Abstract Measure Theory

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Contents

Intro

1 Measure Algebras

1.1 Subject

1.1.1 Definition and Basic Property

$\texttt{MeasureAlgebra} :: ?\sum A : \sigma\text{-}\texttt{DedekindComplete} \,.\, A \to \overset{\infty}{\mathbb{R}}_+$

$(A,\mu) : \texttt{MeasureAlgebra} \iff \forall a \in A \,.\, \mu(a) = 0 \iff a = 0 \;\&$

$\& \;\forall a : \texttt{PairwiseDisjointElements}(\mathbb{N}, A) \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$

$\texttt{MeasureMonotonicity} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, a \le b \Rightarrow \mu(a) \le \mu(b)$

Proof =

Write $\mu(b) = \mu(a) + \mu(b \setminus a) \ge \mu(a)$.

□

$\texttt{MeasureStrictMonotonicity} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, a > b \Rightarrow \mu(a) > \mu(b)$

Proof =

Definition of measure algebra implies that $\mu(b \setminus a) > 0$.

Write $\mu(b) = \mu(a) + \mu(b \setminus a) > \mu(a)$.

□

$\texttt{MinkovskyIneq} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a, b \in A \,.\, \mu(a \vee b) \le \mu(a) + \mu(b)$

Proof =

Write $\mu(a) + \mu(b) = \mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab) + \mu(ab) \ge mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab = \mu(a \vee b)$.

□

$\texttt{MonotonicSupremumAsLimit} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \uparrow A \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$

Proof =

Construct disjoint sequence $b_n = a_n \setminus \bigvee_{k=1}^{n-1} a_k$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} \mu(b_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(b_n) = \lim_{n\to\infty} \mu\left(\bigvee_{k=1}^{n} b_k\right) = \lim_{n\to\infty} \mu(a_n)$.

□

SupremumIneq :: $\forall (A, \mu) :$ MeasureAlgebra . $\forall a : \mathbb{N} \to A \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) \le \sum_{n=1}^{\infty} \mu(a_n)$

Proof =

Construct increasing sequence $b_n = \bigvee_{k=1}^{n} a_n$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) = \lim_{n\to\infty} \mu\left(\bigvee_{k=1}^{n} a_k\right) \le \lim_{n\to\infty} \sum_{k=1}^{n} \mu(a_k) = \sum_{n=1}^{\infty} \mu(a_n)$.

$\square$

MonotonicInfimumAsLimit ::

:: $\forall (A, \mu) :$ MeasureAlgebra . $\forall a : \mathbb{N} \downarrow A \,.\, \forall \aleph : \inf_{n\in\mathbb{N}} \mu(a_n) < \infty \,.\, \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$

Proof =

Without loss of generality assume that $\mu(a_1) < \infty$.

Then construc the increasing sequence $b_n = a_1 \setminus a_n$.

Then $\mu(a_1) - \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(a_1 \setminus \bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} a_1 \setminus a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) =$

$= \lim_{n\to\infty} \mu\left(a_1 \setminus a_n\right) = \lim_{n\to\infty} \mu(a_1) - \mu(a_n) = \mu(a_1) - \lim_{n\to\infty} \mu(a_n)$.

So basic algebraic manipulations $\mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$.

$\square$

SupremumExistance ::

:: $\forall (A, \mu) :$ MeasureAlgebra . $\forall C :$ UpwardsDirected(A) . $\forall \aleph : \sup_{c\in C} \mu(c) < \infty \,.\, \exists a \in A : a = \sup C$

Proof =

1 Assume $\gamma = \sup_{c\in C} \mu(c)$.

2 Then there exists a sequrnce of $a : \mathbb{N} \to C$ such that $\mu(a_n) \ge \gamma - 2^{-n}$.

3 As C is upwards closed, it is possible to find $c : \mathbb{N} \to C$ such that $c_{n+1} \ge a_n \vee c_n$.

4 Then c is monotonic-nondecreasing and so it has $\mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \lim_{n\to\infty} \mu(c_n) = \gamma$.

4.1 Note that $\gamma \ge \mu(c_n) \ge \gamma - 2^{-n}$.

5 let $d = \bigvee_{n=1}^{\infty} c_n$.

6 $d \ge f$ for everty $f \in C$.

6.1 Assume this is false.

6.2 Then $f \setminus d \ne 0$ and so $\alpha = \mu(f \setminus d) > 0$.

6.3 Then there exists n such that $\gamma - \mu(c_n) < \alpha$.

6.4 As C is upwards derected there is $g \in C$ such that $g \ge f \vee c_n$.

6.5 But $\mu(g) \ge \mu(f \vee c_n) = \mu(c_n) + \mu(f \setminus c_n) \ge \mu(c_n) + \mu(f \setminus d) > \gamma$ which is impossible.

7 If there is another f with the property (6), then $d = \bigvee_{n=1}^{\infty} c_n \le f$ as $c_n \le f$ for each $n \in \mathbb{N}$.

$\square$

`UpperContinuity` ::

:: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall C : \texttt{UpwardsDirected}(A) \,.\, \forall \aleph : \exists a \in A : a = \sup C \,.\, \sup_{c \in C} \mu(c) = \mu(\sup C)$

`Proof` =

Case $\sup_{c \in C} \mu(c) = \infty$ is trivial.

Finite case follows from the cconstruction in the previous theorem.

$\square$

`DisjointUpperContinuity` ::

:: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall C : \texttt{PairwiseDisjointElements}(A) \,.\, \forall \aleph : \exists a \in A : a = \sup C \,.$

$. \ \mu(\sup C) = \sum_{c \in C} \mu(c)$

`Proof` =

Construct a new set $D = \left\{ \bigvee_{n=1}^{\infty} c_k \middle| c : \mathbb{N} \to C \right\}$.

Then D is upwards directed and $\sup C = \sup D$.

But this is evedent that $\mu(\sup D) = \sup_{d \in D} \mu(d) = \sup_{c : \mathbb{N} \to C} \mu \left(\bigvee_{n=1} c_n \right) = \sup_{n \in \mathbb{N}, c : \{1,\ldots,n\} \to C} \sum_{k=1}^{n} \mu(c_k) = \sum_{c \in C} \mu(c)$.

$\square$

`InfimumExistance` ::

:: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall C : \texttt{DownwaedDirected}(A) \,.\, \forall \aleph : \inf_{c \in C} \mu(c) < \infty \,.\, \exists a \in A : a = \inf C$

`Proof` =

1 There exists some $a \in C$ such that $\mu(a) < \infty$.

2 Construct another set $D = a \setminus C$.

3 Then D is upwards directed and $\sup_{d \in D} \mu(d) \leq \mu(a) < \infty$.

4 So there is $d = \sup d$.

5 Define $f = a \setminus d$.

6 $f \leq c$ for any $c \in C$ as $a \setminus f \geq a \setminus c$.

7 if some g has property (6) then $a \setminus g \geq d$ and so $g \leq f$.

$\square$

`LowerContinuity` ::

:: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall C : \texttt{DownwardsDirected}(A) \,.\, \forall \aleph : \exists a \in A : a = \inf C \,.$

$. \ \forall \beth : \inf_{c \in C} \mu(c) < \infty \,.\, \inf_{c \in C} \mu(c) = \mu(\inf C)$

`Proof` =

Use the construction in the previous theorem.

$\square$

1.1.2 Measure Algebras Generated by Measure Spaces

$\texttt{measureAlgebra} :: \mathsf{MEAS} \to \texttt{MeasureAlgebra}$

$\texttt{measureAlgebra}\,(X, \Sigma, \mu) = (A_\mu, \bar{\mu}) := \left(\frac{\Sigma}{\Sigma \cap \mathcal{N}_\mu}, [E] \mapsto \mu(E) \right)$

This is obviously well defined as $[E] = [F]$ iff $\mu(E \triangle F) = 0$.

$\texttt{canononicalProjection} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \sigma\text{-}\mathsf{BOOL}(\Sigma, A_\mu)$

$\texttt{canonicalProjection}\,(E) = \pi_\mu(E) := [E]$

1 The algebraic properites are obvious as $\Sigma \cap \mathcal{N}_\mu$ is an ideal.

2 In order to prove sigma-continuity assume $E : \mathbb{N} \to \Sigma$.

2.1 Let $Z : \mathbb{N} \to \Sigma \cap \mathcal{N}_\mu$.

2.2 Then $F_Z = \bigvee_{n=1}^{\infty} (E_n \triangle Z_n) = \left(\bigvee_{n=1}^{\infty} E_n \right) \triangle \left(\bigvee_{n=1}^{\infty} Z_n \right)$.

2.3 Note that $\mu\left(\bigvee_{n=1}^{\infty} Z_n \right) \leq \sum_{n=1}^{\infty} \mu(Z_n) = 0$.

2.4 So $\bigvee_{n=1}^{\infty} Z_n \in \Sigma \cap \mathcal{N}_\mu$ as $\mu \geq 0$.

2.5 Thus $[F_Z] = \left[\bigcap_{n=1}^{\infty} E_n \right]$ for any selection of Z.

2.6 This means that $\pi_\mu \left(\bigcap_{n=1}^{\infty} E_n \right) = \bigvee_{n=1}^{\infty} \pi_\mu(E_n)$.

□

$\texttt{MeasureAlgebraMonotonicity} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \forall T \subset_\sigma \Sigma \,.\, \pi_\mu(T) \subset_\sigma A_\mu$

$\texttt{Proof} =$

1 Clearly $B = \pi_\mu(T) \subset A_\mu$.

2 Also as T is σ-algebra and $\pi - \mu$ is a σ-continuous homomorphism B is again.

□

$\texttt{MeasureAlgebraInverseMonotonicity} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \forall B \subset_\sigma A_\mu \,.\, \pi_\mu^{-1}(B) \subset_\sigma \Sigma$

$\texttt{Proof} =$

1 Clearly $T = \pi_\mu^{-1}(B) \subset \Sigma$.

2 Assume F is a set constructed by applying σ-algebra operations to setes $E_1, E_2, \ldots \in T$.

3 Then $\pi_\mu(F)$ can be constructed by applying same operations to $\pi(E_1), \pi(E_2), \ldots$.

4 This implies that $\pi_\mu(F) \in B$ and reciprorary $F \in T$.

5 Thus T is a σ-algebra.

□

1.1.3 Stone Representation Theorem

$\texttt{StoneRepresentationTheorem} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \exists (X, \Sigma, \nu) \in \mathsf{MEAS} \,.\, (A, \mu) = (A_\nu, \bar{\nu})$

$\texttt{Proof} \; =$

1 By Loomis-Sikorski representation there exists a set X with a sigma-algebra Σ and sigma-ideal I such that $\dfrac{\Sigma}{I} \cong_{\mathsf{BOOL}} A$.

2 Then there is a canonical projetion $\pi_I \in \mathsf{BOOL}(\Sigma, A)$.

3 Define $\nu = \pi_I \mu$.

4 ν is measure on Σ.

4.1 $\nu(\emptyset) = \mu(0) = 0$.

4.2 Assume E is a disjoint sequence in Σ.

4.3 Then $\pi_I(E_n)\pi_I(E_m) = \pi_i(E_n \cap E_m) = \pi_i(\emptyset) = 0$, so $\pi_I(E)$ is disjoint in A.

4.4 Thus, $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \pi_I \mu \left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigvee_{n=1}^{\infty} \pi_I(E_n)\right) = \sum_{n=1}^{\infty} \pi_I \mu(E_n) = \sum_{n=1}^{\infty} \nu(E_n)$.

5 Also by consytuction $\mathcal{N}_\nu \cap \Sigma = I$, so $(A, \mu) = (A_\nu, \bar{\nu})$.

$\square$

$\texttt{spaceOfStone} :: \texttt{MeasureAlgebra} \to \mathsf{MEAS}$

$\texttt{SpaceOfStone}\,(A, \mu) = (Z_A, \dot{\Sigma}_\mu, \dot{\mu}) := \texttt{StoneRepresentationTheorem}(A, \mu)$

1.1.4 Ideals

PrincipleIdealRestriction :: $\forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a \in A \,.\, \texttt{MeasureAlgebra}\Big((a), \mu_{|(a)}\Big)$

Proof $=$

This is obvious.

$\square$

measureQuotient ::

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall [a] \in \frac{A}{I} \,.\, \exists \gamma \in \overset{\infty}{\mathbb{R}}_{++} \,.\, \gamma = \min\{\mu(b) | b \in A, \pi_I(b) = [a]\}$$

Proof $=$

1 $\gamma = \inf\{\mu(b)|b \in A, \pi_I(b) = [a]\}$ exists as a set is bounded by below by 0.

2 If $\gamma = \infty$ then the result is obvious.

3 Otherwise there is a decreasing sequence $b : \mathbb{N} \to A$ such that $\pi_I(b_n) = [a]$ for any n and $\lim_{n\to\infty} \mu(b_n) = \gamma$.

4 Then $c = \bigwedge_{n=1}^{\infty} b_n$ is such that $\mu(c) = \gamma$ and $\pi_I(c) = a$.

4.1 Clearly $\pi_I\left(\bigwedge_{n=1}^{\infty} b_n\right) = \bigwedge_{n=1}^{\infty} \pi_I(b_n) = \bigwedge_{n=1}^{\infty} [a] = [a]$.

5 So the infimum is atteined.

$\square$

measureQuotient :: $\prod (A,\mu) : \texttt{MeasureAlgebra} \,.\, \prod I : \texttt{Ideal}(A) \,.\, \frac{A}{I} \to \mathbb{R}_{++}$

measureQuotient $(a) = \mu_I(a) := \min\{\mu(b)|b \in A, \pi_I(b) = a\}$

finiteElementsIdeal :: $\prod (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Ideal}(A)$

finiteElementsIdeal $() = A^f := \{a \in A | \mu(a) < \infty\}$

$\texttt{MeasureIdealQuotient} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \texttt{MeasureAlgebra}\left(\frac{A}{I}, \mu_I\right)$

$\texttt{Proof} \; =$

1 Clearly $\mu_I(0) = 0$.

2 Assume that $[a] \neq 0$.

2.1 Then there exists $b \in A$ such that $\pi_I(a) = [a]$ and $\mu(b) = \mu_I[a]$.

2.2 As $[a] \neq 0$, then $b \neq 0$, and henceforth $\mu(b) \neq 0$.

2.3 Thus, $\mu_I[a] \neq 0$.

3 Assume $[a] : \mathbb{N} \to \dfrac{A}{I}$ is disjoint.

3.1 It is possible to select representatives b_n for each $[a_n]$ such that $\mu(b_n) = \mu_I[a_n]$.

3.2 Then $b_n b_m \in I$ if $n \neq m$.

3.3 Construct a new sequence $c_n = b_n + \sum_{k=1}^{n-1} b_n b_k$ is a disjoint represintative sequance for $[a_n]$.

3.3.1 In fact $c = b$.

3.4 $\bigvee_{n=1}^{\infty} c_n$ is the minimal representative of $\bigvee_{n=1}^{\infty} [a_n]$.

3.4.1 Assume d is a representative for $\bigvee_{n=1}^{\infty} a_n$.

3.4.2 If $\mu(d) < \mu\left(\bigvee_{n=1}^{\infty} c_n\right)$ then we may construct $c_n \wedge d$ which is smaller then c.

3.4.3 But this is a contradiction.

3.5 So $\mu_I\left(\bigvee_{n=1}^{\infty} [a_n]\right) = \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \mu_I[a_n]$.

$\square$

1.1.5 Measure Properties

ProbabilityAlgebra :: ?MeasureAlgebra
(A, π) : ProbabilityAlgebra $\iff \pi(e) = 1$

FiniteMeasureAlgebra :: ?MeasureAlgebra
(A, μ) : FiniteMeasureAlgebra $\iff \mu(e) < \infty$

σ-FiniteMeasureAlgebra :: ?MeasureAlgebra

(A, μ) : σ-FiniteMeasureAlgebra $\iff \exists a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, \mu(a_n) < \infty \,\&\, \bigvee_{n=1}^{\infty} a_n = e$

SemifiniteMeasureAlgebra :: ?MeasureAlgebra
(A, μ) : SemifiniteMeasureAlgebra $\iff \forall a \in A \,.\, \mu(a) = \infty \Rightarrow \exists b \in A \,.\, b < a \,\&\, 0 < \mu(b) < \infty$

LocalizableMeasureAlgebra := OrderDedekindComplete & SemifiniteMeasureAlgebra : Type;

ProbabilityConstruction :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS}$. Probability$(X, \Sigma, \mu) \iff$ ProbabilityAlgebra$(A_\mu, \bar{\mu})$
Proof =
This is obvious.
□

FiniteConstruction :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS}$. Finite$(X, \Sigma, \mu) \iff$ FiniteMeasureAlgebra$(A_\mu, \bar{\mu})$
Proof =
This is obvious.
□

SigmaFiniteConstruction :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS}$. σ-Finite$(X, \Sigma, \mu) \iff$ σ-FiniteMeasureAlgebra$(A_\mu, \bar{\mu})$
Proof =
This is obvious.
□

SemifiniteConstruction ::
:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS}$. Semifinite$(X, \Sigma, \mu) \iff$ SemifiniteMeasureAlgebra$(A_\mu, \bar{\mu})$
Proof =
This is obvious.
□

LocalizableConstruction ::
:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS}$. Localizable$(X, \Sigma, \mu) \iff$ LocalizableMeasureAlgebra$(A_\mu, \bar{\mu})$
Proof =
This is obvious.
□

AtomInConstruction ::

:: $\forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \forall E \in \Sigma \,.\, E \in \mathrm{Atom}(X,\Sigma,\mu) \iff [E] \in \mathrm{Atom}(A_\mu, \bar\mu)$

Proof =

This is obvious.

$\square$

AtomlessConstruction ::

:: $\forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \forall E \in \Sigma \,.\, E \in \mathtt{Atomless}(X,\Sigma,\mu) \iff [E] \in \mathtt{Atomless}(A_\mu, \bar\mu)$

Proof =

This is obvious.

$\square$

PurelyAtomicConstruction ::

:: $\forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \forall E \in \Sigma \,.\, E \in \mathtt{PurelyAtomic}(X,\Sigma,\mu) \iff [E] \in \mathtt{PurelyAtomic}(A_\mu, \bar\mu)$

Proof =

This is obvious.

$\square$

FinitenessPropertiesIerarchy ::

:: $\forall (A,\mu) : \mathtt{MeasureAlgebra} \,.\, \mathtt{PobabilityAlgebra}(A,\mu) \Rightarrow \mathtt{FiniteMeasureAlgebra}(A,\mu) \Rightarrow$

$\Rightarrow \sigma\text{-}\mathtt{FiniteMeasureAlgebra}(A,\mu) \Rightarrow \mathtt{LocalizableMeasureAlgebra}(A,\mu) \Rightarrow \mathtt{Semifinite}(A,\mu)$

Proof =

1 Most implications here are obvious expect the one deriving Localizability from σ-finiteness.

2 So assume that (A,μ) is σ-finite .

2.1 Then the corresponding Stone space $(\mathsf{Z}A, \Sigma_\mu, \bar\mu)$ is σ-finite.

2.2 But then $(\mathsf{Z}A, \Sigma_\mu, \bar\mu)$ is localizable .

2.3 So (A,μ) is also localizable.

$\square$

MeasureAlgebraOfCompletion :: $\forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, A_\mu \cong_{\mathsf{BOOL}} A_{\hat\mu}$

Proof =

This is basically follows from definitions.

$\square$

MeasureAlgebraOfLocallyDeterminedCompletion ::

:: $\forall (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \exists A_\mu \xrightarrow{\phi} A_{\tilde\mu} : \mathsf{BOOL} \,.\, \forall a \in A_{\tilde\mu} \,.\, \hat{\tilde\mu}(a) < \infty \Rightarrow \exists b \in A_\mu \,.\, \phi(b) = a\ \&$

$\&\ \forall b \in A_\mu \,.\, \hat\mu(b) < \infty \Rightarrow \hat{\tilde\mu}(\phi(b)) = \hat\mu(b)$

Proof =

...

$\square$

localDeterminationMorphism :: $\prod (X,\Sigma,\mu) \in \mathsf{MEAS} \,.\, \mathsf{BOOL}(A_\mu, A_{\tilde\mu})$

localDeterminationMorphism $() = \phi_\mu :=$ MeasureAlgebraOfLocallyDeterminedCompletion

$\texttt{localDeterminationMorhismInjectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \ . \ \texttt{Semifinite}(X, \Sigma, \mu) \iff \texttt{Injective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

$\texttt{Proof} =$

...

$\square$

$\texttt{localDeterminationMorhismBijectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \ . \ \texttt{Localizable}(X, \Sigma, \mu) \iff \texttt{Bijective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SemifinitenessCriterion} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \ .$

$. \ \texttt{SemifiniteMeasureAlgebra}(A, \mu) \iff \exists P : \texttt{PartitionOfUnity}(A) \ . \ \forall p \in P \ . \ \mu(p) < \infty$

$\texttt{Proof} =$

1 $(\Rightarrow)$ assume first that (A, μ) is semifinite.

1.1 Then A^f is order dense in A.

1.2 By order density theorem there is a desired partition of unity.

2 $(\Leftarrow)$ Let P be the partition of unity.

2.1 Assume $a \in A$ is such that $\mu(a) = \infty$.

2.2 Then there exists $p \in P$ such that $pa \neq 0$.

2.3 Note that this means that $\mu(pa) > 0$.

2.4 Also it is clear that $\mu(pa) \leq \mu(p) < \infty$.

$\square$

$\texttt{SemifiniteneSupElementExpression} ::$

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \ . \ \forall a \in A \ . \ a = \bigvee \{b \in A : b \leq a, \mu(b) < \infty\}$

$\texttt{Proof} =$

This follows from the previous theorem.

$\square$

$\texttt{SemifiniteneSupMeasureComputation} ::$

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \ . \ \forall a \in A \ . \ \mu(a) = \bigvee \{\mu(b) \in A : b \leq a, \mu(b) < \infty\}$

$\texttt{Proof} =$

This follows from the previous theorem.

$\square$

1.1.6 Connections with other Boolean Properties

SemifiniteIsWeaklyDistributive ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra(A, μ) . (σ, ∞)-WeaklyDistributive(A, μ)

Proof =

1 Assume $X : \mathbb{N} \to 2^A$ is a sequence of downwards selected sets with $\inf X_n = 0$ for every $n \in \mathbb{N}$.

2 Let $C = \{a \in A : \forall n \in \mathbb{N} . \exists x \in X_n . a \geq x\}$.

3 Assume $d \in A$ is such that $d \neq 0$.

4 Then there is an element $d' \leq d$ such that $0 < \mu(d') < 0$.

5 $\inf_{x \in X} d'x = 0$ for each $n \in N$.

6 Select a sequence $x : \prod_{n=1}^{\infty} X_n$ suc that $\mu(d'x_n) \leq 2^{-n-2}\mu(d')$.

7 Define $c = \sup_{n=1} a_n \in C$.

8 Then $\mu(d'c) \leq \sum_{n=0}^{\infty} \mu(cx_n) < \mu(d')$.

9 This means that $d \not\leq c$.

10 And as d was arbitrary $\inf C = 0$.

$\square$

SemifiniteIffCCC :: $\forall (A, \mu)$: SemifiniteMeasureAlgebra(A, μ) .

. σ-FiniteMeasureAlgebra(A, μ) $\iff$ WithCountableChainCondition(A)

Proof =

1 $(\Leftarrow)$ assume that A has ccc.

1.1 Then there is a partition of unitity P in A consisting of finite elements as A is semifinite.

1.2 But as A has ccc P must be atmost countable.

1.3 This proves that A is σ-finite.

2 $(\Rightarrow)$ assume that (A, μ) is σ-finite .

2.1 Then there exists a countable partition of unity P of A wirh finite elements.

2.2 If A is not ccc, then there exists an uncountable refinement Q of A with finite elements.

2.3 Then by pigionhole principle there exists $p \in P$

such that set $Q' = \{q \in Q : q \subset p\}$ such that Q' is uncountable.

2.4 as for $\mu(q) > 0$ for any $q \in Q'$ by pigionhole principle there exists some $n \in \mathbb{Z}$

such that there are an infinite number of $q \in Q'$ with $\mu(q) \in [2^{-n-1}, 2^{-n}]$.

2.5 So $\mu(p) \geq \sum_{q \in Q'} \mu(q) = \infty$, but this is a contradiction.

$\square$

SemifiniteIffProbabilityRenormalizationExists ::

:: $\forall(A,\mu) :$ SemifiniteMeasureAlgebra(A,μ) . $A \neq \{0\} \Rightarrow$

$\Rightarrow \exists \pi : A \to \overset{\infty}{\mathbb{R}}_+$. ProbabilityAlgebra(A,π)

Proof =

1 Corresponding Stone space is σ-finite.

2 So there exists a proper renormalization of $\bar{\mu}$ to a probability π with the same sets of measure zero.

3 Then the measure algebra of $(\mathsf{Z}A,\pi)$ is a probability algebra and $A_\pi \cong_{\mathsf{BOOL}} A$.

□

1.1.7 Subspace Measures and Indefinite Integrals

MeasurableEnvelopePrincipleIdealIsomorphism ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \;.\; \forall Y \subset X \;.\; \forall E : \texttt{MeasurableEnvelope}(X,\Sigma,\mu,Y) \;.\; (A_{\mu|Y}, \widehat{\mu|Y}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

This result is technically convoluted but actually is pretty intuituve.

□

PrincipleIdealIsomorphism ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \;.\; \forall E \in \Sigma \;.\; (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

A straightforward application of a previous theorem.

□

ThickEquivalence ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \;.\; \forall Y : \texttt{Thick}(X,\Sigma,\mu) \;.\; (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} (X, \hat{\mu})$$

Proof =

A straightforward application of a previous theorem.

□

IndefiniteIntegralPrincipleIdealIsomorphism ::

$$:: \forall (X,\Sigma,\mu) \in \mathsf{MEAS} \;.\; \forall f \in \mathsf{I}_+(X,\Sigma,\mu) \;.\; \exists E \in \Sigma \;.\; A_{f\,d\mu} \cong_{\mathsf{BOOL}} ([E])$$

Proof =

We may assume that $\operatorname{supp} f$ has a measurable envelope E.

Then the result is obvious as $\mathcal{N}_\mu \subset \mathcal{N}_{f\,d\mu}$.

□

1.1.8 Simple Products

$$\texttt{simpleProduct} :: \prod_{I \in \mathsf{SET}} (I \to \texttt{MeasureAlgebra}) \to \texttt{MeasureAlgebra}$$

$$\texttt{simpleProduct}\,(A,\mu) = \prod_{i\in I} (A_i,\mu_i) := \left(\prod_{i\in I} A_i, \sum_{i\in I} \mu_i\right)$$

Obviously $\sum_{i\in I} \mu_i(0) = \sum_{i\in I} 0 = 0$.

Also assume $a : \mathbb{N} \to \prod_{i\in I} A_i$ is disjoint.

Then $\sum_{i\in I} \mu_i \left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{i\in I}\sum_{n=1}^{\infty} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty}\sum_{i\in I} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty}\sum_{i\in I} \mu_i(a_n)$.

□

`PrincipleIdealsInMeasureAlgebras` ::

$$:: \forall I \in \mathsf{SET}\ .\ \forall (A,\mu) : I \to \texttt{MeasureAlgebra}\ .\ (A_i,\mu_i) \cong_{\mathsf{MA}} \left((e_i), \left(\sum_{i\in I} \mu_i\right)_{|(e_i)}\right)$$

`Proof` =

This is pretty ovious.

□

`SimpleProductCoproductCorrespondance` ::

$$:: \forall I \in \mathsf{SET}\ .\ \forall (X,\Sigma,\mu) : I \to \mathsf{MEAS}\ .\ \prod_{i\in I} (A_{\mu_i}, \hat{\mu}_i) \cong \texttt{measureAlgebra} \coprod_{i\in I} (X_i, \Sigma_i, \mu_i)$$

`Proof` =

Obvious by Stone Theory.

□

`SimpleProductOfSemifinite` ::

$$:: \forall I \in \mathsf{SET}\ .\ \forall (A,\mu) : I \to \texttt{SemifiniteMeasureAlgebra}\ .\ \texttt{SemifiniteMeasureAlgebra}\left(\prod_{i\in I} (A,\mu)\right)$$

`Proof` =

Assume a has infinite measure in (A,μ).

Then there exists $i \in I$ such that $a_i \neq 0$.

As (A_i,μ_i) is semifinite there is $b \le a_i$ such that $0 < \mu_i(b) < \infty$.

Then $be_i \le a$ and $0 < \sum_{j\in I} \mu_j(be_i) = \mu_i(b) < \infty$.

□

SimpleProductOfLocalizable ::

$$:: \forall I \in \mathsf{SET} . \forall (A,\mu) : I \to \texttt{LocalizableMeasureAlgebra} . \texttt{LocalizableMeasureAlgebra}\left(\prod_{i\in I}(A,\mu)\right)$$

Proof =

Let J be a set and $a : J \to \prod_{i\in I}(A_i,\mu_i)$.

Then $\sup_{j\in J} a_j = (\sup_{j\in J} a_{j,i})_{i\in I}$.

□

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} . \forall (e_n)_{n=1}^{\infty} : \texttt{PartitionOfUnity}(A) . (A,\mu) \cong_{\mathsf{MA}} \prod_{n=1}^{\infty}\Big((e_n), \mu_{|(e_m)}\Big)$$

Proof =

This is pretty obvious.

□

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} . \exists I \in \mathsf{SET} . \exists (B,\nu) : I \to \texttt{FiniteMeasureAlgebra} .$$

$$. (A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$$

Proof =

It is possible to select a partition of unity P of A consisting of finite elements.

Then by previous theorem $(A,\mu) \cong \prod_{p\in P}\Big((p), \mu_{|(p)}\Big)$.

And each $\Big((p), \mu_{|(p)}\Big)$ are obviously finite.

□

LocalizableMeasureAlgebrasHasLocallyDeterminedRepresentations ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} . \exists (X,\Sigma,\nu) : \texttt{LocallyDetermined} . (A,\mu) \cong_{\mathsf{MA}} (A_\nu, \hat{\nu})$$

Proof =

Represent $(A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$.

Then Stone's spaces $\mathsf{Z}\ B_i$ correspond to finite measure spaces.

And Stone's space of product correspond to a disjoint union of $\mathsf{Z}\ B_i$.

But such spaces are trivially locally determined.

□

1.1.9 Strictly Localizable Spaces

$\texttt{StrictlyLocalizableSpacePoU} ::$
$:: \forall (X, \Sigma, \mu) : \texttt{StrictlyLocalizable} \;.\; \forall P : \texttt{PartitionOfUnity}(A_\mu) \;.$
$.\; \exists E : P \to \Sigma \;.\; \forall p \in P \;.\; [E_p] = p \;\&\; \texttt{Decomposition}(X, \Sigma, \mu, \operatorname{Im} E)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

1.1.10 Subalgebras

SubalgebaMeasureAlgebra :: $\forall (A, \mu) :$ MeasureAlgebra . $\forall B \subset_\sigma A$. MeasureAlgebra$(B, \mu_{|B})$

Proof =

This is obvious.

□

SubalgebaFinifteMeasureAlgebra ::

:: $\forall (A, \mu) :$ FiniteMeasureAlgebra . $\forall B \subset_\sigma A$. FiniteMeasureAlgebra$(B, \mu_{|B})$

Proof =

This is obvious.

□

SigmaFiniteSubalgebraMeasureAlgebra ::

:: $\forall (A, \mu) : \sigma$-FiniteMeasureAlgebra . $\forall B \subset_\sigma A$.

. SemifiniteMeasureAlgebra$(B, \mu_{|B}) \Rightarrow \sigma$-FiniteMeasureAlgebra$(B, \mu_{|B})$

Proof =

1 The set B^f is order-dense in B.

2 But then B^f is also order-dense in A.

3 Select a finite-measured countable partition of unity P in A.

4 If B is not σ-finite, then there is a subordinate uncountale partition of unity Q.

5 Then there would exist a uncountable refinement of P subordinate to Q.

6 Then P must contain an infinite element, but this is imposible!.

7 So Q must be countable, and so $(B, \mu_{|B})$ must be countable.

□

FinifteMeasureAlgebraBySubalgebra ::

:: $\forall (A, \mu) :$ MeasureAlgebra . $\forall B \subset_\sigma A$. FiniteMeasureAlgebra$(B, \mu_{|B}) \Rightarrow$ FiniteMeasureAlgebra(A, μ)

Proof =

This is obvious.

□

ProbabilityAlgebraBySubalgebra ::

:: $\forall (A, \mu) :$ MeasureAlgebra . $\forall B \subset_\sigma A$.

. ProbabilityAlgebra$(B, \mu_{|B}) \Rightarrow$ ProbabilityAlgebra(A, μ)

Proof =

This is obvious.

□

$\texttt{SigmaFiniteAlgebraBySubalgebra} ::$

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall B \subset_\sigma A \;.$

$.\; \sigma\text{-}\texttt{Finite}(B, \mu_{|B}) \Rightarrow \sigma\text{-}\texttt{Finite}(A, \mu)$

$\texttt{Proof} \;=$

This is obvious.

$\square$

1.1.11 Localization

MeasureAlgebraCompletion ::

:: $\forall (A,\mu) :$ SemifiniteMeasureAlgebra . $\exists! \hat{\mu} : \tau(A) \to \overset{\infty}{\mathbb{R}}_{++}$.

. $\hat{\mu}_{|A} = \mu$ & LocalizableMeasureAlgebra$(\tau(A), \hat{\mu})$

Proof =

1 Define $\hat{\mu}(t) = \sup\{\mu(a) | a \in A, a \leq t\}$.

2 As A is order dense in $\tau(A)$, it holds that $\hat{\mu}(a) = 0 \iff a = 0$ for any $a \in \tau(A)$.

3 If $t : \mathbb{N} \to \tau(A)$ is disjoint then $\hat{\mu}\left(\bigvee_{n=1}^{\infty} t_n\right) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

3.1 Write $S = \{a \in A : \exists c : \mathbb{N} \to A \,.\, a = \lim_{n\to\infty} c_n \,\&\, c \leq t\}$.

3.2 Then there is $s = \sup S \in \tau(A)$.

3.3 We write $\hat{\mu}(s) = \sup_{c \leq t} \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sup_{c \leq t} \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \sup_{c \leq t_n} \mu(c) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

4 Obviously $(\tau(A), \hat{\mu})$ is semifinite and order-complete, and hence Localizable.

□

localization :: SemifiniteMeasureAlgebra → LocalizableMeasureAlgebra

localization $(A,\mu) = \Big(\tau(A), \tau(\mu)\Big)$:= MeasureAlgebraCompletion

LocalizationFiniteEmbedding ::

:: $\forall (A,\mu) :$ SemifiniteMeasureAlgebra . $\iota_\tau(A^f) = \tau^f(A)$

Proof =

1 Assume $t \in \tau(A)$ such that $\hat{\mu}(t) < \infty$.

2 Note, $\hat{\mu}(t) = \sup_{a \leq t} \mu(a)$.

3 So we may select an increasing $a : \mathbb{N} \to A$ such that $\lim_{n\to\infty} \mu(a_n) = \hat{\mu}(t)$.

4 Then $b = \bigvee_{n=1}^{\infty} a_n \in A$ and $\hat{\mu}(b) = \mu(b) = \hat{\mu}(t)$.

5 So $\mu(t \setminus b) = 0$, and so $t = b \in A$ as clearly $b < t$.

□

1.1.12 Stone Spaces

LocallalizableMeasureAlgebraHasStrictlyLocalizableStoneSpace ::

$:: \forall (A, \mu) :$ LocalizableMeasureAlgebra . StrictlyLocalizable$(\mathsf{Z}\ A, \Sigma_\mu, \bar{\mu})$

Proof =

1 We already proved that $\bar{\mu}$ is locally determined.

2 As (A, μ) is semifinite there is a partition of unity P consisting of finite elements.

3 Use Stone representation $S_A(P)$ to construct a corresponding set in $\mathsf{Z}\ A$.

4 Assume $E \in \Sigma_\mu$ such that $\bar{\mu}(E) > 0$.

5 By definition of Stone's Space there is a clopen set $F \in \mathsf{Z}\ A$ such that $E \triangle F$ is meager.

6 And there is a Stone representation $a \in A$ such that $F = S_A(a)$.

7 Then $\mu(a) = \nu(S_A(a)) = \nu(E) > 0$.

8 So, there exists $p \in P$ such that $ap \neq 0$.

9 Ths means that $\nu(E \cap S_A(p)) > 0$.

10 As E was arbitrary this means that $S_A(P)$ provides a strict localization for $\bar{\mu}$.

$\square$

MeagerSetsAreNowhereDense ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra . $\forall M \in \mathbf{MGR}(\mathsf{Z}\ A)$. NowhereDense$(\mathsf{Z}\ A, M)$

Proof =

1 As it was shown A is (σ, ∞)-WeaklyDistributive boolean algebra.

2 And this is a property of (σ, ∞)-WeaklyDistributive boolean algebra.

$\square$

StoneSpaceMeasurableExpression ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\exists U :$ Clopen$(\mathsf{Z}\ A)$. $\exists F :$ NowhereDense$(\mathsf{Z}\ A)$. $E = U \cap F$

Proof =

1 This is clear from the previous theorem.

$\square$

StoneSpaceMeasureComputation ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\bar{\mu}(E) = \sup \left\{ \mu(U) \middle| U : \text{Clopen}(\mathsf{Z}\ A), U \subset E \right\}$

Proof =

1 This is clear from the previous theorem.

$\square$

StoneSpaceCLDIsStrictlyLocalizable ::

$:: \forall (A, \mu) :$ SemifiniteMeasureAlgebra . StrictlyLocalizable$(\mathsf{Z}\ A, \bar{\Sigma}_\mu, \bar{\bar{\mu}})$

Proof =

...

$\square$

`StoneSpaceCLDZeroSets` ::

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra} \, . \, \mathcal{N}_{\bar{\bar{\mu}}} = \mathcal{N}_{\bar{\mu}}$

`Proof` $=$

...

□

`FiniteStoneSpaceMeasureComputation` ::

$:: \forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \, . \, \forall E \in \Sigma_\mu \, .$

$. \, \bar{\mu}(E) = \inf \left\{ \mu(U) \middle| U : \texttt{Clopen}(\mathsf{Z}\ A), E \subset U \right\}$

`Proof` $=$

1 This is clear from the previous theorem.

□

1.1.13 Purely Infinite Elements

$\texttt{purelyInfiniteElements} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \sigma\texttt{-Ideal}(A)$

$\texttt{purelyInfiniteElements}\,() = I_\infty(\mu := \{a \in A : \forall b \in A \,.\, b \leq a \;\&\; \mu(b) < \infty \Rightarrow b = 0\}$

$\texttt{semifiniteMeasure} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \dfrac{A}{I_\infty(\mu)} \to \overset{\infty}{\mathbb{R}}_+$

$\texttt{semifiniteMeasure}\,([a]) = \mu_{\mathrm{sf}} := \sup\{\mu(b) | b \in A : b \leq a \;\&\; \mu(b) < \infty\}$

If $[a] = [b]$, then $a \triangle b \in I_\infty(\mu)$.

So μ_{sf} is well-defined.

$\texttt{SemifiniteMeasureIsMeasure} ::$

$:: \forall(A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{SemifiniteMeasureAlgebra}\left(\dfrac{A}{I}, \mu_{\mathrm{sf}}\right)$

Proof $=$

1 If $\mu_{\mathrm{sf}}[a] = 0$, then clearly $a \in I_\infty$.

2 Assume $[a] : \mathbb{N} \to A$ is disjoint.

2.1 Then $a_n a_m \in I_\infty$ if $n \neq m$.

2.2 Select increasing $b : \mathbb{N} \to A^f$ such that $b_n \leq \bigvee_{k=1}^{\infty} a_k$ and $\lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}}\left[\bigvee_{k=1}^{\infty} a_k\right] = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k]$.

2.3 By (2.1) we mat assert that ab_n is disjoint and then $\bigvee_{k=1}^{\infty} a_k b_n = b_n$ for any $n \in \mathbb{N}$.

2.4 So $\mu(b) = \sum_{k=1}^{\infty} \mu(a_k b_n)$.

2.5 By taking limits and using monotonic convergence theorem

$$\sum_{k=1}^{\infty} \mu_{\mathrm{sf}}[a_k] = \sum_{k=1}^{\infty} \lim_{n\to\infty} \mu(a_k b_n) = \lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k].$$

3 Clearly $\mu_{\mathrm{sf}}[a] < \mu(a)$.

3.1 If $\mu_{\mathrm{sf}}[a] = \infty$, then $a \notin I_\infty$.

3.2 So it is possible to select $b \in A$ such that $b \leq a$ and $0 < \mu(b) \leq a$.

3.3 $0 < \mu_{\mathrm{sf}}[b] \leq \mu(b) < \infty$.

3.4 This proves that $\left(\dfrac{A}{I}, \mu_{\mathrm{sf}}\right)$ is semifinite.

□

1.2 Topology

1.2.1 Subject

$\texttt{measureAlgebraAsTopologicalSpace} :: \texttt{MeasureAlgebra} \to \mathsf{TOP}$

$\texttt{measureAlgebraAsTopologicalSpace}\,((A,\mu)) = (A,\mu) :=$

$$:= \Big(A, \mathcal{W}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f \,.\, \Lambda b \in A^f \,.\, \Lambda c \in A \,.\, \mu(ac + ab)\big)\Big)$$

$\texttt{measureAlgebraAsUniformlSpace} :: \texttt{MeasureAlgebra} \to \mathsf{UNI}$

$\texttt{measureAlgebraAsUniformSpace}\,((A,\mu)) = (A,\mu) :=$

$$:= \Big(A, \mathcal{I}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f \,.\, \Lambda b \in A^f \,.\, \Lambda c \in A \,.\, \mu(ac \triangle ab)\big)\Big)$$

$\texttt{metricOfFrechetNikodym} :: \prod (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Metric}(A^f)$

$\texttt{metricOfFrechetNikodym}\,() = \rho_\mu := \Lambda a, b \in A^f \,.\, \mu(a \triangle b)$

$\texttt{BooleanOperationsAreUniformlyContinuous} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, (*), (\setminus), (\vee), (\wedge) \in \mathsf{UNI}(A \times A, A)$

$\texttt{Proof} =$

1 Let $\circ$ stay for any binary operation above.

2 Select $c, d \in A$.

3 Then $\mu\Big(a(c \circ d) + b\Big) \leq \mu\Big(a(c \vee d) + b\Big) \leq \mu(ac + d) + \mu(ad + b)$.

4 So μ is bounded by the sum of uniform functions and is uniformly continuous.

$\square$

$\texttt{FiniteElementsAreDense} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{Dense}(A, A^f)$

$\texttt{Proof} =$

1 Select $c \in A$.

2 Then c has a base of neighborhoods of form $U = \{u \in A : \mu(au + ac) \leq r\}$ with $a \in A^f, r \in \mathbb{R}_{++}$.

3 But then $ac \in U$ and $ac \in A^f$.

$\square$

$\texttt{FiniteMeasureAlgebraHasUniformlyContinuousMeasure} ::$

$\forall (A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

$\texttt{Proof} =$

This is pretty obvious as $\mu = \rho_\mu(0, a)$.

$\square$

$\texttt{FiniteMeasureAlgebraHasUniformlyContinuousMeasure} ::$

$\forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \,.\, \mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

$\texttt{Proof} \ =$

This is pretty obvious as $\mu = \rho_\mu(0, a)$.

□

$\texttt{SemifinitMeasureAlgebraHasLowerSemicontinuousMeasure} ::$

$\forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \mu \in \texttt{LowerSemicontinuous}(A, \overset{\infty}{\mathbb{R}}_{++})$

$\texttt{Proof} \ =$

1 Assume $a \in A$ and $\alpha \in \mathbb{R}_+$ such that $\mu(a) > \alpha$.

2 As A is semifinite there exists $b \leq a$ such that $\infty > \mu(b) > \alpha$.

3 Assume $c \in A$ is such that $\mu(b + cb) < \mu(b) - \alpha$.

4 Then $\mu(c) \geq \mu(cb) = \mu(b) - \mu(b(a \setminus c)) = \mu(b) - \mu(b + cb) > \alpha$.

□

$\texttt{MeasureAlgebraHasUniformlyContinuousFinitisedMeasure} ::$

$\forall (A, \mu) : \texttt{MeasureAlgebra} \,.\, \forall a \in A^f \,.\, (\Lambda c \in A \,.\, \mu(ac)) \in \mathsf{UNI}(A, \mathbb{R}_{++})$

$\texttt{Proof} \ =$

This is simmilar to the case of finite measure space.

□

$\texttt{finiteElementMetric} :: \prod A : \texttt{MeasureAlgebra} \,.\, A^f \to \texttt{Semimetric}(A)$

$\texttt{finiteElementMetric}\,(a) = \rho_a := \Lambda x, y \in A \,.\, \mu(ax + ay)$

$\texttt{MeasurAlgebraProductTopology} ::$

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A, \mu) : I \to \texttt{MeasureAlgebra} \,.\, \prod_{i \in I} (A, \mu) =_{\mathsf{TOP}} \left(\prod_{i \in I} A_i, \sum_{i \in I} \mu_i \right)$$

$\texttt{Proof} \ =$

. . .

□

1.2.2 Relations with an Order Structure

upwardDirectedFilter ::

$:: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{NonEmpty} \,\&\, \texttt{UpwardsDirected}(A) \to \texttt{CauchyFilerbase}(A)$

$\texttt{upwardDirectedFilter}\,(D) = \mathcal{F}(\uparrow D) := \Big\{\{c \in D : d \le c\}\Big| d \in D\Big\}$

1 Write $F_d = \{c \in D : d \le c\}$.

2 $\mathcal{F}(\uparrow D)$ is a filter.

2.1 As D is non empty, $\mathcal{F}(\uparrow D)$ is also non-empty.

2.2 $d \in F_d$, so $F_d \ne \emptyset$ and henceforth $\emptyset \notin \mathcal{F}(\uparrow D)$.

2.3 Assume $F_d, F_f \in \mathcal{F}(\uparrow D)$.

2.3.1 Then there is an element $g \in D$ such that $g \ge f \vee d$.

2.3.2 Note, that $F_g \subset F_d \cap F_f$ and $F_g \in \mathcal{F}(\uparrow D)$.

3 $\mathcal{F}(\uparrow D)$ is Cauchy.

3.1 Assume U is some measure connector for (A,μ).

3.2 then there is an element $a \in A^f$ and $r \in \mathbb{R}_{++}$ such that $\{(f,g) \in A \times A : \mu(af + ag) < r\} \subset U$.

3.3 The set $\{\mu(ad)|d \in D\}$ is bounded by $\mu(a)$, so supremum is attained.

3.4 So there is $f \in D$, so $\mu(ad) < \mu(af) + r/2$ for any $d \in D$.

3.5 Assume $g, h \in F_f$.

3.5 Then $\mu(ag + ah) \le \mu(ag \setminus af) + \mu(ah \setminus af) = \Big(\mu(ag) - \mu(af)\Big) + \Big(\mu(ah) - \mu(af)\Big) < r$.

3.6 Thus, $(g,h) \in U$ and $F_f \times F_f \subset U$.

$\square$

UpwardsDirectedSup ::

$:: \forall(A,\mu) : \texttt{SemifiniteMeasureAlgebra} \,.\, \forall D : \texttt{UpwardsDirected}(A) \to \texttt{CauchyFilerbase}(A) \,.\, \forall a \in A \,.$

$.\, a = \sup D \Rightarrow a = \lim \mathcal{F}(\uparrow D)$

Proof =

1 Assume $a = \sup D$.

2 Assume U is an uniformity fo (A,μ).

3 then there is an element $c \in A^f$ and $r \in \mathbb{R}_{++}$ such that $\{g \in A \times A : \mu(ca + cg) < r\} \subset U(a)$.

4 Consider set $M = \{\mu(cd)|d \in D\}$.

5 Then $\sup M = \mu(ca)$.

6 So there is $d \in D$ such that $\mu(ca + cd) < r$.

7 But $d \le f \le a$for any $f \in F_d$.

8 Thus $\mu(cf + cd) < r$ and $F_d \subset U(a)$.

9 Thus, $da = \lim \mathcal{F}(\uparrow D)$.

$\square$

UpwardsDirectedLimit ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall D$: NonEmpty & UpwardsDirected(A) . $\forall a \in A$.

. $a = \sup D \Rightarrow a \in \operatorname*{cl}_A D$

Proof =

...

□

UpwardsDirectedFilterLimit ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall D$: NonEmpty & UpwardsDirected(A) . $\forall a \in A$.

. $a = \lim \mathcal{F}(\uparrow D) \iff a = \sup D$

Proof =

1 $(\Rightarrow)$ $a = \lim \mathcal{F}(\uparrow D)$.

1.1 Then for any connector U of (A,μ) There is some $F \in \mathcal{F}(\uparrow F)$ such that $F \subset U(a)$.

1.2 Assume $d \in D$.

1.3 Assume $d \not\leq a$.

1.4 Then there is $f \in A$ such that $f \leq d \setminus a$ and $0 < \mu(f) < \infty$.

1.5 Thus $\mu(fh + fa) \geq \mu(f)$ for every $h \in F_s$.

1.6 But $G \cap F_d \neq \emptyset$ for any $G \in \mathcal{F}(\uparrow D)$ so this contradicts (1.1).

□

lowerDirectedFilter ::

:: $\prod (A,\mu)$: MeasureAlgebra . NonEmpty & LowerDirected$(A) \to$ CauchyFilerbase(A)

loweDirectedFilter $(D) = \mathcal{F}(\uparrow D) := \Big\{ \{c \in D : d \geq c\} \Big| d \in D \Big\}$

LowerDirectedInf ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall D$: NonEmpty & LowerDirected(A) . $\forall a \in A$.

. $a = \inf D \Rightarrow a = \lim \mathcal{F}(\uparrow D)$

Proof =

By duality.

□

UpwardsDirectedLimit ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall D$: NonEmpty & LowerDirected(A) . $\forall a \in A$.

. $a = \inf D \Rightarrow a \in \operatorname*{cl}_A D$

Proof =

By duality.

□

UpwardsDirectedFilterLimit ::

:: $\forall (A,\mu)$: SemifiniteMeasureAlgebra . $\forall D$: NonEmpty & LowerDirected(A) . $\forall a \in A$.

. $a = \lim \mathcal{F}(\uparrow D) \iff a = \inf D$

Proof =

By duality.

□

$\texttt{ClosedSetsAreOrderClosed} :: \forall (A, \mu) : \texttt{MeasureAlgebra} . \forall F : \texttt{Closed}(A) . \texttt{OrderContinuous}(A, F)$

$\texttt{Proof} =$

Follows from previous theorems in this chapter.

$\square$

$\texttt{DenseSetsAreOrderDense} :: \forall (A, \mu) : \texttt{MeasureAlgebra} . \forall F : \texttt{Dense}(A) . \texttt{OrderDense}(A, F)$

$\texttt{Proof} =$

Follows from previous theorems in this chapter.

$\square$

$\texttt{ClosedRays} :: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra} . \forall a \in A . \texttt{Closed}\Big(A, \{c \in A : c \leq a\} \ \& \ \{c \in A : c \geq a\}\Big)$

$\texttt{Proof} =$

1 Let $F = \{c \in A : c \leq a\}$.

2 Assume $d \in F^{\complement}$.

3 Then $d \setminus a \neq 0$.

4 As A is semifinite there is an $g \in A^f$ such that $g \leq d \setminus a$ and $0 < \mu(g)$.

5 $\rho_g(d, f) \geq \mu(g)$ fo any $f \in F^{\complement}$.

6 And this means that $F^{\complement}$ and F is closed.

$\square$

$\texttt{SupremumConvergence} :: \forall A : \texttt{MeasureAlgebra} . \forall a : \mathbb{N} \uparrow A . \forall s \in A . s = \sup_{n=1} a_n \Rightarrow s = \lim_{n=1} a_n$

$\texttt{Proof} =$

This is obvious now.

$\square$

$\texttt{InfimumConvergence} :: \forall A : \texttt{MeasureAlgebra} . \forall a : \mathbb{N} \downarrow A . \forall s \in A . s = \inf_{n=1} a_n \Rightarrow s = \lim_{n=1} a_n$

$\texttt{Proof} =$

This is obvious now.

$\square$

$\texttt{SummableIncrements} :: \prod A : \texttt{MeasureAlgebra} . ?(\mathbb{N} \to A)$

$$a : \texttt{SummableIncrements} \iff \forall n \in \mathbb{N} . \sum_{n=1}^{\infty} \mu(a_n + a_{n+1}) < \infty$$

`SummableIncrementsLimSupLimInfEq` ::

:: $\forall A$: `MeasureAlgebra` . $\forall a$: `SummableIncrements`(A) . $\inf_{n=1} \sup_{m=n} a_n = \sup_{n=1} \inf_{m=n} a_n$

`Proof` =

1 Let $\alpha_n = \mu(a_n + a_{n+1}), \beta_n = \sum_{m=n}^{\infty} \alpha_n$.

2 As a has summable increments this means $\beta \downarrow 0$.

3 Let $b_n = \sup_{m \geq n} a_m + a_{m+1} = \bigvee_{m=n}^{\infty} a_m + a_{m+1}$.

4 Then $\mu(b_n) \leq \sum_{m=n}^{\infty} \mu(c_m + c_{m+1}) = \beta_n$.

5 Assume $m \leq n$.

6 And also $a_m + a_n \leq \sup_{m \leq k \leq n} a_k + a_{k+1} \leq b_n$.

7 So $a_n \setminus b_n \leq a_m \leq a_n \vee b_n$.

8 Thus $a_n \setminus b_n \leq \inf_{k \geq m} a_k \leq \sup_{k \geq m} a_k \leq a_n \vee b_n$.

9 By taking limits in m one gets $a_n \setminus b_n \leq \inf_{m=1} \sup_{k=n} a_k \leq \sup_{m=1} \inf_{k=m} a_k \leq a_n \vee b_n$.

10 $a_n + \inf_{m=1} \sup_{k=m} a_k \leq b_n$.

11 $a_n + \sup_{m=1} \inf_{k=m} a_k \leq b_n$.

12 From (10) and (11) $\inf_{m=1} \sup_{k=m} a_k \setminus \sup_{m=1} \inf_{k=m} a_k \leq b_n$.

13 But $\lim_{n \to \infty} b_n = 0$.

14 So $\inf_{m=1} \sup_{k=m} a_k = \sup_{m=1} \inf_{k=m} a_k$.

□

`SummableIncrementsLim` ::

:: $\forall A$: `MeasureAlgebra` . $\forall a$: `SummableIncrements`(A) . $\forall x \in A$.

. $x = \lim_{n \to \infty} a_n \Rightarrow \inf_{n=1} \sup_{m=n} a_n = x = \sup_{n=1} \inf_{m=n} a_n$

`Proof` =

This follows from the previous proof.

□

1.2.3 Classification Theorems

SemifiniteIffHausdorff $:: \forall (A,\mu) :$ MeasureAlgebra . SemifiniteMeasureAlgebra$(A,\mu) \iff$ T2(A)

Proof =

1 $(\Rightarrow)$ assume that (A,μ) is semifinite.

1.1 Take $x, y \in A$ such that $x \neq y$.

1.2 Then $x + y \neq 0$ so there is $a \in A^f$ such that $\mu(a) > 0$ and $a < x + y$.

1.3 So $\rho_a(x,y) = \mu(a) > 0$.

1.4 And cells of form $\mathbb{B}_{\rho_a}(x, \mu(a)/2)$ and $\mathbb{B}_{\rho_a}(y, \mu(a)/2)$ produce the separation.

2 $(\Leftarrow)$ assume that A is Hausdorff in the topology of (A,μ).

2.1 Assume $x \in A$ such that $\mu(x) = \infty$.

2.2 Then $x \neq 0$.

2.3 Assume $a \in A^f$.

2.4 If $xa = 0$ then $\rho_a(x, 0) = 0$.

2.5 So, as A is Hausdorff there mustb some $a \in A^f$ such that $xa \neq 0$.

2.6 But this means that (A,μ) is semifinite.

□

SigmaFiniteIffMetrizable ::

$:: \forall (A,\mu) :$ MeasureAlgebra . σ-FiniteMeasureAlgebra$(A,\mu) \iff$ Metrizable(A)

Proof =

1 $(\Rightarrow)$ assume that (A,μ) is σ-finite.

1.1 Then there is a countable partition of unity a with finite elements.

1.2 define $\sigma : A^2 \to \mathbb{R}_{++}$ as $\sigma(x,y) = \displaystyle\sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{a_n}(x,y)}{\mu(a_n)}$.

1.3 Then σ is a metic for A .

1.4 So the topology of (A,μ) is metrizable.

2 $(\Leftarrow)$ assume that (A,μ) is metrizable.

2.1 Let σ be an metrizing metric.

2.2 Then there exists a system of elements $k : \mathbb{N} \to \mathbb{N}, a : \displaystyle\prod_{n=1}^{\infty} \{1, \ldots, k_n\} \to A^f$ and $\delta : \mathbb{N} \to \mathbb{R}_{++}$

such that $\rho_{a_{n,i}}(b, e)$ for any $1 \leq i \leq k_n$ imply that $\sigma(b, e) < 2^{-n}$ for any $b \in A$.

2.3 Then $e = \displaystyle\bigvee_{n=1}^{\infty} \bigvee_{i=1}^{k_n} a_{n,i}$.

2.4 So (A,μ) is σ-finite.

□

LocalizableIffComplete ::

:: $\forall (A,\mu)$: MeasureAlgebra . LocalizableMeasureAlgebra$(A,\mu) \iff$ T2 & Complete(A)

Proof =

1 ($\Rightarrow$) Assume (A,μ) is localizable measure algebra.

1.2 Then A is Hausdorff as (A,μ) is semifinite.

1.3 Assume $\mathcal{F}$ is a Cauchy filter in A.

1.4 Take $a \in A^f$.

1.5 Then there is $d_a \leq a$ and a cauchy sequence c_a subordinate to $\mathcal{F}$ such that $\lim_{n\to\infty} \rho_a(d_a, c_{a,n}) = 0$.

1.5.1 select a sequence $F_a : \mathbb{N} \to \mathcal{F}$ such that $\rho_a(x,y) \leq 2^{-n}$ for $x, y \in F_{a,n}$ and $n \in \mathbb{N}$.

1.5.2 Then select a sequence $c_{a,n} \in \bigcap_{k=1}^{n} F_{a,k}$.

1.5.3 Then $\rho(c_{a,n}, c_{a,n+1}) \leq 2^{-n}$.

1.5.4 Construct $d_a = \liminf ac_a$.

1.5.5 Then $\lim_{n\to\infty} \rho_a(d_a, c_{an}) = \lim_{n\to\infty} \mu(d_a + ac_a) = 0$.

1.6 Assume $a, b \in A^f$ are such that $a \leq b$.

1.7 Then $d_a = ad_b$.

1.7.1 $F_{n,a} \cap F_{n,b} \neq \emptyset$.

1.7.2 So select $f \in F_{n,a} \cap F_{n,b}$.

1.7.3 Then $\rho_a(d_a, d_b) \leq \rho_a(d_a, c_{a,n}) + \rho_a(c_{a,n}, f) + \rho_a(f, c_{b,n}) + \rho_a(c_{b,n}, d_b) \leq$

$\leq \rho_a(d_a, c_{a,n}) + 2^{-n} + 2^{-n} + \rho_a(c_{b,n}, d_b) \to 0$ as $n \to \infty$.

1.8 Let $f = \bigvee_{a\in A^f} d_a$.

1.9 Then $\lim \mathcal{F} = f$.

1.9.1 $ad_a = af$ for any $a \in A^f$.

1.9.2 and there is a $\mathcal{F}$ subordinate Cauchy sequence c_a such that $\rho_a(f, c_a) = \rho_a(d_a, c_a) \to 0$.

1.9.3 Then there is $n \in \mathbb{N}$ such that $\rho_a(d_a, c_{a,n}) + 2^{-n} < \varepsilon$.

1.9.4 Take any $g \in F_{a,n}$.

1.9.5 But $\rho_a(f, g) \leq \rho_a(f, c_{a,n}) + \rho_{c_{a,n}} \leq \rho_a(d_a, c_{a,n}) + 2^{-n} < \varepsilon$.

1.9.6 This $F_{a,n} \subset \mathbb{B}_{\rho_a}(f, \varepsilon)$.

2 ($\Leftarrow$) now Assume that A is Hausdorff and complete.

2.1 As A is Hausdorff (A,μ) must be semifinite.

2.2 As A is complete (A,μ) is order complete and hence localizable.

2.2.1 Think about order filters $\mathcal{F}(\uparrow D)$ and $\mathcal{F}(\downarrow D)$.

□

1.2.4 Closed Subalgebras

ClosedSubalgebraTHM ::

:: $\forall (A,\mu) : $ LocalizableMeasureAlgebra . $\forall B \subset_{\mathsf{RING}} A$. Closed$(A,B) \iff$ OrderClosed(A,B)

Proof $=$

1 ($\Rightarrow$) follows from the general theory.

2 ($\Leftarrow$) Assume now that B is order-closed.

2.1 Assume $g \in \mathrm{cl}_A B$.

2.2 Assume $a \in A^f$ and $n \in \mathbb{N}$.

2.3 Then there exists a sequence $c_a : \mathbb{N} \to B$ such that $\rho_a(c_{a,n}, g) < 2^{-n}$.

2.4 Note, $\sum_{n=1}^{\infty} \mu(ac_{a,n} + ac_{a,n+1}) \le \sum_{n=1}^{\infty} \mu(ac_{a,n} + ag) + \mu(ag + ac_{a,n+1}) < \sum_{n=1}^{\infty} 2^{-n} + 2^{-n-1} = \dfrac{3}{2}$.

2.5 So, sequence ac_a has summable increments .

2.6 Define $d_a = \liminf c_a$.

2.7 As ac_a has finite increments $\lim_{n\to\infty} \rho_a(c_{a,n}, d_n) = 0$.

2.8 Furthermore, $\rho_a(d_a, g) = 0$, so $ag = d_a$.

2.9 As B is order-closed $d_a \in B$ for each $a \in A^f$.

2.10 Set $d'_a = \inf\{d_b : b \in A^f, a \le b\} \in B$.

2.11 $d'_a a = \bigwedge_{a\le b} d_b a = \bigwedge_{a\le b} d_b ba = \bigwedge_{a\le b} gba = ga$.

2.12 Let $D = \{d'_a | a \in A\}$.

2.13 Clearly D is upwards directed as $d'_a \vee d'_b = d'_{a\wedge b}$.

2.14 Then $\sup D = \{ad'_a | a \in A\} = \{ag | a \in A\} = g$ as (A,μ) is semifinite.

2.15 so $g \in B$ as B is order-closed.

2.16 Thus B is closed.

□

SubalgebraClosure :: $\forall (A,\mu) : $ LocalizableMeasureAlgebra . $\forall B \subset_{\mathsf{RING}} A$. $\overline{B} = \tau(B)$

Proof $=$

1 Note that $\overline{B}$ is a subgroup of A.

2 Also it must be order-closed as $\overline{B}$ is closed.

3 Also $\tau(B)$ is an order-closed subalgebra, and hence a closed subalgebra.

4 So both objects can be realized as intersections of closed subalgebras containing B, and hence they are equal.

□

ClosedMeasureSubalgebra :: $\prod (A,\mu)$: MeasureAlgebra . Subalgebra(A)

B : ClosedMeasureSubalgebra $\iff B \subset_{\mathsf{MA}} A \iff$ Closed(A,B)

`OrderClosedExtension` ::

:: $\forall (A, \mu) :$ `LocalizableMeasureAlgebra` $. \forall B \subset_{\mathsf{MA}} A . \forall a \in A . \langle B \cup \{a\} \rangle_{\mathsf{BOOL}} \subset_{\mathsf{MA}} A$

`Proof` $=$

This follows from order-closed subalgebra extension theorem for boolean algebras.

$\square$

1.2.5 Metric Space of Finite Elements

BooleanOperationsAreUniformlyContinuous ::

:: $\forall (A, \mu)$: MeasureAlgebra . $(*), (\setminus), (\vee), (\wedge) \in \mathsf{UNI}(A^f \times A^f, A^f)$

Proof =

This is obvious.

□

MeasureIs1Lip ::

:: $\forall (A, \mu)$: MeasureAlgebra . $\mu_{|A^f} \in$ 1-Lip(A^f)

Proof =

This is obvious.

□

FiniteElementsAreComplete ::

:: $\forall (A, \mu)$: MeasureAlgebra . Complete(A^f)

Proof =

1 Assume a is a cauchy sequence in A^f.

2 without loss of generality we may assume that a has summable differences .

2.1 Just select a subsequence.

3 Define $x = \liminf a \in A$.

4 Then $\lim_{n \to \infty} a_n = x$.

5 So, there is some $n \in \mathbb{N}$ such that $\mu(x \setminus a_n) < \infty$.

6 Thus $\mu(x) < \infty$ and $x \in A^f$.

□

1.2.6 Relation with Convergence In Measure

1.3 Category

1.4 Radon-Nikodym Parallels

2 Maharam's Theory

3 Abstract Ergodic Theory

4 Measurable Algebras

Sources:

1. D. H. Fremlin — Measure Theory (32,33,34) 2016