1 Problem about seventh roots of unity

Question a)

We show that polynomial $P(X) = \sum_{k=0}^{p-1} X^k$ is irreducible for prime p. fitstly w will make a substitution $X = Y + 1$. Then coeficient of Y^k will have form

$$\sum_{n=k}^{p-1} \binom{n}{k} = \binom{p}{k+1} = \frac{p!}{(k+1)!(p-k-1)!}$$

by Christmass stocking theorem. So Y^{p-1} will have coefficient 1, Y^0 will have coefficient p, and all other coeficients will be divisible by p. This means that by Eisenstein's criterion P is irreducible over $\mathbb{Q}$.

Question b)

$P(X) = \sum_{k=0}^{6} X^k = \frac{X^7-1}{X-1}$ is the minimal polynomial of ζ (irreducible by (a),monic and has ζ as root). This means that $[L : \mathbb{Q}] = \deg P = 6$

Question c)

We will use the fact that $\zeta^{-1} = \zeta^6$,$\zeta^6 + \zeta = 2\cos(2\pi/7) \in$ Note that we can factor P over M:

$$P(X) = \prod_{i=k}^{6}(X - \zeta^k) = (X^2 - (\zeta + \zeta^6)X + 1)(X^2 - (\zeta^2 + \zeta^5)X + 1)(X^2 - (\zeta^3 + \zeta^4)X + 1) =$$
$$= (X^2 - 2\cos(2\pi/7)X + 1)(X^2 - 2\cos(4\pi/7)X + 1)(X^2 - 2\cos(6\pi/7)X + 1).$$

So the minimal polynomial for ζ over M is $Q(X) = X^2 - 2\cos(2\pi/7)X + 1$. So $[L : M] = 2$ and as $[L : \mathbb{Q}] = [L : M][M : \mathbb{Q}]$ it is clear that $[M : \mathbb{Q}] = 3$.

Question d)

The group of automorphisms of P is generated by $\zeta \mapsto \zeta^2$, so there exist following options for $f(\zeta)$:

$$\{\zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6\}$$

and following options for $f(\cos(2\pi/7))$

$$\{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\}$$

because automorphisms always map inverse into inverse

$$f\left(\frac{1}{2}(\zeta + \zeta^{-1})\right) = \frac{1}{2}(f(\zeta) + (f(\zeta))^{-1})$$