1) K is a splitting field of P, hence it is normal. $P'(X) = -a \neq 0$ which means that P is separable, hence K is Galois.

We can write

$$P(X) = \prod_{i=1}^{p} (X - \alpha^i).$$

By using structure of P, it is known that

$$\sum_{I \in S(n,p)} (-1)^n \prod_{i \in I} \alpha_i = 0$$

for $n : 1 \leq n < p - 1$, and

$$\sum_{I \in S(p-1,p)} (-1)^{p-1} \prod_{i \in i} \alpha_i = -a$$

Where $S(n, p)$ is set of subsets of p of size n.

By expressing this value of α_i as functions of $\alpha_{i+1}, \ldots, \alpha_p$ as substituting them, it must finally yield

$$a = (\alpha_1 - \alpha_2)^{p-1} = \beta^{p-1}$$

Other roots of $X^{p-1} - a$ will have form $z\beta$ for all nonzero elements $z \in \mathbb{F}_p$ as $(z\beta)^{p-1} = z^{p-1}\beta^{p-1} = \beta^{p-1}$.

2) Hence, this polynomial must have cyclic Galois group generated by action $\beta \mapsto 2\beta$ and isomorphic to multiplicative group $\mathbb{F}_p^*$.

3) As order of α_i was arbitrary, $gx - x = z\beta$ in case $g \neq \mathrm{id}$ or otherwise $gx - x = 0$.

In case $g \in H$ when it is stable on roots $X^{p-1} - a$. So if $g = \mathrm{id}$, it yields 0 on all x. Otherwise $gx - x = gy - y$ for all y in the orbit of x. But as splitting field of P the group H must be cyclic (for root α element $\alpha + z\beta$ is also a root, so it must be spawned by map $\alpha \mapsto \alpha + \beta$), so the result doesn't really depends on choice of x.

4) In case P splits over L group H is trivial ($|H| = 1$). In the other case group H must be cyclic group rotating all roots of P so the only choice is $\mathbb{Z}/p\mathbb{Z}$ ($|H| = p$).

5) If $H = \mathbb{Z}pZ$ as it was said before this indicates that P is irreducible over L (otherwise H is trivial), moreover as $k \subset L$ this means that P is irreducible over k.

Now assume that E is trivial. As

$$(X + \beta)^p - a(X + \beta) - b = X^p + \beta^p - aX - a\beta - b = X^p - aX - b + \beta(\beta^{p-1} - a) = X^p - aX - b$$

this means that $0 = \beta = a_i - a_j$. As P is separable this means that P is reducible over k. So if P is irreducible over k the group $H = \mathbb{Z}/p\mathbb{Z}$.

6) Polynomial $X^{p-1} - T$ is irreducible in $\mathbb{F}_p(T)$. So L is nontrivial extension of k. P also does not split over $\mathbb{F}_p(S)$ so $H \cong \mathbb{Z}/p\mathbb{Z}$.