Topological Groups

Uncultured Tramp

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1 Uniform Spaces

Uniform spaces generalizes many concept of metric topology beyond real numbers.

1.1 Connector Uniformities

One tool defining Uniformities are connectors or entourages.

1.1.1 Connectors

Connectors can be thought as large binary matrices with ones on diagonals.

$$\texttt{Connector} :: \prod_{X \in \mathsf{SET}} ?(X \to ?X)$$

$$U : \texttt{Connector} \iff \forall x \in X \; . \; x \in U(x)$$

$$\texttt{SemimetricConnector} :: \prod X \in \mathsf{SMS} \; . \; \mathbb{R}_{++} \to \texttt{Connector}(X)$$

$$\texttt{SemimetricConnector}\,(\varepsilon) = \mathbb{B}_\varepsilon := \Lambda x \in X \; . \; \mathbb{B}(x, \varepsilon)$$

$$\texttt{connectorAsSubset} :: \prod_{X \in \mathsf{SET}} \texttt{Connector}(X) \to ?X^2$$

$$\texttt{connectorAsSubset}\,(U) = U := \left\{ (x,y) \in X^2 \middle| y \in U(x) \right\}$$

$$\texttt{ConnectorContainsDiagonal} :: \forall X \in \mathsf{SET} \; . \; \forall U : \texttt{Connector}(X) \; . \; \Delta(X) \subset U$$

$$\texttt{Proof} \; =$$

$$\ldots$$

$$\square$$

$$\texttt{SubsetWithDiagonal} :: \prod_{X \in \mathsf{SET}} ??X^2$$

$$V : \texttt{SubsetWithDiagonal} \iff \Delta(X) \subset U$$

$$\texttt{connectorFromDiagonalSubset} :: \prod_{X \in \mathsf{SET}} \texttt{SubsetWithDiagonal}(X) \to \texttt{Connector}(X)$$

$$\texttt{connectorFromDiagonalSubset}\,(V) = V := \Lambda x \in X \; . \; V_x$$

$$\texttt{transpose} :: \prod_{X \in \mathsf{SET}} \texttt{Connector}(X) \to \texttt{Connector}(X)$$

$$\texttt{transpose}\,(U) = U^\top := \left\{ (y,x) \middle| (x,y) \in U \right\}$$

$$\texttt{IdepmpotenTransposition} :: \forall X \in \mathsf{SET} \; . \; \forall U : \texttt{Connector}(X) \; . \; (U^\top)^\top = U$$

$$\texttt{Proof} \; =$$

$$\ldots$$

$$\square$$

$\texttt{ConnectorContainment} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, U \subset V \Rightarrow \forall x \in X \,.\, U(x) \subset V(x)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConnectorTransposeContainment} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, U \subset V \Rightarrow \forall x \in X \,.\, U^\top \subset V^\top$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ConnectorSetAlgebra} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, \texttt{Connector}(X, U \cap V \ \& \ X \cup V)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TransposeIntersection} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, (U \cap V)^\top = U^\top \cap V^\top$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TransposeComposition} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, (U \circ V)^\top = V^\top \circ U^\top$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SemimetricConnectorComposition} :: \forall X \in \mathsf{SMS} \,.\, \forall t, s \in \mathbb{R}_{++} \,.\, \mathbb{B}_t \circ \mathbb{B}_s \subset \mathbb{B}_{t+s}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ (x, z) \in \mathbb{B}_t \circ \mathbb{B}_s,$

$\Big(y, [1]\Big) := \texttt{Ecomposition}(\mathbb{B}_t, \mathbb{B}_s) : \sum_{y \in X} (x, y) \in \mathbb{B}_t \ \& \ (y, z) \in \mathbb{B}_s,$

$[2] := \mathrm{E}\mathbb{B}_t[1.1] : d(x, y) < t,$

$[3] := \mathrm{E}\mathbb{B}_s[1.2] : d(y, z) < s,$

$[4] := \texttt{TriangleIneq}(X)[2][3] : d(x, z) \leq d(x, y) + d(y, z) < t + s,$

$\Big[(x, z).*\Big] := \mathrm{I}\mathbb{B}_{t+s} : (x, z) \in \mathbb{B}_{t+s};$

$\rightsquigarrow [*] := \mathrm{I} \subset : \mathbb{B}_t \circ \mathbb{B}_s \subset \mathbb{B}_{s+t};$

$\square$

$\texttt{CompositionContainment} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, U \subset U \circ V \ \& \ U \subset V \circ U$

$\texttt{Proof} \ =$

$[1] := \texttt{ConnectorContainsDiagonal}(X, V) : \Delta(X) \subset V,$

$[*] := \texttt{Ecomposition}[1] : U \subset U \circ V \ \& \ U \subset V \circ U;$

$\square$

$\texttt{SemimetricConnectorIsMonotonic} :: \forall X \in \mathsf{SMS} \,.\, \mathbb{R}_{++} \xrightarrow{\mathbb{B}} ?X^2 : \mathsf{POSET}$

$\texttt{Proof} \ =$

$\texttt{SemimetricConnectorIntersection} :: \forall X \in \mathsf{SMS} \,.\, \forall t, s \in \mathbb{R}_{++} \,.\, \mathbb{B}_t \cap \mathbb{B}_s = \mathbb{B}_{s \wedge t}$

$\texttt{Proof} \ =$

$[1] := \texttt{TosetMeet}(\mathbb{R}_{++}, t, s) : s \wedge t = t | s \wedge t = s,$

$[2] := \texttt{SemimetricConnectorIsMonotonic}(X)\texttt{NestedIntersection}(X)\texttt{E} =_{\mathbb{R}}:$

$\quad : s \wedge t = t \Rightarrow \mathbb{B}_t \cap \mathbb{B}_s \ = \mathbb{B}_t = \mathbb{B}_{t \wedge s} \ \& \ s \wedge t = s \Rightarrow \mathbb{B}_s \cap \mathbb{B}_t = \mathbb{B}_s = \mathbb{B}_{t \wedge s},$

$[*] := \texttt{E}|[1][2] : \mathbb{B}_s \cap \mathbb{B}_t = \mathbb{B}_{t \wedge s};$

$\square$

$\texttt{ConnectorProductContainment} :: \forall X \in \mathsf{SET} \,.\, \forall U, V : \texttt{Connector}(X) \,.\, \forall x \in X \,.\, U(x) \times V(x) \subset V \circ U^\top$

$\texttt{Proof} \ =$

$\texttt{Assume} \ (a, b) \in U(x) \times V(x),$

$[1] := \texttt{EConnector}(X)\texttt{E}(a, b) : (x, a) \in U \ \& \ (x, b) \in V,$

$\Big[(a, b).*] := \texttt{E} \circ [1] : (a, b) \in V \circ U^\top;$

$\leadsto [*] := \texttt{I} \subset : U(x) \times V(x) \subset V \circ U^\top;$

$\square$

$\texttt{ConnectorProductContainment2} ::$

$\quad :: \forall X \in \mathsf{SET} \,.\, \forall V : \texttt{SymmetricConnector}(X) \,.\, \forall x \in X \,.\, V(x) \times V(x) \subset V \circ V$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.1.2 Uniform Topology

Unifirmities generalize concept of cells or balls from metric topology. They are filters of connectors with nice properties. Also, they define topology similarly to a distance metric.

$$\mathtt{Uniformity} :: \prod_{X \in \mathsf{SET}} ?\mathtt{Filter}(X^2)$$

$$\mathcal{U} : \mathtt{Uniformity} \iff \forall U \in \mathcal{U} \,.\, \mathtt{WithDiagonal}(X, U) \,\&\, U^\top \in \mathcal{U} \,\&\, \exists V \in \mathcal{U} \,.\, V \circ V \subset U$$

$$\mathtt{UniformityBase} :: \prod_{X \in \mathsf{SET}} ?\mathtt{Filterbase}(X^2)$$

$$\mathcal{B} : \mathtt{UniformityBase} \iff \forall U \in \mathcal{B} \,.\, \mathtt{WithDiagonal}(X, U) \,\&\, \exists V \in \mathcal{B} \,.\, V \subset U^\top \,\&\, \exists V \in \mathcal{B} \,.\, V \circ V \subset U$$

$$\mathtt{UniformSpace} := \sum_{X \in \mathsf{SET}} \mathtt{Uniformity}(X) : \mathtt{Type};$$

$$\mathtt{UniformBaseGeneratesUnifomity} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{B} : \mathtt{UniformBase}(X) \,.\, \mathtt{Uniformity}\Big(X, \langle \mathcal{B} \rangle\Big)$$

$\mathtt{Proof}\ =$

$\ldots$

$\square$

$$\mathtt{BasesOfUniformityAreUniformityBases} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{U} : \mathtt{Uniformity}(X) \,.\, \forall \mathcal{B} : \mathtt{FilterBase}(X) \,.$$
$$.\, (\mathcal{B} \subset \mathcal{U} \,\&\, \forall U \in \mathcal{U} \,.\, \exists B \in \mathcal{B} \,.\, B \subset U) \Rightarrow U = \langle B \rangle \,\&\, \mathtt{UniformBase}(X, \mathcal{B})$$

$\mathtt{Proof}\ =$

$$\mathtt{SemimetricFilterbase} :: \forall X \in \mathsf{SMS} \,.\, \exists X \Rightarrow \mathtt{UniformBase}\Big(X, \{\mathbb{B}_\varepsilon | \varepsilon \in \mathbb{R}_{++}\}\Big)$$

$\mathtt{Proof}\ =$

$\mathcal{B} := \{\mathbb{B}_\varepsilon | \varepsilon \in \mathbb{R}_{++}\} : ??X^2,$

$[1] := \mathtt{SemimetricConnectorInersection}(X)\mathtt{I}\mathcal{B} : \forall U, V \in \mathcal{B} \,.\, U \cap V \in \mathcal{B},$

$[2] := \mathtt{E}\mathcal{B}[0]\mathtt{I}\mathcal{B} : \exists \mathcal{B},$

$[3] := \mathtt{IFilterbase}[1][2] : \mathtt{Filtrerbase}(X, \mathcal{B}),$

$[4] := \mathtt{E}\mathcal{B}\mathtt{ESymmetric}(X, \mathbb{R}, d)\mathtt{I}\mathcal{B} : \forall U \in \mathcal{B} \,.\, U^\top \in \mathcal{B},$

$\mathtt{Assume}\ U \in \mathcal{B},$

$$\Big(t, [5]\Big) := \mathtt{E}\mathcal{B}(U) : \sum_{t \in \mathbb{R}_{++}} (U = \mathbb{B}_t),$$

$$[U.*] := \mathtt{SemimetricConnectorComposition}\left(X, \frac{t}{2}, \frac{t}{2}\right) : \mathbb{B}_{\frac{t}{2}} \circ \mathbb{B}_{\frac{t}{2}} \subset \mathbb{B}_t;$$

$\rightsquigarrow [5] := \mathtt{E}\mathcal{B}\mathtt{I}\forall : \forall U \in \mathcal{B} \,.\, \exists V \in \mathcal{B} \,.\, V \circ V \subset U,$

$[*] := \mathtt{IUniformBase}[3]\mathtt{E}\mathcal{B}\mathtt{E}\mathbb{B}\mathtt{I}\mathcal{B}[4][5] : \mathtt{UniformBase}(X, \mathcal{B});$

$\square$

$$\texttt{uniformTopology} :: \prod_{X \in \mathsf{SET}} \texttt{Uniformity}(X) \to \texttt{Topology}(X)$$

$$\texttt{uniformTopology}\,(\mathcal{U}) = \mathcal{T}_{\mathcal{U}} := \left\{ O \subset X \middle| \forall x \in O \,.\, \exists U \in \mathcal{U} \,.\, U(x) \subset O \right\}$$

$\mathcal{T} := \texttt{uniformTopology}(\mathcal{U}) :??X,$

$[1] := \mathtt{E}\mathcal{T}\texttt{ExNihilo}(X)\mathtt{I}\mathcal{T} : \emptyset \in X,$

$[2] := \mathtt{E}_1\texttt{Filter}(X,\mathcal{U}) : \mathcal{U} \neq \emptyset,$

$[3] := \mathtt{E}\mathcal{T}[2]\texttt{UniversumContainsAll}(X) : X \in \mathcal{T},$

$[4] := \mathtt{E}\mathcal{T}[1]\texttt{UnionContainsSubsets}(X) : \forall \mathcal{O} \subset \mathcal{T} \,.\, \bigcup \mathcal{O} \in \mathcal{T},$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ V : \{1, \ldots, n\} \to \mathcal{T},$

$\texttt{Assume}\ x \in \bigcap_{k=1}^{n} V_k,$

$\Big(U, [5]\Big) := \mathtt{E}\mathcal{T}(V) : \sum U : \{1, \ldots, n\} \to \mathcal{U} \,.\, \forall k \in n \,.\, U_k(x) \subset V_k,$

$W := \bigcap_{k=1}^{n} U_k \in \mathcal{U},$

$[n.*] := \mathtt{E}W\texttt{SubsetIntersection}\mathtt{I}W : W(x) \subset \bigcap_{k=1}^{n} V_k;$

$\rightsquigarrow [5] := \mathtt{E}\mathcal{T} : \forall n \in \mathbb{N} \,.\, \forall V : \{1, \ldots, n\} \to \mathcal{T} \,.\, \bigcap_{k=1}^{n} V_k \in \mathcal{T},$

$[*] := \mathtt{I}\texttt{Topology}[1][3][4][5] : \texttt{Topology}(X);$

$\square$

$$\texttt{uniformity} :: \prod (X,\mathcal{U}) : \texttt{UniformSpace} \,.\, \texttt{Uniformity}(X)$$

$$\texttt{Uniformity}\,() = \mathcal{U}_X := \mathcal{U}$$

$$\texttt{UniAsTop} :: \texttt{UniformSpace} \to \mathsf{TOP}$$

$$\texttt{UniAsTop}\,(X) = X := \Big(X, \texttt{uniformTopology}(\mathcal{U}_X)\Big)$$

$$\texttt{Stronger} :: \prod_{X \in \mathsf{SET}} ?\texttt{Uniformity}^2(X)$$

$$(\mathcal{U}, \mathcal{V}) : \texttt{Stronger} \iff \mathcal{U} \geq \mathcal{V} \iff \mathcal{T}_{\mathcal{V}} \subset \mathcal{T}_{\mathcal{U}}$$

$$\texttt{EquivalentUniformities} :: \prod_{X \in \mathsf{SET}} ?\texttt{Uniformity}^2(X)$$

$$(\mathcal{U}, \mathcal{V}) : \texttt{EquivalenUniformities} \iff \mathcal{U} \cong \mathcal{V} \iff \mathcal{T}_{\mathcal{V}} = \mathcal{T}_{\mathcal{U}}$$

$$\texttt{BaseOfUniformity} :: \prod X : \texttt{UniformSpace} \,.\, ?\texttt{UniformBase}(X)$$

$$\mathcal{B} : \texttt{BaseOfUniformity} \iff \langle \mathcal{B} \rangle = \mathcal{U}_X$$

$$\texttt{discreteUniformuty} :: \prod_{X \in \mathsf{SET}} \texttt{Uniformity}(X)$$

$$\texttt{discreteUniformity}\,() := \left\{U \in X^2 : \Delta(X) \subset U\right\}$$

$$\texttt{codiscreteUniformuty} :: \prod_{X \in \mathsf{SET}} \texttt{Uniformity}(X)$$

$$\texttt{codiscreteUniformity}\,() := \{X \times X\}$$

$\texttt{DiscreteUniformityTopology} :: \forall X \in \mathsf{SET} \,.\, \mathcal{T}(X, \texttt{discreteUniformity}(X)) = 2^X$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CodiscreteUniformityTopology} :: \forall X \in \mathsf{SET} \,.\, \mathcal{T}(X, \texttt{codiscreteUniformity}(X)) = \{\emptyset, X\}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$$\texttt{relativeUniformity} :: \prod X : \texttt{UniformSpace} \,.\, 2^X \to \texttt{UniformSpace}$$

$$\texttt{relativeUniformity}\,(A) = A := \left(A, \{U \cap A^2 | U \in \mathcal{U}_X\}\right)$$

1.1.3 Symmetric Connectors

Symmetric connectors are particularly nice. Every uniformity has a base of symmetic connectors.

$$\texttt{SymmetricConnector} :: \prod_{X \in \mathsf{SET}} ?\texttt{Connector}(X)$$

$$U : \texttt{SymmetricConnector} \iff U^\top = U$$

$$\texttt{SymmetricBase} :: \prod_{X \in \mathsf{SET}} ?\texttt{UniformBase}(X)$$

$$\mathcal{B} : \texttt{SymmetricBase} \iff \forall U \in \mathcal{B} \,.\, \texttt{SymmetricConnector}(X, U)$$

$\texttt{SymmetricBaseExists} :: \forall X : \texttt{UniformSpace} \,.\, \exists \mathcal{B} : \texttt{SymmetricBase}(X) \,.\, \mathcal{U}_X = \langle \mathcal{B} \rangle$

$\texttt{Proof} \ =$

$\mathcal{B} := \{U \in \mathcal{U}_X : \texttt{SymmetricConnector}(X, U)\} : ?\mathcal{U}_X,$

$[1] := \texttt{EFilter}(X, \mathcal{U}_X) : \exists \mathcal{U}_X,$

$[2] := \texttt{E}\top\texttt{ISymmetricConnector} : \forall U \in \mathcal{U}_X \,.\, \texttt{SymmetricConnector}\Big(X, U \cap U^\top\Big),$

$[3] := [1][2] : \exists \mathcal{B},$

$[4] := \texttt{E}\mathcal{B}\texttt{ESymmetricConnector}(X)\texttt{E}\cap\texttt{ISymmetricConnector}(X)\texttt{I}\mathcal{B} : \forall U, V \in \mathcal{B} \,.\, U \cap V \in \mathcal{B},$

$[5] := \texttt{IFilterbase}[3][4] : \texttt{Filterbase}(X, \mathcal{B}),$

$[6] := \texttt{E}\mathcal{B}\texttt{ESymmetricConnector}(X)\texttt{I}\mathcal{B} : \forall U \in \mathcal{B} \,.\, U^\top \in \mathcal{B},$

$\texttt{Assume}\ U \in \mathcal{B},$

$\Big(V, [7]\Big) := \texttt{E}_3\texttt{Uniformity}(X, \mathcal{U}, U) : \sum V \in \mathcal{U} \,.\, V \circ V \subset U,$

$[8] := \texttt{CompositionContainment}(X, V, V)[7] : V \subset V \circ V \subset U,$

$[9] := \texttt{ESymmetricConnector}(X, U)[8] : V^\top \subset U,$

$W := V \cap V^\top : \texttt{SymmetricConnector}(X),$

$[10] := \texttt{E}W\texttt{E}V\texttt{I}W : W \in \mathcal{U},$

$[11] := \texttt{E}W[8][9]\texttt{I}W : W \subset U,$

$[U.*] := [11][7]\texttt{E}W\texttt{IntersectionSubset}\texttt{I}W : W \circ W \subset U;$

$\leadsto [7] := \texttt{ISymmetricBase}[5][6] : \texttt{SymmetricBase}(X, \mathcal{B}),$

$[*] := \texttt{E}\mathcal{B}[2]\texttt{GeneratingFilterbase}\texttt{I}\mathcal{B} : \mathcal{U}_X = \langle \mathcal{B} \rangle;$

$\square$

1.1.4 Neighborhoods

Every base of uniformities has a corresponding base of neighborhoods.

$\texttt{uniformityAssociatedBase} :: \prod X : \texttt{UniformSpace} \,.\, \prod_{x \in X} \texttt{BaseOfUniformity}(X) \to \texttt{NeighborhoodBase}(X, x$

$\texttt{uniformityAssociatedBase}\,(\mathcal{B}) = \widetilde{\mathcal{B}}_x := \Big\{ \{y \in X : \exists V \in \mathcal{U}_X : V(y) \subset U(x)\} \Big| U \in \mathcal{B} \Big\}$

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$G := \{y \in X : \exists V \in \mathcal{U}_X : V(y) \subset U(x)\} :?X,$

$[1] := \texttt{SelfSubset}\Big(X, U(x)\Big) : U(x) \subset U(x),$

$[2] := \texttt{E}G[1]\texttt{I}G : x \in G,$

$[3] := \texttt{EConnectorE}G : G \subset U(x),$

$\texttt{Assume}\ g \in G,$

$\Big(V, [3]\Big) := \texttt{E}G(g) : \sum V : \texttt{Connector} \,.\, V(g) \subset U(x),$

$\Big(W, [4]\Big) := \texttt{EUniformity}(X, U) : \sum W \in \mathcal{U}_X \,.\, W \circ W \subset V,$

$\texttt{Assume}\ y \in W(g),$

$\texttt{Assume}\ z \in W(y),$

$[5] := \texttt{EConnector}(W)\texttt{E}y : (g, y) \in W,$

$[6] := \texttt{EConnector}(W)\texttt{E}z : (y, z) \in W,$

$[7] := \texttt{I}(W \circ W)[5][6] : (g, z) \in W \circ W,$

$[z.*] := [7][4][3] : z \in V(g) \subset U(x);$

$\rightsquigarrow [5] := \texttt{I} \subset : W(y) \subset U(x),$

$[y.*] := \texttt{E}G[5] : y \in G;$

$\rightsquigarrow [U.*] := \texttt{EuniformTopology} : G \in \mathcal{T}(X);$

$\rightsquigarrow [1] := \texttt{I}\forall\texttt{I}\exists : \forall U \in \mathcal{B} \,.\, \exists G \in \mathcal{T}(X) \,.\, x \in G \subset U,$

$\texttt{Assume}\ N \in \mathcal{U}(x),$

$\Big(U, [2]\Big) := \texttt{EuniformTopology}(X)\texttt{E}N : \sum U \in \mathcal{U}_X \,.\, x \in U(x) \subset N,$

$\Big(B, [3]\Big) := \texttt{EUniformBase}(\mathcal{U}_X, \mathcal{B}) : \sum B \in \mathcal{B} \,.\, x \in B(x) \subset U(x),$

$\Big(G, [N.*]\Big) := [1][2][3] : \sum G \in \widetilde{\mathcal{B}}_x \,.\, G \subset N;$

$\rightsquigarrow [*] := \texttt{INeighborhoodBase} : \texttt{NeighborhoodBase}(X, \widetilde{\mathcal{B}}_x);$

$\square$

$\texttt{UniformNeighborhood} :: \prod X : \texttt{UniformSpace} \,.\, ?X \to ??X$

$B : \texttt{UniformNeighborhood} \iff \Lambda A \subset X \,.\, \exists U \in \mathcal{U}_X \,.\, U(A) \subset B$

$\texttt{UniformNeighborhoodIsANeighborhood} :: \forall X : \texttt{UniformSpace} . \forall A \subset X .$
$\quad . \forall N : \texttt{UniformNeighborhood}(X, A) . \texttt{Neighborhood}(X, A, N)$

$\texttt{Proof} =$

$\Big(U, [1]\Big) := \texttt{EUniformNeighborhood}(X, A, N) : \sum U \in \mathcal{U}_X . U(A) \subset B,$

$[2] := \mathtt{E}\mathcal{U} : \forall a \in A . \exists O \in \mathcal{U}(a) . O \subset U(a) \subset U(A) \subset B,$

$O := \bigcup_{a \in A} [2](a) : \mathcal{T}(X),$

$[3] := \mathtt{E}O : A \subset O \subset B,$

$[*] := \texttt{INeighborhood}[3] : \texttt{Neighborhood}(X, A, N);$

$\square$

$\texttt{EveryCompactNeighborhoodIsUniform} :: \forall X : \texttt{UniformSpace} . \forall A : \texttt{CompactSubset}(X) .$
$\quad . \forall N : \texttt{Neighborhood}(X, A) . \texttt{UniformNeighborhood}(X, A, N)$

$\texttt{Proof} =$

$\Big(O, [1]\Big) := \texttt{ENeighborhood}(X, A, N) : \sum O \in \mathcal{T}(X) . A \subset O \subset N,$

$\Big(U, [2]\Big) := \texttt{EuniformTopology}[1] : \prod U' : A \to \mathcal{U}_X . \forall a \in A . U'_a(a) \subset O,$

$\Big(U, [22]\Big) := \texttt{EUniformity}[2] : \prod U : A \to \mathcal{U}_X . \forall a \in A . U_a \circ U_a(a) \subset O,$

$\Big(V, [3]\Big) := \mathtt{E}\widetilde{B}[22] : \prod V : \prod_{a \in A} \mathcal{U}(a) . \forall a \in A . V_a \subset U_a(a),$

$[4] := \mathtt{E}V : \texttt{OpenCover}(X, A, \operatorname{Im} V),$

$\Big(n, a, [5]\Big) := \texttt{ECompactSubset}\Big(X, A, [4]\Big) : \sum_{n=1}^{\infty} \sum \{1, \ldots, n\} \to A . \texttt{OpenCover}(X, A, \operatorname{Im} V_a),$

$[6] := [5][3] : \texttt{Cover}\Big(X, A, U_a(a)\Big),$

$W := \bigcap_{k=1}^{n} U_{a_k} \in \mathcal{U}_X,$

$\texttt{Assume } w \in W(A),$

$\Big(b, [7]\Big) := \mathtt{E}w : \sum b \in A . w \in W(b),$

$\Big(k, [8]\Big) := \texttt{ECover}[6](b) : \sum_{k=1}^{n} b \in U_{a_k}(a_k),$

$[9] := \mathtt{E}W[7](k) : (b, w) \in U_{a_k},$

$[w.*] := \texttt{EConnector}(X, U_{a_k})[7][9] : w \in U_{a_k} \circ U_{a_k}(a_k);$

$\leadsto [7] := \mathtt{I} \subset [22][2][1] : W(A) \subset \bigcup_{a \in A} U_a \circ U_a(a) \subset N,$

$[*] := \texttt{IUniformNeighborhood}[7] : \texttt{UniformNeighborhood}(X, A, N);$

$\square$

$\texttt{ConentorAsNeighborhoodOfDiagonal} :: \forall X : \texttt{UniformSpace} . \forall U \in \mathcal{U}_X . \texttt{Neighborhood}\Big(X^2, \Delta(X), U\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

1.1.5 Closures and regularity

There is a special way to compute closures in uniform spaces. Also, any separeted uniform space is regular

$$\texttt{ClosureFormula} :: \forall X : \texttt{UniformSpace} \,.\, \forall \mathcal{B} : \texttt{BaseOfUniformity}(X) \,.\, \forall A \subset X \,.\, \overline{A} = \bigcap_{V \in \mathcal{B}} \bigcup_{a \in A} V(a)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in \overline{A},$

$\texttt{Assume}\ V \in \mathcal{B},$

$[1] := \texttt{SymmetrocBaseExists}(x) : \sum W : \texttt{SymmetricConnector}(X) \,.\, W \subset V,$

$[2] := \texttt{ClosureAltDef}(X, A)\texttt{E}\widetilde{B}_x : \exists A \cap W(x),$

$y := \texttt{E}\exists[2] \in W(x),$

$[3] := \texttt{ESymmetricConnector}(X, W)\texttt{E}y : x \in W(y),$

$[4] := [1][3] : x \in V(x),$

$[x.*] := \texttt{UnionSubset}[4] : x \in \bigcup_{a \in A} V(a);$

$\rightsquigarrow [1] := \texttt{I} \subset : \overline{A} \subset \bigcup_{a \in A} V(a),$

$\texttt{Assume}\ x \in \overline{A}^{\complement},$

$\Big(U, [2]\Big) := \texttt{SymmetrocBaseExists}(X)\texttt{ClosureAltDef}(X, A)\texttt{E}\widetilde{B}_x :$

$\quad : \sum U : \texttt{SymmetricConnector}(X) \,.\, U(x) \cap A = \emptyset,$

$\Big(V, [3]\Big) := \texttt{EUniformBase}(X, \mathcal{B}, U) : \sum V \in \mathcal{B} \,.\, V \subset U,$

$\texttt{Assume}\ [4] : x \in \bigcup_{a \in A} V(a),$

$\Big(a, [5]\Big) := \texttt{EUnionE}x : \sum_{a \in A} x \in V(a),$

$[6] := [3][5] : x \in U(a),$

$[7] := \texttt{ESymmetricConnector}(X, U)[6] : a \in U(x),$

$[8] := [2][7] : \perp;$

$\rightsquigarrow [x.*] := \texttt{E}\perp : x \notin \bigcup_{a \in A} V(a);$

$\rightsquigarrow [*] := [1]\texttt{ISetEq} : \overline{A} = \bigcap_{V \in \mathcal{B}} \bigcup_{a \in A} V(a);$

$\square$

$\texttt{EveryUniformSpaceIsRegular} :: \forall X : \texttt{UniformSpace} \,.\, \texttt{T0}(X) \Rightarrow \texttt{T3}(X)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ N \in \mathcal{U}(x),$

$\Big(U, [1]\Big) := \texttt{EuniformTopology}(X) : \sum U \in \mathcal{U}_X \,.\, U \subset N,$

$\Big(W, [2]\Big) := \texttt{EUniformSpace}(X, U) : \sum W \in \mathcal{U}_X \,.\, W \circ W \subset U,$

$[x.*] := \texttt{ClosureFormula}(X, V(x))[2][1] : \overline{V(x)} \subset \bigcup_{a \in V(x)} V(a) \subset U(x) \subset N;$

$\rightsquigarrow [1] := \mathtt{I}\forall : \forall x \in X \,.\, \forall N \in \mathcal{U}(x) \,.\, \exists V \in \mathcal{U}(x) \,.\, \overline{V} \subset N,$

$[*] := \texttt{RegularT0IsT3}[1] : \texttt{T3}(X);$

$\square$

$\texttt{UniformT3SpaceIntersectsToDiagonal} :: \forall X : \texttt{UniformSpace} \,.\, \texttt{T3}(X) \iff \bigcap \mathcal{U}_X = \Delta(X)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CloppenByConnector} :: \forall X : \texttt{UniformSpace} \,.\, \forall A \subset X \,.\, \forall U : \texttt{Connector}(X) \,.$

$\quad . \bigcup_{a \in A} U(a) \subset A \Rightarrow \texttt{Clopen}(X, A)$

$\texttt{Proof} \ =$

$[1] := \Lambda a \in A \,.\, [0]\texttt{UnionSubset} A, U(a) : \forall a \in A \,.\, U(a) \subset A,$

$[2] := \texttt{EuniformTopology}[1] : A \in \mathcal{T}(X),$

$[3] := \texttt{ClosureFormula}(X, A)[0] : \overline{A} \subset A,$

$[4] := \texttt{Eclosure}(X, A)[3] : \overline{A} = A,$

$[*] := \texttt{IClopen}[2][4] : \texttt{Clopen}(X, A);$

$\square$

$\texttt{ConnectorsClopenAggregation} :: \forall X : \texttt{UniformSpace} \,.\, \forall U \in \mathcal{U}_X \,.\, \forall A \subset \mathcal{U}_X \,.\, \texttt{Clopen}\left(X, \bigcup_{n=1}^{\infty} \bigcup_{a \in A} U^{\circ n}(a)\right)$

$\texttt{Proof} \ =$

$B := \bigcup_{n=1}^{\infty} \bigcup_{a \in A} U^{\circ n}(a) :?X,$

$[1] := \texttt{E}B\texttt{EConnector}(X, U)\texttt{CompositionContainment}(X)\mathtt{I}B :$

$\quad : U(B) = U \bigcup_{n=1}^{\infty} U^{\circ n}(A) = \bigcup_{n=2}^{\infty} U^{\circ n}(A) = \bigcup_{n=1}^{\infty} U^{\circ n}(A) = B,$

$[*] := \texttt{CloppenByConnector}[1] : \texttt{Clopen}(X, B);$

$\square$

1.1.6 Closed Connectors

There are always a base of closed connectors

$\texttt{ConnectorClosureFormula} :: \forall X : \texttt{UniformSpace} . \forall \mathcal{B} : \texttt{BaseOfUniformity}(X) . \forall U \in \mathcal{U}_X .$
$. \overline{U} = \bigcap \{V \circ U \circ V : V \in \mathcal{B}\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (a,b) \in \overline{U}^{\complement},$

$\Big(V, [1]\Big) := \texttt{ClosureAltDef}(X^2, U)\texttt{E}(a,b) : \sum V \in \mathcal{U}(a,b) . V \cap U = \emptyset,$

$\Big(W, [2]\Big) := \texttt{EproductTopology}(X, V)\texttt{EuniformTopology}(X)\texttt{E}\widetilde{B}\texttt{SymmetricBaseExists}(X) :$
$: \sum W : \texttt{SymmetricConnector}(X) . W(a) \times W(b) \subset V,$

$\Big(O, [3]\Big) := \texttt{EUniformBase}(X, \mathcal{B}, W) : \sum O \in \mathcal{B} . O \subset W,$

$\texttt{Assume}\ [4] : O \circ U \circ O(a,b),$

$\Big(x, y, [5]\Big) := \texttt{EConnector}(X, O \ \&\ U)[4] : \sum x, y \in X . x \in O(a) \ \&\ y \in U(x) \ \&\ b \in O(y),$

$[6] := [5.1][3] : W(a,x),$

$[7] := [5.2][3]\texttt{ESymmetricConnector}(X, W) : W(b,y),$

$[8] := [5.3] : U(x,y),$

$[9] := \texttt{I} \times [6][7][2] : (x,y) \subset W(a) \times W(b) \subset V,$

$[4.*] := [9][1] : \bot;$

$\rightsquigarrow [4] := \texttt{E}\bot : \neg O \circ U \circ O(a,b),$

$\Big[(a,b).*\Big] := \texttt{CompositionContainment}[4] : \neg O(a,b);$

$\rightsquigarrow [1] := \texttt{I}\exists\texttt{I}\forall : \forall (a,b) \in \overline{V}^{\complement} . \exists B \in \mathcal{B} . \neg B \circ U \circ B(a,b),$

$\texttt{Assume}\ (a,b) \in \overline{U},$

$\texttt{Assume}\ B \in \mathcal{B},$

$\Big(W, [2]\Big) := \texttt{SymmetricBaseExists}(X, B) : \sum W : \texttt{SymmetricConnector}(X) . W \subset B,$

$[3] := \texttt{EproductTopology}(X, V)\texttt{EuniformTopology}(X)\texttt{E}\widetilde{B}[2] : \exists \Big(W(a) \times W(b)\Big) \cap U,$

$(x,y) := \texttt{E}\exists[3] \in \Big(W(a) \times W(b)\Big) \cap U,$

$[*] := \texttt{EConnector}(X, U \ \&\ W)\texttt{ESymmetricConnector}(X, W) : (a,b) \in W \circ U \circ W \subset B \circ U \circ B;$

$\rightsquigarrow [*] := [1] : \overline{U} = \bigcap \{V \circ U \circ V : V \in \mathcal{B}\};$

$\square$

$\texttt{UniformityTrisection} :: \forall X : \texttt{UniformSpace} . \forall U \in \mathcal{U}_X . \exists V : \texttt{SymmetricConnector}(X) . V \circ V \circ V \subset U$

$\texttt{Proof} \ =$

$\Big(V, [1]\Big) := \texttt{EUniformity}(X, U) : \sum V \in \mathcal{U}_X . V \circ V \subset U,$

$\Big(W, [2]\Big) := \texttt{EUniformity}(X, V) : \sum W \in \mathcal{U}_X . W \circ W \subset V,$

$[3] := \texttt{MonotonicContainMent}(X)[1][2] : W \circ W \circ W \subset W \circ W \circ W \circ W \subset V \circ V \subset U,$

$\Big(O, [4]\Big) := \texttt{SymmetricBaseExists}(X, W) : \sum O : \texttt{SymmetricConnector}(X) . O \subset W,$

$[5] := [3][4] : O \circ O \circ O \subset U;$

$\square$

$\texttt{ClosedConnector} :: \prod X : \texttt{UniformSpace} \,.\, ?\mathcal{U}_X$

$U : \texttt{ClosedConnector} \iff \texttt{Closed}(X^2, U)$

$\texttt{ClosedConnectorsBaseExists} :: \forall X : \texttt{UniformSpace} \,.\, \exists \mathcal{B} : \texttt{BaseOfUniformity}(X) \,.$

$\quad . \forall U \in \mathcal{B} \,.\, \texttt{ClosedConnector} \,\&\, \texttt{SymmetricConnector}(X, U)$

$\texttt{Proof} =$

$\mathcal{B} := \left\{U \in \mathcal{U}_X : \texttt{ClosedConnector} \,\&\, \texttt{SymmetricConnector}(X, U)\right\} : ?\mathcal{U}_X,$

$S := \left\{U \in \mathcal{U}_X : \texttt{SymmetricConnector}(X, U)\right\} : \texttt{BaseOfUniformity}(X),$

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$\Big(V, [1]\Big) := \texttt{UniformityTrisection}(X, U) : \sum V : \texttt{SymmetricConnector}(X) \,.\, V \circ V \circ V \subset U,$

$[2] := \texttt{ConnectorClosureFormula}(X, V)[1] : \overline{V} \subset V \circ V \circ V \subset U,$

$[4] := \texttt{CinnectorClosureFormula}(X, V, S) : \overline{V} = \bigcap \{W \circ V \circ W | W \in S\},$

$[U.*] := \texttt{ESymmetricConnector}(X, V)[4]\texttt{ISymmetricConnector} : \texttt{SymmetricConnector}\Big(X, \overline{V}\Big);$

$\rightsquigarrow [*] := \texttt{I}\mathcal{B}\texttt{IBaseOfUniformity} : \texttt{BaseOfUniformity}\Big(X, \mathcal{B}\Big);$

$\square$

$\texttt{UniformityTrisection2} :: \forall X : \texttt{UniformSpace} \,.\, \forall U \in \mathcal{U}_X \,.$

$\quad . \exists V : \texttt{SymmetricConnector} \,\&\, \texttt{ClosedConnector}(X) \,.\, V \circ V \circ V \subset U$

$\texttt{Proof} =$

$\dots$

$\square$

1.1.7 Uniform Convergence

Uniform convergence of continuous functions preserves continuoty.

$$\texttt{UniformlyConvergent} :: \prod X \in \mathsf{SET} \;.\; \prod Y : \texttt{UniformSpace} \;.\; ?\Big(\texttt{Net}(X \to Y) \times (X \to Y)\Big)$$

$$\Big((\Delta, f), g\Big) : \texttt{UniformlyConvergent} \iff f_{\delta\in\Delta} \rightrightarrows g \iff$$

$$\iff \forall U \in \mathcal{U}_Y \;.\; \exists \delta_0 \in \Delta \;.\; \forall \delta \geq \delta_0 \;.\; \forall x \in X \;.\; \Big(f_\delta(x), g(x)\Big) \in U$$

$$\texttt{UniformConvergencePreservesContinuity} :: \forall X \in \mathsf{TOP} \;.\; \forall Y : \texttt{UniformSpace} \;.\; \forall g : X \to Y \;.$$

$$.\; \forall (\Delta, f) : \texttt{Net}\Big(\mathsf{TOP}(X,Y)\Big) \;.\; f_{\delta\in\Delta} \rightrightarrows g \Rightarrow g \in \mathsf{TOP}(X,Y)$$

$\texttt{Proof} \; =$

$\texttt{Assume}\; x \in X,$

$\texttt{Assume}\; O \in \mathcal{U}(g(x)),$

$\Big(U, [1]\Big) := \texttt{EuniformTopology}\Big(Y, O, g(x)\Big) : \sum U \in \mathcal{U}_Y \;.\; U\Big(g(x)\Big) \subset O,$

$\Big(V, [2]\Big) := \texttt{UniformityTrisection2}(X, U) :$

$\quad : \sum V : \texttt{SymmetricConnector} \;\&\; \texttt{ClosedConnector}(Y) \;.\; V \circ V \circ V \subset U,$

$\Big(\delta, [3]\Big) := \mathtt{E}[0](V) : \sum_{\delta\in\mathcal{D}} \forall t \geq \delta \;.\; \forall u \in X \;.\; \Big(f_t(u), g(u)\Big) \in V,$

$\Big(O', [4]\Big) := \mathtt{E}\widetilde{B}(V, g(x)) : \sum O' \in \mathcal{T}(Y) \;.\; g(x) \in O' \subset V\Big(g(x)\Big),$

$[5] := \mathtt{E}\mathsf{TOP}(X, Y, f_\delta, O') : f_\delta^{-1}(O') \in \mathcal{T}(X),$

$\texttt{Assume}\; u \in f_\delta^{-1}(O'),$

$[6] := \texttt{EuEpreimage} : f_\delta(u) \in O' \subset V\Big(g(x)\Big),$

$[7] := [3](\delta, u) : \Big(f_\delta(u), g(u)\Big) \in V,$

$[x.*] := \texttt{ESymmetricConnector}(Y, V)[6][7][2][1] : \Big(g(u), g(x)\Big) \in O;$

$\rightsquigarrow [*] := \texttt{ContinuityIsLocal}(X, Y) : g \in \mathsf{TOP}(X, Y);$

$\square$

1.1.8 Pseudo-Uniformities [∞]

There are more general concepts then uniformities. Will be written on demand.

1.1.9 v-Closure [∞]

This is about columns in connectors. Will be written on demand.

1.1.10 Transitive Uniformities[∞]

Transitive uniformities are very special. Will be written on demand.

1.2 Coverering Uniformities[∞]

One can also define uniformities as families of covers. Will be written on demand.

1.3 Uniform Continuity

1.3.1 Uniform Maps

The notion of uniform map nicely generalizes frim metric spaces.

$$\mathtt{UniformlyContinuous} :: \prod X, Y : \mathtt{UniformSpace} \, . \, ?(X \to Y)$$

$$\varphi : \mathtt{UniformlyContinuous} \iff \forall U \in \mathcal{U}_Y \, . \, \exists V \in \mathcal{U}_X \, . \, (\varphi \times \varphi)(V) \subset U$$

$$\mathtt{MetricUniformContinuity} :: \forall X, Y \in \mathsf{SMS} \, . \, \forall \varphi : X \to Y \, . \, \mathtt{UniformlyContinuous}(X, Y, \varphi) \iff$$

$$\iff \forall \varepsilon \in \mathbb{R}_{++} \, . \, \exists \delta \in \mathbb{R}_{++} \, . \, \forall a, b \in X \, . \, d(a, b) < \delta \Rightarrow d\Big(\varphi(a), \varphi(b)\Big) < \varepsilon$$

$\mathtt{Proof} \ =$

$\dots$

$\square$

$$\mathtt{UniformContinuousIsContinuous} :: \forall X, Y : \mathtt{UniformSpace} \, . \, \forall \varphi : \mathtt{UniformlyContinuous}(X, Y) \, .$$

$$. \, \varphi \in \mathsf{TOP}(X, Y)$$

$\mathtt{Proof} \ =$

$\mathtt{Assume} \ x \in X,$

$\mathtt{Assume} \ O \in \mathcal{U}\Big(\varphi(x)\Big),$

$\Big(U, [1]\Big) := \mathtt{EuniformTopology}\Big(Y, \varphi(x)\Big) : \sum U \in \mathcal{U}_Y \, . \, U\Big(\varphi(x)\Big) \subset O,$

$\Big(V, [2]\Big) := \mathtt{EUniformlyContinuous}(X, Y, \varphi, U) : \sum V \in \mathcal{U}_X \, . \, (\varphi \times \varphi)(V) \subset U,$

$\Big(W, [3]\Big) := \mathtt{E}\widetilde{\mathcal{B}}_x(V) : \sum W \in \mathcal{U}(x) \, . \, W \subset V(x),$

$[x.*] := [1][2][3] : \varphi(W) \subset O;$

$\leadsto [*] := \mathtt{ContinuityIsLocal} : \varphi \in \mathsf{TOP}(X, Y);$

$\square$

$$\mathtt{UniformityInclusionByIdentityContinuity} ::$$

$$: \forall X \in \mathsf{SET} \, . \, \mathcal{U}, \mathcal{V} : \mathtt{Uniformity}(X) \, . \, \mathcal{U} \le \mathcal{V} \iff \mathtt{UniformlyContinuous}\Big((X, \mathcal{V}), (X, \mathcal{U}), \mathrm{id}\Big)$$

$\mathtt{Proof} \ =$

$\dots$

$\square$

$$\mathtt{Unimorphism} :: \prod X, Y : \mathtt{UniformSpace} \, . \, ?\mathtt{Homeomorphis}(X, Y)$$

$$\varphi : \mathtt{Unimorphism} \iff \mathtt{UniformlyContinuous}(X, Y, \varphi) \ \& \ \mathtt{UniformlyContinuous}(Y, X, \varphi^{-1})$$

$\texttt{UniformConvergencePreservesContinuity} :: \forall X, Y \in \mathsf{UNI} \,.\, \forall g : X \to Y \,.$

$\quad .\, \forall (\Delta, f) : \texttt{Net}\Big(\mathsf{UNI}(X,Y)\Big) \,.\, f_{\delta\in\Delta} \rightrightarrows g \Rightarrow g \in \mathsf{UNI}(X,Y)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ V \in \mathcal{U}_Y,$

$\Big(W, [1]\Big) := \texttt{UniformityTrisection}(Y, V) : \sum W : \texttt{SymmetricConnector}(Y) \,.\, W \circ W \circ W \subset V,$

$\Big(\delta, [2]\Big) := \texttt{EUniformConvergence}[0](W) : \sum_{\delta\in\Delta} \forall x \in X \,.\, \forall t > \delta \,.\, \Big(f_t(x), g(x)\Big) \in VW,$

$\Big(U, [3]\Big) := \texttt{EUniformlyContinuous}(X, Y, f_\delta, W) : \sum U \in \mathcal{U}_X \,.\, f_\delta \times f_\delta(U) \subset W,$

$\texttt{Assume}\ (a, b) \in U,$

$[4] := [3](a, b) : \Big(f_\delta(a), f_\delta(b)\Big) \in W,$

$[5] := [2](a, \delta) : \Big(f_\delta(a), g(a)\Big) \in W,$

$[6] := [3](b, \delta) : \Big(f_\delta(b), g(b)\Big) \in W,$

$\Big[(a, b).*\Big] := [4][5][6]\texttt{EConnector}(Y, W)[1] : \Big(g(a), g(b)\Big) \in W \circ W \circ W \subset V;$

$\rightsquigarrow [V.*] := \texttt{I} \subset : (g \times g)(U) \subset V;$

$\rightsquigarrow [*] := \texttt{IUniformlyContinuous} : g \in \mathsf{UNI}(X, Y);$

$\square$

1.3.2 Category of Uniform Spaces, Initial and Final Uniformity

With uniform maps as morphism we have a bicomplete category. The notion of final and initial uniformity is very useful for constructing limits. Note, that embedding in TOP reflects limits.

$$\texttt{UniformCategory} :: \mathsf{CAT}$$

$$\texttt{UniformCategory}\,() = \mathsf{UNI} := \Big(\texttt{UniformSpace}, \texttt{UniformlyContinuous}, \circ, \mathrm{id}\Big)$$

$$\texttt{UniformSeparetadCategory} :: \mathsf{CAT}$$

$$\texttt{UniformSeparatedCategory}\,() = \mathsf{UNIS} := \Big(\texttt{UniformSpace} \,\&\, \texttt{T3}, \texttt{UniformlyContinuous}, \circ, \mathrm{id}\Big)$$

$$\texttt{initialUniformity} :: \prod_{X,I \in \mathsf{SET}} \prod_{Y : I \to \mathsf{UNI}} \left(\prod_{i \in I} X \to Y_i\right) \to \texttt{Uniformity}(X)$$

$$\texttt{initialUniformity}\,(\phi) = \mathcal{I}_X(I, Y, \phi) := \min \left\{ \mathcal{U} : \texttt{Uniformity}(X) : \forall i \in I \,.\, \phi_i \in \mathsf{UNI}\Big((X, \mathcal{U}), Y\Big) \right\}$$

$$\texttt{InitialUniformityExpression} :: \forall X, I \in \mathsf{SET} \,.\, \forall Y : I \to \mathsf{UNI} \,.\, \forall \phi : \prod_{i \in I} X \to Y_i \,.$$

$$.\, \mathcal{I}_X(I, Y, \phi) = \left\langle \left\{ (\phi_i \times \phi_i)^{-1}(V) \Big| i \in I, V \in \mathcal{U}_{Y_i} \right\} \right\rangle_{\mathcal{F}}$$

Proof =

...

□

$$\texttt{productUniformSpace} :: \prod_{I \in \mathsf{SET}} (I \to \mathsf{UNI}) \to \mathsf{UNI}$$

$$\texttt{productUniformSpace}\,(X) = \prod_{i \in I} X_i := \left(\prod_{i \in I} X_i, \mathcal{I}(I, X, \pi) \right)$$

$$\texttt{UniformCategoryIsComplete} :: \texttt{Complete}(\mathsf{UNI})$$

Proof =

...

□

$$\texttt{LimitsOfUniAgreeWithTop} :: \forall (I, X, \phi) : \texttt{Diagram}(\mathsf{UNI}) \,.\, \lim_{\mathsf{UNI}}(I, X, \phi) \cong_{\mathsf{TOP}} \lim_{\mathsf{TOP}}(I, X, \phi)$$

Proof =

...

□

$$\texttt{supremumUniformity} :: \prod_{X,I \in \mathsf{SET}} \Big(I \to \texttt{Uniformity}(X)\Big) \to \texttt{Uniformity}(X)$$

$$\texttt{supremumUniformity}\,(\mathcal{U}) = \bigvee_{i \in I} \mathcal{U}_i := \mathcal{I}_X\Big(I, (X, \mathcal{U}), \mathrm{id}\Big)$$

$\texttt{InitialUniformityUniversalProperty} ::$

$$:: \forall I, X \in \mathsf{SET} \,.\, \forall Y : I \to \mathsf{UNI} \,.\, \forall \phi : \left(\prod_{i \in I} X \to Y_i\right) \,.\, \forall Q \in \mathsf{UNI} \,.\, \forall \psi : Q \to X \,.$$

$$.\, \psi \in \mathsf{UNI}\Big(Q, \big(X, \mathcal{I}_X(I, Y, \phi)\big)\Big) \iff \forall i \in I \,.\, \psi\phi_i \in \mathsf{UNI}(Q, Y_i)$$

$\texttt{Proof} =$

...

$\square$

$$\texttt{finalUniformity} :: \prod_{Y, I \in \mathsf{SET}} \prod_{X : I \to \mathsf{UNI}} \left(\prod_{i \in I} X_i \to Y\right) \to \texttt{Uniformity}(X)$$

$$\texttt{finalUniformity}(\phi) = \mathcal{F}_Y(I, X, \phi) := \max\left\{\mathcal{V} : \texttt{Uniformity}(Y) : \forall i \in I \,.\, \phi_i \in \mathsf{UNI}\Big(X, (Y, \mathcal{V})\Big)\right\}$$

$$\mathfrak{V} := \left\{\mathcal{V} : \texttt{Uniformity}(Y) : \forall i \in I \,.\, \phi_i \in \mathsf{UNI}\Big(X_i, (Y, \mathcal{V})\Big)\right\} :?\texttt{Uniformity}(Y),$$

$$\mathcal{U} := \bigvee_{V \in \mathfrak{V}} V : \texttt{Uniformity}(X),$$

$\texttt{Assume}\ i \in I,$

$[1] := \mathtt{E}\mathfrak{V}(i) : \forall \mathcal{V} \in \mathfrak{V} \,.\, \phi_i \in \mathsf{UNI}\Big(X_i, (Y, \mathcal{V})\Big),$

$[2] := \mathtt{E}\mathcal{U} : \mathcal{U} = \mathcal{I}_Y\Big(\mathfrak{V}, \Lambda\mathcal{V} \in \mathfrak{V} \,.\, (Y, V), \mathrm{id}\Big),$

$[i.*] := \texttt{InitialUniformityUniversalProperty}[1][2] : \phi_i \in \mathsf{UNI}\Big(X_i, (Y, \mathcal{U})\Big);$

$\rightsquigarrow [1] := \mathtt{I}\mathfrak{V} : \mathcal{U} \in \mathfrak{V},$

$[*] := \mathtt{E}\mathcal{U}[1]\mathtt{I}\mathcal{F}_Y(I, X, \phi) : \mathcal{U} = \mathcal{F}_Y(I, X, \phi);$

$\square$

$\texttt{FinalUniformityUniversalProperty} ::$

$$:: \forall I, Y \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{UNI} \,.\, \forall \phi : \left(\prod_{i \in I} X_i \to Y\right) \,.\, \forall Z \in \mathsf{UNI} \,.\, \forall \psi : Y \to Z \,.$$

$$.\, \psi \in \mathsf{UNI}\Big(\big(Y, \mathcal{F}_Y(I, X, \phi)\big), Z\Big) \iff \forall i \in I \,.\, \phi_i\psi \in \mathsf{UNI}(X_i, Z)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{UniformCategoryIsBicomplete} :: \texttt{Bicomplete}(\mathsf{UNI})$

$\texttt{Proof} =$

...

$\square$

$$\texttt{quotientUniformity} :: \prod_{X \in \mathsf{UNI}} \texttt{Equivalence}(X) \to \mathsf{UNI}$$

$$\texttt{quotientUniformity}(\sim) = \frac{X}{\sim} := \left(\frac{X}{\sim}, \mathcal{F}(\star, \star \mapsto X, \pi_\sim)\right)$$

$$\texttt{infimumUniformity} :: \prod_{I,X\in\mathsf{SET}} \Big(I \to \texttt{Uniformity}(X)\Big) \to \texttt{Uniformity}(X)$$

$$\texttt{infimumUniformity}\,(\mathcal{U}) = \bigwedge_{i\in\mathcal{I}} \mathcal{U}_i := \bigg(X, \mathcal{F}_X\Big(I, (X,\mathcal{U}), \mathrm{id}\Big)\bigg)$$

$\texttt{UniformSeparatedCategoryIsBicomplete} :: \texttt{Bicomplete}(\mathsf{UNIS})$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MinUniformityBase} ::$

$$:: \forall X \in \mathsf{SET}\ .\ \forall \mathcal{U},\mathcal{V} : \texttt{Uniformity}(X)\ .\ \texttt{BaseOfUniformity}\Big(X,\mathcal{U}\wedge\mathcal{V}, \{U\circ V | U \in \mathcal{U}, V \in \mathcal{V}\}\Big)$$

$\texttt{Proof} \;=$

$\dots$

$\square$

1.3.3 Uniform Covers

Notion of uniform covers is useful for identifying uniform maps.

$$\texttt{UniformCover} :: \prod_{X \in \mathsf{UNI}} ?\texttt{Cover}(X)$$

$$\mathcal{C} : \texttt{UniformCover} \iff \exists U \in \mathcal{U}_X \,.\, \forall x \in X \,.\, \exists C \in \mathcal{C} \,.\, U(x) \subset C$$

$$\texttt{UniformContinuityByPreimages} :: \forall X, Y \in \mathsf{UNI} \,.\, \forall \varphi : X \to Y \,.$$

$$.\, \varphi \in \mathsf{UNI}(X, Y) \iff \forall V \in \mathcal{U}_Y \,.\, \texttt{UniformCover}\left(X, \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\}\right)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \varphi \in \mathsf{UNI}(X, Y),$

$\texttt{Assume}\ V \in \mathcal{U}_Y,$

$\mathcal{C} := \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\} :??X,$

$\Big(U, [2]\Big) := \texttt{EUniformlyContinuous}(X, Y, \varphi)[1] : \sum U \in \mathcal{U}_X \,.\, (\varphi \times \varphi)(U) \subset V,$

$\texttt{Assume}\ x \in X,$

$[3] := [2](x) : \varphi\Big(U(x)\Big) \subset V\Big(\varphi(x)\Big),$

$[x, *] := \varphi^{-1}[3] : U(x) \subset \varphi^{-1}\Big(V\big(\varphi(x)\big)\Big);$

$\leadsto [1.*] := \texttt{IUniformCover} : \texttt{UniformCover}(X, \mathcal{C});$

$\leadsto [1] := \texttt{I} \Rightarrow : \varphi \in \mathsf{UNI}(X, Y) \Rightarrow \forall V \in \mathcal{U}_Y \,.\, \texttt{UniformCover}\left(X, \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\}\right),$

$\texttt{Assume}\ [2] : \forall V \in \mathcal{U}_Y \,.\, \texttt{UniformCover}\left(X, \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\}\right),$

$\texttt{Assume}\ V \in \mathcal{U}_Y,$

$\mathcal{C} := \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\} : \texttt{UniformCover}(X),$

$\Big(U, [3]\Big) := \texttt{EUniformCover}(X) : \forall x \in X \,.\, \exists C \in \mathcal{C} \,.\, U(x) \subset C,$

$[V.*] := \varphi\Big(\texttt{E}\mathcal{C}[3]\Big)\texttt{I}\times : (\varphi \times \varphi)(U) \subset V;$

$\leadsto [2.*] := \texttt{IUniformlyContinuous} : \varphi \in \mathsf{UNI}(X, Y);$

$\leadsto [*] := \texttt{I}(\iff)[1] : \varphi \in \mathsf{UNI}(X, Y) \iff \forall V \in \mathcal{U}_Y \,.\, \texttt{UniformCover}\left(X, \left\{\varphi^{-1}\Big(V(y)\Big)\Big| y \in Y\right\}\right);$

$\square$

1.3.4 Compact Uniform Spaces

All continuous maps with compact uniform domains are uniformly continuous.

$\texttt{EveryCoverOfCompactSpaceIsUniform} ::$

$:: \forall X \in \mathsf{UNI} \,.\, \texttt{Compact}(X) \Rightarrow \forall \mathcal{O} : \texttt{OpenCover}(X) \,.\, \texttt{UniformCover}(X, \mathcal{O})$

$\texttt{Proof} \ =$

$\Big(\mathcal{O}', [1]\Big) := \texttt{ECompact}(X, \mathcal{O}) : \sum \mathcal{O}' : \texttt{FiniteSubset}(\mathcal{O}) \,.\, \texttt{Cover}(X, \mathcal{O}'),$

$\Big(V, [2]\Big) := \texttt{EuniformTopology}[1] : \sum V : \prod_{O \in \mathcal{O}'} \prod_{x \in O} \mathcal{U}_X \,.\, \forall O \in \mathcal{O}' \,.\, \forall x \in O \,.\, V_{O,x}(x) \subset O,$

$\Big(W, [21]\Big) := \texttt{SymmetricBaseExists}(X, V) :$

$: \sum W : \prod_{O \in \mathcal{O}'} \prod_{x \in O} \texttt{SymmetricConnector}(X) \,.\, \forall O \in \mathcal{O}' \,.\, \forall x \in O \,.\, W_{O,x} \circ W_{O,x} \subset U_{O,x},$

$\Big(O', [3]\Big) := \mathtt{E}\widetilde{\mathcal{B}}(W)[2] : \sum O' : \prod_{O \in \mathcal{O}'} \prod_{x \in O} x \in O'_{O,x} \subset W_{O,x}(x),$

$[4] := \texttt{ECover}(X, \mathcal{O}')[1][3] : \texttt{OpenCover}(X, \operatorname{Im} O'),$

$\Big(n, x, \theta, [5]\Big) := \texttt{ECompact}(X, \operatorname{Im} O') : \sum_{n=1}^{\infty} \sum x : \{1, \ldots, n\} \to X \,.$

$. \sum \theta : \{1, \ldots, n\} \to \mathcal{O}' \,.\, \Big(\forall k \in \{1, \ldots, n\} \,.\, x_k \in \theta_k\Big) \ \&\ \texttt{OpenCover}(X, \operatorname{Im} O'_{\theta,x}),$

$[6] := [5][3] : \texttt{Cover}(X, \operatorname{Im} W_{\theta,x}(x)),$

$U := \bigcap_{k=1}^{n} W_{\theta_k, x_k} \in \mathcal{U}_X,$

$\texttt{Assume}\ y \in X,$

$\Big(k, [7]\Big) := \texttt{ECover}[6] : \sum_{k=1}^{n} y \in W_{\theta_k, x_k}(x_k),$

$\texttt{Assume}\ u \in U(y),$

$[8] := \mathtt{E}u\mathtt{E}U(k) : u \in W_{\theta_k, x_k}(y),$

$[y.*] := \texttt{ESymmetricConnector}(X, W_{\theta_k, x_k})[21][2] : u \in W_{\theta_k, x_k} \circ W_{\theta, x_k}(x_k) \subset V_{\theta_k, x_k} \subset \theta_k;$

$\rightsquigarrow [*] := \texttt{IUniformCover} : \texttt{UniformCover}(X, \mathcal{O});$

$\square$

$\texttt{CompactUniformContinuity} :: \forall X, Y \in \mathsf{UNI} \,.\, \texttt{Compact}(X) \Rightarrow \mathsf{TOP}(X, Y) = \mathsf{UNI}(X, Y)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CompactUniformityEquivalence} :: \forall X, Y \in \mathsf{UNI} \,.\, \texttt{Compact}(X \ \&\ Y) \Rightarrow (X \cong_{\mathsf{TOP}} Y \Rightarrow X \cong_{\mathsf{UNI}} Y)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.4 Completeness

As metric spaces, the uniform spaces can be complete.

1.4.1 Cauchy Filterbase

Cauchy Filterbases are natural generalizations of Cauchy sequences.

$$\texttt{CauchyFilterbase} :: \prod_{X \in \mathsf{UNI}} ?\texttt{Filterbase}(X)$$

$$\mathcal{F} : \texttt{CauchyFilterbase} \iff \forall U \in \mathcal{U}_X \,.\, \exists F \in \mathcal{F} \,.\, F \times F \subset U$$

$\texttt{EveryConvergentFilterbaseIsCauchy} ::$

$:: \forall X \in \mathsf{UNI} \,.\, \forall \mathcal{F} : \texttt{ConvergentFilterbase}(X) \,.\, \texttt{CauchyFilterbase}(X, \mathcal{F})$

$\texttt{Proof} \ =$

$\Big(x, [1]\Big) := \texttt{EConvergentFilterbase}(X, \mathcal{F}) : \sum_{x \in X} \lim \mathcal{F} = x,$

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$(V, [3]) := \texttt{SymmetricBaseExists}(X) : \sum V : \texttt{SymmetricConnector}(X) \,.\, V \circ V \subset X,$

$\Big(O, [4]\Big) := \texttt{E}\widetilde{\mathcal{B}}_x(V) : \sum O \in \mathcal{U}(x) \,.\, O \subset V(x),$

$\Big(F, [5]\Big) := \texttt{E}\lim[1](O) : \sum F \in \mathcal{F} \,.\, F \subset O,$

$[U.*] := [4][5]\texttt{ProductSubset}(X) : F \times F \subset V \circ V \subset U;$

$\rightsquigarrow [*] := \texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}(X, \mathcal{F});$

$\square$

$\texttt{UniformMapsPreserveCauchyFilters} ::$

$:: \forall X, Y \in \mathsf{UNI} \,.\, \forall \varphi \in \mathsf{UNI}(X, Y) \,.\, \forall \mathcal{F} : \texttt{CauchyFilterbase}(X) \,.\, \texttt{CauchyFilterbase}\Big(Y, \varphi(\mathcal{F})\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ V \in \mathcal{U}_Y,$

$\Big(U, [1]\Big) := \texttt{EUniformlyContinuous}(X, Y, f, V) : \sum U \in \mathcal{U}_X \,.\, \varphi \times \varphi(U) \subset V,$

$\Big(F, [2]\Big) := \texttt{ECauchyFilterbase}(X, \mathcal{F}, U) : \sum F \in \mathcal{F} \,.\, F \times F \subset U,$

$[V.*] := [1][2] : \varphi(F) \times \varphi(F) \subset V;$

$\rightsquigarrow [*] := \texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}\Big(Y, \varphi(\mathcal{F})\Big);$

$\square$

$\mathtt{CauchyClustersAreLimits} :: \forall X \in \mathsf{UNI} \,.\, \forall \mathcal{F} : \mathtt{CauchyFilterbase}(X) \,.\, \forall x : \mathtt{Cluster}(X, \mathcal{F}) \,.\, x \in \lim \mathcal{F}$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ O \in \mathcal{U}(x),$

$\Big(U, [1]\Big) := \mathtt{EuniformTopology}(U, O) : \sum U \in \mathcal{U}_X \,.\, U(x) \subset O,$

$\Big(V, [2]\Big) := \mathtt{ClosedBaseExists}(X, U) : \sum V : \mathtt{ClosedConnector}(X) \,.\, V \subset U,$

$\Big(F, [3]\Big) := \mathtt{ECauchyFilterbase}(X, \mathcal{F}, V) : \sum F \in \mathcal{F} \,.\, F \times F \subset V,$

$\mathtt{Assume}\ f \in F,$

$[5] := \mathtt{ECluster}(X, \mathcal{F}, x, F)\mathtt{EClosedConnector}(X, V)[3] : (x, f) \in \overline{F} \times F \subset V,$

$[f.*] := [2]\mathtt{EConnector}(X, U)[1] : f \in O;$

$\rightsquigarrow [O.*] := \mathtt{I} \subset : F \subset O;$

$\rightsquigarrow [*] := \mathtt{I} \lim : x \in \lim \mathcal{F};$

$\square$

$\mathtt{SupUniformityCauchyFilterbase} ::$

$:: \forall X, I \in \mathsf{SET} \,.\, \forall \mathcal{U} : I \to \mathtt{Uniformity}(X) \,.\, \forall \mathcal{F} : \mathtt{Filterbase}(X) \,.$

$$. \mathtt{CauchyFilterbase}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), \mathcal{F}\right) \iff \forall i \in I \,.\, \mathtt{CauchyFilterbase}\Big((X, \mathcal{U}_i), \mathcal{F}\Big)$$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [1] : \mathtt{CauchyFilterbase}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), \mathcal{F}\right),$

$[2] := \mathtt{E} \bigvee_{i \in I} \mathcal{U}_i : \forall i \in I \,.\, \mathrm{id} \in \mathsf{UNI}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), (X, \mathcal{U}_i)\right),$

$[1.*] := \mathtt{UniformMapsPreserveCauchyFilters}[1][2] : \forall i \in I \,.\, \mathtt{CauchyFilterbase}\Big((X, \mathcal{U}_i), \mathcal{F}\Big);$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow : \mathtt{CauchyFilterbase}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), \mathcal{F}\right) \Rightarrow \forall i \in I \,.\, \mathtt{CauchyFilterbase}\Big((X, \mathcal{U}_i), \mathcal{F}\Big),$

$\mathtt{Assume}\ [1] : \forall i \in I \,.\, \mathtt{CauchyFilterbase}\Big((X, \mathcal{U}_i), \mathcal{F}\Big),$

$\mathtt{Assume}\ U \in \bigvee_{i \in I} \mathcal{U}_i,$

$\Big(n, i, V, [2]\Big) := \mathtt{E} \bigvee_{i \in I} \mathcal{U}_i(U)\mathtt{InitialUnifomityExpression} : \sum_{n=0}^{\infty} \sum i : \{1, \ldots, n\} \to I \,.\, \sum V : \prod_{k=1}^{\infty} \mathcal{U}_{i_k} \,.\, \bigcap_{k=1}^{n} V_k \subset$

$\Big(F, [3]\Big) := \Lambda k \in \{1, \ldots, n\} \,.\, \mathtt{ECauchyFilterbase}(X, \mathcal{U}_{i_k}, V_k) : \sum F : \{1, \ldots, n\} \to \mathcal{F} \,.\, \forall k \in \{1, \ldots, n\} \,.\, F_k \times$

$\Big(G, [4]\Big) := \mathtt{EFilterbase}(X, F) \in \sum G \in \mathcal{F} \,.\, G \subset \bigcap_{k=1}^{n} F_k,$

$[U.*] := [4][3]\mathtt{SubsetIntersect}[2] : G \times G \subset U;$

$\rightsquigarrow [2.*] := \mathtt{ICauchyFilterbase} : \mathtt{CauchyFilterbase}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), \mathcal{F}\right);$

$\rightsquigarrow [*] := \mathtt{I} \iff [*] : \mathtt{CauchyFilterbase}\left(\left(X, \bigvee_{i \in I} \mathcal{U}_i\right), \mathcal{F}\right) \iff \forall i \in I \,.\, \mathtt{CauchyFilterbase}\Big((X, \mathcal{U}_i), \mathcal{F}\Big);$

$\square$

1.4.2 Complete Uniform Spaces

Now complete spaces are those, where all Cauchy filters are converging

$$\texttt{CompleteUniformSpace} :: ?\mathsf{UNI}$$
$$X : \texttt{CompleteUniformSpace} \iff \forall \mathcal{F} : \texttt{CauchyFilterbase}(X) \; . \; \texttt{ConvergentFilterbase}(X, \mathcal{F})$$

$$\texttt{CauchySequence} :: \prod_{X \in \mathsf{UNI}} \mathbb{N} \to X$$
$$x : \texttt{CauchySequence} \iff \texttt{CauchyFilterbase}\Big(X, \big\{\{x_n, n \geq m\} \big| m \in \mathbb{N}\big\}\Big)$$

$$\texttt{SequenceCompleteUniformSpace} :: ?\mathsf{UNI}$$
$$X : \texttt{SequenceCompleteUniformSpace} \iff \forall x : \texttt{CauchySequence}(X) \; . \; \texttt{Convergent}(X, x)$$

$$\texttt{IsomorphicCompleteness} :: \forall X, Y \in \mathsf{UNI} \; . \; X \cong_{\mathsf{UNI}} Y \Rightarrow$$
$$\Rightarrow \Big(\texttt{CompleteUniformSpace}(X) \iff \texttt{CompleteUniformSpace}(Y)\Big)$$

Proof $=$

$\dots$

$\square$

$$\texttt{IsomorphicSequenceCompleteness} :: \forall X, Y \in \mathsf{UNI} \; . \; X \cong_{\mathsf{UNI}} Y \Rightarrow$$
$$\Rightarrow \Big(\texttt{SequenceCompleteUniformSpace}(X) \iff \texttt{SequenceCompleteUniformSpace}(Y)\Big)$$

Proof $=$

$\dots$

$\square$

$$\texttt{LargerEqUniformityIsComplete} :: \forall X : \texttt{CompleteUniformSpace} \; . \; \forall \mathcal{V} \geq \mathcal{U}_X \; .$$
$$. \; \mathcal{V} \cong \mathcal{U}_X \Rightarrow \texttt{CompleteUniformSpace}(X, \mathcal{V})$$

Proof $=$

$\dots$

$\square$

$$\texttt{LargerEqUniformityIsComplete} :: \forall X : \texttt{CompleteUniformSpace} \; . \; \forall \mathcal{V} \geq \mathcal{U}_X \; .$$
$$. \; \mathcal{V} \cong \mathcal{U}_X \Rightarrow \texttt{CompleteUniformSpace}(X, \mathcal{V})$$

Proof $=$

$\dots$

$\square$

$$\texttt{LargerEqUniformityIsSeqComplete} :: \forall X : \texttt{SequenceCompleteUniformSpace} \; . \; \forall \mathcal{V} \geq \mathcal{U}_X \; .$$
$$. \; \mathcal{V} \cong \mathcal{U}_X \Rightarrow \texttt{SequenceCompleteUniformSpace}(X, \mathcal{V})$$

Proof $=$

$\dots$

$\square$

$\texttt{ClosedOfCompleteIsComplete} ::$

$:: \forall X : \texttt{CompleteUniformSpace} \,.\, \forall A : \texttt{Closed}(X) \,.\, \texttt{CompleteUniformSpace}(A)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase}(A),$

$\mathcal{F}' := \{F \cup B | F \in \mathcal{F}, B \subset X\} : \texttt{Filter}(X),$

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$U' := U \cap A \times A \in \mathcal{U}_A,$

$\Big(F, [1]\Big) := \texttt{E}\texttt{CauchyFilterbase}(A, \mathcal{F})\texttt{E}U'\texttt{IntersectionIsSubset}(X, U', U, A \times A) : \sum_{F \in \mathcal{F}} F \times F \subset U' \subset U,$

$[U.*] := \texttt{E}F\texttt{UnionWithEmpty}(X)\texttt{I}\mathcal{F}' : F \in \mathcal{F}';$

$\rightsquigarrow [1] := \texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}\Big(X, \mathcal{F}'\Big),$

$x := \texttt{ECompleteUniformSpace}(X, \mathcal{F}') \in \lim \mathcal{F}',$

$[2] := \texttt{EClosed}(X, A)\texttt{E}\mathcal{F}'\texttt{ClosedFilterLimit} : x \in A,$

$[\mathcal{F}.*] := \texttt{E}\mathcal{F}'\texttt{E}x[2] : x \in \lim \mathcal{F};$

$\rightsquigarrow [*] := \texttt{ICompleteUniformSpace} : \texttt{CompleteUniformSpace}(A);$

$\square$

$\texttt{CompleteProductTHM} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{UNI} \,.\, \Big(\forall i \in I \,.\, \exists X_i\Big) \Rightarrow$

$$\Rightarrow \texttt{CompleteUniformSpace}\left(\prod_{i \in I} X_i\right) \iff \forall i \in I \,.\, \texttt{CompleteUniformSpace}(X_i)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \texttt{CompleteUniformSpace}\left(\prod_{i \in I} X_i\right),$

$\texttt{Assume}\ i \in I,$

$J := I \setminus \{i\} :?I,$

$x := \texttt{Choice}\Big(J, X, [0]\Big) : \prod_{j \in J} X_j,$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase}(X_i),$

$\mathcal{F}' := \left\{F \times_i \prod_{j \in J} \{x_j\} \middle| F \in \mathcal{F}\right\} : \texttt{Filterbase}\left(\prod_{i \in I} X_i\right),$

$\phi := \Lambda u \in X_i \,.\, \Lambda k \in I \,.\, \texttt{if}\ i == k\ \texttt{then}\ u\ \texttt{else}\ x_k : X_i \to \prod_{i \in I} X_i,$

$[1] := \texttt{E}\phi\texttt{E}\mathcal{F}' : \mathcal{F}' = \phi_*(\mathcal{F}),$

$[2] := \texttt{E}\phi\texttt{E}\pi_i\texttt{I}\,\mathrm{id}\,\texttt{E}\mathsf{CAT}(\mathsf{UNI}) : \phi\pi_i = \mathrm{id} \in \mathsf{UNI}(X_i, X_i),$

$[3] := \Lambda j \in J \,.\, \texttt{E}\phi\texttt{E}\pi_j\texttt{EUniformity}(\mathcal{U}_{X_j}) : \forall j \in J \,.\, \phi\pi_j = x_j \in \mathsf{UNI}(X_i, X_j),$

$[4] := \texttt{E}\prod_{i \in I} X_i\texttt{InitialUniformityUniversalProperty}[2][3] : \phi \in \mathsf{UNI}\left(X_i, \prod_{i \in I} X_i\right),$

$[5] := \texttt{UniformMapsPreserveCauchyFilters}[1][4] : \texttt{CauchyFilterbase}\left(\prod_{i \in I} X_i, \mathcal{F}'\right),$

$f := \lim \mathcal{F}' \in \prod_{i \in I} X_i,$

$\texttt{Assume}\ O \in \mathcal{U}(f_i),$

$O' := O \times_i \prod_{j \in J} X_j \in \mathcal{U}(f),$

$\Big(F', [6]\Big) := \texttt{E}\lim[5](O') : \sum F' \in \mathcal{F}' \;.\; F' \subset O',$

$F := \pi_i(F') :?X,$

$[7] := \texttt{E}F\texttt{E}\mathcal{F}' : F \in \mathcal{F},$

$[O.*] := \texttt{E}F\texttt{E}O'[6] : F \subset O;$

$\rightsquigarrow [1.*] := \texttt{I}\lim : f_i \in \lim \mathcal{F};$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow: \texttt{CompleteUniformSpace}\left(\prod_{i \in I} X_i\right) \Rightarrow \forall i \in I \;.\; \texttt{CompleteUniformSpace}(X_i),$

$\texttt{Assume}\ [2] : \forall i \in I \;.\; \texttt{CompleteUniformSpace}(X_i),$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase}\left(\prod_{i \in I} X_i\right),$

$[3] := \Lambda i \in I \;.\; \texttt{UniformMapsPreserveCauchyFilters}\left(\prod_{i \in I} X_i, X_i, \pi_i, \mathcal{F}\right) :$

$\quad : \forall i \in \mathcal{I} \;.\; \texttt{CauchyFilterbase}\Big(X_i, \pi_i(\mathcal{F})\Big),$

$f := \Lambda i \in I \;.\; \lim \pi_i(\mathcal{F}) \in \prod_{i \in I} X_i,$

$\texttt{Assume}\ O : \mathcal{U}(f),$

$\Big(J, E, [4]\Big) := \texttt{EproductTopology}(I, X, O) : \sum J : \texttt{Finite}(I) \;.\; \sum E : \prod_{j \in J} \mathcal{T}(X_j) \;.\; f \in \prod_{j \in J} E_j \times \prod_{j \in J^\complement} X_j \subset O,$

$\Big(F, [5]\Big) := \texttt{E}f(E) : \sum F \in \prod_{j \in J} \pi_j(\mathcal{F}) \;.\; F_j \subset E_j,$

$[6] := \Lambda j \in J \;.\; \texttt{E}F\texttt{E}\pi_j\texttt{EFilter}\left(\prod_{i \in I} X_i, \mathcal{F}\right) : \forall j \in J \;.\; F_j \times_j \prod_{i \in \{j\}^\complement} X_i \in \mathcal{F},$

$[7] := \texttt{EFilter}\left(\prod_{i \in I} X_i, \mathcal{F}\right)[6] : \prod_{j \in J} F_j \times \prod_{j \in J^\complement} X_j \in \mathcal{F},$

$[O.*] := [7][5][4] : \prod_{j \in J} F_j \times \prod_{j \in J^\complement} X_j \subset O;$

$\rightsquigarrow [2.*] := \texttt{I}\lim : f \in \lim \mathcal{F};$

$\rightsquigarrow [*] := \texttt{I} \iff [1] : \texttt{CompleteUniformSpace}\left(\prod_{i \in I} X_i\right) \iff \forall i \in I \;.\; \texttt{CompleteUniformSpace}(X_i);$

$\square$

There is also an extension theorem.

1.4.3 Extension Theorem

$\texttt{UCExtensionTheorem} :: \forall X \in \mathsf{UNI} \;.\; \forall Y : \texttt{CompleteUniformSpace} \;.\; \forall D : \texttt{Dense}(X) \;.\; \forall \phi \in \mathsf{UNI}(D, Y) \;.$
$\;.\; \exists \Phi \in \mathsf{UNI}(X, Y) \;.\; \Phi_{|D} = \phi$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x \in X,$

$\mathcal{F}_x := \left\{ U(x) \cap D \middle| U \in \mathcal{U}_X \right\} :??D,$

$[1] := \mathtt{E}\mathcal{F}_x\mathtt{EUniformity}(X) : \texttt{Filterbase}(X, \mathcal{F}_x),$

$[2] := \mathtt{E}\mathcal{F}_x\mathtt{EuniformTopology}(X)\mathtt{EDense}(X, D) : \lim \mathcal{F}_x = x,$

$[3] := \texttt{ConvergenrFilterbaseIsCauchy}[2] : \texttt{CauchyFilterbase}(X, \mathcal{F}_x),$

$[4] := \texttt{UniformBasePreservesCauchyFilters}\Big(X, Y, \phi\Big)[3] : \texttt{CauchyFilterbase}\Big(Y, \phi(\mathcal{F}_x)\Big),$

$\Phi(x) := \texttt{ECompleteUniformSpace}\Big(Y, \phi(\mathcal{F}_x)\Big)\texttt{EConvergentFilterbase} \in \lim \phi(\mathcal{F}_x);$

$\rightsquigarrow \Phi := \mathtt{I}(\to) : X \to Y,$

$[1] := \mathtt{E}\Phi\texttt{ContinuousPreserveLimits} : \Phi_{|D} = \phi,$

$\texttt{Assume}\; V \in \mathcal{U}_Y,$

$\Big(W, [2]\Big) := \texttt{UniformityTrisection2}(Y, V) : \sum W \in \texttt{SymmetricConnector} \;\&\; \texttt{ClosedConnector}(Y) \;.$
$\;.\; W \circ W \circ W \subset V,$

$\Big(U, [3]\Big) := \mathtt{E}\mathsf{UNI}(D, X, \phi, W) : \sum U \in \mathcal{U}_D \;.\; (\phi \times \phi)(U) \subset W,$

$\Big(U', [4]\Big) := \texttt{EsubsetUniformity}(X, D, U) : \sum U' \in \mathcal{U}_X \;.\; U = U' \cap (D \times D),$

$\Big(O, [5]\Big) := \texttt{UniformityTrisection2}(X, U') : \sum O : \texttt{SymmetricConnector} \;\&\; \texttt{ClosedConnector}(X) \;.$
$\;.\; O \circ O \circ O \subset U',$

$\texttt{Assume}\; (a, b) \in O,$

$r := \texttt{EDense}(X, D)\mathtt{E}\widetilde{\mathcal{B}}_a(O) \in O(a) \cap D,$

$s := \texttt{EDense}(X, D)\mathtt{E}\widetilde{\mathcal{B}}_b(O) \in O(b) \cap D,$

$[6] := \texttt{ESymmetricConnector}(Y, O)[5] : (r, s) \in U',$

$[7] := [6][4][4] : \Big(\phi(r), \phi(s)\Big) \in U',$

$\texttt{Assume}\; N : \mathcal{U}\Big(\Phi(a)\Big),$

$\Big(F, [8]\Big) := \mathtt{E}\Phi(N) : \sum F \in \mathcal{F}_a \; \phi(F) \subset N,$

$\Big(I, [9]\Big) := \mathtt{E}\mathcal{F}_a(F) : \sum I \in \mathcal{U}_X \;.\; F = I(a) \cap D,$

$d := \texttt{EUniformity}(\mathcal{U}_X)\texttt{EDense}(X, D) \in I(a) \cap O(a) \cap D,$

$[10] := \mathtt{E}d[9][8] : \phi(d) \in N,$

$[11] := \mathtt{E}d\mathtt{E}r\texttt{ESymmetricConnector}(X, O)[5] : (d, r) \in O \circ O \subset U',$

$[12] := [11][3][4] : \Big(\phi(d), \phi(r)\Big) \in W,$

$[13] := \mathtt{I}\exists[12][10] : \exists N \cap W\Big(\phi(r)\Big);$

$\rightsquigarrow [8] := \texttt{EClosedConnector}(X, O) : \Phi(a) \in W\Big(\phi(r)\Big),$

$\texttt{Assume}\ N : \mathcal{U}\Big(\Phi(b)\Big),$

$\Big(F, [9]\Big) := \texttt{E}\Phi(N) : \sum F \in \mathcal{F}_b\ \phi(F) \subset N,$

$\Big(I, [10]\Big) := \texttt{E}\mathcal{F}_b(F) : \sum I \in \mathcal{U}_X\ .\ F = I(b) \cap D,$

$d := \texttt{EUniformity}(\mathcal{U}_X)\texttt{EDense}(X, D) \in I(b) \cap O(b) \cap D,$

$[11] := \texttt{E}d[10][9] : \phi(d) \in N,$

$[12] := \texttt{E}d\texttt{E}s\texttt{ESymmetricConnector}(X, O)[5] : (d, s) \in O \circ O \subset U',$

$[13] := [12][3][4] : \Big(\phi(d), \phi(s)\Big) \in W,$

$[14] := \texttt{I}\exists[13][10] : \exists N \cap W\Big(\phi(s)\Big);$

$\rightsquigarrow [9] := \texttt{EClosedConnector}(X, O) : \Phi(b) \in W\Big(\phi(s)\Big),$

$\Big[(x, y). *\Big] := \texttt{ESymmetricConnector}(Y, W)[7][8][9] : \Big(\Phi(a), \Phi(b)\Big) \in W \circ W \circ W \subset V;$

$\rightsquigarrow [*] := \texttt{I}\mathsf{UNI} : \Phi \in \mathsf{UNI}(X, Y);$

$\square$

$\texttt{UCUniqueExtensionTheorem} ::$

$:: \forall X \in \mathsf{UNI}\ .\ \forall Y : \texttt{CompleteUniformSpace}\ \&\ \texttt{T3}\ .\ \forall D : \texttt{Dense}(X)\ .\ \forall \phi \in \mathsf{UNI}(D, Y)\ .$

$.\ \exists! \Phi \in \mathsf{UNI}(X, Y)\ .\ \Phi_{|D} = \phi$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{UCbyUCRestricton} :: \forall X, Y \in \mathsf{UNI}\ .\ \forall \varphi \in \mathsf{TOP}(X, Y)\ .\ \forall D : \texttt{Dense}(X)\ .\ \varphi_{|D} \in \mathsf{UNI}(D, Y) \Rightarrow \varphi \in \mathsf{UNI}(X, Y)$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{UnimorphismExtension} :: \forall X, Y : \texttt{CompleteUniformSpace}\ \&\ \texttt{T3}\ .\ \forall A : \texttt{Dense}(X)\ .\ \forall B : \texttt{Dense}(Y)\ .\ A \cong_{\mathsf{UNI}} B$

$\texttt{Proof}\ =$

$\dots$

$\square$

There is also a notions of total boundednes. Subsets of uniform space are compacts iff they complete and totally bounded.

1.4.4 Total Boundednes

$$\texttt{Small} :: \prod_{X \in \mathsf{UNI}} \mathcal{U}_X \to ??X$$

$$A : \texttt{Small} \iff \Lambda U \in \mathcal{U}_X \,.\, A \times A \subset U$$

$\texttt{SmallSymmetricConnector}$ ::

$$:: \forall X \in \mathsf{UNI} \,.\, \forall U \in \mathcal{U}_X \,.\, \forall V : \texttt{SymmetricConnector}(X) \,.\, \forall x \in X \,.\, V \circ V \subset U \Rightarrow \texttt{Small}\Big(X, U, V(x)\Big)$$

$\texttt{Proof}$ =

$[1] := \texttt{ConnectrorProductContainment}(X, V, x) : V(x) \times V(x) \subset V \circ V,$

$[2] := [1][0] : V(x) \times V(x) \subset U,$

$[*] := \texttt{ISmall}[2] : \texttt{Small}\Big(X, U, V(x)\Big);$

$\square$

$$\texttt{TotallyBounded} :: \prod_{X \in \mathsf{UNI}} ??X$$

$$A : \texttt{TotallyBounded} \iff \forall U \in \mathcal{U}_X \,.\, \exists n \in \mathbb{N} \,.\, \exists C : \{1, \ldots, n\} \to \texttt{Small}(X, U) \,.\, A \subset \bigcap_{k=1}^{n} C_k$$

$\texttt{InTBEveryUltrafilterIsCauchy} :: \forall X : \mathsf{UNI} \,.\, \texttt{TotallyBounded}(X, X) \Rightarrow$

$$\Rightarrow \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{CauchyFilterbase}\Big(X, \mathcal{F}\Big)$$

$\texttt{Proof}$ =

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$\Big(n, C, [1]\Big) := \texttt{ETotallyBounded}(X, X, U) : \sum_{n=1}^{\infty} \sum \{1, \ldots, n\} \to \texttt{Small}(X, U) \,.\, X = \bigcup_{k=1}^{n} C_k,$

$\Big(k, [2]\Big) := \texttt{UltrafilterUnion}[1] : \sum_{k=1}^{n} C_k \in \mathcal{F},$

$[U.*] := \texttt{ESmall}(X, U, C_k) : C_k \times C_k \subset U;$

$\leadsto [*] := \texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}(X, \mathcal{F});$

$\square$

$\texttt{TotallyBoundedClosure} :: \forall X \in \mathsf{UNI} . \forall A : \texttt{TotallyBounded}(X) . \texttt{TotallyBounded}\left(X, \overline{A}\right)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U \in \mathcal{U}_X,$

$\left(V, [1]\right) := \texttt{ClosedConnectorsBaseExists} : \sum V : \texttt{ClosedConnector}(X) . V \subset U,$

$\left(n, C, [2]\right) := \texttt{ETotallyBounded}(X, A, V) : \sum_{n=1}^{\infty} \sum C : \{1, \dots, n\} \to \texttt{Small}(X, V) . A \subset \bigcup_{k=1}^{n} C_k,$

$[3] := \texttt{ESmall}(C) : \forall k \in \{1, \dots, n\} . C_k \times C_k \subset V,$

$[4] := \texttt{EClosedConnector}(X, V)[3][1] : \forall k \in \{1, \dots, n\} . \overline{C}_k \times \overline{C}_k \subset V \subset U,$

$[5] := \texttt{ISmall}[4] : \forall k \in \{1, \dots, n\} . \texttt{Small}\left(X, U, \overline{C}_k\right),$

$[U.*] := \texttt{FiniteClosureUnion}[4] : \overline{A} \subset \bigcup_{k=1}^{n} \overline{C}_k;$

$\leadsto [*] := \texttt{ITotallyBounded} : \texttt{TotallyBounded}\left(X, \overline{A}\right);$

$\square$

$\texttt{UCPreservesTB} :: \forall X, Y \in \mathsf{UNI} . \forall \varphi \in \mathsf{UNI}(X, Y) . \forall A : \texttt{TotallyBounded}(X) . \texttt{TotallyBounded}\left(Y, f(A)\right)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ V \in \mathcal{U}_X,$

$\left(U, [1]\right) := \texttt{EUniformlyContinuous}(X, Y, \varphi, V) : \sum_{U \in \mathcal{U}_X} (\varphi \times \varphi)(U) \subset V,$

$\left(n, C, [2]\right) := \texttt{ETotallyBounded}(X, A, U) : \sum_{n=1}^{\infty} \sum C : \{1, \dots, n\} \to \texttt{Small}(X, U) . A \subset \bigcap_{k=1}^{n} C_k,$

$[3] := \texttt{ESmall}(C) : \forall k \in \{1, \dots, n\} . C_k \times C_k \subset U,$

$[4] := \varphi[3][1] : \forall k \in \{1, \dots, n\} . \varphi(C_k) \times \varphi(C_k) \subset V,$

$[5] := \texttt{ISmall}[4] : \forall k \in \{1, \dots, n\} . \texttt{Small}\left(Y, V, \varphi(C_k)\right),$

$[V.*] := \texttt{UnionMap}\left(X, Y, \varphi\right)[2] : \varphi(A) \subset \bigcap_{k=1}^{n} \varphi(C_k);$

$\leadsto [*] := \texttt{ITotallyBounded} : \texttt{TotallyBounded}\left(Y, \varphi(A)\right);$

$\square$

$\texttt{TBByUltrafilters} :: \forall X : \mathsf{UNI} \,.\, \Big(\forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{CauchyFilterbase}\Big(X, \mathcal{F}\Big) \Rightarrow$

$\quad \Rightarrow \texttt{TotallyBounded}(X, X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \neg\texttt{TotallyBounded}(X, X),$

$\Big(U, [2]\Big) := \texttt{ETotallyBounded}[1] : \sum U : \texttt{SymmetricConnector}(X) \,.\, \forall A : \texttt{Finite}(X) \,.\, X \neq U(A),$

$\Big(x, [3]\Big) := \mathtt{E}U[2] : x : \mathbb{N} \to X \,.\, \forall n, m \in \mathbb{N} \,.\, n \neq m \Rightarrow (x_n, x_m) \notin U,$

$A := \Lambda n \in \mathbb{N} \,.\, \{x_m | m \geq n\} : \mathbb{N} \to ?X,$

$\texttt{Assume}\ \mathcal{F} : \texttt{Filter}(X),$

$\texttt{Assume}\ [4] : \operatorname{Im} A \subset \mathcal{F},$

$\texttt{Assume}\ [5] : \texttt{CauchyFilterbase}(X, \mathcal{F}),$

$\Big(F, [6]\Big) := \texttt{ECauchyFilterbase}(X, \mathcal{F}, U) : \sum_{F \in \mathcal{F}} F \times F \subset U,$

$\Big(N, [7]\Big) := \texttt{EFilter}(X, \mathcal{F}, F)[4]\mathtt{E}A : \sum N : \texttt{Infinite}(\mathbb{N}) \,.\, \forall n \in N \,.\, x_n \in F,$

$[\mathcal{F}.*] := [7][3] : \bot;$

$\rightsquigarrow [4] := \mathtt{I}\forall : \forall \mathcal{F} : \texttt{Filter}(X) \,.\, \operatorname{Im} A \subset \mathcal{F} \Rightarrow \neg\texttt{CauchyFilterbase}(X, \mathcal{F}),$

$\Big(\mathcal{F}, [5]\Big) := \texttt{UltrafilterTHM}(X, \operatorname{Im} A) : \sum \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \operatorname{Im} A \subset \mathcal{F},$

$[6] := [4][5] : \neg\texttt{CauchyFilterbase}(X, \mathcal{F}),$

$[7] := [0](\mathcal{F}) : \texttt{CauchyFilterbase}(X, \mathcal{F}),$

$[1.*] := [6][7] : \bot;$

$\rightsquigarrow [*] := \mathtt{E}\bot : \texttt{TotallyBounded}(X, X);$

$\square$

$\texttt{UniCompactenessTHM} :: \forall X \in \mathsf{UNI} \,.\, \texttt{Compact}(X) \iff \texttt{TotallyBounded} \,\&\, \texttt{CompleteUniformSpace}(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{Compact}(X),$

$[2] := \texttt{CompactnessByUltrafilters} : \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{ConvergentFilterbase}(X, \mathcal{F}),$

$[3] := \texttt{EveryConvergenFilterbaseIsCauchy}[2] : \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{CauchyFilterbase}(X, \mathcal{F}),$

$[1.*.1] := \texttt{TBByUltrafilters}[3] : \texttt{TotallyBounded}(X, X),$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase}(X),$

$\Big(x, [4]\Big) := \texttt{inCompactFilterHasCluster} : \sum_{x \in X} \texttt{Cluster}\Big(X, \mathcal{F}, x\Big),$

$[\mathcal{F}.*] := \texttt{CuychyClustersAreLimits} : x \in \lim \mathcal{F};$

$\rightsquigarrow [4] := \texttt{IConvergentFilterbaseI}\forall : \forall \mathcal{F} : \texttt{CauchyFilterbase}(X) \,.\, \texttt{ConvergentFilterbase}(X, \mathcal{F}),$

$[1.*.2] := \texttt{ICompleteUniformSpace}[4] : \texttt{CompleteUniformSpace}(X, \mathcal{F});$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow : \texttt{Compact}(X) \Rightarrow \texttt{TotallyBounded} \,\&\, \texttt{CompleteUniformSpace}(X),$

$\texttt{Assume}\ [2] : \texttt{TotallyBounded} \,\&\, \texttt{CompleteUniformSpace}(X),$

$[3] := \texttt{InTBEveryUltrafilterIsCauchy}(X) : \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{CauchyFilterbase}(X, \mathcal{F}),$

$[4] := \texttt{ECompleteUniformSpace}(X)[2] : \forall \mathcal{F} : \texttt{Ultrafilter}(X) \,.\, \texttt{ConvergentFilterbase}(X, \mathcal{F}),$

$[2.*] := \texttt{CompactByUltrafilters}[4] : \texttt{Compact}(X);$

$\rightsquigarrow [*] := \mathtt{I} \Rightarrow \mathtt{I} \iff [1] : \texttt{Compact}(X) \iff \texttt{TotallyBounded} \,\&\, \texttt{CompleteUniformSpace}(X);$

$\square$

1.4.5 Bounded Sets [∞]

There is also boundedness, but this property is wierd. This chapter will be written then there is a demand for it.

$$\texttt{Bounded} :: \prod_{X \in \mathsf{UNI}} ??X$$

$$A : \texttt{Bounded} \iff \forall U \in \mathcal{U}_X \;.\; \exists n \in \mathbb{N} \;.\; \exists F : \texttt{Finite}(X) \;.\; A \subset U^{\circ n}(F)$$

1.5 Special Constructions

Many operation which can be used to construct metric spaces from given topological data, can be also extended for uniformities.

1.5.1 Uniformization

As asnalogy of metrization there is an uniformization.

$$\texttt{ringUniformity} :: \prod_{X \in \mathsf{TOP}} \texttt{Uniformity}(X)$$

$$\texttt{ringUniformity}\,() = \mathcal{C}_X := \mathcal{I}_X\Big(C(X), \mathbb{R}, \mathrm{id}_{C(X)}\Big)$$

$$\texttt{CompletelyRegularUniformization} :: \forall X : \texttt{CompletelyRegular} \,.\, (X, \mathcal{C}_X) \cong_{\mathsf{TOP}} X$$

Proof $=$

...

□

$$\texttt{EveryMetricSpaceAdmitsCompleteStruct} :: \forall X \in \mathsf{MS} \,.\, \exists \mathcal{U} : \texttt{Uniformity}(X) \,.$$
$$.\, \texttt{CompleteUniformSpace}(X, \mathcal{U}) \;\&\; X \cong_{\mathsf{TOP}} (X, \mathcal{U})$$

Proof $=$

$\mathcal{U} := \mathbb{B}_X \vee \mathcal{C}_X : \texttt{Uniformity}(X),$

Assume $\mathcal{F} : \texttt{CauchyFilterbase}(X, \mathcal{U}),$

$[1] := \texttt{E}\mathcal{U}\texttt{SupUniformityCauchyFilterbase}(X, \mathcal{U}, \mathcal{F}) : \texttt{CauchyFilterbase}(X, \mathcal{B}_X),$

$\Big(f, [2]\Big) := \texttt{ECompletion}(X, \widehat{X}, \mathcal{F}) : \sum f \in \widehat{X} \,.\, f = \lim \mathcal{F},$

$\phi := \Lambda x \in X \,.\, d(x, f) \in C(X),$

Assume $[3] : f \notin X,$

$[4] := \texttt{E}\mathcal{U}\texttt{E}\phi\texttt{continuousInverse} : \dfrac{1}{\phi} \in \mathsf{UNI}(X, \mathbb{R}),$

$[5] := \texttt{E}f : \lim \dfrac{1}{\phi(\mathcal{F})} = \infty,$

$[6] := \texttt{UniformMapsPreserveFilters}[5][4]\texttt{ECauchyFilterbase} : \bot;$

$\leadsto [\mathcal{F}.*] := \texttt{E}\bot : f \in X;$

$\leadsto [*] := \texttt{ICompleteUniformSpace} : \texttt{CompleteUniformSpace}(X, \mathcal{U});$

□

$$\texttt{CompactUniformityIsExists} :: \forall X \in \mathsf{HC} \,.\, \exists! \mathcal{U} : \texttt{Uniformity}(X) \,.\, (X, \mathcal{U}) \cong_{\mathsf{TOP}} X$$

Proof $=$

...

□

$$\texttt{betaUniformity} :: \prod X : \texttt{Tychonoff} \,.\, \texttt{Uniformity}(X)$$

$$\texttt{betaUniformity}\,() = \mathcal{B}_X := \mathcal{U}_{\beta X} \cap (X \times X)$$

$$\texttt{alphaUniformity} :: \prod X : \texttt{LocallyComapct} \;\&\; \texttt{T2} \,.\, \texttt{Uniformity}(X)$$

$$\texttt{alphaUniformity}\,() = \mathcal{A}_X := \mathcal{U}_{\omega X} \cap (X \times X)$$

$\texttt{NormalBetaUniformity} :: \forall X : \texttt{T4} \ \& \ \neg\texttt{Compact} \ .$
$\quad . \ \neg\texttt{CompleteUniformSpace}(X, \mathcal{B}_X) \ \& \ \texttt{SequenceCompleteUniformSpace}(X, \mathcal{B}_X)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{MetrizableUniforSpace} :: ?\mathsf{UNI}$

$X : \texttt{MetrizableUniformSpace} \iff \exists d : \texttt{Metric}(X) \ . \ \mathcal{U}_X = \mathbb{B}_{(X,d)}$

$\texttt{SemimetrizableUniforSpace} :: ?\mathsf{UNI}$

$X : \texttt{SemimetrizableUniformSpace} \iff \exists d : \texttt{Semimetric}(X) \ . \ \mathcal{U}_X = \mathbb{B}_{(X,d)}$

$\texttt{NormalBetaIsNotMetrizable} :: \forall X : \texttt{T4} \ \& \ \neg\texttt{Compact} \ . \ \neg\texttt{MetrizableUniformSpace}(X, \mathcal{B}_X)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{TychonoffBetaIsNotMetrizable} :: \forall X : \texttt{Tychonoff} \ \& \ \neg\texttt{Compact} \ . \ \neg\texttt{MetrizableUniformSpace}(X, \mathcal{B}_X)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{RingUniformotyCompleteIffRealCompact} :: \forall X : \texttt{Tychonoff} \ . \ \texttt{CompleteUniformSpace}(X, \mathcal{C}_X) \iff \texttt{Realcor}$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{Uniformazable} :: ?\mathsf{TOP}$

$X : \texttt{Uniformazable} \iff \exists \mathcal{U} \in \texttt{Uniformity}(X) \ . \ (X, \mathcal{U}) \cong_{\mathsf{TOP}} X$

Gages is a different kind of generalizations of metric spaces. It uses a set of metrics to define its topology. Gage spaces are completely regular, but not neccesarly normal.

1.5.2 Gages

$$\texttt{GageSubbase} :: \forall X \in \mathsf{SET} \,.\, \forall \mathfrak{R} :?\texttt{Semimetric}(X) \,.\, \exists \mathfrak{R} \Rightarrow \texttt{Subbase}\Big(\big\{\mathbb{B}_\rho(x,\varepsilon)\big|\rho \in \mathfrak{R}, x \in X, \varepsilon \in \mathbb{R}_{++}\big\}\Big)$$

$\texttt{Proof} \;=$

...

$\square$

$$\texttt{gageTopology} :: \prod X \in \mathsf{SET} \,.\, \texttt{NonEmpty}\;\texttt{Semimetric}(X) \to \texttt{Topology}(X)$$

$$\texttt{gageTopology}\,(\mathfrak{R}) = \mathcal{T}_{\mathfrak{R}} := \Big\langle \big\{\mathbb{B}_\rho(x,\varepsilon)\big|\rho \in \mathfrak{R}, x \in X, \varepsilon \in \mathbb{R}_{++}\big\} \Big\rangle$$

$$\texttt{gageUniformity} :: \prod X \in \mathsf{SET} \,.\, \texttt{NonEmpty}\;\texttt{Semimetric}(X) \to \texttt{Uniformity}(X)$$

$$\texttt{gageUniformity}\,(\mathfrak{R}) = \mathcal{U}_{\mathfrak{R}} := \bigvee_{\rho \in \mathfrak{R}} \mathbb{B}_{(X,\rho)}$$

$$\texttt{GageTopology} :: \forall X \in \mathsf{SET} \,.\, \forall \mathfrak{R} :?\texttt{Semimetric}(X) \,.\, (X, \mathcal{T}_{\mathfrak{R}}) \cong_{\mathsf{TOP}} (X, \mathcal{U}_{\mathfrak{R}})$$

$\texttt{Proof} \;=$

...

$\square$

$$\texttt{GageSpace} :: ?\mathsf{TOP}$$

$$X : \texttt{GageSpace} \iff \exists \mathfrak{R} : \texttt{NonEmpty}\;\texttt{Semimetric}(X) \,.\, X \cong_{\mathsf{TOP}} (X, \mathcal{T}_{\mathfrak{R}})$$

$$\texttt{EveryGageSpaceIsUniformizable} :: \forall X : \texttt{GageSpace} \,.\, \texttt{Uniformizable}(X)$$

$\texttt{Proof} \;=$

...

$\square$

$$\texttt{EveryGageSpaceIsCompletelyRegular} :: \forall X : \texttt{GageSpace} \,.\, \texttt{CompletelyRegular}(X)$$

$\texttt{Proof} \;=$

...

$\square$

1.5.3 Metrization

Uniform spaces can be metrized if they have countable base of uniformity. It turnes out that gage topologies and uniform topologies are the same thing.

$$\texttt{triangulization} :: \prod_{X \in \mathsf{SET}} (X \times X \to \mathbb{R}_+) \to \texttt{TriangleIneq}(X)$$

$$\texttt{triangulization}\,(f) = d_f := \Lambda x, y \in X\ .\ \inf \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \middle| n \in \mathbb{N}, u : \{1, \dots, n\} \to X, u_1 = x, u_n = y \right\}$$

$\texttt{Assume}\ x, y, z \in X,$

$$A := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \middle| n \in \mathbb{N}, u : \{1, \dots, n\} \to X, u_1 = x, u_n = y \right\} :?\mathbb{R}_+,$$

$$B := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \middle| n \in \mathbb{N}, u : \{1, \dots, n\} \to X, u_1 = y, u_n = z \right\} :?\mathbb{R}_+,$$

$$C := \left\{ \sum_{i=1}^{n-1} f(u_i, u_{i+1}) \middle| n \in \mathbb{N}, u : \{1, \dots, n\} \to X, u_1 = x, u_n = z \right\} :?\mathbb{R}_+,$$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ b \in B,$

$$\Big(n, u, [1]\Big) := \mathtt{E}A(a) : \sum_{n=1}^{\infty} \sum u : \{1, \dots, n\} \to X\ .\ u_1 = x\ \&\ u_n = y\ \&\ a = \sum_{i=1}^{n-1} f(u_i, u_{i+1}),$$

$$\Big(m, v, [2]\Big) := \mathtt{E}B(b) : \sum_{m=1}^{\infty} \sum u : \{1, \dots, m\} \to X\ .\ v_1 = y\ \&\ v_n = z\ \&\ b = \sum_{i=1}^{m-1} f(v_i, v_{i+1}),$$

$$\Big[(a, b).*\Big] := [1][2]\mathtt{E}C : a + b = \sum_{i=1}^{n-1} f(u_i, u_{i+1}) + \sum_{i=1}^{m-1} f(v_i, v_{i+1}) \in C;$$

$$\rightsquigarrow [*] := \mathtt{I}d_f : d_f(x, z) \le d_f(x, y) + d_f(y, z);$$

$\square$

$\texttt{CountableUniformSpace} :: ?\mathsf{UNI}$

$X : \texttt{CountableUniformSpace} \iff \exists \mathcal{B} : \texttt{BaseOfUniformity}(X)\ .\ |\mathcal{B}| \le \aleph_0$

$\texttt{SemimetrizationOfUniformSpace} :: \forall X : \texttt{CountableUniformSpace} \;.\; \texttt{SemimetrizableUniformSpace}(X)$

$\texttt{Proof} \;=$

$\Big(\mathcal{B}, [1]\Big) := \texttt{ECountableUniformSpace}(X) : \sum \mathcal{B} : \texttt{BaseOfUniformity}(X) \;.\; |\mathcal{B}| \le \aleph_0,$

$B := \texttt{enumerate}(\mathcal{B}) : \mathbb{N} \twoheadrightarrow \mathcal{B},$

$\Big(V, [2]\Big) := \texttt{recursion}(\mathbb{Z}_+, X \times X, \Lambda V \in \mathcal{U}_X \;.\; \Lambda n \in \mathbb{N} \;.\; \texttt{UniformityTrisection}(X, V \cap B_n)) :$

$\quad : \sum V : \mathbb{N} \to \texttt{SymmetricConnector}(X) \;.\; \forall n \in \mathbb{Z}_+ \;.\; V_{n+1} \circ V_{n+1} \circ V_{n+1} \subset V_n \cap B_{n+1},$

$\lambda := \Lambda x, y \in X \;.\; \inf\{2^{-n} | n \in \mathbb{Z}_+, (x, y) \in V_n\} : X \times X \to \mathbb{R}_{++},$

$\rho := d_\lambda : \texttt{TriangleIneq}(X),$

$[3] := \texttt{E}\lambda\texttt{E}\rho : \texttt{Semimetric}(X, \rho),$

$\texttt{Assume}\; x, y, z, w \in X,$

$r := \max\Big(\lambda(w, x), \lambda(x, y), \lambda(y, z)\Big) : \mathbb{R}_{++},$

$\Big(n, [4]\Big) := \texttt{E}r\texttt{E}\lambda : \sum n \in \mathbb{Z}_+ \cup \{\infty\} \;.\; 2^{-n} = r,$

$[5] := [4]\texttt{E}r : \max\Big(\lambda(w, x), \lambda(x, y), \lambda(y, z)\Big) \le 2^{-n},$

$[6] := \texttt{E}\lambda[5] : (w, x), (x, y), (y, z) \in V_n,$

$[7] := \texttt{ESymmetricConnector}(X, V_n)[2][6] : (w, z) \in V_{n-1},$

$[8] := \texttt{I}\lambda[7] : \lambda(w, z) \le 2^{-n+1},$

$\Big[(x, y, z, w).*] := [8][4]\texttt{E}r : \lambda(w, z) \le 2\max\Big(\lambda(w, x), \lambda(x, y), \lambda(y, z)\Big);$

$\rightsquigarrow [4] := \texttt{I}\forall : \forall x, y, z, w \in X \;.\; \lambda(w, z) \le 2\max\Big(\lambda(w, x), \lambda(x, y), \lambda(y, z)\Big),$

$[5] := \texttt{E}\max[4] : \forall n \in \mathbb{N} \;.\; \forall x : \{1, \dots, n\} \to X \;.\; \lambda(x_1, x_n) \le 2\sum_{i=1}^{n-1} \lambda(x_i, x_{i+1}),$

$[6] := \texttt{E}\rho[5] : \rho \le \lambda \le 2\rho,$

$\texttt{Assume}\; U \in \mathcal{U}_X,$

$\Big(n, [7]\Big) := \texttt{EBaseOfUniformity}(X, \mathcal{B})[2] : \sum_{n=1}^{\infty} V_n \subset B_n \subset U,$

$[U.*] := [6][7] : \mathbb{B}_\rho(2^{-n}) \subset U;$

$\rightsquigarrow [8] := \texttt{EUniformity}(X, \mathbb{B}_\rho) : \mathcal{U}_X \subset \mathbb{B}_\rho,$

$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$

$\Big(n, [8]\Big) := \texttt{ExponentialLimit}(2, \varepsilon) : \sum n \in \mathbb{N} \;.\; 2^{-n} < \varepsilon,$

$[\varepsilon.*] := \texttt{E}\rho[8] : V_n \subset \mathbb{B}_\rho(\varepsilon);$

$\rightsquigarrow [9] := \texttt{EUniformity}(X, \mathcal{U}_X : \mathbb{B}_\rho \subset \mathcal{U}_X,$

$[*] := \texttt{ISetEq}[8][9] : \mathbb{B}_\rho = \mathcal{U}_X;$

$\square$

$\texttt{UniformGageSpace} :: ?\mathsf{UNI}$

$X : \texttt{UniformGageSpace} \iff \exists \mathfrak{R} : \texttt{NonEmptyp}\;\texttt{Semimetric}X \;.\; \mathcal{U}_X = \mathcal{U}_{\mathfrak{R}}$

$\texttt{EveryUniformSpaceIsAGage} :: \forall X \in \mathsf{UNI} \,.\, \texttt{GageSpace}(X)$

$\texttt{Proof} \;=$

$\mathfrak{U} := \{\mathcal{V} \in \texttt{Uniformity}(X) : \texttt{CountableUniformSpace}(X, \mathcal{V}) \;\&\; \mathcal{V} \subset \mathcal{U}_X\} :?\texttt{Uniformity}(X),$

$\Big(\rho, [1]\Big) := \texttt{SemimetrizationOfUniformSpace} : \rho : \mathfrak{U} \to \texttt{Semimetric}(X) \,.\, \forall \mathcal{U} \in \mathfrak{U} \,.\, \mathbb{B}_{\rho_{\mathcal{U}}} = \mathcal{U},$

$\mathfrak{R} := \operatorname{Im} \rho : \texttt{NonEmpty Semimetric}(X),$

$[2] := \mathtt{E}\mathfrak{U}\mathtt{I}\sup[1]\mathtt{I}U_{\mathfrak{R}} : \mathcal{U}_X = \bigvee \mathfrak{U} = \bigvee_{\rho \in \mathfrak{R}} \mathbb{B}_\rho = \mathcal{U}_{\mathfrak{R}},$

$[*] := \texttt{IGageUniformSpace}[2] : \texttt{GageUniformSpace}(X);$

$\square$

$\texttt{UniformizableIffCompletelyRegular} :: \forall X \in \mathsf{TOP} \,.\, \texttt{CompletelyRegular}(X) \iff \texttt{Uniformizable}(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{UnimorphicEmbedding} :: \prod X, Y \in \mathsf{UNI} \,.\, ?\mathsf{UNI}(X, Y)$

$\varphi : \texttt{UnimorphicEmbedding} \iff \texttt{Unimorphism}\Big(X, \varphi(X), \varphi^{|\varphi(X)}\Big)$

$\texttt{UnimorphicEmbeddingToAProduct} ::$

$$:: \forall X \in \mathsf{UNI} \,.\, \exists I \in \mathsf{SET} \,.\, \exists Y : I \to \mathsf{SMS} \,.\, \exists \texttt{UnimorphicEmbedding}\left(X, \prod_{i \in I} Y_i\right)$$

$\texttt{Proof} \;=$

$\Big(\mathfrak{R}, [1]\Big) := \texttt{EveryUniformSpaceIsAGage}(X) : \sum \mathfrak{R} : \texttt{NonEmpy Semimetric}(X) \,.\, \mathcal{U}_X = \mathcal{U}_{\mathfrak{R}},$

$\varphi := \Lambda x \in X \,.\, \Lambda \rho \in \mathfrak{R} \,.\, x : X \to X^{\mathfrak{R}},$

$[2] := \mathtt{E}\varphi : \forall \rho \in \mathfrak{R} \,.\, \varphi \pi_\rho = \mathrm{id}_X \in \mathsf{UNI}\Big(X, (X, \rho)\Big),$

$$[3] := \texttt{InitialUniformityUniversalProperty}[1] : \varphi \in \mathsf{UNI}\left(X, \prod_{\rho \in \mathfrak{R}} (X, \rho)\right),$$

$[4] := \mathtt{E}\varphi : \varphi(X) = \Delta^{\mathfrak{R}}(X),$

$\texttt{Assume } n \in \mathbb{N},$

$\texttt{Assume } \rho : \{1, \dots, n\} \to \mathfrak{R},$

$\texttt{Assume } \varepsilon : \{1, \dots, n\} \to \mathfrak{R},$

$R := \operatorname{Im} \rho : \texttt{Finite}(\mathfrak{R}),$

$$[5] := \mathtt{E}\texttt{productUniformity} : \Big(\Delta^{\mathfrak{R}}(X) \times \Delta^{\mathfrak{R}}(X)\Big) \cap \prod_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i) \times X^{R^{\complement}} \in \mathcal{U}(\dots),$$

$$[n.*] := \mathtt{E}\varphi : (\varphi^{-1} \times \varphi^{-1})\Big(\Delta^{\mathfrak{R}}(X) \times \Delta^{\mathfrak{R}}(X)\Big) \cap \prod_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i) \times X^{R^{\complement}} \subset \bigcap_{i=1}^n \mathbb{B}_{\rho_i}(\varepsilon_i);$$

$$\leadsto [*] := \mathtt{E}\texttt{gageUniformity}[1]\mathtt{I}\texttt{UnimorphicEmbedding} : \texttt{UnimorphicEmbedding}\left(X, \prod_{\rho \in \mathfrak{R}} (X, \rho), \varphi\right);$$

$\square$

Also there is a completion.

1.5.4 Completion

$$\texttt{Completion} :: \prod_{X \in \mathsf{UNI}} ?\sum Y : \texttt{CompleteUniformSpace} \,.\, \texttt{UnimorphicEmbedding}(X, Y)$$

$$\iota : \texttt{Completion} \iff \texttt{Dense}\Big(Y, \iota(X)\Big)$$

$$\texttt{SeparableCompletion} :: \prod_{X \in \mathsf{UNI}} ?\texttt{Completion}(X)$$

$$(Y, \iota) : \texttt{SeparableCompletion} \iff Y \in \mathsf{UNIS}$$

$$\texttt{EveryUniformSpaceHasACompltion} :: \forall X \in \mathsf{UNI} \,.\, \exists \texttt{Completion}(X)$$

$\texttt{Proof} =$

$$\Big(I, Y, \varphi\Big) := \texttt{UnimorphicEmbeddingToAProduct} : \sum_{I \in \mathsf{SET}} \sum_{Y : I \to \mathsf{SMS}} \sum \varphi : \texttt{UniomorphicEmbedding}\left(X, \prod_{i \in I} Y_i\right),$$

$$\Big(\widehat{Y}, \iota, [1]\Big) := \Lambda i \in I \,.\, \texttt{SemimetricCompletionExists}(Y_i) : \prod_{i \in I} \texttt{Completion}(Y_i),$$

$$Z := \mathrm{cl}_{\left(\prod_{i \in I} \widehat{Y}_i\right)} \varphi(X) : \texttt{Closed}\left(\prod_{i \in I} \widehat{Y}_i\right),$$

$$[2] := \texttt{CompleteProductTHM}(I, \widehat{Y}_i)\texttt{ClosedOfCompleteIsComplete} : \texttt{CompleteUniformSpace}(Z),$$

$$\psi := \varphi \prod_{i \in I} \iota_i : \texttt{HomeomorphicEmbedding}(X, Z),$$

$$[3] := \texttt{E}Z\texttt{IDense} : \texttt{Dense}\Big(Z, \psi(X)\Big),$$

$$[*] := \texttt{ICompletion} : \texttt{Completion}(X, Z, \psi);$$

$\square$

$$\texttt{UnimorphicEmbeddingToAMetricProduct} ::$$

$$:: \forall X \in \mathsf{UNIS} \,.\, \exists I \in \mathsf{SET} \,.\, \exists Y : I \to \mathsf{MS} \,.\, \exists \texttt{UnimorphicEmbedding}\left(X, \prod_{i \in I} Y_i\right)$$

$\texttt{Proof} =$

$$\Big(I, Y, \varphi\Big) := \texttt{UnimorphicEmbeddingToAProduct} : \sum_{I \in \mathsf{SET}} \sum_{Y : I \to \mathsf{SMS}} \sum \varphi : \texttt{UniomorphicEmbedding}\left(X, \prod_{i \in I} Y_i\right),$$

$$\Big(Z, \phi\Big) := \Lambda i \in I \,.\, \texttt{MetricQuotient} : \prod_{i \in I} \sum_{Z_i \in \mathsf{MS}} \texttt{Isometry}(Y_i, Z_i),$$

$$\psi := \varphi \prod_{i \in I} \psi_i \in \mathsf{UNI}\left(X, \prod_{i \in I} Z_i\right),$$

$$[1] := \texttt{E}\psi\texttt{E}\mathsf{UNIS}(X)\texttt{EUnimorphicEmbedding}(\varphi) : \texttt{Injective}\left(X, \prod_{i \in I} Z_i, \psi\right),$$

$$[2] := [1] \mathit{Elim}\texttt{Isometry}(\phi) : \texttt{IsometricEmbedding}\left(\varphi(X), \prod_{i \in I} Z_i \left(\prod_{i \in I} \phi_i\right)_{|\varphi(X)}\right),$$

$$[*] := \texttt{E}\psi[2] : \texttt{UnimorphicEmbedding}\left(X, \prod_{i \in I} Z_i, \psi\right);$$

$\square$

$\texttt{SeparatedSpaceHasSeparetedComplition} :: \forall X \in \mathsf{UNIS} \;.\; \exists \texttt{SeparableCompletion}(X)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SeparebleCompletionAreUnique} :: \forall X \in \mathsf{UNIS} \;.\; \forall (Y, \iota), (Y', \iota') : \texttt{SeparableCompletion}(X) \;.\; Y \cong_{\mathsf{UNI}} Y'$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{separableCompletion} :: \prod X \in \mathsf{UNIS} \;.\; \texttt{SeparableCompletion}(X)$

$\texttt{separableCompletion}\,() = (\gamma X, \iota_{\gamma X}) := \texttt{SeparedSpaceHasSeparatedComplition}(X)$

$\texttt{UniformlyContinuousByDenseSubset} ::$

$:: \forall X \in \mathsf{UNI} \;.\; \forall Y \in \mathsf{UNIS} \;.\; \forall \varphi \in \mathsf{TOP}(X, Y) \;.\; \forall D : \texttt{Dense}(X) \;.\; \varphi_{|D} \in \mathsf{UNI}(D, Y) \Rightarrow \varphi \in \mathsf{UNI}(X, Y)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UniformityEqualityTHM} :: \forall Y \in \mathsf{SET} \;.\; \forall X \subset X \;.\; \forall \mathcal{U}, \mathcal{V} \in \texttt{Uniformity}(X) \;.$

$(Y,\mathcal{U}), (Y, \mathcal{V}) \in \mathsf{UNIS} \;\&\; \texttt{Dense}\Big((Y,\mathcal{U}) \;\&\; (Y, \mathcal{V}), X\Big) \;\&\; \mathcal{U} \cap X \times X = \mathcal{V} \cap X \times X \Rightarrow \mathcal{U} = \mathcal{V}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

1.5.5 Uniformly Continuous Metric [*]

Will be written on demand.

1.6 Function Spaces

Many sets of functions have natural uniformities.

1.6.1 Pointwise Unifomity

Functions with their values in an uniform space can be given an uniformity which corresponds to a pointwise convergence. Turns out it corresponds to the product uniformity.

$$\texttt{evaluation} :: \prod_{X,Y \in \mathsf{SET}} X \to (X \to Y) \to Y$$

$$\texttt{evaluation}\,(x, f) = \epsilon_x(f) := f(x)$$

$$\texttt{pointwiseUniformSpace} :: \mathsf{SET} \times \mathsf{UNI} \to \mathsf{UNI}$$

$$\texttt{pointwiseUniformSpace}\,(X, Y) = X \to_{\mathrm{pt}} Y := \Big(X \to Y, \mathcal{I}(X, Y, \epsilon)\Big)$$

$\texttt{PointwiseUniformSpaceIsAProduct} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, (X \to_{\mathrm{pt}} Y) \cong_{\mathsf{UNI}} Y^X$

Proof $=$

...

□

$\texttt{PointwisePreservesSeparation} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNIS} \,.\, (X \to_{\mathrm{pt}} Y) \in \mathsf{UNIS}$

Proof $=$

...

□

$\texttt{PointwisePreservesCompleteness} :: \forall X \in \mathsf{SET} \,.\, \forall Y : \texttt{CompleteUniformSpace} \,.$

$\quad . \texttt{CompleteUniformSpace}(X \to_{\mathrm{pt}} Y)$

Proof $=$

...

□

$\texttt{PointwisePreservesSeparatedCompleteness} :: \forall X \in \mathsf{SET} \,.\, \forall Y : \texttt{SequenceCompleteUniformSpace} \,.$

$\quad . \texttt{SequenceCompleteUniformSpace}(X \to_{\mathrm{pt}} Y)$

Proof $=$

...

□

$\texttt{PointwiseFilterConvergence} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall f : X \to Y \,.\, \forall \mathcal{F} : \texttt{Filter}(X \to_{\mathrm{pt}} Y) \,.$

$\quad . f \in \lim \mathcal{F} \iff \forall x \in X \,.\, f(x) \in \lim \epsilon_x(\mathcal{F})$

Proof $=$

...

□

$\texttt{PointwiseCauchyFilters} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall f : X \to Y \,.\, \forall \mathcal{F} : \texttt{Filter}(X \to_{\mathrm{pt}} Y) \,.$

$\quad . \texttt{CauchyFilterbase}(X \to_{\mathrm{pt}} Y, \mathcal{F}) \iff \forall x \in X \,.\, \texttt{CauchyFilterbase}\Big(X \to_{\mathrm{pt}} Y, \epsilon_x(\mathcal{F})\Big)$

Proof $=$

...

□

$\texttt{PointwiseCompactness} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall K \subset (X \to_{\mathrm{pt}} Y) \,.$

$. \, \texttt{CompactSubset}(X \to_{\mathrm{pt}} Y, K) \iff \texttt{Closed}(X \to_{\mathrm{pt}} Y, K) \;\&\; \forall x \in X \,.\, \texttt{CompactSubset}\left(Y, \overline{\epsilon_x(K)}\right)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{PointwiseCompleteness} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall K \subset (X \to_{\mathrm{pt}} Y) \,.$

$. \, \texttt{CompleteUniformSpace}(K) \iff \texttt{Closed}(X \to_{\mathrm{pt}} Y, K) \;\&\; \forall x \in X \,.\, \texttt{CompleteUniformSpace}\left(\overline{\epsilon_x(K)}\right)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.6.2 Uniformity of Uniform Convergence

It is also possible to define uniformity of uniform convergence.

$$\texttt{uniformUniformity} :: \prod_{X \in \mathsf{SET}} \prod_{Y \in \mathsf{UNI}} \texttt{Uniformity}(X \to Y)$$

$$\texttt{uniformUniformity}\,() = \mathcal{UU}(X,Y) := \left\{ \left\{ (f,g) \in (X \to Y)^2 \middle| \forall x \in X \,.\, \big(f(x), g(x)\big) \in U \right\} \middle| U \in \mathcal{U}(Y) \right\}$$

$$\texttt{uniformConvergenceSpace} :: \mathsf{SET} \to \mathsf{UNI} \to \mathsf{UNI}$$

$$\texttt{uniformConvergenceSpace}\,(X,Y) = (X \to_{\mathcal{U}} Y) := \Big(X \to Y, \mathcal{UU}(X,Y) \Big)$$

$$\texttt{PointwiseFilterConvergence} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall f : X \to Y \,.\, \forall \mathcal{F} : \texttt{Filter}(X \to_{\mathcal{U}} Y) \,.$$
$$.\; f \in \lim_{\mathcal{U}} \mathcal{F} \iff \texttt{CauchyFilterbase}(X \to_{\mathcal{U}} Y, \mathcal{F}) \;\&\; f \in \lim_{\text{pt}} \mathcal{F}$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : f \in \lim_{\mathcal{U}} \mathcal{F},$

$[2] := \texttt{ConvergentIsCauchy}[1] : \texttt{CauchyFilterbase}(X \to_{\mathcal{U}} Y, \mathcal{F}),$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ O : \mathcal{U}\Big(f(x)\Big),$

$[3] := \texttt{EuniformTopologyE}\mathcal{UU}(X,Y) : O^X \in \mathcal{T}(X \to_{\mathcal{U}} Y),$

$\Big(F, [4]\Big) := [1][3] : \sum F \in \mathcal{F} \,.\, F \subset O^X,$

$[x.*] := \epsilon_x[4] : \epsilon_x(F) \subset O;$

$\rightsquigarrow [3] := \texttt{I}\forall : \forall x \in X \,.\, f(x) \in \lim \epsilon_x \mathcal{F},$

$[1.*] := \texttt{PointwiseFilterConvergence}[3] : f \in \lim_{\text{pt}} \mathcal{F};$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : f \in \lim_{\mathcal{U}} \mathcal{F} \Rightarrow \texttt{CauchyFilterbase}(X \to_{\mathcal{U}} Y, \mathcal{F}) \;\&\; f \in \lim_{\text{pt}} \mathcal{F},$

$\texttt{Assume}\ [2] : \texttt{CauchyFilterbase}(X \to_{\mathcal{U}} Y, \mathcal{F}),$

$\texttt{Assume}\ [3] : f \in \lim_{\text{pt}} \mathcal{F},$

$\texttt{Assume}\ O \in \mathcal{U}_{X \to_{\mathcal{U}} Y}(f),$

$\Big(U, [4]\Big) := \texttt{EuniformTopologu}(O) : \sum U \in \mathcal{UU}(X,Y) \,.\, U(f) \subset O,$

$\Big(W, [5]\Big) := \texttt{SymmetricConnectorBaseExists}(Y, V)\texttt{ClosedConnectorBaseExists}(Y, V) :$
$: \sum W : \texttt{SymmetricConnector} \;\&\; \texttt{ClosedConnector}(X \to_{\mathcal{U}} Y) \,.\, W \subset U,$

$\Big(V, [6]\Big) := \texttt{E}\mathcal{UU}(X, Y, U) : \sum V \in \texttt{SymmetricConnector} \;\&\; \texttt{ClosedConnector}(Y) \,.$
$.\; W = \left\{ (f,g) \in (X \to Y)^2 \middle| \forall x \in X \,.\, \big(f(x), g(x)\big) \in V \right\},$

$\Big(F, [7]\Big) := \texttt{ECauchyFilterbase}(X \to_{\mathcal{U}} Y, \mathcal{F}, U) : \sum F \in \mathcal{F} \,.\, F \times F \subset W,$

$\texttt{Assume}\ g \in F,$

$[9] := [7][8]\texttt{ESymmetricConnector}(Y, V) : \forall x \in X \,.\, F(x) \subset V\Big(g(x)\Big),$

$\texttt{Assume}\ x \in X,$

$[10] := [3](x) : \lim \mathcal{F}(x) = f(x),$

$\texttt{Assume}\ E \in \mathcal{U}_Y,$

$\Big(G, [11]\Big) := \texttt{EConvergent}[10] : \sum G \in \mathcal{F} \,.\, G(x) \subset E\Big(f(x)\Big),$

$[E.*] := [11][9](x) : F(x) \cap G(x) \subset V\Big(g(x)\Big) \cap E\Big(f(x)\Big);$

$\rightsquigarrow [g.*] :=$ `E`$\mathcal{U}_Y$`EClosedConnector`$(Y, V) : f(x) \in V\Big(g(x)\Big);$

$\rightsquigarrow [O.*] := [4][5][6] : G \subset O;$

$\rightsquigarrow [2.*] :=$ `I` $\lim : \lim\limits_{\mathcal{U}} \mathcal{F} = f;$

$\rightsquigarrow [*] :=$ `I` $\iff : \lim\limits_{\mathcal{U}} \mathcal{F} = f \iff$ `CauchyFilterbase`$(X \to_{\mathcal{U}} Y, \mathcal{F})\ \&\ f \in \lim\limits_{\mathrm{pt}} \mathcal{F};$

$\square$

1.6.3 Uniform Convergence over S

For any family $\mathcal{S}$ of subset of the domain, there is possible to define a uniformity for uniform convergence over $\mathcal{S}$. It turns out that both pointwise and uniform convergence are special cases of this.

$$\texttt{fUniformity} :: \prod X \in \mathsf{SET} \,.\, \prod Y \in \mathsf{UNI} \,.\, ??X \to \texttt{Uniformity}(X \to Y)$$

$$\texttt{fUniformity}\,(\mathcal{S}) = \mathcal{F}(X,Y,\mathcal{S}) := \mathcal{I}\Big(\mathcal{S}, \Lambda S \in \mathcal{S} \,.\, \big(S \to Y, \mathcal{UU}(S,Y)\big), \Lambda S \in \mathcal{S} \,.\, \Lambda f : X \to Y \,.\, f_{|S}\Big)$$

$$\texttt{FiniteFUniformityIsPointwise} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \Big(X \to Y, \mathcal{F}(X,Y,\texttt{Finite}(Y))\Big) \cong_{\mathsf{UNI}} X \to_{\mathrm{pt}} Y$$

Proof $=$

. . .

□

$$\texttt{GlobaFUniformityIsUniform} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \Big(X \to Y, \mathcal{F}(X,Y,\{Y\})\Big) \cong_{\mathsf{UNI}} X \to_{\mathcal{U}} Y$$

Proof $=$

$$\texttt{compactConvergence} :: \mathsf{TOP} \to \mathsf{UNI} \to \mathsf{UNI}$$

$$\texttt{compactConvergence}\,(X,Y) = X \to_{\mathbb{K}} Y := \Big(X \to Y, \mathcal{F}\big(X,Y,\texttt{CompactSubset}(X)\big)\Big)$$

$$\texttt{precompactConvergence} :: \mathsf{UNI} \to \mathsf{UNI} \to \mathsf{UNI}$$

$$\texttt{precompactConvergence}\,(X,Y) = X \to_{\lambda} Y := \Big(X \to Y, \mathcal{F}\big(X,Y,\texttt{TotallyBounded}(X)\big)\Big)$$

$$\texttt{ClosedContinousCriterion} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} :??X \,.$$
$$.\, \forall x \in X \,.\, \exists S \in \mathcal{S} \,.\, x \in \operatorname{int} S \Rightarrow \texttt{Closed}\Big(\big(X \to Y, \mathcal{F}(X,Y,\mathcal{S})\big), C(X,Y)\Big)$$

Proof $=$

Assume $f \in \overline{C(X,Y)}$,

Assume $x \in X$,

$\Big(S, [2]\Big) := [0](x) : \sum S \in \mathcal{S} \,.\, x \in \operatorname{int} S,$

$[x.*] := \mathrm{E}\mathcal{F}(X,Y,\mathcal{S})\texttt{UniformLimitIsContinuous}(f) : f_{|S} \in C(S,Y);$

$\leadsto [f.*] := \texttt{ContinuityIsLocal} : f \in C(S,Y);$

$\leadsto [*] := \texttt{ClosedByLimits} : \texttt{Closed}\Big(\big(X \to Y, \mathcal{F}(X,Y,\mathcal{S})\big), C(X,Y)\Big);$

□

$$\texttt{FunctionalTopologyCompleteness} :: \forall X \in \mathsf{SET} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} :??X \,.$$
$$.\, \texttt{CompleteUniformSpace}(Y) \,\&\, \texttt{Cover}(X,\mathcal{S}) \iff \texttt{CompleteUniformSpace}\Big(X \to Y, \mathcal{F}(X,Y,\mathcal{S})\Big)$$

Proof $=$

. . .

□

1.6.4 Equicontinuiuty and Uniform Equicontinuiuty[!]

Notions of equicontinuity and Arzello-Ascolli theorem also generalizes nicely. There is no proofs in this chapter. They may be provided on demand.

$$\texttt{EquicontinousAtAPoint} :: \prod X \in \mathsf{TOP} \,.\, \prod Y \in \mathsf{UNI} \,.\, X \to ??(X \to Y)$$

$$F : \texttt{EquicontinousAtAPoint} \iff \Lambda x \in X \,.\, \forall U \in \mathcal{U}_Y \,.\, \exists O \in \mathcal{U}(x) \,.\, \forall f \in F \,.\, f(O) \subset V\Big(f(x)\Big)$$

$$\texttt{UniformlyEquicontinousAtAPoint} :: \prod X, Y \in \mathsf{UNI} \,.\, X \to ??(X \to Y)$$

$$F : \texttt{UniformlyEquicontinousAtAPoint} \iff \Lambda x \in X \,.\, \forall U \in \mathcal{U}_Y \,.\, \exists V \in \mathcal{U}_X \,.\, \forall f \in F \,.\, f\Big(V(x)\Big) \subset V\Big(f(x)\Big)$$

$$\texttt{Equicontinous} :: \prod X \in \mathsf{TOP} \,.\, \prod Y \in \mathsf{UNI} \,.\, ??(X \to Y)$$

$$F : \texttt{Equicontinous} \iff \forall x \in X \,.\, \texttt{EquicontinuousAtAPoint}(X, Y, F, x)$$

$$\texttt{UniformlyEquicontinous} :: \prod X, Y \in \mathsf{UNI} \,.\, ??(X \to Y)$$

$$F : \texttt{UniformlyEquicontinous} \iff \forall x \in X \,.\, \texttt{UniformlyEquicontinuousAtAPoint}(X, Y, F, x)$$

$\texttt{UniformlyEquicontinuousAltDef} :: \forall X, Y \in \mathsf{UNI} \,.\, \forall F :??(X \to Y) \,.$

$\quad . \texttt{UnifomlyEquicontinuous}(X, Y, F) \iff \forall V \in \mathcal{U}_Y \,.\, \exists U \in \mathcal{U}_X \,.\, \forall f \in F \,.\, (f \times f)(U) \subset V$

Proof $=$

$\dots$

$\square$

$\texttt{BourbakiJointEquiontinuityTheorem} :: \forall T \in \mathsf{SET} \,.\, \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} :??T \,.$

$\quad . \forall f : T \times X \to Y \,.\, \Lambda x \in X \,.\, \Lambda t \in T \,.\, f(t, x) \in \mathsf{TOP}\Big(X, \Big(T \to Y, \mathcal{F}(T, Y, \mathcal{S})\Big)\Big) \iff$

$\quad \iff \forall S \in \mathcal{S} \,.\, \texttt{Equicontinuous}\Big(X, Y\Big\{\Lambda x \in X \,.\, f(t, x)\Big| t \in S\Big\}\Big)$

Proof $=$

$\dots$

$\square$

$\texttt{BourbakiJointUniformEquiontinuityTheorem} :: \forall T \in \mathsf{SET} \,.\, \forall X, Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} :??T \,.$

$\quad . \forall f : T \times X \to Y \,.\, \Lambda x \in X \,.\, \Lambda t \in T \,.\, f(t, x) \in \mathsf{UNI}\Big(X, \Big(T \to Y, \mathcal{F}(T, Y, \mathcal{S})\Big)\Big) \iff$

$\quad \iff \forall S \in \mathcal{S} \,.\, \texttt{UnifomlyEquicontinuous}\Big(X, Y\Big\{\Lambda x \in X \,.\, f(t, x)\Big| t \in S\Big\}\Big)$

Proof $=$

$\dots$

$\square$

$\texttt{EquicontinuityClosureTHM} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall F :?(X \to Y) \,.$

$\quad . \, \texttt{Equicontinuous}(X, Y, F) \iff \texttt{Equicontinuous}(X, Y, \mathrm{cl}_{\mathrm{pt}} F)$

$\texttt{Proof} \ =$

...

□

$\texttt{UniformEquicontinuityClosureTHM} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall F :?(X \to Y) \,.$

$\quad . \, \texttt{UnifomlyEquicontinuous}(X, Y, F) \iff \texttt{UnifomlyEquicontinuous}(X, Y, \mathrm{cl}_{\mathrm{pt}} F)$

$\texttt{Proof} \ =$

...

□

$\texttt{EquicontinuitySClosureTHM} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} : \texttt{Cover}(X) \,.\, \forall F :?(X \to Y) \,.$

$\quad . \, \texttt{Equicontinuous}(X, Y, F) \iff \texttt{Equicontinuous}(X, Y, \mathrm{cl}_{\mathcal{S}} F)$

$\texttt{Proof} \ =$

...

□

$\texttt{UniformEquicontinuitySClosureTHM} :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall \mathcal{S} : \texttt{Cover}(X) \,.\, \forall F :?(X \to Y) \,.$

$\quad . \, \texttt{UnifomlyEquicontinuous}(X, Y, F) \iff \texttt{UnifomlyEquicontinuous}(X, Y, \mathrm{cl}_{\mathcal{S}} F)$

$\texttt{Proof} \ =$

...

□

$\texttt{PointwiseConvergenceIsCompact} ::$

$\quad :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall F : \texttt{Equicontinuous}(X, Y) \,.\, (F, \mathrm{pt}) \cong_{\mathsf{UNI}} (F, \mathbb{K})$

$\texttt{Proof} \ =$

...

□

$\texttt{OnCompactPpintwiseConvergenseIsUniform} ::$

$\quad :: \forall X : \texttt{Compact} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall F : \texttt{Equicontinuous}(X, Y) \,.\, \Big(F, \mathcal{UU}(X, Y)\Big) \cong_{\mathsf{UNI}} (F, \mathrm{pt})$

$\texttt{Proof} \ =$

...

□

$\texttt{PointwiseAndCompactClosureAgree} ::$

$\quad :: \forall X \in \mathsf{TOP} \,.\, \forall Y \in \mathsf{UNI} \,.\, \forall F : \texttt{Equicontinuous}(X, Y) \,.\, \mathrm{cl}_{\mathrm{pt}} F = \mathrm{cl}_{\mathbb{K}} F$

$\texttt{Proof} \ =$

...

□

EquicontinuousSquareTHM ::

$$:: \forall X : \texttt{Compact} \;.\; \forall Y \in \mathsf{UNI} \;.\; \forall F : \texttt{Equicontinuous}(X,Y) \;.\; \left(F^2, \mathcal{UU}^2(X,Y)\right) \cong_{\mathsf{UNI}} (F^2, \mathrm{pt}^2)$$

Proof =

...

□

SConvergenceTotallBoundnessImplyEq :: $\forall X \in \mathsf{TOP} \;.\; \forall Y \in \mathsf{UNI} \;.\; \forall \mathcal{S} : \texttt{Cover}(X) \;.$

$$.\; \forall F : \texttt{TotallyBounded}\Big(X \to Y, \mathcal{F}(X,Y,\mathcal{S})\Big) \;.\; \Big(\forall S \in \mathcal{S} \;.\; F_{|S} \subset \mathsf{TOP}(S,Y)\Big) \Rightarrow$$

$$\Rightarrow \forall S \in \mathcal{S} \;.\; \texttt{Equicontinuous}\Big(S, Y, F_{|S}\Big)$$

Proof =

...

□

SConvergenceTotallBoundnessImplyTB :: $\forall X \in \mathsf{TOP} \;.\; \forall Y \in \mathsf{UNI} \;.\; \forall \mathcal{S} : \texttt{Cover}(X) \;.$

$$.\; \forall F : \texttt{TotallyBounded}\Big(X \to Y, \mathcal{F}(X,Y,\mathcal{S})\Big) \;.\; \Big(\forall S \in \mathcal{S} \;.\; F_{|S} \subset \mathsf{TOP}(S,Y)\Big) \Rightarrow$$

$$\Rightarrow \forall x \in X \;.\; \texttt{TotallyBounded}\Big(Y, F(x)\Big)$$

Proof =

...

□

ArzeloAscolliTHM :: $\forall X \in \mathsf{TOP} \;.\; \forall Y \in \mathsf{UNIS} \;.\; \forall F \subset C(X,Y) \;.$

$$.\; \texttt{CompactSubset}\Big(X \to_{\mathbb{K}} Y, F\Big) \iff \texttt{Closed}\Big(X \to_{\mathbb{K}} Y, F\Big) \;\&\; \forall x \in X \;.\; \texttt{CompactSubset}\Big(Y, \overline{F(x)}\Big) \;\&$$

$$\&\; \forall K : \texttt{CompactSubset}(X) \;.\; \texttt{Equicontinuous}(X, Y, F_{|K})$$

Proof =

...

□

2 Topological Groups Basics

Groups can be equipid with a topology in such a way that their alebraic and topological structrure interplay.

2.1 Group Topology

There are different ways to equip a group with a group topology, but all of them have some necessary properties.

2.1.1 Category of Topological Groups

Topological groups form a complete category.

$$\texttt{TopologicalGroup} :: ?\sum G \in \mathsf{GRP} \,.\, \texttt{Topology}(G)$$

$$(G, \mathcal{T}) : \texttt{TopologicalGroup} \iff$$

$$\iff \circ_G \in \mathsf{TOP}\Big((G,\mathcal{T}) \times (G,\mathcal{T}), (G,\mathcal{T})\Big) \;\&\; \Lambda g \in G \,.\, g^{-1} \in \mathsf{TOP}\Big((G,\mathcal{T}), (G,\mathcal{T})\Big)$$

$$\texttt{topologicalGroupAsGroup} :: \texttt{TopologicalGroup} \to \mathsf{GRP}$$

$$\texttt{topologicalGroupAsGroup}\,(G,\mathcal{T}) = (G,\mathcal{T}) := G$$

$$\texttt{topologicalGroupAsTopologicalSpace} :: \texttt{TopologicalGroup} \to \mathsf{TOP}$$

$$\texttt{topologicalGroupAsTopologicalSpace}\,(G,\mathcal{T}) = (G,\mathcal{T}) := (G,\mathcal{T})$$

$$\texttt{categoryOfTopologicalGroups} :: \mathsf{CAT}$$

$$\texttt{categoryOfTopologicalGroups}\,() = \mathsf{TGRP} := (\texttt{TopologicalGroup}, \mathsf{TOP} \,\&\, \mathsf{GRP}, \circ, \mathrm{id})$$

$$\texttt{HomomorphismWeakTopologyIsGroupTopology} ::$$

$$:: \forall I \in \mathsf{SET} \,.\, \forall G \in \mathsf{GRP} \,.\, \forall H : I \to \mathsf{TGRP} \,.\, \forall \phi : \prod_{i \in I} \mathsf{GRP}(G, H_i) \,.\, \Big(G, \mathcal{W}(I,H,\phi)\Big) \in \mathsf{TGRP}$$

$$\texttt{Proof} =$$

$$[1] := \Lambda i \in I \,.\, \mathtt{E}_2\mathsf{GRP}(G,H_i,\phi_i)\mathtt{E}\mathcal{W}_G(I,H,\phi)\mathtt{E}\mathsf{TGRP}(H_i)\mathtt{E}\mathsf{CAT}(\mathsf{TOP}) :$$

$$: \forall i \in I \,.\, \mathrm{inv}_G \phi_i = \phi_i \mathrm{inv}_{H_i} \in \mathsf{TOP}\bigg(\Big(G, \mathcal{W}(I,H,\phi)\Big), H_i\bigg),$$

$$[2] := \texttt{WeakTopologyUniversalProperty}[1] : \mathrm{inv}_G \in \mathrm{Aut}_{\mathsf{TOP}}\Big(G, \mathcal{W}(I,H,\phi)\Big),$$

$$[3] := \Lambda i \in I \,.\, \mathtt{E}_1\mathsf{GRP}(G,H_i,\phi_i)\mathtt{E}\mathcal{W}_G(I,H,\phi)\mathtt{E}\mathsf{TGRP}(H_i)\mathtt{E}\mathsf{CAT}(\mathsf{TOP}) :$$

$$: \forall i \in I \,.\, (\cdot_G)(\phi_i \times \phi_i) = (\phi_i \times \phi_i)(\cdot_{H_i}) \in \mathsf{TOP}\bigg(\Big(G, \mathcal{W}(I,H,\phi)\Big)^2, H_i\bigg),$$

$$[4] := \texttt{WeakTopologyUniversalProperty}[3] : \cdot_G \in \mathsf{TOP}\bigg(\Big(G, \mathcal{W}(I,H,\phi)\Big)^2, \Big(G, \mathcal{W}(I,H,\phi)\Big)\bigg),$$

$$[3] := \mathtt{I}\mathsf{TGRP}[2][4] : \Big(G, \mathcal{W}(I,H,\phi)\Big) \in \mathsf{TGRP},$$

$\square$

$$\texttt{SupTopologyIsGroup} :: \forall G \in \mathsf{GRP} \,.\, \forall I \in \mathsf{SET} \,.\, \forall \mathcal{T} : \mathcal{T} \to \mathsf{TGRP}(G) \,.\, \left(G, \bigvee_{i \in I} \mathcal{T}_i\right) \in \mathsf{TGRP}$$

$$\texttt{Proof} =$$

$\ldots$

$\square$

TopologicalGroupsAreComplete :: Complete(TGRP)

Proof =

...

$\square$

2.1.2 Absolute Values and Invariant Metrics

Group topology can be determinde by an absolute value function, or by an invariant metric. Absolute value functions and invariant metrics are the same.

$\texttt{AbsoluteValue} :: \prod_{G \in \mathsf{GRP}} (G \to \mathbb{R}_+)$

$\alpha : \texttt{AbsoluteValue} \iff \alpha(e) = 0 \,\&\, \forall g \in G \,.\, \alpha(g) = \alpha(g^{-1}) \,\&\, \forall g, h \in G \,.\, \alpha(gh) \leq \alpha(g) + \alpha(h) \,\&$
$\quad \& \, \forall x : \mathbb{N} \to G \,.\, \forall g \in G \,.\, \lim_{n \to \infty} \alpha(x_n) = 0 \Rightarrow \lim_{n \to \infty} \alpha(g x_n g^{-1}) = 0$

$\texttt{absoluteValueAsSemimetric} :: \prod_{G \in \mathsf{GRP}} \texttt{AbsoluteValue}(G) \to \texttt{Semimetric}(G)$

$\texttt{absoluteValueAsSemimetric}\,(\alpha) = d_\alpha := \Lambda a, b \in G \,.\, \alpha(ab^{-1})$

$[1] := \Lambda g \in G \,.\, \mathtt{E}d_\alpha(g,g)\texttt{InverseMeaning}(G)\mathtt{E}_1\texttt{AbsoluteValue}(G,\alpha) :$
$\quad : \forall g \in G \,.\, d_\alpha(g,g) = \alpha(gg^{-1}) = \alpha(e) = 0,$

$[2] := \Lambda g, h \in G \,.\, \mathtt{E}d_\alpha(g,h)\texttt{ProductInverse}(G)\mathtt{E}_2\texttt{AbsoluteValue}(G,\alpha,g^{-1})\mathtt{I}d_\alpha(h,g) :$
$\quad : \forall g, h \in G \,.\, d_\alpha(g,h) = \alpha(gh^{-1}) = \alpha((hg^{-1})^{-1}) = \alpha(hg^{-1}) = d_\alpha(h,g),$

$[3] := \Lambda f, g, h \in G \,.\, \mathtt{E}d_\alpha(g,h)\texttt{InverseMeaning}(G,g)\mathtt{E}_3\texttt{AbsoluteValue}(G,\alpha,g^{-1},fg^{-1},gh^{-1})\mathtt{I}d_\alpha(f,g)\mathtt{I}d_\alpha(g,h) :$
$\quad : \forall f, g, h \in G \,.\, d_\alpha(f,h) = \alpha(fh^{-1}) = \alpha(fg^{-1}gh^{-1}) \leq \alpha(fg^{-1}) + \alpha(gh^{-1}) = d_\alpha(f,g) + d_\alpha(g,h),$

$[*] := \texttt{ISemimetiric}(G)[1][2][3] : \texttt{Semimetric}(G, d_\alpha);$

$\square$

$\texttt{LeftInvariantMetric} :: \prod_{G \in \mathsf{GRP}} ?\texttt{Semimetric}(X)$

$\rho : \texttt{LeftInvariantMetric} \iff \forall a, b, g \in G \,.\, d(ag, bg) = d(a,b)$

$\texttt{RightInvariantMetric} :: \prod_{G \in \mathsf{GRP}} ?\texttt{Semimetric}(X)$

$\rho : \texttt{RightInvariantMetric} \iff \forall a, b, g \in G \,.\, d(ga, gb) = d(a,b)$

$\texttt{TwosidedInvariantMetric} := \prod_{G \in \mathsf{GRP}} \texttt{LeftInvariantMetric} \,\&\, \texttt{RightInvariantMetric} : \mathsf{GRP} \to \texttt{Type};$

$\texttt{AbsoluteValueMetricIsRightInvariant} ::$
$\quad :: \forall G \in \mathsf{GRP} \,.\, \forall \alpha : \texttt{AbsoluteValue}(G) \,.\, \texttt{RightInvariantMetric}(G, d_\alpha)$

$\texttt{Proof} =$

$[1] := \Lambda a, b, g \in G \,.\, \mathtt{E}d_\alpha(ag,bg)\texttt{InverseProduct}(G)\texttt{InverseMeaning}(G,g)\mathtt{I}d_\alpha(ag,bg) :$
$\quad : \forall a, b, g \in G \,.\, d_\alpha(ag,bg) = \alpha(agg^{-1}b^{-1}) = \alpha(ab^{-1}) = d_\alpha(a,b),$

$[*] := \texttt{IRightInvariantMetric}[1] : \texttt{RightInvariantMetric}(G, d_\alpha);$

$\square$

$\texttt{AbsoluteValueMetrizesATopologicalGroup} :: \forall G \in \mathsf{TGRP} \, . \, \forall \alpha : \texttt{AbsoluteValue}(G) \, . \, (G, d_\alpha) \in \mathsf{TGRP}$

$\texttt{Proof} \; =$

$\texttt{Assume} \; g \in G,$

$\texttt{Assume} \; x : \mathbb{N} \to G,$

$\texttt{Assume} \; [1] : \lim_{n\to\infty} x_n = g,$

$[2] := \texttt{MetricLimit}[1] : \lim_{n=1} d_\alpha(x_n, g) = 0,$

$[3] := \texttt{E}d_\alpha : 0 = \lim_{n\to\infty} \alpha(x_n g^{-1}),$

$[4] := \texttt{E}_4\texttt{AbsoluteValue}(G, \alpha, g^{-1})[4]\texttt{InverseProperty}(G, \alpha) :$

$: 0 = \lim_{n\to\infty} \alpha(g^{-1} x_n g^{-1} g) = \lim_{n\to\infty} \alpha(g^{-1} x_n) = \lim_{n\to\infty} d_\alpha(g^{-1}, x_n^{-1}),$

$[g.*] := \texttt{MetricLimit}[4] : \lim_{n=1} x_n^{-1} = g^{-1};$

$\rightsquigarrow [1] := \texttt{ContinuousByLimits} : \Lambda g \in G \, . \, g^{-1} \in \mathrm{Aut}_{\mathsf{TOP}}(G, d_\alpha),$

$\texttt{Assume} \; g, h \in G,$

$\texttt{Assume} \; x, y : \mathbb{N} \to G,$

$\texttt{Assume} \; [2] : \lim_{n\to\infty} x_n = g,$

$\texttt{Assume} \; [3] : \lim_{n\to\infty} y_n = h,$

$[4] := \texttt{MetricLimit}[2]\texttt{E}d_\alpha : \lim_{n=1} d_\alpha(x_n, g) = \lim_{n=1} \alpha(x_n g^{-1}) = 00,$

$[5] := \texttt{MetricLimit}[3]\texttt{E}d_\alpha : \lim_{n=1} d_\alpha(y_n, h) = \lim_{n=1} \alpha(y_n h^{-1})0,$

$[6] := \Lambda n \in \mathbb{N} \, . \, \texttt{E}d_\alpha\texttt{E}_2\texttt{AbsoluteValue}(G, \alpha, x_n^{-1})\texttt{InverseMeaning}\texttt{E}_3\texttt{AbsoluteValue}(G, \alpha, y_n h^{-1}, g^{-1} x_n)$

$\texttt{ESymmetric}(G, G, d_\alpha)\texttt{LimitSum}[4][5] : \lim_{n=1} d_\alpha(x_n y_n, gh) = \lim_{n=1} \alpha(x_n y_n h^{-1} g^{-1}) = \lim_{n=1} \alpha(x_n^{-1} x_n y_n h^{-1} g^{-1} x_n) =$

$= \lim_{n=1} \alpha(y_n h^{-1} g^{-1} x_n) \leq \lim_{n=1} \alpha(y_n h^{-1}) + \alpha(g^{-1} x_n) = \lim_{n=1} d_\alpha(y_n, h) + \lim_{n=11} d_\alpha(x_n, g) = 0;$

$[7] := \texttt{NonNegtiveZeroBound}[8] : \lim_{n=1} d_\alpha(x_n y_n, gh) = 0,$

$\Big[8.*\Big] := \texttt{MetricLimit}[7] : \lim_{n\to\infty} x_n y_n = gh;$

$\rightsquigarrow [2] := \texttt{ContinuousByLimits} : \circ \in \mathsf{TOP}\Big((X, d_\alpha)^2, (X, d_\alpha)\Big),$

$[*] := \texttt{I}\mathsf{TGRP}[1][2] : (X, d_\alpha) \in \mathsf{TGRP};$

$\square$

$\texttt{absoluteValueFromRIM} :: \prod_{G \in \mathsf{TGRP}} \texttt{RightInvariantMetric}(G) \to \texttt{AbsoluteValue}(F)$

$\texttt{absoluteValueFromRIM}\,(\rho) = \alpha_\rho := \Lambda g \in G\ .\ \rho(g, e)$

$[1] := \mathtt{E}\alpha_\rho(e)\mathtt{E}_1\texttt{Semimetric}(\rho) : \alpha_\rho(e) = \rho(e, e) = 0,$

$[2] := \Lambda g \in G\ .\ \mathtt{E}\alpha_\rho(g^{-1})\mathtt{E}\texttt{RightInvariantMetric}(G, \rho)\mathtt{E}\texttt{Symmetric}(G, \rho)\mathtt{I}\alpha_\rho(g^{-1}) :$

$: \forall g \in G\ .\ \alpha_\rho(g^{-1}) = \rho(g^{-1}, e) = \rho(e, g) = \rho(g, e) = \alpha_\rho(g),$

$[3] := \Lambda g, h \in G\ .\ \mathtt{E}\alpha_\rho(gh)\mathtt{E}\texttt{RightInvariantMetric}(G, \rho, gh, e, h^{-1})\mathtt{E}\texttt{TriangleIneq}(G, \rho, g, e, h^{-1})$

$\mathtt{E}\texttt{RightInvariantMetric}(G, \rho, e, h^{-1}, h)\mathtt{I}\alpha_\rho : \forall g, h \in G\ .\ \alpha_\rho(gh) = \rho(gh, e) = \rho(g, h^{-1}) \le$

$\le \rho(g, e) + \rho(e, h^{-1}) = \rho(g, e) + \rho(h, e) = \alpha_\rho(g) + \alpha_\rho(h),$

$\texttt{Assume}\ x : \mathbb{N} \to G,$

$\texttt{Assume}\ [4] : \lim_{n\to\infty} \alpha_\rho(x_n) = 0,$

$\texttt{Assume}\ g \in G,$

$[5] := [4]\mathtt{E}\alpha_\rho\texttt{MetricLimit}(G, \rho) : \lim_{n\to\infty} x_n = e,$

$[6] := \mathtt{E}\mathsf{TGRP}(F)g[5]g^{-1} : \lim_{n\to\infty} g x_n g^{-1} = e,$

$[x.*] := [6]\texttt{MetricLimit}(G, \rho)\mathtt{I}\alpha_\rho : \lim_{n\to\infty} \alpha_\rho(g x_n g^{-1}) = 0;$

$\leadsto [*] := \mathtt{I}\texttt{AbsoluteValue}[1, 2, 3] : \texttt{AbsoluteValue}(G, \alpha_\rho);$

$\square$

$\texttt{AbsoluteValueGeneratingMetricCondition} ::$

$:: \forall G \in \mathsf{GRP}\ .\ \forall \alpha : \texttt{AbsoluteValue}(G)\ .\ \texttt{Metric}(G, d_\alpha) \iff \forall g \in G\ .\ g \ne e \Rightarrow \alpha(g) > 0$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b \in G,$

$\texttt{Assume}\ [1] : a \ne b,$

$[2] := \mathtt{E}\mathsf{GRP}(G)[1] : ab^{-1} \ne e,$

$\Big[(a, b).*\Big] := \mathtt{E} : d_\alpha(a, b) = \alpha(ab^{-1}) > 0;$

$\leadsto [*] := \mathtt{I}\texttt{Metric} : \texttt{Metric}\Big(G, d\Big),$

$\square$

$\texttt{LEMByIsometry} :: \forall G \in \mathsf{GRP}\ .\ \forall \texttt{Semimetric}(G, \rho)\ .$

$.\ \texttt{LeftInvariantMetric}(G, \rho) \iff \forall a \in A\ .\ \texttt{Isometry}(G, G, \lambda_a)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{metricInversion} :: \prod_{G \in \mathsf{GRP}} \texttt{Semimetric}(G) \to \texttt{Semimetric}(G)$

$\texttt{metricInversion}\,(\rho) = \rho^{-1} := \Lambda g, h \in G\ .\ \rho(g^{-1}, h^{-1})$

$\texttt{InvariantInversion} :: \forall G \in \mathsf{GRP}\ .\ \forall \rho : \texttt{LeftInvariantMetric}(G)\ .\ \texttt{RightInvariantMetric}(G, \rho^{-1})$

$\texttt{Proof} =$

$\dots$

$\square$

2.1.3 Neighbourhoods of Unity

Topology of a topological group is fully determined by the neighborhood system of its unity.

$\texttt{SymmetricSet} :: \prod_{G \in \mathsf{GRP}} ?G$

$S : \texttt{SymmetricSet} \iff \mathrm{inv}(S) = S$

$\texttt{UnityHasSymmetricHoodBase} :: \forall G \in \mathsf{TGRP} \,.\, \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) : V \subset U \;\&\; \texttt{SymmetricSet}(G, V)$

$\texttt{Proof} \;=$

$V := U \cap \mathrm{inv}(U) :?G,$

$[1] := \texttt{unityInverse}(G)\texttt{E}V : e \in V,$

$[2] := \texttt{E}\mathsf{TGRP}(G)\texttt{E}V : V \in \mathcal{T}(G),$

$[*] := \texttt{E}V : \mathrm{inv}(V) = V;$

$\square$

$\texttt{InverseContinuityAtUnityCriterion} ::$

$\quad : \forall G \in \mathsf{TOP} \;\&\; \mathsf{GRP} \,.\, \mathrm{inv}_G \in C_e(G, G) \iff \forall U \in \mathcal{U}(e) \,.\, \mathrm{inv}(U) \in \mathcal{U}(e)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{MultContinuityAtUnityCriterion} ::$

$\quad : \forall G \in \mathsf{TOP} \;\&\; \mathsf{GRP} \,.\, (\cdot_G) \in C_{(e,e)}(G^2, G) \iff \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) : VV \subset U$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : (\cdot_G) \in C_{(e,e)}(G^2, G),$

$\texttt{Assume}\ U \in \mathcal{U}(e),$

$W := (\cdot_G)^{-1}(U) \in \mathcal{U}(e, e),$

$\Big(A, B, [2]\Big) := \texttt{ProductTopologyBae}(G, G, W) : \sum A, B \in \mathcal{U}(e) \,.\, A \times B \subset W,$

$V := A \cap B \in \mathcal{U}(e),$

$[1*] := \texttt{E}V[2] : VV \subset U;$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : (\cdot_G) \in C_{(e,e)}(G^2, G) \Rightarrow \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) : VV \subset U,$

$\texttt{Assume}\ [2] : \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) : VV \subset U,$

$\texttt{Assume}\ U \in \mathcal{U}(e),$

$\Big(V, [3]\Big) := [2](U) : \sum V \in \mathcal{U}(e) \,.\, VV \subset U,$

$[U.*] := \texttt{ProductTopologyBase}(G, G, V) : V \times V \in \mathcal{U}(e, e);$

$\rightsquigarrow [2.*] := \texttt{I}C_{(e,e)} : (\cdot_G) \in C_{(e,e)}(G^2, G);$

$\rightsquigarrow [*] := \texttt{I} \iff [1] : (\cdot_G) \in C_{(e,e)}(G^2, G) \iff \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) : VV \subset U;$

$\square$

$\texttt{TopologicalGroupAltDef} :: \forall G \in \mathsf{GRP} \,.\, \forall \mathcal{T} : \texttt{Topology}(G) \,.\, (G, \mathcal{T}) \in \mathsf{TGRP} \iff$

$\iff \Big(\forall g, h \in G \,.\, \forall U \in \mathcal{U}(h) \,.\, gU \in \mathcal{U}(gh)\Big) \;\&\; \Big(\forall U \in \mathcal{U}(e) \,.\, \forall \mathrm{inv}\, U \in \mathcal{U}(e) \,.\,\Big)$

$\&\;\&\; \Big(\forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) \,.\, VV \subset U\Big) \;\&\; \Big(\forall U \in \mathcal{U}(e) \,.\, \forall g \in G \,.\, \exists V \in \mathcal{U}(e) \,.\, aVa^{-1} \subset U\Big)$

$\texttt{Proof} \;=$

$[1] := \texttt{MultContinuityAtUnityCriterion}[0.3] : (\cdot_G) \in C_{(e,e)}(G^2, G),$

$\texttt{Assume}\ (\Delta, g) : \texttt{Net}(G),$

$\texttt{Assume}\ [2] : e \in \lim_{\delta \in \Delta} g_\delta,$

$\texttt{Assume}\ a \in G,$

$\texttt{Assume}\ U \in \mathcal{U}(e),$

$\Big(V, [3]\Big) := [0.4](U) : \sum V \in \mathcal{U}(e) \,.\, aVa^{-1} \subset U,$

$\Big(\delta, [4]\Big) := \texttt{E}[3](V) : \sum \delta \in \Delta \,.\, \forall \sigma \geq \delta \,.\, g_\sigma \in V,$

$\Big[(\Delta, g).3)\big] := [3][4] : \forall \sigma \geq \delta \,.\, a g_\sigma a^{-1} \subset U;$

$\rightsquigarrow [2] := \texttt{I}C_e : \forall a \in G \,.\, (\Lambda g \in G \,.\, aga^{-1}) \in C_e(G, G),$

$\texttt{Assume}\ (\Delta, x), (\Delta, y) : \texttt{Net}(G),$

$\texttt{Assume}\ g, h \in G,$

$\texttt{Assume}\ [3] : g \in \lim_{\delta \in \Delta} x_\delta,$

$\texttt{Assume}\ [4] : h \in \lim_{\delta \in \Delta} y_\delta,$

$[5] := [0.1][3] : e \in \lim_{\delta \in \Delta} g^{-1} x_\delta,$

$[6] := [0.1][4] : e \in \lim_{\delta \in \Delta} y_\delta h^{-1},$

$[7] := [1][5][6] : e \in \lim_{\delta \in \Delta} g^{-1} x_\delta y_\delta h^{-1},$

$[8] := [2](g)[7] : e \in \lim_{\delta \in \Delta} x_\delta y_\delta h^{-1} h^{-1} g^{-1},$

$[\ldots *] := [0.1][8] : hg \in \lim_{\delta \in \Delta} x_\delta y_\delta;$

$\rightsquigarrow [3] := \texttt{ContinuityByNets} : (\cdot_G) \in \mathsf{TOP}(G^2, G),$

$\texttt{Assume}\ g \in G,$

$\texttt{Assume}\ U \in \mathcal{U}(g),$

$[4] := [0.1](U, g^{-1}) : g^{-1}(U) \in \mathcal{U}(e),$

$[5] := [0.2][4] : \mathrm{inv}\Big(g^{-1}(U)\Big) = \mathrm{inv}(U)g \in \mathcal{U}(e),$

$[6] := [0.1][5] : g^{-1}\mathrm{inv}(U)g \in \mathcal{U}(g^{-1}),$

$[g.*] := [3][6] : \mathrm{inv}(U) \in \mathcal{U}(g^{-1});$

$\rightsquigarrow [4] := \texttt{I}\mathsf{TOP} : \mathrm{inv} \in \mathsf{TOP}(G, G),$

$[*] := \texttt{I}\mathsf{TGRP}[3][4] : (G, \mathcal{T}) \in \mathsf{TGRP};$

$\square$

$\texttt{ConjugationIsAutomorphism} :: \forall G \in \mathsf{TGRP} \,.\, \forall g \in G \,.\, \gamma_g \in \mathrm{Aut}_{\mathsf{TGRP}}(G)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TopologicalGroupAltDef2} :: \forall G \in \mathsf{GRP} \,.\, \forall \mathcal{T} : \texttt{Topology} \,.\, G \in \mathsf{TGRP} \iff$

$$\iff \Big(\Lambda g, h \in G \,.\, gh^{-1}\Big) \in \mathsf{TOP}\Big((G, \mathcal{T})^2, (G, \mathcal{T})\Big)$$

$\texttt{Proof} \; =$

$\phi := \Lambda g, h \in G \,.\, gh^{-1} : \mathsf{TOP}\Big((G, \mathcal{T})^2, (G, \mathcal{T})\Big),$

$[1] := \texttt{E}\phi\texttt{I}\mathrm{inv} : \mathrm{inv} = \Lambda g \in G \,.\, \phi(e, g),$

$[2] := \texttt{I}\mathsf{TOP}[1] : \mathrm{inv} \in \mathsf{TOP}\Big((G, \mathcal{T}), (G, \mathcal{T})\Big),$

$[3] := \texttt{E}\phi\texttt{I}\cdot : (\cdot_G) = (\mathrm{id} \times \mathrm{inv})\phi,$

$[4] := \texttt{I}\mathsf{TOP}[3] : (\cdot_G) \in \mathsf{TOP}\Big((G, \mathcal{T})^2, (G, \mathcal{T})\Big),$

$[*] := \texttt{I}\mathsf{TGRP}[2][4] : (G, \mathcal{T}) \in \mathsf{TGRP};$

$\square$

2.1.4 Uniformity and Regularity

Topological groups are uniform spaces and, hence completely regular.

$$\texttt{leftGroupConnector} :: \prod_{G \in \mathsf{TGRP}} \mathcal{U}(e) \to \texttt{Connector}(G)$$

$$\texttt{leftGroupConnector}\,(U) = U_L := \{(a,b) \in G^2 \;.\; a^{-1}b \in U\}$$

$\texttt{LeftConnectorsIntersection} :: \forall G \in \mathsf{TGRP} \;.\; \forall U, V \in \mathcal{U}(e) \;.\; (U \cap V)_L = U_L \cap V_L$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{LeftConnectorTranspose} :: \forall G \in \mathsf{TGRP} \;.\; \forall U \in \mathcal{U}(e) \;.\; (U_L)^\top = \Big(\mathrm{inv}(U)\Big)_L$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{LeftConnectorsCompose} :: \forall G \in \mathsf{TGRP} \;.\; \forall U, V \in \mathcal{U}(e) \;.\; U_L \circ V_L = (VU)_L$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (a,c) \in U_L \circ V_L,$

$\Big(b, [2]\Big) := \texttt{E}(\circ)(U_L \circ V_L)(a,c) : \sum b \in G \;.\; (a,b) \in U_L \;\&\; (b,c) \in V_L,$

$[3] := \texttt{E}U_L[2] : a^{-1}b \in U \;\&\; b^{-1}c \in V,$

$[4] := [3.1][3.2]\texttt{InverseMeaning}(G,G) : a^{-1}c = a^{-1}bb^{-1}c = a^{-1}c \in UV,$

$\Big[(a,c).* \Big] := \texttt{I}(VU)_L[4] : (a,c) \in (VU)_L;$

$\rightsquigarrow [1] := \texttt{I} \subset : U_L \circ V_L \subset (VU)_L,$

$\texttt{Assume}\ (a,c) \in (VU)_L,$

$[2] := \texttt{E}(VU)_L(a,c) : a^{-1}c \in VU,$

$\Big(v, u, [3]\Big) := \texttt{E}VU[2] : \sum v \in V \;.\; \sum u \in U \;.\; a^{-1}c = vu,$

$[4] := [3]u^{-1} : a^{-1}cu^{-1} = v \in V,$

$[5] := \texttt{I}V_L[4] : (a, cu^{-1}) \in V_L,$

$[6] := v^{-1}[3] : v^{-1}a^{-1}c = u \in U,$

$[7] := \texttt{I}U_L[6] : (av, c) \in U_L,$

$[8] := [3]\texttt{InverseMeaning}(G,u) : cu^{-1} = avuu^{-1} = av,$

$\Big[(a,c).* \Big] := \texttt{I}U_L \circ V_L[5][7][8] : (a,c) \in U_L \circ V_L;$

$\rightsquigarrow [*] := \texttt{ISetEq}[1] : U_L \circ V_L = (VU)_L;$

$\square$

$$\texttt{leftGroupUniformity} :: \prod_{G \in \mathsf{TGRP}} \texttt{Uniformity}(G)$$

$$\texttt{leftGroupUniformity}\,() = \mathcal{L}_G := \Big\langle \{U_L | U \in \mathcal{U}_G(e)\} \Big\rangle_{\mathcal{F}}$$

$\texttt{TopologicalGroupIsUniformizableByLeftUniformities} :: \forall G \in \mathsf{TGRP} \,.\, G \cong_{\mathsf{TOP}} (G, \mathcal{L}_G)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ g \in G,$

$\texttt{Assume}\ U \in \mathcal{U}(g),$

$\texttt{Assume}\ u \in U,$

$\varphi := \Lambda x \in G \,.\, u^{-1}x : \mathrm{Aut}_{\mathsf{TOP}}(G),$

$V := \varphi(U) : \mathcal{U}(e),$

$[1] := \texttt{E}\mathsf{GRP}(G)\texttt{E}V\texttt{I}V_L : U = V_L(u),$

$[u.*] := \texttt{EqIsSubset}[1] : V_L(u) \subset U;$

$\rightsquigarrow [U.*] := \texttt{EuniformTopology} : U \in \mathcal{U}_{\mathcal{L}_G}(g);$

$\rightsquigarrow [1] := \texttt{I} \subset : \mathcal{U}(g) \subset \mathcal{U}_{\mathcal{L}_G}(g),$

$\texttt{Assume}\ U : \mathcal{U}_{\mathcal{L}_G}(g),$

$\Big(V, [2]\Big) := \texttt{E}\mathcal{L}_G : \sum V \in \mathcal{U}(e) \,.\, U = V_L(g),$

$[3] := \texttt{E}V_L : U = gV,$

$[U.*] := \texttt{E}\mathsf{TGRP}(G)[3] : U \in \mathcal{U}(g);$

$\rightsquigarrow [x.*] := \texttt{ISetEq}[1] : \mathcal{U}(g) = \mathcal{U}_{\mathcal{L}_G}(g);$

$\rightsquigarrow [*] := \texttt{TopologyEqByHoods} : G \cong_{\mathsf{TOP}} (G, \mathcal{L}_G);$

$\square$

$$\texttt{rightGroupConnector} :: \prod_{G \in \mathsf{TGRP}} \mathcal{U}(e) \to \texttt{Connector}(G)$$

$\texttt{rightGroupConnector}\,(U) = U_R := \{(a,b) \in G^2 \,.\, ab^{-1} \in U\}$

$\texttt{rightConnectorsIntersection} :: \forall G \in \mathsf{TGRP} \,.\, \forall U, V \in \mathcal{U}(e) \,.\, (U \cap V)_R = U_R \cap V_R$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{RightConnectorTranspose} :: \forall G \in \mathsf{TGRP} \,.\, \forall U \in \mathcal{U}(e) \,.\, (U_R)^\top = \Big(\mathrm{inv}(U)\Big)_R$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{RightConnectorsCompose} :: \forall G \in \mathsf{TGRP} \,.\, \forall U, V \in \mathcal{U}(e) \,.\, U_R \circ V_R = (VU)_R$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{rightGroupUniformity} :: \prod_{G \in \mathsf{TGRP}} \texttt{Uniformity}(G)$$

$\texttt{rightGroupUniformity}\,() = \mathcal{R}_G := \Big\langle \{U_R | U \in \mathcal{U}_G(e)\} \Big\rangle_{\mathcal{F}}$

TopologicalGroupIsUniformizableByRightUniformities :: $\forall G \in \mathsf{TGRP} \,.\, G \cong_{\mathsf{TOP}} (G, \mathcal{R}_G)$

Proof $=$

...

$\square$

TopologicalGroupsAreCompletelyRegular :: $\forall G \in \mathsf{TGRP} \,.\, \mathtt{CompletelyRegular}(G)$

Proof $=$

...

$\square$

SeparatedTopologicalGroupsAreTychonoff :: $\forall G \in \mathsf{TGRP} \,.\, \mathtt{T0}(G) \iff \mathtt{Tychonoff}(G)$

Proof $=$

...

$\square$

upperTwoSidedUniformity :: $\prod_{G \in \mathsf{TGRP}} \mathtt{Uniformity}(G)$

upperTwoSidedUniformity $() = \mathcal{S}^{\vee}_G := \mathcal{L}_G \vee \mathcal{R}_G$

TopologicalGroupIsUniformizableByTwoSidedUniformities :: $\forall G \in \mathsf{TGRP} \,.\, G \cong_{\mathsf{TOP}} (G, \mathcal{S}^{\vee}_G)$

Proof $=$

...

$\square$

supConnector :: $\prod G \in \mathsf{TGRP} \,.\, \mathcal{U}(e) \to \mathtt{Connector}(G)$

supGroupConnector $(U) = U_{\vee} := U_L \cap U_R$

TwoSidedUniformityBase :: $\forall G \in \mathsf{TGRP} \,.\, \mathtt{BaseOfUniformity}\Big(G, \mathcal{S}^{\vee}_G, \{U_{\vee} | U : \mathtt{SymmetricSet}(G) \,\&\, \mathcal{U}(e)\}\Big)$

Proof $=$

Assume $U \in \mathcal{S}^{\vee}_G$,

$\Big(O, [1]\Big) := \mathtt{E}\mathcal{S}^{v} ee_G \mathtt{E} U \mathtt{E} \mathcal{L}_G : \sum O \in \mathcal{U}(e) \,.\, O_L \subset U,$

$E := O \cap \mathrm{inv} O' \in \mathtt{SymmetricSet}(G) \,\&\, \mathcal{U}(e),$

$[U.*] := \mathtt{E}E_{\vee}\mathtt{IntersectionIsSubset}(G \times G, E_L, E_R)\mathtt{E}E\mathtt{LeftConnectorIntersection}(G)\mathtt{IntersectionIsSubs}$

$\rightsquigarrow [*] := \mathtt{IBaseOfUniformity} : \mathtt{BaseOfUniformity}\Big(G, \mathcal{S}^{\vee}_G, \{U_{\vee} | U : \mathtt{SymmetricSet}(G) \,\&\, \mathcal{U}(e)\}\Big),$

$\square$

ClosureInTopologicalGroup :: $\forall G \in \mathsf{TGRP} \,.\, \forall A \subset G \,.\, \overline{A} = \bigcap \Big\{ AU \Big| U \in \mathcal{U}(e) \Big\}$

Proof $=$

...

$\square$

$\texttt{ClosureInTopologicalGroup1} :: \forall G \in \mathsf{TGRP} \,.\, \forall A \subset G \,.\, \overline{A} = \bigcap \left\{ UA \middle| U \in \mathcal{U}(e) \right\}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ClosureInversion} :: \forall G \in \mathsf{TGRP} \,.\, \forall A \subset G \,.\, \overline{A^{-1}} = \overline{A}^{-1}$

$\texttt{Proof} \; =$

$[*] := \texttt{ClosureInTopologicalGroup}(G, A)\mathrm{E}\mathrm{Aut}_{\mathsf{TOP}}(G, \mathrm{inv})\mathrm{E}\mathsf{GRP}(G)$

$\quad \texttt{Iimage}(\mathrm{inv})\texttt{ClosureInTopologicalGroup2}(G, A) :$

$$: \overline{A^{-1}} = \bigcap \left\{ A^{-1}U \middle| U \in \mathcal{U}(e) \right\} = \bigcap \left\{ A^{-1}U^{-1} \middle| U \in \mathcal{U}(e) \right\} = \bigcap \left\{ (UA)^{-1} \middle| U \in \mathcal{U}(e) \right\} =$$

$$= \left(\bigcap \left\{ AU \middle| U \in \mathcal{U}(e) \right\} \right)^{-1} = \overline{A}^{-1};$$

$\square$

$\texttt{ConjugationClosure} :: \forall G \in \mathsf{TGRP} \,.\, \forall A \subset G \,.\, \forall g \in G \,.\, g\overline{A}g^{-1} = \overline{gAg^{-1}}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ClosureMult} :: \forall G \in \mathsf{TGRP} \,.\, \forall A, B \subset G \,.\, (\overline{A})(\overline{B}) \subset \overline{AB}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{OpenProduct} :: \forall G \in \mathsf{TGRP} \,.\, \forall U \in \mathcal{T}(G) \,.\, \forall A \subset G \,.\, UA, UA \in \mathcal{T}(G)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ClosedProduct} :: \forall G \in \mathsf{TGRP} \,.\, \forall x \in G \,.\, \forall A : \texttt{Closed}(G) \,.\, \texttt{Closed}(G, xA \,\&\, Ax)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{InverseIsUniformlyContinuousLR} :: \forall G \in \mathsf{TGRP} \,.\, \mathrm{inv}_G \in \mathsf{UNI}\Big((G, \mathcal{L}), (G, \mathcal{R}) \Big)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{InverseIsUniformlyContinuousRL} :: \forall G \in \mathsf{TGRP} \,.\, \mathrm{inv}_G \in \mathsf{UNI}\Big((G, \mathcal{R}), (G, \mathcal{L}) \Big)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ClosureOfSubgroup} :: \forall G \in \mathsf{TGRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \overline{H} \subset_{\mathsf{GRP}} G$

$\texttt{Proof} \; =$

$[1] := \texttt{SetMultSubset}\Big(G, \overline{H}, (\overline{H})^{-1}\Big)\texttt{ClosureInversion}\Big(G, H\Big)\texttt{ClosureNult}(G, H, H^{-1})\texttt{ESubgroup}(H) :$

$: \overline{H} \subset \overline{H}(\overline{H})^{-1} = \overline{H}(\overline{H^{-1}}) \subset \overline{HH^{-1}} = \overline{H},$

$[2] := \texttt{DoubleIneqLemma}\Big(?G, \overline{H}, \overline{H}(\overline{H})^{-1}\Big)[1] : \overline{H} = \overline{H}(\overline{H})^{-1},$

$[*] := \texttt{SubgroupAltDef}[2] : \overline{H} \subset_{\mathsf{GRP}} G;$

$\square$

$\texttt{ClosureOfNormalSubgroup} :: \forall G \in \mathsf{TGRP} \,.\, \forall H \lhd G \,.\, \overline{H} \lhd G$

$\texttt{Proof} \; =$

$[1] := \Lambda g \in G \,.\, \texttt{ConjugationClosure}(G, H, g)\texttt{ENormalSubgroup}(G, H) : \forall g \in G \,.\, g\overline{H}g^{-1} = \overline{gHg^{-1}} = \overline{H},$

$[*] := \texttt{I} \lhd [1] : \overline{H} \lhd G;$

$\square$

$\texttt{AbelianClosureIsAbelian} :: \forall G \in \mathsf{TGRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \texttt{T2}(G) \;\&\; H \in \mathsf{ABEL} \Rightarrow \overline{H} \in \mathsf{ABEL}$

$\texttt{Proof} \; =$

$\varphi := \Lambda g, h \in G \,.\, ghg^{-1}h^{-1} \in \mathsf{TOP}(G^2, G),$

$[1] := \texttt{T2HasClosedPoints}(G)\texttt{ClosedPreimage}(G^2, G, \varphi, \{e\}) : \texttt{Closed}\Big(G \times G, \varphi^{-1}\{e\}\Big),$

$[2] := \texttt{AbelianHasTrivialCommutator}(G, H)\texttt{E}\varphi : H \times H \subset \varphi^{-1}\{e\},$

$[3] := [1][2]\texttt{EClosure}(G \times G)\texttt{ClosureProduct}(G) : \overline{H} \times \overline{H} \subset \overline{H \times H} \subset \varphi^{-1}\{e\}\texttt{E}\varphi,$

$[*] := \texttt{AbelianByTrivialCommutot}[3] : \overline{H} \in \mathsf{ABEL};$

$\square$

$\texttt{OpenGroupsAreClopen} :: \forall G \in \mathsf{TGRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, H \in \mathcal{T}(G) \Rightarrow \texttt{Clopen}(G, H)$

$\texttt{Proof} \; =$

$[1] := \texttt{Outproduct}(G, H)\texttt{EsetProduct}(G, H^{\complement}, H)\texttt{E}\mathsf{TOP}(G)[0] : H^{\complement} = H^{\complement}H = \bigcap_{x \in H^{\complement}} xH \in \mathcal{T}(G),$

$[2] := \texttt{IClosed}[1] : \texttt{Closed}(G, H),$

$[*] := \texttt{IClopen}[2][0] : \texttt{Clopen}(G, H);$

$\square$

$\texttt{OpenGroupProduceClopenSets} :: \forall G \in \mathsf{TGRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \forall A \subset G \,.\, H \in \mathcal{T}(G) \Rightarrow \texttt{Clopen}(G, AH \;\&\; HA)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; x \in (AH)^{\complement},$

$[1] := \texttt{E}x : \forall a \in A \,.\, \forall h \in H \,.\, ah \neq x,$

$\texttt{Assume}\; h \in H,$

$\texttt{Assume}\; [2] : xh \in AH,$

$[3] := [2]h^{-1}\texttt{ESubgroup}(G, H) : x \in AHh^{-1} = AH,$

$[1.*] := [1][3] : \bot;$

$\rightsquigarrow [1] := \texttt{E}\bot\texttt{OpenProduct}(G, H, (AH)^{\complement}) : (AH)^{\complement} = (AH)^{\complement}H \in \mathcal{T}(G),$

$[2] := \texttt{IClosed}[1] : \texttt{Closed}(G, AH),$

$[3] := \texttt{OpenProduct}(G, H, A) : AH \in \mathcal{T}(G),$

$[*] := \texttt{IClopen}[2][3] : \texttt{Clopen}(G, AH);$

$\square$

$\texttt{OpenGroupIntersection} :: \forall G \in \mathsf{TGRP} \,.\, \bigcap \texttt{Subgroup} \,\&\, \mathcal{T}(G) \lhd G$

$\texttt{Proof} \;=$

$Z := \bigcap \texttt{Subgroup} \,\&\, \mathcal{T}(G) : \texttt{Subgroup}(G),$

$\texttt{Assume}\; g \in G,$

$\texttt{Assume}\; z \in Z,$

$\texttt{Assume}\; H : \texttt{Subgroup} \,\&\, \texttt{Open}(G),$

$[1] := \mathsf{E}\mathsf{TGRP} : \texttt{Subgroup} \,\&\, \texttt{Open}(G, g^{-1}Hg),$

$[2] := \mathtt{E}Z\mathtt{E}x[1] : z \in g^{-1}Hg,$

$[H.*] := g[2]g^{-1} : gzg^{-1} \in H;$

$\rightsquigarrow [g.*] := \mathtt{I}z : gzg^{-1} \in Z;$

$[*] := \texttt{INormalSubgroup} : Z \lhd G;$

$\square$

$\texttt{ClosedGroupIntersection} :: \forall G \in \mathsf{TGRP} \,.\, \bigcap \left\{ H \subset_{\mathsf{GRP}} G : \texttt{Closed}(G, H) \,\&\, H \neq \{e\} \right\} \lhd G$

$\texttt{Proof} \;=$

$\mathcal{A} := \left\{ H \subset_{\mathsf{GRP}} H : \texttt{Closed}(G, H) \,\&\, H \neq \{e\} \right\} :?\texttt{Subgroup}(G),$

$Z := \bigcap \mathcal{A} : \texttt{Subgroup}(G),$

$\texttt{Assume}\; g \in G,$

$\texttt{Assume}\; z \in Z,$

$\texttt{Assume}\; H \in \mathcal{A},$

$[1] := \mathsf{E}\mathsf{TGRP}\texttt{BijectionPreservesCardinality}(G)\mathtt{E}\mathcal{A} : g^{-1}Hg \in \mathcal{A},$

$[2] := \mathtt{E}Z\mathtt{E}x[1] : z \in g^{-1}Hg,$

$[H.*] := g[2]g^{-1} : gzg^{-1} \in H;$

$\rightsquigarrow [g.*] := \mathtt{I}z : gzg^{-1} \in Z;$

$[*] := \texttt{INormalSubgroup} : Z \lhd G;$

$\square$

$\texttt{ClosureAnnihilation} :: \forall G \in \mathsf{TGRP} \,.\, \forall A, B \subset G \,.\, \forall U \in \mathcal{T}(G) \,.\, \overline{A}U\overline{B} = AUB$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{DoubleClosureExpession} :: \forall G \in \mathsf{TGRP} \,.\, \forall A \subset G \,.\, \forall B : \texttt{CompactSubset}(G) \,.\, \overline{AB} = \bigcap_{U \in \mathcal{U}(e)} AUB$

$\texttt{Proof} \;=$

$\dots$

$\square$

2.1.5 SIN Groups and Uniform Continuity

$$\texttt{LowerTwoSidedUniformity} :: \prod_{G \in \mathsf{TGRP}} \texttt{Uniformity}(G)$$

$$\texttt{LowerTwoSidedUniformity}\,() = \mathcal{S}^{\wedge}_G := \mathcal{R}_G \wedge \mathcal{L}_G$$

$$\texttt{SinGroup} :: ?\mathsf{TGRP}$$

$$G : \texttt{SinGroup} \iff \texttt{SIN}(G) \iff \forall U \in \mathcal{U}_G(e) \,.\, \exists V \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, gVg^{-1} = V \subset U$$

$$\texttt{SINThm} :: \forall G \in \mathsf{TGRP} \,.\, \texttt{SIN}(G) \iff \mathcal{L}_G = \mathcal{R}_G$$

Proof $=$

($\Leftarrow$) By the hypothesis there is $V \in \mathcal{U}(e)$ such that $V_L \subset U_R$ for every $U \in \mathcal{U}(e)$.

This means that $gV \subset Ug$ for any $g \in G$.

But this can be rewritten as $gVg^{-1} \subset U$.

$W = \bigcup_{g \in G} gVg^{-1}$ is invariant.

And as U was arbitraty then G is SIN .

($\Rightarrow$) Use simmilar derivation, but in the inverse directions.

$\square$

$$\texttt{AbelianIsSIN} :: \forall G \in \mathsf{TGRP} \,.\, G \in \mathsf{ABEL} \Rightarrow \texttt{SIN}(G)$$

Proof $=$

$\ldots$

$\square$

$$\texttt{CompactIsSIN} :: \forall G \in \mathsf{TGRP} \,.\, \texttt{Compact}(G) \Rightarrow \texttt{SIN}(G)$$

Proof $=$

$\ldots$

$\square$

$$\texttt{DiscreteIsSIN} :: \forall G \in \mathsf{TGRP} \,.\, \texttt{Discrete}(G) \Rightarrow \texttt{SIN}(G)$$

Proof $=$

$\ldots$

$\square$

$$\texttt{CodiscreteIsSIN} :: \forall G \in \mathsf{TGRP} \,.\, \texttt{Codiscrete}(G) \Rightarrow \texttt{SIN}(G)$$

Proof $=$

$\ldots$

$\square$

UniformlyContinuousMult $:: \forall G \in \mathsf{TGRP} \,.\, (\cdot_G) \in \mathsf{UNI}\Big((G,\mathcal{L}) \times (G,\mathcal{R}).(G,\mathcal{L} \wedge \mathcal{R})\Big)$

Proof $=$

$\dots$

$\square$

SINByIdUniformContinuityLeft $:: \forall G \in \mathsf{TGRP} \,.\, \mathtt{SIN}(G) \iff \mathrm{id}_G \in \mathsf{UNI}\Big((G,\mathcal{L}),(G,\mathcal{R})\Big)$

Proof $=$

$\dots$

$\square$

SINByInvUniformContinuityRight $:: \forall G \in \mathsf{TGRP} \,.\, \mathtt{SIN}(G) \iff \mathrm{id}_G \in \mathsf{UNI}\Big((G,\mathcal{R}),(G,\mathcal{L})\Big)$

Proof $=$

$\dots$

$\square$

SINByInvUniformContinuityLeft $:: \forall G \in \mathsf{TGRP} \,.\, \mathtt{SIN}(G) \iff \mathrm{inv}_G \in \mathsf{UNI}\Big((G,\mathcal{L}),(G,\mathcal{L})\Big)$

Proof $=$

$\dots$

$\square$

SINByInvUniformContinuityRight $:: \forall G \in \mathsf{TGRP} \,.\, \mathtt{SIN}(G) \iff \mathrm{inv}_G \in \mathsf{UNI}\Big((G,\mathcal{R}),(G,\mathcal{R})\Big)$

Proof $=$

$\dots$

$\square$

typicalUniformity $:= \Lambda G \in \mathsf{TGRP} \,.\, \mathbb{U}_G = \mathcal{L}_G|\mathcal{R}_G|\mathcal{L}_G \vee \mathcal{R}_G|\mathcal{L}_G \wedge \mathcal{R}_G : \prod G \in \mathsf{TGRP} \,.\, \mathtt{Uniformity}(G);$

SinByCommutativeConvergence ::

$$:: \forall G : \mathsf{TGRP} \,.\, \left(\forall x, y : \mathbb{N} \to G \,.\, \lim_{n\to\infty} x_n y_n = e \iff \lim_{n\to\infty} y_n x_n = e\right) \iff \mathtt{SIN}(G)$$

Proof $=$

$(\Leftarrow)$: Assume that $\lim_{n\to\infty} x_n y_n = e$ and take $U \in \mathcal{U}_G(e)$.

Then as G is SIN there is neighborhood $V \subset U$ of e such that $gVg^{-1} = V$ for every $g \in G$.

There is $N \in \mathbb{N}$ such that $x_n y_n \in V$ for all $n \geq N$.

This means that $y_n x_n \in y_n V y_n^{-1} = V \subset U$ for all $n \geq N$, so $\lim_{n\to\infty} y_n x_n = e$ as U was arbitrary.

The reverse deduction for $y_n x_n$ is trivially the same.

$(\Rightarrow)$:? .

$\dots$

$\square$

2.1.6 Metrics for Twosided Uniformities

There is a special form of metrics for twosided uniformities.

$$\vee\texttt{-Semimetric} :: \prod_{G \in \mathsf{GRP}} ?\texttt{Semimetric}(G)$$

$$\rho : \vee\texttt{-Semimetric} \iff \vee\texttt{-Semimetric} \iff \exists \sigma : \texttt{LeftInvariantMetric}(G) \;.\; \forall g, h \in G \;.$$
$$\rho(g,h) = \sigma(g,h) + \sigma(g^{-1}, h^{-1})$$

$\texttt{VeeMetricConstructionIsUnique}$ ::

$:: \forall G \in \mathsf{GRP} \;.\; \forall \rho : \vee\texttt{-Semimetric}(G) \;.\; \exists! \sigma : \texttt{LeftInvariantMetric}(G) \;.\; \forall g, h \in G \;.$

$.\; \rho(g,h) = \sigma(g,h) + \sigma(g^{-1}, h^{-1})$

$\texttt{Proof} =$

$\Big(\sigma, [1]\Big) := \texttt{E}\vee\texttt{-Semimetric}(G, \rho) :$

$: \sum \sigma : \texttt{LeftInvariantMetric}(G) \;.\; \forall g, h \in G \;.\; \rho(g,h) = \sigma(g,h) + \sigma(g^{-1}, h^{-1}),$

$[2] := q\Lambda g \in G \;.\; \texttt{E}\alpha_\rho(g)[1]\texttt{ELeftInvariantMetric}(G, \sigma)\texttt{I}\alpha_\sigma :$

$: \alpha_\rho(g) = \rho(g, e) = \sigma(g, e) + \sigma(g^{-1}, e) = 2\sigma(g, e) = 2\alpha_\sigma(g),$

$[3] := \texttt{I}(=, \rightarrow)[2] : \alpha_\rho = 2\alpha_\sigma,$

$[*] := d\left(\frac{[3]}{2}\right) : \sigma = d_{\frac{\alpha_\rho}{2}};$

$\square$

$\texttt{VeeMetricMetrisesUpperTwoSidedUniformity} :: \forall G \in \mathsf{GRP} \;.\; \forall \rho : \vee\texttt{-Semimetric}(G) \;.\; \mathcal{S}^{\vee}_{(G,\rho)} = \mathbb{B}_\rho$

$\texttt{Proof} =$

...

$\square$

2.1.7 Ellis Theorem

If group has a locally compact Haussdorff topology then it is only enough to have continuous multiplication to show that the topolgy is a group toplogy!

$$\texttt{EllisTopology} :: \prod_{G \in \mathsf{GRP}} ?\texttt{Topology}(G)$$

$$\mathcal{T} : \texttt{EllisTopology} \iff \texttt{T2}(G, \mathcal{T}) \ \& \ \texttt{LocallyCompact}(G, \mathcal{T}) \ \& \ \cdot_G \in \mathsf{TOP}(G^2, G)$$

$\texttt{EllisCompactInversionLemma}$ ::

$$:: \forall G \in \mathsf{GRP} \ . \ \forall \mathcal{T} : \texttt{EllisTopology}(G) \ . \ \forall K : \texttt{CompactSubset}(G, \mathcal{T}) \ . \ \texttt{Closed}\Big((G, \mathcal{T}), K^{-1}\Big)$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ b \in \overline{K^{-1}},$

$\Big(\mathcal{F}, [1]\Big) := \texttt{ClosureByLimits}(G, K^{-1}, b) : \sum \mathcal{F} : \texttt{Filter}(K^{-1}) \ . \ b = \lim \mathcal{F},$

$a := \texttt{FilterCompact}(K, \mathrm{inv}_* \, \mathcal{F}) : \texttt{Cluster}\Big(K, \mathrm{inv}_* \, \mathcal{F}\Big),$

$\Big(\mathcal{G}, [2]\Big) := \texttt{ClusterConvergingFilter} : \sum \mathcal{G} : \texttt{Filter}(K) \ . \ a = \lim \mathcal{G} \ \& \ \mathrm{inv}_* \mathcal{F} \subset \mathcal{G},$

$[3] := [2.1][1] : b = \lim \mathrm{inv}_* \mathcal{G},$

$\mathcal{L} := \mathcal{G} \times \mathrm{inv}_* \mathcal{G} : \texttt{Filterbase}(K \times K^{-1}),$

$[4] := \mathtt{E}\mathcal{L}[1][3] : \lim \mathcal{L} = (a, b),$

$[5] := \mathtt{E}_3\texttt{EllisTopology}(G, \mathcal{T})[4] : \lim(\cdot_G)(\mathcal{L}) = ab,$

$[6] := \mathtt{E}\mathcal{L}\texttt{FilterLimit} : \texttt{Cluster}\Big(G, (\cdot_G)(\mathcal{L}), e\Big),$

$[7] := \mathtt{E}_1\texttt{EllisTopology}(G, \mathcal{T})\texttt{T2HasUniqueClusters} : ab = e,$

$[b.*] := a^{-1}[7] : b \in K^{-1};$

$\leadsto [1] := \mathtt{I} \subset : \overline{K^{-1}} \subset K^{-1},$

$[*] := \mathtt{E}\texttt{Closure}[1] : \overline{K^{-1}} = K^{-1},$

$\square$

$\texttt{EllisCountableInversionLemma} ::$

$:: \forall G \in \mathsf{GRP} \;.\; \forall \mathcal{T} : \texttt{EllisTopology}(G) \;.\; \forall A : \texttt{Countable}(G) \;.\; \forall b \in \overline{A} \;.\; b^{-1} \in \overline{A^{-1}}$

$\texttt{Proof} \;=$

$\Big(\mathcal{F}, [1]\Big) := \texttt{ClosureByLimits}(G, K^{-1}, b) : \sum \mathcal{F} : \texttt{Filter}(K^{-1}) \;.\; b = \lim \mathcal{F},$

$H := \langle A \cup \{b\} \rangle_{\mathsf{GRP}} : \texttt{Subgroup}(G),$

$[2] := \texttt{GeneratedByConutableIsCountable}(G)\mathtt{E}H : |H| \le \aleph_0,$

$\texttt{Assume}\ K \in \mathcal{K}(e),$

$\texttt{Assume}\ y \in \overline{H},$

$[3] := \mathtt{E}_3\texttt{EllisTopology}(G, \mathcal{T}) : yK \in \mathcal{K}(y),$

$[4] := \texttt{ClosureAltDef}(G, H)\mathtt{E}\mathcal{K}[3] : \exists yK \cap H,$

$\Big(k, [5]\Big) := \mathtt{E}\exists[4] : \sum k \in K \;.\; yk \in H,$

$[y.*] := [5]k^{-1} : y \in HK^{-1};$

$\rightsquigarrow [3] := \mathtt{I} \subset : \overline{H} \subset HK^{-1},$

$[4] := \Lambda g \in H \;.\; \mathtt{E}_3\texttt{EllisTopology}(G, \mathcal{T})\texttt{HomeoClusureEq}\Big((G, \mathcal{T}), H, \lambda_g\Big)\texttt{ESubgroup}(G, H) :$

$: \forall g \in G \;.\; g\overline{H} = \overline{gH} = \overline{H},$

$[5] := [3]\Lambda x \in H \;.\; \texttt{InverseMeaning}(G, x)[4] :$

$$: \overline{H} = \bigcup_{x \in H} xK^{-1} \cap \overline{H} = \bigcup_{x \in H} x(K^{-1} \cap x^{-1}\overline{H}) = \bigcup_{x \in H} x(K^{-1} \cap \overline{H}),$$

$[6] := \texttt{EllisCompactInversionLemma}(G, \mathcal{T}, K) : \texttt{Closed}\Big((G, \mathcal{T}), K^{-1}\Big),$

$[7] := \mathtt{E}_{1,2}\texttt{EllisTopology}(G, \mathcal{T})\texttt{BaireCategoryTHM} : \texttt{Baire}(\overline{H}, \mathcal{T} \cap \overline{H}),$

$\Big(h, [8]\Big) := \mathtt{E}\texttt{Baire}(\overline{H}, \mathcal{T} \cap \overline{H})[5] : \sum h \in H \;.\; \exists^* x \in \overline{H} \;.\; x \in h(K^{-1} \cap \overline{H}),$

$[9] := \mathtt{E}\exists^*[8] : \neg\texttt{Dense}\Big(\overline{H}, \overline{H} \setminus h(K^{-1} \cap \overline{H})\Big),$

$\Big(U, [10]\Big) := \mathtt{E}\texttt{Dense}[9] : \sum U \in \mathcal{T} \;.\; U \subset h(K^{-1} \cap \overline{H}) \;\&\; \exists U \cap \overline{H},$

$u := \texttt{ClosureAltDef}\Big((G, \mathcal{T}), H, U\Big)[10] \in H \cap U,$

$[11] := [4]\mathtt{E}u : bu^{-1}(U \cap \overline{H}) = bu^{-1}U \cap \overline{H} \in \mathcal{U}_{\overline{H}}(b),$

$\Big(F, [12]\Big) := \mathtt{E}\texttt{FilterConvergece}[1][11][10][4]\texttt{IntersectionIsSubset} :$

$: F \subset bu^{-1}(U \cap \overline{H}) \subset bu^{-1}h(K^{-1} \cap \overline{H}) \subset bu^{-1}hK^{-1},$

$\Big(F, [12]\Big) := \mathtt{E}\texttt{FilterConvergece}[1][11][10][4]\texttt{IntersectionIsSubset} :$

$: F \subset bu^{-1}(U \cap \overline{H}) \subset bu^{-1}h(K^{-1} \cap \overline{H}) \subset bu^{-1}hK^{-1},$

$[13] := [12]^{-1} : F^{-1} \subset Kb^{-1}uh^{-1},$

$a := \texttt{CompactHasCluseter}\Big((A, \mathcal{T} \cap A), Kb^{-1}uh^{-1}, \mathcal{F}^{-1}\Big) : \texttt{Cluster}\Big((A, \mathcal{T} \cap A), Kb^{-1}uh^{-1}, \mathcal{F}^{-1}\Big),$

$[14] := [13]\mathtt{E}a : a \in Kb^{-1}uh^{-1} \cap \overline{A^{-1}},$

$[*] := \texttt{ByAnalogy}\Big(\texttt{EllisCompactInversionLemma}\Big)[14] : b^{-1} = a \in \overline{A^{-1}};$

$\square$

$\texttt{EllisInversCompactnessLemma} ::$

$$:: \forall G \in \mathsf{GRP} \,.\, \forall \mathcal{T} : \texttt{EllisTopology} \,.\, \forall K : \texttt{CompactSubset}(G,\mathcal{T}) \,.\, \texttt{CompactSubset}\Big((G,\mathcal{T}), K^{-1}\Big)$$

$\texttt{Proof} \ =$

$[1] := \texttt{EllisCompactInversionLemma}(G,\mathcal{T},K) : \texttt{Closed}\Big((G,\mathcal{T}), K^{-1}\Big),$

$U := \mathrm{E}_2\texttt{EllisTopology}(G,\mathcal{T})\texttt{ELocallyCompact}(G,e) \in \mathcal{K}(e),$

$\texttt{Assume}\ [2] : \forall A : \texttt{Finite}(G) \,.\, K^{-1} \not\subset AU,$

$\Big(k,[3]\Big) := \mathrm{E}\mathbb{N}[2] : \sum k : \mathbb{N} \to K^{-1} \,.\, \forall n \in \mathbb{N} \,.\, k_{n+1} \notin \bigcup_{i=1}^{n} k_i U,$

$\Big(b,[4]\Big) := \texttt{CompactHasClusetrs}\Big((G,\mathcal{T}), K, k^{-1}\Big) : \sum b \in K \,.\, \texttt{Cluster}\Big((G,\mathcal{T}), k^{-1}, b\Big),$

$\Big(V,[5]\Big) := \mathrm{E}_1\texttt{EllisTopology}(G,\mathcal{T})\texttt{ProducTopologyBase}(\mathcal{T},\mathcal{T},U) : \sum V \in \mathcal{U}(e) \,.\, V^2 \subset U,$

$\Big(n,[6]\Big) := \texttt{ECluster}\Big((G,\mathcal{T}), k^{-1}, b, Vb\Big) : \sum_{n=1}^{\infty} k_n^{-1} \in Vb,$

$[7] := k_n[6]b^{-1} : b^{-1} \in k_n V,$

$A := \{k_m^{-1} | m > n\} :?K,$

$[8] := \texttt{ECluster}\Big((G,\mathcal{T}), k^{-1}, b)\mathrm{E}A : b \in \overline{A},$

$[9] := \texttt{EllisCountableInversionLemma}[8] : b^{-1} \in \overline{A^{-1}},$

$\Big(m,[10]\Big) := \mathrm{E}A[9]\texttt{ClosureAltdef}(b^{-1}V) : \sum m > n \,.\, k_m \in b^{-1}V,$

$[11] := [10][7][5] : k_m \in k_n V^2 \subset k_n U,$

$[2.*] := [2][11] : \bot;$

$\rightsquigarrow \Big(A,[2]\Big) := \mathrm{E}\bot : \exists A : \texttt{Finite}(G) \,.\, K^{-1} \subset AU,$

$[3] := \mathrm{E}_1\texttt{EllisTopology}(G,\mathcal{T})\texttt{FiniteCompactUnion} : \texttt{CompactSubset}\Big((G,\mathcal{T}), AU\Big),$

$[*] := \texttt{ClosedSubsetOfCompactIsCompact}[2][3] : \texttt{CompactSubset}\Big((G,\mathcal{T}), K^{-1}\Big),$

$\square$

$\texttt{EllisTheorem} :: \forall G \in \mathsf{GRP} \,.\, \forall \mathcal{T} : \texttt{EllisTopology}(G) \,.\, (G, \mathcal{T}) \in \mathsf{TGRP}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U \in \mathcal{U}(e),$

$\texttt{Assume}\ [1] : \forall K \in \mathcal{K}(e) \,.\, K^{-1} \not\subset U,$

$\mathcal{F} := \left\{ K^{-1} \cap U^{\complement} \middle| K \in \mathcal{K}(e) \right\} :?\texttt{CompactSubset}(K, \mathcal{T}),$

$[2] := [1]\mathtt{E}\mathcal{F} : \emptyset \notin \mathcal{F},$

$[3] := \mathtt{E}_2\texttt{EllisTopologyELocallyCompact}(G, \mathcal{T})\mathtt{E}\mathcal{F} : \mathcal{F} \neq \emptyset,$

$[4] := \mathtt{E}\mathcal{K}(e)\mathtt{E}\mathrm{Aut}_{\mathsf{SET}}(G, \mathrm{inv})\mathtt{E}\mathcal{F} : \forall A, B \in \mathcal{F} \,.\, A \cap B \in \mathcal{F},$

$[5] := \texttt{IFilterbase}[2-4] : \texttt{Filterbase}\Big((G, \mathcal{T}), \mathcal{F}\Big),$

$[6] := \texttt{CantorIntersectionTHM}[5] : \exists \bigcap \mathcal{F},$

$[7] := \mathtt{E}_{1,2}\texttt{EllisTopology}(G, \mathcal{T})\texttt{RegularNbhdBaseIntersection} : \bigcap \mathcal{K}(e) = \{e\},$

$[8] := \texttt{BijectionOfIntersection}(G, G, \mathrm{inv})[7] : \bigcap \mathrm{inv}_* \mathcal{K}(e) = \{e\},$

$[9] := [8]\mathtt{E}\mathcal{F} : \bigcap \mathcal{F} = \emptyset,$

$[1.*] := [9][6] : \bot;$

$\leadsto \Big(K, [U.*]\Big) := \mathtt{E}\bot : \sum K \in \mathcal{K}(e) \,.\, K^{-1} \subset U;$

$\leadsto [1] := \mathtt{I}C_e : \mathrm{inv} \in C_e\Big((G, \mathcal{T}), (G, \mathcal{T})\Big),$

$[2] := [1]\mathtt{E}_3\texttt{EllisTopology}(G, \mathcal{T}) : \mathrm{inv} \in \mathrm{Aut}_{\mathsf{TOP}}(G, \mathcal{T}),$

$[*] := \mathtt{I}\mathsf{TGRP}[2]\mathtt{E}_3\texttt{EllisTopology}(G, \mathcal{T}) : (G, \mathcal{T}) \in \mathsf{TGRP};$

$\square$

2.1.8 Topological Groups with Ultrametrics

Balls around the unity producesed by ultrametric are subgroups, and hence clopen.

$$\texttt{Ultravalue} :: \prod_{G \in \mathsf{GRP}} ?\texttt{AbsoluteValue}(G)$$

$$\alpha : \texttt{Ultravalue} \iff \forall a, b \in A \,.\, \alpha(ab) \leq \max\Big(\alpha(a), \alpha(b)\Big)$$

$\texttt{UlravalueProduceUltrametric} :: \forall A \in \mathsf{ABEL} \,.\, \forall \alpha : \texttt{Ultravalue}(A) \,.\, \texttt{Ultrametric}(A, d_\alpha)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UltrametricCellsAreSubgroups} :: \forall A \in \mathsf{ABEL} \,.\, \forall \alpha : \texttt{Ultravalue}(A) \,.\, \forall r \in \mathbb{R}_{++}\ \mathbb{B}(0, r) \lhd A$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a, b \in \mathbb{B}(0, r),$

$[1] := \texttt{E}d_\alpha\texttt{EUltravalue}(A)\texttt{I}d_\alpha\texttt{E}a, b \in \mathbb{B}(0, r) :$

$\quad : d_\alpha(0, a + b) = \alpha(a + b) \leq \max\Big(\alpha(a), \alpha(b)\Big) = \max\Big(d_\alpha(0, a), d_\alpha(0, b)\Big) < r,$

$\Big[(a, b).*\Big] := \texttt{E}\mathbb{B}(0, r)[1] : a + b \in \mathbb{B}(0, r);$

$\leadsto [1] := \texttt{I}\forall : \forall a, b \in \mathbb{B}(0, r) \,.\, a + b \in \mathbb{B}(0, r),$

$[2] := \texttt{EAbsoluteValue}(A, \alpha) : \forall a \in \mathbb{B}(0, r) \,.\, a^{-1} \in \mathbb{B}(0, r),$

$[*] := \texttt{ISubgroup} : \mathbb{B}(0, r) \subset_{\mathsf{GRP}} G;$

$\square$

$\texttt{UltrametrizableGroupHasBaseOfSubgroups} ::$

$\quad :: \forall A \in \mathsf{ABEL} \,.\, \forall \alpha : \texttt{Ultravalue}(A) \,.\, \exists \mathcal{N} : \texttt{NeighborhoodBase}(A, \rho) \,.\, \forall N \in \mathcal{N} \,.\, N \lhd A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UltrametrizableGroupHasClopenBalls} ::$

$\quad :: \forall G \in \mathsf{TGRP} \,.\, \forall \alpha : \texttt{Ultravalue}(G) \,.\, \forall r \in \mathbb{R} \,.\, \texttt{Clopen}(G, \mathbb{B}_\alpha(e, r))$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UltrametrizableGroupsAreZeroDim} ::$

$\quad :: \forall G \in \mathsf{GRP} \,.\, \forall \alpha : \texttt{Ultravalue}(G) \,.\, \dim_{\mathsf{TOP}}(G, d_\alpha) = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.1.9 Some Interesting Examples

Sometimes our basic expectations fail.

$$\texttt{SumOfIntegersIsNotClosed} :: \alpha \in \mathbb{R} \setminus \mathbb{Q} \,.\, \neg\texttt{Closed}(\mathbb{R}, \mathbb{Z} + \alpha\mathbb{Z})$$

$\texttt{Proof} \ =$

$A := \mathbb{Z} + \alpha\mathbb{Z} :?\mathbb{R},$

$[1] := \texttt{IrrationalGenDense}(\alpha)\texttt{I}A : \texttt{Dense}(\mathbb{R}, A),$

$\texttt{Assume}\ [2] : \texttt{Closed}(\mathbb{R}, A),$

$[3] := \texttt{EDense}(\mathbb{R}, A)[2] : A = \mathbb{R},$

$\Big(n, m, [4]\Big) := \texttt{E}A[3]\left(\frac{\alpha}{2}\right) : \sum n, m \in \mathbb{Z} \,.\, \frac{\alpha}{2} = n\alpha + m,$

$[5] := [4] - n\alpha : \frac{1-2n}{2}\alpha = m,$

$[6] := \frac{2}{1-2n}[5]\texttt{I}\mathbb{Q} : \alpha = \frac{2m}{1-2n} \in \mathbb{Q},$

$[2.*] := \texttt{E}\alpha[6] : \perp;$

$\leadsto [*] := \texttt{E}\perp : \neg\texttt{Closed}(\mathbb{R}, A),$

$\square$

$$\texttt{PositiveRaysTopolgy} :: \texttt{Topology}(\mathbb{Z})$$

$$\texttt{PositiveRaysTopology}\,() = \mathcal{T}_{+\infty} := \Big\{[n, \ldots, +\infty)\Big| n \in \mathbb{Z}\Big\} \cup \{\emptyset, \mathbb{Z}\}$$

$$\texttt{PositiveRayTopologyHasContinuousAddition} :: (+_{\mathbb{Z}}) \in \mathsf{TOP}\Big((\mathbb{Z}, \mathcal{T}_{+\infty})^2, (\mathbb{Z}, \mathcal{T}_{+\infty})\Big)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ U \in \mathcal{T}_{+\infty},$

$\Big(n, [1]\Big) := \texttt{E}\mathcal{T}_{+\infty} : \sum n \in \mathbb{Z} \,.\, U = [n, +\infty),$

$[2] := \texttt{E}(+_{\mathbb{Z}})[1] : (+_{\mathbb{Z}})^{-1}U = \Big\{(k, l) \in \mathbb{Z}^2 \Big| k + l \geq n\Big\},$

$[3] := \Lambda(k, l) \in (+_{\mathbb{Z}})^{-1}U[2]\texttt{SumIneq}[2] : \forall (k, l) \in (+_{\mathbb{Z}})^{-1}U \,.\, [k, \ldots, +\infty) \times (l, \ldots, +\infty) \subset (+_{\mathbb{Z}})^{-1}U,$

$[U.*] := \texttt{ProductTopologyBase}[3] : (+_{\mathbb{Z}})^{-1}U \in \mathcal{T}\Big((\mathbb{Z}, \mathcal{T}_{+\infty})^2\Big);$

$\leadsto [*] := \texttt{E}\mathsf{TOP} : (+_{\mathbb{Z}}) \in \mathsf{TOP}\Big((\mathbb{Z}, \mathcal{T}_{+\infty})^2, (\mathbb{Z}, \mathcal{T}_{+\infty})\Big);$

$\square$

$$\texttt{PositiveRaysIsNotGroupTopology} :: (\mathbb{Z}, \mathcal{T}_{+\infty}) \notin \mathsf{TGRP}$$

$\texttt{Proof} \ =$

$[1] := \texttt{E}\mathcal{T}_{+\infty}(0) : [0, \ldots, +\infty) \in \mathcal{T}_{+\infty},$

$[2] := \texttt{E}\mathrm{inv}_{\mathbb{Z}} : \mathrm{inv}_{\mathbb{Z}}[0, \ldots, +\infty) = (-\infty, \ldots, 0] \notin \mathcal{T}_{+\infty},$

$[3] := [1][2] : \mathrm{inv}_{\mathbb{Z}} \notin \mathrm{Aut}_{\mathsf{TOP}}(\mathbb{Z}, \mathcal{T}_{+\infty}),$

$[*] := \texttt{E}\mathsf{TGRP}[3] : (\mathbb{Z}, \mathcal{T}_{+\infty}) \notin \mathsf{TGRP};$

$\square$

$$G := \mathbf{GL}(\mathbb{R}, 2) : \mathsf{TGRP};$$

$$x := \begin{pmatrix} \frac{1}{n} & \frac{1}{n^2} \\ 0 & 1 \end{pmatrix} : \mathbb{N} \to G;$$

$$x := \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{N} \to G;$$

$$x_n y_n = \begin{pmatrix} 1 & \frac{1}{n} + \frac{1}{n^2} \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$y_n x_n = \begin{pmatrix} 1 & \frac{1}{n} + 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

2.2 Further Topological Properties

Topological group structure simplifies work with some topological and metric concepts

2.2.1 Continuous Homomorphism

For topological groups it is especially easy to prove that morphism is continuous.

$\texttt{PointContinuityImplyContinuity} ::$
$: \forall G, H \in \mathsf{TGRP} \, . \, \forall g \in G \, . \, \forall \varphi \in \mathsf{GRP} \,\&\, C_g(G,H) \, . \, \varphi \in \mathsf{TGRP}(G,H)$
$\texttt{Proof} \;=$
$\texttt{Assume}\; h \in \operatorname{Im}\varphi,$
$\Big(p, [1]\Big) := \texttt{E}\operatorname{Im}\varphi h : \sum p \in G \, . \, h = \varphi(p),$
$\texttt{Assume}\; U \in \mathcal{U}(h),$
$[2] := \texttt{E}\mathsf{TGRP}(H) : \varphi(g)h^{-1}U \in \mathcal{U}\Big(\varphi(g)\Big),$
$[3] := \texttt{E}C_h(G,H)[2] : \varphi^{-1}\Big(\varphi(g)h^{-1}U\Big) \in \mathcal{U}(g),$
$[h.*] := \texttt{E}\mathsf{TGRP}(G)\texttt{E}\mathsf{GRP}(G,H,\varphi)[1][3] : \varphi^{-1}(U) = pg^{-1}\varphi^{-1}\Big(\varphi(g)h^{-1}U\Big) \in U(p);$
$\rightsquigarrow [*] := \texttt{ContinuityIsLocal} : \varphi \in \mathsf{TGRP}(G,H);$
$\square$

$\texttt{IdentityOpenessImplyOpeness} :: \forall G, H \in \mathsf{TGRP} \, . \, \forall \varphi \in \mathsf{GRP}(G,H) \, .$
$. \Big(\forall U \in \mathcal{U}_G(e) \, . \, \varphi(U) \in \mathcal{U}_H(e)\Big) \Rightarrow \texttt{OpenMap}(G,H,\varphi)$
$\texttt{Proof} \;=$
$\texttt{Assume}\; g \in G,$
$\texttt{Assume}\; U \in \mathcal{U}_G(g),$
$[1] := \texttt{E}\mathsf{TGRP} : g^{-1}U \in \mathcal{U}_G(e),$
$[2] := [0][1] : \varphi\Big(g^{-1}U\Big) \in \mathcal{U}_H(e),$
$[g.*] := \texttt{E}\mathsf{GRP}(G,H,\varphi)\texttt{E}\mathsf{TGRP}(H) : \varphi(U) = \varphi(g)\Big(g^{-1}U\Big) \in \mathcal{U}_H\Big(\varphi(g)\Big);$
$\rightsquigarrow [*] := \texttt{OpennessIsLocal} : \texttt{OpenMap}(G,H,\varphi);$
$\square$

$\texttt{LeftUnifomityUCCriterion} :: \forall G, H \in \mathsf{GRP} \,.\, \forall \varphi : G \to H \,.\, \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big) \iff$

$\iff \forall V \in \mathcal{U}_H(e) \,.\, \exists U \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, \varphi\Big(gU\Big) \subset \varphi(g)V$

$\texttt{Proof} =$

$\texttt{Assume } [1] : \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big),$

$\texttt{Assume } B : \mathcal{U}_H(e),$

$[2] := \texttt{E}\mathcal{L}_H(V) : V_L \in \mathcal{L}_H,$

$\Big(U, [3]\Big) := \texttt{E}\mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big)(\varphi) : \sum U \in \mathcal{L}_G \,.\, (\varphi \times \varphi)U \subset V_L,$

$\Big(W, [4]\Big) := \texttt{EBaseOfUniformity}\Big(G, \mathcal{L}_G, \big(\mathcal{U}_G(e)\big)_L, U\Big) : \sum W \in \mathcal{U}_G(e) \,.\, W_L \subset U,$

$\texttt{Assume } g \in G,$

$[5] := \texttt{E}W_L(g) : gW = W_L(g),$

$[1.*] := \texttt{E}(=)\bigg([5], \varphi\Big(gW\Big)\bigg)\texttt{EConnector}(G, W_L \,\&\, U)\texttt{MonotonicImage}\Big(G, H, W_L(g), U(g)[4]\Big)[3]\texttt{E}V_L\Big(\varphi(g)\Big) :$

$\varphi\Big(gW\Big) = \varphi\Big(W_L(g)\Big) \subset \varphi\Big(U(g)\Big) \subset V_L\Big(\varphi(g)\Big) = \varphi(g)V;$

$\rightsquigarrow [1] := \texttt{I}(\Rightarrow) : \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big) \Rightarrow \forall V \in \mathcal{U}_H(e) \,.\, \exists U \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, \varphi\Big(gU\Big) \subset \varphi(g)V,$

$\texttt{Assume } [2] : \forall V \in \mathcal{U}_H(e) \,.\, \exists U \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, \varphi\Big(gU\Big) \subset \varphi(g)V,$

$\texttt{Assume } V \in \mathcal{L}_H,$

$\Big(W, [3]\Big) := \texttt{EBaseOfUniformity}\Big(H, \mathcal{L}_H, \big(\mathcal{U}_H(e)\big)_L, V\Big) : \sum W \in \mathcal{U}_H(e) \,.\, W_L \subset V,$

$\Big(U, [4]\Big) := [2](W) : \sum U \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, \varphi\Big(gU\Big) \subset \varphi(g)W,$

$[V.*] := [4][3] : (\varphi \times \varphi)U_L \subset W_L \subset V;$

$\rightsquigarrow [2.*] := \texttt{I}\mathsf{UNI} : \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big);$

$\rightsquigarrow [*] := \texttt{I}(\iff)[1] :$

$: \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big) \iff \forall V \in \mathcal{U}_H(e) \,.\, \exists U \in \mathcal{U}_G(e) \,.\, \forall g \in G \,.\, \varphi\Big(gU\Big) \subset \varphi(g)V;$

$\square$

$\texttt{TopologicalHomomorphismsAreUniformlyContinuous} ::$

$:: \forall G, H \in \mathsf{TGRP} \,.\, \forall \varphi \in \mathsf{TGRP}(G, H) \,.\, \varphi \in \mathsf{UNI}\Big((G, \mathcal{L}_G), (H, \mathcal{L}_H)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

2.2.2 Metrization

Topologigal groups with a countable base of neighborhood can be metrized in a way compatible with their algebraic structure.

$\texttt{TGRPTrisection} ::$
$:: \forall G \in \mathsf{TGRP} \,.\, \forall U \in \mathcal{U}(e) \,.\, \exists V \in \mathcal{U}(e) \;\&\; \texttt{SymmetricSet}(G) \,.\, VVV \subset U$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{LeftGroupMetrization} :: \forall G \in \mathsf{TGRP} \,.\, \forall \mathcal{N} : \texttt{NeighborhoodBase}(G, e) \,.$
$. \, |\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \texttt{LeftInvariantMetric}(G) \,.\, (G, \rho) \cong_{\mathsf{TOP}} G$
$\texttt{Proof} \;=$
$N := \texttt{enumerate}(\mathcal{N}) : \mathbb{N} \twoheadrightarrow \mathcal{N},$
$\Big(V, [2],\Big) := \texttt{rec2}\Big(G, \Lambda n \in \mathbb{N} \Lambda U \in \mathcal{U}(e) \,.\, \texttt{TGRPTrisection}(G, U \cap N_n)\Big) :$
$: \sum \mathbb{Z}_+ \to \mathcal{U}(e) \;\&\; \texttt{SymmetricSet}(G) \,.\, \forall n \in \mathbb{N} \,.\, V_n V_n V_n \subset U_{n-1},$
$\alpha := \Lambda g \in G \,.\, \inf\{2^{-n} | n \in \mathbb{Z}_+, g \in V_n\} : G \to \mathbb{R}_+,$
$[3] := \mathtt{E}\alpha\mathtt{E}\Lambda n \in \mathbb{Z}_+ \,.\, \mathtt{E}\texttt{SymmetricSet}(G, V_n) : \forall g \in G \,.\, \alpha(g) = \alpha(g^{-1}),$
$[4] := \mathtt{E}\alpha\mathtt{E}\Lambda n \in \mathbb{Z}_+ \,.\, \mathtt{E}V_n\mathcal{U}_G(e) : \alpha(e) = 0,$
$[5] := \mathtt{E}\alpha[2] : \forall a, b, c \in G \,.\, \alpha(a, b, c) \leq 2 \max \alpha\Big(\alpha(a), \alpha(b), \alpha(c)\Big),$
$\beta := \Lambda g \in G \,.\, \inf \left\{ \sum_{i=1}^{n} \alpha(h_i h_{i-1}^{-1}) \middle| n \in \mathbb{N}, h : \{0, \dots, n\} \to G, h_0 = e, h_n = g \right\} : G \to \mathbb{R}_{++},$
$\texttt{Assume}\; h, g : G,$
$\texttt{Assume}\; \varepsilon : \mathbb{R}_{++},$
$\Big(n, a, [6]\Big) := \mathtt{E}\beta\left(g, \frac{\varepsilon}{2}\right) : \sum_{n=1}^{\infty} \sum_{\{0,\dots,n\} \to G} a_0 = e \;\&\; a_n = g \;\&\; \frac{\varepsilon}{2} + \sum_{i=1}^{n} \alpha(a_i a_{i-1}^{-1}) = \beta(g),$
$\Big(m, b, [7]\Big) := \mathtt{E}\beta\left(h, \frac{\varepsilon}{2}\right) : \sum_{m=1}^{\infty} \sum_{\{0,\dots,m\} \to G} b_0 = e \;\&\; b_n = h \;\&\; \frac{\varepsilon}{2} + \sum_{i=1}^{m} \alpha(b_i b_{i-1}^{-1}) = \beta(h),$
$c := \texttt{concat}\Big(b, a_{|\{1,\dots,n\}} h\Big) : \{0, \dots, n+m\} \to G,$
$\Big[(h, g). *\Big] := \mathtt{E}\beta(gh)\mathtt{E}c[6][7] : \beta(gh) \leq \sum_{i=1}^{n+m} \alpha(c_i c_{i+1}^{-1}) = \sum_{i=1}^{n} \alpha(a_i a_{i+1}^{-1}) + \sum_{i=1}^{m} \alpha(b_i b_{i+1}^{-1}) \leq \beta(g) + \beta(h) + \varepsilon;$
$\rightsquigarrow [6] := \mathtt{I}\forall : \forall h, g \in G \,.\, \beta(gh) \leq \beta(g) + \beta(h),$
$[7] := \mathtt{E}\beta\mathtt{E}\texttt{BaseOfUniformity}(G, \mathcal{L}_G, \mathcal{N}) : \forall \mathcal{F} : \texttt{Filter}(G) \,.\, e \in \lim \mathcal{F} \iff 0 = \lim \beta(\mathcal{F}),$
$[*] := \texttt{TopologyByFilters} : (G, d_\beta) \cong_{\mathsf{TOP}} G;$
$\square$

$\texttt{RightGroupMetrization} :: \forall G \in \mathsf{TGRP} \,.\, \forall \mathcal{N} : \texttt{NeighborhoodBase}(G, e) \,.$
$. \, |\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \texttt{RightInvariantMetric}(G) \,.\, (G, \rho) \cong_{\mathsf{TOP}} G$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{VeeGroupMetrization} :: \forall G \in \mathsf{TGRP} \; . \; \forall \mathcal{N} : \texttt{NeighborhoodBase}(G, e) \; .$

$\quad . \; |\mathcal{N}| \leq \aleph_0 \Rightarrow \exists \rho : \vee\text{-}\texttt{Semimetric}(G) \; . \; (G, \rho) \cong_{\mathsf{TOP}} G$

$\texttt{Proof} \; =$

$\ldots$

$\square$

2.2.3 Completeness

Completeness showcases some symmetry between key uniformities

$\texttt{CauchyFilterInversion} :: \forall G \in \mathsf{TGRP} \,.\, \forall \mathcal{F} : \texttt{FilterBase} \,.$
$\quad . \forall \texttt{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}) \iff \texttt{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}^{-1})$

$\texttt{Proof} \; =$

$\texttt{Assume}\ [1] : \texttt{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}),$

$\texttt{Assume}\ U \in \mathcal{R},$

$\Big(V, [2]\Big) := \texttt{E}\mathcal{R}(U) : \sum V \in \mathcal{U}(e) \,.\, V_R \subset U,$

$\Big(F, [3]\Big) := \texttt{ECauchyFilterbase}(G, \mathcal{L}, \mathcal{F}, V_L^\top) : \sum F \in \mathcal{F} \,.\, F \times F \subset V_L^\top,$

$\texttt{Assume}\ (x, y) : V_L^\top,$

$[4] := \texttt{E}V_L : y^{-1}x \in V,$

$\Big[(x, y).*\Big] := \texttt{I}V_R[4] : (x^{-1}, y^{-1}) \in V_R;$

$\rightsquigarrow [4] := \texttt{I} \subset : (\mathrm{inv} \times \mathrm{inv})(V_L^\top) \subset V_R,$

$[U.*] := [3][4][2] : F^{-1} \times F^{-1} \subset U;$

$\rightsquigarrow [*] := \texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}^{-1});$

$\square$

$\texttt{TwoSidedCauchyFilters} :: \forall G \in \mathsf{TGRP} \,.\, \forall \mathcal{F} : \texttt{FilterBase} \,.$
$\quad . \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}) \iff \texttt{CauchyFilterbase}(G, \mathcal{R} \,\&\, \mathcal{R}, \mathcal{F})$

$\texttt{Proof} \; =$

$[*] := \texttt{E}\mathcal{S}^\vee\texttt{SupUniformityCauchyFilterbase}(G, (\mathcal{L}, \mathcal{R})) :$
$\quad : \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}) \iff \texttt{CauchyFilterbase}(G, \mathcal{R} \,\&\, \mathcal{R}, \mathcal{F});$

$\square$

$\texttt{TwoSidedCauchyFilterInversion} ::$
$\quad :: \forall G \in \mathsf{TGRP} \,.\, \forall \mathcal{F} \in \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee) \,.\, \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}^{-1})$

$\texttt{Proof} \; =$

Combain two previous results.

$\square$

$\texttt{LeftRightMutualCompleteness} ::$
$\quad :: \forall G \in \mathsf{TGRP} \,.\, \texttt{CompleteUniformSpace}(G, \mathcal{L}) \iff \texttt{CompleteUniformSpace}(G, \mathcal{R})$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{LeftOrRightCompletenessImplyTwoSided} ::$
$\quad :: \forall G \in \mathsf{TGRP} \,.\, \texttt{CompleteUniformSpace}(G, \mathcal{L}|\mathcal{R}) \Rightarrow \texttt{CompleteUniformSpace}(G, \mathcal{S}^\vee)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{CompleteByNbhd} ::$

$:: \forall G \in \mathsf{TGRP} \,.\, \texttt{CompleteUniformSpace}(G, \mathcal{L}_G) \iff \exists N \in \mathcal{N}(e_G) \,.\, \texttt{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2)$

$\texttt{Proof} \;=$

$[1] := \Lambda T : \texttt{CompleteUniformSpace}(G, \mathcal{L}_G) \,.\, T :$

$: \texttt{CompleteUniformSpace}(G, \mathcal{L}_G) \Rightarrow \exists N \in \mathcal{N}(e_G) \,.\, \texttt{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2),$

$\texttt{Assume}\ N \in \mathcal{N}_G(e_G),$

$\Big(U, [0]\Big) := \mathrm{E}\mathcal{N}_G(e_G) : \sum U \in \mathcal{U}_G(e_G) \,.\, U \subset N,$

$\texttt{Assume}\ [2] : \texttt{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2),$

$\texttt{Assume}\ \mathcal{F} : \texttt{CauchyFilterbase}(G, \mathcal{L}_G),$

$\Big(F, [3]\Big) := \texttt{ECauchyFilterbase}(G, \mathcal{L}_G, \mathcal{F}, U_L) : \sum F \in \mathcal{F} \,.\, F \times F \subset U_L,$

$f := \texttt{EFilterbase}(G, \mathcal{F}, F) \in F,$

$[4] := \mathrm{E}U_L[3] : f^{-1}F \subset U,$

$[5] := \texttt{EFilterbase}(G, \mathcal{F})[4] : \forall F' \in \mathcal{F} \,.\, f^{-1}F' \cap N \neq \emptyset,$

$[6] := \mathrm{E}\mathsf{TGRP}(G)\texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}\Big(N, \mathcal{L}_G \cap N^2, f^{-1}\mathcal{F} \cap N\Big),$

$\Big(g, [7]\Big) := \texttt{ECompleteUniformSpace}(N, \mathcal{L}_G \cap N^2, f^{-1}\mathcal{F} \cap N) : \sum g \in N \,.\, g \in \lim f^{-1}\mathcal{F} \cap N,$

$[\mathcal{F}.*] := \mathrm{E}\mathsf{TGRP}(G)[7] : fg \in \lim \mathcal{F};$

$\rightsquigarrow [N.*] := \texttt{ICompleteUniformSpace} : \texttt{CompleteUniformSpace}(G, \mathcal{L}_G);$

$\rightsquigarrow [*] := \texttt{I} \iff [1] :$

$: \texttt{CompleteUniformSpace}(G, \mathcal{L}_G) \iff \exists N \in \mathcal{N}(e_G) \,.\, \texttt{CompleteUniformSpace}(N, \mathcal{L}_G \cap N^2);$

$\square$

$\texttt{LocallyCompactGroupIsComplete} :: \forall G \in \mathsf{TGRP}\ \&\ \texttt{LocallyCompact} \,.\, \texttt{CompleteUniformSpace}(G, \mathcal{L})$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{LocallyCompactGroupRegularity} :: \forall G \in \mathsf{TGRP}\ \&\ \texttt{LocallyCompact}\ \&\ \texttt{T0} \,.\, \texttt{T4}(G)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ E : \texttt{Closed}(G),$

$\texttt{Assume}\ \phi \in \mathsf{TOP}(E, \mathbb{R}),$

$\Big(\hat{\mathrm{id}}_E, [1]\Big) := \texttt{OnePointCompactificationExtenstion}(E, \mathrm{id}) : \sum \hat{\mathrm{id}}_E \in \mathsf{UNI}(E^{\mathrm{pt}}, E) \,.\, \hat{\mathrm{id}}_{E|E} = \mathrm{id}_E,$

$\Phi := \hat{\mathrm{id}}\phi \in \mathsf{TOP}(E^{\mathrm{pt}}, \mathbb{R}),$

$\Big(\hat{\Phi}, [2]\Big) := \texttt{TietzeExtensionTheorem}(E^{\mathrm{pt}}, \mathbb{R}, G^{\mathrm{pt}}, \Phi) : \sum \hat{\Phi} \in \mathsf{UNI}(G^{\mathrm{pt}}, \mathbb{R}) \,.\, \hat{\Phi}_{|E^{\mathrm{pt}}} = \Phi,$

$\Big(\hat{\mathrm{id}}_G, [4]\Big) := \texttt{OnePointCompactificationExtenstion}(E, \mathrm{id}) : \sum \hat{\mathrm{id}}_G \in \mathsf{UNI}(G^{\mathrm{pt}}, G) \,.\, \hat{\mathrm{id}}_{G|G} = \mathrm{id}_G,$

$\varphi := \hat{\mathrm{id}}_G\hat{\Phi} : \mathsf{UNI}(G, \mathbb{R}),$

$[\phi.*] := \mathrm{E}\varphi : \varphi_{|E} = \phi;$

$\rightsquigarrow [*] := \texttt{TietzeExtensionTHM} : \texttt{T4}(G);$

$\square$

2.2.4 Completion

One can use symmety properties of Cauchy filters mentioned above to show that separable completion of a sepatable group in its two-sided uniformity is a topological group with a two-sided uniformities again.

$\texttt{LeftCauchyLemma} :: \forall G : \mathsf{TGRP} \ . \ \forall \mathcal{F} : \texttt{CauchyFilterbase}(G, \mathcal{L}) \ . \ \forall U \in \mathcal{U}(e) \ . \ \exists F \in \mathcal{F} \ . \ F^{-1}F \subset U$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{LeftCauchyFilterProduct} ::$

$:: \forall G \in \mathsf{TGRP} \ . \ \forall \mathcal{F}, \mathcal{F}' : \texttt{CauchyFilterbase}(G, \mathcal{L}) \ . \ \texttt{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}\mathcal{F}')$

$\texttt{Proof} \ =$

$\texttt{Assume} \ U' \in \mathcal{L},$

$\Big(U, [1]\Big) := \texttt{E}\mathcal{L}(U') : \sum U \in \mathcal{U}(e) \ . \ U_L \subset U',$

$\Big(V, [2]\Big) := \texttt{TGRPTrisection}(G, U) : V : \texttt{SymmetricSet}(G) \ . \ V \in \mathcal{U}(e) \ \& \ VVV \subset U,$

$(F', [3]) := \texttt{LeftCauchyLemma}(G, \mathcal{F}', V) : \sum F' \in \mathcal{F}' \ . \ F'^{-1}F' \subset V,$

$f := \texttt{EFilterbase}(G, \mathcal{F}', F')\texttt{E}\exists \in F',$

$\Big(W, [4]\Big) := \texttt{TopologicalGroupAltDef}_4(G, V, f^{-1}) : \sum W \in \mathcal{U}(e) \ . \ f^{-1}Wf \subset V,$

$(F, [5]) := \texttt{LeftCauchyLemma}(G, \mathcal{F}, W) : \sum F \in \mathcal{F} \ . \ F^{-1}F \subset W,$

$[U'.*.1] := \texttt{EsetProduct}(F, F') : FF' \in \mathcal{F}\mathcal{F}',$

$[U'.*.2] := \texttt{ProductInverse}(G)\texttt{InverseMeaning}^2(G, f)[5][4][3][2][1] :$

$: (FF')^{-1}FF' = F'^{-1}F^{-1}FF'^{-1} = F'^{-1}ff^{-1}F^{-1}Fff^{-1}F'^{-1} \subset F'^{-1}ff^{-1}Wff^{-1}F'^{-1} \subset F'^{-1}fVf^{-1}F'^{-1} \subset$

$\subset VVV \subset U \subset U'(e);$

$\rightsquigarrow [*] := \texttt{E}\mathcal{L}\texttt{ICauchyFilterbase} : \texttt{CauchyFilterbase}(G, \mathcal{L}, \mathcal{F}\mathcal{F}');$

$\square$

$\texttt{RightCauchyFilterProduct} ::$

$:: \forall G \in \mathsf{TGRP} \ . \ \forall \mathcal{F}, \mathcal{F}' : \texttt{CauchyFilterbase}(G, \mathcal{R}) \ . \ \texttt{CauchyFilterbase}(G, \mathcal{R}, \mathcal{F}\mathcal{F}')$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{UpperTwoSidedCauchyFilterProduct} ::$

$:: \forall G \in \mathsf{TGRP} \ . \ \forall \mathcal{F}, \mathcal{F}' : \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee) \ . \ \texttt{CauchyFilterbase}(G, \mathcal{S}^\vee, \mathcal{F}\mathcal{F}')$

$\texttt{Proof} \ =$

$\dots$

$\square$

$$\texttt{TopologicalGroupCompletion} :: \prod_{G \in \mathsf{TGRP}} \sum_{H \in \mathsf{TGRP}} \mathsf{TGRP}(G, H)$$

$$\iota : \texttt{TopologicalGroupCompletion} \iff \texttt{Completion}\Big((G, \mathcal{S}^\vee_G), (H, \mathcal{S}^\vee_H), \iota\Big)$$

$\texttt{ContinuityByDenseSetAndPoints} :: \forall X \in \mathsf{TOP} \,.\, \forall Y : \texttt{Regular} \,.\, \forall \phi : X \to Y \,.\, \forall D : \texttt{Dense}(X) \,.$

$. \left(\forall x \in X \,.\, \phi_{|D \cup \{x\}} \in \mathsf{TOP}(D \cup \{x\}, Y) \right) \Rightarrow \phi \in \mathsf{TOP}(X, Y)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in X,$

$\texttt{Assume}\ V \in \mathcal{U}(x\ \phi),$

$C := \overline{V} : \texttt{Closed}(Y),$

$U := \phi^{-1}(V) \cap (D \cup \{x\}) : \mathcal{U}_{D \cup \{x\}}(x),$

$\left(W, [2]\right) := \texttt{SubspaceTopology}(X, U) : \sum W \in \mathcal{U}(x) \,.\, U = W \cap (D \cup \{x\}),$

$[3] := \mathtt{E}W\mathtt{E}C : \phi(W \cap D) \subset V \subset C,$

$\texttt{Assume}\ w \in W,$

$\texttt{Assume}\ O \in \mathcal{U}(w\ \phi),$

$I := \phi^{-1}(O) \cap (D \cup \{x\}) : \mathcal{U}_{D \cup \{w\}}(w),$

$\left(E, [4]\right) := \texttt{SubspaceTopology}(X, U) : \sum E \in \mathcal{U}(x) \,.\, I = E \cap (D \cup \{w\}),$

$[5] := \mathtt{E}E : \phi(E \cap D) \subset O,$

$[6] := \mathtt{E}E\mathtt{E}w\mathtt{E}\texttt{Dense}(X, D) : \exists (E \cap W \cap D),$

$[7] := \texttt{ImageIntersection}[3][5] : \phi(W \cap E \cap D) \subset O \cap V,$

$[w.*] := [4][7]\mathtt{I}E : \exists O \cap V;$

$\rightsquigarrow [x.*] := \texttt{Iimage}\mathtt{E}C\texttt{ClosureAltDef} : \phi(W) \subset C,$

$\rightsquigarrow [*] := \texttt{ContinuityIsLocalBase}(X, Y, \phi)\texttt{RegularHasClosedNbhdBase}(Y) : \phi \in \mathsf{TOP}(X, Y);$

$\square$

$\texttt{SeparetedTopologicalGroupHasComplition} :: \forall G \in \mathsf{TGRP}\ \&\ \texttt{T0} \,.\, \exists \texttt{TopologicalGroupCompletion}(G)$

$\texttt{Proof} \ =$

$\left(H, \iota\right) := \texttt{SeparatedHasSeparatedCompletion}(G, \mathcal{S}_G^{\vee}) : \texttt{SeparatedCompletion}(G, \mathcal{S}_G^{\vee}),$

$\texttt{Assume}\ f, f' \in H,$

$\left(\mathcal{F}, F', [1]\right) := \mathtt{E}\texttt{SeparatedCompletion}(G, H, \iota) :$

$: \sum \mathcal{F}, \mathcal{F}' : \texttt{CauchyFilterbase}(G, \mathcal{S}_G^{\vee}) \,.\, f = \lim \mathcal{F}\ \&\ f' = \lim \mathcal{F}',$

$[2] := \texttt{UpperTwoSidedCauchyFilterProduct}(G, \mathcal{F}, \mathcal{F}') : \texttt{CauchyFilterbase}\left((G, \mathcal{S}_G^{\vee}), \mathcal{F}\mathcal{F}'\right),$

$f \cdot_H f' := \lim \mathcal{F}\mathcal{F}' \in H;$

$\rightsquigarrow \cdot_H := \mathtt{E}\mathsf{TGRP}(G)\mathsf{UNI}(G, H, \iota) : H \times H \to H,$

$\texttt{Assume}\ f \in H,$

$\left(\mathcal{F}, [1]\right) := \mathtt{E}\texttt{SeparatedCompletion}(G, H, \iota) : \sum \mathcal{F} : \texttt{CauchyFilterbase}((G, \mathcal{S}_G^{\vee})) \,.\, f = \lim \mathcal{F},$

$[2] := \texttt{TwoSidedCauchyFilterInversion}(G, \mathcal{F}) : \texttt{CauchyFilterbase}\left((G, \mathcal{S}_G^{\vee}), \mathcal{F}^{-1}\right),$

$\mathrm{inv}_H f := \lim \mathcal{F}^{-1} \in H;$

$\rightsquigarrow \mathrm{inv}_H := \mathtt{E}\mathsf{TGRP}(G)\mathsf{UNI}(G, H, \iota) : H \to H,$

$[1] := \texttt{ContinuityByDenseSetAndPoints}(H \times H, H)\mathtt{E}\cdot_H : \cdot_H \in \mathsf{TOP}(H \times H, H),$

$[2] := \texttt{ContinuityByDenseSetAndPoints}(H, H)\mathtt{E}\mathrm{inv}_H : \mathrm{inv}_H \in \mathsf{TOP}(H, H),$

$[*.1] := \mathtt{E}\mathsf{TGRP}(G)\texttt{ContinuityByDenseSetAndPoints}(\ldots)\mathtt{I}\mathsf{TGRP} : H \in \mathsf{TGRP},$

$[*.2] := \mathtt{E} \cdot_H \mathtt{E}\mathrm{inv}_H : \iota \in \mathsf{TGRP}(G, H),$

$[*.3] := \mathtt{E} \cdot_H \mathtt{E}\mathrm{inv}_H : \mathcal{U}_H = \mathcal{S}_H^{\vee};$

$\square$

2.2.5 Baire's Category

Being Baire and topological group structure interplay nicely.

$\texttt{ClopenSubgroupByNonemptyInterior} :: \forall G : \mathsf{TGRP} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \exists \operatorname{int} H \Rightarrow \texttt{Clopen}(G, H)$

$\texttt{Proof} \;=$

$u := \mathtt{E}\exists[0] \in \operatorname{int} H,$

$\Big(U, [1]\Big) := \mathtt{E}\operatorname{int} H(u) : \sum U \in \mathcal{U}(u) \,.\, U \subset H,$

$[2] := \Lambda h \in H \,.\, \mathtt{E}\mathrm{Aut}_{\mathsf{TOP}}(G, \lambda_{hu^{-1}}, U) : \forall h \in H \,.\, hu^{-1}U \in \mathcal{U}(h),$

$[3] := \Lambda h \in H \,.\, \texttt{ESubgroup}(G, H, hu^{-1}, U)[1] : \forall h \in H \,.\, hu^{-1}U \subset H,$

$[4] := \texttt{OpenByCover}[2][3] : \texttt{Open}(G, H),$

$[5] := \texttt{OpenSubgroupsAreClopen}[5] : \texttt{Open}(G, H);$

$\square$

$\texttt{ClopenSubgroupGeneration} :: \forall G \in \mathsf{TGRP} \,.\, \forall U : \texttt{SymmetricSet} \,\&\, \texttt{Open}(G) \,.\, \texttt{Clopen}\Big(G, \langle U \rangle_{\mathsf{GRP}}\Big)$

$\texttt{Proof} \;=$

$H := \langle U \rangle_{\mathsf{GRP}} : \texttt{Subgroup}(G),$

$[1] := \texttt{GeneratedSubgroupIsSuper}(G, U)\mathtt{I}H : U \subset H,$

$[2] := \texttt{Iinterion}[1] : \exists \operatorname{int} H,$

$[*] := \texttt{ClopenSubgroupByNonemptyInterior}[2] : \texttt{Clopen}(G, H);$

$\square$

$\texttt{FirstCategoryByClopenSets} :: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{C} : \texttt{Disjoint}\Big(\texttt{Clopen} \,\&\, \texttt{Meager}(X)\Big) \,.\, X = \bigcup \mathcal{C} \Rightarrow \neg\exists^* X$

$\texttt{Proof} \;=$

$\Big(N, [1]\Big) := \mathtt{E}?\texttt{Meager}(X) : \sum N : \mathcal{C} \to \mathbb{N} \to \texttt{NowhereDense}(X) \,.\, \forall C \in \mathcal{C} \,.\, C = \bigcup_{n=1}^{\infty} N_{C,n},$

$M := \Lambda n \in \mathbb{N} \,.\, \bigcup_{C \in \mathcal{C}} N_{C,n} : \mathbb{N} \to ?X,$

$[2] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}M_n\mathtt{E}\operatorname{cl}\mathtt{E}\operatorname{int}\Lambda C \in \mathcal{C} \,.\, \texttt{EClopen}(X, C)\texttt{ENowhereDense}(X, N_{C,n})\texttt{EmptySum}(X) :$

$: \forall n \in \mathbb{N} \,.\, \operatorname{int}\operatorname{cl} M_n = \operatorname{int}\operatorname{cl} \bigcap_{C \in \mathcal{C}} N_{C,n} = \bigcap_{C \in \mathcal{C}} \operatorname{cl}\operatorname{int} N_{C,n} = \bigcap_{C \in \mathcal{C}} \emptyset = \emptyset,$

$[3] := \texttt{INowhereDense}[2] : \forall n \in \mathbb{N} \,.\, \texttt{NowhereDense}(X, M_n),$

$[4] := [0][1]\texttt{UnionCommutativity}(X)\mathtt{I}M_n : X = \bigcup_{C \in \mathcal{C}} C = \bigcup_{C \in \mathcal{C}} \bigcap_{n=1}^{\infty} N_{C,n} = \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{C}} N_{C,n} = \bigcap_{n=1}^{\infty} M_n,$

$[*] := \texttt{IMeager}[3][4] : \texttt{Meager}(X, X);$

$\square$

$\texttt{TGRPBaireCondition} :: \forall G \in \mathsf{TGRP} \,.\, \texttt{BaireSpace}(X) \iff \exists^* X$

$\texttt{Proof} \;=$

$[1] := \Lambda T : \texttt{BaireSpace}(X) \,.\, \texttt{E}\texttt{BaireSpace}(X, X)\texttt{I}\exists^* : \texttt{BaireSpace}(X) \Rightarrow \exists^* X,$

$\texttt{Assume}\ [2] : \exists^* X,$

$\texttt{Assume}\ [3] : \neg\texttt{BaireSpace}(X),$

$\Big(U, [3]\Big) := \texttt{E}\texttt{BaireSpace}(X) : \sum U \in \mathcal{T}(X) \,.\, \exists U \;\&\; \neg\exists^* U,$

$u := \texttt{E}\exists[3.1] \in U,$

$V := u^{-1}U \in \mathcal{U}(e),$

$W := V \cap V^{-1} : \texttt{SymmetricSet}(X),$

$[4] := \texttt{E}W : W \in \mathcal{U}(e) \;\&\; \texttt{Meager}(X, W),$

$H := \langle W \rangle_{\mathsf{GRP}} : \texttt{Subgroup}(G, H),$

$[5] := \texttt{ClopenSubgroupGeneration}(G, W)\texttt{I}H : \texttt{Clopen}(G, H),$

$[6] := \Lambda n \in \mathbb{N} \,.\, \texttt{MeagerOpenImage}\left(G^n, G, \prod, W^{\times n}\right) : \forall n \in \mathbb{N} \,.\, \texttt{Meager}(G, W^n),$

$[7] := \texttt{E}H\texttt{MeagerCountableUnion}(G)[6] : \texttt{Meager}(G, H),$

$\mathcal{C} := \{gH | g \in G\} :?\Big(\texttt{Clopen} \;\&\; \texttt{Meager}(G)\Big),$

$[8] := \texttt{DisjointCosets}(G)\texttt{E}\mathcal{C} : \texttt{Disjoint}\Big(\texttt{Clopen} \;\&\; \texttt{Meager}(G), \mathcal{C}\Big),$

$[9] := \texttt{E}\mathsf{GRP}(G)\texttt{E}\mathcal{C} : G = \bigcap_{C \in \mathcal{C}} C,$

$[10] := \texttt{FirstCategoryByClopenSets}(G, \mathcal{C}) : \neg\exists^* G,$

$[3.*] := [2][10] : \bot;$

$\rightsquigarrow [2.*] := \texttt{E}\bot : \texttt{BaireSpace}(G, H);$

$\rightsquigarrow [*] := \texttt{I} \iff [1] : \texttt{BaireSpace}(G, H) \iff \exists^* G;$

$\square$

$\texttt{BaireGroup} := \mathsf{TGRP} \;\&\; \texttt{BaireSpace} : \texttt{Type};$

$\texttt{DenseGdeltaSugGroupsAreUnique} :: \forall G : \texttt{BaireGroup} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \texttt{Dense} \;\&\; G_\delta(G, H) \,.\, H = G$

$\texttt{Proof} \;=$

$[1] := \texttt{E}\texttt{Dense} \;\&\; G_\delta(G, H)\texttt{I}\texttt{Meager}(G) : \texttt{Meager}\Big(G, H^{\complement}\Big),$

$\texttt{Assume}\ [2] : G \neq H,$

$[3] := \texttt{I}\lambda_G[2] : |\lambda_G G| > 1,$

$[4] := \texttt{E}\mathsf{TGRP}\texttt{MeagerSubset}(G)[1][3] : \texttt{Meager}(G, H),$

$[5] := \texttt{MeagerUnion}[1][4] : \texttt{Meager}(G, G),$

$[2] := \texttt{NPGRBairConditionEBaireGroup}(G)[5] : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot : G = H;$

$\square$

$\texttt{GDeltaSubgroupIsClosed} :: \forall G : \texttt{BaireGroup} \ \& \ \texttt{CompleteMetricSpace} \ . \ \forall H \subset_{\mathsf{GRP}} G \ .$

$. \ G_\delta(H) \Rightarrow \texttt{Closed}(G, H)$

$\texttt{Proof} \ =$

$[1] := \texttt{ClosureOfSubgroup}(G, H) : \overline{H} \subset G,$

$[2] := \texttt{ClosedSubsetsAreComplete}(G, \overline{H}) : \texttt{CompleteMetricSpace}(\overline{H}),$

$[3] := \texttt{BairCategoryTHM2}[2] : \texttt{BaireSpace}(\overline{H}),$

$[4] := \texttt{DenseGdeltaSubgroupsAreUnique}(\overline{H}, H) : H = \overline{H},$

$[5] := \texttt{ClosedByClosure}[4] : \texttt{Closed}(G, H);$

$\square$

$\texttt{DiscontiniousRealEndomorphismsHaveDenseGraphs} ::$

$:: \forall \phi \in \mathrm{End}_{\mathsf{GRP}}(\mathbb{R}, +) \ . \ \phi \notin \mathrm{End}_{\mathsf{TOP}}(\mathbb{R}) \Rightarrow \texttt{Dense}\Big(\mathbb{R}^2, G(\phi)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ContinuityByGraph} :: \forall \phi \in \mathrm{End}_{\mathsf{GRP}}(\mathbb{R}, +) \ . \ \phi \in \mathrm{End}_{\mathsf{TOP}}(\mathbb{R}) \iff G_\delta\Big(\mathbb{R}^2, G(\phi)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

2.2.6 Connectedness

Connected groups can be generated by any neighborhood of unity. There is no proofs in this chapter, results are pretty trivial.

$\texttt{ConnectedGroupGeneration} :: \forall G : \mathsf{TGRP} \ \& \ \texttt{Connected} \ . \ \forall N \in \mathcal{N}(e) \ . \ G = \langle N \rangle_{\mathsf{GRP}}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ConnectedGroupCardinality} :: \forall G : \mathsf{TGRP} \ \& \ \texttt{Connected} \ \& \ \texttt{T0} \ . \ G \neq \star \Rightarrow |G| > \aleph_0$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ConnectedSubgroupSeparabilityCondition} :: \forall G \in \mathsf{TGRP} \ \& \ \texttt{Connected} \ . \ \forall U \in \mathcal{U}(e) \ .$

$. \ \texttt{Separable}(U) \Rightarrow \texttt{Separable}(G)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ConnectedGroupCountability} :: \forall G : \mathsf{TGRP} \ \& \ \texttt{Connected} \ \& \ \texttt{FirstCountable} \ \& \ \texttt{LocallyCompact} \ .$

$. \ \texttt{SecondCountable}(G)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

2.2.7 Group of an Interval's Homeomorphisms

Group of an Interval's Homeomorphisms is an intersting exmple. it is complete in thir upper two-sided uniformity. But not in the on-sided ones. Theis shows that they don't have corresponding group completions.

$\texttt{homeoAbsoluteValue} :: \texttt{AbsoluteValue}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1]\Big)$

$\texttt{homeoAbsoluteValue}\,(f) = \upsilon(f) := \sup_{t\in[0,1]} \big|t - f(t)\big|$

$[1] := \mathtt{E}\upsilon(\mathrm{id})\Lambda t \in [0,1]\ .\ \texttt{InverseMeaning}(\mathbb{R}, t)\mathtt{E}|\bullet|\mathtt{E}\sup : \upsilon(\mathrm{id}) = \sup_{t\in[0,1]} \big|t - t\big| = \sup_{t\in[0,1]} 0 = 0,$

$[2] := \Lambda f \in \mathrm{Aut}_{\mathsf{TOP}}[0,1]\ .\ \mathtt{E}\upsilon(f^{-1})\texttt{Substitution}([0,1], f, y \mapsto f(x))\texttt{EAbsValue}\big(\mathbb{R}, |\bullet|\big)\mathtt{I}\upsilon(f) :$

$: \upsilon(f^{-1}) = \sup_{y\in[0,1]} \big|y - f^{-1}(y)\big| = \sup_{x\in[0,1]} \big|f(x) - x\big| = \sup_{x\in[0,1]} \big|x - f(x)\big| = \upsilon(f),$

$[3] := \Lambda f, g \in \mathrm{Aut}_{\mathsf{TOP}}[0,1]\mathtt{E}\upsilon(fg)\texttt{Substitution}([0,1], f^{-1}, x \mapsto f^{-1}(y))\texttt{TriangleIneq}\Big(\mathbb{R}, |\bullet|, f^{-1}(y), y, g(y)\Big)$

$\texttt{SupSumIneq}(\mathbb{R})\mathtt{I}\upsilon(f^{-1})\mathtt{I}\upsilon(g)[2] : \upsilon(fg) = \sup_{x\in[0,1]} \big|x - fg(x)\big| = \sup_{y\in[0,1]} \big|f^{-1}(y) - g(y)\big| \le$

$\le \sup_{y\in[0,1]} \big|f^{-1}(y) - y\big| + \big|y - g(y)\big| \le \sup_{y\in[0,1]} \big|y - f^{-1}(y)\big| + \sup_{y\in[0,1]} \big|y - g(y)\big| \le \upsilon(f^{-1}) + \upsilon(g) \le \upsilon(f) + \upsilon(g),$

$\texttt{Assume}\ f : \mathbb{N} \to \mathrm{Aut}_{\mathsf{TOP}}[0,1],$

$\texttt{Assume}\ [4] : \lim_{n\to\infty} \upsilon(f_n) = 0,$

$\texttt{Assume}\ g \in \mathrm{Aut}_{\mathsf{TOP}}[0,1],$

$[f.*] := \lim_{n\to\infty} \mathtt{E}\upsilon(g^{-1}f_n g)\texttt{Substitution}([0,1], g, x \mapsto g(y))\mathtt{E}\mathrm{Aut}_{\mathsf{UNI}}\Big([0,1], g\Big)[4] :$

$: \lim_{n\to\infty} \upsilon(g^{-1}f_n g) = \lim_{n\to\infty} \sup_{x\in[0,1]} \big|x - g^{-1}f_n g(x)\big| = \lim_{n\to\infty} \sup_{x\in[0,1]} \big|g(y) - f_n g(y)\big| = 0;$

$\rightsquigarrow [*] := \texttt{IAbsoluteValue} : \texttt{AbsoluteValue}\Big(\mathrm{Aut}_{\mathsf{UNI}}[0,1], \upsilon\Big);$

$\square$

$\texttt{HomeoAbsValueProducesUniformMetric} :: \forall f, g \in \mathrm{Aut}_{\mathsf{TOP}}[0,1]\ .\ d_\upsilon(f,g) = \sup_{t\in[0,1]} \Big|f(t) - g(t)\Big|$

$\texttt{Proof}\ =$

$[*] := \mathtt{E}d_\upsilon \mathtt{E}\upsilon\texttt{Substitution}([0,1], f, f(x) \mapsto x) :$

$: d_\upsilon(f,g) = \upsilon(g^{-1}f) = \sup_{x\in[0,1]} \big|x - g^{-1}f(x)\big| = \sup_{y\in[0,1]} \big|g(y) - f(y)\big|;$

$\square$

$\texttt{HomeoAreNotLeftComplete} :: \neg\texttt{CompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{L}\Big)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : \texttt{CompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{L}\Big),$

$[2] := \texttt{HomeoAbsValueProducesUniformMetricECompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{L}\Big)\texttt{IClosed} :$

$: \texttt{Closed}\Big(\mathrm{End}_{\mathsf{TOP}}[0,1], \mathrm{Aut}_{\mathsf{TOP}}[0,1]\Big),$

$f := \Lambda n \in \mathbb{N}\ .\ \Lambda t \in [0,1]\ .\ 2\sqrt[n]{\frac{1}{2}t}\left[t < \frac{1}{2}\right] + \sqrt[n]{t}\left[t \ge \frac{1}{2}\right] : \mathbb{N} \to \mathrm{Aut}_{\mathsf{TOP}}[0,1],$

$[3] := \mathtt{E}\lim_{n\to\infty} f_n : \lim_{n\to\infty} f_n = 2t\left[t < \frac{1}{2}\right] + \left[t \ge \frac{1}{2}\right] \notin \mathrm{Aut}_{\mathsf{TOP}}[0,1],$

$[1.*] := \texttt{ClosedHasLimits}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1]\Big)[2][3] : \perp;$

$\rightsquigarrow [*] := \mathtt{E}\perp : \neg\texttt{CompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{L}\Big),$

$\square$

$\texttt{HomeoIsUpperComplete} :: \texttt{CompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{S}^{\vee}\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : \texttt{CauchySeq}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{S}^{\vee}\Big),$

$[1] := \texttt{TwoSidedCauchyFilters}(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, f) :$

$\quad : \texttt{CauchySeq}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \|\bullet\|_\infty, f\Big)\ \&\ \texttt{CauchySeq}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \|\bullet\|_\infty, f^{-1}\Big),$

$\Big(g, [2]\Big) := \texttt{ECompleteUniformSpace}\Big(\mathrm{End}_{\mathsf{TOP}}[0,1], \|\bullet\|_\infty, f\Big) : \sum g \in \mathrm{End}_{\mathsf{TOP}}[0,1] \ .\ \lim_{n\to\infty} f_n = g,$

$\Big(h, [3]\Big) := \texttt{ECompleteUniformSpace}\Big(\mathrm{End}_{\mathsf{TOP}}[0,1], \|\bullet\|_\infty, f^{-1}\Big) : \sum h \in \mathrm{End}_{\mathsf{TOP}}[0,1] \ .\ \lim_{n\to\infty} f_n^{-1} = h,$

$\texttt{Assume}\ x : [0,1],$

$\texttt{Assume}\ \varepsilon : \mathbb{R}_{++},$

$\Big(N, [4]\Big) := \mathrm{E}\upsilon[3](\varepsilon) : \sum N \in \mathbb{N} \ .\ \forall n \geq N \ .\ \Big|gh(x) - gf_n^{-1}(x)\Big| < \frac{\varepsilon}{2},$

$\Big(M, [5]\Big) := \mathrm{E}\upsilon[2](\varepsilon) : \sum M \in \mathbb{N} \ .\ \forall m \geq M \ .\ \Big|gf_m^{-1} - x\Big| < \frac{\varepsilon}{2},$

$n := \max(N, M) \in \mathbb{N},$

$[x.*] := \texttt{TriangleIneq}[4][5] : \Big|gh(x) - x\Big| \leq \Big|gh(x) - gf_n^{-1}(x)\Big| + \Big|gf_n^{-1} - x\Big| < \varepsilon;$

$\rightsquigarrow [4] := \mathrm{Iinv} : gh = \mathrm{id},$

$[5] := \texttt{ByAnalogy}[4] : hg = \mathrm{id},$

$[f.*] := \mathrm{EAut}_{\mathsf{TOP}}[0,1][4][5] : g \in \mathrm{Aut}_{\mathsf{TOP}}[0,1];$

$\rightsquigarrow [*] := \texttt{ICompleteUniformSpace} : \texttt{CompleteUniformSpace}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{S}^{\vee}\Big);$

$\square$

$\texttt{HomeoHasDistinctUniformities} :: \mathcal{L}_\upsilon \neq \mathcal{S}^{\vee}_\upsilon\ \&\ \mathcal{L}_\upsilon \neq \mathcal{R}_\upsilon$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{HomeoHasNoLeftGroupCompletion} :: \neg\exists\texttt{TopologicalGroupCompletion}\Big(\mathrm{End}_{\mathsf{TOP}}[0,1], \upsilon, \mathcal{L}\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

2.3 Further Group Properties[0]

This chapter will include topics such initial and final construcures, quotients, group action, free groups. It will be written on demand in a first return.

2.4 Some Analytic Properties[1]

Many famous theorems of functional analysis can be proved for topological groups. This chapter will be written on demand in a first return. Prerequisite: Further Group Properties

2.5 Representation[5]

Results about topological groups can be applied to their representations. Prerequisite: Further Group Properties, Some Analytic Properties,Haar measure, group representations

2.5.1 Continuous Characters[5]

Characters of topological groups are Also Continuous

$$\texttt{dualGroup} :: \mathsf{TGRP} \to \mathsf{TGRP}$$
$$\texttt{dualGroup}(G) = G^* := \mathsf{TGRP}(G, \mathbb{S}^1)$$

2.5.2 Keller's Theorem[6]

Keller's theorem states that every compact convex body in a Hilbert space is homeomorphic to a Hilbert cube. In 1993 Agaev published a proof, which uses representation of topological groups as a main tool. Prerequisite: Spectral theory

2.6 Almost Metrizable Groups[∞]

Almost metrizable spaces are those where every compact admits a countable system of neighborhood. Almost metrizable groups are of some esoteric interest.

3 Polish Groups

3.1 Basics

3.1.1 Definition and Examples

$\texttt{PolishGroup} := \mathsf{TGRP} \ \& \ \texttt{Polish} : \texttt{Type};$

$\texttt{cli} :: ?\texttt{PolishGroup}$

$G : \texttt{cli} \iff \exists \rho : \texttt{LeftInvariantMetric}(G) \ . \ \texttt{Complete}(G, \rho) \ \& \ (G, \rho) \cong_{\mathsf{TOP}} G$

$\texttt{DiscreteIsCli} :: \forall G \in \mathsf{TGRP} \ . \ |G| \leq \aleph_0 \ \& \ \texttt{Discrete}(G) \Rightarrow \texttt{cli}(G)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NiceIsCli} :: \forall G \in \mathsf{TGRP} \ . \ \texttt{FirstCountable} \ \& \ \texttt{LocallyCompact} \ \& \ \texttt{T0} \ . \ \texttt{cli}(G)$

$\texttt{Proof} =$

First countable topological groups are LIM metrizable.

First countable topological groups are second countable.

Second countable and locally compact metrizable spaces are polish.

Locally compact topological groups are complete in their left uniformity, and hence LIM-complete. $\square$

$\texttt{RealsAndComplexAreCli} :: \texttt{cli}\Big((\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{R} \setminus \{0\}, *), (\mathbb{C} \setminus \{0\}, *)\Big)$

$\texttt{Proof} =$

Result for additive groups is well known fact of Reals Analysis

For multiplicative groups nice is cli.

$\square$

$$\texttt{ProductOfPolishGroupsIsPolish} :: \forall n \in \sigma(\omega) \ . \ \forall G : n \to \texttt{PolishGroup} \ . \ \texttt{PolishGroup}\left(\prod_{i=1}^{n} G\right)$$

$\texttt{Proof} =$

product of topological groups is a topological group.

Countable product of polish spaces is polish.

$\square$

$\texttt{UnitaryOperatorsAreCli} :: \forall V \in \mathbb{C}\text{-}\mathsf{HIL} \ \& \ \texttt{Separable} \ . \ \texttt{cli}\Big(\mathbf{U}(V)\Big)$

$\texttt{Proof} \ =$

$[1] := \Lambda A, B, C \in \mathbf{U}(V) \ . \ \Lambda x \in V \ . \ \mathsf{E}\mathsf{RING}\Big(\mathbf{B}(V)\Big)\texttt{EIsometry}(V, C) :$

$: \forall A, B, C \in \mathbf{U}(V) \ . \ \forall x \in V \ . \ \Big\|(CA - CB)x\Big\| = \|C(A-B)x\| = \Big\|(A-B)x\Big\|,$

$[2] := \texttt{IoperatorNorm}(V)[1] : \forall A, B, C \in \mathbf{U}(V) \ . \ \Big\|C(A-B)\Big\| = \Big\|A - B\Big\|,$

$\rightsquigarrow [3] := \texttt{ILeftInvariantMetric}[2] : \texttt{LeftInvariantMetric}\Big(\mathbf{U}(V), \Lambda A, B \in \mathbf{U}(V) \ . \ \|A - B\|\Big),$

$[4] := \texttt{MultiplicationContinuousOnBoundedSets}(V, \mathbf{U}(V))\texttt{AdjoiningIsContinuous}(V)\mathtt{I}\mathsf{TGRP} :$

$. \ \mathbf{U}(V) \in \mathsf{TGRP},$

$[5] := \mathtt{E}\mathbf{U}(V) : \mathbf{U}(V) = \Big(\Lambda A \in \mathbb{S}\Big(\mathcal{B}(V)\Big) \ . \ A^*A\Big)^{-1}(\mathrm{id}) \cap \Big(\Lambda A \in \mathbb{S}\Big(\mathcal{B}(V)\Big) \ . \ A^*A\Big)^{-1}(\mathrm{id}),$

$[6] := \texttt{MultiplicationContinuousOnBoundedSets}(V, \mathbf{U}(V))\texttt{IClosed} : \texttt{Closed}\Big(\mathcal{B}(V), \mathbf{U}(V)\Big),$

$[7] := \texttt{ClosedSetsAreGDelta}[5] : G_\delta\Big(\mathcal{B}(V), \mathbf{U}(V)\Big),$

$[8] := \texttt{CompleteSubsetIsGDelta} : \texttt{Complete}\Big(\mathbf{U}(V)\Big),$

$[*] := \texttt{Icli}[7][4][3] : \texttt{cli}\Big(\mathbf{U}(V)\Big);$

$\square$

$\texttt{UnitaryOperatorsAreNotLocallyCompact} :: \forall V : \mathbb{C}\text{-}\mathsf{HIL} \ \& \ \texttt{Separable} \ . \ \neg\texttt{LocallyCompact}\Big(\mathbf{U}(V)\Big)$

$\texttt{Proof} \ =$

Fix any $v \in V, \|v\| = 1$, then the evaluation map ϵ_v is a bounded, surjective operator on $\mathcal{B}(V)$.
Then by open mapping theorem, ϵ_v is an open mapping.
As ϵ_v also surjective it preserves local compactness.
ϵ_v maps $\mathbf{U}(V)$ onto $\mathbb{S}(0,1)$, but Spheres are not locally Compact in V

$\square$

$\texttt{CompactMetrizableHomeoArePolish} :: \forall X : \texttt{Compact} \ \& \ \texttt{Metrizable}(X) \ . \ \texttt{PolishGroup}\Big(\mathrm{Aut}_{\mathsf{TOP}}(X)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{HomeoOfIntervalAreNotCli} :: \neg\texttt{cli}\Big(\mathrm{Aut}_{\mathsf{TOP}}[0,1]\Big)$

$\texttt{Proof} \ =$

$\square$

$\texttt{IsometryGroupIsCLI} ::$

$:: \forall X \in \mathsf{MS} \ . \ \texttt{Complete} \ \& \ \texttt{Separable}(X) \Rightarrow \texttt{cli}\Big(\mathrm{Aut}_{\mathsf{MS}_{\circ\to\cdot}}(X), \mathcal{W}(X, X, \epsilon)\Big)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

`InfinitePermutationsArePolishGroup` :: `PolishGroup`(S_∞)

`Proof` =

...

$\square$

`IsometryGroupIsCompact` :: $\forall X \in \mathsf{MS}$. `Compact`$(X) \Rightarrow$ `CompactSubset`$\Big(\mathrm{Aut}_{\mathsf{TOP}}(X), \mathrm{Aut}_{\mathsf{MS}_{\circ\to\cdot}}(X)\Big)$

`Proof` =

Assume f is a sequence of isometries.

Let $(E_n)_{n=1}^\infty$ be a sequence of finite $\dfrac{1}{n}$-nets, such that $E_{n+1} \subset E_n$.

Then, it is possible to select a collection $(g^n)_{n=1}^\infty$ of subsequences of f

Let g^n be converging on each $x \in X_n$ to some $L(x)$ and $\forall m \in \mathbb{N}$. $d\Big(L(x), g^n_m\Big) \leq \dfrac{1}{m}$.

This is possible as X is compact and E_n is finite.

Set $h_n = g^n_n$, Then h is converging $D = \bigcup\limits_{n=1}^\infty E_n$ to an isometry L.

But D is dense in X and L is uniformly continuous, so there is an extension $\widehat{L}$ over whole X. .

$\widehat{L}$ is an isometry as metric is continuous. .

We show that h converges to L pointwise. .

Assume $x \in X$, assume $\varepsilon \in \mathbb{R}_{++}$. .

Then there is an $y \in D$ such that $d(x, y) < \varepsilon$.

Also there is an $N \in \mathbb{N}$, such that $d\Big(L(y), h_n(y)\Big) < \varepsilon$ for all $n \geq N$. Let n be such .

So, $d\Big(\widehat{L}(x), h_n(x)\Big) \leq d\Big(\widehat{L}(x), \widehat{L}(y)\Big) + d\Big(\widehat{L}(y), h_n(y)\Big) + d\Big(h_n(y), h_n(x)\Big) = 2d(y, x) + d\Big(L(y), h_n(y)\Big) < 3\varepsilon$.

Hence, f has a subseqequence which converges to L pointwise..

As f was arbitraty, it means that group of isometries is compact. .

$\square$

`NatIdealIsGDelta` :: $\forall I :$ `Ideal`$(\mathbb{N})$. $G_\delta(\mathcal{C}, I) \Rightarrow$ `Closed`$(\mathcal{C}, I)$

`Proof` =

Cantor's space seen as $\mathcal{C} = ?\mathbb{N} = \prod\limits_{n=1}^\infty \dfrac{\mathbb{Z}}{2\mathbb{Z}}$ is a polish group, hence Baire .

Note, that ideals are subgroups for this structure .

So, from the theorem in chapter 2.2.5 of this treatise the result follows .

$\square$

`FrechetIdealOfNatIsOnlyFSigma` :: $F_\sigma\Big(\mathcal{C},$ `Finite`$(\mathbb{N})\Big)$ & $\neg G_\delta\Big(\mathcal{C},$ `Finite`$(\mathbb{N})\Big)$

`Proof` =

Set of finite subsets is countable and $\mathcal{C}$ is separated,so it is F_σ .

By previous result, if Frechet Ideal was G_δ it would be Closed .

But Finite subsets are dense in $\mathcal{C}$ so this contradicts the definition of Frechet's ideal .

$\square$

3.1.2 Baire Groups Redux

Polish groups are Baier, so Baire property will be useful.

$\texttt{PettisBPTheorem} :: \forall G \in \texttt{BaireGroup} \,.\, \forall A \in \mathbf{BP}(G) \,.\, \exists^* A \Rightarrow \exists U \in \mathcal{U}(e) \,.\, U \subset A^{-1}A$

$\texttt{Proof} \ =$

$\Big(U, E, [1]\Big) := \mathtt{E}\exists^* A : \sum U \in \mathcal{T}(G) \,.\, \sum E : \texttt{Meager}(G) \,.\, A = U \triangle E \,\&\, \exists U,$

$\Big([2]\Big) := \mathtt{E}\exists^* A : \sum U \in \mathcal{T}(G) \,.\, \sum E : \texttt{Meager}(G) \,.\, A = U \triangle E \,\&\, \exists^* U,$

$g := \mathtt{E}\exists^* U \in U \cap A,$

$\Big(V, [3]\Big) := \texttt{TopologicalGroupAltDef}(G, g^{-1}U) : \sum V \in \mathcal{U}(e) \,.\, VV^{-1} \subset g^{-1}U,$

$[4] := \texttt{ESetProduct}[3] : \forall v \in V \,.\, gV \subset U \cap Uv,$

$\texttt{Assume}\ v \in V,$

$[6] := \texttt{CheckingBooleanTables} : (U \cap Uv) \triangle (A \cap Av) \subset (U \triangle A) \cup (U \triangle A)v = E \cup Ev,$

$[7] := \texttt{MeagerUnion}(G)\texttt{MeagerSubset}(G)[6] : \texttt{Meager}\Big(G, (U \cap Uv) \triangle (A \cap Av)\Big),$

$[v.*] := \Lambda T : A \cap Av = \emptyset \,.\, T[7]\texttt{EBaireSpace}(G)\mathtt{E}\bot : A \cap Av \neq \emptyset;$

$\leadsto [5] := \mathtt{I}\forall : \forall v \in V \,.\, A \cap Av \neq \emptyset,$

$[*] := \texttt{ISetPtoduct} : V \subset A^{-1}A;$

$\square$

$\texttt{BaireGroupMeasurableIsContinuous} ::$

$: \forall G : \texttt{BaireGroup}.\forall H \in \mathsf{TGRP} \,\&\, \texttt{Separable} \,.\, \forall \phi \in \mathsf{GRP} \,\&\, \texttt{BairMeasurable}(G, H) \,.\, \phi \in \mathsf{TOP}(G, H)$

$\texttt{Proof} \ =$

$\Big(h, [1]\Big) := \texttt{ESeparable}(H) : \sum \mathbb{N} \to h \,.\, \texttt{Dense}(H, \operatorname{Im} h),$

$\texttt{Assume}\ U : \mathcal{U}_H(e),$

$\Big(V, [2]\Big) := \texttt{TopologicalGroupAltDef}(H, U) : \sum V \in \mathcal{U}_H(e) \,.\, V^{-1}V \subset U,$

$$[3] := \texttt{EDense}[1](V) : H = \bigcup_{n=1}^{\infty} h_n V,$$

$[4] := \texttt{UniversalPreimage}(G, H, \phi)[3]\texttt{PreimageUnion}(G, H, \phi) :$

$$: G = \phi^{-1}(H) = \phi^{-1}\left(\bigcup_{n=1}^{\infty} h_n V\right) = \bigcup_{n=1}^{\infty} \phi^{-1}(h_n V),$$

$$\Big(n, [5]\Big) := \texttt{EBaireSpace}(G)[4] : \sum_{n=1}^{\infty} \exists^* \phi^{-1}(h_n V),$$

$\Big(W, [6]\Big) := \texttt{PettisBPTheorem}(G, \phi^{-1}(h_n V)) : \sum W \in \mathcal{U}_G(e) \,.\, W \subset \Big(\phi^{-1}(h_n V)\Big)^{-1} \phi^{-1}(h_n V),$

$[U.*] := \phi([6])[2] : \phi(W) \subset V^{-1}V \subset [U];$

$\leadsto [2] := \mathtt{I}C_e : \phi \in C_e(G, H),$

$[*] := \texttt{PointContinuityImplyContinuity} : \phi \in \mathsf{TOP}(G, H);$

$\square$

$\texttt{NonMeagerBPSubgroupIsClopen} :: \forall G : \texttt{BaireGroup} \,.\, \forall H \subset_{\mathsf{GRP}} G \,.\, \mathbf{BP}(H) \,\&\, \exists^* H \Rightarrow \texttt{Clopen}(H)$

$\texttt{Proof} \ =$

By Pettis Theorem there is an open $U \subset HH^{-1} = H$.

But, then $H = UH$ is open, and hence clopen .

$\square$

RealSetWithoutBPExists :: $\mathbf{BP}^{\complement}(\mathbb{R}) \neq \emptyset$

Proof =

Let h be a Hamel basis for $\mathbb{R}$ taken as $\mathbb{Q}$-vector space. .

Define $A = \{a \in \mathbb{R} | a_1 = 0\}$.

Then $\mathbb{R}$ is a countable union of translates of A, so A can't be meager .

Then, if A has Baire property it must be clopen by the previous remark .

But $\mathbb{R}$ are connected, producing a contradiction .

$\square$

ContinuousActionTheorem ::

$\forall G \in \mathsf{GRP} . \forall \mathcal{T} : \texttt{Topology}(G) . \forall X : \texttt{Metrizable} . \forall \alpha \in \mathsf{GRP} \ \& \ \mathsf{TOP}\Big((G, \mathcal{T}), \mathrm{Aut}_{\mathsf{TOP}}(X)\Big) .$

$. \texttt{BaireSpace} \ \& \ \texttt{Metrizable}(X, \mathcal{T}) \ \& \ \Big(\forall g \in G . \lambda_g \in \mathsf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big)\Big) \Rightarrow \alpha \in \mathsf{TOP}\Big((G, \mathcal{T}) \times X, X\Big)$

Proof =

Use jpoint continuity theorem from descreptive set theory..

$\square$

TopologicalGroupBySeparateContinuity ::

$\forall G \in \mathsf{GRP} . \forall \mathcal{T} : \texttt{Topology}(G) .$

$. \texttt{BaireSpace} \ \& \ \texttt{Metrizable}(X, \mathcal{T}) \ \& \ \mathrm{inv}_G \in \mathsf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big) \ \&$

$\& \ \Big(\forall g \in G . \lambda_g, \rho_g \in \mathsf{TOP}\big((X, \mathcal{T}), (X, \mathcal{T})\big)\Big) \Rightarrow (G, \mathcal{T}) \in \mathsf{TGRP}$

Proof =

...

$\square$

$\texttt{MillerStabilizerTHM} :: \forall G : \texttt{BaireGroup} . \forall X : \texttt{T1} \ \& \ \texttt{SecondCountable} . \forall \alpha \in \mathsf{GRP}\Big(G, \mathrm{Aut}_{\mathsf{SET}}(X)\Big) .$

$. \Big(\forall H \subset_{\mathsf{GRP}} G . \texttt{Closed}(G, H) \Rightarrow \texttt{BaireSpace}(H)\Big) \ \&$

$\& \Big(\forall x \in X . \forall H \subset_{\mathsf{GRP}} G . \texttt{Closed}(G, H) \Rightarrow \texttt{BairMeasurable}\big(H, X, \Lambda h \in H . \alpha(h)(x)\big)\Big) \Rightarrow$

$\Rightarrow \forall x \in X . \texttt{Closed}\big(G, \mathrm{Stab}(\alpha, x)\big)$

$\texttt{Proof} \ =$

$H := \overline{\mathrm{Stab}(\alpha, x)} : \texttt{Closed}(G),$

$[1] := \texttt{SubgroupClosure}(H) : \texttt{Subgroup}(G, H),$

$[2] := [0.1][1] : \texttt{BaireGroup}(H),$

$[4] := \texttt{DenseInAClosure}\Big(G, \mathrm{Stab}(\alpha, x)\Big) : \texttt{Dense}\Big(H, \mathrm{Stab}(\alpha, x)\Big),$

$\texttt{Assume} \ [5] : \exists^* \mathrm{Stab}(\alpha, x),$

$[6] := \texttt{NonMeagerBPSubgroupIsClopen}[5] : \texttt{Clopen}(H, \mathrm{Stab}(\alpha, x)),$

$[5.*] := [4][6] : H = \mathrm{Stab}(\alpha, x);$

$\rightsquigarrow [5] := \texttt{I}(\rightarrow) : \exists^* \mathrm{Stab}(\alpha, x) \rightarrow H = \mathrm{Stab}(\alpha, x),$

$\texttt{Assume} \ [6] : \neg \exists^* \mathrm{Stab}(\alpha, x),$

$\Big(V, [7]\Big) := \texttt{ESecondCountable}(X) : \sum V : \mathbb{N} \rightarrow \mathcal{T}(X) . \texttt{BaseOfTopology}(X, \mathrm{Im}\, V),$

$[8] := \texttt{BaseSeparation}(X, V) : \forall x, y \in X . x \neq y \rightarrow \exists n, m \in \mathbb{N} . x \in V_n \ \& \ y \in V_m \ \& \ V_m \cap V_n = \emptyset,$

$\phi := \Lambda h \in H . \alpha(h)(x) : G \rightarrow X,$

$A := \phi^{-1}(V) : \mathbb{N} \rightarrow \mathbf{BP}(H),$

$[9] := \mathrm{EStabE}A : \forall h \in H . \Big(\forall n \in \mathbb{N} . A_n h = A_n\Big) \iff h \in \mathrm{Stab}(\alpha, x),$

$[10] := \mathrm{E}A : \forall h \in H . h\mathrm{Stab}(\alpha, x) = \bigcap \{A_n | n \in \mathbb{N}, g \in A_n\},$

$[11] := \texttt{FirstTopological01Law}[9] : \forall n \in \mathbb{N} . \forall^* A_n \Big| \neg \exists^* A_n,$

$[12] := \texttt{EBaireGroup}(H)[6][11][10] : \forall h \in H . \exists n \in \mathbb{N} . h \in A_n \ \& \ \neg \exists^* A_n,$

$[13] := \texttt{MeagerUnion} : \neg \exists H,$

$[6.*] := \texttt{EBaireGroup}(H)[12] : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot\texttt{E}|[5] : H = \mathrm{Stab}(\alpha, x);$

$\square$

3.1.3 Universal Polish Group

The proof of this result is rather convoluted and we may need Keller's theorm for it.

$\texttt{UspenskiTHM} :: \forall G : \texttt{PolishGroup} \,.\, \exists H \subset_{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{TOP}}[0,1]^{\mathbb{N}} \,.\, H \cong_{\mathsf{TGRP}} G$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$G^\star := \mathrm{Aut}_{\mathsf{TOP}}[0,1]^{\mathbb{N}} : \mathsf{TGRP};$

3.1.4 Selectors and Transversal

$$\texttt{Selector} :: \prod_{X \in \mathsf{SET}} \texttt{Equivalence}(X) \to ?(X \to X)$$

$$\sigma : \texttt{Selector} \iff \Lambda E : \texttt{Equivalence}(X) \,.\, \forall C : \texttt{EquivalenceClass}\,(X, E) \,.\, \forall x, y \in C \,.\, \sigma(x) = \sigma(y) \;\&\; \sigma(x) \in$$

$$\texttt{Transversal} :: \prod_{X \in \mathsf{SET}} \texttt{Equivalence}(X) \to ??X$$

$$T : \texttt{Transversal} \iff \Lambda E : \texttt{Equivalence}(X) \,.\, \forall C : \texttt{EquivalenceClass}\,(X, E) \,.\, |C \cap T| = 1$$

$$\texttt{selectorAsTransversal} :: \prod_{X \in \mathsf{SET}} \prod E : \texttt{Equivalence}(X) \,.\, \texttt{Selector}(X, E) \to \texttt{Transversal}(X, E)$$

$$\texttt{selectorAsTransversal}\,(\sigma) = \sigma := \operatorname{Im} \sigma$$

$$\texttt{transversalAsSelector} :: \prod_{X \in \mathsf{SET}} \prod E : \texttt{Equivalence}(X) \,.\, \texttt{Transversal}(X, T) \to \texttt{Transversal}(X, T)$$

$$\texttt{transversalAsSelector}\,(T) = T := \Lambda x \in X \,.\, \texttt{ESingleton}(X)\texttt{ETransversal}\big(T, [x]\big)$$

$$\texttt{saturation} :: \prod_{X \in \mathsf{SET}} \texttt{Equivalence}(X) \to ?X \to ?X$$

$$\texttt{saturation}\,(E, A) = [A]_E := \{x \in X : \exists a \in A : xEa\}$$

$\texttt{BorelSelectors} :: \forall X : \texttt{Polish} \,.\, \forall E : \texttt{Equivalence} \,.$

$\quad . \forall [0.1] : \forall C : \texttt{EquivalenceClass}\,(X, E) \,.\, \texttt{Closed}(X, C) \,.$

$\quad . \forall [0.2] : \forall U \in \mathcal{T}(X) \,.\, [U]_E \in \mathcal{B}(X) \,.$

$\quad . \exists \sigma : \texttt{Selector}(X) \,.\, \sigma \in \operatorname{End}_{\mathsf{BOR}}(X)$

$\texttt{Proof} =$

$\varphi := \Lambda x \in X \,.\, [x]_E : X \to \mathsf{EFF}(X),$

$[1] := \Lambda U \in \mathcal{T}(X) \,.\, \mathtt{E}\varphi\mathtt{I}[U]_E[0.2] : \forall U \in \mathcal{T}(X) \,.\, \varphi^{-1}\Big\{A \in \mathsf{EFF}(X) : \exists A \cap U\Big\} = [U]_E \in \mathcal{B}(X),$

$[2] := \mathtt{I}\mathsf{BOR}[1] : \varphi \in \operatorname{End}_{\mathsf{BOR}}(X);$

$\Big(\delta, [3]\Big) := \texttt{SelectionTHM}(X) : \sum \delta : \mathbb{N} \to \mathsf{BOR}\Big(X, \mathsf{EFF}(X)\Big) \,.\, \forall A \in \mathsf{EFF}(X) \,.\, \texttt{Dense}\Big(A, \delta_{\mathbb{N}}(A)\Big),$

$\sigma := \varphi\delta_1 : \mathsf{BOR}(X, X),$

$[*] := \mathtt{E}\sigma[3] : \texttt{Selector}(X, E, \sigma);$

$\square$

$\texttt{SubgroupSelector} :: \forall G : \texttt{PolishGroup} \,.\, \forall H : \texttt{Closed} \;\&\; \texttt{Subgroup}(G) \,.\, \exists \sigma \in \operatorname{Aut}_{\mathsf{BOR}}(G) \,.$

$\quad . \texttt{Selector}\Big(G, \texttt{Coset}(G, H), \sigma\Big)$

$\texttt{Proof} =$

$[1] := \mathtt{E}\mathsf{TGRP}(G) : \forall g \in G \,.\, \texttt{Closed}(G, gH),$

$[2] := \Lambda U \in \mathcal{T}(G)\mathtt{E}\mathsf{TGRP}(G)\mathtt{E}\mathsf{TOP}(G) : \forall U \in \mathcal{T}(G) \,.\, [U]_H = \bigcup_{g \in G} gU \in \mathcal{T}(G),$

$[3] := \texttt{BorelSelector}(G, \sim_H)[1][2] : \exists \sigma \in \operatorname{Aut}_{\mathsf{BOR}}(G) \,.\, \texttt{Selector}\Big(G, \texttt{Coset}(G, H), \sigma\Big);$

$\square$

3.1.5 Borel Space of Polish Groups

$\texttt{ClosedSubgroupsAreBorel} :: \texttt{Closed} \,\&\, \texttt{Subgroup}(G^\star) \in \mathcal{B}\Big(\mathsf{EFF}(G^\star)\Big)$

$\texttt{Proof} \ =$

$S := \texttt{Closed} \,\&\, \texttt{Subgroup}(G^\star) :??G^\star,$

$\Big(\delta, [1]\Big) := \texttt{SelectionTHM}(G^\star) : \sum \delta : \mathbb{N} \to \mathsf{BOR}\Big(G^\star, \mathsf{EFF}(G^\star)\Big) \,.\, \forall A \in \mathsf{EFF}(G^\star) \,.\, \texttt{Dense}\Big(A, \delta_{\mathbb{N}}(A)\Big),$

$[*] :=: S = \Big\{A \in \mathsf{EFF}(G^\star) \Big| e \in A\Big\} \cap \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (\delta_n \delta_m^{-1})^{-1}\{(x, A) \in G^\star \times \mathsf{EFF}(G^\star) \Big| x \in A\};$

$\square$

3.1.6 Standard Borel Groups

$$\texttt{BorelGroup} :: ?\sum G \in \mathsf{GRP} \;.\; \sigma\texttt{-Algebra}(G)$$

$$(G, \mathcal{A}) : \texttt{BorelGroup} \iff$$

$$\iff \circ_G \in \mathsf{BOR}\Big((G, \mathcal{T}) \times (G, \mathcal{T}), (G, \mathcal{T})\Big) \;\&\; \Lambda g \in G \;.\; g^{-1} \in \mathsf{BOR}\Big((G, \mathcal{T}), (G, \mathcal{T})\Big)$$

$$\texttt{groupOfBorelAsGroup} :: \texttt{BorelGroup} \to \mathsf{GRP}$$

$$\texttt{groupOfBorelAsGroup}\,(G, \mathcal{A}) = (G, \mathcal{A}) := G$$

$$\texttt{groupOfBorelAsMeasurableSpace} :: \texttt{BorelGroup} \to \mathsf{BOR}$$

$$\texttt{groupOfBorelAsMeasurablelSpace}\,(G, \mathcal{A}) = (G, \mathcal{A}) := (G, \mathcal{A})$$

$$\texttt{StandardBorelGroup} := \texttt{BorelGroup} \;\&\; \texttt{StandardBorelSpace} : \texttt{Type};$$

$$\texttt{StandardGroupTopologyUniqueness} :: \forall G : \texttt{StandardBorelGroup} \;.\; \forall \mathcal{A}, \mathcal{B} : \texttt{Topology}(G) \;.$$

$$.\; \forall [0.1] : \texttt{PolishGroup}\Big((G, \mathcal{A}) \;\&\; (G, \mathcal{B})\Big) \;.\; \forall [0.2] : \mathsf{A}(G) = \sigma(\mathcal{A}) = \sigma(\mathcal{B}) \;.\; \mathcal{A} = \mathcal{B}$$

$\texttt{Proof} =$

$$[1] := \texttt{BorelIsBairMeasurable}[0.2] :$$

$$: \texttt{BairMeasurable}\Big((G, \mathcal{A}), (G, \mathcal{B}), \mathrm{id}_G\Big) \;\&\; \texttt{BairMeasurable}\Big((G, \mathcal{B}), (G, \mathcal{A}), \mathrm{id}_G\Big),$$

$$[2] := \texttt{BairMeasurableIsContinuous}[1][0.1] : \mathsf{TOP}\Big((G, \mathcal{A}), (G, \mathcal{B}), \mathrm{id}_G\Big) \;\&\; \mathsf{TOP}\Big((G, \mathcal{B}), (G, \mathcal{A}), \mathrm{id}_G\Big),$$

$$[3] := \texttt{E}\,\mathrm{id}[2] : \mathcal{A} = \mathcal{B};$$

$\square$

$$\texttt{Polishable} :: ?\texttt{StandardBorelGroup}$$

$$G : \texttt{Polishable} \iff \exists \mathcal{T} : \texttt{Topology}(G) \;.\; \texttt{PolishGroup}(G, \mathcal{T}) \;\&\; \mathsf{A}(G) = \sigma(\mathcal{A})$$

$$\texttt{eventuallyOneGroup} :: \mathsf{TGRP}$$

$$\texttt{eventuallyOneGroup}\,() = \mathbb{T}^{\mathbb{N}}_1 := \Big\{ s \in \mathbb{T}^{\mathbb{N}} : \big|\{n \in \mathbb{N} \;.\; s_n \neq 1\}\big| < \infty \Big\}$$

$$\texttt{EventuallyOneGroupIsBorel} :: \mathbb{T}^{\mathbb{N}}_1 \in \mathcal{B}(\mathbb{T}^{\mathbb{N}})$$

$\texttt{Proof} =$

$$\mathbb{T}^{\mathbb{N}}_1 = \bigcup_{n=1}^{\infty} \mathbb{T}^n \times \{1\}^{\mathbb{N}}$$

$\square$

$$\texttt{EventuallyOneGroupIsStandard} :: \texttt{StandardBorelGroup}(\mathbb{T}^{\mathbb{N}}_1)$$

$\texttt{Proof} =$

The space $\bigsqcup_{n=1} \mathbb{T}^n$ is standard Borel.

Factorizing by closed sets should produce $\mathbb{T}^{\mathbb{N}}_1$ with polish topology.

$\square$

$\texttt{EventuallyOneGroupIsNotPolishable} :: \neg\texttt{Polishable}\left(\mathbb{T}_1^{\mathbb{N}}\right)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \texttt{Polishable}\left(\mathbb{T}_1^{\mathbb{N}}\right),$

$\Big(\mathcal{T}, [2]\Big) := \mathtt{E}[1] : \sum \mathcal{T} : \texttt{Topology}(X)\ .\ \texttt{PolishGroup}(\mathbb{T}_1^{\mathbb{N}}, \mathcal{T})\ \&\ \mathsf{A}(\mathbb{T}_1^{\mathbb{N}}) = \sigma(\mathcal{T}),$

$[3] := \texttt{PolishIsGDelta}[2.1] : G_\delta(\mathbb{T}^{\mathbb{N}}, \mathbb{T}_1^{\mathbb{N}}),$

$[4] := \texttt{GDeltaSubgroupIsClosed}[3] : \texttt{Closed}(\mathbb{T}^{\mathbb{N}}, \mathbb{T}_1^{\mathbb{N}}),$

$x := \Lambda n \in \mathbb{N}\ .\ \Lambda m \in \mathbb{N}\ .\ \exp\left(\frac{\mathrm{i}}{n}\right) : \mathbb{N} \to \mathbb{T}^{\mathbb{N}},$

$[6] := \mathtt{E}x\mathtt{E}\mathbb{T}_1^{\mathbb{N}} : \forall n \in \mathbb{N}\ .\ x_n \notin \mathbb{T}_1^{\mathbb{N}},$

$[7] := \mathtt{E}x\mathtt{E}\mathbb{T}^{\mathbb{N}}\mathtt{E}\mathbb{T}_1^{\mathbb{N}} : \lim_{n\to\infty} x_n = 1 \in \mathbb{T}_1^{\mathbb{N}},$

$[1.*] := \texttt{ClosedBySequences}[4][7] : \perp;$

$\rightsquigarrow [*] := \mathtt{E}\perp : \neg\texttt{Polishable}(\mathbb{T}^{\mathbb{N}});$

$\square$

$\texttt{L2SequencesArePolishable} :: \texttt{Polishable}(l_2)$

$\texttt{Proof} \ =$

l_2 is polish as a separable Hilbert space.

$\square$

3.2 Borel Action

3.2.1 E Separation

$\texttt{EInvariant} :: \prod X \in \texttt{Set} \,.\, \texttt{Equivalence}(X) \to ??X$

$A : \texttt{EInvariant} \iff \forall x \in A \,.\, \forall y \in [x]_E \,.\, y \in A$

$\texttt{ESeparation} :: \forall X : \texttt{StandardBorelSpace} \,.\, \forall E : \texttt{Equivalence}(X) \,.\, \forall [0.1] : E \in \Sigma^1_1(X^2) \,.$
$\quad .\, \forall A, B \in \Sigma^1_1(X) \;\&\; \texttt{EInvariant}(X, E) \,.\, \forall [0.2] : \texttt{DisjointPair}(X, A, B) \,.$
$\quad .\, \exists C \in \mathcal{B}(X) \;\&\; \texttt{EInvariant}(X, E) \,.\, A \subset C \;\&\; B \cap C = \emptyset$

$\texttt{Proof} =$

Assume X is Polish without loss of generality..

It is possible to give strong topology to the set $\dfrac{X}{E}$, so the projection π_E is continuous.

With this structure $\pi_E(A), \pi_E(B)$ are anlytic in $\dfrac{X}{E}$.

They also disjoint as they were E-Invariant.

As E is analytic set itself, we can realize by a pair of continuous maps $\phi_1, \phi_2 : \mathcal{B} \to X$.

Then, $x \sim_E y$ iff there is a $b \in \mathcal{B}$ such that $\phi_1(b) = x$ and $\phi_2(b) = y$.

Thus, $\dfrac{X}{E}$ is equivalent to pushout $X \sqcup_{\mathcal{B},\phi} X$.

So, $\dfrac{X}{E}$ must be Polish .

Now apply Suslin separation theorem in $\dfrac{X}{E}$ to separate A and B by some C .

Then $\pi_E^{-1}(C)$ is Borel and E-invariant, it also separates A and B.

...

□

$\texttt{PolishTopologicalGroupCondition} :: \forall G \in \mathsf{GRP} \,.\, \forall \mathcal{T} : \texttt{PolishTopology}(G) \,.$
$\quad .\, \forall [0] : \forall g \in G \,.\, \lambda_g, \rho_g \in \mathrm{Aut}_{\mathsf{TOP}}(G, \mathcal{T}) \,.\, (G, \mathcal{T}) \in \mathsf{TGRP}$

$\texttt{Proof} =$

...

□

$\texttt{BlackwellTHM} ::$

$:: \forall X : \texttt{StandardBorelSpace} \, . \, \forall A : \mathbb{N} \to \mathcal{S}_X \, . \, \forall S \subset X \, . \, \texttt{EInvariant}(X, E, S) \,\&\, \mathcal{B}(X) \iff S \in \sigma(\operatorname{Im} A)$

$\texttt{where} \quad E = \left\{(x, y) \in X^2 : \forall n \in \mathbb{N} \, . \, x \in A_n \iff y \in A_n\right\}$

$\texttt{Proof} =$

By properties of logical $\iff$ it is obvious that E is equivalence.

Firstly, we show that each A_n is E-invariant.

consider $x \in A_n$ and $y \in [x]_E$, then by definition of E we also have $y \in A_n$.

It is clear that union of E-invariant sets is E-invarinat.

Now assume that B is E-invariant.

Assume $x \in B^{\complement}$ and $y \in [x]_E$.

If y was in B then by symmetry x would also be in B, so $y \in B^{\complement}$.

So, invariant subsets form a σ-algebra containing all A_n.

Thus, $\sigma(\operatorname{Im} A)$ is all E-invariant.

Now assume B is Borel amd E-invariant.

Note, that equivalence class of E are Borel and correspond to elements of $\mathcal{C}$.

For $c \in \mathcal{C}$ the set $\alpha_c = \bigcap_{c_n=1} A_n \cap \bigcap_{c_n=0} A_n^{\complement}$ is equivalence class of E.

And every equivalence class of E can be expressed as som α_c .

Thus, every equivalence class belongs to $\sigma(\operatorname{Im} A)$.

Nevertheless, we always can express $B = \bigcup_{c \in C} \alpha_c$ for some $C \subset \mathcal{C}$.

Consider a mapping $\psi : \dfrac{X}{E} \to \mathcal{C}$ defined by $\psi(\alpha_c) = c$.

This mapping is a measurable (see argument obout prebase next) injection.

So $C = \psi[B]_E$ must be measurable in $\mathcal{C}$.

but topology on $\mathcal{C}$ can be generated by prebase of sets of form $\{c \in \mathcal{C} | c_n = j\}$, where $j = 1, 0$ and $n \in \mathbb{N}$.

And such sets also generate the Borel algebra of $\mathcal{C}$.

Now sets $C = \{c \in \mathcal{C} | c_n = 1\}$ corresponds directly to A_n .

So, B must belong to $\sigma(\operatorname{Im} A)$.

$\square$

3.2.2 Subject

$\texttt{BorelAction} := \Lambda X : \texttt{StandardBorelSpace} . \Lambda G : \texttt{StandardBorelGroup} . G \curvearrowright_{\mathsf{BOR}} X =$

$= \Lambda X : \texttt{StandardBorelSpace} . \Lambda G : \texttt{StandardBorelGroup} . \mathsf{GRP}\Big(G, \mathrm{Aut}_{\mathsf{BOR}}(X)\Big) \ \& \ \mathsf{BOR}(G \times X, X) :$

$: \texttt{StandardBorelSpace} \to \texttt{StandardBorelGroup} \to \texttt{Type};$

$\texttt{BorelOrbitRelationIsAnalytic} ::$

$:: \forall X : \texttt{StandardBorelSpace} . \forall G : \texttt{StandardBorelGroup} . \forall \alpha : G \curvearrowright_{\mathsf{BOR}} X . E_\alpha \in \Sigma^1_1(X^2)$

$\texttt{Proof} \ =$

Enrich topology on $X \times G$ so α is continuous.

Then, mapping $\beta : (x, g) \mapsto \Big(x, \alpha(x, g)\Big)$ is also continuous.

But its image is E_α.

$\square$

$\texttt{FreeBorelOrbitIsBorel} ::$

$:: \forall X : \texttt{StandardBorelSpace} . \forall G : \texttt{StandardBorelGroup} . \forall \alpha : G \curvearrowright_{\mathsf{BOR}} X . \texttt{Free}(G, X, \alpha) \Rightarrow E_\alpha \in \mathcal{B}(X^2)$

$\texttt{Proof} \ =$

In this case β will be injective.

So E_α is Borel by Injective Image Theorem.

$\square$

$\texttt{LocallyCompactContinuousOrbitIsFSigma} ::$

$:: \forall X : \texttt{Polish} . \forall G : \texttt{PolishGroup} \ \& \ \texttt{LocallyCompact} . \forall \alpha : G \curvearrowright_{\mathsf{TOP}} X . E_\alpha \in F_\sigma(X^2)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{MillerBorelOrbitTHM} ::$

$:: \forall G : \texttt{PolishGroup} . \forall X : \texttt{StandardBorelSpace} . \forall \alpha : G \curvearrowright_{\mathsf{BOR}} X . \forall x \in X . \ O_\alpha(x) \in \mathcal{S}_X$

$\texttt{Proof} \ =$

$[1] := \texttt{MillerStabilizerTHM}(G, \alpha) : \forall x \in X . \texttt{Closed}\Big(G, \mathrm{Stab}(\alpha, x)\Big),$

$\Big(T, [2]\Big) := \texttt{SubgroupSelector}[1](x) : T \in \mathcal{B}(G) . \forall g \in G . \Big|T \cap g\mathrm{Stab}(\alpha, x)\Big| = 1,$

$[3] := \texttt{EGroupAction}(G, X, \alpha) : \forall g, h \in G . gx = hx \iff \exists f \in G . g, h \in f\mathrm{Stab}(\alpha, x),$

$\varphi := \Lambda g \in G . gx \in \mathsf{BOR}(G, X),$

$[4] := \texttt{E}\varphi[2][3] : \texttt{Injective}\Big(T, X, \varphi_{|T}\Big),$

$[5] := \texttt{E}\varphi\texttt{I}O_\alpha : \varphi(T) = O_\alpha(x),$

$[6] := \texttt{InjectiveImageTHM} : O_\alpha(x) \in \mathcal{S}_X;$

$\square$

BorelHomo :: $\forall G$: PolishGroup . $\forall H$: StandardBorelGroup . $\forall \varphi \in \mathsf{BOR} \cap \mathsf{GRP}(G, H)$. $\varphi(G) \in \mathcal{S}_H$

Proof $=$

Define Borel action $\alpha : G \curvearrowright H$ by $\alpha(g, h) = \varphi(g)h$.

Then $\varphi(G) = O_\alpha(e)$.

By Miller's Theorem it must be measurable.

$\square$

3.2.3 Vaught Transfom

$\texttt{actionSaturation} :: \prod G \in \mathsf{GRP} \,.\, \prod X \in \mathsf{SET} \,.\, \prod \alpha : G \curvearrowright X \,.\, ?X \to ?X$

$\texttt{actionSaturation}\,(A) = [A]_\alpha := \{x \in X : \exists g \in G \,.\, gx \in A\}$

$\texttt{actionHull} :: \prod G \in \mathsf{GRP} \,.\, \prod X \in \mathsf{SET} \,.\, \prod \alpha : G \curvearrowright X \,.\, ?X \to ?X$

$\texttt{actionHull}\,(A) = (A)_\alpha := \{x \in X : \forall g \in G \,.\, gx \in A\}$

$\texttt{SaturationAndHullRelation} :: \forall G \in \mathsf{GRP} \,.\, \forall X \in \mathsf{SET} \,.\, \forall \alpha : G \curvearrowright X \,.\, \forall A \subset X \,.\, (A)_\alpha \subset A \subset [A]_\alpha$

$\texttt{Proof} =$

This is obvious.

□

$\texttt{AnalyticSaturation} ::$

$:: \forall G : \texttt{StandardBorelGroup} \,.\, \forall X : \texttt{StandardBorelSpace} \,.\, \forall \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, \forall A \in \mathcal{S}_X \,.\, [A]_\alpha \in \Sigma^1_1(X)$

$\texttt{Proof} =$

View $[A]_\alpha$ as an image of $G \times A$ under α.

□

$\texttt{CoanalyticHull} ::$

$:: \forall G : \texttt{StandardBorelGroup} \,.\, \forall X : \texttt{StandardBorelSpace} \,.\, \forall \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, \forall A \in \mathcal{S}_X \,.\, (A)_\alpha \in \Pi^1_1(X)$

$\texttt{Proof} =$

View $(A)^{\complement}_\alpha$ as an image of $G \times A^{\complement}$ under α.

□

$\texttt{nonmeagerVaughtTransform} :: \prod G \in \texttt{StandardBorelGroup} \,.\, \prod X \in \texttt{StandardBorelSpace} \,.$

$\,.\, \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, ?X \to ?X$

$\texttt{nonmeagerVaughtTransform}\,(A) = A^\star_\alpha := \{x \in X : \exists^* g \in G \,.\, gx \in A\}$

$\texttt{comeagerVaughtTransform} :: \prod G \in \texttt{StandardBorelGroup} \,.\, \prod X \in \texttt{StandardBorelSpace} \,.$

$\,.\, \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, ?X \to ?X$

$\texttt{comeagerVaughtTransform}\,(A) = A^{\triangle}_\alpha := \{x \in X : \forall^* g \in G \,.\, gx \in A\}$

$\texttt{nonmeagerLocalVaughtTransform} :: \prod G \in \texttt{StandardBorelGroup} \,.\, \prod X \in \texttt{StandardBorelSpace} \,.$

$\,.\, \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, \mathcal{T}(G) \to ?X \to ?X$

$\texttt{nonmeagerLocalVaughtTransform}\,(A, U) = A^{\star U}_\alpha := \{x \in X : \exists^* g \in U \,.\, gx \in A\}$

$\texttt{comeagerLocalVaughtTransform} :: \prod G \in \texttt{StandardBorelGroup} \,.\, \prod X \in \texttt{StandardBorelSpace} \,.$

$\,.\, \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X \,.\, \mathcal{T}(G) \to ?X \to ?X$

$\texttt{comeagerLocalVaughtTransform}\,(A, U) = A^{\triangle U}_\alpha := \{x \in X : \forall^* g \in U \,.\, gx \in A\}$

VaughtTransformsRelation ::

$:: \forall G \in \texttt{StandardBorelGroup} . \forall X \in \texttt{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X . \forall A \subset X . (A)_\alpha \subset A_\alpha^\triangle \subset A_\alpha^\star \subset A \subset [A]_\alpha$

Proof $=$

Obvious.

$\square$

VaughtTransformsInvariant $:: \forall G \in \texttt{StandardBorelGroup} . \forall X \in \texttt{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X . \forall A \subset X . \texttt{Invariant}(G, X, \alpha, A_\alpha^\triangle \ \& \ A_\alpha^\star)$

Proof $=$

...

$\square$

VaughtTransformInvarianceCriterion $:: \forall G \in \texttt{StandardBorelGroup} . \forall X \in \texttt{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X . \forall A \subset X . \texttt{Invariant}(G, X, \alpha, A) \iff A_\alpha^\triangle = A_\alpha^\star$

Proof $=$

...

$\square$

LocalVaughtTransformIsBorel $:: \forall G \in \texttt{StandardBorelGroup} . \forall X \in \texttt{StandardBorelSpace} .$

$. \prod \alpha : G \curvearrowright_{\mathsf{BOR}} X . \forall A \subset \mathcal{S}_X . \forall U \in \mathcal{T}(G) . A_\alpha^{\triangle U}, A_\alpha^{\star U} \in \mathcal{S}_X$

Proof $=$

Follows from Novikov-Montgomery theorem.

$\square$

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