Topological Vector Spaces 2

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Contents

1 Abstract Topological Vector Spaces

1.1 Minkowski's Theory

1.1.1 Intro and Definition

$$\texttt{TopologicalVectorSpace} :: \prod k : \texttt{TopologicalField} \ . \ ?\sum_{V \in k\text{-}\mathsf{VS}} \texttt{Topology}(V)$$

$$(V,\tau) : \texttt{TopologicalVectorSpace} \iff \cdot_V \in \mathsf{TOP}\Big(k \times (V,\tau), (V,\tau)\Big) \ \& \ +_V \in \mathsf{TOP}\Big((V,\tau) \times (V,\tau), (V,\tau)\Big)$$

$k :: \texttt{TopologicalField};$

$$\texttt{VectorTopology} := \Lambda V \in k\text{-}\mathsf{VS} \ . \ \texttt{TopologicalVectorSpace}(V) : \prod_{V \in k\text{-}\mathsf{VS}} V \ . \ ?\texttt{Topology}(V);$$

$$\texttt{categoryOfTopologicalVectorSpaces} :: \texttt{TopologicalField} \to \mathsf{CAT}$$
$$\texttt{categoryOfTopologicalVectorSpaces}\,(k) = k\text{-}\mathsf{TVS} :=$$
$$:= (\texttt{TopologicalVectorSpace}(k), k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$$

$$\texttt{categoryOfTopologicalVectorSpaces} :: \texttt{TopologicalField} \to \mathsf{CAT}$$
$$\texttt{categoryOfHausdorffTopologicalVectorSpaces}\,(k) = k\text{-}\mathsf{HTVS} :=$$
$$:= (\texttt{TopologicalVectorSpace}(k) \ \& \ \texttt{T2}, k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$$

$$\texttt{asTopologicalGroup} :: k\text{-}\mathsf{TVS} \to \mathsf{TGRP}$$
$$\texttt{asTopologicalGroup}\,(V) = V := V$$

$$\texttt{asVectorSpace} :: k\text{-}\mathsf{TVS} \to k\text{-}\mathsf{VS}$$
$$\texttt{asVectorSpace}\,(V) = V := V$$

1.1.2 Absorbent and Balanced Sets

$k :: \texttt{AbsoluteValueField}(\mathbb{R});$

$$\texttt{Balanced} :: \prod_{V : k\text{-TVS}} ??V$$

$$B : \texttt{Balanced} \iff \mathbb{D}_k(0,1)B \subset B$$

$$\texttt{Absorbent} :: \prod k : \texttt{AbsoluteValueField}(\mathbb{R}) \,.\, \prod_{V : k\text{-TVS}} ??V$$

$$A : \texttt{Absorbent} \iff \forall v \in V \,.\, \exists \rho \in \mathbb{R}_{++} \,.\, \forall \alpha \in \mathbb{D}_k(0,\rho) \,.\, \alpha v \in A$$

$\texttt{VectorSubspaceIsBalanced} :: \forall V \in k\text{-TVS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \texttt{Balanced}(V,U)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{AbsorbentVectorSubspaceIswhole} :: \forall V \in k\text{-TVS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \texttt{Absorbent}(V,U) \Rightarrow V$

$\texttt{Proof} =$

Take $v \in V$.

By definition of absorbent there is $\alpha \in k_*$ such that $\alpha v \in U$.

But then $v = \alpha^{-1}\alpha v \in U$.

So, $U = V$.

□

$\texttt{BalancedSetsAreDedikindComplete} :: \forall V \in k\text{-TVS} \,.\, \texttt{OrderDedekindComplete}\Big(\texttt{Balanced}(V)\Big)$

$\texttt{Proof} =$

Assume β is a set of balanced sets in V.

If $v \in \bigcup \beta$, then there is a $B \in \beta$ such that $v \in B$.

And by definition of balanced $\alpha v \in B \subset \bigcup \beta$ for any $\alpha \in \mathbb{B}_k(0,1)$.

So $\bigcup \beta$ is Balanced.

if $v \in \bigcap \beta$, then $v \in B$ for any $B \in \beta$.

And by definition of balanced $\alpha v \in B \subset \bigcup \beta$ for any $\alpha \in \mathbb{B}_k(0,1)$ and for all $B \in \beta$.

So $\bigcap \beta$ is Balanced.

□

$\texttt{AbsorbentAreClosedUnderUnions} :: \forall V \in k\text{-TVS} \,.\, \forall \alpha :? \texttt{Absorbent}(V) \,.\, \texttt{Absorbent}\left(V, \bigcup \alpha\right)$

$\texttt{Proof} =$

This is obvious.

□

AbsorbentAreClosedUnderFiniteIntersections ::

$$:: \forall V \in k\text{-TVS} \ . \ \forall \alpha : \texttt{Finite}\Big(\texttt{Absorbent}(V)\Big) \ . \ \texttt{Absorbent}\left(V, \bigcap \alpha\right)$$

Proof =

Say $n = |\alpha|$.

if $n = 0$, then $\bigcap \alpha = V$ which is always absorbent.

otherwisr represent $\alpha = \{A_1, \dots, A_n\}$ and assume $v \in V$.

Select a finite sequence $\rho : \{1, \dots, n\} \to \mathbb{R}_{++}$, with ρ_i absorbing v for A_i.

Let $\sigma = \min\{\rho_1, \dots, \rho_n\}$.

Then σ is absorbing for every A_i, so it is absorbing for $\bigcap \alpha$.

□

In case of infinite intersiction the minimum may not exit.

$$\texttt{balancedHull} :: \prod_{V : k\text{-TVS}} 2^V \to \texttt{Balanced}(V)$$

$$\texttt{balancedHull}\,(A) = \operatorname{bal} A := \bigcap \Big\{ B : \texttt{Balanced}(V), A \subset B \Big\}$$

BalancedHullProductExpression :: $\forall_{V \in k\text{-TVS}} \forall A \subset V \ . \ \operatorname{bal} A = \mathbb{B}_k(0,1)A$

Proof =

Clearly $\mathbb{B}_k(0,1)A$ is balanced.

Assume that B is a balanced set such that $A \subset B$.

Then $\mathbb{B}_k(0,1)A \subset \mathbb{B}_k(0,1)B \subset B$ as B as balanced.

This proves the result as balanced hull of A may beviewed as the smallest balanced set containing A.

□

$$\texttt{balancedCore} :: \prod_{V : k\text{-TVS}} 2^V \to \texttt{Balanced}(V)$$

$$\texttt{balancedCore}\,(A) = A^{\text{bal}} := \bigcup \Big\{ B : \texttt{Balanced}(V), B \subset A \Big\}$$

BalancedCoreAsIntersction :: $\forall_{V \in k\text{-TVS}} \forall A \subset V \ . \ \operatorname{bal} A = \bigcap_{\alpha \in \mathbb{B}_k^{\complement}(0,1)} \alpha A$

Proof =

Firstly, I show that $B = \bigcap_{\alpha \in \mathbb{B}_k^{\complement}(0,1)} \alpha A$ is balanced.

Assume $v \in B$.

Then, $v \in \alpha A$ for all $\alpha \in \mathbb{B}_k^{\complement}(0,1)$.

Thus $\mathbb{B}_k(0,1)v \subset A$.

By definition A^{bal} as a union this means, that $v \in A^{\text{bal}}$, so $B \subset A^{\text{bal}}$.

Assume now that $v \in A^{\text{bal}}$.

Then $\mathbb{B}_k(0,1)v \subset \mathbb{B}_k(0,1)A^{\text{bal}} \subset A^{\text{bal}} \subset A$ As A^{bal} is a union of subsets.

But this mean that $v \in B$, so $A = B$.

□

ClosedBalancedCoreIsOpen :: $\forall V : k\text{-}\mathsf{TVS} \,.\, \forall F : \texttt{Closed}(V) \,.\, \texttt{Closed}(V, F^{\mathrm{bal}})$

Proof =

Multiplication by non-zero scalar is a homeomorphism.

So result follows from intersection representation as αF will be closed.

□

LinearMapsBalancedToBalanced ::

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall B : \texttt{Balanced}(V) \,.\, \texttt{Balanced}\Big(W, T(B)\Big)$$

Proof =

Assume $w \in T(B)$ and $\alpha \in \mathbb{D}_k(0,1)$.

Then there is $v \in B$ such that $T(v) = w$.

as B is balanced $\alpha v \in B$.

Thus $\alpha w = \alpha T(v) = T(\alpha v) \in T(B)$.

This proves that $T(B)$ is balanced.

□

LinearSurjectiveMapsAbsorbentToAbsorbent ::

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS} \;\&\; \texttt{Surjective}(V, W) \,.\, \forall A : \texttt{Absorbent}(V) \,.\, \texttt{Absorbent}\Big(W, T(A)\Big)$$

Proof =

Assume $w \in W$.

Then there is $v \in V$ such that $T(v) = w$ as T is surjective.

Then there exists $\rho \in \mathbb{R}_{++}$ such that $\mathbb{D}(0, \rho)v \subset A$ as A is absorbent.

Take $\alpha \in \mathbb{D}(0, \rho)$.

Then $\alpha w = \alpha T(v) = T(\alpha v) \in T(A)$.

This proves that $T(A)$ is absorbent.

□

BalancedPreimageIsBalanced ::

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall B : \texttt{Balanced}(W) \,.\, \texttt{Balanced}\Big(V, T^{-1}(B)\Big)$$

Proof =

Take $v \in T^{-1}(B)$ and $\alpha \in \mathbb{D}_k(0,1)$.

Then $T(v) \in B$, but also $T(\alpha v) = \alpha T(v) \in B$ as B is balanced .

But this means that $\alpha v \in T^{-1}(B)$.

□

BalancedPreimageIsBalanced ::

$$:: \forall V, W : k\text{-}\mathsf{TVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, \forall A : \texttt{Absorbent}(W) \,.\, \texttt{Absorbent}\Big(V, T^{-1}(A)\Big)$$

Proof =

Take $v \in V$.

Then there is $\rho \in \mathbb{R}_{++}$ such that $T(\alpha v) = \alpha T(v) \in A$ for any $\alpha \in \mathbb{D}_k(0, \rho)$ as A is absorbent.

But this means that $\alpha v \in T^{-1}(A)$.

□

1.1.3 Topology and Convexity

$$\texttt{Disc} := \Lambda V \in k\text{-}\mathsf{TVS} \, . \, \texttt{Convex} \,\&\, \texttt{Balanced}(V) : \prod_{V \in k\text{-}\mathsf{TVS}} ??V;$$

$\texttt{DiscCharacterization}$ ::

:: $\forall V \in k\text{-}\mathsf{TVS} \, . \, \forall D \subset V \, . \, \texttt{Disc}(V, D) \iff \forall v, w \in D \, . \, \forall \alpha, \beta \in k \, . \, |\alpha| + |\beta| \leq 1 \Rightarrow \alpha v + \beta w \in D$

$\texttt{Proof}$ =

Firstly, assume that D is a Disc.

Take $v, w \in D$ and $\alpha, \beta \in k$ such that $|\alpha| + |\beta| \leq 1$.

$\alpha v, \beta w \in D$ as D is balanced.

So if $\alpha = 0$ or $\beta = 0$ then $\alpha v + \beta w = \alpha v \in V$ or $\alpha v + \beta w = \beta w \in V$.

Otherwise, $|\alpha| + |\beta| \neq 0$ and $\dfrac{|\alpha|}{|\alpha| + |\beta|} + \dfrac{|\beta|}{|\alpha| + |\beta|} = 1$.

Also, $\dfrac{|\alpha| + |\beta|}{|\alpha|}\alpha v, \dfrac{|\alpha| + |\beta|}{|\beta|}\beta w \in D$ as $|\alpha| + |\beta| \leq 1$ and D is absorbent.

Then $\alpha v + \beta w = \dfrac{|\alpha|}{|\alpha| + |\beta|}\dfrac{|\alpha| + |\beta|}{|\alpha|}\alpha v + \dfrac{|\beta|}{|\alpha| + |\beta|}\dfrac{|\alpha| + |\beta|}{|\beta|}\beta w \in D$ as D is convex.

Now assume that the condition holds.

Then convexity and being balanced is obvious.

□

$\texttt{DiskedHull}$ :: $\forall V \in K\text{-}\mathsf{TVS} \, . \, \forall A \subset V \, . \, \bigcap \Big\{ D : \texttt{Disc}(V), A \subset D \Big\} = \operatorname{conv} \operatorname{bal} A$

$\texttt{Proof}$ =

Firstly we need to show that $\operatorname{conv} \operatorname{bal} A$ is balanced.

Assume $v \in \operatorname{conv} \operatorname{bal} A$ and $\alpha \in \mathbb{D}_k(0, 1)$.

If $\alpha = 0$ then $\alpha v = 0 \in \operatorname{bal} A \subset \operatorname{conv} \operatorname{bal} A$.

Otherwise, if C is convex in V, then $\dfrac{\alpha}{|\alpha|}C$ is also convex.

Also if $\operatorname{bal} A \subset C$ then $\operatorname{bal} A = \dfrac{\alpha}{|\alpha|} \operatorname{bal} A \subset \dfrac{\alpha}{|\alpha|}C$ as $\operatorname{bal} A$ is balanced.

Thus, $\dfrac{\alpha}{|\alpha|}v \in \operatorname{conv} \operatorname{bal} A$.

Also, as it was said $0 \in \operatorname{bal} A \subset \operatorname{conv} \operatorname{bal} A$.

So $\alpha v = \dfrac{|\alpha|}{|\alpha|}\alpha v + \Big(1 - |\alpha|\Big)0 \in \operatorname{conv} \operatorname{bal} A$ as $\operatorname{conv} \operatorname{bal} A$ is convex.

So $\operatorname{conv} \operatorname{bal} A$ is a disk and $B = \bigcap \Big\{ D : \texttt{Disc}(V), A \subset D \Big\} \subset \operatorname{conv} \operatorname{bal} A$.

Now assume that D is a disk such that $A \subset D$.

Then $\operatorname{bal} A \subset D$ as D is balanced.

Furthermore, $\operatorname{conv} \operatorname{bal} A \subset D$ as D is convex.

Thus $\operatorname{conv} \operatorname{bal} A = B$.

□

$\texttt{TVSIsConnected} :: \forall V \in k\text{-TVS} . \texttt{Connected}(k) \Rightarrow \texttt{Connected}(V)$

Proof $=$

Note that $V = \bigcup_{v \in V} kv$.

Each kv is connected as continuous image of connected k.

Then all lines kv intersect at 0, so V is connected.

□

$\texttt{AbsorbentNeighborhoodsOfZero} :: \forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \texttt{Absorbent}(V, U)$

Proof $=$

Assume $v \in V$.

Then $\lim_{\alpha \to 0} \alpha v = 0$.

So, there exists $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}_k(0, \rho)v \subset U$.

Then $\mathbb{D}_k\left(0, \frac{\rho}{2}\right) v \subset \mathbb{B}_k(0, \rho)v \subset U$.

Thus, U is absorbent.

□

$\texttt{NeighborhoodsOfZeroScaling} :: \forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \forall \alpha \in k_* . \alpha U \in \mathcal{U}_V(0)$

Proof $=$

$\alpha \cdot \bullet$ is a homeomorphism, so αU i s open .

Obviously, $0 = \alpha 0 \in \alpha U$ as $0 \in U$.

Thus, $U \in \mathcal{U}_V(0)$.

□

$\texttt{EachNeighborhoodsOfZeroContainsBalancedNeighborhoods} ::$

$:: \forall V \in k\text{-TVS} . \forall U \in \mathcal{U}_V(0) . \exists W \in \mathcal{U}_V(0) . W \subset U \ \& \ \texttt{Balanced}(V, W)$

Proof $=$

$(\cdot)^{-1}(U)$ is open in $k \times V$.

So there exist $W \in \mathcal{U}_V(0)$ and $\rho \in \mathbb{R}_{++}$ such that $\mathbb{B}_k(0, \rho) \times W \subset (\cdot)^{-1}(U)$ as $0 \in (\cdot)^{-1}(U)$.

This means that $\mathbb{B}_k(0, \rho)W \subset U$.

Also, note that $\mathbb{B}_k(0, \rho)W = \bigcup_{|\alpha| < \rho} \alpha W \in \mathcal{U}_V(0)$.

Assume $v \in \mathbb{B}_k(0, \rho)W$ and $\alpha \in \mathbb{D}_k(0, 1)$.

Then there is $w \in W$ and $\beta \in \mathbb{B}_k(0, \rho)$ such that $v = w\beta$.

But $\alpha\beta$ is also in $\mathbb{B}_k(0, \rho)$ and so $\alpha v = \alpha\beta w \in \mathbb{B}_k(0, \rho)W$.

Thus, $\mathbb{B}_k(0, \rho)W$ is balanced.

□

$\texttt{ClosedAndBlancedNeighborhoodBase} ::$

$:: \forall V \in k\text{-TVS} . \exists \mathcal{F} : \texttt{Filterbase}(V, \mathcal{U}_V(0)) . \forall F \in \mathcal{F} . \texttt{Closed} \ \& \ \texttt{Balanced}(V, F)$

Proof $=$

Pretty obvious.

□

$\texttt{LocallyConvexSpace} :: ?k\text{-}\mathsf{TVS}$

$$V : \texttt{LocallyConvexSpace} \iff \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{N}_V(0)\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{Convex}(F, \mathcal{F})$$

$\texttt{categoryOfLocallyConvexSpaces} :: \texttt{AbsoluteValueField}(\mathbb{R}) \to \mathsf{CAT}$

$\texttt{categoryOfLocallyConvexSpaces}\,(k) = k\text{-}\mathsf{LCS} :=$

$:= (\texttt{LocallyConvexSpace}(k), k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$

$\texttt{categoryOfTopologicalVectorSpaces} :: \texttt{AbsoluteValueField}(\mathbb{R}) \to \mathsf{CAT}$

$\texttt{categoryOfHausdorffTopologicalVectorSpaces}\,(k) = k\text{-}\mathsf{LCHS} :=$

$:= (\texttt{LocallyConvexSpace}(k)\ \&\ \texttt{T2}, k\text{-}\mathsf{VS} \cap \mathsf{TOP}, \circ, \mathrm{id})$

$\texttt{NormedSpaceIsLocallyConvex} :: \mathsf{NORM}(k) \subset k\text{-}\mathsf{LCHS}$

Proof $=$

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

$\square$

$\texttt{NormedSpaceIsLocallyConvex} :: \mathsf{NORM}(k) \subset k\text{-}\mathsf{LCHS}$

Proof $=$

Balls in normed spaces are convex.

Also they are metric space, hence Hausdorff.

$\square$

$\texttt{LCSHasDiscBase} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{Disc}(V, F)$

Proof $=$

Take $U \in \mathcal{N}_V(0)$.

Then there exists a convex neighborhood $C \in \mathcal{N}_V(0)$ with $C \subset U$ as V is locally convex.

Then there is $B \subset C$ which is a balanced neiborhood which was proved for all topological vector spaces.

Then $\operatorname{conv} B \subset C$ is convex and still an neighborhood of zero.

But convex hull of the balanced set is balanced,hence $\operatorname{conv} B$ is a disc .

$\square$

$\texttt{LCSHasOpenDiscBase} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{Disc}\ \&\ \texttt{Open}(V, F)$

Proof $=$

$\dots$

$\square$

$\texttt{LCSHasClosedDiscBase} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{N}_V(0), \mathcal{F}\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{Disc}\ \&\ \texttt{Closed}(V, F)$

Proof $=$

$\dots$

$\square$

VectorTopologyByAbsorbentAndBalancedSets ::

$$:: \forall V \in k\text{-}\mathsf{VS} \;.\; \forall \mathcal{F} : \mathtt{GroupFilterbase}(V) \;.\; \forall \aleph : \mathcal{F} \subset \mathtt{Balanced} \,\&\, \mathtt{Absorbent}(V) \;.\; \Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big) \in k\text{-}\mathsf{TVS}$$

Proof =

As $F \in \mathcal{F}$ is balanced, then $F = -F$, so $\langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ is a group topology for $(V, +)$.

Now assume $F \in \mathcal{F}$ and $\alpha \in k_*$.

Then there exists balanced $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ such that $0 \in U$ and $2U \subset U + U \subset F$.

Then there exists balanced $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ such that $0 \in U$ and $2U \subset U + U \subset F$.

This can be generalized to th case when $U \in \langle \mathcal{F} \rangle_{\mathsf{TGRP}}$ and $2^n U \subset F$.

So, we can take such U that $|\alpha| \leq 2^n$ and $\alpha U \subset 2^n U \subset F$ for any $\alpha \in k_*$ and $F \in \mathcal{F}$.

Now consider $\alpha \in k_*$, $v \in V$ and $F \in \mathcal{F}$.

There exists $U \in \mathcal{F}(0)$ such that $U + U + U \subset F$.

As U is absorbent there is $\rho \in (0, 1)$ such that $\mathbb{B}(0, \rho)v \subset U \subset F$.

Thus, $Cell(0, \rho)(v + U) = \mathbb{B}(0, \rho)v + \mathbb{B}(0, \rho)U = U + U \subset F$.

Now, assume $\alpha \neq 0$.

There is $U' \in \mathcal{F}$ such that $\alpha U' \subset U$.

Then there is also a $W \in \mathcal{F}$ such that $W \subset U' \cap U$.

Thus, $\mathbb{B}(\alpha, \rho)(v + W) = \alpha v + \alpha W + \mathbb{B}(0, \rho)(v + W) \subset \alpha v + U + U + U \subset \alpha v + F$.

This proves that scalar multiplication is continuous.

□

LocallyConvexTopologyByDiscFilterbase ::

$$:: \forall V \in k\text{-}\mathsf{VS} \;.\; \forall \mathcal{F} : \mathtt{Filterbase}(V) \;.\; \forall \aleph : \mathcal{F} \subset \mathtt{Disc} \,\&\, \mathtt{Absorbent}(V) \;.$$

$$. \forall \beth : \forall F \in \mathcal{F} \;.\; \exists \alpha \in (0, 1/2) \;.\; \alpha F \in \mathcal{F} \;.\; \Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big) \in k\text{-}\mathsf{LCS}$$

Proof =

We need to show that $\mathcal{F}$ is a group filterbase.

Assume $F \in \mathcal{F}$.

By assumption there are $\alpha \in (0, 1/2)$ such that $\alpha F \in \mathcal{F}$.

Then, as αF is convex and F is absorbent $\alpha F + \alpha F = 2\alpha F \subset F$.

Thus, by previous theorem $\Big(V, \langle \mathcal{F} \rangle_{\mathsf{TGRP}}\Big)$ is a topolofical vector space .

And it is locally convex as there is a filterbase consising of disks by construction.

□

1.1.4 Semimetrization

$\texttt{FSeminorm} :: \prod V \in k\text{-VS} \,.\, ?(V \to \mathbb{R}_+)$

$\sigma : \texttt{FSeminorm} \iff \Big(\forall \alpha \in \mathbb{D}_k(0,1) \,.\, \forall v \in V \,.\, \sigma(\alpha v) \leq \sigma(v)\Big) \,\&$

$\& \left(\forall v \in V \,.\, \lim_{n\to\infty} \sigma\left(\frac{v}{n}\right)\right) \,\&\, (\forall v, w \in V \,.\, \sigma(v+w) \leq \sigma(v) + \sigma(w))$

$\texttt{FNorm} :: \prod V \in k\text{-VS} \,.\, ?\texttt{FSeminorm}(V)$

$\sigma : \texttt{FNorm} \iff \forall v \in V \,.\, \sigma(v) = 0 \iff v = 0$

$\texttt{FSeminormSemimetrization} :: \forall V \in k\text{-VS} \,.\, \forall \sigma : \texttt{FSeminorm} \,.\, \exists \tau : \texttt{VectorTopology}(V) \,.\, \sigma \in C(V, \tau)$

$\texttt{Proof} =$

I will show that σ is a value.

Firstly, note that $\sigma(-v) \leq \sigma(v)$ and $\sigma(v) \leq \sigma(-v)$, so $\sigma(v) = \sigma(-v$.

Also $\sigma(0) = \sigma\left(\frac{0}{n}\right) \to 0$, so $\sigma(0)$.

Other properties of value follows trivially by commutativity of $+_V$.

Now I show that scalar multiplication is continuous in topology defined by semimetric $\rho(v,w) = \sigma(v-w)$.

There are neighborgoods of zero defined by relation $\sigma(v) < \varepsilon$.

By first property of F-seminorm these balls are ballanced .

And by second property of F-seminorm these balls are absorbent.

So produced topology of ρ is a vector space topplogy.

$\square$

$\texttt{FNormSemimetrization} :: \forall V \in k\text{-VS} \,.\, \forall \sigma : \texttt{FNorm} \,.\, \exists \tau : \texttt{VectorTopology}(V) \,.\, \sigma \in C(V, \tau) \,\&\, \texttt{T2}(V, \tau)$

$\texttt{Proof} =$

In this case ρ is a metric, so resulting topology musy be Hausdorff.

$\square$

$\texttt{subspaceSeminorm} :: \prod V \in k\text{-VS} \,.\, \prod U \subset_{k\text{-VS}} V \,.\, \texttt{FSeminorm}(V) \to \texttt{FSeminorm}\left(\frac{V}{U}\right)$

$\texttt{subspaceSeminorm}\,(\sigma) = [\sigma]_U := \Lambda[v] \in \frac{V}{U} \,.\, \inf_{u\in U} \sigma(v+u)$

$\texttt{SubspaceSemimetrization} :: \forall V \in k\text{-TVS} \,\&\, \texttt{Semimetrizable} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \texttt{Semimetrizable}\left(\frac{V}{U}\right)$

$\texttt{Proof} =$

...

$\square$

1.1.5 Completion

$$\texttt{Completion} :: \prod_{V \in k\text{-TVS}} ? \sum_{W \in k\text{-TVS}} \texttt{TopologicalEmbedding}(V, W)$$

$$(W, \iota) : \texttt{Completion} \iff \texttt{Complete}(V)\ \&\ \texttt{Dense}\Big(W, \iota(V)\Big)$$

$\texttt{EveryTVSHasACompletion} :: \forall V \in k\text{-TVS} . \exists \texttt{Completion}(V)$

$\texttt{Proof} =$

As with topological Groups.

$\square$

$$\texttt{TopologicalVectorSpaceSubset} :: \prod_{V \in k\text{-TVS}} ??V$$

$$U : \texttt{TopologicalVectorSpaceSubset} \iff U \subset_{k\text{-TVS}} V \iff U \subset_{k\text{-VS}} V\ \&\ \texttt{Closed}(V, U)$$

$$\texttt{CompleteteQuotient} :: \forall V \in k\text{-TVS} . \forall U \subset k\text{-TVS}V . \texttt{Complete}(V) \Rightarrow \texttt{Complete}\left(\frac{V}{U}\right)$$

$\texttt{Proof} =$

As with topological groups.

$\square$

$\texttt{BalancedHullOfTotallyBoundedIsTotallyBounded} ::$

$:: \forall V \in k\text{-TVS} . \forall B : \texttt{TotallyBounded}(V) . \texttt{TotallyBounded}(V, \operatorname{bal} B)$

$\texttt{Proof} =$

Embed B in a completion of $\hat{V}$ of V.

Then $\operatorname{cl} B$ is a compact in $\hat{V}$.

As $\mathbb{D}_k(0,1)$ is comapct in k, then $\mathbb{D}_k(0,1)\operatorname{cl}_{\hat{V}} B$ is compact is continuous image of compact $\mathbb{D}_k(0,1) \times \operatorname{cl}_{\hat{V}} B$.

Then $\operatorname{bal} B = \mathbb{D}_k(0,1)B$ is totally bounded as a subset of compact $\mathbb{D}_k(0,1)\operatorname{cl}_{\hat{V}} B$.

$\square$

$\texttt{BalancedHullOfCompactIsCompacts} ::$

$:: \forall V \in k\text{-TVS} . \forall K : \texttt{CompactSubset}(V) . \texttt{CompactSubset}(V, \operatorname{bal} K)$

$\texttt{Proof} =$

$\mathbb{D}_k(0,1)K$ is compact as am image of compact $\mathbb{D}_k(0,1) \times K$.

$\square$

ConvexHullofTotallyBoundedAsTotallyBounded ::

:: $\forall V \in k\text{-}\mathsf{LCS} \,.\, \forall B : \texttt{TotallyBounded}(V) \,.\, \texttt{TotallyBounded}(V, \operatorname{conv} B)$

Proof =

In order to show that $\operatorname{conv} B$ is totally bounded we need to show that $convB$ can be covered by finite number of translates $(U + v_i)_{i=1}^n$ for any $U \in \mathcal{U}_V(0)$.

Select disc $D \in \mathcal{U}_V(0)$ such that $D + D \subset U$.

This is possible as V is loclly convex.

As K totally bounded there are a finite set of translates such that $K \subset (D + v_i)_{i=1}^n \subset \operatorname{conv}\{v_1, \ldots, v_n\} + D$.

As sum of convex sets is convex $\operatorname{conv} K \subset \operatorname{conv}\{v_1, \ldots, v_n\} + D$.

As $\operatorname{conv}\{v_1, \ldots, v_n\}$ is compact it is possible to select a finite set of m translates u_i of D such that

$\operatorname{conv} K \subset \bigcup_{i=1}^{m} (D + u_i)$.

So $\operatorname{conv} K$ is totally bounded.

□

ConvexHullofTotallyBoundedAsTotallyBounded ::

:: $\forall V \in k\text{-}\mathsf{LCS}\texttt{Complete} \,.\, \forall K : \texttt{CompactSubset}(V) \,.\, \texttt{CompactSubset}(V, \operatorname{conv} K)$

Proof =

$\operatorname{conv} K$ is closed.

And as it was shown in the previous theorem $\operatorname{conv} K$ is also totally bounded, hence compact.

□

1.1.6 Continuous Decompositions

$\texttt{TopologicalComplement} :: \prod V : k\text{-TVS} \,.\, ?\texttt{LinearComplement}(V)$

$(U,W) : \texttt{TopologicalComplement} \iff V =_{k\text{-TVS}} U \oplus W \iff$

$\iff \texttt{Homeomorphism}\Big(U \oplus W, V, \Lambda(u,w) \in U \oplus W \,.\, u + w\Big)$

$\texttt{TopologicalComplementsByContinuousProjection} ::$

$:: \forall V \in k\text{-TVS} \,.\, \forall U, W : \texttt{LinearComplement}(V) \,.\, U \oplus W =_{k\text{-TVS}} V \iff P_{U,W} \in \mathrm{End}_{\mathsf{TOP}}(V)$

$\texttt{Proof} =$

Define $T : U \oplus W \to V$ by $T(u,w) = u + w$.

$(\Rightarrow)$: Assume that T is a homeomorphism.

There is an expression $P_{U,W} = T^{-1} P_1 I_U$, where $P_1 : U \oplus W \to U$ is a projection,

and $I_U : U \to V$ is a natural embedding.

Thus, $P_{U,W}$ is continuous as composition of continuous functions.

$(\Leftarrow)$:Assume $(\Delta, u_\delta + w_\delta)$ is a net in V converging to 0 .

Then by continuity $0 = P_{U,W}(0) = P_{U,W}(\lim_{\delta \in \Delta} u_\delta + w_\delta) = \lim_{\delta \in \Delta} P_{U,W}(u_\delta + w_\delta) = \lim_{\delta \in \Delta} u_\delta$.

Also $E - P_{U,W} = P_{W,U}$ is continuous.

So by the argument similar to one above $\lim_{\delta \in \Delta} w_\delta = 0$.

Thus, $\lim_{\delta \in \Delta}(u_\delta, w_\delta) = 0$ and T^{-1} is continuous meaning that T is homeomorphism.

$\square$

$\texttt{TopologicalComplementsByIsomorphicQuotient} ::$

$$:: \forall V \in k\text{-TVS} \,.\, \forall U, W : \texttt{LinearComplement}(V) \,.\, U \oplus W =_{k\text{-TVS}} V \iff \texttt{Homeomorphism}\left(W, \frac{V}{U}, \pi_{U|W}\right)$$

$\texttt{Proof} =$

π_U is a quotient map, and hence continuous.

$(\Rightarrow)$:Assume $(\Delta, [U + w_\delta])$ is a net in $\dfrac{V}{U}$ converging to zero.

But this means that $\lim_\delta w_\delta = 0$ and $\lim_\delta \pi^{-1}_{U|W}[U + \mathbf{w}_\delta] = \lim_\delta w_\delta = 0$.

So $\pi_{U|W}$ is homeomorphism.

$(\Leftarrow)$: write $P_{U,W} = \pi_U \pi^{-1}_{U|W} I_W$.

This is continuous a as composition of continuous functions.

So by the previous theorem $V = U \oplus_{k\text{-TVS}} W$.

$\square$

$\texttt{ComplementedImpliesClosed} :: \forall V \in k\text{-TVS} \forall (U,W) : \texttt{TopologicalComplement}(V) \,.\, \texttt{Closed}(V,U)$

$\texttt{Proof} =$

By previous theorem $P_{W,U}$ is continuous.

Thus, $U = \ker P_{W,U}$ is closed.

$\square$

$\texttt{MaximalSubspace} :: \prod_{V \in k\text{-VS}} ?\texttt{VectorSubspace}(V)$

$U : \texttt{MaximalSubspace} \iff \forall W \subset_{k\text{-VS}} V \;.\; U \subsetneq W \Rightarrow W = V$

$\texttt{MaximalClosedSubspace} ::$

$:: \forall V \in k\text{-TVS} \;.\; \forall U \subset_{k\text{-VS}} V \;.$

$. \;\texttt{MaximalSubspace} \;\&\; \texttt{Closed}(V, U) \iff \exists f \in \mathsf{TOP}(V, k) \;.\; U = \ker f \;\&\; f \neq 0$

$\texttt{Proof} \;=$

$(\Rightarrow)$: Assume U is closed and maximal subspace in V.

As U is maximal it should have a codimension 1.

So where exists $v \in U^{\complement}$ such that $V = U \oplus \langle v \rangle$.

As U is closed, where exists a balanced open subset $O \in \mathcal{U}_V(0)$ such that $(O + v) \cap U = \emptyset$.

assume $u + \alpha v \in O$ is such that $|\alpha| > 1$ and $u \in U$.

Then, as O is balanced, $\alpha^{-1}u + v \in O$.

But, then $(\alpha^{-1}u + v) - v = \alpha^{-1}u \in (O + v) \cap U$, which is a contradiction.

Thus, $u + \alpha t \in \sigma O$ implies that $|\alpha| < |\sigma|$.

Define $f(u + \alpha v) = \alpha : V \to k$.

Consider a net $v_\delta = u_\delta + \alpha_\delta v$ converging to zero with u_δ in U.

But the previous remark shows that $f(v_\delta) = \alpha_\delta$ converges to zero.

$\texttt{SchroederBernsteinTHM} ::$

$:: \forall V, V' \in k\text{-TVS} \;.\; \forall \aleph : V \cong_{k\text{-TVS}} V \oplus V \;.\; \forall \beth : V' \cong_{k\text{-TVS}} V' \oplus V' \;.$

$. \;\forall \gimel : \texttt{TopologicalComplement}(V, V') \;.\; \forall \daleth : \texttt{TopologicalComplement}(V', V') \;.\; V \cong_{k\text{-TVS}} V'$

$\texttt{Proof} \;=$

Write $V \cong V' \oplus U = (V' \oplus V') \oplus U \cong V' \oplus (V' \oplus U) \cong V' \oplus V$.

Symmetricaly, $V' \cong V' \oplus V$.

Thus, $V \cong V \oplus V' \cong V'$.

$\square$

1.1.7 Finite Dimension Conditions

OneDimTVS :: $\forall V \in k\text{-HTVS} \;.\; \dim V = 1 \iff V \cong_{k\text{-TVS}} k$

Proof $=$

As dimension is invarint for linear isomorphism $(\Leftarrow)$ is obvious .

$(\Rightarrow)$: As $\dim V = 1$ there is a $v \in V$ such that $v \neq 0$ and $V = kv$.

Then the map $T(\alpha v) = \alpha$ is a linear isomorphism .

fix some $\rho \in \mathbb{R}_{++}$.

As V is Hausdorff there must exist an open set $U \in \mathcal{U}_V(0)$ such that $\rho v \notin U$.

Furthermore, U must have a balanced subset $W \in \mathcal{U}_V(0)$.

As W is balanced, $W \subset \mathbb{B}(0, \rho)v$.

So, $\alpha_\delta v \to 0 \iff \alpha_\delta \to 0$.

Thus, T must be a homeomorphism.

$\square$

FinDimIsomorphism ::

$\forall V \in k\text{-HTVS} \;.\; \forall n \in \mathbb{N} \;.\; \dim V = n \iff V \cong_{k\text{-TVS}} (k^n, \|\bullet\|_\infty)$

Proof $=$

I modify the proof of the previous theorem.

By algebraic there must exist a base $\mathbf{e} = (e_1, \ldots, e_n)$ of V .

fix ρ in $\mathbb{R}_{++}$.

As V is Hausdorff and each $e_i \neq 0$ there $U \subset \mathcal{U}_V(0)$ such $\rho e_i \notin U$ for any $i \in \{1, \ldots, n\}$.

So there exists a blanced subset W of U such that $W \subset \mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \rho) \cdot \mathbf{e}$.

Thus, the mapping $\alpha \cdot \mathbf{e} \mapsto \alpha$ is continuous.

Also, if $U \in \mathcal{U}_V(0)$ the set U must be absorbent,

so there is a sequence $\rho_1, \ldots, \rho_n \in \mathbb{R}_{++}$ such that $\mathbb{D}_k(0, \rho_i)e_i \subset U$.

Let $\sigma = \min(e_1, \ldots, e_n) \in \mathbb{R}_{++}$.

Then $\mathbb{B}_{k^n, \|\bullet\|_\infty}(0, \sigma) \cdot \mathbf{e} \subset U$.

So, the inverse $\alpha \mapsto \alpha \cdot \mathbf{e}$ is also continuous.

$\square$

FDimdSubspaceIsClosed :: $\forall V \in k\text{-HTVS} \;.\; \forall U \subset_{k\text{-VS}} V \;.\; \dim U < \infty \Rightarrow \mathtt{Closed}(V, U)$

Proof $=$

U is Hausdorff as a subset of Hausdorff space.

Then U is isomorphic to $\ell^\infty_{k, \dim U}$ which is complete.

So, U can be viewed as an uniform embedding of complete space into V, and hence must be closed.

$\square$

`ClosedFDimSum` :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall U \subset_{k\text{-}\mathsf{TVS}} V \,.\, \forall W \subset_{k\text{-}\mathsf{VS}} V \,.\, \dim W < \infty \Rightarrow \mathtt{Closed}(V, U + W)$

`Proof` =

As U is closed in V the quotient $\dfrac{V}{U}$ must be Hausdorff.

As $\dim P_U(W) \leq \dim \dim W$ the image $P_U(W)$ is still finite dimensional.

So by previous theorem $P_U(W)$ is closed in $\dfrac{V}{U}$.

But then the preimage $U + W = P_U^{-1} P_U(W)$ is closed as quotient map P_U is continuous.

□

`FiniteDimensionalDomain` :: $\forall V, U \in k\text{-}\mathsf{HTVS} \,.\, \forall T \in k\text{-}\mathsf{VS}(V, U) \,.$

$. \dim V < \infty \Rightarrow T \in k\text{-}\mathsf{TVS}(V, U)$

`Proof` =

$\dim T(V) \leq \dim V$, thus $T(V)$ must be finite dimensional.

Thus both V and $T(V)$ are isomorphic to copies of l_k^∞ with coresponding finite dimensions.

And T must be continuous as any mapping between such spaces does.

`FiniteDimensionalCodomain` :: $\forall V, U \in k\text{-}\mathsf{HTVS} \,.\, \forall T \in k\text{-}\mathsf{TVS} \,\&\, \mathtt{Surjective}(V, U) \,.$

$. \dim U < \infty \Rightarrow \mathtt{Open}(V, U, T)$

`Proof` =

By isomorphism theorem $\dfrac{V}{\ker T} \cong_{k\text{-}\mathsf{VS}} T(V) = U$.

So $\dim \dfrac{V}{\ker T} < \infty$.

Also $\dfrac{V}{\ker T}$ is Haussdorf as T is continuous .

So by prvious theorem the isomorphism must $\dfrac{V}{\ker T} \cong_{k\text{-}\mathsf{VS}} U$ must be continuous.

So U is also finite dimensional Hausdorff this bijection is homeomorphism and so $\dfrac{V}{\ker T} \cong_{k\text{-}\mathsf{TVS}} U$.

Denote this homeomorphism by S.

Then T factors as $P_{\ker T} S$ and both these maps are open.

□

FDimIffLocallyCompact :: $\forall V \in k\text{-HTVS} .\ \dim V < \infty \iff$ LocallyCompact(V)

Proof $=$

$(\Rightarrow)$: V is homeomorphic to $l^\infty_{k,\dim V}$ and this space is locally compact..

This can be easily shown by considering a base of closed cubes.

So V is locally compact.

$(\Leftarrow)$: now consider the case when V is locally compact.

Then there exists a compact balansed neighborhood of zero, say K.

Take K to be any another open neighborhood and choose $W \in \mathcal{U}_V(0)$ such balanced set that $W + W \subset U$.

As K is compact, it is totally bounded and hence can be covered by a finite set of translates $K \subset \bigcup_{i=1}^{n} W + v_i$.

As W is absorbent and balanced there is $\rho \in (1, +\infty)$ such that each $v_i \in \rho U$.

Then $K \subset \bigcup_{i=1}^{n} W + v_i \subset W + \rho W \subset \rho W + \rho W = \rho(W + W) \subset \rho U$.

Thus, sets of form $2^{-n}K$ form base at zero.

As K is totally bounded it can can be covered by a finite set of translates $K \subset \bigcup_{i=1}^{n} \frac{1}{2}K + e_i$.

$F = \operatorname{span} e$ is finite-dimensional and hence closed.

$K \subset \bigcup n_{i=1} \frac{1}{2}K + e_i \subset \frac{1}{2}K + F$.

But also $\alpha F = F$ or any non-zero scalar α.

So $\frac{1}{2}K \subset \frac{1}{4}K + F$.

Iterating this relation and substituting we get the result that $K \subset \frac{1}{2^n}K + F$ for any $n \in \mathbb{N}$.

This can be rewriten as $K \subset \bigcap_{n=1}^{\infty} \frac{1}{2^n}K + F = F$.

But K spans whole V, and so $V = F$ which is finite dimensional.

$\square$

FDimCompactConvexHullIsCompact ::

:: $\forall V \in k\text{-TVS} .\ \forall K :$ CompactSubset(V) . $\dim V < \infty \Rightarrow$ CompactSubset$(V, \operatorname{conv} K)$.

Proof $=$

Let $n = \dim V$.

$\operatorname{conv} K$ consists of convex combination of form $\sum_{i=1}^{2n+1} \lambda_i x_i$ where $\lambda \geq 0$ and $\sum_{i=1}^{2n+1} \lambda_i = 1$ and $x_i \in K$.

This condition can be express as $\lambda \in \triangle_{2n+1} \subset k^{2n+1}$.

But $\triangle_{2n+1}$ is also compact, ans so is $\triangle_{2n+1} \times K^{2n+1}$ by Tychonoff's theorem.

So $\operatorname{conv} K = (\cdot)(\triangle_{2n+1} \times K^{2n+1})$ is compact as a continuous image of a compact.

$\square$

1.1.8 Case of Ultravalued Field

k : `UltravaluedField`;

`AbsolutelyKConvex` :: $\prod_{V:k\text{-TVS}} ??V$

A : `AbsolutelyKConvex` $\iff \mathbb{D}_k(0,1)A + \mathbb{D}_k(0,1)A = A$

`KConvex` :: $\prod_{V:k\text{-TVS}} ??V$

V : `KConvex` $\iff \exists v \in V \; . \; \exists A :$ `AbsolutelyKConvex`$(V) \; . \; C = A + v$

`AbsolutelyKConvexByZeroContaintment` :: $\forall V \in k\text{-TVS} \; . \; \forall C :$ `KConvex`$(V) \; . \; 0 \in C \Rightarrow$ `AbsolutelyKConvex`$(V$

`Proof` =

C must be a translate of absolutely K-Convex set, so write $C = A + v$.

As A is absolutely K-Convex, then $\alpha(x+v) + \beta(y+v) - v \in C$ for any $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

Take $\alpha = \beta = 1, y = 0$.

Then the expression above reduces to $x + v \in C$.

But this means that $A \subset C$.

On the other hand, $\alpha(x+v) + \beta(y+v) \in A$ for any $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

Taking $\alpha = 1, \beta = -1, y = 0$, produces $x \in A$.

Thus $C \subset A$ and $C = A$ is absolutely K-convex.

□

`TripleCombinationKConvexityCondition` ::

:: $\forall V \in k\text{-TVS} \; . \; \forall C \subset V$.

. `KConvex`$(V, C) \iff \forall x, y, z \in C \; . \; \forall \alpha, \beta, \gamma \in \mathbb{D}_k(0,1) \; . \; \alpha + \beta + \gamma = 1 \Rightarrow \alpha x + \beta y + \gamma z \in C$

`Proof` =

1 $(\Rightarrow)$: assume that C is K-convex.

1.1 C must be a translate of absolutely K-Convex set, so write $C = A + v$.

1.2 Then $\alpha x + \beta y + \gamma z = \alpha(x - v) + \beta(y - v) + \gamma(z - v) + v \in C$.

2 $(\Leftarrow)$.

2.1 If $C = \emptyset$ then it is trivially K-convex, so assume the contrary.

2.2 Take $v \in V$ and let $A = C - v$.

2.3 A is absolutely K-convex.

2.3.1 Assume $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

2.3.2 $1 - \alpha - \beta \in \mathbb{D}_k(0,1)$.

2.3.2.1 $|1 - \alpha - \beta| \leq \max\Big(1, |\alpha|, |\beta|\Big) = 1$.

2.3.3 Then by the hypothesis $\alpha x + \beta y + (1 - \alpha - \beta)v \in C$.

2.3.4 Translating by $-v$ gives $\alpha(x - v) + \beta(y - v) = \alpha x + \beta y + (1 - \alpha - \beta)v - v \in A$.

□

convexCombinationKConvexityCondition ::

:: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall \aleph : \text{res char } k \neq 2 \,.\, \forall C \subset V \,.$

$. \,\texttt{KConvex}(V, C) \iff \forall x, y \in C \,.\, \forall \alpha \in \mathbb{D}_k(0,1) \,.\, \alpha x + (1-\alpha)y + \gamma z \in C$

Proof $=$

1 ($\Rightarrow$) This direction is obvious.

1.1 The convex combination is a weaker form of triple combination in the previous result.

2 ($\Leftarrow$) .

2.1 If $C = \emptyset$ then it is trivially K-convex, so assume the contrary.

2.2 Take $v \in V$ and let $A = C - v$.

2.3 A is absolutely K-convex.

2.3.1 Assume $x, y \in C$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

2.3.2 Rewrite $\alpha(x-v) + \beta(y-v) + v = \frac{1}{2}\big(2\alpha x + (1-2\alpha)v\big) + \frac{1}{2}\big(2\beta y + (1-2\beta)v\big)$.

2.3.3 Both $\frac{1}{2}\big(2\alpha x + (1-2\alpha)v\big)$ and $\frac{1}{2}\big(2\beta y + (1-2\beta)v\big)$ in C.

2.3.3.1 for ultravalue $|2\alpha| = |\alpha + \alpha| \leq |\alpha| = 1$.

2.3.3.2 Same holds for β.

2.3.3.3 So the convex combination hypothesis can be applied.

2.3.4 clearlly $\frac{1}{2} + \frac{1}{2} = 1$, so $\alpha(x-v) + \beta(y-v) \in A$.

2.3.4.1 $\left|\frac{1}{2}\right| = 1$ as residual characteristic of the field is not 2.

□

AbsolutelyKConvexIntersection :: $\forall V : k\text{-}\mathsf{TVS} \,.\, \forall I \in \mathsf{SET} \,.$

$. \,\forall A : I \to \texttt{AbsolutelyKConvex}(V) \,.\, \texttt{AbsolutelyKConvex}\left(V, \bigcap_{i \in I} A_i\right)$

Proof $=$

Obvious.

□

$\texttt{KConvexIntersection} :: \forall V : k\text{-}\mathsf{TVS} \,.\, \forall I \in \mathsf{SET} \,.$

$$. \forall C : I \to \texttt{KConvex}(V) \,.\, \texttt{KConvex}\left(V, \bigcap_{i\in I} C_i\right)$$

$\texttt{Proof} =$

1 Assume that $\bigcap_{i\in I} C_i \neq \emptyset$.

1.1 Otherwise the condition is trivial.

2 Take any $v \in \bigcap_{i\in I} C_i$.

3 Then $\left(\bigcap_{i\in I} C_i\right) - v$ is absolutely K-convex and $\bigcap_{i\in I} C_i$ is K-convex.

3.1 $\left(\bigcap_{i\in I} C_i\right) - v = \bigcap_{i\in I}(C_i - v)$ as translation by v is bijective.

3.2 Then every $C_i - v$ are K-convex sets, which contain zero, so they are absolutely K-Convex.

3.3 So, the intersection $\bigcap_{i\in I}(C_i - v)$ is also absoluterly K-Convex.

□

$\texttt{kConvexHull} :: \prod_{V:k\text{-}\mathsf{TVS}} (?V) \to \texttt{KConvex}(V)$

$\texttt{kConvexHull}\,(X) = K\text{-conv } X := \bigcap \left\{C : \texttt{KConvex}(V), X \subset C\right\}$

$\texttt{KConvexHullByLinearCombinations} ::$

$$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall X \subset V \,.$$

$$. K\text{-conv } X = \left\{ x_{n+1} + \sum_{i=1}^{n} \alpha_i (x_i - x_{n+1}) \middle| n \in \mathbb{Z}_+, \alpha : \{1,\dots,n\} \to \mathbb{D}_k(0,1), x : \{1,\dots,n+1\} \to X \right\}$$

$\texttt{Proof} =$

1 Let B denote the set defined above.

2 B is K-Convex.

2.1 Note, that x_{n+1} in definition can be fixed.

2.2 Then $B - x_{n+1}$ is obviously absolutely K-convex.

3 $X \subset B$.

3.1 Just take $n = 1, \alpha_1 = 1$.

4 So K-conv $X \subset B$.

5 If C is K-convex, then $B \subset C$.

5.1 Some $x_{n+1} \in X$ must also be contained in C .

5.2 So $C - x_{n+1}$ is absolutely K-convex. .

5.3 So by induction $\sum_{i=1}^{n} \alpha_i (x_i - x_{n+1}) \in C - x_{n+1}$.

6 Thus, $B \subset K$-conv X, and so $B = K$-conv X .

□

$\texttt{kDiskHull} :: \prod_{V:k\text{-TVS}} (?V) \to \texttt{AbsolutelyKConvex}(V)$

$\texttt{kDiscHull}\,(X) = K\text{-disc}\,X := \bigcap \Big\{ C : \texttt{AbsolutelyKConvex}(V), X \subset C \Big\}$

$\texttt{AbsolutelyKConvexInterior} :: \forall V : k\text{-TVS} \,.\, \forall A : \texttt{AbsolutelyKConvex}(V) \,.\, \operatorname{int} A = \emptyset | \operatorname{int} A = A$

Proof $=$

1 assume $\operatorname{int} A \neq \emptyset$.

2 Take $v \in \operatorname{int} A$.

3 Without loss of generality assume $v = 0$.

3.1 Then $A - v$ is an isomorphic absolutely convex set with $0 \in \operatorname{int} A$.

4 Take any $U \in \mathcal{U}_V(0)$ such that $U \subset \operatorname{int} A \subset A$.

5 Now take arbitrary $v \in A$.

6 Then $U + v \subset A$.

6.1 $U + v$ consists of elements $u + v$ with $u \in U \subset A$.

6.2 As $v \in A$ also and A is absolutely K-convex it must be the case that $u + v \in A$.

7 As translation is a homeomorphism $U + v$ is open and so $v \in \operatorname{int} A$.

$\square$

$\texttt{OpenKDiscHull} :: \forall V : k\text{-TVS} \,.\, \forall U : \texttt{Open}(V) \,.\, \texttt{Open}(V, K\text{-disc}\,U)$

Proof $=$

1 K-disc U is absolutely K-convex.

2 $U \subset K$-disc U, so $\operatorname{int} K$-disc $U \neq 0$.

3 But this means that K-disc U is open.

$\square$

$\texttt{LocallyKConvexSpace} :: ?k\text{-TVS}$

$V : \texttt{LocallyKConvexSpace} \iff \exists \mathcal{F} : \texttt{Filterbase}\Big(V, \mathcal{U}_V(0)\Big) \,.\, \forall F \in \mathcal{F} \,.\, \texttt{KConvex}(V, F)$

$\texttt{NonarchimedeanVSHasZeroTopDim} :: \forall V : \texttt{LocallyKConvexSpace}(k)\ \&\ \texttt{T2}\ .\ \dim_{\mathsf{TOP}} V = 0$

$\texttt{Proof}\ =$

1 V has a base of closed K-discs.

1.1 Consider $U \in \mathcal{U}_V(0)$.

1.2 Then there exists an open K-disic D such that $0 \in D \subset \overline{D} \subset U$.

1.3 Then $\overline{D}$ is a K-disk.

1.3.1 If $u, v \in \overline{D}$ it means that every their open neighborhood meet D .

1.3.2 Assume $\alpha, \beta \in \mathbb{D}_k(0,1)$.

1.3.3 Consider an open neighborhood W of $\alpha u + \beta v$.

1.3.4 Then there is an open neighborhood of zero $O + O \subset W - \alpha u - \beta v$.

1.3.5 Consider the case $\alpha \neq 0 \neq \beta$.

1.3.6 Then there must be some $u' \in D \cap \dfrac{1}{\alpha}(O + \alpha u)$.

1.3.7 Then there is also $v' \in D \cap \dfrac{1}{\beta}(O + \beta v)$.

1.3.8 Then $\alpha u' + \beta v' \in D$ as D is absoluterly K-convex.

1.3.9 Also $\alpha u' + \beta v' \in O + O + \alpha u + \beta v \subset W$.

1.3.10 As W was arbitrary this means that $\alpha u + \beta v \in \overline{D}$.

1.4 $\overline{D} \subset U$.

1.4.1 This is true as V is Hausdorff, and Hence regular.

2 But then every K-disc in this base is clopen.

2.1 To be in base every K-disc D should contain an element of $U_V(0)$.

2.2 Hence D has non-empty interior.

2.3 But This means that D is open.

3 Thus $\dim_{\mathsf{TOP}} V = 0$.

$\square$

$\texttt{RelativelyKConvex} :: \prod_{V_k\text{-}\mathsf{TVS}} \prod_{A \subset V} ??A$

$R : \texttt{RelativelyKConvex} \iff \exists C : \texttt{KConvex}(K)\ .\ R = C \cap A$

$\texttt{KConvexFilterbase} :: \prod V : k\text{-}\mathsf{TVS}\ .\ \prod_{A \subset V} ?\texttt{Filterbase}(A)$

$\mathcal{F} : \texttt{KConvexFilterbase} \iff \forall F \in \mathcal{F}\ .\ \texttt{RelativelyKConvex}(V, A, F)$

$\texttt{CCompact} :: \prod_{V_k\text{-}\mathsf{TVS}} ??V$

$K : \texttt{CCompact} \iff \forall \mathcal{F} : \texttt{KConvexFilterbase}(V, K)\ .\ \exists \texttt{AdherencePoint}\Big(V, \mathcal{F}\Big)$

$|\cdot| \neq \Lambda \alpha \in k\ .\ [\alpha \neq 0]$

$\texttt{EveryCompactIsCCompact} :: \forall V : k\text{-TVS} \ . \ \forall K : \texttt{Compact}(V, K) \ . \ \texttt{CCompact}(V, K)$

$\texttt{Proof} \ =$

1 Assume $\mathcal{F}$ is a K-Convex filterbase on K.

2 Then associated ultrafilter must have a limit.

3 This limit is an adherence point of $\mathcal{F}$.

□

$\texttt{ClosedSubsetOfCCompact} :: \forall V : k\text{-HTVS} \ . \ \forall K : \texttt{CCompact}(V) \ . \ \forall L : \texttt{Closed}(K) \ \& \ \texttt{KConvex}(V) \ .$
$. \ \texttt{CCompact}(V, L)$

$\texttt{Proof} \ =$

1 Assume $\mathcal{F}$ is a K-Convex filterbase on L.

2 Then the $\mathcal{F}$ is also a K-Convex filterbase for K.

3 Then, there is an adherence poiint $p \in K$ fo $\mathcal{F}'$.

4 p is also an adherence point for $\mathcal{F}$.

4.1 Take any $U \in \mathcal{U}_V(p)$.

4.2 Then $F \cap K \cap U \neq \emptyset$ for any $F \in \mathcal{F}$.

4.3 Bat all these $F \subset L$.

4.4 Thus $p \in \operatorname*{cl}_K L = L$.

□

$\texttt{MaximalConvexFilterbase} ::$

$:: \forall V : \texttt{LocallyKConvexSpace}(k) \ . \ \forall C : \texttt{KConvex}(V) \ . \ \forall \mathcal{F} \in \max \texttt{KConvexFilterbase}(V, C) \ .$
$. \ \forall p \in \mathcal{C} \ . \ \texttt{AherencePoint}(C, \mathcal{F}, p) \iff \lim \mathcal{F} = p$

$\texttt{Proof} \ =$

1 $(\Rightarrow)$: Assume p is an adherence point for $\mathcal{F}$ in $\mathcal{C}$.

1.1 Then $\forall F \in \ . \ \forall U \in \mathcal{U}_V(p) \ . \ U \cap F \neq \emptyset$.

1.2 Assume that $U \in \mathcal{U}_C(p)$.

1.3 Then there exist a K-convex D and open $W \in \mathcal{U}_C(p)$ such that $W \subset D \subset V$.

1.4 Then $\forall F \in \mathcal{F} \ . \ D \cap F \neq \emptyset$.

1.4.1 $\forall F \in \mathcal{F} \ . \ W \cap F \neq \emptyset$.

1.4.2 $W \subset D$.

1.5 As $\mathcal{F}$ is maximal $D \in \mathcal{F}$.

1.6 Thus, $p = \lim \mathcal{F}$.

2 $(\Leftarrow)$: Now Assume $p = \lim \mathcal{F}$.

2.1 Then $\forall U \in \mathcal{U}_C(p) \ . \ \exists F \in \mathcal{F} \ . \ F \subset U$.

2.2 Take arbitrary $U \in \mathcal{U}_C(p)$ and $F \in \mathcal{F}$.

2.3 Then by (2.1) there exits $G \in \mathcal{F}$ such that $G \subset Y$.

2.4 As $\mathcal{F}$ is a filterbase $G \cap F \neq \emptyset$.

2.5 Thus $F \cap U \neq \emptyset$.

2.6 This proves that p is and adherence point for $\mathcal{F}$.

□

$\texttt{KConvexAndCcompactIsClosed} ::$

$:: \forall V : \texttt{LocallyKConvexSpace}(k) \,.\, \forall K : \texttt{CCompact} \,\&\, \texttt{KConvex}(V) \,.\, \texttt{Closed}(V, K)$

$\texttt{Proof} =$

1 Assume p is a Limit point for K.

2 Then there exists an filter $\mathcal{F}$ in K such that $p = \lim \mathcal{F}$.

2.1 Take $\mathcal{N}_V(p) \cap K$ for example.

3 Then p is an adherence point of $\mathcal{F}$.

4 construct a K-convex filterbase $\mathcal{C}$ from $\mathcal{F}$.

4.1 For example, use the fact that V is locally K-convex.

4.2 Let C be the intersections of K and K-convex neighborhoods of p.

5 Then p is still a limit point of $\mathcal{C}$ in V.

6 There also must exist an adherence point of $\mathcal{C}$ in K, say q.

7 But as V is Hausdorff and $\mathcal{C}$ has a limit it must be the case $q = p$.

8 Thus K has all its limit points and must be closed.

$\square$

$\texttt{CCompactProduct} :: \forall I \in \texttt{Set} \,.\, \forall V : I \to k\text{-}\mathsf{TVS} \,.\, \forall C : \prod_{i \in I} \texttt{CCompact}(V_i) \,.\, \texttt{CCompact}\left(\prod_{i \in I} V_i, \prod_{i \in I} C_i\right)$

$\texttt{Proof} =$

Same proof as Tychonoff's theorem's proof with filters, but with k-convex sets.

$\square$

$\texttt{CCompactCombination} :: \forall V : \texttt{LocallyKConvexSpace}k \,.\, \forall n \in \mathbb{Z}_+ \,.$

$. \forall D : \{1, \ldots, n\} \to \texttt{AbsolutelyKConvex} \,\&\, \texttt{CCompact}(V) \,.\, \texttt{CCompact}\left(V, K\text{-conv} \bigcup_{i=1}^{n} D_i\right)$

$\texttt{Proof} =$

1 I will give a proof by induction.

2 K-conv $\bigcup_{i=1}^{n} D_i = \emptyset$ in case $n = 0$ and is trivially c-compact.

3 K-conv $\bigcup_{i=1}^{n+1} D_i = K$-conv $\left(D_{n+1} + \bigcup_{i=1}^{n} D_i\right)$ by the result expressing K-convex hulls by linear combinations.

4 So for the induction step we need to prove case of two c-compacts D_1 and D_2.

5 assume $\mathcal{F}$ is a closed k-convex filterbase on K-conv $D_1 \cup D_2$.

6 Let $\mathcal{F}' = \left\{ \{(x, y) \in D_1 \times D_2 : \exists \alpha, \beta \in \mathbb{D}_k(0, 1) \,.\, \alpha x + \beta y \in F\} \middle| F \in \mathcal{F} \right\}$.

7 Then $\mathcal{F}'$ is a k-convex fiterbase on $D_1 \times D_2$.

8 $D_1 \times D_2$ is c-compact.

9 So there is an adherence point (x, y) of $\mathcal{F}'$.

10 Let $C = K$-disc$\{x, y\}$.

11 Then C is c-compact K-disc.

12 Then $\overline{F} \cap C \neq \emptyset$ fo all $F \in \mathcal{F}$.

13 So $\mathcal{F}'' = \{\overline{F} \cap C | F \in \mathcal{F}\}$ is a filterbas on C.

14 So there exists and adherence point P of $\mathcal{F}''$.

15 But p is als an adherence point of $\mathcal{F}$ then.

$\square$

CCompactIffSphericallyComplete :: CCompact(k) $\iff$ SphericallyComplete(k)

Proof =

1 ($\Rightarrow$) : Assume that k is c-compact.

1.1 Let $B : \mathbb{N} \to 2^k$ be a deacrising sequence of closed balls.

1.2 Then $\mathcal{B} = \{B_i | i \in \mathbb{N}\}$ is a k-convex filter.

1.3 So there must exist and adherence point β of $\mathcal{B}$.

1.4 Then $\beta \in B_n$ for every $n \in \mathbb{N}$.

1.4.1 $B_n \cap U \neq \emptyset$for every $U \in \mathcal{U}_k(\beta)$.

1.4.2 This means that $\beta \in \overline{B}_n$.

1.4.3 But $B_n = \overline{B}_n$ as B_n is closed.

1.5 Which can be rendered as $\beta \in \bigcap_{n=1}^{\infty} B$.

2 ($\Rightarrow$) : Assume that k is sphercally complete.

2.1 we claim that every k-convex set in k is either $\emptyset$ or a ball.

2.1.1 Assume A is an absolutely k-convex set such that $\emptyset \neq A \neq k$.

2.1.2 Take $\omega \in A^\complement$.

2.1.3 Then $\omega \neq 0$.

2.1.4 Then every ω' such that $|\omega| \leq |\omega'|$ is not in A.

2.1.4.1 Assume there is some $\omega' \in A$ such that $|\omega| \leq |\omega'|$.

2.1.4.2 Then $\left|\frac{\omega}{\omega'}\right| \leq 1$.

2.1.4.3 Thus, as A is a k-disc, $\omega = \frac{\omega}{\omega'}\omega' \in A$.

2.1.5 So the set $R = \left\{|\omega| \middle| \omega \in A^\complement\right\}$ is bounded from above.

2.1.6 Let $r = \sup R$.

2.1.7 Take $\alpha \in A$ and $\beta \in k$ with $|\beta| \leq |\alpha|$.

2.1.8 Then $\beta \in A$.

2.1.9 so A is a ball of radius r open or closed depending on iclusion of r to R.

2.2 Also note, that in non-archimedian space any balls are either disjoin or contained in one or another.

2.3 So any k-convex filterbase $\mathcal{F}$ in k can be represented as a decreasing sequence of balls, closed or open.

2.4 Construct sequence of closed balls $\mathcal{B}$ by taking closures.

2.4.1 radii of balls will form a set R bounded from below by 0.

2.4.2 let $\delta = \inf R$.

2.4.3 Then there exists a decreasing sequence of balls B with respective radi r such that $\lim_{n\to\infty} r_n = \delta$.

2.4.3.1 This is true as all elements in the filterbase $\mathcal{F}$ must have non-empty intersecion.

2.5 Then there exists $\beta \in \bigcap \mathcal{B}$.

2.4.4 Take $\mathcal{B} = \{B_n | n \in \mathbb{N}\}$.

2.6 β is an adherence point of $\mathcal{F}$.

2.6.1 There is some $B \in \mathcal{B}$ such $\beta \in B \subset \overline{F}$ for very element $F \in \mathcal{F}$.

2.6.2 Then $F \cap U \neq \emptyset$ for every $U \in \mathcal{U}_k(\beta)$.

$\square$

1.1.9 Some Interesting Examples

$k :: \texttt{AbsoluteValueField}(\mathbb{R})$

$\texttt{NonLocallyConvexSpace} :: \exists V : k\text{-}\mathsf{TVS} \,.\, \neg\texttt{LocallyConvexSpace}(V)$

$\texttt{Proof} =$

1 Let $V = L^p(\mathbb{R}, \lambda)$ for $p \in (0, 1)$.

2 Its topology can be metrized by the metroc $\rho(f, g) = \int |f - g|^p$.

2.1 we use inequelity of form $\left(\sum_{i=1}^{n} \alpha_i\right)^p \le \sum_{i=1}^{n} \alpha_i$ for $\alpha_i > 0$.

3 on the other hand $\operatorname{conv} \mathbb{B}_V(0, \sigma) \subset \mathbb{B}_V(0, 2^{p-1}\sigma)$.

3.1 Assume $f \in \mathbb{B}_V(0, \sigma)$.

3.2 Define $F(t) = \int_{-\infty}^{t} |f|^p$.

3.3 Then F is a continuos function on $[-\infty, +\infty]$ such that $F(-\infty) = 0$ and $F(+\infty) = \rho(0, f)$.

3.4 By intermidieat value theorem there exists $t \in \mathbb{R}$ such that $F(t) = \dfrac{\rho(0, f)}{2}$.

3.5 Let $g(x) = f(x)\delta_x(-\infty, t), h(x) = f(x)\delta_x(t, +\infty)$.

3.6 Then $\rho(g, 0) \le \dfrac{\sigma}{2}$ and $\rho(h, 0) \le \dfrac{\sigma}{2}$ and $f = h + g = \dfrac{2}{2}f + \dfrac{2}{2}g$.

3.7 But $2g, 2h \in \mathbb{B}_V(0, 2^{2p-1}\sigma)$, so $f \in \operatorname{conv} \mathbb{B}_V(0, 2^{2p-1}\sigma)$.

4 By iterating one gets $\operatorname{conv} \mathbb{B}_V(0, \sigma) = V$.

5 So there are no non-trivial convex neighborhoods of 0.

$\square$

$\texttt{NonCompactConvexHullOfTheCompact} :: \exists V : k\text{-}\mathsf{TVS} \,.\, \exists K : \texttt{CompactSubset}(V) \,.\, \neg\texttt{CompactSubset}(V, \operatorname{conv} K)$

$\texttt{Proof} =$

1 Let $V = \ell^1$.

2 Let $K = \left\{0, \delta^\bullet_1, \ldots, \dfrac{1}{n}\delta^\bullet_n, \ldots\right\}$.

3 Define $\xi_n = \dfrac{1}{\sum_{i=1}^{n} 2^{-i}} \sum_{t=1}^{n} \dfrac{2^{-t}}{t}\delta^\bullet_t \in \operatorname{conv} K$.

4 Then $\zeta = \lim_{n\to\infty} \xi_n = \sum_{t=1}^{\infty} \dfrac{2^{-t}}{t}\delta^\bullet_t$.

5 But then $\zeta_i \ne 0$ for all $i \in \mathbb{N}$, but this means that $\zeta \notin \operatorname{conv} K$, so K is not compact.

$\square$

NoncomplimentedClosedSubpaceExist :: $\exists V : k\text{-TVS} \,.\, \exists U \subset_{k\text{-TVS}} V \,.\, \neg \mathtt{TopologicalComplement}(V, U)$

Proof =

1 Let $V = \ell^\infty$.

2 Let $U = c_0$.

...

$\square$

k :: UltravaluedField

PathologicalConvexSet ::

:: res $k = \mathbb{F}_2 \Rightarrow \exists V : k\text{-TVS} \,.\, \exists A : \neg \mathtt{KConvex}(V) \,.\, \forall a, b \in A \,.\, \forall \lambda \in \mathbb{D}_k(0,1) \,.\, \lambda a + (1-\lambda)b \in A$

Proof =

1 Let $V = k^3$ and let $A = \Big\{a \in \mathbb{D}_k(0,1) : \exists i \in \{1,2,3\} \,.\, a_i \in \mathbb{B}_k(0,1)\Big\}$.

2 A has desired property for convex combinations of two elements.

2.1 Assume $\lambda \in \mathbb{D}_k(0,1)$ and $a, b \in A$.

2.2 Note, either $|\lambda| = 1$ or $|1-\lambda| = 1$.

2.2.1 $1 = [1] = [1-\lambda+\lambda] = [1-\lambda] + [\lambda]$ in a residue1 field $\mathbb{F}_2$.

2.3 There exists some $i, j \in \{1,2,3\}$ such that $|a_i| < 1$ and $|b_j| < 1$.

2.4 So $|\lambda a_i| = |\lambda||a_i| < 1$ and $|(1-\lambda)b_j| = |1-\lambda||b_j| < 1$.

2.5 so either $|\lambda a_i + (1-\lambda)b_i| < 1$ or $|\lambda a_j + (1-\lambda)b_j| < 1$.

3 A is not K-convex.

3.1 $(-1,1,1) \notin A$.

3.1.1 $|-1| = |1| = 1$.

3.2 on the othe hand $(-1,1,1) = -1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3 \in K\text{-conv } A$.

$\square$

1.1.10 Seminorms

$k :: \texttt{AbsoluteValueField}(\mathbb{R})$

$\texttt{Seminorm} :: \prod V : k\text{-}\mathsf{VS} \,.\, ?(V \to \mathbb{R}_{++})$

$\nu : \texttt{Seminorm} \iff \forall v, w \in V \,.\, \nu(v+w) \leq \nu(v) + \nu(w) \;\&\; \forall v \in V \,.\, \forall \lambda \in k \,.\, \nu(\lambda v) = |\lambda|\nu(v)$

$\texttt{ZeroSeminorm} :: \forall V : k\text{-}\mathsf{VS} \,.\, \forall \nu : \texttt{Seminorm}(V) \,.\, \nu(0) = 0$

Proof $=$

1 $\nu(0) = \nu(\lambda 0) = |\lambda|\nu(0)$ for any $\lambda \in k$.

2 This means that $\nu(0)$ is not invertible in k.

3 So $\nu(0) = 0$.

□

$\texttt{SymmetricSeminorm} :: \forall V : k\text{-}\mathsf{VS} \,.\, \forall \nu : \texttt{Seminorm}(V) \,.\, \forall v \in V \,.\, \nu(-v) = \nu(v)$

Proof $=$

1 $\nu(-v) = |-1|\nu(v) = \nu(v)$.

□

$\texttt{SumOfSeminorms} :: \forall V : k\text{-}\mathsf{VS} \,.\, \forall n \in \mathbb{N} \,.\, \forall \nu : \{1, \ldots, n\} \to \texttt{Seminorm}(V) \,.\, \texttt{Seminorm}\left(V, \sum_{i=1}^{n} \nu_i\right)$

Proof $=$

Obvious.

□

$\texttt{MaxOfSeminorms} :: \forall V : k\text{-}\mathsf{VS} \,.\, \forall n \in \mathbb{N} \,.\, \forall \nu : \{1, \ldots, n\} \to \texttt{Seminorm}(V) \,.\, \texttt{Seminorm}(v, \max_{1 \leq i \leq n} \nu_i)$

Proof $=$

Obvious.

□

Note: this means that seminorms over V form an ordered tropical semirng with $0 = -\infty$.

$\texttt{seminormsFunctor} :: \texttt{Contravariant}(k\text{-}\mathsf{VS}, \mathsf{TSRING})$

$\texttt{seminormsFunctor}\,(V) = \mathsf{SMN}(V) := \texttt{Seminorm}(V)$

$\texttt{seminormsFunctor}\,(V, W, T) = \mathsf{SMN}_{V,W}(T) := T^*$

$\texttt{seminormCell} :: \prod V \in k\text{-}\mathsf{VS} \,.\, \texttt{Seminorm}(V) \to ?V$

$\texttt{seminormCell}\,(\nu) = \mathbb{B}(\nu) := \{v \in V : \nu(v) < 1\}$

$\texttt{seminormDisc} :: \prod V \in k\text{-}\mathsf{VS} \,.\, \texttt{Seminorm}(V) \to ?V$

$\texttt{seminormDisc}\,(\nu) = \mathbb{D}(\nu) := \{v \in V : \nu(v) \leq 1\}$

$\texttt{SeminormIneq} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall \nu, \nu' : \texttt{Seminorm}(V) \,.\, \nu \leq \nu' \iff \mathbb{B}(\nu') \subset \mathbb{B}(\nu)$

$\texttt{Proof} =$

Obvious.

□

Note: This means that $\mathbb{B}$ is an antitone map or functor $\mathsf{SMN}(V) \to 2^V$.
Moreover, both $\mathbb{B}$ and $\mathbb{D}$ are natural transform from SMN to the lattice of absorbent discs.

$\texttt{SeminormScalling} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall \nu \in \mathsf{SMN}(V) \,.\, \forall \lambda \in \mathbb{R}_{++} \,.\, \lambda\mathbb{B}(\nu) = \mathbb{B}(\lambda^{-1}\nu)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{SeminormCellIsAbsobentDisc} :: \forall V \in k\text{-}\mathsf{VS} \forall \nu \in \mathsf{SMN}(V) \,.\, \texttt{Absorbent} \,\&\, \texttt{Disc}\Big(V, \mathbb{B}(\nu)\Big)$

$\texttt{Proof} =$

Obvious.

□

$\texttt{SeminormCellClosureTheorem} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall \nu \in \mathsf{SMN} \,\&\, C(V) \,.\, \mathrm{cl}_V \mathbb{B}(\nu) = \mathbb{D}(\nu)$

$\texttt{Proof} =$

1 Assume $v \in \mathbb{D}(\nu)$.

2 then the sequence $u_n = \left(1 - \frac{1}{n}\right) v \in \mathbb{B}(\nu)$ has limit v.

3 So $\mathbb{D}(\nu) \subset \mathrm{cl}_V \mathbb{B}(\nu)$.

4 On the other hand $\mathbb{D}(\nu) = \nu^{-1}[0, 1]$ is closed.

5 So $\mathrm{cl}_V \mathbb{B}(\nu) \subset \mathbb{D}(\nu)$ and $\mathbb{D}(\nu) = \mathrm{cl}_V \mathbb{B}(\nu)$.

□

SeminormContinuity :: $\forall V : k\text{-}\mathsf{TVS} \,.\, \forall \nu \in \mathsf{SMN}(V)\,.$

(1) $\nu \in \mathsf{UNI}(V, \mathbb{R}) \iff$

(2) $\mathbb{B}(\nu) \in \mathcal{T}(V) \iff$

(3) $\mathbb{D}(\nu) \in \mathcal{N}(V) \iff$

(4) $\mathtt{ContinuousAt}(V, \mathbb{R}, 0, \nu)$

Proof $=$

1 (1) $\Rightarrow$ (2) $\Rightarrow$ (3) obvious.

2 (3) $\Rightarrow$ (4).

2.1 As non-zero scalar multiplication is a homeomorphism $\lambda \mathbb{D}(\nu) \in \mathcal{N}(V)$ for all $\lambda \in \mathbb{R}_{++}$.

2.2 consider a net v such that $\lim_\delta v_\delta = 0$.

2.3 Eventualy $v_\delta \in \lambda \mathbb{D}(\nu)$ for any $\lambda \in \mathbb{R}_{++}$.

2.4 This means that $\lim_\delta \nu(v_\delta) = 0$.

3 (4) $\Rightarrow$ (1).

3.1 $\nu^{-1}[0, \lambda)$ is open for any $\lambda \in \mathbb{R}_{++}$.

3.2 As V is a topological group there is $U \in \mathcal{U}_V(0)$ such that $U - U \subset \nu^{-1}[0, \lambda)$.

3.3 Thus, $\nu(x - y) < \lambda$ for any $x, y \in U$.

3.4 Let $v \in V$ be arbitraty .

3.5 Take $u \in v + U$.

3.6 Then $\nu(u) = \nu(u + v - v) \le \nu(u - v) + \nu(v) \le \nu(v) + \lambda)$.

3.7 On the other hand $\nu(u) \ge \nu(v) - \nu(u - v) \ge \nu(v) - \lambda$ as $\nu(v) = \nu(v - u + u) \le \nu(u) + \nu(u - v)$.

3.8 So $\Big|\nu(u) - \nu(v)\Big| \le \lambda$.

□

SeminormContinuityByDomination ::

$:: \forall V : k\text{-}\mathsf{TVS} \,.\, \forall \nu \in \mathsf{SMN}(V) \,.\, \forall \mu \in \mathsf{SMN} \,\&\, C(V) \,.\, \nu \le \mu \Rightarrow \nu \in \mathsf{UNI}(V, \mathbb{R})$

Proof $=$

By antitonicity $\mathbb{B}(\mu) \subset \mathbb{B}(\nu) \subset \mathbb{D}(\nu)$.

But $\mathbb{B}(\mu)$ is open, so $\mathbb{D}(\nu) \in \mathcal{N}_V(0)$.

Thus ν is uniformly continuous.

□

GaugesOfDiscsProduceSeminorms :: $\forall V \in k\text{-}\mathsf{VS} . \forall D : \mathtt{Disc}\ \&\ \mathtt{Absorbent}(D) . \gamma(\bullet|D) \in \mathsf{SMN}(V)$

Proof =

1 Discs are convex, so $\gamma(\bullet|D)$ is a convex function.

2 Take some $v \in V$.

2.1 Let $I_v = \{\lambda \in \mathbb{R}_{++} : \lambda^{-1}v \in D\}$.

2.2 As D is absorbent, $I_v \neq \emptyset$.

2.3 As D is balanced then if $\alpha \in I_v$ and $\beta \geq \alpha$, then $\beta \in I$.

2.4 Thus, $I_v = \Big(\gamma(v|D), +\infty\Big)$.

2.5 Then it is clear that $I_{\lambda v} = \lambda I_v = \Big(\lambda\gamma(v|D), +\infty\Big) = \Big(\gamma(\lambda v|D), +\infty\Big)$.

3 So $\gamma(\bullet|D)$ is positively homogeneous.

4 $\gamma(\bullet|D)$ is subadditive.

4.1 Take some $v, w \in V$.

4.2 Write $\gamma(v + w|D) = \gamma\left(\frac{2}{2}v + \frac{2}{2}w|D\right) \leq \frac{1}{2}\gamma(2v|D) + \frac{1}{2}\gamma(2w|D) = \gamma(v|D) + \gamma(w|D)$.

□

Note: Cells and gauges produce a Functor isomorphism.
This isomorphism is between $\mathsf{SMN} : k\text{-}\mathsf{VS} \to \mathsf{ORD}$ and some absorbent disc functor, open or closed.

GaugeContinuity :: $\forall V \in k\text{-}\mathsf{TVS} . \forall D : \mathtt{Disc}\ \&\ \mathtt{Absorbent}(D) . \gamma(\bullet|D) \in C(V) \iff D \in \mathcal{N}_V(0)$

Proof =

1 This follows from seminorm continuity theorem as $\mathbb{B}(\gamma(\bullet|D)) \subset D \subset \mathbb{D}(\gamma(\bullet|D))$.

□

Sublinear :: $\prod V : k\text{-}\mathsf{VS} . ?(V \to \mathbb{R})$

$\phi : \mathtt{Sublinear} \iff \phi \in \mathcal{SL}(V) \iff \forall v, w \in V . \phi(v + w) \leq \phi(v) + \phi(w)\ \&\ \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . \phi(\alpha v) = \alpha\phi$

seminormFromSublinear :: $\prod V : k\text{-}\mathsf{VS} . \mathtt{Sublinear}(V) \to \mathsf{SMN}(V)$

seminormFromSublinear $(\phi) = \nu_\phi := \Lambda v \in V . \max\Big(\phi(v), \phi(-v)\Big)$

1 Either $\phi(v) \geq 0$ or $\phi(-v) \geq 0$.

1.1 From positive homogenity $\phi(0) = 0$.

1.2 Write $0 = \phi(0) = \phi(v - v) \leq \phi(v) + \phi(-v)$.

2 So ν_ϕ has positive range .

3 Minkowsky Inequality holds also.

3.1 $\nu_\phi(v + w) = max\Big(\phi(v + w), \phi(-v - w)\Big) \leq \max\Big(\phi(v) + \phi(w), \phi(-v) + \phi(-w)\Big) \leq$

$\leq \max\Big(\phi(v), \phi(-v)\Big) + \max\Big(\phi(w), \phi(-w)\Big) = \nu_\phi(v) + \nu_\phi(w)$.

□

1.1.11 Topology of Locally Convex Space

$$\texttt{seminormTopology} :: \prod_{V \in k\text{-VS}} ?\mathsf{SMN}(V) \to \texttt{VectorTopology}(V)$$

$$\texttt{seminormTopology}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) := \mathcal{W}_V(\mathcal{N}, \mathbb{R}, \mathrm{id})$$

`HausdorffSeminormTopology` ::

$$:: \forall V \in k\text{-VS} \, . \, \forall \mathcal{N} \subset \mathsf{SMN}(V) \, . \, \texttt{T2}\Big(V, \mathcal{T}(\mathcal{N})\Big) \iff \forall v \in \mathcal{V} \, . \, v \neq 0 \Rightarrow \exists \nu \in \mathcal{N} \, . \, \nu(v) \neq 0$$

`Proof` =

1 If such norm ν exists then v can be sparated from 0 by an open set.

2 For topological group $(V, +)$ this is enough.

□

`SeminormTopologyBase` ::

$$:: \forall V \in k\text{-VS} \, . \, \forall \mathcal{N} \subset \mathsf{SMN}(V) \, . \, \texttt{Base}\left(V, \mathcal{T}(\mathcal{N}), \left\{ \lambda \mathbb{B}(\nu) \middle| \lambda \in \mathbb{R}_{++}, \nu \in \mathcal{N} \right\}\right)$$

`Proof` =

1 Seems obvious by weak topology definition.

□

`SeminormTopologyIsLC` :: $\forall V \in k\text{-VS} \, . \, \forall \mathcal{N} \subset \mathsf{SMN}(V) \, . \, \Big(V, \mathcal{T}(\mathcal{N})\Big) \in k\text{-LCS}$

`Proof` =

1 This holds as the base is convex.

□

`EveryLCSHasSeminormTopology` :: $\forall V \in k\text{-LCS} \, . \, \exists \mathcal{N} \subset \mathsf{SMN}(V) \, . \, \mathcal{T}_V = \mathcal{T}(\mathcal{N})$

`Proof` =

1 As we working with froup topologies it is enough to work with zero equivalence.

2 Take $U \in \mathcal{U}_V(0)$.

3 Then there exists a disc $D \subset U$.

4 $\gamma(\bullet|D)$ is continuous gauge for V.

5 So $U \in \mathcal{T}\Big(\Big\{\gamma(\bullet|D)\Big\}\Big)$.

6 Define $\mathcal{N}$ to be set of all such gauges.

7 Then $\mathcal{T}_V \subset \mathcal{T}(\mathcal{N})$.

8 On the other hand $\mathcal{T}(\mathcal{N}) \subset \mathcal{T}_V$ as all gauges are continuous.

□

Note: There should exists a k-VS $\to$ ORD functor equivalence.
Take functors of saturated seminorm cones an locally convex topologies.

$\texttt{Saturated} :: \prod_{V \in k\text{-}\mathsf{VS}} ??\mathsf{SMN}(k)$

$\mathcal{N} : \texttt{Saturated} \iff \forall \nu, \mu \in \mathcal{N} \ . \ \max(\nu, \mu) \in \mathcal{N} \iff$

$\texttt{saturatedSeminormCones} :: \texttt{Covariant}(k\text{-}\mathsf{VS}, \mathsf{ORD})$

$\texttt{saturatedSeminormCones}\,(V) = \mathsf{SSC}(V) := \texttt{Saturated}(V) \ \& \ \texttt{ConvexCone}\Big(\mathcal{SL}(V)\Big)$

$\texttt{saturatedSeminormCones}\,(V, W, *) = \mathsf{SSC}_{V,W}(T) := (T^*)^{-1}$

$\texttt{SeminormedProductTopolgy} ::$

$$\forall I \in \mathsf{SET} \ . \ \forall V : I \to k\text{-}\mathsf{TVS} \ . \ \forall \mathcal{N} : \prod_{i \in I} ?\mathsf{SMN}(V) \ . \ \prod_{i \in \mathcal{I}} \Big(V_i, \mathcal{T}(\mathcal{N}_i)\Big) \cong_{\mathsf{TOP}} \left(\prod_{i \in I} V_i, \left\{\pi_i^* \nu \middle| i \in I, \nu \in \mathcal{N}_i\right\}\right)$$

$\texttt{Proof} =$

1 This may be seen as functorial eqiavalence interacting with limits.

2 And weak topologies are limits.

□

$\texttt{LocallyConvexProduct} ::$

$$\forall I \in \mathsf{SET} \ . \ \forall V : I \to k\text{-}\mathsf{LCS} \ . \ \prod_{i \in I} V_i \in k\text{-}\mathsf{LCS}$$

$\texttt{Proof} =$

1 Now this is obvious.

□

LocallyConvexSemimetrizability ::

:: $\forall V \in k\text{-LCS} \,.\, \texttt{Semimetrizable}(V) \iff \exists \nu : \mathbb{N} \uparrow C(V) \,\&\, \mathsf{SMN}(V) \,.\, \mathcal{T}_V = \mathcal{T}(\operatorname{Im} \nu)$

Proof =

1(⇒) assume V is semimetrizable.

1.1 Then there exists a decreasing sequence of disked neighborhoods of unity D which generate the toplogy.

1.2 Then $\gamma(\bullet|D_n)$ is clearly a sequence of seminorms we seek.

2(⇐) assume ν are seminorms of the hypothesis.

2.1 Define $\mu(x) = \sum_{n=1}^{\infty} 2^{1-n} \frac{\nu_n(x)}{1+\nu_n(x)}$.

2.2 Then μ is an F-seminorm.

2.2.1 Assume $\alpha \in \mathbb{D}_k(0,1)$ and $v \in V$.

2.2.2 Then $\frac{\nu_n(\alpha v)}{1+\nu_n(\alpha v)} = \frac{|\alpha|\nu_n(v)}{1+|\alpha|\nu_n(v)} \le \frac{\nu_n(v)}{1+\nu_n(v)}$ for any $n \in \mathbb{N}$.

2.2.2.1 Note, that $f(x) = \frac{x}{1+x}$ is increasing for $x > 0$.

2.2.2.1.1 $f'(x) = \frac{1}{(1+x)^2} > 0$.

2.2.2.2 And $|\alpha|\nu_n(v) \le \nu_n(v)$ for any $n \in \mathbb{N}$.

2.2.3 Thus $\mu(\alpha v) \le \mu(v)$.

2.2.4 Also $\lim_{m\to\infty} \mu\left(\frac{v}{m}\right) = \lim_{m\to\infty} \sum_{n=1}^{\infty} 2^{1-n} \frac{\nu_n(v/m)}{1+\nu_n(v/m)} = \sum_{n=1}^{\infty} \lim_{m\to\infty} \frac{2^{1-n}}{m} \frac{\nu_n(v)}{1+\nu_n(v/m)} = 0$

by dominated convergence theorem with dominator $x_n = 2^{2-n}$.

2.2.5 The Minkowsky inequality for μ is obvious from metric topology

2.3 By construction μ is continuous in a topology defined by $(\nu_n)_{n=1}^{\infty}$ by construction.

2.3.1 μ is a uniform limit of continuous functions.

2.4 Also F-seminorm $2^{1-n} \frac{\nu_n}{\nu_n + 1} \le \mu$ for each n.

2.5 so each F-seminorm $2^{1-n} \frac{\nu_n}{\nu_n + 1}$ is continuous in the topology defined by μ.

2.6 But this means that each ν_n is also continuous in this topology .

□

continuousDual :: $\prod k : \texttt{TopologicalField} \,.\, k\text{-TVS} \to k\text{-VS}$

continiousDual $(V) = V' := V^* \cap \mathsf{TOP}(V, k)$

DiscontinuousFunctionalExists :: $\forall V \in k\text{-LCS} \,.\, \forall \aleph : \texttt{Semimetrizable}(V) \,.\, \forall \beth : \dim V = \infty \,.\, \exists (V^* \setminus V')$

Proof =

1 Let ρ be a semimetric for V .

2 Then there exists an infinite linearly independent sequence $(e_n)_{n=1}^{\infty}$.

3 Extend $(e_n)_{n=1}^{\infty}$ to a Hamel basis H.

4 As V is semimetrizable it is possible to select a countables decreasing base of absorbent discs $(D_n)_{n=1}^{\infty}$.

5 Then it is possible to selecect λ_n such that $\lambda_n e_n \in D_n$.

6 Obviously, then $\lim_{n\to\infty} \lambda_n e_n = 0$.

7 Define linear functional f by $f(e_n) = \frac{1}{\lambda_n}$ and and $f(h) = 0$ if h is linearly independent from all e_n.

8 Then clearly $lim_{n\to\infty} f(\lambda_n e_n) = 1$, so f can't be contiuous.

□

FinitieDimensionByContinuousFunctionals ::

:: $\forall V : \texttt{NormedSpace}(k) \ . \ \dim V < \infty \iff V' = V^*$

Proof $=$

1 As V is metric and locally convex this follows from the precious result.

$\square$

FinestLocallyConvexSpaceIsNotMetrizable ::

:: $\forall V \in k\text{-}\mathsf{VS} \ . \ \forall \aleph : \dim V = \infty \ . \ \neg\texttt{Metrizable}\Big(V, \mathcal{W}_V(V^*, k, \mathrm{id})\Big)$

Proof $=$

1 As V is locally convex this follows from the precious result.

$\square$

defininigSeminorms :: $\prod V \in k\text{-}\mathsf{LCS} \ . \ \mathsf{SSC}(V)$

definingSeminorms $() = \mathrm{ssc}(V) := \mathsf{SMN}(V) \cap \mathsf{TOP}(V, \mathbb{R})$

ConvergenceInLocallyConvexSpace ::

:: $\forall V : k\text{-}\mathsf{LCS} \ . \ \forall (\Delta, x) : \texttt{Net}(V) \ . \ \forall v \in V \ . \ \lim_{\delta \in \Delta} x_\delta = v \iff \forall \nu \in \mathrm{ssc}(V) \ . \ \lim_{\delta \in \Delta} \nu(x_\delta - v) = 0$

Proof $=$

1 $(\Rightarrow)$ This is obvious as each ν is continuous.

2 $(\Leftarrow)$ Assume D is an open disc in V.

2.1 as D is open disc then $\gamma(\bullet|D) \in \mathrm{ssc}(V)$ is continuous.

2.2 But this meand that $\lim_{\delta \in \Delta} \gamma(x_\delta - v|D) = 0$.

2.3 So $x_\delta - v$ is eventually inside D.

2.4 As D was arbitraty this means that $\lim_{\delta \in \Delta} x_\delta = v$.

$\square$

CauchyPropertyInLocallyConvexSpace ::

:: $\forall V : k\text{-}\mathsf{LCS} \ . \ \forall (\Delta, x) : \texttt{Cauchy}(V) \ . \ \forall \nu \in \mathrm{ssc}(V) \ . \ \texttt{Cauchy}(V, \Delta, \nu(x))$

Proof $=$

1 This is true as every ν is uniformly continuous.

$\square$

LocallyConvexContinuityCriterion ::

:: $\forall V, W : k\text{-}\mathsf{LCS} \ . \ \forall T \in k\text{-}\mathsf{VS}(V, W) \ . \ T \in k\text{-}\mathsf{LCS} \iff \forall \nu \in \mathrm{ssc}(W) \ . \ \exists \mu \in \mathrm{ssc}(V) \ . \ T^*\nu \leq \mu$

Proof $=$

1 $(\Rightarrow)$ True as $T^*\nu$ is continuous as composition and $T^*\nu \leq T^*\nu$.

2 $(\Leftarrow)$ As $T * \nu \leq \mu$ the seminorm $T^*\nu$ is continuous by domination.

2.1 Then the result follows by universal property of weak topology.

$\square$

`ContinuousIfBounded` ::

:: $\forall V, W : \texttt{NormedSpace}(k) \,.\, \forall T \in k\text{-}\mathsf{VS}(V, W) \,.\, T \in \mathsf{TOP}(V, W) \iff T \in \mathcal{B}(V, W)$

`Proof` $=$

1 Now this is obvious specification of the previous result.

$\square$

Note: This is intersting how the fundamental theorem of elementary functional analysis can be seen as application of the universal property of weak topology.

`KernelSeparationLemma` :: $\forall V : k\text{-}\mathsf{VS} \,.\, \forall f \in V^* \,.\, \forall v \in V \,.\, \forall \aleph : f(v) = 1 \,.$

$.\, \forall U : \texttt{Balanced}(V) \,.\, (v + U) \cap \ker f = \emptyset \iff \forall u \in U \,.\, |f(u)| < 1$

`Proof` $=$

1 $(\Rightarrow)$ Assume $x + U \cap \ker f = \emptyset$.

1.1 Assume there is $u \in U$ such that $|f(u)| \geq 1$.

1.2 As U is balanced, then $w = -\dfrac{u}{f(u)} \in U$.

1.3 But $f(v + w) = f(v) + f(w) = 1 - 1 = 0$, a contradiction !.

2 $(\Leftarrow)$ Assume $\forall u \in U \,.\, |f(u)| < 1$ is the case.

2.1 $f(v) \neq -f(u)$ for any $u \in U$.

2.2 So $f(v + u) = f(v) + f(u) \neq 0$.

$\square$

`ContinuousByClosedKernel` :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall f \in V^* \,.\, f \in V' \iff \texttt{Closed}(V, \ker f)$

`Proof` $=$

1 $(\Rightarrow)$ This direction is obvious as k is Hausdorff.

2 $(\Leftarrow)$ Now assume $\ker f$ is closed.

2.1 If $f = 0$ then continuity is trivial.

2.2 So assume there is x such that $f(x) \neq 0$.

2.2.1 Without loss of generality assume $f(x) = 1$.

2.2.2 Then there is some balanced open U such that $U_\gamma + x \cap \ker f = \emptyset$.

2.2.3 But this means that $\forall u \in U \,.\, |f(u)| < 1$.

2.2.4 This means that $\mathbb{D}(|f|) \in \mathcal{N}_V(0)$.

2.3 So f is continuous.

$\square$

`ContinuousByRealPart` :: $\forall V \in \mathbb{C}\text{-}\mathsf{TVS} \,.\, \forall f \in V^* \,.\, f \in V' \iff \operatorname{Re} f \in C(V)$

`Proof` $=$

1 write $f(v) = \operatorname{Re} f(v) - \mathrm{i}\operatorname{Re} f(\mathrm{i}v)$.

$\square$

`ContinuousFunctionalIsOpen` :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall f \in V' \,.\, f \neq 0 \Rightarrow \texttt{Open}(V, k, f)$

`Proof` $=$

1 As $f \neq 0$ this musbe the case that f is surjective.

2 So f is open as it linear, continuous and surjective.

$\square$

ContinuityOfMultilinearMap ::

$$:: \forall n \in \mathbb{N} \,.\, \forall V : \{1, \dots, n\} \to k\text{-}\mathsf{LCS} \,.\, \forall W \in k\text{-}\mathsf{LCS} \,.\, \forall A : \bigotimes_{i=1}^{n} V_i \to W \,.$$

$$. \, A \in k\text{-}\mathsf{TVS}\left(\bigotimes_{i=1}^{n} V_i, W\right) \iff \forall \nu : \prod_{i \in I} \mathrm{ssc}(V_i) \,.\, \forall \mu \in \mathrm{ssc}(W) \,.\, \exists \lambda \in \mathbb{R}_{++} \,.\, A\mu \leq \lambda \prod_{i=1}^{n} \nu_i$$

Proof $=$

This follows from the theory of norms on tensor spaces.

$\square$

1.1.12 Spaces of Continuous Functions

$$\texttt{compactOpenTopology} :: \prod X \in \mathsf{TOP} . \texttt{Topology}\Big(\mathsf{TOP}(X,k)\Big)$$

$$\texttt{compactOpenTopology}\,() = \kappa_X := \mathcal{T}\Big(\Big\{\Lambda f \in \mathsf{TOP}(X,k) . \sup_{x\in K}|f(x)|\Big| K \in \mathsf{K}(X)\Big\}\Big)$$

$$\texttt{SpaceWithCompactOpenTopology} :: \forall X \in \mathsf{TOP} . V = \Big(\mathsf{TOP}(X,k), \kappa_X\Big) \in k\text{-}\mathsf{LCHS}$$

Proof =

1 Topology on V is generated by seminorms, so V is locally convex.

2 As sets $\{x\}$ are dcompact, the evaluation seminorm $\epsilon_x : f \mapsto |f(x)|$ is continuous for V.

3 If $f \neq 0$ then there is some $x \in X$ such that $f(x) \neq 0$.

4 So $\epsilon_x(f) \neq 0$ and this means that V is Hausdorff.

□

$$\texttt{Hemicompact} :: ?\mathsf{TOP}$$

$$X : \texttt{Hemicompact} \iff \exists \mathcal{C} : \texttt{Countable}\Big(\mathsf{K}(X)\Big) . \forall K \in \mathsf{K}(X) . \exists F \in \mathcal{C} . K \subset F$$

$$\texttt{CompactOpenTopologyMetrization} :: \forall X \in \texttt{T3.5} . \texttt{Hemicompact}(X) \iff \texttt{Metrizable}\Big(\mathsf{TOP}(X,k), \kappa_X\Big)$$

Proof =

1 $(\Rightarrow)$ Assume X is hemicompact.

1.1 Then let F be an enumeration of the set $\mathcal{C}$ from the definition of hemicompact.

1.2 Without loss of generality we may assume that F is increasing.

1.3 Then $\nu_n(f) = \sup_{x\in F_n} |f(x)|$ is an increasing family of seminorms.

1.4 By hemicompactness ν_n defines κ_X.

1.5 So the κ_X is metrizable.

2 $(\Leftarrow)$ now assume κ_X is metrizable.

2.1 Then there is a countable base defined by sup-functionals for some compacts F_n.

2.2 Then for any compact K its sup-functional is less then a scalar multiple of a sup-functional of some F_n.

2.3 Assune This is the case, but $K \not\subset F_n$.

2.4 Then there is some $x \in K \setminus F_n$.

2.5 Also there is some $f \in \mathsf{TOP}(X,k)$ such that $f(x) = 1$ and $f(F_n) = \{0\}$.

2.5.1 This is true as X is Tychonoff and Hausdorff.

2.6 Then $\sup_{x\in K}|f(x)| \geq \sup_{x\in\mathcal{F}_n} |f(x)|$ which is a contradiction.

2.7 So X must be hemicompact.

□

$$\texttt{KRSpace} :: \mathsf{TOP} \to ?\mathsf{TOP}$$

$$X : \texttt{KRSpace} \iff \Lambda Y \in \mathsf{TOP} \forall f : X \to Y . \Big(\forall K \in \mathsf{K}(X) . f_{|K} \in \mathsf{TOP}(K,Y)\Big) \Rightarrow f \in \mathsf{TOP}(X,Y)$$

$\texttt{CompactOpenTopologyCompleteness} :: \forall X : \texttt{T3.5} \,.\, \texttt{KRSpace}(k, X) \iff \texttt{Complete}\Big(\mathsf{TOP}(X, k), \kappa_X\Big)$

$\texttt{Proof} =$

1 ($\Rightarrow$): Assume X is a KRSpaces for k.

1.1 Take f to be a Cauchy sequence for κ_X.

1.2 Then $f(x)$ is also Cauchy as $\{x\}$ is compact for any $x \in X$.

1.3 Thus, as k is complete $F = \lim_{n\to\infty} f_n$ exists.

1.4 On every compact K the convergence of $f_{|K}$ towards $F_{|K}$ is uniform so $F_{|K}$ is continuous.

1.5 But as X is KRSpace the whole F must be continuous.

1.6 So κ_X is complete.

2 ($\Leftarrow$): Now assume that κ_X is complete.

2.1 Take some $f : X \to k$ such that $f_{|K}$ is continuous for any comapct K.

2.2 Then by Tietze extension theorem $f_{|K}$ can extended to a continuous function $F_K : \beta X \to k$.

2.3 By properties of Tietze-Urysohn extension we may assume that $\sup F_K = \sup f_{|K}$.

2.4 Define $g_K = F_{K|X}$.

2.5 The set $\mathsf{K}(X)$ is directed.

2.6 Then g_K is a Cauchy net.

2.6.1 Take K be a compact in X and let $\nu_K(f) = \sup_{x\in K} |f|$.

2.6.2 Then $\nu_K(g_L - g_H) = 0$ for any $L, H \in \mathsf{K}(X)$ such that $K \subset L$ and $K \subset H$.

2.6.3 So $g_L - g_H \in \mathbb{B}(\nu_K)$ in this case.

2.7 Thus there exists a continuous limit G for κ_X.

2.8 But $G = f$.

2.8.1 If $x \in X$ then $g_K(x) = f(x)$ for any $K \in \mathsf{K}(X)$ such that $x \in K$.

2.9 Thus f is continuous.

$\square$

$\texttt{pointwiseConvergenceTopology} :: \prod X \in \mathsf{TOP} \,.\, \texttt{Topology}\Big(\mathsf{TOP}(X, k)\Big)$

$\texttt{pointwiseConvergenceTopology}\,() = \pi_X := \mathcal{T}\Big(\big\{\Lambda f \in \mathsf{TOP}(X, k) \,.\, |f(x)| \big| x \in X\big\}\Big)$

$\texttt{SpaceWithPointeisConvergenceTopology} :: \forall X \in \mathsf{TOP} \,.\, V = \Big(\mathsf{TOP}(X, k), \kappa_X\Big) \in k\text{-}\mathsf{LCHS}$

$\texttt{Proof} =$

1 Topology on V is generated by seminorms, so V is locally convex.

2 If $f \neq 0$ then there is some $x \in X$ such that $f(x) \neq 0$.

3 So $\epsilon_x(f) \neq 0$ and this means that V is Hausdorff.

$\square$

$\texttt{PointwiseConvergence} ::$

$$:: \forall X \in \mathsf{TOP} \,.\, \forall (\Delta, f) : \texttt{Net}\Big(\mathsf{TOP}(X, k)\Big) \,.\, \forall g \in \mathsf{TOP}(X, k) \,.\, \lim_{\delta\in\Delta} f_\delta =_{\pi_X} g \iff \forall x \in X \,.\, \lim_{\delta\in\Delta} f_\delta(x) = g(x)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{Equicontinuous} :: \prod X \in \mathsf{TOP} \,.\, \prod G \in \mathsf{TGRP} \,.\, ??\mathsf{TOP}(X, G)$

$\mathcal{F} : \texttt{Equicontinuous} \iff \forall x \in X \,.\, \forall V \in \mathcal{U}_G(e) \,.\, \exists U \in \mathcal{U}_X(x) \,.\, \forall f \in \mathcal{F} \,.\, f(U) \subset f(x)V$

$\texttt{Equibounded} :: \prod X \in \mathsf{TOP} \,.\, ??\mathsf{TOP}(X, k)$

$\mathcal{F} : \texttt{Equibounded} \iff \forall x \in X \,.\, \exists \beta \in \mathbb{R}_{++} \,.\, \forall f \in \mathcal{F} \,.\, |f(x)| \le \beta$

$\texttt{EquicontinuousTopologyEquality} :: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{F} : \texttt{Equicontinuous}(X, k) \,.\, (\mathcal{F}, \kappa_X) = (\mathcal{F}, \pi_X)$

Proof $=$

1 Firstly, $\kappa_X \subset$.

1.1 Take $g \in \mathcal{F}$.

1.2 Assume $U \in \kappa_X(g)$ has form $U = \left\{ f \in \mathsf{TOP}(X, k) : \sup_{x \in K} |f(x) - g(x)| < \alpha \right\}$ for some compact K and $\alpha \in \mathbb{R}_{++}$.

1.3 Then for each $x \in K$ there is some $W_x \in \mathcal{U}_X(x)$ such that $f(W_x) \subset f(x) + \mathbb{B}_k(0, \alpha/4)$ for each $f \in \mathcal{F}$.

1.4 As K is compact and W is an open cover we can select a finite family of points $(x_i)_{i=1}^n$ such that $K \subset \bigcup_{i=1}^n W_{x_i}$.

1.5 Let ϵ_y stand for evaluation seminorm $\epsilon_y(f) = |f(y)|$.

1.6 Then $V = \bigcap_{i=1}^n \frac{\alpha}{2} \mathbb{B}(\epsilon_{x_i}) + g \in \pi_X$ and $V \subset U$ in $\mathcal{F}$.

1.6.1 Take some $f \in V \cap \mathcal{F}$ and some $y \in K$.

1.6.2 Then there is some $i \in \{1, \dots, n\}$ such that $y \in W_{x_i}$.

1.6.3 $|f(y) - g(y)| \le |f(y) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(y)| < \alpha$.

1.6.4 So $\sup_K |f - g| < \alpha$.

1.7 This means that U is open in π_X.

2 This is obvious from definition that $\pi_X \subset \kappa_X$ and $\pi_X = \kappa_X$.

□

$\texttt{PointwiseClosureEquicontinuous} ::$

$$:: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{F} : \texttt{Equicontinuous}(X, k) \,.\, \texttt{Equicontinuous}\left(X, k, \operatorname{cl}_{\pi_X} \mathcal{F}\right)$$

Proof $=$

1 Take $x \in X$ and $V \in \mathcal{U}_k(0)$.

2 Then by equicontinuity there is $U \in \mathcal{U}_X(x)$ such that $f(U) \subset f(x) + V$ for any $f \in \mathcal{F}$.

3 Take g to be a limit point in $\mathcal{F}$.

4 Then there is sequence f such that $\lim_{n \to \infty} f_n = g$ pointwise.

5 Take some $u \in U$.

6 Then $g(u) = \lim_{n \to \infty} f_n(u)$.

7 Then $|g(u) - g(x)| \le |g(u) - f_n(u)| + |f_n(u) - f_n(x)| + |f_n(x) - g(x)| \le 3\varepsilon$ for suitably choosen n.

8 So $\operatorname{cl}_{\pi_X} \mathcal{F}$ is equicontinuous.

□

ArzeloAscoli1 ::

:: $\forall X \in \mathsf{TOP}\ .\ \forall \mathcal{F} : \texttt{Equicontinuous}(X, k)\ \&\ \texttt{Equibounded}(X)\ \&\ \texttt{Closed}\Big(\mathsf{TOP}(X, k), \kappa_X, \mathcal{F}\Big)\ .$

$.\ \texttt{CompactSubset}\Big(\mathsf{TOP}(X, k), \pi_X, \mathcal{F}\Big)$

Proof $=$

1 Eeach $\mathcal{F}(x)$ is a compact subset of k by Heine-Borel Lemma.

2 So by Tychonoff theorem $\prod \mathcal{F}(x)$ is compact in the product topology.

3 But $\mathcal{F}$ is a closed subset of $\prod \mathcal{F}(x)$ in π_X, so $\mathcal{F}$ is also compact in π_X.

4 As $\mathcal{F}$ is equicontinuous π_X is equal to κ_X on $\mathcal{F}$, so $\mathcal{F}$ is also compact in κ_X.

$\square$

ArzeloAscoli2 ::

:: $\forall X : \texttt{LocallyCompact}\ .\ \forall \mathcal{F} : \texttt{CompactSubset}\Big(\mathsf{TOP}(X, k), \kappa_X, \mathcal{F}\Big)\ .$

$.\ \texttt{Equicontinuous}(X, k, \mathcal{F})\ \&\ \texttt{Equibounded}(X, \mathcal{F})\ \&\ \texttt{Closed}\Big(\mathsf{TOP}(X, k), \pi_X, \mathcal{F}\Big)$

Proof $=$

...

$\square$

1.1.13 Constructions

SubspaceQuotientSeminorm ::

$$:: \forall V \in k\text{-LCS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \mathcal{T}\left(\frac{V}{U}\right) = \mathcal{T}\left(\left\{\Lambda[v] \in \frac{V}{U} \,.\, \inf_{u \in U} \nu(v+u) \middle| \nu \in \mathrm{ssc}(V)\right\}\right)$$

Proof =

1 Let $\nu \in \mathrm{ssc}(V)$.

2 define $\mu = \Lambda[v] \in \frac{V}{U} \,.\, \inf_{u \in U} \nu(v+u)$.

3 Then μ is a seminorm.

3.1 $[v] = 0$ imply $v \in U$.

3.2 So $\mu = 0$ as $\nu(w) \geq 0$ and $\nu = 0$.

3.3 Take $[v] \in \frac{V}{U}$ and $\alpha \in k$.

3.4 Then $\mu[\alpha v] = \inf_{u \in U} \nu(\alpha v + u) = \inf_{u \in U} \nu(\alpha v + \alpha u) = |\alpha| \inf_{u \in U} \nu(v+u) = |\alpha|\mu[v]$.

3.5 Now take $v, w \in V$.

3.6 Then $\mu[v+w] = \inf_{u \in U} \nu(v+w+u) = \inf_{u,o \in U} \nu(v+w+u+o) \leq \inf_{u,o \in U} \nu(v+u) + \nu(w+o) =$

$= \inf_{u \in U} \nu(v+u) + \inf_{o \in U} \nu(v+o)\mu[v] + \mu[w]$.

4 Then $\mathbb{B}(\mu) = \pi_U \mathbb{B}(\nu)$.

5 As open cells as above form a base of topology on V,

and quotien topology is an image topology, the result follows.

□

LocallyConvexQuotient :: $\forall V \in k\text{-LCS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \forall \frac{V}{U} \in k\text{-LCS}$

Proof =

1 This is True as topology on $\frac{V}{U}$ is generated by seminorms.

□

kernelOfSeminorm :: $\prod_{V \in k\text{-VS}} \mathsf{SMN}(V) \to$ VectorSubspace(V)

kernelOfSeminorm $(\nu) = \ker \nu := \nu^{-1}\{0\}$

SeminormedCompletion :: $\forall V$: SeminormedSpace(k) . $\exists(\hat{V}, \iota)$: TVSCompletion(V) . $\mathsf{SMS}(k, \hat{V})$

Proof =

1 Take $[v] \in \hat{V}$.

2 Then $[v]$ can associated with Cauchy sequence v.

3 Define $\nu_{\hat{V}}[v] = \lim_{n \to \infty} \nu_V(v_n)$.

3.1 As ν_V is uniformly continuous the $\nu_V(v_n)$ must be again Cauchy, and hence convergent as k is complete.

3.2 Use completion metric argument to see that $\nu_{\hat{V}} is Uniquely determined.$

3.2.1 Assume x an y are both Cauchy sequences for $[v]$.

3.2.2 Then $\lim_{n \to \infty} |\nu_V(x_n) - \nu_V(y_n)| \leq \lim_{n \to \infty} \nu_V(x_n - y_n) = \lim_{n \to \infty} \rho_V(x_n, y_n) = 0$.

□

$\texttt{SeminormedSpaceProductEmbedding} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \exists I \in \mathsf{SET} \,.\, \exists W : I \to \texttt{SeminormedSpace} \,.$

$$. \exists U \subset_{k\text{-}\mathsf{VS}} \prod_{i \in I} W_i \,.\, V \cong_{k\text{-}\mathsf{TVS}} W$$

Proof $=$

1 For $\nu \in \mathrm{ssc}(V)$ define $W = (V, \nu)$.

2 Then the mapping $x \mapsto (x)_{\nu \in \mathrm{ssc}(V)}$ is an isomorphism.

$\square$

$\texttt{BanachSpaceProductEmbedding} :: \forall V \in k\text{-}\mathsf{LCHS} \,.\, \exists I \in \mathsf{SET} \,.\, \exists W : I \to \mathsf{BAN}(k) \,.$

$$. \exists U \subset_{k\text{-}\mathsf{VS}} \prod_{i \in I} W_i \,.\, V \cong_{k\text{-}\mathsf{TVS}} W$$

Proof $=$

1 For $\nu \in \mathrm{ssc}(V)$ define $W = \widehat{\left(\frac{V}{\ker \nu}\right)}$.

2 Then each W_ν is an Banach space.

3 Then the mapping $\phi : x \mapsto ([x]_{\ker \nu})_{\nu \in \mathrm{ssc}(V)}$ is an isomorphism.

3.1 ϕ is one-to-one as V is hausdorff.

3.1.1 For any $v \in V$ such that $v \neq 0$ exists $\nu \in \mathrm{ssc}(V)$ such that $\nu(v) \neq 0$.

3.1.2 So $[v]_{\ker \nu} \neq 0$.

$\square$

$\texttt{LCSCompletion} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \exists (\hat{V}, \iota) : \texttt{TVSCompletion}(V) \,.\, \hat{V} \in k\text{-}\mathsf{LCS}$

Proof $=$

1 Construct product emedding $\phi : V \hookrightarrow \prod_{\nu \in \mathrm{ssc}(V)} W_\nu$ as in the previous theorem.

3 This embedding can be extended to the embedding into a complete vecor space $\prod_{\nu \in \mathrm{ssc}(V)} \hat{W}_\nu$.

3.1 The product of complete spaces is complete.

4 Then $\operatorname{cl}_{\hat{W}} \phi(V)$ is a closed subset of the complete space.

5 So $\hat{V} = \operatorname{cl}_{\hat{W}} \phi(V)$ is the sought completion.

$\square$

$\texttt{LCHSCompletion} :: \forall V \in k\text{-}\mathsf{LCHS} \,.\, \exists (\hat{V}, \iota) : \texttt{TVSCompletion}(V) \,.\, \hat{V} \in k\text{-}\mathsf{LCHS}$

Proof $=$

1 Same argument as above.

$\square$

1.1.14 Non-Archimedean Spaces

k : UltravaluedField;

Ultraseminorm :: $\prod_{V \in k\text{-VS}} ?\mathsf{SMN}(V)$

ν : Ultraseminorm $\iff \forall v, w \in V \;.\; \nu(v+w) \leq \max\Big(\nu(v), \nu(w)\Big)$

UltraseminormMaximumPrinciple ::

:: $\forall V \in k\text{-VS} \;.\; \forall v, w \in V \;.\; \forall \nu :$ Ultraseminorm$(V) \;.\; \nu(v) < \nu(w) \Rightarrow \nu(v+w) = \nu(w)$

Proof =

1 $\nu(w+v) \leq \max\Big(\nu(w), \nu(v)\Big) = \nu(w)$.

2 $\nu(w) = \nu\Big(v - (w+v)\Big) \leq \max\Big(\nu(v), \nu(w+v)\Big) = \nu(w+v)$.

2.1 This must be the case as $\nu(v) < \nu(w)$.

3 $\nu(w) = \nu(w+v)$.

$\square$

Ultradisc ::

:: $\forall V \in k\text{-VS} \;.\; \forall \nu :$ Ultraseminorm(V) . AbsolutelyKConvex & Absorbent$\Big(V, \mathbb{B}(\nu)\Big)$

Proof =

1 Assume $v, w \in \mathbb{B}(\nu)$ and $\alpha, \beta \in \mathbb{D}_k(0,1)$.

2 Then $\nu(\alpha v + \beta w) \leq |\alpha|\nu(v) + |\beta|\nu(w) < 1$.

3 So $\mathbb{B}(\nu)$ is K-convex.

4 Take $v \in V$ such that $\nu(v) \neq 0$.

5 Then $\alpha v \in \mathbb{B}(\nu)$ for any $\alpha \in k$ such that $|\alpha| < \nu^{-1}(v)$.

6 So $\mathbb{B}(\nu)$ is absorbent.

$\square$

ultragauge :: $\prod_{V \in k\text{-VS}}$ AbsolutelyKConvex & Absorbent$(V) \to$ Ultraseminorm(V)

ultragauge $(D) = \upsilon(\bullet|D) := \lambda v \in V \;.\; \inf\Big\{|\alpha|\Big|\alpha \in k : v \in \alpha D\Big\}$

1 It is obvious that the ultragauge is a seminorm.

2 Now take $v, w \in V$.

3 Then as D is K-convex $\upsilon(v+w|D) \leq \max\Big(\upsilon(v|D), \upsilon(w|D)\Big)$.

3.1 Take a sequence $\alpha, \beta : \mathbb{N} \to k_*$ such that $\alpha_n v \in D, \beta_n w \in D, \lim_{n\to\infty} |\alpha_n|^{-1} = \upsilon(v|D), \lim_{n\to\infty} |\beta_n|^{-1} = \upsilon(w|D)$.

3.2 Define $\gamma_n = \arg\max_{\tau \in \{\alpha_n, \beta_n\}} |\tau$.

3.3 Then $\gamma_n(v+w) \in D$ as D is K-Convex.

3.4 Then $\upsilon(v+w|D) \leq |\gamma_n| \leq \max\Big(|\alpha_n|, |\beta_n|\Big)$.

3.5 Taking limits gives $\upsilon(v+w|D) \leq \max\Big(\upsilon(v|D), \upsilon(w|D)\Big)$.

$\square$

UltragaugeBound ::

$$:: \forall V \in k\text{-}\mathsf{VS} \ . \ \forall D : \texttt{AbsolutelyKConvex} \ \& \ \texttt{Absorbent}(V) \ . \ \mathbb{B}\Big(\upsilon(\bullet|D)\Big) \subset D \subset \mathbb{D}\Big(\upsilon(\bullet|D)\Big)$$

Proof =

Pretty obvious.

□

UltragaugeContinuity ::

$$:: \forall V \in k\text{-}\mathsf{TVS} \ . \ \forall D : \texttt{AbsolutelyKConvex} \ \& \ \texttt{Absorbent}(V) \ . \ D \in \mathcal{N}_V \iff \upsilon(\bullet|D) \in C(V)$$

Proof =

1 ($\Rightarrow$) Assume D has non-empty interior.

1.1 By previous result this implies that D is open.

1.2 Then $\upsilon^{-1}\Big([0,\rho), D\Big) = \bigcup_{\alpha \in \mathbb{D}(0,\rho)} \alpha D$.

1.3 But αD is also open as multiplication by α is a homeomorphism.

1.4 So the ultraguage must be continuous.

2 ($\Leftarrow$) Assume that ultragauge is continuous.

2.1 Then $\upsilon^{-1}\Big([0,\rho), D\Big) \subset D$.

2.2 So D has non-empty interior.

□

$$\texttt{topologyOfUltraseminorms} :: \prod_{V \in k\text{-}\mathsf{VS}} ?\texttt{Ultraseminorm}(V) \to \texttt{VectorTopology}(V)$$

$$\texttt{topologyOfUltraseminorms}\,(\Upsilon) = \mathcal{T}(\Upsilon) := \mathcal{W}_V(\Upsilon, \mathbb{R}, \mathrm{id})$$

UltraseminormsDefineLocallyKConvexTopology ::

$$:: \forall V \in k\text{-}\mathsf{VS} \ . \ \forall \Upsilon : ?\texttt{Ultraseminorm}(V) \ . \ \texttt{LocallyKConvexSpace}(k, V, \mathcal{T}(\Upsilon))$$

Proof =

1 Take $\upsilon \in \Upsilon$.

2 Then $\mathbb{B}(\upsilon)$ is absolutely K-convex.

2.1 See ultradisc theorem.

□

LocallyKConvexTopologyIsGeneratedByUltraseminorms ::

$$:: \forall V : \texttt{LocallyKConvexSpace}(k) \ . \ \exists \Upsilon : ?\texttt{Ultraseminorm}(V) \ . \ \mathcal{T}_V = \mathcal{T}(\Upsilon)$$

Proof =

Take ultragauges for the K-discs generating the locally K-convex topology.

□

$$\texttt{definingUltraseminorms} :: \prod V : \texttt{LocallyKConvexSpace}(k) \ . \ ?\texttt{Ultraseminorm}(V)$$

$$\texttt{definingUltraseminorms}\,(V) = \mathrm{suc} := C(V) \cap \texttt{Ultraseminorm}(V)$$

Ultrasemimetrization ::

:: $\forall V \in$ LocallyKConvexSpace(k) . Ultrasemimetrizable$(V) \iff$

$\iff \exists : \upsilon : \mathbb{N} \uparrow$ Ultraseminorm(V) . $\mathcal{T}_V = \mathcal{T}(\operatorname{Im} \upsilon)$

Proof $=$

1 This is simmilar to normal semimetrization theorem .

2 Define an F-seminorm $\mu(v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\upsilon_n(v)}{1 + \upsilon_n(v)}$.

3 The only difference is in the proving the ulrametric property.

3.1 Take some $v, w \in V$.

3.2 Then $\upsilon_n(v + w) \leq \max\Big(\upsilon_n(v), \upsilon_n(w)\Big)$.

3.3 But as th function $\frac{x}{x+1}$ is increasing $\frac{\upsilon_n(v+w)}{1+\upsilon_n(v+w)} \leq \max\left(\frac{\upsilon_n(w)}{1+\upsilon_n(v)}, \frac{\upsilon_n(w)}{1+\upsilon_n(w)}\right)$.

4 Thus $\mu(v + w) \leq \max\Big(\mu(v), \mu(w)\Big)$ for any $v, w \in V$.

5 So μ defines an ultrasemimetric.

$\square$

LocallyCCompact $:: ?k\text{-TVS}$

V : LocallyCCompact $\iff \exists \mathcal{F}$: Filterbase$(\mathcal{N}_0(V))$. $\forall F \in \mathcal{F}$. CCompact & AbsolutelyKConvex(V, F)

Ultranorm $:: \prod_{V \in k\text{-VS}} ?$Ultraseminorm$(V)$

υ : Ultranorm $\iff \forall v \in V . \upsilon(v) = 0 \iff v = 0$

UltranormedSpace $:: ? \sum_{V \in k\text{-TVS}}$ Ultraseminorm(V)

(V, υ) : UltranormedSpace $\iff \mathcal{T}_V = \mathcal{T}\{\upsilon\}$

$|\cdot|_k \neq \Lambda\alpha \in k . [k \neq 0]$

LocallyCCompactHasLocllyCCompactField :: $\forall V$: LocallyCCompact(k) . $\dim V > 0 \Rightarrow$ LocallyCCompact$(k,$

Proof $=$

1 As k has non-trivial valuation Every one-dimensional subspace of V is isomorphic to k.

2 Let L be such one-dimensional subspace.

3 And let C be a C-compact neighborhood of 0 in V.

4 The $C \cap L$ is C-compact and and reltively open in L.

5 So L is locally compact.

6 And so is k as it is isomorphic to L.

$\square$

$\texttt{LocallyCCompactIsFinDim} :: \forall V : \texttt{Ultranormed} \;\&\; \texttt{LocallyCCompact}(k)\;.\;\dim V < \infty$

$\texttt{Proof} \;=$

1 Assume that $\dim V = \infty$.

2 As V is Locally C-compact there is a C-compact neighborhood C of 0 in V.

3 Then there is a ball $D \subset C$ if radius ρ.

4 Select a topologically linearly independent seqyence $(e_i)_{i=1}^{\infty}$ such that $\|e_i\| = \rho$.

5 Define $F_i = \operatorname*{cl}_V K\text{-conv}\,(e_j)_{j=i}^{\infty}$.

6 Then $(F_i)_{i=1}^{\infty}$ is a closed k-convex filterbase on C.

6.1 The K-convex filterbase property is obvious.

6.2 As all points e_i, e_j are separated, sets $(e_j)_{j=i}^{\infty}$ are closed.

6.3 And k-convex hull of closed sets must be closed.

7 This mean that $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$ as C is C-convex.

8 On the other hand, clearly $\bigcap_{i=1}^{\infty} F_i = \emptyset$, a contradiction!

8.1 Assume $v \in \bigcap_{i=1}^{\infty} F_i = \emptyset$.

8.2 Then $v \in \operatorname{span}(e_i)_{i=1}^{\infty}$ by construction.

8.3 But as $v \in F_{i=1}$ it means that its e_i coefficient must be 0.

8.4 So it must be the case that $v = 0$.

8.5 But 0 do not belong to any F_i.

$\square$

$\texttt{LocallyCCompactIsCCompact} :: \forall V : \texttt{LocallyCCompact} \;\&\; \texttt{LocallyKConvexSpace}(k)\;.\;\texttt{CCompact}(V)$

$\texttt{Proof} \;=$

1 k is C-compact.

1.1 Let $\mathcal{F}$ be a K-convex Filter on k .

1.2 Then $\mathcal{F}$ can be structured as a monotonic sequence of balls.

1.3 If $\mathcal{F}$ there is a ball D such tha all small enough elements $F \in \mathcal{F}$ are in D.

1.4 but all closed discs are isomorphic in k.

1.5 Thus D is C-compact.

1.6 So $\mathcal{F}$ must have an adherence point in D.

1.7 So it also has an adherence point in k, and k is C-compact.

2 Then $V \cong k^n$ as V must be finite-dimensional.

3 And k^n is C-compact as a product of C-compact sets.

$\square$

1.2 Towards Bornology

1.2.1 Bounded Sets

k : AbsoluteValueField($\mathbb{R}$) | UltravaluedField;

Bounded :: $\prod V \in k\text{-TVS} \,.\, ??V$

B : Bounded $\iff \forall U \in \mathcal{U}_V(0) \,.\, \exists \lambda \in \mathbb{R} \,.\, \forall \alpha \in k \,.\, |\alpha| \geq \lambda \Rightarrow B \subset \alpha U$

BoundedByBase ::

:: $\forall V \in k\text{-TVS} \,.\, \forall B \subset V \,.\, \forall \beta :$ BalancedBaseBase(V) . Bounded$(V) \iff \forall U \in \mathcal{B} \,.\, \exists \alpha \in k \,.\, \alpha B \subset U$

Proof =

Obvious.

□

BoundedBySeminirms ::

:: $\forall V \in k\text{-LCS} \,.\, \forall B \subset V \,.\, \forall \beta :$ BalancedBaseBase(V) . Bounded$(V) \iff \forall \nu \in \mathrm{ssc}$. Bounded$(B, \nu_{|B})$

Proof =

Obvious.

□

TotallyBoundedIsBounded :: $\forall V \in k\text{-TVS} \,.\, \forall B :$ TotallyBounded(V) . Bounded(V, B)

Proof =

1 Assume $U \in \mathcal{U}_V(0)$.

2 Then where exists a balanced and absorbing $W \in \mathcal{U}_V(0)$ such that $W + W \subset V$.

3 As B is totally bounded there is a finite subset $F \subset V$ such that $B \subset W + F$.

4 As W is absorbing there exists $\alpha \in k$ such that $F \subset \alpha W$.

5 Without loss of generality we may assume that $|\alpha| > 1$.

5 So, as W is balanced $W \subset \alpha W$.

6 Thus, $B \subset \alpha W + \alpha W = \alpha(W + W) \subset \alpha U$.

□

KolomogorovsBoundednessCriterion ::

:: $\forall V \in k\text{-TVS} \,.\, \forall B \subset V \,.\,$ Bounded$(V, B) \iff \forall \alpha : \mathbb{N} \to k \,.\, \forall b : \mathbb{N} \to B \,.\, \left(\lim_{n\to\infty} \alpha_n = 0 \Rightarrow \lim_{n\to\infty} \alpha_n b_n = 0 \right)$

Proof =

1 ($\Rightarrow$) This direction is obvious.

2 ($\Leftarrow$) Assume B is not bounded.

2.1 Then there is an $U \in \mathcal{U}_V(0)$ such that for any $\rho \in \mathbb{R}_{++}$ there is $\alpha \in k$ such that $|\alpha| \geq \rho$ an $U \not\subset \alpha B$.

2.2 So there exists sequences α with $|\alpha_n| \leq \dfrac{1}{n}$ and $b : \mathbb{N} \to B$ such that $\alpha_n b_n \notin U$.

2.3 $|\alpha_n| \leq \dfrac{1}{n}$ imply that $\lim_{n\to\infty} \alpha_n = 0$.

2.4 On the other hand $\alpha_n b_n \notin U$ imply that $\lim_{n\to\infty} \alpha_n bbb_n \neq 0$.

2.5 This conradicts an initial assumption.

□

$\texttt{BoundednesByCountableSubsets} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall B \subset V \,.\, \texttt{Bounded}(V, B) \iff \forall C : \texttt{CountableSubset}(V, B$

$\texttt{Proof} =$

This follows from Kolmogorov's ctiterion.

$\square$

$\texttt{BoundedMetrizationTHM} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall N \in \mathcal{N}_V(0) \,.\, \texttt{Bounded}(V, N) \Rightarrow \texttt{Semimetrizable}(V)$

$\texttt{Proof} =$

$\left(\frac{1}{n}N\right)_{n=1}^{\infty}$ is a countable base of vector topology for V.

$\square$

$\texttt{BoundedNormizationTHM} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall N \in \mathcal{N}_V(0) \,.\, \texttt{Bounded} \,\&\, \texttt{Disc}(V, N) \Rightarrow \texttt{Seminormable}(V)$

$\texttt{Proof} =$

Topology may be determined by $\gamma(\bullet|N)$.

$\square$

1.2.2 Stability under Operations

SubsetOfBounded :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall B : \mathtt{Bounded}(V) \,.\, \forall C \subset B \,.\, \mathtt{Bounded}(V, B)$

Proof =

Obvious.

$\square$

BoundedUnion :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall B, C : \mathtt{Bounded}(V) \,.\, \mathtt{Bounded}(V, B \cap C)$

Proof =

Select max.

$\square$

BoundedScale :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall B : \mathtt{Bounded}(V) \,.\, \forall \alpha \in k \,.\, \mathtt{Bounded}(V, \alpha B)$

Proof =

Rescale.

$\square$

BoundedSum :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall B, C : \mathtt{Bounded}(V) \,.\, \forall \alpha \in k \,.\, \mathtt{Bounded}(V, B + C)$

Proof =

Assume $U \in \mathcal{U}_V(0)$.

Select $V \in \mathcal{U}_V(0)$ such that $V + V \subset U$.

Then there are two V-absorbtion factors ρ and σ for B and C respectively.

If $\alpha \in k$ is such that $|\alpha| \geq \max(\rho, \sigma)$, then $B + C \subset \alpha V + \alpha V = \alpha(V + V) \subset \alpha U$.

$\square$

BoundedQuotient :: $\forall V \in k\text{-}\mathsf{TVS} \,.\, \forall W \subset_{k\text{-}\mathsf{VS}} \,.\, \forall B : \mathtt{Bounded}(V) \,.\, \forall \mathtt{Bounded}\left(\frac{V}{W}, \pi_W(B)\right)$

Proof =

Use the preimage to determine the absorbtion factor.

$\square$

BoundedProducts :: $\forall I \in \mathsf{SET} \,.\, \forall V : I \to k\text{-}\mathsf{TVS} \,.\, \forall B : \prod_{i \in \mathcal{I}} \mathtt{Bounded}(V_i) \,.\, \mathtt{Bounded}\left(\prod_{i \in I} V_i, \prod_{i \in I} B_i\right)$

Proof =

Assume $U \in \mathcal{U}_{\prod_{i \in I} V_i}(0)$.

Then there exists $W \in \prod_{i \in I} \mathcal{T}(V_i)$ such that that $W_i \neq V_i$ only for a finite set of indices $J \subset I$ and $\prod_{i \in I} W_i \subset U$.

Then find a W_i-absorbtion factor ρ_i for each $i \in J$.

Then $\prod_{i \in I} B_i \subset \alpha \prod_{i \in I} W_i \subset \alpha U$ for any $\alpha \in k$ with $|\alpha| \geq \max_{i \in J} \rho_i$.

$\square$

$\texttt{BoundedClosure} :: \forall V \in k\text{-}\mathsf{TVS} \ . \ \forall B : \texttt{Bounded}(V) \ . \ \texttt{Bounded}(V, \overline{B})$

$\texttt{Proof} \ =$

$\overline{B}$ is bounded for the base of closed neighborhoods of unity.

Thus,$\overline{B}$ is bounded in a general sence.

$\square$

$\texttt{BoundedBalancedHull} :: \forall V \in k\text{-}\mathsf{TVS} \ . \ \forall B : \texttt{Bounded}(V) \ . \ \texttt{Bounded}(V, \operatorname{bal} B)$

$\texttt{Proof} \ =$

$\operatorname{bal} B$ is bounded for the base of balanced neighborhoods of unity.

Thus,$\overline{B}$ is bounded in a general sence.

$\square$

$\texttt{BoundedConvexHull} :: \forall V \in k\text{-}\mathsf{LCS} \ . \ \forall B : \texttt{Bounded}(V) \ . \ \texttt{Bounded}(V, \operatorname{conv} B)$

$\texttt{Proof} \ =$

$\operatorname{conv} B$ is bounded for the base of disced neighborhoods of unity.

Thus,$\overline{B}$ is bounded in a general sence.

$\square$

$$\texttt{BoundedBase} :: \prod_{V \in k\text{-}\mathsf{TVS}} ??\texttt{Bounded}(V)$$

$\beta : \texttt{BoundedBase} \iff \forall B : \texttt{Bounded}(V) \ . \ \exists B' \in \beta \ . \ B \subset B'$

$\texttt{ClosedDiscsAsBoundedBase} :: \forall V \in k\text{-}\mathsf{LCS} \ . \ \texttt{BoundedBase}\Big(V, \texttt{Closed} \ \& \ \texttt{Disc}(V)\Big)$

$\texttt{Proof} \ =$

Assume B is bounded in V.

Then the disced hull of B is also bounded.

$\square$

1.2.3 Locally Bounded Maps

$k : \texttt{AbsoluteValueField}(\mathbb{R});$

$$\texttt{LocallyBounded} :: \prod_{V,W:k\text{-}\mathsf{TVS}} ?(V \to W)$$

$$f : \texttt{LocallyBounded} \iff \forall B : \texttt{Bounded}(V) \ . \ \texttt{Bounded}\Big(W, f(B)\Big)$$

$$\texttt{Homogeneous} :: \prod_{V,W:k\text{-}\mathsf{VS}} ?(V \to W)$$

$$f : \texttt{Homogeneous} \iff \exists \delta \in \mathbb{R}_{++} \ . \ \forall v \in V \ . \ \forall \rho \in \mathbb{R}_{++} \ . \ f(\rho v) = \rho^{\delta} f(v)$$

$\texttt{ContunuousHomogenuousIsLocallyBounded}$ ::

$$:: \forall V, W \in k\text{-}\mathsf{TVS} \ . \ \forall f : \mathsf{TOP} \ \& \ \texttt{Homogeneous}(V, W) \ . \ \texttt{LocallyBounded}(V, W, f)$$

$\texttt{Proof}$ =

Pretty obvious if you use basic properties.

□

$$\texttt{BoundedProductsConverse} :: \forall I \in \mathsf{SET} \ . \ \forall V : I \to k\text{-}\mathsf{TVS} \ . \ \forall B \subset \prod_{i \in \mathcal{I}} V_i \ .$$

$$. \ \Big(\forall i \in I \ . \ \texttt{Bounded}(V_i, \pi_i(B))\Big) \iff \texttt{Bounded}\left(\prod_{i \in I} V_i, \prod_{i \in I} B_i\right)$$

$\texttt{Proof}$ =

1 $(\Rightarrow)$.

1.1 As $\pi_i(B)$ is bounde, so is $\prod_{i \in I} \pi_i(B)$.

1.2 Then B is bounded as $B \subset \prod_{i \in I} \pi_i(B)$.

2 $(\Leftarrow)$.

2.1 In product topology each π_i is continuous linear and so locally bounded.

□

$\texttt{MultilinearIsLocallyBounded}$ ::

$$:: \forall n \in \mathbb{N} \ . \ \forall V : \{1, \ldots, n\} \to k\text{-}\mathsf{TVS} \ . \ \forall W \in k\text{-}\mathsf{TVS} \ .$$

$$. \ \forall T \in \mathcal{L}(V; W)\mathsf{TOP}\left(\prod_{i=1}^{n} V_i, W\right) \ . \ \texttt{LocallyBounded}\left(\prod_{i=1}^{n} V_i, W, T\right)$$

$\texttt{Proof}$ =

Multilinear maps are homogeneous of degree n.

□

BoundedSetsInWeakTopology :: $\forall V \in k\text{-VS} . \forall I \in \mathsf{SET} . \forall W : I \to k\text{-TVS}$.

$$. \forall T : \prod_{i \in I} k\text{-VS}(V, W_i) . \forall B \subset V . \mathtt{Bounded}\Big((V, \mathcal{W}(I, W, T)), B\Big) \iff \forall i \in I . \mathtt{Bounded}\Big(W_i, T_i(B)\Big)$$

Proof =

1 This is simmilar to the case with products.

2 We may assume that topology is determined by one map $T : V \to \prod_{i \in I} W_i$.

3 Then $\prod_{i \in I} T_i(B)$ is bounded in $\prod_{i \in I} W_i$.

4 Assume U is a neighborhood in the weak topology .

5 Then it must be a preimage of some open $O \in \prod_{i \in I} W_i$.

6 So find an O-absorbing scale ρ for $\prod_{i \in I} T_i(B)$ and use it as U-absorbing scale for B.

6.1 Take some $\alpha \in k$ such that $|\alpha| \geq \rho$.

6.2 Then $T(b) \in \alpha O$ for any $b \in B$.

6.3 By thaking inverse image $b \in T^{-1}(\alpha O) = \alpha T^{-1}(O) = \alpha U$.

□

ContinuityByBoundedImage ::

$$:: \forall V, W \in k\text{-TVS} . \forall T \in k\text{-VS}(V, W) . \forall U \in \mathcal{U}_V(0) . \mathtt{Bounded}\Big(W, T(U)\Big) \Rightarrow T \in k\text{-TVS}(V, W)$$

Proof =

Assume $O \in \mathcal{U}_W(0)$.

Then there exists an $\rho \in \mathbb{R}_{++}$ such that $T(U) \subset rO$.

But this means that $T(r^{-1}U) \subset O$.

Then by topological group theory T is continuous.

□

LocallyBoundedWithSemimetrizableDomainIsContinuous ::

$$:: \forall V, W \in k\text{-TVS} . \forall T \in k\text{-VS} \ \& \ \mathtt{LocallyBounded}(V, W) . \mathtt{Semimetrizable}(V) \Rightarrow T \in k\text{-TVS} .$$

Proof =

1 Let U be a decreasing countable base of neighborhoods of 0 in V.

2 Assume that T is discontinuous.

2.1 By group topology T must be discontinuous at 0.

2.2 Then there is a $O \in \mathcal{U}_W(0)$ such that $T^{-1}(O)$ is not a neighborhood of 0 in V.

2.3 So $\frac{1}{n} U_n \not\subset T^{-1}(O)$.

2.4 It must be possible select a sequence u such that $u_n \in U_n$ and $Tu_n \notin O$.

2.5 As U is a neighborhood baseit follows that $\lim_{n \to \infty} nu_n = 0$.

2.6 This means that $\{nu_n | n \in \mathbb{N}\}$ is bounded.

2.6.1 Given $E \in \mathcal{U}_V(0)$ there is only finite amount of numbers n such that $nu_n \notin E$.

2.6.2 So it is posible to find E-absorbtion scales for this finite number and take max.

2.7 So nTu_n can be also be viewed as a sequence in a bounded subset of W.

2.8 So there exist an O-absorbtion scale α for nTu_n.

2.9 That is $nTu_n \in \alpha O$ for all $n \in \mathbb{N}$.

2.10 By archemedian property there exists $n \in \mathbb{N}$ such that $n \geq \alpha$, so $Tu_n \in O$, a contradiction!

□

1.2.4 Liouville's Theorem

$$\texttt{Bounded} :: \prod_{X \in \texttt{Set}} \prod_{V \in k\text{-TVS}} ?(X \to V)$$

$$f : \texttt{Bounded} \iff \texttt{Bounded}\Big(V, f(X)\Big)$$

$$\texttt{Analytic} :: \prod_{U \in \mathcal{T}(\mathbb{C})} \prod_{V \in \mathbb{C}\text{-TVS}} ?(X \to V)$$

$$f : \texttt{Analytic} \iff \forall u \in U \,.\, \exists v \in V \,.\, \lim_{z \to u} \frac{f(z) - f(u)}{z - u} = v$$

$$\texttt{Entire} := \Lambda V \in \mathbb{C}\text{-TVS} \,.\, \texttt{Analytic}(\mathbb{C}, V) : \mathbb{C}\text{-TVS} \to \texttt{Type};$$

$\texttt{ContinuousComposition}$ ::

$$:: \forall U \in \mathcal{T}(\mathbb{C}) \,.\, \forall V \in \mathbb{C}\text{-TVS} \,.\, \forall v : \texttt{Analytic}(U, V) \,.\, \forall f \in V' \,.\, \texttt{Analytic}\Big(U, \mathbb{C}, f(v)\Big)$$

$\texttt{Proof}$ =

Use the continuity of f on the limit, which defines derivative.

□

$$\texttt{Total} :: \prod_{V \in k\text{-VS}} ??V$$

$$A : \texttt{Total} \iff \forall v \in V \,.\, \Big(\forall f \in A \,.\, f(v) = 0\Big) \Rightarrow v = 0$$

$\texttt{LiouvillesTheorem}$ ::

$$:: \forall V \in \mathbb{C}\text{-TVS} \,.\, \forall v : \texttt{Bounded}(\mathbb{C}, V) \,\&\, \texttt{Entire}(V) \,.\, \forall \aleph : \texttt{Total}(V, V') \,.\, \texttt{Type}Constant(\mathbb{C}, V, v)$$

$\texttt{Proof}$ =

$f(v)$ is an entire bounded function for every $f \in V'$.

So $f(v)$ must be constant by classical Liouville theorem.

But this means that $f\Big(v(\alpha) - v(\beta)\Big) = f\Big(v(\alpha)\Big) - f\Big(f(\beta)\Big) = 0$ for every $\alpha, \beta \in \mathbb{C}$.

But as V' is total this means that v is constant.

□

1.2.5 p-convexity

$$\texttt{PConvex} :: \prod_{V \in \mathbb{R}\text{-VS}} \mathbb{R}_{++} \to ??V$$

$$A : \texttt{PConvex} \iff \Lambda p \in \mathbb{R}_{++} \,.\, \forall \alpha, \beta \in \mathbb{R}_{++} \,.\, \forall v, w \in A \,.\, \alpha^p + \beta^p = 1 \Rightarrow \alpha v + \beta w \in A$$

$$\texttt{AbsolutelyPConvex} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \to ??V$$

$$A : \texttt{AbsolutelyPConvex} \iff \Lambda p \in \mathbb{R}_{++} \,.\, \forall \alpha, \beta \in k \,.\, \forall v, w \in A \,.\, |\alpha|^p + |\beta|^p \le 1 \Rightarrow \alpha v + \beta w \in A$$

$$\texttt{PSeminorm} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \to ?\texttt{Sublinear}(V, \mathbb{R})$$

$$\nu : \texttt{PSeminorm} \iff \Lambda p \in \mathbb{R}_{++} \,.\, \forall \alpha \in k \,.\, \forall v \in A \,.\, \|\alpha v\| = |\alpha|^p \|v\|$$

$$\texttt{PSeminorm} :: \prod_{V \in k\text{-VS}} \mathbb{R}_{++} \to ?\texttt{Sublinear}(V, \mathbb{R})$$

$$\nu : \texttt{PSeminorm} \iff \Lambda p \in \mathbb{R}_{++} \,.\, \forall \alpha \in k \,.\, \forall v \in A \,.\, \|\alpha v\| = |\alpha|^p \|v\|$$

$$\texttt{pSeminormedTopology} :: \prod_{V \in k\text{-VS}} \texttt{PSeminorm}(V, k, p) \to \texttt{Top}\vec{\texttt{o}}\texttt{logy}(V)$$

$$\texttt{pSeminormedTopology}\,(\nu) = \mathcal{T}(\nu) := \left\langle \left\{ \{w \in W : \nu(v - w) < \rho\} \middle| v \in V, \rho \in \mathbb{R}_{++} \right\} \right\rangle$$

$$\texttt{PSeminormable} :: \mathbb{R}_{++} \to ?k\text{-TVS}$$

$$V : \texttt{PSeminormable} \iff \Lambda p \in \mathbb{R}_{++} \,.\, \exists \nu : \texttt{PSeminorm}(V) \,.\, \mathcal{T}(V) = \mathcal{T}(\nu)$$

$\texttt{PSeminormableSpace}$ ::

$$:: \forall V \in k\text{-TVS} \,.\, \forall p \in \mathbb{R}_{++}\texttt{PSeminormable}(V, p) \iff \exists U \in \mathcal{U}_V(0) \,.\, \texttt{Bounded}(V, U) \;\&\; \texttt{PConvex}(V, p, U)$$

$\texttt{Proof}$ =

$\left(\frac{1}{n}U\right)_{n=1}^{\infty}$ is a countable base of vector topology for V.

The gauges defined by U are p-seminorms.

□

1.2.6 Bornology

$k : \texttt{AbsoluteValueField}(\mathbb{R}) \Big| \texttt{UltravaluedField};$

$\texttt{Bornology} := \Lambda X \in \mathsf{SET} \; . \; \texttt{Ideal}(2^X) : \mathsf{SET} \to \texttt{Type};$

$$\texttt{BoundedStructrure} := \sum_{X \in \mathsf{SET}} \texttt{Bornology}(X) : \texttt{Type};$$

$\texttt{asSet} \; :: \; \texttt{BoundedStructure} \to \texttt{Set}$
$\texttt{asSet}\,(X, \beta) = (X, \beta) := X$

$\texttt{bornology} \; :: \; \prod (X, \beta) : \texttt{BoundedStructure} \; . \; \texttt{Bornology}(X)$
$\texttt{bornology}\,() = \mathcal{B}(X, \beta) := \beta$

$\texttt{Bounded} \; :: \; \prod X : \texttt{BoundedStructure} \; . \; ??X$
$B : \texttt{Bounded} \iff B \in \mathcal{B}(X)$

$\texttt{CompactsAreBornology} \; :: \; \forall X \in \mathsf{TOP} \; . \; \texttt{Bornology}\Big(X, \texttt{RelativeCompcts}(X)\Big)$

Proof $=$

This is obvious.

$\square$

$\texttt{standardBornology} \; :: \; k\text{-}\mathsf{TVS} \to \texttt{BoundedStructure}$
$\texttt{standardBornology}\,(V) = V := \Big(V, \texttt{Bounded}(V)\Big)$

$\texttt{BornologyBase} \; :: \; \prod X : \texttt{BoundedStructure} \; . \; ??\mathcal{B}(X)$
$\mathcal{C} : \texttt{BornologyBase} \iff \forall B \in \mathcal{B}(X) \; . \; \exists C \in \mathcal{C} \; . \; B \subset C$

$$\texttt{generateBornology} \; :: \; \prod_{X \in \mathsf{SET}} ??X \to \texttt{Bornology}(X)$$
$$\texttt{generateBornology}\,(\alpha) = \langle \alpha \rangle_{\mathsf{BORN}} := \left\{ A \subset X : \exists n \in \mathbb{N} \; . \; \exists C : \{1, \ldots, n\} \to \alpha \; . \; A \subset \bigcup_{i=1}^{n} C_i \right\}$$

$\texttt{bornologicalCategory} \; :: \; \mathsf{CAT}$
$\texttt{bornologicalCategory}\,() = \mathsf{BORN} :=$
$$:= \Big(\texttt{BoundedStructure}, \Lambda X, Y : \texttt{BoundedStructure} \; . \; \{f : X \to Y \; . \; \forall B \in \mathcal{B}(X) \; . \; f(B) \in \mathcal{B}(Y)\}, \circ, \mathrm{id}\Big)$$

$$\texttt{strongBornology} :: \prod_{X \in \mathsf{SET}} \prod Y : \texttt{BoundedStructure} \ . \ (X \to Y) \to \texttt{Bornology}(X)$$

$$\texttt{strongBornology}\,(f) = \mathcal{S}(Y, f) := \langle f^{-1}\mathcal{B}(Y)\rangle_{\mathsf{BORN}}$$

$$\texttt{weekBornology} :: \prod_{Y \in \mathsf{SET}} \prod X : \texttt{BoundedStructure} \ . \ (X \to Y) \to \texttt{Bornology}(Y)$$

$$\texttt{weakBornology}\,(f) = \mathcal{W}(X, f) := \langle f\mathcal{B}(X)\rangle_{\mathsf{BORN}}$$

By use of week and strong notions, we may define subset bornology, quotient bornology or any kind of limit bornologies.

$$\texttt{supBornology} :: \prod_{X, I \in \mathsf{SET}} \Big(I \to \texttt{Bornology}(X)\Big) \to \texttt{Bornology}(X)$$

$$\texttt{supBornology}\,(\beta) = \bigvee_{i \in I} \beta_i := \left\langle \bigcup_{i \in I} \beta_i \right\rangle_{\mathsf{BORN}}$$

$$\texttt{infBornology} :: \prod_{X, I \in \mathsf{SET}} \Big(I \to \texttt{Bornology}(X)\Big) \to \texttt{Bornology}(X)$$

$$\texttt{infBornology}\,(\beta) = \bigwedge_{i \in I} \beta_i := \left\langle \bigcap_{i \in I} \beta_i \right\rangle_{\mathsf{BORN}}$$

This shows that a set of bornologies forms acomplete lattice.

$$\texttt{VectorBornology} :: \prod V \in k\text{-}\mathsf{VS} \ . \ ?\texttt{Bornology}(V)$$

$$\beta : \texttt{VectorBornology} \iff +_V \in \mathsf{BORN}\Big((V, \beta) \times (V, \beta), (V, \beta)\Big) \ \& \ \cdot_V \in \mathsf{BORN}\Big(k \times (V, \beta), (V, \beta)\Big)$$

$$\texttt{ConvexBornology} :: \prod V \in k\text{-}\mathsf{VS} \ . \ ?\texttt{VectorBornology}(V)$$

$$\beta : \texttt{ConvexBornology} \iff \exists \gamma : \texttt{BornologyBase}(V, \beta) \ . \ \forall B \in \gamma \ . \ \texttt{Convex}(V, B)$$

$\texttt{VectorBornologyCharacterisation}$ ::

:: $\forall V \in k\text{-}\mathsf{VS} \ . \ \forall \beta : \texttt{Bornology}(V)$.

. $\texttt{VectorBornology}(V, \beta) \iff \forall A, B \in \beta \ . \ A + B \in \beta \ \& \ \forall A \in \beta \ . \ \mathrm{bal}\, A \in \beta$

$\texttt{Proof}$ =

1 ($\Rightarrow$).

1.1 $A + B \in \beta$ as addition is locally bounded.

1.2 bal $A = \mathbb{D}_k(0, 1)A$ and scalar multiplication is uniformly bounded.

2 ($\Leftarrow$) .

2.1 $A + B \in \beta$ implies that addition is locally bounded .

2.2 $\mathbb{D}_k(0, r)A \in \beta$.

2.2.1 By archimedean property of $\mathbb{R}$ there is $n \in \mathbb{N}$ such that $n \geq r$.

2.2.2 But $\mathbb{D}_k(0, r)A \subset \mathbb{D}_k(0, n)A \subset \sum_{i=1}^{n} \mathbb{D}(0, 1)A = \sum_{i=1}^{n} \mathrm{bal}\, A \in \beta$.

2.2.3 As β is ideal $\mathbb{D}_k(0, r)A$.

2.3 As k has bornology base of discs the scalar multiplication must be continuous.

□

$\texttt{EquicontinuousBornology} :: \forall X \in \mathsf{TOP} \ . \ \texttt{VectorBornology}\Big(\mathsf{TOP}(X,k), \texttt{Equicontinuous}(X,k)\Big)$

$\texttt{Proof} \ =$

1 Denote by η the set of equicontinuous subsets of $\mathsf{TOP}(X,k)$.

2 It is obvious that η is downwards closed.

3 η is also closed under finite unions.

3.1 Assume $A, B \in \eta$, also assume $U \in \mathcal{U}_k(0)$ and $x \in X$.

3.2 Then there exists $V \in \mathcal{U}_X(x)$ such that $f(V) \subset U + f(x)$ for all $f \in A$.

3.3 Also there is $W \in \mathcal{U}_X(x)$ such that $f(W) \subset U + f(x)$ for all $f \in B$.

3.4 Then taking $V \cap W$ should for $A \cup B$.

4 Also η is closed under addition.

4.1 Assume $A, B \in \eta$, also assume $U \in \mathcal{U}_k(0)$ and $x \in X$.

4.2 Then there exists $O \in \mathcal{U}_k(0)$ such that $O + O \subset U$.

4.3 Then there exists $V \in \mathcal{U}_X(x)$ such that $f(V) \subset O + f(x)$ for all $f \in A$.

4.4 Also there is $W \in \mathcal{U}_X(x)$ such that $f(W) \subset O + f(x)$ for all $f \in B$.

4.5 A function $h \in A + B$ can be expressed as $h = f + g$ for $f \in A$ and $g \in B$.

4.6 So $h(V \cap W) = f(V \cap W) + g(V \cap W) \subset O + O + f(x) + g(x) \subset U + h(x)$.

5 Scalar multiplication is locally bounded.

5.1 Assume $A \in \eta$, also assume $U \in \mathcal{U}_k(0)$ and $x \in X$.

5.2 Then there exist a balanced $W \in \mathcal{U}_k(0)$ such that $W \subset U$.

5.3 Then there exists $V \in \mathcal{U}_X(x)$ such that $f(V) \subset W + f(x)$ for all $f \in A$.

5.4 Then for any $f \in \operatorname{bal} A = \mathbb{D}_k(0,1)A$ there is $g \in A$ and $\alpha \in \mathbb{D}_k(0,1)$ such that $f = \alpha g$.

5.5 Then $f(V) = \alpha g(V) \subset \alpha W + \alpha g(x) = W + f(x) \subset U + f(x)$.

$\square$

$$\texttt{closure} :: \prod_{X \in \mathsf{TOP}} \texttt{Bornology}(X) \to \texttt{Bornology}(X)$$

$$\texttt{closure}\,(\beta) = \operatorname{cl}\beta := \Big\langle \{\operatorname{cl} B | B \in \beta\} \Big\rangle_{\mathsf{BORN}}$$

$$\texttt{interior} :: \prod_{X \in \mathsf{TOP}} \texttt{Bornology}(X) \to \texttt{Bornology}(X)$$

$$\texttt{interior}\,(\beta) = \operatorname{int}\beta := \Big\langle \{\operatorname{int} B | B \in \beta\} \Big\rangle_{\mathsf{BORN}}$$

$\texttt{InteriorClosureOrder} :: \forall X \in \mathsf{TOP} \ . \ \forall \beta : \texttt{Bornology}(X) \ . \ \operatorname{int}\beta \subset \beta \subset \operatorname{cl}\beta$

$\texttt{Proof} \ =$

This follows from the fact that β is closed under taking susets.

And $\operatorname{int} A \subset A \subset \operatorname{cl} A$ for any $A \subset X$.

$\square$

$\texttt{MonotonicInterior} :: \forall X \in \mathsf{TOP} \ . \ \forall \alpha, \beta : \texttt{Bornology}(X) \ . \ \alpha \subset \beta \Rightarrow \operatorname{int}\alpha \subset \operatorname{int}\beta$

$\texttt{Proof} \ =$

Obvious.

$\square$

$\texttt{MonotonicClosure} :: \forall X \in \mathsf{TOP} . \forall \alpha, \beta : \texttt{Bornology}(X) . \alpha \subset \beta \Rightarrow \operatorname{cl} \alpha \subset \operatorname{cl} \beta$

$\texttt{Proof} =$

Obvious.

□

$\texttt{Open} :: \forall X \in \mathsf{TOP} . ?\texttt{Bornology}(X)$

$\beta : \texttt{Open} \iff \operatorname{int} \beta = \beta$

$\texttt{Closed} :: \forall X \in \mathsf{TOP} . ?\texttt{Bornology}(X)$

$\beta : \texttt{Closed} \iff \operatorname{cl} \beta = \beta$

$\texttt{Proper} := \texttt{Closed} \,\&\, \texttt{Open} : \prod_{X \in \mathsf{TOP}} ?\texttt{Bornology}(X);$

$\texttt{ClodednessAltDef} ::$

$. \forall X \in \mathsf{TOP} . \forall \beta : \texttt{Bornology}(X) . \texttt{Closed}(X, \beta) \iff \texttt{BornologyBase}\Big(X, \beta, \beta \cap \texttt{Closed}(X)\Big)$

$\texttt{Proof} =$

1 $(\Rightarrow)$.

1.1 If $A \in \beta$, then $A \subset \operatorname{cl} A$.

1.2 Also $\operatorname{cl} A \in \beta$.

2 $(\Leftarrow)$.

2.1 Assume $A \in \beta$.

2.2 Then there is a closed set $F \in \beta$ such that $A \subset F$.

2.3 But $A \subset \operatorname{cl} A \subset F$.

2.4 So $\operatorname{cl} A \in \beta$ as β is closed under taking subsets.

□

$\texttt{LocallyBounded} :: ?\mathsf{TOP} \,\&\, \mathsf{BORN}$

$X : \texttt{LocallyBounded} \iff \forall x \in X . \mathcal{N}_V(x) \cap \beta \neq \emptyset$

$\texttt{CompactsAreBoundedInLocallyBoundedSpace} :: \forall X : \texttt{LocallyBounded} . \mathsf{K}(X) \subset \mathcal{B}(X)$

$\texttt{Proof} =$

1 Take K to be compact in X.

2 Select a bounded Neighborhood U_x for each point $x \in K$.

3 As K is compact there is a finite subcover $(x_i)_{i=1}^n$.

4 Then $\bigcup_{i=1}^n U_{x_i} \in \mathcal{B}(X)$ as $\mathcal{B}(X)$ is an ideal.

5 But $K \subset \bigcup_{i=1}^n U_{x_i} \in \mathcal{B}(X)$, so $K \in \mathcal{B}(X)$, as $\mathcal{B}(X)$ is an ideal.

□

$\texttt{semimetricBornology} :: \prod_{X \in \mathsf{SET}} \texttt{Semimetric}(X) \to \texttt{Bornology}(X)$

$\texttt{semimetricBornology}\,(\rho) = \mathcal{B}(\rho) := \langle \mathbb{B}_X(X, \mathbb{R}_{++}) \rangle_{\mathsf{BORN}}$

$\texttt{Semimetrizable} :: ?\mathsf{TOP} \,\&\, \mathsf{BORN}$

$X : \texttt{Semimetrizable} \iff \exists \rho : \texttt{Semimetric}(X)\ .\ \mathcal{T}(X) = \mathcal{T}(\rho) \,\&\, \mathcal{B}(X) = \mathcal{B}(\rho)$

$\texttt{SemimetrizationTHM} ::$

$\quad :: \forall (X, \tau, \beta) \in \mathsf{TOP} \,\&\, \mathsf{BORN}\ .$

$\quad .\ \texttt{Semimetrizable}(X, \tau, \beta) \iff$

$\quad \iff \texttt{Semimetrizable}(X, \tau) \,\&\, \texttt{LocallyBounded} \,\&\, \texttt{Proper}(X, \beta) \,\&\, \exists \beta' : \texttt{BornologyBase}(X)\ .\ |\beta'| \le \aleph_0$

$\texttt{Proof} =$

$\dots$

$\square$

1.2.7 Interesting Examples and Facts

1.3 Infinite Dimensional Geometry

1.3.1 Dominated Extension

OneDimensionalExtension ::

:: $\forall V \in \mathbb{R}\text{-VS} . \forall U \subset_{\mathbb{R}\text{-VS}} V . \forall \sigma : \texttt{Sublinear}(V) \forall f \in U^* . \forall v \in U^{\complement}$.

. $\forall \aleph : f \leq \sigma_{|U} . \exists F \in (U \oplus v)^* . F_{|U} = f \ \& \ F \leq \sigma_{|U \oplus v}$

Proof =

1 $\alpha = \sup_{u \in U} -\sigma(-u - v) - f(u) \leq \inf_{u \in U} \sigma(u + v) - f(u) = \beta$.

1.1 Assume $u, w \in U$.

1.2 Then $f(u) - f(w) = f(u - w) \leq \sigma(u - w) = \sigma(u + v - v - w) = \sigma(u + v) + \sigma(-v - w)$.

1.3 By rearenging one gets $-\sigma(-v - w) - f(w) \leq \sigma(u + v) - f(u)$.

1.4 Not, that both α and β must be finite by inf and sup definition .

2 So $-\sigma(-v - u) \leq \gamma \leq \sigma(v + u)$ for any $\gamma \in [\alpha, \beta]$ and $u \in U$.

3 Select $\gamma \in [\alpha, \beta]$.

4 Define $F(u + \delta v) := f(u) + \delta\gamma$ on $U \oplus v$, which is linear.

5 $F \leq \sigma$ on $U \oplus v$.

5.1 Assume $\delta > 0$.

5.1.1 Then $F(u + \delta v) \leq f(u) + \delta\sigma\left(\frac{u}{\delta} + v\right) - f(u) = \sigma(u + \delta v)$ by construction of γ.

5.1.2 Here we used thed fact that σ is conic.

5.2 Assume $\delta < 0$.

5.2.1 Then $F(u + \delta v) \leq f(u) - \delta\sigma\left(-\frac{u}{\delta} - v\right) - f(u) = \sigma(u + \delta v)$ by construction of γ.

5.3 The case $\delta = 0$ is evident.

□

HahnBanachTheorem1 ::

:: $\forall V \in \mathbb{R}\text{-VS} . \forall U \subset_{\mathbb{R}\text{-VS}} V . \forall \sigma : \texttt{Sublinear}(V) \forall f \in U^* . \forall \aleph : f \leq \sigma_{|U} . \exists F \in V^* . F_{|U} = f \ \& \ F \leq \sigma$

Proof =

1 Define $\phi \subset \sum W : \texttt{VectorSubspace}(V) . W^*$ to be the set of all extensions of f bounded by σ.

2 Order ϕ by saying $(W, g) \leq (O, h)$ iff $W \subset_{k\text{-VS}} O$ and $h_{|W} = g$.

3 By Zorn Lemma extract an upper bound (W, F) of ϕ.

3.1 Clearly $(U, f) \in \phi$, so $\phi \neq \emptyset$.

3.2 If $\mathcal{C}$ is a chain in ϕ, then $\bigcup \mathcal{C} \in \phi$ is an upper bound of $\mathcal{C}$.

4 If $W \neq V$ then the extension F can be extended furtherly, but this contradicts the maximality.

□

k :: $\texttt{AbsoluteValueField}(\mathbb{R})$;

HahnBanachExtension ::

:: $\forall V \in k\text{-TVS} . \forall U \subset_{k\text{-VS}} V . \forall \sigma \in \mathsf{SMN}(V) . \forall f \in U^* . \forall \aleph : f \leq \sigma_{|U} . \exists F \in V^* . F_{|U} = f \ \& \ |F| \leq \sigma$

Proof =

This is a modification of Hahn-Banach.

□

ContinuousExtension ::

:: $\forall V \in k\text{-LCS} . \forall U \subset_{k\text{-VS}} V . \forall f \in U' . \forall \aleph : f \leq \sigma_{|U} . \exists F \in V' . F_{|U} = f$

Proof =

1 The family of seminorms $\mathrm{ssc}(V)$ generates the topology of V.

2 The restrictions $\sigma_{|U}$ for $\sigma \in \mathrm{ssc}(V)$ generate the locally convex topology of U.

3 So there exists $\sigma \in \mathrm{ssc}(V)$ such that $|f| \leq \sigma_{|U}$.

3.1 This is a continiuity ctiterion for locally convex spaces.

4 By Hahn-Banach there is an extension F of f such that $|F| \leq \sigma$.

5 So by same continuity criterion $F \in V'$.

□

SublinearFunctionalSupport :: $\forall V \in k\text{-TVS} . \forall \sigma : \mathtt{Sublinear}(V) . \forall v \in V . \exists f \in V^* .$

$. f(v) = \sigma(v) \ \& \ \forall w \in V . -\sigma(-w) \leq f(w) \leq \sigma(w)$

Proof =

1 define g on kv by setting $g(\alpha v) = \alpha\sigma(v)$.

2 Obviously g is linear.

3 $g \leq \sigma_{kv}$.

3.1 Assume $\alpha \geq 0$.

3.1.1 Then by defineition $g(\alpha v) = \alpha\sigma(v) = \sigma(\alpha v)$.

3.1.2 So $g(\alpha v) \leq \sigma(\alpha v)$.

3.2 Assume $\alpha < 0$.

3.2.1 Then $f(\alpha v) = \alpha = -(-\alpha)\sigma(v) = -\sigma(-\alpha v) \leq \sigma(\alpha v)$.

3.2.2 Last Inequelity follow from the fact that $0 = \sigma(0) = \sigma(u - u) \leq \sigma(u) + \sigma(-u)$ for any $u \in V$.

3.2.3 So $-\sigma(-u) \leq \sigma(u)$.

4 By Hahn Banach there is a dominated extendion $f \in V^*$ of g.

5 By one-dimensional extension proof's construction it must be the case that $-\sigma(w) \leq f(w) \leq \sigma(w)$.

5.1 Apply statement (1) to the construction with $u = 0$.

□

SeminormFunctionalSupport :: $\forall V \in \mathbb{R}\text{-VS} . \forall \sigma \in \mathsf{SMN}(V) . \forall v \in V . \exists f \in V^* .$

$. f(v) = \sigma(v) \ \& \ |f| \leq \sigma$

Proof =

This is an obvious modification of the previous result.

□

ContinuousFunctionalSupport :: $\forall V \in \mathbb{R}\text{-TVS} . \forall \sigma : \text{Sublinear}(V) \cap \text{TOP}(V, \mathbb{R}) . \forall v \in V . \exists f \in V' .$
$. f(v) = \sigma(v) \ \& \ -\sigma(-w) \leq f(w) \leq \sigma(w)$

Proof =

1 Assume $U \in \mathcal{U}_{\mathbb{R}}(0)$.

2 Then there is a balanced $W \in \mathcal{U}_k(0)$ such that $W \subset U$.

3 By continuity there is $O \in \mathcal{U}_V(0)$ such that $\sigma(O) \subset W$.

4 Let $E \in \mathcal{U}_V(0)$ be a balanced subset of O .

5 Then $f(E) \subset U$.

5.1 Select $w \in E$.

5.2 Then $w, -w \in O$, so $-\sigma(-w), \sigma(w) \in W$.

5.3 But $-\sigma(-w) \leq f(w) \leq \sigma(w)$.

5.4 As W is balanced $f(w) \in E$.

5.4.1 Think about W as open interval $(-\alpha, \alpha)$.

6 By continuity at zero, the general continiuity follows.

$\square$

FiniteDimIsComplemented :: $\forall V \in k\text{-LCHS} . \forall U \subset_{k\text{-VS}} V . \ \dim U < \infty \Rightarrow \exists W \subset_{k\text{-VS}} \ . V =_{k\text{-TVS}} U \oplus W$

Proof =

Let $(e_i)_{i=1}^n$ be a finite base of U.

Then functionals $f_i\,(\alpha e) = \alpha_i$ are continuous .

So there exist a continuous extensions $F_i \in V'$ of each f_i.

Define continuous operator $P(v) = F_i(v)e_i$.

Obviously, $P^2 = P$, so P is a continuous projector.

This means that P must be complemented.

$\square$

NormPreservingFunctionalExtension :: $\forall V : \text{NormedSpace}(k) . \forall U \subset_{k\text{-VS}} V . \forall f \in U' . \exists F \in V' . \|f\| = \|F\|$

Proof =

1 Define a sublinear function $\sigma(v) = \|f\|\|v\|$ on V.

2 Then, by the definition of dual normed space $|f| \leq \sigma_{|U}$.

3 Construct F as Hahn-Banach dominated extension of f dominated by σ.

4 Then F is continuous.

5 As $|F| \leq \sigma$ it must be the case that $\|F\| \leq \|f\|$.

6 On the other hand there must exist a sequence $u : \mathbb{N} \to U$ such that $|f(u_n)| \to \|f\|$.

7 But this means that $|F(u_n)| = |f(u_n)| \to \|f\|$, so $\|F\| = \|f\|$.

$\square$

FunctionalAbundence :: $\forall V : \text{NormedSpace}(k) . \forall v \in V \exists f \in \mathbb{S}(V') . f(v) = \|v\|$

Proof =

1 Define a a function $g : kv \to k$ by $g(\alpha v) = \alpha\|v\|$.

2 Then g is linear and has norm $\|g\| = 1$.

3 By the previous result there exists an extension f of g on V.

$\square$

$\texttt{DualZeroCritetion} :: \forall V : \texttt{NormedSpace}(k) \,.\, \forall v \in V \,.\, v = 0 \iff \forall f \in \mathbb{S}(V') \,.\, f(v) = 0$

$\texttt{Proof} \; =$

Obvious.

$\square$

$\texttt{DualNormConstruction} :: \forall V : \texttt{NormedSpace}(k) \,.\, \forall v \in V \,.\, \|v\| = \sup \left\{ |f(v)| \middle| f \in \mathbb{S}(V') \right\}$

$\texttt{Proof} \; =$

There must be $f \in \mathbb{S}(V')$ such that $f(v) = \|v\|$.

On the other hand by definition of the dual norm $|f(v)| \le \|f\|\|v\| = \|v\|$.

$\square$

$\texttt{SubspaceSeparatingFunctionalExists} ::$

$:: \forall V : \texttt{NormedSpace}(k) \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \forall v \in (\operatorname{cl} U)^{\complement} \,.\, \forall \delta \in \mathbb{R}_{++} \,.\, \forall \aleph : d_V(v, U) = \delta \,.$

$.\, \exists f \in \mathbb{S}(V') \,.\, f(U) = \{0\} \;\&\; f(v) = \delta$

$\texttt{Proof} \; =$

1 Define $g(u + \alpha v) = \alpha\delta$ over $U \oplus kv$.

2 Then g is linear.

3 g is continuous and has $\|g\| \le 1$.

3.1 Assume $u + \alpha v$ is sach that $\|u + \alpha v\| = 1$.

3.2 Then $f(u + \alpha v) = \alpha\delta$.

3.3 If $\alpha = 0$, then $|f(u + \alpha v)| = |0| = 0 \le 1$.

3.4 So assume $\alpha \ne 0$.

3.5 write $1 = \|u + \alpha v\| = \left\| -\alpha \dfrac{-u}{\alpha} + \alpha v \right\| = |\alpha| \left\| v - \dfrac{-u}{\alpha} \right\| \ge |\alpha|\delta$.

3.5.1 Here the last inequality holds by the definition of a distance to a set.

3.6 Also $|f(u + \alpha v)| = |\alpha\delta| = |\alpha|\delta \le 1$.

4 Actually $\|g\| = 1$.

4.1 Select a sequence $u : U \to \mathbb{N}$ such that $\lim_{n\to\infty} \|v - u_n\| = \delta$.

4.2 But $g(v - u_n) = \delta$, so $\|g\| \ge 1$.

5 Define f to be a Hahn-Banach extension of g.

$\square$

$\texttt{LinearlyIndependendFunctionSeparation} ::$

$:: \forall V : \texttt{NormedSpace}(k) \,.\, \forall n \in \mathbb{Z}_+ \,.\, \forall v : \texttt{LinearlyIndependent}\Big(\{1, \dots, n\}, V\Big)$

$.\, \exists f : \{1, \dots, n\} \to V' \,.\, \forall i, j \in \{1, \dots, n\} \,.\, f_i(v_j) = \delta_{i,j}$

$\texttt{Proof} \; =$

Define functionals on $\operatorname{span}\{v_1, \dots, v_n\}$ and then extend the to the whole space.

$\square$

GreaterNormExtension ::

:: $\forall V : \text{NormedSpace}(k) \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \forall f \in U' \,.\, \exists F \in V' \,.\, F_{|U} = f \;\&\; \|F\| \geq \|f\|$

Proof =

1 If $V = \operatorname{cl} U$ the result holds trivially.

2 So take $v \in (\operatorname{cl} U)^{\complement}$.

3 Let $\delta = d_V(U, v)$.

4 define $g(u + \alpha v) = f(u) + \alpha\beta$ with $\beta \geq \|v\|\|f\|$ on $U \oplus kv$.

5 This functional is continuous as g is sum of f

and the fucntional of the theorem SubspaceSeparatingFunctionalExists.

6 $\|g\| \geq \|f\|$.

6.1 See that $g\left(\frac{u}{\|u\|}\right) = \frac{\beta}{\|u\|} \geq \|f\|$.

7 Extend g By Hahn-Banach to get the result.

□

1.3.2 Mazur-Orlich Theorem

MazurOrlichTHM ::

$:: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall \sigma : \mathtt{Sublinear}(V) \,.\, \forall X \in \mathsf{SET} \,.\, \forall v : X \to V \,.\, \forall \rho : X \to \mathbb{R} \,.$

$. \left(\exists f \in V^* \,.\, f \le \sigma \,\&\, \rho \le vf\right) \iff$

$$\iff \forall n \in \mathbb{N} \,.\, \forall \alpha : \{1,\ldots,n\} \to \mathbb{R}_+ \,.\, \forall x : \{1,\ldots,n\} \to X \,.\, \sum_{i=1}^n \alpha_i \rho(x_i) \le \sigma\left(\sum_{i=1}^n \alpha_i v(x_i)\right)$$

Proof $=$

1 $(\Rightarrow)$.

1.1 $\displaystyle\sum_{i=1}^n \alpha_i \rho(x_i) \le \sum_{i=1}^n \alpha_i f(v(x_i)) = f\left(\sum_{i=1}^n \alpha_i v(x_i)\right) \le \sigma\left(\sum_{i=1}^n \alpha_i v(x_i)\right)$.

2 $(\Leftarrow)$.

2.1 Take some $n \in \mathbb{N}$ and $u \in V$.

2.2 Define $\displaystyle\Gamma_n(u) = \left\{\sigma\left(u + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) \middle| \alpha : \{1,\ldots,n\} \to \mathbb{R}_+, x : \{1,\ldots,n\} \to X\right\}$.

2.3 Also Define $\gamma(u) = \inf_{n\in\mathbb{N}} \inf \Gamma_n(u)$.

2.3.1 $\gamma(u)$ is well defined and bounded below by $-\sigma(-u)$.

2.3.1.1 $\displaystyle\sum_{i=1}^n \alpha_i \rho(x_i) \le \sigma\left(\sum_{i=1}^n \alpha_i v(x_i)\right) \le \sigma\left(u + \sum_{i=1}^n \alpha_i v(x_i)\right) + \sigma(-u)$ for any α and x.

2.3.1.2 By rearanging inequality one gets the bound.

2.3.2 γ is subadditive.

2.3.2.1 Take some $u, w \in V$.

2.3.2.2 Then $\displaystyle\gamma(u+w) = \inf_{n,\alpha,x} \sigma\left(u + w + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) =$

$$= \inf_{n,\alpha,\beta,x,y} \sigma\left(u + w + \sum_{i=1}^n \alpha_i v(x_i) + \sum_{i=1}^n \beta_i v(y_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) - \sum_{i=1}^n \beta_i \rho(y_i) \le$$

$$\le \inf_{n,\alpha,\beta,x,y} \sigma\left(u + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) + \sigma\left(w + \sum_{i=1}^n \beta_i v(y_i)\right) - \sum_{i=1}^n \beta_i \rho(y_i) =$$

$$= \inf_{n,\alpha,x} \sigma\left(u + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) + \inf_{n,\beta,y} \sigma\left(w + \sum_{i=1}^n \beta_i v(y_i)\right) - \sum_{i=1}^n \beta_i \rho(y_i) = \gamma(u) + \gamma(v)\ .$$

2.3.3 γ is positively homogeneous.

2.3.3.1 Take some $u \in V$ and $\lambda \in \mathbb{R}_{++}$.

2.3.3.2Then $\displaystyle\gamma(\lambda u) = \inf_{n,\alpha,x} \sigma\left(\lambda u + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) =$

$$= \inf_{n,\alpha,x} \sigma\left(\lambda u + \sum_{i=1}^n \lambda\alpha_i v(x_i)\right) - \sum_{i=1}^n \lambda\alpha_i \rho(x_i) = \lambda \inf_{n,\alpha,x} \sigma\left(u + \sum_{i=1}^n \alpha_i v(x_i)\right) - \sum_{i=1}^n \alpha_i \rho(x_i) = \lambda\gamma(u).$$

2.4 Define f as Hahn-Banach extension of 0 dominated by γ.

2.5 Clearly $f \le \gamma \le \sigma$.

2.6 $\rho \le fv$.

2.6.1 Select $x \in X$.

2.6.2 Then by construction $\gamma(-v(x)) \le \sigma(-v(x) + v(x)) - \rho(x) = -\rho(x)$.

2.6.3 But $f(v(x)) \geq -\gamma(-v(x)) \geq \rho(x)$.

$\square$

1.3.3 Subliniar Functionals

$\texttt{sublinear} = \Lambda V \in k\text{-VS} \; . \; V^{\#} := \Lambda V \in k\text{-VS} \; . \; \texttt{Sublinear}(V) : k\text{-VS} \to \texttt{Type};$

$\texttt{LinearityCriterion} :: \forall V \in \mathbb{R}\text{-VS} \; . \; \forall \sigma \in V^{\#} \; . \; \sigma \in V^{*} \iff \forall v \in V \; . \; \sigma(v) + \sigma(-v) = 0$

$\texttt{Proof} \;=$

1 $(\Rightarrow)$ is obvious.

2 $(\Leftarrow)$.

2.1 Assume $\sigma(v) + \sigma(-v) = 0$ holds.

2.2 Then $\sigma(v) = -\sigma(-v)$.

2.3 So σ is homogeneous.

2.4 σ is additive.

2.4.1 Assume $v, w \in V$.

2.4.2 Then $\sigma(v) = \sigma(v + w - w) \leq \sigma(v + w) + \sigma(-w) = \sigma(v + w) - \sigma(w)$.

2.4.3 By rearanging inequalities $\sigma(v) + \sigma(w) \leq \sigma(v + w)$.

2.4.4 But this is an inverse of Minkowski's inequality, so $\sigma(v + w) = \sigma(v) + \sigma(w)$.

□

$$\texttt{auxiliaryConjugate} :: \prod_{V \in \mathbb{R}\text{-VS}} \; . \; V^{\#} \to V^{\#}$$

$\texttt{auxiliaryConjugate}\,(\sigma) = \sigma^{\#} := \Lambda v \in V \; . \; \inf\{\sigma(v + w) - \sigma(w) | w \in V\}$

$\texttt{LinearIfMinimal} :: \forall V \in \mathbb{R}\text{-VS} \; . \; V^{*} = \min V^{\#}$

$\texttt{Proof} \;=$

1 Take $f \in V^{*}$.

1.1 Assume $\sigma \in V^{\#}$ is such that $\sigma \leq f$.

1.2 Then $f(v) \geq \sigma(v) \geq -\sigma(-v) \geq -f(-v) = f(v)$ for every $v \in V$.

1.3 So $f = \sigma$.

1.4 As σ was arbitrary, this proves that f is minimal.

2 Take $\sigma \in \min V^{\#}$.

2.1 Then $\sigma^{\#} = \sigma$.

2.1.1 This holds as $\sigma^{\#} \leq \sigma$ and σ is minimal.

2.2 Note, that this implies that $\sigma(v) \leq \sigma\left(\frac{1}{2}v\right) - \sigma\left(-\frac{1}{2}v\right)$ for any $v \in V$.

2.3 which can be rewritten as $\sigma\left(\frac{1}{2}v\right) \leq \sigma\left(-\frac{1}{2}v\right)$.

2.4 Or as v was arbitrary $\sigma(v) \leq -\sigma(-v)$ which proves that $\sigma \in V^{*}$.

□

$\texttt{MinimalSublinearAlwaysExists} :: \forall V \in \mathbb{R}\text{-VS} \; . \; \forall \sigma \in V^{\#} \; . \; \exists \tau \in \min V \; . \; \tau \leq \sigma$

$\texttt{Proof} \;=$

This is Equivalent to Hahn-Banach Theorem.

□

$\texttt{sublinearCell} :: \prod V \in k\text{-}\mathsf{VS} \,.\, V^{\#} \to \texttt{Convex}(V)$

$\texttt{sublinearCell}\,(\sigma) = \mathbb{B}(\sigma) := \{v \in V : \sigma(v) < 1\}$

$\texttt{SeparationAndDomination} :: \forall V \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \forall f \in V^* \setminus \{0\} \,.\, \forall \sigma \in V_+^{\#} \,.\, f \leq \sigma \iff \mathbb{B}(\sigma) f^{-1}\{1\} = 1$

Proof $=$

1 $(\Rightarrow)$.

1.1 This is straightforward by inequality $f(v) \leq \sigma(v) < 1$.

2 $(\Leftarrow)$.

2.1 Assume $v \in V$ such that $f(v) > \sigma(v) \geq 0$.

2.2 Then there is a scale λ such that $f(\lambda v) = 1$.

2.3 But this means that $\lambda\sigma(v) < \lambda f(v) = f(\lambda v) \leq \sigma(\lambda v) = \lambda\sigma(v)$.

2.4 But this is impossible!

$\square$

$\texttt{ConrinuityAndDomination} :: \forall V \in \mathbb{R}\text{-}\mathsf{TVS} \,.\, \forall f \in V^* \setminus \{0\} \,.\, \forall \sigma \in V_+^{\#} \cap \mathsf{TOP}(V, \mathbb{R}) \,.\, f \leq \sigma \Rightarrow f \in V'$

Proof $=$

1 Take $U \in \mathcal{U}_{\mathbb{R}}(0)$.

2 Then there is a balanced $W \in \mathcal{U}_{\mathbb{R}}(0)$ such that $W \subset U$.

3 By continuity there is $O \in \mathcal{U}_V(0)$ such that $\sigma(O) \subset W$.

4 Select Balanced $E \in \mathcal{U}_V(0)$ such that $E \subset O$.

5 Then $f(E) \subset W \subset U$.

5.1 Assume $v \in E$.

5.2 If $f(v) = 0$ then $f(v) \in W$, so assume that $f(v) \neq 0$.

5.3 Then either $f(v) > 0$ or $f(-v) > 0$.

5.4 So either $0 \leq f(v) \leq \sigma(v)$ or $0 \leq f(-v) \leq \sigma(-v)$.

5.5 And as E is balanced this means that either $f(v) \in W$ or $f(-v) \in W$.

5.6 But as W is also balanced and $-f(v) = f(-v)$ it always must be thes case that $f(v) \in W$.

6 Continuity at 0 of f proves global continuity.

$\square$

$\texttt{InverseMinkowskiIneq} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall \sigma \in V^{\#} \,.\, \forall v, w \in V \,.\, \sigma(v) - \sigma(w) \leq \sigma(v - w)$

Proof $=$

1 write $\sigma(v) = \sigma(v - w + w) \leq \sigma(v - w) + \sigma(w)$.

2 By rearanging the inequlity $\sigma(v) - \sigma(w) \leq \sigma(v - w)$.

$\square$

$\texttt{SublinearUniformContinuityCriterion} ::$

$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall \sigma \in V^{\#} \,.\, \sigma \in C_0(V) \Rightarrow \sigma \in \mathsf{UNI}(V, \mathbb{R})$

Proof $=$

Obvious.

$\square$

PositiveSublinearContinuity ::

:: $\forall V \in k\text{-}\mathsf{TVS} . \forall \sigma \in V_+^\# . \sigma \in \mathsf{UNI}(V, \mathbb{R}) \iff \mathbb{B}(\sigma) \in \mathcal{T}(V)$

Proof =

1 ($\Rightarrow$).

1.1 This follows directly from topological definition of continuity.

2 ($\Leftarrow$).

2.1 Assume (Δ, v) is a net in V such that $\lim_{\delta \in \Delta} v_\delta = 0$.

2.2 Then $v_\delta \in \lambda \mathbb{B}_V$ for all sufficiently large δ and any $\delta \in \mathbb{R}_{++}$.

2.3 But this means that $\sigma(v_\delta) < \lambda$, so $\lim_{\delta \in \Delta} \sigma(v_\delta) = 0$.

2.4 This proves uniform continuity .

□

ContinuousGauge :: $\forall V \in k\text{-}\mathsf{TVS} . \forall C :$ Convex$(V) \cap \mathcal{U}_0(V) . \gamma(\bullet|C) \in \mathsf{UNI}(V, \mathbb{R})$

Proof =

This follows by the previous theorem.

□

OpenConvexRepresentation ::

:: $\forall V \in k\text{-}\mathsf{TVS} . \forall C :$ Convex & NonEmpty & $\mathcal{T}(V) . \exists \sigma \in V_+^\#$ & $\mathsf{UNI}(V, \mathbb{R}) . \exists v \in V . C = v + \mathbb{B}(\sigma)$

Proof =

This follows by the previous theorem.

□

1.3.4 Geometric Interpretation

GeometricRealHahnBanachTheorem ::

:: $\forall V \in \mathbb{R}\text{-TVS} \,.\, \forall C : \texttt{Convex} \,\&\, \mathcal{T}(V) \,.\, \forall A \subset_{k\text{-AFF}} V \,.\, \forall \aleph : CA = \emptyset \,.$

$.\, \exists H : \texttt{Hyperplane}(V) \,.\, A \subset H \,\&\, CH = \emptyset$

Proof =

1 Without loss of generality assume $A \subset_{k\text{-VS}} V$.

2 Represent C as $v + \mathbb{B}(\sigma)$ with $\sigma \in V_+^\# \,\&\, \mathsf{UNI}(V, \mathbb{R})$ and $v \in V$.

3 Note, that (1) implies that $v \neq 0$, furthermore $v \notin A$.

4 By separation and domination theorem $\sigma(a - v) \geq 1$ for any $a \in A$.

5 define $f(a + \alpha v) = -\alpha$ on $A \oplus kv$.

6 $f \leq \sigma$ on $A \oplus kv$.

6.1 $f(a + \alpha v) = -\alpha \leq 0 \leq \sigma(a + \alpha v)$ if $\alpha \leq 0$.

6.2 $f(a + \alpha v) = -\alpha \leq -\alpha \left(\alpha^{-1} a + v\right) \leq \sigma f(a + \alpha v)$ if $\alpha > 0$.

7 Construct an extension F of f dominated by σ by Hahn-Banach.

8 Then using $H = \ker F$ produces the desired result.

□

GeometricComplexHahnBanachTheorem ::

:: $\forall V \in \mathbb{C}\text{-TVS} \,.\, \forall C : \texttt{Convex} \,\&\, \mathcal{T}(V) \,.\, \forall A \subset_{k\text{-AFF}} V \,.\, \forall \aleph : CA = \emptyset \,.$

$.\, \exists H : \texttt{Hyperplane}(V) \,.\, A \subset H \,\&\, CH = \emptyset$

Proof =

1 Treat V as a real vector space and construct H' as in the previous theorem.

2 Then $H = H' \cap \mathrm{i}H'$ is a desired complex hyperplane.

□

PlaneOpenConvexSetSeparationReal ::

:: $\forall V \in \mathbb{R}\text{-TVS} \,.\, \forall C : \texttt{Convex} \,\&\, \mathcal{T}(V) \,.\, \forall A \subset_{k\text{-VS}} V \,.\, \forall \aleph : CA = \emptyset \,.$

$.\, \exists f \in V' \,.\, f(A) = 0 \,\&\, \forall x \in C \,.\, f(x) > 0$

Proof =

1 Just use the functional $-F$ of geometric Hahn-Banach theorem.

2 $F(H) = 0$, so $F(A) = 0$.

3 $-F$ is positive on C.

3.1 $v \in C$ and we know that $-F(v) = 1$.

3.2 If $x \in C$, then $[v, x] \subset C$.

3.3 So, if $f(x) < 0$ there exists a midpoint $u \in [v, x]$ such that $f(u) = 0$ by intermidiate value theorem.

3.4 But this means that $u \in CH$, which is imposible by construction.

□

PlaneOpenConvexSetSeparationComplex ::

:: $\forall V \in \mathbb{C}\text{-TVS} \,.\, \forall C : \texttt{Convex} \,\&\, \mathcal{T}(V) \,.\, \forall A \subset_{k\text{-VS}} V \,.\, \forall \aleph : CA = \emptyset \,.$

$.\, \exists f \in V' \,.\, f(A) = 0 \,\&\, \forall x \in C \,.\, \operatorname{Re} f(x) > 0$

Proof =

...

□

$\texttt{PlanePointSeparationTheorem} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \forall A \subset_{k\text{-}\mathsf{TVS}} V \,.\, \forall v \in A^{\complement} \,.\, \exists f \in V' \,.\, f(A) = 0 \;\&\; f(v) \neq 0$

$\texttt{Proof} \;=$

1 As A is closed and V is locally convex there exists a convex set $C \in \mathcal{U}_V(A)$ such that $CA = \emptyset$.

2 Apply separation theorem to A and C.

$\square$

$\texttt{AbundanceOfContinuousFunctionals} :: \forall V \in k\text{-}\mathsf{LCS} \,.\, \forall v \in \left(\mathrm{cl}\{0\}\right)^{\complement} \,.\, \exists f \in V' \,.\, f(v) \neq 0$

$\texttt{Proof} \;=$

Apply previous theorem to $\mathrm{cl}\{0\}$ and v.

$\square$

$\texttt{ContinuousDualSeparetesLocallyConvexSpace} :: \forall V \in k\text{-}\mathsf{LCHS} \,.\, \texttt{Separates}(V, V')$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ContinuousDualIsTotal} :: \forall V \in k\text{-}\mathsf{LCHS} \,.\, \texttt{Total}(V, V')$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{NontrivialDual} :: \forall V \in k\text{-}\mathsf{TVS} \,.\, V' \neq \{0\} \iff \exists U \in \mathcal{U}_V(0) \,.\, \texttt{Convex}(V, U) \;\&\; U \neq V$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{VectorValuedCauchyIntegralTheorem} ::$

$$\forall V \in \mathbb{C}\text{-}\mathsf{LCHS} \,.\, \forall (D, C) : \texttt{JordanArc} \,.\, \forall v \in \mathsf{TOP}(D \cup C, V) \,.\, \forall \aleph : \texttt{Analytic}(v, D) \,.\, \int_C v(s)ds = 0$$

$\texttt{Proof} \;=$

Take $f \in V'$.

Then $f(v)$ is analytic.

Then $f\left(\int_C v(s)ds\right) = \int_C f\Big(v(s)\Big)ds = 0$ by normal cauchy intergral theorem.

But as V' is total this means that $\int_C v(s)ds = 0$.

$\square$

1.3.5 From Geometry to Analysis

GeneralHahnBanachTheorem $:: \forall V \in \mathbb{R}\text{-VS} . \forall p : \texttt{Convex}(V,V) . \forall U \subset_{\mathbb{R}\text{-VS}} V . \forall f \in U^* .$
$. \forall [0] : f \leq p . \exists F \in V^* : F_{|U} = f \,\&\, F \leq p$

Proof $=$

$C := \Big\{(v,\alpha) \in V \times \mathbb{R} : \alpha > p(v)\Big\} :?(V \times \mathbb{R}),$

Assume $(v,\alpha),(w,\beta) \in C,$

Assume $t \in [0,1],$

$[1] := \eth\texttt{CauchyFilterbase}(V,V)(p)(v,w,t) : p\Big(tv + (1-t)w\Big) \leq tp(v) + (1-t)p(w) < t\alpha + (1-t)\beta,$

$\Big[\big((v,\alpha),(w,\beta)\big).*\Big] := \jmath C[1] : t(v,\alpha) + (1-t)(w,\beta) \in C;$

$\rightsquigarrow [1] := \eth^{-1}\texttt{Convex} : \texttt{Convex}(C),$

$A := \Big\{(u,f(u))\Big| u \in f(u)\Big\} : \texttt{VectorSubspace}(V \times \mathbb{R}),$

$[2] := \jmath A \jmath C : A \cap \operatorname{core} C = \emptyset,$

$[3] := \eth\texttt{CauchyFilterbase}(V,V)(p)\jmath C : \operatorname{core} C \neq \emptyset,$

$\Big(H,[4]\Big) := \texttt{SeparationTHM}(V \times \mathbb{R}, C, A)[2][3] : \sum H : \texttt{Hyperplane}(V \times \mathbb{R}) . \texttt{Separates}(V \times \mathbb{R}, H, C, A),$

$[5] := \eth\texttt{Separates}[4] : A \subset H,$

$\Big(g,r,[6]\Big) := \eth\texttt{Hyperplane}(V \times \mathbb{R}, H) : \sum g \in (V \times \mathbb{R})^* . H = \mathrm{H}(g,r),$

$\Big(h,\gamma,[7]\Big) := \eth(V \times \mathbb{R})^* : \sum h \in V^* . \sum \gamma \in \mathbb{R}^\times . \forall (v,\alpha) \in V \times \mathbb{R} . g(v,\alpha) = h(v) + \gamma\alpha,$

$[8] := [5][6][7] : \forall u \in U . h(u) + \gamma f(u) = r,$

$[9] := [8](0) : r = 0,$

$[10] := \eth\texttt{Field}\mathbb{R}[8] : \forall u \in U . f(u) = \frac{1}{\gamma}(r - h(u)),$

$F := -\frac{1}{\gamma}h : V^*,$

Assume $v : V,$

$[11] := \eth\texttt{Separates}[4][6][7][9] : h(v) + \gamma p(v) \geq 0,$

$[v.*] := \jmath F[11] : F(v) = -\frac{1}{\gamma}h(v) \leq p(v);$

$\rightsquigarrow [*] := \eth^{-1}\texttt{Ineq} : F \leq p;$

$\square$

$\texttt{DieodonneConvexExtensionTHM} :: \forall V \in \mathbb{R}\text{-VS} . \forall p : \texttt{Convex}(V, V) . \forall U \subset_{\mathbb{R}\text{-VS}} V .$
$. \forall f : \texttt{Convex}(V, U) . \forall [0] : f \le p . \exists F \in \texttt{Convex}(V, V) : F_{|U} = f \;\&\; F \le p$

$\texttt{Proof} \;=$

$C := \Big\{ (v, \alpha) \in V \times \mathbb{R} : \alpha > p(v) \Big\} : \texttt{Convex}(V \times \mathbb{R}),$

$C' := \Big\{ (u, \alpha) \in U \times \mathbb{R} : \alpha > f(u) \Big\} : \texttt{Convex}(U \times \mathbb{R}),$

$\texttt{Assume}\; u \in U,$

$[1] := \jmath C : (u, f(u)) \in \partial C',$

$\Big(H', [2]\Big) := \texttt{ClosedSupportExists}(U \times \mathbb{R}, C', u) : \sum H' : \texttt{Hyperplane}(U \times \mathbb{R}) . \texttt{Supports}(V, H', C'),$

$[3] := \jmath C \eth \texttt{Supports}[2] : C \cap H' = \emptyset,$

$H, [4] := \texttt{ConvexBodyBound}(V, H') : \sum H : \texttt{Hyperplane}(V \times \mathbb{R}) . \texttt{Bounds}(V, H, C),$

$E_u := \Big\{ (v, \alpha) \in V \times \mathbb{R} \Big| \exists (v, h) \in H : \alpha \le h \Big\} : \texttt{Convex}(V \times \mathbb{R});$

$\rightsquigarrow \Big(E, [1]\Big) := \texttt{I}\left(\prod\right) : \prod_{u \in U} \sum E_u : \texttt{Convex}(V \times \mathbb{R}) . \texttt{Bounds}(V, \operatorname{lin} E_u, C) \;\&$
$\& \; \texttt{Supports}\Big(V, \operatorname{lin} E_u \cap U \times \mathbb{R}, \operatorname{lin} C', (u, f(u))\Big),$

$D := \bigcap_{u \in U} E_u : \texttt{Convex}(V),$

$[2] := \jmath D[1] : C \subset D,$

$F := \Lambda v \in V \;.\; \inf \Big\{ \alpha \in \mathbb{R} \Big| (v, \alpha) \in D \Big\} : V \to \mathbb{R},$

$[3] := \jmath F \jmath D[1] : F_{|U} = f,$

$[4] := \jmath F \jmath D[2] : F \le p,$

$[*] := \eth \texttt{Convex}(V, D) \jmath F : \texttt{CauchyFilterbase}(V, V, F);$

$\square$

1.3.6 Smooth Norms

$$\texttt{SmoothAtPoint} :: \prod V \in k\text{-}\mathsf{TVS} \,.\, \prod C \in \texttt{Convex} \,.\, ?\,\partial C$$

$$p : \texttt{SmoothAtPoint} \iff \#\texttt{Support}(V, C, p) = 1$$

$$\texttt{SmoothBody} :: \prod V \in k\text{-}\mathsf{TVS} \,.\, ?\texttt{Convex}$$

$$C : \texttt{SmoothBody} \iff \forall p \in \partial C \,.\, \texttt{SmoothAtPoint}(p)$$

$$\texttt{SmoothNorm} :: \prod V \in k\text{-}\mathsf{TVS} \,.\, \texttt{Norm}(V) \to ?V$$

$$\nu : \texttt{SmoothNorm} \iff \texttt{SmoothBody}(V, \mathbb{B}(\nu))$$

$\texttt{SmmothNormIfDifferentiable}$ ::

$$:: \forall V \in k\text{-}\mathsf{TVS} \,.\, \forall \nu : \texttt{Norm}(V) \,.\, \texttt{SmoothNorm}(V, \nu) \iff \nu \in \mathsf{Diff}(V, V \setminus \{0\}, \mathbb{R})$$

$\texttt{Proof} =$

$\ldots$

$\square$

1.3.7 Sandwich Theorems

combinedAuxilarlyFunctional :: $\prod_{V\in\mathbb{R}\text{-VS}} \prod_{\sigma\in V^{\#}} \prod_{X\subset V} \prod_{f:X\to\mathbb{R}} (f\le\sigma)\to(V\to\mathbb{R})$

combinedAuxilarlyFunctional $(\aleph) = (\sigma,f)^{\#}_X := \Lambda v\in V\ .\ \inf\{\sigma(v+\lambda x)-\lambda f(x)|x\in X,\lambda\in\mathbb{R}_{++}\}$

$\sigma(v+\lambda x)-\lambda f(x)\ge\sigma(\lambda v)-\sigma(-v)-\lambda f(x)=-\sigma(-v)+\lambda(\sigma(x)-f(x))\ge-\sigma(-v)$.

So $(\sigma,f)^{\#}_X(v)\ge-\sigma(-v)>-\infty$ and this means that $(\sigma,f)^{\#}_X$ is well defined .

$\square$

combinedAuxilarlyFunctionalBound1 ::

:: $\forall V\in\mathbb{R}\text{-VS}\ .\ \forall\sigma\in V^{\#}\ .\ \forall X\subset V\ .\ \forall f:X\to\mathbb{R}\ .\ \forall\aleph: f_{|X}\le\sigma\ .\ (\sigma,f)^{\#}_X\le\sigma$

Proof =

$(\sigma,f)^{\#}_X(x)\le\sigma(v+\lambda x)-\lambda f(x)\le\sigma(v)+\lambda\sigma(x)-\lambda f(x)=\sigma(v)+\lambda\big(\sigma(x)-f(x)\big)$.

By taking $\lambda\to0$ one gets $(\sigma,f)^{\#}_X(v)\le\sigma(v)$.

$\square$

combinedAuxilarlyFunctionalBound2 ::

:: $\forall V\in\mathbb{R}\text{-VS}\ .\ \forall\sigma\in V^{\#}\ .\ \forall X\subset V\ .\ \forall f:X\to\mathbb{R}\ .\ \forall\aleph: f_{|X}\le\sigma$.

. $\forall h\in V^*\ .\ h\le(\sigma,f)^{\#}_X\iff f\le h_{|X}\ \&\ h\le\sigma$

Proof =

1 $(\Rightarrow)$ assume $h\le(\sigma,f)^{\#}_X$.

1.1 $h(x)=-h(-x)\ge-(\sigma,f)^{\#}_X(-x)\ge-\sigma(-x+x)+f(x)=f(x)$ for any $x\in X$.

1.2 $h\le(\sigma,f)^{\#}_X\le\sigma$.

2 $(\Leftarrow)$ assume $f\le h_{|X}$ anf $h\le\sigma$.

2.1 Write $\sigma(v+\lambda x)-\lambda f(x)\ge h(v+\lambda x)-\lambda h(x)=h(v)$.

2.2 Then by taking infimum $(\sigma,f)^{\#}_X(v)\ge h(v)$.

$\square$

combinedAuxilarlyFunctionalBound2 ::

:: $\forall V\in\mathbb{R}\text{-VS}\ .\ \forall\sigma\in V^{\#}\ .\ \forall X:\mathtt{Convex}(V)\ .\ \forall f:\mathtt{Concave}(V,V)\ .\ \forall\aleph: f_{|X}\le\sigma$.

. $(\sigma,f)^{\#}_X\in V^{\#}$

Proof =

1 Positive homogenety is obvious.

2 So we investigate subadditivity.

2.1 Select $v,w\in V$.

2.2 Then $(\sigma,f)^{\#}_X(v+w)=\inf_{x,\lambda}\sigma(v+w+\lambda x)-\lambda f(x)\le\inf_{x,\lambda}\sigma\left(v+\frac{\lambda}{2}x\right)-\frac{\lambda}{2}f(x)+\sigma\left(w+\frac{\lambda}{2}x\right)-\frac{\lambda}{2}f(x)$.

. . .

$\square$

$\texttt{SandwichTheorem} :: \forall V \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \forall \sigma \in V^{\#} \,.\, \forall X : \texttt{Convex}(V) \,.\, \forall f : \texttt{Concave}(V, V) \,.\, \forall \aleph : f_{|X} \leq \sigma \,.$
$\quad . \, \exists h \in V^* \,.\, f \leq h_{|X} \;\&\; h \leq \sigma$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.3.8 Paired Spaces

1.3.9 Polar Sets

1.4 Barelled Spaces

1.5 Bornological Spaces

1.6 Towards Approximation Theory

2 Spaces of Distributions

3 Ordered Topological Vector Spaces

3.1 Reisz Spaces and Banach Lattices

3.1.1 Order Unit Norm

OrderUnitDefinesASublinear ::

$:: \forall V : \texttt{OrderedVectorSpace}(\mathbb{R}) \,.\, \forall u : \texttt{OrderUnit}(V) \,.\, \texttt{Sublinear}(V, \Lambda v \in V \,.\, \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\})$

Proof =

1 Write $\omega(v) = \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\}$.

2 Obviously ω is positively homogeneous.

3 Now take $v, w \in V$.

3.1 Define $\alpha = \omega(v) + \omega(w)$.

3.2 Then $v + w \leq \Big(\omega(v) + \omega(w)\Big)u = \alpha u$.

3.3 So $\omega(v + w) \leq \alpha = \omega(v) + \omega(w)$.

□

orderUnitFunctional :: $\prod V : \texttt{OrderedVectorSpace}(\mathbb{R}) \,.\, \texttt{OrderUnit}(V) \to \texttt{Sublinear}(V)$

orderUnitFunctional $(u) = \omega_u := \inf\{\lambda \in \mathbb{R}_{++} : v \leq \lambda u\}$

orderUnitSeminorm :: $\prod V : \texttt{ArchemedeanVectorSpace}(\mathbb{R}) \,.\, \texttt{OrderUnit}(V) \to \mathsf{SMN}(V)$

orderUnitFunctional $(u) = \nu_u := \Lambda v \in V \,.\, \max\Big(\omega_u(v), \omega_u(-v)\Big)$

UnitDiscIsAnInterval :: $\forall V : \texttt{ArchemedeanVectorSpace}(\mathbb{R}) \,.\, \forall u : \texttt{OrderUnit}(V) \,.\, \mathbb{D}(\nu_u) = [-u, u]$

Proof =

1 Obvious.

□

3.1.2 Topological Vector Lattices

$$\texttt{TopologicalVectorLattice} :: ?\mathbb{R}\text{-TVS} \;\&\; \texttt{RieszSpace}$$

$$V : \texttt{TopologicalVectorLattice} \iff \texttt{Closed}(V, \mathcal{C}_V) \;\&$$
$$\&\; \exists \mathcal{B} : \texttt{NeighborhoodBase}(V, 0) \,.\, \forall B \in \mathcal{B} \,.\, \texttt{OrderConvex}(V, B)$$

$$\texttt{BanachLattice} :: ?\texttt{NormedSpace} \;\&\; \texttt{RieszSpace}$$

$$V : \texttt{BanachLattice} \iff \forall v, w \in V \,.\, |v| \leq |w| \Rightarrow \|v\| \leq \|w\|$$

$$\texttt{MSpace} :: ?\texttt{NormedSpace} \;\&\; \texttt{RieszSpace}$$

$$V : \texttt{MSpace} \iff \forall v, w \in V_+ \,.\, \|v \vee w\| = \|v\| \vee \|w\|$$

$$\texttt{LSpace} :: ?\texttt{NormedSpace} \;\&\; \texttt{RieszSpace}$$

$$V : \texttt{LSpace} \iff \forall v, w \in V_+ \,.\, \|v + w\| = \|v\| + \|w\|$$

3.1.3 Lattice of Continuous Functions

ExtremallyDisconnected ::

$:: \forall X \in \mathsf{TOP}\ .\ \texttt{ExtemellyDisconnected}(X) \iff \forall U, V \in \mathcal{T}(X)\ .\ UV = \emptyset \Rightarrow \underset{X}{\mathrm{cl}}\, U\ \underset{X}{\mathrm{cl}}\, V = \emptyset$

Proof $=$

OrderCompletenessOfContinuousFunctions ::

$:: \forall X \in \texttt{ExtemellyDisconnected}\ .\ \texttt{OrderDedekindComplete}\Big(C(X)\Big)$

Proof $=$

$\dots$

$\square$

OrderCompletenessOfContinuousFunctions ::

$:: \forall X : \mathtt{T3.5}\ .\ \texttt{OrderDedekindComplete}\Big(C(X)\Big) \Rightarrow \texttt{ExtemellyDisconnected}(X)$

Proof $=$

$\dots$

$\square$

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