Convex Analysis

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May 26, 2022

Contents

1 Convex Functions

1.1 Subject

$\texttt{epigraph} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \left(D \to \overset{\infty}{\mathbb{R}}\right) \to ?(V \oplus \mathbb{R})$

$\texttt{epigraph}\,(f) = \operatorname{epi} f := \{(x,\phi) | x \in D, \phi \in \mathbb{R}, \phi \geq f(x)\}$

$\texttt{Convex} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . ?\left(D \to \overset{\infty}{\mathbb{R}}\right)$

$f : \texttt{Convex} \iff \texttt{Convex}(V \oplus \mathbb{R}, \operatorname{epi} f)$

$\texttt{effectiveDomain} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \texttt{Convex}(V, D) \to ?D$

$\texttt{effectiveDomain}\,(f) = \operatorname{dom} f := \pi_1 \operatorname{epi} f$

$\texttt{DomainIsConvex} :: \forall V \in \mathbb{R}\text{-VS} . \forall D \subset V . \forall f : \texttt{Convex}(V, D) . \texttt{Convex}(V, \operatorname{dom} f)$

$\texttt{Proof} =$

As a linear image of convex set.

$\square$

$\texttt{ProperConvexFunction} :: \prod V : \mathbb{R}\text{-VS} . ?\texttt{Convex}(V, V) .$

$f : \texttt{ProperConvexFunction} \iff \forall x \in V . f(x) > -\infty \;\&\; \exists x \in V . f(x) < +\infty$

$\texttt{InterpolationProperty} ::$

$:: \forall V : \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$

$. \texttt{Convex}(V, C, f) \iff \forall x, y \in C . \forall \lambda \in [0,1] .$

$. f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda) f(y)$

$\texttt{Proof} =$

$(\Rightarrow)$: assume that f is convex.

Then f has convex epigraph.

Take arbitrary $x, y \in C$ and $\lambda \in [0,1]$.

If f takes value $+\infty$ either in x or y, then the inequality follows, so assume the contrary.

Then $\Big(x, f(x)\Big), \Big(y, f(y)\Big)$ trivially belong to the epigraph,

so by convexity $\Big(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)\Big)$ is also in epigraph.

The definition of epigraph produces the inequality $f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda) f(y)$.

$(\Leftarrow)$: now assume that inequality always hold.

Assume $(x,\phi), (y,\psi)$ belong to the epigraph and $\lambda \in [0,1]$.

Then $\lambda\phi + (1-\lambda)\psi \geq \lambda f(x) + (1-\lambda) f(y) \geq f\Big(\lambda x + (1-\lambda)y\Big)$.

So $\lambda(x,\phi) + (1-\lambda)(y,\psi)$ belong to the epigraph.

Thus, epigraph is convex and f is convex.

JensensIneq ::

$$:: \forall V : \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$$

$$. \forall n \in \mathbb{N} . \forall \lambda \in \mathbb{R}^n_+ . \forall \aleph : \sum_{k=1}^n \lambda_k = 1 . \forall v \in V^n . f\left(\sum_{k=1}^n \lambda_k v_k\right) \leq \sum_{k=1}^n \lambda_k f(v_k)$$

Proof =

Iterate the interpolation property.

□

SecondDerivativeConvexityTest :: $\forall I : \texttt{OpenInterval}(\mathbb{R}) . \forall f \in C^2(I)$.

$. \texttt{Convex}(\mathbb{R}, I, f) \iff f'' \geq 0$

Proof =

$(\Rightarrow)$: assume there is a $t \in I$ such that $f''(t) < 0$.

As f'' must be continous there is whole open interval (a, b) such that $f''(j) < 0$ for all $j \in (a, b)$.

Take some $x, y \in (a, b)$ with $x < y$ and define$z = \lambda x + (1 - \lambda)y$ for siome $\lambda \in (0, 1)$.

Then $f(z) - f(x) = \displaystyle\int_x^z f'(t)\, dt > f'(z)(z - x)$ and $f(y) - f(z) = \displaystyle\int_z^y f'(t)\, dt < f'(z)(y - z)$.

Then from definiton of z we get $f(z) > f(x) - (1 - \lambda)f'(z)(y - x)$ and $f(z) > f(y) + \lambda f'(z)(y - x)$.

By adding two inequalities with multipliers λ and $(1 - \lambda)$ one gets $f\Big(\lambda x + (1 - \lambda)y\Big) > \lambda f(x) + (1 - \lambda)f(y)$.

But this contradicts a convexity. .

$(\Rightarrow)$: use same inequalities but with different sign to prove the convexity.

□

ExponentIsConvexity :: $\forall \alpha \in \mathbb{R} . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}, \Lambda t \in \mathbb{R} . e^{\alpha t}\Big)$

Proof =

write $f(t) = e^{\alpha t}$.

Then $f''(t) = \alpha^2 e^{\alpha t} \geq 0$.

□

MonomialConvexity1 :: $\forall p \in [1, +\infty) . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . t^p\Big)$

Proof =

Write $f(t) = t^p$.

Then $f''(t) = p(p - 1)t^{p-2} \geq 0$ for $t > 0$.

□

MonomialConvexity2 :: $\forall p \in [0, 1) . \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} . -t^p\Big)$

Proof =

Write $f(t) = t^p$.

Then $f''(t) = p(1 - p)t^{p-2} \geq 0$ for $t > 0$.

□

$\texttt{MonomialConvexity3} :: \forall p \in (-\infty, 0] \;.\; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t^p\Big)$

$\texttt{Proof} \;=$

write $f(t) = t^p$.

Then $f''(t) = p(p-1)t^{p-2} \geq 0$ for $t > 0$.

$\square$

$\texttt{GeneralizedArcsinDerivativeIsConvex} :: \forall \alpha \in \mathbb{R}_{++} \;.\; \texttt{Convex}\left(\mathbb{R}, (-\alpha, \alpha), \Lambda t \in \mathbb{R} \;.\; \frac{1}{\sqrt{\alpha^2 - t^2}}\right)$

$\texttt{Proof} \;=$

Write $f(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}}$.

Then $f'(t) = \dfrac{t}{\sqrt{\alpha^2 - t^2}^3}$.

And $f''(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}^3} + \dfrac{3t^2}{\sqrt{\alpha^2 - t^2}^5} > 0$ for $t \in (-\alpha, \alpha)$.

$\square$

$\texttt{NegativeLogIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; -\ln(t)\right)$

$\texttt{Proof} \;=$

Write $f(t) = -\ln(t)$.

Then $f''(t) = \dfrac{1}{t^2} > 0$ for $t > 0$.

$\square$

$\texttt{NegativeEntropyIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t\ln(t)\right)$

$\texttt{Proof} \;=$

Write $f(t) = t\ln(t)$.

Then $f'(t) = \ln(t) + 1$.

And $f''(t) = \dfrac{1}{t} > 0$ for $t > 0$.

$\square$

$\texttt{Concave} :: \prod V : \mathbb{R}\text{-VS} \;.\; \prod D \subset V \;.\; ?\Big(D \to \overset{\infty}{\mathbb{R}}\Big)$

$f : \texttt{Concave} \iff \texttt{Convex}(V, D, -f)$

$\texttt{SecondDerivativeConvexityTest2} :: \forall V : \texttt{EucledeanSpace} \;.\; \forall U : \texttt{Open} \;\&\; \texttt{Convex}(V) \;.\; \forall f \in C^2(U) \;.$
$\quad . \;\texttt{Convex}(\mathbb{R}, U, f) \iff \mathbf{D}^2 f \geq 0$

$\texttt{Proof} \;=$

For $x \in U$ and $v \in V \setminus \{0\}$ define $\phi_{x,v}(t) = f(x + tv)$ with a domain $I_{x,v} = \{t \in \mathbb{R} | x + tv \in C\}$.

Then f is convex iff every $\phi_{x,v}$ does.

But $\phi''_{x,v}(t) = \langle v, \mathbf{D}^2 f|_y v\rangle$, where $y = x + tv$.

So f is convex iff $\mathbf{D}^2 f$ is positive-semidefinite.

$\square$

`GeometricMeanIsConcave` ::

$$:: \forall V : \texttt{EucledeanSpace} \,.\, \texttt{Concave}\left(V, V_{++}, \Lambda x \in V \,.\, \prod_{k=1}^{n} \sqrt[n]{x_k}\right) \quad \texttt{where} \quad n = \dim V$$

`Proof` =

write $f(x) = \prod_{k=1}^{n} \sqrt[n]{x_k}$.

Then $\nabla f|_x = \left(\frac{1}{n\sqrt[n]{x_i}^{n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}\right)_{i=1}^{n}$.

And $\mathbf{D}^2_{i,j} f|_x = \frac{1}{n^2 \sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k}$ when $i \neq j$, and $\mathbf{D}^2_{i,i} f|_x = -\frac{n-1}{n^2 \sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}$.

So, $\mathbf{D}^2 f|_x(v,v) = -\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{\sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j} + \frac{1}{n^2} \sum_{i \neq j}^{n} \frac{v_i v_j}{\sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k} =$

$= f(x)\left(-\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} + \frac{1}{n^2} \sum_{i \neq j}^{n} \frac{v_i v_j}{x_i x_j}\right) = -\frac{f(x)}{n^2}\left(n \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} - \left(\sum_{i=1}^{n} \frac{v_i}{x_i}\right)^2\right) \leq 0$.

This follows from obvious matching schema.

□

`NormsAreConvex` :: $\forall V : \mathbb{R}\text{-VS} \,.\, \forall \eta : \texttt{Norm}(V)\texttt{Convex}(V, V, \eta)$

`Proof` =

Write $\eta(v) = \|v\|$.

Just use triangle inequality $\Big\|\lambda x + (1-\lambda)y\Big\| \leq \|\lambda x\| + \Big\|(1-\lambda)y\Big\| = \lambda\|x\| + (1-\lambda)\|y\|$.

□

`convexIndicator` :: $\forall V : \mathbb{R}\text{-VS} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

`convexIndicator` $(C) = \Lambda x \in V \,.\, \chi(x|C) := \Lambda x \in V \,.\, \infty\left[x \in C^{\complement}\right]$

`supportFunction` :: $\forall V : \mathbb{R}\text{-HIL} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

`supportFunction` $(C) = \Lambda x \in V \,.\, \chi^*(x|C) := \sup_{y \in C} \langle x, y \rangle$

`gauge` :: $\forall V : \mathbb{R}\text{-VS} \,.\, \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

`gauge` $(C) = \Lambda x \in V \,.\, \gamma(x|C) := \Lambda x \in V \,.\, \inf\left\{\lambda \in \mathbb{R}_{++} \middle| x \in \lambda C\right\}$

`ConvexFunctionHasConvexLevelSets` ::

$$:: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall f : \texttt{Convex}(V, V) \,.\, \forall \alpha \in \overset{\infty}{\mathbb{R}} \,.\, \texttt{Convex}\Big(V, \{v \in V : f(v) \geq \alpha\}\Big)$$

`Proof` =

...

□

`ConvexFunctionHasConvexStrictLevelSets` ::

$$:: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall f : \texttt{Convex}(V, V) \,.\, \forall \alpha \in \overset{\infty}{\mathbb{R}} \,.\, \texttt{Convex}\Big(V, \{v \in V : f(v) > \alpha\}\Big)$$

`Proof` =

...

ConvexlyBoundedRegionIsConvex ::

$$:: \forall V \in \mathbb{R}\text{-VS} . \forall I \in \mathsf{SET} . \forall \alpha : I \to \overset{\infty}{\mathbb{R}} . \forall f : I \to \mathtt{Convex}(V, V) . \mathtt{Convex}\Big(V, \{v \in V : \forall i \in I . f_i(v) > \alpha_i\}\Big)$$

Proof =

GeneralizedAMGMIneq $:: \forall n \in \mathbb{N} . \forall \lambda : \mathbb{R}^n_+ . \forall x : \mathbb{R}^n_{++} . \forall \aleph : \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$

Proof =

By Jensen inequality for natural logarithm $\ln\left(\sum_{i=1}^n \lambda_i x_i\right) \geq \sum_{i=1}^n \lambda_i \ln(x_i)$.

Then by exponentiating both parts $\sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$.

□

PositivelyHomogeneous $:: \prod V : \mathbb{R}\text{-VS} . ?\Big(V \to (-\infty, +\infty]\Big)$

$f : \mathtt{PositivelyHomogeneous} \iff \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . f(\alpha v) = \alpha f(v)$

PositiveHomogeneousZeroPositivity $:: \forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V) . f(0) \geq 0$

Proof =

Note that $f(0) = f(t0) = tf(0)$ for all $t \in \mathbb{R}_{++}$.

This means that $f(0)$ is either 0 or $+\infty$.

□

PositiveHomogeneousConvexity $:: \forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V) .$

$. \mathtt{Convex}(V, V, f) \iff \forall x, y \in V . f(x + y) \leq f(x) + f(y)$

Proof =

$(\Rightarrow)$: assume f is convex.

Then $f(x + y) = f\left(\frac{2}{2}x + \frac{2}{2}y\right) \leq \frac{1}{2}f(2x) + \frac{1}{2}f(2y) = f(x) + f(y)$ for any $x, y \in V$.

$(\Leftarrow)$: assume the inequality holds .

Then $f\Big(\lambda x + (1 + \lambda)y\Big) \leq f(\lambda x) + f\Big((1 - \lambda)y\Big) = \lambda f(x) + (1 - \lambda)f(y)$ when $\lambda \in (0, 1)$ and $x, y \in V$.

Otherwise, when $\lambda = 0, 1$, convexity condition holds trivially.

□

Conic $:= \lambda V \in \mathbb{R}\text{-VS} . \mathtt{Convex}(V, V) \times \mathtt{PositivelyHomogeneous}(V) : \mathbb{R}\text{-VS} \to \mathtt{Type}$;

ConicIneq $:: \forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{Convex}(V, V) \,\&\, \mathtt{PositivelyHomogeneous}(V) . \forall n \in \mathbb{N} . \forall x \in V^n .$

$$. \forall \lambda \in \mathbb{R}^n_{++} . f\left(\sum_{i=1}^n \lambda_i x\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

Proof =

Iterate previous theorem.

□

$\texttt{ConicEpigraph} :: \forall V \in \mathbb{R}\text{-VS} . \forall f : V \to (-\infty, +\infty) . \texttt{Conic}(V, f) \iff \texttt{ConvexCone}(V, \operatorname{epi} f)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ConicIsSupersymmetric} :: \forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . \forall v \in V . f(v) \geq -f(-v)$

$\texttt{Proof} =$

Write $f(x) + f(-x) \geq f(x - x) = f(x) \geq 0$.

So $f(x) \geq -f(-x)$.

$\square$

$\texttt{ConicIsLinearIffSymmetric} :: \forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . f \in V^* \iff \forall v \in V . f(-v) = -f(v)$

$\texttt{Proof} =$

$(\Rightarrow)$: this is trival.

$(\Leftarrow)$: assume that the property holds .

Let $x, y \in V$.

Then $f(x) + f(y) \geq f(x + y) \geq -f(-x - y) \geq -f(-x) - f(-y) = f(x) + f(y)$.

This mean $f(x) + f(y) = f(x + y)$.

But as x and y were arbitrary f must be additive and hence linear.

$\square$

$\texttt{StrictlyConvexFunction} :: \prod V : \mathbb{R}\text{-VS} . ?\texttt{ProperConvexFunction}(V)$

$f : \texttt{StrictlyConvexFunction} \iff$

$$\iff \forall \lambda \in [0, 1] . \forall x, y \in V . \forall \aleph : x \neq y . f\Big(\lambda x + (1 - \lambda) y\Big) < \lambda f(x) + (1 - \lambda) f(y)$$

$\texttt{SquareNoremIsStrictlyConvex} :: \forall V : \texttt{NormedSpace}(\mathbb{R}) . \texttt{StrictConvexFunction}\Big(V, \| \bullet \|^2\Big)$

$\texttt{Proof} =$

$\forall x, y \in V, \lambda \in (0, 1) . \Big\|\lambda x + (1 - \lambda) y\Big\|^2 \leq \Big(\lambda \|x\| + (1 - \lambda) \|y\|\Big)^2 < \lambda^2 \|x\|^2 + (1 - \lambda)^2 \|y\|^2 <$
$< \lambda \|x\|^2 + (1 - \lambda) \|y\|^2$.

$\square$

1.2 Convexity Preserving Operations

$\texttt{ConvexComposition} :: \forall V \in \mathbb{R}\text{-VS} . \forall D \subset V \forall f : \texttt{Convex}(V, D) . \forall \phi : \texttt{Convex} \ \& \ \texttt{Increasing}(\mathbb{R}, \mathbb{R}) . \texttt{Convex}(V,$

$\texttt{Proof} =$

Assume $x, y \in \operatorname{dom} f, \lambda \in [0, 1]$.

Then $\phi\Big(f\big(\lambda x + (1-\lambda)y\big)\Big) \leq \phi\Big(\lambda f(x) + (1-\lambda)f(y)\Big) \leq \lambda\phi \circ f(x) + (1-\lambda)\phi \circ f(y)$.

$\square$

$\texttt{ConvexFunctionFromSet} :: \forall V \in \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V \oplus \mathbb{R}) . \texttt{Convex}\Big(V, V, \Lambda v \in V \ . \ \inf\big\{t|(v,t) \in C\big\}\Big)$

$\texttt{Proof} =$

This is function has cinvex epigraph.

$\square$

$\texttt{InfimalConvolutionIsConvex} :: \forall V \in \mathbb{R}\text{-VS} . \forall n \in \mathbb{N} . \forall f : \{1, \dots, n\} \to \texttt{ProperConvexFunction}(V) .$

$$. \texttt{Convex}\left(V, V, \Lambda x \in V \ . \ \inf\left\{\sum_{k=1}^{n} f_k(v_k)\middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}\right)$$

$\texttt{Proof} =$

Let $g = \inf\left\{\sum_{k=1}^{n} f_k(v_k)\middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$.

$C = \sum_{k=1}^{n} \operatorname{epi} f_k$ is convex.

A tuple $(x, \phi) \in C$ if there is a sequence $(v, \psi) \in (V \oplus \mathbb{R})^n$ such that $x = \sum_{k=1}^{n} v_k, \phi = \sum_{k=1}^{n} \psi_k$

and $f(v_k) \leq \psi_k$ for every $k \in \{1, \dots, n\}$.

Thus $\phi = \sum_{k=1}^{n} \psi_k \geq \sum_{k=1}^{n} f(v_k) \geq g(x)$, so $(x, \phi) \in \operatorname{epi} g$.

Then g can be constructed from set C.

$\square$

$$\texttt{infimalConvolution} :: \prod_{V \in \mathbb{R}\text{-VS}} \prod_{n=1}^{\infty} \Big(\{1, \dots, n\} \to \texttt{ProperConvexFunction}(V)\Big) \to \texttt{Convex}(V, V)$$

$$\texttt{infimalConvolution}(f) = \mathop{\square}_{k=1}^{n} f_i := \inf\left\{\sum_{k=1}^{n} f_k(v_k)\middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$$

$$\texttt{ConvexDelta} :: \prod_{V \in \mathbb{R}\text{-VS}} V \to \texttt{ProperConvexFunction}(V)$$

$$\texttt{convexDelta}(a) = \delta_a := \Lambda x \in V \ . \ \texttt{if} \ x = a \ \texttt{then} \ 0 \ \texttt{else} \ +\infty$$

$\texttt{GraphTranslationByInfimalConvolution} :: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{ProperConvexFunction}(V) . \forall a, v \in V . (\delta_a \Box f$

$\texttt{Proof} =$

Clearly $(\delta_a \Box f)(v) = \min \left\{ f(v-a), +\infty \right\}$.

□

$\texttt{InfimalConvolutionDomain} :: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{ProperConvexFunction}(V) . \forall a, v \in V . \operatorname{dom}(f \Box g) = \operatorname{dom} f$

$\texttt{Proof} =$

Obvious .

□

$\texttt{DistanceExpression} :: \forall V : \texttt{NormedSpace}(\mathbb{R}) . \forall C : \texttt{Convex}(V) . d_V(C, \bullet) = \| \bullet \| \Box \chi(\bullet | C)$

$\texttt{Proof} =$

□

$\texttt{InfimalConvolutionDefinesCommutativeMonoid} ::$

$:: \forall V \in \mathbb{R}\text{-VS} .$

$. \texttt{CommutativeMonoid}\Big(\texttt{Convex}(V, V), \Lambda f, g \in \texttt{Convex}(V, V) . \Lambda x \in V . \inf \left\{ \phi \middle| (v, \phi) \in (\operatorname{epi} f + \operatorname{epi} g) \right\} \Big)$

$\texttt{Proof} =$

δ_0 is a neutral element, comutativity and associativity is almost obvious.

□

$$\texttt{rightScalarMultiplication} :: \prod_{V \in \mathbb{R}\text{-VS}} . \texttt{Convex}(V, V) \to \mathbb{R}_+ \to \texttt{Convex}(V, V)$$

$$\texttt{rightScalarMultiplication}\,(f, \lambda) = f\lambda := \texttt{ConvexFunctionFromSet}(V, \lambda \operatorname{epi} f)$$

$\texttt{RightScalarMultiplicationExpression} ::$

$:: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) . \forall \lambda \in \mathbb{R}_{++} . \forall x \in X . f\lambda(x) = \lambda f(\lambda^{-1} x)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{RightScalarMultiplicationByZero} :: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) . f0(x) = \delta(0)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{ConicityByRightMultiplication} :: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) . \texttt{Conic}(V, f) \iff \forall \lambda \in \mathbb{R}_{++} . f\lambda = f$

$\texttt{Proof} =$

Follows from the expression for right multiplication.

□

$\texttt{conicClosure} :: \prod_{V \in \mathbb{R}\text{-VS}} \texttt{ConvexFunction}(V) \to \texttt{Conic}(V)$

$\texttt{conicClosure}\,(\operatorname{cone} f) := \texttt{ConvexFunctionFromSet}\Big(V.\operatorname{cone}\operatorname{epi} f\Big)$

$\texttt{GaugeExpression} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall C : \texttt{Convex} \,\&\, \texttt{NonEmpy}(V) \,.\, \gamma(\bullet|C) = \operatorname{cone}\Big(\chi(\bullet|C) + 1\Big)$

Proof $=$

$(x, \phi) \in \operatorname{epi}\gamma(\bullet|C)$ iff $x \in \lambda C$ and $0 < \lambda \le \phi$.

This means that $(x, \lambda) \in \operatorname{cone} C \times \{1\} \subset \operatorname{cone}\operatorname{epi}\Big(\chi(\bullet|C) + 1\Big)$.

So $(x, \phi) \in \operatorname{cone} C \times \{\phi/\lambda\} \subset \operatorname{cone}\operatorname{epi}\Big(\chi(\bullet|C) + 1\Big) = \operatorname{epi}\operatorname{cone}\Big(\chi(\bullet|C) + 1\Big)$.

On the other hand id $(x, \psi) \in \operatorname{epi}\operatorname{cone}\Big(\chi(\bullet|C) + 1$ then there exists $\lambda \in \mathbb{R}_{++}$ such that $\lambda x \in C$ and $\lambda\psi \ge 1$.

But this means that $\psi \ge \lambda^{-1} \ge \gamma(x|C)$.

Thus $(x, \psi) \in \gamma(\bullet|C)$.

And both functions are equal by equality of epigraphs.

□

$\texttt{SupremumIsConvex} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to \texttt{ConvexFunction}(V) \,.\, \texttt{ConvexFunction}(V, \sup_{i \in I} f_i)$

Proof $=$

$\operatorname{epi}\sup_{i \in I} f_i = \bigcap_{i \in I} \operatorname{epi} f_i$ is convex.

$\texttt{convexHull} :: \prod_{V \in \mathbb{R}\text{-VS}} \prod_{I \in \mathsf{SET}} \Big(I \to V \to \overset{\infty}{\mathbb{R}}\Big) \to \texttt{ConvexFunction}(V)$

$\texttt{convexHull}\,(f) = \operatorname{conv}_{i \in I} f_i := \texttt{ConvexFunctionFromSet}\left(V, \operatorname{conv}\bigcup_{i \in \mathcal{I}} \operatorname{epi} f_i\right)$

$\texttt{ConvexHullExpression} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to V \to (-\infty, +\infty] \,.\, \forall v \in V \,.$

$$. \operatorname{conv}_{i \in I} f_i(x) = \inf\left\{ \sum_{i \in I} \lambda_i f_i(v_i) \middle| \lambda \in \mathbb{R}_+^{\oplus I}, v : I \to V, \sum_{i \in I} \lambda_i = 1, \sum_{i \in I} \lambda_i v_i = x \right\}$$

Proof $=$

This follows from the thorough examination of the definition.

$\texttt{convexPullback} :: \prod V, W \in \mathbb{R}\text{-VS} \,.\, \mathbb{R}\text{-VS}(V, W) \to \texttt{ConvexFunction}(W) \to \texttt{ConvexFunction}(V)$

$\texttt{convexPullback}\,(f, T) = fT := f \circ T$

$\texttt{convexPushforward} :: \prod V, W \in \mathbb{R}\text{-VS} \,.\, \mathbb{R}\text{-VS}(V, W) \to \texttt{ConvexFunction}(V) \to \texttt{ConvexFunction}(W)$

$\texttt{convexPullback}\,(f, T) = T_* f := \Lambda w \in W \,.\, \inf\{f(v) | w = Tv\}$

1.3 Metric and Topological Properties

`SphericalBound` ::

$:: \forall V \in \mathsf{BAN}(\mathbb{R}) \;.\; \forall f : \texttt{ProperConvexFunction}(V) \;.\; \forall c \in \operatorname{dom} f \;.\; \forall \rho \in \mathbb{R}_{++} \;.\; \forall \aleph : \eta < +\infty \;.\; \forall \alpha \in (0,1) \;.$

$. \;\forall x \in \mathbb{B}_V(c, \alpha\rho) \;.\; \Big|f(x) - f(c)\Big| \le \alpha\Big(\eta - f(c)\Big)$

`where` $\eta = \sup f\Big(\mathbb{B}_V(c,\rho)\Big)$

`Proof` =

By convexity $f(x) - f(c) = f\left((1-\alpha)c + \alpha\left(\frac{x-(1-\alpha)c}{\alpha}\right)\right) - f(c) \le \alpha\left(f\left(c + \frac{x-c}{\alpha}\right) - f(c)\right).$

If $x \in \mathbb{B}_V(c, \alpha\rho)$ then $c + \dfrac{x-c}{\alpha} \in \mathbb{B}_V(c,\rho)$.

So $f(x) - f(c) \le \alpha\Big(\eta - f(c)\Big).$

On the other hand $f(c) - f(x) = f\left(\frac{x}{1+\alpha} + \frac{\alpha}{1-\alpha}\frac{(1+\alpha)c - x}{\alpha}\right) - f(x) \le$

$$\le \frac{\alpha}{1+\alpha}\left(f\left(c + \frac{c-x}{\alpha}\right) - f(x)\right) \le \frac{\alpha}{1+\alpha}(\eta - f(x)) = \frac{\alpha}{1+\alpha}\Big(\eta - f(c)\Big) + \frac{\alpha}{1+\alpha}\Big(f(c) - f(x)\Big).$$

So by rearanging inequalities one gets $f(c) - f(x) \le \alpha\Big(\eta - f(x)\Big).$

`LocalLipschitzContinuity` ::

$:: \forall V \in \mathsf{BAN}(\mathbb{R}) \;.\; \forall f : \texttt{ProperConvexFunction}(V) \;.\; \forall c \in \operatorname{dom} f \;.\; \forall \rho \in \mathbb{R}_{++} \;.\; \forall \aleph : \delta < +\infty \;.$

$. \left(\frac{\delta}{\rho}\right)\text{-Lip}\Big(\mathbb{B}(c,\rho), \mathbb{R}, f_{|\mathbb{B}(c,\rho)}\Big)$

`where` $\eta = \operatorname{diam} f\Big(\mathbb{B}_V(c,\rho)\Big)$

`Proof` =

Assume $x, y \in \mathbb{B}_V(c,\rho)$ sych that $x \ne y$.

Let $\alpha = \dfrac{\|x-y\|}{\|x-y\| + \rho} < \dfrac{\|x-y\|}{\rho}$ and $z = x + \dfrac{1-\alpha}{\alpha}(x-y)).$

Then, $\|z - c\| \le \|z - x\| + \|x - c\| \le \dfrac{1-\alpha}{\alpha}\|x-y\| + \rho \le 2\rho$, so $z \in \mathbb{B}_V(c, 2\rho)$.

Thus, by convexity $f(x) = f\Big(\alpha z + (1-\alpha)y\Big) \le f(y) + \alpha(f(z) - f(y) \le f(y) + \alpha\delta \le f(y) + \dfrac{\delta}{\rho}\|x-y\|$.

From symmetry $|f(x) - f(y)| \le \dfrac{\delta}{\rho}\|x-y\|$.

□

$\texttt{ContinuityByBound} ::$

$:: \forall V \in \mathsf{BAN}(\mathbb{R}) \ . \ \forall f : \texttt{ProperConvexFunction}(V) \ . \ \forall x \in V \ . \ \forall U \in \mathcal{U}(x) \ .$

$. \ \forall \aleph : \texttt{Bounded}(V, U, f_{|U}) \ . \ \operatorname{int}\operatorname{dom} f \xrightarrow{f_{|\operatorname{int}\operatorname{dom} f}} \mathbb{R} : \mathsf{TOP}$

$\texttt{Proof} =$

Assume $v \in U \cap \operatorname{rel}\operatorname{int}\operatorname{dom} f$.

As v belongs to relative intrior there are $w \in \operatorname{dom} f, \rho \in Reals_+, \lambda \in (0,1)$

such that $v \in \lambda\mathbb{B}(x,\rho) + (1-\lambda)w$ and $\mathbb{B}(x,\rho) \subset U$.

Then $f(x) \leq \lambda f(y) + (1-\lambda)f(w) \leq \lambda\beta + (1-\lambda)f(w)$,

where $x \in \lambda\mathbb{B}(x,\rho) + (1-\lambda)w, y \in \mathbb{B}(y,\rho)$ and β is the bound for U.

So by the previous theorem f is locally Lipshitz and continuous on $\operatorname{rel}\operatorname{int}\operatorname{dom} f$

and so f is actually continuous on $\operatorname{rel}\operatorname{int}\operatorname{dom} f$.

$\square$

$\texttt{ContinuityByFiniteDimension} ::$

$:: \forall V \in \texttt{EucledeanSpace} \ . \ \forall f : \texttt{ProperConvexFunction}(V) \ . \ \operatorname{int}\operatorname{dom} f \xrightarrow{f_{|\operatorname{int}\operatorname{dom} f}} \mathbb{R} : \mathsf{TOP}$

$\texttt{Proof} =$

Let $n = \dim V$.

Assume $x \in V$ and $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}(x,\rho) \subset \operatorname{dom} f$.

Then there exists a simplicital set $\{v_1, \ldots, v_{n+1}\}$ such that $\mathbb{B}(x,\rho) \subset \operatorname{conv}(v_1, \ldots, v_{n+1}) \subset \operatorname{dom} f$.

But this means that $\sup f\Big(\mathbb{B}(x,\rho)\Big) \leq \max_{i=1,\ldots,n+1} f(v_i)$.

So by the previous theorem f is locally Lipshitz and continuous on $\operatorname{rel}\operatorname{int}\operatorname{dom} f$

and so f is actually continuous on $\operatorname{rel}\operatorname{int}\operatorname{dom} f$.

$\square$

$\texttt{NonEmptyInteriorCondition} :: \forall V \in \mathsf{BAN}(\mathbb{R}) \ . \ \forall f : \texttt{ProperConvexFunction}(V) \ . \ \forall x \in V \ . \ \forall U \in \mathcal{U}(x) \ .$

$. \ \forall \aleph : \texttt{Bounded}(V, U, f_{|U}) \ . \ \operatorname{int}\operatorname{epi} f \neq \emptyset$

$\texttt{Proof} =$

We can find radius ρ Lipschitz constant β such that $\Big|f(v) - f(w)\Big| \leq \beta\|v-w\| \leq \beta\rho$ for every $v, w \in \mathbb{B}(x,\rho)$.

Select $\delta \in (2\beta\rho, +\infty)$ and set $\gamma = \min(\rho, \delta/2) > 0$.

And let $(y,\phi) \in V \oplus \mathbb{R}$ such that $\Big\|(y,\phi) - \big(x, f(x)+\rho\big)\Big\|^2 \leq \gamma^2$.

Then $\|y - x\| \leq \gamma \leq \rho$ and $\Big|\phi - \big(f(x)+\delta\big)\Big| \leq \gamma \leq \delta/2$.

So, $f(y) < f(x) + \delta/2 = f(x) + \delta - \delta/2 \leq f(x) + \delta - \gamma \leq \phi$.

Thus $(y,\phi) \in \operatorname{epi} f$.

As (y,ϕ) was arbitrary, the whole sphere is a subset of $\operatorname{epi} f$.

So $\operatorname{int}\operatorname{epi} f \neq \emptyset$.

$\square$

$\texttt{InteriorLevelSet} ::$

$:: \forall V : \mathbb{R}\text{-}\mathsf{BAN} \,.\, \forall f : \texttt{LowerSemicontinuous}\Big(V, (-\infty, +\infty]\Big) \,.\, \forall x \in V \,.\, \forall \aleph : f(x) < 0 \,.\, x \in \operatorname{int} f^{-1}(-\infty, 0]$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConvexLevelSetInterior} :: \forall V : \mathbb{R}\text{-}\mathsf{BAN} \,.\, \forall f : \texttt{ConvexFunction}(V) \,.\, \forall x \in V \,.\, \forall \aleph : f(x) < 0 \,.$

$. \operatorname{int} f^{-1}(-\infty, 0] \subset f^{-1}(-\infty, 0)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConvexUsCLevelSetInteriorEq} ::$

$. \forall V : \mathbb{R}\text{-}\mathsf{BAN} \,.\, \forall f : \texttt{ConvexFunction}(V) \,.\, \forall \aleph : \texttt{LowerSemicontinuous}\Big(f_{|f^{-1}(-\infty,0)}, (-\infty, 0)\Big) \,.$

$. \operatorname{int} f^{-1}(-\infty, 0] = \operatorname{int} f^{-1}(-\infty, 0)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConvexEucLevelSetInteriorEq} ::$

$. \forall V \in \texttt{EucledeanSpace} \,.\, \forall f : \texttt{ConvexFunction}(V) \,.\, \forall \aleph : \operatorname{dom} f \in \mathcal{T}(V) \,.\, \operatorname{int} f^{-1}(-\infty, 0] = \operatorname{int} f^{-1}(-\infty, 0)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.4 Closures

$\texttt{LowerSemicontinuityByEpigraph} :: \forall V \in \texttt{EucledeanSpace} . \forall f : V \to \mathbb{R}\text{-TOPVS} .$

$$\texttt{LowerSemicontinuous}(V, \overset{\infty}{\mathbb{R}}, f) \iff \texttt{Closed}(V \oplus \mathbb{R}, \operatorname{epi} f)$$

$\texttt{Proof} \;=$

If f is lower semicontinuous then $\liminf_{x \to v} f(x) = f(v)$.

So the epigraph must be closed.

Thus, result is basically obvious.

$$\texttt{closure} :: \prod_{V \in \mathbb{R}\text{-TOPVS}} \texttt{ConvexFunction}(V) \to \Big(\texttt{ConvexFunction}(V) \;\&\; \texttt{LowerSemicontinuous}(V, \mathbb{R})\Big)$$

$$\texttt{closure}\,(f) = \operatorname{cl} f := \textbf{if } f > -\infty \textbf{ then } \texttt{FunctionFromSet}(V, \operatorname{cl}\operatorname{epi} f) \textbf{ else } -\infty$$

$\texttt{ClosedFunction} :: \prod V \in \mathbb{R}\text{-TOPVS} . ?\texttt{ConvexFunction}(V)$

$f : \texttt{ClosedFunction} \iff \operatorname{cl} f = f$

$\texttt{ImproperDomain} :: \forall V \in \mathbb{R}\text{-TOPVS} . \forall f : \texttt{ConvexFunction}(V) .$

$. \forall \aleph : -\infty \in \operatorname{Im} f . \forall x \in \operatorname{rel}\operatorname{int}\operatorname{dom} f . f(x) = -\infty$

$\texttt{Proof} \;=$

If there is a point $p \in V$ such that $f(p) = -\infty$ then $p \in \operatorname{dom} f$.

Also assume that $u \in \operatorname{rel}\operatorname{int}\operatorname{dom} f$.

By propertirs of relative interiot there exists $x \in \operatorname{dom} f$ suxh that $u \in (p, x)$.

So there is $\lambda \in (0, 1)$ suxh that $u = \lambda u + (1 - \lambda)x$.

But $(p, \alpha) \in \operatorname{epi} f$ for any arbitrary $\alpha \in \mathbb{R}$.

Thus $\Big(u, \lambda\alpha + (1 - \lambda)f(x)\Big) \in \operatorname{epi} f$ fo any α .

And by taking the limit $\alpha \to -\infty$ we see that it must be the case that $f(u) = -\infty$.

$\square$

$\texttt{ContinuityByClosedness} ::$

$:: \forall V \in \mathsf{BAN}(\mathbb{R}) . \forall f : \texttt{ClosedFunction}(V) . \operatorname{int}\operatorname{dom} f \xrightarrow{f_{|\operatorname{int}\operatorname{dom} f}} \mathbb{R} : \mathsf{TOP}$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ConvexLsCLevelSetInteriorEq} ::$

$. \forall V \in \texttt{EucledeanSpace} . \forall f : \texttt{ClosedFunction}(V) . \forall \aleph : \operatorname{dom} f \in \mathcal{T}(V) .$

$. \operatorname{int} f^{-1}(-\infty, 0] = \operatorname{int} f^{-1}(-\infty, 0)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{ClosedFunctionSupremum} :: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to \texttt{ClosedFunction}(V) \,.$

$\quad .\ \sup f \in \texttt{ClosedFunction}(V)$

$\texttt{Proof} \ =$

The epigraph of the supremum is the intersection of epigraphs.

Then use the fact that intersection of closed sets is closed.

$\square$

$\texttt{ClosedFunctionSum} ::$

$\quad .\ \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f, g \in \texttt{ClosedFunction} \ \&\ \texttt{ProperConvexFunction}(V) \,.$

$\quad .\ f + g \in \texttt{ClosedFunction} \ \&\ \texttt{ProperConvexFunction}(V)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{ClosedFunctionSum2} ::$

$\quad .\ \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : \mathbb{N} \to \texttt{ClosedFunction}(V) \,.$

$\quad .\ \forall \aleph : \inf_{i \in I} f_i \geq 0 \,.\, \sum_{i \in I} f_i \in \texttt{ClosedFunction}(V)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{DegenerateClosedForm} ::$

$\quad .\ \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \texttt{ClosedFunction}(V) \,.\, \forall \aleph : -\infty \in \operatorname{Im} f \,.\ \operatorname{Im} f = \{-\infty, +\infty\}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{ClosureLevelSets} ::$

$\quad : \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \texttt{ConvexFunction}(V) \,.\ \operatorname{cl}\left(f^{-1}(-\infty, 0)\right) = \operatorname{cl}\left(f^{-1}(-\infty, 0]\right) = \left(\operatorname{cl} f\right)^{-1}(-\infty, 0)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{ProperClosedFunction} := \texttt{ProperConvexFunction} \ \&\ \texttt{ClosedFunction} : \mathbb{R}\text{-}\mathsf{TOPVS} \to \texttt{Type};$

$\texttt{ConvexLimit} :: \forall V \in \mathbb{R}\text{-}\mathsf{TOPVS} \,.\, \forall f : \texttt{ProperClosedFunction}(V) \,.\, \forall x \in \operatorname{dom} f \,.\, \forall y \in V \,.\ \lim_{\lambda \downarrow 0} g(\lambda) = f(y)$

$\qquad \texttt{where} \quad g = \Lambda \lambda \in [0, 1] \,.\, f\Big(\lambda x + (1 - \lambda) y\Big)$

$\texttt{Proof} \ =$

In $f\Big(\lambda x + (1-\lambda)y\Big) \leq \lambda f(x) + (1-\lambda) f(y)$ the majorant is continuous in *lambda*.

So in $f(x) \leq \lim_{\lambda \to \infty} f\Big(\lambda x + (1-\lambda)y\Big) \leq f(x)$ holds.

$\square$

ProperConvexContinuity :: $f \in$ ProperClosedFunction$(\mathbb{R})$. $f_{|\operatorname{cl}\operatorname{dom} f} \in \mathsf{TOP}\left(\operatorname{cl}\operatorname{dom} f, \overset{\infty}{\mathbb{R}}\right)$

Proof =

Use convex limits as above.

□

ConvexExtension ::

$\forall V \in \mathbb{R}\text{-}\mathsf{TOPVS}$. $\forall f, g \in$ ProperClosedFunction$(V \oplus \mathbb{R})$.

. $\forall \aleph : \operatorname{dom} f \cup \operatorname{dom} g \subset V \oplus \mathbb{R}_+$. $\forall \beth : f_{|V \times \mathbb{R}_{++}} = g_{|V \times \mathbb{R}_{++}}$. $f = g$

Proof =

...

□

1.5 Recession

2 Duality

2.1 Conjugate Functions

$$\texttt{conjugateFunction} :: \prod V \in \mathbb{R}\text{-TOPVS} \,.\, \left(V \to \overset{\infty}{\mathbb{R}}\right) \to \left(V^* \to \overset{\infty}{\mathbb{R}}\right)$$

$$\texttt{conjugateFunction}\,(\phi) = \phi^* := \Lambda f \in V^* \,.\, \sup_{x \in V} \; f(x) - \phi(x)$$

$\texttt{ConjugationIneq} :: \forall V \in \mathbb{R}\text{-TOPVS} \,.\, \forall \phi : V \to \overset{\infty}{\mathbb{R}} \,.\, \forall x \in V \,.\, \forall f \in V^* \,.\, \phi(x) + \phi^*(f) \geq f(x)$

$\texttt{Proof} \;=$

Just note that $\phi(x) + \phi^*(f) = \phi(x) + \sup_{y \in V} \; f(y) - \phi(y) \geq f(x)$.

□

$\texttt{DualOfIndicatorIsSupport} :: \forall V \in \mathbb{R}\text{-TOPVS} \,.\, \forall C : \texttt{Convex}(V) \,.\, \Big(\chi(\bullet|C)\Big)^* = \chi^*(\bullet|C)$

$\texttt{Proof} \;=$

$\Big(\chi(\bullet|C)\Big)^*(f) = \sup_{x \in V} f(x) - \chi(x|C) = \sup_{x \in C} f(x) = \chi^*(f|C).$

□

$\texttt{PolarBySupportExpression} :: \forall V \in \mathbb{R}\text{-TOPVS} \,.\, \forall C : \texttt{Convex}(V) \,.\, C^\wedge = \{f \in V^* : \chi^*(f|C) \leq 1\}$

$\texttt{Proof} \;=$

See convex geometry or consider this a definition.

□

$\texttt{ConjugateIsConvex} :: \forall V \in \mathbb{R}\text{-TOPVS} \,.\, \forall \phi : V \to \overset{\infty}{\mathbb{R}} \,.\, \texttt{ClosedFunction}\Big((V, \mathbf{w}^*_V), \phi^*\Big)$

$\texttt{Proof} \;=$

$\phi^*(f) = \sup_{x \in V} f(x) - \phi(x)$ which is supremum of affine functions in f.

So, ϕ^* must be continuous.

Clearly, the weakest topology there each $f(x)$ is continuous is weak-star topology.

□

$\texttt{DoubleConjugateIneq} :: \forall V \in \mathbb{R}\text{-TOPVS} \,.\, \forall \phi : V \to \overset{\infty}{\mathbb{R}} \,.\, \phi^{**}_{|V} \leq \phi$

$\texttt{Proof} \;=$

Assume $x \in V$.

$\phi^{**}(x) = \sup_{f \in V^*} \; f(x) - \phi^*(x) = \sup_{f \in V^*} \inf_{y \in V} f(x) - f(y) + \phi(y) = \sup_{f \in V^*} \inf_{y \in V} f(x - y) + \phi(y) \leq$

$\leq \sup_{f \in V^*} \; f(x - x) + \phi(x) = \sup_{f \in V^*} \; \phi(x) = \phi(x).$

□

ConjugateIneq :: $\forall V \in \mathbb{R}\text{-TOPVS} . \forall \phi, \psi : V \to \overset{\infty}{\mathbb{R}} . \forall \aleph : \phi \leq \psi . \phi^* \geq \psi^*$

Proof =

$$\phi^*(f) = \sup_{x \in V} f(v) - \phi(x) \geq \sup_{x \in V} f(v) - \psi(x) = \psi^*(f).$$

□

EvenConjugatePowerStability :: $\forall V \in \mathbb{R}\text{-TOPVS} . \forall \phi : V \to \overset{\infty}{\mathbb{R}} . \forall n : \text{Even} . \phi^{n*}_{|V} = \phi^{**}$

Proof =

Write $n = 2m$ for $m \in \mathbb{N}$.

From previous results $f^{2m*}_{|V} \leq f^{2(m-1)*}_{|V} \leq \ldots \leq f^{**}_{|V} \leq f$.

And also $f^{(2m-1)*}_{|V^*} \leq f^{(2m-3)*}_{|V^*} \leq \ldots \leq f^*$.

But Taking the dual of the last inequelity gives $f^{2m*}_{|V} \geq f^{**}$.

So the equality holds $f^{2m*}_{|V} = f^{**}$.

□

OddConjugatePowerStability :: $\forall V \in \mathbb{R}\text{-TOPVS} . \forall \phi : V \to \overset{\infty}{\mathbb{R}} . \forall n : \text{Odd} . \phi^{n*}_{|V} = \phi^*$

Proof =

Write $n = 2m - 1$ for $m \in \mathbb{N}$.

From previous results $f^{2(m-1)*}_{|V} \leq f^{2(m-2)*}_{|V} \leq \ldots \leq f^{**}_{|V} \leq f$.

And also $f^{(2m-1)*}_{|V^*} \leq f^{(2m-3)*}_{|V^*} \leq \ldots \leq f^*$.

But Taking the dual of the first inequelity gives $f^{(2m-1)*}_{|V} \geq f^*$.

So the equality holds $f^{n*}_{|V} = f^*$.

□

2.2 Affine Minorization

AffineMinorization $::$

$:: \forall V : \texttt{LocallyConvexSpace}(\mathbb{R})\ .\ \forall \phi : \texttt{ProperClosedFunction}(V)\ .\ \forall v \in V\ .$

$.\ \phi(v) = \sup \left\{ A(v) | A \in \mathsf{TAFF}(\mathbb{R}, V, \mathbb{R}), A \leq \phi \right\}$

Proof $=$

As ϕ is proper we may assume that there is some $v \in V$ such that $\phi(v) \neq +\infty$.

So $v \in \operatorname{dom} \phi$ so $\Big(v, \phi(v) - \varepsilon\Big) \notin \operatorname{epi} \phi$, so we may apply Hahn-Banach theorem .

So there exists a support hyperplane H for $\operatorname{epi} \phi$ at $\Big(v, \phi(v)\Big)$ such that $H = \ker A$ and $A(x, \alpha) = B(x) + \alpha\beta$.

Then for any $x \in V$ it holds that $B(x) + \beta\phi(x) \geq 0$.

β must be nonnegative, otherwise the affine hyperplane H will intersect $\operatorname{epi} f$.

Thus, $-\dfrac{1}{\beta} B(x) \leq \phi(x)$.

So $-\dfrac{1}{\beta} B(x)$ is the affine minorant .

$\square$

3 (Sub)differential Calculus

4 From Optimization to Convex Algebra

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