Order Theory

Uncultured Tramp

September 7, 2016

Contents

1 Orders

$$\texttt{PartialOrder} :: \prod A : \texttt{Set} \mathrel{.} ?\texttt{Relation}(A)$$

$$R : \texttt{PartialOrder} := \texttt{Reflexive}\&\texttt{Antisymmetric}\&\texttt{Transitive}$$

$$\texttt{Poset} :: \sum A : \texttt{Set} \mathrel{.} \texttt{PartialOrder}$$

$$\texttt{implicit} :: \texttt{Poset} \to \texttt{Set}$$

$$\texttt{implicit}(A, \leq) = (A, \leq) := A$$

$$\texttt{order} :: \prod (A, \leq) : \texttt{Poset} \mathrel{.} \texttt{PartialOrder}(A)$$

$$\texttt{order} = \leq := \leq$$

$$\texttt{TotalOrder} :: ?\texttt{PartialOrder}(A)$$

$$R : \texttt{TotalOrder} \iff \forall a, b \in A \mathrel{.} (a, b) \in R | (b, a) \in R$$

$$\texttt{Chain} :: \prod A : \texttt{Poset} \mathrel{.} ??A$$

$$C : \texttt{Chain} \iff \leq_{A|C} : \texttt{TotalOrder}(C)$$

$$\texttt{Chain} :: \prod A : \texttt{Poset} \mathrel{.} ??A$$

$$C : \texttt{Chain} \iff \leq_{A|C} : \texttt{TotalOrder}(C)$$

$$\texttt{UpperBound} :: \prod A : \texttt{Poset} \mathrel{.} ?A \to ?A$$

$$a : \texttt{UpperBound}(X) \iff \forall x \in X \mathrel{.} x \leq_A a$$

$$\texttt{LeastUpperBound} :: \prod A : \texttt{Poset} \mathrel{.} \prod X \subset A \mathrel{.} ?\texttt{UpperBound}(X)$$

$$a : \texttt{LeastUpperBound} \iff \forall b : \texttt{LeastUpperBound}(X) \mathrel{.} a \leq x$$

$$\texttt{Maximal} :: \prod P : \texttt{Poset} \mathrel{.} ?P$$

$$m : \texttt{Maximal} \iff \forall a \in P : m \leq a \mathrel{.} m = a$$

$$\texttt{Minimal} :: \prod P : \texttt{Poset} \ . \ ?P$$

$$m : \texttt{Minimal} \iff \forall a \in P : a \leq m \ . \ m = a$$

$$\texttt{LowerBound} :: \prod A : \texttt{Poset} \ . \ ?A \rightarrow ?A$$

$$a : \texttt{LowerBound}(X) \iff \forall x \in X \ . \ a \leq_A x$$

$$\texttt{GreatestLowerBound} :: \prod A : \texttt{Poset} \ . \ \prod X \subset A \ . \ ?\texttt{LowerBound}(X)$$

$$a : \texttt{LeastUpperBound} \iff \forall b : \texttt{LeastUpperBound}(X) \ . \ x \leq a$$

$$\texttt{Top} :: \prod A : \texttt{Poset} \ . \ ?A$$

$$1 : \texttt{Top} \iff \forall a \in A \ . \ a \leq 1$$

$$\texttt{bottom} :: \prod A : \texttt{Poset} \ . \ ?A$$

$$0 : \texttt{bottom} \iff \forall a \in A \ . \ 0 \leq 1$$

2 Lattices

$$\texttt{Lattice} :: ?\texttt{Poset}$$

$$L : \texttt{Lattice} \iff \forall a, b \in L \;.\; \exists \texttt{LeastUpperBound}\{a, b\} \& \exists \texttt{GreaterUpperBound}\{a, b\}$$

$$\texttt{join} :: \prod L : \texttt{Lattice} \;.\; L \to L \to L$$

$$\texttt{join}(a, b) = a \vee b := \eth\texttt{Lattice}(L)(a, b)_1\texttt{Extract}$$

$$\texttt{meet} :: \prod L : \texttt{Lattice} \;.\; L \to L \to L$$

$$\texttt{meet}(a, b) = a \wedge b := \eth\texttt{Lattice}(L)(a, b)_2\texttt{Extract}$$

$$\texttt{CompleteLattice} :: ?\texttt{Lattice}$$

$$L : \texttt{CompleteLattice} \iff \forall A \subset L \;.\; \exists \texttt{LeastUpperBound}\, A \;\&\; \exists \texttt{GreaterUpperBound}\, A$$

$$\texttt{Sublattice} :: \prod L : \texttt{Lattice} \;.\; ??L$$

$$X : \texttt{Sublattice} \iff \forall a, b \in X \;.\; a \vee b \in X \;\&\; a \wedge b \in X$$

$$\texttt{CompleteSubsetComplete} :: \forall L : \texttt{CompleteLattice} \;.\; \forall S \subset L : 1_L \in S : \forall T \subset S : T \neq \emptyset \;.\; \bigwedge T \in S \;.\; S : \texttt{CompleteLattice}$$

Proof $\approx$

Assume hypothesis. By assumption S is closed under joins. Now let $\mathrm{UB}(T)$ be set of all upper bounds of T inside S. As $1 \in S$ $\mathrm{UB}(T) \neq \emptyset$. But this means that $\bigvee T = \bigwedge \mathrm{UB}(T) \in S$ and this is exactly what we need.

$\square$