

Metric Topology

Uncultured Tramp

May 28, 2022

Contents

1	Metric Spaces	3
1.1	Distance Metric	3
1.2	Topology	6
1.3	Calculus of Convergence	11
1.4	Compacta	12
1.5	Completeness and Completion	19
1.6	Baire Category	26
1.7	Hausdorff Metric	28
1.8	Geodesic Paths and Hopf-Rinow Theorem	33
1.9	Lipschitz Connected Spaces	37
2	Metrization	38
2.1	Metrization Theorems	38
2.2	Metrization Preserving Operations	42
3	Metric Dimension	43
3.1	Covering Dimension	43
3.2	Embedding Theorem	45

1 Metric Spaces

1.1 Distance Metric

Semimetric :: $\prod X \in \text{SET} . \text{Symmetric}(X, X; \mathbb{R}_+)$

$d : \text{Semimetric} \iff \forall x, y, z \in X . d(x, x) = 0 \ \& \ d(x, y) \leq d(x, z) + d(z, y)$

SemimetricSpace := $\sum_{X \in \text{SET}} \text{Semimetric}(X) : \text{Type};$

Metric :: $\prod x \in \text{SET} . \text{Semimetric}(X, X; \mathbb{R}_+)$

$d : \text{Metric} \iff \forall x, y \in X . d(x, y) = 0 \iff x = y$

MetricSpace := $\sum_{X \in \text{SET}} \text{Metric}(X) : \text{Type};$

semimetric :: $\prod (X, \rho) : \text{SemimetricSpace} . \text{Semimetric}(X)$

$\text{semimetric}() = d_X := \rho$

synecdoche :: $\prod (X, \rho) : \text{SemimetricSpace} . \text{SET}$

$\text{synecdoche}() = (X, \rho) := X$

TriangleInequality :: $\forall (X, d) : \text{SemimetricSpace} . \forall x, y, z \in X . d(x, y) \leq d(x, z) + d(z, y)$

Proof =

...

□

ReversedTriangleInequality :: $\forall (X, d) : \text{SemimetricSpace} . \forall x, y, z \in X . \left| d(x, y) - d(y, z) \right| \leq d(x, z)$

Proof =

[1] := **TriangleInequality**(x, y, z) : $d(x, y) \leq d(x, z) + d(z, y)$,

[2] := $\text{Symmetric}(d)(z, y)[1] : d(x, y) \leq d(x, z) + d(y, z)$,

[3] := [2] - d(y, z) : $d(x, y) - d(y, z) \leq d(x, z)$,

[4] := **TriangleInequality**(y, z, x) : $d(y, z) \leq d(y, x) + d(x, z)$,

[5] := $\text{Symmetric}(d)(x, y)[4] : d(y, z) \leq d(y, x) + d(x, z)$,

[6] := [5] - d(y, z) : $d(y, z) - d(x, y) \leq d(x, z)$,

[*] := $\text{absValue}[3][6] : \left| d(x, y) - d(y, z) \right| \leq d(x, z)$;

□

$\text{Lipschitz} :: \prod X, Y : \text{SemimetricSpace} . \mathbb{R}_+ \rightarrow ?(X \rightarrow Y)$

$f : \text{Lipschitz} \iff f \in L\text{-Lip} \iff \Lambda L \in \mathbb{R}_+ . \forall x, y \in X . d(f(x), f(y)) \leq Ld(x, y)$

$\text{SemiisometryCategory} :: \text{CAT}$

$\text{SemiisometryCategory} () = \text{SMS}_{\text{o}\rightarrow} . := \left(\text{SemimetricSpace}, 1\text{-Lip}, \circ, \text{id} \right)$

$\text{IsometryCategory} :: \text{CAT}$

$\text{IsometryCategory} () = \text{MS}_{\text{o}\rightarrow} . := \left(\text{MetricSpace}, 1\text{-Lip}, \circ, \text{id} \right)$

$\text{Semiisometry} :: \prod X, Y : \text{SemimetricSpace} . ?(X \rightarrow Y)$

$f : \text{Semiisometry} \iff \forall a, b \in X . d(f(a), f(b)) = d(a, b)$

$\text{Isometry} :: \prod X, Y : \text{MetricSpace} . ?(X \rightarrow Y)$

$f : \text{Isometry} \iff \forall a, b \in X . d(f(a), f(b)) = d(a, b)$

$\text{distance} :: \prod X : \text{SemimetricSpace} . ?X \rightarrow X \rightarrow \mathbb{R}_+$

$\text{distance} (A, x) = d(A, x) := \inf_{a \in A} d(a, x)$

$\text{distanceBetweenSets} :: \prod X : \text{SemimetricSpace} . ?X \rightarrow ?X \rightarrow \widehat{\mathbb{R}}_+$

$\text{distanceBetweenSets} (A, B) = d(A, B) := \sup_{b \in B} d(A, b)$

$\text{diameter} :: \prod X : \text{SemimetricSpace} . ?X \rightarrow \widehat{\mathbb{R}}_+$

$\text{diameter} (A) = \text{diam } A := \sup_{x, y \in A} d(x, y)$

$\text{Bounded} :: \prod X \in \text{SMS}_{\text{o}\rightarrow} . ?_{\text{Semiiso}} ?X$

$A : \text{Bounded} \iff \text{diam } A < \infty$

$\text{BoundedSubset} :: \forall X \in \text{SMS}_{\text{o}\rightarrow} . \forall A : \text{Bounded}(X) . \forall B \subset A . B : \text{Bounded}(X)$

Proof =

[1] := $\text{Bounded}(X)(A) \text{diam } \text{SubsetSupremum}(A, B) \text{diam}^{-1} \text{diam} :$

$: \infty > \text{diam } A = \sup_{x, y \in A} d(x, y) \geq \sup_{x, y \in B} d(x, y) = \text{diam } B,$

[2] := $\text{Bounded}^{-1}[1] : \left(B : \text{Bounded}(X) \right);$

□

disk :: $\prod X \in \text{SMS}_{\text{o} \rightarrow} . X \rightarrow \mathbb{R}_+ \rightarrow ?X$

disk $(x, r) = \mathbb{D}(r, x) := \left\{ y \in X : d(x, y) \leq r \right\}$

cell :: $\prod X \in \text{SMS}_{\text{o} \rightarrow} . X \rightarrow \mathbb{R}_+ \rightarrow ?X$

cell $(x, r) = \mathbb{B}(r, x) := \left\{ y \in X : d(x, y) < r \right\}$

sphere :: $\prod X \in \text{SMS}_{\text{o} \rightarrow} . X \rightarrow \mathbb{R}_+ \rightarrow ?X$

sphere $(x, r) = \mathbb{S}(r, x) := \left\{ y \in X : d(x, y) = r \right\}$

Cellbound :: $\forall X \in \text{SMS}_{\text{o} \rightarrow} . \forall A \subset X . A : \text{Bounded}(X) \iff \exists x \in X : \exists r \in \mathbb{R}_+ : A \subset \mathbb{B}(x, r)$

Proof =

...

□

BoundedMetric1 :: $\forall X \in \text{SET} . \forall d : \text{Semimetric}(X) . \frac{d}{1+d} : \text{Semimetric}$

Proof =

Assume $x, y, z : X$,

$[\dots *] := \text{PlusUnityRecipricol}(d(x, z)) \text{TrinagleIneq}(X, d) \text{PlusUnityRecipricol}(d(x, y) + d(y, z))$

$$\begin{aligned} \text{PositiveSumIsGreater} : \frac{d(x, z)}{1 + d(x, z)} &= 1 - \frac{1}{1 + d(x, z)} \leq 1 - \frac{1}{1 + d(x, y) + d(y, z)} = \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}; \end{aligned}$$

$$\leadsto [*] := \mathfrak{D}^{-1} \text{Semimetric} : \left(\frac{d}{1+d} : \text{Semimetric}(X) \right),$$

□

BoundedMetric2 :: $\forall X \in \text{SET} . \forall d : \text{Semimetric}(X) . \min(1, d) : \text{Semimetric}$

Proof =

...

□

1.2 Topology

`metricTopology` :: $\text{SMS}_{\text{op} \rightarrow} \rightarrow \text{TOP}$

`metricTopology` (X) = $X := \left\langle \left\{ \mathbb{B}(x, r) \mid r \in \mathbb{R}_{++}, x \in X \right\} \right\rangle_{\text{TOP}}$

`semimetricSpaceCategory` :: `Category`

`semimetricSpaceCategory` () = $\text{SMS} := \left(\text{SemimetricSpace}, C, \circ, \text{id} \right)$

`metricSpaceCategory` :: `Category`

`metricSpaceCategory` () = $\text{MS} := \left(\text{metricSpace}, C, \circ, \text{id} \right)$

`OpenCells` :: $\forall X \in \text{SMS} . \forall x \in X . \forall r \in \mathbb{R}_{++} . \mathbb{B}(x, r) \in \mathcal{T}(X)$

`Proof` =

...

□

`MetricOpennessCriterion` :: $\forall X \in \text{SMS} . \forall U \subset X . U \in \mathcal{T}(X) \iff \forall x \in X . \exists r \in \mathbb{R}_{++} : \mathbb{B}(x, r) \subset U$

`Proof` =

...

□

`ClosureByDistance` :: $\forall X \in \text{SMS} . \forall A \subset X . \overline{A} = \{x \in X : d(x, A) = 0\}$

`Proof` =

$B := \{x \in X : d(x, A) = 0 : \text{Subset}(X),$

`Assume` $x : B,$

$[1] := jB(x) : d(x, A) = 0,$

`Assume` $U : \mathcal{U}(x),$

$(r, [2]) := \text{MetricOpennessCriterion}(U, x) : \sum r \in \mathbb{R}_{++} . \mathbb{B}(x, r) \subset U,$

$[U.*] := [1] \text{distanceToSet}[2] : U \cap A \neq \text{Emptyset};$

$\leadsto [x.*] := \text{AltClosure}(A) : x \in \overline{A};$

$\leadsto [1] := bd^{-1} \text{Subset} : B \subset \overline{A},$

`Assume` $x : \overline{A},$

`Assume` $r : \mathbb{R}_{++},$

$[2] := \text{OpenCell}(x, r) \text{AltClosure}(A) : \mathbb{B}(x, r) \cap A = \emptyset,$

$[r.*] := \text{Functor}(\text{distanceToSet}, () x, A)[2] : d(x, A) < r;$

$\leadsto [2] := \text{InfimumInduction} : d(x, A) = 0,$

$[x.*] := jB[2] : x \in B;$

$\leadsto [*] := \text{SetEq}[1] : A = B;$

□

DiscIsClosed :: $\forall X \in \mathbf{SMS} . \forall x \in X \forall r \in \mathbb{R}_{++} \mathbb{D}(x, r) : \mathbf{Closed}(X)$

Proof =

$D := \mathbb{D}(x, r) : \mathbf{Subset}(X),$

Assume $y : X,$

Assume $[1] : d(y, D) = 0,$

$(a, [2]) := \mathfrak{d}\mathbf{distanceToSet}(y, D)[1]\mathfrak{d}\mathbf{infimum} : \sum a : \mathbb{N} \rightarrow D . \lim_{n \rightarrow \infty} d(y, a_n) = 0,$

Assume $\varepsilon : \mathbb{R}_{++},$

$(n, [3]) := \mathfrak{d}\mathbf{Limit}[2](\varepsilon) : \sum n \in \mathbb{N} d(y, a_n) < \varepsilon,$

$[\varepsilon.*] := \mathbf{TriangleIneq}\mathfrak{d}\mathbf{disc}(D)[2]\mathbf{IneqSum} : d(x, y) \leq d(x, a_n) + d(a_n, y) < r + \varepsilon;$

$\leadsto [3] := \mathbf{IneqLim} : d(x, y) \leq r,$

$[y.*] := \mathfrak{d}\mathbf{disc}(D) : y \in D;$

$\leadsto [1] := \mathbf{ClosureByDistance} : D = \overline{D},$

$[*] := \mathfrak{d}\mathbf{closure} : (D : \mathbf{Closed}(X));$

□

SphereIsClosed :: $\forall X \in \mathbf{SMS} . \forall x \in X \forall r \in \mathbb{R}_{++} \mathbb{S}(x, r) : \mathbf{Closed}(X)$

Proof =

$S := \mathbb{S}(x, r) : \mathbf{Subset}(X),$

$[1] := \mathbf{TrichotomyPrinciple}\mathfrak{d}\mathbf{sphere}(S) : S^{\mathbb{C}} = \mathbb{B}(x, r) \cup \mathbb{D}^{\mathbb{C}}(x, r),$

$[2] := \mathfrak{d}\mathbf{ClosedDiscIsClosed}(x, r) : \mathbb{D}^{\mathbb{C}}(x, r) \in \mathcal{T}(X),$

$[3] := \mathbf{OpenCell}(x, r) : \mathbb{B}(x, r) \in \mathcal{T}(X),$

$[4] := [1]\mathbf{OpenUnion}[3][2] : S^{\mathbb{C}} \in \mathcal{T}(X),$

$[*] := \mathfrak{d}^{-1}\mathbf{Closed}(X)[4] : (S : \mathbf{Closed}(X));$

□

SemimetricBalance :: $\forall X \in \mathbf{SMS} . \forall x, y \in X . \forall U \in \mathcal{U}(x) . y \in U \iff \forall U \in \mathcal{U}(y) . x \in U$

Proof =

...

□

BoundedMetricTopology1 :: $\forall (X, d) \in \mathbf{SMS} . (X, d) \cong_{\mathbf{SMS}} \left(X, \frac{d}{1+d} \right)$

Proof =

...

□

BoundedMetricTopology2 :: $\forall (X, d) \in \mathbf{SMS} . (X, d) \cong_{\mathbf{SMS}} \left(X, \min(d, 1) \right)$

Proof =

...

□

MetricSpaceRegularity :: $\forall X \in \mathbf{MS} . X : \mathbf{T4}$

Proof =

Assume $A, B : \mathbf{Closed}(X)$,

Assume $[1] : A \cap B = \emptyset$,

Assume $a : A$,

$(r, [2]) := \mathbf{ClosureByDistance}[1] : \sum r \in \mathbb{R}_{++} . d(a, B) > r$,

$O(a) := \mathbb{B}\left(a, \frac{r}{4}\right) : \mathcal{T}(X)$;

$\leadsto O := I(\rightarrow) : A \rightarrow \mathcal{T}(X)$,

$U := \bigcap_{a \in A} O(a) : \mathcal{T}(X)$,

$[1] := jU : A \subset U$,

$[2] := jU : B \cap U = \emptyset$,

Assume $b : B \cap \overline{U}$,

$(u, [3]) := \mathbf{ClosureByDistance} \mathfrak{d} \mathbf{distanceToSet} : \sum u : \mathbb{N} \rightarrow U . \lim_{n \rightarrow \infty} d(b, u_n) = 0$,

Assume $n : \mathbb{N}$,

$(a_n, [4]) := jU[3] : \sum a_n \in A . u_n \in O(a_n)$,

$r := d(a_n, B) : \mathbb{R}_{++}$,

$[5] := jO(a_n)jr : d(a_n, b) \geq r$,

$[6] := \mathfrak{d} \mathbf{cell} : d(a_n, u_n) < \frac{r}{2}$,

Assume $[7] : \frac{r}{2} \geq d(u_n, b)$,

$[8] := \mathbf{TriangleIneq}(a_n, b)[7][6] : d(a_n, b) \leq d(a_n, u_n) + d(u_n, b) < r$,

$[7.*] := \mathbf{TrichtomyPrinciple}[5][8] : \perp$;

$\leadsto [7] := E(\perp) : \frac{r}{2} < d(u_n, b)$,

$[n.*] := \mathbf{ReverseTriangularIneq}(a_n, u_n, b)[6][7] : \left| d(a_n, b) - d(u_n, b) \right| \leq d(a_n, u_n) < \frac{r}{2} < d(u_n, b)$;

$\leadsto (a, [4]) := \sum a : \mathbb{N} \rightarrow A . \lim_{n \rightarrow \infty} d(a_n, b) = 0$,

$[5] := \mathbf{ClodureByDistancr} : b \in A$,

$[b.*] := \mathbf{EmptyNonempty}[1][5] : \perp$;

$\leadsto [3] := E(\perp) : B \cap \overline{U} = \emptyset$,

$V := (\overline{U})^c : \mathbf{Open}(X)$,

$[*] := \mathfrak{d}^{-1} \mathbf{T4} : (X : \mathbf{T4})$;

□

SemimetricSeparability :: $\forall X \in \mathbf{SMS} . X : \mathbf{T0} \Rightarrow X \in \mathbf{MS}$

Proof =

...

□

SemimetricSeparability :: $\forall X \in \mathbf{SMS} . X : \mathbf{T0} \Rightarrow X : \mathbf{T4}$

Proof =

...

□

SemimetricCountability :: $\forall X \in \text{SMS} . X : \text{FirstCountable}$

Proof =

...

□

SemimetricCountability :: $\forall X \in \text{SMS} . X : \text{Separable} \Rightarrow X : \text{SecondCountable}$

Proof =

...

□

DeltaEpsilonFormalism :: $\forall X, Y \in \text{SMS} . \forall f : X \rightarrow Y . f \in C(X, Y) \iff \forall \epsilon \in \mathbb{R}_+ . \forall x \in X .$
 $. \exists \delta \in \mathbb{R}_+ : \forall a, b \in \mathbb{B}(x, \delta) . d(f(a), f(b)) < \epsilon$

Proof =

Assume [1] : $f \in C(X, Y)$,

Assume $x : X$,

Assume $\epsilon : \mathbb{R}_+$,

$y := f(x) : \text{Element}(Y)$,

$B := \mathbb{B}(y, \epsilon) : \text{Open}(Y)$,

$U := f^{-1}(B) : \text{Open}(X)$,

[1] := $jU : x \in U$,

$(\delta, [2]) := \text{MetricOpennesCriterion}[1] : \sum \delta \in \mathbb{R}_+ . \mathbb{B}(x, \delta) \subset U$,

[1.*] := $jU[2] : f\mathbb{B}(x, \delta) \subset \mathbb{B}(y, \epsilon)$;

$\leadsto [1] := \text{Cell}I(\Rightarrow) : \text{Left} \Rightarrow \text{Right}$,

Assume [2] : **Right**,

Assume $U : \text{Open}(Y)$,

[U.*] := $\text{MetricOpennesCriterion} \text{Right}[2](U) : f^{-1}(U) \in \mathcal{T}(X)$;

$\leadsto [2.*] := \text{Cell}^{-1}C(X, Y) : f \in C(X, Y)$;

$\leadsto [*] := I(\iff) : \text{THIS}$;

□

LipschitzIsContinuous :: $\forall X, Y \in \text{SMS} . \forall L \in \mathbb{R}_{++} . \forall f \in \text{L-Lip}(X, Y) . f \in C(X, Y)$

Proof =

...

□

IsometricBijectionIsHomeo :: $\forall X, Y \in \text{SMS} . \forall f : \text{IsometryBijection}(X, Y) . f : \text{Home}(X, Y)$

Proof =

...

□

SemiisometryFullEmbedding :: $\text{SMS}_{\text{om}} : \text{FullEmbedding}(\text{SMS})$

Proof =

...

□

IsometryFullEmbedding :: MS_{o→.} : FullEmbedding(MS)

Proof =

...



1.3 Calculus of Convergence

$$\text{NEpsilonFormalism} :: \forall X \in \text{SMS} . \forall D : \text{DirectedSet} . \forall x : \text{Net}(D, x) . \forall L \in X . \lim_{n \in D} x_n = L \iff \\ \iff \forall \epsilon \in \mathbb{R}_{++} . \exists N \in D : \forall n \in D . n \geq N \Rightarrow x_n \in \mathbb{B}(L, \epsilon)$$

Proof =

...

□

$$\text{SemimetricConvergence} :: \forall X \in \text{SMS} . \forall D : \text{DirectedSet} . \forall x : \text{Net}(D, x) . \forall L \in X . \lim_{n \in D} x_n = L \iff \\ \iff \lim_{n \in D} d(x_n, L) = 0$$

Proof =

...

□

$$\text{FrechetCombination} :: \forall D : \text{DirectedSet} . \forall x : \mathbb{N} \rightarrow \text{Net}(D, \mathbb{R}_+) . \lim_{\delta \in D} \sum_{n=1}^{\infty} \frac{x_{n,\delta}}{2^n(1+x_{n,\delta})} = 0 \iff \\ \iff \forall n \in \mathbb{N} . \lim_{\delta \in D} x_{n,\delta} = 0$$

Proof =

...

□

$$\text{FrechetWeakSemimetric} :: \forall X \in \text{Set} . \forall d : \mathbb{N} \rightarrow \text{Semimetric}(X) . \forall D : \text{DirectedSet} . \\ . \forall x : \text{Net}(X, D) . x : \text{Convergent} \left(D, \left(X, \sum_{i=1}^n \frac{d}{2^n(1+d)} \right) \right) \iff \forall n \in \mathbb{N} . x : \text{Convergent}(D, (X, d_n))$$

Proof =

...

□

$$\text{ConvergenceByDenseSubset} :: \forall X \in \text{SMS} . \forall A : \text{Dense}(X) . \forall L \in X . \forall D : \text{DirectedSet} . \forall x : \text{Net}(D, X) . \\ . \lim_{n \in D} x_n = L \iff \forall a \in A . \lim_{n \in D} d(x_n, a) = d(L, a)$$

Proof =

...

□

1.4 Compacta

`TotallyBounded` :: ?SMS

$$X : \text{TotallyBounded} \iff \forall \epsilon \in \mathbb{R} . \exists n \in \mathbb{N} : \exists x : n \rightarrow X : X \subset \bigcup_{i=1}^n \mathbb{B}(x_i, \epsilon)$$

`EmptySpaceIsTotallyBounded` :: $\emptyset : \text{TotallyBounded}$

`Proof` =

...

□

`CountablyCompactIsTotallyBounded` :: $\forall X \in \text{SMS} . X : \text{CountablyCompact} \Rightarrow X : \text{TotallyBounded}$

`Proof` =

`Assume` [1] : $X ! \text{TotallyBounded}$,

$$(\epsilon, [2]) := \text{dTotalyBounded} : \sum \epsilon \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . \forall x : n \rightarrow X . \bigcup_{i=1}^n \mathbb{B}(x_i, \epsilon) \neq X,$$

$$(x_0, [3]) := \text{dNonEmptyTotallyBounded} : \sum x_0 : \text{type}(X) . x_0 \in X,$$

`Assume` $n : \mathbb{N}$,

$$(x_n, [4_n]) := \sum x_n \in X : \forall i \in \mathbb{Z}_+ . n - 1 . d(x_i, x_n) > \epsilon;$$

$$\leadsto (x, [4]) := I \left(\sum \right) : \sum x : \mathbb{N} \rightarrow X . \forall m, n \in \mathbb{Z}_+ . n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon,$$

$c := \text{CountablyCompactHasClusters}(X, x) : \text{Cluster}(X, x)$,

$$(n, m, [6, 7, 8]) := \text{dCluster}(X, x, c) : \sum n, m \in \mathbb{Z}_+ . n \neq m \ \& \ d(x_n, c) < \frac{\epsilon}{2} \ \& \ d(x_m, c) < \frac{\epsilon}{2},$$

$$[9] := \text{TriangleIneq}(x_n, x_m, c) \text{IneqSum}[7][9] : d(x_n, x_m) \leq d(x_n, c) + d(c, x_m) < \epsilon,$$

$$[1.*] := [4](n, m)[6][9] \text{TrichtomyPrinciple}(\mathbb{R}_+) : \perp;$$

$$\leadsto [*] := E(\perp) : (X : \text{TotallyBounded});$$

□

TotallyBoundedIsSecondCountable :: $\forall X \in \mathbf{SMS} . X : \mathbf{TotallyBounded} \Rightarrow X : \mathbf{SecondCountable}$

Proof =

Assume $q : \mathbb{Q} \cap (0, 1)$,

$$(n, x, [1]) := \mathfrak{d}\mathbf{TotallyBounded}(q) : \sum n \in \mathbb{N} . \sum x : n \rightarrow X . X = \bigcup_{i=1}^n \mathbb{B}(x_i, q),$$

$$\beta_q := \left\{ \mathbb{B}(x_i, q) \mid i \in n \right\} : \mathbf{Finite} \mathcal{T}(X);$$

$$\leadsto \beta := I(\rightarrow) : \mathbb{Q} \cap (0, 1) \rightarrow \mathbf{Finite} \mathcal{T}(X),$$

$$\mathcal{B} := \bigcup_{q \in \mathbb{Q} \cap (0, 1)} \beta_q : ?\mathcal{T}(X),$$

$$[1] := \mathbf{CountableFiniteUnion}(\mathcal{B}, \mathbb{Q} \cap (0, 1), \beta, j\mathcal{B}) : |\mathcal{B}| \leq \aleph_0,$$

Assume $U : \mathbf{Open}(X)$,

Assume $u : U$,

$$(r, [2]) := \mathbf{MetricOpennessCriterion}(U, x) : \sum r \in \mathbb{R}_{++} . \mathbb{B}(u, r) \subset U,$$

$$(q, [3]) := \mathbf{ArchimedianLimit}(r) : \sum q \in (0, 1) \cap \mathbb{Q} . q < \frac{r}{2},$$

$$(x, [4]) := j\beta_{q}(u) : \sum x \in X . u \in \mathbb{B}(x, q),$$

Assume $y : \mathbb{B}(x, q)$,

$$[5] := \mathbf{TriangleInequality}(u, y, x)[4]\mathfrak{d}\mathbf{cell}(X, x, q)[3] : d(u, y) \leq d(u, x) + d(x, y) < 2q < r,$$

$$[y.*] := \mathfrak{d}\mathbf{cell}(X, u, r)[2] : y \in U;$$

$$\leadsto [5] := \mathfrak{d}^{-1}\mathbf{Subset} : \mathbb{B}(x, q) \subset U,$$

$$[u.*] := I(\exists)(\mathcal{B})[5][4] : \exists B \in \mathcal{B} : u \in B \subset U;$$

$$\leadsto [2] := I(\forall) : \forall u \in U . \exists B \in \mathcal{B} : u \in B \subset U,$$

$$[U.*] := \mathbf{InnerCover}[2] : \exists \mathcal{A} \subset \mathcal{B} . \bigcup \mathcal{A} = U;$$

$$\leadsto [2] := \mathfrak{d}^{-1}\mathbf{Base} : \left(\mathcal{B} : \mathbf{Base}(X) \right),$$

$$[*] := \mathfrak{d}^{-1}\mathbf{SecondCountable}(\mathcal{B})[1][2] : \left(X : \mathbf{SecondCountable} \right);$$

□

CountablyCompactIsCompact :: $\forall X : \mathbf{SMS} . X : \mathbf{CountablyCompact} \Rightarrow X : \mathbf{Compact}$

Proof =

$$[1] := \mathbf{CountablyCompactIsTotallyBounded}(X) : \left(X : \mathbf{TotallyBounded} \right),$$

$$[2] := \mathbf{TotallyBoundedIsSecondCountable}(X) : \left(X : \mathbf{SecondCountable} \right),$$

$$[3] := \mathbf{SecondCountableIsLindelof}(X) : \left(X : \mathbf{Lindelof} \right),$$

$$[*] := \mathbf{LindelofAndCountablyCompactIsCompact} : \left(X : \mathbf{Compact} \right);$$

□

SequentiallyCompactIffCompact :: $\forall X : \mathbf{SMS} . X : \mathbf{SequentiallyCompact} \iff X : \mathbf{Compact}$

Proof =

...

□

TotallyBoundedByDenseSubset :: $\forall X \in \mathbf{SMS} . \forall A : \mathbf{Dense}(X) . A : \mathbf{TotallyBounded} \Rightarrow X : \mathbf{TotallyBounded}$
Proof =
Assume $[1] : X ! \mathbf{TotallyBounded}$,
 $(\epsilon, [2]) := \mathfrak{d}\mathbf{TotallyBounded} : \sum \epsilon \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . \forall x : n \rightarrow X . \bigcup_{i=1}^n \mathbb{B}(x_i; \epsilon) \neq X$,
 $(x_0, [3]) := \mathfrak{d}\mathbf{NonEmptyTotallyBounded} : \sum x_0 : \mathbf{type}(X) . x_0 \in X$,
Assume $n : \mathbb{N}$,
 $(x_n, [4_n]) := \sum x_n \in X : \forall i \in \mathbb{Z}_+ . n - 1 . d(x_i, x_n) > \epsilon$;
 $\leadsto (x, [4]) := I \left(\sum \right) : \sum x : \mathbb{N} \rightarrow X . \forall m, n \in \mathbb{Z}_+ . n \neq m \Rightarrow d(x_n, x_m) \geq \epsilon$,
 $(a, [5]) := \mathfrak{d}\mathbf{Dense}(A) \left(x, \frac{\epsilon}{3} \right) : \sum a : \mathbb{Z}_+ \rightarrow A . \forall n \in \mathbb{Z}_+ . d(a_n, x_n) < \frac{\epsilon}{3}$,
 $(n, [6, 7]) := \mathfrak{d}\mathbf{TotallyBounded}(A) \left(q, \frac{\epsilon}{3} \right) : \sum n \in \mathbb{N} . d(a_n, a_0) < \frac{\epsilon}{3} \ \& \ n \neq 0$,
 $[8] := \mathbf{TriangleIneq}[6][5]\mathbf{SumIneq} : d(x_n, x_0) \leq d(x_n, a_n) + d(a_n, a_0) + d(a_0, x_0) < \epsilon$,
 $[1.*] := [8][4](0, n)[7]\mathbf{TrichtomyPrinciple}(\mathbb{R}_{++}) : \perp$;
 $\leadsto [*] := E(\perp) : (X : \mathbf{TotallyBounded})$;
 \square

Equicontinuous :: $\prod X, Y \in \mathbf{SMS} . ??C(X, Y)$
 $\Phi : \mathbf{Equicontinuous} \iff \Phi \in \mathcal{EC}(X, Y) \iff \forall x \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists U \in \mathcal{U}(x) : \forall f \in \Phi . \text{diam } f(U) < \varepsilon$

EquicontinuousAtAPoint :: $\prod X, Y \in \mathbf{SMS} . X \rightarrow ??C(X, Y)$
 $\Phi : \mathbf{Equicontinuous} \iff \Phi \in \mathcal{EC}_x(X, Y) \iff \Lambda x \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists U \in \mathcal{U}(x) : \forall f \in \Phi . \text{diam } f(U) < \varepsilon$

FiniteContinuousIsEquicontinuous :: $\forall X, Y \in \mathbf{SMS} . \forall \Phi : \mathbf{Finite} \ C(X, Y) . \Phi \in \mathcal{EC}(X, Y)$
Proof =
 \dots
 \square

FiniteEquicontinuousUnion :: $\forall X, Y \in \mathbf{SMS} . \forall n \in \mathbb{N} . \forall \Phi : n \rightarrow \mathcal{EC}(X, Y) . \bigcup \Phi \in \mathcal{EC}(X, Y)$
Proof =
 \dots
 \square

EquicontinuityIsLocal :: $\forall X, Y \in \mathbf{SMS} . \forall \Phi : ??C(X, Y) . \left(\forall x \in X . \Phi \in \mathcal{EC}_x(X, Y) \right) \Rightarrow \Phi \in \mathcal{EC}(X, Y)$
Proof =
 \dots
 \square

EquicontinuityByMonotonicConvergence :: $\forall X \in \text{SMS} . f : \mathbb{N} \rightarrow C(X) . \forall \varphi \in C(X) .$

$f \downarrow \varphi \Rightarrow \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$

Proof =

Assume $x : \text{In}(X),$

Assume $\varepsilon : \mathbb{R}_{++},$

$(N, [1]) := \text{Limit}(f(x), \varphi(x), \varepsilon/10) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \frac{\varepsilon}{10},$

$(\delta_1, [2]) := \text{C}(X)(\varphi, \varepsilon/10, x) : \sum \delta_1 \in \mathbb{R}_{++} . \varphi(\mathbb{B}(x, \delta_1)) \subset \left(\varphi(x) - \frac{\varepsilon}{10}, \varphi(x) + \frac{\varepsilon}{10}\right),$

$(\delta_2, [3]) := \text{C}(X)(f_N, \varepsilon/10, x) : \sum \delta_2 \in \mathbb{R}_{++} . f_N(\mathbb{B}(x, \delta_2)) \subset \left(f_N(x) - \frac{\varepsilon}{10}, f_N(x) + \frac{\varepsilon}{10}\right),$

$\delta := \min(\delta_1, \delta_2) : \mathbb{R}_{++},$

Assume $n : \mathbb{N},$

Assume $[4] : n \geq N,$

Assume $y : \mathbb{B}(x, \delta),$

$[5] := \text{MonotonicConvergence}(f, \varphi, y)[4] : |f_N(y) - \varphi(y)| > |f_n(y) - \varphi(x)|,$

$[n.*] := \text{TriangleIneq}^4[1, 2, 3, 4, 5] : |f_n(y) - f_n(x)| \leq |f_n(y) - \varphi(y)| + |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| <$

$|f_N(y) - \varphi(y)| + \frac{2\varepsilon}{10} \leq |f_N(y) - f_N(x)| + |f_N(x) - \varphi(x)| + |\varphi(x) - \varphi(y)| + \frac{2\varepsilon}{10} < \frac{\varepsilon}{2};$

$\sim [4] := \text{C}_x : \{f_n | n \geq N\} \in \mathcal{EC}_x(X, \mathbb{R}),$

$[x.*] := \text{FiniteContinuousIsEquiContinuous}\{f_n | n < N\} \text{FiniteEqyiContinuousUnion}[4] :$

$: \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}_x(X, \mathbb{R});$

$\sim [*] := \text{C}_x : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, \mathbb{R});$

□

maricFunction :: $\prod X \in \text{SMS} . \text{Pointwise}(X) \rightarrow \mathbb{R}_{++} \rightarrow X \rightarrow \mathbb{R}$

maricFunction $(f, \varepsilon) = M_{f, \varepsilon} := \Lambda x \in X . \sum_{n=1}^{\infty} \min \left(0, |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| - \varepsilon\right)$

MaricFunctionContinuity :: $\forall X \in \text{SMS} . \forall \varphi \in C(X) . f : \text{Pointwise}(X, \varphi) . \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, Y)$

Proof =

$\varphi := \lim_{n \rightarrow \mathbb{N}} f_n : X \rightarrow \mathbb{R},$

Assume $x : X,$

$(N, [1]) := \text{Pointwise}(X) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow |f_n(x) - \varphi(x)| < \frac{\varepsilon}{3} \ \& \ n < N \Rightarrow |f_n(x) - \varphi(y)| >$

$\varepsilon' := \min(|f_{N-1}(x) - \varphi(x)| - \varepsilon, \varepsilon) : \mathbb{R}_{++},$

$(\delta_1, [2]) := \text{EC} : \prod \delta \in \mathbb{R}_{++} . \forall n \in \mathbb{N} . f_n \mathbb{B}(x, \delta) \subset \mathbb{B}\left(f(x), \frac{\varepsilon'}{3}\right),$

$(\delta_2, [3]) := \text{C}(X) : \prod \delta \in \mathbb{R}_{++} . \varphi \mathbb{B}(x, \delta) \subset \mathbb{B}\left(\varphi(x), \frac{\varepsilon'}{3}\right),$

$\delta := \min(\delta_1, \delta_2, \delta_3) : \mathbb{R}_+,$

Assume $y : \mathbb{B}(x, \delta),$

Assume $n : \mathbb{N},$

Assume $[4] : n \geq N,$

$[y.*] := \text{TriangleIneq}[1][2][3] : |\varphi(y) - f_n(y)| \leq |\varphi(y) - \varphi(x)| + |\varphi(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon;$

$\sim [4] := I(\forall) : \forall n \geq N . \forall y \in \mathbb{B}(x, \delta) . |\varphi(y) - f_n(x)| < \varepsilon,$

Assume $y : \mathbb{B}(x, \delta),$

Assume $n : \mathbb{N}$,

Assume $[5] : n < N$,

$[y.*] := [1][2][3][5] : |\varphi(y) - f_n(y)| > |\varphi(x) - f_n(x)| - \varepsilon' > \varepsilon;$

$\leadsto [5] := I(\forall) : \forall y \in \mathbb{B}(x, \delta) . \forall n \in \mathbb{N} . n < N \Rightarrow |\varphi(x) - f_n(x)| < \varepsilon,$

$[6] := [4][5]\breve^{-1}\text{maricFunction} : \forall y \in \mathbb{B}(x, \delta) . M_{f,\varepsilon}(y) = \sum_{n=1}^{N-1} |f_n(y) - \varphi(y)| - \varepsilon,$

$[y.*] := \text{ContinuousSum} : M_{f,\varepsilon|\mathbb{B}(x,\delta)} \in C(\mathbb{B}(x, \delta));$

$\leadsto [*] := \text{ContinuityIsLocal} : M_{f,\varepsilon} \in C(X);$

□

DiniLemma :: $\forall X : \text{Pseudocompact} \ \& \ \text{SMS} . \forall f : \mathbb{N} \rightarrow C(X) . \forall \varphi \in C(X) .$

$. f \downarrow \varphi \Rightarrow \varphi : \text{Limit} \left(\left(C(X), \|\cdot\|_\infty \right), \varphi \right)$

Proof =

$[1] := \text{EquicontinuityByMonotonicConvergence} : \{f_n | n \in \mathbb{N}\} \in \mathcal{EC}(X, \mathbb{R}),$

$[2] := \text{MaricFunctionContinuity} : \forall \varepsilon \in \mathbb{R}_{++} . M_{f,\varepsilon} \in C(X),$

$[a, 3] := \breve{\text{Paracompact}}(M_f) : \sum \mathbb{R}_{++} \rightarrow \mathbb{R}_+ . \forall \varepsilon \in \mathbb{R}_{++} . \max_{x \in X} M_{f,\varepsilon}(x) = a_\varepsilon,$

Assume $\varepsilon : \mathbb{R}_{++}$,

Assume $n : \mathbb{N}$,

Assume $[4] : n > \left\lceil \frac{a_\varepsilon}{\varepsilon} \right\rceil ,$

Assume $[5] : \|f_n - \varphi\|_\infty \geq \varepsilon,$

$(x, [6]) := \breve{\text{uniformNorm}}[5]\breve{\text{Paracompact}}(X) : \sum x \in X : |f_n(x) - \varphi(x)| \geq \varepsilon,$

$[7] := \breve{\text{MonotonicConvergence}}[5] : \forall m \in \mathbb{N} . m \leq n \Rightarrow |f_m(x) - \varphi(x)| \geq \varepsilon,$

$[8] := \breve{\text{maricFunction}}[4][7] : M_{f,\varepsilon}(x) \geq \sum_{i=1}^n \|f(x) - \varphi(x)\| > a_\varepsilon,$

$[n.*] := \breve{\text{Maximum}}(a_n, M_{\varepsilon,x})[8] : \perp;$

$\leadsto [\varepsilon.*] := E(\perp) : \forall n \in \mathbb{N} . n > \left\lceil \frac{a_\varepsilon}{\varepsilon} \right\rceil \Rightarrow \|f_n - \varphi\|_\infty < \varepsilon;$

$\leadsto [*] := \text{MetricLimit} : \text{Limit} \left(\left(C(X), \|\cdot\|_\infty \right), f, \varphi \right);$

□

DiniSpace :: ?TOP

$X : \text{DiniSpace} \iff \forall f : \mathbb{N} \rightarrow C(X) . \forall \varphi \in C(X) .$

$. f \downarrow \varphi \Rightarrow \varphi : \text{Limit} \left(\left(C(X), \|\cdot\|_\infty \right), \varphi \right)$

InverseDiniLemma :: $\forall X : \text{DiniSpace} . X : \text{Paracompact}$

Proof =

...

□

EquicontinuousNet :: $\prod X, Y \in \text{SMS} . ?\text{Net Pointwise}(X, Y)$

$(D, f) : \text{EquicontinuousNet} \iff \{f_\delta | \delta \in D\} \in \mathcal{EC}(X, Y)$

EquicontinuousLimitIsContinuous :: $\forall X, Y \in \text{SMS} . \forall (D, f) : \text{EquicontinuousNet}(X) . \forall F : \text{Limit}(D, f) .$

Proof =

Assume $x : X,$

Assume $\varepsilon : \mathbb{R}_+,$

$(\delta, [1]) := \mathfrak{d}\text{EquicontinuousNet}(D, f) \left(x, \frac{\varepsilon}{y} \right) : \sum \delta' \in \mathbb{R}_{++} . \forall n \in D . \text{diam } f_n \mathbb{B}(x, \delta) < \frac{\varepsilon}{3},$

Assume $y : Y,$

$(N, [2]) := \mathfrak{d}\text{Limit}(f, d)(F) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow d(f_n(x), F(x)) < \frac{\varepsilon}{3},$

$(M, [3]) := \mathfrak{d}\text{Limit}(f, d)(F) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow d(f_n(y), F(y)) < \frac{\varepsilon}{3},$

$K := \max(N, M) : D,$

$[y.*] := \text{TriangleIneq}[1][2][3] : d(F(x), F(y)) \leq d(F(x), f_K(x)) + d(f_K(x), f_K(y)) + d(f_K(y), F(y)) < \varepsilon;$

$\leadsto [x.*] := I(\forall) : \forall y \in \mathbb{B}(x, \delta) . d(F(x), F(y)) < \varepsilon;$

$\leadsto [*] := \mathfrak{d}^{-1}C(X, Y) : F \in C(X, Y);$

□

ConvergentContinuously :: $\prod X, Y \in \text{TOP} . ?\text{PointwiseNet}(X, Y)$

$(D, f) : \text{ConvergentContinuously} \iff \forall x : \text{ConvergentNet}(D, X) . \lim_{n \in D} f_n(x_n) = \left(\lim_{n \in D} f \right) \left(\lim_{n \in D} x_n \right)$

UniformIsConvergentContinuously :: $\forall X, Y \in \text{SMS} . \forall (D, f) : \text{UniformNet}(X, Y) . (D, f) : \text{ConvergentCont}$

Proof =

...

□

UniformIsConvergentContinuously :: $\forall X, Y \in \text{SMS} . \forall (D, f) : \text{EquicontinuousNet}(X, Y) . (D, f) : \text{ConvergentCont}$

Proof =

...

□

UniformConvergenceOnCompact :: $\forall Y \in \mathbf{SMS} . \forall X : \mathbf{Compact} \ \& \ \mathbf{SMS} . \forall (D, f) : \mathbf{Net} \ C(X, Y) .$
 $. \forall F \in C(X, Y) . \forall [1] : \mathbf{PointwiseLimit}(D, f, F) . \mathbf{UniformLimit}(D, f, F)$

Proof =

Assume $\varepsilon : \mathbb{R}_{++}$,

Assume $x : X$,

$\left(r', [1]\right) := \mathfrak{d}\mathbf{EquicontinuosNet}(D, f) \left(x, \frac{\varepsilon}{3}\right) : \sum r' \in \mathbb{R}_{++} . \forall n \in D . \text{diam } f_n \mathbb{B}(x, r') < \frac{\varepsilon}{3},$

$\left(r'', [2]\right) := \mathfrak{d}C(X, Y)(F) \left(x, \frac{\varepsilon}{3}\right) : \sum r'' \in \mathbb{R}_{++} . \text{diam } F \mathbb{B}(x, r'') < \frac{\varepsilon}{3},$

$r(x) := \min(r', r'') : \mathbf{In} \ \mathbb{R}_{++},$

$\left(N(x), [3]\right) := \mathfrak{d}\mathbf{PointwisLimit}(D, f, F) \left(x, \frac{\varepsilon}{3}\right) : \sum N(x) \in D : \forall n \in D . n \geq N \Rightarrow d(f_n(x), F(x)) < \frac{\varepsilon}{3},$

Assume $y : \mathbf{In} \ \mathbb{B}\left(x, r(x)\right),$

Assume $n : \mathbf{In}(D),$

Assume $[4] : n \geq N(x),$

$[y.*] := \mathbf{TriangleIneq}[1, 2, 3] : d\left(f_n(y), F(y)\right) \leq d(f_n(y), f_n(x)) + d(f_n(x), F(x)) + d(F(x), F(y)) < \varepsilon;$

$\leadsto [x.*] := I(\forall) : \forall y \in \mathbb{B}\left(x, r(x)\right) . d\left(f_n(y), F(y)\right) < \varepsilon;$

$\leadsto \left(r, N, [1]\right) := I\left(\prod\right) : \prod_{x \in X} \left(r(x), N(x)\right) : \mathbb{R}_{++} \times D . \forall y \in \mathbb{B}\left(x, r(x)\right) . \forall n \in D .$

$. n \geq N(x) \Rightarrow d\left(f_n(y), F(y)\right) < \varepsilon,$

$[2] := \mathbf{FullCover}\mathbb{B}(\cdot, r) : X = \bigcup_{x \in X} \mathbb{B}\left(x, r(x)\right),$

$\left(n, x, [3]\right) := \mathfrak{d}\mathbf{Compact}(X) : \sum n \in \mathbb{N} . \sum x : n \rightarrow X . X = \bigcup_{i=1}^n \mathbb{B}\left(x_i, r(x_i)\right),$

$\Delta := \max_{i \in n} N(x_i) : D;$

Assume $y : X,$

Assume $\delta : \mathbf{In} \ D,$

Assume $[4] : \delta \geq \Delta,$

$\left(i, [5]\right) := [2](y) : \sum i \in n . y \in \mathbb{B}\left(x_i, r(x_i)\right),$

$[6] := [4]j\Delta(i) : \delta \geq N(x_i),$

$[\varepsilon.*] := [1][5][6] : d\left(f_\delta(y), F(y)\right) < \varepsilon;$

$\leadsto [*] := \mathfrak{d}^{-1}\mathbf{UniformLimit} : \mathbf{UniformLimit}(D, f, F);$

□

1.5 Completeness and Completion

$\text{CauchyFilterBase} :: \prod X \in \text{SMS} . ?\text{FilterBase}(X)$

$\mathcal{F} : \text{CauchyFilterBase} \iff \inf_{F \in \mathcal{F}} \text{diam } F = 0$

$\text{CauchyFilter} := \prod X \in \text{SMS} . \text{Filter} \ \& \ \text{CauchyFilterBase}(X) : \text{SMS} \rightarrow \text{Type};$

$\text{ConvergentFilterBaseIsCauchy} :: \forall X \in \text{SMS} . \forall \mathcal{F} : \text{ConvergentFilterBase}(X) .$
 $\quad . \mathcal{F} : \text{CauchyFilterBase}(X)$

Proof =

...

□

$\text{CauchyNet} :: \prod X \in \text{SMS} . \prod D : \text{DirectedSet} . ?\text{Net}(D, x)$

$x : \text{CauchyNet} \iff \forall \varepsilon \in \mathbb{R} . \exists \Delta \in D : \forall \delta, \delta' \in D . d(x_\delta, x_{\delta'}) < \varepsilon$

$\text{CauchySequence} := \prod X \in \text{SMS} . \text{CauchyNet}(\mathbb{N}, X) : \prod_{X \in \text{SMS}} ?(\mathbb{N} \rightarrow X);$

$\text{CauchyFilterAndNetEquivalence} :: \forall X \in \text{SMS} . \forall (D, x) : \text{Net}(X) .$
 $\quad . x : \text{CauchyNet}(D, X) \iff \mathcal{F}(D, x) : \text{CauchyFilter}(X)$

Proof =

...

□

$\text{Complete} :: ?\text{SMS}$

$X : \text{Complete} \iff \forall \mathcal{F} : \text{CauchyFilterBase}(X) . \mathcal{F} : \text{ConvergentFilterBase}(X)$

$\text{CompleteAltDef} :: \forall X \in \text{SMS} . X : \text{Complete} \iff \forall x : \text{CauchySequence}(X) . x : \text{Convergent}(X)$

Proof =

Assume $R : \forall x : \text{CauchySequence}(X) . x : \text{Convergent},$

Assume $\mathcal{F} : \text{CauchyFilterbase},$

$\left(F, [1] \right) := \Lambda n \in \mathbb{N} . \delta \text{CauchyFilter}(X, \mathcal{F}) \left(\frac{1}{n} \right) : \prod_{n \in \mathbb{N}} \sum_{F_n \in \mathcal{F}} \text{diam } F_n < \frac{1}{n},$

$x := \delta \text{CauchyFilterbase}(\mathcal{F})(F) : \prod_{n \in \mathbb{N}} \bigcap_{i=1}^n F_i,$

$[2] := \delta^{-1} \text{CauchySequence}(X)[1] : \text{CauchySequence}(X, x),$

$[3] := R[2] : \text{Convergent}(X, x),$

$L := \lim_{n \rightarrow \infty} x_n : X,$

Assume $\varepsilon : \mathbb{R}_{++}$,

$$(N, [4]) := \text{Limit}(X, x, L) \left(\frac{\varepsilon}{2} \right) : \sum_{N \in \mathbb{N}} : \forall n \in \mathbb{N} . d(L, x_n) < \frac{\varepsilon}{2},$$

$$(M, [5]) := \text{InverseArchimedeanLimit} : \sum M \in \mathbb{N} . \frac{1}{M} < \frac{\varepsilon}{2},$$

$$k := \max(M, N) : \mathbb{N},$$

Assume $y : F_k$,

$$[\varepsilon.*] := \text{TriangleIneq}(X, L, y, x_k)[1, 4, 5] : d(L, y) \leq d(L, x_k) + d(x_k, y) < \varepsilon;$$

$$\leadsto [4] := \text{FilterbaseLimit} : L = \lim \mathcal{F},$$

$$[\mathcal{F}.*] := \text{ConvergentFilterBase}[4] : \text{ConvergentFilterBase}(X, \mathcal{F});$$

$$\leadsto [*] := \text{Complete} : \text{Complete}(X);$$

□

$$\text{CauchyFilterbaseClustersAreLimits} :: \forall X : \text{SMS} . \forall \mathcal{F} : \text{CauchyFilterBase}(X) . \\ . \forall c : \text{Cluster}(\mathcal{F}) . c : \text{Limit}(\mathcal{F})$$

Proof =

Assume $\varepsilon : \mathbb{R}_{++}$,

$$(F, [1]) := \text{CauchyFilterBase}(\mathcal{F}) \left(\frac{\varepsilon}{2} \right) : \sum_{F \in \mathcal{F}} \text{diam } F < \frac{\varepsilon}{2},$$

$$(x, [2]) := \text{Cluster}(\mathcal{F}, c) \left(F, \mathbb{B} \left(\frac{\varepsilon}{2} \right) \right) : \sum_{x \in F} d(x, c) < \frac{\varepsilon}{2},$$

Assume $y : F$,

$$[3] := \text{TriangleIneq}(X)(c, y, x) : d(c, y) \leq d(c, x) + d(x, y) < \varepsilon,$$

$$[y.*] := \text{cell}[3] : y \in \mathbb{B}(c, \varepsilon);$$

$$\leadsto [F.*] := \text{Subset} : F \subset \mathbb{B}(c, \varepsilon);$$

$$\leadsto [*] := \text{Limit} : \text{Limit}(X, \mathcal{F}, c);$$

□

$$\text{CauchyPartialLimitTHM} :: \forall X \in \text{SMS} . \forall x : \text{CauchySequence}(X) . \forall L : \text{PartialLimit}(X, x) . \lim_{n \rightarrow \infty} x_n = L$$

Proof =

Assume $\varepsilon : \mathbb{R}_{++}$,

$$(N, [1]) := \text{CauchySequence}(X, x) \left(\frac{\varepsilon}{2} \right) : \sum_{N \in \mathbb{N}} \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2},$$

$$(M, [2]) := \text{PartialLimit}(X, x, p) \left(p, \frac{\varepsilon}{2} \right) : \sum_{M \in \mathbb{N}} M \geq N \ \& \ d(x_M, L) < \frac{\varepsilon}{2},$$

Assume $n : \mathbb{N}$,

Assume $[3] : n \geq N$,

$$[4] := \text{TriangleIneq}[1][2] : d(x_n, L) < d(x_n, x_M) + d(x_M, L) < \varepsilon;$$

$$\leadsto [*] := \text{Limit} : \lim_{n \rightarrow \infty} x_n = L;$$

□

CompactIffCompleteAndTotallyBounded :: $\forall X \in \text{SMS} . \text{Compact}(X) \iff \text{Complete} \ \& \ \text{TotallyBounded}(X)$

Proof =

Assume $L : \text{Compact}(X)$,

$[1] := \text{CountablyCompactIsTotallyBounded}(X, L) : \text{TotallyBounded}(X)$,

$[2] := \text{SequantiallyCompactIffCompact}(X, L) : \text{SequantiallyCompact}(X)$,

Assume $x : \text{CauchySequence}(X)$,

$p := \text{SequantiallyCompact}(X)(x) : \text{PartialLimit}(X, x)$,

$[x.*] := \text{CauchyPartialLimitTHM}(X, x, p) : \lim_{n \rightarrow \infty} x_n = p$;

$\rightsquigarrow [x.*] := \text{CompleteAltDef} : \text{Complete}(X)$;

$\rightsquigarrow [1] := I(\Rightarrow) : \text{Left} \Rightarrow \text{Right}$,

Assume $R : \text{Complete} \ \& \ \text{TotallyBounded}(X)$,

Assume $x : \mathbb{N} \rightarrow X$,

$u^1 := x : \mathbb{N} \rightarrow X$,

Assume $n : \mathbb{N}$,

$[m, c, [2]] := \text{TotallyBounded}(X) \left(\frac{1}{n} \right) : \sum_{m \in \mathbb{N}} \sum_{c: m \rightarrow X} X = \bigcup_{i=1}^m \mathbb{B} \left(c_i, \frac{1}{n} \right)$,

$[i, [3]] := \text{PigionholePrinciple}[2] : \sum_{i \in m} \left| \left\{ k \in \mathbb{N} : u_k^n \in \mathbb{B} \left(c_i, \frac{1}{n} \right) \right\} \right| = \aleph_0$,

$(u^{n+1}, [n.*]) := \text{subsequence}[3] : \sum u^{n+1} : \text{Subsequence}(X, u^{n+1}) . \text{Im } u^{n+1} \subset \mathbb{B} \left(c_i, \frac{1}{n} \right)$;

$\rightsquigarrow (u, [2]) := I \left(\prod \right) : \prod_{u: \mathbb{N} \rightarrow \mathbb{N} \rightarrow X} . \forall n \in \mathbb{N} . u^{n+1} : \text{Subsequence}(X, u^n) \ \& \ \exists c \in X : \text{Im } u^{n+1} \mathbb{B} \left(c, \frac{1}{n} \right)$,

$\Delta := \Lambda n \in \mathbb{N} . u_n^n : \text{Subsequence}(X, x)$,

$[3] := \jmath \Delta[2] : \forall N \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m > N \rightarrow d(x_n, x_m) < \frac{2}{N}$,

$[4] := \text{SequantiallyCompact}[3] : \text{CauchySequence}(X, \Delta)$,

$[5] := \text{CompleteAltDef}(X)[4] : \text{Convergent}(X, \Delta)$,

$\delta := \lim_{n \rightarrow \infty} \Delta_n : X$,

$[x.*] := \text{PartialLimit}(x) : \text{PartialLimit}(X, x, \delta)$;

$\rightsquigarrow [* . R] := \text{SequantiallyCompactIffCompact} : \text{Compact}(X)$;

$\rightsquigarrow [*] := I(\iff) : \text{Left} \iff \text{Right}$;

□

ConvergentByCompleteSubset :: $\forall X \in \text{SMS} . \forall \mathcal{F} : \text{CauchyFilter}(X) . \left(\exists F \in \mathcal{F} : F : \text{Complete} \right) \Rightarrow$
 $\Rightarrow \text{Convergent}(X, \mathcal{F})$

Proof =

...

□

ConvergentByCompactSubset :: $\forall X \in \text{SMS} . \forall \mathcal{F} : \text{CauchyFilter}(X) . \left(\exists F \in \mathcal{F} : F : \text{Compact} \right) \Rightarrow$
 $\Rightarrow \text{Convergent}(X, \mathcal{F})$

Proof =

...

□

CompletionCategory :: $\text{SMS} \rightarrow \text{CAT}$

CompletionCategory $(X) = \text{COMP}(X) := \left(\sum Y : \text{Complete} . f : \text{Isometry}(X, Y), \right.$
 $\left. , \left((Y, f), (Z, g) \right) \mapsto \left\{ \varphi : \text{Isometry}(Y, Z) \mid f\varphi = g \right\}, \circ, \text{id} \right)$

Completion := $\Lambda X \in \text{SMS} . \text{Initial COMP}(X) : \text{SMS} \rightarrow \text{Type};$

CompleteByDenseSubset :: $\forall X \in \text{SMS} . \forall A : \text{Dense}(X) . \left(\forall a : \text{CauchySequence}(A) . a : \text{Convergent}(X) \right) \Rightarrow$
 $\Rightarrow X : \text{Complete}(X)$

Proof =

Assume $x : \text{CauchySequence}(X),$

$(a, [1]) := \Lambda n \in \mathbb{N} . \text{Dense}(X)(x_n) : \sum a : \mathbb{N} \rightarrow A . \forall n \in \mathbb{N} . d(a_n, x_n) < \frac{1}{n},$

Assume $\varepsilon : \mathbb{R}_{++},$

$(K, [2]) := \text{CauchySequence}(X, x) \left(\frac{\varepsilon}{3} \right) : \sum K \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m \geq K \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{3},$

$(L, [3]) := \text{ReducioInfinuma} \left(\frac{\varepsilon}{3} \right) : \sum L \in \mathbb{N} . \frac{1}{L} < \frac{\varepsilon}{3},$

$N := \max(K, L) : \mathbb{N},$

Assume $n, m : \mathbb{N},$

Assume $[4] : n, m \geq N,$

$[\varepsilon, *] := \text{TriangleIneq}^2(\dots)[1, 2, 3, 4] : d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(a_m, x_m) < \varepsilon;$

$\leadsto [4] := \text{CauchySequence}^{-1} : \text{CauchySequence}(A, a),$

$L := \lim_{n \rightarrow \infty} a_n : \text{In } X,$

Assume $\varepsilon : \mathbb{R}_{++},$

$(K, [5]) := \text{Limit}(X, a, L) \left(\frac{\varepsilon}{2} \right) : \sum K \in \mathbb{N} . \forall n, m \in \mathbb{N} . n \geq K \Rightarrow d(a_n, L) < \frac{\varepsilon}{2},$

$(M, [6]) := \text{ReducioInfinuma} \left(\frac{\varepsilon}{2} \right) : \sum M \in \mathbb{N} . \frac{1}{M} < \frac{\varepsilon}{2},$

$N := \max(K, M) : \mathbb{N},$

Assume $n : \mathbb{N},$

Assume $[7] : n > N,$

$[\varepsilon, *] := \text{TriangleIneq}[1, 5, 6, 7] : d(x_n, L) \leq d(x_n, a_n) + d(a_n, L) < \varepsilon;$

$\leadsto [x, *] := \text{Limit}^{-1} : \lim_{n \rightarrow \infty} x_n = L;$

$\leadsto [*] := \text{CompleteAltDef} : \text{Complete}(X);$

□

CompletionIsUnique :: $\forall X \in \text{SMS} . \forall (A, a), (B, b) : \text{Completion}(X) . (A, B) : \text{Semiisometric}$

Proof =

...

□

RealsAreComplete :: $\mathbb{R} : \text{Complete}$

Proof =

...

□

CauchySequanceDistanceExists :: $\forall X \in \text{SMS} . \forall x, y : \text{CauchySequance}(X) . d(x, y) : \text{Convergent}(\mathbb{R}_+)$

Proof =

Assume $\varepsilon : \mathbb{R}_{++}$,

$(L, [1]) := \text{CauchySequance}(X, x) \left(\frac{\varepsilon}{2} \right) : \sum L \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m > L \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2},$

$(M, [2]) := \text{CauchySequance}(X, y) \left(\frac{\varepsilon}{2} \right) : \sum M \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m > M \Rightarrow d(y_n, y_m) < \frac{\varepsilon}{2},$

$N := \max(L, M) : \mathbb{N}$,

Assume $n, m : \mathbb{N}$,

Assume $[3] : n, m \geq N$,

$[\varepsilon.*] := \text{TriangleIneq}(d(x_n, y_n), d(x_m, y_m), d(x_n, y_m))$

$\text{ReversedTriangleIneq}(x_n, y_n, y_m) \text{ReversedTriangleIneq}(x_n, x_m, y_m)[1, 2, 3] :$

$: \left| d(x_n, y_n) - d(x_m, y_m) \right| \leq \left| d(x_n, y_n) - d(x_n, y_m) \right| + \left| d(x_n, y_m) - d(x_m, y_m) \right| \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon;$

$\leadsto [4] := \text{CauchySequance} : \text{CauchySequance}(\mathbb{R}, d(x, y)),$

$[*] := \text{RealsAreComplete CompleteAltDef}(\mathbb{R}) : \text{Convergent}(\mathbb{R}, d(x, y));$

□

CauchyMetric :: $\forall X \in \text{SMS} . \lim d : \text{Semimetric CauchySequance}(X)$

Proof =

...

□

completion :: $\text{SMS} \rightarrow \text{MS}$

$\text{completion}(X, d) = (\hat{X}, \hat{d}) := \frac{(\text{CauchySequance}(X), \lim d)}{\lim d}$

CompletionTHM :: $\forall X \in \mathbf{SMS} . \left(\hat{X}, \Lambda x \in X . \Lambda n \in \mathbb{N} . x \right) : \mathbf{Completion}(X)$

Proof =

Assume $p : \hat{X}$,

$\left(x, [1] \right) := \mathfrak{d}\hat{X}(p) : \sum x : \mathbf{CauchySequance}(X) . p = [x],$

$A := \Lambda n \in \mathbb{N} . \Lambda m \in \mathbb{N} . \text{if } m < n \text{ then } x_m \text{ else } x_n : \mathbb{N} \rightarrow \mathbf{Convergent}(X),$

Assume $\varepsilon : \mathbb{R}_{++}$,

$\left(N, [2] \right) := \mathfrak{d}\mathbf{CauchySequance}(X, x)(\varepsilon) : \sum N \in \mathbb{N} . \forall n, m \in \mathbb{N} n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon,$

Assume $n : \mathbb{N}$,

Assume $[3] : n \geq N$,

$[p.*] := \mathfrak{d}\mathbf{completion}[1]jA[2, 3] : \hat{d}(p, [A_n]) = \lim_{m \rightarrow \infty} d(x_m, A_{n,m}) = \lim_{m \rightarrow \infty} d(x_m, x_n) < \varepsilon;$

$\leadsto [1] := \mathfrak{d}^{-1}\mathbf{Dense} : \mathbf{Dense}(\hat{X}, X),$

$[2] := \mathbf{CompleteByDenseSubset}(\hat{X}, X) : \mathbf{Complete}(\hat{X}),$

$\phi := \Lambda x \in X . \left[\Lambda n \in \mathbb{N} . x \right] : \mathbf{Isometry}(X, \hat{X}),$

Assume $(Y, f) : \mathbf{InCOMP}(X),$

$\psi := \Lambda [x] \in \hat{X} . \lim_{n \rightarrow \infty} f(x_n) : \mathbf{Isometry}(\hat{X}, Y),$

Assume $x : X$,

$[x.*] := j\phi j\psi \mathbf{ConstantSeq} : x\phi\psi = [n \mapsto x]\psi = \lim_{n \rightarrow \infty} f(x) = f(x);$

$\leadsto [3] := \mathfrak{d}^{-1}\mathbf{COMP}(X) : \left(\psi : \hat{X} \xrightarrow{\mathbf{COMP}(X)} Y \right),$

Assume $\psi' : \hat{X} \xrightarrow{\mathbf{COMP}(X)} Y,$

Assume $p : \hat{X}$,

$\left(x, [4] \right) := \mathfrak{d}\hat{X}(p) : \sum x : \mathbf{CauchySequance}(X) . p = [x],$

$A := \Lambda n \in \mathbb{N} . \Lambda m \in \mathbb{N} . \text{if } m < n \text{ then } x_m \text{ else } x_n : \mathbb{N} \rightarrow \mathbf{Convergent}(X),$

$\left[(Y, f) \right] := [4]j^{-1}A\mathbf{ContinuousLimit}j^{-1}\phi\mathfrak{d}\mathbf{COMP}(X) : \psi'(p) = \psi'[x] = \psi' \lim_{n \rightarrow \infty} [A_n] = \lim_{n \rightarrow \infty} \psi'[A_n] =$
 $= \lim_{n \rightarrow \infty} \psi'\phi(x_n) = \lim_{n \rightarrow \infty} f(x_n) \lim_{n \rightarrow \infty} \psi\phi(x_n) = \psi(p);$

$\leadsto [*] := \mathfrak{d}^{-1}\mathbf{Completion} : \mathbf{Completion}\left(X, (\hat{X}, \varphi) \right);$

□

Contraction := $\Lambda X \in \mathbf{MS} . \sum \alpha \in (0, 1) . \alpha\text{-Lip}(X) : \mathbf{MS} \rightarrow \mathbf{Type};$

BanachFixedPointTHM :: $\forall X \in \mathbf{MS} \ \& \ \mathbf{Complete} \ \& \ \mathbf{NonEmpty} . \forall f : \mathbf{Contraction}(X) . \exists ! x \in X : f(x) = x$

Proof =

$$(\alpha, [1]) := \mathfrak{d}\mathbf{Contraction}(f) : \sum_{\alpha \in (0,1)} \forall x, y \in X . d(f(x), f(y)) \leq \alpha * d(f(x), f(y)),$$

$$x := \mathfrak{d}\mathbf{NonEmpty}(X) : \mathbf{In}(X),$$

$$\delta := d(x, f(x)) : \mathbb{R}_+,$$

$$\mathbf{Assume} \ \varepsilon : \mathbb{R}_{++},$$

$$(N, [2]) := \mathbf{PowerReduction} : \sum_{N \in \mathbb{N}} \alpha^N < \frac{(1 - \alpha)\varepsilon}{\delta},$$

$$\mathbf{Assume} \ n, m : \mathbb{N},$$

$$\mathbf{Assume} \ [3] : n, m \geq N,$$

$$\mathbf{Assume} \ [4] : n \geq m,$$

$$[\varepsilon.*] := \mathbf{TriangleInequality}[4]\mathbf{NonNegSum}[3][1]\mathbf{PowerSum}[2][3] :$$

$$: d(f^n(x), f^m(x)) \leq \sum_{i=0}^{n-m} d(f^{m+i}(x), f^{m+i+1}(x)) \leq \sum_{i=0}^{\infty} d(f^{N+i}(x), f^{N+i+1}(x)) \leq \sum_{i=0}^{\infty} \delta \alpha^{N+i} = \frac{\delta \alpha^N}{1 - \alpha} < \varepsilon;$$

$$\leadsto [2] := \mathfrak{d}^{-1}\mathbf{CauchySequance} : \mathbf{CauchySequance}(X, \Lambda n \in \mathbb{N} . f^n(x)),$$

$$[3] := \mathbf{AltCompleteDef}[2] : \mathbf{Convergent}(X, \Lambda n \in \mathbb{N} . f^n(x)),$$

$$p := \lim_{n \rightarrow \infty} f^n(x) : \mathbf{In}X,$$

$$\mathbf{Assume} \ \varepsilon : \mathbb{R}_{++},$$

$$(M, [4]) := \mathfrak{d}\mathbf{Limit}(X, \Lambda n \in \mathbb{N} . f^n(x), L) : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \geq N \Rightarrow d(f^n(x), p) < \frac{\varepsilon}{3},$$

$$(K, [5]) := \mathfrak{d}\mathbf{CauchySequance}(X, \Lambda n \in \mathbb{N} . f^n(x)) : \sum K \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(f^n(x), f^m(x)) < \frac{\varepsilon}{3},$$

$$N := \max(M, K) : \mathbb{N},$$

$$\mathbf{Assume} \ n : \mathbb{N},$$

$$\mathbf{Assume} \ [6] : n \geq N,$$

$$[\varepsilon.*] := \mathbf{TriangleIneq}[1](f^n x, p)[4, 5, 6] : d(p, f x) \leq d(p, f^n x) + d(f^n x, f^{n+1} x) + d(f^{n+1}(x), f p) < 2d(p, f^n x) + d(f^n x, f^{n+1} x) < \varepsilon;$$

$$\leadsto [4] := \mathbf{ZeroBound} : d(L, f p) = 0,$$

$$[5] := \mathfrak{d}\mathbf{MS}[4] : p = f p;$$

$$\mathbf{Assume} \ q : \mathbf{TypeIn}X,$$

$$\mathbf{Assume} \ [6] : q = f q,$$

$$[7] := [1](p, q)[5][6] : d(p, q) \leq \alpha d(f, f q) = \alpha d(p, q),$$

$$[8] := \mathbf{FixedMultiolicationIsZero}[7] : d(p, q) = 0,$$

$$[q.*] := \mathfrak{d}^{-1}\mathbf{MS}[8] : p = q;$$

$$\leadsto [*] := \mathfrak{d}^{-1}\mathbf{Unique} : \exists ! x \in X : f(x) = x;$$

□

1.6 Baire Category

CantorIntersectionTheorem :: $\forall X \in \text{SMS} \ \& \ \text{Complete} . \forall A : \text{Decreasing} \left(\text{Closed}(X) \ \& \ \text{NonEmpty} \right) .$

$$. \forall [0] : \lim_{n \rightarrow \infty} \text{diam } A_n = 0 . \bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

Proof =

$$a := \text{NonEmpty}(A) : \prod_{n \in \mathbb{N}} A_n,$$

$$[1] := \text{CauchySequence}[0] a : \text{CauchySequence}(X, a),$$

$$[2] := \text{AltCompleteDef} : \text{Convergent}(X, a),$$

$$K := \lim_{n \rightarrow \infty} a_n : \text{NonEmpty}(X),$$

Assume $x : K$,

$$[3] := \Lambda n \in \mathbb{N} . \text{ClosedLimit}(X, A_n, a_{+n}, x) : \forall n \in \mathbb{N} . x \in A_n,$$

$$[x.*] := \text{Intersection} : x \in \bigcap_{n=1}^{\infty} A_n;$$

$$\leadsto [3] := \text{Subset} : K \subset \bigcap_{n=1}^{\infty} A_n,$$

$$[*] := \text{NonEmpty}[3] : \bigcap_{n=1}^{\infty} A_n \neq \emptyset;$$

□

MetricBaireCategoryTheorem :: $\forall X \in \text{SMS} \ \& \ \text{Complete} . X : \text{Baire}$

Proof =

Assume $U : \mathbb{N} \rightarrow \text{Dense} \ \& \ \text{Open}(X)$,

Assume $x : \text{In } X$,

Assume $\varepsilon : \mathbb{R}_{++}$,

$$u_1 := x : \text{In } X,$$

$$r_1 := \varepsilon : \mathbb{R}_{++},$$

$$\delta_1 := 1 : \mathbb{R}_{++},$$

Assume $n : \mathbb{N}$,

$$[1] := \text{OpenDenseIntersection} : \bigcap_{i=1}^n U_i : \text{Open} \ \& \ \text{Dense}(X),$$

$$V := \mathbb{B}(x_n, r_n) \cap \bigcap_{i=1}^n U_i : \text{Open} \ \& \ \text{NonEmpty}(X),$$

$$v := \text{NonEmpty}(X, V : \text{In}(V)),$$

$$\Delta := d(v, \partial V) : \mathbb{R}_{++},$$

$$\delta_{n+1} := \frac{\min(\Delta, \delta_n)}{2} : \mathbb{R}_{++},$$

$$K_n := \{v \in V : d(v, \partial V) \leq \delta_{n+1}\} : \text{NonEmpty}(V),$$

$$[2] := \text{ContinuousDistance}(\overline{V}, \partial V) \text{ClosedPreimage}_J K_n : \text{Closed}(X, K_n),$$

$$u_{n+1} := \text{NonEmpty} : K_n,$$

$$r_{n+1} := \min \left(\frac{r_n}{2}, \delta_{n+1} \right) : \mathbb{R}_{++};$$

$$\leadsto \left(K, [1] \right) := I \left(\sum \right) : \sum K : \text{Decreasing Closed}(X) . \lim_{n \rightarrow \infty} \text{diam } K_n = 0 \ \& \ \forall n \in \mathbb{N} . K_n \subset \bigcap_{i=1}^n U_n \cap \mathbb{B}(x, \varepsilon),$$

$$[2] := \text{CantorIntersectionTheorem}(X, K, [1]) : \bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

$$[x.*] := \text{IntersectionSubset}(X, K, [1], [2]) : \bigcap_{n=1}^{\infty} K_n \subset \mathbb{B}(x, \varepsilon) \cap \bigcap_{n=1}^{\infty} U_n;$$

$$\leadsto [U.*] := \mathfrak{d}^{-1} \text{Dense} : \text{Dense} \left(X, \bigcup_{n=1}^{\infty} U_n \right);$$

$$\leadsto [*] := \mathfrak{d}^{-1} \text{Baire} : \text{Baire}(X);$$

□

$$\text{FirstCategory} :: \prod X \in \text{TOP} . ??X$$

$$A : \text{FirstCategory} \iff A \neq \emptyset \ \& \ \exists P : \mathbb{N} \rightarrow \text{NowhereDense}(X) . \bigcup_{n=1}^{\infty} P_n = A$$

$$\text{FirstCategoryTheorem1} :: \forall X : \text{Baire} . \forall U \in \mathcal{T}(X) . U ! \text{FirstCategory}$$

Proof =

$$\text{Assume } [1] : \text{FirstCategory}(X, U),$$

$$[2] := \mathfrak{d}_1 \text{FirstCategory}(X, U)[1] : U \neq \emptyset,$$

$$\left(A, [3] \right) := \mathfrak{d}_2 \text{FirstCategory}(X, U)[1] : \sum A : \mathbb{N} \rightarrow \text{NowhereDense}(X) . U = \bigcap_{n=1}^{\infty} A_n,$$

$$[3] := \text{DualBaireProperty}[3] : \text{Codense}(U),$$

$$[4] := \mathfrak{d} \text{Codense}(X, U) : \text{Dense}(X, U^c),$$

$$[5] := \text{DenseOpenIntersection}[4][2] : U \cap U^c \neq \emptyset,$$

$$[1.*] := \text{ComplementIntersection}(U) \text{EmptyIsNonEmpty} : \perp;$$

$$\leadsto [*] := E(\perp) : ! \text{FirstCategory}(X, U);$$

□

$$\text{FirstCategoryTheorem2} :: \forall X \in \text{TOP} . \forall [0] : \forall U \in \mathcal{T}(X) . U ! \text{FirstCategory} . X : \text{Baire}$$

Proof =

$$\text{Assume } U : \mathbb{N} \rightarrow \text{Open} \ \& \ \text{Dense}(X),$$

$$\text{Assume } [1] : \bigcap_{n=1}^{\infty} U_n ! \text{Dense}(X),$$

$$\left(V, [2] \right) := \mathfrak{d} \text{Dense}[1] : \sum V \in \mathcal{T}(X) . V \cap \bigcap_{n=1}^{\infty} U_n = \emptyset,$$

$$A := V \cap U^c : \mathbb{N} \rightarrow \text{Codense}(X),$$

$$[3] := jA[2] : V = \bigcup_{i=1}^{\infty} A_i,$$

$$[1.*] := [0](V) \mathfrak{d}^{-1} \text{FirstCategory}[3] : \perp;$$

$$\leadsto [U.*] := E(\perp) : \bigcap_{n=1}^{\infty} U_n = \text{Dense}(X);$$

$$\leadsto [*] := \mathfrak{d}^{-1} \text{Baire} : \text{Baire}(X);$$

□

1.7 Hausdorff Metric

$\text{metricOfHausdorff} :: \text{MS} \rightarrow \text{MS}$

$\text{metricOfHausdorff}(X) = \mathcal{H}(X) = \left(\mathcal{H}(X), d_H \right) := \left(\text{Closed} \ \& \ \text{Bounded} \ \& \ \text{Nonempty}(X), A, B \mapsto \max_{x \in A} \left(\sup_{y \in B} d(x, y) \right) \right)$

$\text{Assume } A, B, C : \text{NonEmpty} \ \& \ \text{Compact}(X),$

$\text{Assume } [1] : d(A, B) = \sup_{x \in A} d(x, B),$

$\left(a, [2] \right) := \text{supremum} : \sum a : \mathbb{N} \rightarrow A . d(A, B) = \lim_{n \rightarrow \infty} d(a_n, B),$

$\text{Assume } n : \mathbb{N},$

$\text{Assume } c : C,$

$[c.*] := [1] \text{setDistance}(B) \forall x \in A . \forall y \in B \text{TriangleIneq}(x, y, c) \text{InfReduction}(B) \text{SupReduction}(B)$

$\text{SumIneq}(\mathbb{R}, \dots) \text{SupremumIntroduction}(C) \text{supremum}^{-1} \text{metricOfHausforff} :$

$: d(a_n, B) = \inf_{b \in B} d(a_n, b) \leq d(a_n, c) + \inf_{b \in B} d(c, b) \leq d(a_n, c) + \sup_z \inf_b d(z, b) \leq d(a_n, c) + d(C, B);$

$\leadsto [1.*] := \text{UniversalIneq} : d(a_n, B) \leq \inf_{z \in C} d(a_n, z) + d(C, B) \leq d(A, C) + d(C, B),$

$[1] := I(\Rightarrow) : d(A, B) = \sup_{x \in A} d(x, B) \Rightarrow d(A, B) \leq \sup_{x \in A} d(x, B) + d(C, B);$

□

$\text{blowUp} :: \prod X \in \text{SMS} . \mathbb{R}_{++} \rightarrow (?X) \rightarrow (?X)$

$\text{blowUp}(\varepsilon, A) = A_\varepsilon := \bigcup_{a \in A} \mathbb{B}(a, \varepsilon)$

$\text{HausdorffMetricAltDef} :: \forall X \in \text{MS} . \forall A, B : \text{Compact}(X) . d(A, B) = \inf \left\{ \varepsilon \in \mathbb{R}_{++} \mid B \subset A_\varepsilon \ \& \ A \subset B_\varepsilon \right\}$

$\text{Proof} =$

...

□

$\text{HausdorffFunctor} :: \text{MS}_{\text{o} \rightarrow \cdot} \xrightarrow{\text{CAT}} \text{MS}_{\text{o} \rightarrow \cdot}$

$\text{HausdorffFunctor}(X) = \mathcal{H}(X) := \mathcal{H}(X)$

$\text{HausdorffFunctor}(X, Y, f) = \mathcal{H}(f) := \Lambda A \in \mathcal{H}(X) . \text{cl } f(A)$

$\text{CauchyDiamConverge} :: \forall X \in \text{MS} . \forall A : \text{Cauchy}(X) . \text{diam } A : \text{Convergent}(\mathbb{R})$

$\text{Proof} =$

...

□

HausdorffMetrizationANdCompletionCommute :: $\forall X \in \mathbf{MS}.\mathcal{H}(\widehat{X}) \cong_{\mathbf{MS}_{\circ \rightarrow}} \widehat{\mathcal{H}(X)}$

Proof =

Assume $A : \mathcal{H}(\widehat{X})$,

Assume $a : A$,

$(x^a, [1]) := \widehat{\partial}(X)(a) : \sum x : \mathbf{Cauchy}(X) . a = [x^a] \ \& \ \forall N \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(x_m^a, x_n^a) \leq N^{-1}$;

$\leadsto (x, [1]) := I(A) : \prod_{a \in A} \sum x : \mathbf{Cauchy}(X) . a = [x^a] \ \& \ \forall N \in \mathbb{N} . \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(x_m^a, x_n^a) \leq N^{-1}$,

$B := \Lambda n \in \mathbb{N} . \text{cl}_X \{x_n^a | a \in A\} : \mathbb{N} \rightarrow \mathbf{Closed}(X)$,

Assume $n : \mathbb{N}$,

Assume $b, b' : \mathbf{In}(B_n)$,

Assume $\varepsilon : \mathbb{R}_{++}$,

$(a', a, [2]) := \widehat{\partial} \mathbf{closure}_j B_n(b, b') : \sum a', a : A . d(b, x_n^a) < \varepsilon \ \& \ d(b', x_n^{a'}) < \varepsilon$,

$[n.*] := \mathbf{TriangleIneq}^4[2][1] : d(b', b) \leq d(b, x_n^a) + d(x_n^a, a) + d(a, a') + d(a', x_n^{a'}) + d(x_n^{a'}, b') < \text{diam } A + \frac{2}{n} + 2\varepsilon$;

$\leadsto [2] := I(\forall) : \forall n \in \mathbb{N} . \text{diam } B_n \leq \text{diam } A + \frac{2}{n}$,

Assume $\varepsilon : \mathbb{R}$,

$(N, [3]) := \mathbf{ReductionInfinuma}(\varepsilon) : \prod N \in \mathbb{N} . \varepsilon < \frac{1}{N}$,

Assume $n, m : \mathbb{N}$,

Assume $[4] : n, m \geq N$,

$[\varepsilon.*] := \widehat{\partial} \mathcal{H}(X) \widehat{\partial} \mathbf{supremum}[1][3][4] : d(B_n, B_m) = \max \left(\sup_{a \in A} d(x_n^a, B_m), \sup_{a \in A} d(x_m^a, B_n) \right) \leq \sup_{a \in A} d(x_n^a, x_m^a) \leq \frac{1}{N} < \varepsilon$;

$\leadsto [3] := \widehat{\partial}^{-1} \mathbf{CauchySequance} : \mathbf{CauchySequance}(\mathcal{H}(X), B)$,

$\varphi(X) := \lim_{n \rightarrow \infty} X_n : \widehat{\mathcal{H}(X)}$;

$\leadsto \varphi := I(\rightarrow) : \mathcal{H}(\widehat{X}) \rightarrow \widehat{\mathcal{H}(X)}$,

Assume $A, B : \mathcal{H}(\widehat{X})$,

$(\alpha, [4]) := j\varphi(A) : \sum \alpha : \mathbf{Cauchy}(\mathcal{H}(X)) \varphi(A) = [\alpha] \ \& \ \lim_{n \rightarrow \infty} \text{cl}_{\widehat{X}} \alpha_n = A$,

$(\beta, [5]) := j\varphi(B) : \sum \beta : \mathbf{Cauchy}(\mathcal{H}(X)) \phi(B) = [\beta] \ \& \ \lim_{n \rightarrow \infty} \text{cl}_{\widehat{X}} \beta_n = B$,

$[A, B, [*]] := [\widehat{\partial} \mathbf{Complement}4][5] : d(\varphi(A), \varphi(B)) = \lim_{n \rightarrow \infty} d(\alpha_n, \beta_n) = d(A, B)$;

$\leadsto [4] := \widehat{\partial}^{-1} \mathbf{Isometry} : \mathbf{Isometry}(\mathcal{H}(\widehat{X}), \widehat{\mathcal{H}(X)})$,

Assume $A : \widehat{\mathcal{H}(X)}$,

$(K, [1]) := \widehat{\partial} \mathbf{Completion}(\mathcal{H}(X)) : \prod K : \mathbf{CachySequance}(\mathcal{H}(X)) . A = [K]$,

$C := \left\{ x : \mathbf{CauchySequance}(X) | \forall n, m \in \mathbb{N} . x_n \in K_n \ \& \ d(x_n, x_m) \leq d(K_n, K_m) \right\} : \mathbf{Cauchy}(X)$,

$\psi(A) := \text{cl} \{ \lim_{n \rightarrow \infty} x_n | x \in C \} : \mathbf{Closed}(\widehat{X})$,

$[A.*] := \mathbf{CauchyDiamConverge} : \mathbf{Bounded}(\widehat{X}. \psi(A))$;

$\leadsto \psi := I(\rightarrow) : \widehat{\mathcal{H}(X)} \rightarrow \mathcal{H}(\widehat{X})$,

Assume $A, B : \widehat{\mathcal{H}(X)}$,

$$(\alpha, [4]) := j\varphi(A) : \sum \alpha : \text{Cauchy}(\mathcal{H}(X)) . A = [\alpha],$$

$$(\beta, [5]) := j\varphi(B) : \sum \beta : \text{Cauchy}(\mathcal{H}(X)) . B = [\beta],$$

$$C_A := \left\{ x : \text{CauchySequence}(X) \mid \forall n, m \in \mathbb{N} . x_n \in \alpha_n \ \& \ d(x_n, x_m) \leq d(\alpha_n, \alpha_m) \right\} : ?\text{Cauchy}(X),$$

$$C_B := \left\{ x : \text{CauchySequence}(X) \mid \forall n, m \in \mathbb{N} . x_n \in \beta_n \ \& \ d(x_n, x_m) \leq d(\beta_n, \beta_m) \right\} : ?\text{Cauchy}(X),$$

$$[(A, B).*] := \text{HausdorffMetric} \text{ completion } \text{UniformCauchyExchange}[4][5] \text{ completion} ::$$

$$d(\psi(A), \psi(B)) = \max \left(\sup_{a \in \psi(A)} \inf_{b \in \psi(B)} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right) =$$

$$= \max \left(\lim_{n \rightarrow \mathbb{N}} \lim_{m \rightarrow \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m), \lim_{n \rightarrow \mathbb{N}} \lim_{m \rightarrow \mathbb{N}} \sup_{x \in C_A} \inf_{y \in C_B} d(x_n, y_m) \right) = \lim_{n \in \mathbb{N}} \lim_{m \in \mathbb{N}} d(\alpha_n, \beta_m) = d(A, B);$$

$$\leadsto [5] := \text{Isometry}^{-1} : \text{Isometry}(\widehat{\mathcal{H}(X)}, \mathcal{H}(\widehat{X})),$$

Assume $A : \mathcal{H}(\widehat{X})$,

$$(\alpha, [6]) := j\varphi(A) : \sum \alpha : \text{Cauchy}(\mathcal{H}(X)) \varphi(A) = [\alpha] \ \& \ \lim_{n \rightarrow \infty} \text{cl}_{\widehat{X}} \alpha_n = A,$$

$$[7] := [6] \text{ContinuousLimitCommute} j\psi : d(\psi\varphi A, A) = d(\psi \lim_{n \rightarrow \infty} \alpha_n, A) = \lim_{n \rightarrow \infty} d(\psi\alpha_n, A) = \lim_{n \rightarrow \infty} d(\text{cl}_{\widehat{X}} \alpha_n, A) = d(A, A)$$

$$[A.*] := \text{MS}[7][8] : \psi\varphi A = A;$$

$$\leadsto [6] := I(\rightarrow, =) : \psi\varphi = \text{id},$$

$$[*] := \text{IsometricSurjectionIsBijection}(\psi) \text{RightInverseIsBijective}[6] : \psi = \varphi^{-1};$$

□

$$\text{HausdorffCompleteIffComplete} :: \forall X \in \text{MS} . \text{Complete}(X) \iff \text{Complete } \mathcal{H}(X)$$

Proof =

...

□

$$\text{CompactHausdorffConvergence} :: \forall X : \text{Complete} . \forall K : \text{CauchySequence}(\mathcal{H}(X)) .$$

$$. \forall [0] : \forall n \in \mathbb{N} . \text{Compact}(K_n) . \lim_{n \rightarrow \infty} (\text{Compact})$$

Proof =

$$A := \text{cl} \bigcup_{n=1}^{\infty} K_n : \text{Closed}(X),$$

$$[1] := \text{ClosedSubsetOfComplete}(X, A) : \text{Complete}(A),$$

Assume $\varepsilon : \mathbb{R}_{++}$,

$$(N, [2]) := \text{CauchySequence}(\mathcal{H}(X), K) \left(\frac{\varepsilon}{8} \right) : \sum_{N \in \mathbb{N}} \forall n, m \in \mathbb{N} . n, m \geq N \Rightarrow d(K_n, K_m) < \frac{\varepsilon}{8},$$

$$(k, E, [3]) := \text{CompactIsTotallyBounded} \left((K_n)_{n=1}^N \right) \text{TotallyBounded} :$$

$$: \prod_{n=1}^N \sum_{k \in \mathbb{N}} \sum_{E: k \rightarrow ? K_n} \forall i \in k_n . \text{diam } E < \frac{\varepsilon}{2} \ \& \ \bigcup_{i=1}^{k_n} E_i = K_n,$$

$$F := \Lambda n \in N . \Lambda i \in k_n . \left\{ x \in A . d(x, E_i) \leq \frac{\varepsilon}{4} \right\} : \prod_{n=1}^N k_n \rightarrow ? A,$$

$$[\varepsilon.1.*] := jF \text{HausdorffMetric}^{-1} \text{diam} : \forall n \in N . \forall i \in k_n . \text{diam } F_{n,i} \leq \varepsilon,$$

Assume $a : A$,

$$(x, [5]) := \text{MetricClosure}_J A : \sum x \in \bigcup_{n=1}^{\infty} K_n . d(x, a) < \frac{\varepsilon}{8},$$

$$(n, [6]) := \text{Union}(K, x) : \sum n \in \mathbb{N} . x \in K_n,$$

$$(m, [7]) := \text{DirechletPrinciple}[2] : \sum m \in N . d(K_m, K_n) < \frac{\varepsilon}{8},$$

$$(y, [8]) := \text{distanceOfHausdorff}[7] : \sum y \in K_m . d(y, x) < \frac{\varepsilon}{8},$$

$$[9] := \text{TriangleIneq}[5][8] : d(a, y) \leq d(a, x) + d(x, y) < \frac{\varepsilon}{4},$$

$$(i, [10]) := [3](y) : \sum i \in k_i . y \in E_i,$$

$$[a.*] := j[10][9] : a \in F_{n,i};$$

$$\leadsto [\varepsilon.2.*] := \text{Union} : \bigcup_{n=1}^N \bigcup_{i=1}^{k_n} F_i = A;$$

$$\leadsto [2] := \text{TotallyBounded} : \text{TotallyBounded}(A),$$

$$[3] := \text{CompleteAndTotallyBoundedIsCompact}[2] : \text{Compact}(A),$$

$$[4] := \text{H}(X) : \lim_{n \rightarrow \infty} K_n \subset A,$$

$$[*] := \text{ClosedSubsetCompact}[3, 4] : \text{Compact}(\lim_{n \rightarrow \infty} K_n);$$

□

$$\text{SingletonHausdorffLimit} :: \forall X \in \text{MS} . \forall p : \text{Converging}(\mathcal{X}) . \forall n \in \mathbb{N} . |p_n| = 1 \Rightarrow \left| \lim_{n \rightarrow \infty} x_n \right|$$

Proof =

...

□

$$\text{CompactHausdorffMetrIffCompact} :: \forall X \in \text{MS} . \text{Compact}(X) \iff \text{Compact}(\mathcal{H}(X))$$

Proof =

Assume $[0] : \text{Compact}(X)$,

$$[1] := \text{HausdorffMetrizationAndCompletionCommute}(X) : \text{Complete}(\mathcal{X}),$$

Assume $\varepsilon : \mathbb{R}_{++}$,

$$(n, A, [2]) := \text{CompactTotallyBounded}(X) \text{Union} : \sum_{n=1}^{\infty} \sum_{A: n \rightarrow ?X} \bigcup_{i=1}^n A_i = X \ \& \ \forall i \in n . \text{diam } A \leq \varepsilon,$$

$$B := \Lambda b \in 2^n . \{K \in \mathcal{H}(X) : \forall i \in . b_i = 1 \iff A_i \cap K \neq \emptyset\} : 2^n \rightarrow ?\mathcal{H}(X),$$

Assume $b : 2^n$,

Assume $K, K' : B_b$,

$$[b.*] := jB \text{distanceOfHausdorff}[2] : d(K, K') \leq \max_{i \in b} d(K \cap A_i, K' \cap A_i) \leq \varepsilon;$$

$$\leadsto [\varepsilon.*.1] := I(\forall) : \forall b \in 2^n . \text{diam } B_b \leq \varepsilon,$$

$$[\varepsilon.*.2] := jB[2] : \bigcup_{b \in 2^n} = \mathcal{H}(X);$$

$$\leadsto [2] := \text{TotallyBounded} : \text{TotallyBounded}(\mathcal{H}(X)),$$

$$[0.*] := \text{CompactIfCompleteAndTotallyBounded}(\mathcal{H}(X)) : \text{Compact}(\mathcal{H}(X));$$

$$\leadsto LR := I(\Rightarrow) : \text{Compact}(X) \Rightarrow \text{Compact}(\mathcal{H}(X)),$$

Assume $R : \text{Compact}(\mathcal{H}(X),$

Assume $x : \mathbb{N} \rightarrow X,$

$[P',] := \text{SequanceCompactIffCompact}(\mathcal{H}(X))\tilde{\text{SequanceCompact}} : \sum P' : \text{Subsequence}\{x\} . P' : \text{Converging}$

$P := \lim_{n \rightarrow \infty} p' : \text{In}\mathcal{H}(X),$

$(p, [1]) := \text{SingletonHausdorffLimit}(P) : \sum p \in X . P = \{p\},$

$(x', [x.*])) := [1]p\tilde{\text{distanceOfHausdorff}} : \sum x' : \text{Subsequence}(X) . p = \lim_{n \rightarrow \infty} x_n;$

$\sim [R.*] := \text{SequanceCompactIffCompact}(X) : \text{Compact}(X);$

$\sim [R.*] := I(\iff) : \text{Compact}(X) \iff \text{Compact}(\mathcal{H}(X));$

□

1.8 Geodesic Paths and Hopf-Rinow Theorem

RectifiablePath :: $\prod X \in \mathbf{MS} . \prod x, y \in X . ?\Omega(x, y)$

$\gamma : \mathbf{RectifiablePath} \iff \gamma \in R(x, y) \iff$

$$\iff \mathbf{ConvergingNet} \left(\mathbb{R}_+, M(\gamma), \Lambda(n, \omega) \in M(\gamma) . \sum_{i=1}^n d_X(\omega_i(0), \omega_i(1)) \right)$$

LengthSpace :: ?**MS**

$X : \mathbf{LengthSpace} \iff \forall x, y \in X . R(x, y) \neq \emptyset \iff$

arclength :: $\prod X \in \mathbf{MS} . \prod x, y \in X . R(x, y) \rightarrow \mathbb{R}_+$

$\mathbf{arclength}(\gamma) = |\gamma| := \lim_{(n, \omega) \in M(\gamma)} \sum_{i=1}^n d_X(\omega_i(0), \omega_i(1))$

ArclengthSum :: $\forall X \in \mathbf{MS} . \forall x, y, z \in X . \forall \alpha \in R(x, y) . \forall \beta \in R(y, z) . |\alpha\beta| = |\alpha| + |\beta|$

Proof =

...

□

ArclengthInversion :: $\forall X \in \mathbf{MS} . \forall x, y \in X . \forall \gamma \in R(x, y) . |\gamma^\circ| = \gamma$

Proof =

...

□

ConstantArclength :: $\forall X \in \mathbf{MS} . \forall x \in X . \left| \mathbf{constant}([0, 1], X, x) \right| = 0$

Proof =

IntrinsicMetric :: $\prod X : \mathbf{LengthSpace} . \mathbf{Metric}(X)$

$\mathbf{IntrinsicMetric}(x, y) = d_I(x, y) := \inf_{\gamma \in R(x, y)} |\gamma|$

Geodesic :: $\prod X \in \mathbf{MS} . ?R(X, Y)$

$\gamma : \mathbf{Geodesic} \iff |\gamma| = \inf_{\gamma \in R(x, y)} |\gamma|$

GeodesicSpace :: ?**LengthSpace**(X)

$X : \mathbf{GeodesicSpace} \iff \forall x, y \in X . \exists \gamma : \mathbf{Geodesic}(x, y)$

GeodesicSpaceIsPathConnected :: $\forall X : \text{Geodesic} . X : \text{PathConnected}$

Proof =

...

□

HopfRinow :: ?MS

$X : \text{HopfRinow} \iff \forall x, y \in X . \forall a, b \in \mathbb{R}_+ . a + b < d(x, y) \Rightarrow d(\mathbb{B}(x, a), \mathbb{B}(y, b)) = d(x, y) - a - b$

HopfRinowLemma :: $\forall X : \text{Complete} \ \& \ \text{HopfRinow} . \forall x \in X . \forall r \in \mathbb{R}_{++} \left(\forall r' \in \mathbb{R}_{++} . \text{Compact}(\mathbb{D}(x, r'), \varepsilon) \Rightarrow \text{Compact}(\mathbb{D}(x, r), \varepsilon) \right)$

Proof =

Assume $y : X$,

Assume $[1] : d(x, y) = r$,

Assume $t : (0, r)$,

Assume $s : (0, r - t)$,

$(u, v[2]) := \text{HopfRinov}(X, x, y, t, s) : \forall t \in (0, r) . \forall s \in (0, r - t) . \exists u, v \in X : d(x, u) = t \ \& \ d(y, v) = s \ \& \ d(u, v) > r - t$

$[y.3] := \text{TriangleIneq} : d(y, u) \leq d(y, v) + d(u, v) = r - t$;

$\leadsto [1] := \text{metricSpace}^{-1} : \forall t \in (0, r) . d(\mathbb{D}(x, t), \mathbb{D}(x, r)) \leq r - t$,

$[2] := \text{CompactHausdorffConvergence}(X)[1] : \text{Compact}(X, \mathbb{D}(x, r))$;

□

CompactBlowUp :: $\forall X \in \text{MS} \ \& \ \text{LocallyComapct} . \forall K : \text{Compact}(X) . \exists U \in \mathcal{U}(K) : \text{Compact}(\overline{U})$

Proof =

...

□

HopfRinowTHM1 :: $\forall X : \text{LocallyCompact} \ \& \ \text{HopfRinow} \ \& \ \text{Complete} . \forall x \in X . \forall r \in \mathbb{R}_{++} . \text{Compact}(\mathbb{D}(x, r))$

Proof =

$R := \left\{ r \in \mathbb{R}_{++} \mid \text{Compact}(X, \mathbb{D}(x, r)) \right\} : ?\mathbb{R}_{++}$,

$[1] := \text{LocallyCompact}(X)_j R : R \neq \emptyset$,

Assume $t : \sup R$,

$[2] := \text{HopfRinowLemma}(X, x, t) \text{HopfRinov}(X, x, t) : t \in R$,

$K := \mathbb{D}(x, t) : \text{Compact}$,

$(U, [3]) := \text{CompactBlowUp}_j R[2] \text{HopfRinov}(X, x, t) : \sum U \in \mathcal{U}(K) . \overline{U} : \text{Compact}$,

$A := X \setminus U : \text{Closed}(X)$,

$(s, [4]) := \text{DisjointClosedDistance}_j A[3] : \sum s \in \mathbb{R}_{++} . d(K, A) > s$,

$[5] := [4]_j K[3] : \mathbb{D}(x, t + s) \subset \overline{U}$,

$[6] := \text{CompactClosedSubset}[5] : \text{Compact}(X, \mathbb{D}(x, t + s))$,

$[4] := \text{HopfRinov}(X, x, t) : \perp$;

$\leadsto [2] := \text{CompactSubset} E(\perp) : R = \mathbb{R}_{++}$;

□

CompleteByCompactDiscs :: $\forall X \in \mathbf{MS} . \forall [0] : \left(\forall x \in X . \forall r \in \mathbb{R}_{++} . \forall \text{Compact} \left(X, \mathbb{D}(x, r) \right) . \right) . \text{Complete}(X)$

Proof =

Assume $\mathcal{F} : \text{CauchyFilterBase}(X)$,

Assume $F : \mathcal{F}$,

Assume $[1] : \text{diam } F < \infty$,

$\delta := \text{diam } F : \mathbb{R}_{++}$,

$\mathcal{G} := \{G \in \mathcal{F} | G \subset F\} : \text{CauchyFilterBase}(X)$,

$x := \text{NonEmpty}(F) : F$,

$[2] := [0](x, \delta) : \text{Compact} \left(X, \mathbb{D}(x, r) \right)$,

$[3] := \text{CompactIsComplete}[2] : \text{Complete} \left(X, \mathbb{D}(x, r) \right)$,

$[F.*] := \text{NonComplete}[3](\mathcal{G}) : \text{ConvergingFilterBase} \left(X, \mathcal{G} \right)$;

$\leadsto [\mathcal{F}.*] := \text{NonCauchyFilterBase} \text{NonConvergingFilterBase} : \text{ConvergingFilterBase}(X, \mathcal{F})$;

$\leadsto [*] := \text{NonComplete} : \text{Complete}(X)$;

□

MidpointTHM :: $\forall X : \text{Complete} \ \& \ \text{LocallyCompat} \ \& \ \text{HopfRinow} . \forall x, y \in X . \exists z \in X :$

$$d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$$

Proof =

Assume $[1] : x \neq y$,

$\delta := d(x, y) : \mathbb{R}_{++}$,

Assume $t : (0, \delta)$,

$(a, b, [2]) := \text{HopfRinowNonHausdorffDistance} :$

$$\sum a : \mathbb{N} \rightarrow \mathbb{D} \left(x, \frac{\delta - t}{2} \right) . \sum b : \mathbb{N} \rightarrow \mathbb{D} \left(x, \frac{\delta - t}{2} \right) .$$

$$. \lim_{n \rightarrow \infty} d(a_n, b_n) = t \ \& \ \lim_{n \rightarrow \infty} d(a_n, x) = \frac{\delta - t}{2} \ \& \ \lim_{n \rightarrow \infty} d(b_n, y) = \frac{\delta - t}{2},$$

$[3] := \text{HopfRinowTHM1}(X) \text{CompactIsSequenceCompact} : \text{PartiallyConverging}(X, (a, b))$,

$$\alpha_t := \text{partial} \lim_{n \rightarrow \infty} a_n : \mathbb{D} \left(x, \frac{\delta - t}{2} \right),$$

$$\beta_t := \text{partial} \lim_{n \rightarrow \infty} b_n : \mathbb{D} \left(y, \frac{\delta - t}{2} \right),$$

$$[t.*] := [2]j(\alpha, \beta) : d(x, \alpha_t) = \frac{\delta - t}{2} \ \& \ d(\alpha_t, \beta_t) = t \ \& \ d(y, \beta_t) = \frac{\delta - t}{2};$$

$$\leadsto (\alpha, \beta, [2]) := I \left(\prod \right) I \left(\sum \right) : \prod t \in (0, \delta) . \sum \alpha_t \in \mathbb{D}(x, \delta) . \sum \beta_t \in \mathbb{D}(y, \delta) . d(x, \alpha_t) = \frac{\delta - t}{2} \ \& \ d(\alpha_t, \beta_t) = t$$

$$a := \Lambda n \in \mathbb{N} . \alpha_{\frac{1}{n}} : \mathbb{N} \rightarrow \mathbb{D}(x, \delta),$$

$$b := \Lambda n \in \mathbb{N} . \beta_{\frac{1}{n}} : \mathbb{N} \rightarrow \mathbb{D}(y, \delta),$$

$[3] := \text{HopfRinowTHM1}(X) \text{CompactIsSequenceCompact} : \text{PartiallyConverging}(X, (a, b))$,

$$\alpha_t := \text{partial} \lim_{n \rightarrow \infty} a_n : X,$$

$$\beta_t := \text{partial} \lim_{n \rightarrow \infty} b_n : X,$$

$$[4] := [2]j(\alpha, \beta) : d(x, \alpha) = \frac{\delta}{2} \ \& \ d(\alpha, \beta) = 0 \ \& \ d(y, \beta) = \frac{\delta}{2},$$

$$[*] := \text{NonComplete}[4] : \alpha = \beta;$$

□

GeneralizedMidpointTHM :: $\forall X : \text{Complete} \ \& \ \text{LocallyCompat} \ \& \ \text{HopfRinow} .$

. $\forall x, y \in X . \exists \xi : \mathbb{Q}_2 \cap [0, 1] \rightarrow X :$

: $\forall a, b \in \mathbb{Q}_2 \cap [0, 1] . d(\xi(a), \xi(b)) = |a - b|d(x, y) \ \& \ \xi(0) = x \ \& \ \xi(1) = y$

Proof =

...

□

HopfRinowTHM2 :: $\forall X : \text{Complete} \ \& \ \text{LocallyCompat} \ \& \ \text{HopfRinow} .$

. $\forall x, y \in X . \exists \gamma : \Omega(x, y) :$

: $\forall a, b \in [0, 1] . d(\gamma(a), \gamma(b)) = |a - b|d(x, y)$

Proof =

$\delta := d(x, y) : \mathbb{R}_{++},$

$(\xi, [1]) := \text{GeneralizedMidpointTHM}(X, x, y) : \sum \xi : \mathbb{Q}_2 \cap [0, 1] \rightarrow X . \forall a, b \in \mathbb{Q}_2 \cap [0, 1] .$

. $d(\xi(a), \xi(b)) = |a - b|d(x, y) \ \& \ \xi(0) = x \ \& \ \xi(1) = y,$

$[2] := \text{LipschitzIsUC} : \text{UniformlyContinuous}(\mathbb{Q}_2, X, \xi),$

$(\gamma, [3]) := \text{UniformlyContinuousExtension}[1, 2] : \sum \gamma : \Omega(x, y) :$

: $\forall a, b \in [0, 1] . d(\gamma(a), \gamma(b)) = |a - b|d(x, y);$

□

HopfRinowGeodesic :: $\forall X : \text{Complete} \ \& \ \text{LocallyCompat} \ \& \ \text{HopfRinow} . X : \text{GeodesicSpace}$

Proof =

...

□

1.9 Lipschitz Connected Spaces

$\text{LipschitzConnected} :: \mathbb{R}_{++} \rightarrow ?\text{MS}$

$X : \text{LipschitzConnected} \iff \exists C \in \mathbb{R}_{++} . 1\text{-Lip}(X) \iff$

$\iff \forall x, y \in X . \forall \varepsilon \in \mathbb{R}_{++} . \exists n \in \mathbb{N} : \exists q : n \rightarrow X .$

$. q_1 = x \ \& \ q_n = y \ \& \ \& \ \forall i \in (n - 1) . d(q_i, q_j) < \varepsilon \ \& \ \sum_{i=1}^n \leq C d(x, y)$

$\text{OneLipschitzConnectedIsGeodesic} :: \forall X : 1\text{-Lip}(X) . \text{GeodesicSpace}(X)$

$\text{Proof} =$

...

□

$\text{GeodesicIsOneLipschitz} :: \forall X : \text{GeodesicSpace}(X) . 1\text{-Lip}(X)$

$\text{Proof} =$

...

□

2 Metrization

2.1 Metrization Theorems

Metrizable :: ?TOP

$X : \text{Metrizable} \iff \exists d : \text{Metric}(X) . (X, d) \cong_{\text{TOP}} X$

UrysohnMetrization :: $\forall X : \text{SecondCountable} \ \& \ \text{T3} . \text{Metrizable}(X)$

Proof =

$\mathcal{B} := \mathfrak{B}\text{SecondCountable}(X) : \text{Countable} \ \& \ \text{Base}(X),$

Assume $V, W : \mathcal{B},$

Assume $[1] : \overline{V} \subset W,$

$\left(f_{V,W}, [2]\right) := \text{UrysohnLemma}(\overline{V}, W^c, [1]) : \sum f_{V,W} : C(X, [0, 1]) . W^c = f^{-1}\{0\} \ \& \ \overline{V} = f^{-1}\{1\};$

$\rightsquigarrow \left(f, [1]\right) := I\left(\prod\right) : \prod_{V, W \in \mathcal{B} : \overline{V} \subset W} \sum_{f : X \xrightarrow{\text{TOP}} [0, 1]} W^c = f^{-1}\{0\} \ \& \ \overline{V} = f^{-1}\{1\},$

$\Phi := \text{Im } f : ?C(X, [0, 1]),$

$[2] := \mathcal{J}\Phi \mathfrak{B} : |\Phi| \leq \aleph_0,$

$\phi := \text{enumerate}(\Phi) : \mathbb{N} \leftrightarrow \Phi,$

$d := \Lambda x, y \in X . \sum_{n=1}^{\infty} \frac{|\phi_n(x) - \phi_n(y)|}{n!} : \text{Semimetric}(X),$

$[3] := \mathcal{J}d[1] \mathfrak{B}\text{T1}(X) : \text{Metric}(X, d),$

$[4] := \text{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$[*] := \mathfrak{B}\text{T3} : \mathcal{T}(X) = \mathcal{T}(X, d);$

□

SeparableMetrization :: $\forall X : \text{Separable} . \text{Metrizable}(X) \iff \text{SecondCountable} \ \& \ \text{T3}(X)$

Proof =

...

□

ContinuousByLocallyFinite :: $\forall X, Y \in \mathbf{TOP} . \forall f : X \rightarrow Y . \forall \mathcal{A} : \mathbf{LocallyFinite} \ \& \ \mathbf{Cover}(X) .$

$$. \left(\forall A \in \mathcal{A} . f|_{\overline{A}} \in C(\overline{A}, X) \right) \Rightarrow f \in C(X, Y)$$

Proof =

Assume $x : X$,

$$\mathcal{A}' := \left\{ A \in \mathcal{A} \mid x \in A \right\} : ?\mathcal{A},$$

$$[2] := j\mathcal{A}'\delta\mathbf{LocallyFinite}(X, \mathcal{A}) : |\mathcal{A}'| < \infty,$$

$$(U, [3]) := \delta\mathbf{LocallyFinite}(X) : \sum U \in \mathcal{U}(x) . \forall A \in \mathcal{A} . A \cap U \neq \emptyset \Rightarrow A \in \mathcal{A}',$$

$$[4] := \delta^{-1}\mathbf{Cover}(X)(\mathcal{A}') : \mathcal{A}' \neq \emptyset,$$

Assume $V : \mathcal{U}(f(x))$,

Assume $A : \mathcal{A}'$,

$$W := f|_A^{-1}(V) : ?A,$$

$$[5] := \delta\mathbf{TOP} : W_A \in \mathcal{T}(A),$$

$$(O_A, [A.*]) := \delta\mathbf{subsetTopology}[5] : \sum O_A \in \mathcal{U}_X(x) . W = O_A \cap A;$$

$$\leadsto (O, [5]) := I \left(\sum \right) I(\rightarrow) : \sum O : \mathcal{A} \rightarrow \mathcal{U}_X(x) . \forall A \in \mathcal{A}' . O_A \cap A = f|_A^{-1}(V),$$

$$\theta := \bigcap_{A \in \mathcal{A}'} O_A \cap U : \mathcal{U}(x),$$

$$[*].1 := j\theta[3][5] : f(\theta) \subset U;$$

$$\leadsto [*] := \mathbf{ContinuityLocalProof}(X, Y, f) : f \in C(X, Y);$$

□

CountableSupport :: $\prod X \in \mathbf{SET} . ?(X \rightarrow \mathbb{R})$

$$f : \mathbf{CountableSupport} \iff \left| \{x \in X : f(x) \neq 0\} \right| \leq \aleph_0$$

CountableSupport :: $\prod X \in \mathbf{SET} . ?\mathbf{CountableSupport}(X)$

$$f : \mathbf{CountableSupport} \iff f \in L_1(X, \#) \iff \sum_{x \in x} |f(x)| < \infty$$

ContinuousIndicators :: $\prod X \in \mathbf{TOP} . \prod S : ??X . ?(S \rightarrow C(X))$

$$f : \mathbf{ContinuousIndicators} \iff \forall A \in S . f_A(A^c) = \{0\}$$

NagataSmirnovFunc :: $\prod X \in \mathbf{TOP} . \prod S : ??X . \mathbf{ContinuousIndicators}(X) \rightarrow S \rightarrow X \rightarrow \mathbb{R}$

$$\mathbf{NagataSmirnovFunc}(f, A) = \mathbf{NS}_{f,x}(A) := f_A(x)$$

SemimetricBaseTheorem :: $\forall X \in \text{SMS} . \exists \sigma\text{-LocallyFinite} \ \& \ \text{Base}(X)$

Proof =

$(\leq) := \text{WellOrderingTHM} : \text{WellOrder}(X),$

$U := \Lambda x \in X . \Lambda k \in \mathbb{N} . \Lambda n \in \text{after}(k) . \mathbb{B} \left(x, \frac{1}{k} - \frac{1}{n} \right) \setminus \overline{\bigcap_{y < x} \mathbb{D} \left(y, \frac{1}{k} - \frac{1}{n+1} \right)} :$

$: X \rightarrow \prod_{k=1}^{\infty} \left(\text{after}(k) \rightarrow \mathcal{T}(X) \right),$

$\mathcal{A} := \Lambda k \in \mathbb{N} . \left\{ U(x, k, n) \mid x \in X, n \in \text{after}(k) \right\} : \mathbb{N} \rightarrow ?\mathcal{T}(X),$

Assume $k : \mathbb{N},$

Assume $z : X,$

$R := \left\{ x \in X : z \in \mathbb{B} \left(x, \frac{1}{k} \right) \right\} : ?X,$

$[1] := \mathfrak{D}\text{cell}_j R : z \in R,$

$x := \min R : R,$

$\left(n, [2] \right) := jRjx : \sum_{n=k+1}^{\infty} \frac{1}{n} \leq \frac{1}{k} - d(x, z),$

$[k.*] := jxjRjU[2] : z \in U(z, k, n);$

$\leadsto [1] := I(1) : \forall k \in \mathbb{N} . \text{Cover} \left(X, \mathcal{A}_k \right),$

$[2] := \mathfrak{D}^{-1} \text{Base}_j \mathcal{A} : \text{Base} \left(X, \bigcup_{1=k}^{\infty} \mathcal{A}_k \right),$

Assume $k : \mathbb{N},$

Assume $n : \text{after}(k),$

Assume $x, y : X,$

Assume $[3] : x \neq y,$

$[*] := jU : d \left(U(x, k, n), U(z, k, n) \right) \geq \frac{1}{n(n+1)};$

$\leadsto [3] := I(\forall) : \forall x \in X . \forall k \in \mathbb{N} . \forall n : \text{after}(k) . d \left(U(x, k, n), U(z, k, n) \right) \geq \frac{1}{n(n+1)},$

$[4] := \mathfrak{D}^{-1} \text{LocallyFinite}[3] : \forall k \in \mathbb{N} . \text{LocallyFinite}(X, \parallel),$

$[*] := \mathfrak{D}^{-1} \sigma\text{-LocallyFinite}(X)[4] : \sigma\text{-LocallyFinite}(X) \left(X, \bigcap_{k=1}^{\infty} \mathcal{A}_k \right);$

□

NagataSmirnovLemma :: $\forall X \in \mathbf{TOP} . \forall S : \mathbf{LocallyFinite}(X) . \forall f : \mathbf{ContinuousIndicators}(X, S) .$
 $. \mathbf{NS}_f \in C(X, L_1(X, \#))$

Proof =

Assume $(D, x) : \mathbf{ConvergingNet}(X),$

$L := \lim_{n \rightarrow \infty} x_n : X,$

$(U, [1]) := \mathfrak{d}\mathbf{LocallyFinite}(X, S, L) : \sum U \in \mathcal{U}(L) . \left| \{s \in S : s \cap U \neq \emptyset\} \right| < \infty,$

$(\delta, [2]) := \mathfrak{d}\mathbf{ConvergingNet}(X, D, x, U) : \sum \delta' \in D . \forall \delta' \in D . \delta'' \geq \delta \Rightarrow x_{\delta'} \in U,$

$S' := \{s \in S : s \cap U \neq \emptyset\} : \mathbf{Finite}(S),$

$[3] := \mathfrak{d}L_1[1][2]\mathfrak{d}\mathbf{ContinuousIndicators}(X, S, f)\mathbf{ContinuousSum}(X, S', \dots)$

ContinuousConvergence $(X, f_s, D, x) :$

$$: \lim_{n \in D} \left\| \mathbf{NS}_{f, x_n} - \mathbf{NS}_{f, L} \right\|_1 = \lim_{n \in D} \sum_{s \in S} \left| f_s(x_n) - f_s(L) \right| = \lim_{n \in D} \sum_{s \in S'} \left| f_s(x_n) - f_s(L) \right| =$$

$$= \sum_{s \in S'} \lim_{n \in D} \left| f_s(x_n) - f_s(L) \right| = 0,$$

$* \dots] := \mathbf{NormConvergence} : \lim_{n \in D} \mathbf{NS}_{f, x_n} = \mathbf{NS}_{f, L};$

$\leadsto [*] := \mathbf{ContinuousConvergence} : \mathbf{NS}_f \in C(X, L_1(X, \#));$

□

SmirnoffNagataMetrizationTHM :: $\forall X : \mathbf{T4} . \forall \mathcal{B} : \mathbf{Base} \ \& \ \sigma\text{-}\mathbf{LocallyFinite}(X) . X : \mathbf{Metrizable}(X)$

Proof =

$(\mathcal{B}', [1]) := \mathfrak{d}\sigma\text{-}\mathbf{LocallyCompact}(X) : \sum \mathcal{B}' : \mathbb{N} \rightarrow \mathbf{LocallyFinite}(X) . \mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}',$

Assume $m, n : \mathbb{N},$

Assume $V : \mathcal{B}_n,$

$A := \bigcup \{W \in \mathcal{B}_m : \overline{W} \subset V\} : \mathcal{T}(X),$

$[2] := \mathbf{LocallyFiniteClosure} : \overline{A} \subset V,$

$(f_V, [3]) := \mathbf{UrysohnLemma}(X, \overline{A}, V^c) : \sum f \in C(X) . f^{-1}\{1\} = \overline{A} \ \& \ f^{-1}\{0\} = V^c;$

$\leadsto (f_{n,m}, [4]) := I\left(\sum\right) : \sum f_{n,m} : \mathcal{B}_n \rightarrow C(X) . \forall V \in \mathcal{B}_n . f_{n,m,V}^{-1}\{1\} = \overline{A} \ \& \ f_{n,m,V}^{-1}\{0\} = V^c,$

$[2] := \mathfrak{d}^{-1}\mathbf{ContinuousIndicators} : \mathbf{ContinuousIndicators}(X, \mathcal{B}_n, f_{n,m}),$

$h_{n,m} := \mathbf{NS}_{f_{n,m}} : X \xrightarrow{\mathbf{TOP}} L_1(\mathcal{B}_n, \#);$

$\leadsto h := I(\rightarrow) : \mathbb{N} \times \mathbb{N} \rightarrow X \xrightarrow{\mathbf{TOP}} L_1(\mathcal{B}_n, \#),$

$H := \text{Im } h : ?C(X, L_1(\mathcal{B}_n, \#)),$

$[2] := \mathbf{ImageCardinality} : |H| \leq \aleph_0,$

$f := \mathbf{enumerate}(H) : \mathbb{N} \rightarrow C(X, L_1(\mathcal{B}_n, \#);$

$[4] := \mathbf{WeakTopologyMetrization}[3] : \mathcal{T}(X, d) \subset \mathcal{T}(X),$

$d := \Lambda x, y \in X . \sum_{n=1}^{\infty} \frac{\|f_n(x) - f_n(y)\|_1}{n!} : \mathbf{Metric}(X),$

...

□

2.2 Metrization Preserving Operations

`MetrizationIsPreservedBySupTopology` ::

$$: \forall X \in \text{SET} . \forall \tau : \mathbb{N} \rightarrow \text{Metrizable}(X) . \text{Metrizable} \left(X, \bigvee_{n \in \mathbb{N}} \tau_n \right)$$

`Proof` =

Let ρ_n be a metrizing distance metric for topology τ_n .

Then the metric $d = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n}{1 + \rho_n}$ metrizes $\bigvee_{n \in \mathbb{N}} \tau_n$ by considerations of convergence.

□

3 Metric Dimension

3.1 Covering Dimension

$$\begin{aligned} \text{CoveringDimensionLE} &:: \prod_{X \in \text{MS}} ?(\text{Compact}(X) \times \mathbb{Z}_+) \\ (A, n) : \text{CoveringDimensionLE} &\iff \dim A \leq n \iff \forall r \in \mathbb{R}_{++} . \exists k \in \mathbb{N} : \exists x : k \rightarrow X : \\ &: A \subset \bigcup_{i=1}^k \mathbb{B}(x_i, r) \ \& \ \forall a \in A . \left| \{i \in K : a \in \mathbb{B}(x_i, r)\} \right| \leq n + 1 \end{aligned}$$

$$\begin{aligned} \text{CoveringDimensionGreater} &:: \prod_{X \in \text{MS}} ?(\text{Compact}(X) \times \mathbb{Z}_+) \\ (A, n) : \text{CoveringDimensionGreater} &\iff \dim A > n \iff !(\dim A \leq n) \end{aligned}$$

$$\begin{aligned} \text{FiniteDimensionalCompact} &:: \prod_{X \in \text{MS}} ?\text{Compact}(X) \\ K : \text{FiniteDimensionalCompact} &\iff \exists n \in \mathbb{Z}_+ . \dim X \leq n \end{aligned}$$

$$\begin{aligned} \text{coveringDimension} &:: \prod_{X \in \text{MS}} \text{FiniteDimensionalCompact} \rightarrow \mathbb{Z}_+ \\ \text{coveringDimension}(K) &= \dim K := \min\{n \in \mathbb{Z}_+ : \dim X \leq n\} \end{aligned}$$

IntervalDimension :: $\dim[0, 1] = 1$

Proof =

Assume [1] : $\dim[0, 1] \leq 1$,

$(k, x, [2], [3]) := \text{CoveringDimensionLE}[1] \left(\frac{1}{3} \right) : \sum k \in \mathbb{N} . \sum x : k \rightarrow [0, 1] .$

$. [0, 1] \subset \bigcup_{i=1}^k \mathbb{B} \left(x_i, \frac{1}{3} \right) \ \& \ \forall a \in [0, 1] . \left| \{i \in k : a \in \mathbb{B}(x_i, r)\} \right| \leq 1,$

[4] := $\text{DisjointUnion} : [0, 1] = \bigsqcup_{i=1}^k \mathbb{B} \left(x_i, \frac{1}{3} \right),$

Assume [5] : $k = 1$,

[6] := $\text{diam}[0, 1] \text{BallDiam} \left(x_i, \frac{1}{3} \right) : 1 = \text{diam}[0, 1] = \text{diam} \mathbb{B} \left(x_i, \frac{1}{3} \right) < \frac{2}{3},$

[5.*] := **OneIsNotZero**[6] : \perp ;

\leadsto [5] := $E(\perp) \text{UnitIsMinimal}(\mathbb{N}) : k > 1,$

[6] := **DisjointOpenUnionDisconnected** : **Disconnected**[0, 1],

[1.*] := **IntervalIsConnected**[6] : \perp ;

\leadsto [1] := $E(\perp) : \dim[0, 1] > 0,$

Assume $r : \mathbb{R}_{++}$,

[2] := $\text{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > 1\} \neq \emptyset,$

$k := \min\{N \in \mathbb{N} : Nr > 1\} : \mathbb{N},$

$x := \Lambda i \in k . (i - 1)r : k \rightarrow [0, 1],$

Assume $t : [0, 1],$

[2] := $\text{Archimedean}(\mathbb{R}) : \{N \in \mathbb{N} : Nr > t\} \neq \emptyset,$

$i := \min\{N \in \mathbb{N} : Nr > t\} \neq \emptyset : \mathbb{N},$

[3] := $j \text{d}t : i \leq k,$

$[t.*.1] := j \text{d}x : t \in [(i - 1)r, ir) \subset \mathbb{B}(x_i, r),$

Assume $j, l : k,$

Assume [4] : $t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r),$

[5] := $j^{-1}x : r|l - j| \text{TriangleIneq}([0, 1], x_j, x_l, t) \leq |x_j - x_l| \leq |x_l - t| + |t - x_i| < 2r,$

$[(j, l).*] := \frac{[5]}{r} : |l - j| \leq 1;$

\leadsto [4] := $I(\forall) : \forall j, l \in k . t \in \mathbb{B}(x_j, r) \cap \mathbb{B}(x_l, r) \Rightarrow |l - j| \leq 1,$

$[t.*.2] := \text{dN}[2] : \left| \{t \in k : a \in \mathbb{B}(x_i, r)\} \right| \leq 2;$

\leadsto [2] := $\text{CoveringDimensionLE} : \dim[0, 1] \leq 1,$

[*] := $\text{dN}[1][2] : \dim[0, 1] = 1;$

□

3.2 Embedding Theorem

MengerMap :: $\prod X, Y \in \mathbf{MS} . \mathbb{R} \rightarrow ? C(X, Y)$

$f : \mathbf{MengerMap} \iff \Lambda \varepsilon \in \mathbb{R}_{++} . \forall a, b \in X . f(a) = f(b) \Rightarrow d(a, b) < \varepsilon$

MengerMapsIsOpen :: $\forall K : \mathbf{MS} \ \& \ \mathbf{Compact} . \forall n \in \mathbb{N} . \forall \varepsilon \in \mathbb{R}_{++} . \mathbf{MengerMap}(X, \mathbb{R}^n, \varepsilon) \in \mathcal{T}(C(X, \mathbb{R}^n))$

Proof =

[1] := **CompactProduct**(K, K) : **Compact**($K \times K$),

Assume $f : \mathbf{MengerMap}(X, \mathbb{R}^n, \varepsilon)$,

$A := \left(d\right)^{-1}[\varepsilon, +\infty) : \mathbf{Closed}(K \times K)$,

[2] := **ClosedCompactSubset** : **Compact**($K \times K, A$),

$\delta := \inf_{(a,b) \in A} \|f(a) - f(b)\| : \mathbb{R}_{++}$,

Assume $g : \mathbb{B}\left(f, \frac{\delta}{2}\right)$,

Assume $x, y : K$,

Assume [4] : $g(x) = g(y)$,

[5] := **TriangleIneq**($f(x), g(x), g(y), g(y)$)[4] $\tilde{\partial} g$:

$: |f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$,

$[(x, y).*] := j\delta j A[5] : d(x, y) < \varepsilon$;

$\leadsto [g.*] := \tilde{\partial}^{-1} \mathbf{MengerMap} : \mathbf{MengerMap}(X, \mathbb{R}^n, \varepsilon, g)$;

$\leadsto [f.*] := I(\exists) \tilde{\partial}^{-1} \mathbf{Subset} : \exists U \in \mathcal{U}(f) . U \subset \mathbf{MengerMap}(X, \mathbb{R}^n, \varepsilon)$;

$\leadsto [*] := \mathbf{OpenByCover} : \mathbf{Open} C(X, \mathbb{R}^n) \ \mathbf{MengerMap}(X, \mathbb{R}^n, \varepsilon)$;

□

CoveringSequence :: $\prod K : \mathbf{FiniteDimensionalCompact}(X) . \mathbb{R}_{++} \rightarrow ? \sum_{n=1}^{\infty} n \rightarrow X$

$x : \mathbf{CoveringSequence} \iff \Lambda r \in \mathbb{R}_{++} . X \subset \bigcup_{i=1}^k \mathbb{B}(x_i, r) \ \& \ \forall X \in A . \left| \{i \in K : a \in \mathbb{B}(x_i, r)\} \right| \leq n + 1$

PontryaginTwinSequence :: $\prod X : \mathbf{Compact} \ \& \ \mathbf{MS} . \prod r \in \mathbb{R}_{++} . \prod (n, x) : \mathbf{CoveringSequence}(X, r) .$
 $. (X \xrightarrow{\text{TOP}} \mathbb{R}^{2*\dim K+1}) \rightarrow \mathbb{R} \rightarrow ? (n \rightarrow \mathbb{R}^{2\dim X+1})$

$y : \mathbf{PontryaginTwinSequence} \iff \Lambda f : (X \xrightarrow{\text{TOP}} \mathbb{R}^{2*\dim K+1}) . \Lambda s \in \mathbb{R}_{++} . \forall i \in n . d(f \mathbb{B}(x_i, r), y_i) < s \ \& \$
 $\ \& \ \forall V : \mathbf{ProperAffineSubspace}(\mathbb{R}^{2\dim K}) . |\text{Im } y \cap V| \leq \dim V + 2$

PontryaginTwinSequenceExists :: $\forall X : \text{FiniteDimensionalCompact} . \forall f \in C(X, \mathbb{R}^{1+2 \dim X}) . \forall r, s \in \mathbb{R}_{++} .$
 $. \forall (n, x) : \text{CoveringSequence}(X, r) . \exists \text{PontryaginTwinSequence}(X, f, x, s)$

Proof =

$(y^0, [1]) := \text{RealOpenIsInfinite}(\dots) : \sum y^0 : n \hookrightarrow X . \forall i \in \|y_i^0 - f(x_i)\| < \frac{s}{1+2^n},$

$\mathcal{A}^0 := \left\{ A : \text{ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) : \|A \cap \Im y^0\| > \dim A + 1 \right\} :$

$: \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}),$

$w := \Lambda \mathcal{A} : \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) . \sum_{A \in \mathcal{A}} |A \cap \Im y^0| - \dim A - 1 :$

$: \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) \rightarrow \mathbb{Z}_+,$

$M := w(\mathcal{A}^0) : \mathbb{Z}_+,$

$[0] := j M j \mathcal{A}_0 : M \leq 2^n,$

Assume $m : (M - 1)_+,$

Assume $[2] : \mathcal{A}^m \neq \emptyset,$

$(k, A) := \text{enumerate}(\mathcal{A}^n) : \sum k \in \mathbb{N} . A : k \leftrightarrow \mathcal{A}^m,$

$i := \max\{i \in n : x_i^m \in A_1\} : n,$

$\delta := \min \left\{ d(y_i^n, A) \middle| A : \text{ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) \ \& \ \text{Im } y^n \cap A \neq \emptyset \ \& \ y_i^n \notin A \right\} : \mathbb{R}_{++},$

$(u, [3]) := \text{RealOpenIsInfinite}(\dots) : \sum u \in \mathbb{R}^{1+2 \dim X} . d(y_i^n, u) = \min \left(\delta, \frac{s}{1+2^n} \right),$

$y^{m+1} := \Lambda j \in n . \text{if } i == j \text{ then } u \text{ else } y_i^n : n \rightarrow \mathbb{R}^{1+2 \dim X},$

$\mathcal{A}^{m+1} := \left\{ A : \text{ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) : \|A \cap \Im y^{m+1}\| > \dim A + 1 \right\} :$

$: \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}),$

$[2.*] := j W j \delta j u j y^{m+1} j \mathcal{A}^0 : w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m);$

$\leadsto [2] := I(\Rightarrow) : \mathcal{A}^m \neq \emptyset \Rightarrow w(\mathcal{A}^{m+1}) < w(\mathcal{A}^m),$

Assume $[2] : \mathcal{A}^m = \emptyset,$

$y^{m+1} := y^m : n \rightarrow \mathbb{R}^{1+2 \dim X},$

$\mathcal{A}^{m+1} := \mathcal{A}^m : \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X});$

$\leadsto y, \mathcal{A}, [2] := I \left(\prod \right) : \prod_{m=0}^M \sum y^m : n \rightarrow \mathbb{R}^{1+2 \dim X} . \sum \mathcal{A}^m : \text{Finite ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) .$

$. \mathcal{A}^m = \left\{ A : \text{ProperAffineSubspace}(\mathbb{R}^{1+2 \dim X}) : \|A \cap \Im y^m\| > \dim A + 1 \right\} \ \&$

$\forall m \in (M - 1) . \mathcal{A}^m \neq \emptyset w(\mathcal{A}^m) < w(\mathcal{A}^{m+1}),$

$[3] := [2] j w \text{InverseFiniteInduction} : \mathcal{A}^M = \emptyset,$

$[4] := j y^M : \forall i \in n . d(f(x_i), y_i^M) \leq \|f(x_i) - y_i^1\| + \sum_{j=1}^M \|y_i^j - y_i^{j+1}\| < s,$

$[*] := \breve{\text{O}}^{-1} \text{PontryaginTwinSequence} : \text{PontryaginTwinSequence}(X, f, x, s, y^M);$

□

$$\begin{aligned}
& \text{functionOfNebeling} :: \prod X : \text{FiniteDimensionalCompact} . \prod f \in C(X, \mathbb{R}^{1+2 \dim X}) . \prod r, s \in \mathbb{R}_{++} . \\
& . \prod (n, x) : \text{CoveringSequence}(X, r) . \prod p : \text{PontryaginTwinSequence}(X, f, x, s) . X \xrightarrow{\text{TOP}} \mathbb{R}^{1+2 \dim X} \\
& \text{functionOfNebeling}(u) = N_p(u) := \frac{\sum_{i=1}^n d(u, \mathbb{B}^{\mathbb{C}}(x_i, r), r \text{Big}) p_i}{\sum_{i=1}^n d(u, \mathbb{B}^{\mathbb{C}}(x_i, r))}
\end{aligned}$$

$$\begin{aligned}
& \text{FunctionOfNebelingIsClose} :: \forall X : \text{FiniteDimensionalCompact} . \forall f \in C(X, \mathbb{R}^{1+2 \dim X}) . \forall r, s \in \mathbb{R}_{++} . \\
& . \forall (n, x) : \text{CoveringSequence}(X, r) . \forall p : \text{PontryaginTwinSequence}(X, f, x, s) . \\
& [0] . \forall [0] : \forall i \in . \omega_f(x_i, 2r) < s . \|N_p - f\| < 2s
\end{aligned}$$

$$\text{Proof} =$$

$$\text{Assume } u : X,$$

$$\begin{aligned}
& (n', k, [1]) := \text{CoveringSequence}(x) : \sum n' \in 1 + \dim X . \sum k : n' \rightarrow n . \forall i \in n . \\
& . u \in \mathbb{B}(x_i, r) \iff \exists j \in n' : i = k_i,
\end{aligned}$$

$$y := N_p(u) : \text{In } \mathbb{R}^{1+2 \dim X},$$

$$[2] := \text{yy} N_p[1] : y \in \text{conv Im } p_k,$$

$$(v, [3]) := \text{PontryaginTwinSequence}(p) : \prod_{i=1}^{n'} \mathbb{B}(x_{k_i}, r) . \forall i \in n' . d(f(v_i), p_{k_i}) < s,$$

$$y' := \frac{\sum_{i=1}^{n'} d(u, \mathbb{B}^{\mathbb{C}}(x_i, r)) f(v_i)}{\sum_{i=1}^n \sum_{i=1}^{n'} d(u, \mathbb{B}^{\mathbb{C}}(x_i, r))} : \text{In } \mathbb{R}^{1+2 \dim X},$$

$$[4] := [3][2] \text{yy} N_p \text{yy}' : \|y - y'\| < s,$$

$$[5] := [0] \text{yy}' : \|f(u) - y'\| < s,$$

$$[u.*] := \text{TriangleIneq}[4][5] : d(f(u), N_p(u)) < 2s;$$

$$\leadsto [*] := \text{uniformNorm}^{-1} : \|f - N_p\| < 2s;$$

□

NebelingsAreMenger :: $\forall X : \text{FiniteDimensionalCompact} . \forall f \in C(X, \mathbb{R}^{1+2\dim X}) . \forall r, s \in \mathbb{R}_{++} .$

$. \forall (n, x) : \text{CoveringSequence}(X, r) . \forall p : \text{PontryaginTwinSequence}(X, f, x, s) .$

$[0] . N_p : \text{MengerMap}(X, \mathbb{R}^{2*\dim(X)+1}, 2r)$

Proof =

Assume $u, v : X,$

Assume $[1] : N_p(u) = N_p(v),$

$y := N_p(u) : \text{In } \mathbb{R}^{1+2*\dim X},$

$(m, k, [1]) := \text{CoveringSequence}(X, n, x, u) :$

$\sum m \in 1 + \dim X . \sum k : m \hookrightarrow n . \forall i \in n . u \in \mathbb{B}(x_i, r) \iff \exists j \in m : i = k_i,$

$(m', k', [1']) := \text{CoveringSequence}(X, n, x, v) :$

$\sum m' \in 1 + \dim X . \sum k' : m' \hookrightarrow n . \forall i \in n . v \in \mathbb{B}(x_i, r) \iff \exists j \in m' : i = k'_i,$

$[2] := jyjN_p[1] : y \in \text{conv Im } p_k,$

$[2'] := jyjN_p[1'] : y \in \text{conv Im } p_{k'},$

$(\alpha, [3], [4]) := \text{conv}[2] : \sum \alpha : m \rightarrow \mathbb{R} . \sum_{i=1}^m \alpha_i = 1 \ \& \ \sum_{i=1}^m \alpha_i p_{k_i} = y,$

$(\alpha', [3'], [4']) := \text{conv}[2'] : \sum \alpha' : m' \rightarrow \mathbb{R} . \sum_{i=1}^{m'} \alpha'_i = 1 \ \& \ \sum_{i=1}^{m'} \alpha'_i p_{k'_i} = y,$

$[5] := [4][4'] : \sum_{i=1}^m \alpha_i p_{k_i} = y = \sum_{i=1}^{m'} \alpha_i p_{k'_i},$

$(i, [6]) := [3]\text{ZeroSumIsZero} : \sum i \in m . \alpha_i \neq 0,$

$m'' := m - 1 : \mathbb{Z}_+,$

$k'' := \hat{k}_i : m'' \rightarrow n,$

$\alpha'' := \hat{\alpha}_i : m'' \rightarrow \mathbb{R},$

$[7] := [6][5] \dots : p_{k_i} = \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} p_{k'_j} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} p_{k''_j},$

$[8] := [3][3'][6]j\alpha''\text{Field}(\mathbb{R}) : \sum_{j=1}^{m'} \frac{\alpha'_j}{\alpha_i} - \sum_{j=1}^{m''} \frac{\alpha''_j}{\alpha_i} = \frac{1}{\alpha_i} + \frac{\alpha_i - 1}{\alpha_i} = \frac{\alpha_i}{\alpha_i} = 1,$

$[9] := \text{conv}^{-1}[7][8] : p_{k_i} \in \text{conv } p_{k'' \oplus k'},$

$[10] := \text{AffineCombinationDim}(p_{k'' \oplus k'})\text{conv} m' \text{conv} m'' :$

$: \dim \text{conv } p_{k \oplus k'} \leq m' + m'' - 1 \leq 1 + \dim X + \dim X - 1 = 2 \dim X,$

$(j, [11]) := \text{PontryaginTwinSequence}(p)[9][10] : \sum j \in m' + m'' . (k' \otimes k'')_j = k_i,$

$[12] := [11]\text{injective}(k) : k'_j = k_i,$

$l := k'_j : n,$

$[13] := j[12][1][1'] : u, v \in \mathbb{B}(x_l, r),$

$\left[(u, v) . * \right] := [13]\text{cell}(X)\text{TriangleIneq}(X) : d(u, v) < 2r;$

$\rightsquigarrow [*] := \text{MengerMap} : \text{MengerMap}(X, \mathbb{R}^{2*\dim(X)+1}, 2r),$

□

AllMengerIsInjective :: $\forall X, Y \in \mathbf{MS} . \forall f : X \xrightarrow{\mathbf{MS}} Y . \left(\forall q \in \mathbb{Q}_{++} . f : \mathbf{MengerMap}(X, Y, q) \right) .$

. **Injective**(X, Y, f)

Proof =

...

□

MengerIsDense :: $\forall X : \mathbf{FiniteDimensionalCompact} . \forall \varepsilon \in \mathbb{R}_{++} .$

. **Dense** $\left(C(X, \mathbb{R}^{1+2 \dim X}) \mathbf{MengerMap}(X, \mathbb{R}^{1+2 \dim X}, \varepsilon) \right)$

Proof =

Assume $f : C(X, \mathbb{R}^n),$

Assume $s : \mathbb{R}_{++},$

$[1] := \mathbf{UCByCompact}(X, f) : f \in \mathbf{UC}\left(X, \mathbb{R}^{1+2 \dim X}\right),$

$\left(\delta, [2] \right) := \mathfrak{D}\mathbf{UC}\left(X, \mathbb{R}^{1+2 \dim X}\right)(f) \left(\frac{s}{2} \right) : \sum \delta \in \mathbb{R}_{++} . \forall x \in X . \text{diam } f\mathbb{B}(x, \delta) < \frac{s}{2},$

$r := \min \left(\delta, \frac{\varepsilon}{2} \right) : \mathbb{R}_{++},$

$(n, x) := \mathfrak{D}\mathbf{FiniteDimensionalCompact}(X) (r) : \mathbf{CoveringSequence}(X, r),$

$[2] := \mathfrak{Jr}\mathfrak{D}\mathbf{CoveringSequence}(X, r, n, x) : \forall i \in n . \omega_f(x_i, r) < \frac{s}{2},$

$p := \mathbf{PontryaginDualSequenceExists}\left(X, f, x, \frac{s}{2}\right) : \mathbf{PontryaginDualSequence}\left(X, f, x, \frac{s}{2}\right),$

$[f.*] := \mathbf{FunctionOfNebelingIsClose}(X, p, [2]) : d(N_p, f) < s;$

$\leadsto [*] := \mathfrak{D}^{-1}\mathbf{Dense} : \mathbf{Dense}\left(C(X, \mathbb{R}^{1+2 \dim X}) \mathbf{MengerMap}(X, \mathbb{R}^{1+2 \dim X}, \varepsilon);$

...

□

MengerNebelingPontryaginEmbeddingTHM :: $\forall X : \mathbf{FiniteDimensionalCompact} .$

$\exists \mathbf{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2 \dim X})$

Proof =

$[1] := \mathbf{DualBaireProperty}\left(C(X, \mathbb{R}^{1+2 \dim X}), \Lambda q \in \mathbb{Q}_{++} . \mathbf{MengerMap}(X, \mathbb{R}^{1+2 \dim X}, q)\right)$

$\mathbf{AllMengerIsInjective}(X, \mathbb{R}^{1+2 \dim X}) : \mathbf{Dense}\left(C(X, \mathbb{R}^{1+2 \dim X}), \mathbf{Injective}(X, \mathbb{R}^{1+2 \dim X})\right),$

$[2] := \mathbf{ConstantIsContinuous}(X, \mathbb{R}^{1+2 \dim X}, 0) : 0 \in C(X, \mathbb{R}^{1+2 \dim X}),$

$[3] := \mathfrak{D}^{-1}\mathbf{NonEmpty} : C(X, \mathbb{R}^{1+2 \dim X}) \neq \emptyset,$

$f := [3][1] : \mathbf{Injective}(X, \mathbb{R}^{1+2 \dim X}),$

$[*] := \mathbf{CompactHausdorffIsClosed} \mathfrak{D}^{-1}\mathbf{HomeomorphicEmbedding} : \mathbf{HomeomorphicEmbedding}(X, \mathbb{R}^{1+2 \dim X}, f);$

□