \in \mathbb{N} \;.\; \prod E : m \to \mathsf{BAN}(K) \;.\; \prod F \in \mathsf{BAN}(K) \;.$$

$$.\; \prod U : \texttt{Open}\left(\prod_{i=1}^{n} E\right) \;.\; \prod n \in \mathbb{N} \;.\; \prod J : (n \to m) \;.$$

$$.\; U \to \texttt{NPartiallyDiffentiable}(E, U, F)(n) \to \mathcal{B}\Big((E_{J_i})_{i=1}^{n}; F\Big)$$

$$\texttt{nPartialDerivative}\,(u, f, h) = \mathrm{D}_J f|_u h := \mathrm{D}_{J_n} \mathrm{D}_{J_{|n-1}} f|_u|_u h$$

$$\texttt{Having} \quad \mathrm{D}_{[i]} = \mathrm{D}_i$$

$$\texttt{NSmooth} :: \mathbb{N} \to ?\Big((E, U) \to_{\mathsf{DIFF}} (F, V)\Big)$$

$$f : \texttt{NSmooth}(1) \iff \texttt{True}$$

$$f : \texttt{NSmooth}(n) \iff f \in C^n(U, V) \iff \mathrm{D}f \in C^{n-1}(U, \mathcal{B}(E, F))$$

$$\texttt{InfitlySmooth} :: ?\Big((E, U) \to_{\mathsf{DIFF}} (F, V)\Big)$$

$$f : \texttt{InfinitlySmooth} \iff f \in C^\infty(U, V) \iff \forall n \in \mathbb{N} \;.\; f \in C^n(U, V)$$

$$\texttt{categoryOfNSmoothMaps} :: \prod K : \texttt{AbsValField} \;\&\; \texttt{Complete} \;.\; \overset{\infty}{\mathbb{N}} \to \texttt{Category}$$

$$\texttt{categoryOfNSmoothMaps}\,(n) = \mathsf{DIFF}(n) :=$$

$$= \left(\sum E : \mathsf{BAN}(K) \;.\; \texttt{Open}(E), \Big((E, U), (F, V)\Big) \mapsto C^n(U, V), \circ\right)$$

$\texttt{NDifferentiabilityByLimit} :: \forall E, F : \mathsf{BAN}(K) \;.\; \forall U : \texttt{Open}(E) \;.\; \forall n \in \mathbb{N} \;.$

$$. \forall f : \texttt{NDiffernriable}(E, F, U)(n) \;.\; \forall u \in U \;.\; \forall A \in \mathcal{B}\Big((E)_{i=1}^{n}, F\Big) \;.$$

$$. A = \mathrm{D}^n f|_u \iff \lim_{h \to 0} \frac{\left\| \sum_{i=0}^{n} (-1)^{n-i} \sum_{S \in 2^n : |S| = i} f\left(u + \sum_{j \in S} h_j\right) - A(h) \right\|}{\|h\|^n} = 0$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SymmetricDifferentials} :: \forall E, F : \mathsf{BAN}(K) \;.\; \forall U : \texttt{Open}(E) \;.\; \forall n \in \mathbb{N} \;.$

$$. \forall f : \texttt{NDifferentiable}(E, F, U)(n) \;.\; \forall u \in U \;.\; \mathrm{D}^n f|_u : \texttt{Symmetric}\Big((E)_{i=1}^{n}; F\Big)$$

$\texttt{Proof} =$

$(r, 1) := \eth\texttt{Open}(E)(U)(u) : \sum r \in \mathbb{R}_{++} \;.\; \mathbb{B}(u, r) \subset U,$

$$\varphi := \Lambda h \in \mathbb{B}_{E^n}(0, r) \;.\; \sum_{i=0}^{n} (-1)^{n-i} \sum_{S \in 2^n : |S| = i} f\left(u + \sum_{j \in S} h_j\right) : \mathbb{B}_E(0, r) \to F,$$

$(2) := \eth^{-1}\texttt{Symmetric}\eth\varphi : \Big(\varphi : \texttt{Symmetric}\big((E)_{i=1}^{n}; F\big)\Big),$

$\texttt{Assume}\ \sigma : S_n,$

$$(3) := \texttt{NDifferentiabilityByLimit}(n, f) : \lim_{h \to 0} \frac{\left\|\varphi(h) - \mathrm{D}^n f|_u h\right\|}{\|h\|^n} = 0,$$

$$(4) := \texttt{PermutationIsometry}(3)(\sigma)(4) : 0 = \lim_{\sigma h \to 0} \frac{\left\|\varphi(\sigma h) - \mathrm{D}^n f|_u \sigma h\right\|}{\|\sigma h\|^n} = \lim_{h \to 0} \frac{\left\|\varphi(h) - \mathrm{D}^n f|_u \sigma h\right\|}{\|h\|^n},$$

$(5) := \texttt{NDifferentiabilityByLimit}(4) : \mathrm{D}^n f|_u = \mathrm{D}^n f|_u \circ \sigma;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Symmetric} : \Big(\mathrm{D}^n f|_u : \texttt{Symmetric}\big((E)_{i=1}^{n}, F\big)\Big);$

$\square$

$\texttt{SchwarzTheorem} :: \forall F \in \mathsf{BAN}(K) \;.\; \forall m, n \in \mathbb{N} \;.\; \forall E : m \to \mathsf{BAN}(K) \;.$

$$. \forall f : \texttt{NDifferentiable}\left(\prod_{i=1}^{m} E, F, U\right)(n) \;.\; \forall I \in n \to m \;.\; \forall \sigma \in S_n \;.\; \forall u \in U \mathrm{D}_I f|_u = \mathrm{D}_{\sigma I} f|_u \cdot \sigma$$

$\texttt{Proof} =$

Use the fact that

$\mathrm{D}_I f|_u h = \mathrm{D}^n f|_u (\iota_{E, I_j} h_j)_{j=1}^{n}$

But $\mathrm{D}^n f|_u$ is symmetric, so the result follows

$\square$

$\texttt{NSmoothByPartial} :: \forall n, m \in \mathbb{N} \;.\; \forall E : m \to \mathsf{BAN}(K) \;.\; \forall F : \mathsf{BAN}(K) \;.\; \forall U : \texttt{Open}(U) \;.$

$$. \forall f : \texttt{PartiallyDifferentiable}(E, F, U) \;.\; f \in C^n(U, F) \iff \forall i \in m \;.\; \mathrm{D}_i f \in C^{n-1}(U, F)$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{MultilinearDifferentiation} :: \forall n \in \mathbb{N} \,.\, \forall E : n \to \mathsf{BAN}(K) \,.\, \forall F : \mathsf{BAN}(K) \,.\, \forall T : \mathcal{B}(E, F) \,.$

$T \in C^\infty(E, F) \;\&\; \forall (m, 1) : \sum m \in \mathbb{N} \,.\, m > n \,.\, \mathrm{D}^m T = 0 \;\&\; \forall m \in n \,.\, \forall p, h \in E \,.$

$$. \, \mathrm{D}^m T|_p h = m! \sum_{S \in 2^n : |S| = m} T\Big((h_i)_{i \in S} \oplus (p_i)_{i \in S^\complement}\Big)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{InverseDifferentiation} :: \forall B : \texttt{BanachAlgebra}(K) \,.\, \texttt{inv} \in C^\infty\Big(\texttt{Invertible}(B), \texttt{Invertible}(B)\Big) \,.$

$$. \, \forall n \in \mathbb{N} \,.\, \forall h \in B^n \,.\, \forall p : \texttt{Invertible}(B) \,.\, \mathrm{D}^n \texttt{inv}|_p h = (-1)^n p^{-1} \sum_{\sigma \in S_n} \prod_{i=1}^n \left(h_{\sigma(i)} p^{-1}\right)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{NDifferentiableComposition} :: \forall E, F, G : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall V : \texttt{Open}(F) \,.\, \forall n \in \mathbb{N} \,.$

$. \, \forall f : \texttt{NDifferentiable}(E, F, U)(n) \,.\, \forall g : \texttt{NDifferentiable}(F, G, V)(n) \,.\, \forall (1) : \operatorname{Im} f \subset V \,.$

$. \, g \circ f : \texttt{NDifferentiable}(E, G, U)(n)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{NSmoothCompsition} :: \forall E, F, G : \mathsf{BAN}(K) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall V : \texttt{Open}(F) \,.\, \forall n \in \mathbb{N} \,.$

$. \, \forall f \in C^n(U, V) \,.\, \forall g \in C^n(V, G) \,.\, g \circ f \in C^n(U, G)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{NSmoothDiffeomorphism} :: \forall (E, U), (F, V) \in \mathsf{DIFF}(n) \,.\, \forall f : (E, U) \to_{\mathsf{DIFF}(n)} (F, V) \;\&$

$\& \; (E, U) \leftrightarrow_{\mathsf{DIFF}(1)} (F, V) \,.\, f : (E, U) \leftrightarrow_{\mathsf{DIFF}(n)} (F, V)$

$\texttt{Proof} \;=$

...

$\square$

1.8 Taylor Expansion

$\texttt{TaylorFormulaWithTheIntegralReminder} :: \forall E, F \in \mathsf{BAN}(K) . \forall U : \texttt{Open}(E) .$

$$. \forall n \in \mathbb{N} . \forall f \in C^{n+1}(U, F) . \forall (a, h, 1) : \sum (a, h) \in U \times E . [a, a+h] \subset U .$$

$$f(a+h) = \sum_{k=0}^{n} \frac{\mathrm{D}^k f|_a (h)_{i=1}^{k}}{k!} + \int_0^1 \frac{(1-t)^n}{n!} \mathrm{D}^{n+1} f|_{a+th} (h)_{i=1}^{n+1} \, \mathrm{d}t$$

$\texttt{Proof} \ =$

...

□

$\texttt{TaylorFormulaWithLagrangeReminder} :: \forall E, F \in \mathsf{BAN}(K) . \forall U : \texttt{Open}(E) .$

$$. \forall n \in \mathbb{N} . \forall f \in C^{n+1}(U, F) . \forall (a, h, 1) : \sum (a, h) \in U \times E . [a, a+h] \subset U .$$

$$\forall (M, 2) : \sum M \in \mathbb{R}_{++} . \forall t \in [0, 1] . \left\| \mathrm{D}^n f|_{a+th} \right\| \leq M .$$

$$. \left\| f(a+h) - \sum_{k=0}^{n} \frac{\mathrm{D}^k f|_a (h)_{i=1}^{k}}{k!} \right\| \leq M \frac{\|h\|^{n+1}}{(n+1)!}$$

$\texttt{Proof} \ =$

...

□

1.9 Analytic Polynomials

$(K.0) : \sum K : \texttt{Field} \ . \ \texttt{char}\, K = 0$

$\texttt{HomogeneusPolynomial} :: \prod V, W \in \mathsf{VS}(K) \ . \ \mathbb{Z}_+ \to ?(V \to W)$

$p \in \texttt{HomogeneusPolynomials}(n) \iff p \in \mathcal{HP}(V, W, n) \iff$

$\iff \exists A \in \mathcal{L}\Big((V)_{i=1}^n, W\Big) \ . \ \forall v \in V \ . \ p(v) = A(v)_{i=1}^n$

$\texttt{HomgeneusPolynomialIsNHomogeneus} :: \forall p \in \mathcal{HP}(V, W, n) \ . \ \forall a \in K \ . \ \forall v \in V \ . \ p(av) = a^n p(v)$

$\texttt{Proof} =$

...

$\square$

$\texttt{HomogeneusPolynomialsAreVectorSpace} :: \mathcal{HP}(V, W, n) \in \mathsf{VS}(K)$

$\texttt{Proof} =$

...

$\square$

$\texttt{prodHP} :: \prod V, W, E, F \in \mathsf{VS}(K) \ . \ \prod m, n \in \mathbb{N} \ . \ \mathcal{L}\big([V, W], F\big) \to \mathcal{HP}(E, V, n) \to \mathcal{HP}(E, W, m) \to \mathcal{HP}(E, F, r$

$\texttt{prodHP}\,(\Phi, p, q) = p *_\Phi q := \Lambda h \in E \ . \ \Phi\Big(p(h), q(h)\Big)$

...

$\square$

$\texttt{PreanalyticPolynomial} :: \prod V, W \in \mathsf{VS}(K) \ . \ ?(V \to W)$

$p : \texttt{PreanalyticPolynomial} \iff p \in \mathcal{P}(V, W) \iff p = 0 | \exists n \in \mathbb{Z}_+ \ . \ \exists q \in \prod i \in n \ . \ \mathcal{HP}(V, W, i) \ . \ p = \sum_{i=1}^n q_i$

$\texttt{degree} :: \mathcal{P}(V, W) \to \mathbb{Z}_+ \cup \{-\infty\}$

$\texttt{degree}\,(0) = \deg 0 := -\infty$

$\texttt{degree}\,(p) = \deg p := \Big(\eth\mathcal{P}(V, W)(p)\Big)_1$

$\texttt{degreewisePolynomialVS} :: \Big(\mathsf{VS}(K)\Big)^2 \to \mathbb{N} \to \mathsf{VS}(K)$

$\texttt{degreewisePolynomialVS}\,(V, W, n) = \mathcal{P}^n(V, W) := \{p \in \mathcal{P}(V, W) : \deg p \leq n\}$

$\texttt{prodP} :: \prod V, W, E, F \in \mathsf{VS}(K) \ . \ \prod m, n \in \mathbb{N} \ . \ \mathcal{L}\big([V, W], F\big) \to \mathcal{P}^n(E, V) \to \mathcal{P}^m(E, W) \to \mathcal{P}^{n+m}(E, F)$

$\texttt{prodP}\,(\Phi, p, q) = p *_\Phi q := \Lambda h \in E \ . \ \sum_{i,j=1}^{n',m'} \Phi\Big(f_i(h), g_j(h)\Big)$

where

$(n', f, 1) = \eth\mathcal{P}^n(E, V)(p) : \sum n' \in n \ . \ \sum f : \prod i \in n' \ . \ \mathcal{HP}(E, V, i) \ . \ p = \sum_{i=1}^{n'} f_i$

$(m', g, 2) = \eth\mathcal{P}^m(E, W)(q) : \sum m' \in n \ . \ \sum g : \prod j \in m' \ . \ \mathcal{HP}(E, W, i) \ . \ q = \sum_{j=1}^{m'} g_j$

$\texttt{difference} :: \prod V, W \in \mathsf{VS}(K) \ . \ V \to (V \to W) \to (V \to W)$

$\texttt{difference}\,(h, f) = \Delta_h f := \Lambda v \in V \ . \ f(v + h) - f(v)$

$\texttt{nDifference} :: \prod V, W \in \mathsf{VS}(K) \ . \ \prod n \in \mathbb{N} \ . \ (n \to V) \to (V \to W)$

$\texttt{nDifference}\,([h], f) = \Delta^1_{[h]} f := \Delta_h f$

$\texttt{nDifference}\,(h, f) = \Delta^n_h f := \Delta_{h_n} \Delta^{n-1}_{h_{|n-1}} f$

$\texttt{DifferenceOfPolynomials} :: \forall V, W \in \mathsf{VS}(k) \ . \ \forall p \in \mathcal{P}(V, W) \ . \ \forall h \in V \ . \ \Delta_h p \in \mathcal{P}(V, W)$

Proof $=$

$\ldots$

$\square$

$\texttt{DifferenceDegree} :: \forall V, W \in \mathsf{VS}(K) \ . \ \forall (p, 1) : \sum p \in \mathcal{P}(V, W) \ . \ \deg p > 0 \ . \ \forall n \in \deg p \ . \ \forall h : n \to V \ .$

$\deg \Delta^n_h p = (\deg p) - n$

Proof $=$

$\ldots$

$\square$

$\texttt{DifferenceDegreeII} :: \forall V, W \in \mathsf{vs}(K) \ . \ \forall p : \sum p \in \mathcal{P}(V, K) \ . \ \forall (n, 1) : \sum n \in \mathbb{N} \ . \ n > \deg p \ . \ \forall h : n \to V \ .$

$. \ \Delta^n_h p = 0$

Proof $=$

$\ldots$

$\square$

$\texttt{PolynomialRepresentationIsUnique} :: \forall (p, 1) : \sum p \in \mathcal{P}(V, W) \ . \ p \neq 0 \ .$

$. \ \exists! q : \prod n \in \deg p \ . \ \mathcal{HP}(V, W, n) \ . \ p = \sum_{n=1}^{\deg p} q_n$

Proof $=$

$\ldots$

$\square$

$\texttt{AnalyticPolynomial} :: \prod V, W : \mathsf{TOPVS}(K) \,.\, ?\mathcal{P}(V, W)$

$p : \texttt{AnalyticPolynomial} \iff p \in \mathcal{AP}(V, W) \iff p \in C(V, W)$

$\texttt{AnalyticPolynomialOfDegree} :: \prod V, W : \mathsf{TOPVS}(K) \,.\, \mathbb{N} \to ?\mathcal{AP}(V, W)$

$\texttt{AnalyticPolynomialsOfDegree}\,(n) = \mathcal{AP}^n(V, W) := \mathcal{P}^n(V, W) \cap C(V, W)$

Theorems about continuity of polynomials go here but not stated explicitely in this printing.

1.10 Finite Expansion

$$\texttt{NTangentToZero} :: \prod E, F : \mathsf{BAN}(K) . \prod U \in \mathcal{U}(0) . \mathbb{N} \to ?(U \to F)$$

$$f : \texttt{NTangentToZero}(n) \iff f(x) = O(\|x\|^n) \iff \lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0$$

$$\texttt{NTangentDifference} :: \forall E, F : \mathsf{BAN}(K) . \prod U \in \mathcal{U}(0) . \forall n \in \mathbb{N} .$$
$$. \forall (f, 1) : \sum f : U \to F . f(x) = O\Big(\|x\|^n\Big) . \Delta^n_x f(0) = O\Big(\|x\|^n\Big)$$

Proof $=$

...

□

$$\texttt{FiniteExpansion} :: \prod E, F : \mathsf{BAN}(K) . \prod U : \texttt{Open}(E) . \prod n \in \mathbb{N} . (U \to F) \to U \to ?\mathcal{AP}^n(E, F)$$

$$p : \texttt{FiniteExpansion}(f, u) \iff f(u + x) - p(x) = O(\|x\|^n)$$

$$\texttt{FiniteExpansionIsUnique} :: \forall p, q : \texttt{FiniteExpansion}(E, F, U, n)(f, u) . p = q$$

Proof $=$

...

□

$$\texttt{NthDifferenceFinitExpansion} :: \forall p : \texttt{FiniteExpansion}(E, F, U, n)(f, u) . \Delta^n_x f(u) - n!T(x) = O\Big(\|x\|^n\Big)$$

where

$$(n', f, 1) = \eth\mathcal{P}^n(E, F)(p) : \sum n' \in n . \sum f : \prod i \in n' . \mathcal{HP}(E, V, i) . p = \sum_{i=1}^{n'} f_i$$

$$q = \texttt{if} \quad n' < n \quad \texttt{then} \quad 0 \quad \texttt{else} \quad f_n : \mathcal{HP}(E, U, n)$$

$$(T, 2) = \eth\mathcal{HP}(E, U, n)(f_n) : \sum T \in \mathcal{B}\Big((E)^n_{i=1}, F\Big) . \forall x \in E . q(x) = T(x)^n_{i=1}$$

Proof $=$

...

□

$\texttt{truncation} :: \prod E, F : \mathsf{BAN}(K) \;.\; \prod n \in \mathbb{N} \;.\; \prod m \in n \;.\; \mathcal{AP}^n(E,F) \to \mathcal{AP}^m(E,F)$

$$\texttt{truncation}\,(p) = \mathrm{trunc}(p, m) := \sum_{i=1}^{\min(m,n')} f_i$$

where

$$(n', f, 1) = \eth\mathcal{P}^n(E, V)(p) : \sum n' \in n \;.\; \sum f : \prod i \in n' \;.\; \mathcal{HP}(E, V, i) \;.\; p = \sum_{i=1}^{n'} f_i$$

$\texttt{FiniteExpansionTruncation} :: \forall p : \texttt{FiniteExpansion}(E, F, U, n)(f, u) \;.\; \forall m \in n \;.$
$\;.\; \mathrm{trunc}(p, m) : \texttt{FiniteExpansion}(E, F, U, m)(f, u)$

Proof $=$

$\dots$

$\square$

$$\texttt{TaylorCollorarly} :: \forall f \in C^{n+1}(U, V) \;.\; \forall u \in U \;.\; \sum_{k=0}^{n} \frac{\mathrm{D}^k f|_u}{k!} : \texttt{FiniteExpansion}(\dots, U, n)(f, u)$$

Proof $=$

$\dots$

$\square$

$\texttt{FiniteExpansionAddition} :: \forall p : \texttt{FiniteExpansion}(E, F, U, n)(f, u) \;.$
$\forall q : \texttt{FiniteExpansion}(E, F, U, n)(g, u) \;.\; p + q : \texttt{FiniteExpansion}(E, F, U, n)(g + f, u)$

Proof $=$

$\dots$

$\square$

$\texttt{FiniteExpansionMultiplication} :: \forall \Phi : \mathcal{B}\Big([V, W], F\Big) \;.\; \forall p : \texttt{FiniteExpansion}(E, V, U, n)(f, u) \;.$
$\;.\; \forall q : \texttt{FiniteExpansion}(E, W, U, n)(g, u) \;.\; \mathrm{trunc}(p *_\Phi q, n) : \texttt{FiniteExpansion}(E, F, U, n)(f *_\Phi q, u)$

Proof $=$

$\dots$

$\square$

$\texttt{FiniteExpansionComposition} :: \forall p : \texttt{FiniteExpansion}(E, F, U, n)(f, u)$.

$. \forall q : \texttt{FiniteExpansion}(F, G, V, n)(g, v) . \forall (1) : \operatorname{Im} f \subset V \;\&\; f(u) = v$.

$$. g(f(u)) + \sum_{i=1}^{n} \sum_{J:i \to n:|J| \leq n} T_i(a_{J_j})_{j=1}^{i} : \texttt{FiniteExpansion}(E, G, U, n)(g \circ f, u)$$

Where

$$(a, 1) = \eth\mathcal{AP}^n(E, F)(p) : \sum a : \prod i \in n . \mathcal{HP}(E, F, i) \cup C(E, F) . p = \sum_{i=0}^{n} a_i$$

$$(b, 2) = \eth\mathcal{AP}^n(F, G)(q) : \sum b : \prod i \in n . \mathcal{HP}(F, G, I) \cup C(F, E) . q = \sum_{i=0}^{n} q_i$$

$$(T, 3) = \forall i \in n . \eth\mathcal{HP}^n(F, G)(b) : \sum T : \prod i \in n . \mathcal{B}\Big((F)_{j=1}^{i}, G\Big) . \forall x \in F . \forall i \in n . b_i(x) = T_i(x)_{j=1}^{i}$$

Proof $=$

...

$\square$

1.11 Extremal Points Theorems

$\texttt{LocalMinimum} :: \prod X \in \mathsf{TOP} \,.\, \prod Y : \texttt{Ordered} \,.\, (X \to Y) \to ?X$

$x : \texttt{LocalMinimum}(f) \iff \exists U \in \mathcal{U}(x) \,.\, \forall u \in U \,.\, f(x) \le f(u)$

$\texttt{LocalMaximum} :: \prod X \in \mathsf{TOP} \,.\, \prod Y \in \texttt{Ordered} \,.\, (X \to Y) \to ?X$

$x : \texttt{LocalMaximum}(f) \iff \exists U \in \mathcal{U}(x) \,.\, \forall u \in U \,.\, f(x) \ge f(u)$

$\texttt{StrictLocalMinimum} :: \prod X \in \mathsf{TOP} \,.\, \prod Y \in \texttt{Ordered} \,.\, (X \to Y) \to ?X$

$x : \texttt{StrictLocalMinimum}(f) \iff \exists U \in \mathcal{U}(x) \,.\, \forall u \in U \,.\, f(x) < f(u)$

$\texttt{StrictLocalMaximum} :: \prod X \in \mathsf{TOP} \,.\, \prod Y : \texttt{Ordered} \,.\, (X \to Y) \to ?X$

$x : \texttt{StrictLocalMaximum}(f) \iff \exists U \in \mathcal{U}(x) \,.\, \forall u \in U \,.\, f(x) > f(u)$

$\texttt{LocalExtremum} = \texttt{LocalMinimum}|\texttt{LocalaMaximum}$

$\texttt{StrictLocalExremum} = \texttt{StrictLocalMinimum}|\texttt{StrictLocalMaximum}$

$\texttt{FirstOrderExtrmalPointTheorem} :: \forall E : \mathsf{BAN}(\mathbb{R}) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall f : U \to \mathbb{R} \,.$
$\quad .\, \forall p : \texttt{LocalExtrmum}(U, \mathbb{R})(f) \,.\, f : \texttt{DifferentiableAt}(E, \mathbb{R}, U)(p) \Rightarrow \mathrm{D}f|_p = 0$

Proof $=$

$\ldots$

$\square$

$\texttt{LocalMininimumCriterion} :: \forall E : \mathsf{BAN}(\mathbb{R}) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall f : U \to \mathbb{R} \,.$
$\quad .\, \forall p : \texttt{LocalMinimum}(U, \mathbb{R})(f) \,.\, f : \texttt{NDifferentiable}(R, \mathbb{R}, U)(2) \Rightarrow \mathrm{D}^2 f|_p \ge 0$

Proof $=$

$\ldots$

$\square$

$\texttt{LocalMaximumCriterion} :: \forall E : \mathsf{BAN}(\mathbb{R}) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall f : U \to \mathbb{R} \,.$
$\quad .\, \forall p : \texttt{LocalMaximum}(U, \mathbb{R})(f) \,.\, f : \texttt{NDifferentiable}(R, \mathbb{R}, U)(2) \Rightarrow \mathrm{D}^2 f|_p \le 0$

Proof $=$

$\ldots$

$\square$

$\texttt{StrictLocalMinimumCriterion} :: \forall E : \mathsf{BAN}(\mathbb{R}) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall f : U \to \mathbb{R} \,.$
$\quad .\, \forall p : \texttt{StrictLocalMinimum}(U, \mathbb{R})(f) \,.\, f : \texttt{NDifferentiable}(R, \mathbb{R}, U)(2) \Rightarrow \mathrm{D}^2 f|_p > 0$

Proof $=$

$\ldots$

$\square$

$\texttt{StrictLocalMaximumCriterion} :: \forall E : \mathsf{BAN}(\mathbb{R}) \,.\, \forall U : \texttt{Open}(E) \,.\, \forall f : U \to \mathbb{R} \,.$

$\quad .\, \forall p : \texttt{StrictLocalMaximum}(U, \mathbb{R})(f) \,.\, f : \texttt{NDifferentiable}(R, \mathbb{R}, U)(2) \Rightarrow \mathrm{D}^2 f|_p < 0$

$\texttt{Proof} \ =$

$\ldots$

$\square$