Set Theory

Uncultured Tramp

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Contents

1 Locally Naive Typed Set Theory

1.1 Sets ansd Subsest

$$\texttt{Set} := \prod T : \texttt{Type} \,.\, T \to \texttt{Bool} : \texttt{Type} \to \texttt{Type},$$
$$S : \texttt{Set}(T) \iff S : ?T$$

$$\texttt{In} := \prod S : \texttt{Set}(T) \,.\, \prod a : T \,.\, S(a) =_{\texttt{Bool}} 1 : \prod T : \texttt{Type} \,.\, ?T \to T \to \texttt{Type},$$
$$x : \texttt{In}(S, a) \iff x : a \in S$$

$$\texttt{NotIn} := \prod S : \texttt{Set}(T) \,.\, \prod a : T \,.\, S(a) =_{\texttt{Bool}} 0 : \prod T : \texttt{Type} \,.\, ?T \to T \to \texttt{Type},$$
$$x : \texttt{NotIn}(S, a) \iff x : a \notin S$$

$$\texttt{implicit} :: \prod T : \texttt{Type} \,.\, ?T \to \texttt{Type}$$
$$\texttt{implicit}\,(S) := \sum a : T \,.\, \exists a \in S$$

$$\texttt{emptySet} :: \prod T : \texttt{Type} \,.\, \texttt{Set}(T)$$
$$\texttt{emptySet}\,(T) = \emptyset := \Lambda a : T \,.\, 0$$

$$\texttt{universum} :: \prod T : \texttt{Type} \,.\, \texttt{Set}(T)$$
$$\texttt{universum}\,(T) = U(T) := \Lambda a : T \,.\, 1$$

$$\texttt{singleton} :: \prod T \,.\, T \to ?T$$
$$\texttt{singleton}\,(a) = \{a\} := \Lambda b : T \,.\, a == b$$

$$\texttt{Subset} :: \prod S : ?T \,.\, ??T$$
$$A : \texttt{Subset} \iff A \subset S \iff \forall a : T \,.\, a \in A \Rightarrow a \in S$$

$$\texttt{StrictSubset} :: \prod S : ?T \,.\, ?\texttt{Subset}(S)$$
$$A : \texttt{StrictSubset} \iff A \subsetneq S \iff \exists \sum a \in S \,.\, a \notin A$$

$\mathtt{SetEq} :: \forall A, B : \mathtt{Set}(T) \;.\; A = B \iff A \subset B \;\&\; B \subset A$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ (1) : A = B,$

$\mathtt{Assume}\ x : T,$

$\mathtt{Assume}\ (2) : x \in A,$

$(3) := \eth\mathtt{In}(a, A)(2) : A(x) = 1,$

$(4) := E(=, \to)(1)(3) : B(x) = 1,$

$() := \eth^{-1}\mathtt{In}(a, B)(4) : x \in B;$

$\rightsquigarrow () := I(\Rightarrow) : x \in A \Rightarrow x \in B;$

$\rightsquigarrow (2) := I(\forall) : \forall x : T \;.\; x \in A \Rightarrow x \in B,$

$(3) := \eth^{-1}\mathtt{Subset}(2) : A \subset B,$

$\mathtt{Assume}\ (4) : x \in B,$

$(5) := \eth\mathtt{In}(a, B)(4) : B(x) = 1,$

$(6) := E(=, \to)(1)(5) : A(x) = 1,$

$() := \eth^{-1}\mathtt{In}(a, A)(6) : x \in A;$

$\rightsquigarrow () := I(\Rightarrow) : x \in B \Rightarrow x \in A;$

$\rightsquigarrow (4) := I(\forall) : \forall x : T \;.\; x \in B \Rightarrow x \in A,$

$(5) := \eth^{-1}\mathtt{Subset}(4) : B \subset A,$

$(6) := I(\;\&\;)(3)(5) : A \subset B \;\&\; B \subset A;$

$\rightsquigarrow (1) := I(\Rightarrow) : A = B \Rightarrow A \subset B \;\&\; B \subset A,$

$\mathtt{Assume}\ (2) : A \subset B \;\&\; B \subset A,$

$(3) := \pi_l(2) : A \subset B,$

$(4) := \pi_r(2) : B \subset A,$

$\mathtt{Assume}\ x : T,$

$(5) := \mathtt{LEM}(A(x)) : A(x) = 0 | A(x) = 1,$

$\mathtt{Assume}\ (6) : A(x) = 1,$

$(7) := \eth^{-1}\mathtt{In}(6) : x \in A,$

$(8) := \eth\mathtt{Subset}(B)(A)(3)(7) : x \in B,$

$(9) := \eth\mathtt{In}(8) : B(x) = 1,$

$(10) := I(=)(6)(9) : A(x) = B(x);$

$\rightsquigarrow (6) := I(\Rightarrow) : A(x) = 1 \Rightarrow A(x) = B(x),$

$\mathtt{Assume}\ (7) : A(x) = 0,$

$\mathtt{Assume}\ (8) : B(x) = 1,$

$(9) := \eth^{-1}\mathtt{In}(B)(x)(8) : x \in B,$

$(10) := \eth\mathtt{Subset}(A)(B)(9) : x \in A,$

$(11) := \eth\mathtt{In}(A)(x)(10) : A(x) = 1,$

$(12) := I(=)(7)(11) : 0 = 1,$

$(13) := \mathtt{TruthIsFalsehoodContradiction}(12) : \bot;$

$\rightsquigarrow (8) := E(\bot) : B(x) \neq 1,$

$(9) := \mathtt{EqLEM}(8) : B(x) = 0,$

$(10) := E(=)(7)(9) : A(x) = B(x);$

$\rightsquigarrow (7) := I(\Rightarrow) : A(x) = 0 \Rightarrow A(x) = B(x),$

$(8) := E(|)(5)(6)(7) : A(x) = B(x);$

$\rightsquigarrow (2) := I(\Rightarrow) : A \subset B \;\&\; B \subset A \Rightarrow A = B,$

$(*) := I(\iff)(1)(2) : A = B \iff A \subset B \;\&\; B \subset A;$

$\square$

$\texttt{EmptySetRule} :: \forall S : \texttt{Set}(T) \,.\, \emptyset \subset S$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : T,$

$\texttt{Assume}\ (1) : x \in \emptyset,$

$(2) := \eth\emptyset(x) : \emptyset(x) = 0,$

$(3) := \eth\texttt{In}(x) : \emptyset(x) = 1,$

$(4) := I(=)(2)(3) : 1 = 0,$

$(5) := \texttt{TruthIsFalsehoodContradiction}(4) : \bot,$

$(6) := E(\bot)(x \in S) : x \in S;;$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall x \in T \,.\, x \in \emptyset \Rightarrow x \in S,$

$(*) := \eth\texttt{Subset}(1) : \emptyset \subset S;$

$\square$

$\texttt{UniverseRule} :: \forall S : \texttt{Set}(T) \,.\, S \subset U(T)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : T,$

$\texttt{Assume}\ (1) : x \in S,$

$(2) := \eth U(T)(x) : U(T)(x) = 1,$

$(3) := \eth^{-1}\texttt{In}(2) : x \in U(T);$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \forall x \in T \,.\, x \in S \Rightarrow x \in U(T),$

$(*) := \eth\texttt{Subset}(1) : S \subset U(T);$

$\square$

$\texttt{SingletonRule} :: \prod T : \texttt{Type} \,.\, \forall x \in T \,.\, x \in \{x\}$

$\texttt{Proof} \ =$

$(1) := \eth(==)(x) : x == x = 1,$

$(2) := \eth\texttt{singleton}(x)(x) : \{x\}(x) = x == x,$

$(3) := E(=)(1)(2) : \{x\}(x) = 1,$

$(*) := \eth^{-1}\texttt{In}(\{x\})(3) : x \in \{x\};$

$\square$

$\texttt{SingletonEq} :: \prod T : \texttt{Type} \,.\, \forall a, b : T \,.\, a = b \iff \{a\} = \{b\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (1) : a = b,$

$() := I(=, \rightarrow)\texttt{singleton}(a, b)(1) : \{a\} = \{b\};$

$\rightsquigarrow (1) := I(\Rightarrow) : a = b \Rightarrow \{a\} = \{b\},$

$\texttt{Assume}\ (2) : \{a\} = \{b\},$

$(4) := \texttt{SingletonRule}(a) : a \in \{a\},$

$(5) := \eth\texttt{In}(4) : \{a\}(a) = 1,$

$(6) := E(=, \rightarrow)(2)(5) : \{b\}(a) = 1,$

$(7) := \eth\texttt{singleton}(6) : a == b = 1,$

$() := \eth(==)(7) : a = b;$

$\rightsquigarrow (2) := I(\Rightarrow) : \{a\} = \{b\} \Rightarrow a = b,$

$(*) := I(\iff)(1)(2) : a = b \iff \{a\} = \{b\};$

$\square$

1.2 Inner Set Algebra

$$\texttt{union} :: \prod T : \texttt{Type} \,.\, ??T \to ?T$$

$$\texttt{union}\,(\mathcal{A}) = \bigcup \mathcal{A} := \bigvee_{A \in \mathcal{A}} A$$

$$\texttt{unionFunc} :: \prod T, I : \texttt{Type} \,.\, (I \to ?T) \to T$$

$$\texttt{unionFunc}\,(A) = \bigcup_{i:I} A_i := \bigvee_{i:I} A_i$$

$$\texttt{binaryUnion} :: \prod T : \texttt{Type} \,.\, ?T?T \to ?T$$

$$\texttt{binaryUnion}\,(A, B) = A \cap B := A \vee B$$

$$\texttt{intersect} :: \prod T : \texttt{Type} \,.\, ??T \to ?T$$

$$\texttt{intersect}\,(\mathcal{A}) = \bigcap \mathcal{A} := \bigwedge_{A \in \mathcal{A}} A$$

$$\texttt{intersectFunc} :: \prod T, I : \texttt{Type} \,.\, (I \to ?T) \to T$$

$$\texttt{intersectFunc}\,(A) = \bigcap_{i:I} A_i := \bigwedge_{i:I} A_i$$

$$\texttt{binaryIntersect} :: \prod T : \texttt{Type} \,.\, ?T?T \to ?T$$

$$\texttt{binaryIntersect}\,(A, B) = A \cap B := A \wedge B$$

$$\texttt{setDifference} :: \prod T : \texttt{Type} \,.\, ?T?T \to ?T$$

$$\texttt{SetDifference}\,(A, B) = A \setminus B := A \wedge !B$$

$$\texttt{complement} :: \prod T : \texttt{Type} \,.\, ?T \to ?T$$

$$\texttt{complement}\,(A) = A^{\complement} := U(T) \setminus A$$

$$\texttt{symmetricDifference} :: \prod T : \texttt{Type} \,.\, ?T?T \to ?T$$

$$\texttt{SetDifference}\,(A, B) = A \triangle B := A \oplus B$$

$$\texttt{DisjointPair} :: \prod T : \texttt{Type} \,.\, ?(?T?T)$$

$$(A, B) : \texttt{DisjointPair} \iff A \cap B = \emptyset$$

$$\texttt{Disjoint} :: \prod T : \texttt{Type} \,.\, ???T$$

$$\mathcal{A} : \texttt{Disjoint} \iff \forall A, B \in \mathcal{A} \,.\, \Big(A = B | (A, B) : \texttt{DisjointPair}\Big)$$

$\texttt{UnionRule} :: \forall \mathcal{A} :??T \,.\, \forall A \in \mathcal{A} \,.\, A \subset \bigcup \mathcal{A}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : T,$

$\texttt{Assume}\ (1) : a \in A,$

$(2) := \eth\texttt{In}(1) : A(a) = 1,$

$(3) := \eth A \eth \bigvee(\mathcal{A}, 2)(a) : \bigvee_{b \in \mathcal{A}} b(a) = 1,$

$(4) := \eth\texttt{In}\eth \bigcup \mathcal{A}(a) : a \in \bigcup_{A \in \mathcal{A}} A;$

$\rightsquigarrow (*) := \eth\texttt{Subset}I(\Rightarrow) : A \subset \bigcup \mathcal{A};$

$\square$

$\texttt{IntersectionRule} :: \forall \mathcal{A} :??T \,.\, \forall A \in \mathcal{A} \,.\, \bigcap \mathcal{A} \subset A$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : T,$

$\texttt{Assume}\ (1) : a \in \bigcap \mathcal{A},$

$(2) := \eth\texttt{In}(1) : \bigcap \mathcal{A}(a) = 1,$

$(3) := \eth \bigcap \mathcal{A}(2) : \bigwedge_{A \in \mathcal{A}} A(a) = 1,$

$(4) := \eth \bigwedge \eth A(3) : A(a) = 1,$

$(5) := \eth^{-1}\texttt{In} : a \in A;$

$\rightsquigarrow (*) := \eth\texttt{Subset}I(\Rightarrow) : \bigcap \mathcal{A} \subset A;$

$\square$

1.3 Outer Set Algebra

$$\texttt{cartesianProduct} :: \prod I : \texttt{Type} \,.\, \prod T : I \to \texttt{Type} \,.\, \left(\prod i : I \,.\, ?T_i\right) \to ? \left(\prod i : I \,.\, T_i\right)$$

$$\texttt{cartesianProduct}\,(A) = \prod_{i:I} A := \Lambda f : \prod i : I \,.\, T_i \,.\, \bigwedge_{i:I} A_i\ f\ i$$

$$\texttt{binaryProduct} :: \prod T, S : \texttt{Type} \,.\, ?T?S \to ?TS$$

$$\texttt{binaryProduct}\,(A, B) = A \times B := \Lambda(t, s) : TS \,.\, A(t) \wedge B(s)$$

$$\texttt{disjointUnion} :: \prod I : \texttt{Type} \,.\, \prod T : I \to \texttt{Type} \,.\, \left(\prod i : I \,.\, ?T_i\right) \to ? \left(\sum i : I \,.\, T_i\right)$$

$$\texttt{disjoinUnion}\,(A) = \bigsqcup_{i:I} A := \Lambda(i, x) : \sum i : I \,.\, T_i \,.\, A_i\ x$$

$$\texttt{binaryDUnion} :: \prod T, S : \texttt{Type} \,.\, ?T?S \to ?(T|S)$$

$$\texttt{binaryDUnion}\,(A, B) = A \sqcup B := \Lambda(i, x) : T|S \,.\, \texttt{if}\ i == 1\ \texttt{then}\ A(x)\ \texttt{else}\ B(x)$$

$\texttt{EmptyProductRight} :: \forall A :?T \,.\, A \times \emptyset_S = \emptyset_{TS}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (x, y) : TS,$

$(1) := \eth\emptyset(y) : \emptyset(y) = 0,$

$(2) := \eth \wedge (A(x))(1) : A(x) \wedge \emptyset(y) = 0,$

$(3) := \eth A \times \emptyset(2) : A \times \emptyset(x, y) = 0;$

$\rightsquigarrow (1) := I(\forall) : \forall(x, y) : TS \,.\, A \times \emptyset(x, y) = 0,$

$(*) := \eth^{-1}\emptyset(1) : A \times \emptyset = \emptyset;$

$\square$

$\texttt{EmptyProductLeft} :: \forall A :?S \,.\, \emptyset_T \times A = \emptyset_{TS}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (x, y) : TS,$

$(1) := \eth\emptyset(y) : \emptyset(x) = 0,$

$(2) := \eth \wedge (A(x))(1) : \emptyset(x) \wedge A(y) = 0,$

$(3) := \eth\emptyset \times A(2) : \emptyset \times A(x, y) = 0;$

$\rightsquigarrow (1) := I(\forall) : \forall(x, y) : TS \,.\, \emptyset \times A(x, y) = 0,$

$(*) := \eth^{-1}\emptyset(1) : \emptyset \times A = \emptyset;$

$\square$

1.4 Set Functions

$$\texttt{SetFunction} :: \prod T, S : \texttt{Type} \;.\; ??TS$$

$$F : \texttt{SetFunction} \iff F \in S^T \iff \forall x \in T \;.\; (\exists y \in S \;.\; (x, y) \in F) \;\&$$
$$\& \;\forall y, z \in S \;.\; \Big(\big((x, y) \in F \;\&\; (x, z)\big) \in F \to y = z\Big)$$

$$\texttt{graph} :: (T \to S) \to S^T$$
$$\texttt{graph}\,(f) := \Lambda(x, y) : TS \;.\; f(x) == y$$

$$\texttt{image} :: (X \to Y) \to ?X \to ?Y$$
$$\texttt{image}\,(f, A) = f(A) := \Lambda y : Y \;.\; \bigvee_{x:X} f(x) == y$$

$$\texttt{preimage} :: (X \to Y) \to ?Y \to ?X$$
$$\texttt{preimage}\,(f, A) = f^{-1}(A) := \Lambda x : X \;.\; \bigvee_{y:Y} f(x) == y$$

$$\texttt{degraph} :: S^T \to T \to S$$
$$\texttt{degraph}\,(F) := \Lambda x : T \;.\; \eth F(x)$$

$$\texttt{compose} :: S^T R^S \to R^T$$
$$\texttt{compose}\,(F, G) := \Lambda(x, y) : TR \;.\; \bigvee_{z:S} (x, z) \in F \wedge (z, y) \in G$$

1.5 Category SET

$$\mathsf{SET} :: \texttt{Category}$$
$$\mathcal{O}(\mathsf{SET}) = \sum T : \texttt{Type} \;.\; ?T$$
$$\mathcal{M}_{\mathsf{SET}}(T, A)(S, B) = \texttt{SetFunction}(A, B)$$
$$\mathrm{id}_{(T,A)} = \Lambda(x, y) : T^2 \;.\; x == y$$
$$\circ_{\mathsf{SET}} = \texttt{compose}$$

2 Relations

2.1 Types of Relations

$$\texttt{Relation}(A) = ?(A \times A) : \prod T : \texttt{Type} \; . \; ?T \to ?(T \times T)$$

$$\texttt{Reflexive} :: ?\texttt{Relation}(A)$$
$$R : \texttt{Reflexive} \iff \forall a \in A \; . \; (a, a) \in R$$

$$\texttt{Antireflexive} :: ?\texttt{Reflection}(A)$$
$$R : \texttt{Antireflexive} \iff \forall a \in A \; . \; (a, a) \notin R$$

$$\texttt{Symmetric} :: ?\texttt{Relation}(A)$$
$$R : \texttt{Symmetric} \iff \forall a, b \in A \; . \; (a, b) \in R \iff (b.a) \in R$$

$$\texttt{Antisymmetric} :: ?\texttt{Relation}(A)$$
$$R : \texttt{Antisymmetric} \iff \forall a, b \in A \; . \; \Big((a, b) \in R \;\&\; (b, a) \in R\Big) \iff a = b$$

$$\texttt{Transitive} :: ?\texttt{Relation}(A)$$
$$R : \texttt{Transitive} \iff \forall a, b, c \in A \; . \; (a, b), (b, c) \in R \Rightarrow (a, c) \in R$$

$$\texttt{Total} :: ?\texttt{Relation}(A)$$
$$R : \texttt{Total} \iff \forall a, b \in A \; . \; (a, b) \in A | (b, a) \in A$$

$$\texttt{compose} :: \texttt{Relation}(A) \times \texttt{Relation}(A) \to \texttt{Relation}(A)$$
$$\texttt{compose}\,(R, S) = R \circ S := \Lambda(x, y) \in A^2 \; . \; \bigvee_{a \in A} R(x, a) \wedge S(a, y)$$

2.2 Equivalence Relation

$$\texttt{EquivalenceRelation}(A) = \texttt{Reflexive}(A) \;\&\; \texttt{Symmetric}(A) \;\&\; \texttt{Transitive}$$

2.3 Factor Sets

$$\texttt{equivalenceClass} :: \prod A : \texttt{Set}(T) \; . \; \texttt{EquivalenceRelation}(A) \to A \to ?A$$
$$\texttt{equivalenceClass}\,(E, a) := \{x \in A : a = x\}$$

$$\texttt{factorSet} :: \prod A : \texttt{Set}(T) \; . \; \texttt{EquivalenceRelation}(A) \to ??A$$
$$\texttt{factorSet}\,(E) = \frac{A}{E} := \Big\{\texttt{equivalenceClass}(E, a) \Big| a \in A\Big\}$$

3 Locally Naive Cardinals

3.1 Inner Cardinals

$$\texttt{SetIsoclass} :: \prod T : \texttt{Type} \,.\, ???T$$
$$\mathcal{A} : \texttt{SetIsoclass} \iff \forall A, B \in \mathcal{A} \,.\, \exists A \leftrightarrow_{\mathsf{SET}} B$$

$$\texttt{Cardinal} :: \prod T : \texttt{Type} \,.\, ?\texttt{SetIsoclass}(T)$$
$$\kappa : \texttt{Cardinal} \iff \kappa \in \mathcal{K}(T) \iff \forall \alpha : \texttt{SetIsoclass}(T) \,.\, \big(\alpha \cap \kappa = \emptyset | \alpha \subset \kappa\big)$$

$$\texttt{HasCardinality} :: \prod T : \texttt{Type} \,.\, ?((?T)\mathcal{K}(T))$$
$$(A, \kappa) : \texttt{HasCardinality} \iff |A| = \kappa \iff A \in \kappa$$

$$\texttt{SameCardinality} :: \prod T : \texttt{Type} \,.\, ?(?T \times ?T)$$
$$(A, B) : \texttt{SameCardinality} \iff |A| = |B| \iff \exists \kappa \in \mathcal{K}(T) \,.\, |A| = \kappa \,\&\, |B| = \kappa$$

$$\texttt{IsBigger} :: \prod T : \texttt{Type} \,.\, ?(?T \times ?T)$$
$$(A, B) : \texttt{IsBigger} \iff |A| \geq |B| \iff \exists S : \texttt{Subset}(A) \,.\, |S| = |B|$$

$\texttt{CardinalsExist} :: \forall A : ?T \,.\, \exists \kappa \in \mathcal{K}(T) \,.\, |A| = \kappa$

$\texttt{Proof} \; =$

$X := \left\{ B : ?T \middle| \exists A \leftrightarrow_{\mathsf{SET}} B \right\} : ??T,$

$\texttt{Assume}\ I, J : X,$

$f := \eth I \eth X : A \leftrightarrow_{\mathsf{SET}} I,$

$g := \eth J \eth X : A \leftrightarrow_{\mathsf{SET}} J,$

$(1) := \texttt{IsoComposition}(g, f^{-1}) : g \circ f^{-1} : I \leftrightarrow_{\mathsf{SET}} J;$

$\rightsquigarrow (2) := \eth \texttt{SetIsoclass} I(\forall) : \Big(X : \texttt{SetIsoclass}(T)\Big),$

$\texttt{Assume}\ Y : \texttt{SetIsoclass}(T),$

$\texttt{Assume}\ (3) : X \cap Y \neq \emptyset,$

$B := \eth \emptyset (3) : \texttt{In}(X \cap Y),$

$\texttt{Assume}\ C : \texttt{In}(Y),$

$f := \eth \texttt{SetIsoclass}(T)(Y)(B, C) : B \leftrightarrow_{\mathsf{SET}} C,$

$g := \eth \texttt{SetIsoclass}(T)(X)(A, B) : A \leftrightarrow_{\mathsf{SET}} C,$

$(4) := \texttt{IsoComposition}(f, g) : f \circ g : (A \leftrightarrow_{\mathsf{SET}} C),$

$(5) := \eth X(4) : C \in X;$

$\rightsquigarrow (4) := \eth^{-1} \texttt{Subset} I(\Rightarrow) : Y \subset X;$

$\rightsquigarrow (3) := \eth^{-1} \mathcal{K}(T) I(\forall) : X \in \mathcal{K}(T),$

$(*) := \eth^{-1} \texttt{HasCardinality}(A, X) : |A| = X;$

$\square$

$$\texttt{cardinality} :: \prod T : \texttt{Type} \,.\, ?T \to \mathcal{K}(T)$$
$$\texttt{cardinality}(A) = |A| := \texttt{CardinalsExist}(A)$$

3.2 Global Cardinality

$$\texttt{EquellCardinals} :: \prod T, S : \texttt{Type} \,.\, ?(\mathcal{K}(T) \times \mathcal{K}(S))$$
$$(\alpha, \beta) : \texttt{EquallCardinals} \iff \alpha = \beta \iff \forall A \in \alpha \,.\, \forall B \in \beta \,.\, \exists A \leftrightarrow_{\mathsf{SET}} B$$

$$\texttt{IsBigger} :: \prod T, S : \texttt{Type} \,.\, ?(?T \times ?S)$$
$$(A, B) : \texttt{IsBigger} \iff |A| \geq |B| \iff \exists S : \texttt{Subset}(A) \,.\, |S| = |B|$$

$$\texttt{zeroCardinal} :: \prod T : \texttt{Type} \,.\, \mathcal{K}(T)$$
$$\texttt{zeroCardinal}\,() = 0_T := \{\emptyset\}$$

$$\texttt{finiteCardinal} :: \mathbb{N} \to \mathcal{K}(\mathbb{N})$$
$$\texttt{finiteCardinal}\,(n) = n := \Big|\{m : \mathbb{N} : m \leq n\}\Big|$$

$$\texttt{Finite} :: \prod T : \texttt{Type} \,.\, ??T$$
$$A : \texttt{Finite} \iff |A| < \infty \iff \exists n \in \mathbb{N} \,.\, |A| = n$$

$$\texttt{Infinite} :: \prod T : \texttt{Type} \,.\, ??T$$
$$A : \texttt{Infinite} \iff |A| = \infty \iff \forall n \in \mathbb{N} \,.\, |A| \geq n$$

$$\texttt{countableInfinity} :: \mathcal{K}(\mathbb{N})$$
$$\texttt{countableInfinity}\,() = \aleph_0 := |U(\mathbb{N})|$$

$$\texttt{Countable} :: \prod T : \texttt{Type} \,.\, ??T$$
$$A : \texttt{Countable} \iff |A| \leq |U(\mathbb{N})|$$

$$\texttt{IsStrictlyBigger} :: ?\texttt{IsBigger}(T, S)$$
$$(A, B) : \texttt{IsStriclyBigger} \iff |A| > |B| \iff !|A| = |B|$$

3.3 Cantor Theorems

$\texttt{CantorTHM} :: \forall A :?T . \left|2^A\right| > |A|$

$\texttt{Proof} \ =$

$(1) := \eth ?A \eth 2^A \eth \texttt{singleton} : \left(\texttt{singleton} : A \leftrightarrow_{\mathsf{SET}} \{\{a\}|a \in A\}\right),$

$(2) := \eth^{-1}\texttt{IsBigger}(2^A, A)(1) : |2^A| \geq |A|,$

$\texttt{Assume } (3) : |2^A| = |A|,$

$f := \eth|2^A| = |A| : 2^A \leftrightarrow_{\mathsf{SET}} A,$

$Z := \{a \in A : a \notin f(a)\} :?A,$

$(z, 4) := \eth f(Z) : \sum z \in A . f(z) = Z,$

$\texttt{Assume } (6) : z \in Z,$

$(7) := \eth Z(6) : z \notin f(z),$

$(8) := E(=)(4)(6) : z \notin Z,$

$(9) := (8)(6) : \bot;$

$\rightsquigarrow (6) := E(\bot) : z \notin Z,$

$(7) := E(=)(4)(6) : z \notin f(z),$

$(8) := \eth Z(7) : z \in Z,$

$(9) := (6)(8) : \bot;$

$\rightsquigarrow (3) := E(\bot) :! |2^A| = |A|,$

$(*) := \eth^{-1}\texttt{IsStrictlyBigger}(2^A, A)(2)(3) : |2^A| > |A|;$

$\square$

$\texttt{continum} :: \mathcal{K}(?\mathbb{N})$

$\texttt{continum}\,() = \mathfrak{c} := \left|2^{\mathbb{N}}\right|$

$\texttt{CantorBernsteinTHM} :: \forall A :?T \;.\; \forall B :?T \;.\; |A| \geq |B| \;\&\; |B| \geq |A| \iff |A| = |B|$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : |A| \geq |B| \;\&\; |B| \geq |A|,$

$(S, f) := \eth\texttt{IsBigger}(A, B)(1) : \sum S :?B \;.\; f : A \leftrightarrow_{\mathsf{SET}} S,$

$(R, g) := \eth\texttt{IsBigger}(B, A)(2) : \sum R :?A \;.\; g : B \leftrightarrow_{\mathsf{SET}} R,$

$\alpha_0 := A :?A,$

$\beta_0 := B :?B,$

$\texttt{Assume}\ n : \mathbb{N},$

$\alpha_n := g(\beta_{n-1}) :?A,$

$\beta_n := f(\alpha_{n-1}) :?B,$

$(2_n) := \texttt{ConstrictBijection}(g, \beta_{n-1})\eth\alpha_n : \alpha_n \cong_{\mathsf{SET}} \beta_{n-1},$

$(3_n) := \texttt{ConstricBijecction}(f, \alpha_{n-1})\eth\beta_n : \beta_n \cong_{\mathsf{SET}} \alpha_{n-1},$

$(4_n) := (3_{n-1})5_{n-1} : \alpha_n \subset \alpha_{n-1},$

$(5_n) := 2_{n-1}4_{n-1} : \beta_n \subset \beta_{n-1};$

$\rightsquigarrow (\alpha, \beta, 2) := I(\sum) : \sum(\alpha, \beta) : \mathbb{Z}_+ \to (?A)(?B) \;.\; \forall n \in \mathbb{N} \;.$

$\quad \alpha_n \cong_{\mathsf{SET}} \beta_{n-1} \;\&\; \beta_n \cong_{\mathsf{SET}} \alpha_{n-1} \;\&\; \alpha_n \subset \alpha_{n-1} \;\&\; \beta_n \subset \beta_{n-1},$

$(3) := \eth(\alpha, \beta)\eth\texttt{compose} : \forall n \in \mathbb{Z}_+ \;.\; \alpha_n \cong_{\mathsf{SET}} \alpha_{n+2},$

$\delta := \bigcap_n^\infty \alpha_n :?A,$

$\texttt{Assume}\ n : \mathbb{Z}_+,$

$\gamma_n := \alpha_n \setminus \alpha_{n-1} :?A;$

$\rightsquigarrow \gamma := I(\to) : \mathbb{Z}_+ \to ?A,$

$\gamma_\infty := \delta :?A,$

$(4) := \texttt{DifferenceIsomorphism}(\eth\gamma)(3) : \forall n \in \mathbb{Z}_+ \;.\; \gamma_n \cong_{\mathsf{SET}} \gamma_{n+1},$

$(5) := \eth^{-1}\texttt{Partition}\eth\alpha\eth\gamma\eth\delta : \gamma : \texttt{Partition}(A)),$

$\texttt{Assume}\ x : A,$

$(n, 6) := \eth\texttt{Partition}(A)(x) : \sum n \in \mathbb{Z}_+ \;.\; x \in \gamma_n,$

$\texttt{Assume}\ (7) : (n : \texttt{Odd}),$

$\varphi(x) := x : A;$

$\rightsquigarrow (7) := I(\Rightarrow) : (n : \texttt{Odd}) \Rightarrow \varphi(x) = x,$

$\texttt{Assume}\ (8) : (n : \texttt{Even}),$

$\varphi(x) := g \circ f(x) : A;$

$\rightsquigarrow (8) := I(\Rightarrow) : (n : \texttt{Even}) \Rightarrow g \circ f(x);$

$\rightsquigarrow \varphi := \eth\gamma\eth f\eth g\texttt{ConditionalFunction} : \alpha_0 \leftrightarrow_{\mathsf{SET}} \alpha_1,$

$(6) := \eth\alpha_1\eth\varphi E(=) : A \cong_{\mathsf{SET}} B,$

$(7) := \eth^{-1}\texttt{EquallCardinals}(6) : |A| = |B|;$

$\rightsquigarrow (1) := I(\Rightarrow) : |A| \geq |B| \;\&\; |B| \geq |A| \Rightarrow |A| = |B|,$

$\texttt{Assume}\ (2) : |A| = |B|,$

$(3) := \eth^{-1}\texttt{IsBigger}(A, B)(2) : |A| \geq |B|,$

$(4) := \eth^{-1}\texttt{IsBiffer}(B, A)(2) : |B| \geq |A|,$

$(5) := I(\;\&\;)(3)(4) : |A| \geq |B| \;\&\; |B| \geq |A|;$

$\rightsquigarrow (2) := I(\Rightarrow) : |A| = |B| \Rightarrow |A| \geq |B| \;\&\; |B| \geq |A|,$

$(*) := I(\iff)(1)(2) : |A| \geq |B| \;\&\; |B| \geq |A| \iff |A| = |B|;$

$\square$

3.4 Cardinal Algebra

$$\texttt{cardSum} :: \mathcal{K}(T) \times \mathcal{K}(S) \to \mathcal{K}(T|S)$$
$$\texttt{cardSum}\,(A, B) = |A| + |B| := |A \sqcup B|$$

$$\texttt{cardProduct} :: \mathcal{K}(T) \times \mathcal{K}(S) \to \mathcal{K}(T \times S)$$
$$\texttt{cardProduct}\,(A, B) = |A||B| := |A \times B|$$

$$\texttt{cardPower} :: \mathcal{K}(T) \times \mathcal{K}(S) \to \mathcal{K}\big(T^S\big)$$
$$\texttt{cardPower}\,(A, B) = |A|^{|B|} := |A^B|$$

3.5 Category CARD