# Descriptive Set Theory

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# Contents

# 1 Polish Topology

## 1.1 Trees

### 1.1.1 Finite lists

$$\texttt{List} := \Lambda A \in \mathsf{SET} \ . \ A^* = \Lambda A \in \mathsf{SET} \ . \ \bigsqcup_{n=0}^{\infty} A^n : \mathsf{SET} \to \mathsf{SET};$$

$$\texttt{length} :: \prod_{A \in \mathsf{SET}} A^* \to \mathbb{Z}_+$$

$$\texttt{length}\,((n,a)) = \mathrm{len}(n,a) := n$$

$$\texttt{values} :: \prod_{A \in \mathsf{SET}} \prod_{(n,a) \in A^*} A^n$$

$$\texttt{values}\,() = (n,a) := a$$

$$\texttt{InitialSegment} :: \prod_{A \in \mathsf{SET}} A^* \to ?A^*$$

$$s : \texttt{InitialSegement} \iff \Lambda x \in A^* \ . \ s \subset x \iff \Lambda x \in A^* \ . \ \mathrm{len}(x) \geq \mathrm{len}(s) \ \& \ x_{|[1,\dots,\mathrm{len}(s)]} = s$$

$$\texttt{Extension} :: \prod_{A \in \mathsf{SET}} A^* \to ?A^*$$

$$x : \texttt{Extension} \iff \Lambda s \in A^* \ . \ \texttt{InitialSegement}(A, x, s)$$

$$\texttt{Compatible} :: \prod_{A \in \mathsf{SET}} ?(A^* \times A^*)$$

$$x, y : \texttt{Compatible} \iff x \subset y | y \subset x$$

$$\texttt{Incompatible} :: \prod_{A \in \mathsf{SET}} ?(A^* \times A^*)$$

$$x, y : \texttt{Incompatible} \iff x \perp y \iff \neg\texttt{Compatible}(A, x, y)$$

$$\texttt{concatination} :: \prod_{A \in \mathsf{SET}} A^* \times A^* \to A^*$$

$$\texttt{concatination}\,(x,y) = xy := \Bigg( \mathrm{len}(x) + \mathrm{len}(y), \\ \Lambda i \in \Big[1, \dots, \mathrm{len}(x) + \mathrm{len}(y)\Big] \ . \ \texttt{if } i \leq \mathrm{len}(x) \texttt{ then } x_i \texttt{ else } y_{i - \mathrm{len}(x)} \Bigg)$$

$$\texttt{InitialSegment} :: \prod_{A \in \mathsf{SET}} A^{\mathbb{N}} \to ?A^*$$

$$s : \texttt{InitialSegement} \iff \Lambda x \in A^{\mathbb{N}} \ . \ s \subset x \iff \Lambda x \in A^{\mathbb{N}} \ . \ x_{|[1,\dots,\mathrm{len}(s)]} = s$$

$$\texttt{Extension} :: \prod_{A \in \mathsf{SET}} A^* \to ?A^{\mathbb{N}}$$

$$x : \texttt{Extension} \iff \Lambda s \in A^* \; . \; \texttt{InitialSegement}(A, x, s)$$

$$\texttt{Incompatible} :: \prod_{A \in \mathsf{SET}} ?(A^* \times A^{\mathbb{N}})$$

$$x, y : \texttt{Incompatible} \iff x \perp y \iff \neg\texttt{InitialSegement}(A, x, y)$$

$$\texttt{infConcatination} :: \prod_{A \in \mathsf{SET}} \left(\mathbb{N} \to A^*\right) \to \left(A^* \sqcup A^{\mathbb{N}}\right)$$

$$\texttt{infConcatination}\,(x) = \prod_{n=1}^{\infty} x_n := \left( \sum_{n=1}^{\infty} \mathrm{len}(x_m), \right.$$

$$\left. \Lambda i \in \left[1, \ldots, \sum_{i=1}^{n} \mathrm{len}(x_i)\right] \; . \; \texttt{if } i \leq \mathrm{len}(x_1) \texttt{ then } x_{1,i} \texttt{ else } \left(\prod_{n=2}^{\infty} x_n\right)_{i - \mathrm{len}(x)} \right)$$

$$\texttt{Tree} :: \prod_{A \in \mathsf{SET}} ??A^*$$

$$T : \texttt{Tree} \iff \forall t \in T \; . \; \forall n \in \left[1, \ldots, \mathrm{len}(t)\right] \; . \; t_{|[1, \ldots, n]} \in T$$

$$\texttt{Body} :: \prod_{A \in \mathsf{SET}} \texttt{Tree}(A) \to A^{\mathbb{N}}$$

$$x : \texttt{Body} \iff x \in [A] \iff \forall n \in \mathbb{N} \; . \; x_{|[1, \ldots, n]} \in T$$

$$\texttt{ExstensionComplete} :: \prod_{A \in \mathsf{SET}} ??A^*$$

$$X : \texttt{ExtensionComplete} \iff \forall x \in X \; . \; \forall s : \texttt{Extension}(A, x) \; . \; s \in X$$

$$\texttt{Pruned} :: \prod_{A \in \mathsf{SET}} ?\texttt{Tree}(A)$$

$$T : \texttt{Pruned} \iff \forall t \in T \; . \; \exists s \in T : \texttt{InitialSegement}(A, s, t) \;\&\; \mathrm{len}(s) > \mathrm{len}(t)$$

### 1.1.2 Discrete Topology

$\texttt{discreteProductMetric} :: \prod_{A \in \mathsf{SET}} \texttt{Metric}(A^{\mathbb{N}})$

$\texttt{discreteProducMetric}\,() = d := \Lambda x, y \in A^{\mathbb{N}}\ .\ \texttt{if}\ x == y\ \texttt{then}\ 0\ \texttt{else}\ 2^{-1-\min\{n \in \mathbb{N} : x_n \neq y_n\}}$

$\texttt{Assume}\ x, y, z \in A^{\mathbb{N}},$

$\texttt{Assume}\ [1] : x = z,$

$[1.*] := \texttt{E}d\texttt{EifThenElse}[1]\texttt{NonNegSumIsNotLess}\Big(d(x,y)\ \&\ d(y,z)\Big) : d(x,z) = 0 \le d(x,y) + d(y,z);$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : x = z \Rightarrow d(x,z) \le d(x,y) + d(y,z);$

$\texttt{Assume}\ [2] : x \neq z,$

$n := \min\{n \in \mathbb{N} : x_n \neq z_n\} \in \mathbb{N},$

$[3] := \texttt{E}(n) : x_n \neq z_n,$

$[4] := \texttt{ETransitive}(A, =)[3]\texttt{I}y_n : (x_n \neq y_n)|(y_n \neq z_n),$

$[5] := \texttt{I}d[4] : d(x,z) \le d(x,y)|d(x,z) \le d(y,z),$

$[2.*] := \texttt{OrMaxIneq}[5]\texttt{NonegMaxNonGreatetThenSum} : d(x,y) \le \max\Big(d(x,y), d(y,z)\Big) \le d(x,y) + d(y,z);$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : x \neq z \Rightarrow d(x,z) \le d(x,y) + d(y,z);$

$[*] := \texttt{E}(|)\texttt{LEM}(x = z)[1][2] : d(x,z) \le d(x,y) + d(y,z);$

$\square$

$\texttt{DiscreteProductMetricMetrizeDiscreteProduct} :: \Big(A^{\mathbb{N}}, d\Big) \cong_{\mathsf{TOP}} \prod_{n=1}^{\infty} A$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{DiscreteProductMetricIsUltrametric} :: \texttt{Ultrametric}\Big(A^{\mathbb{N}}, d\Big)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{standardBase} :: \prod_{A \in \mathsf{SET}} A^* \to \mathcal{T}\Big(A^{\mathbb{N}}\Big)$

$\texttt{standardBase}\,(s) = N_s := \Big\{a \in A^{\mathbb{N}} : a_{|[1,\ldots,\mathrm{len}(s)]} = s\Big\}$

$\texttt{StandardBaseIsBase} :: \texttt{Type}Base\Big(\operatorname{Im} N, \mathcal{T}(A^{\mathbb{N}})\Big)$

$\texttt{Proof} =$

$\texttt{Assume}\ U \in \mathcal{T}(A^{\mathbb{N}}),$

$\Big(m, S, [1]\Big) := \texttt{EdiscreteTopologyEproductTopology} : \sum_{m=0}^{\infty} \sum_{S \subset A^n} U = S \times A^{\mathbb{N}},$

$[U.*] := \texttt{I}(N)[1] : U = \bigcup_{s \in S} N_s;$

$\rightsquigarrow [*] := \texttt{IBase} : \texttt{Base}\Big(\operatorname{Im} N, \mathcal{T}(A^{\mathbb{N}})\Big);$

$\square$

$\texttt{MinimalSpanningStandardBase} :: \forall U \in \mathcal{T}(A^{\mathbb{N}}) \ .\ \exists S \subset A^* : U = \bigcup_{s \in S} N_s \ \&\ \forall t, s \in S \ .\ t \neq s \Rightarrow t \perp s$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{StandardBaseDensityCondition} :: \forall A \in \mathsf{SET} \ .\ \forall U \in \mathcal{T}(A^{\mathbb{N}}) \ .\ \forall S : \texttt{ExtensionComplete}(A) :$

$: U = \bigcup_{s \in S} N_s \Rightarrow \texttt{Dense}(A^{\mathbb{N}}, U) \iff \texttt{Dense}(A^*, S)$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{Dense}(A^*, S),$

$[2] := \texttt{EDense}[1] : \forall s \in A^* \ .\ \exists t \in S : s \subset t,$

$\texttt{Assume}\ x \in A^{\mathbb{N}},$

$\texttt{Assume}\ n \in \mathbb{N},$

$s := x_{|[1,\ldots,n]} \in A^*,$

$\Big(t, [3]\Big) := [2](s) : \sum t \in S \ .\ s \subset t,$

$[4] := \texttt{E}N[3] : N_t \subset N_s,$

$[x.*] := [0][4]\texttt{UnionIntersect} : U \cap N_s \neq \emptyset;$

$\rightsquigarrow [1.*] := \texttt{DenseByNeighborhoodBase} : \texttt{Dense}(A^{\mathbb{N}}, U);$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : \texttt{Dense}(A^*, S) \Rightarrow \texttt{Dense}(A^{\mathbb{N}}, U),$

$\texttt{Assume}\ [2] : \texttt{Dense}(A^{\mathbb{N}}, U),$

$\texttt{Assume}\ s \in A^*,$

$[3] := \texttt{EDense}(A^{\mathbb{N}}, U) : N_s \cap U \neq \emptyset,$

$[4] := \texttt{Eunion}[0][4] : \exists t \in S : N_s \cap N_t \neq \emptyset,$

$[5] := \texttt{E}N[4] : \exists t \in S : t \subset s | s \subset t,$

$[s.*] := \texttt{EExtensionComplete}(A, S)[5] : \exists t \in S : s \subset t;$

$\rightsquigarrow [2.*] := \texttt{IDense} : \texttt{Dense}(A^*, S);$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : \texttt{Dense}(A^{\mathbb{N}}, U) \Rightarrow \texttt{Dense}(A^*, S),$

$[*] := \texttt{I}(\iff)[1][2] : \texttt{Dense}(A^{\mathbb{N}}, U) \iff \texttt{Dense}(A^*, S);$

$\square$

$\texttt{DiscreteProductConvergence} :: \forall A \in \mathsf{SET} \,.\, \forall x : \mathbb{N} \to A^{\mathbb{N}} \,.$

$.\, \forall L \in A^{\mathbb{N}} \,.\, L = \lim_{n\to\infty} x_n \iff \forall m \in \mathbb{N} \,.\, \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq N \Rightarrow x_{n,m} = L_m$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{tree} :: \prod_{A \in \mathsf{SET}} ?A^{\mathbb{N}} \to \texttt{Pruned}(A)$

$\texttt{tree}\,(X) = T(X) := \{x_{|[1,\ldots,n]} | x \in X, n \in \mathbb{Z}_+\}$

$\texttt{BodyBijection} :: \texttt{Bijection}\Big(\Lambda T : \texttt{Pruned}(A) \,.\, [T], \texttt{Pruned}(A), \texttt{Closed}(A^{\mathbb{N}})\Big)$

$\texttt{Proof} =$

$F := \Lambda T : \texttt{Pruned}(A) \,.\, [T] : \texttt{Pruned}(A) \to ?A^{\mathbb{N}},$

$\texttt{Assume}\ T : \texttt{Pruned}(A),$

$\texttt{Assume}\ x \in F^{\complement}(T),$

$\Big(n, [1]\Big) := \texttt{E}x\texttt{E}F : \sum n \in \mathbb{N} \,.\, x_{|[1,\ldots,n]} \in T^{\complement},$

$[2] := \texttt{ETree}(T)[1] : \forall t \in A^* \,.\, s \subset t \Rightarrow t \notin T^{\complement},$

$[x.*] := \texttt{I}N_s\texttt{I}F[2] : N_s \cap F(T) = \emptyset;$

$\leadsto [T.*] := \texttt{CloseByNeigborhoodBase} : \texttt{Closed}\Big(A^{\mathbb{N}}, F(T)\Big));$

$\leadsto [1] := \texttt{I}\,\mathrm{Im} : \mathrm{Im}\,F \subset \texttt{Closed}(A^{\mathbb{N}}),$

$\texttt{Assume}\ X : \texttt{Closed}(A^{\mathbb{N}}),$

$Y := TF(X) : \texttt{Closed}(A^{\mathbb{N}}),$

$\texttt{Assume}\ x \in X,$

$[2] := \texttt{E}T(X)(x) : \forall n \in \mathbb{N} \,.\, x_{|[1,\ldots,n]} \in T(X),$

$[x.*] := \texttt{E}F[2]\texttt{I}Y : x \in Y;$

$\leadsto [2] := \texttt{ISubset} : X \subset Y,$

$\texttt{Assume}\ y \in Y,$

$[2] := \texttt{E}Y\texttt{E}Fy : \forall n \in \mathbb{N} \,.\, y_{|[1,\ldots,n]} \in T(X),$

$[3] := \texttt{E}T(X)\texttt{I}N[2] : \forall n \in \mathbb{N} \,.\, N_{y_{|[1,\ldots,n]}} \cap X \neq \emptyset,$

$[4] := \texttt{ClosureByNeighborhoodBase}[3] : y \in \overline{X},$

$[y.*] := \texttt{Eclosure}[4]\texttt{EClosed}(A^{\mathbb{N}}, X) : y \in X;$

$\leadsto [3] := \texttt{ISubset} : Y \subset X,$

$[X.*] := \texttt{ISetEq}[2][3] : Y = X;$

$\leadsto [2] := \texttt{ISurjective} : \texttt{Surjective}(F, \texttt{Pruned}(A), \texttt{Closed}(A^{\mathbb{N}}));$

$\texttt{Assume}\ \alpha, \beta : \texttt{Pruned}(A),$

$\texttt{Assume}\ [3] : F(\alpha) = F(\beta),$

$\Big[(\alpha, \beta)\Big] := \texttt{E}F\texttt{EPruned}(A, \alpha \,\&\, \beta) : \alpha = \beta;$

$\leadsto [3] := \texttt{IInjective} : \texttt{Injective}(F, \texttt{Pruned}(A), \texttt{Closed}(A^{\mathbb{N}})),$

$[*] := \texttt{IBijective}[2][3] : \texttt{Bijective}(F, \texttt{Pruned}(A), \texttt{Closed}(A^{\mathbb{N}}));$

$\square$

$$\texttt{rootedTree} :: \prod_{A\in\mathsf{SET}} \texttt{Tree}(A) \to A^* \to \texttt{Tree}(A)$$

$$\texttt{rootedTree}\,(T,s) = T_s := \{t \in A^* : st \in T\}$$

$$\texttt{filteredTree} :: \prod_{A\in\mathsf{SET}} \texttt{Tree}(A) \to A^* \to \texttt{Tree}(A)$$

$$\texttt{filteredTree}\,(T,s) = T_{[s]} := \{t \in A^* : \neg(t \perp s)\}$$

$$\texttt{finiteSequencesAsPoset} :: \prod_{A\in\mathsf{SET}} \mathsf{POSET}$$

$$\texttt{finiteSequencesAsPoset}\,() = A^* := \Big(A^*, \subset\Big)$$

$$\texttt{residualBody} :: \prod_{A,B\in\mathsf{SET}} \prod T : \texttt{Tree}(A)\,.\, \prod S : \texttt{Tree}(B)\,.\, \mathsf{POSET}(T,S) \to ?[T]$$

$$\texttt{residualBody}\,(f) = D(f) := \Big\{x \in [T] : \lim_{n\to\infty} \operatorname{len} f(x_{|[1,\dots,n]}) = \infty\Big\}$$

$$\texttt{bodyPushforward} :: \prod_{A,B\in\mathsf{SET}} \prod T : \texttt{Tree}(A)\,.\, \prod S : \texttt{Tree}(B)\,.\, \prod f \in \mathsf{POSET}(T,S)\,.\, D(f) \to [S]$$

$$\texttt{bodyPushforward}\,(x) = f^*(x) := \bigcup_{n=1}^{\infty} f(x_{|[1,\dots,n]})$$

$$\texttt{ProperTreeMorphism} :: \prod_{A,B\in\mathsf{SET}} \prod T : \texttt{Tree}(A) \prod S : \texttt{Tree}(B) ? \mathsf{POSET}(A,B)$$

$$f : \texttt{ProperTreeMorphism} \iff D(f) = [T]$$

$\texttt{ResidualBodyIsGDelta} :: \forall A,B \in \mathsf{SET}\,.\, \forall T : \texttt{Tree}(A)\,.\, \forall S : \texttt{Tree}(B)\,.\, \forall f \in \mathsf{POSET}(T,S)\,.\, D(f) \in G_\delta[T]$

$\texttt{Proof} =$

$U := \Lambda m \in \mathbb{N}\,.\, \Big\{x \in [T] : \exists n \in \mathbb{N}\,.\, \operatorname{len} f(x_{|[1,\dots,n]}) \geq m\Big\} : \mathbb{N} \to ?[T],$

$\texttt{Assume}\ m \in \mathbb{N},$

$\texttt{Assume}\ u \in U_m,$

$\big(n,[1]\big) := \mathtt{E}U_m(u) : \sum n \in \mathbb{N}\,.\, \operatorname{len} f(x_{|[1,\dots,n]}) \geq m,$

$s := x_{|[1,\dots,n]} : A^n,$

$[m.*] := \mathtt{E}\mathsf{POSET}(T,S)(f)[1]\mathtt{I}N : N_s \cap [T] \subset U_m;$

$\rightsquigarrow [1] := \texttt{OpenByOpenCoverI}(\forall) : \forall m \in \mathbb{N}\,.\, U_m \in \mathcal{T}[T],$

$[2] := \mathtt{E}D(f)\mathtt{E}\lim \texttt{SetBuilderUniversalI}U_m : D(f) = \Big\{x \in [T] : \lim_{n\to\infty} \operatorname{len} f(x_{|[1,\dots,n]}) = \infty\Big\} =$

$$= \Big\{x \in [T] : \forall m \in \mathbb{N}\,.\, \exists n \in \mathbb{N} : f(x_{|[1,\dots,n]}) \geq m\Big\} = \bigcap_{m=1}^{\infty} \Big\{x \in [T] : \exists n \in \mathbb{N} : f(x_{|[1,\dots,n]}) \geq m\Big\} = \bigcap_{m=1}^{\infty} U_m,$$

$[*] := \mathtt{I}G_\delta[2] : D(f) \in G_\delta[T];$

$\square$

$\texttt{BodyPushforwardIsContinuous} :: \forall A, B \in \mathsf{SET} \,.\, \forall T : \texttt{Tree}(A) \,.\, \forall S : \texttt{Tree}(B) \,.\, \forall f \in \mathsf{POSET}(T, S) \,.$

$\quad . \, f^* \in \mathsf{TOP}\Big(D(f), [S]\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InversePushforwardTheorem} :: \forall A, B \in \mathsf{SET} \,.\, \forall T : \texttt{Tree}(A) \,.\, \forall S : \texttt{Tree}(B) \,.\, \forall D \in G_\delta[T] \,.$

$\quad . \, \forall \varphi \in \mathsf{TOP}(D, [S]) \,.\, \exists! f \in \mathsf{POSET}(T, S) : D = D(f) \ \& \ \varphi = f^*$

$\texttt{Proof} \ =$

$$\Big(U, [1]\Big) := \mathtt{E}G_\delta[T](D) : \sum U : \mathbb{Z}_+ \downarrow \mathcal{T}[T] \,.\, U_0 = [T] \ \& \ \bigcap_{n=1}^{\infty} U_n = D,$$

$$k := \Lambda t \in T \,.\, \min\Big(\operatorname{len}(s), \max\{k \in \mathbb{Z}_+ : N_t \cap [T] \subset U_k\}\Big) : T \to \mathbb{Z}_+,$$

$f := \Lambda t \in T \,.\, \texttt{if } N_t \cap D \neq \emptyset \texttt{ then } \max\{u \in S \,.\, \operatorname{len}(u) \leq k(s) \ \& \ \varphi(N_t \cap D) \subset N_u\}$

$\quad \texttt{else } f(t_{|[1,\dots,m]}) \quad \texttt{where} \quad m = \max\{m \in [0, \dots, \operatorname{len} t] : N_{t_{|[1,\dots,m]}} \cap D \neq \emptyset\} : T \to S,$

$\texttt{Assume } t, t' \in T,$

$\texttt{Assume } [2] : t \subset t',$

$\texttt{Assume } [3] : \varphi(N_{t'} \cap D) \neq \emptyset,$

$[4] := \texttt{NonemptyImage}[3] : N_{t'} \cap D \neq \emptyset,$

$[5] := \mathtt{E}N[3] : N_{t'} \subset N_t,$

$[6] := \texttt{MonotonicIntersect}[3][5] : N_t \cap D\emptyset,$

$[7] := \mathtt{E}U[4](\operatorname{len}(t')) : N_{t'} \cap U_{\operatorname{len} t'} \neq \emptyset,$

$[8] := \mathtt{E}U[6](\operatorname{len}(t)) : N_t \cap U_{\operatorname{len} t} \neq \emptyset,$

$[9] := \mathtt{I}k(t')[7] : k(t') = \operatorname{len}(t'),$

$[10] := \mathtt{I}k(t')[8] : k(t) = \operatorname{len}(t),$

$[11] := [9][2][11] : k(t) \leq k(t'),$

$[3.*] := \mathtt{E}f(t)\mathtt{E}f(t')[5][11]\mathtt{I}f(t)\mathtt{I}f(t') : f(t) \subset f(t');$

$\rightsquigarrow [3] := \mathtt{I}(\Rightarrow) : \varphi(N_{t'} \cap D) \neq \emptyset \Rightarrow f(t) \subset f(t'),$

$\texttt{Assume } [4] : \varphi(N_{t'} \cap D) = \emptyset,$

$[4.*] := \mathtt{E}f\texttt{EifElseThen}[4][3] : f(t) \subset f(t');$

$\rightsquigarrow [4] := \mathtt{I}(\Rightarrow) : \varphi(N_{t'} \cap D) = \emptyset \Rightarrow f(t) \subset f(t'),$

$\Big[(t, t').*\Big] := \mathtt{E}(|)\texttt{LEM}\Big(\varphi(N_{t'} \cap D) = \emptyset\Big)[4][3] : f(t) \subset f(t');$

$\rightsquigarrow [2] := \mathtt{I}\mathsf{POSET} : f \in \mathsf{POSET}(T, S),$

$\texttt{Assume } x \in D(f),$

$$[3] := \mathtt{E}D(f)(x) : \lim_{n\to\infty} f\Big(x_{|[1,\dots,n]}\Big) = \infty,$$

$$[4] := \mathtt{E}f[3] : \Big(\forall n \in \mathbb{N} \,.\, \varphi(N_{x_{|[1,\dots,n]}} \cap D) \neq \emptyset\Big) \ \&$$

$$\quad \& \ \Big(\forall m \in \mathbb{N} \,.\, \exists b \in B^m : \exists n \in \mathbb{N} : N_{x_{|[1,\dots,n]}} \cap [T] \subset U_m \ \& \ \varphi(N_{x_{|[1,\dots,n]}} \cap D) \subset N_b\Big),$$

$[5] := \texttt{EmptyImage}[4.1] : \forall n \in \mathbb{N} \,.\, N_{x_{|[1,\dots,n]}} \cap D \neq \emptyset,$

$$\Big(y, [6]\Big) := \mathtt{E}N[5][4.2]\texttt{LimByNeighbourhoodBase} : \sum y : \prod_{i=1}^{n} U_n \,.\, x = \lim_{n\to\infty} y_n,$$

$$[x.*] := \mathtt{E}U[1]\mathtt{E}y[6] : x = \lim_{n\to\infty} y_n \in D;$$

$\rightsquigarrow [3] := \texttt{ISubset} : D(f) \subset D,$

Assume $x \in D$,

$[4] := \texttt{ENeighborhoodBase}(N)\texttt{E}x : \forall n \in \mathbb{N} \,.\, N_{x_{|[1,\dots,n]}} \cap D \neq \emptyset$,

$[5] := \varphi[4] : \forall n \in \mathbb{N} \,.\, \varphi(N_{x_{|[1,\dots,n]}} \cap D) \neq \emptyset$,

$[6] := [4][1] : \forall n \in \mathbb{N} \,.\, N_{x_{|[1,\dots,n]}} \cap U_n \neq \emptyset$,

$[7] := \texttt{I}k[6] : \forall n \in \mathbb{N} \,.\, k(x_{|[1,\dots,n]}) = n$,

$[8] := \mathsf{ETOP}\Big([T],[S]\Big)(\varphi)\texttt{ET1}[S] : \bigcap_{i=1}^{n} \varphi(N_{x_{|[1,\dots,n]}} \cap D) = \{\varphi(x)\}$,

$[9] := \texttt{MonotonicIntersectSubset}[8] : \forall m \in \mathbb{N} \,.\, \exists n \in \mathbb{N} : \varphi(N_{x_{|[1,\dots,n]}} \cap D) \subset N_{\varphi(x)_{|[1,\dots,m]}}$,

$[10] := \texttt{I}f[7][9] : \lim_{n\to\infty} \operatorname{len} f(x_{|[1,\dots,n]}) = \infty$,

$[x.*] := [10]\texttt{I}D(f) : x \in D(f)$;

$\leadsto [4] := \texttt{ISubset} : D \subset D(f)$,

$[5] := \texttt{ISetEq}[3][4] : D = D(f)$,

Assume $x \in D$,

$[6] := \texttt{E}x \dots : \forall n \in \mathbb{N} \,.\, \varphi(N_{x_{|[1,\dots,n]}} \cap D) \neq \emptyset$,

Assume $m \in \mathbb{N}$,

$\Big(n, [7]\Big) := \mathsf{ETOP}\Big([T],[S],\varphi\Big)(N_{\varphi(x)_{|[1,\dots,m]}})\texttt{EBase}(N)(x) : \sum n \in \mathbb{N} \,.\, \varphi(N_{x_{|[1,\dots,n]}} \cap D) \subset N_{\varphi(x)_{|[1,\dots,m]}}$,

$[m.*] := \texttt{I}f[7] : \varphi(x)_{|[1,\dots,m]} \subset f(x_{|[1,\dots,n]})$,

$\leadsto [7] := \texttt{I}\forall : \forall m \in \mathbb{N} \,.\, \varphi(x)_{|[1,\dots,m]} \subset f(x_{|[1,\dots,n]})$,

$[x.*] := \texttt{IUnion}[7]\texttt{SetFunctionUnion}(\varphi(x))\texttt{I}f^* : \varphi(x) = \bigcup_{n=1}^{\infty} f(x_{|[1,\dots,n]}) = f^*$;

$\leadsto [*] := \texttt{I}(\to,=) : \varphi = f^*$;

$\square$

$\texttt{LipschitzTreeMorphism} :: \prod_{A,B\in\mathsf{SET}} \prod T : \texttt{Tree}(A) \,.\, \prod S : \texttt{Tree}(B) \,.\, ?\mathsf{POSET}(T,S)$

$f : \texttt{LipschitzTreeMorphism} \iff \forall t \in T \,.\, \operatorname{len} f(t) = \operatorname{len} t$

$\texttt{LipschitzTreeMorphismPushforwardIsLipschitzMap} ::$

$:: \forall A,B \in \mathsf{SET} \,.\, \forall T : \texttt{Tree}(A) \,.\, \forall S : \texttt{Tree}(B) \,.\, \forall f : \texttt{LipschitzTreeMorphism}(T,S) \,.$

$.\; f^* \in \operatorname{Lip}\Big(([T],d),([S],d)\Big)$

Proof $=$

Assume $x, y \in [T]$,

Assume $[1] : x \neq y$,

$\Big(n, [2]\Big) := \texttt{E}(\to,\#)[1]\texttt{EWellFounded}(\mathbb{N}) : \sum n \in \mathbb{N} \,.\, x_n \neq y_n \,\&\, x_{|[1,\dots,n-1]} = y_{|[1,\dots,n-1]}$,

$[3] := \texttt{I}d[2] : d(x,y) = 2^{-n}$,

$[4] := \texttt{E}(\to,=)[2.1]\texttt{ELipschitzTreeMorphism}(f)\texttt{I}f^* : f(x_{|[1,\dots,n-1]}) = f^*(x)_{|[1,\dots,n-1]} = f^*(x)_{|[1,\dots,n-1]} = f(x_{|[1,\dots,n-}$

$\Big[(x,y).*\Big] := \texttt{I}d[4][2] : d\Big(f^*(x), f^*(y)\Big) \leq 2^{-n} = d(x,y)$;

$\leadsto * := \texttt{ILip} : f^* \in \operatorname{Lip}([T],[S])$;

$\square$

$\texttt{InBairlikeSpaceAllClosedAreRetracts} ::$

$:: \forall A \in \mathsf{SET} \,.\, \forall H, K : \texttt{Closed} \,\&\, \texttt{NonEmpty}(A^{\mathbb{N}}) \,.\, H \subset K \,.\, \Rightarrow \texttt{Retract}(K, H)$

$\texttt{Proof} \ =$

$(\leq) := \texttt{ZermelosTHM}(H) : \texttt{WellOrdering}(H),$

$f := \Lambda k \in K \,.\, \texttt{if}\ k \in H\ \texttt{then}\ k\ \texttt{else}$

$\texttt{E}\min\left\{h \in H : h_{|[1,\dots,n]} = k_{|[1,\dots,n]} \middle| n = \max\{n \in \mathbb{N} : \exists h \in H : h_{|[1,\dots,n]} = k_{|[1,\dots,n]}\}\right\} : K \to H,$

$\texttt{Assume}\ h \in H,$

$\texttt{Assume}\ n \in H,$

$\texttt{Assume}\ k \in f^{-1}(N_{h_{|[1,\dots,n]}} \cap H),$

$[1] := \texttt{Epreimage}k : f(k) \in N_{h_{|[1,\dots,n]}} \cap H,$

$[2] := \texttt{E}N_{h_{|[1,\dots,n]}}[1] : f(k)_{|[1,\dots,n]} = h_{|[1,\dots,n]},$

$\texttt{Assume}\ [3] : k_{|[1,\dots,n]} = h_{|[1,\dots,n]},$

$[3.*] := \texttt{E}f[3] : N_{k_{|[1,\dots,n]}} \cap K \subset f^{-1}(N_{h_{|[1,\dots,n]}});$

$\leadsto [3] := \texttt{I}(\Rightarrow) : k_{|[1,\dots,n]} = h_{|[1,\dots,n]} \Rightarrow N_{k_{|[1,\dots,n]}} \cap K \subset f^{-1}(N_{h_{|[1,\dots,n]} \cap H}),$

$\texttt{Assume}\ [4] : k_{|[1,\dots,n]} \neq h_{|[1,\dots,n]},$

$[5] := \texttt{E}f[4][2] : f(N_{k_{|[1,\dots,n]}} \cap K) = \{h\},$

$[4.*] := \texttt{IPreimage}[5] : N_{k_{|[1,\dots,n]}} \cap K \subset f^{-1}(N_{h_{|[1,\dots,n]}} \cap H);$

$\leadsto [4] := \texttt{I}(\Rightarrow) : k_{|[1,\dots,n]} \neq h_{|[1,\dots,n]} \Rightarrow N_{k_{|[1,\dots,n]}} \cap K \subset f^{-1}(N_{h_{|[1,\dots,n]}} \cap H),$

$[k.*] := \texttt{E}(|)\texttt{LEM}[3][4] : N_{k_{|[1,\dots,n]}} \cap K \subset f^{-1}(N_{h_{|[1,\dots,n]}} \cap H);$

$\leadsto [h.*] := \texttt{OpenByOpenCover} : f^{-1}(N_{h_{|[1,\dots,n]}} \cap H) \in \mathcal{T}(K);$

$\leadsto [1] := \texttt{I}\mathsf{TOP} : f \in \mathsf{TOP}(K, H),$

$[2] := \texttt{E}f\texttt{IRetraction} : \texttt{Retraction}(K, H, f),$

$[*] := \texttt{IRetract} : \texttt{Retract}(K, H);$

$\square$

### 1.1.3 Category

$$\texttt{treeCategory} :: \mathsf{CAT}$$

$$\texttt{treeCategory}\,() = \mathsf{TREE} := \left(\sum_{X \in \mathsf{SET}} \texttt{Tree}(X), \texttt{ProperTreeMorphism}, \circ, \mathrm{id}\right)$$

$$\texttt{FullTreeFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{TREE})$$

$$\texttt{FullTreeFunctor}\,(X) = \mathsf{FS}(X) := \left(X, X^*\right)$$

$$\texttt{FullTreeFunctor}\,(X, Y, f) = \mathsf{FS}_{X,Y}(f) := \Lambda\omega \in \mathsf{FS}(X)\ .\ \texttt{if}\ \omega = \emptyset\ \texttt{then}\ \emptyset$$
$$\texttt{else}\ \mathsf{FS}_{X,Y}(f)(\omega_{|[1,\dots,\mathrm{len}(\omega)-1]})f(\omega_{\mathrm{len}(\omega)})$$

$$\texttt{CroneFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{TREE})$$

$$\texttt{CroneFunctor}\,(X) = \mathsf{CRONE}(X) := \left(X, X \cup \{\emptyset\}\right)$$

$$\texttt{CroneFunctor}\,(X, Y, f) = \mathsf{CRONE}_{X,Y}(f) := \Lambda\omega \in \mathsf{CRONE}(X)\ .\ \texttt{if}\ \omega = \emptyset\ \texttt{then}\ \emptyset\ \texttt{else}\ f(\omega)$$

$$\texttt{TreeEmbedding} :: \texttt{ReflexiveEmbedding}(\mathsf{TREE}, \mathsf{POSET})$$
$$\texttt{TreeEmbedding}\,(A, T) = (A, T) := T$$
$$\texttt{TreeEmbedding}\,(X, Y, f) = (X, Y, f) := f$$

$$\texttt{BodyFunctor} :: \texttt{Covariant}\ \&\ \texttt{Full}(\mathsf{TREE}, \mathsf{MS})$$
$$\texttt{BodyFunctor}\,(A, T) = \mathsf{BODY}(A, T) := [T]$$
$$\texttt{BodyFunctor}\,(X, Y, f) = \mathsf{BODY}_{X,Y}(f) := f^*$$

$$\texttt{treeProduct} :: \prod_{\mathcal{I} \in \mathsf{SET}} (\mathcal{I} \to \mathsf{TREE}) \to \mathsf{TREE}$$

$$\texttt{treeProduct}\,((X, T)) = \prod_{i \in \mathcal{I}} (X_i, T_i) := \left(\prod_{i \in \mathcal{I}} X_i, \left\{x \in \left(\prod_{i \in \mathcal{I}} X_i\right)^* \middle| \forall i \in\ .\ \pi_i^*(x) \in X_i{}^*\right\}\right)$$

$$\texttt{PrunedTreeProduct} :: \forall \mathcal{I} \in \mathsf{SET}\ .\ \forall X : \mathcal{I} \to \mathsf{SET}\ .\ \forall T : \prod_{i \in \mathcal{I}} \texttt{Pruned}(X_i)\ .\ \texttt{Pruned}\left(\prod_{i \in \mathcal{I}} X_i, \prod_{i \in \mathcal{I}} T_i\right)$$

$\texttt{Proof}$ =

...

□

$$\texttt{ProductBody} :: \forall \mathcal{I} : \texttt{Finite}\ .\ \forall X : \mathcal{I} \to \mathsf{SET}\ .\ \forall T : \prod_{i \in \mathcal{I}} \texttt{Tree}(X_i)\ .\ \left[\prod_{i \in \mathcal{I}} T_i\right] \cong_{\mathsf{TOP}} \prod_{i \in \mathcal{I}} [T_i]$$

$\texttt{Proof}$ =

...

□

$$\texttt{treeSection} :: \prod A, B \in \mathsf{SET}\ .\ \texttt{Tree}(A \times B) \to A^{\mathbb{N}} \to \texttt{Tree}(B)$$

$$\texttt{treeSection}\,(T, a) = T(a) := \left\{b \in b^* \middle| \left(a_{|[1,\dots,\mathrm{len}\,b]}, b\right) \in T\right\}$$

### 1.1.4 Well Foundness

$$\texttt{IllFounded} :: \prod_{A \in \mathsf{SET}} ?\texttt{Tree}(A)$$

$$T : \texttt{IllFounded} \iff [T] \neq \emptyset$$

$$\texttt{WellFounded} :: \prod_{A \in \mathsf{SET}} ?\texttt{Tree}(A)$$

$$T : \texttt{WellFounded} \iff [T] = \emptyset$$

$$\texttt{leftmostBranch} :: \prod A \in \mathsf{ORD} \;.\; \texttt{IllFounded}(A) \to A^{\mathbb{N}}$$

$$\texttt{leftmostBranch}\,(T, n) = (\operatorname{lb} T)_n := \min\{a \in A : [T_{\operatorname{lb} T_{|[1,\dots,n-1]} a}] \neq \emptyset\}$$

$$\texttt{branchRank} :: \prod_{A \in \mathsf{SET}} \prod T : \texttt{WellFounded} \;.\; T \to \mathbb{Z}_+ \cup \{\infty\}$$

$$\texttt{branchRank}\,(t) = \operatorname{rank} t := \texttt{if } \{a \in A : ta \in T\} = \emptyset \texttt{ then } 0 \texttt{ else } \sup\{1 + \operatorname{rank}(ta) | a \in A : ta \in T\}$$

$$\texttt{treeRank} :: \prod_{A \in \mathsf{SET}} \texttt{WellFounded} \to \mathbb{Z}_+ \cup \{\infty\}$$

$$\texttt{treeRank}\,(\emptyset) = \operatorname{rank} \emptyset := 0$$

$$\texttt{treeRank}\,(T) = \operatorname{rank} T := 1 + \operatorname{rank}_T \emptyset$$

$\texttt{TreeRankAndWellFoundness} :: \forall A, B \in \mathsf{SET} \forall T : \texttt{WellFounded}(A) \;.\; \forall S \in \texttt{Tree}(B) \;.$

$\;.\; \Big(\texttt{WellFounded}(B, S) \;\&\; \operatorname{rank} S \leq \operatorname{rank} T\Big) \iff \exists \texttt{StrictlyMonotonic}(S, T)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{WellFounded}(B, S),$

$\texttt{Assume}\ [2] : \operatorname{rank} S \leq \operatorname{rank} T,$

$\texttt{Assume}\ [0] : S \neq \emptyset,$

$[00] := \texttt{E}[2] \operatorname{rank} S[0] : T \neq \emptyset,$

$f(\emptyset) := \emptyset_{\mathbb{N}\times A} \in T,$

$[4.0] := [2]\texttt{E} f(\emptyset) : \operatorname{rank}_T f(\emptyset) = (\operatorname{rank} T) - 1 \geq (\operatorname{rank} S) - 1 = \operatorname{rank}_S \emptyset,$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ s \in S,$

$\texttt{Assume}\ [3] : \operatorname{len} s = n,$

$[5] := \texttt{E}\Big[4.(n-1)\Big](s_{|[1,\ldots,n-1]}) : \operatorname{rank} f(s_{|[1,\ldots,n-1]}) \geq \operatorname{rank} s_{|[1,\ldots,n-1]},$

$\Big(a, [6]\Big) := \texttt{E} \operatorname{rank}[5] : \sum a \in A \;.\; \operatorname{rank} f(s_{|[1,\ldots,n-1]}) = 1 + \operatorname{rank} f(s_{|[1,\ldots,n-1]})a,$

$f(s) := f(s_{|[1,\ldots,n-1]}) \in T,$

$[7] := \texttt{E} \operatorname{rank} s : \operatorname{rank} s_{|[1,\ldots,n+1]} \geq 1 + \operatorname{rank} s,$

$4.n := [6][5][7] : \operatorname{rank} f(s) = (\operatorname{rank} f(s_{|[1,\ldots,n-1]})) - 1 \geq (\operatorname{rank} s_{|[1,\ldots,n-1]}) - 1 \geq \operatorname{rank} s,$

$[n.*] := \texttt{E} f(s) : f(s) > f(s_{|[1,\ldots,n-1]});$

$\leadsto \Big(f, [4]\Big) := \texttt{IStrictlyMonotonic} : \sum f : \texttt{StrictlyMonotonic}(S, T) \;.\; \forall n \in \mathbb{N} \;.\; \forall s \in S \;.$

$\;.\; \operatorname{len} s = n \Rightarrow \operatorname{rank} f(s) \geq \operatorname{rank} s;$

$\leadsto [1] := \texttt{I}(\Rightarrow) : \texttt{WellFounded}(S) \;\&\; \operatorname{rank} S \leq \operatorname{rank} T \Rightarrow \exists \texttt{StrictlyMonotonic}(S, T),$

$\texttt{Assume}\ f : \texttt{StrictlyMonotonic}(S, T),$

$\texttt{Assume}\ [2] : \texttt{IllFounded}(S),$

$b := \texttt{EIllFounded}(S) \in [S],$

$[3] := \texttt{EStrictlyMonotonic}(f) : \lim_{n\to\infty} f(b_{|[1,\ldots,n]}) \in [T],$

$[2.*] := \texttt{EWellFounded}(T)[3] : \bot;$

$\leadsto [2] := \texttt{IWellFounded}(S)[4] : \texttt{WellFounded}(S),$

$[3] := \Lambda s \in S \;.\; \texttt{EStrictlyMonotonic}(S, T, f)\texttt{I} \operatorname{rank} s : \forall s \in S \;.\; \operatorname{rank} f(s) \geq \operatorname{rank} s,$

$[2.*] := \texttt{E} \operatorname{rank} S[3]\texttt{E} \sup \texttt{I} \operatorname{rank} T :$

$: \operatorname{rank} S = \sup\{1 + \operatorname{rank} s | s \in S\} \leq \sup\{1 + \operatorname{rank} f(s) | s \in S\} \leq \sup\{1 + \operatorname{rank} t | t \in T\} = \operatorname{rank} T;$

$\leadsto [2] := \texttt{I} \Rightarrow : \exists \texttt{StrictlyMonotonic}(S, T) \Rightarrow \texttt{WellFounded}(S) \;\&\; \operatorname{rank} S \leq \operatorname{rank} T,$

$* := \texttt{I}(\iff)[1][2] : \texttt{WellFounded}(S) \;\&\; \operatorname{rank} S \leq \operatorname{rank} T \iff \exists \texttt{StrictlyMonotonic}(S, T);$

$\square$

$\texttt{orderTree} :: \prod A \in \mathsf{SET} \;.\; \texttt{Order}(A) \to \texttt{Tree}(A)$

$\texttt{orderTree}\,(\leq) := \Big\{a : [1, \ldots, n] \to A \Big| n \in \mathbb{Z}_+, \forall i \in [1, \ldots, n-1] \;.\; a_{i+1} < a_i\Big\}$

$\texttt{OrderTreeWellFoundness} :: \forall A \in \mathsf{SET} \;.\; \forall (\leq) : \texttt{Order}(A) \;.$

$\;.\; \texttt{WellFounded}(A, \leq) \iff \texttt{WellFounded}\Big(\texttt{orderTree}(A, \leq)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{wellFoundedPart} :: \prod_{A \in \mathsf{SET}} \texttt{Tree}(A) \to ?A$

$\texttt{wellFoundedPart}\,(T) = \mathrm{WF}_T := \{t \in T : \texttt{WellFounded}(T_t)\}$

$\texttt{rankOfWellFoundedPart} :: \prod_{A \in \mathsf{SET}} \prod T : \texttt{Tree}(A)\ .\ \mathrm{WF}_T \to \mathbb{Z}_+ \cup \{\omega\}$

$\texttt{rankOfWellFoundedPart}\,(t) = \operatorname{rank} t := \texttt{if}\ \{a \in A : ta \in T\} = \emptyset\ \texttt{then}\ 0\ \texttt{else}$
$\quad \sup\{1 + \operatorname{rank}(ta) | a \in A : ta \in T\}$

$\texttt{rankOfIllFoundedBranch} :: \prod_{A \in \mathsf{SET}} \prod T : \texttt{Tree}(A)\ .\ T \to \mathbb{Z}_+ \cup \{\omega\} \cup \{\alpha\}$

$\texttt{rankOfIllFoundedBranch}\,(t) = \operatorname{rank} t := \texttt{if}\ t \in \texttt{WellFounded}_T\ \texttt{then}\ \operatorname{rank} t\ \texttt{else}\ \infty$
$\quad \texttt{where}\quad \infty = \min\{a \in \mathsf{ORD} : |a| > \infty\ \&\ |a| > |A|\}$

$\texttt{KleeneBrouwerOrder} :: \prod A : \texttt{Toset}\ .\ \texttt{TotalOrder}(A^*)$

$(s,t) : \texttt{KleeneBrouwerOrder} \iff s \le_{\mathrm{KB}} t \iff t \subset s \Big|$
$\quad \exists i \in \Big[1, \dots, \min\Big(\operatorname{len}(s), \operatorname{len}(t)\Big)\Big] : s_i < t_i\ \&\ \forall j \in [1, \dots, i-1]\ .\ s_j = t_j$

$\texttt{KleeneBrouweTHM} :: \forall A : \texttt{WellOrdered}\ .\ \forall T : \texttt{Tree}(A)\ .\ \texttt{WellFounded}(T) \iff \texttt{WellOrdered}(T, \le_{\mathrm{KB}})$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : \texttt{WellFounded}(T),$

$\texttt{Assume}\ X : ?T,$

$\texttt{Assume}\ [2] : \min_{\mathrm{KB}} X = \emptyset,$

$\Big(x, [3]\Big) := \mathtt{E}\min[2] : \sum x : \mathbb{N} \to X : \forall n, m \in \mathbb{N}\ .\ n > m \Rightarrow x_n <_{\mathrm{KB}} x_m,$

$\Big(m, i, [4]\Big) := \texttt{EWellFounded}(T)[3] : \sum m : \mathbb{N} \to \mathbb{N}\ .\ \prod_{n=1}^{\infty} i_n \in \Big[1, \dots \min(\operatorname{len} x_{m_n}, \operatorname{len} x_{m_n+1})\Big]\ .$
$\quad .\ \forall n \in \mathbb{N}\ .\ x_{m_n, i_n} > x_{m_n+1, i_n}\ \&\ \forall j \in [1, \dots, i_n - 1]\ .\ x_{m_n, j} = x_{m_n+1, j},$

$[5] := \texttt{EWellOrdered}(A)[4] : \lim_{n \to \infty} i_n = \infty,$

$[6] := [5][3] : [T] \ne \emptyset,$

$[*] := \texttt{EWellFounded}(T) : \bot;$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow : \texttt{WellFounded}(T) \Rightarrow \texttt{WellOrdered}(T, \le_{\mathrm{KB}}),$

$\texttt{Assume}\ [2] : \texttt{WellOrdered}(T, \le_{\mathrm{KB}}),$

$\texttt{Assume}\ x \in [T],$

$[3] := \mathtt{E}[T](x)\mathtt{I}\min_{\mathrm{KL}} : \min_{\mathrm{KL}}\{x_{|[1,\dots,n]} | n \in \mathbb{N}\} = \emptyset,$

$[x.*] := [2][3] : \bot;$

$\rightsquigarrow [3] := \texttt{IEmptyset} : [T] = \emptyset,$

$[2.*] := \texttt{IWellFounded}(T)[3] : \texttt{WellFounded}(T);$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow : \texttt{WellOrdered}(T, \le_{\mathrm{KB}}) \Rightarrow \texttt{WellFounded}(T),$

$[*] := \mathtt{I}(\iff)[1][2] : \texttt{WellFounded}(T) \iff \texttt{WellOrdered}(T, \le_{\mathrm{KB}});$

$\square$

## 1.2 Polish Topology

### 1.2.1 Definition and examples

$\texttt{CompletelyMetrizable} :: ?\mathsf{TOP}$
$X : \texttt{CompletelyMetrizable} \iff \exists d : \texttt{Metric}(X) : \texttt{Complete}(X, d)$

$\texttt{Polish} := \texttt{Separable} \;\&\; \texttt{CompletelyMetrizable} : \texttt{Type};$

$\texttt{RealSpacesArePolish} :: \forall n \in [1, \dots, \omega]_{\mathsf{ORD}} \,.\, \texttt{Polish}(\mathbb{R}^n)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{ComplexSpacesArePolish} :: \forall n \in [1, \dots, \omega]_{\mathsf{ORD}} \,.\, \texttt{Polish}(\mathbb{C}^n)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{IntervalsArePolish} :: \forall n \in [1, \dots, \omega]_{\mathsf{ORD}} \,.\, \texttt{Polish}(I^n)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{TorusIsPolish} :: \forall n \in [1, \dots, \omega]_{\mathsf{ORD}} \,.\, \texttt{Polish}(\mathbb{T}^n)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{DiscreteCountableIsPolish} :: \forall A : \texttt{Countable} \,.\, \texttt{Polish}(A)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{DiscreteInfiniteProductIsPolish} :: \forall A : \texttt{Countable} \,.\, \texttt{Polish}(A^{\mathbb{N}})$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{spaceOfCantor} :: \texttt{Polish}$
$\texttt{spaceOfCantor}\,() = \mathcal{C} := 2^{\mathbb{N}}$

$\texttt{spaceOfBair} :: \texttt{Polish}$
$\texttt{spaceOfBair}\,() = \mathcal{B} := \mathbb{N}^{\mathbb{N}}$

### 1.2.2 Extension of continuous functions

$$\texttt{oscilationAt} :: \prod_{X \in \mathsf{TOP}} \prod_{Y \in \mathsf{MS}} \prod_{A \subset X} (A \to Y) \to X \to \hat{\mathbb{R}}$$

$$\texttt{oscilationAt}\,(f, x) = \operatorname{osc}_f(x) := \inf \left\{ \operatorname{diam} f(U \cap A) | U \in \mathcal{U}(x) \right\}$$

$$\texttt{ContinuityPoint} :: \prod X \in \mathsf{TOP} \,.\, \prod Y \in \mathsf{MS} \,.\, (X \to Y) \to ?X$$

$$x : \texttt{ContinuityPoint} \iff \Lambda f : X \to Y \,.\, x \in \mathcal{C}_f \iff \Lambda f : X \to Y \,.\, \operatorname{osc}_f(x) = 0$$

$\texttt{ContinuityPointsAreGDelta} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{MS} \,.\, \mathcal{C}_f \in G_\delta(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ClosedSetsAreGDelta} :: \forall X : \texttt{Metrizable} \,.\, \forall F : \texttt{Closed}(X) \,.\, F \in G_\delta(X)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{KuratowskiExtensionTHM} :: \forall X : \texttt{Metrizable} \,.\, \forall Y : \texttt{CompletelyMetrizable} \,.\, \forall A \subset X \,.$

$\quad .\, \forall A \xrightarrow{f} Y : \mathsf{TOP} \,.\, \exists G \in G_\delta(X) \,.\, A \subset G \subset \overline{A} \,\&\, \exists G \xrightarrow{F} Y : \mathsf{TOP} \,.\, F_{|A} = f$

$\texttt{Proof} =$

$G := \mathcal{C}_f \cap \overline{A} : G_\delta(X),$

$\texttt{Assume}\ x \in G,$

$[1] := \mathtt{E}\mathcal{C}_f(G)(x) : \forall a : \mathbb{N} \to A \,.\, \lim_{n \to \infty} a_n = x \Rightarrow \texttt{Cauchy}(Y, f(a)),$

$\left(a, [2]\right) := \mathtt{E}\overline{A}(G)(x) : \sum a : \mathbb{N} \to A \,.\, x = \lim_{n \to \infty} a_n,$

$F(x) := \lim_{n \to \infty} f(a_n) : Y;$

$\rightsquigarrow F := \mathtt{I}(\to) : G \to Y,$

$[1] := \mathtt{E}F\mathtt{E}\mathcal{C}_f : F \in \mathsf{TOP}(G, Y),$

$[*] := \mathtt{E}F : F_{|A} = f;$

$\square$

$\texttt{LavrentievTHM} :: \forall X, Y : \texttt{CompletelyMetrizable} \,.\, \forall A \subset X \,.\, \forall B \subset Y \,.\, \forall A \overset{f}{\leftrightarrow} B : \mathsf{TOP} \,.$

$\quad . \exists G \in G_\delta(X) : \exists H \in G_\delta(Y) : \exists G \overset{F}{\leftrightarrow} H : \mathsf{TOP} \,.\, A \subset G \,\&\, B \subset H \,\&\, F_{|A} = f$

$\texttt{Proof} \;=$

$\Big(A', F, [1]\Big) := \texttt{KuratowskiExtenstionTHM}(X, Y, A, f) : \sum A' \in G_\delta(X) \,.\, \sum A' \xrightarrow{F'} Y : \mathsf{TOP} \,.$

$\quad . F_{|A} = f \,\&\, A \subset A' \subset \overline{A},$

$\Big(B', F', [1]\Big) := \texttt{KuratowskiExtenstionTHM}(Y, X, B, f^{-1}) : \sum B' \in G_\delta(Y) \,.\, \sum B' \xrightarrow{F'} X : \mathsf{TOP} \,.$

$\quad . F'_{|B} = f^{-1} \,\&\, B \subset B' \subset \overline{B},$

$Z := F \cap \texttt{swap}\; F' :?(X \times Y),$

$[3] := \mathtt{E}Z[1][2] : f \subset Z \subset A' \times B',$

$G := \pi_X(Z) :?X,$

$H := \pi_Y(Z) :?Y,$

$[4] := \mathtt{E}G[3] : A \subset G \subset A',$

$[5] := \mathtt{E}H[3] : B \subset H \subset B',$

$[6] := \mathtt{E}\mathsf{TOP}(A', Y, F)\mathtt{E}\overline{A}\mathtt{E}\mathsf{TOP}(X, B', F') : \forall x \in G \,.\, F'(F(x)) = x,$

$[7] := \mathtt{E}\mathsf{TOP}(X, B', F)\mathtt{E}\overline{B}\mathtt{E}(X, B', F') : \forall y \in H \,.\, F(F'(y)) = y,$

$h := F_{|G} : G \to H,$

$[8] := \mathtt{E}h[7][6]\mathtt{E}F\mathtt{E}F' : G \overset{h}{\leftrightarrow} H : \mathsf{TOP},$

$[9] := \texttt{GDeltaPreimage}(F \times \mathrm{id}, B') : G = (F \times \mathrm{id})^{-1}(B') \in G_\delta(X),$

$[*] := \texttt{GDeltaPreimage}(\mathrm{id} \times F', A') : H = (F \times \mathrm{id})^{-1}(A') \in G_\delta(Y);$

$\square$

$\texttt{OneSetLavrentievTHM} :: \forall X : \texttt{CompletelyMetrizable} \,.\, \forall A \subset X \,.\, \forall A \overset{f}{\leftrightarrow} A : \mathsf{TOP} \,.$

$\quad . \exists G \in G_\delta(X) : \exists G \overset{F}{\leftrightarrow} G : \mathsf{TOP} \,.\, A \subset G \,\& \;\; \& \, F_{|A} = f$

$\texttt{Proof} \;=$

$(G, H, F, [1]) := \texttt{LavrentevTHM}\Big(X, X, A, A, f\Big) : \sum G, H \in G_\delta(X) \,.\, \sum F : G \xleftrightarrow{\mathsf{TOP}} H \,.\, A \subset G \,\&\, A \subset H,$

$G'_0 := G : G_\delta(X),$

$[2.0] := [1]\mathtt{E}G_1 : A \subset G_1,$

$\texttt{Assume}\; n : \mathbb{N},$

$G'_n := G'_{n-1} \cap F(G'_{n-1}) \cap F^{-1}(G'_{n-1}) \in G_\delta(X),$

$[2.n] := [2.n-1][1] : A \subset G'_n;$

$\rightsquigarrow \Big(G', [3]\Big) := \mathtt{I}\Big(\sum\Big) : \sum G' : \mathbb{Z}_+ \to G_\delta(X) \,.\, \forall n \in \mathbb{N} \,.\, A \subset G'_n,$

$H := \bigcup_{n=1}^{\infty} G'_n : G_\delta(X);$

$[4] := \texttt{IntersectionSubset}[2] : A \subset H,$

$\texttt{Assume}\; x \in H,$

$\Big([5]\Big) := \texttt{EIntersectionE}H\mathtt{E}x : \forall n \in \mathbb{N} \,.\, x \in G'_n,$

$[6] := \mathtt{E}G'_n[5] : \forall n \in \mathbb{N} \,.\, F(x) \in G'_n \,\&\, \exists y \in G'_{n-1} : x = F(y),$

$[x.*] := \mathtt{E}H\texttt{EIntersection}[6] : F(x) \in H \,\&\, \exists y \in H : x = F(y);$

$\rightsquigarrow [5] := \mathtt{I}(\forall) : \forall x \in H \,.\, F(x) \in H \,\&\, \exists y \in H : x = F(y),$

$\Big([6]\Big) := \texttt{Iimage}[5] : F(H) = H,$

$F' := F_{|H} : \mathrm{Aut}_{\mathsf{TOP}}(H),$

$\square$

### 1.2.3 Subsets of Polish spaces

$\texttt{CompleteSubsetIsGDelta} :: \forall X : \texttt{Metrizable} \,.\, \forall Y \subset X \,.\, \texttt{CompletelyMetizable}(Y) \Rightarrow Y \in G_\delta(X)$

$\texttt{Proof} \;=$

$[1] := \mathtt{E}\mathsf{CAT}(\mathsf{TOP}, Y) : \mathrm{id}_Y \in \mathrm{Aut}_{\mathsf{TOP}}(Y),$

$\Big(G, F, [2]\Big) := \texttt{KuraroveskyExtensionTHM}(X, Y, Y, \mathrm{id}) :$

$\quad : \sum G \in G_\delta(X) \,.\, \sum F \in \mathrm{Aut}_{\mathsf{TOP}}(G) \,.\, F_{|Y} = \mathrm{id} \;\&\; Y \subset G \subset \overline{Y},$

$[3] := \texttt{CompleteDenseExtension}(G, Y, F)[2] : F = \mathrm{id}_G,$

$[*] := \mathtt{E}\,\mathrm{id}[3] : G = Y;$

$\square$

$\texttt{GDeltaSubsetIsComplete} :: \forall X : \texttt{CompletelyMetrizable} \,.\, \forall Y \in G_\delta(X) \,.\, \texttt{CompletelyMetrizable}(Y)$

$\texttt{Proof} \;=$

$(U, [1]) := \mathtt{E}G_\delta(X) : \sum U : \mathbb{N} \to \mathcal{T}(X) \,.\, Y = \bigcap_{n=1}^{\infty} U_n,$

$F := U^{\complement} : \mathbb{N} \to \texttt{Closed}(X),$

$\Big(d, [2]\Big) := \texttt{EMetrizable}(X) : \sum d : \texttt{Metric}(X) \,.\, (X, d) \cong_{\mathsf{TOP}} X,$

$d' := \Lambda x, y \in Y \,.\, d(x, y) + \sum_{n=0}^{\infty} \min\left\{2^{-n}, \left|\frac{1}{d(x, F_n)} - \frac{1}{d(y, F_n)}\right|\right\} : \texttt{Metric}(Y),$

$[3] := \mathtt{E}d' : (Y, d') \cong_{\mathsf{TOP}} Y,$

$\texttt{Assume}\; y : \texttt{Cauchy}(Y, d'),$

$[4] := [3]\mathtt{E}d'(y) : \texttt{Cauchy}\Big((X, d), y\Big),$

$\Big(L, [5]\Big) := \texttt{EComplete}(X, d) : \sum L \in X \,.\, L = \lim_{n \to \infty} y_n,$

$[6] := \mathtt{E}d'[5] : \forall n \in \mathbb{N} \,.\, \lim_{i,j \to \infty} \left|\frac{1}{d(y_i, F_n)} - \frac{1}{d(y_j, F_n)}\right| = 0,$

$\Big(r, [7]\Big) := \texttt{EComplete}[6] : \sum r : \mathbb{N} \to \mathbb{R} \,.\, \forall n \in \mathbb{N} \lim_{i \to \infty} \frac{1}{d(y_i, F_n)} = r_n,$

$[8] := \texttt{ReciprocalLimit}[7]\texttt{EMetric}(X, d) : \forall n \in \mathbb{N} \,.\, \lim_{i \to \infty} d(y_i, F_n) \in \mathbb{R}_{++},$

$[y.*] := \mathtt{E}F_n[1][8][5] : L \in Y;$

$\rightsquigarrow [*] := \texttt{IComplete} : \texttt{Complete}(Y, d'),$

$\square$

$\texttt{PolishSubset} :: \forall X : \texttt{Polish} \,.\, \forall Y \subset X \,.\, Y \in G_\delta(X) \iff \texttt{Polish}(Y)$

$\texttt{Proof} \;=$

$\dots$

$\square$

### 1.2.4 Compacts and trees

$\texttt{PolishCompact} :: \forall X : \texttt{Compact} \,\&\, \texttt{Metrizable} \,.\, \texttt{Polish}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$$\texttt{FiniteSplitting} :: \prod_{A \in \mathsf{SET}} ?\texttt{Tree}(A)$$

$$T : \texttt{FiniteSplitting} \iff \forall t \in T \,.\, \Big|\{a \in A : ta \in T\}\Big| < \infty$$

$\texttt{FiniteSplittingIffCompact} :: \forall A \in \texttt{Set} \,.\, \forall T : \texttt{Pruned}(A) \,.\, \texttt{Compact}[T] \iff \texttt{FiniteSplitting}(T)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \texttt{Compact}[T],$

$\texttt{Assume}\ t \in T,$

$\mathcal{O} := \{N_s | s \in T : \operatorname{len} s = 1 + \operatorname{len} t\} : \texttt{OpenCover}[T],$

$\Big(\mathcal{O}', [2]\Big) := \texttt{ECompact}[T](\mathcal{O}) : \sum \mathcal{O}' : \texttt{Subcover}\Big([T], \mathcal{O}\Big) \,.\, |\mathcal{O}'| < \infty,$

$[t.*] := \texttt{InjectiveCodomainCardinalityBoundESubcover}\big([T], \mathcal{O}, \mathcal{O}'\big)\texttt{EPruned}(T)\texttt{E}\mathcal{O}\texttt{E} :$

$\quad : |\{a \in A : ta \in T\}| \le |\{s \in T : \operatorname{len} s = 1 + \operatorname{len} t\}| = |\mathcal{O}'| < \infty;$

$\rightsquigarrow [1.*] := \texttt{IFiniteSplitting} : \texttt{FiniteSplitting}(T);$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : \texttt{Compact}[T] \Rightarrow \texttt{FiniteSplitting}(T),$

$\texttt{Assume}\ [2] : \texttt{FiniteSplitting},$

$[3] := \texttt{E}[T]\texttt{EproductTopologyIClosed} : \texttt{Closed}\Big(A^{\mathbb{N}}, [T]\Big),$

$[4] := \texttt{CountableDiscreteProductIsComplete}(A) : \texttt{Complete}(A^{\mathbb{N}}, d),$

$[5] := \texttt{EFinitieSplittingE}d\texttt{ITotallyBounded} : \texttt{TotallyBounded}\Big([T], d\Big),$

$[6] := \texttt{ClosedCompleteSubset}[3][4] : \texttt{Complete}\Big([T], d\Big),$

$[2.*] := \texttt{TotallyBoundedCompleteIsCompact}[5][6] : \texttt{Compact}[T];$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow : \texttt{FiniteSplitting}(T) \Rightarrow \texttt{Compact}[T],$

$[*] := \texttt{I} \iff [1][2] : \texttt{Compact}[T] \iff \texttt{FiniteSplitting}(T);$

$\square$

$\texttt{BairSpaceIsNotSigmaCompact} :: \neg\sigma\text{-}\texttt{Compact}(\mathcal{B})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \sigma\text{-}\texttt{Compact}(\mathcal{B}),$

$$\Big(K,[2]\Big) := \texttt{E}\sigma\text{-}\texttt{Compact}(\mathcal{B}) : \sum K : \mathbb{N} \to \texttt{CompactSubset}(\mathcal{B}) \ . \ \mathcal{B} = \bigcup_{n=1}^{\infty} K_n,$$

$[3] := \texttt{CompactIsClosed}(\mathcal{B}, K) : \forall n \in \mathbb{N} \ . \ \texttt{Closed}(\mathcal{B}, K_n),$

$$\Big(T,[4]\Big) := \texttt{BodyBijection}[2] : \sum T : \mathbb{N} \to \texttt{Pruned}(\mathbb{N}) \ . \ \forall n \in \mathbb{N} \ . \ K_n = [T_n],$$

$[5] := \texttt{FiniteSplittingIffCompact}[4]\texttt{E}K : \forall n \in \mathbb{N} \ . \ \texttt{FiniteSplitting}(\mathbb{N}, T_n),$

$k := \Lambda n, m \in \mathbb{N} \ . \ 1 + \max\{t_m | t \in T_n\} : \mathbb{N} \to \mathbb{N} \to \mathbb{N},$

$[6] := \texttt{E}k\texttt{EFiniteSplitting}(T)[4] : \forall n \in \mathbb{N} \ . \ \forall x \in K_n \ . \ k_n > x,$

$\Delta := \Lambda n \in \mathbb{N} \ . \ k_{n,n} \in \mathcal{B},$

$[7] := \texttt{TrichtomyPrinciple}[6](\Delta) : \forall n \in \mathbb{N} \ . \ \Delta \notin K_n,$

$[8] := \texttt{Eunion}[2][7] : \Delta \notin \mathcal{B},$

$[9] := \texttt{E}\mathcal{B}(\Delta)[8] : \bot;$

$\rightsquigarrow [10] := \texttt{E}(\bot) : \neg\sigma\text{-}\texttt{Compact}(\mathcal{B});$

$\square$

$\texttt{KönigsLemma} :: \forall T : \texttt{FiniteSplitting}(A) \ . \ [T] \neq \emptyset \iff |T| = \infty$

$\texttt{Proof} \ =$

$\dots$

$\square$

### 1.2.5 Universality of the Hilbert's Cube

$\texttt{UniversalityOfTheHilbertsCube} :: \forall X : \texttt{Separable} \,\&\, \texttt{Metrizable} \,.\, \exists \texttt{TopologicalEmbedding}(X, I^{\mathbb{N}})$

$\texttt{Proof} =$

$(d, [1]) := \texttt{BoundedRemtrizationECompletelyMetrizable}(X) : \sum d : \texttt{Metric}(X) \,.\, d < 1 \,\&\, \texttt{Complete}(X, d),$

$D, [2] := \texttt{ESeparable}(X) : \sum D : \texttt{Dense}(X) \,.\, |D| \leq \aleph_0,$

$\delta := \texttt{enumerate}(D) : \mathbb{N} \leftrightarrow D,$

$f := \Lambda x \in X \,.\, \Lambda n \in \mathbb{N} \,.\, d(x, \delta_n) \in \mathsf{TOP}(X, I^{\mathbb{N}}),$

$\texttt{Assume}\ x, y \in X,$

$\texttt{Assume}\ [3] : f(x) = f(y),$

$\Big(a, [4]\Big) := \texttt{EDense}(X, D)(x) : \sum a : \mathbb{N} \to D \,.\, x = \lim_{n\to\infty} a_n,$

$[5] := \texttt{E}f[3][4]\texttt{ConvergenceInMetricSpace} : y = \lim_{n\to\infty} a_n,$

$\Big[(x, y).*\Big] := \texttt{T1HasUniqueLimit}[4][5] : x = y;$

$\leadsto [3] := \texttt{IInjective} : \texttt{Injective}(X, I^{\mathbb{N}}, f),$

$\texttt{Assume}\ K : \texttt{Closed}(X),$

$\texttt{Assume}\ L \in \partial f(K) \cap f(X),$

$\Big(y, [4]\Big) := \texttt{E}\partial f(K) : \sum y : \mathbb{N} \to f(K) \,.\, L = \lim_{n\to\infty} y_n,$

$\Big(A, [5]\Big) := \texttt{Eimage}(f)(L) : \sum A \in X \,.\, L = f(X),$

$\Big(x, [6]\Big) := \texttt{Eimage}(f)(y) : \sum x : \mathbb{N} \to K \,.\, \forall n \in \mathbb{N} \,.\, y_n = f(n),$

$\Big(a, [7]\Big) := \texttt{EDense}(X, D)(A) : \sum a : \mathbb{N} \to D \,.\, A = \lim_{n\to\infty} a_n,$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(n, [8]\Big) := \texttt{ELimit}[4]\left(\frac{\varepsilon}{3}\right) : \sum n \in \mathbb{N} \,.\, d(a_n, A) = L_n \leq \frac{\varepsilon}{3},$

$\Big(m, [9]\Big) := \texttt{E}\delta[5] : \sum m \in \mathbb{N} \,.\, \delta_m = a_n,$

$\Big(N, [10]\Big) := \texttt{ELimit}[4] : \forall k \in \mathbb{N} \,.\, k \geq N \Rightarrow \Big|L_n - y_{k,m}\Big| \leq \frac{\varepsilon}{3},$

$[11] := [10][8][5] : \forall k \in \mathbb{N} \,.\, k \geq N \Rightarrow y_{k,m} \leq \frac{2\varepsilon}{3},$

$[\varepsilon.*] := \Lambda k \in \mathbb{N} \,.\, \Lambda[0] : k \geq N \,.\, \texttt{TriangleIneq}(X, d)(x_k, A, a_n)[9][5][6]\texttt{E}f[8][11][0] : \forall k \in \mathbb{N} \,.\, k \geq N \Rightarrow$
$\Rightarrow d(x_k, A) \leq d(x_k, a_n) + d(a_n, A) = d(x_k, \delta_m) + d(\delta_m, A) = L_m + y_{k,m} \leq \varepsilon;$

$\leadsto [8] := \texttt{MetricConvergenceCriterion} : A = \lim_{n\to\infty} x_n,$

$[9] := \texttt{ClosedSetHasLimits}[8] : A \in K,$

$[L.*] := \texttt{IImage}(A) : L \in f(K);$

$\leadsto [11] := \texttt{ClosedByBoundary} : \texttt{Closed}(I^{\mathbb{N}}, f(K));$

$\leadsto [*] := \texttt{ITopologicalEmbedding} : \texttt{TopologicalEmbedding}(X, I^{\mathbb{N}}, f);$

$\square$

$\texttt{PolishSpacesAreHilbertCubesSubspaces} :: \forall X : \texttt{Polish} \,.\, \exists A : G_\delta(I^{\mathbb{N}}) : X \cong_{\mathsf{TOP}} A$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CantorSetIsACompactificationOfBairSet} :: \texttt{Compactification}(\mathcal{B}, \mathcal{C})$

$\texttt{Proof} \ =$

$(a, b) := \texttt{enumerate}(\mathbb{N} \times \mathbb{N}) : \mathbb{N} \leftrightarrow \mathbb{N} \times \mathbb{N},$

$\texttt{Assume}\ n \in \mathcal{B},$

$\beta := \Lambda k \in \mathbb{N} \,.\, \texttt{binaryExpansion}(n_k) : \mathbb{N} \to \mathbb{N} \to \{0, 1\},$

$f(n) := \Lambda k \in \mathbb{N} \,.\, \beta_{a_k, b_k} : \mathcal{C};$

$\rightsquigarrow \mathcal{B} \xrightarrow{f} \mathcal{C} := \mathtt{I}\mathsf{TOP} : \mathsf{TOP},$

$[1] := \mathtt{E}f : f(\mathcal{B}) = \left\{ b \in \mathcal{C} : \left| b^{-1}(0) \right| = \infty \right\},$

$[*] := \texttt{IDense}[1] : \texttt{Dense}\Big(\mathcal{C}, f(\mathcal{B})\Big);$

$\square$

$\texttt{UnitIntervalIsACompactificationOfBairSet} :: \texttt{Compactification}(\mathcal{B}, I)$

$\texttt{Proof} \ =$

$(a, b) := \texttt{enumerate}(\mathbb{N} \times \mathbb{N}) : \mathbb{N} \leftrightarrow \mathbb{N} \times \mathbb{N},$

$\texttt{Assume}\ n \in \mathcal{B},$

$\beta := \Lambda k \in \mathbb{N} \,.\, \texttt{binaryExpansion}(n_k) : \mathbb{N} \to \mathbb{N} \to \{0, 1\},$

$$f(n) := \sum_{k=1}^{\infty} 2^{-k} \beta_{a_k, b_k} : I;$$

$\rightsquigarrow \mathcal{B} \xrightarrow{f} I := \mathtt{I}\mathsf{TOP} : \mathsf{TOP},$

$[1] := \mathtt{E}f : \mathbb{Q}_2 \cap I \subset f(\mathcal{B}),$

$[*] := \texttt{IDense}[1] : \texttt{Dense}\Big(I, f(\mathcal{B})\Big);$

$\square$

$\texttt{PolishSpacesAsClosedSubset} :: \forall X : \texttt{Polish} \,.\, \exists A : \texttt{Closed}(X) \,.\, A \cong_{\mathsf{TOP}} X$

$\texttt{Proof} \ =$

$(G, [1]) := \texttt{PolishSpacesAreSubsetsOfHilbertCube} : \sum G \in G_\delta(I^{\mathbb{N}}) \,.\, G \cong_{\mathsf{TOP}} X,$

$$(U, [2]) := \mathtt{E}G_\delta(G) : \sum U : \mathbb{N} \to \mathcal{T}(I^{\mathbb{N}}) \,.\, G = \bigcap_{n=1}^{\infty} U_n,$$

$F := U^{\complement} : \mathbb{N} \to \texttt{Closed}(I^{\mathbb{N}}),$

$$f := \Lambda x \in X \,.\, \Lambda n \in \mathbb{N} \,.\, \texttt{if}\ n\ \texttt{is Even then}\ x_{n/2}\ \texttt{else}\ \frac{1}{d(x, F_{(n+1)/2})} : X \xrightarrow{\mathsf{TOP}} \mathbb{R}^{\mathbb{N}},$$

$[3] := \mathtt{E}f\texttt{IInjective} : \texttt{Injective}(X, \mathbb{R}^{\mathbb{N}}, f),$

$\texttt{Assume}\ L : \texttt{Limit}\Big(f(X)\Big),$

$$\Big(y, [4]\Big) := \texttt{ELimit}(f(X), L) : \sum y : \mathbb{N} \to f(X) \,.\, L = \lim_{n \to \infty} y_n,$$

$\Big(x, [5]\Big) := \texttt{Eimage}(y) : \sum x : \mathbb{N} \to X \,.\, y = f(x),$

$[6] := \mathtt{E}f[5][4] : \texttt{Convergent}(I^{\mathbb{N}}, x),$

$$A := \lim_{n \to \infty} x_n \in I^{\mathbb{N}},$$

$[7] := \mathtt{EAE}f[5][4] : \forall n \in \mathbb{N} \,.\, d(A, F_n) \neq 0,$

$[8] := [2][7] : A \in G,$

$[L.*] := \texttt{Iimage}[8][4]\texttt{ContinuousImage} : L = f(A) \in f(G);$

$\rightsquigarrow [*] := \texttt{ClosedByLimits} : \texttt{Closed}(\mathbb{R}^{\mathbb{N}}, f(X));$

$\square$

### 1.2.6 Universality of Cantor's Set

$\texttt{CantorsSetUniversality} :: \forall X : \texttt{Polish} \,\&\, \texttt{Compact} \;.\; \exists f \in \mathsf{TOP}(\mathcal{C}, X) \;.\; X = f(\mathcal{C})$

$\texttt{Proof} \;=$

$\Big(A, [1]\Big) := \texttt{PolishSpacesAreHilbertSpaceSubsets}(X) : \sum A : G_\delta(I^{\mathbb{N}}) \;.\; A \cong_{\mathsf{TOP}} X,$

$A \overset{\varphi}{\leftrightarrow} X := \texttt{EIsomorphic}(A, X)[1] : \mathsf{TOP},$

$$g := \Lambda b \in \mathcal{C} \;.\; \sum_{n=1}^{\infty} b_n 2^{-n} \in \mathsf{TOP}(\mathcal{C}, I),$$

$[2] := \texttt{RealsBinaryExpansionE}\psi \in \texttt{Surjective}(\mathcal{C}, I, g),$

$\mathcal{C}^{\mathbb{N}} \overset{\psi}{\leftrightarrow} \mathcal{C} := \texttt{CantorSetPowerHomeo}(\mathbb{N}) : \mathsf{TOP},$

$h := \psi g^{\mathbb{N}} : \mathsf{TOP}(\mathcal{C}, I^{\mathbb{N}}),$

$[3] := \texttt{SurjectiveCompositionE}h[2]\texttt{Homeomorphism}(\psi) : \texttt{Surjective}(\mathcal{C}, I^{\mathbb{N}}, h),$

$[4] := \texttt{CompactImage}[1] : \texttt{CompactSubset}(I^{\mathbb{N}}, A),$

$B := h^{-1}(A) : \texttt{Compact}(\mathcal{C}),$

$[5] := \texttt{E}B[1][3] : B \neq \emptyset,$

$r := \texttt{InBairlikespaceAllClosedAreRetracts}(\mathcal{C}, \mathcal{C}, B)[5] : \texttt{Retraction}(\mathcal{C}, B),$

$f := rh\varphi \in \mathsf{TOP}(X, \mathcal{C}),$

$[*] := \texttt{SurjectiveCompositonE}f\texttt{ERetraction}(r)[4]\texttt{EHomeomorphism}(\varphi)\texttt{ESurjection} : f(\mathcal{C}) = X;$

$\square$

### 1.2.7 More Examples

$\texttt{ContinuousFunctionsArePolish} :: \forall X : \texttt{Compact} \,\&\, \texttt{Polish} \,.\, \forall Y : \texttt{Polish} \,.\, \texttt{Polish}\Big(C(X,Y)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \texttt{Cauchy}\ C(X,Y),$

$[1] := \texttt{ECauchy}\Big(C(X,Y), f\Big) : \forall x \in X \,.\, \lim_{n,m\to\infty} d(f_n(x), f_m(x)) = 0,$

$[2] := \texttt{EComplete}(Y,d)[1] : \forall x \in X \,.\, \texttt{Convergent}\Big(Y, f(x)\Big),$

$\varphi := \Lambda x \in X \,.\, \lim_{n\to\infty} f_n(x) : X \to Y,$

$[3] := \texttt{ECauchy}\Big(C(X,Y), f\Big)\texttt{E}d_u : \lim_{n,m\to\infty} \sup_{x\in X} d(f_n(x), f_m(x)) = 0,$

$:=: d(\varphi(x_m), \varphi(L)) \le d(\varphi(x_m), f_n(x_m)) + d(f_n(x_m), f_n(L)) + d(f_n(L), \varphi(L)),$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(N, [4]\Big) := \texttt{ELimit}[3]\left(\frac{\varepsilon}{2}\right) : \sum N \in \mathbb{N} \,.\, \forall n,m \in \mathbb{N} \,.\, n,m \ge N \Rightarrow \sup_{x\in X} d(f_n(x), f_m(x)) < \frac{\varepsilon}{2},$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ [5] : n \ge N,$

$\texttt{Assume}\ x \in X,$

$[6] := \texttt{E}\varphi(x) : \lim_{n\to\infty} f_n(x) = \varphi(x),$

$\Big(M, [7]\Big) := \texttt{ELimit}[6]\left(\frac{\varepsilon}{2}\right) : \sum M \in \mathbb{N} \,.\, \forall m \in \mathbb{N} \,.\, m \ge M \Rightarrow d(f_m(x), \varphi(x)) < \frac{\varepsilon}{2},$

$m := \max(M, N) \in \mathbb{N},$

$\texttt{Assume}\ x \in X,$

$[x.*] := \texttt{TriablgeIneq}(Y)\Big(f_n(x), \varphi(x), f_m\Big)[7][4] : d\Big(f_n(x), \varphi(x)\Big) \le d\Big(f_n(x), f_m(x)\Big) + d\Big(f_m(x), \varphi(x)\Big) < \varepsilon;$

$\rightsquigarrow [\varepsilon.*] := \texttt{I}d_u : \sup_{x\in X} d(f_n, \varphi) < \varepsilon;$

$\rightsquigarrow [4] := \texttt{I}\forall : \forall \varepsilon \in \mathbb{R}_{++} \,.\, \sup_{x\in X} d(f_n, \varphi) < \varepsilon,$

$\texttt{Assume}\ L \in X,$

$\texttt{Assume}\ x : \mathbb{N} \to X,$

$\texttt{Assume}\ [5] : \lim_{n\to\infty} x_n = L,$

$[6] := \Lambda n \in \mathbb{N} \,.\, \texttt{ContinuousByLimits}(f_n, x, L) : \forall n \in \mathbb{N} \,.\, \lim_{m\to\infty} f_n(x_m) = f_n(L),$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(n, [7]\Big) := \texttt{ELimit}\left(\frac{\varepsilon}{3}\right) : \sum_{n=1}^{\infty} \sup_{x\in X} d\Big(f_n(x), \varphi(x)\Big) < \frac{\varepsilon}{3},$

$\Big(N, [8]\Big) := \texttt{ELimit}[6](n) : \sum_{N=1}^{\infty} \forall m \in \mathbb{N} \,.\, m \ge N \Rightarrow d\Big(f_n(x_m), f_n(L)\Big) < \frac{\varepsilon}{3},$

$\texttt{Assume}\ m \in \mathbb{N},$

$\texttt{Assume}\ [9] : m \ge N,$

$[\varepsilon.*] := \texttt{TriangleIneq}(Y,)[7]^2[8] : d(\varphi(x_n), \varphi(L)) \le d(\varphi(x_n), f(x_n)) + d(f(x_n), f(L)) + d(f(L), \varphi(L)) < \varepsilon;$

$\rightsquigarrow [L.*] := \texttt{ILimit} : \lim_{n\to\infty} \varphi(x_n) = \varphi(L);$

$\rightsquigarrow [7] := \texttt{ContinuousByLimits} : \varphi \in C(X,Y),$

$[f.*] := \texttt{ILimit}[6] : \lim_{n\to\infty} f_n = \varphi;$

$\rightsquigarrow [1] := \texttt{IComplete} : \texttt{Complete}\Big(C(X,Y), d_u\Big),$

$C := \Lambda n, m \in \mathbb{N} \,.\, \left\{ f \in C(X,Y) : \forall x, y \in X \,.\, d(x,y) < \frac{1}{m} \Rightarrow d(f(x), f(y)) < \frac{1}{n} \right\} : \mathbb{N} \to \mathbb{N} \to ?C(X,Y),$

$\Big(A, [1]\Big) := \texttt{CompactHasEpsilonNets}(A) : \sum A : \mathbb{N} \to \texttt{Finite}(X) \,.\, \forall n \in \mathbb{N} \,.\, \forall x \in X \,.\, \exists a \in A : d(a,x) < \frac{1}{n},$

$D := \texttt{countableRefinementAt}(C, X) : \prod_{n,m=1}^{\infty} \texttt{Countable}(C_{n,m}),$

$[2] := \mathtt{E}D : \forall n, m \in \mathbb{N} \,.\, \forall f \in C_{n,m} \,.\, \forall \varepsilon \in \mathbb{R}_{++} \,.\, \exists g \in D_{n,m} : \forall a \in A_n \,.\, d(f(a), g(a)) < \varepsilon,$

$E := \bigcup_{n,m=1}^{\infty} D_{n,m} : \texttt{Countable}\Big(C(X,Y)\Big),$

$\texttt{Assume } f \in C(X,Y),$

$\Big(m, [3]\Big) := \texttt{ArchemedeanProperty}\left(\frac{3}{\varepsilon}\right) : \sum m \in \mathbb{N} \,.\, m > \frac{3}{\varepsilon},$

$\Big(n, [4]\Big) := \texttt{EUniformlyContinuous}(f)\mathtt{E}C : \sum n \in \mathbb{N} \,.\, f \in C_{n,m},$

$[f.*] := \mathtt{E}C_{n,m}\mathtt{E}A_n[4][2] : \exists g \in D_{n,m} \,.\, d(f,g) < \varepsilon;$

$\rightsquigarrow [2] := \texttt{IDense} : \texttt{Dense}\Big(C(X,Y), E\Big),$

$[*] := \texttt{IPolish}[1][2] : \texttt{Polish}\Big(C(X,Y)\Big);$

$\square$

$\texttt{compactsWithVietorisTopology} :: \mathsf{TOP} \to \mathsf{TOP}$

$\texttt{compactsWithVietorisTopology}\,(X) = \mathsf{K}(X) := \Big(\texttt{CompactSubset}(X),$

$\quad \texttt{fromBase}\Big\{\{K : \texttt{CompactSubset} : K \subset U_0 \,\&\, \forall i \in [1, \dots, n] \,.\, K \cap U_i \neq \emptyset\} \Big| n \in \mathbb{Z}_+, U : [0, \dots, n] \to \mathcal{T}(X)\Big\}\Big)$

$\texttt{EmptySetIsIsolatedInVietorisTopology} :: \forall X \in \mathsf{TOP} \,.\, \texttt{Isolated}\Big(\mathsf{K}(X), \emptyset_X\Big)$

$\texttt{Proof} =$

$[1] := \mathtt{E}\emptyset : \forall A : \texttt{CompactSubset}(X) \,.\, A \subset \emptyset \Rightarrow A = \emptyset,$

$[2] := \mathtt{E}\mathsf{K}(X)[1] : \{\emptyset\} \in \mathcal{T}\Big(\mathsf{K}(X)\Big),$

$[*] := \texttt{IIsolated}[2] : \texttt{Isolated}\Big(\mathsf{K}(X), \emptyset_X\Big);$

$\square$

$\texttt{distanceOfHausdorff} :: \prod X : \texttt{BoundedMetricSpace} \,.\, \texttt{Metric}\Big(\mathsf{K}(X)\Big)$

$\texttt{distanceOfHausdorff}\,(\emptyset, \emptyset) = d_{\mathrm{H}}(\emptyset, \emptyset) := 0$

$\texttt{distanceOfHausdorff}\,(A, \emptyset) = d_{\mathrm{H}}(A, \emptyset) := 1$

$\texttt{distanceOfHausdorff}\,(\emptyset, B) = d_{\mathrm{H}}(\emptyset, B) := 1$

$\texttt{distanceOfHausdorff}\,(A, B) = d_{\mathrm{H}}(A, B) := \max\left(\max_{a \in A} \min_{b \in B} d(a,b), \max_{b \in B} \min_{a \in A} d(a,b)\right)$

$\texttt{VietorisTopologyIsMetrizedByHausdorffMetric} :: \forall X \in \mathsf{MS} \,.\, \mathsf{K}(X) \cong_{\mathsf{TOP}} \Big(\mathsf{K}(X), d_{\mathrm{H}}\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{PolishCompacts} :: \forall X : \texttt{Separable} \,\&\, \texttt{Metrizable} \,.\, \texttt{Separable} \,\&\, \texttt{Metrizable}\Big(\mathsf{K}(X)\Big)$

$\texttt{Proof} \;=$

$[1] := \texttt{BoundedRemetrization}(X)\texttt{VietorisTopologyIsMetrizedByHausdorffMetric} : \texttt{Metrizable}\Big(\mathsf{K}(X)\Big),$

$\Big(D, [2]\Big) := \texttt{ESeparable}(X) : \sum D : \texttt{Dense}(X) \,.\, |D| \le \aleph_0,$

$A := \texttt{Finite}(D) :?\mathsf{K}(X),$

$[3] := \texttt{FiniteSetsCardinalityE}A : |A| \le \aleph_0,$

$[4] := \texttt{E}d_{\mathrm{H}}\texttt{E}A : \texttt{Dense}\Big(\mathsf{K}(X), A\Big),$

$[*] := \texttt{ISeparable}[4] : \texttt{Separeble}\Big(\mathsf{K}(X)\Big);$

$\square$

$$\texttt{topologicalLowerLimit} :: \prod_{X \in \mathsf{TOP}} (\mathbb{N} \to ?X) \to ?X$$

$$\texttt{topologicalLowerLimit}\,(A) = \mathrm{T}\varliminf_{n\to\infty} A_n := \left\{ x \in X : \Big|\{ n \in \mathbb{N} : \forall U \in \mathcal{U}(x) \,.\, U \cap A_n \ne \emptyset \}\Big| = \infty \right\}$$

$$\texttt{topologicalUpperLimit} :: \prod_{X \in \mathsf{TOP}} (\mathbb{N} \to ?X) \to ?X$$

$$\texttt{topologicalUpperLimit}\,(A) = \mathrm{T}\varlimsup_{n\to\infty} A_n := \left\{ x \in X : \Big|\{ n \in \mathbb{N} : \exists U \in \mathcal{U}(x) \,.\, U \cap A_n = \emptyset \}\Big| < \infty \right\}$$

$\texttt{TopologicalLimitsRelation} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \mathbb{N} \to ?X \,.\; \mathrm{T}\varliminf_{n\to\infty} A_n \subset \mathrm{T}\varlimsup_{n\to\infty} A_n$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{TopologicalLimitsAreClosed} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \mathbb{N} \to ?X \,.\, \texttt{Closed}\Big(X, \mathrm{T}\varliminf_{n\to\infty} A_n \;\&\; \mathrm{T}\varlimsup_{n\to\infty} A_n\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{TopologicalLimit} :: \prod_{X \in \mathsf{TOP}} \,.\, (\mathbb{N} \to ?X) \to ?\texttt{Closed}(X)$$

$$L : \texttt{TopologicalLimit} \iff \Lambda A : \mathbb{N} \to ?X \,.\; \mathrm{T}\lim_{n\to\infty} A_n = L \iff \Lambda A : \mathbb{N} \to ?X \,.\, L = \mathrm{T}\varliminf_{n\to\infty} A_n \,\&\, L = \mathrm{T}\varlimsup_{n\to\infty} A_n$$

$\texttt{HausdorffConvergenceAsTopologicalLimit} :: \forall X : \mathsf{MS} \,.\, \forall K : \mathbb{N} \to \mathsf{K}(X) \,.\, \forall L \in \mathsf{K}(X) \,.$

$\quad .\; L = \lim_{n\to\infty} K_n \Rightarrow L = \mathrm{T}\lim_{n\to\infty} K_n$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CompactHausdorffConvergence} :: \forall X : \texttt{Compact} \ \& \ \texttt{BoundedMetricSpace} \ . \ \forall K : \mathbb{N} \to \mathsf{K}(X) \ . \ \forall L \in \mathsf{K}(X) \ .$

$\quad . \ L = \mathrm{T}\lim_{n\to\infty} K_n \iff L = \lim_{n\to\infty} K_n$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PolishHausdforffisPolish} :: \forall X : \texttt{Polish} \ . \ \texttt{Polish}(\mathsf{K}(X))$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{ECompletelyMetrizable}(X)\texttt{BoundedRemetrization} : \sum d : \texttt{Metric}(X) \ .$

$\quad . \ \texttt{Complete}(X, d) \ \& \ (X, d) \cong_{\mathsf{TOP}} X \ \& \ d < 1,$

$\texttt{Assume} \ K : \texttt{Cauchy} \ \mathsf{K}(X),$

$L := \mathrm{T}\overline{\lim_{n\to\infty}} K_n : \texttt{Closed}(X),$

$\Big(F, [2]\Big) := \Lambda n \in \mathbb{N} \ . \ \texttt{ETotallyBounded} : \prod_{n=1}^{\infty} \prod_{m=1}^{\infty} \sum F_{n,m} : \texttt{Finite}(K_n) \ . \ \forall x \in K_n \exists f \in F_{n,m} : d(x, f) < 2^{-m-2},$

$\Big(p, [3]\Big) := \texttt{ECauchy}\Big(\mathsf{K}(X), K\Big) : \prod_{n=1}^{\infty} \sum_{p_n=1}^{\infty} \ . \ \forall i, j \in \mathbb{N} \ . \ i, j \geq p_n \Rightarrow d_{\mathrm{H}}(K_i, K_j) < 2^{-m-2},$

$J := \Lambda n \in \mathbb{N} \ . \ \bigcup_{k=n}^{p_n} F_{k,n} : \mathbb{N} \to \texttt{Finite}(X),$

$\texttt{Assume} \ n \in \mathbb{N},$

$\texttt{Assume} \ x \in L,$

$(m, [4]) := \mathtt{E}L(x, n) : \sum_{m=1}^{\infty} m \geq p_n \ \& \ \forall U \in \mathcal{U}(x) \ . \ U \cap K_m \neq \emptyset,$

$[5] := [3](n, p_n, m) : d_{\mathrm{H}}(K_{p_n}, K_m) < 2^{-n-2},$

$\Big(u, [6]\Big) := [4] \left(\left(\mathbb{B}_d \left(x, 2^{-n-2}\right)\right)\right) : \sum u \in K_m \ . \ d(u, x) < 2^{-n-2},$

$\Big(v, [7]\Big) := \mathtt{E}d_{\mathrm{H}}[5](u) : \sum v \in K_{p_n} \ . \ d(u, v) < 2^{-n-2},$

$\Big(f, [8]\Big) := [2](p_n, n)(v) : \sum f \in F_{p_n,n} \ . \ d(v, f) < 2^{-n-2},$

$[9] := \mathtt{E}J_n\texttt{Eunion}(f) : f \in J_n,$

$[n.*] := \texttt{TriangleIneq}(X, d)(x, u, v, f)[6, 7, 8] : d(x, f) \leq d(x, u) + d(u, v) + d(v, f) < 2^{-n};$

$\rightsquigarrow [4] := \texttt{ITotallyBounded} : \texttt{TotallyBounded}(X, L),$

$[5] := \texttt{ClosedIsComplete}(X, L) : \texttt{Complete}(X),$

$[6] := \texttt{CompactIffCompleteAndTotallyBounded}[4][5]\mathtt{I}\mathsf{K}(X) : L \in \mathsf{K}(X),$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(N, [7]\Big) := \texttt{ECauchy}(K) : \sum N \in \mathbb{N} \,.\, \forall i, j \in \mathbb{N} \,.\, i, j > N \Rightarrow d(K_i, K_j) < \frac{\varepsilon}{2},$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [8] : n \geq N,$

$\texttt{Assume}\ x \in L,$

$\Big(k, y, [9]\Big) := \texttt{E}L : \sum k : \mathbb{N} \uparrow \mathbb{N} \,.\, \sum y : \prod_{n=1}^{\infty} K_{k_n} \,.\, x = \lim_{n \to \infty} y_n,$

$\Big(M, [10]\Big) := \texttt{ELimit}(x, y)[9]\left(\frac{\varepsilon}{2}\right) : \sum M \in \mathbb{N} \,.\, \forall m \in \mathbb{N} \,.\, m \geq M \Rightarrow k_m \geq N \,\&\, d(y_m, x) < \frac{\varepsilon}{2},$

$\Big(z, [11]\Big) := [7](y_{k_M}) : \sum z \in K_n \,.\, d(z, y_{k_M}) < \frac{\varepsilon}{2},$

$[x.*] := \texttt{TrinagleIneq}(x, y_{k_M}, z)[10][11] : d(x, z) \leq d(x, y_{k_M}) + d(y_{k_M}, z) < \varepsilon;$

$\leadsto [9] := \texttt{I}\sup : \sup_{x \in L} \inf_{y \in K_n} d(x, y) < \varepsilon,$

$\texttt{Assume}\ y \in K_n,$

$\Big(k, [10]\Big) := \texttt{ECauchy}(K) : \sum k : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) \,.\, k_1 = n \,\&\, \forall i, m \in \mathbb{N} \,.\, m \geq k_i \Rightarrow d(K_{k_i}, K_m) < 2^{-i-1}\varepsilon,$

$x_1 := y \in K_n,$

$\texttt{Assume}\ i \in \mathbb{N},$

$\Big(x_{i+1}, [11]\Big) := [10](i, k_{i+1}) : \sum x_{i+1} \in K_{k_{i+1}} \,.\, d(x_{i+1}, x_i) < 2^{-i-1}\varepsilon;$

$\leadsto \Big(x, [11]\Big) := \texttt{I}\left(\prod\right) : \sum x : \prod_{i=1}^{\infty} K_{k_i} \,.\, \forall i \in \mathbb{N} \,.\, d(x_{i+1}, x_i) < 2^{-i-1}\varepsilon,$

$[12] := \texttt{ICauchy}[11] : \texttt{Cauchy}(X, x),$

$[13] := \texttt{EComplete}(X)[12] : \texttt{Convergent}(X, x),$

$z := \lim_{n \to \infty} x_n \in X,$

$[14] := \texttt{E}L\texttt{E}\,\mathrm{T}\,\overline{\lim}\,\texttt{E}z : z \in L,$

$[y.*] := \texttt{E}z[11] : d(y, z) < \varepsilon;$

$\leadsto [10] := \texttt{I}\sup : \sup_{x \in K_n} \inf_{y \in L} d(x, y) < \varepsilon,$

$[\varepsilon.*] := \texttt{I}d_H[9][10] : d(K_n, L) < \varepsilon;$

$\leadsto [K.*] := \texttt{ILimit} : \lim_{n \to \infty} K_n = L;$

$\leadsto [*] := \texttt{EComplete} : \texttt{Complete}(\mathsf{K}(X));$

$\square$

$\texttt{CompactHaudorffMetricIsCompact} :: \forall X : \texttt{Compact} \ \& \ \texttt{Polish} \ . \ \texttt{Compact} \ \& \ \texttt{Polish}\Big(\mathsf{K}(X)\Big)$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{ECompletelyMetrizable}(X)\texttt{BoundedRemetrization} : \sum d : \texttt{Metric}(X) \ .$

$\quad . \ \texttt{Complete}(X, d) \ \& \ (X, d) \cong_{\mathsf{TOP}} X \ \& \ d < 1,$

$\Big(F, [2]\Big) := \texttt{ETotallyBounded}(X) : \sum F : \mathbb{N} \to \texttt{Finite}(X) \ . \ \forall n \in \mathbb{N} \ . \ \forall x \in X \ . \ \exists f \in F_n : d(x, f) < \frac{1}{n},$

$[3] := \texttt{PolishHausdorffPolish}(X) : \texttt{Polish}\Big(\mathsf{K}(X)\Big),$

$[4] := \texttt{EPolish}\Big(\mathsf{K}(X)\Big) : \texttt{Complete}(\mathsf{K}(X), d_H),$

$F' := \Lambda n \in \mathbb{N} \ . \ 2^{F'} : \mathbb{N} \to \texttt{Finite}\Big(\texttt{Finite}(X)\Big),$

$[5] := \texttt{FiniteIsCompact}(F)\texttt{I}F' : \forall n \in \mathbb{N} \ . \ \texttt{Finite}\Big(\mathsf{K}(X), F'_n\Big),$

$\texttt{Assume} \ n \in \mathbb{N},$

$\texttt{Assume} \ K \in \mathsf{K}(X),$

$G := \left\{ f \in F_n : d(x, K) < \frac{1}{n} \right\} \in F'_n,$

$[n.*] := \texttt{E}G[2] : d(K, G) < \frac{1}{n};$

$\leadsto [6] := \texttt{ITotallyBounded} : \texttt{TotallyBounded}\Big(X, d_{\mathrm{H}}\Big),$

$[*] := \texttt{CompactIffCompleteAndTotallyBounded}[4][6] : \texttt{Compact}(\mathsf{K}(X));$

$\square$

$\texttt{SingletonIsTopologicalEmbedding} :: \forall X \in \mathsf{MS} \,.\, \texttt{IsometricEmbedding}\Big(X, \mathsf{K}(X), \Lambda x \in X \,.\, \{x\}\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{HausdorffConvergenceByIntersection} :: \forall X \in \mathsf{MS} \,.\, \forall K : \texttt{Decreasing}\Big(\mathsf{K}(X)\Big) \,.\, \lim_{n\to\infty} K_n = \bigcap_{n=1}^{\infty} K_n$

$\texttt{Proof} \;=$

$[1] := \texttt{CompactHausdorffMetricIsCompact}(K_1) : \texttt{Compact}\Big(\mathsf{K}(K_1)\Big),$

$\Big(k, [2]\Big) := \texttt{CompactIsSequinceCompcat} : \sum k : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) \,.\, \texttt{Converging}(\Big(\mathsf{K}(K_1), d_{\mathrm{H}}\Big), K_k),$

$[3] := \texttt{ConvergingIsCauchy}[2] : \texttt{Cauchy}\Big(\big(\mathsf{K}(K_1), d_{\mathrm{H}}\big), K_k\Big),$

$[4] := \texttt{EDecreasing}\Big(\mathsf{K}(X), K\Big)\texttt{ECauchy}[2] : \texttt{Cauchy}\Big(\big(\mathsf{K}(K_1), d_{\mathrm{H}}\big), K\Big),$

$[5] := \texttt{EComplete}\Big(\mathsf{K}(K_1)\Big)[4] : \texttt{Convergent}\Big(\big(\mathsf{K}(K_1), d_{\mathrm{H}}\big), K\Big),$

$L := \lim_{n\to\infty} K_n \in \mathsf{K}(X),$

$\texttt{Assume}\; l \in L,$

$\Big(x, [6]\Big) := \texttt{CompactHausdorffConvergence}(K) : \sum x : \prod_{n=1}^{\infty} K_n \,.\, l = \lim_{n\to\infty} x_n,$

$[7] := \texttt{EDecreasing}(K)\texttt{CompactIsClosed}(K)\texttt{ClosedHasLimits}(K)[6](l) : \forall n \in \mathbb{N} \,.\, l \in K_n,$

$[l.*] := \texttt{I}\bigcup[7] : l \in \bigcup_{n=1}^{\infty} K_n;$

$\leadsto [6] := \texttt{I} \subset: L \subset \bigcup_{n=1}^{\infty} K_n,$

$[7] := \texttt{CompactHausdorffConvergence}(K)\texttt{E}\bigcup_{n=1}^{\infty} K_n\texttt{I} \subset: \bigcup_{n=1}^{\infty} K_n \subset L,$

$[*] := \texttt{ISetEq}[7][6] : \bigcup_{n=1}^{\infty} K_n = L;$

$\square$

$\texttt{ContainmentIsClosedRelation} :: \forall X : \texttt{Metrizable} \,.\, \texttt{Closed}\Big(X \times \mathsf{K}(X), \big\{(x, K) \in X \times \mathsf{K}(X) : x \in K\big\}\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SubsetIsClosedRelation} :: \forall X : \texttt{Metrizable} \,.\, \texttt{Closed}\Big(\mathsf{K}(X) \times \mathsf{K}(X), \big\{(K, L) \in \mathsf{K}(X) \times \mathsf{K}(X) : K \subset L\big\}\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{NonEmptyIntersectionIClosedRelation} :: \forall X : \texttt{Metrizable} .$

$. \texttt{Closed}\Big(\mathsf{K}(X) \times \mathsf{K}(X), \big\{(K, L) \in \mathsf{K}(X) \times \mathsf{K}(L) : K \cap L \neq \emptyset\big\}\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{UnionIsContinuous} :: \forall X : \texttt{Metrizable} . \Big(\Lambda(A, B) \in \mathsf{K}(X) \times \mathsf{K}(X) . A \cap B\Big) \in \mathsf{TOP}\Big(\mathsf{K}(X) \times \mathsf{K}(X), \mathsf{K}(X)\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CompactUnionIsContinuous} :: \forall X : \texttt{Metrizable} . \Big(\Lambda\mathcal{A} \in \mathsf{K}^2(X) . \bigcup\mathcal{A}\Big) \in \mathsf{TOP}\Big(\mathsf{K}^2(X), \mathsf{K}(X)\Big)$

$\texttt{Proof} =$

$\Big(d, [1]\Big) := \texttt{E}\texttt{Metrizable}(X)\texttt{BoundedRemetrization} : \sum d : \texttt{Metric}(X) .$

$. (X, d) \cong_{\mathsf{TOP}} X \ \& \ d < 1,$

$\texttt{Assume}\ \mathcal{A} \in \mathsf{K}^2(X),$

$\texttt{Assume}\ x : \texttt{Cauchy}\left(\bigcup\mathcal{A}, d\right),$

$\Big(K, [1]\Big) := \texttt{E}\bigcup\mathcal{A}(x) : \sum K : \mathbb{N} \to \mathcal{A} . \forall n \in \mathbb{N} . x_n \in K_n,$

$\Big(k, [2]\Big) := \texttt{CompactIffSequenceCompact}\left(\bigcup\mathcal{A}, K\right) : \sum k : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) . \texttt{Converging}\Big(\mathsf{K}(X), K_k\Big),$

$L := \lim_{n\to\infty} K_{k_n} \in \mathsf{K}(X),$

$\Big(y, [3]\Big) := \texttt{E}L\texttt{ELimit}\Big(\mathsf{K}(X), d_{\mathrm{H}}\Big)(y) : \sum y \in L . \lim_{n\to\infty} d(y_n, x_{k_n}) = 0,$

$\Big(l, [3]\Big) := \texttt{CompactIffSequenceCompact}\Big(L, y\Big) : \sum l : \texttt{Increasing}(\mathbb{N}, \mathbb{N}) . \texttt{Converging}(L, y_l),$

$z := \lim_{n\to\infty} y_{l_n} \in L,$

$[4] := \texttt{E}z\texttt{MetricLimitAgrees}[4] : \lim_{n\to\infty} x_{k_{l_n}} = z,$

$[*] := \texttt{CauchyHasSubsequenceLimit}\Big((X, d), x\Big)[4] : \lim_{n\to\infty} x_n = z;$

$\rightsquigarrow [1] := \texttt{IComplete} : \texttt{Complete}\left(\bigcap_{n=1}^{\infty}\mathcal{A}\right),$

$\texttt{Assume}\ n : \mathbb{N},$

$\Big(\mathcal{B}, [2]\Big) := \texttt{CompactIsTotallyBounded}\Big((\mathsf{K}(X), d_{\mathrm{H}}), A\Big)\texttt{ETotallyBounded}\Big((\mathsf{K}(A), d_{\mathrm{H}}), A\Big) :$

$: \sum \mathcal{B} : \texttt{Finite}(A) \ . \ \forall K \in \mathcal{A} \ . \ \exists L \in \mathcal{B} : d_{\mathrm{H}}(K, L) < \frac{1}{2n},$

$\Big(F, [3]\Big) := \texttt{CompactIsTotallyBounded}\Big((X, d), B\Big)\texttt{ETotallyBounded}\Big((X, d), B\Big) :$

$: \sum F : \prod_{K \in \mathcal{B}} \texttt{Finite}(X) \ . \ \forall K \in \mathcal{B} \ . \ \forall x \in K \ . \ \exists y \in F_K \ . \ d(x, y) < \frac{1}{2n},$

$G := \bigcup_{K \in B} F_B : \texttt{Finite}(X),$

$\texttt{Assume}\ x : \bigcup \mathcal{A},$

$\Big(K, [4]\Big) := \texttt{Eunion}(\mathcal{A})(x) : \sum_{K \in \mathcal{A}} \ . \ x \in K,$

$\Big(L, [5]\Big) := [2](K) : \sum L \in \mathcal{B} \ . \ d_H(K, L) < \frac{1}{2n},$

$\Big(y, [6]\Big) := \mathtt{E}d_H[5](x) : \sum y \in L \ . \ d(x, y) < \frac{1}{2n},$

$\Big(z, [7]\Big) := [3](L, y) : \sum z \in F_L \ . \ d(y, z) < \frac{1}{2n},$

$[8] := \mathtt{E}G(z) : z \in G,$

$[n.*] := \texttt{TriangleIneq}(X, x, y, z)[6][7] : d(x, z) \le d(x, y) + d(y, z) < \frac{1}{2n};$

$\rightsquigarrow [2] := \texttt{ITotallyBounded} : \texttt{TotallyBounded}\left(\bigcup \mathcal{A}\right),$

$[\mathcal{A}.*] := \texttt{CompactIffCompleteAndTotallyBounded}[2] : \texttt{CompactSubset}\left(X, \bigcup \mathcal{A}\right);$

$\rightsquigarrow [1] := \mathtt{I}(\forall) : \forall \mathcal{A} \in \mathsf{K}^2(X) \ . \ \bigcup \mathcal{A} \in \mathsf{K}(X),$

$\texttt{Assume}\ \mathcal{A} : \texttt{Converging}\Big(\mathsf{K}^2(X)\Big),$

$\mathcal{B} := \lim_{n \to \infty} \mathcal{A}_n \in \mathsf{K}^2(X),$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(N, [2]\Big) := \mathtt{E}\mathcal{B}\texttt{ELimit}\Big(\mathsf{K}^2(X)\Big) : \sum N \in \mathbb{N} \ . \ \forall n \in \mathbb{N} \ . \ n \ge N \Rightarrow d_{\mathrm{H}}(\mathcal{A}_n, \mathcal{B}) < \varepsilon,$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ [3] : n \ge N,$

$[4] := [2]\Big(n, [3]\Big) : d_{\mathrm{H}}(\mathcal{A}_n, \mathcal{B}) < \varepsilon,$

$[5] := \Lambda K \in \mathcal{A}_n \ . \ \mathtt{E}d_{\mathrm{H}}[4](K) : \forall K \in \mathcal{A}_n \ . \ \exists L \in \mathcal{B} \ . \ d_{\mathrm{H}}(K, L) < \varepsilon,$

$[6] := \Lambda L \in \mathcal{B} \ . \ \mathtt{E}d_{\mathrm{H}}[4](L) : \forall L \in \mathcal{B} \ . \ \exists K \in \mathcal{A}_n \ . \ d_{\mathrm{H}}(K, L) < \varepsilon,$

$[\mathcal{A}.*] := \mathtt{E}d_{\mathrm{H}}[5][6] : d_{\mathrm{H}}\left(\bigcup \mathcal{A}_n, \mathcal{B}\right) < \varepsilon;$

$\rightsquigarrow [*] := \texttt{ContinuousByLimits} : \left(\Lambda \mathcal{A} \in \mathsf{K}^2(X) \ . \ \bigcup \mathcal{A}\right) \in \mathsf{TOP}(\mathsf{K}^2(X), \mathsf{K}(X)),$

$\square$

$$\texttt{HausdorffMetricImageContinuity} :: \forall X, Y : \texttt{Metrizable} \,.\, \forall f \in \mathsf{TOP}(X, Y) \,.\, f^* \in \mathsf{TOP}\Big(\mathsf{K}(X), \mathsf{K}(Y)\Big)$$

$\texttt{Proof} \; =$

$$\Big(\alpha, [1]\Big) := \texttt{EMetrizable}(X)\texttt{BoundedRemetrization} : \sum \alpha : \texttt{Metric}(X) \,.$$
$$.\, (X, \alpha) \cong_{\mathsf{TOP}} X \;\&\; \alpha < 1,$$
$$\Big(\beta, [2]\Big) := \texttt{EMetrizable}(X)\texttt{BoundedRemetrization} : \sum \alpha : \texttt{Metric}(Y) \,.$$
$$.\, (Y, \beta) \cong_{\mathsf{TOP}} Y \;\&\; \beta < 1,$$
$$\texttt{Assume}\; K \in \mathsf{K}(X),$$
$$\texttt{Assume}\; \varepsilon \in \mathbb{R}_{++},$$
$$[1] := \texttt{CompactImage}(f, K) : \texttt{CompactSubset}\Big(Y, f(K)\Big),$$
$$\Big(F, [2]\Big) := \texttt{ETotallyBounded}\Big(Y, f(K)\Big)\left(\frac{\varepsilon}{2}\right) : \sum F : \texttt{Finite}\Big(f(K)\Big) \,.\, \forall y \in f(K) \,.\, \exists z \in F : \beta(y, z) < \frac{\varepsilon}{2},$$
$$\mathcal{U} := \left\{ f^{-1}\left(\mathbb{B}_\beta\left(y, \frac{\varepsilon}{2}\right)\right) \middle| y \in \mathcal{F} \right\} :?\mathcal{T}(X),$$
$$V := \bigcup \mathcal{U} \in \mathcal{T}(X),$$
$$W := \Big\{ L \in \mathsf{K}(X) : L \subset V \;\&\; \forall U \in \mathcal{U} \,.\, U \cap V \neq \emptyset \Big\} : \mathcal{T}\Big(\mathsf{K}(X)\Big),$$
$$[3] := \texttt{E}W[1] : X \in W,$$
$$\Big(\delta, [4]\Big) := \texttt{MetricOpenCriterion}[2] : \sum \delta \in \mathbb{R}_{++} \,.\, \mathbb{B}_{\alpha_{\mathrm{H}}}(K, \delta) \subset W,$$
$$\texttt{Assume}\; L \in \mathbb{B}_{\alpha_{\mathrm{H}}}(K, \delta),$$
$$[5] := [4](L) : L \in W,$$
$$\texttt{Assume}\; y \in f(L),$$
$$\Big(x, [6]\Big) := \texttt{Eimage}(y) : \sum x \in L \,.\, y = f(x),$$
$$[7] := [5](x) : x \in V,$$
$$[8] := \texttt{E}V[7][6][2] : \exists z \in F \,.\, \beta(z, y) = \beta(z, f(x)) < \frac{\varepsilon}{2} < \varepsilon;$$
$$\rightsquigarrow [6] := \texttt{I}\forall : \forall y \in f(L) \,.\, \exists z \in f(K) : \beta(y, z) < \varepsilon,$$
$$\texttt{Assume}\; y \in f(K),$$
$$\Big(z, [7]\Big) := [2](y) : \sum z \in F \,.\, d(z, y) < \frac{\varepsilon}{2},$$
$$\Big(U, [8]\Big) := \texttt{E}U(z) : \sum U \in \mathcal{U} \,.\, z \in f(U),$$
$$[9] := \texttt{E}W[5](U) : U \cap L \neq \emptyset,$$
$$\Big(x, [10]\Big) := \texttt{ENonEmpty}[9]\texttt{E}U[8] : \sum x \in L \,.\, \beta\Big(f(x), z\Big) < \frac{\varepsilon}{2},$$
$$[y.*] := \texttt{TriangleInrq}\Big((Y, \beta), y, z, f(x)\Big)[7][10] : d\Big(y, f(x)\Big) \leq d(y, z) + d\Big(z, f(x)\Big) < \varepsilon;$$
$$\rightsquigarrow [7] := \texttt{I}\forall : \forall y \in f(K) \,.\, \exists z \in f(L) \,.\, \beta(y, z) < \varepsilon,$$
$$[\varepsilon.*] := \texttt{I}\beta_{\mathrm{H}}[6][7] : \beta_{\mathrm{H}}\Big(f(K), f(L)\Big) < \varepsilon;$$
$$\rightsquigarrow [*] := \texttt{EpsilonDeltaContinuity} : f^* \in \mathsf{TOP}\Big(\mathsf{K}(X), \mathsf{K}(Y)\Big);$$

$\square$

$\texttt{HausdorffMetricProduct} :: \forall X, Y : \texttt{Metrizable} \,.\, (\times) \in \mathsf{TOP}\Big(\mathsf{K}(X) \times \mathsf{K}(Y), \mathsf{K}(X \times Y)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{FiniteSetsSetIsFSigma} :: \forall X : \texttt{Metrizable} \,.\, \texttt{Finite}(X) \in F_\sigma\Big(\mathsf{K}(X)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{PerfectComapctsSetIsGDelta} :: \forall X : \texttt{Metrizable} \,.\, \texttt{Perfect} \,\&\, \texttt{CompactSubset}(X) \in G_\delta\Big(\mathsf{K}(X)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$$\texttt{treeSet} :: \prod_{A \in \mathsf{SET}} ?\Big(A^* \to \mathbb{B}\Big)$$

$$\texttt{treeSet}\,() = \mathrm{Tr}(A) := \texttt{set}\Big(\texttt{Tree}(A)\Big)$$

$$\texttt{prunedTreeSet} :: \prod_{A \in \mathsf{SET}} ?\Big(A^* \to \mathbb{B}\Big)$$

$$\texttt{prunedTreeSet}\,() = \mathrm{PTr}(A) := \texttt{set}\Big(\texttt{Pruned}(A)\Big)$$

$\texttt{NatTreeSetIsClosed} :: \texttt{Closed}\Big(\mathrm{Tr}(\mathbb{N}), \mathbb{B}^{\mathbb{N}^*}\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ T : \mathbb{N} \to \mathrm{Tr}(\mathbb{N}),$

$\texttt{Assume}\ [1] : \texttt{Converging}\Big(\mathbb{B}^{\mathbb{N}^*}, T\Big),$

$S := \lim_{n \to \infty} T_n \in \mathbb{B}^{\mathbb{N}^*},$

$\texttt{Assume}\ x \in \mathbb{B}^{\mathbb{N}^*},$

$\texttt{Assume}\ [2] : S(x) = 1,$

$\Big(N, [3]\Big) := \texttt{EproductTopology}[2]\texttt{E}S : \sum N \in \mathbb{N} \,.\, \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow T(x) = 1,$

$[4] := [3]\texttt{ETree}(T) : \forall y \in \mathbb{N}^* \,.\, y \subset x \Rightarrow \forall n \in \mathbb{N} \,.\, n \geq N \Rightarrow T(y) = 1,$

$[5] := \texttt{EproductTopology}[4]\texttt{I}S : \forall y \in \mathbb{N}^* \,.\, y \subset x \Rightarrow S(y) = 1,$

$[6] := \texttt{E}\,\mathrm{Tr}(\mathbb{N})[5] : X \in \mathrm{Tr}(\mathbb{N});$

$\rightsquigarrow [*] := \texttt{ClosedByConvergence} : \texttt{Closed}\Big(\mathbb{B}^{(*\mathbb{N})}, \mathrm{Tr}(\mathbb{N})\Big);$

$\square$

NatPrunedTreeSetIsGDelta :: $\mathrm{PTr}(\mathbb{N}) \in G_\delta\left(\mathbb{B}^{\mathbb{N}^*}\right)$

Proof $=$

$[1] := $ NatTreeSetIsClosed ClosedIsGDeltaInPolish : $\mathrm{Tr}(\mathbb{N}) \in G_\delta\left(\mathbb{B}^{\mathbb{N}^*}\right)$,

$[2] := $ E $\mathrm{PTr}(\mathbb{N}) : \mathrm{PTr}(\mathbb{N}) = \mathrm{Tr}(\mathbb{N}) \cap \bigcap_{\emptyset \neq w \in \mathbb{N}^*} N_{w=0} \cup \bigcap_{w \subset u} N_{u=1}$,

$[3] := [1][2]$GdeltaIntersectionIsGdelta FiniteGdeltaUnionIsGdelta : $\mathrm{PTr}(\mathbb{N}) \in G_\delta\left(\mathbb{B}^{\mathbb{N}^*}\right)$;

□

BoolTreeSetIsClosed :: Closed$\left(\mathrm{Tr}(\mathbb{B}), \mathbb{B}^{\mathbb{B}^*}\right)$

Proof $=$

. . .

□

BoolPrunedTreeSetIsClosed :: Closed$\left(\mathrm{PTr}(\mathbb{B}), \mathbb{B}^{\mathbb{B}^*}\right)$

Proof $=$

. . .

□

BodyBijectionIsHomeo :: $\mathsf{K}(\mathcal{C}) \cong_{\mathsf{TOP}} \mathrm{PTr}(\mathbb{B})$

Proof $=$

. . .

□

### 1.2.8 Locally Compact Spaces

$\texttt{LocallyCompactIsPolishIffSecondCountable} :: \forall X : \texttt{LocallyCompact}\ .$
$\quad . \texttt{Polish}(X) \iff \texttt{Metrizable}\ \&\ \texttt{SecondCountable}(X)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{LocallyCompactIsPolishIffMetAndSigma} :: \forall X : \texttt{LocallyCompact}\ .$
$\quad . \texttt{Polish}(X) \iff \texttt{Metrizable}(X)\ \&\ \sigma\texttt{-Compact}(X)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{LocallyCompactPolishIffCompactlyMetrizable} :: \forall X : \texttt{LocallyCompact}\ .$
$\quad . \texttt{Polish}(X) \iff \texttt{CompactlyMetrizable}(X)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{LocallyCompactPolishIffOpenSubset} :: \forall X : \texttt{LocallyCompact}\ .$
$\quad . \texttt{Polish}(X) \iff \texttt{Metrizable}(X)\ \&\ \exists Y : \texttt{CompactlyMetrizable} : \exists U \in \mathcal{T}(X)\ .\ U \cong_{\mathsf{TOP}} Y$

$\texttt{Proof}\ =$

...

$\square$

### 1.2.9 Cantor's schemes

$$\texttt{CantorSchema} :: \prod_{X \in \mathsf{SET}} \mathbb{B}^* \to ?X$$

$$A : \texttt{CantorSchema} \iff \Big(\forall s \in \mathbb{B}^* \;.\; A_{s0} \cap A_{s1} = \emptyset\Big) \;\&\; \Big(\forall t, s \in \mathbb{B}^* \;.\; t \subset s \Rightarrow A_s \subset A_t\Big)$$

$$\texttt{TopologicalSchema} :: \prod (X, d) \in \mathsf{MS} \;.\; ?\texttt{CantorSchema}(X)$$

$$U : \texttt{TopologicalSchema} \iff \Big(\forall s \in \mathbb{B}^* \;.\; U_s \in \mathcal{T}(X) \;\&\; U_s \neq \emptyset \;\&\; \operatorname{diam}(U_s) \leq 2^{-\operatorname{len}(s)}\Big) \;\&$$
$$\& \;\forall s, t \in \mathbb{B}^* \;.\; s \subset t \Rightarrow \overline{U}_t \subset U_s$$

$\texttt{EmbeddingOfCantorSetBySchema} :: \forall X : \texttt{Polish} \;.\; \forall U : \texttt{MetricSchema}(X) \;.\; \exists \texttt{TopologicalEmbedding}(\mathcal{C}, X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x \in \mathcal{C},$

$[1] := \Big(\Lambda n \in \mathbb{Z}_+ \;.\; \texttt{EMetricSchema}(X, U)(x_{|[1,\ldots,n]})[2]\Big)\texttt{ReductioInfima} : \lim_{n\to\infty} \operatorname{diam} \overline{U}_{x_{|[1,\ldots,n]}} = 0,$

$\Big(f(x), [x.*]\Big) := \texttt{CantrorIntersectionTHM}[1] : \sum f(x) \in X \;.\; f(x) = \bigcup_{n=0}^{\infty} \overline{U}_{x_{|[1,\ldots,n]}};$

$\rightsquigarrow \Big(f, [1]\Big) := \texttt{I} \to \texttt{I} \sum : \sum f : \mathcal{C} \to X \;.\; \forall x \in \mathcal{C} \;.\; f(x) = \bigcup_{n=0}^{\infty} \overline{U}_{x_{|[1,\ldots,n]}},$

$[2] := [1]\texttt{EMetricSchema}(U)\texttt{ContinuousByConvergence} : f \in \mathsf{TOP}(\mathcal{C}, X),$

$[3] := [1]\texttt{ECantorSchema}(U)\texttt{IInjective} : \texttt{Injective}\Big(\mathcal{C}, X, f\Big),$

$[4] := \texttt{ProperByCompactDomain}(\mathcal{C}, X, f) : \texttt{ProperMap}(\mathcal{C}, X, f),$

$[5] := \texttt{FirstCountableIsCG}(X) : X \in \mathsf{CG},$

$[*] := \texttt{InjectiveProperIsEmbedding}[2][3][4][5] : \texttt{TopologicalEmbedding}(\mathcal{C}, X, f);$

$\square$

$\texttt{MetricSchemaExists} :: \forall X : \texttt{Polish} \;\&\; \texttt{Perfect} \;.\; \forall d : \texttt{CompleteMetric}(X) \;.\; X \neq \emptyset \Rightarrow \exists \texttt{MetricSchema}(X, d)$

$\texttt{Proof} \;=$

$U_\emptyset := X \in \mathcal{T}(X) \;\&\; \texttt{NonEmpty},$

$\texttt{Assume}\ n \in \mathbb{Z}_+,$

$\texttt{Assume}\ s \in \mathcal{B}^*,$

$\texttt{Assume}\ [1] : \operatorname{len}(s) = n,$

$x := \texttt{ENonEmpty}(s) \in U_s,$

$\Big(y, [2]\Big) := \texttt{EPerfect}(X)(x)\texttt{EIsolatedPoint}(x, U_s) : \sum y \in U_s \;.\; x \neq y,$

$r := \min\left(2^{-1-n}, \dfrac{d(x, y}{3}, d(x, \partial U_s), d(y, \partial U_t)\right) : \mathbb{R}_{++},$

$U_{s0} := \mathbb{B}_d(x, r) \in \mathcal{T}(X) \;\&\; \texttt{NonEmpty},$

$U_{s1} := \mathbb{B}_d(y, r) \in \mathcal{T}(X) \;\&\; \texttt{NonEmpty},$

$[s.1] := \texttt{E}r\texttt{E}U_{s0}\texttt{E}U_{s1} \in U_{s0} \cap U_{s1} = \emptyset,$

$[s.2] := \texttt{E}r\texttt{E}U_{s0} : \overline{U}_{s0} \subset U_s,$

$[s.3] := \texttt{E}r\texttt{E}U_{s1} : \overline{U}_{s1} \subset U_s,$

$[s.4] := \texttt{E}r\texttt{E}U_{s0} : \operatorname{diam} U_{s0} < 2^{-n};$

$[n.s.*] := \texttt{E}r\texttt{E}U_{s1} : \operatorname{diam} U_{s1} < 2^{-n};$

$\rightsquigarrow [*] := \texttt{IMetricSchema} : \texttt{MetricSchema}(U);$

$\square$

$\texttt{CantorSetEmbedding} :: \forall X : \texttt{Polish} \;\&\; \texttt{Perfect} \;.\; X \neq \emptyset \Rightarrow \exists \texttt{TopologicalEmbedding}(X, \mathcal{C})$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ParfectPolishCardinality} :: \forall X : \texttt{Polish} \;.\; \forall A : \texttt{PerfectSubset}(X) \;.\; |A| = 2^{\aleph_0}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{PolishContinuumTHM} :: \forall X : \texttt{Polish} \;.\; |X| > \aleph_0 \Rightarrow |X| = 2^{\aleph_0}$

$\texttt{Proof} \;=$

$\dots$

$\square$

### 1.2.10 Cantor-Bendixson's ranks

$\texttt{MonotonicicOrdinalTopologicalBound} ::$

$:: \forall X : \texttt{SecondCountable} \,.\, \forall a \in \mathsf{ORD} \,.\, \forall F : \texttt{StrictlyDecreasing}\Big(a, \texttt{Closed}(X)\Big) \,.\, |a| \leq \aleph_0$

$\texttt{Proof} \;=$

$\mathcal{V} := \texttt{ESecondCountable}(X) : \sum \mathcal{V} : \texttt{Base}(X) \,.\, |\mathcal{V}| \leq \aleph_0,$

$V := \texttt{enumerate}(\mathcal{V}) : \Big[0, \ldots |\mathcal{V}|\Big) \leftrightarrow \mathcal{V},$

$N := \Lambda i \in a \,.\, \{n \in \mathbb{N} | F \cap V_n \neq \emptyset\} : a \to ?\mathbb{N},$

$[1] := \texttt{E}NEF : \texttt{StrictlyDecreasing}(a, ?\mathbb{N}, N),$

$M := \Lambda i \in a \,.\, \texttt{if } i = \max(a) \texttt{ then } N_i \texttt{ else } N_i \setminus N_{\sigma(i)} : a \to ?\mathbb{N},$

$[2] := \texttt{EStrictlyDereasing}(a, ?\mathbb{N}, N)\texttt{E}M : \forall i \in a \,.\, i \neq \max a \Rightarrow M_a \neq \emptyset,$

$[3] := \texttt{E}M\texttt{Esetminus} : \forall i, j \in a \,.\, i \neq j \Rightarrow M_i \cap M_j = \emptyset,$

$m := \texttt{Choice}[2] \in \prod_{i<a} M_i,$

$[4] := \texttt{E}m[3]\texttt{IInjective} : \texttt{Injective}(a, \mathbb{N}, m),$

$[*] := \texttt{InjectionCardinalityBound}[4]\texttt{NaturalNumbersAreCountable} : |a| \leq \aleph_1;$

$\square$

$\texttt{derivativeOfCantorBendixon} :: \mathsf{ORD} \to \mathsf{TOP} \to \mathsf{TOP}$

$\texttt{derivativeOfCantorBendixon}\,(0, X) = \mathrm{d}^0 X := X$

$\texttt{derivativeOfCantorBendixon}\,(\sigma(a), X) = \mathrm{d}^{\sigma(a)} X := \lim \mathrm{d}^a X$

$\texttt{derivatveOfCantorBendixon}\,(a, X) = \mathrm{d}^a X := \lim \bigcap_{n<a} \mathrm{d}^n X$

$\texttt{CantorBendixonRankExists} :: \forall X : \texttt{Polish} \, . \, \exists a \in [0, \epsilon_0) \, . \, \forall b \geq a \, . \, \mathrm{d}^b X = \mathrm{d}^a X$

$\texttt{Proof} \ =$

$[1] := \Lambda a \in [0, \epsilon_0] \, . \, \mathrm{d}^a X : \texttt{Decreasing}\Big([0, \epsilon_0], \texttt{Closed}(X)\Big),$

$[2] := \texttt{MonotonicOrdinalTopologicalBound}[1] : \neg\texttt{StrictlyDecreasing}\Big([0, \epsilon_0], \texttt{Closed}(X), \Lambda a \in [0, \epsilon_0] \, . \, \mathrm{d}^a X\Big)$

$\Big(a, [3]\Big) := \texttt{EStrictlyDecreasing} : \sum a \in \mathsf{ORD} \, . \, \mathrm{d}^{\sigma(a)} X = \mathrm{d}^a X,$

$\texttt{Assume}\ b \in \mathsf{ORD},$

$\texttt{Assume}\ [0] : b \geq a,$

$\texttt{Assume}\ [4] : \forall c \in [a, b] \, . \, \mathrm{d}^c X = \mathrm{d}^a X,$

$[b.*] := \texttt{E}\mathrm{d}[4](b)\texttt{E}\mathrm{d}[3] : \mathrm{d}^{\sigma(b)} X = \mathrm{d}\mathrm{d}^b X = \mathrm{d}\mathrm{d}^a X = \mathrm{d}^{\sigma(a)} X = \mathrm{d}^a X;$

$\rightsquigarrow [4] := \texttt{I}\forall\texttt{I} \Rightarrow : \forall b \in \mathsf{ORD} \, . \, (b \geq a) \Rightarrow \Big(\forall c \in [a, b])\mathrm{d}^c X = \mathrm{d}^c X\Big) \Rightarrow \mathrm{d}^{\sigma(b)} X = \mathrm{d}^a X,$

$\texttt{Assume}\ b : \texttt{Limit},$

$\texttt{Assume}\ [0] : b \geq a,$

$\texttt{Assume}\ [5] : \forall c \in [a, b) \, . \, \mathrm{d}^{\sigma(c)} X = \mathrm{d}^a X,$

$b.* := \texttt{E}\mathrm{d}\texttt{EDecreasing}[5]\texttt{E}\mathrm{d} : \mathrm{d}^b X = \mathrm{d} \bigcap_{c<b} \mathrm{d}^c X = \mathrm{d}\mathrm{d}^a X = \mathrm{d}^{\sigma(a)} X = \mathrm{d}^a X;$

$\rightsquigarrow [5] := \texttt{I}\forall\texttt{I} \Rightarrow : \forall b \in \texttt{Limit} \, . \, (b \geq a) \Rightarrow \Big(\forall c \in [a, b) \, . \, \mathrm{d}^c X = \mathrm{d}^c X\Big) \Rightarrow \mathrm{d}^b X = \mathrm{d}^a X,$

$[6] := \texttt{I}(=)(\mathrm{d}^a X) : \mathrm{d}^a X = \mathrm{d}^a X,$

$[*] := \texttt{TransfinitieInduction}[6][4][5] : \forall b \geq a \, . \, \mathrm{d}^b = \mathrm{d}^a X;$

$\square$

$\texttt{rankOfCantorBendixson} :: \texttt{Polish} \to \epsilon_0$

$\texttt{rankOfCantorBendixon}\,(X) = \mathrm{rank}_{\mathrm{CB}}\, X := \min\{a \in \mathsf{ORD} : \mathrm{d}^{\sigma(a)} X = \mathrm{d}^a X\}$

$\texttt{PerfectTree} :: \prod_{A \in \mathsf{SET}} \texttt{Tree}(A)$

$T : \texttt{PerfectTree} \iff \forall t \in T \, . \, \exists a, b \in T : t \subset a \,\&\, t \subset b \,\&\, a \perp b$

$\texttt{PerfectIsPruned} :: \forall A \in \mathsf{SET} \, . \, \forall T : \texttt{PerfectTree}(A) \, . \, \texttt{Prunded}(A, T)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PerfectBodyTHM} :: \forall A \in \mathsf{SET} \, . \, \forall T : \texttt{Pruned}(A) \, . \, \texttt{PerfectTree}(A, T) \iff \texttt{Perfect}\big([A]\big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

## 1.3 Zero Dimensional Spaces and Schemas

### 1.3.1 Dimension Zero

$\texttt{ZeroDimensional} :: ?\mathsf{TOP}$

$X : \texttt{ZeroDimensional} \iff \dim_{\mathsf{TOP}} X = 0 \iff \exists \mathcal{V} : \texttt{Base}\big(\mathcal{T}(X)\big) : \forall V \in \mathcal{V} \,.\, \texttt{Clopen}(X, V)$

$\texttt{UltrametricTrianglesAreEquiliterals} ::$

$:: \forall X : \texttt{UltrametricSpace} \,.\, \forall x, y, z \in X \,.\, d(x,z) \neq d(y,z) \Rightarrow d(x,y) = \max\Big(d(x,z), d(y,z)\Big)$

$\texttt{Proof} =$

$[1] := \texttt{EUltrametric}(X, d, x, y, z)\texttt{ESymmetric}(x, d, y, z) : d(x,y) \leq \max\Big(d(x,z), d(y,z)\Big),$

$\texttt{Assume}\ [2] : d(x,y) < \max\Big(d(x,z), d(y,z)\Big),$

$[3] := \texttt{EUltrametric}(X, d, x, z, y) : d(x,z) \leq \max\Big(d(x,y), d(y,z)\Big),$

$[4] := \texttt{EUltrametric}(X, d, y, z, x) : d(y,z) \leq \max\Big(d(y,x), d(x,z)\Big),$

$[5] := [0][3][4] : d(x,z) \leq d(x,y) | d(y,z) \leq d(x,y),$

$[6] := [5][2] : d(x,z) \leq d(x,y) < d(y,z) | d(y,z) \leq d(x,y) < d(x,z),$

$[2.*] := [6][3][4] : \bot;$

$\leadsto [*] := \texttt{TrichtomyPrinciple} : d(x,y) = \max\Big(d(x,z), d(y,z)\Big),$

$\square$

$\texttt{UltrametricAreZeroDim} :: \forall X : \texttt{UltrametricSpace} \,.\, \dim_{\mathsf{TOP}} X = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ r \in \mathbb{R}_{++},$

$\texttt{Assume}\ b : \mathbb{N} \to \mathbb{B}_X(x,r),$

$\texttt{Assume}\ [1] : \texttt{Converging}(X,b),$

$L := \lim_{n\to\infty} b_n \in X,$

$t := d(L,x) : \mathbb{R}_{++},$

$\texttt{Assume}\ [2] : t = 0,$

$[3] := \texttt{EMetric}[2] : x = L,$

$[*] := [3]\texttt{Ecell}(X)(x,r) : L \in \mathbb{B};$

$\rightsquigarrow [2] := \texttt{I}(\Rightarrow) : t = 0 \Rightarrow L \in \mathbb{B}_X(x,r),$

$\texttt{Assume}\ [3] : t > 0,$

$\Big(n,[4]\Big) := \texttt{ELimit}(L) : \sum n \in \mathbb{N} \,.\, d(u_n,L) < t,$

$[5] := \texttt{Ecell}(X)(x,r)(u_n) : d(u_n,x) < r,$

$[6] := \texttt{UltrametricTriangleAreEquilitertal}(X,u_n,x,L) : d(L,x) = d(u_n,x) \Big| d(L,x) = d(u_n,L),$

$[7] := \texttt{LimitMetricE}t : t \le r,$

$[8] := [6][4][5][7] : d(L,x) < r,$

$[*] := \texttt{Ecell}(X)(X,d)[6] : L \in \mathbb{B}_X(x,r);$

$\rightsquigarrow [3] := \texttt{I}(\Rightarrow) : t \neq 0 \Rightarrow L \in \mathbb{B}_X(x,r),$

$[u.*] := \texttt{ELEM}(t=0)[2][3] : L \in \mathbb{B}_X(x,r);$

$\rightsquigarrow [1] := \texttt{ClosedByLimits} : \texttt{Closed}\Big(X, \mathbb{B}_X(x,r)\Big),$

$[x.*] := \texttt{IClopen} : \texttt{Clopen}\Big(X, \mathbb{B}_X(x,r)\Big);$

$\rightsquigarrow [*] := \texttt{IZeroDimensional} : \dim_{\mathsf{TOP}} X = 0;$

$\square$

$\texttt{InUltrametricAllBallPointsAreCenters} ::$

$:: \forall X : \texttt{UltrametricSpace} \,.\, \forall x \in X \,.\, \forall r \in \mathbb{R}_{++} \,.\, \forall y \in \mathbb{B}_d(x,r) \,.\, \mathbb{B}_X(x,r) = \mathbb{B}_X(y,r)$

$\texttt{Proof} \;=$

$t := d(x,y) : \mathbb{R}_+,$

$[1] := \mathrm{E}\mathbb{B}_d(x,r)(y)\mathrm{E}t : t < r,$

$\texttt{Assume}\ u \in \mathbb{B}_X(x,r),$

$s := d(x,u) : \mathbb{R}_+,$

$[2] := \mathrm{E}\mathbb{B}_d(x,r)(u)\mathrm{E}s : s < r,$

$[3] := \texttt{UltrametricTriangleAreAllEquiliteral}(X,x,y,u)\texttt{IsI}t : d(y,u) = s \Big| d(y,u) = t,$

$[4] := [1][2][3] : d(y,u) < r,$

$[u.*] := \mathrm{E}\mathbb{B}_X(y,r)[4] : u \in \mathbb{B}_X(y,r);$

$\rightsquigarrow [2] := \texttt{I} \subset: \mathbb{B}_X(x,r) \subset \mathbb{B}_X(y,r),$

$\texttt{Assume}\ u \in \mathbb{B}_X(y,r),$

$s := d(y,u) : \mathbb{R}_+,$

$[3] := \mathrm{E}\mathbb{B}_d(y,r)(u)\mathrm{E}s : s < r,$

$[4] := \texttt{UltrametricTriangleAreAllEquiliteral}(X,x,y,u)\texttt{IsI}t : d(x,u) = s \Big| d(x,u) = t,$

$[5] := [1][3][4] : d(y,u) < r,$

$[u.*] := \mathrm{E}\mathbb{B}_X(x,r)[4] : u \in \mathbb{B}_X(x,r);$

$\rightsquigarrow [2] := \texttt{I} \subset: \mathbb{B}_X(y,r) \subset \mathbb{B}_X(x,r),$

$[*] := \texttt{ISetEq}[1][2] : \mathbb{B}_X(y,r) = \mathbb{B}_X(x,r);$

$\square$

$\texttt{UltrametricIntersectionImplyContainment} ::$

$:: \forall X : \texttt{UltrametricSpace} \,.\, \forall x,y \in X \,.\, \forall r,s \in \mathbb{R}_{++} \,.\, \mathbb{B}_X(x,r) \cap \mathbb{B}_X(y,s) \,.$

$\mathbb{B}_X(x,r) \subset \mathbb{B}_X(y,s) \Big| \mathbb{B}_X(y,s) \subset \mathbb{B}_X(x,r)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UltrametricCauchySequences} ::$

$:: \forall X : \texttt{UltrametricSpace} . \forall x : \mathbb{N} \to X . \texttt{Cauchy}(X, x) \iff \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(N, [2]\Big) := [1]\texttt{ELimit} : \sum N \in \mathbb{N} . \forall n \in \mathbb{N} . n \ge N \Rightarrow d(x_n, x_{n+1}) < \varepsilon,$

$\texttt{Assume}\ m \in \mathbb{N},$

$\texttt{Assume}\ [3] : m > N,$

$\texttt{Assume}\ [4] : \forall k \in [n+1, \dots, m-1] . d(x_N, x_k) < \varepsilon,$

$[5] := [4](m-1) : d(x_N, x_{m-1}) < \varepsilon,$

$[6] := [2](m-1) : d(x_{m-1}, x_m) < \varepsilon,$

$[m.*] := \texttt{EUltrametric}[5][6] : d(x_N, x_m) < \varepsilon;$

$\rightsquigarrow [3] := \mathrm{E}\mathbb{N} : \forall n \in \mathbb{N} . n \ge N \Rightarrow d(x_N, x_n) < \varepsilon;$

$\texttt{Assume}\ n, m \in \mathbb{N},$

$\texttt{Assume}\ [4] : n, m \ge N,$

$[5] := [3](n) : d(x_N, x_n) < \varepsilon,$

$[6] := [3](m) : d(x_N, x_m) < \varepsilon,$

$\Big[(n, m). *\Big] := \texttt{EUltrametric}[5][6] : d(x_n, x_m) < \varepsilon;$

$\rightsquigarrow [\varepsilon.*] := \mathrm{E}\forall : \forall n, m \in \mathbb{N} . n, n \ge N \Rightarrow d(x_n, x_m) < \varepsilon;$

$\rightsquigarrow [x.*] := \texttt{ICauchy} : \texttt{Cauchy}(X, x);$

$\rightsquigarrow [*] := \mathrm{I} \Rightarrow: \lim_{n\to\infty} d(x_n, x_{n+1}) \Rightarrow \texttt{Cauchy}(X, x);$

$\square$

$\texttt{ClopenSeparation} :: \forall X : \texttt{SecondCountable}\ \dim_{\mathsf{TOP}} X = 0 \Rightarrow$

$\Rightarrow \forall A, B : \texttt{ClosedSet}(X) . A \cap B = \emptyset \Rightarrow \exists C : \texttt{Clopen}(C) . A \subset C \ \&\ C \cap B = \emptyset$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{KuratowskiZeroDimensionalChar} ::$

$:: \forall X \in \mathsf{TOP} . \dim_{\mathsf{TOP}} X = 0 \iff \forall A : \texttt{Closed}(X) . \texttt{Retract}(X, A)$

$\texttt{Proof} =$

$\dots$

$\square$

### 1.3.2 Cantor space

$\texttt{BrouwerSchema} :: \prod_{X \in \mathsf{MS}} ?\texttt{MetricSchema}$

$U : \texttt{BrouwerSchema} \iff U_\emptyset \forall s \in \mathbb{B}^* \,.\, \texttt{Clopen}(X, U_s) \;\&\; U_s = U_{s1} \cup U_{s2}$

$\texttt{BrouwerSchemaInducesHomeomorphism} :: \forall X : \texttt{Polish} \;\&\; \texttt{MetricSpace} \,.\, \forall U : \texttt{BrouwerSchema}(X) \,.\, X \cong_{\mathsf{TOP}} \mathcal{C}$

$\texttt{Proof} \;=$

$\texttt{BrouwerSchemaExists} :: \forall X : \texttt{Perfect} \;\&\; \texttt{CompactMetrizable} \;\&\; \texttt{NonEmpty} \,.\, \forall d : \texttt{Metric}(X) \,.$

$\quad .\, (X, d) \cong_{\mathsf{TOP}} X \;\&\; \dim_{\mathsf{TOP}} X = 0 \Rightarrow \exists \texttt{BrouwerSchema}(X)$

$\texttt{Proof} \;=$

$U_\emptyset := X : \texttt{Clopen}(X),$

$S_1 := \{\emptyset\} :?\mathbb{B}^*,$

$S_0 := \emptyset :?\mathbb{B}^*,$

$\texttt{Assume}\; n : \mathbb{N},$

$s := \min S_n \in \mathbb{B}^*,$

$\Big(\mathcal{V}, [1]\Big) := \texttt{ECompact}(X)\texttt{E}\dim_{\mathsf{TOP}} X = 0 : \sum \mathcal{V} : \texttt{Finite}\Big(\texttt{Clopen}(X)\Big) \,.\, \forall V \in \mathcal{V} \,.\; \operatorname{diam} V < \frac{1}{n},$

$m := |\mathcal{V}| \in \mathbb{N},$

$V := \texttt{enumerate}(\mathcal{V}) : [1, \dots, m] \leftrightarrow \mathcal{V},$

$\texttt{Assume}\; i \in [1, \dots, m],$

$U_{s0^i} := \bigcup_{j=i}^{n} V_i : \texttt{Clopen}(X),$

$U_{s0^{i-1}1} := V_i : \texttt{Clopen}(X);$

$\rightsquigarrow U := \texttt{I} \to: \{s0^i | i \in [1, \dots, m]\} \cup \{s0^{i-1}1 | i \in [1, \dots, m]\} \to \texttt{Clopen}(X),$

$S_{n+1} := \Big(S_n \setminus \{s\}\Big) \cup \{s0^m\} \cup \Big\{s0^{i-1}i | i \in [1, \dots, m]\Big\} :?;^*$

$\rightsquigarrow U := \texttt{IBrouwerSchema} : \texttt{BrouwerSchema}(X);$

$\square$

$\texttt{BrouwerTopologicalCharOFCantorSpace} ::$

$\quad :: \forall X : \texttt{Perfect} \;\&\; \texttt{CompactMetrizable} \;\&\; \texttt{NonEmpty} \,.\; \dim_{\mathsf{TOP}} X = 0 \Rightarrow X \cong_{\mathsf{TOP}} \mathcal{C}$

$\texttt{Proof} \;=$

$\dots$

$\square$

### 1.3.3 Lusin's schema

$$\texttt{LusinSchema} :: \prod_{X \in \mathsf{TOP}} \mathbb{N}^* \to ?X$$

$$L : \texttt{LusinSchema} \iff \Big(\forall s \in \mathbb{N}^* \,.\, \forall n, m \in \mathbb{N} \,.\, n \neq m \Rightarrow L_{sn} \cap L_{sm} = \emptyset\Big) \;\&\; \Big(\forall s, t \in \mathbb{N}^* \,.\, s \subset t \Rightarrow L_t \subset L_s\Big) \Longleftarrow$$

$$\texttt{VanishingDiameter} :: \prod_{X \in \mathsf{MS}} \texttt{LusinSchema}(X)$$

$$L : \texttt{VanishingDiameter} \iff \forall b \in \mathcal{B} \,.\, \lim_{n \to \infty} \operatorname{diam} L_{b_{|[1,\dots,n]}} = 0$$

$$\texttt{domainOfLusin} :: \prod_{X \in \mathsf{MS}} \texttt{VanishingDiameter}(X) \to ?\mathcal{B}$$

$$\texttt{LusinDomain}\,(L) = D(L) := \left\{ b \in \mathcal{B} : \bigcap_{n=1}^{\infty} L_{b_{|[1,\dots,n]}} \neq \emptyset \right\}$$

$$\texttt{associatedMap} :: \prod_{X \in \mathsf{MS}} \prod L : \texttt{VanishingDiameter}(X) \,.\, D(L) \to X$$

$$\texttt{associatedMap}\,(b) = f_L(b) := \texttt{ESingleton}(X) \bigcap_{n=1}^{\infty} L_{b_{|[1,\dots,n]}}$$

$$\texttt{associatedMapIsContinuousInjection} :: \forall X \in \mathsf{MS} \,.\, \forall L : \texttt{VanishingDiameter}(X) \,.$$
$$.\, f_L \in \mathsf{TOP} \;\&\; \texttt{Injective}\Big(D(L), X\Big)$$

Proof $=$

...

□

$$\texttt{AssociatedMapIsContinuousInjection} :: \forall X \in \mathsf{MS} \,.\, \forall L : \texttt{VanishingDiameter}(X) \,.$$
$$.\, \mathsf{TOP} \;\&\; \texttt{Injective}\Big(D(L), X, f_L\Big)$$

Proof $=$

...

□

$\texttt{ClosedLusinDomain} :: \forall X : \texttt{Complete} . \forall L : \texttt{VanishingDiameter}(X) . \left(\forall t \in L^* . \texttt{Closed}(X, L_t)\right) \Rightarrow$

$\Rightarrow \texttt{Closed}\Big(\mathcal{B}, D(L)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ b \in D^{\complement}(L),$

$[1] := \texttt{E}D(L)(K) : \bigcap_{n=1}^{\infty} L_{b_{|[1,\dots,n]}} = \emptyset,$

$\Big(n, [2]\Big) := \texttt{CantorIntersectionTHM}(L_{b_{|[1,\dots,\bullet]}})[2] : L_{b_{|[1,\dots,n]}} = \emptyset,$

$t := b_{|[1,\dots,n]} : \mathbb{N}^*,$

$[b.*] := [2]\texttt{E}t : b \in N_t \subset D^{\complement}(L);$

$\rightsquigarrow [1] := \texttt{OpenByOpenCover} : D^{\complement}(L) \in \mathcal{T}(\mathcal{B}),$

$[*] := \texttt{IClosed}[1] : \texttt{Closed}\Big(\mathcal{B}, D(L)\Big);$

$\square$

$\texttt{AssociatedMapEmbeddingCondition} :: \forall X \in \mathsf{MS} . \forall L : \texttt{VanishingDiameter}(X) .$

$. \left(\forall t \in L^* . \texttt{Open}(X, L_t)\right) \Rightarrow \texttt{TopologicalEmbedding}\Big(D(L), X, f\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{AlexandrovUryshonSchema} :: \prod X : \texttt{Complete} . ?\texttt{VanishingDiameter}$

$U : \texttt{AlexandrovUryshonSchema} \iff U_{\emptyset} = X$

$\quad \forall t \in \mathbb{N}^* . U_t \neq \emptyset$

$\quad \forall t \in \mathbb{N}^* . \texttt{Clopen}(U_t)$

$\quad \forall t \in \mathbb{N}^* . U_t = \bigcup_{n=1}^{\infty} U_{tn}$

$\quad \forall t \in \mathbb{N}^* . \ \operatorname{diam} U_t \leq 2^{-\operatorname{len}(t)}$

$\texttt{Wiry} :: ?\mathsf{TOP}$

$X : \texttt{Wiry} \iff \forall K : \texttt{Compact}(X) . \ \operatorname{int} K = \emptyset$

$\texttt{AlexandrovUryshonTHM} :: \forall X : \texttt{Polish} \ \& \ \texttt{NonEmpty} \ \& \ \texttt{Wiry} \ . \ \dim_{\mathsf{TOP}} X = 0 \Rightarrow$

$\Rightarrow \exists \texttt{AlexandrovUryshonSchema}(X)$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{EPolish}(X) : \sum d : \texttt{Metric}(X) \ .$

$. \ (X, \alpha) \cong_{\mathsf{TOP}} X \ \& \ \texttt{Complete}(X, d),$

$U_\emptyset := X : \texttt{Clopen}(X),$

$\texttt{Assume} \ b \in \mathbb{N}^*,$

$[2] := \texttt{EclosureEinteriot} : \emptyset \neq U_b \subset \operatorname{int} \overline{U}_b,$

$[3] := \texttt{EWiry}[2] : \neg\texttt{Compact}\Big(X, \overline{U}_b\Big),$

$[4] := \texttt{MetricCompact}(X, d)\texttt{ClosedIsComplete}(X, d)[3] : \neg\texttt{TotallyBounded}\Big(X, \overline{U}_b\Big),$

$\Big(\mathcal{V}, [5]\Big) := \texttt{ETotallyBounded}[5]\texttt{EZeroDimensional}(X) :$

$: \sum \mathcal{V} : \texttt{ClopenCover} \ \& \ \texttt{IrreducibleCover} \ \& \ \texttt{Disjoint}(X, \overline{U}_b) \ . \ \Big(\forall V \in \mathcal{V} \ . \ \operatorname{diam} V < 2^{-1-\operatorname{len}(b)}\Big),$

$[6] := \texttt{ESecondCountable}(X)\texttt{E}\mathcal{V} : |\mathcal{V}| = \aleph_0,$

$V := \texttt{enumerate}(\mathcal{V}) : \mathbb{N} \hookrightarrow \mathcal{V},$

$\texttt{Assume} \ n \in \mathbb{N},$

$U_{bn} := V_n : \texttt{Clopen}(X) \ \& \ \texttt{NonEmpty};$

$\rightsquigarrow [b.*] := \texttt{Define} : \forall n \in \mathbb{N} \ . \ \texttt{Defined}(U_{bn});$

$\rightsquigarrow [*] := \texttt{IAlexandrovUryshonSchema} : \texttt{AlexandrovUryshonSchema}(X, U);$

$\square$

$\texttt{AlexandrovUryshonSchemaDomain} :: \forall X \in \mathsf{MS} \ . \ \forall U : \texttt{AlexandrovUryshonSchema}(X) \ . \ D(U) = X$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{AlexandrovUryshonSchemaAssociatesHomeomrphism} ::$

$:: \forall X \in \mathsf{MS} \ . \ \forall U : \texttt{AlexandrovUryshonSchema}(X) \ . \ \texttt{Homeomorphism}\Big(\mathcal{B}, X, f_U\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{BairSpaceTopChar} :: \forall X \in \mathsf{TOP} \ \texttt{Polish} \ \& \ \texttt{NonEmpty} \ \& \ \texttt{Wiry}(X) \ \& \ \dim_{\mathsf{TOP}} X = 0 \iff X \cong_{\mathsf{TOP}} \mathcal{B}$

$\texttt{Proof} \ =$

$\dots$

$\square$

### 1.3.4 Universality of Bair space

$$\texttt{EmbeddingInABairSpace} :: \forall X : \texttt{Polish} \;.\; \dim_{\mathsf{TOP}} X = 0 \Rightarrow$$
$$\Rightarrow \exists f : \texttt{TopologicalEmbedding}(X, \mathcal{B}) \;.\; \texttt{Closed}\Big(f(X), \mathcal{B}\Big)$$

$\texttt{Proof} \;=$

$\Big(d, [1]\Big) := \texttt{EPolish}(X) : \sum d : \texttt{Metric}(X) \;.$

$\quad . \; (X, \alpha) \cong_{\mathsf{TOP}} X \;\&\; \texttt{Complete}(X, d),$

$\Big(U, [2]\Big) := \texttt{EZeroDimensional}(X) : \sum U : \texttt{VanishingDiameter}(X, d) \;.\; \forall t \in \mathbb{N}^* \;.\; \texttt{Clopen}(X, U_t),$

$[3] := \texttt{ClosedLusinDomain}[2] : \texttt{Closed}\Big(\mathcal{B}, D(U)\Big),$

$[4] := \texttt{AssociatedMapEmbeddingCondition}[2] : \texttt{TopologicalEmbedding}\Big(D(U), X, f_U\Big),$

$[5] := \texttt{E}U : f(\mathcal{B}) = X,$

$[*] := [4][5] : \texttt{ToplogicalEmbedding}(X, \mathcal{B}, f_U^{-1});$

$\square$

$$\texttt{EmbeddingInABairSpace} :: \forall X : \texttt{Polish} \;.\; \dim_{\mathsf{TOP}} X = 0 \Rightarrow$$
$$\Rightarrow \exists f : \texttt{TopologicalEmbedding}(X, \mathcal{C}) \;.\; f(X) \in G_\delta\Big(\mathcal{C}\Big)$$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{BairImageTHM} :: \forall X : \texttt{Polish} . \exists A : \texttt{Closed}(\mathcal{B}) : \exists \texttt{Continuous} \ \& \ \texttt{Bijective}(A, X)$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{EPolish}(X) : \sum d : \texttt{Metric}(X) \ .$

$\quad . \ (X, \alpha) \cong_{\mathsf{TOP}} X \ \& \ \texttt{Complete}(X, d),$

$B_\emptyset := X \in F_\sigma(X),$

$\texttt{Assume} \ s \in \mathbb{N}^*,$

$\texttt{Assume} \ B_s \in F_\sigma(X),$

$\Big(C, [3]\Big) := \texttt{E}F_\sigma(X) : \sum \mathbb{N} \xrightarrow{C} \texttt{Closed}(X) : \mathsf{POSET} \ . \ B_s = \bigcup_{n=1}^{\infty} C_n,$

$\Big(E, [4]\Big) := \texttt{FSigmaClosedDifferenceDecomp}(C, 2^{-1-\operatorname{len}(s)}) : \sum E : \mathbb{N} \to \mathbb{N} \to F_\sigma(X) \ .$

$\quad \left(\forall n \in \mathbb{N} \ . \ C_{n+1} \setminus C_n = \bigcup_{i=1}^{\infty} E_{n,i}\right) \ \&$

$\quad \& \ \Big(\forall n, m \in \mathbb{N} \ . \ \forall i, j \in \mathbb{N} \ . \ (n, i) \neq (m, j) \Rightarrow E_{n,i} \cap E_{m,j} = \emptyset\Big) \ \&$

$\quad \& \ \forall n, m \in \mathbb{N} \ . \ \operatorname{diam} E_{n,m} < 2^{-1-\operatorname{len}(s)},$

$(n, m) := \texttt{enumerate}(\mathbb{N} \times \mathbb{N}) : \mathbb{N} \leftrightarrow \mathbb{N} \times \mathbb{N},$

$\texttt{Assume} \ k \in \mathbb{N},$

$B_{sk} := E_{n_k, m_k} : F_\sigma(X);$

$\rightsquigarrow [s.*] := \texttt{I}\forall : \forall k \in \mathbb{N} \ . \ B_{sk} \in F_\sigma(X);$

$\rightsquigarrow \Big(B, [3]\Big) := \texttt{ILusinSchema} : \sum B : \texttt{LusinSchema}(X) \ . \ \Big(\forall s \in \mathbb{N}^* \ . \ B_s \in F_\sigma(X)\Big) \ \&$

$\quad \& \ \left(\forall s \in \mathbb{N}^* \ . \ \operatorname{diam} B_s \leq 2^{-1-\operatorname{len}(s)}\right) \ \&$

$\quad \& \ \Big(\forall s, t \in \mathbb{N}^* \ . \ s \subset t \Rightarrow \overline{B}_t \subset B_t\Big) \ \&$

$\quad \& \ \left(\forall s \in \mathbb{N}^* B_t = \bigcup_{k=1}^{\infty} \overline{B}_{tk}\right),$

$[4] := \texttt{I}D(B)[3.4] : f_B\big(D(B)\big) = X,$

$[5] := \texttt{AssociationMapisContinuousInjecttion}[4] : \texttt{Continuous} \ \& \ \texttt{Bijection}(D(B), X, f),$

$\texttt{Assume} \ x : \mathbb{N} \to D(B),$

$\texttt{Assume} \ L \in \mathcal{B},$

$\texttt{Assume} \ [6] : L = \lim_{n \to \infty} x_n,$

$[7] := \texttt{ConvergenIsCauchy}\Big(D(B), x, [6]\Big) : \texttt{Cauchy}\Big(D(B), x\Big),$

$[8] := \texttt{ContinuousPreservesCauchy} : \texttt{Cauchy}\Big((X, d), f(x)\Big),$

$[9] := \texttt{EComplete}(X, d)[8][1] : \texttt{Converging}\Big(X, f(x)\Big),$

$y := \lim_{n \to \infty} f(x_n) : X,$

$[10] := \texttt{E}y[3.3] : y \in \bigcap \overline{B_{L_{|[1,\ldots,n]}}},$

$[x.*] := \texttt{E}D(B)[10] : L \in D(B);$

$\rightsquigarrow [*] := \texttt{ClosedByLimits} : \texttt{Closed}\Big(\mathcal{B}, D(B)\Big);$

$\square$

$\texttt{BairExtensionTHM} :: \forall X : \texttt{Polish} . \exists \texttt{Continuous} \ \& \ \texttt{Surjective}(A, X)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

### 1.3.5 Bair space as subset

$$\mathtt{HurwitzCriterion} :: \forall X : \mathtt{Polish} \,.\, \Big(\exists A : \mathtt{Closed}(X) : A \cong_{\mathsf{TOP}} \mathcal{B}\Big) \iff \neg\sigma\text{-}\mathtt{Compact}(X)$$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ A : \mathtt{Closed}(X),$

$\mathtt{Assume}\ [1] : A \cong_{\mathsf{TOP}} \mathcal{B},$

$[2] := \mathtt{BairSpaceIsNotSigmaCompact}[1] : \neg\sigma\text{-}\mathtt{Compact}\Big(A\Big),$

$[*] := \mathtt{CompacClosedSubset}(X, A)[2] : \neg\sigma\text{-}\mathtt{Compact}\Big(X\Big);$

$\leadsto [1] :\Longrightarrow (\overset{)}{\to}: \Big(\exists A \subset X : A \cong_{\mathsf{TOP}} \mathcal{B}\Big) \Rightarrow \neg\sigma\text{-}\mathtt{Compact}(X),$

$\mathtt{Assume}\ [2] : \neg\sigma\text{-}\mathtt{Compact}(X),$

$C_\emptyset := X : \mathtt{Closed}(X),$

$\mathtt{Assume}\ s \in \mathbb{N}^*,$

$\mathtt{Assume}\ C_s : \mathtt{Closed}\ \&\ \mathtt{NonEmpty}(X)\ \&\ \neg\sigma\text{-}\mathtt{Compact},$

$$H := \left\{x \in C_s : \forall U \in \mathcal{U}(x) \,.\, \neg\sigma\text{-}\mathtt{Compact}\Big(\overline{U \cap C_s}\Big)\right\} :?C_s,$$

$\mathtt{Assume}\ h : \mathtt{Converging}(H),$

$L := \lim_{n\to\infty} h_n \in C_s,$

$\mathtt{Assume}\ U : \mathcal{U}(L),$

$$\Big(n, [4]\Big) := \mathtt{NbhdConvergence}(h, L, U) : \sum_{n=1}^{\infty} h_n \in U,$$

$[U.*] := \mathtt{E}H(h_n)\Big(U, [4]\Big) : \neg\sigma\text{-}\mathtt{Compact}\Big(\overline{U \cap C_s}\Big);$

$\leadsto [h.*] := \mathtt{E}H : L \in H;$

$\leadsto [4] := \mathtt{ClosedByLimits}(X) : \mathtt{Closed}(X, H),$

$[5] := \mathtt{ENonEmpty}(C_s)\mathtt{ESecondCountable}(X)\mathtt{E}H\mathtt{E}\sigma\text{-}\mathtt{Compact} : H \neq \emptyset,$

$[6] := \mathtt{E}H\mathtt{I}\sigma\text{-}\mathtt{CompactIsetminus} : \sigma\text{-}\mathtt{Compact}\Big(C_s \setminus H\Big),$

$[7] := \mathtt{E}\neg\sigma\text{-}\mathtt{Compact}(C_s)[6] : \neg\mathtt{CompactSubset}\Big(X, H\Big),$

$$\Big(h, [8]\Big) := \mathtt{CompactIffSequencCompact}(H) : \sum h : \mathbb{N} \to H \,.\, \mathtt{ConvergingSubsequence}\Big(X, h\Big) = \emptyset,$$

$\Big(U, [9]\Big) := \mathtt{EConvergingSubsequence}(X, h)[8]\mathtt{NbhdConvergence} :$

$$.\ \sum U : \prod_{n=1}^{\infty} \mathcal{U}(h_n) \,.\, \Big(\forall n \in \mathbb{N} \,.\, \operatorname{diam} U < 2^{-1-\operatorname{len}(s)}\Big)\ \&\ \Big(\forall n, m \in \mathbb{N} \,.\, n = m \Rightarrow \overline{U}_n \cap \overline{U}_m = \emptyset\Big),$$

$\mathtt{Assume}\ n \in \mathbb{N},$

$C_{sn} := \overline{C_s \cap U_n} : \mathtt{Closed}\ \&\ \mathtt{NonEmpty}(X)\ \&\ \neg\sigma\text{-}\mathtt{Compact};$

$\leadsto [n.*] := \mathtt{I}\forall : \forall n \in \mathbb{N} \,.\, \mathtt{Closed}\ \&\ \mathtt{NonEmpty}(X)\ \&\ \neg\sigma\text{-}\mathtt{Compact}(C_{sn});$

$\leadsto \Big(C, [3]\Big) := \mathtt{IVanishingDiameterI} \sum :$

$$:\ \sum C : \mathtt{VanishingDiameter}(X) \,.\, \forall s \in \mathbb{N}^* \,.\, \mathtt{Closed}\ \&\ \mathtt{NonEmpty}(X)\ \&\ \neg\sigma\text{-}\mathtt{Compact}(C_{sn}),$$

$[4] := \mathtt{E}C\mathtt{I}D(C)[3.1][3.2] : D(C) = \mathcal{B},$

$A := f_C(\mathcal{B}) :?X,$

$\texttt{Assume}\ L \in \overline{A}$,

$\Big(a, [4]\Big) := \texttt{E}A[4] : \sum a \in \mathbb{N} \to A \;.\; L = \lim_{n\to\infty} a_n$,

$\Big(b, [5]\Big) := \texttt{E}A(a) : \sum b : \mathbb{N} \to \mathcal{B} \;.\; a = f_C(b)$,

$[6] := \texttt{ConvergingIsCauchy}(X, a) : \texttt{Cauchy}(X, a)$,

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++}$,

$\Big(n, [7]\Big) := \texttt{EType}Archemedian(\mathbb{N})(\varepsilon) : \sum n \in \mathbb{N} \;.\; 2^{-n} < \varepsilon$,

$\Big(\delta, [8]\Big) := \texttt{E}C(n) : \sum \delta \in \mathbb{R}_{++} \;.\; \forall s \in \mathbb{N}^* \;.\; \operatorname{len} s = n \Rightarrow \forall x \in C_s \;.\; \forall y \in A \;.\; d(x, y) < \delta \Rightarrow y \in C_s$,

$\Big(N, [9]\Big) := \texttt{ECauchy}(X, a)(\delta : \sum N \in \mathbb{N} \;.\; \forall n, m \in \mathbb{N} \;.\; n, m \geq N \Rightarrow d(a_n, a_m) < \varepsilon$,

$\Big(s, [\varepsilon.*]\Big) := [9][8][5]\texttt{E}C : \sum s \in ({}^*\mathbb{N}) \;.\; \operatorname{len}(s) = n \;\&\; \forall k \geq N \;.\; b_{|[1,\ldots,k]} = s$;

$\rightsquigarrow [7] := \texttt{ICauchy} : \texttt{Cauchy}(\mathcal{B}, b)$,

$b' := \lim_{n=1} b_n \in \mathcal{B}$,

$[8] := \texttt{ContinuousImage}(f_C)\texttt{E}b'[5][4] : L = f_C(b')$,

$[L.*] := \texttt{E}A[8] : L \in A$;

$\rightsquigarrow [4] := \texttt{ClosedByLimits} : \texttt{Closed}(X, A)$,

$\texttt{Assume}\ s \in \mathbb{N}^*$,

$[5] := \texttt{E}C[3.1](s) : \forall b \in N_s \;.\; \exists U \in \mathcal{U}(f_C(b)) \;.\; U \subset C_s$,

$[s.*] := \texttt{E}C[5] : f_C(N_s) \in \mathcal{T}(A)$;

$\rightsquigarrow [*] := \texttt{IHomeo}[3] : \mathcal{B} \xleftrightarrow{f_C} A : \mathsf{TOP}$;

$\square$

## 1.4 Baire Category and Topological Games

### 1.4.1 Recap

$$\texttt{NowhereDense} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$A : \texttt{NowhereDense} \iff \operatorname{int} \overline{A} = \emptyset$$

$$\texttt{Meager} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$B : \texttt{Meager} \iff \exists A : \mathbb{N} \to \texttt{NowhereDense}(X) \;.\; B = \bigcup_{n=1}^{\infty} A_n$$

$$\texttt{Comeager} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$A : \texttt{Comeager} \iff \texttt{Meager}(X, A^{\complement})$$

$$\texttt{SetIdeal} :: \prod_{X \in \mathsf{SET}} ???X$$

$$I : \texttt{SetIdeal} \iff I : \texttt{Ideal}(X) \iff \Big(\emptyset \in I\Big) \;\&\; \Big(\forall A \in I \;.\; \forall B \subset A \;.\; B \in I\Big) \;\&\; \Big(\forall A, B \in I \;.\; A \cup B \in I\Big)$$

$$\texttt{SetSigmaIdeal} :: \prod_{X \in \mathsf{SET}} ?\texttt{Ideal}(X)$$

$$I : \texttt{SetSigmaIdeal} \iff I : \texttt{Ideal}(X) \iff \forall A : \mathbb{N} \to I \;.\; \bigcup_{n=1}^{\infty} A_n \in I$$

$$\texttt{Bair} :: ?\mathsf{TOP}$$

$$X : \texttt{Bair} \iff \forall A : \texttt{Comeager}(X) \;.\; \texttt{Dense}(X, A)$$

$$\texttt{BairCategoryTHM} :: \forall X : \texttt{LocallyCompact} \;\&\; \texttt{T2} \;.\; \texttt{Baire}(X)$$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{MetricBairCategoryTHM} :: \forall X : \texttt{Complete} \;.\; \texttt{Baire}(X)$$

$\texttt{Proof} =$

$\ldots$

$\square$

### 1.4.2 Choquet game

$$\texttt{InfiniteIterativeTwoPlayersGame} := \Lambda X \in \mathsf{SET} \;.\; \sum T : \texttt{Pruned}(X) \;.\; [T] \to \mathbb{B} : \mathsf{SET} \to \texttt{Type};$$

$$\texttt{FirstPlayerStrategy} :: \prod (T, w) : \texttt{InfiniteIterativeTwoPlayersGame} \;.$$
$$.\; ?\Big(\texttt{Subtree} \;\&\; \texttt{NonEmpty}(T)\Big)$$
$$S : \texttt{FirstPlayerStrategy} \iff \forall s \in S \;.\; \texttt{Even}(\mathrm{len}(s)) \Rightarrow \Big|\{x \in X : sx \in S\}\Big| = 1 \;\&$$
$$\&\; \texttt{Odd}(\mathrm{len}(s)) \Rightarrow \{x \in X : sx \in S\} = \{x \in X : sx \in T\}$$

$$\texttt{FirstPlayerWinningStrategy} :: \prod (T, w) : \texttt{InfiniteIterativeTwoPlayersGame} \;.$$
$$.\; ?\texttt{FirstPlayerStrategy}(T, w)$$
$$S : \texttt{FirstPlayerWinningStrategy} \iff \forall x \in [S] \;.\; w(x) = 1$$

$$\texttt{SecondPlayerStrategy} :: \prod (T, w) : \texttt{InfiniteIterativeTwoPlayersGame} \;.$$
$$?\Big(\texttt{Subtree} \;\&\; \texttt{NonEmpty}(T)\Big)$$
$$S : \texttt{SecondPlayerStrategy} \iff \forall s \in S \;.\; \texttt{Odd}(\mathrm{len}(s)) \Rightarrow \Big|\{x \in X : sx \in S\}\Big| = 1 \;\&$$
$$\&\; \texttt{Even}(\mathrm{len}(s)) \Rightarrow \{x \in X : sx \in S\} = \{x \in X : sx \in T\}$$

$$\texttt{SecondPlayerWinningStrategy} :: \prod (T, w) : \texttt{InfiniteIterativeTwoPlayersGame} \;.$$
$$.\; ?\texttt{SecondPlayerStrategy}(T, w)$$
$$S : \texttt{SecondPlayerWinningStrategy} \iff \forall x \in [S] \;.\; w(x) = 0$$

$$\texttt{legalPosition} :: \prod_{X \in \mathsf{SET}} \texttt{InfiniteIterativeTwoPlayersGame}(X) \to \texttt{Pruned}(X)$$
$$\texttt{legalPositions}\,((T, w)) = \mathrm{lp}(T, w) := T$$

$$\texttt{winningCriterion} :: \prod_{X \in \mathsf{SET}} \prod (T, w) : \texttt{InfiniteIterativeTwoPlayersGame}(X) \;.\; [T] \to \mathbb{B}$$
$$\texttt{winningCriterion}\,() = w_{(T,w)} := w$$

$$\texttt{gameOfChoquet} :: \prod X \in \mathsf{TOP} \;\&\; \texttt{NonEmpty} \;.\; \texttt{InfiniteIterativeTwoPlayersGame}\Big(\mathcal{T}(X)\Big)$$
$$\texttt{gameOfChoquet}\,() = \eth_{Ch}(X) := \left(\bigcup_{n=0}^{\infty} \texttt{Decreasing}\Big([1, \dots, n], \mathcal{T} \;\&\; \texttt{NonEmpty}(X)\Big),\right.$$
$$\left.\Lambda U : \mathbb{N} \to \mathcal{T}(X) \;.\; \bigcap_{n=1}^{\infty} U_n ! = \emptyset\right)$$

$\texttt{OxtobyChoquetTHM} :: \forall X \in \mathsf{TOP} \,.\, X \neq \emptyset \Rightarrow \left(\neg\exists\texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X)\big) \iff \texttt{Baire}(X)\right)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \neg\exists\texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X)\big),$

$\texttt{Assume}\ [2] : \neg\texttt{Baire}(X),$

$\Big(U, [3]\Big) := \texttt{EqBairProperty}[2] : \sum U : \mathbb{N} \to \mathcal{T}\ \&\ \texttt{Dense}(X) \,.\, \bigcap_{n=1}^{\infty} U = \emptyset,$

$T_0 := \{\emptyset\} :?\mathcal{T}(X)^*,$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ [4] : \texttt{Odd},$

$\Big(k, [5]\Big) := \texttt{OddnesCriterion}[4] : \sum k \in \mathbb{Z}_+ \,.\, n = 2k + 1,$

$T_n := \texttt{if}\ k == 0\ \texttt{then}\ \{1 \mapsto U_1\}\ \texttt{else}\ \Big\{s(s_{2k} \cap U_k)\Big| s \in T_{2k}\Big\} :?\mathcal{T}^n(X);$

$\rightsquigarrow [4] := \mathtt{I} \Rightarrow : \texttt{Odd}(n) \Rightarrow T_n \in ?\mathcal{T}^n(X),$

$\texttt{Assume}\ [5] : \texttt{Even}(n),$

$T_n := \Big\{sV\Big| s \in T_{n-1}, V \in \mathcal{T}(X)\ \&\ V \subset s_{n-1}\Big\} :?\mathcal{T}^n(X);$

$\rightsquigarrow [5] := \mathtt{I} \Rightarrow : \texttt{Even}(n) \Rightarrow T_n \in ?\mathcal{T}^n(X),$

$[*] := \mathtt{E}(|)\texttt{OddOrEven}[4][5] : T_n \in ?\mathcal{T}^n(X);$

$\rightsquigarrow T := \mathtt{I}\prod : \prod_{n=0}^{\infty} ?\mathcal{T}^n(X),$

$S := \bigcup_{n=0}^{\infty} T_n :?\mathcal{T}(X)^*,$

$[4] := \mathtt{E}S : \texttt{FirstPlayerStrategy}\Big(\eth_{Ch}(X), S\Big),$

$\texttt{Assume}\ V \in [S],$

$[V.*] := \mathtt{E}S[3] : \bigcap_{n=1}^{\infty} V_n \subset \bigcap_{n=1}^{\infty} U_n = \emptyset;$

$\rightsquigarrow [5] := \mathtt{E}\eth_{Ch}(X)\mathtt{I}\texttt{FirstPlayerWinningStrategy} : \texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X), S\big);$

$[2.*] := [1](S) : \bot;$

$\rightsquigarrow [1.*] := \mathtt{LEM} : \texttt{Baire}(X);$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow : \Big(\neg\exists\texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X)\big)\Big) \Rightarrow \texttt{Baire}(X),$

$\texttt{Assume}\ [2] : \texttt{Baire}(X),$

$\texttt{Assume}\ S : \texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X)\big),$

$\Big(U, [3]\Big) := \mathtt{E}\texttt{FirstPlayerStrategy}\big(\eth_{Ch}(X), S\big)\mathtt{E}\texttt{Singleton} : \sum U \in \mathcal{T}(X) \,.\, \{s \in S : \mathrm{len}(s) = 1\} = \{1 \mapsto U\},$

$[4] := \mathtt{E}\eth_{Ch}(X)\mathtt{E}U : U \neq \emptyset,$

$T_0 := \{\emptyset\} :?\mathcal{T}(X)^*,$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ [5] : \texttt{Odd},$

$T_n := \texttt{if}\ n == 0\ \texttt{then}\ \{1 \mapsto U\}\ \texttt{else}\ \Big\{S(s)\Big| s \in T_{2k}\Big\} :?\mathcal{T}^n(X);$

$\rightsquigarrow [5] := \mathtt{I} \Rightarrow : \texttt{Odd}(n) \Rightarrow T_n \in ?\mathcal{T}^n(X),$

$\texttt{Assume}\ [6] : \texttt{Even}(n),$

$\texttt{Assume}\ p \in S,$

$\texttt{Assume}\ [7] : \mathrm{len}(p) = n - 1,$

$V := p_{n-1} : \mathcal{T}(X)\ \&\ \texttt{NonEmpty},$

$\mathcal{V} := \max\left\{\mathcal{U} :?\Big(\mathcal{T}(X)\ \&\ ?V\ \&\ \texttt{NonEmpty}\Big) : \forall W, W' \in \mathcal{U}\ .\ S(sW) \cap S(sW') = \emptyset\right\} :?\mathcal{T}(X),$

$t_s := \{sW | W \in \mathcal{V}\} :?\mathcal{T}^n(X);$

$\rightsquigarrow t := \mathtt{I} \rightarrow: S_{n-1} \rightarrow ?\mathcal{T}^n(X),$

$T_n := \bigcup_{s \in S_{n-1}} t_s :?\mathcal{T}^n(X);$

$\rightsquigarrow [5] := \mathtt{I} \Rightarrow: \texttt{Even}(n) \Rightarrow T_n \in ?\mathcal{T}^n(X),$

$[*] := \mathtt{E}(|)\texttt{OddOrEven}[4][5] : T_n \in ?\mathcal{T}^n(X);$

$\rightsquigarrow T := \mathtt{I} \prod : \prod_{n=0}^{\infty} ?\mathcal{T}^n(X),$

$S' := \bigcup_{n=0}^{\infty} T_n :?S,$

$[5] := \mathtt{E}S' : \forall n : \texttt{Even}\ .\ \forall s, t \in S'\ .\ \mathrm{len}(s) = n = \mathrm{len}(t)\ \&\ s \neq t \Rightarrow s_n \cap t_n = \emptyset,$

$[6] := \mathtt{E}S'\mathtt{E}\max : \forall n : \texttt{Even}\ .\ \forall U \in S'\ .\ \mathrm{len}(U) = n \Rightarrow \texttt{Dense}\Big(U_n, \bigcup\{V_{n+2} | V \in S'\ \&\ \mathrm{len}(V) = n + 2\}\Big),$

$[7] := \mathtt{E}S'\texttt{IPruned} : \texttt{Pruned}\big(\mathcal{T}(X), S'\big),$

$W := \Lambda n \in \mathbb{N}\ .\ \bigcup\{s_n | s \in S', \mathrm{len}(s) = n\} \in \mathcal{T}(X),$

$[8] := \mathtt{E}W\mathtt{E}S'[6] : \forall n \in \mathbb{N}\ .\ \texttt{Dense}\Big(U, W_n\Big),$

$[9] := \mathtt{E}W[7][5]\mathtt{E}\texttt{FirstPlayerWinningStrategy}\big(\eth_{Ch}(X), S\big)\mathtt{E}\texttt{union} :$

$: \bigcap_{n=1}^{\infty} W_n = \bigcap_{n=0}^{\infty} \bigcup_{s \in S'_{2n+1}} s_{2n+1} = \bigcup_{x \in |S'|} \bigcap_{n=1}^{\infty} s_{2n+1} = \bigcup_{x \in |S'|} \emptyset = \emptyset,$

$[2.*] := \mathtt{E}\texttt{Baire}(X)\texttt{BairOpenSubsets}[9] : \bot;$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow: \texttt{Baire}(X) \Rightarrow \neg\exists\texttt{FirstPlayerWinningStrategy}\Big(\eth_{Ch}(X), S\Big),$

$[*] := \mathtt{I} \iff [1][2] : \neg\Big(\exists\Big(\eth_{Ch}(X), S\Big) \iff \texttt{Baire}(X);$

□

$\texttt{ChoquetSpace} :: ?(\mathsf{TOP}\ \&\ \texttt{NonEmpty})$

$X : \texttt{ChoquetSpace} \iff \exists\texttt{SecondPlayerWinningStrategy}\Big(\eth_{Ch}(X)\Big)$

$\texttt{ChoquetIsBair} :: \forall X : \texttt{ChoquetSpace}\ .\ \texttt{Baire}(X)$

$\texttt{Proof}\ =$

...

□

$\texttt{ChoquetSpaceProduct} :: \forall X, Y : \texttt{ChoquetSpace} \,.\, \texttt{ChoquetSpace}(X \times Y)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ChoquetSpaceOpenSubsets} :: \forall X : \texttt{ChoquetSpace} \,.\, \forall U \in \mathcal{T}(X) \,.\, U \neq \emptyset \Rightarrow \texttt{ChoquetSpace}(U)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{strongChoquetGame} :: \prod X \in \mathsf{TOP} \,\&\, \texttt{NonEmpty} \,.\, \texttt{InfiniteIterativeTwoPlayersGame}\left(\sum_{U \in \mathcal{T}(X)} U\right)$$

$$\texttt{strongChoquetGame}\,() = \eth_{sCh}(X) := \left(\bigcup_{n=0}^{\infty}\left\{(U,x) : [1,\dots,n] \to \sum_{U \in \mathcal{T}(X)} U : \texttt{Decreasing}\Big([1,\dots,n], \mathcal{T}(X)\Big) \,\& \right.\right.$$
$$\left.\left. \& \,\forall k \in [1,\dots,n] \,.\, \texttt{Even}(k) \Rightarrow x_k = x_{k-1}\right\}, \Lambda(U,x) : \mathbb{N} \to \sum_{U \in \mathcal{T}(X)} U \,.\, \bigcap_{n=1} U_n = \emptyset\right)$$

$\texttt{StrongChoquetSpace} :: ?(\mathsf{TOP} \,\&\, \texttt{NonEmpty})$

$X : \texttt{StrongChoquetSpace} \iff \exists \texttt{SecondPlayerWinningStrategy}\Big(\eth_{sCh}(X)\Big)$

$\texttt{StrongChoquetIsChoquet} :: \forall X : \texttt{StrongChoquetSpace} \,.\, \texttt{ChoquetSpace}(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ChoquetCategoryTHM} :: \forall X : \texttt{LocallyCompact} \,\&\, \texttt{T2} \,.\, \texttt{StrongChoquetSpace}(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MetricChoquetCategoryTHM} :: \forall X : \texttt{Complete} \,.\, \texttt{StrongChoquetSpace}(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{StrongChoquetGDeltaSubsets} :: \forall X : \texttt{StrongChoquetSpace}(X) \,.\, \forall A \in G_\delta(X) \,.\, A \neq \emptyset \Rightarrow$
$\quad \Rightarrow \texttt{StrongChoquetSpace}(A)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{StrongChoquetMapping} :: \forall X : \texttt{StrongChoquetSpace}(X) \, . \, \forall Y \in \mathsf{TOP} \, .$

$\quad : \forall f : \texttt{Surjective} \,\&\, \texttt{Open}(X, Y) \, . \, \texttt{StrongChoquetSpace}(Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

### 1.4.3 Characterization of polish spaces

$\texttt{OxtobyPolishCharTHM} :: \forall X : \texttt{Polish} \ \& \ \texttt{NonEmpty} \ . \ \forall D : \texttt{Dense}(X) \ .$

$\quad . \ \texttt{ChoquetSpace}(D) \iff \texttt{Comeager}(X, D)$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{EPolish}(X) : \sum d : \texttt{Metric}(X) \ .$

$\quad . \ (X, \alpha) \cong_{\mathsf{TOP}} X \ \& \ \texttt{Complete}(X, d),$

$\texttt{Assume} \ [2] : \texttt{ChoquetSpace}(D),$

$S := \texttt{EChoquetSpace}(D) : \texttt{SecondPlayerWinningStrategy}\Big(\partial_{Ch}(D)\Big),$

$\Big(S', [3]\Big) := \texttt{EDense}(D, X)\texttt{ESecondPlayerStrategy}\Big(\partial_{Ch}(D), S\Big) : \sum S' : \texttt{Pruned}(\mathcal{T}(X)) \ . \ S \neq \emptyset \ \&$

$$\& \left(\forall s \in S' \ . \ \prod_{i=1}^{\mathrm{len}(s)} (s_i \cap D) \in S\right) \ \&$$

$\quad \& \ (\forall s \in S' \ . \ \texttt{Even}(\mathrm{len}\, s) \Rightarrow \texttt{Disjoint}\{V \in \mathcal{T}(X) : \exists U \in \mathcal{T}(X) : sUV \in S'\}) \ \&$

$\quad \& \ \Big(\forall s \in S' \ . \ \texttt{Even}(\mathrm{len}\, s) \Rightarrow \texttt{Dense}\left(s_{\mathrm{len}\, s}, \bigcup\{V \in \mathcal{T}(X) : \exists U \in \mathcal{T}(X) : sUV \in S'\}\right)\Big) \ \&$

$\quad \& \ \Big(\forall s \in S' \ . \ \texttt{Even}(\mathrm{len}\, s) \Rightarrow \mathrm{diam}\, s_{\mathrm{len}\, s} < 2^{-\mathrm{len}\, s}\Big),$

$$W := \Lambda n \in \mathbb{N} \ . \bigcup_{s \in S'_{2n}} s_{2n} : \mathbb{N} \to \mathcal{T}(X),$$

$[4] := \mathtt{E}W[3.1][3.4] : \forall n \in \mathbb{N} \ . \ \texttt{Dense}\Big(X, W_n\Big),$

$$\texttt{Assume} \ x \in \bigcap_{n=1}^{\infty} W_n,$$

$\Big(U, [5]\Big) := \mathtt{E}W(x)\mathtt{E}S' : \sum U \in [S'] \ . \ \forall n \in \mathbb{N} \ . \ x \in U_n,$

$$[6] := \texttt{Iintersect}[5] : x \in \bigcap_{n=1}^{\infty} U_n,$$

$$[7] := [3.5][6] : \{x\} = \bigcap_{n=1}^{\infty} U_n,$$

$[x.*] := [7][3.2]\texttt{ESecondPlayerWinningStrategy}\Big(\partial_{Ch}(X), S\Big) : x \in D;$

$$\rightsquigarrow [5] := \mathtt{I} \subset : \bigcap_{n=1}^{\infty} W_n \subset D,$$

$[2.*] := \texttt{ComeagerByDenseOpenIntersect}\Big(X, D, W, [5]\Big) : \texttt{Comeager}(X, D);$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow : \texttt{ChoquetSpace}(D) \Rightarrow \texttt{Comeager}(X, D),$

$[3] := \mathtt{I} \Rightarrow \texttt{IChoquetSpace}(D)\texttt{ISecondPlayerWinningStrategy}\Big(\partial_{Ch}(D)\Big)\texttt{EComeager}(X, D) : \texttt{Comeager}(X, D)$

$[*] := \mathtt{I}(\iff)[2][3] : \texttt{Comeager}(X, D) \iff \texttt{ChoquetSpace}(D);$

$\square$

$$\texttt{PointFiniteRefinement} :: \prod_{X \in \mathsf{TOP}} \prod_{\mathcal{U} : ?\mathcal{T}(X)} \texttt{Refinement}(X, \mathcal{U})$$

$\mathcal{V} : \texttt{PointFiniteRefinement} \iff \forall x \in X \ . \ \Big|\{V \in \mathcal{V} : x \in V\}\Big| < \infty$

$\texttt{SeparableMetricSpaceHasSmallPointFiniteRefinements}$ ::

$:: \forall X \in \mathsf{MS} \ \& \ \texttt{Separable} \ . \ \forall \mathcal{U} :? \mathcal{T}(X) \ . \ \forall \varepsilon \in \mathbb{R}_{++} \ . \ \exists \mathcal{V} : \texttt{PointFreeRefinement}(X, \mathcal{U}) \ .$

$. \ \forall V \in \mathcal{V} \ . \ \operatorname{diam} V < \varepsilon$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ChoquetPolishCharTHM} :: \forall X : \texttt{Polish} \ \& \ \texttt{NonEmpty} \ . \ \forall D : \texttt{Dense}(X) \ .$

$. \ \texttt{StrongChoquetSpace}(D) \Rightarrow \texttt{Polish}(D)$

$\texttt{Proof} \ =$

$\Big(d, [1]\Big) := \texttt{EPolish}(X) : \sum d : \texttt{Metric}(X) \ .$

$. \ (X, \alpha) \cong_{\mathsf{TOP}} X \ \& \ \texttt{Complete}(X, d),$

$S := \texttt{EStrongChoquetSpace}(D) : \texttt{SecondPlayerWinningStrategy}\Big(\eth_{sCh}(D)\Big),$

$\Big(S', [2]\Big) := \texttt{SeparableMetricSpaceHasSmallPointFiniteRefinements}(X, S) : \sum S' : \texttt{Pruned}\left(\prod_{U \in \mathcal{T}(X)} U\right) :$

$: (\forall (U', x) \in S' \ . \ \exists (U, x) \in S : U' \cap D = U) \ \&$

$\& \ \Big(\forall (U', x) \in S' \ . \ \forall i \in \Big[1, \ldots, \operatorname{len}(U', x)\Big] \ . \ U'_{i+1} \subset U'_i\Big) \ \&$

$\& \ \Bigg(\forall (W, x) \in S' \ . \ \texttt{Even}\Big(\operatorname{len}(W, x)\Big) \Rightarrow W_{\operatorname{len} s} \cap D \subset \bigcup$

$\pi_1 \left\{ (V, y) \in \prod_{V \in \mathcal{T}(X)} V : \exists (U, y) \in \prod_{U \in \mathcal{T}(X)} U : (W, x)(U, y)(V, y) \in S' \right\} \Bigg) \ \&$

$\& \ \Bigg(\forall (W, x) \in S' \ . \ \texttt{Even}\Big(\operatorname{len}(W, x)\Big) \Rightarrow \forall x' \in X \ .$

$\left| \left\{ (V, y) \in \prod_{V \in \mathcal{T}(X)} V : \exists (U, y) \in \prod_{U \in \mathcal{T}(X)} U : (W, x)(U, y)(V, y) \in S', x' \in V \right\} \right| < \infty \Bigg) \ \&$

$\& \ \big(\forall (U, x) \in S' \ . \ \texttt{Even}(\operatorname{len} U) \Rightarrow \operatorname{diam} U_{\operatorname{len} U} < 2^{-\operatorname{len} U}\big) \ ,$

$W := \Lambda n \in \mathbb{N} \ . \ \bigcup_{(U, x) \in S'_{2n}} U_{2n} : \mathbb{N} \to \mathcal{T}(X),$

$[3] := \mathtt{E}\mathbb{N}\mathtt{E}W[2.1][2.3] : X \subset \bigcap_{n=1}^{\infty} W_n,$

$\texttt{Assume} \ x \in \bigcap_{n=1}^{\infty} W_n,$

$[4] := \mathtt{E}W\mathtt{I}S' : |S'_x| = \infty,$

$[5] := \texttt{IFiniteSplitting}[2.4] : \texttt{FiniteSplitting}(S'_x),$

$[6] := \texttt{K/"onigsLema}[5][4] : [S'_x] \neq \emptyset,$

$(U, y) := \texttt{ENonEmpty} \in [S'_x],$

$\Big[(V, y), [7]\Big] := [2.1](U, y) : \sum (U, y) \in [S] \ . \ V = U \cap D,$

$[8] := \texttt{E}\texttt{SecondPlayerWinningStrategy}\left(\text{Ə}_{sCh}, S\right) : \bigcap_{n=1}^{\infty} V_n \neq \emptyset,$

$[9] := \texttt{IntersectionDistributivity}\left(X, \bigcap_{n=1}^{\infty} V_n\right)\texttt{VanishingDiameterIntersection}[8][2.5][7]\texttt{E}X :$

$: D \cap \bigcap_{n=1}^{\infty} W_n = \bigcap_{n=1}^{\infty} W_n \cap D = \bigcap_{n=1}^{\infty} V_n == \{x\},$

$[x.*] := \texttt{E}\texttt{intersection}[9] : x \in D;$

$\rightsquigarrow [4] := \texttt{I} \subset [3]\texttt{ISetEq} : D = \bigcap_{n=1}^{\infty} W_i,$

$[*] := \texttt{I}G_\delta(X)\texttt{PolishSubset}(X) : \texttt{Polish}(X);$

$\square$

### 1.4.4 Bair property

$$\texttt{Bimeager} :: \prod_{X \in \mathsf{TOP}} ?X \times ?X$$

$$(A, B) : \texttt{Bimeager} \iff A =^* B \iff A = B \mod \texttt{Meager}(X)$$

$$\texttt{BairProperty} :: \prod_{X \in \mathsf{TOP}} ?X$$

$$B : \texttt{BairProperty} \iff B \in \mathbf{BP}(X) \iff \exists U \in \mathcal{T}(X) \,.\, B =^* U$$

$$\texttt{BairPropertyAsSmallestSigmaAlgebra} :: \forall X \in \mathsf{TOP} \,.\, \mathbf{BP}(X) = \sigma\Big(\mathcal{T}(X) \cup \texttt{Meager}(X)\Big)$$

$\texttt{Proof} =$

$[1] := \mathtt{E}\mathbf{BP}(X)\mathtt{E}\mathcal{T}(X)\mathtt{I}\emptyset : \emptyset \in \mathbf{BP}(X),$

$\texttt{Assume}\ A : \mathbf{BP}(X),$

$\Big(U, [2]\Big) := \mathtt{E}\mathbf{BP}(X, A) : \sum U \in \mathcal{T}(X) \,.\, A =^* U,$

$[3] := \mathtt{E}(A =^* U)[2] : \texttt{Meager}\Big(X, A \triangle U\Big),$

$V := \operatorname{int} U^{\complement} \in \mathcal{T}(X),$

$[4] := \mathtt{E}V\mathtt{E} \triangle \texttt{InteriorSubsetInteriorClosureDecompositionI} \triangle : A^{\complement} \triangle V = A^{\complement} \triangle \operatorname{int} U^{\complement} =$

$= \Big(A^{\complement} \cap \operatorname{int} U^{\complement}{}^{\complement}\Big) \cup \Big(A \cap \operatorname{int} U^{\complement}\Big) \subset \Big(A^{\complement} \cap \operatorname{int} U^{\complement}{}^{\complement}\Big) \cup \Big(A \cap U^{\complement}\Big) =$

$= \Big(A^{\complement} \cap (\overline{U} \setminus U)\Big) \cup \Big(A^{\complement} \cap \operatorname{int} U^{\complement}{}^{\complement}\Big) \cup \Big(A \cap U^{\complement}\Big) = \Big(A^{\complement} \cap (\overline{U} \setminus U)\Big) \cup A \triangle U,$

$[5] := [3][4]\texttt{NowhereDenseResidual}(U)\texttt{MeagerSubset} : \texttt{Meager}(X, A^{\complement} \triangle V),$

$[6] := \mathtt{I}\texttt{Bimeager}[5] : A^{\complement} =^* V,$

$[A.*] := \mathtt{I}\mathbf{BP}(X)[6] : A^{\complement} \in \mathbf{BP}(X);$

$\leadsto [2] := \mathtt{I}(\forall) : \forall A \in \mathbf{BP}(X) \,.\, A^{\complement} \in \mathbf{BP}(X),$

$\texttt{Assume}\ A : \mathbb{N} \to \mathbf{BP}(X),$

$\Big(U, [3]\Big) := \mathtt{E}\mathbf{BP}(X, A) : \sum U : \mathcal{T}(X) \,.\, \forall n \in \mathbb{N} \,.\, A =^* U,$

$[4] := \mathtt{E}(A =^* U)[3] : \forall n \in \mathbb{N} \,.\, \texttt{Meager}\Big(X, A_n \triangle U_n\Big),$

$[5] := \texttt{DifferenceUnionSubset}(A, U) : \bigcup_{n=1}^{\infty} A_n \triangle \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} A_n \triangle U_n,$

$[6] := [5][4]\texttt{MeagerCountableUnionMeagerSubset} : \texttt{Meager}\left(X, \bigcup_{n=1}^{\infty} A_n \triangle \bigcup_{n=1}^{\infty} U_n\right),$

$[7] := \mathtt{I}\texttt{Bimeager}[6] : \bigcup_{n=1}^{\infty} A_n =^* \bigcup_{n=1}^{\infty} U_n,$

$[8] := \mathtt{I}\mathbf{BP}(X)[7] : \bigcup_{n=1}^{\infty} A_n \in \mathbf{BP}(X);$

$\leadsto [3] := \mathtt{I}\forall : \forall A : \mathbb{N} \to \mathbf{BP}(X) \,.\, \bigcup_{n=1}^{\infty} A_n \in \mathbf{BP}(X),$

$[4] := \mathtt{I}\sigma\texttt{-Algebra}[1][2][3] : \sigma\texttt{-Algebra}\Big(\mathbf{BP}(X)\Big),$

$[5] := \mathtt{I}\sigma\mathtt{E}\sigma\mathtt{E}\mathbf{BP}(X)[4] : \sigma\Big(\mathcal{T}(X) \cup \texttt{Meager}(X)\Big) \subset \mathbf{BP}(X),$

Assume $B \in \mathbf{BP}(X)$,

$\Big(U, [6]\Big) := \mathtt{E}\mathbf{BP}(X) : \sum U \in \mathcal{T}(X) \,.\, B =^* U$,

$[7] := \mathtt{E}(B =^* U) : \mathtt{Meager}(X, B \triangle U)$,

$[8] := \mathtt{OneSideSubsetSymmetric}(B, U) : B \setminus U \subset B \triangle U$,

$[9] := \mathtt{MeagerSubset}[8] : \mathtt{Meager}(X, B \setminus U)$,

$[10] := \mathtt{MeagerSubset}[8] : \mathtt{Meager}(X, U \setminus B)$,

$[11] := \mathtt{DifferenceDecomposition1}(B, U)\mathtt{IntersectDifferenceDecomposion}(U, B) :$

$\quad : B = (U \cap B) \cup (B \setminus U) = \Big(U \setminus (U \setminus B)\Big) \cup (B \setminus U)$,

$[B.*] := \mathtt{E}\sigma\text{-}\mathtt{Algebra}\Big(\sigma\big(\mathcal{T}(X) \cup \mathtt{Meager}(X)\big)[9][10][11]\Big) : B \in \sigma\Big(\mathcal{T}(X) \cup \mathtt{Meager}(X)\Big)$;

$\leadsto [6] := \mathtt{I} \subset : \mathbf{BP}(X) \subset \sigma\Big(\mathcal{T}(X) \cup \mathtt{Meager}(X)\Big)$,

$[*] := \mathtt{ISetEq}[5][6] : \mathbf{BP}(X) = \sigma\Big(\mathcal{T}(X) \cup \mathtt{Meager}(X)\Big)$;

$\square$

$\mathtt{BPAsGDelta} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, A \in \mathbf{BP}(X) \iff \exists E \in G_\delta(X) : \exists M : \mathtt{Meager}(X) \,.\, A = E \cup M$

Proof $=$

...

$\square$

$\mathtt{BPAsFSigma} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, A \in \mathbf{BP}(X) \iff \exists E \in F_\sigma(X) : \exists M : \mathtt{Meager}(X) \,.\, A = E \setminus M$

Proof $=$

...

$\square$

$\mathtt{RealBPIsNotTrivial} :: \mathbf{BP}(\mathbb{R}) \neq ?\mathbb{R}$

Proof $=$

...

$\square$

### 1.4.5 Localization

$$\texttt{Forces} :: \prod_{X \in \mathsf{TOP}} \mathcal{T}(X) \to ?X$$

$$A : \texttt{Forces} \iff \Lambda U \in \mathcal{T}(X) \,.\, U \Vdash A \iff \texttt{Meager}\Big(U, U \setminus A\Big)$$

$\texttt{BairPropertyByForcing} :: \forall X \in \mathsf{TOP} \,.\, \forall A \in \mathbf{BP}(X) \,.\, X \Vdash (X \setminus A) \Big| \exists U \in \mathcal{T}(U) \,.\, U \Vdash A$

$\texttt{Proof} =$

$\Big(U, [1]\Big) := \mathbf{EBP}(X, A) : \sum U \in \mathcal{T}(X) \,.\, \texttt{Meager}\Big(X, A \triangle U\Big),$

$\texttt{Assume}\ [2] : U \neq \emptyset,$

$[3] := \texttt{OneSidedSymmetricDifferenceSubset}(U, A)\texttt{MeagerSubset} : \texttt{Meager}(X, U \setminus A),$

$[4] := \texttt{SubsetMeager}[3] : \texttt{Meager}(U, U \setminus A),$

$[2.*] := \texttt{I} \Vdash [4] : U \Vdash A;$

$\leadsto [2] := \texttt{I}(\Rightarrow) : U \neq \emptyset \Rightarrow U \Vdash A,$

$\texttt{Assume}\ [3] : U = \emptyset,$

$[4] := [2][3] : \texttt{Meager}(X, A),$

$[3.*] := \texttt{I} \Vdash [4] : X \Vdash X \setminus A;$

$\leadsto [3] := \texttt{I}(\Rightarrow) : U = \emptyset \Rightarrow X \Vdash (X \setminus A),$

$[*] := \texttt{LEM}(U = \emptyset)[2][3] : X \Vdash (X \setminus A) \Big| \exists U \in \mathcal{T}(U) \,.\, U \Vdash A;$

$\square$

$\texttt{BairForcing} :: \forall X : \texttt{Baire} \,.\, \forall A \in \mathbf{BP}(X) \,.\, X \Vdash (X \setminus A) \oplus \exists U \in \mathcal{T}(U) \,.\, U \Vdash A$

$\texttt{Proof} =$

$\ldots$

$\square$

$$\texttt{WeakBasis} :: \prod_{X \in \mathsf{TOP}} ??\Big(\mathcal{T}(X)\ \&\ \texttt{NonEmpty}(X)\Big)$$

$$\mathcal{U} : \texttt{WeakBasis} \iff \forall V \in \mathcal{T}(X)\ \&\ \texttt{NonEmpty}(X) \,.\, \exists U \in \mathcal{U} : U \subset V$$

$\texttt{WeakBasisBairPropertyForcing} :: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{U} : \texttt{WeakBasis}(X) \,.\, \forall A \in \mathbf{BP}(X) \,.$
$\quad .\, X \Vdash (X \setminus A) \Big| \exists U \in \mathcal{U} \,.\, U \Vdash A$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ForcingIntersection} :: \forall X \in \mathsf{TOP} \,.\, \forall A : \mathbb{N} \to X \,.\, \forall U \in \mathcal{T}(X) \,.\, U \Vdash \bigcap_{n=1}^{\infty} A_n \iff \forall n \in \mathbb{N} \,.\, U \Vdash A_n$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ForcingComplement} :: \forall X : \texttt{Baire} \,.\, \forall B \in \mathbf{BP}(X) \,.\, \forall V \in \mathcal{T}(X) \,.\, \forall \mathcal{U} : \texttt{WeakBasis} \,.\, V \Vdash B^{\complement} \iff$
$\iff \forall U \in \mathcal{U} \,\&\, \texttt{Subset}(V) \,.\, U \nVdash B$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$$\texttt{ForcingUnion} :: \forall X : \texttt{Baire} \,.\, \forall B : \mathbb{N} \to \mathbf{BP}(X) \,.\, \forall V \in \mathcal{T}(X) \,.\, V \vdash \bigcup_{n=1}^{\infty} B_n \iff$$
$$\iff \forall U \in \mathcal{U} \,\&\, \texttt{Subset}(V) \,.\, \exists n \in \mathbb{N} \,.\, \exists W \in \mathcal{U} \,\&\, \texttt{Subset}(U) \,.\, W \Vdash B_n$$

$\texttt{Proof} \;=$

$$\texttt{openApproximation} :: \prod_{X \in \mathsf{TOP}} ?X \to \mathcal{T}(X)$$
$$\texttt{openApproximation}\,(A) = U_{\Vdash}(A) := \bigcup \{U \in \mathcal{T}(X) : U \Vdash A\}$$

$\texttt{MeagerInOpenApproximation} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, \texttt{Meager}\Big(X, U_{\Vdash}(A) \setminus A\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OpenApproximationWithBairProperty} :: \forall X \in \mathsf{TOP} \forall A \in \mathbf{BP}(X) \,.\, \texttt{Meager}\Big(X, A \setminus U_{\Vdash}(A)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OpenApproximationBimeager} :: \forall X \in \mathsf{TOP} \,.\, \forall A \in \mathbf{BP}(X) \,.\, U_{\Vdash}(A) =^{*} A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OpenApproximationIsOpenDomain} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, \texttt{OpenDomain}(X, U_\Vdash(A))$

$\texttt{Proof} \ =$

$[1] := \mathtt{E}\,\mathrm{int}\,\mathtt{Eclosure} : U_\Vdash(A) \subset \mathrm{int}\,\overline{U}_\Vdash(A),$

$[2] := \texttt{MeagerInOpenApproximation}(X, A) : \texttt{Meager}\Big(X, U_\Vdash(A) \setminus A\Big),$

$[3] := [1]\texttt{SubsetDifferenceDecomposition} : \mathrm{int}\,\overline{U_\Vdash(A)} \setminus A = \Big(U_\Vdash(A) \setminus A\Big) \cup \Big(\mathrm{int}\,\overline{U_\Vdash(A)} \setminus \Big(A \cup U_\Vdash(A)\Big)\Big),$

$[4] := \texttt{InteriorIsSubset}\Big(X, \overline{U_\Vdash(A)}\Big) : \mathrm{int}\,\overline{U_\Vdash(A)} \subset \overline{U_\Vdash(A)},$

$[5] := \texttt{SubsetOfUnion}\Big(X, A, U_\Vdash(A)\Big) : U_\Vdash(A) \subset X \setminus A,$

$[6] := \texttt{DifferenceMonotonicity}[4][5] : \mathrm{int}\,\overline{U_\Vdash(A)} \setminus \Big(A \cup U_\Vdash(A)\Big) \subset \overline{U_\Vdash(A)} \setminus U_\Vdash(A),$

$[7] := \texttt{OpenHasMeagerBoundary}\Big(X, U_\Vdash(A)\Big) : \texttt{Meager}\Big(X, \overline{U_\Vdash(A)} \setminus U_\Vdash(A)\Big),$

$[8] := \texttt{MeagerSubset}[7][8] : \texttt{Meager}\Big(X, \mathrm{int}\,\overline{U_\Vdash(A)} \setminus \big(A \cup U_\Vdash(A)\big)\Big),$

$[9] := \texttt{MeagerUnion}[3][2][8] : \texttt{Meager}(X, \mathrm{int}\,\overline{U_\Vdash(A)} \setminus A),$

$[10] := \mathtt{E}U_\Vdash(A)[9][1] : U_\Vdash(A) = \mathrm{int}\,\overline{U_\Vdash(A)},$

$[*] := \texttt{IOpenDomain}[10] : \texttt{OpenDpmain}\Big(X, U_\Vdash(A)\Big);$

$\square$

$\texttt{OpenApproximationIsUniqueForcingOpenDomain} :: \forall X : \texttt{Baire} \,.\, \forall A \in \mathbf{BP}(X) \,.\, \forall U : \texttt{OpenDomain}(X) \,.$
$\,.\, U =^* A \Rightarrow U = U_\Vdash(A)$

$\texttt{Proof} \ =$

$[1] := \mathtt{E}U_\Vdash(A)[0] : U \subset U_\Vdash(A),$

$[2] := \texttt{OpenApproximationBimeager}(X, A) : U_\Vdash(A) =^* A,$

$[3] := [0][2] : U_\Vdash(A) =^* U,$

$[*] := \texttt{EBimeager}[3]\texttt{EOpenDomain}(U) : U_\Vdash(A) = U;$

$\square$

$$\texttt{meagerIdeal} :: \prod_{X \in \mathsf{TOP}} \sigma\text{-}\texttt{Ideal}(X)$$

$$\texttt{meagerIdeal}\,() = \mathbf{MGR}(X) := \texttt{Meager}(X)$$

$$\texttt{categoryAlgebra} :: \mathsf{TOP} \to \sigma\text{-}\texttt{Algebra}$$

$$\texttt{categoryAlgebra}\,() = \mathsf{CAT}(X) := \frac{\mathbf{BP}(X)}{\mathbf{MGR}(X)}$$

$\texttt{OpenDomainAlgebraTHM} :: \prod X : \texttt{Baire} \,.\, \forall A \in \mathbf{CAT}(X) \,.\, \exists! U : \texttt{OpenDomain}(X) \,.\, A = [U]$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CatAlgebraIsCCC} :: \forall X : \texttt{Baire} \ \& \ \texttt{SecondCountable} \ . \ \texttt{WithCountableChainCondition}\Big(\mathbf{CAT}(X)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ \mathcal{U} : \texttt{PairwiseDisjointElements}\Big(\mathbf{CAT}(X)\Big),$

$\Big(\mathcal{U}', [1]\Big) := \texttt{OpenDomainAlgebra}(\mathcal{U}) : \sum \mathcal{U}' \in ?\mathcal{T}(X) \ . \ \forall u \in \mathcal{U} \ . \ \exists! U \in \mathcal{U}' \ . \ u = [U] \ \& \ \forall U \in \mathcal{U}' \ . \ \exists! U \in \mathcal{U}' : u =$

$[2] := \texttt{EPairwiseDisjointElements}\Big(\mathbf{CAT}(X), \mathcal{U}\Big)[1] : \texttt{PairwiseDisjointElements}\Big(?X, \mathcal{U}'\Big),$

$[3] := \texttt{ESeconCountable})(X)[2] : |\mathcal{U}'| \le \aleph_0),$

$[\mathcal{U}.*] := [1][3] : |\mathcal{U}| \le \aleph_0;$

$\leadsto [*] := \texttt{IWithCountableChainCondition} : \texttt{WithCountableChainCondition}\Big(\mathbf{CAT}(X)\Big);$

$\square$

$\texttt{CatAlgebraIsComplete} :: \forall X : \texttt{Baire} \ . \ \tau\text{-}\texttt{Algebra}\Big(\mathbf{CAT}(X)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ \mathcal{I} \in \texttt{Set},$

$\texttt{Assume} \ u : \mathcal{I} \to \mathbf{CAT}(X),$

$\Big(U, [1]\Big) := \texttt{OpenDomainTHM} : \sum \mathcal{I} \to \texttt{OpenDomain}(X) \ . \ \forall i \in \mathcal{I} \ . \ u_i = [U_i],$

$V := \operatorname{int} \overline{\bigcup_{i \in \mathcal{I}} U_i} : \texttt{OpenDomain}(X),$

$[2] := \Lambda i \in I \ . \ \texttt{SubsetOfUnion}(I, U, i)\texttt{IclosureIinteriorI}V : \forall i \in \mathcal{I} \ . \ U_i \subset \bigcup_{i \in \mathcal{I}} U_i \subset \operatorname{int} \overline{\bigcup_{i \in \mathcal{I}} U_i} = V,$

$[3] := \texttt{I}\mathbf{CAT}(X)[1] : \forall i \in I \ . \ u_i \le [V],$

$\texttt{Assume} \ w \in \mathsf{CAT}(X),$

$\texttt{Assume} \ [4] : \forall i \in I \ . \ u_i \le w,$

$(W, [5]) := \texttt{OpenDomainTHM}(X, w) : \sum W : \texttt{OpenDomain}(X) \ . \ w = [W],$

$[6] := [1][4][5] : \forall i \in \mathcal{I} \ . \ U_i \setminus W \in \mathbf{MGR}(X),$

$[7] := \bigcup [6] : \left(\bigcup_{i \in I} U_i\right) \setminus W \in \mathbf{MGR}(X),$

$[8] := \texttt{NowhereDenseReisdual}[7]\texttt{I}V : V \setminus W \in \mathbf{MGR}(X),$

$[w.*] := \texttt{E}\mathsf{CAT}(X)\texttt{LessByDifference}[8] : [V] \le w;$

$\leadsto [u.*] := \texttt{I} \vee [3] : [V] = \bigvee_{i \in I} u_i;$

$\leadsto [*] := \texttt{CompleteBySupremas} : \tau\text{-}\texttt{Algebra}\Big(\mathbf{CAT}(X)\Big);$

$\square$

### 1.4.6 Banach-Mazur game

$$\texttt{gameOfBanachMazur} :: \prod_{X \in \mathsf{TOP}} ?X \to \texttt{InfiniteIterativeTwoPlayersGame}\Big(\mathcal{T}(X)\Big)$$

$$\texttt{gameOfBanachMazur}\,(A) = \eth^{**}(A) :=$$

$$:= \left(\Big\{U : [1,\dots,n] \downarrow \mathcal{T}(X)\Big| n \in \mathbb{Z}_+, \forall i \in \{1,\dots,n\} \,.\, \exists U_i\Big\}, \Lambda \mathcal{U} \in \mathcal{T}^{\mathbb{N}}(X) \,.\, \bigcap_{n=1}^{\infty} U_n \subset A\right)$$

$$\texttt{SecondPlayerBanachMazurTheorem} :: \forall X \in \mathsf{TOP} \,.\, \forall A \subset X \,.\, \exists X \Rightarrow$$

$$\Rightarrow \left(\texttt{Comeager}(X, A) \iff \exists \texttt{SecondPlayerWinningStrategy}\Big(\eth^{**}(A)\Big)\right)$$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{Comeager}(X, A),$

$\Big(U, [2]\Big) := \texttt{EComeager}(X, A)[1] : \sum U : \mathbb{N} \to \texttt{Dense}\ \&\ \texttt{Open}(X) \,.\, A = \bigcap_{n=1}^{\infty} U_n,$

$\texttt{Assume}\ V : \mathrm{lp}\ \eth^{**}(A),$

$n := \mathrm{len}\, V \in \mathbb{Z}_+,$

$\texttt{Assume}\ [3] : \texttt{Odd}(n),$

$k := \dfrac{n+1}{2} \in \mathbb{N},$

$V_{n+1} := V_n \cap U_k : \texttt{Open}\ \&\ \texttt{NonEmpty}(X);$

$\leadsto V := \texttt{Play}\Big(\eth^{**}(A)\Big) : [\mathrm{lp}\, \eth^{**}(A)],$

$[3] := \texttt{E}V[2] : \bigcap_{n=1}^{\infty} V_n \subset \bigcap_{n=1}^{\infty} U_n = A,$

$[1.*] := \texttt{ISecondPlayerWinningStrategy}[3] : \exists \texttt{SecondPlayerWinningStrategy}\Big(\eth^{**}(A)\Big);$

$\square$

$\texttt{FirstPlayerBanachMazurTheorem} :: \forall X : \texttt{ChoquetSpace} \,.\, \forall A \subset X \,.\, \Big(\exists d : \texttt{Metric}(X) \,.\, \mathcal{T}(X,d) \subset \mathcal{T}(X)\Big) \Rightarrow$

$\Rightarrow \Bigg(\exists U \in \mathcal{T}(X) \,.\, \exists U \,\&\, \texttt{Meager}(U, U \cap A) \iff \exists\texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A)\Big)\Bigg)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U \in \mathcal{T}(X),$

$\texttt{Assume}\ [1] : \exists U,$

$\texttt{Assume}\ [2] : \texttt{Meager}(U, U \cap A),$

$\Big(W, [3]\Big) := \texttt{EMeager}(U, U \cap A) : \sum W : \mathbb{N} \to \texttt{Open} \,\&\, \texttt{Dense}(U) \,.\, \bigcap_{n=1} W_n = U \setminus A,$

$[4] := \texttt{OpenOpenSubset}(X, U, W) : \forall n \in \mathbb{N} \,.\, W_n \in \mathcal{T}(X),$

$[5] := \texttt{ChoquetSpaceOpenSubset}(X, U) : \texttt{ChoquetSpace}(U),$

$\sigma := \texttt{EChoquetSpace}(U) : \texttt{SecondPlayerWinningStrategy}\Big(\eth_{\mathrm{Ch}}(U)\Big),$

$\texttt{Assume}\ V : \mathrm{lp}\,\Big(\eth^{**}(A)\Big),$

$n := \mathrm{len}\, V \in \mathbb{Z}_+,$

$\texttt{Assume}\ [6] : \texttt{Even}(n),$

$\texttt{Assume}\ [7] : n = 0,$

$V_1 := U : \texttt{Open} \,\&\, \texttt{NonEmpty}(X);$

$\rightsquigarrow [7] := \texttt{I} \Rightarrow : (n = 0) \Rightarrow \texttt{Open} \,\&\, \texttt{NonEmpty}(X),$

$\texttt{Assume}\ [8] : n \neq 0,$

$[9] := \texttt{InGame}[7][8] : \forall k \in [1, \dots, n] \,.\, V_k \subset U,$

$V' := \Lambda k \in [1, \dots, n] \,.\, \texttt{if}\ \texttt{Odd}(k)\ \texttt{then}\ V_k \cap W_{(k+1)/2}\ \texttt{else}\ V_k : [1, \dots, n] \to \mathcal{T}(X),$

$[10] := [9]\texttt{E}\,\mathrm{lp}\,\Big(\eth^{**}(A)\Big)\texttt{I}\,\mathrm{lp}\,\Big(\eth_{\mathrm{Ch}}(U)\Big) : V' \in \mathrm{lp}\,\Big(\eth_{\mathrm{Ch}}(U)\Big),$

$[11] := \texttt{InGame}(n-1)[9] : V' \in \sigma,$

$\Big(O, [12]\Big) := \texttt{ESecondPlayerStrategy}\big(\eth_{\mathrm{Ch}}(U), \sigma, V'\big) : \sum O \in \mathcal{T}(U) \,.\, \exists O \,\&\, V'O \in \sigma,$

$[13] := \texttt{OpenOpenSubset}(X, U, O) : O \in \mathcal{T}(X),$

$V_{n+1} := O : \texttt{Open} \,\&\, \texttt{NonEmpty}(X);$

$\rightsquigarrow V := \texttt{Play}\Big(\eth^{**}(A)\Big) : \Big[\mathrm{lp}\, \eth^{**}(A)\Big],$

$[6] := \texttt{E}V\texttt{ESecondPlayerWinningStrategy}(\sigma) : U \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset,$

$[1.*] := \texttt{E}V[3] : A^{\complement} \cap \bigcap_{n=1}^{\infty} V_n;$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : \exists U \in \mathcal{T}(X) \,.\, \exists U \,\&\, \texttt{Meager}(U, U \cap A) \Rightarrow \exists\texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A)\Big),$

$\texttt{Assume}\ \sigma : \texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A)\Big),$

$\Big(U,[2]\Big) := \texttt{EFirstPlayerWinningStrategy}\Big(\eth^{**}(A),\sigma,1\Big) : \sum U \in \mathcal{T}(X) : \exists U \ \&\ \{U\} = \{t_1 | t \in \sigma\},$

$\texttt{Assume}\ V : \mathrm{lp}\ \Big(\eth^{**}(A)\Big),$

$n := \operatorname{len} V \in \mathbb{Z}_+,$

$\texttt{Assume}\ [3] : \texttt{Even}(n),$

$\texttt{Assume}\ [4] : n = 0,$

$V_1 := U : \texttt{Open}\ \&\ \texttt{NonEmpty}(X);$

$\rightsquigarrow [4] := \texttt{I} \Rightarrow : (n = 0) \Rightarrow \texttt{Open}\ \&\ \texttt{NonEmpty}(X),$

$\texttt{Assume}\ [5] : n \neq 0,$

$x := \mathtt{E}\exists V_n \in V_n,$

$W := V_k \cap \mathbb{B}_d(x, 2^{-n-1}) : \texttt{Open}\ \&\ \texttt{NonEmpty}(X),$

$V' := V_{|[1,\dots,1-n]} W : [1,\dots,n] \to \texttt{Open}\ \&\ \texttt{NonEmpty}(X),$

$[6] := \mathtt{E}V'\texttt{InPlay} : V' \in \sigma,$

$\Big(O,[7]\Big) := \texttt{EFirstPlayerStrategy}\Big(\eth^{**}(A),\sigma,V'\Big) : \sum O : \texttt{Open}\ \&\ \texttt{NonEmpty}(X)\ .\ V'O \in \sigma,$

$[8] := \mathtt{E}\eth^{**}(A)[7] : \operatorname{diam} O \leq 2^{-n},$

$V_{n+1} := O : \texttt{Open}\ \&\ \texttt{NonEmpty}(X);$

$\rightsquigarrow V := \texttt{Play}\ \eth^{**}(A) : \Big[\mathrm{lp}\,\eth^{**}(A)\Big],$

$[3] := \mathtt{E}V\texttt{EFirstPlayerWinningStrategy}\Big(\eth^{**}(A),\sigma\Big) : A^{\complement} \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset,$

$[4] := \mathtt{E}V\mathtt{E}V.7 : \lim_{n\to\infty} \operatorname{diam} V_n = 0,$

$\Big(x,[V.*]\Big) := [3][4] : \sum x \in A^{\complement}\ .\ \bigcap_{n=1}^{\infty} V_n = \{x\};$

$\rightsquigarrow \Big(\sigma',[3]\Big) := \texttt{GameOver} : \sum \sigma' : \texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A)\Big)\ .$

$.\ \forall V \in [\sigma']\ .\ \exists x \in A^{\complement}\ .\ \bigcap_{n=1}^{\infty} V_n = \{x\},$

$[4] := \texttt{EFirstPlayerWinningStrategy}\Big(\eth^{**}(A),\sigma'\Big)\mathtt{E}\sigma' : \texttt{Comeager}(U, A^{\complement} \cap U),$

$[2.*] := [4]^{\complement} : \texttt{Meager}(U, A \cap U);$

$\rightsquigarrow [*] := \texttt{I} \iff : \exists U \in \mathcal{T}(X)\ .\ \exists U\ \&\ \texttt{Meager}(U, U \cap A) \iff \exists\texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A)\Big);$

$\square$

$$\texttt{Determined} :: \prod_{X \in \mathsf{SET}} ?\texttt{InfiniteIterativeTwoPlayersGame}(X)$$

$$g : \texttt{Determined} \iff \exists\texttt{FirstPlayerWinningStrategy}(g)\Big|\exists\texttt{SecondPlayerWinningStrategy}(g)$$

$\texttt{BairPropertyByDetermination} :: \forall X : \texttt{ChoquetSpace} \,.\, \forall A \subset X \,.\, \Big(\exists d : \texttt{Metric}(X) \,.\, \mathcal{T}(X,d) \subset \mathcal{T}(X)\Big) \Rightarrow$

$\Rightarrow \Big(A \in \mathbf{BP}(X) \iff \forall U \in \mathcal{T}(X) \,.\, \exists U \Rightarrow \texttt{Determind}\Big(\eth^{**}(A \cup U)\Big)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : A \in \mathbf{BP}(X),$

$\texttt{Assume}\ U \in \mathcal{T}(X),$

$\texttt{Assume}\ [3] : \exists U,$

$[4] := \texttt{LEM}\Big(\texttt{Comeager}(U, A \cap U)\Big) : \texttt{Comeager}(U, A \cap U)\Big|\neg\texttt{Comeager}(U, A \cap U),$

$\texttt{Assume}\ [5] : \neg\texttt{Comeager}(U, A \cap U),$

$\Big(W, [6]\Big) := \texttt{EComeager}[5] : \sum W \in \mathcal{T}(X) \,.\, W \subset U \;\&\; \exists W \;\&\; \texttt{Meager}(W, A \cap W),$

$[8] := \texttt{FirstPlayerBanachMazurTheorem}[6] : \exists\texttt{FirstPlayerWinningStrategy}\Big(\eth^{**}(A \cap U)\Big),$

$[5.*] := \texttt{IDetermined}[8] : \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big);$

$\rightsquigarrow [5] := \texttt{I} \Rightarrow : V \bigtriangleup U \in \mathbf{MGR}(X) \Rightarrow \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big),$

$[6] := \texttt{SecondPlayerBanachMazurTheoremIDetermined} : \texttt{Comeager}(U, A \cap U) \Rightarrow \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big),$

$[1.*] := \texttt{E}|[4][5][6] : \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big);$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : A \in \mathbf{BP}(X) \Rightarrow \forall U \in \mathcal{T}(X) \,.\, \exists U \Rightarrow \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big),$

$\texttt{Assume}\ [2] : \forall U \in \mathcal{T}(X) \,.\, \exists U \Rightarrow \texttt{Determined}\Big(\eth^{**}(A \cap U)\Big),$

$\mathcal{U} := \Big\{U \in \mathcal{T}(X) : \exists U \;\&\; \exists\texttt{SecondPlayerWinningStrategy}\Big(\eth^{**}(A \cap U)\Big)\Big\} :?\mathcal{T}(X),$

$V := \bigcup \mathcal{U} \in \mathcal{T}(X),$

$\texttt{Assume}\ [3] : \neg\texttt{Comeager}(V, V \cap A),$

$\Big(W, [4]\Big) := [2][3] : \sum W \in \mathcal{T}(V) \,.\, \exists W \;\&\; W \cap A \in \mathbf{MGR}(V),$

$\Big(U, [5]\Big) := \texttt{E}V(W) : \sum U \in \mathcal{U} \,.\, U \cap W \neq \emptyset,$

$[6] := [5][4]\texttt{E}\mathcal{U}\texttt{E}\eth^{**} : U \notin \mathcal{U},$

$[3.*] := \texttt{E}U[5] : \perp;$

$\rightsquigarrow [3] := \texttt{E}\perp : \texttt{Comeager}(V, V \cap A),$

$\mathcal{U}' := \Big\{U \in \mathcal{T}(X) : \exists \;\&\; \texttt{Meager}(U, U \cap A)\Big\} :?\mathcal{T}(X),$

$V' := \bigcup \mathcal{U}' \in \mathcal{T}(X),$

$[4] := \texttt{IMeager}(X, A)\texttt{E}V' : \texttt{Meager}(V', A \cap V'),$

$[5] := \texttt{E}V'\texttt{E}V[2] : X = V' \cup \partial V \cup V,$

$[2.*] := \texttt{I}\mathbf{BP}\texttt{MeagerResidual}(X)[3][4][5] : A \in \mathbf{BP}(X);$

$\rightsquigarrow [*] :=: A \in \mathbf{BP}(X) \iff \forall U \in \mathcal{T}(X) \,.\, \exists U \Rightarrow \texttt{Determind}\Big(\eth^{**}(A \cup U)\Big);$

$\square$

$$\texttt{EquivalentGames} :: \prod_{X,Y,Z \in \mathsf{SET}} ?\Big( \big(X \to \texttt{InfiniteIterativeTwoPlayersGame}(Y)\big)$$
$$\big(X \to \texttt{InfiniteIterativeTwoPlayersGame}(Z)\big)\Big)$$
$$(g, g') : \texttt{EquivalentGames} \iff g \cong g' \iff \forall x \in X \; .$$
$$. \left( \left(\exists \texttt{FirstPlayerWinningStrategy}\big(g(x)\big) \iff \exists \texttt{FirstPlayerWinningStrategy}\big(g'(x)\big) \; \& \right.\right.$$
$$\left.\left. \& \; \exists \texttt{SecondPlayerWinningStrategy}\big(g(x)\big) \iff \exists \texttt{SecondPlayerWinningStrategy}\big(g'(x)\big)\right)\right)$$

$$\texttt{weakBasisBanachMazurGame} :: \prod_{X \in \mathsf{TOP}} \prod \mathcal{V} : \texttt{WeakBasis}(X) \; .$$
$$. \; ?X \to \texttt{InfiniteIterativeTwoPlayersGame}(\mathcal{V})$$
$$\texttt{weakBasisBanachMazurGame}\,(A) = \eth^{**}(A)_{\mathcal{V}} :=$$
$$:= \left( \left\{ U : [1, \dots, n] \downarrow \mathcal{V} \middle| n \in \mathbb{Z}_+, \forall i \in \{1, \dots, n\} \; . \; \exists U_i \right\}, \Lambda \mathcal{U} \in \mathcal{V}^{\mathbb{N}}(X) \; . \; \bigcap_{n=1}^{\infty} U_n \subset A \right)$$

$\texttt{WeakBanachMazurGameEquivalence} :: \forall X \in \mathsf{TOP} \; . \; \forall \mathcal{V} : \texttt{WeakBasis} \; . \; \eth^{**} \cong \eth^{**}_{\mathcal{V}}$

Proof $=$

...

□

### 1.4.7 Bair measurable functions

$$\texttt{BairMeasurable} :: \prod_{X,Y \in \mathsf{TOP}} ?\mathsf{SET}(X,Y)$$

$$f : \texttt{BairMeasurable} \iff \forall U \in \mathcal{T}(Y) \,.\, f^{-1}(U) \in \mathbf{BP}(X)$$

$$\texttt{BairMeasurabilityContinuousPart} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{SecondCoubtable} \,.\, \forall f : \texttt{BairMeasurable}(X,Y) \,.$$
$$.\, \exists G \subset X \,.\, \exists U : \mathbb{N} \to \texttt{Open} \,\&\, \texttt{Dense}(X) \,.\, G = \bigcap_{n=1}^{\infty} U_n \,\&\, f_{|G} \in \mathsf{TOP}(G,Y)$$

Proof $=$

$\Big(\mathcal{U}, [1]\Big) := \texttt{ESecondCountable}(Y) : \sum \mathcal{U} : \texttt{Base}(Y) \,.\, |\mathcal{U}| \le \aleph_0,$

$[2] := \texttt{EBairMeasurable}(X,Y) : f^{-1}(\mathcal{U}) \subset \mathbf{BP}(X),$

Assume $U \in \mathcal{U}$,

$\Big(V, E, [3]\Big) := \mathtt{E}\mathbf{BP}(X) : \sum V \in \mathcal{T}(X) \,.\, \sum E \in \mathbf{MGR}(X) \,.\, f^{-1}(U) = V \triangle E,$

$\Big(N, [4]\Big) := \mathtt{E}\mathbf{MGR}(X,V) : \sum N : \mathbb{N} \to \texttt{Closed} \,.\, E \subset \bigcap_{n=1}^{\infty} N_n,$

$F := \bigcap_{n=1}^{\infty} N_n \in \mathbf{MGR}(X),$

$[5] := \mathtt{E}F[3][4] : f^{-1}(U_n) \triangle V_n \subset F_n,$

$G_U := F^{\complement} : \texttt{Comeager}(X);$

$\rightsquigarrow \Big(V, G, [2]\Big) := \mathtt{I}\prod : \prod_{U \in \mathcal{U}} \Big(V_U, G_U\Big) : \mathcal{T}(X) \times \texttt{Comeager}(X) \,.\, f^{-1}(U) \triangle V_U \subset G_U^{\complement},$

$H := \bigcap_{U \in \mathcal{U}} G_U : \texttt{Comeager}(X),$

$[3] := [2]^{\complement}\mathtt{E}\triangle \ : \forall U \in \mathcal{U} \,.\, f^{-1}(U) \cap H = V_U \cap H,$

$[4] := \mathtt{I} f_{|H}^{-1}[3] : \forall U \in \mathcal{U} \,.\, f_{|H}^{-1}(U) = V_U,$

$[*] := \mathtt{I}\mathsf{TOP}[5] : f_{|H} \in \mathsf{TOP}(H,Y);$

$\square$

$$\texttt{BairMeasurableCantorImage} :: \forall X : \texttt{Perfect} \,\&\, \texttt{Polish} \,.\, \forall Y : \texttt{SeconCountable} \,.$$
$$.\, \forall f : \texttt{BairMeasurable} \,\&\, \texttt{Injective}(X,Y) \,.\, \exists C \subset f(X) \,.\, C \cong \mathcal{C}$$

Proof $=$

$\ldots$

$\square$

### 1.4.8 Kuratowski-Ulam theorem

$$\texttt{BairQuantificationForall} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$A : \texttt{BairQuantificationForall} \iff \forall^* x \,.\, A(x) \iff \texttt{Comeager}(X, A)$$

$$\texttt{BairQuantificationExists} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$A : \texttt{BairQuantificationExists} \iff \exists^* x \,.\, A(x) \iff \neg\texttt{Meager}(X, A)$$

$$\texttt{LocalBairQuantificationForall} :: \prod_{X \in \mathsf{TOP}} \mathcal{T}(X) \to ??X$$

$$A : \texttt{BairQuantificationForall} \iff \Lambda U \in \mathcal{T}(U) \,.\, \forall^* x \in U \,.\, A(x) \iff \Lambda U \in \mathcal{T}(U) \,.\, \texttt{Comeager}(U, A \cap U)$$

$$\texttt{LocalBairQuantificationExists} :: \prod_{X \in \mathsf{TOP}} \mathcal{T}(X) \to ??X$$

$$A : \texttt{LocalBairQuantificationExists} \iff \Lambda U \in \mathcal{T}(U) \,.\, \exists^* x \in U \,.\, A(x) \iff$$
$$\iff \Lambda U \in \mathcal{T}(U) \,.\, \neg\texttt{Meager}(U, A \cap U)$$

$\texttt{NowhereDenseSectionLemma} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{SecondCountable}(X) \,.$
$\quad . \forall F : \texttt{NowhereDense}(X \times Y) \,.\, \forall^* x \in X \,.\, \texttt{NowhereDense}(Y, F_x)$

$\texttt{Proof} \;=$

$F' := \overline{F} : \texttt{Closed}(X \times Y),$

$[1] := \texttt{ENowhereDense}(X \times T, F)\texttt{ClosureIsRetraction}(X \times Y)\texttt{INowhereDenseI}F' :$
$\quad : \texttt{NowhereDense}(X \times T, F'),$

$U := (X \times Y) \setminus F' : \texttt{Open} \;\&\; \texttt{Dense}(X \times Y),$

$\big(\mathcal{V}, [2]\big) := \texttt{ESecondCountable}(Y) : \sum \mathcal{V} : \texttt{Base}(Y) \,.\, |\mathcal{V}| \le \aleph_0,$

$\mathcal{U} := \left\{ \pi_X\big(U \cap (X \times V)\big) \middle| V \in \mathcal{V}, \exists V \right\} :?\mathcal{T}(X),$

$\texttt{Assume } O \in \mathcal{U},$

$\big(V, [4]\big) := \texttt{E}\mathcal{U}(O) : \sum V \in \mathcal{V} \,.\, O = \pi_X\big(U \cap (X \times V)\big),$

$\texttt{Assume } W \in \mathcal{T}(X),$

$\texttt{Assume } [3] : \exists W,$

$[5] := \texttt{EDense}(X \times Y, U, W \times V) : \exists\big(U \cap (W \times V)\big),$

$[6] := \texttt{SubsetProduct}(X, Y, W, V)\texttt{SubsetIntersection}\Big(U, X\Big)[4] : U \cap (W \times V) \subset U \cap (X \times V) = O),$

$[O.*] := \texttt{ProjectionIntersection}(X, Y)[6][5] : \exists(O \cap W);$

$\rightsquigarrow [3] := \texttt{IDenseI}\forall : \forall O \in \mathcal{U} \,.\, \texttt{Dense}(X, O),$

$A := \bigcap \mathcal{U} : \texttt{Comeager}(X),$

$[4] := \texttt{EAE}\mathcal{U} : \forall a \in A \,.\, \forall V \in \mathcal{V} \,.\, U_a \cap V \ne \emptyset,$

$[5] := \texttt{DenseByBase}[4] : \forall a \in A \,.\, \texttt{Dense}(Y, U_a),$

$[6] := \texttt{I}\forall^* : \forall^* x \in X \,.\, \texttt{Dense}(Y, U_x),$

$[7] := \texttt{E}U[6] : \forall^* x \in X \,.\, \texttt{NowhereDense}(Y, F'_x),$

$[8] := \texttt{E}F'\texttt{NowhereDenseSubset}(Y)[7] : \forall^* x \in X \,.\, \texttt{NowhereDense}(Y, F_x);$

$\square$

$\texttt{MeagerSectionLemma} :: \forall X \in \mathsf{TOP} . \forall Y : \texttt{SecondCountable}(X) .$
$\quad . \forall F : \texttt{Meager}(X \times Y) . \forall^* x \in X . \texttt{Meager}(Y, F_x)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{KuratowskiUlamTHM1} :: \forall X, Y : \texttt{SecondCountable}(X) . \forall A \in \mathbf{BP}(X \times Y) .$
$\quad . \forall^* x \in X . A_x \in \mathbf{BP}(Y) \ \& \ \forall^* y \in Y . A^y \in \mathbf{BP}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{KuratowskiUlamTHM2} :: \forall X, Y : \texttt{SecondCountable}(X) . \forall A \in \mathbf{MGR}(X \times Y) .$
$\quad . \forall^* x \in X . A_x \in \mathbf{MGR}(Y) \ \& \ \forall^* y \in Y . A^y \in \mathbf{MGR}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{MeagerProductLemma} :: \forall X, Y : \texttt{SecondCountable}(X) . \forall A \subset X . \forall B \subset Y .$
$\quad A \in \mathbf{MGR}(X) \Big| B \in \mathbf{MGR}(Y) \Rightarrow A \times B \in \mathbf{MGR}(X \times Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : A \in \mathbf{MGR}(X),$

$\Big(N, [2]\Big) := \texttt{E}\mathbf{MGR}(X, A) : \sum N : \mathbb{N} \to \texttt{NowhereDense}(X) . A = \bigcup_{n=1}^{\infty} N_n,$

$[3] := \texttt{ProductUnion}(X, Y, A, B)[2] : A \times B = \bigcup_{n=1}^{\infty} N_n \times B,$

$[4] := \Lambda n \in \mathbb{N} . \texttt{NowhereDenseProductProduct}(X, Y, N_n, B) : \forall n \in \mathbb{N} . \texttt{NowhereDense}(X \times Y, N_n \times B),$

$[*] := \texttt{I}\mathbf{MGR}(X \times Y)[3][4] : A \times B \in \mathbf{MGR}(X \times Y);$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : A \in \mathbf{MGR}(X) \Rightarrow A \times B \in \mathbf{MGR}(X \times Y),$

$\texttt{Assume}\ [2] : B \in \mathbf{MGR}(Y),$

$\Big(N, [3]\Big) := \texttt{E}\mathbf{MGR}(Y, B) : \sum N : \mathbb{N} \to \texttt{NowhereDense}(Y) . B = \bigcup_{n=1}^{\infty} N_n,$

$[4] := \texttt{ProductUnion}(X, Y, A, B)[4] : A \times B = A \times \bigcup_{n=1}^{\infty} N_n,$

$[5] := \Lambda n \in \mathbb{N} . \texttt{NowhereDenseProductProduct}(X, Y, A, N_n) : \forall n \in \mathbb{N} . \texttt{NowhereDense}(X \times Y, A \times N_n),$

$[*] := \texttt{I}\mathbf{MGR}(X \times Y)[4][5] : A \times B \in \mathbf{MGR}(X \times Y);$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow : B \in \mathbf{MGR}(Y) \Rightarrow A \times B \in \mathbf{MGR}(X \times Y),$

$[*] := \texttt{E}(|)[0][1][2] : A \times B \in \mathbf{MGR}(X \times Y);$

$\square$

$\texttt{MeagerProductLemma2} :: \forall X, Y : \texttt{SecondCountable}(X) . \forall A \subset X . \forall B \subset Y .$
$\quad A \times B \in \mathbf{MGR}(X \times Y) \Rightarrow A \in \mathbf{MGR}(X) \Big| B \in \mathbf{MGR}(Y)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InverseKuratowskiUlamTHM2} :: \forall X, Y : \texttt{SecondCountable}(X) \,.\, \forall A \in \mathbf{BP}(X \times Y) \,.$

$\quad . \left(\forall^* x \in X \,.\, A_x \in \mathbf{MGR}(Y) \middle| \forall^* y \in Y \,.\, A^y \in \mathbf{MGR}(X)\right) \Rightarrow A \in \mathbf{MGR}(X \times Y)$

$\texttt{Proof} =$

$\left(U, E[-1]\right) := \mathtt{E}\mathbf{BP}(X \times Y, A) : \sum U \in \mathcal{T}(X \times Y) \,.\, \sum E \,\mathrm{Im}\, \mathbf{MGR}(X \times Y) \,.\, A = U \triangle E,$

$\texttt{Assume } [1] : \forall^* x \in X \,.\, A_x \in \mathbf{MGR}(Y),$

$\texttt{Assume } [2] : \exists^* A,$

$[3] := [2][-1] : \exists^* U,$

$\left(V, W, [4]\right) := \texttt{SCProducTopologyProperty}(X, Y, U)[3] : \sum_{V \in \mathcal{T}(X)} \sum_{W \in \mathcal{T}(Y)} V \times W \subset U \,\&\, \exists^* V \times W,$

$[5] := \texttt{MeagerProductLemma}[4] : \exists^* V \exists^* W,$

$\left(x, [6]\right) := \mathtt{E}\forall^*[1](V) : \sum x \in V \,.\, A_x \in \mathbf{MGR}(Y),$

$[7] := [6]\mathtt{E}\exists^*[5] : E_x \in \mathbf{MGR}(Y),$

$[8] := \texttt{DifferenceSubset}(Y, W, E_x, U_x)[4.1]\texttt{SymmetricIsMore}(Y)\texttt{SectionSubset}(X, Y)[-1] :$

$\quad : W \setminus E_x \subset U_x \setminus E_x \subset U_x \triangle E_x \subset A_x,$

$[9] := \texttt{MeagerSubset}(Y)[8][6]\texttt{MeagerDifference}(Y)[7] : W \in \mathbf{MGR}(Y),$

$[1.*] := \mathtt{E}\exists^*[4.2][9] : \bot;$

$\leadsto [1] := \mathtt{I} \Rightarrow : \left(\forall^* x \in X \,.\, A_x \in \mathbf{MGR}(Y)\right) \Rightarrow A \in \mathbf{MGR}(X \times Y),$

$\texttt{Assume } [2] : \forall^* y \in Y \,.\, A^y \in \mathbf{MGR}(y),$

$\texttt{Assume } [3] : \exists^* A,$

$[4] := [3][-1] : \exists^* U,$

$\left(V, W, [5]\right) := \texttt{SCProducTopologyProperty}(X, Y, U)[3] : \sum_{V \in \mathcal{T}(X)} \sum_{W \in \mathcal{T}(Y)} V \times W \subset U \,\&\, \exists^* V \times W,$

$[6] := \texttt{MeagerProductLemma}[5] : \exists^* V \exists^* W,$

$\left(y, [7]\right) := \mathtt{E}\forall^*[2](W) : \sum y \in W \,.\, A^y \in \mathbf{MGR}(X),$

$[8] := [7]\mathtt{E}\exists^*[6] : E^y \in \mathbf{MGR}(X),$

$[9] := \texttt{DifferenceSubset}(X, V, E^y, U^y)[5.1]\texttt{SymmetricIsMore}(X)\texttt{SectionSubset}(X, Y)[-1] :$

$\quad : V \setminus E^y \subset U^y \setminus E^y \subset U^y \triangle E^y \subset A^y,$

$[10] := \texttt{MeagerSubset}(Y)[9][7]\texttt{MeagerDifference}(Y)[6] : V \in \mathbf{MGR}(Y),$

$[2.*] := \mathtt{E}\exists^*[5.2][10] : \bot;$

$\leadsto [2] := \mathtt{I} \Rightarrow : \left(\forall^* x \in X \,.\, A_x \in \mathbf{MGR}(Y)\right) \Rightarrow A \in \mathbf{MGR}(X \times Y),$

$[*] := \mathtt{E}(|)[0][1][2] : A \in \mathbf{MGR}(X \times Y);$

$\square$

$\texttt{UlamKuratowskiTHM} :: \forall X, Y : \texttt{SecondCountable} \,.\, \forall A \in \mathbf{BP}(X \times Y) \,.$

$\quad . \forall^* x \in X \,.\, \forall^* y \in Y \,.\, A(x, y) \iff \forall^* y \in Y \,.\, \forall^* x \in X \,.\, A(x, y) \iff \forall (x, y) \in X \times Y \,.\, A(x, y)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{BairProduct} :: \forall X, Y : \texttt{Baire} \ \& \ \texttt{SecondCountable} \ . \ \texttt{Baire}(X \times Y)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ U : \mathbb{N} \to \texttt{Open} \ \& \ \texttt{Dense}(X \times Y),$

$[1] := \Lambda n \in \mathbb{N} \ . \ \texttt{UlamKuratowskiTHM}(U_n) : \forall n \in \mathbb{N} \ . \ \forall^* x \in X \ . \ \forall^* y \in Y \ . \ U_n(x, y),$

$\Big(F, D, [2]\Big) := \texttt{E}\forall^*[1]\texttt{EBaire}(Y) : \sum F : \texttt{Comeager}(X) \ . \ \sum D : X \to \texttt{Dense}(Y) \ . \ \forall x \in F \ . \ D_x \subset \Big(\bigcap U_n\Big)_x,$

$[3] := \texttt{EBaire}(X, F) : \texttt{Dense}(X, F),$

$\texttt{Assume} \ V \in \mathcal{T}(X \times Y),$

$\texttt{Assume} \ [5] : \exists V,$

$[6] := \pi_x[5] : \exists \pi_x(V),$

$\Big(f, [7]\Big) := \texttt{EDense}(X, F)[6] : \sum f \in F \ . \ f \in \pi_x(V),$

$[8] := \texttt{I}V_f[7] : \exists V_f,$

$[*] := \texttt{EDense}(Y, D_f)[7] : \exists \left(\{f\} \times D_f\right) V;$

$\rightsquigarrow [4] := \texttt{IDense} : \texttt{Dense}\left(X \times Y, \bigcup_{f \in F} \{f\} \times D_f\right),$

$[5] := \bigcup_{x \in X} [2](x) : \bigcup_{f \in F} \{f\} \times D_f \subset \bigcap_{n=1}^{\infty} U_n,$

$[U.*] := \texttt{DenseSubset}[4][3] : \texttt{Dense}\left(X \times Y, \bigcap_{n=1} U_n\right);$

$\rightsquigarrow [*] := \texttt{IBaire} : \texttt{Baire}(X \times Y),$

$\square$

$\texttt{DensePreimageTheorem} :: \forall X, Y \in \mathsf{TOP} \ . \ \forall f \in \texttt{Open}(X, Y) \ . \ \forall D : \texttt{Dense}(Y) \ . \ \texttt{Dense}\Big(X, f^{-1}(D)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ U \in \mathcal{T}(X),$

$\texttt{Assume} \ [1] : \exists U,$

$[2] := f[1] : \exists f(U),$

$[3] := \texttt{EOpen} : f(U) \in \mathcal{T}(Y),$

$[4] := \texttt{EDense}(Y, D) : \exists D \cap f(U),$

$[U.*] := \texttt{ImagePreimage}[4] : \exists f^{-1}(D) \cap U;$

$\rightsquigarrow [*] := \texttt{IDense} : \texttt{Dense}(X, f^{-1}(D));$

$\square$

$\texttt{MeagerPreimageTheorem} :: \forall X, Y \in \mathsf{TOP} \ . \ \forall f \in \texttt{Open} \ \& \ \mathsf{TOP}(X, Y) \ . \ \forall M : \mathbf{MGR}(Y) \ . \ f^{-1}(M) \in \mathbf{MGR}(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{Assume} \ x \in D,$

### 1.4.9 Fun facts

$$\texttt{TopTransitiveGroup} :: \prod_{X \in \mathsf{TOP}} ?_{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{TOP}}(X)$$

$$G : \texttt{TopTransitiveGroup} \iff G : \mathsf{TOP}\text{-}\texttt{Transitive}(X) \iff$$
$$\iff \forall U, V \in \mathcal{T}(X) \;.\; \exists U \;\&\; \exists V \Rightarrow \exists g \in G \;.\; \exists g(U) \cap V$$

$$\texttt{FirstTopological01Law} :: \forall X : \texttt{Baire} \;.\; \forall G : \mathsf{TOP}\text{-}\texttt{Transitive}(X) \;.\; \forall A : \texttt{Invariant}(X, F) \;.$$
$$. \; A \in \mathbf{BP}(X) \Rightarrow \forall^* A \Big| \neg \exists^* A$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : (\neg \forall^* A) \;\&\; \exists^* A,$

$\Big(U, V, [2]\Big) := \texttt{E}\forall^*\texttt{E}\exists^* : U \Vdash A \;\&\; V \Vdash \neg A \;\&\; \exists U \;\&\; \exists V,$

$\Big(g, [3]\Big) := \texttt{E}\mathsf{TOP}\text{-}\texttt{Transitive}(X, G, U, V) : \sum g \in G \;.\; \exists g(U) \cap V,$

$[4] := g[2.1] : g(U) \vdash g(A),$

$[5] := \texttt{EInvariant}(X, G, A)[4] : g(U) \vdash A,$

$[6] := [5][3] : g(U) \cap V \Vdash A,$

$[7] := [2.2][3] : g(U) \cap V \Vdash A^{\complement},$

$[8] := \texttt{E}\mathsf{BOOL}\Big(\mathbf{CAT}(X)\Big)\texttt{E} \Vdash [6][7]\texttt{E}\mathsf{BOOL}\Big(\mathbf{BP}(X), \mathbf{CAT}(X), \pi_{\mathbf{CAT}}\Big)\texttt{E}\complement :$

$$[g(U) \cap V]_{\mathbf{CAT}} = [g(U) \cap V]^2_{\mathbf{CAT}} = [A \cap g(U) \cap V]_{\mathbf{CAT}}[A^{\complement} \cap g(U) \cap V]_{\mathbf{CAT}} =$$
$$= [A]_{\mathbf{CAT}}[A]^{\complement}_{\mathbf{CAT}}[g(U) \cap V]_{\mathbf{CAT}} = 0,$$

$[9] := \texttt{E}\mathbf{CAT}[8] : g(U) \cap V \in \mathbf{MGR}(X),$

$[1.*] := \texttt{EBaire}(X)[9][3] : \perp;$

$\rightsquigarrow [*] := \texttt{E}\perp\texttt{DeMorganaLaw} : \forall^* A \Big| \neg \exists^* A;$

$\square$

$$\texttt{TailSet} :: \prod_{I \in \mathsf{SET}} \prod_{X : I \to \mathsf{SET}} ?? \prod_{i \in I} X_i$$

$$T : \texttt{TailSet} \iff \forall x \in T \;.\; \forall y \in \prod_{i=1} X_i \;.\; \texttt{Finite}\Big(\{i \in I : x_i \neq y_i\}\Big) \Rightarrow y \in T$$

$\texttt{SecondTopolofical01Law} :: \forall X : \mathbb{N} \to \texttt{Baire} \ \& \ \texttt{SecondCountable} \ . \ \forall A : \texttt{TailSet}(\mathbb{N}, X) \ .$

$$. \ A \in \mathbf{BP}\left(\prod_{n=1}^{\infty} X_n\right) \Rightarrow \forall^* A \Big| \neg \exists^* A$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : \neg \exists^* A,$

$\Big(U, [2]\Big) := \mathtt{E}\exists^* \texttt{ProductTopologyRepresentation}(\mathbb{N}, X) :$

$$: \sum U : \prod_{n=1}^{\infty} \mathcal{T}(X_n) \ . \ \exists \prod_{n=1}^{\infty} U_n \ \& \ \prod_{n=1}^{\infty} U_n \Vdash A \ \& \ \texttt{Finite}\Big(\mathbb{N}, \{n \in \mathbb{N} : U_n \neq X_n\}\Big),$$

$N := \{n \in \mathbb{N} : U_n \neq X_n\} : \texttt{Finite}(\mathbb{N}),$

$Y := \prod_{n \in N} X_i \in \mathsf{TOP},$

$Z := \prod_{n \in N^{\complement}} \in \mathsf{TOP},$

$[5] := \mathtt{E}Y\texttt{BaireProduct}\mathtt{I}Y : \texttt{SecondCountable} \ \& \ \texttt{Baire}(Y),$

$[3] := \mathtt{E} \Vdash \mathtt{I}\forall^*\mathtt{I}Y[2.2][5] : \forall^* y \in \prod_{n \in \mathbb{N}} U_n \ . \ \forall^* z \in Z \ . \ A(y, z),$

$[4] := \mathtt{E}\texttt{TailSet}(\mathbb{N}, X, A)[3] : \forall^* y \in Y \ . \ \forall^* x \in X \ . \ A(x, y),$

$[1.*] := \texttt{KuratowskiUlamTHM}[4] : \forall^* A;$

$\rightsquigarrow [*] := \mathtt{I} \Rightarrow \mathtt{I}| : \forall^* A \Big| \neg \exists^* A;$

$\square$

$\texttt{WellOederingIsNotBP} :: \forall X : \texttt{Perfect} \ \& \ \texttt{Polish} \ . \ \forall (<) : \texttt{WellOrdering}(X) \ . \ (<) \notin \mathbf{BP}(X^2)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : (<) \in \mathbf{BP}(X^2),$

$\texttt{Assume} \ [2] : (<) \in \mathbf{MGR}(X^2),$

$[3] := \texttt{KuratovskiUlamTHM2}[2] : \forall^* x \in X \ . \ (<)_x \in \mathbf{MGR}(X) \ \& \ \forall^* x \in X \ . \ (<)^x \in \mathbf{MGR}(X),$

$\Big(x, [4]\Big) := \mathtt{E}\texttt{Baire}(X)[3] : \sum x \in X \ . \ (<)_x, (<)^x \in \mathbf{MGR}(X),$

$[5] := \mathtt{E}\texttt{TotalOrder}(X, \leq) : X = \{x\} \cup (<)_x \cup (<)^x,$

$[6] := \texttt{PerfectHasMeagerPoints}(X)\texttt{MeagerUnion}(X) : X \in \mathbf{MGR}(X),$

$[2.*] := \mathtt{E}\texttt{Baire}(X)[6] : \perp;$

$\rightsquigarrow [2] := \mathtt{I}\exists^* : \exists^*(<),$

$[3] := \texttt{InverseKuratovsliUlamTHM2}[2] : \exists x \in X \ . \ \exists^* <^x,$

$x := \min\{x \in X \ . \ \exists^* <^x\} \in X,$

$Y := <^x :?X,$

$<' := (<) \cap Y^2 :?Y^2,$

$[4] := \mathtt{E}(<')\texttt{KuratovskiUlamTHM1} : (<') \in \mathbf{BP}(Y^2),$

$[5] := \mathtt{E} <' \mathtt{E}x : \forall y \in Y \ . \ (<')^y \in \mathbf{MGR}(Y),$

$[6] := \texttt{KuratovskiUlamTHM2}[5] : (<') \in \mathbf{MGR}(Y),$

$[7] := \texttt{KuratovskiUlamTHM1}[6] : \forall^* y \in Y \ . \ (<')_* \in \mathbf{MGR}(Y),$

$[8] := \mathtt{E}\texttt{Baire}(Y)[7]\texttt{PerfectHasMeagerPoints}(Y)\texttt{MeagerUnion}(Y) : Y \in \mathbf{MGR}(Y),$

$[1.*] := \mathtt{E}\texttt{Baire}(Y)[8] : \perp;$

$\rightsquigarrow [*] := \mathtt{E}\perp : (<) \notin \mathbf{BP}(X^2);$

$\square$

$\texttt{SeparatelyContinuous} :: \prod X, Y, Z \in \mathsf{TOP} \,.\, ?(X \times Y \to Z)$

$f : \texttt{SeparatlyContinuous} \iff \forall x \in X \,.\, f_x \in \mathsf{TOP}(X, Z) \;\&\; \forall y \in Y \,.\, f^y \in \mathsf{TOP}(Y, Z)$

$\texttt{SeparateJointContinuityTHM} :: \forall X, Y, Z \in \mathsf{MS} \,.\, \forall f : \texttt{SeparatlyContinuous}(X, Y, Z) \,.$

$\quad .\, \forall^* \texttt{ContiniutyPoint}(f)$

$\texttt{Proof} \;=$

$$F := \Lambda n, k \in \mathbb{N} \,.\, \left\{(x, y) \in X \times T : \forall u, v \in \mathbb{B}\Big(y, 2^{-k}\Big) \,.\, d\Big(f(x, u), f(x, v)\Big) \le 2^{-n}\right\} : \mathbb{N}^2 \to ?(X \times Y),$$

$$[1] := \texttt{E}F\texttt{ESeparatelyContinuous}(X, Y, Z, f) : X \times Y = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} F_{n,k},$$

$\texttt{Assume}\ n, k \in \mathbb{N},$

$\texttt{Assume}\ (x, y) : \mathbb{N} \to F_{n,k},$

$\texttt{Assume}\ (x', y') \in X \times Y,$

$\texttt{Assume}\ [2] : \lim_{n \to \infty} (x_n, y_n) = (x', y'),$

$\texttt{Assume}\ u, v \in \mathbb{B}(y', 2^{-k}),$

$\Big(N, [3]\Big) := \texttt{ContinuousMetric}(Y)[2](u, v) : \sum N \in \mathbb{N} \,.\, \forall n \ge N \,.\, u, v \in \mathbb{B}(y', 2^{-k}),$

$[4] := \texttt{E}F_{n,k}[3] : \forall n \ge N \,.\, d\Big(f(x_n, u), f(x_n, v)\Big) \le 2^{-n},$

$\Big[(u, v).\, *\Big] := \texttt{ContinuousMetric}(X)[4][2] : d\Big(f(x', u), f(x', v)\Big) \le 2^{-n};$

$\leadsto \Big[(n, k).\, *\Big] := \texttt{E}F_{n,k} : (x', y') \in F_{n,k};$

$\leadsto [3] := \texttt{I} \Rightarrow \texttt{I}\forall\texttt{IClosedI}\forall : \forall n, k \in \mathbb{N} \,.\, \texttt{Closed}(X \times Y, F_{n,k}),$

$$D := \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{(x, y) : x \in F^y_{n,k} \setminus \operatorname{int} F^y_{n,k}\} \in \mathbf{MGR}(X \times Y),$$

$[4] := \texttt{E}D : \forall y \in Y \,.\, D^y \in \mathbf{MGR}(X),$

$G := (X \times Y) \setminus D : \texttt{Comeager}(X \times Y),$

$\texttt{Assume}\ (x, y) \in G,$

$\texttt{Assume}\ \varepsilon \in \mathbb{R}_{++},$

$\Big(n, [5]\Big) := \texttt{EArchimedian}(\mathbb{R}, \varepsilon) : \sum n \in \mathbb{N} \,.\, 2^{-n} \le \varepsilon,$

$\Big(k, [6]\Big) := [1](x, y)(n) : \sum k \in \mathbb{N} \,.\, (x, y) \in F_{n,k},$

$[7] := \texttt{E}G[6]\texttt{E}D^y : x \in F^y_{n,k} \setminus D^y \subset \operatorname{int} F^y_{n,k},$

$[8] := \texttt{E}\mathsf{TOP}(Y, Z, f^y) : \forall V \in \mathcal{U}_X(x) \,.\, V \subset F^y_{n,k} \Rightarrow \exists s \in V \,.\, d\Big(f(x, y), f(s, y)\Big) \le \varepsilon,$

$\texttt{Assume}\ V \in \mathcal{U}_X(x),$

$\texttt{Assume}\ [9] : V \subset F^y_{n,k},$

$\Big(s, [10]\Big) := [8](V)[9] : \sum_{s \in V} d\Big(f(x, y), f(s, y)\Big) \le \varepsilon,$

$\Big[(x, y).\, *\Big] := \texttt{TriangleIneq}(Z)\texttt{E}F^y_{n,k}(s)[5] :$

$\quad : \forall t \in \mathbb{B}(y, 2^{-k}) \,.\, d\Big(f(x, y), f(s, t)\Big) \le d\Big(f(x, y), f(s, y)\Big) + d\Big(f(s, y), f(s, t)\Big) \le 2\varepsilon;$

$\leadsto [*] := \texttt{IContinuityPoint} : \texttt{ContinuityPoint}(f, G);$

$\square$

$\texttt{NamiokaTHM} :: \forall X, Y \in \mathsf{MS} \,\&\, \texttt{Compact} \,.\, \forall Z \in \mathsf{MS} \,.\, \forall f : \texttt{SeparatlyContinuous}(X, Y, Z) \,.$

$\quad .\, \forall^* x \in X \,.\, \forall y \in Y \,.\, \texttt{ContinuityPoint}\Big(f, (x, y)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UltrafilterBairProperty} :: \forall A : \texttt{Ultrafilter}(\mathbb{N}) \,.\, \neg\texttt{Principle}(\mathbb{N}, A) \Rightarrow A \notin \mathbf{BP}(\mathcal{C})$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : A \in \mathbf{BP}(\mathcal{C}),$

$[2] := \texttt{ENonPrinciple}(\mathbb{N}, A) : \forall n \in \mathbb{N} \,.\, \exists b \in A : b_n = 0,$

$[3] := \texttt{EUltrafilter}(\mathbb{N}, A)[2] : \forall n \in \mathbb{N} \,.\, \forall x \in \mathbb{B}^n \,.\, \exists a \in A \,.\, a_{|[1,\ldots,n]} = x,$

$[4] := \texttt{StandardBaseIsBase}(\mathcal{C})[4]\texttt{IDense}(\mathcal{C}) : \texttt{Dense}(A, \mathcal{C}),$

$[5] := [3]\texttt{EUltrafilter}(\mathbb{N}, A)\texttt{IDense}(\mathcal{C}) : \texttt{Dense}(A, \mathcal{C}),$

$[4] := \mathtt{E}\mathsf{BOOL}\Big(\mathbf{CAT}(\mathcal{C})\Big)[5][7]\mathtt{E}\complement : [\mathcal{C}]_{\mathbf{CAT}} = [\mathcal{C}]^2_{\mathbf{CAT}} = [A]_{\mathbf{CAT}}[A^{\complement}]_{\mathbf{CAT}} = 0,$

$[1.*] := \texttt{EBaire}(\mathcal{C})[4] : \bot;$

$\rightsquigarrow [*] := \mathtt{E}\bot : A \notin \mathbf{BP}(\mathcal{C});$

$\square$

# 2 Borel Topology

## 2.1 Measurability

### 2.1.1 Algebras of Sets

$$\mathtt{Algebra} :: \prod_{X \in \mathsf{SET}} ?^3 X$$

$$\mathcal{F} : \mathtt{Algebra} \iff \mathcal{F} \subset_{\mathsf{BOOL}} ?X$$

$$\sigma\text{-}\mathtt{Algebra} :: \prod_{X \in \mathsf{SET}} ?^3 X$$

$$\mathcal{F} : \sigma\text{-}\mathtt{Algebra} \iff \mathcal{F} \subset^{\sigma}_{\mathsf{BOOL}} ?X$$

$$\mathtt{generateSigmaAlgebra} :: \prod_{X \in \mathsf{SET}} ??X \to \sigma\text{-}\mathtt{Algebra}(X)$$

$$\mathtt{generateSigmaAlgebra}\,(S) = \sigma(S) := \bigcap \left\{ \mathcal{A} \middle| \sigma\text{-}\mathtt{Algebra}(X, \mathcal{A}), S \subset \mathcal{A} \right\}$$

$$\mathtt{CountablyGeneratedSigmaAlgebra} :: \prod_{X \in \mathsf{SET}} ?\sigma\text{-}\mathtt{Algebra}(X)$$

$$\mathcal{F} : \mathtt{CountablyGeneratedSigmaAlgebra} \iff \exists S : \mathtt{Countable}(X) \;.\; \mathcal{F} = \sigma(X)$$

$$\mathtt{MonotonicClass} :: \prod_{X \in \mathsf{SET}} ???X$$

$$\mathcal{M} : \mathtt{MonotonicClass} \iff \left( \forall A : \mathbb{N} \uparrow \mathcal{M} \;.\; \bigcup_{n=1}^{\infty} A_n \in \mathcal{M} \right) \;\&\; \left( \forall A : \mathbb{N} \downarrow \mathcal{M} \;.\; \bigcap_{n=1}^{\infty} A_n \in M \right)$$

$\texttt{MonotonicClassLemma} :: \forall X : \mathsf{SET} \,.\, \forall \mathcal{A} : \texttt{Algebra}(X) \,.\, \sigma(\mathcal{A}) = \bigcap \left\{ \mathcal{M} : \texttt{MonotonicClass}(X, \mathcal{M}) \middle| \mathcal{A} \subset \mathcal{M} \right\}$

$\texttt{Proof} \ =$

$\mathcal{B} := \bigcap \left\{ \mathcal{M} \middle| \texttt{MonotonicClass}(X, \mathcal{M}), \mathcal{A} \subset \mathcal{M} \right\} :??X,$

$[1] := \texttt{E}\mathcal{B}\texttt{E}\sigma(\mathcal{A})\texttt{E}\sigma\texttt{-Algebra}\Big(X, \sigma(\mathcal{A})\Big)\texttt{I}\mathcal{B} : \mathcal{B} \subset \sigma(\mathcal{A}),$

$\texttt{Assume}\ A \in \mathcal{A},$

$\mathcal{C}_A := \{C \subset X : C \setminus A, A \setminus C, A \cap C \in \mathcal{B}\} :???X,$

$[2] := \texttt{E}\mathcal{C}_A\texttt{E}\mathcal{B} : \texttt{MonotonicClass}(X, \mathcal{C}_A),$

$[3] := \texttt{E}\mathcal{C}_A\texttt{E}\mathcal{B}\texttt{E}\texttt{Algebra}(X, \mathcal{A}) : \mathcal{A} \subset \mathcal{C}_A,$

$[A.*] := [2][3]\texttt{E}\mathcal{B} : \mathcal{C}_A \subset \mathcal{B};$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall A \in \mathcal{A} \,.\, \forall B \in \mathcal{B} \,.\, A \setminus B, B \setminus A, B \cap A \in \mathcal{B},$

$\texttt{Assume}\ B \in \mathcal{B},$

$\mathcal{C}_B := \{C \subset X : C \setminus B, B \setminus C, B \cap C \in \mathcal{B}\} : \texttt{MonotonicClass}(X),$

$[3] := \texttt{E}\mathcal{C}_B[2] : \mathcal{A} \subset \mathcal{C}_B,$

$[*.1] := [3]\texttt{E}\mathcal{B} : \mathcal{B} \subset \mathcal{C}_B;$

$\rightsquigarrow [3] := \texttt{IAlgebra} : \texttt{Algebra}(X, \mathcal{B}),$

$[4] := \texttt{E}\texttt{MonotonicClass}(\mathcal{B})\texttt{I}\sigma\texttt{-Algebra} : \sigma\texttt{-Algebra}(X, \mathcal{B}),$

$[5] := \texttt{E}\sigma(\mathcal{A})[4] : \sigma(\mathcal{A}) \subset \mathcal{B},$

$[*] := \texttt{ISetEq}[1][5] : \sigma(\mathcal{A}) = \mathcal{B};$

$\square$

$\texttt{PiClass} :: \prod_{X \in \mathsf{SET}} ???X$

$\mathcal{P} : \texttt{PiClass} \iff \pi\texttt{-Class}(\mathcal{P}) \iff \forall A, B \in \mathcal{P} \,.\, A \cap B \in \mathcal{P}$

$\texttt{DisjointSeq} :: \prod_{X \in \mathsf{SET}} \prod_{\mathcal{A} : ??X} \mathbb{N} \to \mathcal{A}$

$A : \texttt{DisjointSeq} \iff \forall n, m \in \mathbb{N} \,.\, n \neq m \to A_n \cap A_m = \emptyset$

$\texttt{LambdaClass} :: \prod_{X \in \mathsf{SET}} ???X$

$\mathcal{L} : \texttt{LambdaClass} \iff \lambda\texttt{-Class}(\mathcal{L}) \iff \forall A \in \mathcal{L} \,.\, A^{\complement} \in \mathcal{L} \ \&\ \forall A : \texttt{DisjointSeq}(\mathcal{L}) \,.\, \bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$

$\texttt{PiLambdaClassLemma} :: \forall X : \mathsf{SET} \,.\, \forall \mathcal{P} : \pi\texttt{-Class}(X) \,.\, \sigma(\mathcal{P}) = \bigcap \left\{ \mathcal{L} : \lambda\texttt{-Class}(X, \mathcal{L}) \middle| \mathcal{P} \subset \mathcal{L} \right\}$

$\texttt{Proof} \ =$

$\mathcal{L} := \bigcap \left\{ \mathcal{L} : \lambda\texttt{-Class}(X, \mathcal{L}) \middle| \mathcal{P} \subset \mathcal{L} \right\} :??X,$

$[1] := \mathtt{E}\mathcal{L}\mathtt{E}\sigma(\mathcal{P})\mathtt{E}\sigma\texttt{-Algebra}\Big(X, \sigma(\mathcal{P})\Big)\mathtt{I}\mathcal{L} : \mathcal{L} \subset \sigma(\mathcal{P}),$

$\texttt{Assume}\ A \in \mathcal{P},$

$\mathcal{C}_A := \{C \subset X : A \cap C \in \mathcal{L}\} :???X,$

$\texttt{Assume}\ C \in \mathcal{C}_A,$

$[2] := \mathtt{E}\mathcal{C}_A(C) : C \cap A \in \mathcal{L},$

$[3] := \texttt{DeMorgannaLaw}(?X)\texttt{UnionDisjoining}(?X)[2]\mathtt{E}\lambda\texttt{-Class}(X, \mathcal{L}) :$

$\quad : C^{\complement} \cap A = (C \cup A^{\complement})^{\complement} = ((C \cap A) \cup A^{\complement})^{\complement} \in \mathcal{L},$

$[C.*] := \mathtt{E}\mathcal{C}_A[3] : C^{\complement} \in \mathcal{C}_A;$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall C \in \mathcal{C}_A \,.\, C^{\complement} \in \mathcal{C}_A,$

$[3] := \mathtt{E}\mathcal{C}_A\mathtt{E}\texttt{AssociativeLattice}(?X)\mathtt{E}\lambda\texttt{-Class}(X, \mathcal{L}) : \forall C : \texttt{DisjointSeq}(\mathcal{C}_A) \,.\, \bigcup_{n=1}^{\infty} C_n \in \mathcal{C}_A,$

$[4] := \mathtt{I}\lambda\texttt{-Class} : \lambda\texttt{-Class}(X, \mathcal{C}_A),$

$[5] := \mathtt{E}\mathcal{C}_A\mathtt{E}\pi\texttt{-Class}(X, \mathcal{P}) : \mathcal{P} \subset \mathcal{C}_A,$

$[A.*] := [4][5]\mathtt{E}\mathcal{L} : \mathcal{C}_A \subset \mathcal{L};$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall A \in \mathcal{P} \,.\, \forall B \in \mathcal{L} \,.\, B \cap A \in \mathcal{B},$

$\texttt{Assume}\ B \in \mathcal{L},$

$\mathcal{C}_B := \{C \subset X : B \cap C \in \mathcal{L}\} : \lambda\texttt{-Class}(X),$

$[3] := \mathtt{E}\mathcal{C}_B[2] : \mathcal{P} \subset \mathcal{C}_B,$

$[*.1] := [3]\mathtt{E}\mathcal{L} : \mathcal{L} \subset \mathcal{C}_B;$

$\rightsquigarrow [3] := \mathtt{I}\texttt{Algebra}\texttt{UnionDisjoinig}(?X) : \texttt{Algebra}(X, \mathcal{L}),$

$[4] := \mathtt{E}\lambda\texttt{-Algebra}(\mathcal{L})\mathtt{I}\sigma\texttt{-Algebra} : \sigma\texttt{-Algebra}(X, \mathcal{L}),$

$[5] := \mathtt{E}\sigma(\mathcal{P})[4] : \sigma(\mathcal{P}) \subset \mathcal{L},$

$[*] := \texttt{ISetEq}[1][5] : \sigma(\mathcal{P}) = \mathcal{L};$

$\square$

$\texttt{SigmaAlgebraGenerationWithDisjoinUinion} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{A} :??X \,.$

$$\sigma(\mathcal{A}) = \bigcap \left\{ \mathcal{B} :??X \middle| \mathcal{A} \subset \mathcal{B}, \mathcal{A}^{\complement} \subset \mathcal{B}, \forall B : \mathbb{N} \to \mathcal{B} \,.\, \bigcap_{n=1}^{\infty} B_n \in \mathcal{B}, \forall B : \texttt{DisjointSeq}(\mathcal{B}) \,.\, \bigcup_{n=1}^{\infty} B_n \in \mathcal{B} \right\}$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

### 2.1.2 Measurable Category

$$\texttt{MeasurableSet} := \sum X \in \mathsf{SET} \;.\; \sigma\texttt{-Algebra}(X) : ;$$

$$\texttt{measurableSetAsSet} :: \texttt{MeasurableSet} \to \mathsf{SET}$$
$$\texttt{measurableSetAsSet}\,((X,\mathcal{S})) = (X,\mathcal{S}) := X$$

$$\texttt{algebra} :: \prod (X,\mathcal{F}) : \texttt{MeasurableSet} \to \sigma\texttt{-Algebra}(X)$$
$$\texttt{algebra}\,() = \mathcal{S}_{(X,\mathcal{F})} := \mathcal{F}$$

$$\texttt{MeasurableMap} :: \prod X, Y : \texttt{MeasurableSet} \;.\; ?(X \to Y)$$
$$f : \texttt{MeasurableMap} \iff \forall A \in \mathcal{S}_Y \;.\; f^{-1}(A) \in \mathcal{S}_X$$

$$\texttt{categoryOfBorel} :: \mathsf{CAT}$$
$$\texttt{categoryOfBorel}\,() = \mathsf{BOR} := (\texttt{MeasurableSet}, \texttt{MeasurableMap}, \circ, \mathrm{id})$$

$$\texttt{forgetfulFunctorBor} :: \texttt{Covariant}(\mathsf{BOR}, \mathsf{SET})$$
$$\texttt{forgetfulFunctorBor}\,(X,\mathcal{S}) = \mathsf{U}_{\mathsf{BOR}}(X,\mathcal{S}) := X$$
$$\texttt{forgetfulFunctorBor}\,(X,Y,f) = \mathsf{U}_{\mathsf{BOR};X,Y}(f) := f$$

$$\texttt{discreteMeasurableStructureFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{BOR})$$
$$\texttt{discreteMeasurableStructureFunctor}\,(X) = \mathsf{F}_{\mathsf{BOR}}(X) := (X, 2^X)$$
$$\texttt{discreteMeasurableStructureFunctor}\,(X,Y,f) = \mathsf{F}_{\mathsf{BOR};X,Y}(f) := f$$

$$\texttt{codiscreteMeasurableStructureFunctor} :: \texttt{Covariant}(\mathsf{SET}, \mathsf{BOR})$$
$$\texttt{codiscreteMeasurableStructureFunctor}\,(X) = \mathsf{F}^{\mathsf{BOR}}(X) := (X, \{\emptyset, X\})$$
$$\texttt{codiscreteMeasurableStructureFunctor}\,(X,Y,f) = \mathsf{F}^{\mathsf{BOR}}_{X,Y}(f) := f$$

$$\texttt{AdjointStructure} :: \mathsf{F}_{\mathsf{BOR}} \dashv \mathsf{U}_{\mathsf{BOR}} \dashv \mathsf{F}^{\mathsf{BOR}}$$
Proof =

...

□

$$\texttt{algebraFunctor} :: \texttt{Contravariant}(\mathsf{BOR}, \mathsf{BOOL}_\sigma)$$
$$\texttt{algebraFunctor}\,(X) = \mathsf{A}(X) := \mathcal{S}_X$$
$$\texttt{algebraFunctor}\,(X,Y,f) = \mathsf{A}_{X,Y}(f) := f_*$$

$$\texttt{embededStoneFunctor} :: \texttt{Contravariant}(\mathsf{BOOL}_\sigma, \mathsf{BOR})$$
$$\texttt{embededStoneFunctor}\,(A) = \mathsf{Z}(A) := \Big(Z_A, S_A(A)\Big)$$
$$\texttt{embededStoneFunctor}\,(A,B,f) = \mathsf{Z}_{A,B}(f) := Z_A(f)$$

$\texttt{initialMeasurableStructure} :: \prod X, I \in \mathsf{SET} \,.\, \prod Y : I \to \mathsf{BOR} \,.\, \left(\prod_{i \in I} X \to Y_i\right) \to \sigma\text{-}\texttt{Algebra}(X)$

$\texttt{initialMeasurableStructure}\,(f) = \mathcal{I}_X(I, Y, f) := \inf\left\{\mathcal{A} : \sigma\text{-}\texttt{Algebra}(X) \middle| \forall i \in I \,.\, f_i \in \mathsf{BOR}\big((X, \mathcal{A}), Y_i\big)\right\}$

$\texttt{finalMeasurableStructure} :: \prod Y, I \in \mathsf{SET} \,.\, \prod X : I \to \mathsf{BOR} \,.\, \left(\prod_{i \in I} Y_i \to X\right) \to \sigma\text{-}\texttt{Algebra}(X)$

$\texttt{initialMeasurableStructure}\,(f) = \mathcal{F}_Y(I, X, f) := \sup\left\{\mathcal{A} : \sigma\text{-}\texttt{Algebra}(Y) \middle| \forall i \in I \,.\, f_i \in \mathsf{BOR}\big(X_i, (Y, \mathcal{A})\big)\right\}$

$\texttt{BorIsBicomplete} :: \texttt{Bicomplete}(\mathsf{BOR})$

$\texttt{Proof} =$

Define all limits and colimits as in $\mathsf{SET}$.

Then equip them with initial or respectively final measurable structure..

It is easy to see that this constructions have universal properties..

This is analogues to what eas done with $\mathsf{TOP}$ in some model theoretic sence..

$\square$

$\texttt{MeasurableSection} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{BOR} \,.\, \forall i \in \mathcal{I} \,.\, \forall x \in \prod_{i \neq j} X_j \,.\, \forall A \in \mathsf{A}\left(\prod_{i \in I} X_i\right) \,.\, \sigma_{i,x}(A) \in \mathsf{A}(X_i)$

$\texttt{Proof} =$

$\mathcal{C} := \left\{\prod_{i \in I} A_i \middle| A \in \prod_{i \in I} \mathsf{A}(X_i)\right\} :?? \prod_{i \in I} X_i,$

$[1] := \mathtt{E}\left(\mathsf{BOR}, \prod_{i \in I} X_i\right) : \mathsf{A}\left(\prod_{i \in I} X_i\right) = \sigma(\mathcal{C}),$

$\mathcal{B} := \left\{A \subset \prod_{i \in I} X_i \middle| \sigma_{i,x}(A) \in \mathsf{A}(X_i)\right\} :?? \prod_{i \in I} X_i,$

$[2] := \mathtt{E}\mathcal{C}\mathtt{E}\mathcal{B}\mathtt{E}\sigma_{i,x} : \mathcal{C} \subset \mathcal{B},$

$[3] := \mathtt{E}\mathcal{B}\mathtt{E}\sigma\text{-}\texttt{Algebra}\big(X_i, \mathsf{A}(X_i)\big)\mathtt{E}\sigma_{i,x}\mathtt{I}\sigma\text{-}\texttt{Algebra} : \sigma\text{-}\texttt{Algebra}\left(\prod_{i \in I} X_i, \mathcal{B}\right),$

$[4] := \mathtt{E}\sigma[1][2][3] : \mathsf{A}\left(\prod_{i \in I} X_i\right) \subset \mathcal{B},$

$[*] := \mathtt{E}\mathcal{B}[4] : \forall A \in \mathsf{A}\left(\prod_{i \in I} X_i\right) \,.\, \sigma_{i,x}(A) \in \mathsf{A}(X_i);$

$\square$

MeasurablePartialComputation :: $\forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{BOR} \,.\, \forall Y \in \mathsf{BOR} \,.\, \forall i \in I \,.\, \forall x \in \prod_{j \neq i} X_j \,.$

$$. \forall f \in \mathsf{BOR}\left(\prod_{i \in I} X_i, Y\right) \,.\, f(x) \in \mathsf{BOR}(X_i, Y)$$

Proof =

Let $A$ be measurable in $Y$.

Then, $(f(x))^{-1}(A) = \sigma_{i,x}\Big(f^{-1}(A)\Big)$.

This is measurable by the previous theorem, and so $f(x)$ is measurable.

□

## 2.2 Borel Basics

### 2.2.1 Sets and the Functor

$$\texttt{functorOfBorel} :: \texttt{Covariant}(\mathsf{TOP}, \mathsf{BOR})$$

$$\texttt{functorOfBorel}\,((X, \mathcal{T})) = \mathsf{B}(X, \mathcal{T}) := \Big(X, \sigma(\mathcal{T})\Big)$$

$$\texttt{functorOfBorel}\,(X, Y, f) = \mathsf{B}_{X,Y}(f) := f$$

$\mathcal{A} := \mathcal{F}_Y\Big(1, \mathsf{B}(X), f\Big) : \sigma\text{-}\texttt{Algebra}(Y),$

$[1] := \mathtt{E}\mathcal{A}\mathtt{E}\mathsf{B}(X)\mathtt{E}\mathsf{TOP}(X, Y, f) : \mathcal{T}(Y) \subset \mathcal{A},$

$[2] := \mathtt{E}\sigma[1] : \sigma\Big(\mathcal{T}(Y)\Big) \subset \mathcal{A},$

$[*] := \mathtt{I}\mathsf{B}[2]\mathtt{E}\mathcal{A}\mathtt{E}\mathcal{F}_Y : f \in \mathsf{BOR}\Big(\mathsf{B}(X), \mathsf{B}(Y)\Big);$

$\square$

$$\texttt{borelAlgebra} :: \texttt{Contravariant}(\mathsf{TOP}, \mathsf{BOOL}_\sigma)$$

$$\texttt{borelAlgebra}\,() = \mathcal{B} := \mathsf{BA}$$

$\texttt{CountablyGeneratedBorel}$ ::

$:: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{U} : \texttt{SubbaseOfTopology}(X) \,.\, \forall |\mathcal{U}| \le \aleph_0 \,.\, \texttt{CountablyGeneratedSigmaAlgebra}\Big(\mathcal{B}(X)\Big)$

$\texttt{Proof}$ =

. . .

$\square$

$\texttt{CountablyGeneratedBorel2} :: \forall X : \texttt{SecondCountable} \,.\, \texttt{CountablyGeneratedSigmaAlgebra}\Big(\mathcal{B}(X)\Big)$

$\texttt{Proof}$ =

. . .

$\square$

$\texttt{BorelContainsOpen} :: \forall X \in \mathsf{TOP} \,.\, \mathcal{T}(X) \subset \mathcal{B}(X)$

$\texttt{Proof}$ =

. . .

$\square$

$\texttt{BorelContainsClosed} :: \forall X \in \mathsf{TOP} \,.\, \texttt{Closed}(X) \subset \mathcal{B}(X)$

$\texttt{Proof}$ =

. . .

$\square$

$\texttt{BorelContainsGdelta} :: \forall X \in \mathsf{TOP} \,.\, G_\delta(X) \subset \mathcal{B}(X)$

$\texttt{Proof}$ =

. . .

$\square$

$\texttt{BorelContainsFSigma} :: \forall X \in \mathsf{TOP} \,.\, F_\sigma(X) \subset \mathcal{B}(X)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CountableBorelCommutesWithCountableProducts} ::$

$$:: \forall n \in \sigma(\omega) \,.\, \forall X : n \to \texttt{SecondCountable} \,.\, \prod_{i=0}^{n} \mathsf{B}(X_i) = \mathsf{B}\left(\prod_{i=0}^{n} X_i\right)$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{BorelMeasurable} := \Lambda X \in \mathsf{BOR} \,.\, \Lambda Y \in \mathsf{TOP} \,.\, \mathsf{BOR}(X,Y) \iff \mathsf{BOR}\Big(X, \mathsf{B}(Y)\Big) : \texttt{Polymorphism};$

$\texttt{MeasurableBySubbase} :: \forall X \in \mathsf{BOR} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall f : X \to Y \,.\, \forall \mathcal{U} : \texttt{SubbaseOfTopology}(Y) \,.$
$\quad . \, \forall [0] : \forall U \in \mathcal{U} \,.\, f^{-1}\, U \in \mathcal{S}_X \,.\, f \in \mathsf{BOR}(X,Y)$

$\texttt{Proof} \ =$

$[1] := \mathtt{E}\mathsf{BOR}(X)\mathtt{E}\texttt{SubbaseOfTopology}(Y,\mathcal{U})[0]\texttt{UnionPreimage}(X,Y,f)$
$\quad \texttt{IntersectionPreimage}(X,Y,f) : \forall U \in \mathcal{T}(Y) \,.\, f^{-1}\, U \in \mathcal{S}_X,$

$\mathcal{A} := \mathcal{F}_Y\Big(1, X, f\Big) : \sigma\texttt{-Algebra}(Y),$

$[2] := \mathtt{E}\mathcal{A}[1] : \mathcal{T}(Y) \subset \mathcal{A},$

$[3] := \mathtt{E}\sigma[2] : \sigma\Big(\mathcal{T}(Y)\Big) \subset \mathcal{A},$

$[*] := \mathtt{I}\mathsf{BOR}[3]\mathtt{E}\mathcal{A}\mathtt{E}\mathcal{F}_Y : f \in \mathsf{BOR}\Big(X, Y\Big);$

$\square$

### 2.2.2 Hierarchi

$$\mathtt{hierarchiOfBorel} :: \prod X : \mathtt{Metrizable} \,.\, \omega_1 \to (?X)^2$$

$$\mathtt{hierchiOfBorel}\,() = (\Sigma^0(X), \Pi^0(X)) := \mathtt{boundedCompleteTransfiniteRecursion}$$

$$\Bigg(\mathsf{ORD}, \Big(\mathtt{Open}(X), \mathtt{Closed}(X)\Big), \lambda\kappa \in (1, \omega_1) \,.\, \lambda\Big(\Sigma, \Pi\Big) : \kappa \to (?X)^2 \,.$$

$$.\, \left(\left\{\bigcup_{n=1}^{\infty} A_n \middle| \xi : \mathbb{N} \to \kappa, A : \prod_{n=1}^{\infty} \Pi_{\xi_n}\right\}, \left\{\bigcap_{n=1}^{\infty} A_n \middle| \xi : \mathbb{N} \to \kappa, A : \prod_{n=1}^{\infty} \Sigma_{\xi_n}\right\}\right)\Bigg)$$

$$\mathtt{ambigiousClass} :: \prod X : \mathtt{Metrizable} \,.\, \mathsf{ORD} \to ?X$$

$$\mathtt{ambigiousClass}\,(\kappa) = \Delta^0_\kappa(X) := \Sigma^0_\kappa(X) \cap \Pi^0_\kappa(X)$$

$\mathtt{DirectBorelHierarchi}$ ::

$:: \forall X : \mathtt{Metrizable}(X) \,.\, \forall \kappa \in \mathsf{ORD} \,.\, \forall \xi \in \kappa \,.\, \Sigma^0_\xi \subset \Sigma^0_\kappa \;\&\; \Pi^0_\xi \subset \Pi^0_\kappa$

$\mathtt{Proof}$ $=$

$[1] := \mathrm{E}\Sigma^0_1(X) : \Sigma^0_1(X) = \mathcal{T}(X),$

$[2] := \mathrm{E}\Sigma^0_2(X) : \Sigma^0_2(X) = F_\sigma(X),$

$[3] := \mathrm{E}\Pi^0_1(X) : \Sigma^0_1(X) = \mathtt{Closed}(X),$

$[4] := \mathrm{E}\Pi^0_2(X) : \Sigma^0_2(X) = G_\delta(X),$

$[5] := \mathtt{OpenIsFSigma}(X)[1][2] : \Sigma^0_1(X) \subset \Sigma^0_2(X),$

$[6] := \mathtt{ClosedIsGDelta}(X)[3][4] : \Pi^0_1(X) \subset \Pi^0_2(X),$

$[*] := \mathrm{E}(\Sigma, \Pi)[5][6] : \forall \kappa \in \mathsf{ORD} \,.\, \forall \xi \in \kappa \,.\, \Sigma^0_\xi \subset \Sigma^0_\kappa \;\&\; \Pi^0_\xi \subset \Pi^0_\kappa;$

□

$\mathtt{DirectAmbiguousClasses} :: \forall X : \mathtt{Metrizable}(X) \,.\, \forall \kappa \in \mathsf{ORD} \,.\, \forall \xi \in \kappa \,.\, \Delta^0_\xi \subset \Delta^0_\kappa$

$\mathtt{Proof}$ $=$

. . .

□

$\mathtt{BorelHierarchiComplementation} :: \forall X : \mathtt{Metrizable}(X) \,.\, \forall A \subset X \,.\, \forall \kappa \in \mathsf{ORD} \,.\, A \in \Sigma^0_\kappa \iff A^\complement \in \Pi^0_\kappa$

$\mathtt{Proof}$ $=$

. . .

□

$$\mathtt{BorelTransfiniteExpression} :: \forall X : \mathtt{Metrizable}(X) \,.\, \exists \xi \in \mathsf{ORD} \,.\, \bigcup_{\kappa<\xi} \Sigma^0_\kappa = \bigcup_{\kappa<\xi} \Pi^0_\kappa = \bigcup_{\kappa<\xi} \Delta^0_\kappa = \mathcal{B}(X)$$

$\mathtt{Proof}$ $=$

$\alpha := \min\left\{\kappa \in \mathsf{ORD} : |\kappa| = 2^{2^{|X|}}\right\} : \mathsf{ORD},$

By cardinality limitation $\Sigma^0, \Pi^0$ and $\Delta_0$ will stabilize until reaching $\alpha$ .

So, their unions are closed under finite unions and intersections.

Hence,This unions are sigma-algebras and contain $\mathcal{B}(X)$.

Also by simple transfinite induction they all consist of members of $\mathcal{B}(X)$.

The result follows.

□

### 2.2.3 Examples

$$\texttt{Simple} :: \prod_{X \in \mathsf{BOR}} \mathsf{BOR}(X, \mathbb{R})$$

$$\varsigma : \texttt{Simple} \iff \exists n \in \mathbb{N} \,.\, S : \{1, \ldots, n\} \to \mathcal{S}_X \,.\, \alpha : \{1, \ldots, n\} \to \mathbb{R} \,.\, \varsigma = \sum_{i=1}^{n} \alpha_i \chi_{S_i}$$

$$\texttt{binaryDigits} :: \mathbb{N} \to \mathsf{BOR}\Big([0,1], \mathbb{B}\Big)$$

$$\texttt{binaryDigits}\,(n, t) = \beta_n(t) := \sum_{k=0}^{2^{n-1}} \delta_t \left(\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right]$$

$$\texttt{NormalNumbers} :: \mathcal{B}[0,1]$$

$$\alpha : \texttt{NormalNumbers} \iff \alpha \in \overline{\mathbb{N}} \iff \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \beta_i(\alpha)}{n} = \frac{1}{2}$$

$$\overline{\mathbb{N}} = \bigcap_{\varepsilon \in \mathbb{Q}_{++}} \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} \left\{ \alpha \in [0,1] : \left| \frac{\sum_{i=1}^{n} \beta_i(\alpha)}{n} - \frac{1}{2} \right| < \varepsilon \right\}$$

□

$$\texttt{ContDiffirientiableIsBorel} :: C^1[0,1] \in \mathcal{B}\Big(C[0,1]\Big)$$

$\texttt{Proof} =$

$$\mathcal{I} = \Lambda n \in \mathbb{N} \,.\, \left\{ I : \{1, \ldots, n\} \to \texttt{OpenInterval}\Big(\mathbb{Q} \cap [0,1]\Big), [0,1] = \bigcup_{i=1}^{n} I_i \right\}$$

$$\forall n \in \mathbb{N} \,.\, |\mathcal{I}_n| \leq \aleph_0$$

$$C^1[0,1] = \bigcap_{\varepsilon \in \mathbb{Q}_{++}} \bigcup_{n=1}^{\infty} \bigcup_{I \in \mathcal{I}_n} \bigcap_{k=1}^{n} \left\{ f \in C[x,y] : \forall a,b,c,d \in I_k \,.\, b > a, d > c \,.\, \left| \frac{f(b) - f(a)}{b - a} - \frac{f(d) - f(c)}{d - c} \right| \leq \varepsilon \right\}$$

□

$$\texttt{ZeroConvergentIsBorel} :: l_0 \in \mathcal{B}(l_\infty)$$

$\texttt{Proof} =$

$$l_0 = \bigcap_{\varepsilon \in \mathbb{Q}_{++}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Big\{ x \in l_\infty \,.\, |x_n| < \varepsilon \Big\}$$

□

$$\texttt{PointsOfDifferentiabilityIsBorel} :: \forall f \in C[0,1] \,.\, D_f \in \mathcal{B}[0,1]$$

$\texttt{Proof} =$

$$D_f = \bigcap_{\varepsilon \in \mathbb{Q}_{++}} \bigcup_{\delta \in \mathbb{Q}_{++}} \bigcap_{a,b \in \mathbb{Q} \cap [0,1]} \left\{ t \in [0,1] : 0 < |a - t| < \delta, 0 < |b - t| < \delta, \left| \frac{f(t) - f(a)}{t - a} - \frac{f(t) - f(b)}{t - b} \right| < \varepsilon \right\}$$

□

### 2.2.4 Functions

BorelMeasurablePointwiseConvergence ::

$$\forall X \in \mathsf{BOR} \,.\, \forall Y : \mathtt{Metrizble} \,.\, \forall \phi : \mathbb{N} \to \mathsf{BOR}(X,Y) \,.\, \forall \varphi : X \to Y \,.\, \forall [0] : (\mathrm{pt})\ \varphi = \lim_{n\to\infty} \phi_n \,.\, \varphi \in \mathsf{BOR}(X,Y)$$

Proof $=$

$d := \mathtt{EMetrizable}(Y) : \mathtt{Metrizes}(Y,d),$

Assume $K \in \mathtt{Closed}(Y),$

$[K.*] := \mathtt{Epreimage}(X,Y,\varphi,K)\mathtt{EpointwiseConvergence}\Big(X,(Y,d),f,\varphi\Big)$

$\Lambda n \in \mathbb{N} \,.\, \mathtt{Ipreimage}(X,Y,\varphi,K)\Lambda n \in \mathbb{N} \,.\, \mathtt{E}\mathsf{BOR}(X,Y,f_n)\mathtt{E}\mathsf{BOR}(X) :$

$$: \varphi^{-1}(K) = \{x \in X \,.\, \varphi(x) \in K\} = \bigcap_{\varepsilon\in\mathbb{Q}_{++}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Big\{x \in X \,.\, d(f_n(x),K) < \varepsilon\Big\} =$$

$$= \bigcap_{\varepsilon\in\mathbb{Q}_{++}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1} \bigcup_{y\in K} \mathbb{B}_d(y,\varepsilon) \in \mathcal{S}_X;$$

$\rightsquigarrow [*] := \mathtt{MeasurableByGenerators} : \varphi \in \mathsf{BOR}(X,Y),$

□

DerivativeIsBorelMeasurable :: $\forall f \in \mathsf{DIFF}\Big([0,1],\mathbb{R}\Big) \,.\, \varphi' \in \mathsf{BOR}\Big([0,1],\mathbb{R}\Big)$

Proof $=$

$\alpha := \Lambda t \in [0,1] \,.\, \Lambda s \in [0,1] \,.\, \min(t+s,1) : [0,1]^2 \to [0,1],$

$$g := \Lambda n \in \mathbb{N} \,.\, \Lambda t \in [0,1] \,.\, \mathtt{if}\ t < 1\ \mathtt{then}\ \frac{\varphi\Big(f(t,2^{-n})\Big) - f(t)}{\alpha(t,2^{-n}) - t}\ \mathtt{else}\ f'(1) : \mathbb{N} \to \mathsf{BOR}\Big([0,1],\mathbb{R}\Big),$$

$[*] := \mathtt{BorelMeasurablePointwiseConvergence} : f' = (\mathrm{pt})\ \lim_{n\to\infty} g_n \in \mathcal{B}\Big([0,1],\mathbb{R}\Big);$

□

SemicontinuousIsMeasurable :: $\forall X \in \mathsf{TOP} \,.\, \forall f : \mathtt{Semicontinuous}(X) \,.\, f \in \mathsf{BOR}(X,\mathbb{R})$

Proof $=$

Half-intervals are expressible as intersections of open rays and their complements.
Open intervals are expressible as countable unions or intersection of half-intervals.
Open subsets of real line are expressible as countable disjoint unions of open intervals..
Preimages of open rays are open for semicontinuous functions are open..
This means that preimage of an open set is Borel.
Hence, the semicontinuous functions are Borel-measurable.

□

$\texttt{BorelByPartialComputaions} ::$

$:: \forall X, Z : \texttt{Metrizable} . \forall Y \in \mathsf{TOP} . \forall D : \texttt{Dense}(X) . \forall f : X \times Y \to Z . \forall [0.1] : |D| \leq \aleph_0 .$

$. \forall [0.2] : \forall y \in Y . f(\bullet, y) \in \mathsf{TOP}(X, Z) . \forall [0.3] : \forall x \in D . f(x, \bullet) \in \mathsf{BOR}(Y, Z) . f \in \mathsf{BOR}(X \times Y, Z)$

$\texttt{Proof} =$

$\rho := \texttt{EMetrizable}(X) : \texttt{Metrizes}(X, \rho),$

$\delta := \texttt{EMetrizable}(Z) : \texttt{Metrizes}(Z, \sigma),$

$d := \texttt{enumerate}(D) : \texttt{Surjective}(\mathbb{N}, D),$

$K := \Lambda n \in \mathbb{N} . \{d_1, \dots, d_n\} : \mathbb{N} \to \texttt{Finite}(X),$

$$\sigma := \Lambda n \in \mathbb{N} . \Lambda x \in X . d\Big(\min\{m \in \operatorname*{arg\,min}_{1 \leq m \leq n} \rho(d_m, x)\}\Big) : \prod_{n=1}^{\infty} (X \to K_n),$$

$$g := \Lambda n \in \mathbb{N} . \Lambda(x, y) \in X \times Y . f\Big(\sigma_n(x), y\Big) : \mathbb{N} \to (X \times Y) \to Z,$$

$\texttt{Assume } y \in Y,$

$\texttt{Assume } x \in X,$

$\texttt{Assume } U \in \mathcal{U}\Big(f(x, y)\Big),$

$V := f^{-1}(\bullet, y)(U) \in \mathcal{U}(x),$

$\Big(r, [1]\Big) := \texttt{MetricTopology}(X, \rho, V) : \sum R \in \mathbb{R}_{++} . \mathbb{B}_\rho(x, r) \subset V,$

$\Big(N, [2]\Big) := \texttt{E}d\texttt{EDense}(X, D)(V) : \sum N \in \mathbb{N} . d_N \in \mathbb{B}_\rho\left(x, r\right),$

$\texttt{Assume } n : \mathbb{N},$

$\texttt{Assume } [3] : n \geq N,$

$$\Big(m, [4]\Big) := \texttt{E}g_n[2][3] : \sum_{m=N}^{\infty} g_n(x, y) = f(d_m, y) \ \& \ \rho(d_m, x) < r,$$

$[5] := [4.2][1] : d_m \in V,$

$[y.*] := \texttt{E}V[4.1][5] : g_n(x, y) \in U;$

$\rightsquigarrow [1] := \texttt{I}(\mathrm{pt}) \ \lim : (\mathrm{pt}) \ \lim_{n \to \infty} g_n = f,$

$$B := \Lambda n \in \mathbb{N} . \Lambda k \in \{1, \dots, n\} \sigma_n^{-1}(d_k) : \prod_{n=1}^{\infty} \{1, \dots, n\} \to \mathcal{B}(X),$$

$[2] := \Lambda n \in \mathbb{N} \Lambda A \in \mathsf{A}(Z) \Lambda k \in \{1, \dots, n\} . \texttt{E}g_n[0.3](d_k, A)$

$\texttt{CountableBorelCommutesWithCountableProducts}\Big(2, (X, Y)\Big) :$

$$: \forall n \in \mathbb{N} . \forall A \in \mathsf{A}(Z) . g_n^{-1}(A) = \bigcup_{k=1}^{n} B_k \times f^{-1}(d_k, \bullet)(A) \in \mathcal{B}(X \times Y),$$

$[3] := \texttt{I}\mathsf{BOR}[2] : \forall n \in \mathbb{N} . g_n \in \mathsf{BOR}(X \times Y, Z),$

$[*] := \texttt{BorelMeasurablePointwiseConvergence}(X \times Y, Z, g, f)[1][3] : f \in \mathsf{BOR}(X \times Y, Z);$

$\square$

VietorisBorelSetsGeneration1 :: $\forall X : \texttt{Polish} \,.\, \mathcal{B}\Big(\mathsf{K}(X)\Big) = \sigma\Big\{\{K \in \mathsf{K}(X) : K \subset U\}\Big| U \in \mathcal{T}(X)\Big\}$

Proof =

We need to express sets of form $\{K \in \mathsf{K}(X) : \exists K \cap U\}$ by sets of form$\{K \in \mathsf{K}(X) : K \subset V\}$ .

Let $\rho$ be a metrization for $X$ and let $(d_n)_{n=1}^{\infty}$ be dense in it.

First, note that $U$ is a $F_\sigma$ set.

So, there is a sequence of closed sets $A$ such that $U = \bigcup_{n=1}^{\infty} A_n$.

So, $\{K \in \mathsf{K}(X) : \exists K \cap U\} = \bigcup_{n=1}^{\infty}\{K \in \mathsf{K}(X) : \exists K \cap A_n\}$.

Now, let $\mathfrak{B}_{n,m}$ stay for a sets of $m$-tuples of rational cells wich are disjoint from $A_n$.

Each $\mathfrak{B}_{n,m}$ is countable.

Every compact $K \subset X$ can be given a cover of open sets disjoined from $A_n$.

This cover can be choosen to consist of rational cells as they form the base of topology.

As $K$ is compact we can find a finite subcover.

So, $K$ would be contained in $\bigcup_{i=1}^{m} B_i$ for some $B \in \mathfrak{B}_{n,m}$.

Thus, $\{K \in \mathsf{K}(X) : \exists K \cap U\} = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{B \in \mathfrak{B}_{n,m}} \left\{K \in \mathsf{K}(X) : K \subset \bigcup_{i=1}^{m} B_i\right\}^{\complement}$ .

□

VietorisBorelSetsGeneration2 :: $\forall X : \texttt{Polish} \,.\, \mathcal{B}\Big(\mathsf{K}(X)\Big) = \sigma\Big\{\{K \in \mathsf{K}(X) : \exists K \cap U\}\Big| U \in \mathcal{T}(X)\Big\}$

Proof =

dually, we express sets of form $\{K \in \mathsf{K}(X) : K \subset U\}$ by sets of form$\{K \in \mathsf{K}(X) : \exists K \cap V\}$ .

Closed set $U^{\complement}$ is $G_\delta$.

So, there is a sequence of open sets $(W_n)_{n=1}^{\infty}$ such that $U^{\complement} = \bigcap_{n=1}^{\infty} W_n$.

Assume, that compact $K \subset U$ meets infinitly many $W_n$..

Then, we can choose a sequence $(x_i)_{i=1}^{\infty}$ and the increasing $n : \mathbb{N} \to \mathbb{N}$, such that $x_i \in K \cap W_{n_i}$.

From sequence-compactness $x$ will have a partial limit in $K$..

And as $X$ is normal it is also in $U^{\complement}$.

But $K \subset U$, a contradiction!

So, $\{K \in \mathsf{K}(X) : K \subset U\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty}\{K \in \mathsf{K}(X) : \exists K \cap W_n\}^{\complement}$ .

□

`projectionOfHausdorff` :: $\prod X : \texttt{Polish} \,.\, \texttt{Closed}(X) \to \mathsf{BOR}\Big(\mathsf{K}(X), \mathsf{K}(X)\Big)$

`projectionOfHausdorff` $(A, K) = \varphi_{A\cap\bullet}(K) := A \cap K$

Let $U$ be open in $A$. .

define $\mathcal{U} = \Big\{V \in \mathcal{T}(X) \,.\, V \cap A = U\Big\}$, .

By definition of subset topology $\exists \mathcal{U}$.

Fix some $V \in \mathcal{U}$.

Then, $\varphi^{-1}_{A\cap\bullet}\Big\{K \in \mathsf{K}(A) \,.\, K \subset U\Big\} = \Big\{K \in \mathsf{K}(X) \,.\, K \subset V \cup A^{\complement}\Big\} \in \mathcal{T}(X)$ .

Note that if $K = \varphi_{A\cap\bullet}(L)$, then $L = K \cup N$ with $N \subset A^{\complement}$ .

If $K \subset U$, then $L \subset V \cup A^{\complement} \in \mathcal{U}$, so $L$ was counted.

If $K$ has points outside $U$, then $L$ will also have them, so it was not counted.

□

`CantorBendixsonDerivativeIsBorel` :: $\forall X : \texttt{Polish} \,.\, \mathrm{d} \in \mathsf{BOR}\Big(K(X), K(X)\Big)$

**Proof** $=$

Let $U$ be open in $X$. .

Then, $A = U^{\complement}$ is closed.

A compact can have only a finite number of isolated points..

$A$ is closed, so it is $G_\delta$ .

It means that there is a decreasing sequence of open sets $V$ such that $A = \bigcap_{n=1}^{\infty} V_n$.

Each $V_n$ can be represented as countable unions of closed sets $D_{n,i}$.

I want the number of points in each $D_{n,i}$ to be bounded by some $j$.

Denote by $K_j(X)$ subsets of $X$ of cardinality atmost $j$..

Write $\mathrm{d}^{-1}\Big\{K \subset \mathsf{K}(X) : K \subset U\Big\} = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{i=1}^{\infty} \varphi^{-1}_{D_{n,i}\cap\bullet} \mathsf{K}_j(D_{n,i})$ .

But projection is Borel measurable, so this set is also Borel.

Sets as above genetate all Borel sets for $\mathsf{K}(X)$..

So, in Vietoris topology the Cantor-Bendixson derivative is Borel measurable .

□

IntersectionIsBorel :: $\forall X : $ Polish . $(\cap) \in \mathsf{BOR}\Big(\mathsf{K}^{\times 2}(X), \mathsf{K}(X)\Big)$

Proof $=$

Let $U$ be open in $X$. .

$U$ is open, so it is $F_\sigma$ .

It means that there is an increasing sequence of closed sets $A$ such that $U = \bigcup_{n=1}^{\infty} A_n$.

A pair of compacts $(K, L)$ have nonempty $K \cap L$ iff $K \times L$ intersects diagonal..

I want this intersection happen at $U$. .

So there must be some $A_n$ with such intersection..

Note that set of compacts intersecting diagonal is closed..

Then write $(\cap)^{-1}\Big\{K \in K(X) : \exists K \cap U\Big\} = \bigcap_{n=1}^{\infty} (\times)^{-1}\Big\{K \in K(X^2) : \exists K \cap \Delta(A_n)\Big\}$ .

As $(\times)$ is a continuous functions the resulting set is closed..

Sets as above genetate all Borel sets for $\mathsf{K}(X)$..

So, in Vietoris topology the intersection is Borel measurable .

□

SectionIsBorel :: $\forall X : $ Polish . $\forall Y : $ CompactMetrizable . $\forall F : $ Closed$(X \times Y)$ . $\sigma_\bullet(F) \in \mathsf{BOR}\Big(X, \mathsf{K}(Y)\Big)$

Proof $=$

Let $A$ be a closed set in $Y$..

As $Y$ is compact, then $A$ also is compact.

Approximate $F \cap X \times A$ by open cells $U_n = \mathbb{B}\left(F \cap X \times A, \frac{1}{n}\right)$.

Then $F \cap X \times A = \bigcap_{n=1}^{\infty} U_n$ .

We claim that $\sigma_\bullet^{-1}(F)\big\{K \in \mathsf{K}(Y) : \exists K \cap A\big\} = \pi_X\Big(F \cap X \times A\Big) = \pi_X\left(\bigcap_{n=1}^{\infty} U_n\right) = \bigcap_{n=1}^{\infty} \pi_X(U_n)$ .

The last equality us somethat questionable.

It would be true iff $\forall x \in X . \pi_X^{-1}(x) \cap \bigcap_{n=1}^{\infty} U_n = \emptyset \Rightarrow \exists m \in \mathbb{N} . \pi_X^{-1}(x) \cap \bigcap_{n=1}^{m} U_n = \emptyset$ .

So there is an $x$ and a sequence of $y_n$ such that $(x, y_n) \in U_n$.

But $Y$ is compact, so there must exists a partial limit $y$.

But this means that $(x, y) \in F \cap X \times A$ as $F \cap X \times A$ is closed..

This means that the fiber of $x$ is non-empty.

So, using the fact that $\pi_X$ is open, we see that $\sigma_\bullet^{-1}(F)\big\{K \in \mathsf{K}(Y) : \exists K \cap A\big\}$ is $G_\delta$, and hence Borel .

As sets of this type generate Borel structure for Vietoris topology, the section is Borel measurable .

□

### 2.2.5 Lebesgue-Hausdorff Theorem

BairHasBP :: $\forall X \in \mathsf{TOP} \,.\, \mathcal{B}(X) \subset \mathbf{BP}(X)$

Proof =

. . .

□

EveryBorelIsBairMeasurable :: $\forall X, Y \in \mathsf{TOP} \,.\, \forall \varphi \in \mathsf{BOR}(X,Y) \,.\, \texttt{BairMeasurable}(X,Y)$

Proof =

. . .

□

LebesgueClass :: $\prod_{X \in \mathsf{SET}} \prod_{Y \in \mathsf{TOP}} ??(X \to Y)$

$\mathcal{C} : \texttt{LebesgueClass} \iff \forall f : \mathbb{N} \to \mathcal{C} \,.\, \forall g : X \to Y \,.\, (\mathrm{pt}) \lim_{n \to \infty} f_n = g \Rightarrow g \in \mathcal{C}$

LebesgueClassIntersection ::

$:: \forall X, I \in \mathsf{SET} \,.\, \forall Y \in \mathsf{TOP} \,.\, \forall \mathcal{C} : I \to \texttt{LebesgueClass}(X,Y) \,.$

$. \, \texttt{LebesgueClass}\left(X, Y, \bigcap_{i \in I} \mathcal{C}_i\right)$

Proof =

. . .

□

generateLebesgueClass :: $\prod_{X \in \mathsf{SET}} \prod_{Y \in \mathsf{TOP}} ?(X \to Y) \to \texttt{LebesgueClass}(X,Y)$

$\texttt{generateLebesgueClass}\,(\mathcal{A}) = \mathrm{LC}(\mathcal{A}) := \bigcap \left\{ \mathcal{C} : \texttt{LebesgueClass}(X,Y), \mathcal{A} \subset \mathcal{C} \right\}$

LebesgueApproximationTHM ::

$:: \forall X \in \mathsf{BOR} \,.\, \forall f \in \mathsf{BOR}(X, \mathbb{R}) \,.\, \exists B : \mathbb{N} \times \mathbb{Z} \to \mathcal{B}(X) \,.\, \exists \alpha : \mathbb{N} \times \mathbb{Z} \to \mathbb{R} \,.\, (\mathrm{pt}) \; f = \lim_{n \to \infty} \sum_{m=-\infty}^{\infty} \alpha_{n,m} \chi_{B_{n,m}}$

Proof =

$I := \Lambda n \in \mathbb{N} \,.\, \Lambda m \in \mathbb{Z} \,.\, \left( \frac{m}{n}, \frac{m+1}{n} \right] : \mathbb{N} \times \mathbb{Z} \to \mathcal{B}(\mathbb{R}),$

$B := f^{-1}(B) : \mathbb{N} \times \mathbb{Z} \to \mathcal{B}(X),$

$\alpha := \Lambda n \in \mathbb{N} \,.\, \Lambda m \in \mathbb{Z} \,.\, \frac{2m+1}{2n} : \mathbb{N} \times \mathbb{Z} \to \mathbb{R},$

$[1] := \mathtt{E}B\mathtt{E}\alpha : \forall n \in \mathbb{N} \,.\, \forall x \in X \,.\, \left| f(x) - \sum_{m=-\infty}^{\infty} \alpha_{n,m} \chi_{B_{n,m}}(x) \right| \leq \frac{1}{2n},$

$[*] := \mathtt{I}(\mathrm{pt})[1] : (\mathrm{pt}) \; f = \lim_{n \to \infty} \sum_{m=-\infty}^{\infty} \alpha_{n,m} \chi_{B_{n,m}};$

□

$\texttt{LebesgueHausdorffTHM} :: \forall X : \texttt{Metrizable} \,.\, \mathrm{LC}\Big(C(X)\Big) = \mathcal{B}(X, \mathbb{R})$

$\texttt{Proof} \ =$

$\mathcal{C} := \mathrm{LC}\Big(C(X)\Big) : \texttt{LebesgueClass}(X, Y),$

$[1] := \mathtt{E}\ \mathrm{LC}\ \Big(C(X)\Big)\mathtt{E}\mathcal{B}(X, \mathbb{R}) : \mathcal{C} \subset \mathcal{B}(X, \mathbb{R}),$

$\texttt{Assume}\ \alpha \in \mathbb{R},$

$\texttt{Assume}\ f \in C(X),$

$\mathcal{C}' := \{g : X \to \mathbb{R} \,.\, \alpha g + f \in \mathcal{C}\} :?(X \to \mathbb{R}),$

$[2] := \mathtt{E}\mathcal{C}\mathtt{E}\mathcal{C}'\mathtt{E}\mathbb{R}\text{-}\mathsf{VS}\Big(C(X)\Big) : C(X) \subset \mathcal{C}',$

$[\alpha.*] := \mathtt{E}\mathcal{C}\mathtt{E}\mathcal{C}'\mathtt{E}\mathrm{LC} : \mathcal{C} \subset \mathcal{C}';$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall \alpha \in \mathcal{C} \,.\, \forall f \in C(X) \,.\, \forall \alpha \in \mathbb{R} \,.\, \forall g \in \mathcal{C} \,.\, f + \alpha g \in \mathcal{C},$

$\texttt{Assume}\ \alpha, \beta \in \mathbb{R},$

$\texttt{Assume}\ f \in \mathcal{C},$

$\mathcal{C}' := \{g : X \to \mathbb{R} \,.\, \alpha g + \beta f \in \mathcal{C}\} :?(X \to \mathbb{R}),$

$[3] := \mathtt{E}\mathcal{C}'[2] : C(X) \subset \mathcal{C}',$

$[\alpha.*] := \mathtt{E}\mathcal{C}\mathtt{E}\mathcal{C}'\mathtt{E}\mathrm{LC} : \mathcal{C} \subset \mathcal{C}';$

$\rightsquigarrow [3] := \mathtt{I}\mathbb{R}\text{-}\mathsf{VS} : \mathcal{C} \in \mathbb{R}\text{-}\mathsf{VS},$

$[4] := \mathtt{E}\chi\mathtt{E}\complement : \forall B \in \mathsf{BOR}(X, \mathbb{R}) \,.\, \chi_{B^\complement} = 1 - \chi_B,$

$[5] := \mathtt{E}\chi\mathtt{E}\bigcup : \forall B : \texttt{DisjointSequence}\Big(\mathsf{BOR}(X, \mathbb{R})\Big) \,.\, \chi_{\bigcup_{n=1}^\infty B_n} = \sum_{n=1}^\infty \chi_{B_n},$

$\mathcal{A} := \Big\{B \in \mathsf{BOR}(X, \mathbb{R}) \Big| \chi_B \in \mathcal{C}\Big\} :?\mathsf{BOR}(X, \mathbb{R}),$

$[6] := \mathtt{E}\mathcal{A}[4][5] : \texttt{LambdaClass}(X, \mathcal{A}),$

$\texttt{Assume}\ U \in \mathcal{T}(X),$

$\Big(A, [7]\Big) := \mathtt{E}F_\sigma(U) : \sum A : \mathbb{N} \to \texttt{Closed}(X) \,.\, A \uparrow X,$

$\Big(f, [8]\Big) := \texttt{UrysohnLemma}(X, A, U^\complement) : \sum f : \mathbb{N} \to \mathsf{TOP}\Big(X, [0,1]\Big) \,.\, f^{-1}(1) = A \,\&\, f^{-1}(0) = U^\complement,$

$[9] := [7][8] : \lim_{n\to\infty} f_n = \chi_U,$

$[10] := \mathtt{E}\mathcal{C}[9] : \chi_U \in \mathcal{C},$

$[*] := \mathtt{E}\mathcal{A}[10] : U \in \mathcal{A};$

$\rightsquigarrow [7] := \mathtt{I} \subset : \mathcal{T}(X) \subset \mathcal{A},$

$[8] := \texttt{PiLambdaLemma}[6][7] : \mathcal{B}(X) = \mathcal{A},$

$[9] := \texttt{LebesgueApproximationTHM}[8][3]\mathtt{E}\mathcal{A} : \mathcal{B}(X, \mathbb{R}) \subset \mathcal{C},$

$[*] := \mathtt{E}\texttt{TypeEq}[8][9] : \mathcal{B}(X) = \mathcal{C};$

$\square$

### 2.2.6 Case of Separable Metrizable Space

$$\mathtt{PointSeparatingAlgebra} :: \prod_{X \in \mathsf{SET}} ?\mathtt{Algebra}(X)$$

$\mathcal{A} : \mathtt{PointSeparatingAlgebra} \iff \forall x, y \in X \,.\, \forall U : x \neq y \,.\, \exists A, B \in \mathcal{A} \,.\, A \cap B = \emptyset \,\&\, x \in A \,\&\, y \in B$

$\mathtt{BorelIsomorphismCondition} ::$

$:: \forall X : \mathsf{BOR} \,.\, \forall [0] : \mathtt{CountablyGeneratedSigmaAlgebra} \,\&\, \mathtt{PointSeparatingAlgebra}(X, \mathcal{S}_X) \,.$

$. \exists Y \subset \mathcal{C} \,.\, X \cong_{\mathsf{BOR}} Y$

$\mathtt{Proof} =$

$\Big(A, [1]\Big) := \mathtt{ECountablyGeneratedSigmaAlgebra}[0.1] : \sum A : \mathbb{N} \to ?X \,.\, \mathcal{S}_X = \sigma(\operatorname{Im} A),$

$\varphi := \Lambda x \in X \,.\, \Lambda n \in \mathbb{N} \,.\, \delta_x(B_n) : X \to \mathcal{C},$

$[2] := \mathtt{E}\varphi\mathtt{EPointSeparatingAlgebra}[0.2] : \mathtt{Injective}(X, \mathcal{C}, \varphi),$

$[3] := \Lambda I : \mathtt{Finite}(\mathbb{N}) \,.\, \Lambda b : I \to \mathsf{BOOL} \,.\, \mathtt{E}\varphi\mathtt{I}\bigcap : \forall I : \mathtt{Finite}(\mathbb{N}) \,.\, \forall b : I \to \mathsf{BOOL} \,.$

$: \varphi^{-1}\Big\{c \in \mathcal{C} : \forall i \in I \,.\, c_i = b_i\Big\} = \{x \in X \,.\, \forall i \in I \,.\, i = 1 \Rightarrow x \in A_i \,\&\, i = 0 \Rightarrow x \notin A_i\} =$

$= \bigcap_{b_i = 1} A_i \cap \bigcap_{b_i = 0} A_i^{\complement} \in \mathcal{S}_X,$

$[4] := \mathtt{I}\mathsf{BOR}(X)[3] : \varphi \in \mathsf{BOR}(X, \mathcal{C}),$

$Y := \varphi(X) : ?\mathcal{C},$

$[5] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}\varphi\mathtt{I}Y : \varphi(A_n) = \{c \in \mathcal{C} : c_n = 1\} \cap Y \in \mathcal{B}(\mathcal{C}),$

$\rightsquigarrow [*] :=: X \cong_{\mathsf{BOR}} Y;$

$\square$

$\mathtt{RealIsomorphismTHM} :: \forall X : \mathtt{Polish} \,.\, \exists A \subset \mathbb{R} \,.\, A \cong_{\mathsf{BOR}} X$

$\mathtt{Proof} =$

$\ldots$

$\square$

$\texttt{KuratowskiMeasurableExtensionTHM} ::$

$:: \forall X \in \mathsf{BOR} . \forall Y : \texttt{Polish} . \forall Z \subset X . \forall f \in \mathsf{BOR}(X,Z) . \exists F \in \mathsf{BOR}(X,Y) . F_{|Z} = f$

$\texttt{Proof} =$

$\Big(V,[1]\Big) := \texttt{ESecondCountable}(Y) : \sum V : \mathbb{N} \to \mathcal{T}(Y) . \texttt{BaseOfTopology}\Big(Y, \operatorname{Im} V\Big),$

$\Big(B,[2]\Big) := \texttt{SubsetMeasurableStructure}\Big(X,Z,f^{-1}(V)\Big) : \sum B : \mathbb{N} \to \mathcal{S}_X . \forall n \in \mathbb{N} . f^{-1}(V_n) = B_n \cap Z,$

$Z' := \Big\{x \in X : \exists y \in Y : \forall n \in \mathbb{N} . x \in B_n \iff y \in V_n\Big\} :?X,$

$f' := \Lambda x \in Z' . \texttt{ESingleton} \bigcap \{V_n | n \in \mathbb{N}, x \in B_n\} : Z' \to Y,$

$[3] := \mathtt{E}Z'(f) : Z \subset Z',$

$[4] := \mathtt{E}f' : \forall n \in \mathbb{N} . f'^{-1}(V_n) = B_n \cap Z',$

$[5] := \mathtt{I}\mathsf{BOR}[4] : f' \in \mathsf{BOR}(Z',Y),$

$\beta := \{(n,x) | x \in B_n\} :?(\mathbb{N} \times X),$

$[6] := \mathtt{E}Z'\mathtt{E}\beta : \forall x \in X . x \in Z' \iff \exists \sigma_x(\beta)\ \&$

$\& \forall k \in \mathbb{N} . \forall n,m \in \sigma_x(\beta) . \exists l \in \sigma_x(\beta) . \overline{V_l} \subset V_n \cap V_m \ \&\ \operatorname{diam}(V_l) < \dfrac{1}{k}\ \&$

$\& \forall n \in \mathbb{N} . \forall m \in \mathbb{N} . m \in \sigma_x(\beta)\ \&\ V_m \subset V_n \Rightarrow n \in \sigma_x(\beta),$

$C := \left\{n,m,k,l \in \mathbb{N} : \overline{V_l} \subset V_n \cap V_m, \operatorname{diam}(V_l) < \dfrac{1}{k}\right\} :?\mathbb{N}^4,$

$D := \{m,n \in \mathbb{N} : V_m \subset V_n\} :?\mathbb{N}^2,$

$[7] := \mathtt{E}Z'\mathtt{E}C\mathtt{E}D : Z' = \bigcup_{n,m,k,l \in C} \Big((B_n \cap B_m)^{\complement} \cup B_l\Big) \cap \bigcup_{n,m \in D} B_m^{\complement} \cup B_n \in \mathcal{B}(X);$

$y := \texttt{ENonEmpty}(Y) \in Y,$

$F := \Lambda x \in X . \texttt{ if } x \in Z' \texttt{ then } f'(z) \texttt{ else } Y : \mathsf{BOR}(X,Y),$

$[*] := \mathtt{E}F\mathtt{E}f' : F_{|Z} = f;$

$\square$

$\texttt{MeasurableLavrentievTHM} :: \forall X,Y : \texttt{Polish}\ \forall A \subset X . \forall B \subset Y . \forall A \overset{f}{\leftrightarrow} B : \mathsf{BOR} .$

$\exists A' \in \mathsf{BOR}(X) . \exists B' \in \mathsf{BOR}(Y) . \exists A' \overset{F}{\leftrightarrow} B' : \mathsf{BOR} . F_A = f$

$\texttt{Proof} =$

Proof by analogy with normal Lavrentiev Theorem..

$\square$

$\texttt{MeasurableGraphTHM} :: \forall X \in \mathsf{BOR} . \forall Y : \texttt{Metrizable}\ \&\ \texttt{Separable} . \forall \varphi : \mathsf{BOR}(X,Y) . G(\varphi) \in \mathcal{S}_X \otimes \mathcal{B}(Y)$

$\texttt{Proof} =$

$\Big(V,[1]\Big) := \texttt{ESecondCountable}(Y) : \sum V : \mathbb{N} \to \mathcal{T}(Y) . \texttt{BaseOfTopology}\Big(Y, \operatorname{Im} V\Big),$

$[*] := \mathtt{E}G(\varphi)[1] : G(\varphi) = \bigcap_{n=1} (X \times V_n)^{\complement} \cup \varphi^{-1}(X) \times Y \in \mathcal{S}_X \otimes \mathcal{B}(Y),$

$\square$

### 2.2.7 Standard and Effros Spaces

$\texttt{StandardBorelSpace} :: ?\mathsf{BOR}$

$X : \texttt{StandardBorelSpace} \iff \exists P : \texttt{Polish} \,.\, P \cong_{\mathsf{BOR}} X$

$\texttt{StandardBorelProduct} :: \forall N \in \sigma(\omega) \,.\, \forall X : N \to \texttt{StandardBorelSpace} \,.\, \texttt{StandardBorelSpace}\left(\prod_{n=1}^{N} X_n\right)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SdandardBorelSum} :: \forall N \in \sigma(\omega) \,.\, \forall X : N \to \texttt{StandardBorelSpace} \,.\, \texttt{StandardBorelSpace}\left(\coprod_{n=1}^{N} X_n\right)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{spaceOfEffros} :: \texttt{Contravariant}(\mathsf{TOP}, \mathsf{BOR})$

$\texttt{spaceOfEffros}\,(X) = \mathsf{EFF}(X) := \left(\texttt{Closed}(X), \sigma\left\{\{K : \texttt{Closed}(X) | \exists (K \cap U)\}\middle| U \in \mathcal{T}(X)\right\}\right)$

$\texttt{spaceOfEffros}\,(X, Y) = \mathsf{EFF}_{X,Y}(f) := f^{-1}$

$\texttt{EffrosRegularityTHM} :: \forall X : \texttt{Polish} \,.\, \texttt{StandardBorelSpace}(\mathsf{EFF}(X))$

$\texttt{Proof} =$

$[1] := \texttt{SubsetTopology}(\beta X, X)\mathtt{E}\beta X : \texttt{Injective}\left(\mathsf{EFF}(X), \mathsf{K}(\beta X), \operatorname*{cl}_{\beta X}\right),$

$\left(V, [2]\right) := \mathtt{E}\texttt{SecondCountable}(\beta X) : \sum V : \mathbb{N} \to \mathcal{T}(\beta X) \,.\, \texttt{BaseOfTopology}\left(\beta X, \operatorname{Im} V\right),$

$\left(U, [3]\right) := \mathtt{E}\beta X\texttt{PolishIsGdelta}(\beta X, X) : \sum U : \mathbb{N} \to \mathcal{T}(\beta X) \,.\, X = \bigcap_{n=1}^{\infty} U_n,$

$G := \left\{ \operatorname*{cl}_{\beta X} F \middle| F \in \mathsf{EFF}(X) \right\} :?\texttt{Closed}(\beta X),$

$\texttt{Assume}\ K \in G,$

$[4] := \mathtt{E}G(K) : \texttt{Dense}(K, K \cap X),$

$[5] := [4][3] : \forall n \in \mathbb{N} \,.\, \texttt{Dense}(K, K \cap U_n),$

$[*] := \mathtt{E}\texttt{Baire}\beta X[2][5] : \forall n \in \mathbb{N} \,.\, \forall m \in \mathbb{N} \,.\, \exists (K \cap V_m) \Rightarrow \exists (K \cap V_m \cap U_n);$

$\rightsquigarrow [4] := \mathtt{I}\bigcap : G = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{K \in \mathsf{K}(\beta X) : \exists K \cap V_m\right\}^{\complement} \cup \left\{K \in \mathsf{K}(\beta X) : \exists K \cap V_m \cap U_n\right\},$

$[5] := \mathtt{I}G_\delta : G \in G_\delta\left(\mathsf{K}(\beta X)\right),$

$[6] := \texttt{GDeltaIsPolish}\left(\mathsf{K}(\beta X), G\right) : \texttt{Polish}(G),$

$[*] := \mathtt{E}\mathsf{EFF}(X)\mathtt{E}G[1][6] : \left(\mathsf{BOR}, \mathsf{EFF}(X), G, \operatorname*{cl}_{\beta X}\right);$

$\square$

$\texttt{spaceOfFell} :: \mathsf{TOP} \to \mathsf{TOP}$

$$\texttt{spaceOfFell}(X) = F(X) := \left(\texttt{Closed}(X), \left\langle \left\{ \left\{ A : \texttt{Closed}(X) \middle| \neg\exists(K \cap A) \ \& \ \forall i \in \{1,\dots,n\} \ . \ \exists(U_i \cap A) \right\} \middle| K \in \mathsf{K}(X), n \in \mathbb{Z}_+, U : \{1,\dots,n\} \to \mathcal{T}(X) \right\} \right\rangle_{\mathsf{TOP}} \right)$$

$\texttt{FellTopologyIsCompact} :: \forall X : \texttt{Polish} \ \& \ \texttt{LocallyCompact} \ . \ \texttt{CompactMetrizable}\big(F(X)\big)$

Proof $=$

$\iota_+ := \Lambda A \in F(X) \ . \ A \cup \{\infty\} : F(X) \to \mathsf{K}(X^+)$,

$d := \texttt{EPolish}(X^+) : \sum d : \texttt{Metric}(X^+) \ . \ \texttt{CompactlyMetrizes}(X^+)$,

$$\rho := \Lambda A, B : \texttt{Closed}(X) \ . \ \max\left( \inf_{x \in \iota_+(A)} \sup_{y \in \iota_+(B)} d(x,y), \inf_{x \in \iota_+(B)} \sup_{y \in \iota_+(A)} d(x,y) \right) : \texttt{Metric}\Big(F(X)\Big),$$

$[*] := \texttt{HausdorffCompactIsCompact}(X^+)\texttt{SubspaceTopology}(X^+, X) : \texttt{Compact}\Big(F(X), \rho\Big)$;

The topology of Fell corresponds to subspace topology for Hausdorff metric on ine point compactification .

The open sets of form $\left\{ A \in F(X) \middle| \neg\exists A \cap K \right\}$ correspond to open sets of form $\left\{ A \in \mathsf{K}(X^+) \middle| A \subset U \right\}$.

Here $U \in \mathcal{U}(\infty)$ .

By the structure of the embedding $\iota_+$ this is enough .

Assume $(A_n)_{n=1}^\infty$ is a sequence of closed set in $F(X)$.

Then $\iota_+(A)$ is a sequence in $\mathsf{K}(X)$.

It will have a partial limit $B$.

Then $B \cap X \in F(X)$ and is a partial limit of $(A_n)_{n=1}^\infty$.

□

$\texttt{FellBorelIsEffros} :: \forall X : \texttt{Polish} \ \& \ \texttt{LocallyCompact} \ . \ \mathsf{B}\Big(F(X)\Big) = \mathsf{EFF}(X)$

Proof $=$

Let $K$ be a compact in $X$.

We need to express sets of form $\left\{ A \in F(X) \middle| \neg\exists A \cap K \right\}$ by sets of form $\left\{ A \in F(X) \middle| \exists A \cap U \right\}$ for $U$ open .

$K$ is a $G_\delta$, so where are open sets $(U_n)_{n=1}^\infty$ such that $K = \bigcap_{n=1}^\infty U_n$ .

Then, $\left\{ A \in F(X) \middle| \neg\exists A \cap K \right\} = \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \left\{ A \in F(X) \middle| \exists A \cap U_n \right\}^{\complement}$ .

□

$\texttt{KSigmaEffrosSpaceIsStandard} :: \forall X : \sigma\texttt{-Compact} \ \& \ \texttt{Separable} \ \& \ \texttt{Metrizable} \ .$

$. \ \texttt{StandardBorelSpace}\Big(\mathsf{EFF}(X)\Big)$

$\texttt{Proof} \ =$

$[1] := \texttt{SubsetTopology}(\beta X, X)\mathtt{E}\beta X : \texttt{Injective}\Big(\mathsf{EFF}(X), \mathsf{K}(\beta X), \operatorname*{cl}_{\beta X}\Big),$

$\Big(K, [2]\Big) := \mathtt{E}\sigma\texttt{-Compact}(X) : \sum K : \mathbb{N} \to \mathsf{K}(X) \ . \ X = \bigcup_{n=1}^{\infty} K_n,$

$\Big(U, [3]\Big) := \texttt{ClosedIsFSigma}(\beta X, \operatorname*{cl}_{\beta X} K) : \sum U : \mathbb{N}^2 \to \mathcal{T}(\beta X) \ . \ \forall n \in \mathbb{N} \ . \ \operatorname*{cl}_{\beta X} K_n = \bigcap_{m=1}^{\infty} U_{n,m},$

$\Big(V, [31]\Big) := \texttt{ESecondCountable}(\beta X) : \sum V : \mathbb{N} \to \mathcal{T}(\beta X) \ . \ \texttt{BaseOfTopology}\Big(\beta X, \operatorname{Im} V\Big),$

$G := \left\{ \operatorname*{cl}_{\beta X} F \middle| F \in \mathsf{EFF}(X) \right\} :?\texttt{Closed}(\beta X),$

$\texttt{Assume} \ K \in G,$

$[4] := \mathtt{E}G(K) : \texttt{Dense}(K, K \cap X),$

$[5] := [4][3] : \forall n \in \mathbb{N} \ . \ \texttt{Dense}(K, K \cap U_n),$

$[*] := \texttt{EBaire}\beta X[2][5] : \forall n, k, l \in \mathbb{N} \ . \ \exists (K \cap V_m) \Rightarrow \exists (K \cap V_m \cap U_{k,l});$

$\rightsquigarrow [4] := \mathtt{I}\bigcap : G = \bigcap_{n=1}^{\infty} \bigcap_{k,l=1}^{\infty} \Big\{ K \in \mathsf{K}(\beta X) : \exists K \cap V_m \Big\}^{\complement} \cup \Big\{ K \in \mathsf{K}(\beta X) : \exists K \cap V_m \cap U_{k,l} \Big\},$

$[5] := \mathtt{I}G_\delta : G \in G_\delta\Big(\mathsf{K}(\beta X)\Big),$

$[6] := \texttt{GDeltaIsPolish}\Big(\mathsf{K}(\beta X), G\Big) : \texttt{Polish}(G),$

$[*] := \mathtt{EEFF}(X)\mathtt{E}G[1][6] : \Big(\mathsf{BOR}, \mathsf{EFF}(X), G, \operatorname*{cl}_{\beta X}\Big);$

$\square$

$\texttt{RaymondsTHM} :: \forall X : \texttt{Separable} \ \& \ \texttt{Metrizable} \ . \ \texttt{StandardBorelSpace}(\mathsf{EFF}(X)) \iff$

$\iff \exists P : \texttt{Polish} \ . \ \exists S : \sigma\texttt{-Compact} \ . \ X = P \cap S$

$\texttt{Proof} \ =$

This is proof is out of the scope of this manuscript.

$\square$

$\texttt{CompactsAreBorelForFell} :: \forall X : \texttt{Polish} \ . \ \mathsf{K}(X) \in \mathcal{B}\Big(F(X)\Big)$

$\texttt{Proof} \ =$

The closed set is compact iff it totally bounded.

Denote by $\mathbb{B}(r)$ the set of open rational cells of radius less then $r$.

For a finite sequence of open rational cells $(U_i)_{i=1}^n$ define $V_{n,U,k}$ to be such open sets that $V_{n,U,k} \downarrow \bigcap_{i=1}^{n} U_i^{\complement}$ .

Then, Express $\mathsf{K}(X) = \bigcap_{\varepsilon \in \mathbb{Q}_{++}} \bigcup_{n=1}^{\infty} \bigcup_{U : n \to \mathbb{B}(\varepsilon)} \bigcup_{k=1}^{\infty} \Big\{ A \in F(X) : \exists A \cap V_{n,U,K} \Big\}^{\complement} \in \mathcal{B}\Big(F(X)\Big)$ .

$\square$

$\texttt{CompactsAreBorelSubspaceOfEffros} :: \forall X : \texttt{Polish} \,.\, \mathsf{K}(X) \subset_{\mathsf{BOR}} \mathsf{EFF}(X)$

$\texttt{Proof} \;=$

Inspect generating sets for respective measurable algebras..

□

$\texttt{CompactsAreEffros} :: \forall X \in \mathsf{HC} \,.\, \mathsf{K}(X) = \mathsf{EFF}(X)$

$\texttt{Proof} \;=$

Obvious.

□

$\texttt{SubsetRelationIsBorel} :: \forall X : \texttt{Polish} \,.\, \left\{(A,B) \in \mathsf{EFF}^2(X) \middle| A \subset B\right\} \in \mathsf{A}\Big(\mathsf{EFF}^2(X)\Big)$

$\texttt{Proof} \;=$

Denote by $\mathbb{B}(X)$ the set of rational cells of $X$.

Express $\left\{(A,B) \in \mathsf{EFF}^2(X) \middle| A \subset B\right\}$.

$$\bigcap_{U \in \mathbb{B}(X)} \left(\left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap X\right\} \times \mathsf{EFF}(X)\right)^{\complement} \cup \mathsf{EFF}(X) \times \left\{B \in \mathsf{EFF}(X) \middle| \exists B \cap U\right\}$$

□

$\texttt{UnionIsBorel} :: \forall X : \texttt{Polish} \,.\, (\cup) \in \mathsf{BOR}\Big(\mathsf{EFF}^2(X), \mathsf{EFF}(X)\Big)$

$\texttt{Proof} \;=$

$$(\cup)^{-1}\left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap U\right\} = \left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap U\right\} \times \mathsf{EFF}(X) \cup \mathsf{EFF}(X) \times \left\{B \in \mathsf{EFF}(X) \middle| \exists B \cap U\right\}$$

□

$\texttt{ProductIsBorel} :: \forall X, Y : \texttt{Polish} \,.\, (\times) \in \mathsf{BOR}\Big(\mathsf{EFF}(X) \times \mathsf{EFF}(Y), \mathsf{EFF}(X \times Y)\Big)$

$\texttt{Proof} \;=$

Let $U$ be open In $X \times Y$.

Denote by $\mathcal{U}$ a set of pairs $(W,V)$ of open rational cells such that $W \times V \subset U$ .

$$(\times)^{-1}\left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap U\right\} = \bigcup_{(U,V) \in \mathcal{U}} \left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap V\right\} \times \left\{B \in \mathsf{EFF}(X) \middle| \exists B \cap W\right\}$$

$\texttt{EffrosPushforward} :: \forall X, Y : \texttt{Polish} \,.\, \forall \varphi \in \mathsf{TOP}(X,Y) \,.\, \Big(\Lambda A \in \mathsf{EFF}(X) \,.\, \overline{f(A)}\Big) \in \mathsf{BOR}\Big(\mathsf{EFF}(X), \mathsf{EFF}(Y)\Big)$

$\texttt{Proof} \;=$

Open set $U$ intersects $\overline{f(A)}$ iff it intersects $f(A)$.

So, $U$ intersects $\overline{f(A)}$ iff open set $f^{-1}(U)$ intersects $A$.

□

$\texttt{ClosedDomainsAreEffrosMeasurable} :: \forall X : \texttt{Polish} \,.\, \texttt{ClosedDomain}(X) \in \mathsf{A}\Big(\mathsf{EFF}(X)\Big)$

$\texttt{Proof} \;=$

$$\bigcap_{U \in \mathbb{B}(X)} \left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap U\right\}^{\complement} \cup \bigcup_{V \leq_{\mathbb{B}} U} \bigcap_{W \leq_{\mathbb{B}} V} \left\{A \in \mathsf{EFF}(X) \middle| \exists A \cap W\right\}$$

□

$\texttt{CatAlgebraIsBorel} :: \forall X : \texttt{Polish} \,.\, \mathbf{CAT}(X) \in \mathcal{B}(X)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SelectionTheorem} :: \forall X : \texttt{Polish} \,.\, \exists \delta : \mathbb{N} \to \mathsf{BOR}\Big(\mathsf{EFF}(X), X\Big) \,.\, \forall A \in \mathsf{EFF}(X) \,.\, \exists A \Rightarrow \texttt{Dense}\Big(A, \delta_{\mathbb{N}}(A)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : X \neq \emptyset,$

$\Big(\rho, [2]\Big) := \texttt{EPolish} : \sum \rho : \texttt{Metrizes}(X) \,.\, \texttt{Complete}(X, \rho),$

$\Big(U, [3]\Big) := \texttt{SouslinSchemaExists}(X, \rho) : \sum U : \mathbb{N}^* \to \mathcal{T}(X) \,.\, \texttt{SouslinSchema}(X, \rho, U),$

$[4] := \texttt{ESouslinSchema}(X, \rho, U) :$

$\quad : \forall w \in \mathbb{N}^* \,.\, U_w \neq \emptyset \;\&$

$\quad \& \; U_\emptyset = X \;\&$

$\quad \& \; \forall w \in \mathbb{N}^* \,.\, \forall n \in \mathbb{N} \,.\, \overline{U_{wn}} \subset U_{wn} \;\&$

$\quad \& \; \forall w \in \mathbb{N}^* \,.\, U_w = \bigcup_{n \in \mathbb{N}} U_{wn} \;\&$

$\quad \& \; \forall w \in \mathbb{N}^* \,.\, \forall b : \mathrm{len}(w) > 0 \,.\; \mathrm{diam}\, U_w \leq 2^{-\mathrm{len}(w)},$

$\alpha := \Lambda s \in \mathcal{B} \,.\, \texttt{ESingleton}\left(\bigcap_{n=1}^{\infty} U_{s_{|[1,\ldots,n]}}, \texttt{EComplete}(X, \rho)[2][4.2][4.5]\right) : \mathcal{B} \to X,$

$[5] := \mathtt{E}\alpha[4.2][4.4][4.5] : \texttt{Surjective}(\mathcal{B}, X, \alpha),$

$[6] := \mathtt{E}\alpha[4.3][4.4][4.5]\mathtt{E}\mathcal{B} : \alpha \in \mathsf{TOP}(\mathcal{B}, X),$

$\texttt{Assume}\ A : \texttt{Closed}(X),$

$\texttt{Assume}\ [7] : \exists A,$

$T := \{w \in \mathbb{N}^* | \exists A \cap U_w\} :?\mathbb{N}^*,$

$[8] := \mathtt{E}T[4.3][4.4] : \texttt{Pruned}(\mathbb{N}, T),$

$[9] := [9][4.2] : \exists T,$

$d(F) := \alpha(\mathrm{lb}\, T) : X,$

$a_F := \mathrm{lb}\, T : \mathcal{B};$

$\rightsquigarrow \Big(d, [7]\Big) := \mathtt{I} \to \texttt{EComplete}(X, \rho)[4.5] : \sum d : \mathsf{EFF}(X) \to X \,.\, \forall A \in \mathsf{EFF}(X) \,.\, A \neq \emptyset \Rightarrow d(A) \in A,$

$\rightsquigarrow \Big(a, [8]\Big) := \mathtt{I} \to: \sum a : \mathsf{EFF}(X) \setminus \{\emptyset\} \to \mathcal{B} \,.\, \forall A \in \mathsf{EFF}(X) \,.\, A \neq \emptyset \Rightarrow d(A) \in \alpha(a_A),$

$[9] := \mathtt{EBaireE}\mathsf{EFF}(X) : \alpha \in \mathsf{BOR}\Big(\mathsf{EFF}(X) \setminus \emptyset, \mathcal{B}\Big),$

$[10] := [9][8] : d \in \mathsf{BOR}\Big(\mathsf{EFF}(X), X\Big),$

$\Big(V, [01]\Big) := \texttt{PolishIsSecondCountable}(X) : \sum V : \mathbb{N} \to \mathcal{T}(X) \,.\, \texttt{BaseOfTopology}(X, \mathrm{Im}\, V),$

$\texttt{Assume}\ m \in \mathbb{N},$

$\texttt{Assume}\ A : \texttt{Closed}(X \cap V_n),$

$\texttt{Assume}\ [07] : \exists A,$

$T := \{w \in \mathbb{N}^* | \exists A \cap U_w\} :?\mathbb{N}^*,$

$[08] := \mathtt{E}T[4.3][4.4] : \texttt{Pruned}(\mathbb{N}, T),$

$[09] := [09][4.2] : \exists T,$

$e(F) := \alpha(\mathrm{lb}\, T) : X \cap V_n,$

$b_F := \mathrm{lb}\, T : \mathcal{B};$

$\rightsquigarrow \Big(e,[11]\Big) := \mathtt{I} \rightarrow \mathtt{EComplete}(X,\rho)[4.5] :$

$: \sum e : \prod_{m=1}^{\infty} : \mathsf{EFF}(X) \to X \;.\; \forall A \in \mathsf{EFF}(X \cap V_n) \;.\; A \neq \emptyset \Rightarrow e_m(A) \in A,$

$\rightsquigarrow \Big(b,[12]\Big) := \mathtt{I} \rightarrow:$

$: \sum b : \prod_{m=1}^{\infty} \mathsf{EFF}(X \cap V_m) \setminus \{\emptyset\} \to \mathcal{B} \;.\; \forall A \in \mathsf{EFF}(X \cap V_m) \;.\; A \neq \emptyset \Rightarrow \alpha(b_m(A)) = e_m(A),$

$\delta := \Lambda n \in \mathbb{N} \;.\; \Lambda A \in \mathsf{EFF}(X) \;.\; \mathtt{if}\ \exists A \cap V_n\ \mathtt{then}\ e_n(A)\ \mathtt{else}\ d(A) : \mathbb{N} \to \mathsf{BOR}\Big(\mathsf{EFF}(X), X\Big),$

$[13] := \mathtt{E}\delta\mathtt{E}e\mathtt{E}d : \forall n \in \mathbb{N} \;.\; \forall A \in \mathsf{EFF}(X) \;.\; A \neq \emptyset \Rightarrow \delta_n(A) \in A,$

$[14] := \mathtt{E}\delta\mathtt{EBaseofTopology}(X, \operatorname{Im} V) : \mathtt{Dense}\Big(A, \delta_{\mathbb{N}}(A)\Big);$

$\square$

$\mathtt{EffrosMeasurabilityCriterion} ::$

$:: \forall X \in \mathsf{BOR} \;.\; \forall Y : \mathtt{Polish} \;.\; \forall \varphi : X \to \mathsf{EFF}(Y) \;.\; \varphi \in \mathsf{BOR}\Big(X, \mathsf{EFF}(Y)\Big) \iff$

$\iff \varphi^{-1}(\emptyset) \in \mathsf{A}(X) \;\&\; \exists \phi : \mathbb{N} \to \mathsf{BOR}(X,Y) \;.\; \forall x \in X \;.\; \varphi(x) \neq \emptyset \Rightarrow \mathtt{Dense}\Big(\varphi(x), \phi_{\mathbb{N}}(x)\Big)$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [1] : \phi \in \mathsf{BOR}\Big(X, \mathsf{EFF}(Y)\Big),$

$\Big(\delta,[2]\Big) := \mathtt{SelectionTHM}(X) : \sum \delta : \mathbb{N} \to \mathsf{BOR}\Big(\mathsf{EFF}(X), X\Big) \;.\; \forall A \in \mathsf{EFF}(X) \;.\; \exists A \Rightarrow \mathtt{Dense}\Big(A, \delta_{\mathbb{N}}(A)\Big),$

$\phi := \varphi\delta : \mathbb{N} \to \mathsf{BOR}(X,Y),$

$[1.*] := \mathtt{E}\phi[2] : \forall x \in X \;.\; \varphi(x) \neq \emptyset \Rightarrow \mathtt{Dense}\Big(\varphi(x), \phi_{\mathbb{N}}(x)\Big);$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow: \varphi \in \mathsf{BOR}\Big(X, \mathsf{EFF}(Y)\Big) \Rightarrow$

$\Rightarrow \varphi^{-1}(\emptyset) \in \mathsf{A}(X) \;\&\; \exists \phi : \mathbb{N} \to \mathsf{BOR}(X,Y) \;.\; \forall x \in X \;.\; \varphi(x) \neq \emptyset \Rightarrow \mathtt{Dense}\Big(\varphi(x), \phi_{\mathbb{N}}(x)\Big),$

$\mathtt{Assume}\ [2] : \varphi^{-1}\{\emptyset\} \in \mathsf{A}(X),$

$\mathtt{Assume}\ \phi : \mathbb{N} \to \mathsf{BOR}(X,Y),$

$\mathtt{Assume}\ [3] : \forall x \in X \;.\; \varphi(x) \neq \emptyset \Rightarrow \mathtt{Dense}\Big(\varphi(x), \phi_{\mathbb{N}}(x)\Big),$

$\mathtt{Assume}\ U \in \mathcal{T}(Y),$

$[U.*] := [2][3] : \varphi^{-1}\Big\{A \in \mathsf{EFF}(Y) \Big| \exists A \cap U\Big\} = \Big(\varphi^{-1}\{\emptyset\}\Big)^{\complement} \cap \bigcup_{n=1}^{\infty} \phi_n^{-1}(U);$

$\rightsquigarrow [2.*] := \mathtt{BorelByGenerators} : \varphi \in \mathsf{BOR}\Big(X, \mathsf{EFF}(Y)\Big);$

$\rightsquigarrow [*] := \mathtt{I} \iff [1] : \varphi \in \mathsf{BOR}\Big(X, \mathsf{EFF}(Y)\Big) \iff$

$\iff \varphi^{-1}(\emptyset) \in \mathsf{A}(X) \;\&\; \exists \phi : \mathbb{N} \to \mathsf{BOR}(X,Y) \;.\; \forall x \in X \;.\; \varphi(x) \neq \emptyset \Rightarrow \mathtt{Dense}\Big(\varphi(x), \phi_{\mathbb{N}}(x)\Big);$

$\square$

## 2.3 Representations and Transformatons

### 2.3.1 Clopen Set Representation

$$\texttt{PolishTopology} :: \prod_{X \in \mathsf{SET}} ?\texttt{Topology}(X)$$

$$\mathcal{T} : \texttt{PolishTopology} \iff \texttt{Polish}(X, \mathcal{T})$$

$\texttt{ClosedSubsetTopologyEnrichment}$ ::

:: $\forall X : \texttt{Polish} \,.\, \forall F : \texttt{Closed}(X) \,.\, \exists \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \sigma(\mathcal{T}) = \mathcal{B}(X) \ \&\ \texttt{Clopen}(X, \mathcal{T}, F)$

$\texttt{Proof}$ =

Repesent $X = F \sqcup F^{\complement}$.

This topology is polish and has $F$ as a clopen set.

As sigma-algebras contain intersections of both closed and open sets the Borel structures coincide.

□

$\texttt{SupTopologyIsPolish}$ ::

:: $\forall X : \texttt{Polish} \,.\, \forall \mathcal{T} : \mathbb{N} \to \texttt{PolishTopology}(X) \,.$

$. \ \forall [0.1] : \forall n \in \mathbb{N} \,.\, \mathcal{T}(X) \subset \mathcal{T}_n \,.\, \texttt{Polish}(X, \sup_{n \in \mathbb{N}} \mathcal{T}_n)$

$\texttt{Proof}$ =

$$\varphi := \Lambda x \in (X, \sup \mathcal{T}_n) \,.\, \Lambda n \in \mathbb{N} \,.\, (\mathcal{T}_n)\ x : (X, \sup_{n \in \mathbb{N}} \mathcal{T}_n) \to \prod_{n \in \mathbb{N}} (X, \mathcal{T}_n),$$

$$[1] := \mathrm{E}\varphi\mathrm{ET}_2(X)[0.1]\texttt{ProductTopologyBase} : \texttt{Closed}\left(\sup_{n \in \mathbb{N}}(X, \mathcal{T}_n), \operatorname{Im} \varphi\right),$$

$[2] := \texttt{GDeltaIsPolish} : \texttt{Polish}\,(\operatorname{Im} \varphi)\,,$

$[*] := \texttt{EHomeomorphis}[2] : \texttt{Polish}(X, \sup_{n=1} \mathcal{T}_n);$

□

$\texttt{SupTopologyBorelPreservation}$ ::

:: $\forall X : \texttt{Polish} \,.\, \forall \mathcal{T} : \mathbb{N} \to \texttt{PolishTopology}(X) \,.$

$$. \ \forall [0.1] : \forall n \in \mathbb{N} \,.\, \mathcal{T}(X) \subset \mathcal{T}_n \,.\, \forall [0.2] : \forall n \in \mathbb{N} \,.\, \sigma(\mathcal{T}_n) \subset \mathcal{B}(X) \,.\, \mathcal{B}\left(X, \sup_{n \in \mathbb{N}} \mathcal{T}_n\right) = \mathcal{B}$$

$\texttt{Proof}$ =

True by topology bases and definition of $\sigma$.

□

$\texttt{ClopenRepresentationTHM} ::$

$: \forall X : \texttt{Polish} \,.\, \forall B \in \mathcal{B}(X) \,.\, \exists \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \texttt{Clopen}\big((X,\mathcal{T}),B\big)\ \&\ \mathcal{B}(X) = \sigma(\mathcal{T})$

$\texttt{Proof} \ =$

$\mathcal{A} := \Big\{ A \subset X : \exists \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \texttt{Clopen}\big((X,\mathcal{T}),B\big)\ \&\ \mathcal{B}(X) = \sigma(\mathcal{T}) \Big\} :??X,$

$[1] := \texttt{ClosedSubsetTopologyEnrichment}(X)\mathtt{I}\mathcal{A} : \mathcal{T}(X) \subset A,$

$[2] := \texttt{ClopenIsAlgebra}(X)\mathtt{I}\mathcal{A} : \forall A \in \mathcal{A} \,.\, A^{\complement} \in \mathcal{A},$

$\texttt{Assume}\ A : \mathbb{N} \to \mathcal{A},$

$\Big(\mathcal{T},[3]\Big) := \mathtt{E}\mathcal{A}(\mathcal{T}) : \sum \mathbb{N} \to \texttt{PolishTopology}(X) \,.\, \forall n \in \mathbb{N} \,.\, \texttt{Clopen}\big((X,\mathcal{T}),A_n\big)\ \&\ \mathcal{B}(X) = \sigma(\mathcal{T}),$

$[4] := \texttt{SupTopologyIsPolish}(X,\mathcal{T}) : \texttt{PolishTopology}\Big(X, \sup_{n\in\mathbb{N}} T_n\Big),$

$[5] := \texttt{SupTopologyBorelPreservation} : \sigma\Big(\sup_{n\in\mathbb{N}} T_n\Big) = \mathcal{B}(X),$

$[6] := \texttt{ClosedIntersection}\bigg(\Big(X, \sup_{n\in\mathbb{N}} \mathcal{T}_n\Big)[4], A\bigg) : \texttt{Closed}\left(\Big(X, \sup_{n\in\mathbb{N}} \mathcal{T}_n\Big), \bigcap_{n=1}^{\infty} A_n\right),$

$[7] := \texttt{ClosedSubsetTopologyEnrichment}[5][6] :$

$: \exists \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \sigma(\mathcal{T}) = \mathcal{B}(X)\ \&\ \texttt{Clopen}\left(X, \mathcal{T}, \bigcap_{n=1}^{\infty} A_n\right),$

$[A.*] := \mathtt{I}\mathcal{A}[7] : \bigcap_{n=1}^{\infty} A_n \in \mathcal{A};$

$\leadsto [4] := \mathtt{I}\sigma\texttt{-Algebra}[2] : \sigma\texttt{-Algebra}(X,\mathcal{A}),$

$[*] := [1][4]\mathtt{I}\mathcal{B}(X) : \mathcal{B}(X) \subset \mathcal{A};$

$\square$

$\texttt{StandardBorelSubset} :: \forall X : \texttt{StandardBorelSpace} \,.\, \forall Y \in \mathcal{S}_X \,.\, \texttt{StandardBorelSpace}(Y, \mathcal{S}_X|Y)$

$\texttt{Proof} \ =$

$\Big(\mathcal{T},[1]\Big) := \mathtt{E}\texttt{StandardBorelSpace}(X)\texttt{ClopenRepresentationTHM} :$

$: \sum \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \mathcal{B}(X,\mathcal{T}) = \mathcal{S}_X\ \&\ \texttt{Clopen}\Big((X,\mathcal{T}),Y\Big),$

$[2] := \texttt{GDeltaIsPolish}(X,Y)[1.2] : \texttt{Polish}(Y, \mathcal{T}|Y),$

$[*] := [1.1][2] : \texttt{StandardBorelSpace}(Y, \mathcal{S}_X|Y);$

$\square$

$\texttt{MultipleClopenRepresentation} ::$

$:: \forall X : \texttt{Polish} \,.\, \forall A : \mathbb{N} \to \mathcal{S}_X \,.\, \exists : \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \mathcal{B}(X) = \sigma(\mathcal{T})\ \&\ \forall n \in \mathbb{N} \,.\, \texttt{Clopen}\Big((X,\mathcal{T}),A_n\Big)$

$\texttt{Proof} \ =$

Construct separate topologies $\mathcal{T}_n$ for every set $A_n$ respectively by clopen representation theorem . Then in $\sup_{n\in\mathbb{N}} \mathcal{T}_n$ all these sets are clopen and the Borel structures coincide .

$\square$

$\texttt{ZeroDimRepresentation} ::$

$:: \forall X : \texttt{Polish} \,.\, \forall A : \mathbb{N} \to \mathcal{S}_X \,.\, \exists : \mathcal{T} : \texttt{PolishTopology}(X) \,.\, \mathcal{B}(X) = \sigma(\mathcal{T})\ \&\ \dim_{\mathsf{TOP}}(X,\mathcal{T}) = 0$

$\texttt{Proof} \ =$

Use base of rational cells as $A_n$ in the previous theorem .

$\square$

$\texttt{PerfectSetTheoremForPerfectSets} :: \forall X : \texttt{Polish} \,.\, \forall A \in \mathcal{B}(X) \,.\, |A| \leq \aleph_0 \Big| \exists C \subset \mathcal{C} \,.\, C \cong_{\mathsf{TOP}} \mathcal{C}$

$\texttt{Proof} \ =$

$\Big(\mathcal{T}, [1]\Big) := \texttt{StandardBorelSubset}(X)\texttt{EStandardBorelSpace}(X) :$

$: \sum \mathcal{T} : \texttt{PolishToplogy}(A) \,.\, \mathcal{T}(X)|A \subset \mathcal{T},$

$\texttt{Assume}\ [2] : |A| > \aleph_0,$

$\Big(C, [3]\Big) := \texttt{CantorSetSubsetTHM}(A, \mathcal{T})[3] : \sum C \subset A \,.\, (\mathcal{T})\ C \cong_{\mathsf{TOP}} \mathcal{C},$

$[2.*] := \texttt{CoarserHomeo} : (X)\ C \cong_{\mathsf{TOP}} \mathcal{C};$

$\rightsquigarrow [*] := \texttt{I}| : |A| \leq \aleph_0 \Big| \exists C \subset \mathcal{C} \,.\, C \cong_{\mathsf{TOP}} \mathcal{C};$

$\square$

$\texttt{StandardBorelSpaceCardinality} :: \forall X : \texttt{Polish} \,.\, \forall [0] : |X| > \aleph_0 \,.\, |X| = 2^{\aleph_0}$

$\texttt{Proof} \ =$

As $X$ contains a copy of $\mathcal{C}$ it at leas has cardinality $2^{\aleph_0}$.

On the other hand $X$ is Polish and has a countable base $\mathcal{U}$.

So, every element $x$ can be identified by a membership $x \in U, U \in \mathcal{U}$ .

Then $X$ can be also embedded into $\mathcal{C}$, so $|X| \leq 2^{\aleph_0}$.

Overall, $|X| = 2^{\aleph_0}$.

$\square$

### 2.3.2 Further Representations

LusinSouslinRepresentation :: $\forall X : \texttt{Polish} . \forall A \in \mathcal{B}(X) . \exists F : \texttt{Closed}(\mathcal{B}) . \exists f : \texttt{Bijective} \ \& \ \mathsf{TOP}(F, A)$

Proof =

$\Big(\mathcal{T}, [1]\Big) := \texttt{StandardBorelSubset}(X)\texttt{EStandardBorelSpace}(X) :$

$: \sum \mathcal{T} : \texttt{PolishToplogy}(A) . \mathcal{T}(X)|A \subset \mathcal{T},$

$\Big(F, f\Big) := \texttt{BaireSpaceUniversalProprty}(A, \mathcal{T}) : \sum F : \texttt{Closed}(\mathcal{B}) . f : \texttt{Bijective} \ \& \ \mathsf{TOP}\Big((F, \mathcal{T}), A\Big),$

$[*] := [1][2] : \texttt{Bijective} \ \& \ \mathsf{TOP}\Big(F, A, f\Big);$

$\square$

LusinSouslinExtension :: $\forall X : \texttt{Polish} . \forall A \in \mathcal{B}(X) . \exists F : \texttt{Closed}(\mathcal{B}) . \exists f : \texttt{Surjective} \ \& \ \mathsf{TOP}(F, A)$

Proof =

$\dots$

$\square$

LusinBorelSchemaExists :: $\forall X : \texttt{Polish} . \forall B \in \mathcal{B}(X) . \exists A : \mathbb{N}^* \to \mathcal{B}(X) .$

$. A_\emptyset = B \ \&$

$\& \ \forall w \in \mathbb{N}^* . A_w = \bigcup_{n \in \mathbb{N}} A_{wn} \ \&$

$\& \ \forall b \in \mathcal{B} . \Big(\forall n \in \mathbb{N} . A_{b_{|[1,\dots,n]}} \neq \emptyset\Big) \Rightarrow \exists L \in X . \{L\} = \bigcap_{n=1}^{\infty} A_{b_{|[1,\dots,n]}} \ \& \ \forall x \in \prod_{n=1}^{\infty} A_{b_{|[1,\dots,n]}} . \lim_{n \to \infty} x_n = L$

Proof =

Extend Topology for $B$ by clopen representation.

Then construct Lusin schema for $B$ in this topology.

$\square$

BaireBorelEncoding :: $\forall X : \texttt{Polish} . \forall A \subset \mathcal{B}(X) . \exists F : \texttt{Closed}(X \times \mathcal{B}) . x \in A \iff \exists! b \in \mathcal{B} . (x, b) \in F$

Proof =

Construct Lusin schema for $A$ in $X$.

Then there is a unique Baire encoding for each $x \in A$.

The last convergence property shows that $F$ is closed.

$\square$

CantorBorelEncoding :: $\forall X : \texttt{Polish} . \forall A \subset \mathcal{B}(X) . \exists G : G_\delta(X \times \mathcal{C}) . x \in A \iff \exists! c \in \mathcal{C} . (x, c) \in G$

Proof =

Construct encodding as in the previous problem.

Then translate encodding to binary.

$G$ can't be taken closed in general, as $\mathcal{B}$ is not compact.

$\square$

$\texttt{BorelMeasurableMapTopologization} :: \forall X : \texttt{Polish} \;.\; \forall Y : \texttt{SecondCountableSpace} \;.\; \forall \varphi \in \mathsf{BOR}(X, Y) \;.$

$\;.\; \exists \mathcal{T} : \texttt{PolishTopology}(X) \;.\; \forall [0] : \mathcal{T}(X) \subset \mathcal{T} \;\&\; \mathcal{B}(\mathcal{T}) = \mathcal{B}\Big(\mathcal{T}(X)\Big) \;\&\; \varphi \in \mathsf{TOP}\Big((X, \mathcal{T}), Y\Big)$

Proof $=$

Let $\mathcal{U}$ be a countable base for $Y$.

Then enrich the topology to make $\varphi^{-1}(\mathcal{U})$ clopen.

Then $\varphi$ will be continuous.

$\square$

$\texttt{BorelMeasurableIsomotphismTopologization} :: \forall X : \texttt{Polish} \;.\; \forall Y : \texttt{SecondCountableSpace} \;.$

$\;.\; \forall \varphi \in \texttt{Isomorphism}(\mathsf{BOR}, X, Y) \;.\; \exists \mathcal{T} : \texttt{PolishTopology}(X) \;.\; \forall [0] : \mathcal{T}(X) \subset \mathcal{T}$

$\& \;\; \& \; \mathcal{B}(\mathcal{T}) = \mathcal{B}\Big(\mathcal{T}(X)\Big) \;\&\; \varphi \in \texttt{Isomorphism}\Big(\mathsf{TOP}, (X, \mathcal{T}), Y\Big)$

Proof $=$

First enrich topology of $Y$, so $\varphi$ is open.

Sencondly, enrich $X$ so it is continuous.

As $\varphi$ is bijection it will be an homeomorphism.

$\square$

$\texttt{BorelMeasurableSequenceTopologization} :: \forall X : \texttt{Polish} \;.\; \forall Y : \texttt{SecondCountableSpace} \;.$

$\;.\; \forall \varphi : \mathbb{N} \to \mathsf{BOR}(X, Y) \;.\; \exists \mathcal{T} : \texttt{PolishTopology}(X) \;.\; \forall [0] : \mathcal{T}(X) \subset \mathcal{T}$

$\& \;\; \& \; \mathcal{B}(\mathcal{T}) = \mathcal{B}\Big(\mathcal{T}(X)\Big) \;\&\; \forall n \in \mathbb{N} \;.\; \varphi \in \mathsf{TOP}\Big((X, \mathcal{T}), Y\Big)$

Proof $=$

$\dots$

$\square$

### 2.3.3 Analytic Sets

$$\texttt{Analytic} :: \prod X : \texttt{Polish} \,.\, ??X$$

$$A : \texttt{Analytic} \iff A \in \Sigma^1_1(X) \iff \exists Y : \texttt{Polish} \,.\, \exists \varphi \in \mathsf{TOP}(X, Y) \,.\, \varphi(Y) = A$$

$$\texttt{BaireUniversalClass} :: \prod T : \prod X : \texttt{Polish} \,.\, ??X \,.\, \prod X : \texttt{Polish} \,.\, ??(\mathcal{B} \times X)$$

$$A : \texttt{BaireUniversalClass} \iff T(\mathcal{B} \times X, A) \,\&\, T(X) = \left\{\sigma_b(A) \middle| b \in \mathcal{B}\right\}$$

$$\texttt{SouslinsCorrection} :: \forall X : \texttt{Polish} \,.\, \forall [0] : |X| > \aleph_0 \,.\, \mathcal{B}(X) \subsetneq \Sigma^1_1(X)$$

$\texttt{Proof} =$

$A := \texttt{LusinSchemaExists}(\mathcal{B}) : \sum A : \mathbb{N}^* \to \mathcal{T}(\mathcal{B})\,.$

$\quad .\, A_\emptyset = \mathcal{B} \,\&$

$\quad \& \,\forall w \in \mathbb{N}^* \,.\, A_w = \bigcup_{n \in \mathbb{N}} A_{wn} \,\&$

$\quad \& \,\forall b \in \mathcal{B} \,.\, \left(\forall n \in \mathbb{N} \,.\, A_{b_{|[1,\dots,n]}} \neq \emptyset\right) \Rightarrow \exists L \in \mathcal{B} \,.\, \{L\} = \bigcap_{n=1}^{\infty} A_{b_{|[1,\dots,n]}} \,\&\, \forall x \in \prod_{n=1}^{\infty} A_{b_{|[1,\dots,n]}} \,.\, \lim_{n\to\infty} x_n = L,$

$w := \texttt{enumerate}(\mathbb{N}^*) : \texttt{Surjective}(\mathbb{N}, \mathbb{N}^*),$

$\mathcal{N} := \left\{(b, x) \in \mathcal{B} \times \mathcal{B} : x \in \bigcup \{A_{w_i} | i \in \mathbb{N} : b_i = 0\}\right\} :?(\mathcal{B} \times \mathcal{B}),$

$\texttt{Assume}\ (b, x) \in \mathcal{N},$

$[1] := \mathrm{E}\mathcal{N}(b, x) : x \in \bigcup \{A_{w_i} | i \in \mathbb{N} : b_i = 0\},$

$\left(i, [2]\right) := \texttt{Eunion}[1] : \sum_{i=1}^{\infty} x \in A_{w_i} \,\&\, b_i = 0,$

$U := \prod_{j=1}^{i-1} \mathbb{N} \times \{0\} \times \prod_{j=i+1}^{\infty} \mathbb{N} : \mathcal{T}(\mathcal{B}),$

$V := U \times A_{w_i} : \mathcal{T}(\mathcal{B} \times X),$

$[*.3] := \mathrm{EVE}\mathcal{N}[2] : (b, x) \in V \subset \mathcal{N};$

$\leadsto [1] := \texttt{OpenByCover}(\mathcal{B} \times \mathcal{B}) : \mathcal{N} \in \mathcal{T}(\mathcal{B} \times \mathcal{B}),$

$[2] := \mathrm{E}\mathcal{N} : \texttt{BaireUniversalClass}(\mathcal{T}, \mathcal{B}, \mathcal{N}),$

$[3] := \texttt{BaireSquerHomeomorphism}[2] : \exists\texttt{BaireUniversalClass}(\mathcal{T}, \mathcal{B}^2),$

$\left(\mathcal{F}, [4]\right) := [3]^{\complement} : \exists\texttt{BaireUniversalClass}(\Pi^0_1, \mathcal{B}^2),$

$\mathcal{A} := \{(x, y) \in \mathcal{B}^2 : \exists z \in \mathcal{B} \,.\, (x, y, z) \in \mathcal{F}\} :?\mathcal{B}^2,$

$[5] := \mathrm{I}\Sigma^1_1\mathrm{E}\mathsf{TOP}(\mathcal{B}^2, \mathcal{B}, \pi) : \mathcal{A} \in \Sigma^1_1, \mathcal{B}^2),$

$[6] := \mathrm{I}\Sigma^1_1\mathrm{E}\mathsf{TOP}(\mathcal{B}^2, \mathcal{B}, \pi) : \forall b \in \mathcal{B} \,.\, \sigma_{1,b}(\mathcal{A}) \in \Sigma^1_1(\mathcal{B}),$

$\texttt{Assume}\ A \in \Sigma^1_1(\mathcal{B}),$

$\left(F, \varphi, [7]\right) := \mathrm{E}\mathcal{S}^1_1(\mathcal{B})\texttt{BaireSpaceEmbedding} : \sum F : \texttt{Closed}(\varphi) \,.\, \sum \varphi : \mathsf{TOP} \,\&\, \texttt{Surjective}(F, A),$

$G := \texttt{swap}\ G(\varphi) :?\mathcal{B}^2,$

$[8] := \texttt{ClosedGraphTHM}(\mathcal{B}, \mathcal{B}, G) : \texttt{Closed}(\mathcal{B} \times \mathcal{B}, G),$

$\left(x, [9]\right) := \mathrm{E}_2\texttt{BaireUniversalClass}(\Pi^0_1, \mathcal{B}^2, \mathcal{F}, G) : \sum x \in \mathcal{B} \,.\, \sigma_{1,x}(\mathcal{F}) = G,$

$[A.*] := \mathrm{E}\mathcal{A}\mathrm{E}G[9] : A = \sigma_{1,x}(A);$

$\leadsto [7] := \texttt{IBaireUniversalClass} : \texttt{BaireUniversalClass}\left(\Sigma^1_1, \mathcal{B}, \mathcal{A}\right),$

$\texttt{Assume}\ [8] : \mathcal{A} \in \mathcal{B}(\mathcal{B}^2)$,

$[9] := \texttt{EAlgebra}[8] : \mathcal{A}^{\complement} \in \mathcal{B}(\mathcal{B})$,

$A := \{x \in \mathcal{B} : (x,x) \notin \mathcal{A}\} :??X$,

$[10] := \texttt{E}\mathcal{A}[9] : A \in \mathcal{B}(\mathcal{B})$,

$[11] := \texttt{E}\Sigma^1_1[10] : A \in \Sigma^1_1(\mathcal{B})$,

$\Big(x, [12]\Big) := \texttt{EBaireUniversalClass}\Big(\Sigma^1_1, \mathcal{B}, \mathcal{A}, x\Big) : \sum x \in \mathcal{B} \,.\, A = \sigma_{1,x}(\mathcal{A})$,

$[13] := \texttt{E}\mathcal{A}\texttt{E}A[12] : (x,x) \in A \iff (x,x) \notin A$,

$[*] := \texttt{LEM}[13] : \bot$;

$\rightsquigarrow [8] := \texttt{E}\bot : \mathcal{A} \notin \mathcal{B}(\mathcal{B}^2)$,

$[*] := \texttt{BaireUniversalProperty}[8] : \forall X : \texttt{Polish} \,.\, \forall [0] : |X| > \aleph_0 \,.\, \mathcal{B}(X) \subsetneq \Sigma^1_1(X)$;

□

$\texttt{AnalyticUnion} :: \forall X : \texttt{Polish} \,.\, \forall A : \mathbb{N} \to \sigma^1_1(X) \,.\, \bigcup_{n=1}^{\infty} A_n \in \sigma^1_1(X)$

$\texttt{Proof}\ =$

There are Polish spaces $(Y_n)_{n=1}^{\infty}$ and continuous maps $\phi_n : Y_n \to X$ such that $A_n = \phi_n(Y_n)$ .

To get union as an image just use disjoint union $\bigsqcup_{n=1} Y_n$ .

□

$\texttt{AnalyticIntersection} :: \forall X : \texttt{Polish} \,.\, \forall A : \mathbb{N} \to \sigma^1_1(X) \,.\, \bigcap_{n=1}^{\infty} A_n \in \sigma^1_1(X)$

$\texttt{Proof}\ =$

There are Polish spaces $(Y_n)_{n=1}^{\infty}$ and continuous maps $\phi_n : Y_n \to X$ such that $A_n = \phi_n(Y_n)$ .

Construct a pushout $Z = \left\{ y \in \prod_{n=1}^{\infty} Y_n \middle| \forall n,m \in \mathbb{N} \,.\, \phi_n(y_n) = \phi_m(y_m) \right\}$ .

Then the limit of $\phi$ will have intersection as its image.

□

$\texttt{AnalyticImage} :: \forall X, Y : \texttt{Polish} \,.\, \forall \varphi \in \mathsf{BOR}(X,Y) \,.\, \forall A \in \Sigma^1_1(X) \,.\, f(A) \in \Sigma^1_1(Y)$

$\texttt{Proof}\ =$

...

□

$\texttt{AnalyticPreimage} :: \forall X, Y : \texttt{Polish} \,.\, \forall \varphi \in \mathsf{BOR}(X,Y) \,.\, \forall A \in \Sigma^1_1(Y) \,.\, f^{-1}(A) \in \Sigma^1_1(X)$

$\texttt{Proof}\ =$

...

□

$\texttt{BorelAnalyticSet} :: \prod X : \texttt{StandardBorelSpace} \,.\, ??X$

$A : \texttt{BorelAnalyticSet} \iff A \in \Sigma^1_1(X) \iff \exists Y : \texttt{Polish} \,.\, \exists X \overset{\varphi}{\leftrightarrow} Y : \mathsf{BOR} \,.\, \varphi(A) \in \Sigma^1_1(Y)$

### 2.3.4 Lusin Separation Theorem

$\texttt{BorelSeparated} :: \prod X \in \mathsf{BOR} \;.\; ?\texttt{DisjointPair}(X)$

$(A, B) : \texttt{BorelSeparated} \iff \exists S \in \mathcal{S}_X \;.\; A \subset S \;\&\; S \cap B = \emptyset$

$\texttt{BorelSeparetedUnion} ::$

$:: \forall X \in \mathsf{BOR} \;.\; \forall P, Q : \mathbb{N} \to ?X \;.\; \forall [0] : \forall n, m \in \mathbb{N} \;.\; \texttt{BorelSeparated}(X, P_n, Q_m) \;.$

$. \;\texttt{BorelSepareted}\left(X, \bigcup_{n=1}^{\infty} P_n, \bigcup_{n=1}^{\infty} Q_n\right)$

$\texttt{Proof} \;=$

Let $B_{n,m}$ be separating sets for a pair $P_n, Q_m$ .

Then $A = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} B_{n,m}$ is Borel.

$\texttt{LusinSeparationTheorem} :: \forall X : \texttt{StandardBorelSpace} \;.\; \forall A, B \in \Sigma^1_1(X) \;.$

$. \;\texttt{DisjointPair}(X, A, B) \Rightarrow \texttt{BorelSepareted}(X, A, B)$

$\texttt{Proof} \;=$

$\Big(\varphi, [1]\Big) := \mathtt{E}\Sigma^1_1(X, A) : \sum \varphi : \mathsf{TOP}(\mathcal{B}, X) \;.\; \varphi(\mathcal{B}) = A,$

$\Big(\psi, [2]\Big) := \mathtt{E}\Sigma^1_1(X, B) : \sum \psi : \mathsf{TOP}(\mathcal{B}, X) \;.\; \psi(\mathcal{B}) = B,$

$\Big(N, [3]\Big) := \texttt{LusinSchemaExists}(\mathcal{B}) : \sum N : \mathbb{N}^* \to \mathcal{T}(\mathcal{B}) \;.$

$. \;N_\emptyset = \mathcal{B} \;\&$

$\& \;\forall w \in \mathbb{N}^* \;.\; N_w = \bigcup_{n \in \mathbb{N}} N_{wn} \;\&$

$\& \;\forall b \in \mathcal{B} \;.\; \Big(\forall n \in \mathbb{N} \;.\; N_{b_{|[1,\dots,n]}} \neq \emptyset\Big) \Rightarrow \exists L \in \mathcal{B} \;.\; \{L\} = \bigcap_{n=1}^{\infty} N_{b_{|[1,\dots,n]}} \;\&\; \forall x \in \prod_{n=1}^{\infty} N_{b_{|[1,\dots,n]}} \;.\; \lim_{n \to \infty} x_n = L,$

$a := \Lambda w \in \mathbb{N}^* \;.\; \varphi(N_w) : \mathbb{N}^* \to ?X,$

$b := \Lambda w \in \mathbb{N}^* \;.\; \psi(N_w) : \mathbb{N}^* \to ?X,$

$[4] := \mathtt{E}a[3] : a_\emptyset = A \;\&\; \forall w \in \mathbb{N}^* \;.\; a_w = \bigcup_{n=1}^{\infty} a_{wn},$

$[5] := \mathtt{E}b[3] : b_\emptyset = B \;\&\; \forall w \in \mathbb{N}^* \;.\; b_w = \bigcup_{n=1}^{\infty} b_{wn},$

$\texttt{Assume}\; [6] : \neg\texttt{BorelSeparated}(X, A, B),$

$\Big(x, y, [7]\Big) := \texttt{BorelSeparatedUnion}[4][5][6] : \sum x, y \in \mathcal{B} \;.\; \forall n \in \mathbb{N} \;.\; \neg\texttt{BorelSeparated}(X, a_{x_{|[1,\dots,n]}}, b_{y_{|[1,\dots,n]}}),$

$[10] := \texttt{EDisjointPair}(X, A, B)(x, y) : \varphi(x) \neq \psi(y),$

$\Big(U, V, [11]\Big) := \mathtt{ET2}(X)[10] : \sum U, V \in \mathcal{T}(X) \;.\; \varphi(x) \in U \;\&\; \psi(y) \in V \;\&\; U \cap V = \emptyset,$

$\Big(n, [12]\Big) := \mathtt{E}\mathsf{TOP}(\mathcal{B}, X, \varphi \;\&\; \psi)[3] : \sum n \in \mathbb{N} \;.\; a_{x_{|[1,\dots,n]}} \subset U \;\&\; b_{y_{|[1,\dots,n]}} \subset V,$

$[6.*] := [12][7](n) : \bot;$

$\rightsquigarrow [*] := \mathtt{E}\bot : \texttt{BorelSeparated}(X, A, B);$

$\square$

LusinSequenceSeparationTheorem ::

$: \forall X : \texttt{StandardBorelSpace} \;.\; \forall A : \texttt{PairwiseDisjoint}\Big(\Sigma^1_1(X)\Big) \;.$

$.\; \exists B : \texttt{PairwiseDisjoint}\Big(\mathcal{B}(X)\Big) \;.\; \forall n \in \mathbb{N} \;.\; A_n \subset B_n$

Proof $=$

Iterate normal Lusin Separation Theorem, using the fact that union of analytic sets is analytic .

$\square$

### 2.3.5 Souslin's Theorem

$\texttt{CoanalyticSet} :: \prod X : \texttt{Polish} . ??X$

$A : \texttt{CoanalyticSet} \iff A \in \Pi^1_1(X) \iff A^\complement \in \Sigma^1_1(X)$

$\texttt{BorelCoanalyticSet} :: \prod X : \texttt{StandardBorelSpace} . ??X$

$A : \texttt{BorelCoanalyticSet} \iff A \in \Pi^1_1(X) \iff A^\complement \in \Sigma^1_1(X)$

$\texttt{BiAnalyticSet} := \Lambda X : \texttt{Polish} . \Delta^1_1(X) = \Lambda X : \texttt{Polish} . \Sigma^1_1(X) \cap \Pi^1_1(X) : \texttt{Polish} \to \texttt{Type};$

$\texttt{BorelBiAnalyticSet} := \Lambda X : \texttt{StandardBorelSpace} . \Delta^1_1(X) =$
$= \Lambda X : \texttt{StandardBorelSpace} . \Sigma^1_1(X) \cap \Pi^1_1(X) : \texttt{Polish} \to \texttt{Type};$

$\texttt{SouslinThm} :: \forall X : \texttt{StandardBorelSpace} . \mathcal{B}(X) = \Delta^1_1(X)$

Proof =

Let $A$ ba a bi-analytic set in $X$ .

Then by Souslin separation theorem there are borel set $B$ which separates $A$ and $A^\complement$.

But, as it were complements, $A = B$ .

□

$\texttt{AnalyticGraphTHM} ::$
$: \forall X, Y : \texttt{StandardBorelSpace} . \forall \phi : X \to Y . G(\phi) \in \Sigma^1_1(X \times Y) \iff \phi \in \mathsf{BOR}(X, Y)$

Proof =

Proof by projections .

□

$\texttt{BorelIsomorphismTrivialityForStadardSpaces} ::$
$:: \forall X, Y : \texttt{StandardBorelSpace} . \forall \varphi : \mathsf{BOR} \;\&\; \texttt{Bijective}(X, Y) . \texttt{Isomorphism}(\mathsf{BOR}, X, Y, \varphi)$

Proof =

The swapped graph is still analytic.

□

$\texttt{PerfectSetTheoremForAnalyticSets} ::$
$:: \forall X : \texttt{StandardBorelSpace} . \forall A \in \Sigma^1_1(X) . |A| > \aleph_0 \Rightarrow |A| = 2^{\aleph_0}$

Proof =

Assume Polish topology on $X$.

There is a Polish space $Z$ and and a continuous map $\phi$ such that $\phi(Z) = A$.

Assuming $A$ is uncountable, construct a cantor schema on $Z$, so .

...

□

### 2.3.6 Injective Images

$\texttt{InjectiveImageTheorem} ::$

$:: \forall X, Y : \texttt{Polish} \,.\, \forall f \in \mathsf{TOP}(X, Y) \,.\, \forall B \in \mathcal{B}(X) \,.\, \forall [0] : \texttt{Injective}(B, Y, f_{|X}) \,.\, f(B) \in \mathcal{B}(Y)$

$\texttt{Proof} \;=$

$[1] := \texttt{LusinSouslinRepresentation}(X, B) : X = \mathcal{B} \;\&\; \texttt{Closed}(X, B),$

$\Big(N, [2]\Big) := \texttt{LusinSchemaExists}(\mathcal{B}) : \sum N : \mathbb{N}^* \to \mathcal{T}(\mathcal{B}) \,.$

$. \; N_\emptyset = \mathcal{B} \;\&$

$$\& \; \forall w \in \mathbb{N}^* \,.\, N_w = \bigcup_{n \in \mathbb{N}} N_{wn} \;\&$$

$$\& \; \forall b \in \mathcal{B} \,.\, \Big(\forall n \in \mathbb{N} \,.\, N_{b_{|[1,\dots,n]}} \neq \emptyset\Big) \Rightarrow \exists L \in \mathcal{B} \,.\, \{L\} = \bigcap_{n=1}^{\infty} N_{b_{|[1,\dots,n]}} \;\&\; \forall x \in \prod_{n=1}^{\infty} N_{b_{|[1,\dots,n]}} \,.\, \lim_{n \to \infty} x_n = L,$$

$A := \Lambda w \in \mathbb{N}^* \,.\, f(N_w \cap B) : \mathbb{N}^* \to \Sigma^1_1(Y),$

$[3] := \mathtt{E}A[2.1] : A_\emptyset = f(B),$

$$[4] := \mathtt{E}A[2.2] : \forall w \in \mathbb{N}^* \,.\, \forall n \in \mathbb{N} \,.\, A_w = \bigcup_{n=1}^{\infty} A_{wn},$$

$\Big(B', [6]\Big) := \texttt{LusinSequenceSeparatiomTHM} : \sum B' : \texttt{BorelLusinSchema}(Y, Y) \,.\, \forall w \in \mathbb{N}^* \,.\, A_w \subset B'_w,$

$B^* := \texttt{lengthRec1}(Y, \Lambda n \in \mathbb{N} \,.\, B'_n \cap \overline{A_n}, \Lambda w \in \mathbb{N}^* \,.\, \Lambda \beta \in \mathcal{B}(Y) \,.\, B'_w \cap \beta \cap \overline{A_w}) : \mathbb{N}^* \to \mathcal{B}(Y),$

$[7] := \mathtt{E}B^*\texttt{IBorelLusinSchema} : \texttt{BorelLusinSchema}(Y, Y, B^*),$

$[8] := \texttt{LengthInduction}\Big([3]\mathtt{E}\mathcal{B}', [6][4]\mathtt{E}\mathcal{B}'\Big) : \forall w \in \mathbb{N}^* \,.\, A_w \subset B^*_w \subset \overline{A_w},$

$$C := \bigcap_{n=1}^{\infty} \bigcup_{w \in \mathbb{N}^n} B^*_w : \mathcal{B}(Y),$$

$\texttt{Assume}\; y \in f(B),$

$\Big(b, [9]\Big) := \texttt{EImage}(X, Y, f, B, y) : \sum b \in \mathcal{B} \,.\, f(b) = y,$

$$[10] := \mathtt{E}A[9] : y \in \bigcap_{n=1}^{\infty} A_{b_{|[1,\dots,n]}},$$

$$[11] := [10][9] : y \in \bigcap_{n=1}^{\infty} B^*_{b_{|[1,\dots,n]}},$$

$[y.*] := \mathtt{E}C[11] : y \in C;$

$\rightsquigarrow [9] := \mathtt{I} \subset : f(B) \subset C,$

$\texttt{Assume}\; y \in C,$

$$\Big(b, [10]\Big) := \mathtt{E}C(y)[2.2][2.3] : \sum b \in \mathcal{B} \,.\, y \in \bigcap_{n=1}^{\infty} B^*_{b_{|[1,\dots,n]}},$$

$$[11] := \mathtt{E}C(y)[2.2] : \sum b \in \mathcal{B} \,.\, y \in \bigcap_{n=1}^{\infty} \overline{A_{b_{|[1,\dots,n]}}},$$

$[12] := [11]\texttt{EclosureI}\exists : \forall n \in \mathbb{N} \,.\, \exists A_{b_{|[1,\dots,n]}},$

$[13] := [12]\mathtt{E}A_{b_{|[1,\dots,n]}} : \forall n \in \mathbb{N} \,.\, \exists B \cap N_{b_{|[1,\dots,n]}},$

$[14] := \texttt{EClosed}(\mathcal{B}, A)[13] : b \in B,$

$$[15] := \mathtt{I}A[14] : f(b) \in \bigcap_{n=1}^{\infty} A_{b_{|[1,\dots,n]}},$$

$[16] := [11][15][2.3][0]\mathtt{E}\mathsf{TOP}(X, Y, f) : f(b) = y;$

$\rightsquigarrow [10] := \texttt{ISetEq}[9] : f(B) = C,$
$[*] := [11]\mathtt{E}C : f(B) \in \mathcal{B}(X);$
$\square$

**BorelInjectiveImageTheorem** ::

$:: \forall X, Y : \texttt{StandardBorelSpace} \ . \ \forall f \in \mathsf{BOR}(X,Y) \ . \ \forall B \in \mathcal{S}_X \ . \ \forall [0] : \texttt{Injective}(B, Y, f_{|B}) \ .$
$f(B) \in \mathcal{B}(Y) \ \& \ B \overset{f_{|B}}{\longleftrightarrow} f(B) : \mathsf{BOR}$

Proof =

. . .

$\square$

**BorelSetsInjectiveChar** ::

$: \forall X : \texttt{Polish} \ . \ \mathcal{B}(X) = \Big\{ f(A) | A : \texttt{Closed}(\mathcal{B}), f \in \mathsf{TOP} \ \& \ \texttt{Injective}(A, X) \Big\}$

Proof =

By previous theorem all such sets are Borel.
On the other hand if $B \in \mathcal{B}(X)$ ther exist an enriched topology on $X$ with $B$ closed.
In this topology $B$ is itself Polish.
By Baire space universal property an embedding of $B$ as a closed set into $\mathcal{B}$.
By taking inverse of this embedding abd combining it with continuous id we get an injective image.
$\square$

**BorelEquivalence** ::

$: \forall X \in \mathsf{SET} \ . \ \forall \mathcal{T}, \mathcal{T}' : \texttt{PolishTopology}(X) \ . \ \mathcal{T} \subset \mathcal{B}(X, \mathcal{T}') \Rightarrow \mathcal{B}(X, \mathcal{T}) = \mathcal{B}(X, \mathcal{T}')$

Proof =

As Borel sets are a minimal sigma-algebra, $\mathcal{B}(X, \mathcal{T}) \subset \mathcal{B}(X, \mathcal{T}')$.
So assume without loss of generality that $\mathcal{T} \subset \mathcal{T}'$ .
Then id is continuous as a mapping from $(X, \mathcal{T}')$ to $(X, \mathcal{T})$.
If $B$ is Borel in $(X, \mathcal{T}')$ then $\mathcal{T}'$ can be furtherly enriched to make $B$ closed.
But by injective image theorem this means that $B$ is Borel in $(X, \mathcal{T})$ .
$\square$

**MeasurableStructureEquivalence** ::

$: \forall X : \texttt{StandardBorelSpace} \ . \ \forall \mathcal{E} \subset \mathcal{S}_X \ . \ |\mathcal{E}| \le \aleph_0 \ \& \ \texttt{SeparatesPoints}(X) \Rightarrow \sigma(\mathcal{E}) = \mathcal{S}_X$

Proof =

Assume that $X$ is Polish.
Generate another topology from $\mathcal{E}$.
Then process as in the previous theorem.
$\square$

### 2.3.7 Isomorphism Theorem

BorelSchroderBernsteinTheorem ::

$\forall X, Y : \mathtt{StandardBorelSpace} \;.\; \forall f : \mathsf{BOR} \;\&\; \mathtt{Injective}(X, Y) \;.\; \forall g : \mathsf{BOR} \;\&\; \mathtt{Injective}(Y, X) \;.$

$.\; \exists A \in \mathsf{BOR}(X) \;.\; \exists B \in \mathsf{BOR}(Y) \;.\; f(A) = Y \setminus B \;\&\; g(B) = X \setminus A$

Proof $=$

$$A := \mathtt{rec}\Big(X, \Lambda A \subset X \;.\; fg(A)\Big) : \mathbb{N} \to \mathcal{B}(X),$$

$$B := \mathtt{rec}\Big(Y, \Lambda B \subset Y \;.\; gf(B)\Big) : \mathbb{N} \to \mathcal{B}(Y),$$

$$A' := \bigcap_{n=1}^{\infty} A_n : \mathcal{B}(X),$$

$$B' := \bigcap_{n=1}^{\infty} B_n : \mathcal{B}(Y),$$

$$[1] := \mathtt{E}A'\mathtt{E}B' : f(A') = B',$$

$$[2] := \mathtt{E}A\mathtt{E}B\mathtt{EInjective}(X, Y, f) : \forall n \in \mathbb{N} \;.\; f\Big(A_n \setminus g(B_n)\Big) = f(A_n) \setminus B_{n+1},$$

$$[3] := \mathtt{E}B\mathtt{E}A\mathtt{EInjective}(Y, X, g) : \forall n \in \mathbb{N} \;.\; g\Big(B_n \setminus f(A_n)\Big) = g(B_n) \setminus A_{n+1},$$

$$Q := A' \cup \bigcup_{n=1} A_n \setminus g(B_n) : \mathcal{B}(X),$$

$$E := \bigcup_{n=1}^{\infty} B_n \setminus f(A_n) : \mathcal{B}(X),$$

$$[*.1] := \mathtt{E}Q[1][3]\mathtt{E}B'\mathtt{I}E :$$

$$: f(Q) = f\left(A' \cup \bigcup_{n=1} (A_n \setminus g(B_n))\right) = B' \cup \bigcup_{n=1} f\Big(A_n \setminus g(B_n)\Big) = B' \cup \bigcup_{n=1} f(A_n) \setminus B_{n+1} =$$

$$= Y \setminus E,$$

$$[*.2] := \mathtt{E}E[2][4]\mathtt{I}Q :$$

$$: g(E) = g\left(\bigcup_{n=1}^{\infty} B_n \setminus f(A_n)\right) = \bigcup_{n=1} g\Big(B_n \setminus f(A_n)\Big) = \bigcup_{n=1} g(B_n) \setminus A_{n+1} = X \setminus Q;$$

□

BorelSchroderBernsteinIsomorphism ::

$\forall X, Y : \mathtt{StandardBorelSpace} \;.\; \forall f : \mathsf{BOR} \;\&\; \mathtt{Injective}(X, Y) \;.\; \forall g : \mathsf{BOR} \;\&\; \mathtt{Injective}(Y, X) \;.$

$.\; X \cong_{\mathsf{BOR}} Y$

Proof $=$

Let $Q$ and $E$ be as in previous theorem.

Then, represent $X = g(E) \sqcup Q$ and $Y = f(Q) \sqcup E$.

But $g(E)$ is Borel isomorphic with $E$.

And $f(Q)$ is Borel isomorphic with $Q$.

So $X$ and $Y$ are indeed Borel isomorphic.

□

IsomorphismTHM :: $\forall X, Y$ : StandardBorelSpace . $|X| = |Y| \Rightarrow X \cong_{\mathsf{BOR}} Y$

Proof =

By universal property of Hilbert cube $X$ can be embedded into $I^{\mathbb{N}}$.

But $\mathcal{C}$ is Borel Isomorphic to $I^{\mathbb{N}}$.

So, $\mathcal{C}$ can be embedded into $X$ and $X$ into $\mathcal{C}$ again.

Then, use Borel-Schroder-Bernstein theorem so $X \cong_{\mathsf{BOR}} \mathcal{C}$ .

Thus, all uncountable standard Borel spaces are isomorphic.

For countable spaces the arguments are more all less trivial.

□

UncountableIsomorphismTHM :: $\forall X, Y$ : StandardBorelSpace . $|X| > \aleph_0$ & $|Y| > \aleph_0 \Rightarrow X \cong_{\mathsf{BOR}} Y$

Proof =

See previous result.

□

DoubleBorelIsomorphismTHM :: $\forall X, Y$ : StandardBorelSpace . $\forall A \in \mathcal{B}(X)$ . $\forall B \in \mathcal{B}(Y)$ .

$. \ |A| = |B| \ \& \ \left|A^{\complement}\right| = \left|B^{\complement}\right| \iff \exists X \overset{\varphi}{\leftrightarrow} B : \mathsf{BOR} \ . \ \varphi(A) = B$

Proof =

Without loss of generality we can choose polish topologies such that both $A$ and $B$ is clopen .

Then $X = A \sqcup A^{\complement}$ and $Y = B \sqcup B^{\complement}$ as topological and measurable spaces.

By isomorphism theorem there are Borel isomorphism $A \overset{\psi}{\leftrightarrow} B$ and $A^{\complement} \overset{\psi'}{\leftrightarrow} B^{\complement}$.

Then $\varphi = \psi \sqcup \psi'$ is an isomorphism with required property.

To see the contrary it is always possible to limit $\varphi$ on $A$.

□

### 2.3.8 Induced Homomorphism

$$\texttt{InducedHomomorphism} :: \prod X, Y \in \mathsf{BOR} \,.\, \prod I : \sigma\texttt{-Ideal}(\mathcal{S}_X) \,.\, \prod Y \xrightarrow{\Phi} \frac{X}{I} : \mathsf{BOOL} \,.\, ?\mathsf{BOR}(X, Y)$$

$$\varphi : \texttt{InducedHomomorphism} \iff \forall B \in \mathcal{B}(X) \,.\, \Phi(B) = \left[\varphi^{-1}(B)\right]$$

$\texttt{SikorskiInducedHomomorphismTheorem}$ ::

:: $\forall X \in \mathsf{BOR} \,.\, \forall Y : \texttt{StandardBorelSpace} \,.\, \forall [0] : \exists Y \,.\, \forall I : \sigma\texttt{-Ideal}(\mathcal{S}_X) \,.$

$. \forall \Phi : \sigma\texttt{-Continuous}\left(\mathcal{B}(Y), \frac{\mathcal{S}_X}{I}\right) \,.\, \exists \texttt{InducedHomomorphism}(X, Y, I, \Phi)$

$\texttt{Proof}$ =

By Isomorphism Theorem assume $Y = [0, 1]$ .

$\Big(B, [1]\Big) := \texttt{Choice}\Big(\mathbb{Q} \cap [0, 1], \Lambda p \in \mathbb{Q} \cap [0, 1] \,.\,\Big) : \sum \mathbb{Q} \cap [0, 1] \to \mathcal{B}(Y) \,.\, \forall p \in \mathbb{Q} \cap [0, 1] \,.\, \Phi[0, p] = \Big[B_p\Big]_I$ &

& $B_1 = X$,

$\varphi := \Lambda x \in X \,.\, \inf\Big\{p \in \mathbb{Q} \cap [0, 1] \Big| x \in B_p\Big\} : X \to [0, 1]$,

$[2] := [1]\texttt{E}\varphi : \forall t \in [0, 1] \,.\, \varphi^{-1}[0, t] = \bigcup_{p<t} B_p$,

$[3] := \texttt{MeasurableByBase}[2] : \varphi \in \mathsf{BOR}\Big(X, [0, 1]\Big)$,

$\Phi' := \Lambda A \in \mathcal{B}(Y) \,.\, [\varphi^{-1}A]_I \in \sigma\texttt{-Continuous}\left(\mathcal{B}(Y), \frac{\mathcal{S}_X}{I}\right)$,

$[4] := \texttt{E}\Phi'[2] : \forall p \in \mathbb{Q} \cap [0, 1] \,.\, \Phi[0, p] = \Phi'[0, p]$,

$[*] := \texttt{E}\sigma\texttt{-Continuous}\left(\mathcal{B}(X), \frac{\mathcal{S}_X}{I}\right) : \Phi = \Phi'$;

$\square$

$\texttt{SikorskiInducedHomomorphismUniqueness}$ ::

:: $\forall X \in \mathsf{BOR} \,.\, \forall Y : \texttt{StandardBorelSpace} \,.\, \forall [0] : \exists Y \,.\, \forall I : \sigma\texttt{-Ideal}(\mathcal{S}_X) \,.$

$. \forall \Phi : \sigma\texttt{-Continuous}\left(\mathcal{B}(Y), \frac{\mathcal{S}_X}{I}\right) \,.\, \forall \varphi, \psi : \texttt{InducedHomomorphism}(X, Y, I, \Phi) \,.\, \Big\{x \in X : \varphi(x) \neq \psi(x)\Big\} \in I$

$\texttt{Proof}$ =

Again assume $Y = [0, 1]$ .

$\texttt{Assume}$ $[1] : \Big\{x \in X : \varphi(x) < \psi(x)\Big\} \notin I$,

$\Big(q, [2]\Big) := \texttt{RationalDensity} : \sum q \in \mathbb{Q} \,.\, \Big\{x \in X : \varphi(x) \leq q < \psi(x)\Big\} \notin I$,

$[3] := [2]\texttt{IpreimageI}\setminus : \varphi^{-1}[0, q] \setminus \psi^{-1}[0, q] \notin I$,

$[4] := \texttt{EInducedHomomorphism}(X, Y, I, \Phi, \varphi \,\&\, \Psi) : \Big[\varphi^{-1}[0, q]\Big]_I = \Phi[0, 1] = \Big[\psi^{-1}[0, q]\Big]_I$,

$[1.*] := \texttt{E}[\bullet]_I[4][3] : \perp$;

Same reasoning works for the case $\psi(x) < \varphi(x)$.

$\rightsquigarrow [*] := \texttt{I}I : \Big\{x \in X : \varphi(x) \neq \psi(x)\Big\} \in I$;

$\square$

DoubleSikorskiInducedIsomorphisTheorem ::

$$:: \forall X, Y : \texttt{StandardBorelSpace} \,.\, \forall I : \sigma\texttt{-Ideal}\big(\mathcal{B}(X)\big) \,.\, \forall J : \sigma\texttt{-Ideal}\big(\mathcal{B}(Y)\big) \,.$$

$$.\, \forall \Phi : \sigma\texttt{-Continuous}\left(\frac{\mathcal{B}(X)}{I}, \frac{\mathcal{B}(Y)}{J}\right) \,.\, \frac{\mathcal{B}(X)}{I} \overset{\Phi}{\leftrightarrow} \frac{\mathcal{B}(Y)}{J} : \mathsf{BOOL} \iff \exists A \in \mathcal{B}(X) \,.\, \exists B \in \mathcal{B}(Y) \,.$$

$$.\, \exists! \varphi \in \mathsf{BOR}(B, A) \,.\, A^{\complement} \in I \;\&\; B^{\complement} \in J \;\&\; \forall [C]_I \in \frac{X}{I} \,.\, \Phi[C]_I = \left[\varphi^{-1}(C \cap A)\right]_J$$

Proof $=$

Apply Sikorski theorem two times in both directions, then combine.

□

SikorskiInducedAutomorphisTheorem ::

$$:: \forall X, : \texttt{StandardBorelSpace} \,.\, \forall I : \sigma\texttt{-Ideal}\big(\mathcal{B}(X)\big) \,.\, \forall \Phi : \sigma\texttt{-Continuous}\left(\frac{\mathcal{B}(X)}{I}, \frac{\mathcal{B}(X)}{I}\right) \,.$$

$$.\, \Phi \in \mathrm{Aut}_{\mathsf{BOOL}}\left(\frac{\mathcal{B}(X)}{I}\right) \,.\, \iff \exists! \varphi \in \mathrm{Aut}_{\mathsf{BOR}}(X) \,.\, \forall [C]_I \in \frac{X}{I} \,.\, \Phi[C]_I = \left[\varphi^{-1}(C \cap A)\right]_I$$

Proof $=$

Apply Sikorski isomorphism theorem to automorphism.

It must be possible to choose $A, B = X$ as $|X| = |X|$.

□

CategoryAlgebraBorelExpression :: $\forall X : \texttt{Polish} \,.\, \mathbf{CAT}(X) = \dfrac{\mathcal{B}(X)}{\mathcal{B}(X) \cap \mathbf{MGR}(X)}$

Proof $=$

...

□

CategoryAlgebraInducedHomo ::

$$\forall X : \texttt{Perfect} \;\&\; \texttt{Polish} \,.\, \forall \Phi \in \mathrm{Aut}_{\mathsf{BOOL}}\Big(\mathbf{CAT}(X)\Big) \,.\, \exists A \in G_\delta(X) \,.\, \exists! \varphi \in \mathrm{End}_{\mathsf{TOP}}(A) \,.$$

$$\forall [B] \in \mathbf{CAT}(X) \,.\, \Phi[B] = \left[\varphi^{-1}(A \cap B)\right]$$

Proof $=$

...

□

### 2.3.9 Definability of Baire Sets

NovikovMontgomeryNonmeagerTHM ::

$: \forall X \in \mathsf{BOR} \,.\, \forall Y : \mathtt{Polish} \,.\, \forall A \in \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \forall U \in \mathcal{T}(X) \,.$

$. \{x \in X : \exists^* u \in U \,.\, A(x,u)\} \in \mathcal{S}_X$

Proof $=$

Assume $[1] : \exists U,$

$\Big(\mathcal{V}, [2]\Big) := \mathtt{ESecondCountable}(Y) : \sum \mathcal{V} : \mathtt{BaseOfTopology}(X) \,.\, |\mathcal{V}| \le \aleph_0,$

$V := \mathtt{enumerate}\Big(\mathcal{V}, [2]\Big) : \mathtt{Surjective}(\mathbb{N}, \mathcal{V}),$

$\mathcal{A} := \Big\{A \in \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \forall U \in \mathcal{T}(Y) \,.\, \{x \in X : \exists^* u \in U \,.\, A(x,u)\} \in \mathcal{S}_X\Big\} :?\Big(\mathcal{S}_X \otimes \mathcal{B}(Y)\Big),$

$C := \Lambda A \in \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \Lambda U \in \mathcal{T}(Y) \,.\, \{x \in X : \exists^* u \in U \,.\, A(x,u)\} : \mathcal{S}_X \otimes \mathcal{B}(Y) \to \mathcal{T}(Y) \to ?X,$

$[3] :=: \forall S \in \mathcal{S}_X \,.\, \forall U, V \in \mathcal{T}(X) \,.\, C\Big(S \times V, U\Big) = \mathtt{if}\ \exists U \times V\ \mathtt{then}\ S\ \mathtt{else}\ \emptyset,$

$[4] := \mathtt{E}\mathcal{A}[3] : \Big\{S \times V | S \in \mathcal{S}_X, V \in \mathcal{T}(Y)\Big\} \subset \mathcal{A},$

$[5] := \mathtt{E}C\mathtt{NonmeagerUnion}(Y) : \forall A : \mathbb{N} \to \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \forall U \in \mathcal{T}(Y) \,.\, C\left(\bigcup_{n=1}^{\infty} A_n, U\right) = \bigcup_{n=1}^{\infty} C(A_n, U),$

$[6] := [5]\mathtt{E}\sigma\text{-}\mathtt{Algebra}(X, \mathcal{S}_X)\mathtt{E}\mathcal{A} : \forall A : \mathbb{N} \to \mathcal{A} \,.\, \bigcup_{n=1}^{\infty} A_n \in \mathcal{A},$

Assume $A \in \mathcal{A},$

Assume $U \in \mathcal{T}(X),$

Assume $x \in X,$

$[7] := \mathtt{E}\Big(x \in C(A^{\complement}, U)\Big)\mathtt{CategoryDeMorganaLaw}(Y)\mathtt{E}\mathbf{BP}\Big(Y, \sigma_{1,x}(S)\Big) :$

$: x \in C(A^{\complement}, U) \iff \exists^* u \in U \,.\, A^{\complement}(x,u) \iff \neg\forall^* u \in U \,.\, A(x,u) \iff \neg\forall n \in \mathbb{N} \,.\, \exists^* v \in V_n \,.\, A(x,v),$

$[A.*] := \mathtt{E}\mathcal{A}(A)[7] : A^{\complement} \in \mathcal{A};$

$\rightsquigarrow [7] := \mathtt{I}\sigma\text{-}\mathtt{Algebra}[6] : \sigma\text{-}\mathtt{Algebra}(X \times Y, \mathcal{A}),$

$[8] := \mathtt{EProduct}[4][7] : \mathcal{S}_X \times \mathcal{B}(Y) \subset \mathcal{A};$

$\square$

NovikovMontgomeryMeagerTHM ::

$: \forall X \in \mathsf{BOR} \,.\, \forall Y : \mathtt{Polish} \,.\, \forall A \in \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \forall U \in \mathcal{T}(X) \,.$

$. \{x \in X : \neg\exists^* u \in U \,.\, A(x,u)\} \in \mathcal{S}_X$

Proof $=$

. . .

$\square$

NovikovMontgomeryComeagerTHM ::

$: \forall X \in \mathsf{BOR} \,.\, \forall Y : \mathtt{Polish} \,.\, \forall A \in \mathcal{S}_X \otimes \mathcal{B}(Y) \,.\, \forall U \in \mathcal{T}(X) \,.$

$. \{x \in X : \forall^* u \in U \,.\, A(x,u)\} \in \mathcal{S}_X$

Proof $=$

$\square$

### 2.4 Uniformization

### 2.5 Partitions

### 2.6 Games

### 2.7 Hierarchi

### 2.8 Applictions

### 2.9 Baire Hierarchi

## 3 Analytic and Projective Sets

# Sources:


1. CLASSICAL DESCRIPTIVE SET THEORY by Alexander S. Kechris Springer Verlag