1 Basic constructions

Types and kinds
$\mathcal{K}$ represent all known kinds of objets.

While $\mathcal{T}$ represent all known types of same objects.

Axiom 1 (Axiom of kindnes)**.**

$$\frac{}{\mathcal{T} : \mathcal{K}}(\text{Kindnes})$$

All types are kinds of certain objects. That is, Type exists only if it is inhabited.

let x be a free sentence in working alphabeth Σ

$$\frac{[x]}{\texttt{IsAtom}(x) : \mathcal{T}}\text{Reserve}$$

We can construct atomic types

$$\frac{\texttt{IsAtom}(x) : \mathcal{T}}{\mathbf{x} : \mathcal{T}, \text{type}(x) : \texttt{IsAtom}}\text{Type}(x)$$

Every atom is inhabited by one and only one construction which is identical to itself.

$$\frac{a : \texttt{isAtom}(x)}{x : \mathbf{x}}\text{I(x)}$$

Tuples
Construction:

$$\frac{A, B : \mathcal{T}}{A \times B : \mathcal{T}}\text{Cons}(\times)$$

Introduction:

$$\frac{a : A; b : B}{(a, b) : A \times B}\text{I}(\times)$$

Left projection:

$$\frac{(a, b) : A \times B}{a : A}\pi_l$$

Right projection:

$$\frac{(a, b) : A \times B}{b : B}\pi_r$$

Functions
Construction:

$$\frac{X, Y : \mathcal{K}; A : X; B : Y}{A \to B : \mathcal{T}}\text{Cons}(\to)$$

Application:

$$\frac{f : A \to B; a : A}{f(a) : B}\text{E}(\to)$$

Abstraction:

$$\frac{b : B}{\lambda\, a \in A\ .\ b : A \to B}\mathrm{I}(\to)$$

Super Functions

Construction:

$$\frac{X, Y : \mathcal{K}; B : Y}{X \to B : \mathcal{T}}\mathrm{Cons}(\to)$$

Application:

$$\frac{X \to B; T : X}{f(X) : B}\mathrm{E}(\to)$$

Abstraction:

$$\frac{b : B}{\lambda\, T \in X\ .\ b : X \to B}I(\to)$$

Alternatives

Construction:

$$\frac{A, B : \mathcal{T}}{A|B : \mathcal{T}}\mathrm{const}(|)$$

Left injection:

$$\frac{a : A}{(0, a) : A|B}i_l$$

Right injection:

$$\frac{b : B}{(1, b) : A|B}i_r$$

Elimination:

$$\frac{x : A|B; f : A \to C; g : B \to C}{(f|g)(x) : C}\mathrm{E}(|)$$

Absurd

Axiom 2 (Axiom of Absurd)**.**

$$\bot : \mathcal{T}$$

Mayhem:

$$\frac{a : \bot; T : \mathcal{T}}{\mu_\bot(a, T) : T}\mathrm{E}(\bot)$$

Generics

Construction:

$$\frac{X, Y : \mathcal{K}}{X \rhd Y : \mathcal{K}}\mathrm{Constr}(\rhd)$$

Application:

$$\frac{F : X \rhd Y; T : X}{F(T) : Y}\mathrm{E}(\rhd)$$

Abstraction:

$$\frac{S : Y}{\lambda\, T \in X\ .\ S : X \rhd Y}\mathrm{I}(\rhd)$$

Membership:

$$\frac{T : F; F : X \rhd Y}{\mu_{\rhd}(T, F) : \sum A : X \;.\; F(A)}\mathrm{M}(\rhd)$$

Predicate

Construction:

$$\frac{X, Y : \mathcal{K}; T : X}{?_Y T : \mathcal{K}}\mathrm{Constr}(?)$$

Application:

$$\frac{P :?_Y T; t : T}{P(t) : Y}\mathrm{E}(?)$$

Abstraction:

$$\frac{S : Y}{\lambda\, t \in T\; .S :?_Y T}\mathrm{I}(?)$$

Membership:

$$\frac{x : P; P :?_Y T}{\mu_?(x, P) : \sum t : T \;.\; P(t)}\mathrm{M}(?)$$

Equality

Construction:

$$\frac{X : \mathcal{K}; T : X; a, b : T}{a =_T b : \mathcal{T}}\mathrm{Constr}(=)$$

Reflexivity:

$$\frac{t : T}{\mathrm{id}\,(t) : t =_T t}\mathrm{I}(=)$$

Elemenation

$$\frac{P : T \to T \to \mathcal{T}, p : P(a, a), e : a =_T b}{c(P, p, e) : P(b, a)}\mathrm{E}(=)$$

Product

Construction:

$$\frac{X, Y : \mathcal{K}; T : X; P :?_{\mathcal{T}} T}{\prod t : T \,.\, P(t) : \mathcal{T}}\mathrm{Constr}(\prod)$$

Definition:

$$\frac{t : T \vdash p(t) : P(t)}{\lambda t \in T \,.\, p(t) : \prod t : X \,.\, P(t)}\mathrm{I}\prod$$

Application:

$$\frac{f : \prod t : T \,.\, P(t); t : T}{f(t) : P(t)}\mathrm{E}\prod$$

Sum

Construction:

$$\frac{X, Y : \mathcal{K}; T : X; P :?_{\mathcal{T}} T}{\sum t : T \, . \, P(t) : \mathcal{T}} \text{Constr} \sum$$

Assembly:

$$\frac{t : T; p : P(t)}{(t; p) : \sum t : T \, . \, P(t)} \text{I} \sum$$

Extraction:

$$\frac{x : \sum t : T \, . \, P(t)}{\pi_l(x) : T; \pi_r(x) : P(\pi_l(x))} \text{E} \sum$$

2 Appartnes

Appartnes

Construction:

$$\frac{T : \mathcal{T}; a, b : T}{a \#_T b : \mathcal{T}} \text{Constr}(\#)$$

Contradiction:

$$\frac{\alpha : a =_T b; \beta : a \#_T b}{c_T(\alpha, \beta) : \bot} \text{E}(\#)$$

Pairs:

$$\frac{\alpha : a \#_A c | b \#_B d}{p(\alpha) : (a, b) \#_{A \times B} (c, d)} I(\times, \#)$$

Alternatives:

$$\frac{(0, a), (1, b) : A|B}{x : (0, a) \#_{A|B} (1, b)} I(|, \#)$$

Functions:

$$\frac{f, g : A \to B; a : A; \alpha : f(a) \#_B g(a)}{\xi(\alpha) : f \#_{A \to B} g} I(\to, \#)$$

Detection:

$$\frac{f : A \to B; a, b : A; \alpha : f(a) \#_B f(b)}{d(\alpha) : a \#_A b} I(\#, \to)$$

3 Univalent Foundations

Statement

$$\frac{T : \mathcal{T}}{\mathtt{Statement}(T) : \mathcal{T}} \quad \frac{T : \mathcal{T}; a, b : T}{u(T, a, b) : \mathtt{Statement}(a =_T b)} \quad \frac{\alpha : \mathtt{Statement}(T), a, b : T}{s(\alpha, a, b) : a =_T b}$$

Existance

$$\frac{A : \mathcal{T}}{\exists A : \mathcal{T}, \alpha(A) : \mathtt{Statement}(\exists A)} \quad \frac{a : A}{[a] : \exists A} \quad \frac{p : \exists A}{w(p) : A}$$

And

$$\frac{A, B : \mathcal{T}, \alpha : \mathtt{Statement}(A), \beta : \mathtt{Statement}(B)}{A \wedge B : \mathcal{T}, \mathrm{and}(\alpha, \beta) : \mathtt{Statement}(A \wedge B)} \quad \frac{a : A, b : B}{(a, b) : A \wedge B} \quad \frac{p : A \wedge B}{w(p) : A \times B}$$

Implication

$$\frac{A, B : \mathcal{T}, \alpha : \mathtt{Statement}(A), \beta : \mathtt{Statement}(B)}{A \Rightarrow B : \mathcal{T}, \mathrm{and}(\alpha, \beta) : \mathtt{Statement}(A \Rightarrow B)}$$

$$\frac{\alpha : \exists A \to B}{[\alpha] : A \Rightarrow B} \quad \frac{p : A \Rightarrow B}{w(p) : A \to B} \quad \frac{p : A \Rightarrow B, a : A}{p(a) : B}$$

Or

$$\frac{A, B : \mathcal{T}; \alpha : \mathtt{Statement}(A); \beta : \mathtt{Statement}(B)}{A \vee B : \mathcal{T}; \mathrm{or}(\alpha, \beta) : \mathtt{Statement}(A \vee B)}$$

$$\frac{c : A|B}{[c] : A \vee B} \quad \frac{p : A \vee B}{w(p) : A|B}$$

Universal

$$\frac{A : \mathcal{T}, \alpha : \mathtt{Statement}(A), P : \prod a : A \;.\; \sum b(a) : B(a) \;.\; \mathtt{Statement}(b)}{\forall a : A \;.\; b(a) : \mathcal{T}, U(A, \alpha, P, B, b) : \mathtt{Statement}\,(\forall a \in A \;.\; b(a))}$$

$$\frac{F : \prod a : A \;.\; b(a)}{[F] : \forall a : A \;.\; b(a)} \quad \frac{p : \forall a : A \;.\; b(a)}{w(p) : \prod a : A \;.\; b(a)}$$

4 Natural numbers

Generic Naturals

$$\begin{aligned}
&\mathrm{Nat} :: \mathcal{T} \rhd \mathcal{T} \\
&\mathrm{Nat}(\mathbb{N}) = \exists 1 : \mathbb{N} \;.\; \exists (<) :?_{\mathcal{T}}(\mathbb{N} \times \mathbb{N}) \;.\; \exists \sigma : \mathbb{N} \to \mathbb{N} \;. \\
&\quad .\; \forall n \in \mathbb{N} \;.\; (n = 1 + (\exists m \in \mathbb{N} \;.\; n = \sigma m)(1 < n))(n < n \to \bot)(n < \sigma n)
\end{aligned}$$

Constructive Naturals

$$\mathbb{N} :: \mathcal{T}$$
$$\text{unit} : \mathbb{N}$$
$$n : \mathbb{N} \mid S(n) : \mathbb{N}$$

$$1 :: \mathbb{N}$$
$$[\text{unit}] : \mathbb{N}$$

$$\text{next} :: \mathbb{N} \to \mathbb{N}$$
$$\text{next}(o) = (1, ((), o))$$

$$\text{priv} :: \overset{\infty}{\mathbb{N}} \to \overset{\infty}{\mathbb{N}} + [\text{overflow}]$$
$$\text{priv}(1) = \text{overflow}$$
$$\text{priv}(1, o) = o$$

$$\mathbb{N} ::?_{\mathcal{T}} \overset{\infty}{\mathbb{N}}$$
$$\mathbb{N}(n) = (\text{next}\, n = n) \to \perp$$

Boolean

$$\mathbb{B} :: \mathcal{T}$$

$$\mathbb{B} = \text{true} + \text{false}$$

Basic Boolean Algebra

$$! :: \mathbb{B} \to \mathbb{B}$$

$$!\text{true} = \text{false}$$

$$!\text{false} = \text{true}$$

$$\wedge :: \mathbb{B}^2 \to \mathbb{B}$$

$$\text{true} \wedge \text{true} = \text{true}$$

$$b \wedge b = \text{false}$$

$$\vee :: \mathbb{B}^2 \to \mathbb{B}$$

$$\text{false} \vee \text{false} = \text{false}$$

$$b \vee b = \text{true}$$

Equate

$$\text{equate} :: \mathbb{N}^2 \to \mathbb{B}$$

$$\text{equate}(1, 1) = \text{true}$$

$$\text{equate}(1, a) = \text{false}$$

$$\text{equate}(a, 1) = \text{false}$$

$$\text{equate}(a, b) = a == b = \text{equate}(\text{priv}\, a, \text{priv}\, b)$$

Equatable class

$$\text{Equatable} :: \mathcal{T} + ?\text{Equatable} \to \mathcal{T}$$

$$\text{Equatable}(T) = \sum (==) : T^2 \to \mathbb{B}. \prod a, b \in T. a == n =_{\mathbb{B}} \text{true} \leftrightarrow a =_T b$$

$$(\mathbb{N}, (==, \ldots)) : \text{Equatable}$$

Set over Type

$$\text{Set} :: \mathcal{T} \triangleright \mathcal{T}$$

$$\text{Set}(T) = T \to \mathbb{B}$$

following structure have non-constructive types and should be banished from the theory:

$$\text{Universe} :: \prod T : \mathcal{T} \ . \ \text{Set}(T)$$

$$\text{Universe}(T) = U_T = \lambda a : T \;.\; \text{true}$$

$$\text{Empty} :: \prod T : \mathcal{T} \;.\; \text{Set}(T)$$

$$\text{Empty}(T) = \emptyset_T = \lambda a : T \;.\; \text{false}$$

However this notation is simple and as it indicates only simple constant functions will be used in the sequel.

Sets and Predicates
Each set forms a predicate:

$$\text{Inside} :: \prod T : \mathcal{T} \;.\; \text{Set(T)} \rightarrow ?_{\mathcal{T}} T$$

$$\text{Inside}(A)(a) = a \in A = A(a) =_{\mathbb{B}} \text{true}$$

However it's not true that all predicates form a set. The most we can is to translate a predicate to a predicate over world of sets.

$$\text{represents} :: \prod T : \mathcal{T}. ?_{\mathcal{T}} T \rightarrow ?_{\mathcal{T}} \text{Set}(T)$$

$$\text{represents}(P)(S) = \prod t : T \;.\; P(t) \leftrightarrow (t \in S)$$

Basic Set Algebra

$$\text{union} :: \prod K : \mathcal{K}. \prod T : \mathcal{T}. \prod I : K.(I \rightarrow \text{Set}(T)) \rightarrow ?\text{Set}(T)$$

$$\text{union}(A)(S) = (S = \bigcup_{i \in I} A_i) = \prod t : T. \Big(t \in S \leftrightarrow \prod i : I.t \in A_i\Big)$$

$$\text{union}' :: \prod T : \mathcal{T}.\text{Set}(T)^2 \rightarrow \text{Set}(T)$$

$$\text{union}'(A, B) = A \cup B = \lambda t : T.A(t) \vee B(t)$$

$$\text{intersect} :: \prod K : \mathcal{K}. \prod T : \mathcal{T}. \prod I : K.(I \rightarrow \text{Set}(T)) \rightarrow ?\text{Set}(T)$$

$$\text{intersect}(A)(S) = (S = \bigcap_{i \in I} A_i) = \prod t : T. \Big(t \in S \leftrightarrow \sum i : I.t \in A_i\Big)$$

$$\text{intersect}' :: \prod T : \mathcal{T}.\text{Set}(T)^2 \rightarrow \text{Set}(T)$$

$$\text{intersect}'(A, B) = A \cap B = \lambda t : T.A(t) \wedge B(t)$$

$$\text{complement} :: \prod T : \mathcal{T}.\text{Set}(T) \rightarrow \text{Set}(T)$$

$$\text{complement } A = A^{\complement} = \lambda t : T.!A(t)$$

$$\text{setminus} :: \prod T : \mathcal{T}.\text{Set}(T)^2 \to \text{Set}(T)$$

$$\text{setminus}(A, B) = A \setminus B = A \cap B^{\complement}$$

Singletons

$$\text{singleton} :: \prod T : \text{Equatable}.T \to \text{Set}(T)$$

$$\text{singleton}(a) = \{a\} = \lambda t : T.t ==_T a$$

Injections, Surjections and Bijections

$$\text{Injection} ::?A \to B$$

$$\text{Injection}(f) = \prod x, y : A.f(x) =_B f(y) \to x =_A y$$

$$\text{Surjection} ::?A \to B$$

$$\text{Surjection}(f) = \prod y : B. \sum x : A.f(x) =_B y$$

$$\text{Bijection} ::?A \to B$$

$$\text{Bijection}(f) = \sum f^{-1} : B \to A. \prod a : A.f^{-1}f(a) = a \times \prod b : B.ff^{-1}(b) = b$$

$$\text{Finite} ::?A \to B$$

$$\text{Bijection}(f) = \sum f^{-1} : B \to A. \prod a : A.f^{-1}f(a) = a \times \prod b : B.ff^{-1}(b) = b$$