Representation Of Finite Groups

Uncultured Tramp

January 8, 2020

Contents

1 Classical Representation Theory

1.1 Category Of Group Representations

$\texttt{groupRepresentationCategory} :: \mathsf{GRP} \to \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{groupRepresentationCategory}\,(G, A) = A\text{-}\mathsf{REPR}\,(G) :=$

$$:= \left(\sum V \in R\text{-}\mathsf{MOD}\,.\, G \xrightarrow{\mathsf{GRP}} \mathbf{GL}(V), \Lambda(V,\rho), (W,\rho')\,.\, \sum T : V \xrightarrow{R\text{-}\mathsf{MOD}} W\,.\, \forall g \in G\,.\, \rho(g)T = T\rho'(g),\right.$$
$$\left., \mathrm{id}, \circ\right)$$

$\texttt{zerothRepresentation} :: \prod G \in \mathsf{GRP}\,.\, \prod A \in \mathsf{ANN}\,.\, A\text{-}\mathsf{REPR}\,(G)$

$\texttt{zerothRepresentation}\,(G, R) = 0_{G,A} := \Big(\{0\}, g \mapsto \mathrm{id}\Big)$

$\texttt{IdentityRepresentation} :: \prod G \in \mathsf{GRP}\,.\, \prod A \in \mathsf{ANN}\,.\, A\text{-}\mathsf{REPR}\,(G)$

$\texttt{IdentityRepresentation}\,(G, R) = e_{G,A} := \Big(A, g \mapsto \mathrm{id}\Big)$

$\texttt{ZerothRepresentationIsZeroObject} :: \forall G \in \mathsf{GRP}\,.\, \forall A \in \mathsf{ANN}\,.\, 0_{G,A} : \texttt{Zero}\Big(A\text{-}\mathsf{REPR}\,(G)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{degreeOfRepresentation} :: A\text{-}\mathsf{REPR}\,(G) \to \mathsf{CARD}$

$\texttt{degreeOfRepresentation}\,(V, \rho) = \deg(V, \rho) := \mathrm{rank}_A V$

$\texttt{GroupInvariantSubspace} :: \prod (V, \rho) \in A\text{-}\mathsf{REPR}\,(G)\,.\, ?\texttt{Submodule}(V)$

$U : \texttt{GroupInvariantSubspace} \iff \forall g \in G\,.\, \rho_g(U) = U$

$\texttt{directSumOfRepresentation} :: \prod I \in \mathsf{SET}\,.\, \Big(I \to A\text{-}\mathsf{REPR}\,(G)\Big) \to A\text{-}\mathsf{REPR}\,(G)$

$$\texttt{directSumOfRepresentations}\,((V, \rho)) = \bigoplus_{i \in I} \rho_i := \left(\bigoplus i \in I V_i, \Lambda g \in G\,.\, \bigoplus_{i \in I} \rho_i(g)\right)$$

$\texttt{RepresentationCoproduct} :: \forall A \in \mathsf{ANN}\,.\, \forall G \in \mathsf{GRP}\,.\, (\oplus, \iota) : \texttt{Coproduct}\Big(A\text{-}\mathsf{REPR}\,(G)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{subrepresentation} :: \prod (\rho, V) \in A\text{-}\mathsf{REPR}\,(G)\,.\, \texttt{GroupInvariantSubspace}(\rho, V) \to A\text{-}\mathsf{REPR}\,(G)$

$\texttt{subrepresentation}\,(U) = \rho_{|U} := \Big(U, \Lambda g \in G\,.\, \rho(g)_{|U}\Big)$

$\texttt{SubrepresentationDirectSum} :: \forall(\rho, V) : A\text{-}\mathsf{REPR}\,(G)\ .\ \forall U, W : \texttt{GroupInvariantSubspace}(\rho, V)\ .$
$\quad .\ \forall[0] : U \cap W = 0\ .\ \rho_{|U} \oplus \rho_{|W} \cong_{A\text{-}\mathsf{REPR}(G)} \rho_{|U \oplus W}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Irreducible} :: ?A\text{-}\mathsf{REPR}\,(G)$

$(V, \rho) : \texttt{Irreducible} \iff \forall U : \texttt{GroupInvariantSubspace}(V, \rho)\ .\ U = V | U = 0$

$\texttt{DegreeOneIsIrreducible} :: \forall k : \texttt{Field}\ .\ \forall \rho : k\text{-}\mathsf{REPR}\,(G)\ .\ \deg \rho = 1 \Rightarrow \rho : \texttt{Irreducible}(k, G)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{EigenvectorIrreducibilityCriterion} :: \forall k : \texttt{Field}\ .\ \forall \rho : k\text{-}\mathsf{REPR}\,(G)\ .\ \forall[0]; \deg \rho = 2\ .$
$$.\ \rho : \texttt{Irreducible}(k, G) \iff \bigcap_{g \in G} \texttt{Eigenvector}(\rho_g) = \emptyset$$

$\texttt{Proof} =$

$\texttt{CompletelyReducible} :: ?A\text{-}\mathsf{REPR}\,(G)$

$(\rho, V) : \texttt{CompletelyReducible} \iff \exists I \in \mathsf{SET} : \exists U : I \to \texttt{Submodule}(A, V) :$
$$: V = \bigoplus_{i \in I} U_i\ \&\ \forall i \in I\ .\ \rho_{|U_i} : \texttt{Irreducible}(A, G)$$

$\texttt{Decomposable} :: ?A\text{-}\mathsf{REPR}\,(G)$

$(\rho, V) : \texttt{Decomposable} \iff \exists U, W : \texttt{GroupInvariantSubspace}(\rho, V)\ .\ U, W \neq 0\ \&\ V = U \oplus W\ \&\ \rho \neq 0_{A,G}$

$\texttt{kernelIsSubrepresentation} :: \forall(V, \alpha), (W, \beta) : A\text{-}\mathsf{REPR}\,(G)\ .\ \forall T : \alpha \xrightarrow{A\text{-}\mathsf{REPR}(G)}\ .$
$\quad .\ \ker T : \texttt{GroupInvariantSubspace}(\alpha)$

$\texttt{Proof} =$

$\texttt{Assume}\ v : \ker T,$

$\texttt{Assume}\ g : G,$

$[1] := \mathcal{I}\,A\text{-}\mathsf{REPR}\,(G)\,(\alpha, \beta)(T)\mathcal{I} \ker T\,\texttt{NeutralImage}(\beta_g) : v\alpha_g T = vT\beta_g = 0\beta_g = 0,$

$[v.*] := \mathcal{I} \ker T[1] : v\alpha_g \in kerT;$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{GroupInvariantSubspace}(\alpha) : \Big(\ker T : \texttt{GroupInvariantSubspace}(\alpha)\Big);$

$\square$

$\texttt{RepresentationsMorphismsAreSubmodule} :: \forall(V, \alpha), (W, \beta) : A\text{-}\mathsf{REPR}\,(G)\ .$
$$\left(\alpha \xrightarrow{A\text{-}\mathsf{REPR}(G)} \beta\right) \subset_{A\text{-}\mathsf{MOD}} \left(V \xrightarrow{A\text{-}\mathsf{MOD}} W\right)$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ImageIsSubrepresentation} :: \forall (V,\alpha),(W,\beta) : A\text{-REPR}\,(G)\ .\ \forall T : \alpha \xrightarrow{A\text{-REPR}(G)}\ .$

$\quad .\ \operatorname{Im} T : \texttt{GroupInvariantSubspace}(\beta)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ w : \operatorname{Im} T,$

$\Big(v,[1]\Big) := \textit{Ɑ}\,\texttt{image}(w) : \sum v \in V\ .\ w = Tv,$

$\texttt{Assume}\ g : G,$

$[1] := \textit{Ɑ}\,A\text{-REPR}\,(G)\,(\alpha,\beta)(T)\textit{Ɑ}\ker T\texttt{NeatralImage}(\beta_g) : w\beta_g = vT\beta_g = v\alpha_g T,$

$[v.*] := \textit{Ɑ}\operatorname{Im} T[1] : w\beta_g \in imT;$

$\rightsquigarrow [*] := \textit{Ɑ}^{-1}\texttt{GroupInvariantSubspace}(\beta) : \Big(\operatorname{Im} T : \texttt{GroupInvariantSubspace}(\beta)\Big);$

$\square$

$\texttt{DecomposableByEquivalence} :: \forall (V,\rho) \in A\text{-REPR}\,(G)\ .\ \forall (V',\rho') : \texttt{Decomposable}(A,G)\ .$

$\quad .\ \forall [0] : \rho \cong_{A\text{-REPR}(G)} \rho'\ .\ (V,\rho) : \texttt{Decomposable}(A,G)$

$\texttt{Proof}\ =$

$\Big(T,[1]\Big) := \textit{Ɑ}\,A\text{-REPR}\,(G)\,[0] : \sum T : V \xleftrightarrow{A\text{-MOD}} V'\ .\ \forall g \in G\ .\ \rho_g T = T\rho'_g,$

$\Big(U',W',[2]\Big) := \textit{Ɑ}\texttt{Decomposable}(A,G)(V',\rho') : \sum U',W' : \texttt{GroupInvariantSubspace}(V',\rho')\ .\ V = U' \oplus W',$

$[3] := \textit{Ɑ}T\textit{Ɑ}\texttt{Decomposable}(A,G)(V',\rho') : (V,\rho) \neq 0_{A,G},$

$U := T^{-1}U' : \texttt{VectorSubspace}(V),$

$W := T^{-1}W' : \texttt{VectorSubspace}(W),$

$[4] := \texttt{DirectSumIsomorphism}\mathcal{O}V\mathcal{O}W : V = U \oplus W,$

$\texttt{Assume}\ u : U,$

$\texttt{Assume}\ g : G,$

$[u.*] := \textit{Ɑ}^{-1}\texttt{inverse}(T)[1]\mathcal{O}U\textit{Ɑ}\texttt{GroupInvariantSubspace}(V',\rho')(U')\mathcal{O}U : u\rho_g = uTT^{-1}\rho_g uT\rho'_g T^{-1} \in U;$

$\rightsquigarrow [5] := \textit{Ɑ}^{-1}\texttt{GroupInvariantSubspace} : \Big(U : \texttt{GroupInvariantSubspace}(V,\rho)\Big),$

$\texttt{Assume}\ w : W,$

$\texttt{Assume}\ g : G,$

$[w.*] := \textit{Ɑ}^{-1}\texttt{inverse}(T)[1]\mathcal{O}W\textit{Ɑ}\texttt{GroupInvariantSubspace}(V',\rho')(W')\mathcal{O}W : w\rho_g = wTT^{-1}\rho_g wT\rho'_g T^{-1} \in W;$

$\rightsquigarrow [6] := \textit{Ɑ}^{-1}\texttt{GroupInvariantSubspace} : \Big(W : \texttt{GroupInvariantSubspace}(V,\rho)\Big),$

$[*] := \textit{Ɑ}^{-1}\texttt{Decomposable}[3][4][5][6] : \Big((V,\rho) : \texttt{Decomposable}(A,G)\Big);$

$\square$

$\texttt{IrreducibleByEquivalence} :: \forall (V,\rho) \in A\text{-REPR}\,(G)\ .\ \forall (V',\rho') : \texttt{Irreducible}(A,G)\ .$

$\quad .\ \forall [0] : \rho \cong_{A\text{-REPR}(G)} \rho'\ .\ (V,\rho) : \texttt{Irreducible}(A,G)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{CompletelyReducibleByEquivalence} :: \forall (V,\rho) \in A\text{-REPR}\,(G)\ .\ \forall (V',\rho') : \texttt{CompletelyReducible}(A,G)\ .$

$\quad .\ \forall [0] : \rho \cong_{A\text{-REPR}(G)} \rho'\ .\ (V,\rho) : \texttt{CompletelyReducible}(A,G)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

1.2 Maschke's Theorem

$\texttt{OrthogonalRepresentation} :: \prod k : \texttt{Field} \ . \ ?(\text{-REPR}(k), G)$

$(\rho, V) : \texttt{OrthogonalRepresentation} \iff V : \texttt{InnerProductSpace}(k) \ \& \ \rho(G) \subset \mathbf{O}(V)$

$\texttt{OrthogonalRepresentationProperty} :: \forall k : \texttt{Field} \ . \ \forall (V, \rho) : \texttt{OrhogonalRepresentation}(k, G) \ .$

$\quad (V, \rho) : \texttt{Irreducible}(k, G) \Big| (V, \rho) : \texttt{Decomposable}(k, G)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [0] : V \ ! \ \texttt{Irreducible}(k, G),$

$\Big(U, [1]\Big) := \mathcal{J}\texttt{Irreducible}[0] : \sum U : \texttt{GroupInvariantSubspace}(V, \rho) \ . \ U \neq 0 \ \& \ U \neq V,$

$W := U^{\perp} : \texttt{VectorSubspace}(V),$

$[2] := \texttt{OrthogonalComplementDecomposition}(U) : V = U \oplus W,$

$[3] := \mathcal{I}W[1] : W \neq 0 \ \& \ W \neq V,$

$\texttt{Assume} \ w : W,$

$\texttt{Assume} \ g : G,$

$\texttt{Assume} \ u : U,$

$\Big(u', [4]\Big) := \mathcal{J}\texttt{GroupInvariantSubspace}(\rho, V)(U)(u)\mathcal{J}\mathbf{GL}(V)(\rho_g) : \sum_{u' \in U} u = u'\rho_g,$

$[u.*] := [4]\mathcal{J}\mathbf{O}(V)(\rho_g)\mathcal{J}\texttt{Orthogonal}(w, u') : \langle w\rho_g, u\rangle = \langle w\rho_g, u'\rho_g\rangle = \langle w, u'\rangle = 0;$

$\rightsquigarrow [w.*] := I(\forall)\mathcal{J}\texttt{OrhogonalComplement}\mathcal{I}W : w\rho_g \in W;$

$\rightsquigarrow [0.*] := \mathcal{J}^{-1}\texttt{GroupInvariantSubspace}(k, G) : \Big(W : \texttt{GroupInvariantSubspace}(k, G)\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big((V, \rho) \ ! \ \texttt{Irreducible}(k, G)\Big) \Rightarrow (V, \rho) : \texttt{Decomposable}(k, G),$

$[*] := \texttt{NegativeLEM}[1] : (V, \rho) : \texttt{Irreducible}(k, G) | (V, \rho) : \texttt{Decomposable}(k, G);$

$\square$

$\texttt{RepresentationOrthogonalization} :: \forall G : \texttt{FiniteGroup} \ . \ \forall k : \texttt{Field} \ . \ \forall (V, \rho) \in k\text{-REPR}(G) \ .$

$\quad . \ \forall [0] : (V : \texttt{InnerProductSpace}(k)) \ . \ . \exists (V', \rho') : \texttt{OrthogonalRepresentation}(k, G) : \rho \cong_{k\text{-REPR}(G)} \rho'$

$\texttt{Proof} \ =$

$Q := \Lambda x, y \in V \ . \ \sum_{g \in G} \langle x\rho_g, y\rho_g\rangle : \texttt{InnerProduct}(V),$

$\texttt{Assume} \ f : G,$

$\texttt{Assume} \ x, y : V,$

$[f.*] := \mathcal{I}Q\mathcal{J}\text{GRP}\Big(G, \mathbf{GL}(V)\Big)(\rho)\texttt{GroupCyclingSum}(G)\mathcal{I}^{-1}Q :$

$\quad : Q(x\rho_f, y\rho_f) = \sum_{g \in G} \langle x\rho_f\rho_g, y\rho_f\rho_g\rangle = \sum_{g \in G} \langle x\rho_{fg}, y\rho_{fg}\rangle \sum_{g \in G} \langle x\rho_g, y\rho_g\rangle = Q(x, y);$

$\rightsquigarrow [1] := \mathcal{J}^{-1}\texttt{OrghogonalRepresentation} : \Big(\big((V, Q), \rho\big) : \texttt{OrhogonalRepresentation}(k, G)\Big),$

$[*] := \mathcal{J}k\text{-REPR}(G) : \big(V, \rho\big) \cong_{k\text{-REPR}(G)} \big(V, \rho\big);$

$\square$

$\texttt{FiniteGroupRepresentationProperty} :: \forall k : \texttt{Field} . \forall G : \texttt{FiniteGroup} . \forall (V, \rho) \in k\text{-REPR}(G) .$
$\quad \forall Q : \texttt{InnerProduct}(V) . (V, \rho) : \texttt{Irreducible}(k, G) | (V, \rho) : \texttt{Decomposable}(k, G)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{AveragingLemma} :: \forall A \in \text{ANN} . \forall G : \texttt{FiniteGroup} .$
$\quad . \forall (V, \rho), (V', \rho') \in A\text{-REPR}(G) . \forall T : V \xrightarrow{A\text{-MOD}} V' . \sum_{g \in G} \rho_g^{-1} T \rho_g : \rho \xrightarrow{A\text{-REPR}(G)} \rho'$

$\texttt{Proof} =$

$T' := \sum_{g \in G} \rho_g^{-1} T \rho_g : V \xrightarrow{A\text{-MOD}} V',$

$\texttt{Assume}\ f : G,$

$\texttt{Assume}\ v : V,$

$[v.*] := \mathcal{D} T' \mathcal{C}^{-1} \texttt{inverse}(\rho_f) \mathcal{C}\, \text{GRP}\Big(G, \mathbf{GL}(V)\Big)(\rho) \mathcal{C}\, \text{GRP}\Big(G, \mathbf{GL}(V)\Big)(\rho') \texttt{GroupSumCycle}(G) \mathcal{D}^{-1} T' :$
$\quad : v T' \rho'_f = \sum_{g \in G} v \rho_g^{-1} T \rho_g \rho_f = \sum_{g \in G} v \rho_f \rho_f^{-1} \rho_g^{-1} T \rho_g \rho_f = \sum_{g \in G} v \rho_f \rho_{gf}^{-1} T \rho_{gf} = \sum_{g \in G} v \rho_f \rho_g^{-1} T \rho_g = v \rho_f T';$

$\rightsquigarrow [f.*] := I(=, \to) : \rho_f T' = T' \rho'_f;$

$\rightsquigarrow [*] := \mathcal{C}\, A\text{-REPR}(G) : \Big(T' : \rho \xrightarrow{A\text{-REPR}(G)} \rho'\Big);$

$\dots$

$\square$

$\texttt{FixedPointsDimensionByAveraging} :: \forall G : \texttt{FiniteGroup} . \forall k : \texttt{Field} . \forall [0] : |G| \neq_k 0 .$
$\quad . \forall (V, \rho) : k\text{-REPR}(G) . \forall [00] : \dim V < \infty . \dim V^\rho = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} \rho_g$

$\texttt{Proof} =$

$P := \frac{1}{|G|} \sum_{g \in G} \rho_g : \operatorname{End}_{k\text{-VS}}(V),$

$[1] := \mathcal{C}^{-1} V^\rho \texttt{GroupSumCycle} : \operatorname{Im} P \subset V^\rho,$

$[2] := \mathcal{C} V^\rho \texttt{ConstantSum} \mathcal{C}\, \texttt{Inverse} : \forall v \in V^\rho . Pv = v,$

$[3] := [1][2] \mathcal{C}\, \texttt{image} : \operatorname{Im} P = V^\rho,$

$[4] := [2] \mathcal{C}\, \texttt{kernel} : \ker P \cap \operatorname{Im} P = 0,$

$[5] := \texttt{KernelRankTHM}(P) \mathcal{C}\, \texttt{rank} : \dim \ker P + \dim \operatorname{Im} P = \dim V,$

$[6] := [4][5] \texttt{SumDimTHM} \mathcal{C}^{-1} \texttt{DirectSum} : V = \ker P \oplus \operatorname{Im} P;$

$[7] := \texttt{StructureOfTheProjection}[2][6] : \Big(P : \texttt{Projetor}(V)\Big),$

$[*] := [3] \texttt{DimensionByProjectorsTrace}(P) \mathcal{D} P \mathcal{C} V Sk(V, k)(P) : \dim V^\rho = \dim \operatorname{Im} P = \operatorname{tr} P = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} \rho_g;$

$\square$

$\texttt{biaveraging} :: \prod G : \texttt{FiniteGroup} \,.\, \prod (V,\alpha),(W,\beta) : A\text{-}\mathsf{REPR}\,(G) \;.\, |G| \in A^* \to$

$\to (V \xrightarrow{A\text{-}\mathsf{MOD}} W) \xrightarrow{A\text{-}\mathsf{MOD}} (\alpha \xrightarrow{A\text{-}\mathsf{REPR}(G)} \beta)$

$\texttt{biaveraging}\,(T) = \operatorname{avg} T := \frac{1}{|G|}\sum_{g\in G} \alpha_g^{-1} T \beta_g$

$\texttt{FiniteGroupRepresentationProperty2} :: \forall k : \texttt{Field} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \forall [0] : |G| \neq_k 0 \,.$

$\quad .\, \forall (V,\rho) \in k\text{-}\mathsf{REPR}\,(G) \;.\, (V,\rho) : \texttt{Irreducible}(k,G) | (V,\rho) : \texttt{Decomposable}(k,G)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [0] : V \,!\, \texttt{Irreducible}(k,G),$

$\Big(U,[1]\Big) := \mathcal{I}\texttt{Irreducible}[0] : \sum U : \texttt{GroupInvariantSubspace}(V,\rho) \,.\, U \neq 0 \;\&\; U \neq V,$

$\Big(W,[2]\Big) := \texttt{LinearComplementExists}(U) : \sum W \subset_{k\text{-}\mathsf{VS}} V \,.\, V = U \oplus W,$

$T := \pi_{U,W} : \mathrm{End}_{k\text{-}\mathsf{VS}}(V),$

$T' := \frac{1}{|G|}\sum_{g\in G} \rho_g^{-1} T \rho_g : \mathrm{End}_{k\text{-}\mathsf{VS}}(V),$

$\texttt{Assume}\ u : U,$

$[u.*] := \mathcal{O}T'\mathcal{I}\texttt{GroupInvariantSubspace}(V,\rho)(U)\mathcal{O}T\mathcal{I}\texttt{projectionOnAlong}\mathcal{I}\texttt{Inverse}$

$\quad \texttt{ConstantSum}(|G|,u)\mathcal{I}\texttt{Inverse}[00] : T'u = \frac{1}{|G|}\sum_{g\in G} u\rho_g^{-1}T\rho_g = \frac{1}{|G|}\sum_{g\in G} u\rho_g^{-1}\rho_g = \frac{1}{|G|}\sum_{g\in G} u = u;$

$\rightsquigarrow [3] := \ldots : \operatorname{Im} T' = U \ker T' \cap U = 0,$

$\texttt{Assume}\ w : \ker T',$

$\texttt{Assume}\ f : G,$

$[4] := \texttt{AveregingLemma} :$

$\quad : w\rho_f T' = wT'\rho_f = 0,$

$[w.*] := \mathcal{I}\texttt{kernel}[5] : w\rho_f \in \ker T';$

$\rightsquigarrow [4] := \mathcal{I}\texttt{GroupInvariantSubspace} : \Big(\ker T' : \texttt{GroupInvariantSubspace}(V,\rho)\Big),$

$[5] := \texttt{kerImLemma}[3] : V = U \oplus \ker T',$

$[1.*] := \mathcal{I}\texttt{Decomposable}[3][4] : \Big((V,\rho) : \texttt{Decomposable}(k,G)\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big((V,\rho) \,!\, \texttt{Irreducible}(k,G)\Big) \Rightarrow (V,\rho) : \texttt{Decomposable}(k,G),$

$[*] := \texttt{NegativeLEM}[1] : (V,\rho) : \texttt{Irreducible}(k,G) | (V,\rho) : \texttt{Decomposable}(k,G);$

$\square$

$\color{red}{\texttt{MaschkeTHM}} :: \forall k : \texttt{Field} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \forall [00] : |G| \neq_k 0 \,.\, \forall (V, \rho) \in k\text{-REPR}\,(G) \,.$
$\quad \forall [0] : \dim V < \infty \,.\, \rho : \texttt{CompletelyReducible}(k, G)$

$\texttt{Proof} \;=$

$\text{♂} := \lambda n \in \mathbb{N} \,.\, \forall (V, \rho) \in k\text{-REPR}\,(G) \dim V \leq n \Rightarrow \rho : \texttt{CompletelyReducible}(k, G) : \mathbb{N} \to \texttt{Type},$

$[1] := \texttt{DegreeOneIsIrreducible}\,\text{Ɑ}^{-1}\texttt{CompletelyReducible}(k, G)\text{Ɔ}^{-1} : \text{♂}(1),$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; [2] : \text{♂}(n),$

$\texttt{Assume}\; (V, \rho) : k\text{-REPR}\,(G)\,,$

$\texttt{Assume}\; [3] : \dim V = n + 1,$

$[4] := \texttt{FiniteGroupRepresentationProperty}(V, \rho) :$
$\quad : \Big((V, \rho) : \texttt{Irreducible}(k, G) \Big| (V, \rho) : \texttt{Decomposable}(k, G) \Big),$

$\texttt{Assume}\; [5] : \Big((V, \rho) : \texttt{Irreducible}(k, G) \Big),$

$[5.*] := \text{Ɑ}^{-1}\texttt{CompletelyReducible}(k, R)[5] : \Big((V, \rho) : \texttt{CompletelyReducible}(k, R) \Big);$

$\rightsquigarrow [5] := I(\Rightarrow) : \Big((V, \rho) : \texttt{Irreducible}(k) \Big) \Rightarrow (V, \rho) : \texttt{CompletelyReducible}(k, R),$

$\texttt{Assume}\; [6] : \Big((V, \rho) : \texttt{Decomposable}(k, G) \Big),$

$\Big(U, W, [7] \Big) := \text{Ɑ}\texttt{Decomposable}(k, G)(V, \rho) :$
$\quad : \sum U, W : \texttt{GroupInvariantSubspace}(V, \rho) \,.\, U, W \neq 0 \;\&\; U \oplus W = V,$

$[8] := \text{Ɑ}^{-1} \dim[7][3] : \dim U \leq n \;\&\; \dim W \leq n,$

$[9] := \text{Ɔ}\text{♂}[2][8](U) : \Big(\rho_{|U} : \texttt{CompletelyReducible}(k, G) \Big),$

$[10] := \text{Ɔ}\text{♂}[2][8](W) : \Big(\rho_{|W} : \texttt{CompletelyReducible}(k, G) \Big),$

$\Big(t, u, [11] \Big) := \text{Ɑ}\texttt{CompletelyReducible}(k, G)(\rho_{|U}) :$
$$: \sum t \in \mathbb{N} \,.\, \sum u : t \to \texttt{Irreducible}(k, G) \,.\, \rho_{|U} = \bigoplus_{i=1}^{t} u_i,$$

$\Big(s, w, [12] \Big) := \text{Ɑ}\texttt{CompletelyReducible}(k, G)(\rho_{|W}) :$
$$: \sum s \in \mathbb{N} \,.\, \sum w : s \to \texttt{Irreducible}(k, G) \,.\, \rho_{|W} = \bigoplus_{i=1}^{s} w_i,$$

$$[13] := [7][11][12] : \rho = \bigoplus_{i=1}^{t} u_i \oplus \bigoplus_{i=1}^{s} w_i,$$

$[6.*] := \text{Ɑ}^{-1}\texttt{CompletelyReducible}(k, R)[13] : \Big((V, \rho) : \texttt{CompletelyReducible}(k, R) \Big);$

$\rightsquigarrow [6] := I(\Rightarrow) : \Big((V, \rho) : \texttt{Decomposeble}(k) \Big) \Rightarrow (V, \rho) : \texttt{CompletelyReducible}(k, R),$

$[n.*] := E(|)[4][5][6] : \Big((V, \rho) : \texttt{CompletelyReducible}(k, R) \Big);$

$\rightsquigarrow [*] := \text{Ɔ}\text{♂}\text{Ɑ}\mathbb{N} : \texttt{This};$

$\square$

1.3 Schur's Lemma

$\texttt{SchurLemma} :: \forall (\alpha, V), (\beta, W) : \texttt{Irreducible}(A, G) \;.\; \forall T : \alpha \xrightarrow{A\text{-REPR}(G)} \beta \;.\; T = 0 | T : \alpha \xleftrightarrow{A\text{-REPR}(G)} \beta$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T : \alpha \xrightarrow{A\text{-REPR}(G)} \beta,$

$\texttt{Assume}\; [0] : T \neq 0,$

$[1] := \texttt{kernelIsSubrepresentation}(T) : \Big(\ker T : \texttt{GroupInvariantSubspace}(\alpha)\Big),$

$[2] := \mathcal{I}\texttt{Irreducible}(V)[1] : \ker T = 0 | \ker T = V,$

$[3] := \texttt{ImageIsSubrepresentation}(T) : \Big(\operatorname{Im} T : \texttt{GroupInvariantSubspace}(\beta)\Big),$

$[4] := \mathcal{I}\texttt{Irreducible}(W)[2] : \operatorname{Im} T = 0 | \operatorname{Im} T = W,$

$[5] := [0][2][4]\texttt{kerRankTHM}(T)\mathcal{I}\texttt{Isomorphic} : \ker T = 0 \;\&\; \operatorname{Im} T = W,$

$[*] := \mathcal{I}^{-1}\texttt{BijectionKenelTHM} : \Big(T : \alpha \xleftrightarrow{A\text{-REPR}(G)} \beta\Big);$

$\rightsquigarrow [5] := \texttt{LEM}E(|) : \texttt{This};$

$\square$

$\texttt{SchurLemma1} :: (\alpha, V), (\beta, W) : \texttt{Irreducible}(A, G) \;.\; \forall [0] : \alpha \ncong_{A\text{-REPR}(G)} \;.\; A\text{-REPR}\,(G)\,(\alpha, \beta) = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SchurLemma2} :: \forall k : \texttt{AlgebraicallyClosedField} \;.\; \forall (\alpha, V) : \texttt{Irreducible}(k, G) \;.\; \forall [0] : \dim V < \infty \;.$

$\;.\; \operatorname{End}_{A\text{-REPR}(G)}(\alpha) = k\,\mathrm{id}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T : \operatorname{End}_{A\text{-REPR}(G)(\alpha)},$

$\lambda := \mathcal{I}\texttt{JordanCellCanonicalJordanForm}(T) : \operatorname{Spec}(T),$

$[1] := \texttt{charPolynomialByDet}\mathcal{I}\texttt{charPolinomial} : \det(\lambda\,\mathrm{id} - T) = 0,$

$[2] := \texttt{RepresentationMorphismsIsSubmodule}(\ldots : \lambda\,\mathrm{id} - T \in \operatorname{End}_{k\text{-REPR}(G)},$

$[T.*] := \texttt{SchurLemma}\mathcal{I}\mathsf{GRP}\Big(\operatorname{End}_{k\text{-VS}}(V), k^*\Big)[1][2] : \lambda\,\mathrm{id} = T;$

$\rightsquigarrow [*] := \mathcal{I}\texttt{SetEq} : \operatorname{End}_{k\text{-REPR}(G)}(T) = k\,\mathrm{id};$

$\square$

$\texttt{IrreducibleAbeleanRepresentation} :: \forall k : \texttt{AlgebraicallyClosedField} \;.\; \forall G \in \mathsf{ABEL} \;.$

$\;.\; \forall \rho : \texttt{Irreducible}(k, G) \;.\; \deg V = 1$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RepresentationDiagonalization} :: \forall k : \texttt{AlgebraicallyClosedField} . \forall G \in \mathsf{ABEL} \,\&\, \texttt{FiniteGroup} .$
$. \forall \rho : \texttt{Irreducible}(k, G) . \forall [0] : |G| \neq_k 0 . \forall [00] : \dim V < \infty . \exists e : \texttt{Basis}(V) : \forall g \in G . \rho_g^{e,e} : \texttt{Diagonal}$

$\texttt{Proof} =$

...

$\square$

$\texttt{FiniteOrderDioganalizability} :: \forall k : \texttt{AlgebraicallyClosedField} . \forall V \in k\text{-}\mathsf{FDVS} . \forall n \in \mathbb{N} .$
$. \forall T : \mathrm{End}_{k\text{-}\mathsf{VS}}(V) . \forall [0] : n \neq_k 0 . \forall [00] : T^n = \mathrm{id} \ . T : \texttt{Diagonalizable}(V)$

$\texttt{Proof} =$

...

$\square$

1.4 Schur Orthogonality Relations

$$\texttt{finiteGroupAlgebraInnerProduct} :: \prod k : \texttt{ConjugationField} \,.\, \prod G : \texttt{FiniteGroup} \,.$$
$$.\, G \neq_k 0 \to \texttt{InnerProduck}G$$
$$\texttt{finitefroupAlgebraInnerProduct}\,(p,q) = \langle p,q\rangle_G := \frac{1}{|G|}\sum_g p(g)\overline{q(g)}$$

$$\texttt{IrreducibleMorphismAveraging} :: \forall k : \texttt{Field} \,.\, \forall G : \texttt{FiniteGroup} \,.$$
$$.\, \forall (V,\alpha),(W,\beta) : \texttt{Irreducible}(k,G) \,.\, \forall T : V \xrightarrow{k\text{-VS}} W \,.\, \forall [0] : |G| \neq_k 0 \,.\, \forall [00] \alpha \ncong_{k\text{-REPR}(G)} \beta \,.$$
$$.\, \operatorname{avg}_{\alpha,\beta} T = 0$$

Proof $=$

...

□

$$\texttt{IrreducibleEndorphismAveraging} :: \forall k : \texttt{Field} \,.\, \forall G : \texttt{FiniteGroup} \,.$$
$$.\, \forall (V,\rho) : \texttt{Irreducible}(k,G) \,.\, \forall T : \mathrm{End}_{k\text{-VS}}(V) \,.\, \forall [0] : |G| \neq_k 0 \,.\, \operatorname{avg}_{\rho,\rho} T = \frac{\operatorname{tr} T}{\dim V}\,\mathrm{id}$$

Proof $=$

...

□

$$\texttt{OrthogonalBasisAveraging} :: \forall k : \texttt{Field} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \forall [0] : |G| \neq_k 0 \,.$$
$$.\, \forall (V,\alpha),(W,\beta) : \texttt{OrthogonalRepresentation}(k,G) \,.$$
$$.\, \forall e : \texttt{Basis}(V) \,.\, \forall f : \texttt{Basis}(W) \,.\, \forall i \in \dim V \,.\, \forall j \in \dim W \,.\, \operatorname{avg}_{\alpha,\beta} f_j \otimes e^i = \langle \beta^f_{s,j}, \alpha^e_{t,i}\rangle_G f_s \otimes e^t$$

Proof $=$

$$[*] := \text{Ꮯ}\texttt{biaveragingMixedTensorAsMap}^2\text{Ꮯ}^{-1}\texttt{finiteGroupAlgebraInnerProduct} :$$
$$: \operatorname{avg}_{\alpha,\beta} f_j \otimes e^i = \frac{1}{|G|}\sum_{g\in G}\alpha_g^{-1}(f_j \otimes e^i)\beta_g = \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)_{i,t}}(f_j\otimes e^t)\beta_g =$$
$$= \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)_{t,i}}\beta(g)_{s,j} f_s \otimes e^t) = \langle \beta_{s,j}, \alpha_{t,i}\rangle_G f_s \otimes e^t;$$

□

$$\texttt{SchurOrthogonalityRelation} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$$
$$.\, \forall (V,\alpha),(W,\beta) : \texttt{Irreducible} \,\&\, \texttt{OrthogonalRepresntation}(k,G) \,.\, \forall [0] : \alpha \ncong_{k\text{-REPR}(G)} \beta \,.$$
$$.\, \forall e : \texttt{Basis}(V) \,.\, \forall f : \texttt{Basis}(W) \,.\, \forall i,j \in \dim V \,.\, \forall t,s \in \dim W \,.\, \langle \alpha^e_{i,j}, \beta^f_{t,s}\rangle_G = 0$$

Proof $=$

$$[1] := \texttt{IrreducibleEndomorphismAveraging}(e^j \otimes f_s) : \operatorname{avg}_{\alpha,\beta} e^j \otimes f_s = 0,$$
$$[2] := \texttt{OrthogonalBasisAveraging}(\alpha,\beta,e,f,j,s) : \operatorname{avg}_{\alpha,\beta} e^j \otimes f_s = \left\langle \alpha^e_{i,j}, \beta^f_{t,s}\right\rangle_G e^i \otimes f_t,$$
$$[*] := [1][2] : \langle \alpha^e_{i,j}, \beta^f_{t,s}\rangle_G = 0;$$

□

$\texttt{SchurOrthogonalityRelation2} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$\quad .\ \forall (V,\rho) : \texttt{Irreducible} \,\&\, \texttt{OrthogonalRepresntation}(k,G) \,.$

$\quad .\ \forall e : \texttt{Basis}(V) \,.\, \forall i,j,t,s \in \dim V \,.\, \langle \alpha^e_{i,j}, \beta^f_{t,s} \rangle_G = \dfrac{\delta^i_t \delta^s_j}{\dim V}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ShurOrthogonalSet} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$\quad .\ \forall (V,\rho) : \texttt{Irreducible} \,\&\, \texttt{OrthogonalRepresntation}(k,G) \,.$

$\quad .\ \forall e : \texttt{Basis}(V) \,.\, \{\alpha^e_{i,j} | i,j \in \dim V\} : \texttt{Orthogonal}(kG)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ShurOrthogonalSet2} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$\quad .\ \forall e : \prod (V,\rho) : \texttt{Irreducible} \,\&\, \texttt{OrthogonalRepresntation}(k,G) \,.\, \texttt{Basis}(V)$

$\quad .\ \{\alpha^{e(\alpha)}_{i,j} | (V,\rho) : \texttt{Irreducible} \,\&\, \texttt{OrthogonalRepresentation}(k,G) i,j \in \dim V\} : \texttt{Orthogonal}(kG)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{RepresentationNumberBound} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$\quad .\ |\texttt{Irreducible}(k,G| \le \sum \rho : \texttt{Irreducible}(k,G) \,.\, \deg^2 \rho \le |G|$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.5 Character Theory

$\texttt{character} :: \prod k : \texttt{Field} \, . \, k\text{-REPR}\,(G) \to kG$

$\texttt{character}\,((V, \rho)) = \chi_\rho := \operatorname{tr} \rho$

$\texttt{IrreducibleCharacters} :: \prod k : \texttt{Field} \, . \, \prod G \in \mathsf{GRP} \, . \, ?kG$

$f : \texttt{IrreducibleCharacter} \iff \exists \rho : \texttt{Irreducible}(k, G) \, . \, f = \chi_\rho$

$\texttt{IdentityCharacter} :: \forall k : \texttt{Field} \, . \, \forall (\rho, V) \in k\text{-REPR}\,(G) \, . \, \chi_\rho(e) = \dim V$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ClassFunction} :: \prod G \in \mathsf{GRP} \, . \, \prod X \in \mathsf{SET} \, . \, ?(X \to G)$

$f : \texttt{ClassFunction} \iff \forall g, h \in G \, . \, f(hgh^{-1}) = f(g)$

$\texttt{CharactersAreClass} :: \forall k : \texttt{Field} \, . \, \forall (\rho, V) \in k\text{-REPR}\,(G) \, . \, \chi_\rho : \texttt{ClassFunction}(G, k)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ClassFunctionIsSubspace} :: \forall k : \texttt{Field} \, . \, \forall G \in \mathsf{GRP} \, . \, \texttt{ClassFunction}(G, k) \subset_{k\text{-VS}} kG$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ClassFunctionDimension} :: \forall k : \texttt{Field} \, . \, \forall G \in \mathsf{GRP} \, . \, \dim \texttt{ClassFunction}(G, k) = \left| \frac{2^G}{\Gamma_G} \right|$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FirstOrhogonalityRelation} :: \forall k : \texttt{Field} \, . \, \forall G \in \mathsf{GRP} \, . \, \forall \alpha, \beta : \texttt{Irreducible}(k, G) \, . \, \langle \chi_\alpha, \chi_\beta \rangle_G = \delta^\alpha_\beta$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{IrreducibleRepresentationClassBound} :: \forall k : \texttt{Field} \, . \, \forall G \in \mathsf{GRP} \, . \, \left| \texttt{Irreducible}(k, G) \right| \le \left| \frac{2^G}{\Gamma_G} \right|$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SumOfCharacters} :: \forall k : \texttt{Field} . \forall G \in \mathsf{GRP} . \forall \alpha, \beta \in k\text{-}\mathsf{REPR}(G) . \chi_{\alpha\oplus\beta} = \chi_\alpha + \chi_\beta$

$\texttt{Proof} =$

...

$\square$

$\texttt{CharacterMultiplicityDerivation} :: \forall k : \texttt{ConjugationField} . \forall G \in \mathsf{GRP} . \forall n \in \mathbb{N} .$

$. \forall \rho : n \hookrightarrow \texttt{Irreducible}(k, G) . \forall \varphi \in k\text{-}\mathsf{REPR}(G) . \forall m : n \to \mathbb{Z}_+ .$

$$. \forall [0] : \varphi \cong \bigoplus_{i=1}^{n} m_i \rho_i . \forall i \in n . \langle \chi_{\rho_i}, \chi_\varphi \rangle = m_i$$

$\texttt{Proof} =$

...

$\square$

$\texttt{IrreducibleByNorm} :: \forall k : \texttt{ConjugationField} . \forall G \in \mathsf{GRP} . \forall \rho \in k\text{-}\mathsf{REPR}(G) .$

$\rho : \texttt{Irreducible}(k, G) \iff \langle \chi_\rho, \chi_\rho \rangle = 1$

$\texttt{Proof} =$

...

$\square$

$\texttt{regularRepresentation} :: \prod R \in \mathsf{ANN} . \prod G \in \mathsf{GRP} . R\text{-}\mathsf{REPR}(G)$

$\texttt{regularRepresentation}() = L_G := \Big(RG, \lambda g \in G . \Lambda f \in RG \Big)$

$\texttt{RegularRepresentationCharacter} :: \forall G : \texttt{FiniteGroup} . \chi_{L_G} = |G| \mathrm{de}$

$\texttt{Proof} =$

$\texttt{Assume}\ g : G,$

$\texttt{Assume}\ [1] : g \neq 1,$

$\texttt{Assume}\ f : G,$

$[f.*] := \mathcal{I} L_G \texttt{BijectiveGroupMultiplication}[1] \mathcal{I} \texttt{Neutral}(G)(e) : L_G(g)(f) = gf \neq ef = f;$

$\rightsquigarrow [g.*] := I(\forall) \mathcal{I}^{-1} \texttt{trace} \mathcal{I}^{-1} \chi_L : \chi_{L_G}(g) = \operatorname{tr} L_G(g) = 0;$

$\rightsquigarrow [1] := I(\forall) I(\Rightarrow) : \forall g \in G . g \neq e \Rightarrow \chi_{L_G}(g) = 0,$

$[2] := \mathcal{I} \chi_{L_G}(e) \mathcal{I} \mathsf{GRP}\Big(G, \mathrm{End}_{k\text{-}\mathsf{VS}}(V)\Big) : \chi_{L_G}(e) = \operatorname{tr} \mathrm{id} = \dim kG = |G|,$

$[*] := \mathcal{I}^{-1} \mathrm{de}[1][2] : \chi_{L_G} = |G| \mathrm{de};$

$\square$

$\texttt{RegularRepresentationStructure} :: \forall G : \texttt{FiniteGroup} . \forall k : \texttt{ConjugationField} . \forall n \in \mathbb{N} .$

$$. \forall [0] : |G| \neq_k 0 . \forall \rho : n \xleftrightarrow{\text{SET}} \texttt{Irreducible}(k, G) . L_G \cong \bigoplus_{i=1}^{n} (\deg \rho_i) \rho_i$$

Proof $=$

$[1] := \texttt{MaschkeTHM}[0] : \Big(L_G : \texttt{CompletelyReducible}(k, G)\Big),$

$\Big(m, [2]\Big) := \daleth\texttt{CompletelyReducible}(k, G\daleth\rho) : \sum m : n \to \mathbb{N} . L \cong \bigoplus_{i=1}^{n} m_i \rho_i,$

Assume $i : n,$

$[i.*] := \texttt{CharacterMulttiplicityDerivation}([2], i)\daleth\texttt{finiteGroupAlgebraInnerProduct}(k, G)$
$\texttt{RegularRepresentationCharacter}(k, G)\daleth\chi_\rho(e)\daleth\texttt{inverse} :$

$$: m_i = \langle \chi_L, \chi_{\rho_i} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_\rho(g)} = \frac{|G|}{|G|} \overline{\chi_\rho(e)} = \deg \rho_i;$$

$\leadsto [3] := I(\forall) : \forall i \in n . m_i = \deg \rho_i,$

$$[*] := [3][2]) : L \cong_{k\text{-REPR}(G)} \bigoplus_{i=1}^{n} (\deg \rho_i) \rho_i;$$

$\square$

$\texttt{GroupSizeByIrreducibleDegrees} :: \forall G : \texttt{FiniteGroup} . \forall k : \texttt{ConjugationField} . \forall n \in \mathbb{N} .$

$$. \forall [0] : |G| \neq_k 0 . \forall \rho : n \xleftrightarrow{\text{SET}} \texttt{Irreducible}(k, G) . |G| = \sum_{i=1}^{n} (\deg \rho_i)^2$$

Proof $=$

...

$\square$

$\texttt{SchurOrthogonalSet2} :: \forall G : \texttt{FiniteGroup} . \forall k : \texttt{ConjugationField} .$

$$. \forall e : \prod (V, \rho) : \texttt{Irreducible} \ \& \ \texttt{OrthogonalRepresntation}(k, G) . \texttt{Basis}(V)$$

$$. \{\alpha_{i,j}^{e(\alpha)} | (V, \rho) : \texttt{Irreducible}(k, G) : i, j \in \dim V\} : \texttt{OrthogonalBasis}(kG)$$

Proof $=$

...

$\square$

$\texttt{CharactersAreOrthogonalBasisOfClass} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$$. \forall [0] : |G| \neq_k 0 \,.\, \{\chi_\rho | \rho : \texttt{Irreducible}(k, G)\} : \texttt{OrhogonalBasis}\Big(\texttt{ClassFunction}(k, G)\Big)$$

$\texttt{Proof} \; =$

$\texttt{Assume}\ f : \texttt{ClassFunction}(k, G),$

$$\Big(e, \alpha, [1]\Big) := \texttt{SchurOrthoginalSet2} : \sum e : \prod (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \texttt{Basis}(V) \,.$$

$$. \sum \alpha : \prod (V, \rho) : \texttt{Irreducible}(k, G) \,.\, f = \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i,j=1}^{\dim V} \alpha_{i,j} \rho_{i,j}^{e,e(\rho,V)},$$

$\texttt{Assume}\ x : G,$

$$[x.*] := \text{Д}\texttt{ClassFunction}(k, G)[1]\text{Д}\texttt{coordinates}(e)\text{Д}\mathsf{GRP}\Big(G, \mathrm{End}_{k\text{-}\mathsf{VS}}(V)\Big)(\rho)$$

$$. \text{Д}^{-1}\texttt{biaveragingEndomorpgismAveraging}\text{Д}^{-1}\texttt{character} :$$

$$. f(x) = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) = \frac{1}{|G|} \sum_{g \in G} \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i,j=1}^{\dim V} \,.\, \alpha_{\rho,i,j} \rho_{i,j}^{e,e(\rho,V)}(g^{-1}xg) =$$

$$= \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i,j=1}^{\dim V} \frac{\alpha_{\rho,i,j}}{|G|} \sum_{g \in G} \Big(\rho^{-1}(g)\rho(x)\rho(g)\Big)_{i,j=1}^{e} =$$

$$= \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i,j=1}^{\dim V} \frac{\alpha_{\rho,i,j}}{|G|} \Big((\mathrm{avg}\ \rho(x)\Big)_{i,j}^{e} =$$

$$= \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i,j=1}^{\dim V} \alpha_{\rho,i,j} \left(\frac{\mathrm{tr}\ \rho(x)}{\deg \rho}\ \mathrm{id}\right)_{i,j}^{e} =$$

$$= \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i=1}^{\dim V} \frac{\alpha_{\rho,i,i}}{\deg \rho} \chi_\rho(x);$$

$$\rightsquigarrow [f.*] := I(\rightarrow) : f = \sum (V, \rho) : \texttt{Irreducible}(k, G) \,.\, \sum_{i=1}^{\dim V} \frac{\alpha_{\rho,i,i}}{\deg \rho} \chi_\rho;$$

$$\rightsquigarrow [1] := \text{Д}^{-1}\texttt{Generating} : \Big(\texttt{Irreducible}(k, G) : \texttt{Generating}\ \texttt{ClassFunction}(k, G)\Big),$$

$[*] := \text{Д}^{-1}\texttt{Basis}[1]\texttt{FirstOrhogonalityRelation} : \texttt{This};$

$\square$

$\texttt{NumberOfIrreducibleRepresentations} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.$

$$. \Big|\{\rho : \texttt{Irreducible}(k, G)\}\Big| = \left|\frac{G}{\Gamma_G}\right|$$

$\texttt{Proof} \; =$

$\dots$

$\square$

$$\texttt{characterMatrix} :: \prod G : \texttt{FiniteGroup} \,.\, \prod k : \texttt{ConjugationField} \,.\, \texttt{Irreducible}(k,G) \times \frac{G}{\Gamma_G} \to k$$

$$\texttt{charactecterMatrix}\,(\rho, A) = \mathbf{Ch}_{\rho,A} := \chi_\rho(A)$$

$$\texttt{SecondOrthogonalityRelation} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationFiels} \,.$$

$$.\ \mathbf{Ch}(G,k)\mathbf{Ch}^\top(G,k) = \texttt{diagonal}\left(\Lambda A \in \frac{G}{\Gamma_G} \,.\, \frac{|G|}{|A|}\right)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A, B : \dfrac{G}{\Gamma_G},$

$$[1] := \mathcal{I}^{-1}\texttt{finiteGrpupAlgebraInnerProduct}\Big(\texttt{GrammSmidtProcess}(\chi)\Big)^2(\delta_a)(\delta_b)$$

$$\mathcal{I}\,\texttt{Orthonormal}(\chi)\mathcal{I}\,\delta_A\mathcal{I}\,\delta_B\mathcal{I}^{-1}\mathbf{Ch}(G,k) :$$

$$: \delta_A^B = \frac{1}{|A|}\langle\delta_A, \delta_B\rangle_G = \frac{1}{|A|}\left\langle \sum_\rho \langle\delta_A, \chi_\rho\rangle\chi_\rho, \sum_\rho \langle\delta_B, \chi_\rho\rangle\chi_\rho \right\rangle_G =$$

$$= \frac{1}{|A|}\sum_\rho \langle\chi_\rho, \delta_A\rangle\langle\chi_\rho, \delta_B\rangle_G = \frac{1}{|A||G|}\sum_\rho\sum_{a\in A}\sum_{b\in B}\chi_\rho(a)\chi_\rho(b) = \frac{|B|}{|G|}\Big(\mathbf{Ch}(G,k)\mathbf{Ch}(G,k)^\top\Big)_{A,B},$$

$$\Big[(A,B).*\Big] := [1]\frac{|G|}{|B|} : \Big(\mathbf{Ch}(G,k)\Big)_{A,B} = \frac{|G|}{|B|}\delta_B^A;$$

$$\rightsquigarrow [*] := I(\forall)\mathcal{I}\,\texttt{deltaOfKronecker}\mathcal{I}^{-1}\texttt{matrixMultiplication} : \texttt{This};$$

$\square$

1.6 Finite Fourier Transform

$$\texttt{finiteGroupFourierTransform} :: \prod G : \texttt{FiniteGroup} \,.\, \prod k : \texttt{Field} \,.$$
$$.\, kG \to \prod (V,\rho) : \texttt{Irreducible}(G,k) \,.\, \mathrm{End}_{k\text{-VS}}(V)$$
$$\texttt{finiteGroupFourierTransform}\,(f) = \widehat{f} := \Lambda(V,\rho) : \texttt{Irreducible}(k,G) \,.\, \sum_{g \in G} f(g)\overline{\rho(g)}$$

$$\texttt{inverseFiniteGroupFourierTransform} :: \prod G : \texttt{FiniteGroup} \,.\, \prod k : \texttt{Field} \,.$$
$$.\, \left(\prod (V,\rho) : \texttt{Irreducible}(k,G) \,.\, \mathrm{End}_{k\text{-VS}}(V)\right) \to kG$$
$$\texttt{finiteGroupFourierTransform}\,(T) = \widehat{T} := \Lambda g \in G \,.\, \frac{1}{n}\sum (\rho,V) : \texttt{Irreducible}(k,G) \,.\, (\dim V)\langle T_\rho, \rho(g)\rangle$$
$$\texttt{where} \quad n = \Big|G\Big|$$

$$\texttt{FourierInversion} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{Field} \,.\, \forall f \in kG \,.\, \hat{\hat{f}} = f$$

Proof $=$

Assume $g : G$,

$$[g.*] := \mathcal{I}\,\texttt{inverseFiniteGroupFourierTransform}(\hat{f})\mathcal{I}\,\texttt{finiteGroupFourierTransform}(f)$$
$$.\, \mathcal{I}^{-1}\texttt{finiteGroupAlgebraInnerProduct}(k,G)\texttt{GrammSchmidtTHM}(f) :$$
$$\hat{\hat{f}}(g) = \frac{1}{n}\sum (\rho,V) : \texttt{Irreducible}(k,G) \,.\, (\dim V)\left\langle \hat{f}(\rho), \rho(g)\right\rangle =$$
$$= \frac{1}{n}\sum (\rho,v) : \texttt{Irreducible}(k,G) \,.\, (\dim V)\sum_{h\in G} f(h)\left\langle \overline{\rho(h)}, \rho(g)\right\rangle =$$
$$= \sum \rho : \texttt{Irreducible}(k,G) \,.\, (\dim V)\langle f, \rho_{i,j}\rangle_G \rho_{i,j}(g) = f(g);$$

$\rightsquigarrow [*] := I(=,\to) : \hat{\hat{f}} = f;$

□

$$\texttt{FourierTransformInversion} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.\, \forall [0] : |G| \neq_k 0 \,.$$
$$.\, \texttt{finiteGroupFourierTransform} : kG \overset{k\text{-VS}}{\longleftrightarrow} \prod (V,\rho) : \texttt{Irreducible}(k,G) \,.\, \mathrm{End}_{k\text{-VS}}(V)$$

Proof $=$

...

□

$\texttt{WedderburnFourierTransformTheorem} :: \forall G : \texttt{FiniteGroup} \,.\, \forall k : \texttt{Field} \,.$

$\,.\, \texttt{finiteGroupFourierTransform} : kG \xleftrightarrow{k\text{-ALGE}} \prod (V, \rho) : \texttt{Irreducibke}(k, G) \,.\, \mathrm{End}_{k\text{-VS}}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y : kG,$

$\texttt{Assume}\; (\rho, V) : \texttt{Irreducible}(k, G),$

$\Big[(\rho, V).*\Big] := \mathit{\Pi}\,\texttt{finiteGroupFourierTransform}(x, y)\mathit{\Pi}\, kG\mathit{\Pi}\,\mathsf{GRP}\Big(G, \mathrm{End}_{k\text{-VS}}(V)\Big)(\rho)\mathit{\Pi}\,\mathsf{GRP}G$

$\mathit{\Pi}^{-2}\texttt{finiteGroupFourierTransform}(x)(y) :$

$$: \widehat{xy}(\rho, V) = \sum_{g \in G} xy(g)\overline{\rho(g)} = \sum_{g \in G}\sum_{hf = g} x(h)y(f)\overline{\rho(g)} = \sum_{g \in G}\sum_{hf = g} x(h)y(f)\overline{\rho(h)\rho(f)} =$$

$$= \sum_{h, f \in G} \Big(x(h)\overline{\rho(h)}\Big)\Big(y(f)\overline{\rho(f)}\Big) = \widehat{x}(\rho, V)\widehat{y}(\rho, V);$$

$\rightsquigarrow \Big[(x, y).*\Big] := I(=, \to) : \widehat{xy} = \widehat{x}\widehat{y};$

$\rightsquigarrow [*] := \mathit{\Pi}^{-1} k\text{-}\mathsf{ALGE} \prod (\rho, V) : \texttt{Irreducible}(k, G) \,.\, \mathrm{End}_{k\text{-VS}}(V) :$

$: \Big(\texttt{finiteGroupFourierTransform} : kG \xrightarrow{k\text{-ALGE}} \prod (\rho, V) : \texttt{Irreducible}(k, G) \,.\, \mathrm{End}_{k\text{-VS}}(V)\Big);$

$\square$

$\texttt{AbeleanGroupAlgebraStructure} :: \forall G : \texttt{FiniteGroup} \,\&\, \texttt{Abelean} \,.\, \forall k : \texttt{Field} \,.\, kG \cong_{k\text{-ALGE}} k^{|G|}$

$\texttt{Proof} \;=$

...

$\square$

1.7 First Burnside's Theorem

$\texttt{CharacterIsAlgebraicInteger} :: \forall G : \texttt{FiniteGroup} \,.\, \forall \chi : \texttt{Character}(\mathbb{C}, G) \,.\, \operatorname{Im}\chi \subset \mathbb{Z}(\mathbb{C})$

$\texttt{Proof} \ =$

$\Big((V,\rho),[1]\Big) := \mathcal{I}\texttt{Character}(k,G)(\chi) : \sum (V,\rho) \in \mathbb{C}\text{-}\mathsf{REPR}\,(G) \,.\, \chi = \chi_\rho,$

$\texttt{Assume}\ g : G,$

$n := o(g) : \mathbb{N},$

$[2] := \mathcal{I}\mathsf{GRP}\Big(G, \mathrm{End}_{k\text{-}\mathsf{ALGE}}(V)\Big)\mathcal{O}n\mathcal{I}\texttt{order}(g)\mathcal{I}\mathsf{GRP}\Big(G, \mathrm{End}_{k\text{-}\mathsf{ALGE}}(V)\Big) : \rho^n(g) = \rho(g^n) = \rho(e) = \mathrm{id},$

$[3] := \mathcal{I}\texttt{MinimalPolynomial}[2] : m_{\rho(g)}(x)\Big| x^n - 1,$

$[4] := \texttt{MinimalPolynomialTHM}(\rho(g))\mathcal{I}\texttt{AlgebraicInteger} : \operatorname{Spec}\rho(g) \subset \mathbb{Z}(\mathbb{C}),$

$[g.*] := [1]\mathcal{I}\texttt{charactrerTraceBySpectre}[4] : \chi(g) = \operatorname{tr}\rho(g) = \sum_{\lambda\in\mathbb{C}} \lambda\sigma_{\rho(g)}(\lambda) \in \mathbb{Z}(\mathbb{C});$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Subset}\mathcal{I}^{-1}\texttt{image} : \operatorname{Im}\chi \subset \mathbb{Z}(\mathbb{C}),$

$\square$

$\texttt{IrreducibleCharacterIsAlgebraicInteger} :: \forall G : \texttt{FiniteGroup} \,.\, \forall (V,\rho) : \texttt{Irreducible}(\mathbb{C}, G) \,.$

$\quad .\, \forall g \in G \,.\, \dfrac{|\gamma_G(g)|}{\deg\rho}\chi_\rho(g) \in \mathbb{Z}(\mathbb{C})$

$\texttt{Proof} \ =$

$T := \lambda A \in \dfrac{G}{\gamma_G} \,.\, \sum_{a\in A}\rho(a) : \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V),$

$\texttt{Assume}\ A : \dfrac{G}{\gamma_G},$

$\Big(g,[0]\Big) := \mathcal{I}\dfrac{2^G}{\Gamma_G} : \sum g \in G \,.\, A = \gamma(g),$

$\texttt{Assume}\ h : G,$

$[h.*] := \mathcal{O}T_A\mathcal{I}\mathbb{C}\text{-}\mathsf{ALGE}(\mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V))\mathcal{I}\mathsf{GRP}\Big(G, \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V)\Big)(\rho)\texttt{ConjugationClassSummation}(A) :$

$\quad : \rho^{-1}(h)T_A\rho(h) = \rho^{-1}(h)\left(\sum_{a\in A}\rho(a)\right)\rho(h) = \sum_{a\in A}\rho^{-1}(h)\rho(a)\rho(h) = \sum_{a\in A}\rho\Big(hah^{-1}\Big) = \sum_{a\in A}\rho(a) = T_A;$

$\rightsquigarrow [1] := I(\forall) : \forall h \in G \,.\, \rho^{-1}(h)T_A\rho(h) = T_A,$

$[2] := \mathcal{I}^{-1}\texttt{biaveraging}[1]\texttt{EndomorphidmAveraging}(\rho) : T_A = \operatorname{avg} T_A = \dfrac{\operatorname{tr}T_A}{\dim V}\mathrm{id}_V,$

$[3] := \mathcal{O}T_A\mathcal{I}\mathbb{C}\text{-}\mathsf{VS}\Big(\mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}, \mathbb{C}\Big)(\mathrm{tr})\mathcal{I}^{-1}\texttt{character}[0] : \operatorname{tr}T = \sum_{a\in A}\operatorname{tr}\rho(a) = \sum_{a\in A}\chi_\rho(a) = |A|\chi_\rho(g),$

$[A.*] := [2][3] : T_A = \dfrac{|A|\chi_\rho(g)}{\dim V}\mathrm{id};$

$\rightsquigarrow [1] := I(\forall) : \forall A \in \dfrac{G}{\gamma_G} \,.\, T_A = \dfrac{|A|\chi_\rho(A)}{\dim V},$

$\texttt{Assume}\ A, B : \operatorname{colim}\gamma_G,$

$n := \Lambda g \in G \,.\, \Big|\Big\{(a,b) \in A \times B : g = ab\Big\}\Big| : G \to \mathbb{Z},$

$[2] := \mathcal{O}T\mathcal{I}\mathsf{GRP}\,(G, \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V))\,(\rho)\mathcal{O}^{-1}(n) : T_AT_B = \sum_{a\in A}\sum_{b\in B}\rho(a)\rho(b) = \sum_{a\in A}\sum_{b\in B}\rho(ab) = \sum_{g\in G}n_g\rho(g),$

$\texttt{Assume } C : \operatorname{colim} \gamma_G,$

$\texttt{Assume } h, f : C,$

$\Big(x, [3]\Big) := \text{Ɑ} \operatorname{colim} \gamma_G(C)(h, f) : \sum x \in G \, . \, f = xhx^{-1},$

$\texttt{Assume } a : A,$

$\texttt{Assume } b : B,$

$\texttt{Assume } [4] : ab = h,$

$[4.*] := \text{Ɑ}\texttt{Inverse}[4][3] : xax^{-1}xbx^{-1} = xabx^{-1} = xhx^{-1} = f;$

$\rightsquigarrow [4] := I(\Rightarrow) : ab = h \Rightarrow xax^{-1}xbx^{-1} = f,$

$\texttt{Assume } [5] : ab = f,$

$[5.*] := \text{Ɑ}\texttt{Inverse}[5][3] : x^{-1}axx^{-1}bx = x^{-1}abx = x^{-1}fx = h;$

$\rightsquigarrow [a.*] := I(\Rightarrow) : ab = f \Rightarrow x^{-1}axx^{-1}bx = h;$

$\rightsquigarrow [C.*] := \text{Ɑ}^{-1} n : n_h = n_f;$

$\rightsquigarrow [3] := I^2(\forall) : \forall C \in \operatorname{colim} \gamma_G \, . \, \forall h, f \in C \, . \, n_h = n_f,$

$$\Big[(A, B). *\Big] := [3][2] : T_A T_B = \sum_{C \in \operatorname{colim} \gamma_G} n_C T_C;$$

$$\rightsquigarrow [2] := I(\forall) I(\exists) : \forall A, B \in \operatorname{colim} \gamma_G \, . \, \exists n : (\operatorname{colim} \gamma_G) \to \mathbb{Z} \, . \, T_A T_B = \sum_{C \in \operatorname{colim} \gamma_G} n_C T_C,$$

$$[*] := \texttt{AlgebraicIntegerByIntegralSums}[1][2] : \frac{|\gamma_G(g)| \chi_\rho(g)}{\dim V} \in \mathbb{Z}(\mathbb{C});$$

□

$\texttt{DimensionTHM} :: \forall G : \texttt{FiniteGroup} \, . \, \forall \rho : \texttt{Irreducible}(\mathbb{C}, G) \, . \, \deg \rho \Big| |G|$

$\texttt{Proof} =$

$$[1] := \texttt{Orthonormal}(\chi_\rho)\text{Ɑ}\texttt{finiteGroupAlgebraInnerProduct} : 1 = \langle \chi_\rho, \chi_\rho \rangle_G = \frac{\dim V}{|G| \dim V} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\rho(g)},$$

$$[2] := \frac{|G|}{\dim V}[1]\texttt{DisjointConjugasyClasses}\text{Ɑ}\texttt{ClassDunction}(\chi_\rho)$$

$$\texttt{CharacterIsAlgebraicInteger}(G)\texttt{IrreducibleCharacterIsAlgebraicInteger}(G)\text{Ɑ}\mathsf{ANN}\Big(\mathbb{Z}(\mathbb{C})\Big) :$$

$$: \frac{|G|}{\dim V} = \frac{1}{\dim V} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\rho(g)} = \frac{1}{\dim V} \sum_{C \in \operatorname{colim} \gamma_G} |C| \chi_\rho(C) \overline{\chi_\rho(C)} = \sum_{C \in \operatorname{colim} \gamma_G} \frac{|C|}{\dim V} \chi_\rho(C) \overline{\chi_\rho(C)} \in \mathbb{Z}(\mathbb{C}),$$

$$[3] := \texttt{RealAlgebraicInteger}[2] : \frac{|G|}{\dim V} \in \mathbb{Z},$$

$[4] := \text{Ɑ}\texttt{Divides}[3]\text{Ɑ}^{-1} \deg \rho : \deg \rho \Big| |G|,$

□

$\texttt{DegreeOneRepresentationsNumber} :: \forall G : \texttt{FiniteGroup} \,.\, \left\{\rho \in \mathbb{C}\text{-REPR}\,(G) \,\middle|\, \deg \rho = 1\right\} \cong_{\mathsf{SET}} \frac{G}{[G,G]}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \rho : \mathbb{C}\text{-REPR}\,(G)\,,$

$\texttt{Assume}\ [1] : \deg \rho = 1,$

$[2] := [1] \mathcal{I} \deg \rho : \operatorname{Im} \rho : \texttt{Cyclic},$

$[3] := \texttt{IsomorphismTHM}(\rho) : \operatorname{Im} \rho \cong_{\mathsf{GRP}} \frac{G}{\ker \rho},$

$[4] := \texttt{AbeleanQuatient}[3] : [G,G] \subset \ker \rho,$

$F(\rho) := \Lambda[g] \in \frac{G}{[G,G]} \,.\, [4](\rho(g)) : \texttt{Irreducible}\left(\mathbb{C}, \frac{G}{[G,G]}\right);$

$\rightsquigarrow F := I(\rightarrow) : \left\{\rho \in \mathbb{C}\text{-REPR}\,(G) \,\middle|\, \deg \rho = 1\right\} \rightarrow \texttt{Irreducible}\left(\mathbb{C}, \frac{G}{[G,G]}\right),$

$\Pi := \Lambda \rho : \texttt{Irreducible}\left(\mathbb{C}, \frac{G}{[G,G]}\right) \,.\, \pi_{[G,G]} \rho : \texttt{Irreducible}\left(\mathbb{C}, \frac{G}{[G,G]}\right) \rightarrow \left\{\rho \in \mathbb{C}\text{-REPR}\,(G) \,\middle|\, \deg \rho = 1\right\},$

$[1] := \mathcal{D} F \mathcal{D} \Pi : \Pi = F^{-1},$

$[*] := \mathcal{I}\,\texttt{isomorphic}[1]\texttt{GroupSizeIrreducibleByDegrees}\left(\frac{G}{[G,G]}, \mathbb{C}\right)$

$\quad : \texttt{IrreducibleAbeleanRepresentation}\left(\frac{G}{[G,G]}, \mathbb{C}\right) :$

$\quad : \left\{\rho \in \mathbb{C}\text{-REPR}\,(G) \,\middle|\, \deg \rho = 1\right\} \cong_{\mathsf{SET}} \texttt{Irreducible}\left(\mathbb{C}, \frac{G}{[G,G]}\right) \cong_{\mathsf{SET}} \frac{G}{[G,G]};$

$\square$

$\texttt{BurnsideScalarLemma} :: \forall G : \texttt{FiniteGroup} \,.\, \forall A \in \operatorname{colim} \gamma_G \,.\, \forall (V,\rho) : \texttt{irreducible}(\mathbb{C}, G) \,.$

$\quad .\, \forall [0] : \Big(|A|, \deg \rho\Big) : \texttt{Coprime} \,.\, \Big(\exists \lambda \in \mathbb{C} \,.\, \forall g \in A \,.\, \rho(g) = \lambda \operatorname{id}\Big) \Big| \chi_\rho(C) = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ g : A,$

$\texttt{Assume}\ [1] : \forall \lambda \in \mathbb{C} \,.\, \rho(g) \neq \lambda \operatorname{id},$

$\Big(a, b, [2]\Big) := \texttt{DivisionWithReminder}(|A|, \deg \rho, [0]) : \sum a, b \in \mathbb{Z} \,.\, a|A| + b \deg \rho = 1,$

$c := \frac{\chi_\rho(g)}{\deg \rho} : \mathbb{C},$

$[3] := \mathcal{D} c [2] \mathcal{I}\,\texttt{inverse}(\mathbb{Q})(\deg \rho)\texttt{CharacterIsAlgebraicInteger}(\rho)$

$\quad \texttt{IrreducibleCharacterIsIrreducibleInnteger}(\rho) \mathcal{I}\,\mathsf{ANN}\mathbb{Z}(\mathbb{C}) :$

$$c = \frac{\chi_\rho(g)}{\deg \rho} = \frac{\Big(a|A| + b \deg \rho\Big)\chi_\rho(g)}{\deg \rho} = a \frac{|A|}{\deg \rho} \chi_\rho(g) + b \chi_\rho(g) \in \mathbb{Z}(\mathbb{C}),$$

$\Big(n, \lambda, e, [4]\Big) := \texttt{UnipotentStrucure}(\rho(g)) : \sum n \in \mathbb{N} \,.\, \sum \lambda : (\deg \rho) \rightarrow \texttt{RootsOfUnity}(\mathbb{C}, n) \,.$

$\quad .\, \sum e : \texttt{Basis}(V) \,.\, \rho(g)^{e,e} = \operatorname{diag}(\lambda),$

$\Big(i, j, [5]\Big) := [1][4] : \sum i, j \in \deg \rho \,.\, \lambda_i \neq \lambda_j,$

$[6] := \mathcal{I}\,\texttt{character}(\rho, g)[4][5]\texttt{IteratedTriangleInequality}(\lambda) : \Big|\chi_\rho(g)\Big| = \left|\sum_{i=1}^{\deg \rho} \lambda_i\right| < \deg \rho_i,$

$[7] := [6] \mathcal{D} \alpha : |\alpha| < 1,$

$[8] := [5] \mathcal{D} \alpha : \alpha \in \mathbb{Q}[\omega_n],$

$\texttt{Assume}\ \sigma : G(\omega_n),$

$[*.1] := \texttt{GaloisActionPreservesAlgebraicInteger} : \sigma(\alpha) \in \mathbb{Z}(\mathbb{C}),$

$[*.2] := \mathcal{O}\alpha \mathcal{C} G(\omega_n) \mathcal{C} \lambda \texttt{IteratedTriangleIneq}(\sigma(\lambda)) : |\sigma(\alpha)| < 1;$

$\rightsquigarrow [9] := I(\forall) : \forall \sigma \in G(\omega_n) \ . \ \sigma(\alpha) \in \mathbb{Z}(\mathbb{C}) \ \& \ \Big|\sigma(\alpha)\Big| < 1,$

$$q := \prod_{\sigma \in G(\omega_n)} \sigma(\alpha) : \mathbb{Z}(\mathbb{C}),$$

$[10] := \Lambda \phi \in G(\omega_n) \ . \ \mathcal{O} q \mathcal{C} \mathbb{C}\text{-}\mathsf{ALGE}\Big(\mathbb{Q}[\omega_n], \mathbb{Q}[\omega_n]\Big)(\phi)\texttt{GroupCyclingProduct}\Big(G(\omega_n)\Big)\mathcal{O}^{-1} q :$

$$: \forall \sigma \in G(\omega_n) \ . \ \phi(q) = \phi\left(\prod_{\sigma \in G(\omega_n)} \sigma(\alpha)\right) = \prod_{\sigma \in G(\omega_n)} \phi\sigma(\alpha) = \prod_{\sigma \in G(\omega_n)} \sigma(\alpha) = q,$$

$[11] := \texttt{GaloisInvariantIsBase}[10] : q \in \mathbb{Q},$

$[12] := \texttt{RationalAlgebraicInteger}[11] : q \in \mathbb{Z},$

$[13] := \texttt{CauchySchwartzTHM}[9] : |q| < 1,$

$[14] := |12||13| : q = 0,$

$[1.*] := \mathcal{O} q \mathcal{C} G(\omega_n)[14]\mathcal{O}\alpha : \chi_\rho(g) = 0;$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big(\forall \lambda \in \mathbb{C} \ . \ \rho(g) \neq \lambda \,\mathrm{id}\Big) \Rightarrow \chi_\rho(g) = 0,$

$\texttt{Assume}\ g, f : A,$

$\texttt{Assume}\ \lambda : \mathbb{C},$

$\texttt{Assume}\ [1] : \rho(g) = \lambda \,\mathrm{id},$

$\Big(x, [2]\Big) := \mathcal{C} \operatorname{colim} \gamma_G(A)(g, f) : \sum x \in G \ . \ xgx^{-1} = f,$

$\Big[(g, f). *\Big] := [2]\mathcal{C}\mathsf{GRP}\Big(G \operatorname{End}_{k\text{-}\mathsf{VS}}(V) Big)[1]\mathcal{C}\texttt{inverse} :$

$: \rho(f) = \rho(xgx^{-1}) = \rho(x)\rho(g)\rho^{-1}(x) = \rho(x)\lambda \,\mathrm{id}\, \rho^{-1}(x) = \lambda \,\mathrm{id};$

$\rightsquigarrow [2] := I^3(\forall)I(\Rightarrow) : \forall g, f \in A \ . \ \forall \lambda \in \mathbb{C} \ . \ \rho(g) = \lambda \,\mathrm{id} \Rightarrow \rho(f) = \lambda \,\mathrm{id},$

$[3] := \mathcal{C}\texttt{ClassFunction}(\chi_\rho) : \forall g, f \in A \ . \ \chi_\rho(g) = 0 \Rightarrow \chi_\rho(f) = 0,$

$[*] := [1][2][3] : \texttt{This};$

$\square$

$\texttt{BurnsideComplexityLemma} :: \forall G \in \texttt{FiniteGroup} \ . \ \forall A \in \operatorname{colim} \gamma_G \ . \ \forall p : \texttt{Prime} \ . \ \forall n \in \mathbb{Z}_+ \ . \ \forall [0] : A \neq \{e\} \ .$
$. \ \forall [00] : |A| = p^n \ . \ \forall [000] : G \,!\, \texttt{Abelean} \ . \ G \,!\, \texttt{Simple}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : (G : \texttt{Simple}),$

$\rho := \texttt{NumberOfIrreducibleRepresentations}(G, \mathbb{C}) : \operatorname{colim} \gamma_G \leftrightarrow \texttt{Irreducible}(G, \mathbb{C}),$

$\Big(i, [1]\Big) := \texttt{TrivialIsIrreducible}(\rho) : \sum i \in \gamma_G \ . \ \rho_i = e_{\mathbb{C},G},$

$[3] := [1][2]\mathcal{C}\texttt{Injective}(\rho)\mathcal{C}\texttt{Simple}(G) : \forall j \in \operatorname{colim} \gamma_G \ . \ j \neq i \Rightarrow \ker \rho_j = \{e\},$

$[4] := \texttt{AbeleanMorphism}[3] : \forall j \in \{i\}^\complement \ . \ \deg \rho_j > 1,$

$[5] := \mathcal{C} A \mathcal{C}\texttt{Simple}(G)\texttt{CentrConjugacyClass}(G) : n > 0,$

$\texttt{Assume}\ g : A,$

$\texttt{Assume}\ j : \{i\}^\complement,$

$X := \{x \in G : \exists \lambda \in \mathbb{C} : \rho_j(x) = \lambda \,\mathrm{id}\} :?G,$

$H := \{\lambda \mathrm{id}_{\operatorname{dom} \rho_j(g)} | \lambda \in \mathbb{C}\} : \texttt{Normal}\Big(\mathbf{GL}\big(\operatorname{dom} \rho_j(g)\big)\Big),$

$[6] := \mathcal{O} X \mathcal{O} H : X = \rho_j^{-1}(H),$

$[7] := \texttt{NormalPreimage}[6] : X \lhd G,$
$[8] := \mathcal{I}\texttt{Simple}(G)[7] : X = \{e\} | X = G,$
$[9] := \texttt{AbeleanInjection}[3](j)[8] : X = \{e\},$
$\texttt{Assume}\ [10] : \Big((\deg \rho_j, p) : \texttt{Coprime}\Big),$
$[10.*] := \texttt{BurnsideScalarLemma}[10][9]\mathcal{O}(X) : \chi_{\rho_j}(A) = 0;$
$\rightsquigarrow [j.*] := I(\Rightarrow) : \Big((\deg \rho_j, p) : \texttt{Coprime}\Big) \Rightarrow \chi_{\rho_j}(g) = 0;$
$\rightsquigarrow [6] := I(\forall) : \forall j \in \{i\}^{\complement} \,.\, (\deg \rho_j, p) : \texttt{Coprime} \Rightarrow \chi_{\rho_j}(g) = 0,$
$[7] := [0]\mathcal{I}g : g \neq e,$
$[8] := \texttt{RegularRepresentationCharacter}(G)[7]\texttt{RegularRepresentationStrucure}(G)$
$\mathcal{I}\texttt{character}(L)(g)[1][6] :$
$$: 0 = \chi_L(g) = \sum_{k \in \operatorname{colim} \gamma_G} \deg \rho_k \chi_{\rho_k}(g) = 1 + \sum_{k \in \{i\}^{\complement}} \deg \rho_k \chi_{\rho_k}(g) = 1 + \sum_{k : p | \deg \rho_k} \deg \rho_k \chi_{\rho_k}(g),$$
$\Big([9], z\Big) := \mathcal{I}\texttt{Divides}[8]\texttt{CharacterIsAlgebraicInteger}(\rho, g) : \sum z \in \mathbb{Z}(\mathbb{C}) \,.\, 0 = 1 + pz,$
$[10] := \dfrac{([9] - 1)}{p} : -\dfrac{1}{p} = z,$
$[1.*] := \texttt{RationAlgebraicInteger}[10] : \perp;$
$\rightsquigarrow [*] := E(\perp) : \Big(G : \texttt{Simple}\Big);$
□

$\texttt{BurnsideFirstTheorem} :: \forall G : \texttt{FiniteGroup} \,.\, \forall p, q : \texttt{Prime} \,.\, \forall a, b \in \mathbb{N} \,.$
$.\, \forall [0] : |G| = p^a q^b \,.\, G \,!\, \texttt{Simple}$
$\texttt{Proof}\ =$
$\texttt{Assume}\ [1] : \Big(G \in \mathsf{ABEL}\Big),$
$[*] := [0]\texttt{AbeleanSimplicity} : \Big(G \,!\, \texttt{Simple}\Big);$
$\rightsquigarrow [1] := I(\Rightarrow) : G \in \mathsf{ABEL} \Rightarrow G \,!\, \texttt{Simple},$
$\texttt{Assume}\ [2] : \Big(G \,!\, \texttt{Abelean}\Big),$
$\Big(H, [3]\Big) := \texttt{SylowTHM1}[2] : \sum H \subset_{\mathsf{GRP}} G \,.\, |H| = q^b,$
$\Big(g, [4]\Big) := \texttt{PrimePowerHasNontrivialCentre}[3] : \sum g \in Z(H) \,.\, g \neq e,$
$[5] := \texttt{ClassTHM}[4] : p^a = [G : H] = [G : N(g)][N(g) : H],$
$\Big(m, [6]\Big) := \mathcal{I}\texttt{Divides}[5] : \sum m \in \mathbb{Z}_+ \,.\, [G : N(g)] = p^m,$
$[7] := \mathcal{I}^{-1}\gamma_G[6] : |\gamma_G(g)| = p^m,$
$[2.*] := \texttt{BurnsideComplexityLemma}[2][4][7] : G \,!\, \texttt{Simple}(G);$
$\rightsquigarrow [2] := I(\Rightarrow) : G \,!\, \texttt{Abelean} \Rightarrow G \,!\, \texttt{Simple},$
$[*] := E(|)\mathsf{LEM}(G : \texttt{Abelean})[1][2] : G \,!\, \texttt{Simple};$
□

1.8 Permutation Representation

$\texttt{orbitals} :: \prod X : \mathsf{SET} \,.\, \prod G : \mathsf{GRP} \,.\, (G \xrightarrow{\mathsf{GRP}} S_X) \to ??(X \times X)$

$\texttt{orbitals}\,(\alpha) := \operatorname{colim}(\alpha \times \alpha)$

$\texttt{rank} :: \prod X : \mathsf{SET} \,.\, \prod G : \mathsf{GRP} \,.\, (G \xrightarrow{\mathsf{GRP}} S_X) \to \mathsf{CARD}$

$\texttt{rank}\,(\alpha) = \operatorname{rank}\alpha := |\texttt{orbitals}(G)|$

$\texttt{DoubleTransitivityTHM} :: \forall G : \mathsf{GRP} \,.\, \forall X : \mathsf{SET} \,.\, \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \,.$

$\quad \alpha : 2\text{-}\texttt{Transitive}(G, X) \iff \alpha : \texttt{Transitive}(G, X) \;\&\; \operatorname{rank}\alpha = 2$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \Big(\alpha : 2\text{-}\texttt{Transitive}(G, X)\Big),$

$[2] := \mathit{C\!I}\texttt{Transitive}(\alpha)\mathit{C\!I}^{-1}\Delta(X) : \Delta(X) \in \texttt{orbitals}(\alpha),$

$[*] := \mathit{C\!I}2\text{-}\texttt{Transitive}(\alpha)\mathit{C\!I}^{-1}\Delta(X) : \Delta^{\complement}(X) \in \texttt{orbitals}(\alpha);$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \Big(\alpha : \texttt{Transitive}(G, X)\Big) \;\&\; \operatorname{rank}\alpha = 2,$

$[3] := \mathit{C\!I}\texttt{Transitive}(\alpha)\mathit{C\!I}^{-1}\Delta(X) : \Delta(X) \in \texttt{orbits}(\alpha),$

$[4] := [3][2] : \Delta^{\complement}(X) \in \texttt{orbits}(\alpha),$

$[2.*] := \mathit{C\!I}^{-1}2\text{-}\texttt{Transitive}(G, X) : \Big(\alpha : \texttt{Transitive}(G, X)\Big);$

$\rightsquigarrow [2] := I(\Rightarrow)[1]I(\iff) : \texttt{This},$

$\square$

$\texttt{FixedSquare} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \,.\, \forall g \in G \,.\, \operatorname{Fix}\Big(\alpha \times \alpha(g)\Big) = \operatorname{Fix}\Big(\alpha(g)\Big) \times \operatorname{Fix}\Big(\alpha(g)\Big)$

$\texttt{Proof} \;=$

Limits commute with limits.

$\square$

$\texttt{permutationRepresentation} :: \prod G \in \mathsf{GRP} \,.\, \prod X \in \mathsf{SET} \,.\, \prod R \in \mathsf{ANN} \,.\, (G \xrightarrow{\mathsf{GRP}} X) \to R\text{-}\mathsf{REPR}\,(G)$

$\texttt{permutationRepresentation}\,(\alpha) = \widetilde{\alpha} := \Big(R[X], \Lambda g \in G \,.\, \Lambda r_x x \in R[X] \,.\, r_x \alpha_g(x)\Big)$

$\texttt{PermutationRepresentationIsOrthogonal} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall k : \texttt{Numeric} \,.$

$\quad . \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \,.\, \widetilde{\alpha} : \texttt{OrthogonalRepresentation}(k, G)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{PermutationRepresentationCharacter} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall k : \texttt{Numeric} \,.$

$\quad . \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \,.\, \forall g \in G \,.\, \chi_{\widetilde{\alpha}}(g) = \left|\mathrm{Fix}_\alpha(g)\right|$

$\texttt{Proof} \ =$

$\texttt{FixedSubsaceDimByInnerProduct} :: \forall G \in \texttt{FiniteGroup} \,.\, \forall k : \texttt{ConjugationField} \,.\, \forall (V, \rho) \in k\text{-}\mathsf{REPR}\,(G) \ .$

$\quad . \ \dim \lim \rho = \langle \chi_{e_{k,G}}, \chi_\rho \rangle_G$

$\texttt{Proof} \ =$

$[*] := \texttt{FixedPointDimensionByAveraging} \mathcal{I}^{-1} \texttt{character} k, G$

$\quad \mathcal{I}^{-1} \texttt{finiteGroupAlgebraInnerProduct}(k, G) \mathcal{I}^{-1} e_{k,G} :$

$$: \dim \lim \rho = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}\, \rho(g) \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \langle g \mapsto 1, \chi_\rho \rangle_G = \langle \chi_{e_{k,G}}, \chi_\rho \rangle_G;$$

$\square$

$\texttt{FixedSubsaceDimByOrbits} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall k : \texttt{Numeric} \,.$

$\quad . \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \ . \ \dim \lim \widetilde{\alpha} = \left| \operatorname{colim} \alpha \right|$

$\texttt{Proof} \ =$

$v := \lambda A \in \operatorname{colim} \alpha \ . \ \sum_{a \in A} a : \operatorname{colim} \alpha \to \lim \widetilde{\alpha},$

$[1] := \mathcal{O}(v) \texttt{DisjointOrbits}(\alpha) : (v : \texttt{Orthogonal}(\operatorname{colim} \widetilde{\alpha}))\,,$

$\texttt{Assume}\ x : \lim \widetilde{\alpha},$

$\texttt{Assume}\ O : \operatorname{colim} \alpha,$

$\texttt{Assume}\ o, o' : O,$

$\Big[g, [1]\Big] := \mathcal{I} \operatorname{colim} \alpha : \sum g \in G \,.\, \alpha_g(o) = o',$

$[2] := \mathcal{I} \lim \widetilde{\alpha}(x)(g) : \widetilde{\alpha}(x) = x,$

$[O.*] := [1][2] : x_o = x_{o'};$

$\rightsquigarrow [1] := I^2(\forall) : \forall O \in \operatorname{colim} \alpha \,.\, \forall o, o' \in O \,.\, x_o = x_{o'},$

$[x.*] := [1]\mathcal{O}v : x = \sum_{O \in \lim \widetilde{\alpha}} x_O v_O;$

$\rightsquigarrow [1] := \mathcal{I}^{-1} \texttt{Basis}(\lim \widetilde{\alpha}) : \Big(v : \texttt{Basis}(\lim \widetilde{\alpha})\Big),$

$[*] := \mathcal{I}^{-1} \dim \mathcal{O}v[1] : \dim \lim \widetilde{\alpha} = \left| \operatorname{colim} \alpha \right|;$

$\square$

$\texttt{BurnsideOrbitalLemma} :: \forall G \in \mathsf{GRP} \,.\, \forall X : \texttt{Finite} \,.\, \forall \alpha : G \xrightarrow{\mathsf{GRP}} S_X \,.\, |\operatorname{colim} \alpha| = \frac{1}{|G|} \sum_{g \in G} \left|\mathrm{Fix}_\alpha(g)\right|$

$\texttt{Proof} \ =$

$[(*)] := \texttt{FixedSubspaceDimByOrbits}(G, X, k) \texttt{FixedSubspaceDimByInnerProduct}(G, k, \widetilde{\alpha})$

$\quad \mathcal{I} \texttt{finiteGroupAlgebraInnerProduct}(k, G) \mathcal{I} e_{k,G} \texttt{PermutationRepresentationCharacter}(G, X, k, \alpha) :$

$$: |\operatorname{colim} \alpha| = \dim \lim \widetilde{\alpha} = \langle \chi_{e_{k,G}}, \chi_{\widetilde{\alpha}} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_{\widetilde{\alpha}}(g) = \frac{1}{|G|} \sum_{g \in G} \left|\mathrm{Fix}_\alpha(g)\right|;$$

$\square$

$\texttt{RankByRepresentation} :: \forall G \in \mathsf{GRP} . \forall X \in \mathsf{SET} . \forall \alpha : \texttt{Transitive}(G, X) . \operatorname{rank} \alpha = \langle \chi_{\widetilde{\alpha}}, \chi_{\widetilde{\alpha}} \rangle_G$

$\texttt{Proof} =$

$[1] := \text{Ɑ} \operatorname{rank} \alpha \texttt{BurnsideOrbitalLemma}(G, X, \alpha \times \alpha)\texttt{FixedSquare}(X, G)$
$\texttt{PermutationRepresentationCharacter}(G, X, k, \alpha)\text{Ɑ}^{-1}\texttt{finiteGroupAlgebraInnerProduct}(k, G) :$

$$: \operatorname{rank} \alpha = |\operatorname{colim} \alpha \times \alpha| = \frac{1}{|G|}\sum_{g \in G} \left|\operatorname{Fix}_\alpha(g)\right|^2 = \frac{1}{|G|}\sum_{g \in G} \chi_{\widetilde{\alpha}}\overline{\chi_{\widetilde{\alpha}}} = \langle \chi_{\widetilde{\alpha}}, \chi_{\widetilde{\alpha}} \rangle_G,$$

$\square$

$\texttt{traceOfAction} :: \prod G \in \mathsf{GRP} . \prod R \in \mathsf{ANN} . \prod X : \texttt{Finite} . \texttt{Transitive}(G, X) \to \texttt{Submodule}\Big(R[X]\Big)$

$$\texttt{traceOfAction}(\alpha) = \operatorname{tr}_R \alpha := R\left(\sum_{x \in X} x\right)$$

$\texttt{augmentationOfAction} :: \prod G \in \mathsf{GRP} . \prod X : \texttt{Finite} . \texttt{Transitive}(G, X) \to \mathbb{C}\text{-}\mathsf{REPR}(G)$

$\texttt{augmentationOfAction}(\alpha) = \dot{\alpha} := \Big(V, \widetilde{\alpha}_{|V}\Big) \quad \texttt{where} \quad V = (\operatorname{tr}_{|\mathbb{C}} \alpha)^\perp$

$\texttt{AugmentationIrreducibleIfDoublyTransitive} :: \forall G \in \mathsf{GRP} . \forall X \in \mathsf{SET} . \forall \alpha : \texttt{Transitive}(G, X) .$
$. \dot{\alpha} : \texttt{Irreducible}(\mathbb{C}, G) \iff \alpha : 2\text{-}\texttt{Transitive}(G, X)$

$\texttt{Proof} =$

$\texttt{Assume } [1] : \Big(\dot{\alpha} : \texttt{Irreducible}(\mathbb{C}, G)\Big),$

$[2] := \texttt{RankByRepresentation}(G, X, \alpha)\texttt{OrthonormalIrreducibleCharacters}(\mathbb{C}, G)\text{Ɑ}\,\dot{\alpha}[2] :$
$: \operatorname{rank} \alpha = \langle \chi_{\widetilde{\alpha}}, \chi_{\widetilde{\alpha}} \rangle_G = 2,$

$[1.*] := \texttt{DoubleTransitivityTHM}[2] : \Big(\alpha : 2\text{-}\texttt{Transitive}(G, X)\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big(\dot{\alpha} : \texttt{Irreducible}(\mathbb{C}, G)\Big) \Rightarrow \Big(\alpha : 2\text{-}\texttt{Transitive}(G, X)\Big),$

$\texttt{Assume } [2] : \Big(\alpha : 2\text{-}\texttt{Transitive}(G, X)\Big),$

$[3] := \texttt{DoublyTransitivityTHM}[2] : \operatorname{rank} \alpha = 2,$

$[4] := \texttt{RankByRepresentation}[3] : 2 = 1 + \langle \chi_{\dot{\alpha}}, \chi_{\dot{\alpha}} \rangle,$

$[5] := [4] - 1 : 1 = \langle \chi_{\dot{\alpha}}, \chi_{\dot{\alpha}} \rangle_G,$

$[2.*] := \texttt{IrreducibleByNorm}[5] : \Big(\dot{\alpha} : \texttt{Irreducible}(R, G)\Big);$

$\rightsquigarrow [*] := I(\Rightarrow)[1]I(\iff) : \texttt{This};$

$\square$

$\texttt{centralizerAlgebra} :: \prod X \in \mathsf{SET} . \prod G \in \mathsf{GRP} . (G \xrightarrow{\mathsf{GRP}} S_X) \to \mathbb{C}\text{-}\mathsf{ALGE}$

$\texttt{centralizerAlgebra}(\alpha) = C(\alpha) := \mathbb{C}\text{-}\mathsf{REPR}(G)(\widetilde{\alpha}, \widetilde{\alpha})$

$\texttt{conjugateMatrixRepresentation} :: \prod X \in \mathsf{SET} . \prod G \in \mathsf{GRP} . (G \xrightarrow{\mathsf{GRP}} S_X) \to \mathbb{C}\text{-}\mathsf{REPR}(G)$

$\texttt{conjugateMatrixRepresentation}(\alpha) = T^\alpha := \Big(\mathbb{C}^{|X| \times |X|}, \Lambda g \in G . \Lambda A \in \mathbb{C}^{n \times n} . \widetilde{\alpha}^{X,X}(g)A\big(\widetilde{\alpha}^{X,X}(g)\big)^{-1}\Big)$

$\texttt{CentralizerAlgebraIsFixedSubspace} :: \forall X \in \mathsf{SET} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall \alpha : (G \xrightarrow{\mathsf{GRP}} S_X) \,.\, \lim T^\alpha = C^{X,X}(\alpha)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{DoublePermutationRepresentationEquivalence} :: \forall X \in \mathsf{SET} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall \alpha : (G \xrightarrow{\mathsf{GRP}} S_X) \,.$

$\quad .\, T^\alpha \cong_{\mathbb{C}\text{-}\mathsf{REPR}(G)} \widetilde{\alpha}^2$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{orbitalMatrix} :: \prod X \in \mathsf{SET} \,.\, \prod G \in \mathsf{GRP} \,.\, \prod \alpha : (G \xrightarrow{\mathsf{GRP}} S_X) \,.\, \texttt{orbital}(\alpha) \to \mathbb{C}^{|X|\times|X|}$

$\texttt{orbitalMatrix}\,(\Omega) = M(\Omega) := \Lambda i, j \in |X| \,.\, \texttt{if}\ (i,j) \in \Omega\ \texttt{then}\ 0\ \texttt{else}\ 1$

$\texttt{OrbitalMatricesAreBasisOfCentralizer} :: \forall X : \texttt{Finite} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall \alpha : \texttt{Transitive}(G, X) \,.$

$\quad .\, \left\{ M(\Omega) \middle| \Omega \in \texttt{orbital}(\alpha) \right\} : \texttt{Basis}\Big(C(\alpha)\Big)$

$\texttt{Proof} \;=$

$[1] := \texttt{FixedSubspaceDimByOrrbits}(G, X, \mathbb{C}, \alpha^2) : \{ \text{☼}_{a \in A} a | A : \texttt{orbital}(\alpha) \} : \texttt{Basis}(\lim \alpha^2),$

$[*] := \texttt{DoublePermutationRepresentationEquivalence}(X, G, \alpha)$

$\quad \texttt{CentralizerAlgebraIsFixedSubspace}(X, G, \alpha)\texttt{IsomorphicBasis} :$

$\quad : \left\{ M(\Omega) \middle| \Omega \in \texttt{orbital}(\alpha) \right\} : \texttt{Basis}\Big(C(\alpha)\Big);$

$\square$

$\texttt{GelfandPair} ::? \left(\sum G : \texttt{FiniteGroup} \,.\, \texttt{Subgroup}(G) \right)$

$(G, H) : \texttt{GelfandPair} \iff C(\Lambda_{G,H}) \in \mathbb{C}\text{-}\mathsf{CALGE}$

$\texttt{MultiplicityFree} :: \prod G : \texttt{FiniteGroup} \,.\, \prod k : \texttt{Field} \,.\, ?k\text{-}\mathsf{REPR}\,(G)$

$$\rho : \texttt{MultiplicityFree} \iff \exists n \in \mathbb{N} : \exists \alpha : n \hookrightarrow \texttt{Irreducible}(k, G) \,.\, \rho = \bigoplus_{i=1}^{n} \alpha_i$$

$\texttt{Symmetric} :: \prod X \in \mathsf{SET} \,.\, ?(X \times X)$

$A : \texttt{Symmetric} \iff \texttt{swap}(A) = A$

$\texttt{SymmetricGelfandPair} ::? \left(\sum G : \texttt{FiniteGroup} \,.\, \texttt{Subgroup}(G) \right)$

$(G, H) : \texttt{SymmeticGelfandPair} \iff \forall O \in \texttt{orbital}(\Lambda_{G,H}) \,.\, O : \texttt{Symmetric}$

$\texttt{SymmetricGelfandPairIsGelfandPair} :: \forall (G,H) : \texttt{SymmetricGelfandPair} \,.\, (G,H) : \texttt{GelfandPair}$

$\texttt{Proof} \ =$

$[1] := \texttt{OrbitalsAreBasisOfCentralizer}\left(G, \frac{G}{H}, \Lambda_{G,H}\right) : \Big(M : \texttt{Basis}\Big(C\big(\Lambda_{G,H}\big), \texttt{orbitals}(\Lambda_{G,H})\Big),$

$[2] := \text{Ɑ}\texttt{SymmetricGelfandPair}\text{Ɑ}^{-1}\texttt{SymmetricMatrix} : \operatorname{Im} M \subset \texttt{SymmetricMatrix}\left(\mathbb{C}, \left|\frac{G}{H}\right|\right),$

$[3] := \text{Ɑ}\texttt{Basis}[1][2] : C(\Lambda_{G,H}) \subset \texttt{SymmetricMatrix}\left(\mathbb{C}, \left|\frac{G}{H}\right|\right),$

$[4] := \texttt{SymmectricAlgebraCommutetes}[3] : C(\Lambda_{G,H}) \in \mathbb{C}\text{-}\mathsf{CALGE},$

$[*] := \text{Ɑ}^{-1}\texttt{GelfandPair}[4] : \Big((G,H) : \texttt{GelfandPair}\Big);$

$\square$

1.9 Induced Representation

$\texttt{ClassFunctionRestriction} :: \forall R \in \mathsf{ANN} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.$

$\quad . \forall f : \texttt{ClassFunction}(R, G) \,.\, f_{|H} : \texttt{ClassFunction}(R, H)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{zeroClassExtension} :: \prod R \in \mathsf{ANN} \,.\, \prod G \in \mathsf{GRP} \,.\, \prod H \subset_{\mathsf{GRP}} G \,.$

$\quad . \texttt{ClassFunction}(R, H) \xrightarrow{R\text{-}\mathsf{MOD}} (G \to R)$

$\texttt{zeroClassFunction}\,(f) = \dot{f} := \Lambda g \in G \,.\, \texttt{if}\ g \in H\ \texttt{then}\ f(g)\ \texttt{else}\ 0$

$\texttt{classInduction} :: \prod k : \texttt{Numeric} \,.\, \prod G : \texttt{FiniteGroup} \,.\, \prod H \subset_{\mathsf{GRP}} G \,.$

$\quad . \texttt{ClassFunction}(k, H) \xrightarrow{k\text{-}\mathsf{VS}} \texttt{ClassFunction}(k, G)$

$$\texttt{classInduction}\,(f) = \mathrm{Ind}_H^G\, f := \Lambda g \in G \,.\, \frac{1}{|H|} \sum_{h \in G} \dot{f}(hgh^{-1})$$

$\texttt{Assume}\ A : \operatorname{colim} \gamma_G,$

$\texttt{Assume}\ a, a' : A,$

$\Big(x, [1]\Big) := \text{Ɑ} \operatorname{colim} \gamma_G(A)(a, a') : \sum x \in G \,.\, a' = xax^{-1},$

$X := \{g \in G : gag^{-1} \in H\} :?X,$

$Y := \{g \in G : ga'g^{-1} \in H\} :?Y,$

$[2] := \texttt{ConjugateIsomorphism}[1](X, Y) : |Y| = |X|,$

$[A.*] := \text{Ↄ}\mathrm{Ind}_H^G(f)\text{Ɑ}\dot{f}\text{Ɑ}\texttt{ClassFunction}(f)[2]\text{Ɑ}\texttt{ClassFunction}(f)\text{Ɑ}^{-1}\dot{f}\text{Ↄ}^{-1}\mathrm{Ind}_H^G(f) :$

$$: \mathrm{Ind}_H^G(f)(a') = \frac{1}{|H|} \sum_{g \in G} \dot{f}\Big(ga'g^{-1}\Big) = \frac{1}{|H|} \sum_{g \in Y} f\Big(ga'g^{-1}\Big) = \frac{1}{|H|} \sum_{g \in Y} f\Big(A\Big) =$$

$$= \frac{1}{|H|} \sum_{g \in X} f\Big(A\Big) = \frac{1}{|H|} \sum_{g \in X} f\Big(gag^{-1}\Big) = \frac{1}{|H|} \sum_{g \in G} \dot{f}\Big(gag^{-1}\Big) = \mathrm{Ind}_H^G(f)(a);$$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{ClassFunction} : \Big(\mathrm{Ind}_H^G(f)(a) : \texttt{ClassFunction}(k, G)\Big),$

$\square$

$\texttt{InductionRestriction} :: \forall k : \texttt{Numeric}.\forall G : \texttt{FiniteGroup} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.$

$\quad . \forall f : \texttt{ClassFunction}(k, G) \,.\, \Big(\mathrm{Ind}_H^G\, f\Big)_{|H} = f$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{FrobeniusReciprocity} :: \forall k : \texttt{Numeric} . \forall G : \texttt{FiniteGroup} . \forall H \subset_{\mathsf{GRP}} G .$

$$. \forall w : \texttt{ClassFunction}(k, G) . \forall v : \texttt{ClassFunction}(k, H) . \langle w_{|H}, v\rangle_H = \Big\langle w, \mathrm{Ind}_H^G(v)\Big\rangle_G$$

$\texttt{Proof} =$

$[*] := \mathcal{I}\texttt{finiteGroupAlgebra}(k, H)\mathcal{I}\texttt{restriciton}(G, H)\mathcal{I}\texttt{ClassFunction}(w)\mathcal{I}^{-1}\texttt{zeroClassExtension}(v)$
$\texttt{ConjugationIsomorphism}(G)\mathcal{I}\mathsf{ANN}(k)\mathcal{I}^{-1}\texttt{classInduction}(k, H)\mathcal{I}^{-1}\texttt{finiteGroupAlgebra}(k, H) :$

$$: \langle w_{|H}, v\rangle_H = \frac{1}{|H|}\sum_{h\in H} w(h)\overline{v(h)} = \frac{1}{|H||G|}\sum_{h\in H}\sum_{g\in G} w(ghg^{-1})\overline{v\Big(h\Big)} = \frac{1}{|H||G|}\sum_{g\in G}\sum_{h\in H} w(ghg^{-1})\overline{\dot{v}\Big(h\Big)} =$$

$$= \frac{1}{|H||G|}\sum_{g\in G}\sum_{h\in G} w(h)\overline{\dot{v}\Big(g^{-1}hg\Big)} = \frac{1}{|G|}\sum_{h\in G} w(h)\frac{1}{|H|}\sum_{g\in G}\overline{\dot{v}\Big(g^{-1}hg\Big)} = \frac{1}{|G|}\sum_{h\in G} w(h)\overline{\Big(\mathrm{Ind}_H^G\, v\Big)}(h) =$$

$$= \Big\langle w, (\mathrm{Ind}_H^G\, v)(h)\Big\rangle_G;$$

□

$\texttt{InductionByCosets} :: \forall k : \texttt{Numeric} . \forall G : \texttt{FiniteGroup} . \forall H \subset_{\mathsf{GRP}} G .$

$$. \forall f : \texttt{ClassFunction}(H, G) . \forall n \in \mathbb{N} . \forall t : n \to G . \forall [0] : \Big(t\pi_H : n \twoheadrightarrow GH^{-1}\Big) .$$

$$. \mathrm{Ind}_H^G(f) = \Lambda g \in G . \sum_{i=1}^{n} \dot{f}\Big(t_i g t_i^{-1}\Big)$$

$\texttt{Proof} =$

$\texttt{Assume } a : G,$

$[*.a] := \mathcal{I}\texttt{classInduction}(f)(a)\texttt{DisjointCosets}(G, H)\mathcal{I}\mathsf{ANN}(k)\mathcal{I}\dot{f}\mathcal{I}\texttt{ClassFunction}(f) :$

$$: \mathrm{Ind}_H^G f(a) = \frac{1}{|H|}\sum_{g\in G}\dot{f}\Big(gag^{-1}\Big) = \frac{1}{|H|}\sum_{i=1}^{n}\sum_{h\in H}\dot{f}\Big(ht_iat_i^{-1}h^{-1}\Big) = \sum_{i=1}^{n}\frac{1}{|H|}\sum_{h\in H}\dot{f}\Big(ht_iat_i^{-1}h^{-1}\Big) =$$

$$= \sum_{i=1}^{n}\dot{f}\Big(ht_iat_i^{-1}h^{-1}\Big);$$

$$\rightsquigarrow [*] := I(=, \to) : \mathrm{Ind}_H^G(f) = \Lambda g \in G . \sum_{i=1}^{n}\dot{f}\Big(t_igt_i^{-1}\Big);$$

□

$\texttt{zeroRepresentationExtension} :: \prod R \in \mathsf{ANN} . \prod G \in \mathsf{GRP} . \prod H \subset_{\mathsf{GRP}} G .$

$$. \prod (V, \rho) \in R\text{-}\mathsf{REPR}(H) . G \to \mathrm{End}_{R\text{-}\mathsf{MOD}}(V)$$

$\texttt{zeroClassFunction}() = \dot{\rho} := \Lambda g \in G . \texttt{if } g \in H \texttt{ then } \rho(g) \texttt{ else } 0$

$\texttt{representationInduction} :: \prod k : \texttt{Numeric} . \prod G : \texttt{FiniteGroup} . \prod H \subset_{\mathsf{GRP}} G .$

$$. \prod n \in \mathbb{N} . \prod t : n \to G . \prod [0] : \Big(t\pi_H : n \twoheadrightarrow GH^{-1}\Big) . k\text{-}\mathsf{REPR}(G) \to k\text{-}\mathsf{REPR}(G)$$

$\texttt{representationInduction}((V, \rho)) = \mathrm{Ind}_{H,t}^G\, \rho := \Big(V^{\oplus[G:H]}, \Lambda g \in G . \Lambda i, j \in [G : H] . \dot{\rho}_{t_i^{-1}gt_j}\Big)$

$\mathtt{InducedCharacter} :: \forall k : \mathtt{Numeric} \,.\, \forall G : \mathtt{FiniteGroup} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.$

$$. \forall n \in \mathbb{N} \,.\, \forall t : n \to G \,.\, \forall [0] : \left(t\pi_H : n \twoheadrightarrow GH^{-1} \right) . \forall \rho \in k\text{-}\mathsf{REPR}\,(G) \;.\; \chi_{\mathrm{Ind}_H^G \rho} = \mathrm{Ind}_H^G \chi_\rho$$

$\mathtt{Proof}\ =$

$[*] := \text{Ɑ}\mathtt{character}\text{Ɑ}\mathtt{representationInduction}\text{Ɑ}^{-1}\mathtt{charactetInductionByCosets} :$

$$: \chi_{\mathrm{Ind}_H^G \rho} = \mathrm{tr}\, \left(\mathrm{Ind}_H^G \rho \right) = \sum_{i=1}^n \gamma_G(t_i)\,\mathrm{tr}\,\dot{\rho} = \sum_{i=1}^n \gamma_G(t_i)\dot{\chi}_\rho = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}\dot{\chi}_\rho;$$

$\square$

$\mathtt{DisjointRepresentation} :: \prod R \in \mathsf{ANN} \,.\, \prod G \in \mathsf{GRP} \,.\, ?R\text{-}\mathsf{REPR}\,(G)^2$

$(\alpha, \beta) : \mathtt{DisjointRepresentation} \iff \alpha \perp \beta \iff \forall \rho : R\text{-}\mathsf{REPR}\,(G) \;.$

$$. \deg \rho \neq 0 \Rightarrow \left(\left(\exists \rho' : \alpha \cong_{R\text{-}\mathsf{REPR}(G)} \rho \oplus \rho' \right) \Rightarrow \forall \rho'' \in R\text{-}\mathsf{REPR}\,(G) \;.\; \beta \ncong_{R\text{-}\mathsf{REPR}(G)} \rho \oplus \rho'' \right)$$

$\mathtt{DisjointAsOrthogonal} :: \forall k : \mathtt{ConjugationField} \,.\, \forall G : \mathtt{FiniteGroup} \,.\, \forall \alpha, \beta \in k\text{-}\mathsf{REPR}\,(G) \;.$

$$. \langle \chi_\alpha, \chi_\beta \rangle = 0 \iff \alpha \perp \beta$$

$\mathtt{Proof}\ =$

$\ldots$

$\square$

$\mathtt{MackeyCrossinductionLemma} :: \forall G : \mathtt{FiniteGroup} \,.\, \forall H, K \subset_{\mathsf{GRP}} G \,.\, \forall S \in ?G \,.$

$$. \forall [0] : \left(\pi_{H,K|S} : S \xleftrightarrow{\mathsf{SET}} H^{-1}GK^{-1} \right) . \forall f : \mathtt{ClassFunction}(\mathbb{C}, K) \;.$$

$$. \left(\mathrm{Ind}_K^G f \right)_{|H} = \sum_{s \in S} \mathrm{Ind}_{H \cap sKs^{-1}}^H \left(\gamma_s f \right)_{|H \cap sKs^{-1}}$$

$\mathtt{Proof}\ =$

$V := \Lambda s \in S \,.\, H(H \cap sKs^{-1})^{-1} : S \to ??H,$

$\mathtt{Assume}\ s : S,$

$[1] := \text{Ɑ}\mathtt{leftCosets}\text{Ɔ}V_s : \bigcup_{[v] \in V_s} v(H \cap sks^{-1}),$

$[2] := [1]\text{Ɑ}\mathsf{GRP}(sKs^{-1})\text{Ɑ}\mathtt{Cosets}[0]\text{Ɔ}V_s\text{Ɑ}^{-1}\mathtt{DisjointUnion} :$

$$: HsK = HsKs^{-1}s = \bigcup_{[v] \in V_s} v\left(H \cap sKs^{-1} \right) sKs^{-1}s = \bigcup_{[v] \in V_s} vsK,$$

$\mathtt{Assume}\ [v], [v'] : V_s,$

$\mathtt{Assume}\ [3] : vsK = v'K,$

$[4] := [3]\text{Ɑ}\mathtt{leftCosets}(G, K) : s^{-1}v^{-1}v's \in K,$

$[5] := s[3]s^{-1} : v^{-1}v' \in sKs^{-1},$

$[6] := \text{Ɑ}^{-1}\mathtt{leftCosts}(H, H \cap sKs^{-1}) : v(H \cap sKs^{-1}) = v'(H \cap sKs^{-1}),$

$\left[\left([v], [v'] \right) . * \right] := \text{Ɔ}V[6] : [v] = [v'];$

$\rightsquigarrow [s.*] := [2]\text{Ɑ}^{-1}\mathtt{DisjointUnion} : Hsk = \bigsqcup_{v \in V_s} vsK;$

$\rightsquigarrow [1] := I(\forall) : \forall s \in S \,.\, Hsk = \bigsqcup_{v \in V_s} vsK,$

$T := \Lambda s \in S \,.\, \{vs | [v] \in V_s\} :?G,$

$A := \bigsqcup_{s \in S} T_s :?G,$

$\texttt{Assume}\ s, s' : S,$

$\texttt{Assume}\ [v], [v'] : V,$

$\texttt{Assume}\ [2] : vs = v's',$

$[3] := [2]\text{Д}\,\texttt{doubleCosets} : HsK = Hs'K,$

$[4] := [0][3] : s = s',$

$\Big[(s, s').*\Big] := \text{Э}V[4][2] : [v] = [v'];$

$\rightsquigarrow [2] := \text{Д}^{-1}\texttt{DisjointUnion} : A = \bigsqcup_{s \in S} T_s,$

$[3] := \texttt{DisjointCosets}[0][1]\text{Э}^{-1}T[2]\text{Э}^{-1}A : G = \bigsqcup_{s \in S} HsK = \bigsqcup_{s \in S} \bigcup_{[v] \in V_s} vsK = \bigsqcup_{s \in S} \bigcup_{t \in T_s} tK = \bigsqcup_{t \in A} tK,$

$[4] := \texttt{DisjointCosets}[3] : \left(\pi_{K|T} : T \overset{\mathsf{SET}}{\longleftrightarrow} GK^{-1}\right),$

$\texttt{Assume}\ h : H,$

$[h.*] := \texttt{InductionByCosets}(G, K, f, A)[4]\text{Э}A\text{Э}T\text{Д}^{-1}\gamma\text{Д}\dot{f}\text{Д}^{-1}\texttt{restrictionInductionByCosets} :$

$: \mathrm{Ind}_H^G\ f(h) = \sum_{t \in A} \dot{f}\left(t^{-1}ht\right) = \sum_{s \in S}\sum_{t \in T_s} \dot{f}\left(t^{-1}ht\right) = \sum_{s \in S}\sum_{[v] \in V_s} \dot{f}\left(s^{-1}v^{-1}hvs\right) = \sum_{s \in S}\sum_{[v] \in V_s} (\gamma_s \dot{f})\left(v^{-1}hv\right) =$

$= \sum_{s \in S}\sum_{[v] \in V_s,} (\gamma_s f)\left(v^{-1}hv\right)[v^{-1}hv \in sKs^{-1}] = \sum_{s \in S}\sum_{[v] \in V_s,} \gamma_s(f)_{|H \cap sKs^{-1}}\left(v^{-1}hv\right)[v^{-1}hv \in sKs^{-1}] =$

$= \left(\sum_{s \in S} \mathrm{Ind}_{H \cap sKs^{-1}}^H\ \gamma_s(f)_{|H \cap sKs^{-1}}\right)(h);$

$\rightsquigarrow [*] := I(=, \to) : \texttt{This};$

$\square$

$\texttt{MackeyIrreducibilityTHM} :: \forall G : \texttt{FiniteGroup}\ .\ \forall H \subset_{\mathsf{GRP}} G\ .\ \forall k : \texttt{ConjugationField}\ .$

$.\ \forall \rho \in k\text{-}\mathsf{REPR}\,(H)\ .\ \mathrm{Ind}_H^G\ \rho : \texttt{Irreducible}(K, G) \iff$

$\iff \rho : \texttt{Irreducible}(k, H)\ \&\ \forall s \in H^{\complement}\ .\ \rho_{|H \cap sHs^{-1}} \perp \gamma_s \rho_{|H \cap sHs^{-1}}$

$\texttt{Proof}\ =$

$\Big(S, [1]\Big) := \texttt{DoubleCosetsRepresentativesExists}(G, H, H) : \sum S \subset G\ .\ e \in S\ \&\ \pi_{H,H|S} : S \overset{\mathsf{SET}}{\longleftrightarrow} G,$

$S' := S \setminus \{e\} :?G,$

$[2] := \texttt{MackeyCrossinductionTHM}[1]\text{Э}^{-1}S' : \left(\mathrm{Ind}_H^G\ \chi_\rho\right)_{|H} = \chi_\rho + \sum_{s \in S'} \mathrm{Ind}_{H \cap sHs^{-1}}^H \sigma_s f_{|H \cap sHs^{-1}},$

$[3] := \texttt{FrobeniusReciprocity}\text{Д}\mathcal{L}(H, H; k)\Big(\texttt{finiteGroupAlgebraInnerproduct}(H)\Big)\texttt{FrobeniusReciprocity} :$

$: \langle \mathrm{Ind}_H^G\ \chi_\rho, \mathrm{Ind}_H^G\ \chi_\rho \rangle_G = \left\langle \left(\mathrm{Ind}_H^G\ \chi_\rho\right)_H, \chi_\rho \right\rangle_H = \langle \chi_\rho, \chi_\rho \rangle_H + \sum_{s \in S'} \left\langle \mathrm{Ind}_{H \cap sHs^{-1}}^H (\gamma_s \chi)_{|H \cap sHs^{-1}}, \chi \right\rangle_H =$

$= \langle \chi_\rho, \chi_\rho \rangle_H + \sum_{s \in S'} \left\langle (\gamma_s \chi)_{|H \cap sHs^{-1}}, \chi_{H \cap sHs^{-1}} \right\rangle_{|H \cap sHs^{-1}},$

$[*] := \texttt{IrreducibleByNorm}[1] : \texttt{This};$

$\square$

1.10 Second Burnside's Theorem

$\texttt{CharacterConjugation} :: \forall k : \texttt{ConjugateField} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \forall \rho \in k\text{-}\mathsf{REPR}\,(G) \;.\; \overline{\chi_\rho} = \chi_{\rho^*}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RealCharacter} :: \prod k : \texttt{ConjugateField} \,.\, \prod G : \texttt{FiniteGroup} \,.\, ?\texttt{Character}(k, G)$

$\chi : \texttt{RealCharacters} \iff \overline{\chi} = \chi$

$\texttt{InverseConjugation} :: \forall k : \texttt{ConjugateField} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \forall \chi : \texttt{Character}(k, G) \,.\, \forall g \in G \,.$

$\quad . \,\chi\big(g^{-1}\big) = \overline{\chi(g)}$

$\texttt{Proof} \;=$

$\Big((V, \rho), [1]\Big) := \mathcal{I}\texttt{CharacterRepresentationOrthogonalization} :$

$\quad : \sum (V, \rho) : \texttt{UnitaryRepresentation}(k, G) \,.\, \chi = \chi_\rho,$

$[*] := [1]\mathcal{I}\texttt{character}\mathcal{I}\mathsf{GRP}\Big(G, \mathbf{U}(V)\Big)(\rho)\mathcal{I}\mathbf{U}(V)\texttt{characterConjugation}(k, G, \rho) :$

$\quad : \chi\big(g^{-1}\big) = \operatorname{tr} \rho\big(g^{-1}\big) = \operatorname{tr} \rho^{-1}(g) = \operatorname{tr} \rho^*(g) = \overline{\chi}(g);$

$\square$

$\texttt{RealConjugacyClass} :: \prod G \in \mathsf{GRP} \,.\, ?(\operatorname{colim} \gamma_G)$

$A : \texttt{RealConjugacyClass} \iff A = A^{-1}$

$\texttt{RealConjugacyClassMotivation} :: \forall G \in \mathsf{GRP} \,.\, \forall \chi : \texttt{Character}(\mathbb{C}, G) \,.\, \forall A : \texttt{RealConjugacyClass}(G) \,.$

$\quad . \,\chi(A) \in \mathbb{R}$

$\texttt{Proof} \;=$

$\square$

$\texttt{BurnsideRealityTHM} :: \forall G : \texttt{FiniteGroup} \,.\, \#\texttt{RealsCharacter} \;\&\; \texttt{IrreducibleCharacter}(\mathbb{C}, G) =$

$\quad = \#\texttt{RealConjugacyClass}(G)$

$\texttt{Proof} \;=$

$\Big(n, \chi, C\Big) := \texttt{NumberOfIrreducibleRepresentations}(\mathbb{C}, G) :$

$\quad : \sum n \in \mathbb{N} \,.\, \sum \chi : n \twoheadrightarrow \texttt{Irreducible}(\mathbb{C}, G) \,.\, \sum C : n \twoheadrightarrow (\operatorname{colim} \gamma_G),$

$(\alpha, [1]) := \texttt{IrreducibleByNorm}(\overline{\chi})\mathcal{I}^{-1}S_n : \sum \alpha \in S_n \,.\, \forall i \in n \,.\, \chi_{\alpha(i)} = \overline{(}\chi_i),$

$(\beta, [2] := \texttt{ConjugacyClassInversion}(\overline{\beta})\mathcal{I}^{-1}S_n : \sum \beta \in S_n \,.\, \forall i \in n \,.\, C_{\beta(i)} = C_i^{-1},$

$[3] := \mathcal{I}^{-1}\texttt{fixedPoints}\mathcal{I}^{-1}\texttt{cardinality}(\alpha) : |\mathrm{Fix}(\alpha)| = \#\texttt{RealsCharacter} \;\&\; \texttt{IrreducibleCharacter}(\mathbb{C}, G),$

$[4] := \mathcal{I}^{-1}\texttt{fixedPoints}\mathcal{I}^{-1}\texttt{cardinality}(\beta) : |\mathrm{Fix}(\beta)| = \#\texttt{RealConjugacyClass}(G),$

$[7] := \texttt{PermutationRepresentationCharacter}(\alpha) : \chi_{\widetilde{\mathrm{id}}}(\alpha) = |\mathrm{Fix}(\alpha)|,$

$[8] := \texttt{PermutationRepresentationCharacter}(\beta) : \chi_{\widetilde{\mathrm{id}}}(\alpha) = |\mathrm{Fix}(\beta)|,$

$[9] := \texttt{PermutationMatrixMult}(\alpha)\mathcal{I}\alpha\mathcal{I}\mathbf{Ch}(G)\mathcal{I}\beta\texttt{InverseConjugation} :$

$\quad : \mathbf{Ch}(G, \mathbb{C})\alpha = \overline{\mathbf{Ch}}(G, \mathbb{C}) = \beta\mathbf{Ch}(G, \mathbb{C}),$

$[10] := \texttt{SecondOrthogonalityRelation}[9] : \alpha = \mathbf{Ch}(G,\mathbb{C})\beta\mathbf{Ch}^{-1}(G,\mathbb{C}),$

$[*] := [3][7]\mathcal{I}\texttt{character}[10]\texttt{ShiftInTrace}\mathcal{I}^{-1}\texttt{character}[8][4] :$

$: \#\texttt{RealsCharacter}\ \&\ \texttt{IrreducibleCharacter}(\mathbb{C},G) = |\mathrm{Fix}(\alpha)| = \chi_{\widetilde{\mathrm{id}}}(\alpha) = \mathrm{tr}\,\alpha =$

$= \mathrm{tr}\,\mathbf{Ch}(G,\mathbb{C})\beta\mathbf{Ch}^{-1}(G,\mathbb{C}) = \mathrm{tr}\,\beta = \chi_{\widetilde{\mathrm{id}}}(\alpha) == |\mathrm{Fix}(\beta)| = \#\texttt{RealConjugacyClass}(G);$

□

$\texttt{Oddity} :: \forall G : \texttt{FiniteGroup} \,.\, \forall[0] : \Big(|G| : \texttt{Odd}\Big) \,.\, \forall\rho : \texttt{RealsCharacter}\ \&\ \texttt{IrreducibleCharacter}(\mathbb{C},G) \,.$

$.\, \rho = e_{k,G}$

$\texttt{Proof} =$

$\texttt{Assume}\ A : \texttt{RealConjugacyClass},$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ [1] : a \neq e,$

$\Big(h,[2]\Big) := \mathcal{I}\texttt{RealConjugacyClass}(A)(a) : \sum h \in G \,.\, hah^{-1} = a^{-1},$

$[3] := [2]\texttt{ProductInverse}[2] : h^2gh^{-2} = hg^{-1}h^{-1} = (hgh^{-1})^{-1} = g,$

$[4] := \mathcal{I}^{-1}\texttt{Normalizer}[3] : h^2 \in N(a),$

$\texttt{Assume}\ [5] : h \in \langle h^2\rangle,$

$[6] := [5][4] : h \in N(a),$

$[7] := [2]\mathcal{I}N(a)[6] : a^{-1} = hah^{-1} = a,$

$[8] := \mathcal{I}^{-1}\texttt{order}[1]\texttt{OrderDivides} : |G| : \texttt{Even},$

$[4.*] := \texttt{OddEven}[0][8] : \bot;$

$\rightsquigarrow [4] := \mathcal{I}^{-1}\texttt{complement} : h \in \langle h^2\rangle^{\complement},$

$[5] := \texttt{GeneratorsByCoprime}(h)[4] : o(h) : \texttt{Even},$

$[6] := \texttt{OrderDivides} : |G| : \texttt{Even},$

$[A.*] := \texttt{OddEven}[0][8] : \bot;$

$\rightsquigarrow [1] := I(\forall) : \forall A : \texttt{RealConjugacyClass} \,.\, A = \{e\},$

$[*] := \texttt{BurnsideRealsityTHM}[1] : [*];$

□

$\texttt{SecondBurnsideTHM} :: \forall G : \texttt{FiniteGroup} \,.\, \forall[0] : \Big(|G| : \texttt{Odd}\Big) \,.\, |G| =_{Z_{16}} |\operatorname{colim}\gamma_G|$

$\texttt{Proof} =$

$\Big(n,d,[1]\Big) := \texttt{GroupSizeByIrreducibleRepresentation}(G)\texttt{Oddity}[0] :$

$$: \sum n \in \mathbb{N} \,.\, \sum d : n \to \mathbb{N} \,.\, \Big(\forall i \in n\, \exists \rho : \texttt{Irreducible}(\mathbb{C},G) : d_i = \deg\rho\Big)\ \&\ |G| = 1 + \sum_{i=1}^{n} 2d_i^2,$$

$\Big(m,[2]\Big) := \texttt{DimensionTHM}(G)[1]\mathcal{I}\mathsf{ANN}(\mathbb{Z}) : \sum m : \mathbb{N} \to \mathbb{Z}_+ \,.\, |G| =$

$$= 1 + \sum_{i=1}^{n} 2(2m_i+1)^2 = 1 + 2n + \sum_{i=1}^{n} 8(m_i^2 + m_i = 1 + 2n + 8\sum_{i=1}^{n} m_i(m_i+1),$$

$[*] := [2]\texttt{IrreducibleRepresentationNumber}(G)\texttt{NextProductIsEven}\mathcal{I}^{-1}Z_16 : |G| =_{Z_{16}} 1 + 2n;$

□

1.11 Artin and Brauer Theorems

2 Semisimple Representation

3 Modular Representation

4 Block Theory