Ring Theory

Uncultured Trump

December 13, 2021

Contents

1 Ring Structure Theory

1.1 Rings

$$\texttt{Ring} :: ?\prod R \in \mathsf{SET} \,.\, (R \times R \to R) \times (R \times R \to R)$$
$$(R, +, \cdot) : \texttt{Ring} \iff (R, +) : \texttt{Abelean} \,\&\, (R, \cdot) : \texttt{Monoid} \,\&\, (R, +, \cdot) : \texttt{Distributive}$$

$$\texttt{addition} :: \prod (R, +, \cdot) : \texttt{Ring} \,.\, R \times R \to R$$
$$\texttt{addition}\,(A) = (+_A) := (+)$$

$$\texttt{multiplication} :: \prod (R, +, \cdot) : \texttt{Ring} \,.\, R \times R \to R$$
$$\texttt{multiplication}\,(A) = (\cdot_A) := (\cdot)$$

$$\texttt{RingGroup} :: \prod (R, +, \cdot) : \texttt{Ring} \,.\, \mathsf{ABEL}$$
$$\texttt{RingGroup}\,(A) = A := (R, +)$$

$$\texttt{zero} :: \prod R : \texttt{Ring} \,.\, R$$
$$\texttt{zero}\,(R) = 0_R := \texttt{neutral}(+_R)$$

$$\texttt{identity} :: \prod R : \texttt{Ring} \,.\, R$$
$$\texttt{identity}\,(R) = 1_R := \texttt{neutral}(\cdot_R)$$

$$\texttt{CommutativeRing} :: ?\texttt{Ring}$$
$$(R, +, \cdot) : \texttt{CommutativeRing} \iff (\cdot) : \texttt{Commutative}(R)$$

$$\texttt{Division} :: ?\texttt{Ring}$$
$$(R, +, \cdot) : \texttt{Division} \iff (\cdot) : \texttt{Invertible}(R)$$

$$\texttt{Field} :: ?(\texttt{Division} \,\&\, \texttt{CommutativeRing})$$
$$k : \texttt{Field} \iff 0_k \neq 1_k$$

$$\texttt{RingHomo} :: \prod A, B : \texttt{Ring} \,.\, A \xrightarrow{\mathsf{ABEL}} B$$
$$f : \texttt{RingHomo} \iff \forall x, y \in A \,.\, f(xy) = f(x)f(y) \,\&\, f(1) = 1$$

$$\texttt{IdIsHomo} :: \forall R : \texttt{Ring} \,.\, \mathrm{id}_R : \texttt{RingHomo}$$
$\texttt{Proof} =$
$\texttt{Assume}\ a, b : R,$
$(*) := \eth\,\mathrm{id} : \mathrm{id}(ab) = ab = \mathrm{id}(a)\,\mathrm{id}(b);$
$\square$

$\texttt{RingHomoCompos} :: \forall A, B, C : \texttt{Ring} . \forall f : \texttt{RingHomo}(A, B) . \forall g : \texttt{RingHomo}(B, C) . g \circ f : \texttt{RingHomo}(A, C)$

$\texttt{Proof} =$

$\texttt{Assume } x, y : R,$

$(*) := \eth\texttt{RingHomo}(f)\eth\texttt{RingHomo}(g) : g \circ f(xx') = g\big(f(x)f(x')\big) = f\big(g(x)\big)f\big(g(x')\big);$

$\texttt{RingCat} :: \mathsf{CAT}$

$\texttt{RingCat}\,() = \mathsf{RING} := \Big(\texttt{Ring}, \texttt{HomoRing}, \circ, \mathrm{id}\Big)$

$\texttt{CommRingCat} :: \mathsf{CAT}$

$\texttt{CommRingCat}\,() = \mathsf{ANN} := \Big(\texttt{CommutativeRing}, \texttt{HomoRing}, \circ, \mathrm{id}\Big)$

$\texttt{Subring} :: \prod R \in \mathsf{RING} . ??R$

$A : \texttt{Subring} \iff A \subset_{\mathsf{RING}} R \iff (A, +_{R|A}, \cdot_{R|A}) \in \mathsf{RING}$

$\texttt{TrivialRing} :: \mathsf{ANN}$

$\texttt{TrivialRing}\,() = \star := \Big(\{\star\}, (\star, \star) \mapsto \star, (\star, \star) \mapsto \star\Big)$

$\texttt{MultZero} :: \forall R \in \mathsf{RING} . \forall a \in R . 0a = a0 = 0$

$\texttt{Proof} =$

$(1) := \eth\texttt{Identity}(1)\eth\texttt{Distrivutive}(R, +, \cdot)\eth\texttt{Identity}(0)\eth\texttt{Identity}(1) : 0a + a = (0 + 1)a = a,$

$(2) := \eth\texttt{Identity}(1)\eth\texttt{Distrivutive}(R, +, \cdot)\eth\texttt{Identity}(0)\eth\texttt{Identity}(1) : a0 + a = a(0 + 1) = a,$

$(*) := \texttt{IdentityIsUnique}(1)(2) : a0 = 0 = 0a;$

$\square$

$\texttt{MultNeg} :: \forall R \in \mathsf{RING} . \forall a \in R . (-1)a = -a = a(-1)$

$\texttt{Proof} =$

$(1) := \eth\texttt{Identity}\eth\texttt{Distributive}(R)\eth\texttt{Inverse}(1) : a + (-1)a = (1 - 1)a = 0,$

$(2) := \eth\texttt{Identity}\eth\texttt{Distributive}(R)\eth\texttt{Inverse}(1) : a + a(-1) = a(1 - 1) = 0,$

$(*) := \texttt{InverseIsUnique}(1)(2) : (-1)a = -a = a(-1);$

$\square$

$\texttt{SubringImage} :: \forall A, B \in \mathsf{RING} . \forall S : \texttt{Subring}(A) . \forall f : A \xrightarrow{\mathsf{RING}} B . f(S) \subset_{\mathsf{RING}} B$

$\texttt{Proof} =$

$(1) := \eth\texttt{Subring}(S)\eth\texttt{RingHomo}(f)\eth^{-1}\texttt{image} : f(1) = 1 \in f(S),$

$(2) := \eth\texttt{Subring}(S)\eth\texttt{Homo}(f)\eth^{-1}\texttt{image} : f(0) = 0 \in f(S),$

$\texttt{Assume } x, y : S,$

$(3) := \eth\texttt{Subgroup}(A)(S)(x, y) : x + y \in S,$

$(4) := \eth\texttt{Homo}(A, B)(f)\eth^{-1}\texttt{image}(3) : f(x) + f(y) = f(x + y) \in f(S),$

$(5) := \eth\texttt{Subgroup}(A)(S)(x) : -x \in S,$

$(6) := \texttt{HomoInverse}(5)\eth^{-1}\texttt{image} : -f(x) = f(-x) \in f(S),$

$(7) := \eth\texttt{Sybring}(A)(S) : xy \in S,$

$(*) := \eth\texttt{RingHomo}(A, B)(f)\eth^{-1}\texttt{image}(7) : f(x)f(y) = f(xy) \in f(S);$

$\square$

$\texttt{SubringPreimage} :: \forall A, B \in \mathsf{RING} \;.\; \forall S : \texttt{Subring}(B) \;.\; \forall f : A \xrightarrow{\mathsf{RING}} B \;.\; f^{-1}(S) \subset_{\mathsf{RING}} A$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Subring}(S)\eth\texttt{RingHomo}(f)\eth^{-1}\texttt{preimage} : 1 \in f^{-1}(S),$

$(2) := \eth\texttt{Subring}(S)\eth\texttt{Homo}(f)\eth^{-1}\texttt{image} : 0 \in f^{-1}(S),$

$\texttt{Assume}\; x, y : f^{-1}(S),$

$(3) := \eth\texttt{Homo}(A, B)(f)\texttt{Subgroup}(A)(S)(x, y) : f(x + y) = f(x) + (y) \in S,$

$(4) := \eth^{-1}\texttt{preimage}(3) : x + y \in f^{-1}(S),$

$(5) := \texttt{HomoInverse}(f)(x)\eth\texttt{Subgroup}(A)(S)(x) : f(-x) = -f(x) \in S,$

$(6) := (5)\eth^{-1}\texttt{Preimagemage} : -x \in f(S),$

$(7) := \eth\texttt{RingHomo}(A, B)(f)(x, y)\eth\texttt{Subring}(A)(S) : f(xy) = f(x)f(y) \in S,$

$(*) := \eth^{-1}\texttt{image}(7) : xy \in f^{-1}(S);$

$\square$

$\texttt{RingOfAbeleanMorphism} :: \forall A \in \mathsf{ABEL} \;.\; \left(\mathrm{End}_{\mathsf{ABEL}}(A), +, \circ\right) \in \mathsf{RING}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RingOfFunctions} :: \forall X \in \mathsf{SET} \;.\; \forall R \in \mathsf{RING} \;.\; \left(\mathcal{M}_{\mathsf{SET}}(X, R), +, \cdot\right) \in \mathsf{RING}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{productRing} :: \prod I \in \mathsf{SET} \;.\; (I \to \mathsf{RING}) \to \mathsf{RING}$

$\texttt{productRing}\,(R) = \prod_{i \in I} R_i := \left(\prod i \in I \;.\; R_i, a, b \mapsto \Lambda i \in I \;.\; a_i + b_i, a, b \mapsto \Lambda i \in I \;.\; a_i b_i\right)$

$\texttt{projection} :: \prod I \in \mathsf{SET} \;.\; \prod R : I \to \mathsf{RING} \;.\; \prod i \in I \;.\; \prod_{i \in I} R_i \xrightarrow{\mathsf{RING}} R_i$

$\texttt{projection}\,(a) = \pi_i(a) := a_i$

$\texttt{rightMultiplication} :: \prod R \in \mathsf{RING} \;.\; R \xrightarrow{\mathsf{RING}} \mathrm{End}_{\mathsf{RING}}(R)$

$\texttt{rightMultiplication}\,(a) = \rho_a := \Lambda b \in R \;.\; ab$

1.2 Multiplicative Identities

$$\texttt{BinomialSum} :: \forall R \in \mathsf{RING} \,.\, \forall (a,b) : \texttt{Commutes}(R,\cdot) \,.\, \forall n \in \mathbb{N} \,.\, (a+b)^n = \sum_{i=1}^{n} C_n^i a^i b^{n-i}$$

$\texttt{Proof} \ =$

$z := \Lambda i \in \{0,1\} \,.\, \texttt{if } i == 0 \texttt{ then } a \texttt{ else } b : \{0,1\} \to R,$

$$(1) := \eth\mathsf{RING}(R) : (a+b)^n = \sum_{i:n \to \{1,0\}} \prod_{j=1}^{n} z(i_j),$$

$$(2) := \eth z \eth \texttt{Commutes}(R,\cdot) : \forall i : n \to \{0,1\} \,.\, \forall k \in n \,.\, |i^{-1}\{0\}| = k \Rightarrow \prod_{i=1}^{n} z(i_j) = a^k b^{n-k},$$

$$(3) := \eth\texttt{binom} : \forall k \in n \,.\, \left|\left\{ i : n \to \{0,1\} : |i^{-1}\{0\}| = k \right\}\right| = C_n^k,$$

$$(*) := (1)(2)(3) : (a+b)^n = \sum_{i=0}^{n} C_n^i a^i b^{n-i};$$

$\square$

$$\texttt{MultinomialSum} :: \forall R \in \mathsf{RING} \,.\, \forall m,n \in \mathbb{N} \, \forall a : \texttt{Commuting}(m,R,\cdot) \,.$$
$$.\, \left(\sum_{i=1}^{m} a_i\right)^n = \sum_{i:m \to \mathbb{Z}_+ : \sum_{j=1}^{m} i_j = n} C_n^i \prod_{j=1}^{m} a_j^{i_j}$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$$\texttt{SumOfPowers} :: \forall R \in \mathsf{RING} \,.\, \forall (a,b) : \texttt{Commutes}(R,\cdot) \,.\, \forall n \in \mathbb{N} \,.\, a^n - b^n = (a-b) \sum_{i=0}^{n-1} a^i b^{n-1-i}$$

$\texttt{Proof} \ =$

$$(*) := \eth\mathsf{RING}(R)\eth\texttt{Commutes}(R,\cdot)\eth\texttt{inverse}(R,+) : (a-b) \sum_{i=1}^{n-1} a^i b^{n-1-i} =$$
$$= a \sum_{i=0}^{n-1} a^i b^{n-1-i} - b \sum_{i=0}^{n-1} a^i b^{n-1-i} = a^n + \left(\sum_{i=1}^{n-1} a^i b^{n-i} - a^i b^{n-i} \right) - b^n = a^n + b^n;$$

$\square$

1.3 Elements of The Ring

$$\texttt{LeftUnit} :: \prod R \in \mathsf{RING} \,.\, ?R$$
$$u : \texttt{LeftUnit} \iff \exists a \in R : au = 1$$

$$\texttt{RightUnit} :: \prod R \in \texttt{Ring} \,.\, ?R$$
$$u : \texttt{RightUnit} \iff \exists a \in R : ua = 1$$

$$\texttt{LeftZeroDivisor} :: \prod R \in \texttt{Ring} \,.\, ?R$$
$$x : \texttt{LeftZeroDivizor} \iff \exists a \in R \,.\, xa = 0 \,\&\, x \neq 0$$

$$\texttt{RightZeroDivisor} :: \prod R \in \texttt{Ring} \,.\, ?R$$
$$x : \texttt{RightZeroDivizor} \iff \exists a \in R \,.\, ax = 0 \,\&\, x \neq 0$$

$$\texttt{ZeroDivisor} := \Lambda R \in \mathsf{RING} \,.\, \texttt{RightZeroDivisor}|\texttt{LeftZeroDivisor}(R) : \mathsf{RING} \to \texttt{Type};$$

$$\texttt{Regular} := \Lambda R \in \mathsf{RING} \,.\, !\texttt{ZeroDivisor} : \mathsf{RING} \to \texttt{Type};$$

$$\texttt{Unit} := \Lambda R \in \mathsf{RING} \,.\, \texttt{LeftUnit} \,\&\, \texttt{RightUnit}(R) : \mathsf{RING} \to \texttt{Type};$$

$\texttt{UnitsAreRegular} :: \forall R \in \mathsf{RING} \,.\, \forall u : \texttt{Unit}(R) \,.\, u : \texttt{Regular}(R)$
$\texttt{Proof} \;=$
$\texttt{Assume}\ a : R,$
$\texttt{Assume}\ (1) : ua = 0,$
$\texttt{Assume}\ (2) : a \neq 0,$
$(3, v) := \eth\texttt{LeftUnit}(u) : \sum v \in R \,.\, vu = 1,$
$(4) := \eth\texttt{Identity}(1)(a)(3)(vua)(1)\texttt{ZeroMult}(v) : a = 1a = vua = v0 = 0,$
$() := (2)(4) : \bot;$
$\rightsquigarrow (1) := \eth^{-1}\texttt{RightZeroDivisor}E(\bot) : [u \,!\, \texttt{RightZeroDivisor}(R)],$
$\texttt{Assume}\ a : R,$
$\texttt{Assume}\ (2) : au = 0,$
$\texttt{Assume}\ (3) : a \neq 0,$
$(4, v) := \eth\texttt{LeftUnit}(u) : \sum v \in R \,.\, uv = 1,$
$(4) := \eth\texttt{Identity}(1)(a)(3)(auv)(1)\texttt{ZeroMult}(v) : a = 1a = auv = 0v = 0,$
$() := (2)(4) : \bot;$
$\rightsquigarrow (2) := \eth^{-1}\texttt{LeftZeroDivisor}E(\bot) : [u \,!\, \texttt{LeftZeroDivisor}(R)],$
$(3) := \eth^{-1}\texttt{Regualar}(1)(2) : [u : \texttt{Regular}];$
$\square$

$\texttt{groupOfUnits} :: \mathsf{RING} \to \mathsf{GRP}$
$\texttt{groupOfUnits}\,(R) = R^* := (\texttt{Unit}(R), \cdot_R)$

$\texttt{RationalIdentity} :: \forall R \in \mathsf{RING} \,.\, \forall x, y \in R \,.\, (1 + xy) \in R^* \Rightarrow (1 + yx) \in R^*$

$\texttt{Proof} \ =$

$z := 1 - y(1 + xy)^{-1}x : R,$

$(1) := (1 + yx)\eth z \eth \texttt{Associative}(\cdot_R)\eth\texttt{Inverse}\cdot_R(1 + xy)\eth\texttt{Inverse}+_R(yx) :$

$\quad : (1 + yx)z = 1 + yx - (1 + yx)y(1 + xy)^{-1}x = 1 + yx - y(1 + xy)(1 + xy)^{-1}x = 1 + yx - yx = 1,$

$(2) := \eth z(1 + yx)\eth\texttt{Associative}(\cdot_R)\eth\texttt{Iverse}\cdot_R(1 + xy)\eth\texttt{Inverse}+_R(yx) :$

$\quad : z(1 + yx) = 1 + yx - y(1 + xy)^{-1}x(1 + yx) = 1 + yx - y(1 + xy)^{-1}(1 + xy)x = 1 + yx - yx = 1,$

$(*) := \eth^{-1}\texttt{Unit}(R)(1)(2) : (1 + yx) \in R^*;$

$\square$

$\texttt{Nillpotent} :: \prod R \in \mathsf{RING} \,.\, ?R$

$a : \texttt{Nillpotent} \iff \exists n \in \mathbb{N} : a^n = 0$

$\texttt{Unipotent} :: \prod R \in \mathsf{RING} \,.\, ?R$

$a : \texttt{Unipotent} \iff a - 1 : \texttt{Nillpotent}(R)$

$\texttt{Idempotent} :: \prod R \in \mathsf{RING} \,.\, ?R$

$a : \texttt{Idempotent} \iff \exists n \in \mathbb{N} : a^2 = a$

$\texttt{Involution} :: \prod R \in \mathsf{RING} \,.\, ?$

$a : \texttt{Involution} \iff a^2 = 1$

$\texttt{NillpotentProduct} :: \forall R \in \mathsf{RING} \,.\, \forall a : \texttt{Nillpotent}(R) \,.\, \forall b : \texttt{Commutes}(R, \cdot_R)(a) \,.\, ab : \texttt{Nillpotent}(R)$

$\texttt{Proof} \ =$

$(1, n) := \eth\texttt{Nillpotent}(a) : \sum n \in \mathbb{N} \,.\, a^n = 0,$

$(2) := \eth\texttt{Commutes}(b)(ab)^n(1)\texttt{ZeroMult}(R)(b^n) : (ab)^n = a^n b^n = 0b^n = 0,$

$() := \eth^{-1}\texttt{Nillpotent}(2) : [ab : \texttt{NillPotent}(R)];$

$\square$

$\texttt{NillpotentSum} :: \forall R \in \mathsf{RING} \,.\, \forall a, b : \texttt{Nillpotent}(R) \,.\, \texttt{Commutes}(R, \cdot_R)(a, b) \Rightarrow a + b : \texttt{Nillpotent}(R)$

$\texttt{Proof} \ =$

$(1, n) := \eth\texttt{Nillpotent}(a) : \sum n \in \mathbb{N} \,.\, a^n = 0,$

$(2, m) := \eth\texttt{Nillpotent}(b) : \sum m \in \mathbb{N} \,.\, b^m = 0,$

$(3) := \texttt{BinomialSum}(b, m, n + m)(1)(2 : (a + b)^{n+m} = \sum_{i=1}^{n+m} C^i_{n+m} a^i b^{n+m-i} = 0,$

$() := \eth^{-1}\texttt{Nillpotent}(3) : [a + b : \texttt{NillPotent}(R)];$

$\square$

$\texttt{UnitDiff} :: \forall R \in \mathsf{RING} \;.\; \forall a \in R^* \;.\; \forall b : \texttt{Nillpotent}(R) \;.\; \texttt{Commutes}(R, \cdot_R)(a, b) \Rightarrow a - b \in R^*$

$\texttt{Proof} \;=$

$(n, 1) := \eth\texttt{Nillpotent}(b) : \sum n \in \mathbb{N} \;.\; b^n = 0,$

$(2) := \texttt{SumOfPowers}(a, b, n)(1)\eth\texttt{Inverse} : (a - b)\left(\sum_{i=0}^{n-1} a^i b^{n-1-i}\right) a^{-n} = (a^n - b^n)a^{-n} = a^n a^{-n} = 1,$

$(*) := \eth^{-1} R^*(2) : a - b \in R^*;$

$\square$

$\texttt{IntegralDomain} :: ?\mathsf{RING}$

$R : \texttt{IntgralDomain} \iff R \neq \star \;\&\; \forall a : \texttt{ZeroDivisor}(R) \;.\; a = 0$

$\texttt{multiplicativeMonoid} :: \texttt{IntegralDomain} \to \texttt{CommutativeMonoid}$

$\texttt{multiplicativeMonoid}\,(R) = R^\times := (R \setminus \{0\}, \cdot_R)$

$\texttt{RightCancelation} :: \forall R : \texttt{IntegralDomain} \;.\; \forall x, y \in R \;.\; \forall a \in R^\times \;.\; \forall (0) : xa = ya \;.\; x = y$

$\texttt{Proof} \;=$

$(1) := \Big((0) - ya\Big)\eth\texttt{Distributive}(R) : 0 = xa - ya = (x - y)a,$

$(2) := \eth\texttt{IntegralDomain}(R)(1)\eth R^\times(a) : x - y = 0,$

$(*) := (2) + y : x = y;$

$\square$

$\texttt{LeftCancelation} :: \forall R : \texttt{IntegralDomain} \;.\; \forall x, y \in R \;.\; \forall a \in R^\times \;.\; \forall (0) : ax = ay \;.\; x = y$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{Divides} :: \prod R : \texttt{IntegralDomain} \;.\; ?R^2$

$a, b : \texttt{Divides} \iff a|b \iff \exists x \in R : ax = b$

$\texttt{Associates} :: \prod R : \texttt{IntegralDomain} \;.\; ?R^2$

$a, b : \texttt{Associates} \iff (a|b) \;\&\; (b|a)$

$\texttt{IrreducibleElement} :: \prod R : \texttt{IntegralDomain} \;.\; ?(R^\times \setminus R^*)$

$a : \texttt{IrreducibleElement} \iff \forall x, y \in R \;.\; a = xy \Rightarrow \Big(x \in R^* \Big| y \in R^*\Big)$

$\texttt{PrimeElement} :: \prod R : \texttt{IntegralDomain} \;.\; ?(R^\times \setminus R^*)$

$a : \texttt{PrimeElement} \iff \forall x, y \in R \;.\; a|xy \Rightarrow \Big(a|x \Big| a|y\Big)$

$\texttt{PropertyOfAssociates} :: \forall R : \texttt{IntegralDomain} \;.\; \forall (a,b) : \texttt{Associates}(R) \;.\; \exists u \in R^* \;.\; a = ub$

$\texttt{Proof} \;=$

$\Big(x, (1)\Big) := \eth\texttt{Divides}(a,b)\eth\texttt{Associates}(a,b) : \sum x \in R : b = xa,$

$\Big(y, (2)\Big) := \eth\texttt{Divides}(b,a)\eth\texttt{Associates}(a,b) : \sum y \in R : a = yb,$

$(3) := (1)(2) : a = yxa,$

$(4) := \texttt{RightCancelation}(3) : 1 = yx,$

$(5) := \eth R^*(4) : y \in R^*,$

$(*) := I(\exists)(2)(5) : \exists y \in R^* \;.\; a = yb;$

$\square$

$\texttt{PrimeElementIsIrreducible} :: \forall R : \texttt{IntegralDomain} \;.\; \forall p : \texttt{PrimeElement}(R) \;.\; p : \texttt{IrreducibleElement}(R)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y : R,$

$\texttt{Assume}\; (1) : p = xy,$

$(2) := \eth^{-1} Divides(p, xy)\Big(1, (1)\Big) : p|xy,$

$(3) := \eth\texttt{PrimeElement}(p)(2) : p|x\Big|p|y,$

$\texttt{Assume}\; (4) : p|x,$

$\Big(z, 5\Big) := \eth\texttt{Divides}(4) : \sum z \in R \;.\; x = zp,$

$(6) := (1)(5) : p = pzy,$

$(7) := \texttt{LeftCancelation}(6) : 1 = zy,$

$(8) := \eth^{-1} R^*(7) : y \in R^*,$

$() := I(|)(8) : x \in R^* | y \in R^*;$

$\rightsquigarrow (4) := I(\Rightarrow) : p|x \Rightarrow \Big(x \in R^*\Big|y \in R^*\Big),$

$\texttt{Assume}\; (5) : p|y,$

$\Big(z, 6\Big) := \eth\texttt{Divides}(4) : \sum z \in R \;.\; y = zp,$

$(7) := (1)(6) : p = xzp,$

$(8) := \texttt{RightCancelation}(7) : 1 = xz,$

$(9) := \eth^{-1} R^*(8) : x \in R^*,$

$() := I(|)(9) : x \in R^* | y \in R^*;$

$\rightsquigarrow (5) := I(\Rightarrow) : p|y \Rightarrow \Big(x \in R^*\Big|y \in R^*\Big),$

$() := E(|)(3)(4)(5) : x \in R^* | y \in R^*;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{IrreducibleElement} : [p : \texttt{Irreducible}];$

$\square$

1.4 Ideals and Quotients

$\texttt{LeftIdeal} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{Subgroup}(R)$
$I : \texttt{LeftIdeal} \iff \forall a \in I \,.\, \forall b \in R \,.\, ba \in I$

$\texttt{RightIdeal} :: \prod R \in \mathsf{RING} \,.\, ?\texttt{Subgroup}(R)$
$I : \texttt{RightIdeal} \iff \forall b \in I \,.\, \forall b \in R \,.\, ab \in I$

$\texttt{TwoSidedIdeal} := \prod R \in \mathsf{RING} \,.\, \texttt{LeftIdeal}(R) \,\&\, \texttt{RightIdeal}(R) : \mathsf{RING} \to \texttt{Type};$

$\texttt{CommutativeIdeal} :: \forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{LeftIdeal}(R) \,.\, I : \texttt{TwoSidedIdeal}(R)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{Ideal} := \prod R \in \texttt{CommutativeRing} \,.\, \texttt{LeftIdeal}(R) : \texttt{CommutativeRing} \to \texttt{Type};$

$\texttt{quotMult} :: \prod R : \mathsf{RING} \,.\, \prod I : \texttt{TwoSidedIdeal} \,.\, \frac{R}{I} \to \frac{R}{I} \to \frac{R}{I}$
$\texttt{quatMult}([a],[b]) = [a][b] := [ab]$
$\texttt{Assume}\ x, y : I,$
$(1) := \eth\texttt{RightIdeal}(a, y) : ay \in I,$
$(2) := \eth\texttt{LeftIdeal}(b, x) : xb \in I,$
$(3) := \eth\texttt{RightIdeal}(x, y) : xy \in I,$
$(*) :=: [a + x][b + y] = [ab + xb + ay + xy] = [ab];$
$\square$

$\texttt{quotientRing} :: \prod R : \mathsf{RING} \,.\, \texttt{TwoSidedIdeal} \to \mathsf{GRP}$
$\texttt{quotientRing}(I) = \frac{R}{I} := \left(\frac{R}{I}, +, \texttt{quatMult}\right)$

$\texttt{LeftIdealPreimage} :: \forall A, B \in \mathsf{RING} \,.\, \forall f : A \xrightarrow{\mathsf{RING}} B \,.\, \forall I : \texttt{LeftIdeal}(B) \,.\, f^{-1} : \texttt{LeftIdeal}(A)$
$\texttt{Proof} =$
$(1) := \texttt{SubgroupPreimage}(I, f) : f^{-1}(I) \subset_{\mathsf{GRP}} A,$
$\texttt{Assume}\ x : f^{-1}(I),$
$(2) := \eth\texttt{preimage}(f, I)(x) : f(x) \in I,$
$\texttt{Assume}\ a : A,$
$(3) := \eth\texttt{RingHomo}(A, B)(f)(a, x)\eth\texttt{Ideal}(B)(I)(2) : f(ax) = f(a)f(x) \in I,$
$() := \eth^{-1}\texttt{preimage}(f, I)(3) : ax \in I;$
$\rightsquigarrow (*) := I(\forall)\eth^{-1}\texttt{LeftIdeal}(A)(1) : \left(f^{-1}(I) : \texttt{LeftIdeal}(A)\right);$
$\square$

$\texttt{RightIdealPreimage} :: \forall A, B \in \mathsf{RING} \,.\, \forall f : A \xrightarrow{\mathsf{RING}} B \,.\, \forall I : \texttt{RightIdeal}(B) \,.\, f^{-1}(I) : \texttt{RightIdeal}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{TwoSidedIdealPreimage} :: \forall A, B \in \mathsf{RING} \,.\, \forall f : A \xrightarrow{\mathsf{RING}} B \,.\, \forall I : \texttt{TwoSidedIdeal}(B) \,.$
$\quad . \, f^{-1}(I) : \texttt{TwoSidedIdeal}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{IdealPreimage} :: \forall A, B \in \texttt{CommutativeRing} \,.\, \forall f : A \xrightarrow{\mathsf{RING}} B \,.\, \forall I : \texttt{Ideal}(B) \,.\, f^{-1}(I) : \texttt{Ideal}(A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LeftIdealIntersection} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{LeftIdeal}(R) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{LeftIdeal}(R)$

$\texttt{Proof} =$

$(1) := \texttt{SubgroupIntersection}(\mathcal{A}, I) : \bigcap_{\alpha \in \mathcal{A}} :\subset_{\mathsf{GRP}} R,$

$\texttt{Assume}\ x : \bigcap_{\alpha \in \mathcal{A}} I_\alpha,$

$(2) := \eth\texttt{Itersect}(\mathcal{A})(I)(x) : \forall \alpha \in A \,.\, x \in I_\alpha,$

$\texttt{Assume}\ a : R,$

$\texttt{Assume}\ \alpha : \mathcal{A},$

$() := \eth^{-1}\texttt{Ideal}(I_\alpha)(2)(x)(a) : ax \in I_\alpha;$

$\rightsquigarrow (3) := I(\forall) : \forall \alpha \in \mathcal{A} \,.\, ax \in I_\alpha,$

$() := \eth^{-1}\texttt{intersect}(\mathcal{A})(I)(3) : ax \in \bigcap_{\alpha \in \mathcal{A}} I_\alpha;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{LeftIdeal}(R)(1) : \left[\bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{LeftIdeal}(R)\right];$

$\square$

$\texttt{RightIdealIntersection} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{RightIdeal}(R) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{RightIdeal}(R)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{TwoSidedtIdealIntersection} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{TwoSidedIdeal}(R) \,.$
$\quad . \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{TwoSidedIdeal}(R)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{IdealIntersection} :: \forall R \in \mathsf{ANN} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{Ideal}(R) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{Ideal}(R)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfLeftIdeals} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{LeftIdeal}(R) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{LeftIdeal}(R)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfRightIdeals} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{RightIdeal}(R) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{RightIdeal}(R)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfTwoSidedIdeals} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{TwoSidedIdeal}(R) \,.\,$
$\sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{TwoSidedIdeal}(R)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfIdeals} :: \forall R \in \mathsf{ANN} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{Ideal}(R) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{Ideal}(R)$

$\texttt{Proof} =$

...

$\square$

$$\texttt{compositeIdeal} :: \prod R \in \mathsf{RING} \,.\, \texttt{LeftIdeal}(R) \times \texttt{RightIdeal}(R) \to \texttt{TwoSidedIdeal}(R)$$

$$\texttt{compositeIdeal}\,(I, J) = IJ := \left\{ \sum_{\alpha=1}^{n} a_\alpha b_\alpha | n \in \mathbb{N}, a : n \to I, b : n \to J \right\}$$

$$\texttt{compositeIdeal2} :: \prod R \in \texttt{CommutativeRing} \,.\, \prod n \in \mathbb{N} \,.\, n \to \texttt{Ideal}(R) \to \texttt{Ideak}(R)$$

$$\texttt{compositeIdeal2}\,(I) = \prod_{\alpha=1}^{n} I_\alpha := \left\{ \sum_{\beta=1}^{m} \prod_{\alpha=1}^{n} a_{\alpha,\beta} | m \in \mathbb{N}, a : \prod \alpha \in n \,.\, m \to I_\alpha \right\}$$

$\texttt{ProperByUnityLeft} :: \forall R \in \mathsf{RING} \,.\, \forall I : \texttt{LeftIdeal} \,.\, I = R \iff 1 \in I$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ProperByUnityRight} :: \forall R \in \mathsf{RING} \,.\, \forall I : \texttt{RightIdeal} \,.\, I = R \iff 1 \in I$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ProperByUnityTwoSided} :: \forall R \in \mathsf{RING} \,.\, \forall I : \texttt{TwoSidedIdeal} \,.\, I = R \iff 1 \in I$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ProperByUnity} :: \forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{Ideal} \,.\, I = R \iff 1 \in I$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{UnionOfLeftIdeals} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} : \texttt{TotallyOrdered} \;\&\; \texttt{NonEmpty} \,.$

$\quad .\, \forall I : \texttt{Nondecreasing}\Big(\texttt{Proper} \;\&\; \texttt{LeftIdeal}(R)\Big) \bigcup_{\alpha \in \mathcal{A}} I_\alpha I_\alpha : \texttt{Proper} \;\&\; \texttt{LeftIdeal}(R)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ (1) : 1 \in \bigcup_{\alpha \in \mathcal{A}} I_\alpha,$

$(\alpha, 2) := \eth\texttt{union} : \sum \alpha \in A \,.\, 1 \in I_\alpha,$

$(3) := \texttt{ProperByUnityLeft}(2) : I_\alpha = R,$

$(4) := \eth\texttt{Proper}(I_\alpha) : I_\alpha \neq R,$

$(5) := I(\bot)(3)(4) : \bot,$

$\rightsquigarrow (1) := E(\bot) : 1 \notin \bigcap_{\alpha \in \mathcal{A}} I_\alpha,$

$(2) := \eth^{-1}\texttt{Proper}(1) : \bigcap_{\alpha \in \mathcal{A}} I_\alpha,$

$\ldots$

$\square$

$\texttt{UnionOfRightIdeals} :: \forall R \in \mathsf{RING} \,.\, \forall \mathcal{A} : \texttt{TotallyOrdered} \;\&\; \texttt{NonEmpty} \,.$

$\quad .\, \forall I : \texttt{Nondecreasing}\Big(\texttt{Proper} \;\&\; \texttt{RightIdeal}(R)\Big) \,.\, \bigcup_{\alpha \in \mathcal{A}} I_\alpha I_\alpha : \texttt{Proper} \;\&\; \texttt{RightIdeal}(R)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{UnionOfTwoSidedIdeals} :: \forall R \in \mathsf{RING} . \forall \mathcal{A} : \texttt{TotallyOrdered} \,\&\, \texttt{NonEmpty} .$

$. \forall I : \texttt{Nondecreasing}\Big(\texttt{Proper} \,\&\, \texttt{TwoSidedIdeal}(R)\Big) . \bigcup_{\alpha\in\mathcal{A}} I_\alpha I_\alpha : \texttt{Proper} \,\&\, \texttt{TwoSidedIdeal}(R)$

$\texttt{Proof} =$

...

□

$\texttt{UnionOfIdeals} :: \forall R \in \mathsf{ANN} . \forall \mathcal{A} : \texttt{TotallyOrdered} \,\&\, \texttt{NonEmpty} .$

$. \forall I : \texttt{Nondecreasing}\Big(\texttt{Proper} \,\&\, \texttt{LeftIdeal}(R)\Big) . \bigcup_{\alpha\in\mathcal{A}} I_\alpha I_\alpha : \texttt{Proper} \,\&\, \texttt{Ideal}(R)$

$\texttt{Proof} =$

...

□

$\texttt{MaximalLeftIdeal} :: \prod R \in \mathsf{RING} . ?\texttt{Proper} \,\&\, \texttt{LeftIdeal}(R)$

$I : \texttt{MaximalLeftIdeal} \iff \forall J : \texttt{LeftIdeal}(R) . I \subset J \Leftarrow J = R$

$\texttt{MaximalRightIdeal} :: \prod R \in \mathsf{RING} . ?\texttt{Proper} \,\&\, \texttt{RightIdeal}(R)$

$I : \texttt{MaximalRightIdeal} \iff \forall J : \texttt{RightIdeal}(R) . I \subset J \Leftarrow J = R$

$\texttt{MaximalTwoSidedIdeal} :: \prod R \in \mathsf{RING} . ?\texttt{Proper} \,\&\, \texttt{TwoSidedIdeal}(R)$

$I : \texttt{MaximalTwoSidedIdeal} \iff \forall J : \texttt{TwoSidedIdeal}(R) . I \subset J \Leftarrow J = R$

$\texttt{MaximalLeftIdeal} :: \prod R \in \mathsf{ANN} . ?\texttt{Proper} \,\&\, \texttt{Ideal}(R)$

$I : \texttt{MaximalLeftIdeal} \iff \forall J : \texttt{Ideal}(R) . I \subset J \Leftarrow J = R$

$\texttt{MaximalLeftIdealExists} :: \forall R \in \mathsf{ANN} . \forall I : \texttt{Proper} \,\&\, \texttt{LeftIdeal}(R) .$

$. \exists M : \texttt{MaximalLeftIdeal}(R) : I \subset M$

$\texttt{Proof} =$

Use $\texttt{UnionOfLeftIdeals}$ and $\texttt{ZornLemma}$

□

$\texttt{MaximalRightIdealExists} :: \forall R \in \mathsf{ANN} . \forall I : \texttt{Proper} \,\&\, \texttt{RightIdeal}(R) .$

$. \exists M : \texttt{MaximalRightIdeal}(R) : I \subset M$

$\texttt{Proof} =$

...

□

$\texttt{MaximalTwoSidedIdealExists} :: \forall R \in \mathsf{ANN} . \forall I : \texttt{Proper} \,\&\, \texttt{TwoSidedIdeal}(R) .$

$. \exists M : \texttt{MaximalTwoSidedIdeal}(R) : I \subset M$

$\texttt{Proof} =$

...

□

$\texttt{MaximalIdealExists} :: \forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{Proper} \,\&\, \texttt{Ideal}(R) \,.$
$\quad .\, \exists M : \texttt{MaximalIdeal}(R) : I \subset M$
$\texttt{Proof} \ =$
$\ldots$
$\square$

$\texttt{genLeftIdeal} :: \prod R \in \mathsf{RING} \,.\, ?R \to \texttt{LeftIdeal}$
$\texttt{genLeftIdeal}\,(S) := \bigcap \{I : \texttt{LeftIdeal}(R) : S \subset R\}$

$\texttt{genRightIdeal} :: \prod R \in \mathsf{RING} \,.\, ?R \to \texttt{RightIdeal}$
$\texttt{genRightIdeal}\,(S) := \bigcap \{I : \texttt{RightIdeal}(R) : S \subset R\}$

$\texttt{genTwoSidedIdeal} :: \prod R \in \mathsf{RING} \,.\, ?R \to \texttt{TwoSidedIdeal}$
$\texttt{genTwoSidedIdeal}\,(S) := \bigcap \{I : \texttt{TwoSidedIdeal}(R) : S \subset R\}$

$\texttt{genIdeal} :: \prod R \in \mathsf{ANN} \,.\, ?R \to \texttt{Ideal}$
$\texttt{genIdeal}\,(S) := \bigcap \{I : \texttt{Ideal}(R) : S \subset R\}$

$\texttt{kernelIdeal} :: \forall A, B \in \mathsf{RING} \,.\, \forall \varphi : A \xrightarrow{\mathsf{RING}} B \,.\, \ker \varphi : \texttt{TwoSidedIdeal}(A)$
$\texttt{Proof} \ =$
$(1) := \texttt{NormalKernel}(R, +)(\varphi) : \ker \varphi \subset_{\mathsf{GRP}} A,$
$\texttt{Assume}\ x : \ker \varphi,$
$\texttt{Assume}\ a : A,$
$(2) := \eth \ker \varphi(x) : \varphi(x) = 0,$
$(3) := \eth\texttt{RingHomo}(A, B)(\varphi)(a, x)(2)\texttt{ZeroMult}(B)(\varphi(a)) : \varphi(ax) = \varphi(a)\varphi(x) = \varphi(a)0 = 0,$
$()_1 := \eth^{-1} \ker \varphi : ax \in \ker \varphi;$
$(4) := \eth\texttt{RingHomo}(A, B)(\varphi)(x, a)(2)\texttt{ZeroMult}(B)(\varphi(a)) : \varphi(xa) = \varphi(x)\varphi(a) = \varphi(a)0 = 0,$
$()_2 := \eth^{-1} \ker \varphi : xa \in \ker \varphi;$
$\rightsquigarrow (*) := \eth^{-1}\texttt{TwoSidedIdeal}((1), I(\forall)) : \Big[\ker \varphi : \texttt{TwoSidedIdeal}(A)\Big],$
$\square$

$\texttt{IdealProjectionIsRingHomo} :: \forall R \in \mathsf{RING} \,.\, \forall I : \texttt{TwoSidedIdeal}(R) \,.\, \pi_I : R \xrightarrow{\mathsf{RING}} \dfrac{R}{I}$
$\texttt{Proof} \ =$
$(1) := \eth\pi_I(1) : \pi_I(1) = [1],$
$\texttt{Assume}\ a, b : R,$
$() := \eth\pi_I(ab)\eth\texttt{quotMult}([a], [b])\eth^{-1}\pi_I(a)\eth^{-1}\pi_I : \pi_I(ab) = [ab] = [a][b] = \pi_I(a)\pi_I(b);$
$\square$

$\texttt{EveryIdealIsRHKernel} :: \prod R \in \mathsf{RING} \,.\, \forall I : \texttt{TwoSidedIdeal}(R) \,.\, I = \ker \pi_I$
$\texttt{Proof} \ =$
$\ldots$
$\square$

1.5 Prime Ideals of Commutative Ring

$$\texttt{Prime} :: \prod R \in \mathsf{ANN} \,.\, ?\texttt{ProperIdeal}(R)$$
$$I : \texttt{Prime} \iff \forall a, b \in R \,.\, ab \in I \Rightarrow a \in I | b \in I$$

$$\texttt{Coprime} :: \prod R \in \mathsf{ANN} \,.\, ?\texttt{ProperIdeal}^2(R)$$
$$I, J : \texttt{Coprime} \iff I + J = R$$

$$\texttt{PrimeQuotientIsID} :: \forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{Prime}(R) \,.\, \frac{R}{I} : \texttt{IntegralDomain}$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [a] : \texttt{ZeroDividor}\ \frac{R}{I},$

$\Big([b], (1)\Big) := \eth^{-1}\texttt{ZeroDivisor}\ \frac{R}{I} : \sum [b] \in \frac{R}{I} \,.\, [b] \neq 0 \ \&\ [b][a] = 0,$

$(2) := \eth\frac{R}{I}(1)_2 : ab \in I,$

$(3) := \eth\texttt{Prime}(R)(I)(2) : a \in I | b \in I,$

$\texttt{Assume}\ (4) : a \in I,$

$() := \eth\frac{R}{I}(4) : [a] = 0;$

$\rightsquigarrow (4) := I(\rightarrow) : a \in I \Rightarrow [a] = 0,$

$\texttt{Assume}\ (5) : b \in I,$

$(6) := \eth\frac{R}{I}(5) : [b] = 0,$

$(7) := (6)(1)_1 : \perp,$

$() := E(\perp)([a] = 0) : [a] = 0;$

$\rightsquigarrow (5) := I(\Rightarrow) : b \in I \Rightarrow [a] = 0,$

$() := E(|)(3)(4)(5) : [a] = 0;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{IntegralDomain} : \left[\frac{R}{I} : \texttt{IntegralDomain}\right];$

$\square$

$$\texttt{PrimePreimage} :: \forall A, B \in \mathsf{ANN} \,.\, \forall I : \texttt{Prime}(B) \,.\, \forall \varphi : A \xrightarrow{\mathsf{RING}} B \,.\, \varphi^{-1}(I) : \texttt{Prime}(A)$$

$\texttt{Proof} \ =$

$(0) := \eth^{-1}\texttt{RingHomo}(\varphi)\eth\texttt{ProperIdeal}(B)(I) : 1 \notin \varphi^{-1}(I),$

$\texttt{Assume}\ x, y : A,$

$\texttt{Assume}\ (1) : xy \in \varphi^{-1}(I),$

$(2) := \eth\texttt{preimage}(A, B)(\varphi)(I)(1) : \varphi(xy) \in I,$

$(3) := \eth\texttt{RingHomo}(A, B)(\varphi)(a, b)(2) : \varphi(x)\varphi(y) \in I,$

$(4) := \eth\texttt{Prime}(B)(I)(3) : \varphi(x) \in I | \varphi(y) \in I,$

$() := \eth^{-1}\texttt{preimage}(A, B)(\varphi)(4) : x \in \varphi^{-1}(I) | y \in \varphi^{-1}(I);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Prime}(I)(1) : \left[\varphi^{-1}(I) : \texttt{Prime}(A)\right];$

$\square$

$\texttt{MaximalIdealIsPrime} :: \forall R \in \mathsf{ANN} . \forall I : \texttt{Maximal}(R) . I : \texttt{Prime}(R)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a, b : R,$

$\texttt{Assume}\ (1) : ab \in I,$

$\texttt{Assume}\ (2) : a \notin I \ \&\ b \notin I,$

$(3) := \eth\texttt{Maximal}(I)(2) : I + \left\langle \{a\} \right\rangle_{\mathsf{RING}} = R,$

$(4) := \eth\texttt{Ring}(R) : 1 \in R,$

$(i, v, 5) := (3)(4) : \sum i \in I . \sum v \in R . 1 = i + va,$

$(6) := b(5) : b = bi + vab,$

$(7) := \eth\texttt{Ideal}(R)(I)(1)(6) : b \in I,$

$() := (2)(7) : \bot;$

$\rightsquigarrow () := E(\bot) : a \in I | b \in I;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Ideal} : [I : \texttt{Prime}(R)];$

$\square$

$\texttt{IdealsProductInIntersection} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall I : n \to \texttt{Ideal}(R) . \prod_{k=1}^{n} I_k \subset \bigcap_{i=1}^{n} I_k$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \prod_{k=1}^{n} I_k,$

$(m, a, 1) := \eth\texttt{ringProfuct}(n, I) : \sum m \in \mathbb{N} . \sum a : \prod k \in n . m \to I_k . x = \sum_{j=1}^{m} \prod_{i=1}^{n} a_{i,j},$

$\texttt{Assume}\ j : m,$

$\texttt{Assume}\ k : n,$

$() := \eth^{-1}\texttt{Ideal}(R)(I_k)(a_{k,j}) : \prod_{i=1}^{n} a_{i,j} \in I_k;$

$\rightsquigarrow (2) := I^2(\forall)) : \forall j \in m . \forall k \in n . \prod_{I=1}^{n} a_{i,j} \in I_k,$

$(3) := \eth^{-1}\texttt{intersect}(n, I)(2) : \forall j \in m . \prod_{I=1}^{n} a_{i,j} \in \bigcap_{k=1}^{n} I_k,$

$(*) := \eth\texttt{Subgroup}(R) \left(\bigcap_{k=1}^{n} I_k \right) (1) : x \in \bigcap_{k=1}^{n} I_k;$

$\square$

$\texttt{ProductInsidePrimeLemma} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \to \texttt{Ideal} \,.\, \forall P : \texttt{Prime}(R) \,.$

$$. \forall (0) : \prod_{k=1}^{n} I_k \subset P \,.\, \exists k \in n : I_k \subset P$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : \prod k \in n \,.\, I_k,$

$\texttt{Assume}\ (1) : \forall k \in \ \,.\, a_k \notin I_k,$

$(2) := \eth\texttt{Prime}(R)(P)(1) : \prod_{k=1}^{n} a_i \notin P,$

$() := (2)(0) : \bot;$

$\rightsquigarrow (1) := I(\forall)E(\bot) : \forall a : \prod k \in n \,.\, I_k \,.\, \exists k \in n : a_k \in P,$

$(*) := \texttt{FiniteChoice}(1) : \exists k \in n : I_k \subset P;$

$\square$

$\texttt{IntersectInsidePrime} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \to \texttt{Ideal} \,.\, \forall P : \texttt{Prime}(R) \,.$

$$. \forall (0) : \bigcap_{k=1}^{n} I_k \subset P \,.\, \exists k \in n : I_k \subset P \,.$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CoprimeFamily} :: \prod R \in \mathsf{ANN} \,.\, \sum n \in \mathbb{N} \,.\, ?\Big(n \to \texttt{Ideal}(R)\Big)$

$(n, I) : \texttt{CoprimeFamily} \iff \forall i, j \in n \,.\, i \neq j \Rightarrow (I_i, I_j) : \texttt{Coprime}(R)$

$\texttt{CoprimeProdIsCoprime} :: \forall R \in \mathsf{ANN} \,.\, \forall J : \texttt{Ideal}(R) \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \to \texttt{Ideal}(n) \,.$

$$. \forall (0) : \forall k \in n \,.\, (J, I_k) : \texttt{Coprime}(R) \,.\, \left(J, \prod_{k=1}^{n} I_k\right) : \texttt{Coprime}(R)$$

$\texttt{Proof} \ =$

$(a, b, 1) := \eth\texttt{Coprime}(0)\texttt{ProperbyUnity} : \sum a : n \to J \,.\, \sum b : \prod k \in n \,.\, I_k \,.\, \forall k \in n \,.\, 1 = a_k + b_k,$

$(2) := \eth\mathsf{ANN}(R)\texttt{Iterate}(n)(1) : 1 = \prod_{k=1}^{n} b_k + \sum_{i=0}^{n-1} a_{i+1} \prod_{k=1}^{i} b_k,$

$(3) := \eth\texttt{Ideal}(R)(J)(\dots) : \sum_{i=0}^{n-1} a_{i+1} \prod_{k=1}^{i} b_k \in J,$

$(4) := \eth\texttt{idealProduct}(n, I)(b) : \prod_{k=1}^{n} b_k \in \prod_{k=1}^{n} I_k,$

$(5) := \eth\texttt{idealSum}(2)(3)(4) : 1 \in J + \prod_{k=1}^{n} I_k,$

$(6) := \texttt{ProperByUnity}(5) : R = J + \prod_{k=1}^{n} I_K,$

$(*) := \eth^{-1}\texttt{Coprime}(6) : \left[\left(J, \prod_{k=1}^{n} I_k\right) : \texttt{Coprime}(n)\right];$

$\square$

$\texttt{CoprimeIntersectIsCoprime} :: \forall R \in \mathsf{ANN} \,.\, \forall J : \texttt{Ideal}(R) \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \to \texttt{Ideal}(n) \,.$

$$. \forall (0) : \forall k \in n \,.\, (J, I_k) : \texttt{Coprime}(R) \,.\, \left(J, \bigcap_{k=1}^{n} I_k\right) : \texttt{Coprime}(R)$$

$\texttt{Proof} \;=$

...

□

$\texttt{CoprimeProductLemma1} :: \forall R \in \mathsf{ANN} \,.\, \forall (J, I) : \texttt{Coprime}(R) \,.\, JI = J \cap I$

$\texttt{Proof} \;=$

$(a, b, 1) := \eth\texttt{Coprime}(J, I)\texttt{ProperByUnity} : \sum a \in J \,.\, \sum b \in I \,.\, 1 = a + b,$

$\texttt{Assume}\ x : J \cap I,$

$(2) := \eth\texttt{Intersect}(J, I)(x) : x \in J,$

$(3) := \eth\texttt{Intersect}(J, I)(x) : x \in I,$

$(4) := \eth\texttt{idealProdut}(J, I)(ax) : ax \in JI,$

$(5) := \eth\texttt{idealProduct}(J, I)(xb)) : xb \in JI,$

$() := \eth\texttt{Identity}(1_R)(x)(1)\eth\mathsf{ANN}(R) : x = (a + b)x = ax + bx \in JI;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{SetEq}\Big(\eth^{-1}\texttt{Subset}, \texttt{IdealsProducitInIntersection}\Big) : JI = I \cap J;$

□

$$\texttt{CoprimeProductLemma2} :: \forall R \in \mathsf{ANN} \,.\, \forall (n, I) : \texttt{CoprimeFamily}(R) \,.\, \prod_{k=1}^{n} I_k = \bigcap_{k=1}^{n} I_k$$

$\texttt{Proof} \;=$

...

□

$$\texttt{MaximalQuatientIsField} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{MaximalIdeal}(R) \,.\, \frac{R}{M} : \texttt{Field}$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [a] : \dfrac{R}{M},$

$\texttt{Assume}\ (1) : [a] \neq 0,$

$(2) := \eth\dfrac{R}{M}(1) : a \notin M,$

$(u, r, 3) := \eth\texttt{MaximaIdeal}(2) : \sum u \in M \,.\, \sum r \in R : \,.\, 1 = u + ra,$

$() := \eth\dfrac{R}{M}(3)\eth^{-1}\dfrac{R}{m} : 1 = [1] = [u + ra] = [ra] = [r][a];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Field} : \left[\dfrac{R}{M} : \texttt{Field}\right];$

□

$\texttt{ChineseReminderTheorem1} :: \forall R \in \mathsf{ANN} \,.\, \forall I, J : \texttt{Coprime}(R) \,.\, \frac{R}{I}\frac{R}{J} \cong_{\mathsf{RING}} \frac{R}{IJ}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \Big([a],[b]\Big) : \frac{R}{I}\frac{R}{J},$

$(u, v, 1) := \eth\texttt{Coprime}(-a+b) : \sum u \in I \,.\, \sum v \in J \,.\, -a+b = u+v,$

$(2) := (1) + a - v : b - v = a + u,$

$x := b - v : R,$

$\varphi\Big([a],[b]\Big) := \pi_{IJ}(x) : \frac{R}{IJ},$

$(3) := \eth x(2)\eth\pi_I : \pi_I(x) = \pi_I(a+u) = [a],$

$(4) := \eth x\eth\pi_J : \pi_J(x) = \pi_I(b-v) = [b],$

$\texttt{Assume}\ y : R,$

$\texttt{Assume}\ (5) : \pi_I(y) = [a],$

$\texttt{Assume}\ (6) : \pi_J(y) = [b],$

$(u', 7) := (5)\eth\pi_I : \sum u' \in I : y = a + u,$

$(v', 8) := (8)\eth\pi_J : \sum v' \in J : y = b + v,$

$(9) := (7)\eth x(2) : x - y = u - u' \in I,$

$(10) := (8)\eth x : x - y = u - u' \in J,$

$() := \texttt{CoprimeProductLemma2}\eth^{-1}\texttt{Intersect}(9)(10) : x - y \in IJ;$

$\rightsquigarrow \varphi := I(\to) : \frac{R}{I}\frac{R}{J} \xrightarrow{\mathsf{RING}} \frac{R}{IJ},$

$\texttt{Assume}\ \Big([a],[b]\Big) : \frac{R}{I}\frac{R}{J},$

$\texttt{Assume}\ (1) : \varphi\Big([a],[b]\Big) = 0,$

$(2) := \eth\varphi(1) : a \in I \ \&\ b \in J,$

$() := \eth\texttt{quotRing}(2) : \Big([a],[b]\Big) = 0;$

$\rightsquigarrow (2) := \texttt{HomoInj} : \left[\varphi : \frac{R}{I}\frac{R}{J} \hookrightarrow \mathsf{RING}\frac{\mathsf{RING}}{IJ}\right],$

$(3) := \eth\varphi\eth\texttt{Surj} : \left[\varphi : \frac{R}{I}\frac{R}{J} \xleftrightarrow{\mathsf{RING}} \frac{R}{IJ}\right],$

$(*) := \eth\texttt{Isomorphic} : \frac{R}{I}\frac{R}{J} \cong \frac{R}{IJ};$

$\square$

$\texttt{ChineseReminderTheorem2} :: \forall R \in \mathsf{ANN} \,.\, \forall (n, I) : \texttt{CoprimeFamily}(R) \,.\, \prod_{k=1}^{n} \frac{R}{I_k} \cong_{\mathsf{RING}} \frac{R}{\prod_{i=1}^{n} I_k}$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.6 Localization

$$\texttt{MultiplivativeSubset} :: \prod R \in \mathsf{RING} \,.\, ?R$$

$$S : \texttt{MultiplivativeSubset} \iff (S, \cdot_R) : \texttt{Submonoid}\,(R, \cdot_R) \iff$$

$$\texttt{localization} :: \prod R \in \mathsf{ANN} \,.\, \texttt{MiltiplicativeSubset}(R) \to \mathsf{RING}$$

$$\texttt{localization}\,(S) = \frac{R}{S} := \left(\frac{R \times S}{\Big\{\big((r,s),(r',s')\big) : \exists z \in S : z(s'r - sr) = 0 \big| r \in R, s, z \in S\Big\}},\right.$$

$$\left., \Lambda[a,b],[c,d] \in \frac{R}{S} \,.\, [ad + bc, bd], \Lambda[a,b],[c,d] \in \frac{R}{S} \,.\, [ac, bd]\right)$$

$$\texttt{fraction} :: \prod A \in \mathsf{ANN} \,.\, \prod S : \texttt{MultiplicativeSubset}(A) \,.\, A \times S \to \frac{A}{S}$$

$$\texttt{fraction}\,(a,s) = \frac{a}{s} := [a,s]$$

$$\texttt{Local} :: ?\mathsf{ANN}$$

$$A : \texttt{Local} \iff \exists! M : \texttt{MaximalIdeal}(A)$$

$$\texttt{localize} :: \prod A \in \mathsf{ANN} \,.\, \texttt{Prime}(A) \to \texttt{Local}$$

$$\texttt{localize}\,(P) = A_P := \frac{A}{P^\complement}$$

$$\texttt{maximalIdeal} :: \prod A : \texttt{Local} \,.\, \texttt{maximalIdeal}(A)$$

$$\texttt{maximalIdeal}\,() = \mathfrak{m}(A) := \eth\texttt{Local}(A)$$

$\texttt{LocalInversion} :: \forall A : \texttt{Local} \,.\, \forall a \in \mathfrak{m}^\complement(A) \,.\, a \in A^*$

$\texttt{Proof} \;=$

$\texttt{Assume}\,(1) : a \notin A^*,$

$(2) := \eth\texttt{genIdeal}\{a\}(1) : \texttt{genIdeal}\{a\} \neq A,$

$(M, 3) := \texttt{MaximalIdealExists}(2) : \sum M : \texttt{MaximalIdeal}(A) \,.\, a \in M,$

$(4) := \texttt{SetIneq}\eth a(3) : \mathrm{m}(A) \neq M,$

$() := \eth\texttt{Local}(4) : \perp;$

$\rightsquigarrow (*) := E(\perp) : a \in A^*;$

$\square$

$\texttt{LocalizationTHM} :: \forall A \in \mathsf{ANN} \;.\; \forall P : \texttt{Prime}(A) \;.\; \frac{A}{P^{\complement}} : \texttt{Local}$

$\texttt{Proof} \;=$

$(1) := \eth^{-1}\texttt{MultiplicativeSet}\eth\texttt{Prime}(A)(P) : [P^{\complement} : \texttt{MultiplicativeSet}(A)],$

$M := \left\{\frac{p}{a} | p \in P, a \in P^{\complement}\right\} :? \frac{A}{P^{\complement}},$

$(2) := \eth\texttt{Ideal}(A)(P) : \left[M : \texttt{Ideal}\;\frac{A}{S}\right],$

$(3) := \eth M \eth \frac{A}{P} : M \neq \frac{A}{P^{\complement}},$

$\texttt{Assume}\; M' : \texttt{MacimalIdeal}\;\frac{A}{P^{\complement}},$

$\texttt{Assume}\; \frac{a}{b} : M',$

$(4) := \eth\texttt{ProperIdeal}\;\frac{A}{P^{\complement}}(M') : 1 \notin M',$

$\texttt{Assume}\; (5) : a \in P^{\complement},$

$(6) := \eth\frac{A}{P^{\complement}} : \frac{b}{a}\frac{a}{b} = 1,$

$(7) := \eth\texttt{Ideal}\;\frac{A}{P^{\complement}}(M')(6) : 1 \in M',$

$() := (7)(4) : \perp;$

$\rightsquigarrow (5) := \eth\texttt{compliment}E(\perp) : a \in P,$

$() := \eth M(5) : \frac{a}{b} \in M;$

$\rightsquigarrow (4) := \eth^{-1}\texttt{Subset} : M' \subset M,$

$() := \eth\texttt{MaximalIdeal}(A)(M')(3)(4) : M' = M;$

$\rightsquigarrow () := \eth^{-1}\texttt{Local} : \left[\frac{A}{P^{\complement}} : \texttt{Local}\right];$

$\square$

$\texttt{invCategoryOfMS} :: \prod R \in \mathsf{ANN} \;.\; \texttt{MultiplicativeSet}(A) \to \mathsf{CAT}$

$\texttt{invCategotyOfMS}\,(S) = \mathcal{C}_R(S) :=$

$$= \left(\left\{(B,\psi) : \sum B \in \mathsf{ANN} \;.\; R \xrightarrow{\mathsf{ANN}} B : \forall s \in S \;.\; \psi(s) \in B^*\right\},\right.$$

$$\left., (B,\psi),(B',\psi') \mapsto \{\varphi : B \xrightarrow{\mathsf{RING}} B' : \psi = \varphi\psi'\}, \circ, \mathrm{id}\right)$$

$\texttt{LocalizationUniversalProperty} :: \forall A \in \mathsf{ANN} . \forall S : \texttt{MultiplicativeSet}(A) .$

$$. \left(\frac{A}{S}, a \mapsto \frac{a}{1}\right) : \texttt{Initial}\Big(\mathcal{C}_A(S)\Big)$$

$\texttt{Proof} =$

$\texttt{Assume}\ (B, \psi) : \mathcal{C}_A(S),$

$\varphi := \Lambda \frac{a}{b} \in \frac{A}{S} . \psi(a)\psi^{-1}(b) : \frac{A}{S} \to B,$

$(1) := \eth\texttt{RingHomo}(\psi) : \varphi(1) = \psi(1)\psi^{-1}(1) = 1,$

$\texttt{Assume}\ \frac{a}{b}, \frac{c}{d} : \frac{A}{S},$

$(2) := \eth\frac{A}{S}\eth\texttt{RingHomo}(psi)\eth\mathsf{ANN}(B)\eth^{-1}\varphi :$

$$: \varphi\left(\frac{a}{b}\frac{c}{d}\right) = \varphi\left(\frac{ac}{bd}\right) = \psi(ac)\psi^{-1}(bd) = \psi(a)\psi(c)\psi^{-1}(b)\psi^{-1}(d) = \psi(a)\psi^{-1}(b)\psi(c)\psi^{-1}(d) = \varphi\left(\frac{a}{b}\right)\varphi\left(\frac{c}{d}\right),$$

$(3) := \eth\frac{A}{S}\eth\varphi\eth\texttt{RingHomo}(\psi)\eth\mathsf{ANN}(B)\eth^{-1}\varphi :$

$$: \varphi\left(\frac{a}{b} + \frac{c}{d}\right) = \varphi\left(\frac{ad + bc}{bd}\right) = \psi(ad + bc)\psi^{-1}(bd) = \psi(ad)\psi^{-1}(bd) + \psi(bc)\psi^{-1}(bd) =$$

$$= \psi(a)\psi(d)\psi^{-1}(b)\psi^{-1}(d) + \psi(b)\psi(c)\psi^{-1}(b)\psi^{-1}(d) = \psi(a)\psi^{-1}(b) + \psi(c)\psi^{-1}(d) = \varphi\left(\frac{a}{b}\right) + \varphi\left(\frac{c}{d}\right);$$

$\rightsquigarrow (2) := \eth^{-1}\texttt{RingHomo} : \left[\varphi : S^{-1}A \xrightarrow{\mathsf{RING}} B\right],$

$(3) := \forall a \in A . \eth\varphi\left(\frac{a}{1}\right) : \forall a \in A . \varphi\left(\frac{a}{1}\right) = \psi(a),$

$(4) := \eth\mathcal{C}_A(S)(2)(3) : \left[\varphi : \left(S^{-1}A, a \mapsto \frac{a}{1}\right) \xrightarrow{\mathcal{C}_A(S)} (B, \psi)\right],$

$\texttt{Assume}\ \varphi' : \left(S^{-1}A, a \mapsto \frac{a}{1}\right) \xrightarrow{\mathcal{C}_A(S)} (B, \psi),$

$(5) := \forall a \in A . \eth\mathcal{C}_A(S)(\varphi')\left(\frac{a}{1}\right) : \forall a \in A . \varphi'\left(\frac{a}{1}\right) = \psi(a),$

$(6) := \forall s \in S . \texttt{RingHomoInverse}(\varphi')\frac{1}{s}(5) : \forall s \in S . \varphi'\left(\frac{1}{s}\right) = (\varphi')^{-1}\left(\frac{s}{1}\right) = \psi^{-1}(s),$

$() := \eth\varphi(5)(6) : \varphi = \varphi';$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Initial}\Big(\mathcal{C}_A(S)\Big) : \left[\left(S^{-1}A, a \mapsto \frac{a}{1}\right) : \texttt{Initial}\Big(\mathcal{C}_A(S)\Big)\right];$

$\square$

$\texttt{idealTransfer} :: \prod A \in \mathsf{ANN} . \prod S : \texttt{MultiplivativeSubset}(A) . \texttt{Ideal}(A) \to \texttt{Ideal}\left(S^{-1}A\right)$

$\texttt{idealTransfer}\,(I) = S^{-1}I := \left\{\frac{a}{s} \in S^{-1}A | a \in I\right\}$

2 Basic Taxonomy of Commutative Rings

2.1 Commutative Noetherian Rings

$\texttt{Noetherian} :: ?\mathsf{ANN}$

$A : \texttt{Noetherian} \iff \forall I : \texttt{Nondescsnding}(\mathbb{N}, \texttt{Ideal}(A)) \;.\; \exists N \in \mathbb{N} : \forall n : \texttt{after}(N) \;.\; I_N = I_n$

$\texttt{FinitelyGeneratedIdeal} :: \prod A \in \mathsf{ANN} \;.\; ?A$

$I : \texttt{FinitelyGeneratedIdeal} \iff \exists F : \texttt{Finite}(A) : I = \texttt{genIdeal}(F)$

$\texttt{NoetherianMax} :: \forall A : \texttt{Noetherian} \;.\; \forall \mathcal{I} :?\texttt{Ideal}(A) \;.\; \mathcal{I} \neq \emptyset \Rightarrow \max \mathcal{I} \neq \emptyset$

$\texttt{Proof} =$

Use Zorn lemma.

$\square$

$\texttt{NoetherianHasFinitelyGeneratedIdeals} :: \forall A : \texttt{Noetherian} \;.\; \forall I : \texttt{Ideal}(A) \;.$
$\;.\; I : \texttt{FinitelyGeneratedIdeal}(A)$

$\texttt{Proof} =$

$\mathcal{F} := \{\texttt{genIdeal}(F) | F : \texttt{Finite}(I)\} :??I,$

$J := \texttt{NotherianMax}(\mathcal{F}) : \max \mathcal{F},$

$(F, 1) := \eth\mathcal{F}(J) : \sum F : \texttt{Finite}(I) \;.\; J = \texttt{genIdeal}(F),$

$\texttt{Assume}\ (2) : J \neq I,$

$(a, 3) := \eth\texttt{StrictSubset}(2) : \sum a \in I \;.\; a \notin J,$

$(4) := \texttt{FiniteUnion}(F, \{a\}) : F \cup \{a\} : \texttt{Finite}(I),$

$(5) := \eth^{-1}(\mathcal{F})(F \cup \{a\}) : F \cup \{a\} \in \mathcal{F},$

$(6) := \eth\texttt{genIdeal}(F, F \cup \{a\})\eth F\eth : J \subsetneq \texttt{genIdeal}(F \cap \{a\}),$

$() := \eth \max \eth J(6)(5) : \bot;$

$\rightsquigarrow (2) := E(\bot) : I = J,$

$() := (2)\eth^{-1}\texttt{FinitelyGeneratedIdeal}\eth\mathcal{F} : [I : \texttt{FinitelyGeneratedIdeal}(A)];$

$\square$

$\texttt{NoetherianByFiniteGeneration} :: \forall A \in \mathsf{ANN} \,.\, \forall (0) : \forall I : \texttt{Ideal}(A) \,.\, I : \texttt{FinitelyGeneratedIdeal}(A) \,.$
$\,.\, A : \texttt{Noetherian}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ I : \texttt{Nondecreasing}(\mathbb{N}, \texttt{Ideal}(A)),$

$(1) := \texttt{IdealUnion}(I) : \bigcup_{n=1}^{\infty} I_n : \texttt{Ideal}(A),$

$(2) := (1)(0) : \left[\bigcup_{n=1}^{\infty} I_n : \texttt{FinitelyGeneratedIdeal}(A)\right],$

$(F, 3) := \eth\texttt{FinitelyGeneratedIdeal}(A)\left(\bigcup_{n=1}^{\infty} I_n\right) : \sum F : \texttt{Finite}(A) \,.\, \bigcup_{n=1}^{\infty} I_n = \texttt{GenIdeal}(F),$

$(n, a) := \texttt{enumerate}(F) : \sum n \in \mathbb{N} \,.\, n \twoheadrightarrow F,$

$(m, 4) := \eth^{-1}\texttt{union} : \sum n \to \mathbb{N} \,.\, \forall i \in n \,.\, a_i \in I_{m_i},$

$M := \max_{i \in n} m(i) : \mathbb{N},$

$(5) := \eth M \eth\texttt{NonDecreasing}(I)\eth\texttt{genIdeal}(3) : \bigcap_{n=1}^{\infty} I_n = I_M,$

$() := \eth\texttt{NonDecreasing}(I)\eth\texttt{union}(5) : \forall n : \texttt{after}(M) \,.\, I_M = I_n;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Noetherian} : [A : \texttt{Noetherian}];$

$\square$

$\texttt{NoetherianQuotient} :: \forall A : \texttt{Noetherian} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \frac{A}{I} : \texttt{Noetherian}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ J : \texttt{Ideal}\ \frac{A}{I},$

$(1) := \texttt{IdealPreimage} : \left[\pi_I^{-1}(J) : \texttt{Ideal}(A)\right],$

$(F, 2) := \eth\texttt{Noetherian}(1) : \sum F : \texttt{Finite}(A) \,.\, \pi_I^{-1}(J) : \texttt{Ideal}(A),$

$(3) := \texttt{FiniteImage}(\pi_I, A) : [\pi_I(F) : \texttt{Finite}],$

$(4) := \eth\texttt{Surjective}\left(A, \frac{A}{I}\right)(1)(2) : J = \texttt{genIdeal}(\pi_I(F)),$

$() := \eth^{-1}\texttt{FinitelyGeneratedIdeal}(3)(4) : \left[J : \texttt{FinitelyGeneratedIdeal}\ \frac{A}{I}\right];$

$\rightsquigarrow (*) := \texttt{NoetherianByFiniteGeneration} : \left[\frac{A}{I} : \texttt{Noetherian}\right];$

$\square$

$\texttt{Factorization} :: \prod R : \texttt{IntegralDomain} \,.\, \prod a \in R \,.\, \sum n \in \mathbb{N} \,.\, n \to \texttt{IrreducibleElement}(I)$

$(n, p) : \texttt{Factorization} \iff a = \prod_{i=1}^{n} p_i$

$\texttt{FactorizationsExistInNoetherian} :: \forall A : \texttt{Noetherian} \,\&\, \texttt{IntegralDomain} \,.\, \forall a \in R^\times \setminus R^* \,.$
$\quad .\, \exists \texttt{Factorization}(R, a)$
$\texttt{Proof} \ =$
$\texttt{Assume}\ a : A^\times \setminus A^*,$
$T_0 := \texttt{root}(0) : \texttt{Tree}(\prod n \in \mathbb{Z} \,.\, n \to \{0,1\}),$
$q^0 := \Lambda 0 \in \texttt{leaves}(T_1) \,.\, a : \texttt{leaves}(T_0) \to A^\times \setminus A^*,$
$U_0 := \eth q^0 \eth^{-1} \texttt{Product} : \prod_{i \in \texttt{leaves}(T_0)} q_i^0 = a,$
$\texttt{Assume}\ n : \mathbb{N},$
$T_n := T_{n-1} : \texttt{Tree}\left(\prod n \in \mathbb{Z} \,.\, n \to \{0,1\}\right),$
$\texttt{Assume}\ i : \texttt{leaves}(T_{n-1}),$
$\texttt{Assume}\ (1) : q_i^n : \texttt{IrreducibleElement}(A),$
$T_n := \texttt{addLeave}(T_n, i, i \oplus 0) : \texttt{Tree}(\prod n \in \mathbb{Z} \,.\, n \to \{0,1\}),$
$q^n_{i\oplus 0} := q_i^{n-1} : A^\times \setminus A^*;$
$\rightsquigarrow (1) := I(\Rightarrow) : q_i^{n-1} : \texttt{IrreducibleElement}(A) \Rightarrow q^n_{i\oplus 0} = q_i^{n-1},$
$\texttt{Assume}\ (2) : q_i^{n-1}\ !\ \texttt{IrreducibleiElement}(A),$
$(q^n_{i\oplus 0}, q^n_{i\oplus 1}, (3)) := \eth^{-1}\texttt{IrreducibleElement}(A)(2) : \sum q^n_{i\oplus 0}, q^n_{i\oplus 1} \in A^\times \setminus A^* \,.\, q^n_{i\oplus 0} q^n_{i\oplus 1} = q_i^{n-1}\ \&$
$\quad \&\ (q^n_{i\oplus 0}, q_i^{n-1})\ !\ \texttt{Associates}(A)\ \&\ (q^n_{i\oplus 0}, q_i^{n-1})\ !\ \texttt{Associates}(A),$
$T_n := \texttt{addLeaves}(T_n, i, (i \oplus 0, i \oplus 1)) : \texttt{Tree}(\prod n \in \mathbb{Z} n \to \{0,1\});$
$\rightsquigarrow (2) := I(\Rightarrow) : q_i^{n-1} \Rightarrow q_i^{n-1} = q^n_{i\oplus 0} q^n_{i\oplus 1};$
$\rightsquigarrow q^n := I\left(\sum\right) : \sum q^n : \texttt{leaves}(T_n) \to A^\times \setminus A^* \,.\, \forall i \in \texttt{leaves}(T_{n-1}) \,.\, \ldots,$
$U_n := \eth q^n \eth U_{n-1} : a = \prod_{i : \texttt{leaves}(T_n)} q_i^n;$
$\rightsquigarrow (T, q, U) := I\left(\sum\right) : \sum T : \texttt{Tree}\left(\prod n \in \mathbb{N} \,.\, n \to \{0,1\}\right) \,.\, q : \prod n \in \mathbb{N} \,.\, \texttt{layer}(n, T) \to A^\times \setminus A^* \,.$
$\quad .\, \forall n \in \mathbb{N} \,.\, a = \prod_{i \in \texttt{layer}(n,T)} q_i^n,$
$(N, (1)) := \eth \texttt{UniqueFactorizationDomain}(A) \eth (T, q, U) : \sum N \in \mathbb{N} : \forall n : \texttt{after}(N) \,.$
$\quad .\, |\texttt{layer}(n, A)| = |\texttt{layer}(N, A)|,$
$(*) := \eth^{-1} \texttt{Factorization} \eth (T, q, U) \eth (N, (1)) : [q^N : \texttt{Factorization}(a)];$
$\square$

$\texttt{NoetherianContainsPrimeProduct} :: \forall A : \texttt{Noetherian} \,.\, \forall I : \texttt{Ideal}(A) \,.$

$$. \exists n \in \mathbb{N} : \exists P : n \to \texttt{Prime}(A) \,.\, \prod_{i=1}^{n} P_i \subset I$$

$\texttt{Proof} \ =$

$$\mathcal{I} := \left\{ I : \texttt{Ideal}(A) : \forall n \in \mathbb{N} \,.\, \forall P : n \to \texttt{Prime}(A) \,.\, \prod_{i=1}^{n} P_i \not\subset I \right\} :?\texttt{Ideal}(A),$$

$\texttt{Assume}\ (1) : \mathcal{I} \neq \emptyset,$

$J := \texttt{NotherianMax}(A)(1) : \max \mathcal{I},$

$(2) := \eth\mathcal{I}(J) : [J \,!\, \texttt{Prime}(A)],$

$(a, b.3) := \eth\texttt{Prime}(2) : \sum a, b \in J^{\complement} : ab \in J,$

$I := J + \texttt{genIdeal}\{a\} : \texttt{Ideal}(A),$

$(4) := \eth I(3) : J \subsetneq I,$

$$(n, P, 5) := \eth J \eth\mathcal{I}(4) : \sum n \in \mathbb{N} \,.\, \sum P : n \to \texttt{Prime}(A) \,.\, \prod_{i=1}^{n} P_i \subset I,$$

$I' := J + \texttt{genIdeal}\{b\} : \texttt{Ideal}(A),$

$(6) := \eth I'(3) : J \subsetneq I',$

$$(m, P', 7) := \eth J \eth\mathcal{I}(6) : \sum m \in \mathbb{N} \,.\, \sum P' : m \to \texttt{Prime}(A) \,.\, \prod_{i=1}^{m} P'_i \subset I',$$

$(8) := \eth I \eth I(3) \eth\texttt{Ideal}(J) : II' = J^2 + aJ + bJ + abA = J,$

$$(9) := (5)(7) : \prod_{i=1}^{n} P_i \prod_{i=1}^{m} P'_i \subset II' = J,$$

$() := \eth\mathcal{I}(J)(9) : \bot;$

$\rightsquigarrow (*) := E(\bot) : \mathcal{I} = \emptyset;$

$\square$

2.2 Unique Factorization Domains

$\texttt{EqFactorization} :: \prod R : \texttt{IntegralDomain} \,.\, \prod a \in R \,.\, ?\texttt{Factorization}^2(R, a)$

$\Big((n,p),(m,q)\Big) : \texttt{EqFactorization} \iff (n,p) \cong (m,q) \iff n = m \;\&\; \exists \sigma \in S^n : \,.$

$\quad .\, \forall i \in n \,.\, (p_i, q_{\sigma(i)}) : \texttt{Associates}(R)$

$\texttt{UniqueFactorizationDomain} :: ?\texttt{IntegralDomain}$

$R : \texttt{UniqueFactorizationDomain} \iff \forall a \in R^\times \setminus R^\times \,.\, \exists \texttt{Factorization}(a, n) \;\&$

$\quad \& \; \forall (n,p),(m,q) : \texttt{Factorization}(R,a) \,.\, (n,p) \cong (m,q)$

$\texttt{factorization} :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, \prod a \in R^\times \setminus R^* \,.\, \texttt{Factorization}(R, a)$

$\texttt{factorization}\,() = \Big(n(a), p(a)\Big) := \eth^{-1}\texttt{UFD}$

$\texttt{length} :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, R \to \mathbb{Z}$

$\texttt{length}\,(a) = L(a) := \texttt{if}\ a == 0\ \texttt{then}\ -1\ \texttt{else}\ \ \texttt{if}\ a \in R^*\ \texttt{then}\ 0\ \texttt{else}\ n(a)$

$\texttt{IrreducibleIsPrimeInUFD} :: \forall R : \texttt{UniqueFactorizationDomain} \,.\, \forall a : \texttt{Irreducible}(R) \,.\, a : \texttt{Prime}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x, y : R,$

$\texttt{Assume}\ (1) : a|xy,$

$(v, 2) := \eth\texttt{Divides}(1) : \sum v \in R \,.\, xy = av,$

$(3) := \eth^{-1}\texttt{Factorization} : \Big[(n(v)+1, p(v) \oplus a) : \texttt{Factorization}(xy)\Big],$

$(4) := \eth^{-1}\texttt{Factorization} : \Big[(n(x)+n(y), p(x) \oplus p(y)) : \texttt{Factorization}(xy)\Big],$

$(5) := \eth\texttt{UniqueFactorizationDomain}(R)(3)(4) : n(v)+1 = n(x)+n(y) \;\&\; \ldots,$

$(6) := (5)(n+1) : \exists i \in n(x) : (a, p_i(x)) : \texttt{Associates}\Big|\exists j \in n(y) : (a, p_j(y)) : \texttt{Associates},$

$() := \eth^{-1}\texttt{Divisible}\eth\texttt{Associates}(6) : a|x\Big|a|y;$

$\rightsquigarrow () := \eth^{-1}\texttt{Prime} : \Big[a : \texttt{Prime}(R)\Big];$

$\square$

$\texttt{CommonDivisor} :: \prod R : \texttt{IntegralDomain} \,.\, prodn \in \mathbb{N} \,.\, \prod a : n \to R \,.\, ?R$

$x : \texttt{CommonDivisor} \iff \forall i \in n \,.\, x | a_i$

$\texttt{GreatestCommonDivisor} :: \prod R : \texttt{IntegralDomain} \,.\, \prod n \in \mathbb{N} \,.\, \prod a : n \to R \,.\, ?\texttt{CommonDivisor}(R, n, a)$

$x : \texttt{GreatestCommonDivisor} \iff x : \texttt{GCD}(R, n, a) \iff \forall y : \texttt{CommonDivisor}(R, n, a) \,.\, y | x$

$\texttt{greatestCommonDivisor} :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, \prod a, b \in R \,.\, \texttt{GCD}\Big(R, 2, [a, b]\Big)$

$\texttt{greatestCommonDivisor}\,() = \gcd(a, b) := \texttt{if } a == 0 \texttt{ then } b \texttt{ else if } b == 0 \texttt{ then } a \texttt{ else}$

$\quad \texttt{if } a \in R^* \texttt{ then } a \texttt{ else if } b \in R^* \texttt{ then } b \texttt{ else}$

$$\texttt{if } I = \emptyset \texttt{ then } 1 \texttt{ else } p_i \gcd\left(\prod_{k=1, k \neq i}^{n(a)} p_k(a), \prod^{n(b)} k = 1, k \neq j p_k(a) \right)$$

$$\texttt{where} \quad I = \left\{ (i, j) \in n(a) \times n(b) : \Big(p_i(a), p_j(b)\Big) : \texttt{Associates} \right\}$$

$$\texttt{where} \quad (i, j) = \min I$$

$\texttt{greatestCommonDivisor2} ::$

$\quad :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, \prod n \in \mathbb{N} \,.\, \prod a : n \to R \,.\, \texttt{GCD}\Big(R, n, a)\Big)$

$\texttt{greatestCommonDivisor2}\,() = \gcd(n, a) := \texttt{if } n == 1 \texttt{ then } a \texttt{ else if } n == 2 \texttt{ then } \gcd(a_1, a_2)$

$\quad \texttt{else else } \gcd(a_n, \gcd\Big(n - 1, a_{|n-1})\Big)$

$\texttt{CommonDenominator} :: \prod R : \texttt{IntegralDomain} \,.\, \prod n \in \mathbb{N} \,.\, \prod a : n \to R \,.\, ?R$

$x : \texttt{CommonDenominator} \iff \forall i \in n \,.\, a_i | x$

$\texttt{LeastCommonDenominator} ::$

$\quad :: \prod R : \texttt{IntegralDomain} \,.\, \prod n \in \mathbb{N} \,.\, \prod a : n \to R \,.\, ?\texttt{CommonDenominator}(R, n, a)$

$x : \texttt{GreatestCommonDenominator} \iff x : \texttt{LCD}(R, n, a) \iff \forall y : \texttt{CommonDenominator}(R, n, a) \,.\, x | y$

$\texttt{leastCommonDenominator} :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, \prod a, b \in R \,.\, \texttt{LCD}\Big(R, 2, [a, b]\Big)$

$\texttt{leastCommonDenominator}\,() = \operatorname{lcd}(a, b) := \texttt{if } a == 0 \texttt{ then } 0 \texttt{ else if } b == 0 \texttt{ then } 0 \texttt{ else } \dfrac{ab}{\gcd(a, b)}$

$\texttt{leastCommonDenominator2} ::$

$\quad :: \prod R : \texttt{UniqueFactorizationDomain} \,.\, \prod n \in \mathbb{N} \,.\, \prod a : n \to R \,.\, \texttt{LCD}\Big(R, n, a)\Big)$

$\texttt{leastCommonDivisor2}\,() = \gcd(n, a) := \texttt{if } n == 1 \texttt{ then } a \texttt{ else if } n == 2 \texttt{ then } \operatorname{lcd}(a_1, a_2) \texttt{ else}$

$\quad \texttt{else } \operatorname{lcd}(a_n, \operatorname{lcd}\Big(n - 1, a_{|n-1})\Big)$

2.3 Principle Ideal Domains

$$\texttt{principle} :: \prod A \in \mathsf{ANN} \,.\, A \to \texttt{Ideal}(A)$$
$$\texttt{principle}\,(a) = \langle a \rangle := aA$$

$$\texttt{Principle} :: \prod A \in \mathsf{ANN} \,.\, ?\texttt{Ideal}(A)$$
$$I : \texttt{Principle} \iff \exists a \in A \,.\, I = \langle a \rangle$$

$$\texttt{PrincipleIdealDomain} :: ?\texttt{IntegralDomain}$$
$$A : \texttt{PrincipleIdealDomain} \iff \forall I : \texttt{Ideal}(A) \,.\, I : \texttt{Principle}(A) \,.$$

$\texttt{PrincipalIdealsOfIrreduciblesAreMaximal} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall p : \texttt{Irreducible} \,.\, \langle p \rangle : \texttt{Maxi}$

$\texttt{Proof} =$

$\texttt{Assume}\ a : (R^\times \setminus R^*),$

$\texttt{Assume}\ (-1) : a \notin \langle p \rangle,$

$\texttt{Assume}\ (0) : \langle a \rangle + \langle p \rangle \neq A,$

$(1) := \eth a \eth^{-1} \texttt{genIdeal} \eth \texttt{NotIn}(-1) : \langle p \rangle \subsetneq \texttt{genIdeal}\{a, p\},$

$(b, 2) := \eth \texttt{PrincipleIdealDomain}(A)\big(\texttt{genIdeal}\{a, p\}\big) : \sum d \in A \,.\, \langle b \rangle = \texttt{genIdeal}\{a, p\},$

$(3) := (2)(1) : \langle p \rangle \subsetneq \langle b \rangle,$

$(4) := \eth^2 \texttt{principle}(p)(b)(3) : b | p,$

$(5) := \eth \texttt{IrreducibleElement}(A)(p)(4) \eth \texttt{Associates}(A)(0) : p | b,$

$(6) := \texttt{principle}(5) : b \in \langle p \rangle,$

$(7) := (2)(6) : a \in \langle p \rangle,$

$() := \eth a (7) : \bot;$

$\rightsquigarrow (1) := I(\forall) I(\to) : \forall a \in R^\times \setminus R^* \,.\, a \notin \langle p \rangle \Rightarrow \langle a \rangle + \langle p \rangle = A,$

$p := \eth^{-1} \texttt{MaximalIdeal}(A) : [p : \texttt{MaximalIdeal}(A)];$

□

$\texttt{IrreduciblesArePrimeInPID} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \forall p : \texttt{IrreducebleElement}(A) \,.\, p : \texttt{PrimeEleme}$

$\texttt{Proof} =$

$\texttt{Assume}\ x, y : A,$

$\texttt{Assume}\ (1) : a | xy,$

$\texttt{Assume}\ (2) : a \not| y,$

$(3) := \eth \texttt{MaximalIdealPrincipalIdealsOfIreduciblesAreMaximal}(A)(a)(y)(2) : \langle a \rangle + \langle y \rangle = A,$

$(u, v, 4) := \eth \texttt{principle}(3) : \sum u, v \in A : ua + vy = 1,$

$(5) := x(4) : uxa + vxy = x,$

$(6) := \eth \texttt{pricnciple} : uxa \in \langle a \rangle,$

$(7) := \eth \texttt{Ideal}(A)\big(\langle a \rangle\big) : uxa \in \langle a \rangle,$

$(9) := \eth \texttt{Subgroup}(A)\big(\langle a \rangle\big)(5)(6)(7) : x \in \langle a \rangle,$

$() := \eth^{-1} \texttt{principle}(9) : a | x;$

□

$\texttt{PIDIsUFD} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, A : \texttt{UniqueFactorizationDomain}$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{PrincipleIdealDomain}(A)\eth^{-1}\texttt{UniqueFactorizationDomain}(A) : [A : \texttt{Noetherian}],$

$\texttt{Assume}\ a : A^\times \setminus A^*,$

$(n,p) := \texttt{FactorizationExistsInNoetherian}(A,a)) : \texttt{Factorization}(A,a),$

$\texttt{Assume}\ (m,q) : \texttt{Factorization}(A,a),$

$() := \eth^{-1}\texttt{EqFactorizations}\eth\texttt{PrimeIrreducibleArePrimeInUFD}(n,p)\eth^2\texttt{Factorization}(n,p)(m,q) :$

$\quad : (n,p) \cong (m,q);$

$\rightsquigarrow () := \eth^{-1}\texttt{UniqueFactorizationDomain} : [A : \texttt{UniqueFactorizationDomain}];$

$\square$

$\texttt{DedikindHasseValuation} :: \forall A : \texttt{IntegralDomain} \,.\, A \to \mathbb{Z}_+$

$v : \texttt{DedikindHasseValuation} \iff \forall a,b \in A \,.\, a|b \Big| \exists r,u,v \in A \,.\, v(r) < v(a)\ \&\ ub = va + r$

$\texttt{DHVimpliesPID} :: \forall A : \texttt{IntegralDomain} \,.\, \forall v : \texttt{DedikindHasseValuation}(A) \,.\, A : \texttt{PrincipleIdealDomain}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ I : \texttt{Ideal}(A),$

$\texttt{Assume}\ (1) : I \neq \{0\},$

$a := \arg\min_{a \in I \cap A^\times} v(a) : I,$

$\texttt{Assume}\ b : I,$

$\texttt{Assume}\ (2) : a \not| b,$

$(r,s,3) := \eth\texttt{DedikindHasseValuation}(b,a) : \sum r,u,v \in A \,.\, ub = va + r\ \&\ v(r) < v(a),$

$(4) := \eth\texttt{Ideal}(A)(I)(3)_1 : r \in I,$

$(5) := (4)(3)_2\eth a : \bot;$

$\rightsquigarrow (2) := I(\forall)E(\bot) : \forall b \in I \,.\, a|b,$

$(3) := \eth\texttt{Ideal}(A)\langle a\rangle(2) : I = \langle a\rangle,$

$() := \eth^{-1}\texttt{Principle}(I) : [I : \texttt{Principle}(A)];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{PrincipleIdealDomain} : [A : \texttt{PrincipleIdealDomain}];$

$\square$

$\texttt{PIDAdmitsDHV} :: \forall A : \texttt{PrincipleIdealDomain} \,.\, \exists v : \texttt{DedikindHasseValuation}(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{PrincipleProduct} :: \forall A \in \mathsf{ANN} \,.\, \forall a,b \in A \,.\, \langle a\rangle\langle b\rangle = \langle ab\rangle$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.4 Euclidean Rings

$$\texttt{EuclideanValuation} :: \prod A : \texttt{IntegralDomain} \,.\, A \to \mathbb{Z}_+$$
$$v : \texttt{EuclideanValuation} \iff \forall a \in R \,.\, \forall b \in R^\times \,.\, \exists s, r \in R : a = sb + r \;\&\; v(r) < v(b)$$

$$\texttt{EuclideanRing} := \sum A : \texttt{IntegralDomain} \,.\, \texttt{EuclideanValuation} : \texttt{Type};$$

$$\texttt{euclideanRingAsRing} :: \texttt{EuclideanRing} \to \texttt{Ring}$$
$$\texttt{euclideanRingAsRing}\,(A, v) = \texttt{implicit} := A$$

$$\texttt{euclideanValuation} :: \prod (A, v) : \texttt{EuclideanRing} \,.\, \texttt{EuclideanValuation}(A)$$
$$\texttt{euclideanValuation}\,(a) = |a| := v(a)$$

$\texttt{ERIsPID} :: \forall A : \texttt{EuclideanRing} \,.\, A : \texttt{PrincipleIdealDomain}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ I : \texttt{Ideal}(A),$

$\texttt{Assume}\ (1) : I \neq \{0\},$

$a := \arg\min_{a \in I \cap A^\times} |a| : I,$

$\texttt{Assume}\ b : I,$

$\texttt{Assume}\ (2) : a \nmid b,$

$(r, s, 3) := \eth\texttt{EuclideanRing}(b, a) : \sum r, s \in A \,.\, b = as + r \;\&\; |r| < |a|,$

$(4) := \eth\texttt{Ideal}(A)(I)(3)_1 : r \in I,$

$(5) := (4)(3)_2\eth a : \perp;$

$\rightsquigarrow (2) := I(\forall)E(\perp) : \forall b \in I \,.\, a|b,$

$(3) := \eth\texttt{Ideal}(A)\langle a\rangle(2) : I = \langle a\rangle,$

$() := \eth^{-1}\texttt{Principle}(I) : [I : \texttt{Principle}(A)];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{PrincipleIdealDomain} : [A : \texttt{PrincipleIdealDomain}];$

$\square$

$$\texttt{euclideanDivisionAlgorithm} :: \prod A : \texttt{EuclideanRing} \,.\, A \times A \to \texttt{List}(A \times A \times A)$$
$$\texttt{euclideanDivisionAlgorithm}\,(a, 0) = \texttt{eda}(a, 0) := [\,]$$
$$\texttt{euclideanDivisionAlgotithm}\,(a, b) = \texttt{eda}(a, b) := \texttt{eda}(b, r) : (b, s, r)$$
$$\quad \texttt{where} \quad (s, r) = \eth\texttt{EuclideanRing}(A)(a, b)$$

$\texttt{EDATerminates} :: \forall A : \texttt{EuclideanRing} \,.\, \forall a, b \in A \,.\, \texttt{len}\ \texttt{eda}(a, b) < \infty$

$\texttt{Proof} \;=$

By definition of Euclidean Valuation and Well-orderedness of $\mathbb{Z}_+$.

$\square$

$\texttt{DivisionWithReminderLemma} :: \prod A \in \mathsf{ANN} \,.\, \forall a, b, u, r \in A \,.$

$\quad .\, a = ub + r \Rightarrow \texttt{genIdeal}\{a, b\} = \texttt{genIdeal}\{b, r\}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{GCDByDivisionWithReminder} :: \prod A : \texttt{UniqueFactorizationDomain} \,.\, \forall a, b, u, r \in A \,.$

$\quad .\, a = ub + r \Rightarrow \Big(\gcd(a, b), \gcd(r, b)\Big) : \texttt{Associates}(A)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{EDADelieversGCD} :: \forall A : \texttt{EuclideanRing} \,.\, \forall a, b \in R^{\times} \,.\, \gcd(a, b) = \texttt{first}\ \texttt{head}\ \texttt{eda}(a, b)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{Normlike} :: \prod A : \texttt{IntegralDomain} \,.\, ?\texttt{EucleadianValuation}(A)$

$v : \texttt{Normlike} \iff \forall a, b \in A^{\times} \,.\, v(ab) \geq v(b)$

$\texttt{ERAdmitsNormlike} :: \forall A : \texttt{EuclideanRing} \,.\, \exists \texttt{Normlike}(A)$

$\texttt{Proof} \ =$

Set $v(b) = \min \left\{ |ab| \,\middle|\, a \in A^{\times} \right\}$

$\square$

$\texttt{DiscreteValuation} :: \prod k : \texttt{Field} \,.\, k^* \xrightarrow{\mathsf{GRP}} \mathbb{Z}$

$v : \texttt{DiscreteValuation} \iff \forall a, b \in k^* \,.\, a + b \in k^* \Rightarrow v(a + b) \geq \min\Big(v(a), v(b)\Big)$

$\texttt{DiscreteValuationRing} :: \prod k : \texttt{Field} \,.\, \texttt{DiscreteValuation}(k) \to \texttt{IntegralDomain}$

$\texttt{DiscreteValuationRing}\,(v) = \mathbb{Z}_k(v) := \Big(\{a, b \in k | v(k) \geq 0\} \cup \{0\}, +_k, \cdot_k\Big)$

$\texttt{DVRIsER} :: \forall k : \texttt{Field} \,.\, \forall v : \texttt{DiscreteValuation}(k) \,.\, \mathbb{Z}_k(v) : \texttt{EuclideanRing}$

$\texttt{Proof} \ =$

...

$\square$

2.5 Graded Rings

$$\texttt{GradedAbelean} :: ?\sum G \in \mathsf{ABEL}\ .\ \sum \Delta \in \mathsf{SET}\ .\ \Delta \to \texttt{Subgroup}(G)$$
$$(G, \Delta, H) : \texttt{GradedAbelean} \iff G = \bigoplus_{\delta \in \Delta} H_\delta$$

$$\texttt{Homogeneous} :: \prod (G, \Delta, H) : \texttt{GradedAbelean}\ .\ ?G$$
$$g : \texttt{Homogeneous} \iff \exists \delta \in \Delta\ .\ g \in H_\delta$$

$$\texttt{homogeneousElement} :: \prod (G, \Delta, H) : \texttt{GradedAbelean}\ .\ G \to \Delta \to \texttt{Homogeneous}(G, \Delta, H)$$
$$\texttt{homogeneousElement}\,(g, \delta) = g_\delta := h_\delta \quad \texttt{where} \quad h = \eth\texttt{DirectSum}\Big(\eth\texttt{GradedAbelean}(G, \Delta, H)\Big)(g)$$

$$\texttt{trivialGraduation} :: \prod \Delta \in \mathsf{SET}\ .\ \prod G \in \mathsf{ABEL}\ .\ \Delta \to \texttt{GradedAbelean}(G, \Delta)$$
$$\texttt{trivialGraduation}\,(\delta) := \Lambda \alpha \in \delta\ .\ \texttt{if}\ \alpha == \delta\ \texttt{then}\ G\ \texttt{else}\ \{0\}$$

$$\texttt{Multigraduaion} :: ?\sum G \in \mathsf{ABEL}\ .\ \sum \mathcal{I} \in \mathsf{SET}\ .\ \Delta : \mathcal{I} \to \mathsf{SET}\ .\ \prod_{i \in \mathcal{I}} \Delta_i \to \texttt{Subgroup}(G)$$
$$(G, \mathcal{I}, \Delta, H) : \texttt{Multigrading} \iff \left(G, \prod_{i \in I}\right) : \texttt{GradedAbelean}$$

$$\texttt{partialGraduation} :: \prod (G, \mathcal{I}, \Delta, H) : \texttt{Multigrading}\ .\ \prod \mathcal{J} \subset \mathcal{I}\ .\ \texttt{Multigrading}$$
$$\texttt{partialGraduation}\,(G, \mathcal{I}, \Delta, H, \mathcal{J}) := \left(G, \mathcal{J}, \lambda \delta' \in \prod_{j \in \mathcal{J}} \Delta_j\ .\ \bigoplus_{\delta \in \prod_{i \in \mathcal{I} \setminus \mathcal{J}} \Delta_i} H_{\delta \oplus_{\mathcal{I}} \delta'}\right)$$

$$\texttt{derivedGraduation} :: \prod G \in \mathsf{ABEL}\ .\ \prod \Delta, \Delta' \in \mathsf{SET}\ .\ \texttt{GradedAbelean}(G, \Delta) \to (\Delta \to \Delta') \to \texttt{GradedAbelea}$$
$$\texttt{derivedGraduation}\,(G, \Delta, H, f) := \left(G, \Delta', \Lambda \delta' \in \Delta'\ .\ \bigoplus_{\delta \in f^{-1}\{\delta'\}} H_\delta\right)$$

$$\texttt{totalGraduation} :: \prod G \in \mathsf{ABEL}\ .\ \prod \mathcal{I} \in \texttt{Set}\ .\ \prod \Delta : \texttt{CommutativeMonoid}\ .$$
$$.\ \texttt{GradedAbelean}(G, \Delta^{\oplus I}) \to \texttt{GradedAbelean}(G, \Delta)$$
$$\texttt{totalGraduation}\,\big(G, \Delta^{\oplus I}, H\big) := \texttt{derivedGraduation}\left(G, \Delta^{\oplus I}, H, \Lambda \delta \in \Delta^{\oplus I}\ .\ \sum_{i \in I} \delta_i\right)$$

$$\texttt{GradedRing} :: ?\sum R : \texttt{Ring}\ .\ \sum \Delta : \texttt{CommutativeMonoid}\ .\ \Delta \to \texttt{Subgroup}(R)$$
$$(R, \Delta, H) : \texttt{GradedRing} \iff (R, \Delta, H) : \texttt{GradedAbelean}\ \&\ \forall a, b \in M\ .\ H_a H_b \subset H_{a+b}$$

$\texttt{TheZerothHomogeneousPart} :: \forall (R,\Delta,H) : \texttt{GraderRing} \,.\, \forall [0] : (\Delta : \texttt{Cancelable}) \,.\, H_0 \subset_{\mathsf{RING}} R$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a, b : H_0,$

$[(a,b).*] := \eth\texttt{GradedRing}(a,b) : ab \in H_0;$

$\rightsquigarrow [1] := I(\forall) : \forall a, b \in H_0 \,.\, ab \in H_0,$

$\Big(n, \delta, h, [2]\Big) := \eth\texttt{GradedAbelean}(e) : \sum n \in \mathbb{N} \,.\, \sum \delta : n \hookrightarrow \Delta \,.\, h : \prod i \in n \,.\, H_{\delta_i} \,.\, e = \sum_{i=1}^{n} h_i,$

$\texttt{Assume}\ \alpha : \Delta,$

$\texttt{Assume}\ x : H_\delta,$

$[3] := \eth e[2] : x = xe = xh_i,$

$(i, [4]) := \eth\texttt{GradedRing}[3] : \sum i \in n \,.\, xh_i = x : \forall j \in n \setminus i \,.\, xh_j = 0,$

$[\delta.*] := [3]\eth\texttt{Cancelable}[0] : \delta^{-1}(0) = i = 0;$

$\rightsquigarrow [3] := I(\forall) : \forall \delta \in \Delta \,.\, \forall x \in H_\delta \,.\, xh_{\delta^{-1}}(1) = x,$

$[4] := \eth\texttt{GradedRing}(R,\Delta,H)[1] : \forall x \in A \,.\, xh_{\delta^{-1}(1)} = x,$

$[5] := \eth^{-1}\texttt{Identity}[4] : e = h_{\delta^{-1}(1)} \in H_{\delta^{-1}(q)},$

$[6] := [0][5] : e \in H_0,$

$[*] := \eth^{-1}\texttt{Ring}[6][1] : H_0 \subset_{\mathsf{RING}} R;$

$\square$

$\texttt{CategoryOfGradedRings} :: \texttt{CommutativeRing} \to \mathsf{CAT}$

$\texttt{CategoryOfGradedRings}\,(\Delta) = \mathsf{GRING}(\Delta) := \Big(\{(R,\Delta,H) : \texttt{GradedRings}\},$

$\quad , (R,\Delta,H), (S,\Delta,G) \mapsto \{f : R \xrightarrow{\mathsf{RING}} S : \forall \delta \in \Delta \,.\, f(H_\delta) \subset G_\delta\}, \circ, \mathrm{id}\Big)$

$\texttt{GradedSubring} :: \mathsf{GRING}(\Delta) \to ?\mathsf{GRING}(\Delta)$

$(R', \Delta, H') : \texttt{GradedSubring} \iff (R', H') \subset_{\mathsf{RING}} (R, H) \iff \forall \delta \in \Delta \,.\, H'_\delta \subset H_\delta$

$\texttt{HomogeneousCentralizersAreGradedSubring} :: \forall (R,H) : \mathsf{GRING}(\Delta) \,.\, \forall \delta \in \Delta \,.$

$\quad .\, \forall [0] : (\Delta : \texttt{Cancelable}) \,.\, \forall x \in H_\delta \,.\, \exists V : \delta \to \texttt{Subgroup}\Big(Z(x)\Big) \,.\, \Big(Z(x), V\Big) :\subset_{\mathsf{GRING}} (R, H)$

$\texttt{Proof} \;=$

$V := \Lambda \delta \in \Delta \,.\, H_\delta \cap Z(x) : \texttt{Subgroup}(R),$

$\texttt{Assume}\ y : Z(x),$

$[1] := \eth Z(x) : 0 = yx - xy = \sum_{\delta \in \Delta} (y_\delta x - x y_\delta),$

$[2] := \eth\texttt{GradedRing}[0][1] : \forall \delta \in \Delta \,.\, y_\delta x - x y_\delta = 0,$

$[y.*] := \eth^{-1} Z(x)[3] : \forall \delta \in \Delta \,.\, V_\delta;$

$\rightsquigarrow [1] := \eth^{-1}\texttt{InnerDirectSum} : Z(x) = \bigoplus_{\delta \in \Delta} V_\delta,$

$[*] := \eth Z(x) \eth^{-1}\texttt{GradedSubring} : \Big(Z(x), V\Big) :\subset_{\mathsf{GRING}} (R, H);$

$\square$

$\texttt{GradedCentralizersAreGradedSubring} :: \forall (R, H) : \mathsf{GRING}(\Delta) \,.\, \forall [0] : (\Delta : \texttt{Cancelable}) \,.$

$\quad .\, \forall (R', H') \subset_{\mathsf{GRING}(\Delta)} (R, H) \,.\, \Big(Z(R'), Z(R') \cap H\Big) \subset_{\mathsf{GRING}(\Delta)} (R, H)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{GradedLeftIdeal} :: \prod (R, H) \in \mathsf{GRING}(\Delta) \,.\, ?\texttt{LeftIdeal}(R)$

$I : \texttt{GradedLeftIdeal} \iff \forall x \in I \,.\, \forall \delta \in \Delta \,.\, x_\delta \in I$

$\texttt{GradedRightIdeal} :: \prod (R, H) \in \mathsf{GRING}(\Delta) \,.\, ?\texttt{RightIdeal}(R)$

$I : \texttt{GradedLeftIdeal} \iff \forall x \in I \,.\, \forall \delta \in \Delta \,.\, x_\delta \in I$

$\texttt{GradedTwoSidedIdeal} :: \prod (R, H) \in \mathsf{GRING}(\Delta) \,.\, ?\texttt{RightIdeal}(R)$

$I : \texttt{GradedTwoSidedIdeal} \iff \forall x \in I \,.\, \forall \delta \in \Delta \,.\, x_\delta \in I$

3 Polynomials Over a Ring

3.1 Algebra of Formal Polinomials

$\texttt{monoidRing} :: \mathsf{RING} \times \texttt{Monoid} \to \mathsf{RING}$

$\texttt{monoidRing}(R, M) = R[M] := \left(\left\{f : M \to R : \left|f^{-1}\{0\}^{\complement}\right| < \infty\right\}, +_{M \to R}, (p, q) \mapsto \Lambda m \in M \;.\; \sum_{ab=m} p(a)p(b)\right)$

$\texttt{Assume}\ f, g, h : R[M],$

$\texttt{Assume}\ m : M,$

$()_1 := \eth\texttt{Monoid}(M)\eth\mathsf{RING}(R) : \Big((fg)h\Big)(m) = \sum_{ab=m}\sum_{cd=a} f(c)g(d)h(b) = \sum_{cdb=m} f(c)g(d)h(b) =$

$= \sum_{ab=m}\sum_{cd=b} f(a)g(c)h(d) = \big((fg)h\big)(m);$

$()_2 := \eth\texttt{Ring}(R) : f(g+h)(m) = \sum_{ab=m} f(a)\Big(g(b)+h(b)\Big) = \sum_{ab=m} f(a)g(b) + \sum_{ab=m} f(a)h(b) = \Big(fg + fh\Big)(m);$

$()_3 := \eth\texttt{Ring}(R) : (g+h)f(m) = \sum_{ab=m} \Big(g(b)+h(b)\Big)f(a) = \sum_{ab=m} g(b)f(a) + \sum_{ab=m} h(b)f(a) = \Big(g+h\Big)f(m);$

$\rightsquigarrow (1) := I(=, \to)\eth^{-1}\texttt{Associative}\eth^{-1}\texttt{Distributive} : \left[(\cdot_{R[M]}) : \texttt{Associative} \;\&\; \texttt{Distributive}\Big(R[M]\Big)\right],$

$u := \Lambda m \in M \;.\; \texttt{if}\ m == e\ \texttt{then}\ 1\ \texttt{else}\ 0 : R[M],$

$\texttt{Assume}\ f : R[M],$

$\texttt{Assume}\ m : M,$

$()_1 := \eth u : uf(m) = \sum_{ab=m} u(a)f(b) = f(m);$

$()_2 := \eth u : fu(m) = \sum_{ab=m} f(a)u(b) = f(m);$

$\rightsquigarrow (2) := I(=, \to)\eth^{-1}\texttt{Unity} : \left[u : \texttt{Unity}\Big(R[M]\Big)\right],$

$(3) := \eth^{-1}\mathsf{RING}\ R[M] : R[M] \in \mathsf{RING};$

$\square$

$\texttt{CommutativeMonoidRing} :: \forall A \in \mathsf{ANN} \;.\; \forall M : \texttt{CommutativeMonoid} \;.\; A[M] \in \mathsf{ANN}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ f, g : A[M],$

$\texttt{Assume}\ m : M,$

$() := \eth\texttt{CommutativeMonoid}(M)\eth\mathsf{ANN}(A) : fg(m) = \sum_{ab=m} f(a)g(b) = \sum_{ba=m} f(a)g(b) = \sum_{ba=m} g(b)f(a) = gf(m);$

$\rightsquigarrow (*) := \eth\eth^{-1}\mathsf{ANN} : A[M] \in \mathsf{ANN};$

$\texttt{polinomial} :: \prod R \in \mathsf{RING} \;.\; \Big(\prod n \in \mathbb{Z}_0 \;.\; n \to R\Big) \to R[\mathbb{Z}_+]$

$\texttt{polinomial}(a) = \sum_{i=0}^{n} a_i x_i := \Lambda i \in \mathbb{Z}_+ \;.\; \texttt{if}\ i \in n\ \texttt{then}\ a_i\ \texttt{else}\ 0$

$\texttt{eval} :: R[\mathbb{Z}_+] \to R \to R$

$\texttt{eval}(f, x) = f(x) := \sum_{i=0}^{n} f_i x^i$

$\texttt{degree} :: \prod R \in \mathsf{RING} \,.\, R[\mathbb{Z}_+] \to \mathbb{Z}_+ \cap \{-\infty\}$

$\texttt{degree}\,(0) = \deg 0 := -\infty$

$\texttt{degree}\,(f) = \deg f := \max\{i \in \mathbb{Z}_+ : f_i \neq 0\}$

$\texttt{DegreeHomo} :: \forall R : \texttt{IntegralDomain} \,.\, \forall f, g \in R[\mathbb{Z}_+] \,.\, \deg fg = (\deg f) + (\deg g)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (0) : f \neq 0 \neq g,$

$n := \deg f : \mathbb{Z}_+,$

$m := \deg g : \mathbb{Z}_+,$

$\texttt{Assume}\ k, l : \mathbb{Z}_+,$

$\texttt{Assume}\ (1) : k + l = m + n,$

$\texttt{Assume}\ (2) : k < m,$

$(3) := (1)(2) : m + n = k + l < m + l,$

$() := (3) - m : n < l;$

$\texttt{Assume}\ (2) : l < n,$

$(3) := (1)(2) : m + n = k + l < k + n,$

$() := (3) - m : m < k;$

$\rightsquigarrow (1) := I(\forall) : \forall k, l \in \mathbb{Z}_+ \,.\, k + l = m + n \Rightarrow (k < m \Rightarrow l > n)\ \&\ (l < n \Rightarrow k > m),$

$(2) := \eth R[\mathbb{Z}_+](1)\eth n \eth m \eth \deg \eth \texttt{IntegralDomain}(R) : (fg)_n = f_n g_m \neq 0,$

$\texttt{Assume}\ N : \mathbb{Z}_+,$

$\texttt{Assume}\ (3) : N > n + m,$

$\texttt{Assume}\ k, l : \mathbb{Z}_+,$

$\texttt{Assume}\ (4) : N = k + l,$

$\texttt{Assume}\ (5) : k \leq n,$

$(6) := (4)(5) : k + l > n + m > k + m,$

$(7) := (6) - K : l > m,$

$(8) := \eth \deg(7) : f_k g_l = 0;$

$\texttt{Assume}\ (6) : k > n,$

$(9) := \eth \deg(6) : f_k g_l = 0;$

$\rightsquigarrow (4) := I(\forall)I(\Rightarrow)E(|)\texttt{Trichtomy} : \forall k, l \in \mathbb{Z}_+ \,.\, k + l = N \Rightarrow f_k g_l = 0,$

$() := (4)\eth R[\mathbb{Z}_=] : (fg)_N = \sum_{k+l=N} f_k g_l = 0;$

$\rightsquigarrow (3) := I(\forall) : \forall N : \texttt{after}(n + m) \,.\, (fg)_N = 0,$

$(*) := \eth \deg fg(3)(2) : \deg fg = \deg g + \deg f;$

$\square$

$\texttt{IntegralPolinomials} :: \forall R : \texttt{IntegralDomain} \,.\, R[\mathbb{Z}_+] : \texttt{IntegralDomain}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f, g : R[\mathbb{Z}_+],$

$\texttt{Assume}\ (1) : f \neq 0\ \&\ g \neq 0,$

$(2) := \eth^2 \deg(f)(g)(1) : \deg f \neq -\infty\ \&\ \deg g \neq -\infty,$

$(3) := \texttt{DegreeHomo}(f, g)(2) : \deg(fg) = \deg(f) + \deg(g) \neq -\infty,$

$() := \eth \deg(fg)(3) : fg \neq 0;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{IntegralDomain} : \Big[R[\mathbb{Z}_+] : \texttt{IntegralDomain}\Big];$

$\square$

$\texttt{MultivariatePolinomials} :: \forall R \in \mathsf{RING} \,.\, \forall n \in \mathbb{N} \,.\, R[\mathbb{Z}_+^{n+1}] \cong_{\mathsf{RING}} R[\mathbb{Z}_+^n][\mathbb{Z}_+]$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MulivariatePolinomialsAreID} :: \forall R : \texttt{IntegralDomain} \,.\, \forall n \in \mathbb{N} \,.\, R[\mathbb{Z}_+^n] : \texttt{IntegralDomain}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{leadingCoefficient} :: R[\mathbb{Z}_+] \to R$

$\texttt{leadingCoefficient}\,(0) = \operatorname{lc} 0 := 0$

$\texttt{leadingCoefficient}\,(f) = \operatorname{lc} f := f_{\deg f}$

$\texttt{Monic} :: \prod R \in \mathsf{RING} \,.\, ?R[\mathbb{Z}_+]$

$f : \texttt{Monic} \iff f \neq 0 \;\&\; f_{\deg f} = 1$

$\texttt{MonicMult} :: \forall R \in \mathsf{RING} \,.\, \forall f : \texttt{Monic}(R) \,.\, \forall g \in R[\mathbb{Z}_+] \,.\, \deg fg = \deg f + \deg g$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MonicRegular} :: \forall R \in \mathsf{RING} \,.\, \forall f : \texttt{Monic}(R) \,.\, f : \texttt{Regular}\; R[\mathbb{Z}_+]$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{DivisionWithReminder} :: \forall R \in \texttt{Ring} \,.\, \forall f : \texttt{Monic}(R) \,.\, \forall g \in R[\mathbb{Z}_+] \,.$

$\quad .\, \exists s, r \in R[\mathbb{Z}_+] \,.\, g = fs + r \;\&\; \deg r < \deg f$

$\texttt{Proof} \;=$

$♂ := \Lambda N \in \mathbb{Z}_+ \,.\, \forall f : \texttt{Monic}(R) \,.\, \forall g \in R[\mathbb{Z}_+] \,.\, (0 \leq \deg f - \deg g \leq N) \Rightarrow$

$\quad \Rightarrow \exists s, r \in R[\mathbb{Z}_+] \,.\, g = fs + r \;\&\; \deg r < \deg f : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\; f : \texttt{Monic}(R),$

$\texttt{Assume}\; g : R[\mathbb{Z}_+],$

$\texttt{Assume}\; (1) : \deg f - \deg g = 0,$

$s := \operatorname{lc} g : R,$

$r := g - sf : R[\mathbb{Z}_+],$

$(2) := \eth \deg \eth r(1) : \deg r < \deg f;$

$\rightsquigarrow (1) := \eth^{-1} ♂ : ♂(0),$

$\texttt{Assume}\; N : \mathbb{Z}_+,$

$\texttt{Assume}\; (2) : ♂(N),$

$\texttt{Assume}\; f : \texttt{Monic}(R),$

$\texttt{Assume}\; g : R[\mathbb{Z}_+],$

$\texttt{Assume}\; (3) : \deg f - \deg g = N + 1,$

$a := \mathrm{lc}\, g : R,$
$g' := g - af : R,$
$(4) := \eth \deg \eth g'(2) : \deg g' - \deg f \le N,$
$(s, r, 5) := (2)(4)(f, g') : \sum r, s \in R[\mathbb{Z}_+] \,.\, g' = sf + r \;\&\; \deg r < \deg f,$
$() := \eth g(5) : g = (ax^{N+1} + s)f + r;$
$\rightsquigarrow (2) := I(\forall)I(\Rightarrow) : \forall N \in \mathbb{Z}_+ \,.\, ♂(N) \Rightarrow ♂(N+1),$
$() := \eth \mathtt{InductiveSet}(\mathbb{Z}_+)(♂) : \mathtt{This};$
$\square$

$$\mathtt{MonicQuotientStructure} :: \forall R \in \mathsf{ANN} \,.\, \forall f : \mathtt{Monic}(R) \,.\, \frac{R[\mathbb{Z}_+]}{\langle f \rangle} \cong_{\mathsf{GRP}} R^n$$

$\quad \mathtt{where} \quad n = \deg f$
$\mathtt{Proof} \;=$

$$\varphi := \Lambda a \in R^n \,.\, \left[\sum_{i=1}^{n} a_i x^{i-1}\right] : R^n \xrightarrow{\mathsf{ABEL}} \frac{R[\mathbb{Z}_+]}{\langle f \rangle},$$

$$\mathtt{Assume}\ [g] : \frac{R[\mathbb{Z}_+]}{\langle g \rangle},$$

$(r, s, 1) := \mathtt{DivisionWithReminder}(g, f) : \sum r, s \in R[\mathbb{Z}_+] \,.\, g = fs + r \;\&\; \deg r < \deg f,$
$() := (2) : \varphi(r) = [g];$

$$\rightsquigarrow (1) := \eth^{-1}\mathtt{Surjective} : \left[\varphi : R^n \twoheadrightarrow \frac{R[\mathbb{Z}_+]}{\langle f \rangle}\right],$$

$\mathtt{Assume}\ a : R^n,$
$\mathtt{Assume}\ (2) : \varphi(a) = 0,$

$$(3) := \eth\varphi(2) : \sum_{i=1}^{n} a^i x^{i-1} | f,$$

$() := \mathtt{DegreeHomo}(f)\eth\mathtt{Divides}(3) : a = 0;$

$$\rightsquigarrow (2) := \eth^{-1}\mathtt{IsoInjHomoByKer} : \left[f : R^n \xleftrightarrow{\mathsf{GRP}} \frac{R[\mathbb{Z}_+]}{\langle f \rangle}\right],$$

$$(*) := \eth\mathtt{Isomotphic}(2) : R^n \cong_{\mathsf{GRP}} \frac{R[\mathbb{Z}_+]}{\langle f \rangle};$$

$\square$

$$\mathtt{eval2} :: \prod n \in \mathbb{N} \,.\, \prod R \in \mathsf{ANN} \,.\, R^n \to R[\mathbb{Z}_+^n] \xrightarrow{\mathsf{RING}} R$$

$$\mathtt{eval2}\,(a, f) = f(a) := \sum_{m \in \mathbb{Z}_+^n} f_m \prod_{i=1}^{m} a_i^{m_i}$$

$\mathtt{Polynomial} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, ?(R^n \to R)$
$F : \mathtt{Polynomial} \iff \exists f \in R[\mathbb{Z}_+^n] \,.\, F = \Lambda a \in R^n \,.\, f(a)$

$\mathtt{EuclideanPolynomials} :: \forall k : \mathtt{Field} \,.\, k[\mathbb{Z}_+] : \mathtt{EuclideanRing}$
$\mathtt{Proof} \;=$
$\ldots$
$\square$

3.2 Hilbert Basis Theorem

$$\texttt{HilbertBasisTheorem} :: \forall A : \texttt{Noetherian} \, . \, A[\mathbb{Z}_+] : \texttt{Northerian}$$

$\texttt{Proof} \;=$

$\texttt{Assume}\; I : \texttt{Ideal}\; A[\mathbb{Z}_+],$

$J := \{\mathrm{lc}\, f | f \in I\} :?A[\mathbb{Z}_+],$

$\texttt{Assume}\; a, b : J,$

$(f, 1) := \eth J(a) : \sum f \in I \, . \, \mathrm{lc}\, f = a,$

$(g, 2) := \eth J(b) : \sum g \in I \, . \, \mathrm{lc}\, g = b,$

$l := \texttt{if}\; \deg f > \deg g \;\texttt{then}\; \deg f - \deg g \;\texttt{else}\; 0 : \mathbb{Z}_+,$

$k := \texttt{if}\; \deg g > \deg f \;\texttt{then}\; \deg f - \deg g \;\texttt{else}\; 0 : \mathbb{Z}_+,$

$(3) := \eth\texttt{Subgroup}(I) : x^k f + x^l g \in I,$

$(4) := \eth k \eth l : \deg x^k f = \deg x^l g,$

$() := \eth^{-1}(J)(3)(4) : \mathrm{lc}\left(x^k f + x^l g\right) = a + b \in J;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Subgroup}(A) : [J : \texttt{Subgroup}(A)],$

$\texttt{Assume}\; a : J,$

$\texttt{Assume}\; b : A,$

$(f, 2) := \eth J(a) : \sum f \in I \, . \, \mathrm{lc}\, f = a,$

$(3) := \eth\texttt{Ideal}(I) : bf \in I,$

$() := \eth J(2)(3)\eth\, \mathrm{lc}\, bf : \mathrm{lc}\, bf = ba \in J;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Ideal}(A)(1) : [J : \texttt{Ideal}(A)],$

$(F, 2) := \eth\texttt{Noetherian}(J) : \sum F : \texttt{Finite} \, . \, J = \texttt{genIdeal}(F),$

$(n, j) := \texttt{enum}(F) : \sum n \in \mathbb{N} \, . \, \sum j : n \xleftrightarrow{\text{SET}} F,$

$f := \Lambda k \in n \, . \, \arg\min \deg\{f \in I : \mathrm{lc}\, f = j_k\} : n \to I,$

$d := \max\limits_{k \in n} \deg f_k : \mathbb{Z}_+,$

$M := \{f \in I : \deg f < d\} : \texttt{Module}(A),$

$(m, g, 3) := \texttt{NotherianModuleTHM}(M) : \sum m \in \mathbb{N} \, . \, g : m \to M \, . \, M = \mathrm{span}(g_i)_{i=1}^m,$

$♀ := \Lambda k \in \mathbb{N} \, . \, \forall h \in I \, . \, \deg h < d + k \Rightarrow$

$$\Rightarrow \exists \alpha : n \to A[\mathbb{Z}_+] : \exists \beta : m \to A\mathbb{Z}_+ : h = \sum_{i=1}^n \alpha_i f_i + \sum_{i=1}^m \beta_i g_i : \mathbb{N} \to \texttt{Type},$$

$\texttt{Assume}\; h : I,$

$\texttt{Assume}\; (4) : \deg h < d,$

$(5) := \eth M(4) : h \in M,$

$(a, 6) := (3)\eth\, \mathrm{span} : \sum a : m \to A \, . \, h = \sum_{i=1}^m a_i g_i;$

$\rightsquigarrow (4) := I(\forall)I(\Rightarrow)I(\exists) : ♀(0),$

$\texttt{Assume}\ h : I,$

$\texttt{Assume}\ k : \mathbb{Z}_+,$

$\texttt{Assume}\ (4) : \female(k),$

$\texttt{Assume}\ (5) : \deg h = d + k,$

$k := \Lambda i \in n\ .\ d - \deg f_i : n \to \mathbb{Z}_+,$

$$(a, 5) := (2)(\operatorname{lc} h) : \sum a : n \to A : \operatorname{lc} h = \sum_{i=1}^{n} a_i \operatorname{lc} f_i,$$

$$h' := h - \sum_{i=1}^{n} a_i x^{k_i} f_i : I,$$

$(6) := (5)\eth h' : \deg h' < d + K,$

$$(\alpha, \beta, 7) := (4)(h') : \sum \alpha : n \to A[\mathbb{Z}_+]\ .\ \sum \beta : m \to A[\mathbb{Z}_+]\ .\ h' = \sum_{i=1}^{n} \alpha_i f_i + \sum_{i=1}^{m} \beta_i h_i,$$

$$() := (7)\eth h' : h = \sum_{i=1}^{n} (\alpha_i + a_i x^{k_i}) f_i + \sum_{i=1}^{m} \beta_i g_i;$$

$\rightsquigarrow (6) := I(\forall)I(\Rightarrow)\eth^{-1} : \forall k \in \mathbb{Z}_+\ .\ \female(k) \Rightarrow \female(k+1),$

$() := \eth^{-1}\texttt{FinitelyGeneratedIdeal}\eth\texttt{InductiveSet}(\mathbb{Z}_+)(\female)(6) : I : \texttt{FinitelyGeneratedIdeal}\left(A[\mathbb{Z}_+]\right);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Noetherian} : [R[\mathbb{Z}_+] : \texttt{Noetherian}];$

□

$\texttt{MultivariatePolynomialsNoetherian} :: \forall A : \texttt{Noetherian}\ .\ A[\mathbb{Z}_+^n] : \texttt{Noetherian}$

$\texttt{Proof}\ =$

...

□

3.3 Primitivity, Content and Gauss Lemma

$\texttt{PolynomialIdeal} :: \prod A \in \mathsf{ANN} \,.\, \texttt{Ideal}(A) \to \texttt{Ideal}\Big(A[\mathbb{Z}_+]\Big)$

$\texttt{PolinomialIdeal}\,(I) = IA[\mathbb{Z}_+] := \left\{\sum_{i=0} a_{i+1}x^i | a : (n+1) \to I\right\}$

$\texttt{PolynomialIdealQuotient} :: \forall A \in \mathsf{ANN} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]} \cong_{\mathsf{RING}} \frac{A}{I}[\mathbb{Z}_+]$

$\texttt{Proof} \;=$

$\varphi := \Lambda[f] \in \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]} \,.\, \sum_{i=0}^{n}[f_i]x^i : \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]} \to \frac{A}{I}[\mathbb{Z}_+],$

$\texttt{Assume}\ f : \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]},$

$\texttt{Assume}\ g : IA[\mathbb{Z}_+],$

$() := \eth\texttt{quotientRing}(A, I) : \varphi[f+g] = \sum_{i=0}[f_i+g_i]x^i = \sum_{i=0}[f_i]x^i;$

$\leadsto (2) := \texttt{WellDefined} : \left[\varphi : \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]} \xrightarrow{\mathsf{RING}} \frac{A}{I}[\mathbb{Z}_+]\right],$

$(*) := \eth\texttt{PolinomialIdeal}\eth\varphi : \left[\varphi : \frac{A[\mathbb{Z}_+]}{IA[\mathbb{Z}_+]} \xleftrightarrow{\mathsf{RING}} \frac{A}{I}[\mathbb{Z}_+]\right];$

□

$\texttt{PrimePolynomialIdeal} :: \forall A \in \mathsf{ANN} \,.\, \forall P : \texttt{Prime}(A) \,.\, PA[\mathbb{Z}_+] : \texttt{Prime}\Big(A[\mathbb{Z}_+]\Big)$

$\texttt{Proof} \;=$

$(1) := \texttt{PolynomialIdealQuotient}(A, P) : \frac{A[\mathbb{Z}_+]}{PA[\mathbb{Z}_+]} \cong_{\mathsf{RING}} \frac{A}{P}[\mathbb{Z}_+],$

$(2) := \texttt{PrimeQuotientIsID}(A, P) : \left[\frac{A}{P} : \texttt{IntegralDomain}\right],$

$(*) := \texttt{IntegralPolinomials}(2)(1)\texttt{PrimeQuotientIsID}(A[\mathbb{Z}_+], PA[\mathbb{Z}_+]) : \left[PA[\mathbb{Z}_+] : \texttt{Prime}\Big(A[\mathbb{Z}_+]\Big)\right];$

□

$\texttt{VeryPrimitive} :: \prod A \in \mathsf{ANN} \,.\, ?A[\mathbb{Z}_+]$

$f : \texttt{VeryPrimitive} \iff \forall P : \texttt{Prime} \,.\, f \notin PA[\mathbb{Z}_+]$

$\texttt{Primitive} :: \prod A \in \mathsf{ANN} \,.\, ?A[\mathbb{Z}_+]$

$f : \texttt{Primitive} \iff \forall P : \texttt{Prime} \,\&\, \texttt{Principle} \,.\, f \notin PA[\mathbb{Z}_+]$

$\texttt{PrimitivePolinimialsLemma} :: \forall A \in \mathsf{ANN} \,.\, \forall f, g \in A[\mathbb{Z}_+] \,.\, fg : \texttt{Primitive}(A) \iff f, g : \texttt{Primitive}$

$\texttt{Proof} \;=$

From properties of prime ideals

□

$\texttt{PrimitivePolinimialsLemma} :: \forall A \in \mathsf{ANN} . \forall f, g \in A[\mathbb{Z}_+] .$
$\quad . fg : \texttt{VeryPrimitive}(A) \iff f, g : \texttt{VeryPrimitive}$
$\texttt{Proof} =$
From properties of prime ideals
$\square$

$\texttt{PropertyOfVeryPrimitive} :: \forall A \in \mathsf{ANN} . \forall f \in A[\mathbb{Z}_+] . f : \texttt{VeryPrimitive}(A) \iff \texttt{genIdeal}(\operatorname{Im} f) = A$
$\texttt{Proof} =$
$\texttt{Assume}\ (1) : [f : \texttt{VeryPrimitive}(A)],$
$\texttt{Assume}\ (2) : \texttt{genIdeal}(\operatorname{Im} f) \neq A,$
$(M, 1) := \texttt{MaximalIdealExists}(\texttt{genIdeal}(\operatorname{Im} f)) : \sum M : \texttt{MaximalIdeal}(A) . \texttt{genIdeal}(\operatorname{Im} f) \subset M,$
$(2) := \texttt{MaximalPrime}(M) : [M : \texttt{Prime}(A)],$
$(3) := (1)\eth\texttt{VeryPrimitive}(f)(2)(3) : \perp;$
$\leadsto (4) := E(\perp) : \texttt{genIdeal}(\operatorname{Im}, f) = A;$
$\square$

$\texttt{PropertyOfPrimitive} :: \forall A : \texttt{UniqueFactorizationDomain} . \forall \sum_{i=0}^{n} a_i x^i \in A[\mathbb{Z}_+] .$
$\quad . \sum_{i=0}^{n} a_i x^i : \texttt{Primitive}(A) \iff \gcd(a) = 1$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{content} :: \prod A : \texttt{UniqueFactorizationDomain} . A[\mathbb{Z}_+] \to A$
$$\texttt{content}\left(\sum_{i=0}^{n} a_i x^i\right) = \operatorname{cont}\left(\sum_{i=0}^{n} a_i x^i\right) := \gcd(a)$$

$\texttt{ContentDecomposition} :: \forall A : \texttt{UniqueFactorizationDomain} . \forall f \in A[\mathbb{Z}_+] .$
$\quad . \exists \bar{f} : \texttt{Primitive}(A) : \langle f \rangle = \langle \operatorname{cont}(f) \rangle \langle \bar{f} \rangle$
$\texttt{Proof} =$
$\bar{f} := \sum_{i=0} \frac{f_i}{\operatorname{cont}(f)} x^i : A[\mathbb{Z}_+],$
$(1) := \eth^{-1}\texttt{Primitive}\eth\bar{f}\eth\operatorname{cont}(f) : [\bar{f} : \texttt{Primitive}(A)],$
$(2) := \eth\bar{f} : f = \operatorname{cont}(f)\bar{f},$
$(*) := \texttt{PrincipleProduct}(2) : \langle f \rangle = \langle \operatorname{cont}(f) \rangle \langle \bar{f} \rangle;$
$\square$

$\texttt{ContentRecomposition} :: \forall A : \texttt{UniqueFactorizationDomain} . \forall f \in A[\mathbb{Z}_+] . \forall c \in A . \forall g : \texttt{Primitive}(A) .$
$\quad . \langle c \rangle \langle g \rangle = \langle f \rangle \Rightarrow \langle c \rangle = \langle \operatorname{cont}(f) \rangle$
$\texttt{Proof} =$
$\ldots$
$\square$

$\texttt{GaussLemma} :: \forall A : \texttt{UniqueFactorizationDomain} \, . \, \forall f, g \in A[\mathbb{Z}_+] \, . \, \langle \mathrm{cont}(f,g) \rangle = \langle \mathrm{cont}(f) \, \mathrm{cont}(g) \rangle$

$\texttt{Proof} =$

$(\bar{f}, 1) := \texttt{ContentDecomposition}(f) : \sum \bar{f} : \texttt{Primitive}(A) \, . \, \langle \mathrm{cont}(f) \rangle \langle \bar{f} \rangle,$

$(\bar{g}, 1) := \texttt{ContentDecomposition}(f) : \sum \bar{g} : \texttt{Pimitive}(A) \, . \, \langle \mathrm{cont}(g) \rangle \langle \bar{g} \rangle,$

$(2) := \texttt{principleProduct}(1)(2)\texttt{principleProduct} :$

$\quad : \langle fg \rangle = \langle f \rangle \langle g \rangle = \langle \mathrm{cont}(f) \rangle \langle \bar{f} \rangle \langle \mathrm{cont}(g) \rangle \langle \bar{g} \rangle = \langle \mathrm{cont}(f) \, \mathrm{cont}(g) \rangle \langle \bar{f} \bar{g} \rangle,$

$(*) := \eth^{-1} \texttt{ContentRecomposition}(2) : \langle \mathrm{cont}(fg) \rangle = \langle \mathrm{cont}(f) \rangle \langle \mathrm{cont}(g) \rangle,$

$\square$

$\texttt{GaussLemmaCorollarly} :: \forall A : \texttt{UniqueFactorizationDomain} \, . \, \forall f, g \in A[\mathbb{Z}_+] \, . \, \forall (0) : f | g \, . \; \mathrm{cont}(f) | \, \mathrm{cont}(g)$

$\texttt{Proof} =$

$\Big(h, (1)\Big) := \eth \texttt{Divides}(f, g) : \sum h \in A[\mathbb{Z}_+] \, . \, g = fh,$

$(2) := \texttt{GaussLemma}(f, h)(1) : \mathrm{cont}(g) = \mathrm{cont}(f) \, \mathrm{cont}(h),$

$(*) := \eth^{-1} \texttt{Divides}(2) : \mathrm{cont}(f) | \, \mathrm{cont}(g);$

$\square$

3.4 Factorization Of Polynomials

$\texttt{DivisibilityInFieldsOfFractions} :: \forall A : \texttt{UniqueFactorizationDomain} . \forall f, g \in A[\mathbb{Z}] .$
$. \forall (0) : \langle \mathrm{cont}(f) \rangle_A \subset \langle \mathrm{cont}(g) \rangle . \forall (00) : \langle f \rangle_{\mathrm{Frac}(A)[\mathbb{Z}_+]} \subset \langle g \rangle_{\mathrm{Frac}(A)[\mathbb{Z}_+]} . \langle f \rangle_{A[\mathbb{Z}_+]} \subset \langle g \rangle_{A[\mathbb{Z}_+]}$

$\texttt{Proof} =$

$\Big(h, (1)\Big) := \eth\texttt{Divides}(00) : \sum h : \mathrm{Frac}(A)[\mathbb{Z}_+] . g = hf,$

$\Big(n, \frac{a}{b}, (2)\Big) := \eth\,\mathrm{Frac}(A)[\mathbb{Z}_+](h) : \sum n \in \mathbb{N} . \sum (n+1) \to \frac{a}{b} . \sum_{i=0}^{n} \frac{a_{i+1}}{b_{i+1}} x^i = h(x),$

$\Big(\tilde{h}, (3)\Big) := \eth\,\mathrm{Frac}(A)(2) : \sum \tilde{h} \in A[\mathbb{Z}_n] . h = \frac{\tilde{h}}{\mathrm{lcd}(b)},$

$\Big(\bar{h}, (4)\Big) := \texttt{ContentDecomposition}(\bar{h})(3) : \sum \bar{h} : \texttt{Primitive}(A) . h = \frac{\mathrm{cont}(\tilde{h})\bar{h}}{\mathrm{lcd}(b)},$

$(5) := \eth\,\mathrm{Frac}(A)[\mathbb{Z}_+](4)(1) : \mathrm{lcd}(b)g = \mathrm{cont}(\tilde{h})\bar{h}f,$

$(6) := \texttt{GaussLemma}(\mathrm{Frac}(A)[\mathbb{Z}_+])(5) : \mathrm{cont}(\mathrm{lcd}(b)g) = \mathrm{cont}(\tilde{h})\,\mathrm{cont}(f),$

$(7) := \eth\texttt{Divides}(00)(6) : \mathrm{cont}(\tilde{h})\,\mathrm{cont}(f) | \,\mathrm{lcd}(b)\,\mathrm{cont}(f),$

$(8) := \texttt{DivisibleProduct}(7) : \mathrm{cont}(\tilde{h}) | \,\mathrm{lcd}(b),$

$(9) := \eth\,\mathrm{Frac}(A)(3)(2)(8) : h \in A[\mathbb{Z}_+],$

$(*) := (1)(9) : \langle f \rangle_{A[\mathbb{Z}_+]} \subset \langle g \rangle_{A[\mathbb{Z}_+]};$

$\square$

$\texttt{IrreducibilityInTheFieldOfFractions} :: \forall A : \texttt{UniqueFactorizationDomain} .$
$. \forall f : \texttt{IrreducibleElement}\ A[\mathbb{Z}_+] . \forall (0) : \deg f > 0 . f : \texttt{IrreducibleElement}\ \mathrm{Frac}(A)[\mathbb{Z}_+]$

$\texttt{Proof} =$

$\texttt{Assume}\ (1) : [f\ !\ \texttt{IrreducibleElement}\ \mathrm{Frac}(A)[\mathbb{Z}_+]],$

$\Big(g, h, (2)\Big) := \eth\texttt{IrreducibleElement}(f) : \sum g, h : \mathrm{Frac}(A)^{\times}[\mathbb{Z}_+] \setminus \mathrm{Frac}(A)^{*}[\mathbb{Z}_+] . f = gh,$

$\Big(n, \frac{a}{b}, (3)\Big) := \eth\,\mathrm{Frac}(A)\mathbb{Z}_+ : \sum n \in \mathbb{N} . \sum \frac{a}{b} : (n+1) \to \mathrm{Frac}(A) . g = \sum_{i=0}^{n} \frac{a_{i+1}}{b_{i+1}} x^i,$

$\Big(m, \frac{c}{d}, (4)\Big) := \eth\,\mathrm{Frac}(A)\mathbb{Z}_+ : \sum m \in \mathbb{N} . \sum \frac{c}{d} : (m+1) \to \mathrm{Frac}(A) . h = \sum_{i=0}^{m} \frac{c_{i+1}}{d_{i+1}} x^i,$

$\Big(\tilde{g}, (4)\Big) := \eth\,\mathrm{Frac}(A)[\mathbb{Z}_+](3) : \sum \tilde{g} : A[\mathbb{Z}_+] . g = \frac{\tilde{g}}{\mathrm{lcd}(b)},$

$\Big(\tilde{h}, (5)\Big) := \eth\,\mathrm{Frac}(A)[\mathbb{Z}_+](4) : \sum \tilde{h} : A[\mathbb{Z}_+] . h = \frac{\tilde{h}}{\mathrm{lcd}(d)},$

$\Big(\bar{g}, (6)\Big) := \texttt{ContDecomposition}(\tilde{g}) : \sum \bar{g} : \texttt{Primitive}(A) .\ \mathrm{cont}(\tilde{g})\bar{g},$

$(7) := \eth\texttt{IrreducibleElement}(A)(f)\eth^{-1}\mathrm{cont}\,\eth^{-1}\texttt{Primitive} : \mathrm{cont}(f) = 1 = \mathrm{cont}(\bar{g}),$

$(8) := \eth A[\mathbb{Z}_+](6)(5) : f = \bar{g}\left(\frac{\mathrm{cont}(\tilde{g})\tilde{(h)}}{\mathrm{lcd}(b)\,\mathrm{lcd}(d)}\right),$

$(9) := \texttt{DivisibilityInFieldsOfFractions}(8) : (f)_{A[\mathbb{Z}_+]} \subset (\bar{g})_{A\mathbb{Z}_+},$

$() := (9)(2)\eth\texttt{IrreducibleElement}(A)(f) : \bot;$

$\rightsquigarrow (*) := E(\bot) : [f : \texttt{IrreducibleElement}\ \mathrm{Frac}(A)[\mathbb{Z}_+]],$

$\square$

$\texttt{IrreducibilityInTheFieldOfFractions2} :: \forall A : \texttt{UniqueFactorizationDomain} \,.\, \forall f \in A[\mathbb{Z}_+] \,.$

$\forall(0) : \deg f > 0 \,.\, f : \texttt{IrreducibleElement}\ \mathrm{Frac}(A)[\mathbb{Z}_+] \iff f : \texttt{IrreducibleElement}\ A[\mathbb{Z}_+]$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{IrreduciblePolynomialsArePrime} :: \forall A : \texttt{UniqueFactorizationDomain} \,.$

$.\, \forall f : \texttt{IrreducibleElement}\Big(A[\mathbb{Z}_+]\Big) \,.\, f : \texttt{PrimeElement}\Big(A[\mathbb{Z}_+]\Big)$

$\texttt{Proof} \;=$

$(1) := \texttt{EuclideanPolynomials}(\mathrm{Frac}(A))\texttt{ERIdPID}\,\texttt{PIDIsUFD} : \Big[\mathrm{Frac}(A)[\mathbb{Z}_+] : \texttt{UniqueFactorizationDomain}\Big],$

$\texttt{Assume}\ (2) : \deg f > 0,$

$(3) := \texttt{IrreducibilityInTheFieldOfFractions}((0), f) : [f : \texttt{IrreducibileElement}\Big(\mathrm{Frac}(A)\Big)],$

$(4) := \texttt{IrreducibleIsPrimeInUFD}((3), f) : [f : \texttt{PrimeElement}\Big(\mathrm{Frac}(A)\Big)],$

$(5) := \eth^{-1}\,\mathrm{cont}(f)\eth\texttt{IrreducibleElement}\Big(A[\mathbb{Z}_+]\Big)(f) : \mathrm{cont}(f) = 1,$

$\texttt{Assume}\ x, y : A[\mathbb{Z}_+],$

$\texttt{Assume}\ (6) : (f|xy)_{A[\mathbb{Z}_+]},$

$(7) := \eth\,\mathrm{Frac}(A)(6) : (f|xy)_{\mathrm{Frac}(A)[\mathbb{Z}_+]},$

$(8) := \eth\texttt{PrimeElement}(7) : (f|x)_{\mathrm{Frac}(A)[\mathbb{Z}_+]}\Big|(f|y)_{\mathrm{Frac}(A)[\mathbb{Z}_+]},$

$() := \texttt{DivisibilityInFieldsOfFractions}(5)(8) : (f|x)_{A[\mathbb{Z}_+]}|(f|y)_{A[\mathbb{Z}_+]};$

$\rightsquigarrow (6) := \eth^{-1}\texttt{PrimeElement} : f : \texttt{PrimeElement}\ A[\mathbb{Z}_+];$

$\rightsquigarrow (2) := I(\Rightarrow) : \deg f > 0 \Rightarrow f : \texttt{PrimeElement}\ A[\mathbb{Z}_+],$

$\texttt{Assume}\ (3) : \deg f = 0,$

$(4) := \eth\,\mathrm{cont}(f)(3) : f = \mathrm{cont}(f),$

$\texttt{Assume}\ x, y : A[\mathbb{Z}_+],$

$\texttt{Assume}\ (5) : (f|xy),$

$(6) := \texttt{GaussLemmaCorollarly}(5)(4) : (f|\,\mathrm{cont}(xy))_A,$

$(7) := \texttt{GaussLemma} : (f|\,\mathrm{cont}(x)\,\mathrm{cont}(y)),$

$(8) := \texttt{IrreducibleIsPrimeInUFD}(7) : f|\,\mathrm{cont}(x)\Big|f|\,\mathrm{cont}(y),$

$() := \eth\,\mathrm{cont}(8)(3) : f|x\Big|f|y;$

$\rightsquigarrow (*) := \eth\deg f E(|)I(\Rightarrow)\eth^{-1}\texttt{PrimeElement}\ A[\mathbb{Z}_+](2) : [f : \texttt{PrimeElement}(A)];$

$\square$

$\texttt{PolynomialsUFD} :: \forall A : \texttt{UniqueFactorizationDomain} \,.\, A[\mathbb{Z}_+] : \texttt{UniqueFactorizationDomain}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \mathbb{N} \to A[\mathbb{Z}_+],$

$\texttt{Assume}\ (1) : \langle f \rangle_{A[\mathbb{Z}_+]} : \texttt{Nondescending}(A),$

$(2) := \texttt{GaussLemmaCorollarly}(1) : [\langle \mathrm{cont}\, f \rangle_A : \texttt{Nondescending}(A)],$

$(N, 3) := \texttt{ACCByFactorization}(A)(2) : \sum N \in \mathbb{N} \,.\, \forall n \in \texttt{after}(N) \,.$

$\quad .\ \langle \mathrm{cont}(f_N) \rangle_A = \langle \mathrm{cont}(f_n) \rangle_A,$

$(M, 4) := \texttt{ACCByFactorization}\Big(\mathrm{Frac}(A)[\mathbb{Z}_+]\Big)(2) : \sum M \in \mathbb{N} \,.\, \forall n \in \texttt{after}(M) \,.$

$\quad .\ \langle f_n \rangle_{\mathrm{Frac}(A)[\mathbb{Z}_+]} = \langle f_M \rangle_{\mathrm{Frac}(A)[\mathbb{Z}_+]},$

$() := \eth\texttt{DivisibilityInFieldsOfFractions}(4.3) : \forall n \in \texttt{after}\Big(\max(M, N)\Big) \,.\, \langle f_n \rangle_{A[\mathbb{Z}_+]} = \langle f_N \rangle_{A[\mathbb{Z}_+]};$

$\rightsquigarrow (*) := \eth^{-1}\texttt{UniqueFactorizationDomain}\Big(\texttt{IrreduciblePolynomialsArePrime}\Big) :$

$\quad : \Big[A[\mathbb{Z}_+] : \texttt{UniqueFactorizationDomain}\Big];$

$\square$

$\texttt{MultivariatePolynomialsUFD} :: \forall A : \texttt{UniqueFactorizationDomain} \,.\, \forall n \in \mathbb{N} \,.$

$\quad .\ A[\mathbb{Z}_+^n] : \texttt{UniqueFactorizationDomain}$

$\texttt{Proof} \ =$

$\dots$

$\square$

3.5 Roots And Irreducibility Criterions

$\texttt{RootDivides} :: \forall A : \texttt{IntegralDomain} \,.\, \forall f \in A[\mathbb{Z}_+] \,.\, \forall a \in A \,.\, f(a) = 0 \Rightarrow (x-a)|f$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (0) : f(a) = 0,$

$(1) := \eth^{-1} : \Big[(x-a) : \texttt{Monic}[\mathbb{Z}_+]\Big],$

$(s, r, (2)) := \texttt{DivisionWithReminder}(f, x-a) : \sum s \in A[\mathbb{Z}_+] \,.\, \sum r \in A[\mathbb{Z}_+] : f = s(x-a) + r,$

$(3) := \eth^{-1}\texttt{eval}(2) : f(a) = r,$

$(4) := (0)(3) : r = 0,$

$() := (4)(2) : f = s(x-a);$

$\rightsquigarrow (1) := I(\Rightarrow) : f(a) = 0 \Rightarrow (x-a)|f,$

$\texttt{Assume}\ (0) : (x-a)|f,$

$(s, (2)) := \eth\texttt{Divides}(0) : \sum s \in A[\mathbb{Z}_+] \,.\, f = s(x-a),$

$() := \eth\texttt{eval}(a, f)(2) : f(a) = 0;$

$\rightsquigarrow (*) := I(\iff) : f(a) = 0 \iff (x-a)|f;$

$\square$

$\texttt{roots} :: \prod A \in \mathsf{ANN} \,.\, A[\mathbb{Z}_+] \to ?A$

$\texttt{roots}\,(f) = \rho(f) := \{a \in A : f(a) = 0\}$

$\texttt{multiplicity} :: \prod A : \texttt{IntegralDomain} \,.\, \prod f \in A[\mathbb{Z}_+] \,.\, \rho(f) \to \mathbb{N} \cup \{+\infty\}$

$\texttt{multiplicity}\,(a) = m_f(a) := \max\Big\{m \in \mathbb{N} : \Big((x-a)^m|f\Big)\Big\}$

$\texttt{ZeroPolynomialTHM} :: \prod A : \texttt{IntegralDomain} \,.\, \forall (0) : |A| = \infty \,.$

$\quad .\, \forall f \in A[\mathbb{Z}] \,.\, \Lambda a \in A \,.\, f(a) = 0 \iff f = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : \Lambda a \in A \,.\, f(a) = 0,$

$\texttt{Assume}\ (2) : f \neq 0,$

$\texttt{Assume}\ n : \mathbb{N},$

$() := \texttt{RootDivides}(1)(2)\eth\deg(n) : \deg f > n;$

$(2) := I(\forall) : \forall n \in \mathbb{N} \,.\, \deg f > n,$

$() := \eth\deg(2) : \bot;$

$\rightsquigarrow (2) := E(\bot) : f = 0,$

$\ldots$

$\square$

$\texttt{polyMap} :: \prod A, B \in \mathsf{ANN} \, . \, (A \xrightarrow{\mathsf{ANN}} B) \to \left(A[\mathbb{Z}_+] \xrightarrow{\mathsf{ANN}} B[\mathbb{Z}_+]\right)$

$\texttt{polyMap}\,(\varphi, f) = \varphi[f] := \Lambda n \in \mathbb{Z}_+ \, . \, \varphi(f_n)$

$\texttt{IrreduciblePolynomial} :: \prod A : \texttt{IntegralDomain} \, . \, ?A[\mathbb{Z}_+]$

$f : \texttt{IrreduciblePlynomial} \iff \exists g, h \in A[\mathbb{Z}] \, . \, f = gh \;\&\; \deg g, \deg h \in \mathbb{Z}$

$\texttt{EisensteinsCriterion} :: \forall A : \texttt{IntegralDomain} \, . \, \forall n \in \mathbb{N} \, . \, \forall \sum_{i=0}^{n} a^i x^i \in A[\mathbb{Z}_+] \, . \, \forall P : \texttt{Prime}(A) \, .$

$\quad . \, \forall (0) : a_n \notin P \, . \, \forall (00) : \forall i \in (n-1) \, . \, a^i \in P \, . \, \forall (000) : a_0 \notin P^2 \, .$

$\quad . \, \sum_{i=0}^{n} a^i x^i : \texttt{IrreducibleElement}\,(A[\mathbb{Z}_+])$

$\texttt{Proof} \; =$

$f := \sum_{i=0}^{n} a^i x^i : A[\mathbb{Z}_+],$

$\texttt{Assume}\;(1) : [f \,!\, \texttt{IrreduciblePolynomial}\,(A[\mathbb{Z}_+])],$

$(h, g, 2) := \eth\texttt{IrreduciblePolynomial}(1)(f) : \sum h, g \in A^\times[\mathbb{Z}_+] \setminus A^*[\mathbb{Z}_+] \, . \, f = hg,$

$(m, b, 3) := \eth A[\mathbb{Z}_+](h) : \sum m \in \mathbb{N} \, . \, \sum b : m \to A \, . \, h = \sum_{i=0}^{m} b_i x^i,$

$(l, c, 4) := \eth A[\mathbb{Z}_+](g) : \sum l \in \mathbb{N} \, . \, \sum c : l \to A \, . \, g = \sum_{i=0}^{l} c_i x^i,$

$(5) := \eth\texttt{Prime}(P)(3)(4)\eth A[\mathbb{Z}_+](000) : b_0 \notin P | c_0 \notin P,$

$(6) := \eth\texttt{polyMap}(0)(00) : \pi_P[f] = [a_n]x^n,$

$(7) := \eth\texttt{Ideal}(P)(3)(4) : b_m, c_l \notin P,$

$(8) := (6)(7) : \pi_P[h] = [b_m]x^m \;\&\; \pi_P[g] = [c_l]x^l,$

$(9) := \eth\texttt{QuotienRing}(5) : [b_0] \neq 0 | [c_0] \neq 0,$

$() := (8)(9) : \bot;$

$\rightsquigarrow () := E(\bot) : [f : \texttt{IrreduciblePolynomial}(A)],$

$\square$

$\texttt{ReductionCriterion} :: \forall A, B \in \texttt{IntegralDomain} \, . \, \forall \varphi : A \xrightarrow{\mathsf{RING}} B \, . \, \forall f \in A[\mathbb{Z}_+] \, .$

$\quad . \, \forall (0) : \deg f > 0 \, . \, \forall (00) : \deg g = \deg \varphi[f] \, . \, \forall (000) : \varphi[f] : \texttt{IrreduciblePolynomial}\;\mathrm{Frac}(B) \, .$

$\quad . \, f : \texttt{IrreduciblePolynomial}(A)$

$\texttt{Proof} \; =$

$\texttt{Assume}\;(1) : [f \,!\, \texttt{IrreduciblePolynomial}\,(A[\mathbb{Z}_+])],$

$(h, g, 2) := \eth\texttt{IrreduciblePolynomial}(1)(f) : \sum h, g \in A[\mathbb{Z}_+] \, . \, f = hg \;\&\; \deg h, g \in \mathbb{N},$

$(3) := \eth \deg(0)(00)(2) : \deg h = \deg \varphi h \;\&\; \deg g = \deg \varphi h,$

$() := (000)(3) : \bot;$

$\rightsquigarrow () := E(\bot) : [f : \texttt{IrreduciblePolynomial}(A)],$

$\square$

3.6 Algebra of Formal Power Serias

$\texttt{MonoidOfFiniteType} :: ?\texttt{Monoid}$

$$M : \texttt{MonoidOfFiniteType} \iff \forall m \in M \,.\, \left|(\cdot_M)^{-1}\{m\}\right| < \infty$$

$\texttt{formalPowerSeriesAlgebra} :: \texttt{MonoidOfFiniteType} \times \mathsf{RING} \to \mathsf{RING}$

$$\texttt{formalPowerSeriesAlgebra}\,(R, M) = R\Big[[M]\Big] := \left(M \to R, +_{M\to R} \,.\, \Lambda a, b : M \to R \,.\, \Lambda m \in M \,.\, \sum_{kl=m} a_k b_l\right)$$

$\texttt{formalPowerSeria} :: \prod M : \texttt{MonoidOfFiniteType} \,.\, \prod R \in \mathsf{RING} \,.\, (M \to R) \to R\Big[[M]\Big]$

$$\texttt{formalPowerSeria}\,(a) = \sum_{i\in M} a_i x^i := a$$

$\texttt{PositiveIntegersAreFiniteType} :: \mathbb{Z}_+ : \texttt{MonoidOfFiniteType}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ m : \mathbb{Z}_+,$

$\texttt{Assume}\ a, b : \mathbb{Z}_+,$

$\texttt{Assume}\ (1) : m = a + b,$

$(2) := \texttt{NondecreasingAddition}(1) : a \le m \ \&\ b \le m,$

$() := \eth^{-1}\texttt{prim}(m) : a, b \in \texttt{prim}(\mathbb{Z}_+)(m);$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall a, b \in \mathbb{Z}_+ \,.\, a + b = m \Rightarrow a, b \in \texttt{prim}(\mathbb{Z}_+)(m),$

$(2) := \eth^{-1}\texttt{preimage}(+)(m)(1) : (+)^{-1}(m) \subset \texttt{prim}^2(m),$

$(3) := \texttt{SubsetCardinality}(2)\ \texttt{FiniteProductCard}\ \texttt{PrimitiveSubsetCardinality}(\mathbb{Z}_+)(m) :$

$\quad : \left|(+)^{-1}_{\mathbb{Z}_+}(m)\right| \le \left|m_{\mathbb{Z}_+}\right|^2 = m^2 + 2m + 1 < \infty;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{MonoidOfFiniteType} : [\mathbb{Z} : \texttt{MonoidOfFiniteType}];$

$\texttt{PositiveLatticeIsFiniteType} :: \forall n \in \mathbb{N} \,.\, \mathbb{Z}_+^n : \texttt{MonoidOfFiniteType}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ m : \mathbb{Z}_+^n,$

$$(1) := \eth\mathbb{Z}_+^n\Big((+)^{-1}(m)\Big) : (+)^{-1}(m) = \prod_{i=1}^{n} (+)^{-1}(m_i),$$

$() := \texttt{ProductCard}(1)\forall i \in n \,.\, \eth\texttt{MonoidOfFiniteType}(\mathbb{Z}_+)(m_I) : \left|(+)^{-1}(m)\right| \le \infty;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{MonoidOfFiniteType} : [\mathbb{Z}_+^n : \texttt{MonoidOfFiniteType}];$

$\square$

$\texttt{Topological} :: \prod A \in \mathsf{ANN} \,.\, ?\texttt{Ideal}(A)$

$$I : \texttt{Topological} \iff \bigcap_{n=1}^{\infty} I^n = \{0\}$$

$\texttt{iadicTopology} :: \prod A \in \mathsf{ANN} \,.\, \texttt{Topological}(A) \to \texttt{Topology}(A)$

$\texttt{iadicTopology}\,(I) = \tau_A(I) := \texttt{genTop}\{a + I^n | n \in \mathbb{Z}_+, a \in A\}$

$\texttt{Cauchy} :: \prod A \in \mathsf{ANN} \,.\, \prod I : \texttt{Ideal}(A) \,.\, ?(\mathbb{N} \to A)$

$a : \texttt{Cauchy} \iff \forall n \in \mathbb{N} \,.\, \exists M \in Nat \,.\, \forall m, m' : \texttt{after}(M) \,.\, a_m - a_{m'} \in I^n$

$\texttt{CompleteLocal} :: ?\texttt{Local}$

$A : \texttt{CompleteLocal} \iff \mathfrak{m}(A) : \texttt{Toplogical} \;\&\; \forall a : \texttt{Cauchy}(A, \mathfrak{m}(A)) \,.\, a : \texttt{Convergent}\Big(A, \tau_A\big(\mathfrak{m}(A)\big)\Big)$

$\texttt{degree} :: \prod A \in \mathsf{ANN} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to \mathbb{Z}_+ \cup \{-\infty, +\infty\}$

$\texttt{degree}\,(a) = \deg a := \max\{i \in \mathbb{Z}_+ : a_i \neq 0\}$

$\texttt{degreeOfWeierstrass} :: \prod A : \texttt{Local} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to \mathbb{Z}_+ \cup \{+\infty\}$

$\texttt{degreeOfWeierstrass}\,(a) = \deg_W a := \min\{i \in \mathbb{Z}_+ : a_i \notin \mathfrak{m}(A)\}$

$\texttt{tail} :: \prod A : \texttt{Local} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to A\Big[[\mathbb{Z}_+]\Big]$

$$\texttt{tail}\,(a) = t(a) := \texttt{if}\ \deg_W a = +\infty\ \texttt{then}\ 0\ \texttt{else} \sum_{i=\deg_W a} a_i x^{i-n}$$

$\texttt{head} :: \prod A : \texttt{Local} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to \mathfrak{m}(A)[\mathbb{Z}_+]$

$$\texttt{head}\,(a) = h(a) := \texttt{if}\ \deg_W a = +\infty\ \texttt{then}\ a\ \texttt{else} \sum_{i=0}^{\deg_W a - 1} a_i x^i$$

$\texttt{tail2} :: \prod A \in \mathsf{RING} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to \mathbb{N} \to A\Big[[\mathbb{Z}_+]\Big]$

$$\texttt{tail}\,(a) = t_n(a) := \sum_{i=n}^{\infty} a_i x^{i-n}$$

$\texttt{head2} :: \prod A \in \mathsf{RING} \,.\, A\Big[[\mathbb{Z}_+]\Big] \to \mathbb{N} \to A[\mathbb{Z}_+]$

$$\texttt{head}\,(a) = h_n(a) := \sum_{i=1}^{n-1} a_i x^i$$

$\texttt{HeadTailDecomposition} :: \forall A \in \mathsf{RING} \,.\, \forall a \in A\Big[[\mathbb{Z}_+]\Big] \,.\, \forall n \in \mathbb{N} \,.\, a = h_n(a) + t_n(a)$

$\texttt{Proof} =$

...

$\square$

$\texttt{CommutativePowerSeries} :: \forall A \in \mathsf{ANN} \,.\, \forall M : \texttt{CommutativeMonoid} \,\&\, \texttt{MonoidOfFiniteType} \,.$

$. \, A\Big[[M]\Big] \in \mathsf{ANN}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{antidegree} :: \prod A \in \mathsf{RING} \,.\, A\Big[[M]\Big] \to \mathbb{Z}_+ \cup +\infty$

$\texttt{antidegree}\,(a) = \operatorname{antideg} a := \min i \in \mathbb{Z}_+ : a_i \neq 0$

$\texttt{antidegHomo} :: \forall A \in \mathsf{RING} \,.\, \forall f, g \in A\Big[[\mathbb{Z}_+]\Big] \,.\, \operatorname{antideg} fg \geq \operatorname{antideg} f + \operatorname{antideg} g$

$\texttt{Proof} \ =$

$\texttt{Assume}\ n : \operatorname{antideg} f + \operatorname{antideg} g,$

$\texttt{Assume}\ k, l : \mathbb{Z}_+,$

$\texttt{Assume}\ (1) : k + l = n - 1,$

$\texttt{Assume}\ (2) : k \geq \operatorname{antideg} f \ \&\ l \geq \operatorname{antideg} g,$

$(3) := \texttt{AddIneq}(2)\eth n\texttt{NextIsGreater}(n-1) : k + l \geq \operatorname{antideg} f + \operatorname{antideg} g \geq n > n - 1,$

$(4) := \eth\texttt{StrictlyLess}(3)(1) : \bot;$

$\rightsquigarrow (2) := E(\bot) : k < \operatorname{antideg} f | l < \operatorname{antideg} g,$

$() := \eth \operatorname{antideg}(2)\texttt{ZeroMult}(A) : f_l g_k = 0;$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall l, k \in \mathbb{Z}_+ \,.\, l + k = n \Rightarrow f_l g_k = 0,$

$() := \eth A\Big[[\mathbb{Z}_+]\Big] : (fg)_n = 0;$

$\rightsquigarrow (1) :=: \forall n \in \operatorname{antideg} f + \operatorname{antideg} g \,.\, (fg)_n = 0,$

$(*) := \eth^{-1}(1) : \operatorname{antideg} fg \geq \operatorname{antideg} f + \operatorname{antideg} g;$

$\square$

$\texttt{ZeroType} :: \prod A \in \mathsf{RING} \,.\, ?A\Big[[\mathbb{Z}_+]\Big]$

$f : \texttt{ZeroType} \iff f_0 = 0$

$\texttt{powerSeriaOfPowerSeria} :: \prod A \in \mathsf{RING} \,.\, \texttt{ZeroType}(A) \to A\Big[[\mathbb{Z}_+]\Big]$

$$\texttt{powerSeriaOfPowerSeria}\,(f) = \sum_{k=0}^{\infty} f^k := \Lambda n \in \mathbb{Z}_+ \,.\, \sum_{i=0}^{n} (f^i)_n$$

$\texttt{InveritiblePowerSeria} :: \prod A \in \mathsf{ANN} \,.\, \forall f \in A\Big[[\mathbb{Z}_+]\Big] \,.\, \forall (0) : f_0 \in A^* \,.\, f \in \left(A\Big[[\mathbb{Z}_+]\Big]\right)^*$

$\texttt{Proof} \ =$

$\Big(g, (1)\Big) := \eth^{-1}\texttt{ZeroType}(f) : \sum g : \texttt{ZeroType}(A) \,.\, f = f_0 + g,$

$$u := f_0^{-1} \sum_{k=0}^{\infty} \Big(-f_0^{-1} g\Big)^k : A\Big[[\mathbb{Z}_+]\Big],$$

$$(*) := \eth u \eth A\Big[[\mathbb{Z}_+]\Big]\eth\texttt{powerSeriaOfPowerSeria} : uf = 1 + \sum_{i=1}^{\infty} f_0^{-i} g^i - f_0^{-i} g^i = 1;$$

$\square$

$\texttt{ManinDivision} :: \forall A : \texttt{CompleteLocal} \,.\, \forall f, g \in A[[\mathbb{Z}_+]] \,.\, \forall (0) : \deg_W f < \infty \,.$

$\quad . \,\exists! q, r \in A[[\mathbb{Z}_+]] : \deg r < \deg_W f \;\&\; g = qf + r$

$\texttt{Proof} \;=$

$n := \deg_W f : \mathbb{N},$

$(1) := \eth \deg_W \texttt{HeadTailDecomposition}(n, f) : f = t_n(f) + h_n(f) = t(f)x^n + h(f),$

$(2) := \eth h_n(g) \eth n : \deg h_n(g) < \deg_W(f),$

$(3) := \texttt{InvertiblePowerSeria}(A)(t(f))\texttt{InvertibleInLocal}(A)(f_n) : \Big[t(f) : \texttt{Invertible}\ A\big[\mathbb{Z}_+\big]\Big],$

$\texttt{Assume}\ q : A\Big[[\mathbb{Z}_+]\Big],$

$\texttt{Assume}\ (4) : t_n(g) = t_n(qf),$

$(5) := (4)(1) : t_n(g) = t_n\Big(qt(f)x^n\Big) + t_n\Big(qh(f)\Big),$

$(6) := \eth t_n(5) : t_n(g) = qt(f) + t_n\Big(qh(f)\Big),$

$Z := qt(f) : A\Big[[\mathbb{Z}]\Big],$

$(7) := \eth^{-1} Z(6) : t_n(g) = Z + t_n\left(Z\dfrac{h(f)}{t(f)}\right),$

$(9) := \eth^{-1}\left(A\Big[[\mathbb{Z}_+]\Big] \to A\Big[[\mathbb{Z}]_+\Big]\right) : t_n(g) = \left(E + t_n \circ \mu\dfrac{h(f)}{t(f)}E\right) Z,$

$T := t_n \circ \mu\left(\dfrac{h(f)}{t(f)}\right) : A\Big[[\mathbb{Z}_+]\Big] \xrightarrow{A\text{-Mod}} A\Big[[\mathbb{Z}_+]\Big],$

$S := \Lambda m \in \mathbb{N} \,.\, \displaystyle\sum_{i=0}^{m}(-T)^i : \mathbb{N} \to A\Big[[\mathbb{Z}_+]\Big] \xrightarrow{A\text{-Mod}} A\Big[[\mathbb{Z}_+]\Big],$

$\texttt{Assume}\ x : A\Big[[\mathbb{Z}_+]\Big],$

$\texttt{Assume}\ p : \mathbb{N},$

$\texttt{Assume}\ k, l : \texttt{after}(m),$

$() := \eth h(f) \eth S_k \eth S_l \eth T : S_k x - S_l x \in \mathfrak{m}^p(A)\Big[[\mathbb{Z}_+]\Big];$

$\leadsto (10) := \eth^{-1}\texttt{Cauchy}\left(A\Big[[\mathbb{Z}_+]\Big], \mathfrak{m}(A)\Big[[\mathbb{Z}_+]\Big]\right) : \left[Sx : \texttt{Cauchy}\left(A\Big[[\mathbb{Z}_+]\Big], \mathfrak{m}(A)\Big[[\mathbb{Z}_+]\Big]\right)\right],$

$(V(x), 11) := \eth^{-1}\texttt{Complete}(A) : \sum V(x) \in A\Big[[\mathbb{Z}_+]\Big] \,.\, V(x) = \lim_{n\to\infty} S_n x;$

$\leadsto (V, 11) := I(\to) : \sum V : A\Big[[\mathbb{Z}_+]\Big] \xrightarrow{A\text{-Mod}} A\Big[[\mathbb{Z}_+]\Big] \,.\, V = \lim_{n\to\infty} S_n,$

$(12) := \eth S \eth V : V = \left(E + t_n \circ \dfrac{h(f)}{t(f)}E\right)^{-1},$

$(13) := (12)\left(E + t_n \circ \dfrac{h(f)}{t(f)}E\right)^{-1} : Z = \left(E + t_n \circ \dfrac{h(f)}{t(f)}E\right)^{-1} t_n(g),$

$() := \eth Z(11) : q = \dfrac{t_n(g)\left(1 - t_n \circ \frac{h(f)}{t(f)}\right)^{-1}}{t(f)};$

$\leadsto (4) := I(\forall)I(\iff) : \forall q \in A\Big[[\mathbb{Z}_+]\Big] \,.\, t_n(g) = t_n(qf) \iff q = \dfrac{\left(E + t_n \circ \frac{h(f)}{t(f)}E\right)^{-1} t_n(g)}{t(f)},$

$(*) := (4)(2)\eth q \eth r : \texttt{This};$

□

$\texttt{WeierstrassPreparation} :: \forall A : \texttt{CompleteLocal} . \forall f \in A\Big[[\mathbb{Z}_+]\Big] . \forall (0) : \deg_W f < \infty .$

$\exists ! p : \texttt{Monic}\ \mathfrak{m}(A) : \exists ! u \in \left(A\Big[[\mathbb{Z}_+]\Big]\right) : f = pu$

$\texttt{Proof} \ =$

$n := \deg_W f : \mathbb{Z}_+,$

$\Big(q, r, (1)\Big) := \texttt{ManinDividion}(A, f, x^n) : \sum q \in A\Big[[\mathbb{Z}_+]\Big] . \sum r \in A[\mathbb{Z}_+] . \deg r < \deg_W f \ \&\ x^n = fq + r,$

$(2) := \eth A\Big[[\mathbb{Z}_+]\Big](1) : 1 = \sum_{i=0}^{n} f_{n-i} q_i = f_n q_0 + \sum_{i=1}^{n} f_{n-i} q_i,$

$(3) := \eth \texttt{Ideal}\Big(\mathfrak{m}(A)\Big) \eth \deg_W f : \sum_{i=1}^{n} f_{n-i} q_i \in \mathfrak{m}(A),$

$(4) := \texttt{LocalInvertivle} \eth \texttt{maximalIdeal}\Big((m)(AA)\Big) : f_n q_0 \in A^*,$

$(5) := \eth A^*(4) : q_0 \in A^*,$

$(6) := \texttt{InvertiblePowerSeria}(5) : q \in \left(A\Big[[\mathbb{Z}_+]\Big]\right)^*,$

$(*) := \Big((1) - r\Big) q^{-1} : (x^n + r) q^{-1} = f;$

$\square$

$\texttt{MultivariatePowerSerias} :: \forall A \in \mathsf{RING} . \forall n \in \mathbb{N} . A\Big[[\mathbb{Z}_+^{n+1}]\Big] \cong_{\mathsf{RING}} A[\mathbb{Z}_+^n][\mathbb{Z}_+]$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{NoetherianPowerSerias} :: \forall A : \texttt{Noetherian} . A\Big[[\mathbb{Z}_+]\Big] : \texttt{Noetherian}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{MultivariateNoetherianPowerSerias} :: \forall A : \texttt{Noetherian} . A\Big[[\mathbb{Z}_+^n]\Big] : \texttt{Noetherian}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

4 Categorical Ring Theory[!!]

4.1 RNG and Adjoining of Unity

4.2 Limits in RNG, RING and ANN

4.3 Adjoints of Forgetful Functors