Vector Spaces

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Contents

1 Structural Theory of Vector Spaces

1.1 Concept of Vector Spaces, Basis Theorem

$\texttt{vectorSpaces} :: \texttt{Field} \to \mathsf{CAT}$
$\texttt{vectorSpaces}(k) := k\text{-}\mathsf{MOD}$

$\texttt{MaximalLIndIsBasis} :: \forall V : k\text{-}\mathsf{VS} . \forall E \in \max \texttt{LinearlyIndependent}(V) . E : \texttt{Basis}(V)$
$\texttt{Proof} =$
$\texttt{Assume}\ v : V,$
$\texttt{Assume}\ [2] : v \neq 0,$
$(\alpha, \beta, [1]) := \mathcal{A} \max \texttt{LinearlyIndependent}(V)(E) : \sum \alpha \in k . \sum \beta : E \to k . \alpha v + \sum_{e \in E} \beta_e e = 0\ \&$
$\quad \& \ (\alpha, \beta) \neq 0,$
$[3.1] := [1][3] : \sum_{e \in E} \beta_e e = 0,$
$[3.2] := \mathcal{A}\texttt{LinearlyIndependent}(V)(E)[3.1] : \beta = 0,$
$[3.3] := \mathcal{A}\texttt{zero}[3.2][3.1][1] : 0 \neq 0,$
$[3.1] := I(\bot)[3.3] : \bot;$
$\rightsquigarrow [3] := E(\bot) : \alpha \neq 0,$
$[v.*] := \mathcal{A} k\text{-}\mathsf{VS}[3][1] : v = \sum_{e \in E} \frac{\beta_e}{\alpha} e;$
$\rightsquigarrow [1] := \mathcal{A}^{-1} \operatorname{span} : V = \operatorname{span}(E),$
$[*] := \mathcal{A}^{-1}\texttt{Basis}[1] : (E : \texttt{Basis}(V));$
$\square$

$\texttt{MinimalGeneratingIsBasis} :: \forall V : k\text{-}\mathsf{VS} . \forall E \in \min \texttt{Generating}(V) . E : \texttt{Basis}(V)$
$\texttt{Proof} =$
$\texttt{Assume}\ \alpha : k^{\oplus E},$
$\texttt{Assume}\ [1] : \alpha E = 0,$
$\texttt{Assume}\ [2] : \alpha \neq 0,$
$(e, [3]) := \mathcal{A} 0 [2] : \sum e \in E . \alpha_e \neq 0,$
$F := E \setminus \{e\} :?V,$
$[4] := [3][1]\mathcal{A} k\text{-}\mathsf{VS}(V) : e = \sum_{f \in F} \frac{\alpha_f}{\alpha_e} f,$
$[5] := \texttt{NonemptyRemoval}(E, F)\texttt{SingletonIsNonEmpty}(e) : F \subsetneq E,$
$[6] := \mathcal{A}\texttt{Generating}(V)(E)[4] : (F : \texttt{Generating}(V)),$
$[\alpha.*] := \mathcal{A} \min \texttt{Generating}(V)[5][6] : (\bot);$
$\rightsquigarrow [1] := \mathcal{A}^{-1}\texttt{LinearlyIndependent} : (E : \texttt{LinearlyIndependent}(E)),$
$[*] := \mathcal{A}^{-1}\texttt{Basis}[1] : (E : \texttt{Basis}(V));$
$\square$

VectorSpaceIsTorsionFree :: $\forall V : k\text{-VS} \,.\; \operatorname{tor} V = \{0\}$

Proof $=$

Assume $v : V$,

Assume $[1] : v \neq 0$,

Assume $\alpha : k$,

$[1.2] := \mathcal{I} k\text{-VS}(\alpha) : \alpha^{-1}\alpha v = v$,

$[1.*] := \mathcal{I} k\text{-VS}[1][2] : \alpha v \neq 0$;

$\leadsto [*] := \mathcal{I}^{-1} \operatorname{tor} : \operatorname{tor} V = \{0\}$;

$\square$

LIndUnionLemma :: $\forall V : k\text{-VS} \,.\, \forall C : \texttt{Chain}\big(\texttt{LinearlyIndependent}(V)\big) \,.\, \bigcup_{n=1}^{\infty} C_i : \texttt{LinearlyIndependent}(V)$

Proof $=$

$X := \bigcup_{n=1}^{\infty} C_i :?V$,

Assume $\alpha : k^{\oplus X}$,

$F := \{x \in X : \alpha_x \neq 0\} : \texttt{Finite}(V)$,

$(n, [1]) := \texttt{FiniteInChain}(C, F) : \sum n \in \mathbb{N} \,.\, F \subset C_n$,

Assume $[2] : \alpha X = 0$,

$[2.1] := [2][1] : \alpha_{|C_n} C_n = 0$,

$[2.2] := \mathcal{I}\texttt{LinearlyIndependent}(V)(C_n)[3] : \alpha_{|C_n} = 0$,

$[2.*] := [2.1][1] : \alpha = 0$;

$\leadsto [\alpha.*] := I(\Rightarrow) : \alpha X = 0 \Rightarrow \alpha = 0$;

$\leadsto [*] := \mathcal{I}^{-1}\texttt{LinearlyIndependent}(V) : [X : \texttt{LinearlyIndependent}(V)]$;

$\square$

HamelBasisTheorem :: $\forall V : k\text{-VS} \,.\, \exists E : \texttt{Basis}(V)$

Proof $=$

Assume $[1] : V = \{0\}$,

$[1.*] := \mathcal{I}\emptyset\mathcal{I}[1]\mathcal{I}^{-1}\texttt{Basis} : (\emptyset : \texttt{Basis}(V))$;

Assume $[1] : V \neq \{0\}$,

$[1.2] := \mathcal{I}\texttt{LinearlyIndependent}(V)\texttt{VectorSpaceIsTorsionFree} : \texttt{LinearlyIndependent}(V) \neq \emptyset$,

$E := \texttt{ZornLemma}[1.2]\texttt{LIndUnionLemma} : \max \texttt{LinearlyIndependent}(V)$,

$[1.*] := \texttt{MaximalLindIsBasis}(E) : (E : \texttt{Basis}(V))$;

$\leadsto [*] := \texttt{EQLEM}(V, \{0\})E(|)I^2(\Rightarrow)I^2E : \exists E : \texttt{Basis}(V)$;

$\square$

VectorSpaceIsFree :: $\forall V : k\text{-VS} \,.\, V : \texttt{FreeModule}(k)$

Proof $=$

...

$\square$

$\texttt{dimension} :: k\text{-}\mathsf{VS} \to \mathsf{CARD}$

$\texttt{dimension}\,(V) = \dim_k V := \operatorname{rank}_k V$

$\texttt{finiteDimensionalVectorSpaces} :: \texttt{Field} \to \mathsf{CAT}$

$\texttt{finiteDimensionVectorSpace}\,(k) = k\text{-}\mathsf{FDVS} := \{V \in k\text{-}\mathsf{VS} : \dim V < \infty\}$

$\texttt{LIBasisExtension} :: \forall V : k\text{-}\mathsf{VS} \,.\, \forall F : \texttt{LinearlyIndependent}(V) \,.\, \exists E : \texttt{Basis}(V) : F \subset E$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.2 Subspaces of the Vector Space

$\texttt{VectorSpaceSubspace} :: \prod V \in k\text{-VS} \,.\, ??V$

$U : \texttt{VectorSpaceSubspace} \iff U \subset_{k\text{-VS}} V \iff U \subset_{k\text{-MOD}} V$

$\texttt{SubspaceDim} :: \forall V \in k\text{-VS} \,.\, \forall U, W \subset_{k\text{-VS}} V \,.\, U \subset_{k\text{-VS}} W \Rightarrow \dim U \leq \dim W$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SumDimTHM} :: \forall V \in k\text{-VS} \,.\, \forall U, W \subset_{k\text{-VS}} V \,.\, \dim U + W = \dim U + \dim W - \dim u \cap W$

$\texttt{Proof} \ =$

$E := \texttt{HamelBasisTHM}(W \cap U) : \texttt{Basis}(W \cap U),$

$\big(F, [1]\big) := \texttt{LIBasisExtension}(E, W) : \sum F : \texttt{LinearlyIndependent}(W) \,.\, F \cup E : \texttt{Basis}(W),$

$\big(G, [2]\big) := \texttt{LIBasisExtension}(E, U) : \sum G : \texttt{LinearlyIndependent}(U) \,.\, G \cup E : \texttt{Basis}(U),$

$[3] := [1] \mathcal{I} \texttt{LinearlyIndependent}(W)(F \cup E) : \operatorname{span}(F) \cap U \cap W = \{0\},$

$[4] := [2] \mathcal{I} \texttt{LinearlyIndependent}(U)(G \cup E) : \operatorname{span}(G) \cap U \cap W = \{0\},$

$[5] := \mathcal{I}^{-1} \texttt{Generating} \mathcal{I} \texttt{sum}(U, W) \dots : E \sqcup F \sqcup G : \texttt{Generating}(U + W),$

$\texttt{Assume} \ \alpha : k^{\oplus E \cup F \cup G},$

$\texttt{Assume} \ [6] : \alpha(E \cup F \cup G) = 0,$

$[6.1] := \mathcal{I} E : \alpha_{|E} E \in U \cap W,$

$[6.2] := [6][6.1] \mathcal{I} F \mathcal{I} G : \alpha_{|F} F = -\alpha_{|G} G - \alpha_{|E} E \in U,$

$[6.3] := [3][6.2] : \alpha_{|F} F = 0,$

$[6.4] := \mathcal{I} \texttt{LinearlyIndependent}(W)(F)[6.3] : \alpha_{|F} = 0,$

$[6.5] := [6.3][6] \mathcal{I} \texttt{LinearlyIndependent}(U)(G \cap E) : \alpha_{|G \cup E} = 0,$

$[6.*] := \mathcal{I} \alpha [6.4][6.5] : \alpha = 0;$

$\leadsto [6] := \mathcal{I}^{-1} \texttt{Basis}(W + U)[5] : E \cup F \cup G : \texttt{Basis}(W + U),$

$[*] := \mathcal{I} \dim(W + U)[6][3][4](\pm |E|)[3][4] \mathcal{I}^{-1} \dim \dots :$

$: \dim(W + U) = |E \cup F \cup G| = |E| + |F| + |G| = \big(|E| + |F|\big) + \big(|E| + |G|\big) - |E| =$

$= |E \cup F| + |E \cup G| - |E| = \dim W + \dim U - \dim W \cap U;$

$\square$

$$\texttt{DirectSumDim} :: \forall n \in \mathbb{N} \,.\, \forall V : n \to k\text{-VS} \,.\, \dim \bigoplus_{i=1}^{n} V_i = \sum_{i=1}^{n} \dim V_i$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InnerDirectSumDim} :: \forall V \in k\text{-VS} \,.\, \forall n \in \mathbb{N} \,.\, \forall U : n \to \texttt{VectorSubspace}(V) \,.$

$$. \ V = \bigoplus_{i=1}^{n} U_i \Rightarrow \dim V = \sum_{i=1}^{n} \dim U_i$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{LinearComplement} :: \sum V \in k\text{-VS} \,.\, \texttt{VectorSubspace}(V) \to ?\texttt{VectorSubspace}(V)$

$W : \texttt{LinearComplement} \iff \Lambda U \subset_{k\text{-VS}} V \,.\, U \oplus W = V$

$\texttt{LinearComplementExists} :: \forall V \in k\text{-VS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \exists \texttt{LinearComplement}(V, U)$

$\texttt{Proof} \;=$

$E := \texttt{HamelBasisTHM}(U) : \texttt{Basis}(U),$

$\big(F, [1]\big) := \texttt{BasisExtension}(V, E) : \sum F \in \texttt{LinearlyIndependent}(V) \,.\, F \cup E : \texttt{Basis}(V),$

$[2] := \mathcal{I}^{-1}\texttt{InnerDirectSum}\mathcal{I}\texttt{Basis}(V)(F \cup E) : V = U \oplus \operatorname{span}(F),$

$[*] := \mathcal{I}^{-1}\texttt{LinearComplement}[2] : (\operatorname{span}(F) : \texttt{LinearComplement}(V, U));$

$\square$

$\texttt{LinearComplementsDimAgree} :: \forall V \in k\text{-VS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, \forall W, W' : \texttt{LinearComplement}(V, U) \,.$

$\quad . \dim W = \dim W'$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}^2\texttt{LinearComplement}(V, U)(W)(W') : U \cong W = V = U \cong W',$

$[2] := \mathcal{I}\texttt{InnerDirectSum}[2]_1 : \pi_U + \pi_W : U \oplus W \xleftrightarrow{k\text{-VS}} V,$

$[3] := \mathcal{I}\texttt{InnerDirectSum}[2]_2 : \pi_U + \pi_{W'} : U \oplus W' \xleftrightarrow{k\text{-VS}} V,$

$T := (\pi_U + \pi_{W'})^{-1}_{|W}\pi_{W'} : W \xrightarrow{k\text{-MOD}} W',$

$\texttt{Assume}\; w : W,$

$\texttt{Assume}\; [w.1] : Tw = 0,$

$[w.2] := \mathcal{I}T[w.1] : w \in U,$

$[w.*] := \mathcal{I}\texttt{InnerDirectSum}[2]_1[w.2] : w = 0;$

$\rightsquigarrow [4] := \texttt{ZeroKernelTHM} : \big(T : \texttt{Injective}(W, W')\big),$

$\texttt{Assume}\; w' : W',$

$\big(w, u, [w'.1]\big) := \mathcal{I}\texttt{InnerSum}(V, U, W)(-w') : \sum w \in W \,.\, \sum u \in U \,.\, -w' = w + u,$

$[w'.2] := [w'.1] - w + w' : -w = w' + u,$

$[*] := \mathcal{I}T[w'.2] : T(-w) = w';$

$\rightsquigarrow [5] := \mathcal{I}^{-1}\texttt{Isomorphic}[4] : W \cong_{k\text{-VS}} W',$

$[6] := \texttt{EqRankTHM} : \dim W = \dim W';$

$\square$

$\texttt{codimension} :: \prod V \in k\text{-VS} \,.\, \texttt{VectorSubspace}(V) \to \mathbb{Z}_+$

$\texttt{codimension}\,(U) = \operatorname{codim} U := \dim W \quad \texttt{where} \quad W = \texttt{LinearComplementExists}(U, W)$

$\texttt{QuotientDirectSum} :: \forall V \in k\text{-VS} \,.\, \forall U \subset_{k\text{-VS}} V \,.\, V \cong_{k\text{-VS}} \dfrac{V}{U}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{QuotientDim} :: \forall V \in k\text{-VS} . \forall U \subset_{k\text{-VS}} V . \dim \frac{V}{U} = \operatorname{codim} U$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ProperVectorSubspace} :: \prod V \in k\text{-VS} . ?\texttt{VectorSubspace}(V)$

$U : \texttt{ProperVectorSubspace} \iff U \subsetneq_{k\text{-VS}} V \iff U \neq V$

$\texttt{ZeroIntersecting} :: \prod V \in k\text{-VS} . ??\texttt{ProperVectorSubspace}(V)$

$S : \texttt{ZerpIntersecting} \iff \forall A, B \in S . A \cap B = \{0\}$

$\texttt{CardOfSubspaceUnionByField} :: \forall V \in k\text{-VS} . \forall S : \texttt{ZeroIntersecting}(V) . \forall [0] : V = \bigcup S . |S| \geq |k|$

$\texttt{Proof} =$

$\big(U, [1]\big) := \mathcal{I} S[0] : \sum U \in S.U \neq \{0\},$

$\big(v, [2]\big) := \mathcal{I}\texttt{ProperVectorSubspace}(v) : \sum v \in V . v \notin U,$

$(u, [3]) := \mathcal{I} U[1] : \sum u \in U . u \neq 0,$

$\texttt{Assume } x, y : k,$

$\texttt{Assume } [4] : x \neq y,$

$(A, B, [5]) := [0](xv + u, yv + u) : \sum A, B \in S . xv + u \in Ayv + u \in B,$

$\texttt{Assume } [6] : A = B,$

$[6.1] := [5][6]\mathcal{I}\texttt{VectorSubspace}(V)(A) : \frac{xv + u - yv - u}{v - u} = v \in A,$

$[6.2] := [5][6]\mathcal{I}\texttt{VectorSuvspace}(V)(A) : \frac{y(xv + u) - x(yv + u)}{y - x} = u \in A,$

$[6.3] := \mathcal{I}\texttt{ZeroIntersecting}(V)(S)[6.2]\mathcal{I} u \mathcal{I} : A = U,$

$[6.*] := [6.3][2] : \perp;$

$\rightsquigarrow [(x, y).*] := E(\perp) : A \neq B;$

$\rightsquigarrow [*] := \texttt{InjCard} : |S| \geq |k|;$

$\square$

$\texttt{EqByDim} :: \forall V \in k\text{-FDVS} . \forall U \subset_{k\text{-VS}} . [0] : \dim U = \dim V \Rightarrow U = V$

$\texttt{Proof} =$

$(W, [1]) := \texttt{LinearComplementExists} : \texttt{LinearComplement}(V, U),$

$[2] := \texttt{InnerDirectSumDim}[1] : \dim V = \dim U + \dim W,$

$[3] := [2][0]\mathcal{I} k\text{-FDVS}(V) : \dim W = 0,$

$[*] := \texttt{VectorSpaceTorsionFree}[3]\mathcal{I} U : U = V;$

$\square$

$\texttt{CardOfLinearComplements} :: \forall V \in k\text{-VS} . \forall U \subsetneq_{k\text{-VS}} V . \forall [0] : U \neq \{0\} \# \texttt{LinearComplement}(U, V) \geq |k|$

$\texttt{Proof} =$

$W := \texttt{LinearComplementExists}(V)(U) : \texttt{LinearComplement}(V, U),$

$E := \texttt{HamelBasisTHM}(W) : \texttt{Basis}(W),$

$(\leq) := \texttt{WellOrderingExists} : \texttt{WellOrdering}(E),$

$e := \min E : E,$

$(u, [1]) := \text{\textit{Д}}\texttt{Singleton}[0] : \sum u \in U . u \neq 0,$

$\texttt{Assume } t : k,$

$W_t := \operatorname{span}(\texttt{swapIn}(E, e, e + tu)) : \texttt{VectorSubspace}(V),$

$[t.1] := \text{\textit{Д}} W_t \text{\textit{Д}}\texttt{LinearComplement}(V, U) : W_t + U = V,$

$\texttt{Assume } v : U \cap W_t,$

$\big(w, \alpha, [v.1]\big) := \text{\textit{Д}} W_t \text{\textit{Д}} v : \sum w \in W . \sum \alpha \in k . v = \alpha t u + w,$

$[v.2] := [v.1] - \alpha t u : w = v - \alpha t u \in U,$

$[v.3] := \text{\textit{Д}}\texttt{InnerDirectSum}(V, W, U)[v.2] : w = 0,$

$[v.*] := \text{\textit{Д}} v \text{\textit{Д}}(w, \alpha)[v.3] : v = 0;$

$\rightsquigarrow [t.*] := \text{\textit{Д}}^{-1}\texttt{LinearComplement} : (W_t : \texttt{LinearComplement}(V, U));$

$\rightsquigarrow W := I(\rightarrow) : k \rightarrow \texttt{LinearComplement}(V, U),$

$\texttt{Assume } x, y : k,$

$\texttt{Assume } [2] : x \neq y,$

$\texttt{Assume } [3] : W_x = W_y,$

$(w, [3.1]) := \text{\textit{Д}} W_x \text{\textit{Д}} W_y : \sum w \in W . \sum \alpha \in k . xu + e = \alpha y u + w,$

$[3.2] := \text{\textit{Д}} : (x - \alpha y)u = w - e \in W,$

$[3.3] := \text{\textit{Д}}\texttt{InnerDirectSum}(V, U, W)[3.2] : x - \alpha y = 0,$

$[3.4] := \text{\textit{Д}}\texttt{Field}(k)[3.3] : \alpha = \frac{x}{y},$

$(w', [3.5]) := [3.4][3.3] : \sum w' \in \operatorname{span}(E \setminus \{e\}) . e = \frac{x}{y} e + w',$

$[3.6] := \text{\textit{Д}}\texttt{Basis}(W, E) : x = y,$

$[3.*] := [2][3.6] : \bot;$

$\rightsquigarrow [*] := \texttt{InjCard} : \#\texttt{LinearComplement}(V, U) \geq |k|;$

$\square$

1.3 Linear Maps between Vector Spaces and their Matrices

$$\texttt{rank} :: \prod V, W : k\text{-VS} \ . \ V \xrightarrow{k\text{-VS}} W \to \mathsf{CARD}$$
$$\texttt{rank}\,(T) = \operatorname{rank} T := \dim \operatorname{Im} T$$

$$\texttt{columnRank} :: \prod \kappa, \kappa' \in \mathsf{CARD} \ . \ k^{\kappa\times\kappa'} \to \mathsf{CARD}$$
$$\texttt{columnRank}\,(A) := \dim \operatorname{span}\Big(\mathcal{C}(A)\Big)$$

$$\texttt{rowRank} :: \prod \kappa, \kappa' \in \mathsf{CARD} \ . \ k^{\kappa\times\kappa'} \to \mathsf{CARD}$$
$$\texttt{rowRank}\,(A) := \dim \operatorname{span}\Big(\mathcal{R}(A)\Big)$$

$\texttt{RankByColumnRank} :: \forall V, W : k\text{-VS} \ . \ \forall T : V \xrightarrow{k\text{-VS}} W \ . \ \forall e : \texttt{Basis}(V) \ . \ \forall f : \texttt{Basis}(W) \ .$

$\quad . \ \operatorname{rank} T = \texttt{columnRank}(T^{e,f})$

$\texttt{Proof} \ =$

$\kappa := \dim V : \mathsf{CARD},$

$\kappa' := \dim W : \mathsf{CARD},$

$C := \texttt{HamelBasisTHM}\Big(\operatorname{span}\big(\mathcal{C}(T^{e,f})\big)\Big) : \texttt{Basis}\Big(\operatorname{span}\big(\mathcal{C}(T^{e,f})\big)\Big),$

$\theta := |C| : \mathsf{CARD},$

$\texttt{Assume}\ c : C,$

$A(c) := cf : W,$

$(\alpha_C, [1]) := \text{Ɑ} C(c) : \sum \alpha \in k^{\oplus\kappa} \ . \ c = \alpha\mathcal{C}(T^{e,f}),$

$[2] := \text{Ɑ} A(c) \text{Ɑ} \texttt{matrixOfLinearTransformation}(e, f, T)[1] \text{Ɑ} T^{e,f} \text{Ɑ} k\text{-VS}(V, W)(T) :$

$\quad : A(c) = cf = \alpha\mathcal{C}(T^{e,f})f = \alpha Te = T\alpha e,$

$[*] := \text{Ɑ}^{-1} \operatorname{Im} T[2] : A(c) \in \operatorname{Im} T;$

$\leadsto A := \texttt{FreeFunctorAdjoint} : \operatorname{span}\big(\mathcal{C}(T^{e,f})\big) \xrightarrow{k\text{-VS}} \operatorname{Im} T,$

$\texttt{Assume}\ \alpha : \operatorname{span}\big(\mathcal{C}(T^{e,f})\big),$

$\big(\beta, [0]\big) := \text{Ɑ}\texttt{Basis}(C)(\alpha) : \sum \beta \in k^{\oplus\theta} \ . \ \alpha = \beta C,$

$\texttt{Assume}\ [1] : A(\alpha) = 0,$

$[2] := [1][0]\text{Ɑ} A : 0 = A(\alpha) = A(\beta C) = \beta A(C) = \beta_c c_j f_j,$

$[3] := \text{Ɑ}\texttt{Basis}(f)[2] : \forall j \in \kappa' \ . \ \beta_c c_j = 0,$

$[4] := \text{Ɑ} k^{\oplus\kappa'}[3] : \beta c = 0,$

$[5] := \text{Ɑ}\texttt{Basis}(c)[4] : \beta = 0,$

$[*] := [0][5] : \alpha = 0;$

$\leadsto [1] := \text{Ɑ}^{-1}\texttt{Iso}(k\text{-VS})\texttt{zeroKernelTHM} : \Big(\operatorname{span}\big(\mathcal{C}(T^{f,e})\big) : V \xleftrightarrow{k\text{-VS}} \operatorname{Im} T\Big),$

$[*] := \texttt{IsoRank}(k\text{-VS})[1] : \dim \operatorname{Im} T = \dim \operatorname{span}\big(\mathcal{C}(T^{f,e})\big);$

$\square$

$\texttt{InvertiblePresevesRank} :: \forall T : V \xrightarrow{k\text{-}\mathsf{VS}} W \ . \ \forall A \in \mathrm{Aut}_{k\text{-}\mathsf{VS}}(V) \ . \ \forall B \in \mathrm{Aut}_{k\text{-}\mathsf{VS}}(V) \ .$
$\quad \operatorname{rank} ATB = \operatorname{rank} T$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{GLPreservesColumnRank} :: \forall n, m \in \mathbb{N} \ . \ \forall T \in k^{n \times m} \ . \ \forall A \in \mathbf{GL}(k, n) \ . \ \forall B \in \mathbf{GL}(k, m) \ .$
$\quad \texttt{columnRank}(ATB) = \texttt{columnRank}(T)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{RowRankByTranspose} :: \forall n, m \in \mathbb{N} \ . \ \forall T \in k^{n \times m} \ . \ \texttt{rowRank}(T) = \texttt{columRank}(T^\top)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{GLPresevesRowRank} :: \forall n, m \in \mathbb{N} \ . \ \forall T \in k^{n \times m} \ . \ \forall A \in \mathbf{GL}(k, n) \ . \ \forall B \in \mathbf{GL}(k, m) \ .$
$\quad \texttt{rowRank}(ATB) = \texttt{rowRank}(T)$

$\texttt{Proof} \ =$

$[1] := \texttt{TransposeInv}(A) : A^\top \in GL(k, n),$

$[2] := \texttt{TrasposeInv}(B) : B^\top \in GL(k, n),$

$[*] := \texttt{RowRankByTranspose}(ATB)\texttt{TransposeMult}\text{Ɑ}\texttt{GLPreservesColumRank}(n, m, T, A, B)[1][2]$
$\quad \texttt{RowRankByTranspose}(T) : \texttt{rowRank}(ATB) = \texttt{columnRank}(B^\top T^\top B^\top) = \texttt{columnRank}T^\top - \texttt{rowRank}(T),$

$\square$

$\texttt{RowRankEqualsColumnRank} :: \forall n, m \in \mathbb{N} \ . \ \forall T \in K^{n \times m} \ . \ \texttt{columnRank}(T) = \texttt{rowRank}(T)$

$\texttt{Proof} \ =$

$\big(A, E, E', [1]\big) := \texttt{SmithNormalFormTheorem}(T) :$
$\quad : \sum A : \texttt{SmithNormalForm}(k, n, m) \ . \ \sum E \in \mathbf{GL}(k, m) \ . \ \sum E' \in \mathbf{GL}(k, n) \ . \ ETE' = A,$

$[2] := \text{Ɑ}\texttt{Field}(k)\text{Ɑ}\texttt{SmithNormalForm}(n, m, A)\text{Ɑ}^{-1}\texttt{rowRank}\text{Ɑ}^{-1}\texttt{columnRank} : \texttt{rowRank}(A) = \texttt{columnRank}(A),$

$[3] := \texttt{GLPreservesRowRank}(n, m, T, E, E')[1] : \texttt{rowRank}(T) = \texttt{rowRank}(A),$

$[4] := \texttt{GLPreservesColumnRank}(n, m, T, E, E')[1] : \texttt{columnRank}(T) = \texttt{columnRank}(A),$

$[5] := [3][2][4] : \texttt{rowRank}(T) = \texttt{columnRank}(A);$

$\square$

$\texttt{matrixRank} :: \prod n, m \in \mathbb{N} \ . \ k^{n \times m} \Rightarrow \min(n, m)$

$\texttt{matrixRank}\,(T) = \operatorname{rank} T := \texttt{columnRank}(T)$

$\texttt{TransposePreservesRank} :: \forall n, m \in \mathbb{N} \ . \ \forall T \in k^{n \times m} \ . \ \operatorname{rank} T^\top = \operatorname{rank} T$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{DualPreseresRank} :: \forall V, W \in k\text{-FDVS} . \forall T \in V \xrightarrow{k\text{-VS}} W . \ \operatorname{rank} T^* = \operatorname{rank} T$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{KernelComplementIsImage} :: \forall V, W \in k\text{-VS} . \forall T \in V \xrightarrow{k\text{-VS}} W .$

$\quad . \forall U : \texttt{LinearComplement}(V, \ker T) . \ U \cong_{k\text{-VS}} \operatorname{Im} T$

$\texttt{Proof} \ =$

$[1] := \texttt{ZeroKernelTHM}(T_{|U}) : (T_{|U} : U \hookrightarrow \operatorname{Im} T),$

$\texttt{Assume} \ w : \operatorname{Im} T,$

$(v, [2]) := \mathcal{C}\!\!\!\mathcal{I} \operatorname{Im} T(w) : \sum v \in V . \ w = Tv,$

$(u, z, [3]) := \mathcal{C}\!\!\!\mathcal{I}\, \texttt{InnerDirectSum}(\ker T, U) : \sum u \in U . \ \sum z \in \ker T . \ v = u + z,$

$[4] := \mathcal{C}\!\!\!\mathcal{I} T_{|U} u + 0 \mathcal{C}\!\!\!\mathcal{I} \ker T(z) \mathcal{C}\!\!\!\mathcal{I} k\text{-VS}(V, W)(T)[3][2] : T_{|U} u = Tu + 0 = Tu + Tz = Tv = w,$

$[5] := \mathcal{C}\!\!\!\mathcal{I}^{-1} \operatorname{Im} T_{|U}[4] : v \in \operatorname{Im} T_{|U};$

$\rightsquigarrow [*] := \mathcal{C}\!\!\!\mathcal{I}^{-1} \texttt{Iso}[1] : \operatorname{Im} T \cong_{k\text{-VS}} U;$

$\square$

$\texttt{RankPlusNullityTHM} :: \forall V, W \in k\text{-VS} . \forall T \in V \xrightarrow{k\text{-VS}} W . \ \dim V = \operatorname{rank} T + \dim \ker T$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FDMorphismDeterminism} :: \forall V, W \in k\text{-FDVS} . \forall T \in V \xrightarrow{k\text{-FDVS}} W . \forall [0] : \dim V = \dim W .$

$\quad . \ T : V \hookrightarrow W \iff T :\twoheadrightarrow W$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{InvertibleByRank} :: \forall n \in \mathbb{N} . \forall A \in k^{n \times n} . \forall [0] : \operatorname{rank} A = n . \ A \in \mathbf{GL}(k, n)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{InvertibleByDet} :: \forall n \in \mathbb{N} . \forall A \in k^{n \times n} . \forall [0] : \det A \neq 0 . \ A \in \mathbf{GL}(k, n)$

$\texttt{Proof} \ =$

$\big(A, E, E', [1]\big) := \texttt{SmithNormalFormTheorem}(T) :$

$\quad : \sum A : \texttt{SmithNormalForm}(k, n, m) . \ \sum E \in \mathbf{GL}(k, m) . \ \sum E' \in \mathbf{GL}(k, n) . \ ETE' = A,$

$[2] := \texttt{detProductInvertibleDet}\mathcal{C}\!\!\!\mathcal{I}\texttt{Field}(k)[1] : 0 \neq \det E \det T \det E' = \det A,$

$[3] := \mathcal{C}\!\!\!\mathcal{I}\texttt{SmithNormalForm}(A) \mathcal{C}\!\!\!\mathcal{I} \det[2] : \operatorname{rank} A = n,$

$[4] := \texttt{GLPreservesColumnRank}[3] : \operatorname{rank} T = n,$

$[*] := \texttt{InvertibleByRank}[4] : T \in \mathbf{GL}(n, k);$

$\square$

$\texttt{InvertibleByDet2} :: \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, \forall [0] : \det A \neq 0 \,.\, A \in \mathrm{Aut}_{k\text{-}\mathsf{VS}}(V)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PowerOfConstantRank} :: \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\; \operatorname{rank} T = \operatorname{rank} T^2 \iff \ker T \cap \operatorname{Im} T = \{0\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \operatorname{rank} T = \operatorname{rank} T^2,$

$\texttt{Assume}\ v : \ker T \cap \operatorname{Im} T,$

$(w, [1.1]) := \mathcal{I} \operatorname{Im} T(v) : \sum w \in V \,.\, v = Tw,$

$[1.2] := \mathcal{I} \ker T(v)[1.1] : T^2 w = Tv = 0,$

$[1.3] := \mathcal{I}^{-1} \ker T^2 [1.2] : w \in \ker T^2,$

$[1.4] := \texttt{RankPlusNullityTHM}(V, T)(V, T^2)[1][1.] : \dim \ker T = \operatorname{dom} \ker T^2,$

$[1.5] := \texttt{CompositionKernel}(T, T) : \ker T^2 \subset \ker T,$

$[1.6] := \texttt{EqByDim}[1.5][1.5] : \ker T = \ker T^2,$

$[1.7] := [1.3][1.6] : w \in \ker T,$

$[1.*] := [1.1]\mathcal{I} \ker T(w) : v = Tw = 0;$

$\texttt{Assume}\ [2] : \ker T \cap \operatorname{Im} T = \{0\},$

$[2.1] := \mathcal{I}^{-1} \texttt{InnerDirectSum}[2]\texttt{EqByDim}(\texttt{DirectSumDim}(\ker T, \operatorname{Im} T, [2])) : V = \ker T \oplus \operatorname{Im} T,$

$[2.2] := \texttt{FirstIsoTHM}[2.1]\mathcal{I}\,\texttt{compose}(T, T) : T_{|\operatorname{Im} T} : \operatorname{Im} T \xleftrightarrow{k\text{-}\mathsf{VS}} \operatorname{Im} T^2,$

$[2.3] := \mathcal{I} \operatorname{rank}[2.2] : \operatorname{rank} T = \operatorname{rank} T^2;$

$\square$

$\texttt{SurjectionDontIncreaseDim} :: \forall V, W \in k\text{-}\mathsf{FDVS} \,.\, \forall T : V \xrightarrow{k\text{-}\mathsf{FDVS}} W \,.\, \forall [0] : T : V \twoheadrightarrow W \,.\; \dim W \leq \dim V$

$\texttt{Proof} \ =$

$[*] := [0]\mathcal{I}^{-1} \operatorname{rank} \texttt{RankPlusNullityTHM}(T)\texttt{NonIncreasingAddition}(\mathbb{Z}_+)(\dim \ker T, \dim V) - \dim \ker T :$

$\quad : \dim W = \dim \operatorname{Im} T = \operatorname{rank} T = \dim V - \dim \ker T \leq \dim V;$

$\square$

$\texttt{CompositionRankBound} :: \forall V, W, U \in k\text{-}\mathsf{FDVS} \,.\, \forall A : V \xrightarrow{k\text{-}\mathsf{FDVS}} W \,.\, \forall B : W \xrightarrow{k\text{-}\mathsf{FDVS}} U \,.$

$\quad .\ \operatorname{rank} AB \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$

$\texttt{Proof} \ =$

$[1] := \texttt{CompositionImage} : \operatorname{Im} AB \subset \operatorname{Im} B,$

$[2] := \texttt{SubspaceDim}[1]\mathcal{I}^{-1} \operatorname{rank} : \operatorname{rank} AB \leq \operatorname{rank} B,$

$[3] := \texttt{SurjictiveToIm}(B) : (B_{|\operatorname{Im} A} : \operatorname{Im} A \twoheadrightarrow \operatorname{Im} AB),$

$[4] := \texttt{SurjectionDontIncreaseDim}[3]\mathcal{I}^{-1} \operatorname{rank} : \operatorname{rank} AB \leq \operatorname{rank} A,$

$[*] := \texttt{MinimalBound}([2], [4]) : \operatorname{rank} AB \leq \min(\operatorname{rank} A, \operatorname{rank} B);$

$\square$

$\texttt{CompositionNullityBound} :: \forall V, W, U \in k\text{-FDVS} \,.\, \forall A : V \xrightarrow{k\text{-FDVS}} W \,.\, \forall B : W \xrightarrow{k\text{-FDVS}} U \,.$
$\quad .\ \dim\ker AB \le \dim\ker A + \dim\ker B$

$\texttt{Proof} \;=$

$Z := \ker A \oplus (\ker B \cap \operatorname{Im} A) : k\text{-FDVS},$

$[1] := \texttt{InnerDirectSumDim}(Z)\texttt{SubspaceDim} : \dim Z \le \dim\ker A + \dim\ker B,$

$(C, [2]) := \texttt{SurjectiveToIm}(A)\texttt{SurjectiveHasLeftInverse}(A) : \sum C : \operatorname{Im} A \xrightarrow{k\text{-VS}} V \,.\, CA = \mathrm{id},$

$T := \mathrm{id} \oplus C_{|\ker B \cap \operatorname{Im} A} : Z \xrightarrow{k\text{-VS}} \ker AB,$

$\texttt{Assume}\ v : \ker AB,$

$[v.1] := \mathcal{I} v[v.2] : Av \in \ker T \cap \operatorname{Im} A,$

$(x, [v.2]) := \mathcal{I} C[v.2.1] : \sum x \in \ker A \,.\, v = CAv + x,$

$[v.3] := \mathcal{I} T(x, Av)[v.2] : T(x, Av) = x + CAv = v,$

$[v.4] := \mathcal{I}^{-1} \operatorname{Im} T[v.3] : v \in \operatorname{Im} T;$

$\rightsquigarrow [3] := \mathcal{I}^{-1}\texttt{Surjective} : T : Z \twoheadrightarrow \ker AB,$

$[*] := \texttt{SurjectionDontIncreaseDimension}[3][1] : \dim\ker AB \le \dim\ker A + \dim\ker B;$

$\square$

$\texttt{SumRankBound} :: \forall V, W \in k\text{-FDVS} \,.\, \forall A, B : V \xrightarrow{k\text{-FDVS}} W \,.\ \operatorname{rank}(A + B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}\texttt{sum} : \operatorname{Im}(A + B) \subset_{k\text{-VS}} \operatorname{Im} A + \operatorname{Im} B,$

$[*] := \mathcal{I}^{-1} \operatorname{rank} \texttt{SubspaceDimSumDimTHM}(\operatorname{Im} A, \operatorname{Im} B)\mathcal{I}^{-1} \operatorname{rank} :$
$\quad : \operatorname{rank}(A + B) \le \dim \operatorname{Im} A + \operatorname{Im} B \le \operatorname{rank}(A) + \operatorname{rank}(B);$

$\square$

$\texttt{InvariantSubspace} :: \prod V \in k\text{-VS} \,.\, \mathrm{End}_{k\text{-VS}}(V) \to ?\texttt{VectorSubspace}(V)$

$U : \texttt{InvariantSubspace} \iff \Lambda T \in \mathrm{End}_{k\text{-VS}} \,.\, T(U) \subset U$

$\texttt{restrictOperator} :: \prod V \in k\text{-VS} \,.\, \prod T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, \prod U : \texttt{InvariantSubspace}(V, T) \,.\, \mathrm{End}_{k\text{-VS}}(U)$

$\texttt{restrictOperator}\,() = T|_U := T_{|U}^{|U}$

$\texttt{ReducingPair} :: \prod V \in k\text{-VS} \,.\, \prod T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, ?\texttt{InvariantSubspace}^2(V, T)$

$(U, W) : \texttt{ReducingPair} \iff T = T|_U \boxplus T|_W \iff V = U \oplus W$

$\texttt{Irreducible} :: ?\sum V \in k\text{-VS} \,.\, ?\mathrm{End}_{k\text{-VS}}(V)$

$(V, \mathcal{A}) : \texttt{Irreducible} \iff V : \mathcal{A}\text{-}\texttt{Irreducible}(k) \iff \forall U : \texttt{InvariantSubspace}(V, \mathcal{A}) \,.\, U = \{0\} | U = V$

$\texttt{SchursLemma} :: \forall V : \mathcal{A}\text{-}\texttt{Irreducible}(k) \,.\, \forall W : \mathcal{B}\text{-}\texttt{Irreducible}(k) \,.\, \forall T : V \xrightarrow{k\text{-VS}} W \,.$

$\quad .\, \forall [0] : \mathcal{A}T = T\mathcal{B} \,.\, T = 0 | T : V \xleftrightarrow{k\text{-VS}} W$

$\texttt{Proof} \;=$

$\texttt{Assume}\ v : \ker T,$

$\texttt{Assume}\ A : \mathcal{A},$

$(B, [A.1]) := [0](A) : \sum B \in \mathcal{B} \,.\, AT = TB,$

$[A.2] := [A.1] ◁ \ker T(v) ◁ k\text{-VS}(W, W)(B) : TA(v) = BT(v) = B(0) = 0,$

$[A.*] := ◁^{-1} \ker T[A.2] : Av \in \ker T;$

$\rightsquigarrow [1] := ◁^{-1}\texttt{InvariantSubspace}(V, \mathcal{A}) : (\ker T : \texttt{InvariantSubspace}(V, \mathcal{A})),$

$\texttt{Assume}\ w : \operatorname{Im} T,$

$\texttt{Assume}\ B : \mathcal{B},$

$(v, [B.0]) := ◁ \operatorname{Im} T(w) : \sum v \in V \,.\, w = Tv,$

$(A, [B.1]) := [0](B) : \sum A \in \mathcal{A} \,.\, AT = TB,$

$[B.2] := B[B.0][B.1] : Bw = BT(v) = TA(v),$

$[B.*] := ◁^{-1} \operatorname{Im} T[B.2] : Bw \in \operatorname{Im} T;$

$\rightsquigarrow [2] := ◁^{-1}\texttt{InvariantSubspace}(W, \mathcal{B}) : (\operatorname{Im} T : \texttt{InvariantSubspace}(W, \mathcal{B})),$

$[*] := ◁^{2}\texttt{Irreducible}(V, \mathcal{A})(W, \mathcal{B})[1][2] : \texttt{This};$

$\square$

$\texttt{DimOfOperators} :: \forall V \in k\text{-FDVS} \,.\, \dim \operatorname{End}_{k\text{-VS}}(V) = \dim^2 V$

$\texttt{Proof} \;=$

$(e_i)_{i=1}^{n} := \texttt{HamelBasisTHM}(V) : \texttt{Basis}(V),$

$E := \texttt{Free}\,(\Lambda i, j \in n \,.\, \Lambda k \in n \,.\, \texttt{if}\ i == k\ \texttt{then}\ e_j\ \texttt{else}\ 0) : n^2 \to \operatorname{End}_{k\text{-VS}}(V),$

$\texttt{Assume}\ T : \operatorname{End}_{k\text{-VS}}(V),$

$\texttt{Assume}\ i : n,$

$(a_i, [i.*]) := ◁\texttt{Basis}(Te_i) : \sum a \in k^n \,.\, a_i e = Te_i;$

$\rightsquigarrow a := I(\to) : n^2 \to T,$

$[T.*] := ◁ a ◁ \texttt{FreeModule}(V) : T = aE;$

$\rightsquigarrow [1] := ◁^{-1} \operatorname{span} : \operatorname{End}_{k\text{-VS}}(V) = \operatorname{span}(E),$

$\texttt{Assume}\ \alpha : k^{n \times n},$

$\texttt{Assume}\ [2] : \alpha E = 0,$

$\texttt{Assume}\ i : n,$

$[3] := [2] : 0 = \alpha E e_i = \alpha_i e,$

$[i.*] := ◁\texttt{Basis}(V)(e)(3) : \alpha_i = 0;$

$\rightsquigarrow [2.*] := ◁ \alpha : \alpha = 0;$

$\rightsquigarrow [*] := ◁^{-1}\texttt{Basis}[1] : \texttt{This};$

$\square$

1.4 Projection Operators

$\texttt{Projection} :: \prod V : k\text{-VS} . ?\mathrm{End}_{k\text{-VS}}(V)$

$P : \texttt{Projection} \iff P^2 = P$

$\texttt{StructureOfTheProjection} :: \forall V \in k\text{-VS} . \forall P : \texttt{Projection}(V) . \exists A, B \subset_{k\text{-VS}} V :$
$\quad : V = A \oplus B \ \&\ \forall a \in A . P(a) = a \ \&\ \forall b \in B . P(b) = 0$

$\texttt{Proof} \ =$

$A := \mathrm{Im}\, P : \texttt{VectorSubspace}(V),$

$B := \ker P : \texttt{VectorSubspace}(V),$

$\texttt{Assume}\ a : A,$

$(v, [1]) := ᗡ A ᗡ \,\mathrm{Im}\, P(a) : \sum v \in V . a = Pa,$

$[a.*] := [1]ᗡ\texttt{Projector}(P)[1] : Pa = P^2 v = Pv = a;$

$\rightsquigarrow [1] := I(\forall) : \forall a \in A . Pa = a,$

$\texttt{Assume}\ v : A \cap B,$

$[2] := [1](v) : Pv = v,$

$[3] := ᗡ B ᗡ \ker P(v) : Pv = 0,$

$[v.*] := [2][3] : v = 0;$

$\rightsquigarrow [2] := ᗡ^{-1}\texttt{Singleton}(V)(0) : A \cap B = \{0\},$

$\texttt{Assume}\ v : V,$

$[3] := I(=, Pv + v) - Pv : v = Pv + v - Pv,$

$[4] := ᗡ\mathsf{ABEL}(V, V)(P)[1](v) : P(v - Pv) = Pv - P^2 v = Pv - Pv = 0,$

$[5] := ᗡ^{-1} \ker P[4] : v - Pv \in \ker P,$

$[v.*] := [3][5]ᗡ^{-1} A ᗡ^{-1} B : v \in A \oplus B;$

$\rightsquigarrow [3] := ᗡ^{-1}\texttt{EqSubset} : V = A + B,$

$[*] := ᗡ^{-1}\texttt{InnerDirectSum}[2][3] : V = A \oplus B;$

$\square$

$\texttt{projectionOnAlong} :: \prod V \in k\text{-VS} . \sum A, B \subset_{k\text{-VS}} V . V = A \oplus B \to \texttt{Projector}(V)$

$\texttt{projecctionOnAlong}\,(A, B, [0]) = P_{A,B} := \Lambda a + b \in A \oplus B . a$

$\texttt{OrthogonalProjections} :: \prod V \in k\text{-VS} . ?\texttt{Projection}^2(V)$

$P, Q : \texttt{OrthogonalProjections} \iff P \perp Q \iff PQ = 0 = QP$

$\texttt{ResolutionOfIdentity} :: \prod V \in k\text{-VS} . \prod \kappa \in \mathsf{CARD} . ?(\kappa \to \texttt{Projection}(V))$

$P : \texttt{ResolutionOfIdentity} \iff \mathrm{id} = \sum_{i \in \kappa} P \ \&\ \forall i, j \in \kappa . i \neq j \Rightarrow P_i \perp P_j$

$\texttt{ResolutionOfIdentityTHM1} :: \forall P : \texttt{ResolutionOfIdentity}(V, n) \; . \; V = \bigoplus_{i \in n} \mathrm{Im}\, P_i$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : V,$

$[1] := \mathcal{Q}\texttt{ResolutionOfIdentity}(V, n)(P) : v = \sum_{i \in n} P_i v,$

$[v.*] := \mathcal{Q}^{-1}\, \mathrm{Im}\, P[1] : v \in \sum_{i \in n} \mathrm{Im}\, P_i;$

$\rightsquigarrow [1] := \mathcal{Q}^{-1}\texttt{SetEq} : V = \sum_{i \in n} \mathrm{Im}\, P_i,$

$\texttt{Assume}\; i : n,$

$\texttt{Assume}\; v : \mathrm{Im}\, P_i \cap \sum_{j \in n, j \neq i} \mathrm{Im}\, P_i,$

$[1] := \mathcal{Q}\texttt{ResolutionOfIdentity}(V, n)(P) : v = \sum_{j \in n} P_j v,$

$[2] := \texttt{StructureOfTheProjection}(P)(v) : v = P_i v,$

$[3] := [1][2] : 0 = \sum_{j \in n, j \neq i} P_j v,$

$\big(w, [4]\big) := \mathcal{Q}\texttt{intersect}\mathcal{Q}v\mathcal{Q}\, \mathrm{Im}\, P : \sum w : n \setminus \{i\} \to V \; . \; v = \sum_{j \in n, j \neq i} P_j w_j,$

$[v.*] := [3][4]\mathcal{Q}\texttt{ResolutionOfIdentity}(V, n)(P)\mathcal{Q}\texttt{OrthogonalProjector}(V)(P)\mathcal{Q}\texttt{Projector}(P)[4] :$
$: 0 = \sum_{j \in n, j \neq i} P_j v = \sum_{j,k \in n; j,k \neq i} P_j P_k w_k = \sum_{j \in n, j \neq i} P_j^2 w_j = \sum_{j \in n, j \neq i} P_j w_j = v;$

$\rightsquigarrow [*] := \mathcal{Q}^{-1}\texttt{InnerDirectSum} : V = \bigoplus_{i \in n} \mathrm{Im}\, P_i;$

$\square$

$\texttt{ResolutionOfIdentityTHM2} :: \forall V \in k\text{-}\mathsf{VS} \; . \; \forall n \in \mathsf{CARD} \; . \; \forall U : n \to \texttt{VectorSubspace}(V) \; .$
$. \; \forall [0] : V = \bigoplus_{i \in n} U_i \; . \; \Lambda i \in n \; . \; P_{U_i, \sum_{j \in n, j \neq i} U_j} : \texttt{ResolutionOfIdentity}(V, n)$

$\texttt{Proof} \;=$

$P := \Lambda i \in n \; . \; P_{U_i, \sum_{j \in n, j \neq i} U_j} : n \to \texttt{Projector}(V),$

$\texttt{Assume}\; v : V,$

$\big(u, [1]\big) := \mathcal{Q}\texttt{InnerDirectSum}[0](v) : \sum u : \prod i \in n \; . \; U_i \; . \; v = \sum_{i \in n} u_i,$

$[v.*] := [1]\mathcal{Q}\texttt{projectionOnAlong}(P)[1] : \sum_{i \in n} P_i v = \sum_{i \in n} P_i \sum_{j \in n} u_j = \sum_{i \in n} u_i = v;$

$\rightsquigarrow [1] := \mathcal{Q}^{-1}\mathrm{id} : \sum_{i \in n} P_i = \mathrm{id},$

$\texttt{Assume}\; i : n,$

$\texttt{Assume}\; j : n,$

$\texttt{Assume}\; [2] : i \neq j,$

$\texttt{Assume}\; v : V,$

$[v.1] := \mathcal{Q}\texttt{projectionOnAlong}(P)[2] : P_i P_j v = 0,$

$[v.2] := \mathcal{Q}\texttt{projectionOnAlong}(P)[2] : P_j P_i v = 0,$

$[i.*] := \mathcal{Q}^{-1}\texttt{OrthogonalProjection}[v.1][v.2] : P_j \perp P_i;$

$\rightsquigarrow [*] := \mathcal{Q}^{-1}\texttt{ResolutionOfIdentity}[1] : (P : \texttt{ResolutionOfIdentity}(n, V));$

$\square$

$\texttt{InvarianceByProjection} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, \forall U \subset_{k\text{-}\mathsf{VS}} V \,.$
$\quad . \, U : \texttt{InvariantSubspace}(V,T) \iff \forall W : \texttt{LinearComplement}(V,U) \,.\, P_{U,W}TP_{U,W} = TP_{U,W}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : (U : \texttt{InvariantSubspace}(V,T)),$

$\texttt{Assume}\ W : \texttt{LinearComplement}(V,U),$

$P := P_{U,W} : \texttt{Projector}(V),$

$[1] := \mathcal{U}\texttt{projectionOnAlong}(P) : \mathrm{Im}\, P \subset U,$

$[2] := \mathcal{U}\texttt{LinearComplement}(V,U)(T)[1] : \mathrm{Im}\, TP \subset U,$

$[1.*] := \mathcal{U}\texttt{projectionOnAlong}(P)[2] : PTP = TP;$

$\rightsquigarrow [1] := I(\Rightarrow) : (U : \texttt{InvariantSubspace}(V,T)) \Rightarrow \forall W : \texttt{LinearComplement}(V,U) \,.\, P_{U,W}TP_{U,W} = TP_{U,W},$

$\texttt{Assume}\ [2] : \forall W : \texttt{LinearComplement}(V,U) \,.\, P_{U,W}TP_{U,W} = TP_{U,W},$

$\texttt{Assume}\ u : U,$

$\texttt{Assume}\ [3] : Tu \notin U,$

$W := \texttt{LinearComplementExists}(U) : \texttt{LinearComplement}(V,U),$

$P := P_{U,W} : \texttt{Projector}(V),$

$[4] := [2](P) : PTP = TP,$

$[5] := \mathcal{U}^3\texttt{projectionAlong}(U,W)(P)(u)[3] : TPu = Tu \neq PTu = PTPu,$

$[2.*] := [5][4] : \perp;$

$\rightsquigarrow [3] := I(\iff)[1]\mathcal{U}^{-1}\texttt{InvariantSubspace} : \texttt{This};$

$\square$

$\texttt{ReducedPairByProjection} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, \forall U, W \subset_{k\text{-}\mathsf{VS}} V \,.\, \forall (0) : V = U \oplus W$
$\quad . \, (U,W) : \texttt{ReducingPair}(V,T) \iff P_{U,W}T = TP_{U,W}$

$\texttt{Proof} \ =$

$P := P_{U,W} : \texttt{Projector}(V),$

$\texttt{Assume}\ [1] : \Big((U,W); \texttt{ReducingPair}(V,T)\Big),$

$\texttt{Assume}\ v : V,$

$(u,w,[2]) := \mathcal{U}\texttt{InnerDirectSum}(V)(U,W)[0] : \sum u \in U \,.\, \sum w \in W \,.\, v = u + w;$

$[3] := [2]\mathcal{U}k\text{-}\mathsf{VS}(V,V)(PT)\mathcal{U}^2\texttt{projectionOnAlong}(P)\mathcal{U}^2\texttt{ReducingPair}(V,T)(U,W)\mathcal{U}k\text{-}\mathsf{VS}(V,V)(TP)[2] :$
$\quad : PTv = PT(u+w) = PTu + PTw = Tu = TPu = TPu + TPw = TP(u+w) = TPv,$

$\rightsquigarrow [1] := I(\Rightarrow) : (U,W) : \texttt{ReducingPair}(V,T) \Rightarrow PT = TP,$

$\texttt{Assume}\ [2] : PT = TP,$

$\texttt{Assume}\ u : U,$

$[u.*] := \mathcal{U}\texttt{projectionOnAlong}(V,U,W)(P)[2]\mathcal{U}\texttt{projectionOnAlong}(V,U,W)(P) : Tu = TPu = PTu \in U;$

$\rightsquigarrow [3] := \mathcal{U}^{-1}\texttt{InvariantSubspace} : \big(U : \texttt{InvariantSubspace}(V,T)\big),$

$\texttt{Assume}\ w : W,$

$[w.1] := \mathcal{U}k\text{-}\mathsf{VS}(V,V)(T)\mathcal{U}\texttt{projectionOnAlong}(V,U,W)(P)[2] : 0 = T0 = TPw = PTw,$

$[w.*] := \mathcal{U}\texttt{projectionOnAlong}(V,U,W)(P)[2] : Tw \in W;$

$\rightsquigarrow [4] := \mathcal{U}^{-1}\texttt{InvariantSubspace} : \big(W : \texttt{InvariantSubspace}(V,T)\big),$

$[2.*] := \mathcal{U}^{-1}\texttt{ReducingPair}[3][4] : \big((U,W) : \texttt{ReducingPair}(V,T)\big);$

$\rightsquigarrow [*] := I(\iff)[1] : \texttt{This};$

$\square$

$[-1] := \mathrm{char}\ k \neq 2 : \mathtt{Type};$

$\mathtt{ProjectionAlgebraI} :: \forall V : k\text{-}\mathsf{VS} \mathbin{.} \forall P, Q : \mathtt{Projector}(V) \mathbin{.} P + Q : \mathtt{Projection}(V) \iff P \perp Q$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [1] : P + Q : \mathtt{Projection}(V),$

$[2] := \mathcal{A}\mathtt{Projection}(P+Q) : P + Q = (P+Q)^2 = P^2 + PQ + QP + Q^2 = P + QP + PQ + Q,$

$[3] := [2] - P - Q : -QP = PQ,$

$[4] := [3]\mathcal{A}\mathtt{Projection}(P) : PQP = -QP^2 = -QP,$

$[5] := [3]\mathcal{A}\mathtt{Projection}(P)[3] : PQP = -P^2Q = -PQ = QP,$

$[6] := [-1][4][5] : QP = 0,$

$[7] := [3]\mathcal{A}\mathtt{Projection}(Q) : QPQ = -PQ^2 = -PQ,$

$[8] := [3]\mathcal{A}\mathtt{Projection}(Q)[3] : QPQ = -Q^2P = -QP = PQ,$

$[9] := [-1][8][7] : PQ = 0,$

$[1.*] := \mathcal{A}^{-1}\mathtt{OrthogonalProjections}[6][9] : P \perp Q;$

$\rightsquigarrow [1] := I(\Rightarrow) : P + Q : \mathtt{Projection}(V) \Rightarrow P \perp Q,$

$\mathtt{Assume}\ [2] : P \perp Q,$

$[2.1] := \mathcal{A}\mathtt{Projection}(V)(P,Q)\mathcal{A}\mathtt{OrthogonalProjection}(P,Q) : (P+Q)^2 = P^2 + PQ + QP + Q^2 = P + Q,$

$[2.*] := \mathcal{A}^{-1}\mathtt{Projection}(P+Q) : \big(P + Q : \mathtt{Projection}(V)\big);$

$\rightsquigarrow [*] := I(\iff)[1] : \mathtt{This},$

$\square$

$\mathtt{ProjectionAlgebraII} :: \forall V : k\text{-}\mathsf{VS} \mathbin{.} \forall P, Q : \mathtt{Projector}(V) \mathbin{.} P - Q : \mathtt{Projection}(V) \iff PQ = QP = Q$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ [1] : P - Q : \mathtt{Projection}(V),$

$[2] := \mathcal{A}\mathtt{Projection}(P-Q) : P - Q = (P-Q)^2 = P^2 - PQ - QP + Q^2 = P - QP - PQ + Q,$

$[3] := [2] - P - Q : 2Q = PQ + QP,$

$[4] := [3]\mathcal{A}\mathtt{Projection}(Q) : PQP = 2QP - Q^2P = QP,$

$[5] := [3]\mathcal{A}\mathtt{Projection}(P) : PQP = 2PQ - P^2Q = PQ,$

$[6] := [4][5] : QP = PQ,$

$[1.*] := [6][3] : QP = Q = PQ;$

$\rightsquigarrow [1] := I(\Rightarrow) : P - Q : \mathtt{Projection}(V) \Rightarrow PQ = QP = Q,$

$\mathtt{Assume}\ [2] : PQ = QP = Q,$

$[2.1] := \mathcal{A}\mathtt{Projection}(V)(P,Q)[2] : (P-Q)^2 = P^2 - PQ - QP - Q^2 = P - Q,$

$[2.*] := \mathcal{A}^{-1}\mathtt{Projection}(P-Q) : \big(P - Q : \mathtt{Projection}(V)\big);$

$\rightsquigarrow [*] := I(\iff)[1] : \mathtt{This},$

$\square$

$\mathtt{ProjectionAlgebraIII} :: \forall V : k\text{-}\mathsf{VS} \mathbin{.} \forall P, Q : \mathtt{Projector}(V) \mathbin{.} \forall [0] : PQ = QP \Rightarrow PQ : \mathtt{Projector}(V)$

$\mathtt{Proof}\ =$

$[1] := \mathcal{A}k\text{-}\mathsf{ALG}\big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V)\big)[0] : (PQ)^2 = PQPQ = P^2Q^2 = PQ,$

$[*] := \mathcal{A}^{-1}\mathtt{Projector}(V) : (PQ : \mathtt{Projector}(V));$

$\square$

1.5 Canonical Rational Form

$\texttt{moduleOfJordan} :: \prod V : k\text{-VS} \,.\, \mathrm{End}_{k\text{-VS}}(V) \to k[\mathbb{Z}_+]\text{-MOD}$

$\texttt{moduleOfJordan}\,(T) = V_T := \left(V, \Lambda f \in k[\mathbb{Z}_+] \,.\, \Lambda v \in V \,.\, \sum_{i=0}^{n} f_i T^i v\right)$

$\texttt{JordanModuleIsTorsion} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.\, V_T : \texttt{Torsion}\ k[\mathbb{Z}_+]$

$\texttt{Proof} =$

$\texttt{Assume}\ v : V,$

$(n, [1]) := \mathcal{A} k\text{-FDVS}(V)(\Lambda i \in \mathbb{Z}_+ \,.\, T^i v) : \sum n \in \mathbb{Z}_+ \,.\, (T^i v)_{i=0}^{n} \,!\, \texttt{LinearlyIndependent}(V),$

$(\alpha, [2]) := \mathcal{A}\texttt{LinearlyIndependent}(V)[1] : \sum \alpha \in k^n \,.\, \alpha_i T^i v = 0 \ \&\ \alpha \neq 0,$

$f := \sum_{i=0}^{n} \alpha_i x^i : k[\mathbb{Z}_+],$

$[3] := \mathcal{A} f[2] : fv = 0,$

$[4] := \mathcal{A}^{-1}\texttt{torsion}[3][2] : v \in \mathrm{tor}\, V_T;$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{Torsion} : V_T : \texttt{Torsion};$

$\square$

$\texttt{JordanSubmodulesAreInvariants} :: \forall V \in k\text{-VS} \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, \forall W \subset_{k[x]\text{-MOD}} V_T \,.$
$\quad .\, W : \texttt{InvariantSubspace}(V, T)$

$\texttt{Proof} =$

$\texttt{Assume}\ \alpha : k,$

$\texttt{Assume}\ w : W,$

$[\alpha.*] := \mathcal{A} V_T \mathcal{A}^{-1}\texttt{Submodule}(V_T, W) : \alpha w = \alpha \odot_{V_T} w \in W;$

$\rightsquigarrow [1] := \mathcal{A}^{-1}\texttt{VectorSubspace} : W \subset_{k\text{-VS}} V,$

$\texttt{Assume}\ w : W,$

$[*] := \mathcal{A} V_T \mathcal{A}^{-1}\texttt{Submodule}(V_T, W) : Tw = x \odot_T w \in W;$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{InvariantSubspace}[1] : (W : \texttt{InvariantSubspace}(V, T));$

$\square$

$\texttt{InvariantsAreJordanSubmodules} :: \forall V \in k\text{-VS} \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, \forall W : \texttt{InvariantSubspace}(V, T) \,.$
$\quad .\, W \subset_{k[x]\text{-MOD}} V_T$

$\texttt{Proof} =$

$\texttt{Assume}\ f : k[x],$

$n := \deg f : \mathbb{Z}_+ \cup \{-\infty\},$

$\texttt{Assume}\ w : W,$

$\texttt{Assume}\ i : n,$

$[i.*] := \mathcal{A}\texttt{InvariantSubspace}(V, T)(W)\mathcal{A}\texttt{VectorSubspace}(V)(W) : f_i T^I w \in W;$

$\rightsquigarrow [1] := I(\forall) : \forall i \in n \,.\, f_i T^i w \in W,$

$[f.*] := \mathcal{A} V_T[1]\mathcal{A}\texttt{Subgroup}(V)(W) : f \odot_T w = \sum_{i=0}^{n} f_i T^i w \in W;$

$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{Submodule} : W \subset_{k[x]\text{-MOD}} V_T;$

$\square$

$\texttt{MinimalPolynomialExists} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.\, \exists! f : \texttt{Monic}(k) \,.\, \mathcal{A}_T(V) = \langle f \rangle$

$\texttt{Proof} =$

$[1] := \texttt{PolynomialOverAFieldArePID} : (k[x] : \texttt{PrincipleIdealDomain}),$

$\big(f, [2]\big) := \sigma\texttt{PrincipleIdealDomain}(k[x])(\mathcal{A}_T(V)) : \sum f \in k[x] \,.\, \mathcal{A}_T(V) = \langle V \rangle,$

$n := \deg f : \mathbb{Z}_+ \cup \{-\infty\},$

$E := (T^n)_{n=0}^{\infty} : \mathbb{Z}_+ \to \mathrm{End}_{k\text{-VS}}(V),$

$[3] := \texttt{DimOfOperators}(V) : \dim \mathrm{End}_{k\text{-FDVS}}(V) = \dim^2 V,$

$[4] := \sigma \dim[3][4] : E \,!\, \texttt{LinearlyIndependent}\mathrm{End}_{k\text{-VS}(V)},$

$[5] := \sigma \mathcal{A}_f[4] : \mathcal{A}_f \neq \{0\},$

$[6] := \underline{\mathrm{n}}[2][5] : n \neq -\infty \;\&\; f \neq 0,$

$g := f_n^{-1} f : \texttt{Monic}(k),$

$[*] := \sigma g[2] : \mathcal{A}_T(V) = \langle g \rangle;$

$\square$

$\texttt{minimalPolynomial} :: \prod V \in k\text{-FDVS} \,.\, \mathrm{End}_{k\text{-FDVS}}(V) \to \texttt{Monic}(k)$

$\texttt{minimalPolynomial}\,(T) = m^T := \texttt{MinimalPolynomialExists}(V, T)$

$\texttt{MinimalPolynomialsOfSimmilarMatricesAgree} :: \prod V \in k\text{-FDVS} \,.\, \forall A, B \in \mathrm{End}_{k\text{-FDVS}}(V) \,.$

$\quad .\, A \sim B \Rightarrow m^A = m^B$

$\texttt{Proof} =$

$[1] := \texttt{AnnihilatorIdealsOfSimmilarMatricesAgree}(\ldots) : \mathcal{A}_A(V) = \mathcal{A}_B(V),$

$[*] := \sigma\texttt{genIdeal}\sigma\texttt{Monic}(k)[1] : m^A = m^B;$

$\square$

$\texttt{CyclicSubmoduleByBasis} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.\, \forall W \subset_{k[X]\text{-MOD}} V_T \,.$

$\quad .\, W : \texttt{Cyclic}(V_T) \iff \exists w \in W \,.\, (T^i w)_{i=0}^{\dim W - 1} : \texttt{Basis}(k, W)$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : [w : \texttt{Cyclic}(V_T)],$

$\big(w, [1]\big) := \sigma\texttt{Cyclic}(W) : \sum w \in W \,.\, W = \mathrm{span}\{w\},$

$E := \{T^i w | i \in \mathbb{Z}_+\} :? W,$

$[2] := \sigma V_T[1] : (E : \texttt{Generating}(W)),$

$[1.*] := \sigma\texttt{Minimal}(m^T(W))[2] : (T_i w)_{i=0}^{\dim W} : \texttt{Basis}(k, W);$

$\rightsquigarrow [1] := I(\to) : W : \texttt{Cyclic}(V_T) \Rightarrow \exists w \in W \,.\, (T^i w)_{i=0}^{\dim W - 1} : \texttt{Basis}(k, W),$

$\texttt{Assume}\ w : W,$

$\texttt{Assume}\ [2] : \Big((T^i w)_{i=0}^{\dim W - 1} : \texttt{Basis}(k, W)\Big),$

$[2.*] := \sigma^{-1}\texttt{Cyclic}[2] : (W : \texttt{Ceclic}(W));$

$\rightsquigarrow [*] := I(\iff)[1] : \texttt{This};$

$\square$

$\texttt{characteristicPolynomial} :: \prod V \in k\text{-FDVS} \,.\, \mathrm{End}_{k\text{-FDVS}}(V_T) \to k[x]$

$$\texttt{characteristPolynomial}\,(T) = \chi^T(V) := \prod_{i=1}^{n} \prod_{j=1}^{m_i} p_i^{t_{i,j}} \quad \texttt{where} \quad (n, m, t, p) : \texttt{primesOfMod}(V_T)$$

$\texttt{companionMatrix} :: \prod f : \texttt{Monic}(k) \,.\, k^{\deg f \times \deg f}$

$\texttt{companionMatrix}\,(f) = \mathbf{C}(f) := \texttt{FromColumns}\Big(\Lambda i \in \deg f \,.\, \texttt{if}\ i == \deg f\ \texttt{then}\ (f_{i-1})_{i=1}^{\deg f}\ \texttt{else}\ e_{i+1}\Big)$

$\texttt{RationalCanonicalForm} :: \prod n \in \mathbb{N} \,.\, k^{n\times n}$

$A : \texttt{RationalCanonicalForm} \iff \exists (m,p) : \texttt{Partition}(n) : \exists f : m \to \texttt{Monic}(k) :$

$\quad : A = \texttt{blockDiagonal}\Big(m, p, (\mathbf{C}(f_i))_{i=1}^m\Big)$

$\texttt{MinimalPolynomialOfCyclic} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, f \in \texttt{Monic}(k) \,.$

$\quad . \forall [0] : V_T \cong_{k[x]\text{-MOD}} \dfrac{k[x]}{\langle f(x) \rangle} \,.\, m^T(V) = f$

$\texttt{Proof} =$

$\varphi := \textit{Ɑ}\,\texttt{isomorphic}[0] : V_T \xleftrightarrow{k[x]\text{-MOD}} \dfrac{k[x]}{\langle f(x) \rangle},$

$[1] := \textit{Ɑ}\varphi\textit{Ɑ}\texttt{ringQuotient}(k[x], \langle f \rangle)\textit{Ɑ}\texttt{prinviple}(f) :$

$\quad : \varphi\Big(f(x) \odot_T (V_T)\Big) = f(x) \cdot \varphi(V_T) = f(x) \cdot \dfrac{k[x]}{\langle f(x) \rangle} = \{0\},$

$[2] := \textit{Ɑ}\,\texttt{Iso}k[x]\text{-MOD}(\varphi)[1] : f(x) \odot_T V_T = \{0\},$

$[3] := \textit{Ɑ}^{-1}\mathcal{A}_T[2] : f(x) \in \mathcal{A}_T,$

$\texttt{Assume}\ g(x) : \mathcal{A}_T,$

$[g.1] := \textit{Ɑ}^{-1}\varphi\textit{Ɑ}\,k[x]\text{-MOD}(\ldots)(\varphi)\textit{Ɑ}\,\mathcal{A}_T(g)\textit{Ɑ}\,\text{ABEL}(\ldots)(\varphi) :$

$\quad : g(x) \cdot \dfrac{k[x]}{\langle f(x) \rangle} = g(x) \cdot \varphi(V_T) = \varphi\Big(g(x) \odot_T V_T\Big) = \varphi\{0\} = \{0\},$

$[g.*] := \textit{Ɑ}\texttt{principle}\Big(\langle f(x) \rangle\Big)[g.1] : f(x) | g(x);$

$\rightsquigarrow [*] := \textit{Ɑ}^{-1}\texttt{MinimalPolynomial}[3] : m^T(V) = f;$

$\square$

$\texttt{RationalCanonicalFormTHM} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, \exists e : \texttt{Basis}(V) :$

$\quad : T^{e,e} : \texttt{RationalCanonicalForm}(k)$

$\texttt{Proof} \ =$

$\Big(n, m, p, t, [1]\Big) := \texttt{primesOfMod}(V_T) : \sum n \in \mathbb{N} \,.\, \sum m : n \to \mathbb{N} \,.\, \sum p : n :\hookrightarrow \texttt{Prime}\Big(k[x]\Big) \,.$

$\quad . \sum t : \prod i \in n \,.\, m_i \to \mathbb{N} \,.\, V_T \cong_{k[x]\text{-MOD}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \dfrac{k[x]}{\langle p_i^{t_{i,j}}(x) \rangle},$

$\varphi := \mathcal{I}\texttt{isomorphic}[1] : V_T \xleftrightarrow{k[x]\text{-MOD}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \dfrac{k[x]}{\langle p_i^{t_{i,j}}(x) \rangle},$

$\texttt{Assume}\ i : n,$

$\texttt{Assume}\ j : m_i,$

$U_{i,j} := \varphi^{-1}\pi_{i,j}^{-1}\left(\dfrac{k[x]}{\langle p^{t_{i,j}}(x)\rangle}\right) : k[x]\text{-MOD},$

$[j.1] := \texttt{JordanSubmodulesAreInvariants}(U_{i,j}) : (U_{i,j} : \texttt{InvariantSubspace}(V, T)),$

$f^{i,j} := \left(\dfrac{p_i}{\mathrm{lc}(p_i)}\right)^{t_{i,j}} : \texttt{Monic}(k),$

$\big(u, [j.2]\big) := \texttt{CyclicSubmoduleByBasis}(U_{i,j}) : \sum u \in U_{i,j} \,.\, (T^{l-1}k)_{l=1}^{\deg f^{i,j}} : \texttt{Basis}(k, V),$

$\varepsilon^{i,j} := (T^{l-1}k)_{l=1}^{\deg f} : \texttt{Basis}(k, V),$

$[j.3] := \texttt{MininalPolynomialOfCyclic}(U_{i,j})\mathcal{I} f : m^{T|_{U_{i,j}}} = f^{i,j},$

$[j.*] := \mathcal{I}^{-1}\texttt{CompanionMatrix}\mathcal{I}\varepsilon^{i,j}[j.3] : T|_{U_{i,j}}^{\varepsilon_{i,j},\varepsilon_{i,j}} = \mathbf{C}(-f^{i,j});$

$\rightsquigarrow \Big(U, \varepsilon, f[2]\Big) := I\left(\prod\right) I\left(\sum\right) : \prod i \in n \,.\, \prod j \in m_i \,.\, \sum U_{i,j} : \texttt{InvariantSubspace}(V, T) \,.$

$\quad . \sum \varepsilon^{i,j} : \texttt{Basis}(U_{i,j}) \,.\, \sum f^{i,j} : \texttt{MonicPolynomial}(k) \,.\, T|_{U_{i,j}}^{\varepsilon^{i,j},\varepsilon^{i,j}} = \mathbf{C}(-f^{i,j}),$

$[3] := \mathcal{I}U[1]\mathcal{I}V_T : V = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} U_{i,j},$

$[4] := \mathcal{I}^{-1}\texttt{ReducingSystem}[3] : T = \boxplus_{i=1}^{n} \boxplus_{j=1}^{m_i} T|_{U_{i,j}},$

$e := \coprod_{i=1}^{n} \coprod_{j=1}^{m_i} \varepsilon^{i,j} : \texttt{Basis}(V),$

$[5] := \texttt{InnerDirectSumDim}[3]\texttt{PolynomialQuotientDim}[1]\mathcal{I}^{-1}\texttt{Partition} :$

$\quad : \Big((n \otimes m, \deg f) : \texttt{Partition}(\dim V)\Big),$

$[6] := \texttt{ReducingSystemMatrix}[4][2][5] : T^{e,e} = \texttt{blockDiagonal}((n \otimes m, \deg f), \mathbf{C}(-f)),$

$[*] := \mathcal{I}^{-1}\texttt{RationalCanonicalForm}(k)[6] : \Big(T^{e,e} : \texttt{RationalCanonicalForm}(k)\Big);$

$\square$

$\texttt{CompanionMatrixDetLemma} :: \forall f : \texttt{Monic}(k) \,.\, \forall \lambda \in k \,.\, \det\Big(\lambda I - \mathbf{C}(f)\Big) = f(\lambda)$

$\texttt{Proof} \ =$

$☿ := \Lambda n \in \mathbb{N} \,.\, \forall f : \texttt{Monic}(k) \,.\, \deg f \leq n \Rightarrow \texttt{This}(f) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ \alpha : k,$

$\texttt{Assume}\ \lambda : k,$

$[\alpha.*] := \text{\textit{Ɑ}}\ \mathbf{C}(x-\alpha)\text{\textit{Ɑ}} \det : \det\Big(\lambda - \mathbf{C}(x-\alpha)\Big) = \lambda - \alpha;$

$\rightsquigarrow [1] := \text{\textit{Ɑ}}^{-1}☿1 : ☿(1),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [2] : ☿(n),$

$\texttt{Assume}\ f : \texttt{Monic}(k),$

$\texttt{Assume}\ [3] : \deg f = n + 1,$

$\texttt{Assume}\ \lambda : k,$

$g := x^n + \sum_{i=1}^{n-1} f_{i+1}x^i : \texttt{Monic}(k),$

$[n.*] := \texttt{DeterminantComputation}(\lambda I - \mathbf{C}(f))[2](g)\texttt{LowerTriangularDet}(\dots)$
$\quad \texttt{EvenPowerOfNegativeOne}(2n+2)\text{\textit{Ɑ}}g : \det(\lambda I - \mathbf{C}(f)) = \lambda\det(\lambda I - \mathbf{C}(g)) + (-1)^{n+2}(-1)^n f_0 =$
$\quad = \lambda g(\lambda) + f_0 = f(\lambda);$

$\rightsquigarrow [1] := \texttt{CompleteInduction} : \forall n \in \mathbb{N} \,.\, ☿(n),$

$[*] := \text{\textit{Ɑ}}☿[1] : \texttt{This};$

$\square$

$\texttt{CharPolynomialByDet} :: \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, \chi_T(\lambda) = \det(\lambda\,\mathrm{id} - T)$

$\texttt{Proof} \ =$

$(e,[1]) := \texttt{RationalCanonicalFormTHM}(V,T) : \sum e : \texttt{Basis}(V) \,.\, T^{e,e} : \texttt{RationalCanonicalForm}(k),$

$(n,f,[2]) := \text{\textit{Ɑ}}\texttt{RationalCanonicalForm}(T) : \sum n \in \mathbb{N} \,.\, \sum f : n \to \texttt{Monic}(k) \,.$
$\quad .\, T^{e,e} = \texttt{blockDiagonal}((n, \deg f), \mathbf{C}(f)),$

$[3] := \texttt{BlockDiagonalDet}([2])\texttt{CompanionMatrixDetLemma}(\mathbf{C}(f)) :$
$\quad : \det(\lambda\,\mathrm{id} - T)\det(\lambda I - T^{e,e}) = \prod_{i=1}^{n}(\lambda I - \mathbf{C}(f_i)) = \prod_{i=1}^{n} f_i(\lambda),$

$[*] := \text{\textit{Ɑ}}\texttt{RationalCanonicalFormTHM}(f)\text{\textit{Ɑ}}^{-1}\chi_T : \det(\lambda\,\mathrm{id} - T) = \chi_T(\lambda);$

$\square$

1.6 Canonical Jordan Form

$$\texttt{cellOfJordan} :: \prod n \in \mathbb{N} \,.\, k \to \texttt{UpperTriangularMatrix}(n,k)$$

$$\texttt{cellOfJordan}\,(\lambda) = \mathbf{J}(n,\lambda) := \texttt{fromColumns}\Big(\Lambda i \in n \,.\, \lambda e_i + e_{i-1}\Big) \quad \texttt{where} \quad e_0 = 0$$

$$\texttt{JordanCellMatrix} :: \forall V \in k\text{-}\mathsf{VS} \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V)\, \forall \lambda \in k \,.\, \forall n \in \mathbb{N} \,.$$
$$.\, \forall [0] : V_T \cong_{k[x]\text{-}\mathsf{MOD}} \frac{k[x]}{\big\langle (\lambda - x)^n \big\rangle} \,.\, \exists e : \texttt{Basis}(V) : T^{e,e} = \mathbf{J}(n,\lambda)$$

$\texttt{Proof} \ =$

$[1] := \texttt{MinimalPolynomialOfCyclic}[0] : m^T(x) = (\lambda - x)^n,$

$\big(v,[2]\big) := \mathcal{A}\texttt{minimalPolynomial}[1] : \sum v \in V : (\lambda\,\mathrm{id} - T)^{n-1} v \neq 0,$

$e_n := v : V,$

$[e.0.1] := \mathcal{A}\texttt{minimalPolynomial}[1](e_n) : (\lambda\,\mathrm{id} - T)^n v = 0,$

$[e.0.2] := [2]\mathcal{D}e_n : (\lambda\,\mathrm{id} - T)^{n-1} e_n \neq 0,$

$\texttt{Assume}\ m : n-1,$

$e_{n-m} := Te_{n+1-m} - \lambda e_{n+1-m} : V,$

$[e.m.1] := \mathcal{D}e_{n-m}[e.(m-1).1] : (\lambda\,\mathrm{id} - T)^{n-m} e_{n-m} = -(\lambda\,\mathrm{id} - T)^{n+1-m} e_{n+1-m} = 0,$

$[e.m.2] := \mathcal{D}e_{n-m}[e.(m-1).2] : (\lambda\,\mathrm{id} - T)^{n-m-1} e_{n-m} = -(\lambda\,\mathrm{id} - T)^{n-m} e_{n+1-m} \neq 0;$

$\rightsquigarrow \big(e[3]\big) := I\left(\sum\right) I\,(\forall) : \sum e : n \to V \,.\, \forall k \in n \,.\, (\lambda\,\mathrm{id} - T)^m e_m = 0 \,\&\, (\lambda\,\mathrm{id} - T)^{m-1} e_m \neq 0,$

$[4] := [3]_2(1) : e_1 \neq 0,$

$[e.0.3] := \mathcal{A}^{-1}\texttt{LinearlyIndependent}(V)[4] : \Big((e_i)_{i=1}^1 : \texttt{LinearlyIndependent}(V)\Big),$

$\texttt{Assume}\ m : n-1,$

$\texttt{Assume}\ [5] : \Big((e_i)_{i=1}^{m+1}\ \ !\ \ \texttt{LinearlyIndependent}(V)\Big),$

$\big(\alpha,[6]\big) := \mathcal{A}\texttt{LinearlyIndependent}(V)[5] : \sum \alpha \in k^{m+1} \,.\, \alpha e = 0 \,\&\, \alpha \neq 0,$

$[7] := [e.(m-1).3][6] : \alpha_{k+1} \neq 0,$

$$[8] := [7][6] : e_{m+1} = \sum_{i=1}^{m} -\frac{\alpha_i}{\alpha_{m+1}} e_i,$$

$$[9] := [3][8] : 0 \neq (\lambda\,\mathrm{id} - T)^m e_{m+1} = \sum \sum_{i+1}^{m} -(\lambda \underset{T}{\mathrm{id}})^m \frac{\alpha_i}{\alpha_{m+1}} e_i = 0,$$

$[5.*] := I(\bot)[9] : \bot;$

$\rightsquigarrow [e.m.3] := E(\bot) : \Big((e_i)_{i=1}^{m+1} : \texttt{LinearlyIndependent}(V)\Big);$

$\rightsquigarrow [5] := I(\forall)(n-1) : \Big((e_i)_{i=1}^{m+1} : \texttt{LinearlyIndependent}(V)\Big),$

$[6] := \texttt{PolynomialQuatienDim}\big(k,(\lambda - x)^n\big)\texttt{IsoDim}[0] : \dim V = n,$

$[7] := \texttt{EqByDim}(\mathrm{span}(e),V)[5][6]\mathcal{A}^{-1}\texttt{Basis} : (e : \texttt{Basis}(V)),$

$[8] := [3](1) + Te_1 : Te_1 = \lambda e_1,$

$\texttt{Assume}\ m : n-1,$

$[m.*] := \mathcal{D}(e_{m+1}) : Te_{m+1} = e_m + \lambda e_{m+1};$

$\rightsquigarrow [*] := \mathcal{D}^{-1}\,\mathbf{J}(n,\lambda)\mathcal{D}^{-1}\texttt{matrixOfOperator} : T^{e,e} = \mathbf{J}(n,\lambda);$

$\square$

$\texttt{geometricMultiplicity} :: \prod V \in k\text{-VS} . \prod T \in \mathrm{End}_{k\text{-VS}}(V) . \texttt{Eigenvalue}(T) \to \mathsf{CARD}$

$\texttt{geometricMultiplicity}(\lambda) = \bar{m}_T(\lambda) := \dim\ker(\lambda \operatorname{id} - T)$

$\texttt{algebraicMultiplicity} :: \prod V \in k\text{-VS} . \prod T \in \mathrm{End}_{k\text{-VS}}(V) . \texttt{Eigenvalue}(T) \to \mathsf{CARD}$

$\texttt{algebraicMultiplicity}(\lambda) = \tilde{m}_T(\lambda) := \texttt{mult}(\lambda, \chi^T)$

$\texttt{JordanCanonicalForm} :: \prod k : \texttt{Field} . ?\texttt{UpperTriangularMatrix}(k)$

$A : \texttt{JordanCanonicalForm} \iff \exists m \in \mathbb{N} : \exists n : m \to \mathbb{N} : \exists \lambda : m \to k : A = \texttt{blockDiagonal}(n, \mathbf{J}(n, \lambda))$

$\texttt{JordanCanonicalFormTHM} :: \forall k : \texttt{AlgebraicallyClosedField} . \forall V \in k\text{-FDVS} . \forall T \in \mathrm{End}_{k\text{-VS}}(V) .$

$. \exists e : \texttt{Basis}(V) : T^{e,e} : \texttt{JordanCanonicalForm}(k)$

$\texttt{Proof} =$

$\Big(n, m, p, t, [1]\Big) := \texttt{primesOfMod}(V_T) : \sum n \in \mathbb{N} . \sum m : n \to \mathbb{N} . \sum p : n :\hookrightarrow \texttt{Prime}\Big(k[x]\Big) .$

$. \sum t : \prod i \in n . m_i \to \mathbb{N} . V_T \cong_{k[x]\text{-MOD}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \frac{k[x]}{\langle p_i^{t_{i,j}}(x) \rangle},$

$\varphi := \mathcal{I}\texttt{isomorphic}[1] : V_T \xleftrightarrow{k[x]\text{-MOD}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \frac{k[x]}{\langle p_i^{t_{i,j}}(x) \rangle},$

$(\lambda, [2]) := \mathcal{I}\texttt{AlgebraicallyClosedField}(k)(p) : \sum \lambda : n \to k . \forall i \in n . p_i = (x - \lambda_i),$

$\texttt{Assume } i : n,$

$\texttt{Assume } j : m_i,$

$U_{i,j} := \varphi^{-1}\pi_{i,j}^{-1}\left(\frac{k[x]}{\langle p^{t_{i,j}}(x) \rangle}\right) : k[x]\text{-MOD},$

$[j.1] := \texttt{JordanSubmodulesAreInvariants}(U_{i,j}) : (U_{i,j} : \texttt{InvariantSubspace}(V, T)),$

$\Big(\varepsilon^{i,j}, [j,2]\Big) := \texttt{JordanCellMatrix}(U_{i,j}, T|_{U_{i,j}}) : \sum \varepsilon^{i,j} : \texttt{Basis}(k, U_{i,j}) . T|_{U_{i,j}}^{\varepsilon^{i,j},\varepsilon^{i,j}} = \mathbf{J}(t_{i,j}, \lambda_i),$

$\rightsquigarrow \Big(U, \varepsilon, [3]\Big) := I\Big(\prod\Big) I\Big(\sum\Big) : \prod i \in n . \prod j \in m_i . \sum U_{i,j} : \texttt{InvariantSubspace}(V, T) .$

$. \sum \varepsilon^{i,j} : \texttt{Basis}(U_{i,j}) . T|_{U_{i,j}}^{\varepsilon^{i,j},\varepsilon^{i,j}} = \mathbf{J}(t_{i,j}, \lambda_i),$

$[4] := \mathcal{I}U[1]\mathcal{I}V_T : V = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} U_{i,j},$

$[3] := \mathcal{I}^{-1}\texttt{ReducingSystem}[3] : T = \boxplus_{i=1}^{n} \boxplus_{j=1}^{m_i} T|_{U_{i,j}},$

$e := \coprod_{i=1}^{n} \coprod_{j=1}^{m_i} \varepsilon^{i,j} : \texttt{Basis}(V),$

$[5] := \texttt{InnerDirectSumDim}[3]\texttt{PolynomialQuotientDim}[1]\mathcal{I}^{-1}\texttt{Partition} :$

$: \Big((n \otimes m, t) : \texttt{Partition}(\dim V)\Big),$

$[6] := \texttt{ReducingSystemMatrix}[4][2][5] : T^{e,e} = \texttt{blockDiagonal}\Big((n \otimes m, t), \mathbf{J}(t, \lambda_{(\cdot)_1})\Big),$

$[*] := \mathcal{I}^{-1}\texttt{JordanCanonicalForm}(k)[6] : \Big(T^{e,e} : \texttt{JordanCanonicalForm}(k)\Big);$

$\square$

$\texttt{AlgebraicMultiplicityByKernel} :: \forall k : \texttt{AlgebraicallyClosedField} \,.\, \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.$

$$. \forall \lambda \in k \,.\, \tilde{m}_T(\lambda) = \dim \bigcup_{k=1}^{\infty} \ker(\lambda \,\mathrm{id} - T)^k$$

$\texttt{Proof} =$

...

$\square$

$\texttt{naturalMatrixCategory} :: \texttt{Field} \to \mathsf{SCAT}$

$\texttt{naturalMatrixCategory}\,(k) = k\text{-}\mathbb{N} := \mathbb{N}, (n, m) \mapsto k^{n \times m}, \texttt{matrixMult}, I$

$\texttt{cellStructureOfJordan} :: \prod k : \texttt{AlgebraicallyClosedField} \,.\, \mathrm{End}_{k\text{-FDVS}} \to ?^* \mathcal{M}_{k\text{-}\mathbb{N}}$

$\texttt{cellStructurOfJordan}\,(V, T) := \{\mathbf{J}(t_{i,j}, \lambda_i)\}$ **where** $(n, m, x - \lambda, t) = \texttt{primesOfMod}(V_T)$

$\texttt{cellStructureOfJordanOfSimmilarAgree} :: \forall k : \texttt{AlgebraicallyClosedField} \,.\, \forall V \in k\text{-FDVS} \,.$

$$. \forall A, B \in \mathrm{End}_{k\text{-FDVS}} A \sim B \iff \texttt{cellStructureOfJordan}(A) = \texttt{cellStructureJordan}(B)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{spectre} :: \prod k : \texttt{Field} \,.\, \prod V \in k\text{-FDVS} \,.\, \mathrm{End}_{k\text{-FDVS}}(V) \to k \to \mathbb{Z}$

$$\texttt{spectre}\,(T) = \mathrm{Spec}(T) = \sigma_T := \Lambda \lambda \in \bar{k} \,.\, \left| \{ i \in \dim V \,.\, \mathrm{diag}(T^{e,e}) = \lambda \} \right|$$

where $e = \texttt{JordanCanonicalFormTHM}(\bar{k}, V, T)$

$\texttt{SizeOfSpectre} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.\, \int_{\bar{k}} d\sigma_T = \dim V$

$\texttt{Proof} =$

...

$\square$

$\texttt{TraceBySpectre} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.\, \int_{\bar{k}} \lambda \, d\sigma_T(\lambda) = \mathrm{tr}\, T$

$\texttt{Proof} =$

$(e, [1]) := \texttt{JordanCanonicalFormTHM}(\bar{k}, V, k\text{-VS}) : \sum e : \texttt{Basis}(\bar{k} \otimes V) \,.\, T^{e,e} : \texttt{JordanCanonicalForm}(\bar{k}),$

$[2] := \mathcal{I}\, \mathrm{tr}\, \mathcal{I}\, \texttt{JordanCanonicalForm}(\bar{k})[1] \mathcal{I}^{-1} \sigma_T : \mathrm{tr}\, T^{e,e} = \int_{\bar{k}} \lambda \sigma_T(\lambda),$

$[*] := \mathcal{I}\, \mathrm{tr}[2] : \mathrm{tr}\, T = \int_{\bar{k}} \lambda \, d\sigma_T(\lambda);$

$\square$

$\texttt{SchurTheorem} :: \forall V \in k\text{-FDVS} \,.\, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \,.$

$$. T : \texttt{UpperTriangulizable}(k) \iff \chi^T : \texttt{Splits}(k)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{DetBySpectre} :: \forall V \in k\text{-FDVS} \, . \, \forall T \in \mathrm{End}_{k\text{-FDVS}}(V) \, . \, \det T = \prod_{\bar{k}} \lambda \, \mathrm{d}\sigma_T(\lambda)$

$\texttt{Proof} \ =$

$(e, [1]) := \texttt{JordanCanonicalFormTHM}(\bar{k}, V, k\text{-VS}) : \sum e : \texttt{Basis}(\bar{k} \otimes V) \, . \, T^{e,e} : \texttt{JordanCanonicalForm}(\bar{k}),$

$(n, \lambda, t, [2]) := \Finv \texttt{JordanCanonicalForm}(\bar{k}) : \sum n \in \mathbb{N} \, . \, \sum \lambda : n \to \bar{k} \, .$

$\quad : \sum t : n \to \mathbb{N} \, . \, T^{e,e} = \texttt{blockDiagonal}(n \times t, \mathbf{J}(t, \lambda)),$

$[3] := \texttt{BlockDiagonalDetUpperTriangularDet}[2]\Finv^{-1}\sigma_T : \det T^{e,e} = \prod_{\hat{k}} \lambda \, \mathrm{d}\sigma(\lambda),$

$[*] := \Finv \det[3] : \det T = \prod_{\hat{k}} \lambda \, \mathrm{d}\sigma(\lambda);$

$\square$

$\texttt{SpectralResolution} :: \prod V \in k\text{-VS} \, . \, \mathrm{End}_{k\text{-VS}}(V) \to$

$\quad \to ? \sum \kappa : \mathsf{CARD} \, . \, \texttt{ResolutionOfIdentity}(\kappa, V) \times \kappa \to k$

$(P, \lambda) : \texttt{SpectralResolution} \iff \Lambda T \in \mathrm{End}_{k\text{-VS}}(V) \, . \, T = \lambda_i P_i$

$\texttt{SpectralResolutionTHM} :: \forall V \in k\text{-VS} \, . \, \forall T : \mathrm{End}_{k\text{-VS}}(V) \, .$

$\quad . \, T : \texttt{Diagonalizable}(V) \iff \exists \texttt{SpectralResolution}(V, T)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CommutingHasCommonEigenvector} :: \forall V \in k\text{-FDVS} \, . \, \forall \mathcal{T} : \texttt{Commuting}\big(\mathbb{N}, \mathrm{End}_{k\text{-VS}}(V)\big) \, .$

$\quad . \, \forall [0] : \forall T \in \mathcal{T} \, . \, \chi^T : \texttt{Splits}(k) \, . \, \exists v \in V \, . \, \forall T \in \mathcal{T} \, . \, \exists \lambda \in k : Tv = \lambda v$

$\texttt{Proof} \ =$

$(\mathcal{I}, T) := \texttt{WellOrderigPrinciple}(\mathcal{T}) : \sum \mathcal{I} : \texttt{WellOrdered} \, . \, T : I \leftrightarrow \mathcal{T},$

$(\lambda_1, v_1, [1]) := \texttt{SchurTheorem}(T_1)[0] : \sum v_1 \in V \, . \, \sum \lambda_1 \in k \, . \, T_1 v_1 = \lambda_1 v_1 \ \& \ v_1 \neq 0,$

$E_1 := \texttt{Eigenspace}(T_1, \Lambda_1) : \texttt{VectorSubspace}(V),$

$\texttt{Assume} \ i : \mathcal{I}_+,$

$[i.1] := \texttt{CommutingHasInvariantEignspaces}(\Finv E_{i-1}, T, T_i) : (E_{i-1} : \texttt{InvariantSubspace}(V, T_i)),$

$[i.2] := \texttt{InvariantIsJordanSubmodule}[i.1] : E_{i-1} \subset_{\mathsf{-MOD}} k[x]V_{T_i},$

$[\lambda_i, v_i, [i.3]] := \texttt{JordanCanonicalFormTHM}[i.2][i.1] : \sum v_i \in E_{i-1} \, . \, \sum \lambda_i \in k \, . \, T_i v_i = \lambda_i v_i \ \& \ v_i \neq 0,$

$E_i := \texttt{Eigenspace}(T_i, \lambda_i) \cap E_{i-1} : \texttt{VectorSubspace}(V);$

$\leadsto (v, \lambda, E, [2]) := I \left(\prod\right) : \prod i \in \mathcal{I} \, . \, \sum (v_i, \lambda_i, E_i) \in V \times k \times \texttt{VectorSubspace}(V) \, . \, T_i v_i = \lambda v_i \ \& \ v_i \neq 0 \ \& \ v_i$

$[3] := [2]\Finv\texttt{Singleton}\{0\} : \forall i \in \mathcal{I} \, . \, E_i \neq \{0\},$

$[4] := \Finv^{-1}\texttt{Nonincreasing}[2] : \Big(E : \texttt{Nonincreasing}\big(\mathcal{I}, \texttt{Vectorsubspace}(V)\big)\Big),$

$(N, [5]) := \texttt{FiniteNonIncreasingStabilizes}(E, \texttt{EqByDim}(V)) : \sum N \in \mathcal{I} : \forall i \in \mathcal{I} \, . \, i \geq N \, . \, E_i = E_N,$

$[*] := \text{Ɔ}E[5] : \texttt{This};$

$\square$

1.7 Finite-Dimensional Vector Spaces Are Natural Numbers

$\texttt{FiniteDimensionalVectorSpacesAreNaturalNumbers} :: \forall k : \texttt{Field} \,.\, k\text{-FDVS} \simeq k\text{-}\mathbb{N}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ V : k\text{-FDVS},$

$e_V := \texttt{FreeHasBasis}(V) : \texttt{Basis}(V);$

$\rightsquigarrow e := I \left(\prod\right) : \prod V \in k\text{-FDVS} \,.\, \texttt{Basis}(V),$

$F := \Big((\mathrm{id}, e), \mathrm{id} \Big) : \texttt{Covariant}(k\text{-FDVS}, k\text{-LMAP}),$

$[1] := \text{Ↄ}F(FU) : FU = \mathrm{id},$

$[2] := \text{Ↄ}F(UF) : UF = \Big((V, f) \mapsto (V, e_V), \mathrm{id} \Big),$

$\alpha := \Lambda(V, f) \in k\text{-LMAP} \,.\, \Big((V, f), (V, e_V), \mathrm{id} \Big) : \texttt{NaturalTransform}(\mathrm{id}, UF),$

$[3] := \text{Ɑ}^{-1}\texttt{EquivalentCategories}(F, U, \alpha)[1][2] : k\text{-FDVS} \simeq k\text{-LMAP},$

$D := \Big(\dim(\cdot)_1, \mathrm{id} \Big) : \texttt{Covariant}(k\text{-MAT}, k\text{-}\mathbb{N}),$

$N := \Big(n \mapsto (k^n, e), \mathrm{id} \Big) : \texttt{Covariant}(k\text{-MAT}, k\text{-}\mathbb{N}),$

$[4] := \text{Ↄ}(ND) : ND = \mathrm{id},$

$[5] := \text{Ↄ}(DN) : DN = ((V, f) \mapsto (k^{\dim V}, e), \mathrm{id}),$

$\beta := \Lambda(V, f) \in k\text{-MAT} \,.\, I : \texttt{NaturalTransform}(\mathrm{id}, DN),$

$[6] := \text{Ɑ}^{-1}\texttt{EquivalentCategories}(D, N, \beta)[1][2] : k\text{-}\mathbb{N} \simeq k\text{-LMAP},$

$[7] := \text{Ɑ}\texttt{Transitive}(\texttt{CategoryEq})([6], \texttt{ChoiceOfBasisDefinesIso}, [3]) : k\text{-FDVS} \simeq k\text{-}\mathbb{N},$

$\square$

1.8 Euler Characteristic and Grothendieck Group

$\texttt{charactereristicOfEuler} :: \prod k : \texttt{Field} \,.\, \texttt{FiniteChain}(k\text{-}\mathsf{FDVS}) \to \mathbb{Z}$

$$\texttt{chararacteristicOfEuler}\,(V,f) = \chi(V,f) := \sum_{n=-\infty}^{\infty} (-1)^n \dim V_n$$

$\texttt{EulerCharHomolog} :: \forall k : \texttt{Field} \,.\, \forall (V,f) : \texttt{FiniteChain}(k\text{-}\mathsf{FDVS}) \,.\, \chi(V,f) = \sum_{n=-\infty}^{\infty} (-1)^n \dim H_n(V,f)$

$\texttt{Proof} =$

$\texttt{Assume}\ n : \mathbb{Z},$

$[n.*] := \texttt{RankPlusNullityTHM}(f_n)\daleth\,\mathrm{rank} : \dim V_n = \dim\ker f_n + \dim \mathrm{Im}\, f_n;$

$\rightsquigarrow [*] := \daleth^{-1}\chi(V,f)\daleth\,\mathsf{ABEL}(\mathbb{Z})\daleth^{-1}H(V,f)\texttt{QuotientDimansion} :$

$$: \chi(V,f) = \sum_{n=-\infty}^{\infty} (-1)^n(\dim\ker f_n + \dim \mathrm{Im}\, f_n) = \sum_{n=-\infty}^{\infty} (-1)^n(\dim\ker f_n - \dim \mathrm{Im}\, f_{n+1}) = \sum_{n=-\infty}^{\infty} H_n(V,f);$$

$\square$

$\texttt{groupOfGrothendiek} :: \texttt{AbeleanCategory} \to \mathsf{ABEL}$

$$\texttt{groupOfGrothendiek}\,(k) = K(\mathcal{A}) := \frac{F_{\mathsf{ABEL}}\ \texttt{Isoclass}(\mathcal{A})}{\{[W]-[V]-[U] \mid 0 \to V \to W \to U \to 0 : \texttt{ShortExact}(\mathcal{A})\}}$$

$\texttt{characteristicOdEulerGrothendiek} :: \prod \mathcal{A} : \texttt{AbeleanCategory} \,.\, \texttt{FiniteChain}(\mathcal{A}) \to K(\mathcal{A})$

$$\texttt{characteristicOdEulerGrothendiek}\,(V,f) = \chi_K(V,f) := \sum_{n=-\infty}^{\infty} (-1)[V_n]$$

$\texttt{EGGroupLemma} :: \forall V, U \in k\text{-}\mathsf{FDVS} \,.\, [V \oplus U] =_{K(k\text{-}\mathsf{FDVS})} [V] + [U]$

$\texttt{Proof} =$

$[1] := \texttt{SplittingIsExact}(V,U) : (0 \to V \to V \oplus U \to U \to 0 : \texttt{ShortExact}((\text{-}\mathsf{FDVS}k))),$

$[*] := \daleth K(k\text{-}\mathsf{FDVS})[1] : [V \oplus U] = [V] + [U];$

$\square$

$\texttt{EGCharHomolog} :: \forall k : \texttt{Field} \,.\, \forall (V,f) : \texttt{FiniteChain}(k\text{-}\mathsf{FDVS}) \,.\, \chi(V,f) = \sum_{n=-\infty}^{\infty} (-1)^n \Big[H_n(V,f)\Big]$

$\texttt{Proof} =$

$\texttt{Assume}\ n : \mathbb{Z},$

$[n.*] := \daleth K(k\text{-}\mathsf{FDVS})(f_n)\daleth\,\mathrm{rank} : [V_n] = [\ker f_n] + [\mathrm{Im}\, f_n];$

$\rightsquigarrow [*] := \daleth^{-1}\chi(V,f)\daleth\,\mathsf{ABEL}\Big(K(k\text{-}\mathsf{FDVS})\Big)\daleth K(k\text{-}\mathsf{FDVS})\daleth^{-1}H(V,f) :$

$$: \chi(V,f) = \sum_{n=-\infty}^{\infty} (-1)^n([\ker f_n] + [\dim \mathrm{Im}\, f_n]) = \sum_{n=-\infty}^{\infty} (-1)^n([\dim\ker f_n] - [\dim \mathrm{Im}\, f_{n+1}]) = \sum_{n=-\infty}^{\infty} H_n(V,f);$$

$\square$

$\texttt{CategoryGroupMapping} :: \prod G \in \mathsf{ABEL} \,.\, \prod \mathcal{A} : \texttt{AbeleanCategory} \,.\, \mathcal{A} \to G$

$\delta : \texttt{CategoryGroupMapping} \iff \forall A, B \in \mathcal{A} \,.\, A \cong_{\mathcal{A}} B \Rightarrow \delta(A) = \delta(B)$

$\quad \& \ \forall 0 \to A \to B \to C \to 0 : \texttt{ShortExact}(\mathcal{A}) \,.\, \delta(C) = \delta(B) - \delta(A)$

$\texttt{groupCharacteristicOfEuler} :: \prod \mathcal{A} : \texttt{AbeleanCategory} \,.\, \prod G \in \mathsf{ABEL} \,.$

$\quad .\, \prod \delta : \texttt{GroupCategoryMapping}(\mathcal{A}, G) \,.\, \texttt{FiniteChain}(V) \to G$

$$\texttt{groupCharacteristicOfEuler}\,(V, f) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(V_n) :=$$

$\texttt{GrothendiekGroupIsomorphism} :: \forall G \in \mathsf{ABEL} \,.\, \forall k : \texttt{Field} \,.\, \forall \delta : \texttt{CategoryGroupMapping}(G, k\text{-}\mathsf{FDVS}) \,.\, \exists! \varphi : \mathcal{}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [V] : \chi_K(V, f),$

$\varphi[V] := \delta(V) : G;$

$\rightsquigarrow \varphi := \text{Ɑ}\texttt{CategoryGroupMapping}(\delta) : K(k\text{-}\mathsf{FDVS}) \to G,$

$[1] := \texttt{EGGroupLemma}(\varphi) : (\varphi : K(k\text{-}\mathsf{FDVS}) \xrightarrow{\mathsf{ABEL}} G),$

$\texttt{Assume}\ \psi : K(k\text{-}\mathsf{FDVS}) \xrightarrow{\mathsf{ABEL}} G,$

$\texttt{Assume}\ [2] : \forall (V, f) : \texttt{FiniteChain}(k\text{-}\mathsf{FDVS}) \,.\, \psi\chi_K = \chi_\delta(V, f),$

$\texttt{Assume}\ [V] : K(k\text{-}\mathsf{FDVS}),$

$C := 0 \to V \to 0 : \texttt{FiniteChain}(k\text{-}\mathsf{FDVS}),$

$[3] := \text{Ɑ}\chi_K(C) : \chi_K(C) = [V],$

$[4] := \text{Ɑ}\chi_\delta(C) : \chi_\delta(C) = \delta(V),$

$[\psi.*] := \text{Ɑ}\phi[2][3][4] : \psi[V] = \phi[V];$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{Unique} : \texttt{This},$

$\square$

$\texttt{GrothendiekGroupOfFDVSIsIntegers} :: \forall k : \texttt{Field} \,.\, K(k\text{-}\mathsf{FDVS}) \cong_{\mathsf{ABEL}} \mathbb{Z}$

$\texttt{Proof} \ =$

$[1] := \texttt{FiniteDimensionalVectorSpacesAreNaturalnambers}(k) :$

$\quad .\, \forall V, W \in k\text{-}\mathsf{FDVS} \,.\, \dim V = \dim W \iff V \cong_{k\text{-}\mathsf{VS}} W,$

$[2] := \texttt{RankPlusNullityTHM}\text{Ɑ}\texttt{ShortExact}(k\text{-}\mathsf{FDVS}) :$

$\quad .\, \forall 0 \to V \to U \to W \to : \texttt{ShortExact}(k\text{-}\mathsf{FDVS}) \,.\, \dim W = \dim U - \dim V,$

$[3] := \text{Ɑ}^{-1}\texttt{CategoryGroupMapping} : \Big(\dim : \texttt{CategoryGroupMapping}(k\text{-}\mathsf{FDVS}, \mathbb{Z})\Big),$

$(\varphi, [4]) := \texttt{GrothendiekGroupIsomorphism}(\dim) : \sum \varphi : Kk\text{-}\mathsf{FDVS} \xrightarrow{\mathsf{ABEL}} \mathbb{Z} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \varphi[V] = \dim V,$

$[5] := [4]\text{Ɑ} \dim : \mathbb{Z}_+ \subset \operatorname{Im} \varphi,$

$[6] := [5]\text{Ɑ}\mathsf{ABEL}(K(k\text{-}\mathsf{FDVS}), \mathbb{Z})(\varphi)[5] : (\varphi : K(k\text{-}\mathsf{FDVS}) \twoheadrightarrow \mathbb{Z}),$

$[7] := [1][4] : (\varphi : K(k\text{-}\mathsf{FDVS}) \hookrightarrow \mathbb{Z}),$

$[*] := \text{Ɑ}^{-1}\texttt{isomorphic}[6][7] : \mathbb{Z} \cong_{\mathsf{ABEL}} K(k\text{-}\mathsf{FDVS});$

$\square$

2 Linear Algebra in Eucledean and Hermitian Spaces

2.1 Real and Complex Structures

$\texttt{RealDimOfComplex} :: \forall V \in \mathbb{C}\text{-VS} \,.\, \dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V$

$\texttt{Proof} =$

$e := \texttt{VSIsFree}(V)\ \texttt{FreeHasBasis}(V) : \texttt{Basis}(\mathbb{C}, V),$

$[1] := \mathcal{C}\texttt{Basis}(\mathbb{C})\mathcal{C}^{-1}\texttt{Basis}(\mathbb{R})\mathcal{C}\mathbb{C} : (e \sqcup ie : \texttt{Basis}(\mathbb{R})),$

$[2] := \mathcal{C}\texttt{cardinalitySum}(e \sqcup ie)\mathcal{C}\mathbb{C}\text{-VS}(V) : |e \sqcup ie| = |e| + |ie| = 2|e|,$

$[*] := \mathcal{C}^{-1}\dim V[1][2] : \dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V;$

$\square$

$\texttt{ConjugateSpace} :: \mathbb{C}\text{-VS} \xrightarrow{\text{CAT}} \mathbb{C}\text{-VS}$

$\texttt{ConjugateSpace}\,(V) = \overline{V} := (V, +, \Lambda z \in \mathbb{C} \,.\, \Lambda v \in V \,.\, \overline{z}v)$

$\texttt{ConjugateSpace}\,(V, W, f) = \overline{f} := f$

$\texttt{ConjugationPreservesDim} :: \forall V \in \mathbb{C}\text{-VS} \,.\, \dim \overline{V} = \dim V$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ConjugationMorphism} :: \prod V \in \mathbb{C}\text{-VS} \,.\, V \xrightarrow{\mathbb{C}\text{-VS}} \overline{V}$

$\sigma : \texttt{ConjugationMorphism} \iff \overline{\sigma}\sigma = \underset{V}{\mathrm{id}}$

$\texttt{realStructure} :: \prod V \in \mathbb{C}\text{-VS} \,.\, \texttt{ConjugationMorphism}(V) \to \mathbb{R}\text{-VS}$

$\texttt{realStructure}\,(\sigma) = \Re_\sigma V := \frac{1}{2}(\mathrm{id} + \sigma)V$

$\texttt{imagenableStructure} :: \prod V \in \mathbb{C}\text{-VS} \,.\, \texttt{ConjugationMorphism}(V) \to \mathbb{R}\text{-VS}$

$\texttt{imagenableStructure}\,(\sigma) = \Im_\sigma V := \frac{1}{2}(\mathrm{id} - \sigma)V$

$\texttt{ImaginableUnitIsomorphism} :: \forall V \in \mathbb{C}\text{-VS} \,.\, \forall \sigma : \texttt{ConjugationMorphism} \,.\, \Re_\sigma V \cong_{\mathbb{R}\text{-VS}} \Im_\sigma V$

$\texttt{Proof} =$

$T := \Lambda v \in \Re_\sigma V \,.\, iv : \Re_\sigma V \xrightarrow{\mathbb{R}\text{-VS}} V,$

$S := \Lambda v \in \Im_\sigma V \,.\, -iv : \Im_\sigma V \xrightarrow{\mathbb{R}\text{-VS}} V,$

$\texttt{Assume}\ v : V,$

$[v.1.*] := \mathcal{C}\texttt{complexConjugation}\mathcal{C}\texttt{ConjugationMorphism}\sigma : i(\mathrm{id} + \sigma)v = iv + \sigma(-iv) = (\mathrm{id} - \sigma)(iv),$

$[v.2.*] := \mathcal{C}\texttt{complexConjugation}\mathcal{C}\texttt{ConjugationMorphism}\sigma : -i(\mathrm{id} - \sigma)v = -iv - \sigma(iv) = \mathrm{id} + \sigma(-iv),$

$\leadsto [1] := \mathcal{C}^{-1}\Re_\sigma V\mathcal{C}^{-1}\Im_\sigma V : (T : \Re_\sigma V \xrightarrow{\mathbb{R}\text{-VS}} \Im_\sigma V\ \&\ S : \Im_\sigma V \xrightarrow{\mathbb{R}\text{-VS}} \Re_\sigma V,$

$[2] := \mathcal{C}T\mathcal{C}S[1] : ST = \mathrm{id}\ \&\ TS = \mathrm{id},$

$[*] := \mathcal{C}^{-1}\texttt{Isomorphic}[2] : \Re_\sigma V \cong_{\mathbb{R}\text{-VS}} \Im_\sigma V;$

$\square$

$\texttt{RealImaginableVSDecomposition} :: \forall V \in \mathbb{C}\text{-VS} . \forall \sigma : \texttt{ConjugationMorphism}(V) . V = \Re_\sigma V \oplus \Im_\sigma V$

$\texttt{Proof} =$

$[1] := \mathcal{I}\texttt{ConjugationMorphism}(V) : \frac{1}{4}(\mathrm{id} + \sigma)(\mathrm{id} + \sigma) = \frac{1}{2}(\mathrm{id} + \sigma),$

$[2] := \mathcal{I}\texttt{ConjgationMorphism}(V) : \frac{1}{4}(\mathrm{id} - \sigma)(\mathrm{id} - \sigma) = \frac{1}{2}(\mathrm{id} - \sigma),$

$[3] := \mathcal{I}^{-1}\texttt{Projector} : \left(\frac{1}{2}(\mathrm{id} + \sigma), \frac{1}{2}(\mathrm{id} - \sigma) : \texttt{Projector}(\Re_\sigma V, \Im_\sigma V)\right),$

$\texttt{Assume}\ v : V,$

$[v.*] := \mathcal{I}\mathbb{C}\text{-VS}(\overline{V}) : v = \frac{1}{2}(\mathrm{id} + \sigma)v + \frac{1}{2}(\mathrm{id} - \sigma)v,$

$\rightsquigarrow [4] := \mathcal{I}^{-1}\texttt{subsetSum}\mathcal{I}^{-1}\Re_\sigma V \mathcal{I}^{-1}\Im_\sigma V : V = \Re_\sigma V + \Im_\sigma V,$

$\texttt{Assume}\ v : V,$

$[2.*] := \mathcal{I}\texttt{ConjugationMorphism}(V)(\sigma) : \frac{1}{4}(\mathrm{id} + \sigma)(\mathrm{id} - \sigma)v = \frac{1}{4}(\mathrm{id} + \sigma - \sigma - \mathrm{id})v = 0,$

$[3.*] := \mathcal{I}\texttt{ConjugationMorphism}(V)(\sigma) : \frac{1}{4}(\mathrm{id} - \sigma)(\mathrm{id} + \sigma)v = \frac{1}{4}(\mathrm{id} - \sigma + \sigma - \mathrm{id})v = 0;$

$\rightsquigarrow [5] := \mathcal{I}^{-1}\texttt{OrthogonalProjections} : \frac{1}{2}(\mathrm{id} + \sigma) \perp \frac{1}{2}(\mathrm{id} - \sigma),$

$[*] := \texttt{ResolutionOfIdentity}[4][5] : V = \Re_\sigma V \oplus_{\mathbb{R}} \Im_\sigma V;$

$\square$

$\texttt{RealStructureDimension} :: \forall V \in \mathbb{C}\text{-VS} . \forall \sigma : \texttt{ConjugationMorphism}(V) .$

$. \dim_{\mathbb{R}} \Re_\sigma V = \dim_{\mathbb{R}} \Im_\sigma V = \dim_{\mathbb{C}} V$

$\texttt{Proof} =$

$[1] := \texttt{RealImagenableVSDecomposition}(V, \sigma) : V =_{\mathbb{R}\text{-VS}} \Re_\sigma V \oplus \Im_\sigma V,$

$[2] := \texttt{SumRank}[1] : \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} \Re_\sigma V + \dim_{\mathbb{R}} \Im_\sigma V,$

$[3] := \texttt{ImaginableUnitDecomposition}(V, \sigma) : \Im_\sigma V \cong_{\mathbb{R}\text{-VS}} \Re_\sigma V,$

$[4] := \texttt{IsoRank}[6] : \dim_{\mathbb{R}} \Re_\sigma V = \dim_{\mathbb{R}} \Im_\sigma V,$

$[*] := \texttt{RealDimOfComplex}(V)[4][2] : \dim_{\mathbb{C}} V = \dim_{\mathbb{R}} \Re_\sigma V = \dim_{\mathbb{R}} \Im_\sigma V;$

$\square$

$\texttt{ComplexStructure} :: \prod V : \mathbb{R}\text{-VS} . ?\mathrm{End}_{\mathbb{R}\text{-VS}}(\mathbb{R})$

$J : \texttt{ComplexStructure} \iff J^2 = -\mathrm{id}$

$\texttt{applyComplexStructure} :: \prod V : \mathbb{R}\text{-VS} . \texttt{ComplexStructure}(V) \to \mathbb{C}\text{-VS}$

$\texttt{applyComplexStructure}(J) = V_J := (V, +, \Lambda v \in V . \Lambda a + bi \in \mathbb{C} . av + bJ(v))$

$\texttt{LinearOperatorWithComplexStructure} :: \forall V, W : \mathbb{R}\text{-VS} .$

$. \forall J : \texttt{ComplexStructure}(V) . \forall J' : \texttt{ComplexStructure}(W) . \forall T : V \xrightarrow{\mathbb{R}\text{-VS}} W .$

$. T : V_J \xrightarrow{\mathbb{C}\text{-VS}} W_{J'} \iff TJ' = JT$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{LinearSubspaceOfComplexStructure} :: \forall V : \mathbb{R}\text{-VS} .$
$\quad . \forall J : \texttt{ComplexStructure}(V) . \forall U \subset_{\mathbb{R}\text{-VS}} V . U \subset_{\mathbb{C}\text{-VS}} V \iff J(U) = U$

$\texttt{Proof} =$

...

$\square$

$$\texttt{QuaternionicStructure} :: \prod V : \mathbb{R}\text{-VS} . ?\texttt{ComplexStructure}^2$$

$$(J, K) : \texttt{QuaternionicStructure} \iff JK = -KJ$$

$$\texttt{applyQuaternionicStructure} :: \prod V : \mathbb{R}\text{-VS} . \texttt{QuaternionicStructure}(V) \to \text{MOD-}\mathbb{H}$$

$$\texttt{applyQuaternionicStructure}\,(J, K) = V_{J,K} :=$$
$$:= (V, +, \Lambda v \in V . \Lambda a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \in \mathbb{H} . av + bJK(v) + cJ(v) + dK(v))$$

$\texttt{LinearOperatorWithQuaternionicStructure} :: \forall V, W : \mathbb{R}\text{-VS} .$
$\quad . \forall (J, K) : \texttt{QuaternionicStructure}(V) . \forall (J', K') : \texttt{QuaternionicStructure}(W) . \forall T : V \xrightarrow{\mathbb{R}\text{-VS}} W .$
$\quad . T : V_J \xrightarrow{\text{MOD-}\mathbb{H}} W_{J'} \iff TJ' = JT \;\&\; TK' = KT$

$\texttt{Proof} =$

...

$\square$

$\texttt{LinearSubspaceOfQuaternionicStructure} :: \forall V : \mathbb{R}\text{-VS} .$
$\quad . \forall (J, K) : \texttt{QuaternionicStructure}(V) . \forall U \subset_{\mathbb{R}\text{-VS}} V . U \subset_{\text{MOD-}\mathbb{H}} V \iff J(U) = U \;\&\; K(U) = U$

$\texttt{Proof} =$

...

$\square$

$$\texttt{AnticonjugationMorphism} :: \prod V \in \mathbb{C}\text{-VS} . \mathrm{End}_{\mathbb{R}\text{-VS}}(\overline{V})$$

$$\alpha : \texttt{AnticonjugationMorphism} \iff \alpha\overline{\alpha} = -\,\mathrm{id}$$

$\texttt{QuaternioniStructureByAnticonjugation} :: \forall V \in \mathbb{C}\text{-VS} . \forall \alpha : \texttt{AnticonjugationMorphism}(V) .$
$\quad . (\alpha, \mathrm{i} \cdot \mathrm{id}) : \texttt{QuaternionicStructure}(V)$

$\texttt{Proof} =$

...

$\square$

2.2 Eucledean and Hermitian Products

$$\texttt{InnerProduct} :: \prod k : \texttt{ConjugationField}(R) \ . \prod V \in k\text{-VS} \ . \ V \otimes \overline{V} \xrightarrow{k\text{-VS}} k$$

$$p : \texttt{InnerProduct} \iff \forall v, w \in V \ . \ p(v \otimes w) = \overline{p(w \otimes v)} \ \& \ p(v \otimes v) \in R_+ \ \& \ (p(v \otimes v) = 0 \Rightarrow v = 0)$$

$$\texttt{InnerProductSpace} := \prod k : \texttt{ConjugationField}(R) \ . \sum V : k\text{-VS} \ . \ \texttt{InnerProduct}(V) :$$

$$: \prod R : \texttt{OrderedField} \ . \ \texttt{ConjugationField}(R) \to \texttt{Type};$$

$$\texttt{innerProductSpaceAsVectorSpace} :: \texttt{InnerProductSpace}(k) \to k\text{-VS}$$

$$\texttt{innerProductSpaceAsVectorSpace}\,(V, p) = (V, p) := k\text{-VS}$$

$$\texttt{innerProduct} :: \prod (V, p) : \texttt{InnerProductSpace}(k) \ . \ \mathcal{L}(V, \overline{V}; k)$$

$$\texttt{innerProduct}\,(v, w) = \langle v, w \rangle := p(v, w)$$

$$\texttt{RealPolarizationId} :: \forall V : \texttt{InnerProductSpace}(\mathbb{R}) \ . \ \forall v, w \in V \ .$$

$$. \ \langle v, w \rangle = \frac{1}{4}\langle v + w, v + w \rangle - \frac{1}{4}\langle v - w, v - w \rangle$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$$\texttt{ComplexPolarizationId} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \ . \ \forall v, w \in V \ .$$

$$. \ \langle v, w \rangle = \frac{1}{4}\langle v + w, v + w \rangle - \frac{1}{4}\langle v - w, v - w \rangle + \frac{\mathrm{i}}{4}\langle v + \mathrm{i}w, v + \mathrm{i}w \rangle - \frac{\mathrm{i}}{4}\langle v - \mathrm{i}w, v - \mathrm{i}w \rangle$$

$\texttt{Proof} \ =$

$$[*] := \texttt{MultiAdditve}^{12}\Big(\langle \cdot, \cdot \rangle)\mathrm{Ɑ}^4\mathcal{L}(V, \bar{(}V); \mathbb{C})(\langle \cdot, \cdot \rangle)\mathrm{Ɑ}^8\mathsf{ABEL}(\mathbb{C})\mathrm{Ɑ}^2\mathcal{L}(V, \bar{(}V); \mathbb{C})(\langle \cdot, \cdot \rangle)\mathrm{Ɑ}^3\texttt{Field}(\mathbb{C}) :$$

$$: \frac{1}{4}\langle v + w, v + w \rangle - \frac{1}{4}\langle v - w, v - w \rangle + \frac{\mathrm{i}}{4}\langle v + \mathrm{i}w, v + \mathrm{i}w \rangle - \frac{\mathrm{i}}{4}\langle v - \mathrm{i}w, v - \mathrm{i}w \rangle =$$

$$= \frac{1}{4}\Big(2\langle v, w \rangle + 2\langle w, v \rangle + 2\mathrm{i}\langle v, \mathrm{i}w \rangle + 2\mathrm{i}\langle \mathrm{i}w, v \rangle\Big) = \frac{1}{4}\Big(2\langle v, w \rangle + 2\langle w, v \rangle + 2\langle v, w \rangle - 2\langle w, v \rangle\Big) = \langle v, w \rangle;$$

$\square$

$$\texttt{Isometry} :: \prod V, W : \texttt{InnerProductSpace}(k) \ . \ ?V \xrightarrow{k\text{-VS}} W$$

$$T : \texttt{Isometry} \iff \forall v \in V \ . \ \langle Tv, Tv \rangle = \langle v, v \rangle$$

$$\texttt{RealIsometryProperty} :: \forall V, W : \texttt{InnerProductSpace}(\mathbb{R}) \ . \ \forall T : \texttt{Isometry}(V, W) \ .$$

$$. \ \forall v, u \in V \ . \ \langle Tv, Tu \rangle = \langle v, u \rangle$$

$\texttt{Proof} \ =$

$$[*] := \texttt{RealPolarizationId}(Tv, Tu)\mathrm{Ɑ}^4\mathbb{R}\text{-VS}(V, W)(T)[1]^2\texttt{RealPolarizationId}(v, w) :$$

$$: \langle Tu, Tv \rangle = \frac{1}{4}\langle Tv + Tw, Tv + Tw \rangle - \frac{1}{4}\langle Tv - Tw, Tv - Tw \rangle =$$

$$= \frac{1}{4}\langle T(v + w), T(v + w) \rangle - \frac{1}{4}\langle T(v - w), T(v - w) \rangle = \frac{1}{4}\langle v + w, v + w \rangle - \frac{1}{4}\langle v - w, v - w \rangle = \langle u, v \rangle;$$

$\square$

$\texttt{ComplexIsometryProperty} :: \forall V, W : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T : \texttt{Isometry}(V, W) \,.$

$\quad . \, \forall v, u \in V \,.\, \langle Tv, Tu \rangle = \langle v, u \rangle$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{IsometryComp} :: \forall V, W, U : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{Isometry}(V, W) \,.\, \forall S : \texttt{Isometry}(W, U) \,.$

$\quad . \, TS : \texttt{Isometry}(V, U)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{InnerProductSum} :: \forall (V, p), (W, q) : \texttt{InnerProductSpace}(k) \,.\, (V \oplus W, p \oplus q) : \texttt{InnerProductSpace}(k)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (a, b), (x, y) : V \oplus W,$

$[1.*] := \Lambda p \oplus q \Lambda \texttt{InnerProduct}(p, q) \Lambda^{-1} p \oplus q : p \oplus q\Big((a, b), (x, y)\Big) = p(a, x) + q(b, y) = \overline{p(x, a)} + \overline{q(y, b)} = \overline{p \oplus}$

$[2.*] := \Lambda p \oplus q \Lambda \texttt{InnerProduct}(p, q) \texttt{NonNegativeSum}(R) : p \oplus q\Big((a, b), (a, b)\Big) = p(a, a) + q(b, b) \in R_+,$

$[3.*] := \Lambda p \oplus q \texttt{PositiveSum}(R) : p \oplus q\Big((a, b), (a, b)\Big) = 0 \iff (a, b) = 0;$

$\rightsquigarrow [*] := \Lambda^{-1} \texttt{InnerProduct} : (p \oplus q : \texttt{InnerProduct});$

$\square$

$\texttt{ParallelagrmaLaw} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall v, u \in V \,.\, \langle v + u, v + u \rangle + \langle v - u, v - u \rangle = \langle v, v \rangle + \langle u, u \rangle$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ApolloniusId} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall v, u, w \in V \,.$

$$. \, \langle v - u, v - u \rangle + \langle v - w, v - w \rangle = \frac{1}{2}\langle u - w, u - w \rangle + 2\left\langle w - \frac{1}{2}(u + w), w - \frac{1}{2}(u + w) \right\rangle$$

$\texttt{Proof} \;=$

$[*] :=:$

$$: \langle v - u, v - u \rangle + \langle v - w, v - w \rangle = 2\langle v, v \rangle + \langle u, u \rangle + \langle w, w \rangle - \langle v, u \rangle - \langle u, v \rangle - \langle v, w \rangle - \langle w, v \rangle =$$

$$= \frac{1}{2}\langle u - w, u - w \rangle + \frac{1}{2}\langle u, v \rangle + \frac{1}{2}\langle v, u \rangle + \frac{1}{2}\langle v, v \rangle + \frac{1}{2}\langle w, w \rangle - \langle v, u \rangle - \langle u, v \rangle - \langle v, w \rangle - \langle w, v \rangle + 2\langle v, v \rangle =$$

$$= \frac{1}{2}\langle u - w, u - w \rangle + 2\left\langle w - \frac{1}{2}(u + w), w - \frac{1}{2}(u + w) \right\rangle;$$

$\square$

2.3 Orthogonality and Orthogonalization

$\texttt{OrthogonalVectors} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?V^2$

$(v, w) : \texttt{OrthogonalVectors} \iff v \perp w \iff \langle v, w \rangle = 0$

$\texttt{OrthogonalSets} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?(?V)^2$

$(A, B) : \texttt{OrthogonalSets} \iff A \perp B \iff \forall a \in A \,.\, \forall b \in B \,.\, \langle a, b \rangle = 0$

$\texttt{orthogonalComlement} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?V \to \texttt{VectorSubspace}(k, V)$

$$\texttt{orthogonalComplement}\,(X) = X^\perp := \bigcap_{x \in X} \ker\langle x, \cdot \rangle$$

$\texttt{OrhogonalComplementOfSpan} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall X \in ?V \,.\, X^\perp = (\operatorname{span} X)^\perp$

$\texttt{Proof} =$

...

$\square$

$\texttt{OrthogonalComplementIntersect} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall X \in ?V \,.\, X \cap X^\perp \subset \{0\}$

$\texttt{Proof} =$

...

$\square$

$\texttt{OrhtogonalDirectSum} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?\sum n \in \texttt{Set} \,.\, n \to \texttt{VectorSubspace}(V)$

$$(n, U) : \texttt{OrthogonalDirectSum} \iff V = \bot_{i \in n} U_i \iff V = \bigoplus_{i \in n} U_i \;\&\; \forall i, j \in n \,.\, i \neq j \Rightarrow U_i \perp U_j$$

$\texttt{OrthogonalSet} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ??(V \setminus \{0\})$

$E : \texttt{OrthogonalSet} \iff \forall v, w \in E \,.\, v \neq w \Rightarrow v \perp w$

$\texttt{OrthonormalSet} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?\texttt{OrthogonalSet}(V)$

$E : \texttt{OrthogonalSet} \iff \forall v, w \in E \,.\, \langle v, w \rangle = \delta^v_w$

$\texttt{sphere} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?V$

$\texttt{sphere}\,() = \mathbb{S}_V := \{v \in V : \langle v, v \rangle = 1\}$

$\texttt{Assume}\ R : \texttt{WithSquareRoots},$

$\texttt{norm} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, V \to R_+$

$\texttt{norm}\,(v) = \|v\| := \sqrt{\langle v, v \rangle}$

$\texttt{normalize} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, V \setminus \{0\} \to \mathbb{S}_V$

$$\texttt{normalize}\,(v) := \frac{v}{\|v\|}$$

$\texttt{PythagorusTHM} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall v, u \in V \,.\, v \perp u \Rightarrow \|v+u\|^2 = \|v\|^2 + \|u\|^u$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{OrhogonalIsLInd} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall E : \texttt{Orthogonal}(V) \,.\, E : \texttt{LinearlyIndependent}(V)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : E \,!\, \texttt{LinearlyIndependent}(V),$

$(v, \alpha, [2]) := \mathcal{A}\texttt{LinearlyIndependent}(V)[1] : \prod v \in E \,.\, \prod \alpha \in k^{\oplus E} \,.\, \alpha_v = 0 \ \&\ v = \alpha E,$

$(w.[3]) := \mathcal{A}\texttt{Orthogonal}(V)(E)\mathcal{A}\texttt{linearCombination}[2] : \sum w \in E \,.\, \alpha_w \neq 0,$

$[4] := [3][2] : w \neq v,$

$[5] := \mathcal{A}\texttt{Orthogonal}(V)[4] : \langle w, v\rangle = 0,$

$[6] := [2]\texttt{MultiAdditive}(\langle\cdot,\cdot\rangle)\texttt{MultiHomogen}(\langle\cdot,\cdot\rangle)$

$\quad \mathcal{A}\texttt{Orthogonal}(V)[3]\mathcal{A}\texttt{InnerProduct}(V)\mathcal{A}\texttt{IntegralDomain}(k) : \langle w, v\rangle = \langle w, \alpha E\rangle = \alpha_w\langle w, w\rangle \neq 0,$

$[1.*] := I(\perp)[5][6] : \perp;$

$\rightsquigarrow [*] := E(\perp) : \Big(E : \texttt{LinearlyIndependent}(V)\Big);$

$\square$

$`\texttt{orthogonalFrames} :: \mathbb{N} \to \texttt{InnerProductSpace}(k) \to \mathsf{SET}$

$\texttt{orthogonalFrames}\,(n, V) = E_n(V) := \Big\{(e : n \to V) : \operatorname{Im} e : \texttt{Orhogonal}(E)\Big\}$

$\texttt{GrammSchmidtAugementation} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall n \in \mathbb{N} \,.\, \forall e \in V_n(k) \,.\, \forall u \notin \operatorname{span}(e) \,.$

$\quad .\, \exists f \in E : e \oplus f \in E_{n+1}(V)$

$\texttt{Proof} \ =$

$$f := u - \sum_{i=1}^{n} \frac{\langle u, e_i\rangle e_i}{\langle e_i, e_i\rangle} : V,$$

$[1] := \mathcal{A}u\mathcal{A}\texttt{span}\mathcal{O}f : f \neq 0,$

$\texttt{Assume}\ i : n,$

$[i.1] := \mathcal{O}f\texttt{MultiAdditive}^{n+1}(\langle\cdot,\cdot\rangle)\mathcal{A}E_n(V)\mathcal{A}\texttt{Orhogonal}\mathcal{A}\texttt{Field}(k) :$

$$: \langle f, e_i\rangle = \langle u, e_i\rangle - \sum_{j=1}^{n} \frac{\langle u, e_i\rangle\langle e_j, e_j\rangle}{\langle u, e_j\rangle} = \langle u, e_i\rangle - \langle u, e_i\rangle = 0,$$

$[i.*] := \mathcal{A}^{-1}\texttt{OrthogonalVectors}[i.1] : f \perp e_i;$

$\rightsquigarrow [2] := I(\forall) : \forall i \in n \,.\, f \perp e_i,$

$[*] := \mathcal{A}E_{n+1}(V)[1][2] : e \oplus f \in E_{n+1}(V);$

$\square$

$\texttt{GrammSchmidtOrgogonalization} :: \forall V : \texttt{InnerProductSpace}(k) \ . \ \forall n \in \mathbb{N} \ . \ \forall f : \texttt{LinearlyIndependent}(n, V)$

$\quad \exists e \in E_n(V) \ . \ \mathrm{span}(e) = \mathrm{span}(f)$

$\texttt{Proof} \ =$

$e^1 := 1 \mapsto f_1 : E_1(V),$

$[1_1] := \mathcal{I} \, \mathrm{span} \, \mathcal{J} e^1 : \mathrm{span}(e^1) = \mathrm{span}(f_{|1}),$

$\texttt{Assume} \ i : n - 1,$

$[2] := [1_i] \mathcal{I} \texttt{LinearlyIndependent}(n, V)(f) : f_{i+1} \notin \mathrm{span}(e^i),$

$\big(u, [3]\big) := \texttt{GrammSchmidtAugemtation}(e^i, f_{i,1}[2]) : \sum u \in V \ . \ e^i \oplus u \in E_{i+1}(V),$

$e^{i+1} := e^i \oplus u : E_{i+1}(V),$

$[4] := \mathcal{I} \texttt{GrammSchmidtAugemntation} \mathcal{J}(e^{i+1}) : f_{i+1} \in \mathrm{span}(e^{i+1}),$

$[1_{i+1}] := \mathcal{I} \texttt{Span}[4][1_i] : \mathrm{span}(e^i) = \mathrm{span}(f_{|i+1});$

$\rightsquigarrow e := I \left(\prod\right) : \prod i \in n \ . \ \sum e^i \in E_i(V) \ . \ \mathrm{span}(e^i) = \mathrm{span}(f_{|i}),$

$e := e^n : E_n(V);$

$\square$

$\texttt{OrthonormalBasis} := \texttt{Basis} \ \& \ \texttt{Orthonormal} : \prod k : \texttt{ConjugationField} \ . \ \texttt{InnerProductSpace}(k) \to \texttt{Type};$

$\texttt{FiniteDimensionalInnerProductSpace}(k) := k\text{-FDVS} \ \& \ \texttt{InnerProductSpace}(k) : \texttt{Type};$

$\texttt{OrthonormalBasisTheorem} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(k) \ . \ \exists \texttt{OrthonormalBasis}(V)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{processOfGrammSchmidt} :: \prod V : \texttt{InnerProductSpace}(k) \ . \ \prod n \in \mathbb{N} \ .$

$\quad .$

$\quad \texttt{LinearlyIndependent}(n, V) \to E_n(V)$

$\texttt{processOfGrammSchmidt}\left((f_i)_{i=1}^1\right) = \mathbf{GS}(f_i)_{i=1}^1 := (f_i)_{i=1}^1$

$$\texttt{processOfGrammSchmidt}(f) = \mathbf{GS}(f) := e \oplus \left(f_n - \sum_{i=1}^{n-1} \frac{\langle f_n, e_i \rangle e_i}{\langle e_i, e_i \rangle}\right) \quad \texttt{where} \quad e = \mathbf{GS}(f_{|n-1})$$

$\texttt{orthonormalFrames} :: \mathbb{N} \to \texttt{InnerProductSpace}(k) \to \mathsf{SET}$

$\texttt{orthonormalFrames}(n, U) = V_n(U) := \Big\{(e : n \to U) : \mathrm{Im}\, e : \texttt{Orthonormal}(U)\Big\}$

$\texttt{normalizedGrammSchmidtProcess} :: \prod U : \texttt{InnerProductSpace}(k) \ . \ \prod n \in \mathbb{N} \ .$

$\quad . \ \texttt{LinearlyIndependent}(n, U) \to V_n(U)$

$\texttt{normalizedGrammSchmidtProcess}(f) = \mathbf{NGS}(f) := \texttt{Normalize}\Big(\mathbf{GS}(f)\Big)$

$\texttt{InnerProductProduct} :: \forall (V,p),(W,q) : \texttt{InnerProductSpace}(k) \,.\, (V\otimes W, p\otimes q) : \texttt{InnerProductSpace}(k)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \sum_{i=1}^n a_i\otimes b_i, \sum_{i=1}^m x_i\otimes y_i : V\otimes W,$

$[1.*] := \daleth p\otimes q \daleth \texttt{InnerProduct} \daleth \texttt{Conjugation}(k)\daleth^{-1} :$

$$: p\otimes q\left(\sum_{i=1}^n a_i\otimes b_i, \sum_{i=1}^m x_i\otimes y_i\right) = \sum_{i=1}^n\sum_{j=1}^m p(a_i,x_j)q(b_i,y_j) = \sum_{i=1}^n\sum_{j=1}^m \overline{p(x_j,a_i)q(y_j,b_i)} =$$

$$= \sum_{j=1}^m\sum_{i=1}^n \overline{p(x_j,a_i)q(y_j,b_i)} = \overline{p\otimes q\left(\sum_{i=1}^n x_i\otimes y_i, \sum_{i=1}^m a_i\otimes b_i\right)},$$

$A := \mathrm{span}(a) : \texttt{FiniteDimensionalInnerProductSpace}(k),$

$B := \mathrm{span}(b) : \texttt{FiniteDimensionalInnerProductSpace}(k),$

$e := \texttt{OrthonormalBasisTHM}(A) : \texttt{OrthonormalBasis}(P),$

$f := \texttt{OrthonormalBasisTHM}(B) : \texttt{OrthonormalBasis}(Q),$

$[2] := \texttt{BasisOfTensorProduct}(A,B,e,f) : e\otimes f : \texttt{Basis}(P\otimes Q),$

$$(\alpha,[3]) := \daleth\texttt{Basis}(e\otimes f)\left(\sum_{i=1}^n a_i\times b_i\right) : \sum \alpha : \dim A\times\dim B\to k : \sum_{i=1}^n a_i\otimes b_i = \sum_{i=1}^{\dim A}\sum_{j=1}^{\dim B}\alpha_{i,j}e_i\otimes f_i,$$

$[2.*] := [3]\daleth p\otimes q\texttt{MultiAdd}(p)(q)\texttt{MultiHomogen}(p)(q)\daleth\texttt{Orhonormal}(A)(e)(B)(f)$

$$\texttt{SumOfSquaresIsNonNegative}(|\alpha|) : p\otimes q\left(\sum_{i=1}^n a_i\otimes b_i, \sum_{i=1}^m a_i\otimes b_i\right) =$$

$$= p\otimes q\left(\sum_{i=1}^{\dim A}\sum^{\dim B}\alpha_{i,j}(e_i\otimes f_j), \sum_{i=1}^{\dim A}\sum^{\dim B}\alpha_{i,j}(e_i\otimes f_j)\right) = \sum_{i',i=1}^{\dim A}\sum_{j',j=1}^{\dim B}\alpha_{i,i'}\alpha_{j,j'}p(e_i,e_{i'})q(f_j,f_{j'}) =$$

$$= \sum_{i',i=1}^{\dim A}\sum_{j=1}^{\dim B}|\alpha_{i,j}|^2 \ge 0,$$

$[3.*] := \ldots\daleth\texttt{ZeroNonnegative}(\alpha^2)\texttt{Basis}(e\otimes f)[3] :$

$$: p\otimes q\left(\sum_{i=1}^n a_i\otimes b_i, \sum_{i=1}^m a_i\otimes b_i\right) = \sum_{i',i=1}^{\dim A}\sum_{j=1}^{\dim B}|\alpha_{i,j}|^2 = 0 \iff \sum_{i=1}^m a_i\otimes b_i = 0;$$

$\rightsquigarrow [*] := \daleth^{-1}\texttt{InnerProduct} : \Big(p\otimes q : \texttt{InnerProduct}(V\otimes W)\Big);$

$\square$

$\texttt{dotProduct} :: \prod k : \texttt{ConjugationField} \,.\, \prod n\in\mathbb{N} \,.\, k^n\times k^n\to k$

$\texttt{dotProduct}\,(a,b) = a\cdot b := a_i\overline{b_i}$

$\texttt{DotProductIsInner} :: \forall k : \texttt{ConjugationField} \,.\, \forall n\in\mathbb{N} \,.\, \big(k^n,(\cdot)\big) : \texttt{InnerProductSpace}(k)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{InnerProductAsDotProduct} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(k) \,.\, \forall e : \texttt{OrthonormalBasis}(V)$

$\quad .\, \forall v,w\in V \,.\, \langle v,w\rangle = v_e\cdot w_e$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.4 Orthogonal Matrices and QR-Decomposition

$\texttt{OrthogonalMatrix} :: \prod k : \texttt{ConjugationField} \,.\, \prod n \in \mathbb{N} \,.\, ?k^{n\times n}$

$A : \texttt{OrthogonalMatrix} \iff \mathcal{C}(A) \in E_n(k^n)$

$\texttt{OrthonormalMatrix} :: \prod k : \texttt{ConjugationField} \,.\, \prod n \in \mathbb{N} \,.\, ?k^{n\times n}$

$A : \texttt{OrthonormalMatrix} \iff \mathcal{C}(A) \in V_n(k^n) \iff$

$\texttt{conjugateTranspose} :: \prod n, m \in \mathbb{N} \,.\, k^{n\times m} \to k^{m\times n}$

$\texttt{conjugateTranspose}\,(A) = A^{\bar{\top}} := \overline{A}^{\top}$

$\texttt{OrthogonalMatrixProperty} :: \forall A \in k^{n\times n} \,.\, A : \texttt{OrthogonalMatrix}(k,n) \iff A^{\bar{\top}}A : \texttt{Diagonal}(k,n)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrthonormalMatrixProperty} :: \forall A \in k^{n\times n} \,.\, A : \texttt{OrthonormalMatrix}(k,n) \iff A^{\bar{\top}}A = I$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrthonormalMatricesFormAGroup} :: \texttt{OrthonormalAtrix}(k,n) \in \mathsf{GRP}$

$\texttt{Proof} =$

$\texttt{Assume}\ A, B : \texttt{OrthonormalMatrix}(k,n),$

$[1] := \texttt{ProductConjugateTranspose}(A,B)\texttt{OrthonormalMatrixProperty}^2(k,n)(A)(B) :$

$\quad : (AB)^{\bar{\top}}AB = B^{\bar{\top}}A^{\bar{\top}}AB = B^{\bar{\top}}B = I,$

$[A.*.1] := \texttt{OrthonormalMatrixProperty}[1] : AB : \texttt{OrthonormalMatrix}(k,n),$

$[2] := \texttt{MatLIIsRI}(k,n)\texttt{OrthonormalMatrixProperty}(A) : A^{-1} = A^{\bar{\top}},$

$[3] := \mathcal{I}\,\texttt{Inverse}[2] : I = AA^{\top} = A^{-\bar{\top}}A^{-1},$

$[A.*.1] := \texttt{OrthonormalMatrixProperty}[3] : A^{-1} : \texttt{OrthonormalMatrix}(k,n);$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\mathsf{GRP} : \texttt{OrthonormalMatrix}(k,n) \in \mathsf{GRP},$

$\square$

$\texttt{orthogonalGroup} = \mathbf{O} := \texttt{orthonormalMatrix}(\mathbb{R}) : \mathbb{N} \to \mathsf{GRP};$

$\texttt{unitaryGroup} = \mathbf{U} := \texttt{orthonormaMatrix}(\mathbb{C}) : \mathbb{N} \to \mathsf{GRP};$

$\texttt{QRDecomposition} :: \forall k : \texttt{ConjugatioField}(S) \;.\; \forall n \in \mathbb{N} \;.\; \forall A \in k^{n \times n} \;.$
$\;.\; \exists Q : \texttt{OrthognanalMatrix}(k,n) \exists R : \texttt{RowEchelonForm}(k,n) : A = QR$

$\texttt{Proof} \;=$

$q := \mathbf{GS}\Big(\mathcal{C}) : \texttt{Orthogonal}(n, k^n),$

$Q := \texttt{fromColumns}(q) : k^{n \times n},$

$[1] := \mathcal{C}\texttt{OrthogonalMatrix}\mathcal{D}(Q) : (Q : \texttt{OrthogonalMatrix}(k,n)),$

$(R, [*]) := \mathcal{C}\,\mathbf{GS}\,\mathcal{D}(Q) : \sum R : \texttt{RowEchelonForm}(k,n) \;.\; A = QR;$

$\square$

$\texttt{NormalQRDecomposition} :: \forall S : \texttt{WithSquareRoots} \;.\; \forall k : \texttt{ConjugationField}(S) \;.\; \forall n \in \mathbb{N} \;.\; \forall A \in k^{n \times n} \;.$
$\;.\; \exists Q : \texttt{OrthonormalMatrix}(k,n) \;.\; \exists R : \texttt{UpperTriangularMatrix}(k,n) : A = QR$

$\texttt{Proof} \;=$

$r := \operatorname{rank} A : \mathbb{Z}_+,$

$p := \mathcal{D}(r)\texttt{enumerateMaxLIExists}(\mathcal{C}(A)) : \texttt{LinearlyIndependent}(r, k^n),$

$q := \mathbf{NGS}(p) : V_r(k^n),$

$(q', [1]) := \texttt{GrammSmidtAugementation}(k^n)(q)\mathcal{C}\dim k^n \mathcal{C}\texttt{rank} : \sum q' \in V_n(k^n) \;.\; q'_{|r} = q,$

$Q := \texttt{FromColumns}(q') : k^{n \times n},$

$[2] := \mathcal{C}\texttt{OrthonormalMatrix}\mathcal{D}(Q) : (Q : \texttt{OrthonormalMatrix}(k,n)),$

$(R, [*]) := \mathcal{C}\,\mathbf{NGS}\,\mathcal{D}(Q) : \sum R : \texttt{RowEchelonForm}(k,n) \;.\; A = QR;$

$\square$

$\texttt{decomposeQR} :: \prod S : \texttt{WithSquareRoots} \;.\; \prod k : \texttt{ConjugationField}(S) \;.\; \prod n, m \in \mathbb{N} \;.$
$\;.\; k^{n \times m} \to \texttt{OrthonormalMatrix}(k,n) \times \texttt{UpperTriangularMatrix}(k,n,m)$
$\texttt{decomposeQR}\,(A) = (Q(A), R(A)) := \texttt{NormedQRDecomposition}(A)$

$\texttt{OrthonormalDet} :: \forall A : \texttt{Orthonormal}(k,n) \;.\; |\det A| = 1$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{QRDet} :: \forall A \in k^{n \times n} \;.\; |\det R(A)| = |\det A|$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OrthogonalTriangulization} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(k) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \;.$
$\;.\; \exists e : \texttt{OrthonormalBasis}(V) \;.\; T^{e,e} : \texttt{UpperTriangularMatrix}(k, \dim V, \dim V)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{specialOrthogonalGroup} = \mathbf{SO} := \Lambda n \in \mathbb{N} \;.\; \{A \in \mathbf{O}(n) | \det A = 1\} : \mathbb{N} \to \mathsf{GRP};$

$\texttt{specialUnitaryGroup} = \mathbf{SU} := \Lambda n \in \mathbb{N} \;.\; \{A \in \mathbf{U}(n) | \det A = 1\} : \mathbb{N} \to \mathsf{GRP};$

2.5 Finite-Dimensional Riez Representation Theorem

$\texttt{asFunctional} :: \prod V : \texttt{InnerProductSpace}(k) \; . \; V \xrightarrow{k\text{-VS}} \overline{V^*}$

$\texttt{asFunctional}\,(v) = \phi_v := \Lambda u \in V \; . \; \langle u, v \rangle$

$\texttt{FDRieszRepresentationTheorem} :: \forall S : \texttt{WithSquareRoots} \; . \; \forall k : \texttt{ConjugationFiels}(S) \; . \; \forall V : \texttt{FiniteDimensi}$

$\quad \exists! v \in V : \phi_v = f$

$\texttt{Proof} \; =$

$e := \texttt{OrthogonalBasisTHM} : \texttt{OrthonormalBasis}(V),$

$(\alpha, [1]) := \mathcal{C}\texttt{Basis}(e^*, f) : \sum \alpha \in k^{\dim V} \; . \; f = \alpha e^*,$

$v := \overline{\alpha} e : V,$

$\texttt{Assume}\; u : V,$

$(\beta, [2]) := \mathcal{C}\texttt{Basis}(e)(u) : \sum \beta \in k^{\dim V} \; . \; u = \beta e,$

$[u.*] := \mathcal{C}\texttt{DualBasis}[1][2]\mathcal{C}^{-1}\texttt{dotProduct}\texttt{InnerProductAsInnerProduct}(V, e)[1][2]\mathcal{C}^{-1}\phi_v :$

$\quad : f(u) = \beta_i \alpha_i = \beta \cdot \overline{\alpha} = \langle u, v \rangle = \phi_v(u);$

$\rightsquigarrow [2] := I(=, \rightarrow) : f = phi_v,$

$\texttt{Assume}\; w : V,$

$\texttt{Assume}\; [3] : f = \phi_w,$

$[4] := [3][2] : \phi_v = \phi_w,$

$\texttt{Assume}\; u : V,$

$[u.*] := [4]\texttt{MultiAdditive}(\langle \cdot, \cdot \rangle) : 0 = \phi_v(u) - \phi_w(u) = \langle u, v \rangle - \langle u, w \rangle = \langle u, v - w \rangle;$

$\rightsquigarrow [5] := I(\forall) : \forall u \in V \; . \; \langle u, v - w \rangle = 0,$

$[w.*] := \mathcal{C}\texttt{Nondegenerate}(\langle \cdot, \cdot \rangle)[5] : v - w = 0;$

$\rightsquigarrow [*] := \mathcal{C}^{-1}\texttt{Unique} : \texttt{This},$

$\ldots$

$\square$

$\texttt{VectorOfRiesz} :: \prod S : \texttt{WithSquareRoots} \; . \; \prod k : \texttt{ConjugationFiels}(S) \; .$

$\quad . \prod V : \texttt{FiniteDimensionalInnerProductSpace}(k) \; . \; V^* \xrightarrow{k\text{-VS}} \overline{V}$

$\texttt{VectorOfRiesz}\,(f) = v_f := \texttt{FDRiezRepresentationTheorem}(f)$

$\texttt{RieszIsomorphism} :: \forall S : \texttt{WithSquareRoots} \; . \; \forall k : \texttt{ConjugationFiels}(S) \; .$

$\quad \forall V : \texttt{FiniteDimensionalInnerProductSpace}(k) \; . \; v : V \xleftrightarrow{k\text{-VS}} \overline{V}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

2.6 Adjoint Operators

Assume S : WithSquareRoots,
Assume k : ConjugationField,
Assume V, W, Y : k-FDVS,

Adjoint :: $\mathcal{M}_{k\text{-VS}}(V,W) \to ?\mathcal{M}_{k\text{-VS}}(W,V)$
T' : Adjoint $\iff \Lambda T \in \mathcal{M}_{K\text{-VS}}(V,W) \,.\, \forall v \in V \,.\, \forall w \in W \langle Tv, w\rangle = \langle v, T'w\angle$

AdjointUnique :: $\forall T \in \mathcal{M}_{k\text{-VS}}(V,W) \,.\, \exists!$Adjoint$(V)$
Proof =
$e :=$ OrthonormalBasisTHM(V) : OrthonormalBasis(V),
$f :=$ OrthonormalBasisTHM(W) : OrthonormalBasis(W),
$n := \dim V : \mathbb{N}$,
$m := \dim W : \mathbb{N}$,
$A :=$ matrixOfOperator$(e, f, T) : k^{n\times n}$,
$T' :=$ FromMatrix$(A^{\bar{\top}}, f, e) : \mathcal{M}_{k\text{-VS}}(W,V)$,
Assume $i, j : m$,
$[(i,j).*] := \mathcal{D}A\mathcal{I}^4$OBasis$(e)^2(f)^2\mathcal{D}^{-1}T' : \langle Te_i, e_j\rangle = A_{j,i} = \langle e_i, T'e_j\rangle$;
$\leadsto [1] := \mathcal{I}\mathcal{L}(V,V;k)\Big(\langle \perp, \perp\rangle\Big)I(\forall) : \forall v, u \in V \,.\, \langle Tv, u\rangle = \langle v, Tu\rangle$,
$[2] := \mathcal{I}^{-1}$Adjoint$(T)(T')$: $\Big(T'$: Adjoint$(T)\Big)$,
Assume T'' : Adjoint(T),
Assume $v : V$,
Assume $u : V$,
$[u.*] := \mathcal{I}$Adjoint$(A)[1]$MultiAdditive$(\langle\cdot,\cdot\rangle)\mathcal{I}^{-1}$mapAdd$(T', T'')$:
$\quad : 0 = \langle v, T'u\rangle - \langle v, T''u\rangle = \langle v, T'u - T''u\rangle = \big\langle v, (T' - T'')u\big\rangle$,
$\leadsto [4] := I(\forall) : \forall u \in V \,.\, \big\langle u, (T' - T'')v\big\rangle = 0$,
$[v.*] := \mathcal{I}$Nondegenerate$(\langle\cdot,\cdot\rangle)[4] : (T' - T'')v = 0$;
$\leadsto [5] := I(=, \to) : T' - T'' = 0$;
$\leadsto [*] := \mathcal{I}^{-1}$Unique$(T') : \exists!$Adjoint$(T)$;

adjointOp :: $\mathrm{End}_{k\text{-VS}} \to \mathrm{End}_{k\text{-VS}}$
adjointOp $(T) = T^\star :=$ AdjointUnique(T)

AdjointAdditive :: $\forall A, B : V \xrightarrow{k\text{-VS}} W \,.\, (A+B)^\star = A^\star + B^\star$
Proof =
...
□

AdjointConjugateHomogen :: $\forall A : V \xrightarrow{k\text{-VS}} \,.\, \forall \alpha \in k \,.\, (\alpha A)^\star = \overline{\alpha} A^\star$
Proof =
...
□

$\mathtt{AdjointOfAdjoint} :: \forall A : V \xrightarrow{k\text{-VS}} W \;.\; A^{\star\star} = A$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ v : V,$

$\mathtt{Assume}\ w : W,$

$[(v,w).*] := Ɑ\mathtt{ConjugateSymmetric}(V)Ɑ\mathtt{Conjugate}(T)(T^\star)Ɑ\mathtt{ConjugateSymmetric}(V) :$

$\quad : \langle T^* v, w\rangle = \overline{\langle w, T^* v\rangle} = \overline{\langle Tw, v\rangle} = \langle v, Tw\rangle;$

$\rightsquigarrow [*] := \mathtt{AdjointUnique} : A^{\star\star} = A;$

$\square$

$\mathtt{AdjointCompose} :: \forall A : V \xrightarrow{k\text{-VS}} W \;.\; \forall B : W \xrightarrow{k\text{-VS}} \;.\; (AB)^\star = B^\star A^\star$

$\mathtt{Proof} \;=$

$\ldots$

$\square$

$\mathtt{AdjointInverse} :: \forall A : V \xleftrightarrow{k\text{-VS}} W \;.\; \left(A^{-1}\right)^\star = \left(A^\star\right)^{-1}$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ v, u : V,$

$[(v,w).*] := Ɑ\mathtt{Inverse}Ɑ\mathtt{Adjoint} : \langle u, v\rangle = \langle A^{-1}Au, v\rangle = \left\langle u, A^*\left(A^{-1}\right)^\star v\right\rangle;$

$\rightsquigarrow [1] := Ɑ\mathtt{Nondegenerate}(\langle\cdot,\cdot\rangle)ƑK\text{-VS}(V,V) : A^*\left(A^{-1}\right)^\star = \mathrm{id},$

$[*] := \mathtt{UniqueInverseInFiniteDimension}[1] : \left(A^{-1}\right)^\star = \left(A^\star\right)^{-1};$

$\square$

$\mathtt{AdjointInvariantCondition} :: \forall A : \mathrm{End}_{k\text{-VS}}(V) \;.\; \forall S \subset_{k\text{-VS}} V \;.\; S : \mathtt{InvariantSubspace}(V,A) \iff S^\perp : \mathtt{Inva}$

$\mathtt{Proof} \;=$

$\mathtt{Assume}\ L : \Big(S : \mathtt{InvariantSubspace}(V,A)\Big),$

$\mathtt{Assume}\ v : S^\perp,$

$\mathtt{Assume}\ u : S,$

$[u.*] := Ɑ\mathtt{InvariantSubspace}(V,A)ƑS^\perp Ɑ^{-1}A^\star : 0 = \langle Au, v\rangle = \langle u, A^\star v\rangle;$

$\rightsquigarrow [v.*] := ƑS^\perp : A^\star v \in S^\perp;$

$\rightsquigarrow [L.*] := Ɑ^{-1}\mathtt{InvariantSubspace} : \left(S^\perp : \mathtt{InvariantSubspace}(V, A^\star)\right);$

$\rightsquigarrow [1] := I(\Rightarrow) : \mathtt{Left} \Rightarrow \mathtt{Right},$

$[*] := I(\iff)\mathtt{AdjointOfAdjoint}(A)\mathtt{OrthogonalOfOrthogonal}(S)[2] : \mathtt{This};$

$\square$

AdjointReduction $:: \forall A : \mathrm{End}_{k\text{-VS}} \ . \ \forall S \subset_{k\text{-VS}} \ .$

$. \ A = S \boxplus S^\perp \iff S :$ InvariantSubspace$(V)(A)(A^\star)$

Proof $=$

$[1] := \mathcal{A}$OrthogonalDirectSum$(A) : V = S \perp S^\perp,$

Assume $L : A = S \boxplus S^\perp,$

$[L.*] := \mathcal{A}$ReducingSystem$(L) : \Big(S :$ InvariantSubspace$(V)(A)(A^\star)\Big);$

$\rightsquigarrow [2] := I(\Rightarrow) :$ Left $\Rightarrow$ Right,

Assume $R : \Big(S :$ InvariantSubspace$(V)(A)(A^\star)\Big),$

$[3] :=$ AdjointOfAdjoint(AdjointInvariantCondition$(R)) : \Big(S^\perp :$ InvariantSubspace$(V, A)\Big),$

$[R.*] := \mathcal{A}^{-1}$ReducingSystem$\big([1], R, [3]\big) : A = S \boxplus S^\perp;$

$\rightsquigarrow [*] := I(\iff)[2]I(\Rightarrow) :$ This,

$\square$

AdjointKernel $:: \forall A : V \xrightarrow{k\text{-VS}} W \ . \ \ker A^\star = (\operatorname{Im} A)^\perp$

Proof $=$

Assume $v : (\operatorname{Im} A)^\perp,$

Assume $u : V,$

$[u.*] := \mathcal{A}$orthogonalCpmplement$\mathcal{A}$Adjoint$(A) : 0 = \langle Au, v\rangle = \langle u, A^\star v\rangle;$

$\rightsquigarrow [1] := \mathcal{A}$Nondegenerate$\big(\langle\cdot,\cdot\rangle\big) : A^\star v = 0,$

$[v.*] := \mathcal{A} \ker A^\star[0] : v \in \ker A^\star;$

$\rightsquigarrow [1] := \mathcal{A}$Subset $: (\operatorname{Im} A)^\perp \subset \ker A^\star,$

Assume $u : \ker A^\star,$

Assume $v : V,$

$[v.*] :=: 0 = \langle v, 0\rangle = \langle v, T^* u\rangle = \langle Tv, u\rangle;$

$\rightsquigarrow [u.*] := \mathcal{A}(\operatorname{Im} T)^\perp : u \in (\operatorname{Im} T)^\perp;$

$\rightsquigarrow [*] := \mathcal{A}$SetEq$\mathcal{A}$Subset$[1] : \ker A^\star = (\operatorname{Im} A)^\perp;$

$\square$

AdjointImage $:: \forall A : V \xrightarrow{k\text{-VS}} W \ . \ \operatorname{Im} A^\star = (\ker A)^\perp$

Proof $=$

$\ldots$

$\square$

AdjointCompositionKernel $:: \forall A : \mathrm{End}_{k\text{-VS}}(V) \ . \ \ker AA^\star = \ker A$

Proof $=$

$\ldots$

$\square$

AdjointCompositionImage $:: \forall A : \mathrm{End}_{k\text{-VS}}(V) \ . \ \operatorname{Im} A^\star A = \operatorname{Im} A$

Proof $=$

$\ldots$

$\square$

$\texttt{AdjointAsDual} :: \forall A : V \xrightarrow{k\text{-VS}} W \;.\; A^\star = \phi A^* v$

$\texttt{Proof} \;=$

$\texttt{Assume}\ u : V,$

$\texttt{Assume}\ w : W,$

$[1] := Ɑ A^* Ɑ v Ɑ \phi_w : \langle u, (\phi A^* v) w\rangle = \langle u, (A^* v)\phi_w\rangle = \langle u, v(A\phi_w)\rangle = A\phi_w(u) = \langle Au, w\rangle,$

$[*] := \texttt{AdjointUnique}[1] : A^\star = \phi A^* v;$

$\square$

$\texttt{LeftInverseByAdjoint} :: \forall A : V \xrightarrow{k\text{-VS}} W \;\&\; V \twoheadrightarrow W \;.\; (A^\star A)^{-1} A^\star : \texttt{RightInverse}(A)$

$\texttt{Proof} \;=$

$[1] := \texttt{AddjointCompositionKernel}(A^\star)\texttt{AdjointKernel}(A) Ɑ \texttt{Surjective}(A) Ɑ \texttt{orthogonalComplement} :$

$\quad : \ker A^\star A = \ker A^\star = (\operatorname{Im} A)^\perp = \{0\},$

$[2] := \texttt{FiniteDimensionalInvertibility}[1] : \Big((A^\star A) : \texttt{Invertible}(V)\Big),$

$[*] := Ɑ\texttt{Inverse} : (A^\star A)^{-1} A^\star A = \mathrm{id};$

$\square$

$\texttt{RightInverseByAdjoint} :: \forall A : V \xrightarrow{k\text{-VS}} W \;\&\; V \hookrightarrow W \;.\; A^\star (A^\star A)^{-1} : \texttt{LeftInverse}(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

2.7 Orhogonal Projectors

$\texttt{OrthogonalProjector} :: ?\texttt{Projector}(V)$
$P : \texttt{OrthogonalProjector} \iff \operatorname{Im} P \perp \ker P$

$\texttt{ProjectionIsOrthogonalIffSelfAdjoint} :: \forall P : \texttt{Projector}(V) \ .$
$\quad . \ P : \texttt{OrthogonalProjector}(V) \iff P = P^\star$
$\texttt{Proof} \ =$
$\texttt{Assume}\ L : \big(P : \texttt{OrthogonalProjector}(V)\big),$
$[1] := \mathcal{I}\texttt{OrthogonalProjector}(V) : \operatorname{Im} P \perp \ker P,$
$[2] := \texttt{OrthogonalDirectSum}(P) : V = \operatorname{Im} P \oplus \ker P,$
$\texttt{Assume}\ u : \operatorname{Im} P,$
$\texttt{Assume}\ v : \operatorname{Im} P,$
$[3] := \mathcal{I}\texttt{Projector}(P)\mathcal{I}\texttt{Adjoint}(P^\star) : \langle v, Pu\rangle = \langle Pv, u\rangle = \langle v, P^\star u\rangle,$
$[v.*] := \mathcal{I}\mathcal{L}(V, V; k)(V)[1] : \langle v, u - P^\star u\rangle = 0;$
$\rightsquigarrow [3] := I(\forall) : \forall v \in \operatorname{Im} P \ . \ \langle v, Pu - P^\star u\rangle = 0,$
$\texttt{Assume}\ v : \ker P,$
$[4] := \mathcal{I}\texttt{Orthogonal}(k)\mathcal{I} \ker P\mathcal{I}\texttt{Adjoint}(P^\star) : \langle v, Pu\rangle = 0 = \langle 0, u\rangle = \langle Pv, u\rangle = \langle v, P^\star u\rangle,$
$[v.*] := \mathcal{I}\mathcal{L}(V, V; k)(V)[2] : \langle v, pu - P^\star u\rangle = 0;$
$\rightsquigarrow [4] := I(\forall) : \forall v \in \ker P \ . \ \langle v, Pu - P^\star u\rangle = 0,$
$[5] := [2][3][4] : \forall v \in V \ . \ \langle v, Pu - P^\star u\rangle = 0,$
$[u.*] := \mathcal{I}\texttt{NonDegenerate}(V)[5] : Pu = P^\star u;$
$\rightsquigarrow [3] := I(\forall) : \forall u \in \operatorname{Im} P \ . \ Pu = P^\star u,$
$\texttt{Assume}\ u : \ker P,$
$\texttt{Assume}\ v : V,$
$[4] := \mathcal{I}\texttt{Orthogonal}(k)\mathcal{I} \ker P\mathcal{I}\texttt{Adjoint}(P^\star) : \langle v, Pu\rangle = \langle v, 0\rangle = 0 = \langle Pv, u\rangle = \langle v, P^\star u\rangle,$
$[(u, v).*] := \texttt{NonDegenerate}(V)[4] : Pu = P^\star u;$
$\rightsquigarrow [4] := I(\forall) : \forall u \in \ker P \ . \ Pu = P^\star u,$
$[5] := [2][3][4] : \forall u \in V \ . \ Pu = P^\star u,$
$[6] := ((=, \rightarrow)[5]) : P = P^\star;$
$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$
$\texttt{Assume}\ R : P = P^\star,$
$\texttt{Assume}\ v : \operatorname{Im} P,$
$\texttt{Assume}\ w : \ker P,$
$[*] := \mathcal{I}\texttt{Projection}(P)(R)\mathcal{I} \ker P : \langle v, w\rangle = \langle Pv, w\rangle = \langle v, Pw\rangle = 0;$
$\rightsquigarrow [2] := I(\forall)\mathcal{I}\texttt{OrthogonalVectors} : \forall v \in \operatorname{Im} P \ . \ \forall w \in \ker P \ . \ v \perp w,$
$\rightsquigarrow [3] := \mathcal{I}^{-1}\texttt{OrthogonalSet}[2] : \operatorname{Im} P \perp \ker P,$
$[R.*] := \mathcal{I}^{-1}\texttt{OrthogonalProjector} : \Big(P : \texttt{Orthogonalprojector}(V)\Big);$
$\rightsquigarrow [*] := I(\iff)I(\Rightarrow)[1] : \texttt{This};$
$\square$

$\texttt{OrthogonalResolutionOfIdentity} := \texttt{ResolutionOfIdentity} \ \& \ \texttt{OrthogonalProjector} :$

$: \prod k : \texttt{ConhugationField} \ . \ \mathsf{SET} \to \texttt{InnerProductSpace}(k) \to \texttt{Type};$

$\texttt{OrthogonalResolutionProperty} :: \forall P : \texttt{OrthogonalResolutionOfIdentity}(X, V) \ . \ V = \bigperp_{x \in X} \operatorname{Im} P_x$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{OrthogonalResolutionCondition} :: \forall V : \texttt{InnerProductSpace}(k) \ . \ \forall X \in \mathsf{SET} \ .$

$. \ \forall S : X \to \texttt{VectorSubspace}(V) \ . \ V = \bigperp_{x \in X} S_x \Rightarrow$

$\Rightarrow \exists P : \texttt{OrthogonaResolutionOfIdentity}(X, V) : \forall x \in X \ . \ \operatorname{Im} P_x = S_x$

$\texttt{Proof} \ =$

...

$\square$

2.8 Normal Operators and Spectral Theorem

$\texttt{UnitaryDiagonalizable} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_{k\text{-}\mathsf{VS}}(V)$

$T : \texttt{UnitaryDaigonalizable} \iff \exists e : \texttt{OrthonormalBasis} : \left(T^{e,e} : \texttt{DiagonalMatrix}\right)$

$\texttt{UDByEigenvectors} :: \forall V : \texttt{InnerProductSpace}(K) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \;.$
$\quad T : \texttt{UnitatyDiagonalizable}(V) \iff \exists E : \texttt{OrthonormalBasis} : (\forall e \in E \;.\; e : \texttt{Eigenvector}(V))$
$\texttt{Proof} =$

$\texttt{NormalOperator} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_{k\text{-}\mathsf{VS}}(V)$

$T : \texttt{NormalOperator} \iff TT^\star = T^\star T$

$\texttt{NormalMatrix} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_{k\text{-}\mathsf{VS}}(V)$

$T : \texttt{NormalMatrix} \iff T\bar{T}^\top = \bar{T}^\top T$

$\texttt{OrthonormalBasisNormality} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \;.$
$\quad . \forall e : \texttt{OrthonormalBasis}(V) \;.\; T : \texttt{NormalOperator}(V) \iff T^{e,e} : \texttt{NormalMatrix}(V)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{NormalRestriction} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{NormalOperator}(V) \;.\; \forall S \subset_{k\text{-}\mathsf{VS}} V \;.$
$\quad . \forall [0] : T = S \boxplus S^\perp \;.\; T_{|S} : \texttt{NormalOperator}(V)$
$\texttt{Proof} =$
$[1] := \texttt{AdjointInvariantCondition}(T)\texttt{AdjointReduction}(T)[0] :$
$\quad : \left(S, S^\perp : \texttt{InvariantSubspace}(V)(T)(T^\star)\right),$
$[*] := [1]\mathcal{I}\texttt{NormalOperator}(T)[1] : T^\star_{|S}T_{|S} = (T^\star T)_{|S} = (TT^\star)_{|S} = T_{|S}T^\star_{|S};$
$\square$

$\texttt{NormalAdjoint} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{NormalOperator}(V) \;.\; T^\star : \texttt{NormalOperator}(V)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{NormalInverse} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{NormalOperator}(V) \;\&\; \mathbf{GL}(V) \;.$
$\quad . T^{-1} : \texttt{NormalOperator}(V)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{IsometricNormalAdjoint} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.\, \forall v, w \in V \,.\, \langle Tv, Tw \rangle =$

$\texttt{Proof} =$

$[*] := \mathcal{I}\,\texttt{Adjoint}(T)\,\mathcal{I}\,\texttt{NormalOperator}(V)\,\texttt{AdjointOfAdjoint}(T) :$

$\quad : \langle Tv, Tw \rangle = \langle v, T^\star T w \rangle = \langle v, T T^\star w \rangle = \langle T^\star v, T^\star w \rangle;$

$\square$

$\texttt{NormalKernel} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.\, \ker T^\star = \ker T$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NormalPowerKernel} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.\, \forall n \in \mathbb{N} \,.\, \ker(T^n) = \ker T$

$\texttt{Proof} =$

$[1] := \texttt{NormalKernel}(T) : \ker T^\star = \ker T,$

$[2] := \texttt{AdjointKernel}[1] : \ker T = (\operatorname{Im} T)^\perp,$

$[3] := \texttt{OrthogonalIntersect}[2] : \ker T \cap \operatorname{Im} T = \{0\},$

$[*] := \mathcal{I}(\ker T)\,\mathcal{I}(\operatorname{Im} T)[3] : \ker T^n = \ker T;$

$\square$

$\texttt{NormalPolynomial} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.\, \forall p \in k[x] \,.$

$\quad . \, p(T) : \texttt{NormalOperator}(V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{NormalMinimalPolynomial} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.$

$$. \, \exists n \in \mathbb{N} \,.\, p(x) : n \to \texttt{Prime}\Big(k[x]\Big) \,.\, m^T(x) = \prod_{i=1}^{n} p_i(x)$$

$\texttt{Proof} =$

$\texttt{Assume}\ p : \texttt{PrimeDivisor}(m^T(X)),$

$(n, q, [1]) := \texttt{PrimeDecomposition}\Big(m^T(x)\Big) : \sum n \in \mathbb{N} \,.\, \sum q \in k[x] \,.$

$\quad . \, p^n(x) q(x) = m^T(x) \;\&\; (p, q) : \texttt{Coprime}(k[x]),$

$[2] := \texttt{NormalPolynomial}(T, p) : \Big(T(p) : \texttt{NormalOperator}(V)\Big),$

$[3] := \texttt{ProductKernel}(m^T)[1]\texttt{NormalPowerKernel}(p(T))[2] : \ker p^n(x) q(x) = \ker p(x) q(x),$

$[p.*] := \mathcal{I}\,\texttt{minimalPolynomial}[3] : n = 1;$

$$\rightsquigarrow [*] := \mathcal{I}\,k[x] : m^T(x) = \prod_{i=1}^{n} p_i(x);$$

$\square$

$\texttt{NormalEigenvalule} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.$
$\quad . \, \forall \lambda : \texttt{Eigenvalue}(T) \,.\, \forall v : \texttt{Eigenvector}(T, \lambda) \,.\, T^\star v = \overline{\lambda} v$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{OrthogonalityByPrimality} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.$
$\quad . \, \forall A, B \subset_{k[x]\text{-MOD}} V_T \,.\, \forall [0] : \Big((m^{T_{|A}}, m^{T_{|B}}) : \texttt{Coprime}(k[x]) \Big) \,.\, A \perp B$

$\texttt{Proof} \ =$

$(a, b, [1]) := \mathit{ȡ}\texttt{Coprime}[0] : \sum a(x), b(x) \in k[x] \,.\, 1 = a(x) m^{T_{|A}}(x) + b(x) m^{T_{|B}}(x),$

$\alpha(x) := a(x) m^{T_{|A}}(x) : k[x],$

$\beta(x) := b(x) m^{T_{|A}}(x) : k[x],$

$\texttt{Assume } v : A,$

$\texttt{Assume } w : B,$

$[(v, w).*] := [1] \mathit{ȡ} \alpha(T) \mathit{ȡ} \texttt{Adjoint}\big(\beta(T)\big) \texttt{NormalPolynomial}(\beta, T) \texttt{NormalKernel}(\beta(T))[1] :$
$\quad : \langle v, w \rangle = \Big\langle \big(\alpha(T) + \beta(T)\big) v, w \Big\rangle = \langle \beta(T) v, w \rangle = \langle v, \beta^\star(T) w \rangle = 0;$

$\rightsquigarrow [*] := I(\forall) \mathit{ȡ} \texttt{OrthogonalSet} : \texttt{This};$

$\square$

$\texttt{NormalEigenspaceOrthogonality} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator}(V) \,.$
$\quad \forall \lambda, \mu : \texttt{Eigenvalue}(T) \,.\, \lambda \neq \mu \Rightarrow \mathcal{E}_T(\lambda) \perp \mathcal{E}_T(\mu)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ComplexSpectralTheorem} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \mathrm{End}_{\mathbb{C}\text{-VS}}(V) \,.$
$\quad T : \texttt{NormalOperator}(V) \iff T : \texttt{UnitaryDiaginalizable}(V)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

2.9 Self-Adjoint and Unitary Operators

$\texttt{SelfAdjoint} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_V(k)$

$T : \texttt{SelfAdjoint} \iff T = T^\star$

$\texttt{SkewSelfAdjoint} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_V(k)$

$T : \texttt{SkewSelfAdjoint} \iff -T = T^\star$

$\texttt{OrthogonalOperator} :: \prod V : \texttt{InnerProductSpace}(k) \;.\; ?\mathrm{End}_V(k)$

$T : \texttt{Orthogonal} \iff \mathbf{O}(V) \iff TT^\star = \mathrm{id} = T^\star T$

$\texttt{SelfAdjointIsNormal} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{SelfAdjoint}(V) \;.\; T : \texttt{NormalOperator}(V)$

Proof $=$

...

□

$\texttt{SkewSelfAdjointIsNormal} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{SkewSelfAdjoint}(V) \;.$
$\;.\; T : \texttt{NormalOperator}(V)$

Proof $=$

...

□

$\texttt{OrthogonalIsNormal} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \mathbf{O}(V) \;.\; T : \texttt{NormalOperator}(V)$

Proof $=$

...

□

$\texttt{SelfAdjointIsVS} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \texttt{SelfAdjoint}(V) \in k\text{-VS}$

Proof $=$

...

□

$\texttt{SkewSelfAdjointIsVS} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \texttt{SkewSelfAdjoint}(V) \in k\text{-VS}$

Proof $=$

...

□

$\texttt{SelAdjointPolynomial} :: \forall V : \texttt{InnerProductSpace}(k) \;.\; \forall T : \texttt{SelfAdjoint}(V) \;.$
$\;.\; \forall p(x) \in k[x] \;.\; p(T) : \texttt{SelfAdjoint}(V)$

Proof $=$

...

□

$\texttt{associateQuadraticForm} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \mathrm{End}_{k\text{-VS}}(V) \to \texttt{QuadraticForm}(V)$
$\texttt{associateQuadraticForm}\,(T) = \mathbf{Q}_T := \langle Tv, v \rangle$

$\texttt{OrderedValuesOfSymmetricForm} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) \,.\, T : \texttt{SelfAdjoint}(V) \Rightarrow$
$\texttt{Proof} \ =$
$\texttt{Assume}\ L : T \in \mathbf{O}(V),$
$\texttt{Assume}\ v : V,$
$[1] := \mathit{\Pi}\texttt{SelfAdjoint} : \overline{\langle v, Tv \rangle} = \langle Tv, v \rangle = \langle v, Tv \rangle,$
$[v.*] := \mathit{\Pi}\texttt{ConjugationField}(k) : \langle Tv, v \rangle \in R;$
$\rightsquigarrow [1] := I(\forall)\mathit{\Pi}^{-1}\,\mathrm{Im}\,\mathbf{Q}_T I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right};$
$\square$

$\texttt{ComplexQuadraticZeroTheorem} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \texttt{SelfAdjoint}(V) \,.\, \mathbf{Q}_T = 0 \Rightarrow T = 0$
$\texttt{Proof} \ =$
$\texttt{Assume}\ v : V,$
$\texttt{Assume}\ w : W,$
$[1] := [0](v+w)\mathit{\Pi}\mathbf{Q}_T\texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[0]\mathit{\Pi}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)$
$\quad \mathit{\Pi}\texttt{SelfAdjoint}(V)(T)\texttt{RealPartExpression} : 0 = \mathbf{Q}_T(v+w) = \Big\langle T(v+w), v+w \Big\rangle =$
$\quad = \langle Tv, v \rangle + \langle Tv, w \rangle + \langle Tw, v \rangle + \langle Tw, w \rangle = \langle Tv, w \rangle + \langle Tw, v \rangle = \langle Tv, w \rangle + \langle w, Tv \rangle =$
$\quad = \langle Tv, w \rangle + \overline{\langle Tv, w \rangle} = 2\Re\langle Tv, w \rangle,$
$[2] := [0](v+w)\mathit{\Pi}\mathbf{Q}_T\texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[0]\mathit{\Pi}\texttt{commplexConjugation}\mathit{\Pi}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)$
$\quad \mathit{\Pi}\texttt{SelfAdjoint}(V)(T)\texttt{ImaginablePartExpression} : 0 = \mathbf{Q}_T(v + \mathrm{i}w) = \Big\langle T(v+\mathrm{i}w), v+\mathrm{i}w \Big\rangle =$
$\quad = \langle Tv, v \rangle + \mathrm{i}\langle Tv, w \rangle - \mathrm{i}\langle Tw, v \rangle - \langle Tw, w \rangle = \mathrm{i}\langle Tv, w \rangle - \mathrm{i}\langle Tw, v \rangle = \mathrm{i}\langle Tv, w \rangle - \mathrm{i}\langle w, Tv \rangle =$
$\quad = \mathrm{i}\langle Tv, w \rangle - \mathrm{i}\overline{\langle Tv.w \rangle} = -2\Im\langle Tv, w \rangle,$
$[*] := \texttt{ZeroComplexNumber}[1][2] : \langle Tv, w \rangle = 0;$
$\rightsquigarrow [*] := \mathit{\Pi}\texttt{NonDegenerate}I(=,\to) : Tv = 0,$
$\square$

$\texttt{RealQuadraticZeroTheorem} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \texttt{SelfAdjoint}(V) \,.\, \mathbf{Q}_T = 0 \Rightarrow T = 0$
$\texttt{Proof} \ =$
$\texttt{Assume}\ v : V,$
$\texttt{Assume}\ w : W,$
$[*] := [0](v+w)\mathit{\Pi}\mathbf{Q}_T\texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[0]\mathit{\Pi}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)$
$\quad \mathit{\Pi}\texttt{SelfAdjoint}(V)(T) : 0 = \mathbf{Q}_T(v+w) = \Big\langle T(v+w), v+w \Big\rangle =$
$\quad = \langle Tv, v \rangle + \langle Tv, w \rangle + \langle Tw, v \rangle + \langle Tw, w \rangle = \langle Tv, w \rangle + \langle Tw, v \rangle = \langle Tv, w \rangle + \langle w, Tv \rangle =$
$\quad = \langle Tv, w \rangle + \langle Tv.w \rangle = 2\langle Tv, w \rangle,$
$\rightsquigarrow [*] := \mathit{\Pi}\texttt{NonDegenerate}I(=,\to) : Tv = 0,$
$\square$

$\texttt{ComplexQuadraticZeroTheorem2} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \texttt{SkewSelfAdjoint}(V) \,.\, \mathbf{Q}_T = 0 \Rightarrow T$

Proof $=$

Assume $v : V$,

Assume $w : W$,

$[1] := [0](v+w) \mathcal{I} \mathbf{Q}_T \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[0]\mathcal{I}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)$

$\mathcal{I}\texttt{SelfAdjoint}(V)(T)\texttt{RealPartExpression} : 0 = \mathbf{Q}_T(v+w) = \Big\langle T(v+w), v+w\Big\rangle =$

$= \langle Tv, v\rangle + \langle Tv, w\rangle + \langle Tw, v\rangle + \langle Tw, w\rangle = \langle Tv, w\rangle + \langle Tw, v\rangle = \langle Tv, w\rangle - \langle w, Tv\rangle =$

$= \langle Tv, w\rangle - \overline{\langle Tv, w\rangle} = 2\mathbf{i}\Im\langle Tv, w\rangle,$

$[2] := [0](v+w) \mathcal{I} \mathbf{Q}_T \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[0]\mathcal{I}\texttt{commplexConjugation}\mathcal{I}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)$

$\mathcal{I}\texttt{SelfAdjoint}(V)(T)\texttt{ImaginablePartExpression} : 0 = \mathbf{Q}_T(v+\mathrm{i}w) = \Big\langle T(v+\mathrm{i}w), v+\mathrm{i}w\Big\rangle =$

$= \langle Tv, v\rangle + \mathrm{i}\langle Tv, w\rangle - \mathrm{i}\langle Tw, v\rangle - \langle Tw, w\rangle = \mathrm{i}\langle Tv, w\rangle + \mathrm{i}\langle Tw, v\rangle = \mathrm{i}\langle Tv, w\rangle + \mathrm{i}\langle w, Tv\rangle =$

$= \mathrm{i}\langle Tv, w\rangle + \mathrm{i}\overline{\langle Tv.w\rangle} = 2\mathrm{i}\Re\langle Tv, w\rangle,$

$[*] := \texttt{ZeroComplexNumber}[1][2] : \langle Tv, w\rangle = 0;$

$\rightsquigarrow [*] := \mathcal{I}\texttt{NonDegenerate}I(=,\to) : Tv = 0,$

$\square$

$\texttt{SelfAdjointByRealValues} :: \forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V) \,.$

$\,.\, \forall[00] : \mathrm{Im}\, \mathbf{Q}_T \subset \mathbb{R} \,.\, T \in \texttt{SelfAdjoint}(V)$

Proof $=$

Assume $v : V$,

$[v.*] := \texttt{RealConjugate}(\ldots)\mathcal{I}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)\mathcal{I}\texttt{AdjointT} : \langle Tv, v\rangle = \overline{\langle Tv, v\rangle} = \langle v, Tv\rangle = \langle T^\star v, v\rangle;$

$\rightsquigarrow [1] := \mathcal{I}^{-1}\mathbf{Q}_T I(=,\to) : \mathbf{Q}_T = \mathbf{Q}_{T^\star},$

$[2] := \mathcal{I}\texttt{adjoint} : \Big(T - T^\star\Big)^\star = T^\star - T,$

$[3] := \mathcal{I}^{-1}\texttt{SkewSelfAdjoint}(V)[2] : \Big(T - T^\star : \texttt{SkewSelfAdjoint}(V)\Big),$

$[4] := \texttt{ComplexQuadraticZeroTheorem2}[1][3] : T = T^\star,$

$[*] := \mathcal{I}^{-1}\texttt{SelfAdjoint}(V)[4] : \Big(T : \texttt{SelfAdjoint}(V)\Big);$

$\square$

$\texttt{SelfAdjointHasRealSpectre} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.$

$\,.\, \forall T \in \texttt{SelfAdjoint}(V) \,.\, \mathrm{supp}\, \sigma_T \subset \mathbb{R}$

Proof $=$

Assume $\lambda : \texttt{Eigenvalue}(T)$,

$\Big(v, [1]\Big) := \mathcal{I}\texttt{Eigenvector}(T) : \sum v \in V \,.\, Tv = \lambda v \,\&\, v \neq 0,$

$[2] := \mathcal{I}^{-1}\|v\|^2[1]\texttt{OrderedValuesOfQuadraticForms}(V, T) : \lambda\|v\|^2 = \langle \lambda v, v\rangle = \langle Tv, v\rangle \in \mathbb{R},$

$[(\lambda.*)] := \mathcal{I}\|v\|^2[2] : \lambda \in \mathbb{R};$

$\rightsquigarrow [*] := \mathcal{I}\texttt{spectre} : \mathrm{supp}\, \sigma_T \subset \mathbb{R};$

$\square$

$\texttt{OrthogonalOperatorsAreGroup} :: \forall V \in \texttt{InnerProductSpace}(k) \,.\, \mathbf{O}(V) \in \mathsf{GRP}$

Proof $=$

...

$\square$

$\texttt{OrthogonalIsStabluUnderUnitScalarAction} :: \forall V \in \texttt{InnerProductSpace}(k) \,.\, \forall T \in \texttt{InnerProductSpace}(V$

$. \forall \sigma \in \mathbb{S} \,.\, \sigma T \in \mathbf{O}(V)$

Proof $=$

...

$\square$

$\texttt{OrthogonalOperatorsAreIsometries} :: \forall V \in \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, T \in \mathbf{O}(V) \iff$

$\iff T : \texttt{Isometry}(V)$

Proof $=$

...

$\square$

$\texttt{OrthogonalBasisProperty} :: \forall V \in \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.\, T \in \mathbf{O}(V) \iff$

$\iff \exists e : \texttt{OrthonormalBasis}(V) : \Big(Te : \texttt{OrthonormalBasis}(V)\Big)$

Proof $=$

...

$\square$

$\texttt{OrthogonalEigenvalues} :: \forall V \in \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathbf{O}(V) \,.\, \sigma_T(V) \subset \mathbb{S}(k)$

Proof $=$

...

$\square$

$\texttt{UnitaryEquivalent} :: \prod n \in \mathbb{N} \,.\, ?\big(k^{n \times n} \times k^{n \times n}\big)$

$(A, B) : \texttt{UnitaryEquivalent} \iff \exists U \in \mathbf{U}(k, n) : B = UAU^\star$

$\texttt{UnitaryEquivalenceCriterion} :: \forall n \in \mathbb{N} \,.\, \forall A, B \in k^{n \times n} \,.\, (A, B) : \texttt{UnitaryEquvalent}(k, n) \iff$

$\iff \exists e : \texttt{OrthonormalBasis}(k^n) : \exists f : \texttt{OrthonormalBasis}(k^n) \,.\, A_{e,e} = B_{f,f}$

Proof $=$

...

$\square$

$\color{red}{\texttt{SelfAdjointAdditiveDecomposition}}$ :: $\forall V \in \texttt{InnerProductSpace}(k) \,.\, \forall A \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.$
$\quad .\, \exists! X, Y : \texttt{SelfAdjoint}(V) \,.\, A = X + \mathrm{i}Y \ \&\ A^\star = X - \mathrm{i}Y$

$\texttt{Proof} \ =$

$X := \frac{1}{2}\Big(A + A^\star\Big) : \texttt{SelfAdjoint}(V),$

$Y := \frac{\mathrm{i}}{2}\Big(A^\star - A\Big) : \texttt{SelfAdjoint}(V),$

$[*.1] := \text{Ↄ}X\text{Ↄ}Y\mathrm{i} : A = X + \mathrm{i}Y,$

$[*.1] := \text{Ↄ}X\text{Ↄ}Y\mathrm{i} : A^\star = X - \mathrm{i}Y;$

$\ldots$

$\square$

$\color{red}{\texttt{SkewSymmetricEigenvalues}}$ :: $\forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.$
$\quad \forall T : \texttt{SkewSelfAdjoint}(V) \,.\, \operatorname{supp} \sigma_T \subset \mathrm{i}\mathbb{R}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \lambda : \texttt{Eigenvalue}(T),$

$(v, [1]) := \text{Ɑ}\texttt{Eigenvalue}(\lambda) : \sum v \in V \,.\, Tv = \lambda v \ \&\ v \neq 0,$

$[2] := \text{Ɑ}^{-1}\|v\|^2\text{Ɑ}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)[1]\text{Ɑ}\texttt{Adjoint}[1]\text{Ɑ}\texttt{InnerProduct}\Big(\langle\cdot,\cdot\rangle\Big)\text{Ɑ}^{-1}\|v\|^2 :$
$\quad : \lambda\|v\|^2 = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, -Tv\rangle = \langle v, -\langle v\rangle = -\overline{\langle}\|v\|^2,$

$[3] := [1]\frac{[2]}{\|v\|^2} : \lambda = -\overline{\lambda},$

$[\lambda.*] := \color{red}{\texttt{ImaginableByConjugation}}[3] : \lambda \in \mathrm{i}\mathbb{R};$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{spectrum} : \operatorname{supp} \sigma_T \subset \mathrm{i}\mathbb{R};$

$\square$

$\color{red}{\texttt{NormalByNorm}}$ :: $\forall V : \texttt{InnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}(V) \,.\, \forall [0] : \|Tv\| = \|T^\star v\| \,.\, T : \texttt{NormalOperator}($

$\texttt{Proof} \ =$

$\texttt{Assume}\ v : V,$

$[v.*] := \text{Ɑ}\texttt{Adjoint}[0] : \langle T^\star T v, v\rangle = \langle Tv, Tv\rangle = \langle T^\star v, T^\star v\rangle = \langle TT^\star v, v\rangle;$

$\rightsquigarrow [1] := \text{Ɑ}^{-1}\mathbf{Q}_T : \mathbf{Q}_{T^\star T} = \mathbf{Q}_{TT^\star},$

$[2] := \color{red}{\texttt{SelfAdjointProduct}}(T^\star) : \Big(T^\star T : \texttt{SelfAdjoint}(V)\Big),$

$[3] := \color{red}{\texttt{SelfAdjointProduct}}(T) : \Big(TT^\star : \texttt{SelfAdjoint}(V)\Big),$

$[4] := \color{red}{\texttt{ComplexquadraticZero}}[1][2][3] : T^\star T = TT^\star,$

$[*] := \text{Ɑ}^{-1}\texttt{NormalOperator}(V) : \Big(T : \texttt{NormalOperator}(V)\Big);$

$\square$

$\texttt{NormalIdempotent} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator} \,\&\, \texttt{Idempotent}(V) \,.$
$T : \texttt{SelfAdjoint}(V)$

Proof $=$

$[1] := \ldots :$
$: (\mathrm{id} - T)(\mathrm{id} - T)^\star = (\mathrm{id} - T)(\mathrm{id} - T^\star) = \mathrm{id} - T - T^\star - TT^\star =$
$= \mathrm{id} - T - T^\star - T^\star T = (\mathrm{id} - T)(\mathrm{id} - T^\star) = (\mathrm{id} - T)(\mathrm{id} - T^\star),$

$[2] := \mathcal{I}^{-1}\texttt{NormalOperator}[1] : \Big(\mathrm{id} - T : \texttt{NormalOperator}(V)\Big),$

$[3] := \mathcal{I}\texttt{NormalOperator}(\mathrm{id} - T)\mathcal{I}^{-1}\|\cdot\| : \forall v \in V \,.\, \Big\|(\mathrm{id} - T)^\star v\Big\| = \Big\|(\mathrm{id} - T)v\Big\|,$

$[4] := \mathcal{I}\texttt{Idemptent}(T) : 0 = T - T = T - T^2 = (\mathrm{id} - T)T,$

$[5] := [3][4] : 0 = (\mathrm{id} - T)^\star T = T - T^\star T,$

$[6] := \mathcal{I}\texttt{NormalOperator}(T)\mathcal{I}^{-1}\|\cdot\| : \Big\|T^\star v\Big\| = \Big\|Tv\Big\|,$

$[7] := \mathcal{I}\texttt{Idempotent} : T(\mathrm{id} - T) = 0,$

$[8] := [7][6] : 0 = T(\mathrm{id} - T) = T^\star(\mathrm{id} - T) = T^\star - T^\star T,$

$[9] := [5][8] : T^\star = T,$

$[*] := \mathcal{I}^{-1}\texttt{SelfAdjoint} : \Big(T : \texttt{SelfAdjoint}(V)\Big);$

$\square$

$\texttt{NormalNilpotent} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T : \texttt{NormalOperator} \,\&\, \texttt{Nilpotent}(V) \,.$
$.\, T = 0$

Proof $=$

$[1] := \mathcal{I}\texttt{NormalOperator}(\mathrm{id} - T)\mathcal{I}^{-1}\|\cdot\| : \forall v \in V \,.\, \Big\|Tv\Big\| = \Big\|T^\star v\Big\|,$

$[2] := \mathcal{I}\texttt{Nilpotent}(T)[1] : 0 = T^\star T,$

$[3] := \texttt{AdjointKer}(T)[2] : \ker T = \ker T^\star T = V,$

$[*] := \mathcal{I} \ker T[3] : T = 0;$

$\square$

2.10 Finite-Dimensional Functional Calculus

$$\texttt{complexFunctionalCalculi} :: \prod V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \ .$$
$$. \ (\mathbb{C} \to \mathbb{C}) \to \texttt{NormalOperator}(V) \to \texttt{NormalOperator}(V)$$
$$\texttt{complexFunctionalCalculi}\,(f, T) = f(T) := f(\lambda_i) e_i \otimes e_i^\star \quad \texttt{where} \quad (\lambda, e) = \texttt{SpectralTHM}(V, T)$$

$$\texttt{realFunctionalCalculi} :: \prod V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{R}) \ .$$
$$. \ (\mathbb{R} \to \mathbb{R}) \to \texttt{SelfAdjoint}(V) \to \texttt{SelfAdjoint}(V)$$
$$\texttt{realFunctionalCalculi}\,(f, T) = f(T) := f(\lambda_i) e_i \otimes e_i^\star \quad \texttt{where} \quad (\lambda, e) = \texttt{SpectralTHM}(\mathbb{C} \otimes_{\mathbb{R}} V, T)$$

$$\texttt{CommutesByInjectiveFunction} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \ .$$
$$. \ \forall A, B \in \texttt{NormalOperator}(V) \ . \ \forall f : \operatorname{supp}(\sigma_A + \sigma_B) \hookrightarrow \mathbb{C} \ .$$
$$. \ (A, B) : \texttt{Commutes} \iff \Big(f(A), f(B)\Big) : \texttt{Commutes}$$

Proof $=$

...

$\square$

$$\texttt{NormalCommutativity} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \ .$$
$$. \ \forall A, B \in \texttt{NormalOperator}(V) \ . \ (A, B) : \texttt{Commutes} \iff \exists p, q \in \mathbb{C}[x] : \exists f \in \mathbb{C}[x, y] :$$
$$: A = p\big(f(A, B)\big) \ \& \ B = q\big(f(A, B)\big)$$

Proof $=$

...

$\square$

2.11 Positive Operators and Polar Decomposition

$\texttt{PositiveDefinite} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?\texttt{SelfAdjoint}(V)$

$T : \texttt{PositiveDefinite} \iff T \in \mathbf{S}_{++}(V) \iff \forall v \in V \,.\, v \neq 0 \Rightarrow \mathbf{Q}_T(v) > 0$

$\texttt{PositiveSemiDefinite} :: \prod V : \texttt{InnerProductSpace}(k) \,.\, ?\texttt{SelfAdjoint}(V)$

$T : \texttt{PositiveSemiDefinite} \iff T \in \mathbf{S}_{+}(V) \iff \forall v \in V \,.\, \mathbf{Q}_T(v) \geq 0$

$\texttt{NonNegativeEigenvalues} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathbf{S}_{++}(V) \,.\, \operatorname{supp} \sigma_T \subset R_{++}$

$\texttt{Proof} =$

...

$\square$

$\texttt{PositiveEigenvalues} :: \forall V : \texttt{InnerProductSpace}(k) \,.\, \forall T \in \mathbf{S}_{+}(V) \,.\, \operatorname{supp} \sigma_T(V) \subset R_{+}$

$\texttt{Proof} =$

...

$\square$

$\texttt{squareRootOfTheOperator} :: \prod R : \texttt{WithSquareRoots} \,.\, \prod V : \texttt{FiniteDimensionalInnerProductSpace}(k) \,.$

$\texttt{squareRootOfTheOperator}(T) = \sqrt{T} := \sqrt{\lambda_i} e_i \otimes e_i^\star \quad \texttt{where} \quad (\lambda, e) = \texttt{SpectralTHM}(V, T)$

$\texttt{SquareRootSquare} :: \forall R : \texttt{WithSquares} \,.\, \forall V : \texttt{FiniteDimensionalInnerProductSpace}(V) \,.$

$\quad .\, \forall T \in \mathbf{S}_{++}(V) \,.\, \sqrt{T}^2 = T$

$\texttt{Proof} =$

...

$\square$

$\texttt{PositiveSemidefiniteProduct} :: \forall V : \texttt{InnerProductSpace}(V) \,.\, \forall T \in \mathrm{End}_{T\text{-}\mathsf{VS}}(V) \,.\, TT^\star \in \mathbf{S}_{+}(V)$

$\texttt{Proof} =$

...

$\square$

$\texttt{PositiveDefiniteIffProduct} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(V) \,.\, \forall T \in \mathbf{GL}(V) \,.$

$\quad .\, T \in \mathbf{S}_{++}(V) \iff \exists A \in \mathbf{GL}(V) : T = AA^\star$

$\texttt{Proof} =$

...

$\square$

$\texttt{PositiveDefiniteIffSquareRoot} :: \forall V : \texttt{InnerProductSpace}(V) \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \,.$
$\quad . \, T \in \mathbf{S}_{++} \iff \exists S \in \mathbf{S}_{++}(V) : T = S^2$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{PositiveComplexProduct} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.\, \forall A, B \in \mathbf{S}_{++} \,.$
$\quad . \, \forall [0] : \Big((A, B) : \texttt{Commutes} \Big) \,.\, AB \in \mathbf{S}_{++}$

$\texttt{Proof} \ =$

$[1] := \texttt{CommutesByInjectiveFunction}(A, B, \sqrt{\cdot}) : \left(\sqrt{A}, \sqrt{B}\right) : \texttt{Commutes},$

$[2] := [1]\texttt{SquareRootSquare}(A)(B) : \left(\sqrt{A}\sqrt{B}\right)^2 = \sqrt{A}\sqrt{B}\sqrt{A}\sqrt{B} = (\sqrt{A})^2(\sqrt{B})^2 = AB,$

$[3] := [2][1]\mathcal{C}\texttt{SelfAdjoint}(\sqrt{A})\texttt{SelfAdjoint}(\sqrt{(B)}) : AB = \sqrt{A}\sqrt{B}\left(\sqrt{A}\sqrt{B}\right)^{\star},$

$[*] := \texttt{PositiveDefiniteIffProduct}[3] : AB \in \mathbf{S}_{++}(V);$

$\square$

$\texttt{PolarDecomposition} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.\, \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}} \,.$
$\quad . \, \exists! R \in \mathbf{S}_{+}(V) \,.\, \exists S \in \mathbf{O}(V) \,.\, T = RS$

$\texttt{Proof} \ =$

$R := \sqrt{T^{\star}T} : \mathbf{S}_{+}(V),$

$\texttt{Assume}\ v : V,$

$[v.*] := \mathcal{C}\texttt{SelfAdjoint}(R)\mathcal{D}R\mathcal{C}\texttt{Asjoint}(T) : \langle Rv, Rv \rangle = \langle R^2 v, v \rangle = \langle T^{\star}Tv, v \rangle = \langle Tv, Tv \rangle;$

$\rightsquigarrow [1] := I(\forall) : \forall v \in V \,.\, \langle Rv, Rv \rangle = \langle Tv, Tv \rangle,$

$\texttt{Assume}\ v, w : V,$

$\texttt{Assume}\ [2] : Tv = Tw,$

$[2.1] := \mathcal{C}k\text{-}\mathsf{VS}(V, V)(T)[2] : T(v - w) = 0,$

$[2.2] := [2.1][1] : 0 = \langle 0, 0 \rangle = \langle T(v - w), T(v - w) \rangle = \langle R(v - w), R(v - w) \rangle,$

$[2.*] := \mathcal{C}\texttt{Nondegenerate}[2.2]\mathcal{C}k\text{-}\mathsf{VS}(V, V)(T) : Rv = Rw;$

$\rightsquigarrow [2] := I(\forall) : \forall v, w \in V \,.\, Tv = Tw \Rightarrow Rv = Rw,$

$\texttt{Assume}\ v : \mathrm{Im}\, R,$

$(w, [1]) := \mathcal{C}\,\mathrm{Im}\, R(v) : \sum w \in V \,.\, v = Rw,$

$S'(v) := T(w) : V;$

$\rightsquigarrow S' := I(\rightarrow)[2] : \mathrm{Im}\, R \rightarrow V,$

$(S, [3]) := \texttt{GrammSmidtAugamentation}[1]\mathcal{C}S' : \sum S \in \mathbf{S}_{++}(V) \,.\, S_{|\Im R} = S,$

$[*.1] := [3]\mathcal{D}S' : T = RS,$

$\texttt{Assume}\ R' : \mathbf{S}_{+}(V),$

$\texttt{Assume}\ S' : \mathbf{O}(V),$

$\texttt{Assume}\ [4] : T = R'S',$

$[4.1] := [4]\mathcal{C}\mathbf{O}(V)(S') : TT^{\star} = R'^2,$

$[4.*] := \mathcal{D}R[4.1] : R = R';$

$\rightsquigarrow [*] := \mathcal{C}^{-1}\texttt{Unique} : \texttt{This};$

$\square$

$\texttt{PolarDecomposition2} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}} \;.$
$\;.\; \exists! R \in \mathbf{S}_+(V) \;.\; \exists S \in \mathbf{O}(V) \;.\; T = SR$

Proof $=$

$\ldots$

$\square$

$\texttt{PolarDecompositionTHM} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}} \;.$
$\;.\; \exists! R \in \mathbf{S}_+(V) \;.\; \exists A \in \texttt{NormalOperator}(V) \;.\; T = R\exp(\mathrm{i}A)$

Proof $=$

$\ldots$

$\square$

$\texttt{PolarNormality} :: \forall V : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \;.\; \forall T \in \mathrm{End}_{k\text{-}\mathsf{VS}} \;.$
$T : \texttt{NormalOperator}(V) \iff (R, A) : \texttt{Commutes}$
where $(R, A) = \texttt{PolarDecompositionTHM}(V, T)$

Proof $=$

$\ldots$

$\square$

2.12 Moore-Penrose Pseudoinverse and the Singular Decomposition

$\texttt{SingularValueTheorem} :: \forall V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.\, \forall T : V \xrightarrow{k\text{-VS}} W \,.$

$\quad . \,\exists \sigma \in \mathbb{R}^r : \exists e : \texttt{OrthonormalBasis}(V) : \exists f : \texttt{OrthonormalBasis}(V) :$

$\quad T^{e,f} = \mathrm{diagonal}(\sigma \oplus 0) \quad \texttt{where} \quad r = \mathrm{rank}\, T$

$\texttt{Proof} \;=$

$A := T^\star T : \mathbf{S}_+(V),$

$(\lambda, e, [1]) := \texttt{SpectralTheorm}(A) : \sum \lambda \in \mathbb{C}^n \,.\, \sum e : \texttt{OrthonormalBasis}(V) \,.\, T = \lambda_i e_i \otimes e^i,$

$[2] := \texttt{NonNegativeEigenvalues}(A) \mathcal{O} \lambda : \lambda \in \mathbb{R}_+,$

$\sigma := \sqrt{\texttt{sort}(\lambda)_r} : r \to \mathbb{R}_{++},$

$f' := \dfrac{1}{\sigma} T e_{|r} : r \to W,$

$\texttt{Assume}\ i, j : r,$

$[(i,j).*] := \mathcal{O} f' \mathcal{O}\, \texttt{Adjoint}(T) \mathcal{O} \lambda \texttt{MultiHomogen}\Big(\langle \cdot, \cdot \rangle\Big) :$

$\quad : \langle f'_i, f'_j \rangle = \dfrac{1}{\sigma_i \sigma_j} \langle T e_i, T e_j \rangle = \dfrac{1}{\sigma_i \sigma_j} \langle T^\star T e_i, e_j \rangle = \langle e_i, e_j \rangle;$

$\rightsquigarrow [3] := \mathcal{O}^{-1} V_r(W) : f' \in V_r(W),$

$f := \texttt{GrammScmidtAugmentation}(f') : \texttt{OrthonormalBasis}(W),$

$[*] := \mathcal{O} f \mathcal{O} \sigma \mathcal{O} e : T^{e,f} = \mathrm{diagonal}(\sigma \oplus 0);$

$\square$

$\texttt{SingularValueDecomposition} :: \forall n, m \in \mathbb{N} \,.\, \forall A \in \mathbb{C}^{n \times m} \,.$

$\quad . \,\exists U : \texttt{OrthogonalMatrix}(\mathbb{C}, n) \,.\, \exists V : \texttt{OrthogonalMatrx}(\mathbb{C}, m) \,.\, \exists \Sigma : \texttt{Diagonal}(\mathbb{C}, n, m) : A = V^{\top} \Sigma U$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{singularValues} :: \prod V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.\, \prod T : V \xrightarrow{k\text{-VS}} W \,.$

$\quad \mathrm{rank}\, T \to \mathbb{C}$

$\texttt{singularValues}\,(i) = \sigma_i(T) := \sigma_i$

$\texttt{leftSingularBasis} :: \prod V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.$

$\quad . \, V \xrightarrow{k\text{-VS}} W \to \texttt{OrthonormalBasis}(V)$

$\texttt{leftSingular}\,() = u(T) := e$

$\texttt{rightSingularBais} :: \prod V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \,.$

$\quad . \, V \xrightarrow{k\text{-VS}} W \to \texttt{OrthonormalBasis}(V)$

$\texttt{singularValues}\,() = v(T) := f$

$\quad \texttt{where} \quad (\sigma, e, f) = \texttt{SingularValuesTHM}(T)$

$\texttt{Pseudoinverse} :: \prod V, W : \texttt{InnerProductSpace}(k) \ . \ (V \xrightarrow{k\text{-VS}} W) \to ?(W \xrightarrow{k\text{-VS}} V)$

$B : \texttt{Pseudoinverse} \iff \Lambda A : V \xrightarrow{k\text{-VS}} W \ . \ ABA = A \ \& \ BAB = B \ \&$
$\quad \& \ AB \in \texttt{SelfAdjoint}(V) \ \& \ BA \in \texttt{SelfAdjoint}(W)$

$\texttt{pseudoinverseMoorePenrose} :: \prod V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \ .$
$\quad . \ (V \xrightarrow{k\text{-VS}} W) \to ?(W \xrightarrow{k\text{-VS}} V)$

$$\texttt{pseudoinverseMoorePenrose}\,(A) = A^\dagger := \Lambda \sum_{i=1}^{\dim W} \alpha_i v_i(T) \in W \ . \ \sum_{i=1}^{\operatorname{rank} T} \frac{\alpha_i}{\sigma_i(T)} u_i(T)$$

$\texttt{MoorePenroseTheorem} :: \forall V, W : \texttt{FiniteDimensionalInnerProductSpace}(\mathbb{C}) \ .$
$\quad . \ \forall T : V \xrightarrow{k\text{-VS}} W \ . \ T^\dagger : \texttt{Pseudoinverse}(T) \ \& \ \forall B : \texttt{Pseudoinverse}(T) \ . \ B = T^\dagger$

$\texttt{Proof} \ =$

$n := \dim V : \mathbb{N},$

$m := \dim V : \mathbb{N},$

$r := \operatorname{rank} T : \mathbb{N},$

$\texttt{Assume} \ \sum_{i=1}^{n} \alpha_i u_i(T) : V,$

$$[\ldots*] := \mathcal{I} U_i(T) \mathcal{I} T^\dagger \mathcal{I} \ker(T) : TT^\dagger T \sum_{i=1}^{n} \alpha_i u_i(T) = TT^\dagger \sum_{i=1}^{r} \sigma_i \alpha_i v_i(T) = T \sum_{i=1}^{r} \alpha_i u_i(T) = T \sum_{i=1}^{n} \alpha_i u_i(T);$$

$\rightsquigarrow [1] := I(=,\to) : TT^\dagger T = T,$

$\texttt{Assume} \ \sum_{i=1}^{m} \alpha_i v_i(T) : V,$

$$[\ldots*] := \mathcal{I} T^\dagger \mathcal{I} v(T) \mathcal{I} \ker T^\dagger : T^\dagger TT^\dagger \sum_{i=1}^{m} \alpha_i v_i(T) = T^\dagger T \sum_{i=1}^{r} \sigma_i^{-1} \alpha_i u_i(T) = T^\dagger \sum_{i=1}^{r} \alpha_i v_i(T) = T^\dagger \sum_{i=1}^{m} \alpha_i v_i(T);$$

$\rightsquigarrow [2] := I(=,\to) : T^\dagger TT^\dagger = T^\dagger,$

$[3] := \texttt{SingularValueTHM} \mathcal{I} T^\dagger : TT^\dagger = \underset{r}{\mathrm{id}} \oplus 0 \ \& \ TT^\dagger = \underset{r}{\mathrm{id}} \oplus 0,$

$[4] := \mathcal{I}^{-1} \texttt{SelfAdjoint}[3] : TT^\dagger \in \texttt{SelfAdjoint}(V) \ \& \ T^\dagger T \in \texttt{SelfAdjoint}(W),$

$[*.1] := \mathcal{I}^{-1} \texttt{Pseudoinverse}(a)[1][2][4] : \Big(T^\dagger : \texttt{Pseudoinverse}(T)\Big),$

$\texttt{Assume} \ B : \texttt{Pseudoinverse}(T),$

$[B.1] := \mathcal{I} \texttt{Pseudoinverse}(T)\Big(T^\dagger, B\Big) : T^\dagger = T^\dagger TT^\dagger = \Big(T^\dagger T\Big)^\star T^\dagger = T^\star T^{\dagger\star} T^\dagger = \Big(TBT\Big)^\star T^{\dagger\star} T^\dagger =$
$\quad = T^\star B^\star T^\star T^{\dagger\star} T^\dagger = BTT^\star T^{\dagger\star} T^\dagger = BTT^\dagger TT^\dagger = BTT^\dagger,$

$[B.2] := \mathcal{I} \texttt{Pseudoinverse}(T)\Big(T^\dagger, B\Big) : B = BTB = B\Big(TB\Big)^\star = BB^\star T^\star = BB^\star\Big(TT^\dagger T\Big)^\star =$
$\quad = BB^\star T^\star T^{\dagger\star} T^\star = BB^\star T^\star TT^\dagger = BTBTT^\dagger = BTT^\dagger,$

$[B.*] := [B.1][B.2] : B = T^\dagger;$

$\rightsquigarrow [*] := I(\forall) : \texttt{This},$

$\square$

3 Linear Algebra in Vector Metric Spaces

3.1 Quadratic Spaces

$$\texttt{QuadraticSpace} := \prod k : \texttt{Field} \,.\, \sum V : k\text{-VS} \,.\, \mathcal{L}(V, V; k) : \texttt{Field}(R) \to \texttt{Type};$$

$$\texttt{quadraticSpaceAsVectorSpace} :: \texttt{QuadraticSpace}(k) \to k\text{-VS}$$
$$\texttt{innerProductSpaceAsVectorSpace}\,(V, p) = (V, p) := k\text{-VS}$$

$$\texttt{quadraticStructure} :: \prod (V, p) : \texttt{QuadraticSpace}(k) \,.\, \mathcal{L}(V, V; k)$$
$$\texttt{quadraticStructure}\,(v, w) = \langle v, w\rangle_V := p(v, w)$$

$$\texttt{OrthogonalVectorSpace} :: ?\texttt{QuadraticSpace}(k)$$
$$V : \texttt{OrthogonalVectorSpace} \iff \langle\cdot,\cdot\rangle_V : \texttt{Symmetric}(V, k)$$

$$\texttt{SymplecticVectorSpace} :: ?\texttt{QuadraticSpace}(k)$$
$$V : \texttt{SymplecticVectorSpace} \iff \langle\cdot,\cdot\rangle_V : \texttt{Alternating}(V, k)$$

$$\texttt{MetricVectorSpace} := \prod k : \texttt{Field} \,.\, \texttt{OrthogonalVectorSpace}|\texttt{SymplecticVectorSpace}(k) :$$
$$: \texttt{Field} \to \texttt{Type};$$

$$\texttt{CongruentMatrix} :: \prod n \in \mathbb{N} \,.\, \prod k : \texttt{Field} \,.\, ?\left(k^{n\times n} \times k^{n\times n}\right)$$
$$(A, B) : \texttt{CongruentMatrix} \iff A \cong B \iff \exists C \in \mathbf{GL}(k, n) \,.\, CAC^\top$$

$$\texttt{MatrixCongruenceMeaning} :: \forall V : k\text{-FDVS} \,.\, \forall A, B \in k^{(\dim V)\times(\dim V)} \,.$$
$$\left(\exists e, f : \texttt{Basis}(V) \,.\, A_e = B_f\right) \iff A \cong B$$

Proof $=$

$\dots$

$\square$

$$\texttt{Discriminant} :: \prod n \in \mathbb{N} \,.\, k^{n\times n} \to ?k$$
$$\texttt{Discriminant}\,(A) = \Delta(A) := k^2 \det A$$

$$\texttt{OrthogonalVectors} :: \prod V : \texttt{QuadraticSpace}(k) \,.\, ?V^2$$
$$(v, w) : \texttt{OrthogonalVectors} \iff v \perp w \iff \langle v, w\rangle = 0$$

$$\texttt{OrthogonalSets} :: \prod V : \texttt{QuadraticSpace}(k) \,.\, ?(?V)^2$$
$$(A, B) : \texttt{OrthogonalSets} \iff A \perp B \iff \forall a \in A \,.\, \forall b \in B \,.\, \langle a, b\rangle = 0$$

$\texttt{orthogonalComlement} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?V \to \texttt{VectorSubspace}(k, V)$

$$\texttt{orthogonalComplement}\,(X) = X^\perp := \bigcap_{x \in X} \ker\langle x, \cdot\rangle$$

$\texttt{IsotropicVector} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?V$

$v : \texttt{IsotropicVector} \iff \langle v, v\rangle = 0 \;\&\; v \neq 0$

$\texttt{Isotropic} :: ?\texttt{QuadraticSpace}(k) \;.\; ?V$

$V : \texttt{Isotropic} \iff \exists \texttt{IsotropicVector}(V)$

$\texttt{Anisotropic} := !\ \texttt{Isotropic} : \texttt{Field} \to \texttt{Type};$

$\texttt{Cone} :: \prod V : k\text{-}\mathsf{VS} \;.\; ?V$

$C : \texttt{Cone} \iff kC = C$

$\texttt{IsotropicCone} :: \forall V : \texttt{QuadraticSpace}(V) \;.\; \texttt{Isotropic}(V) : \texttt{Cone}(V)$

$\texttt{Proof} =$

...

$\square$

$\texttt{Degenerate} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?V$

$v : \texttt{Degenerate} \iff \{v\}^\perp = V$

$\texttt{radical} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?V$

$\texttt{radical}\,(V) = \sqrt{V} := V^\perp$

$\texttt{Nonsingular} :: ?\texttt{QuadraticSpace}(k)$

$V : \texttt{Nonsingular} \iff \sqrt{V} = \{0\}$

$\texttt{Singular} :: ?\texttt{QuadraticSpace}(k)$

$V : \texttt{Singular} \iff \sqrt{V} \neq \{0\}$

$\texttt{OrthogonallySymmetric} :: ?\texttt{QuadraticSpace}$

$V : \texttt{OrthogonallySymmetric} \iff \forall v, w \in V \;.\; v \perp w \Rightarrow w \perp v$

$\texttt{SymmetrycVector} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?V$

$v : \texttt{SymmetryVector} \iff \forall w \in V \;.\; \langle v, w\rangle = \langle w, v\rangle$

$\texttt{OrthogonallySymmetricIsMetric} :: \forall V : \texttt{OrthogonallySymmetric}(k) \,.\, \texttt{MetricVectorSpace}k$

$\texttt{Proof} \; =$

$\texttt{Assume}\; v, w : V \setminus \{0\},$

$\texttt{Assume}\; [1] : \langle v, w\rangle \neq \langle w, v\rangle,$

$\texttt{Assume}\; u : V \setminus \{0\},$

$\texttt{Assume}\; [2] : \langle v, u\rangle = \langle u, v\rangle,$

$[u.1] := [1]\mathit{\Pi}^{-}1\texttt{OrthogonalVectors} : v \perp u \iff \langle v, u\rangle\Big(\langle v, w\rangle - \langle w, v\rangle\Big),$

$[u.2] := [2]\texttt{MultiAdditive}(\langle\cdot,\cdot\rangle) : \langle v, u\rangle\Big(\langle v, w\rangle - \langle w, v\rangle\Big) = \langle v, u\rangle\langle v, w\rangle - \langle v, u\rangle\langle w, v\rangle =$

$\quad \langle u, v\rangle\langle v, w\rangle - \langle v, u\rangle\langle w, v\rangle = \Big\langle v, \langle u, v\rangle w - \langle w, v\rangle u\Big\rangle,$

$[u.3] := \texttt{MultiAdditive}(\langle\cdot,\cdot\rangle)\mathit{\Pi}\mathsf{ABEL}(k) : \Big\langle \langle u, v\rangle w - \langle w, v\rangle u, v\Big\rangle = \langle u, v\rangle\langle wv\rangle - \langle u, v\rangle\langle w, v\rangle = 0,$

$[u.*] := \mathit{\Pi}\texttt{OrthogonallySymmetric}(k)(V)[u.3][u.2][u.1] : v \perp u;$

$\rightsquigarrow [v.1] := I(\forall)I(\Rightarrow) : \forall u \in V \,.\, \langle v, u\rangle = \langle u, v\rangle \Rightarrow v \perp u,$

$[v.2] := I(=)\Big(\langle v, v\rangle\Big) : \langle v, v\rangle = \langle v, v\rangle,$

$[v.*] := \mathit{\Pi}^{-1}\texttt{IsotropicVector}[v.1][v.2] : \Big(v : \texttt{IsotropicVector}(V)\Big);$

$\rightsquigarrow [1] := \mathit{\Pi}^{-1}\texttt{SymmetryVector}(V) : \forall v \in V \,.\, v \,!\, \texttt{SymmetryVector}(V) \Rightarrow v : \texttt{IsotropicVector}(V),$

$\texttt{Assume}\; v : !\; \texttt{SymmetryVector}(V),$

$\Big(u, [2]\Big) := \mathit{\Pi}\texttt{SymmetryVector}(v) : \sum u \in V \,.\, \langle u, v\rangle \neq \langle v, u\rangle,$

$[3] := [1][2](u, v)) : \Big(u, v : \texttt{IsotropicVector}(V)\Big),$

$\texttt{Assume}\; w : \texttt{SymmetryVector}(V),$

$[4] := \ldots : w \perp v \;\&\; w \perp u,$

$[5] := \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)\mathit{\Pi}\texttt{SymmetryVector}(w) : \langle w + u, u\rangle = \langle w, u\rangle + \langle u, u\rangle =$

$\quad = \langle u, w\rangle + \langle u, u\rangle = \langle u, w + u\rangle,$

$[6] := \ldots [5] : w + u \perp u,$

$[7] := \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle)[2] : \langle u + w, v\rangle = \langle w, v\rangle \neq \langle v, w\rangle = \langle v, w + u\rangle,$

$[8] := [1][7] : \Big(u + w : \texttt{Isotropic}(V)\Big),$

$[9] := \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\rangle\Big)[3][8][6] : \langle w, w\rangle = \langle u + w - u, u + w - u\rangle = \langle u, u\rangle + \langle w + u, w + u\rangle = 0,$

$\rightsquigarrow [w.*] := \mathit{\Pi}^{-1}\texttt{IsotropicVector} : \Big(w : \texttt{Isotropic}(V)\Big),$

$\rightsquigarrow [4] := I(\forall) : \forall w : \texttt{SymmetryVector}(v) \,.\, w : \texttt{IsotropicVector}(v),$

$[v.*] := \mathit{\Pi}^{-1}\texttt{SymplecticVectorSpace}[1][4] : \Big(V : \texttt{SymplecticVectorSpace}(k)\Big);$

$\rightsquigarrow [2] := I(\exists)I(\Rightarrow)\mathit{\Pi}^{-1}\texttt{MetricVectorSpace} : \exists\,!\; \texttt{SymmetryVector}(V) \Rightarrow V : \texttt{MetricVectorSpace}(k),$

$[3] := \mathit{\Pi}^{-1}\texttt{MetricVectorSpace}\mathit{\Pi}^{-1}\texttt{OrthogonalVectorSpace} :$

$\quad : \nexists\,!\; \texttt{SymmetryVector}(V) \Rightarrow V : \texttt{MetricVectorSpace}(k),$

$[*] := E(\Rightarrow)\texttt{LEM}[2][3] : \Big(V : \texttt{MetricVectorSpace}(k)\Big);$

$\square$

$\texttt{FiniteDimensionalMetricVectorSpace} :: \prod k : \texttt{Field} \,.\, ?\texttt{MetricVectorSpace}(k)$

$V : \texttt{FiniteDimensionalMetricVectorSpace} \iff \dim V < \infty$

$\texttt{asFunctional} :: \prod V : \texttt{QuadraticSpace}(k) \,.\, V \xrightarrow{k\text{-VS}} V^*$

$\texttt{asFunctional}\,(v) = \phi_v := \Lambda u \in V \,.\, \langle u, v \rangle$

$\texttt{FDRieszRepresentationTheorem2} :: \forall k : \texttt{Field}(V) \,.$

$\quad .\, \forall V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, \phi : V \xleftrightarrow{k\text{-VS}} V^\star$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}\,\dim \texttt{DualBasisTHM}(V) : \dim V = \dim V^*,$

$[2] := \mathcal{I}\phi\mathcal{I}\texttt{Nonsingular}(V) : \dim\ker\phi = 0,$

$[*] := \texttt{RankPlusNullityTHM}[1][2] : \left(\phi : V \xleftrightarrow{k\text{-VS}} V^\star\right);$

$\dots$

$\square$

$\texttt{VectorOfRiesz} :: \prod k : \texttt{Field} \,.$

$\quad .\, \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, V^* \xleftrightarrow{k\text{-VS}} V$

$\texttt{VectorOfRiesz}\,(f) = v_f := \texttt{FDRiezRepresentationTheorem}(f)$

$\texttt{SubspaceRieszRepresentation} :: \forall k : \texttt{Field} \,.$

$\quad .\, \forall V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, \forall U :\subset_{k\text{-VS}} V \,.$

$\quad \phi_{|U} : V \twoheadrightarrow U^\star \;\&\; \ker\phi_{|S} = U^\perp$

$\texttt{Proof} \;=$

‘$\dots$

$\square$

$\texttt{OrthogonalDirectSum} :: \prod V : \texttt{MetricVectorSpace}(k) \,.\, ?\prod X \in \mathsf{SET} \,.\, n \to \texttt{VectorSubspace}(V)$

$(X, U) : \texttt{OrthogonalDirectSum} \iff V = \bot_{x \in X} U_x \iff V = \bigoplus_{x \in X} U_x \;\&\; \forall x, y \in X \,.\, x \neq y \Rightarrow U_x \bot U_y$

$\texttt{RadicalDecompositionTHM} :: \forall V : \texttt{MetricVectorSpace}(k) \,.\, \exists U \subset_{k\text{-VS}} V \,.$

$\quad .\, V = \sqrt{V} \bot U \;\&\; U : \texttt{Nonsingular}(V)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrthogonalComplementDimSum} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace}(k) \,.\, \forall U \subset_{k\text{-VS}} V \,.$

$\quad .\, \forall [0] : (U|V) : \texttt{Nonsingular}(V) \,.\; \dim V = \dim U + \dim U^\perp$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{DoubleOrthogonalComplement} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)\ .$

$\quad .\ \forall U \subset_{k\text{-}\mathsf{VS}} V\ .\ U^{\perp\perp} = U$

$\texttt{Proof} =$

...

$\square$

$\texttt{ComplementRadical} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)\ .$

$\quad .\ \forall U \subset_{k\text{-}\mathsf{VS}} V\ .\ \sqrt{U^{\perp}} = \sqrt{U}$

$\texttt{Proof} =$

...

$\square$

$\texttt{SubspaceNonsingularityCriterion} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)\ .$

$\quad .\ \forall U \subset_{k\text{-}\mathsf{VS}} V\ .\ U : \texttt{Nonsingular}(k) \iff U^{\perp} : \texttt{Nonsingular}$

$\texttt{Proof} =$

...

$\square$

3.2 Isoquadrics and Nonsingular Completion

$$\texttt{Isoquadric} :: \prod V, W : \texttt{QuadraticSpace}(k) \; . \; ?V \xrightarrow{k\text{-VS}} W$$

$$T : \texttt{Isoquadric} \iff \forall v, u \in V \; . \; \langle Tu, Tv \rangle = \langle u, v \rangle$$

$$\texttt{IsoquadricCompostition} :: \prod V, U, W : \texttt{QuadraticSpace}(k) \; . \; \forall T : \texttt{Isoquadric}(V, U) \; .$$
$$. \; \forall S : \texttt{Isoquadric}(U, W) \; . \; TS : \texttt{Isoquadric}(V, W)$$

$\texttt{Proof} =$

...

$\square$

$$\texttt{IsoquadricKernel} :: \prod V, W : \texttt{QuadraticSpace}(k) \; . \; \forall T : \texttt{Isoquadric}(V, W) \; . \; \ker T \subset_{k\text{-VS}} \sqrt{V}$$

$\texttt{Proof} =$

...

$\square$

$$\texttt{OrthogonalGroup} :: \texttt{OrthogonalVectorSpace}(k) \; \& \; \texttt{Nonsingular}(k) \to \mathsf{GRP}$$

$$\texttt{OrthogonalGroup}\,(V) = \mathbf{O}(V) := \Big\{ T \in \mathbf{GL}(V) : (T : \texttt{Isoquadric}(V)) \Big\}$$

$$\texttt{SymplecticGroup} :: \texttt{SymplecticVectorSpace}(k) \; \& \; \texttt{Nonsingular}(k) \to \mathsf{GRP}$$

$$\texttt{SymplecticGroup}\,(V) = \mathbf{Sp}(V) := \Big\{ T \in \mathbf{GL}(V) : (T : \texttt{Isoquadric}(V)) \Big\}$$

$$\texttt{IsoquadricByBasis} :: \forall V, W : \texttt{FiniteDimensionalMetricVectorSpace}(k) \; .$$
$$. \; \forall e : \texttt{Basis}(n, V) \; . \; \forall [0] : \forall i, j \in n \; . \; \langle Te_i, Te_j \rangle = \langle e_i, e_j \rangle \; . \; T : \texttt{Isoquadric}(V, W)$$

$\texttt{Proof} =$

...

$\square$

$$\texttt{IsoquadricAsQuadraticForm} :: \forall k : \texttt{NonBinary} \; .$$
$$. \; \forall V, W : \texttt{FiniteDimensionalMetricVectorSpace} \; \& \; \texttt{OrthogonalVectorSpace}(k) \; .$$
$$. \; \forall [0] : \forall v \in V \; . \; \langle Tv, Tv \rangle = \langle v, v \rangle \; . \; T : \texttt{Isoquadric}(V, W)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{IsoquadricOrthogonalComplementTranslations} :: \forall V, W : \texttt{MetricVectorSpace}(k) .$

$\quad . \forall T : \texttt{Isoquadric} \ \& \ \texttt{Bijection}(V, W) . \forall S \subset_{k\text{-VS}} V . T\left(S^\perp\right) = (TS)^\perp$

$\texttt{Proof} \ =$

$\texttt{Assume} \ v : S^\perp,$

$\texttt{Assume} \ w : TS,$

$\left(u, [1]\right) := \mathcal{I}\texttt{image}(w) : \sum u \in S . w = Tu,$

$[v.*] := [1]\mathcal{I}\texttt{Isoquadric}(V, W)(T)\mathcal{I}\texttt{orthogonalComplement}(S)(v) : \langle Tv, w\rangle = \langle v, u\rangle = 0;$

$\leadsto [1] := \mathcal{I}^{-1}\texttt{Orthogonal} : T\left(S^\perp\right)\perp TS,$

$[2] := \mathcal{I}\texttt{orthogonalComplement}[1] : T\left(S^\perp\right) \subset \left(TS\right)^\perp,$

$\texttt{Assume} \ w : \left(TS\right)^\perp,$

$\left(v, [3]\right) := \mathcal{I}\texttt{Bijection}(V, W)(T) : \sum v \in V . Tv = w,$

$\texttt{Assume} \ u : S,$

$[u.*] := [3]\mathcal{I}\texttt{Isoquadric}(V, W)(T) : \langle u, v\rangle = \langle Tu, w\rangle = 0;$

$\leadsto [4] := \mathcal{I}\texttt{orthogonalComplement} : v \in S^\perp,$

$[w.*] := [3][4] : w \in T\left(S\right)^\perp;$

$\leadsto [*] := \mathcal{I}^{-1}\texttt{SetEq}[2] : T\left(S^\perp\right) = (TS)^\perp;$

$\square$

$\texttt{HyperbolicPair} :: \prod V \in \texttt{MetricVectorSpace}(k) . ?(V \times V)$

$(v, w) : \texttt{HyperbolicPair} \iff \langle v, w\rangle = 1 \ \& \ \langle v, v\rangle = 0 \ \& \ \langle w, w\rangle = 0$

$\texttt{HyperbolicPlane} :: \prod V \in \texttt{MetricVectorSpace}(k) . ?\texttt{VectorSubspace}(V)$

$H : \texttt{HypervolivPlane} \iff \exists (v, w) : \texttt{HyperbolicPair}(V) . H = \mathrm{span}\{v, w\}$

$\texttt{HyperbolicSpace} :: ?\texttt{MetricVectorSpace}(k)$

$V : \texttt{HypervolicSpace} \iff \exists X \in \texttt{Set} : \exists H : X \to \texttt{HyperbolicPlane}(V) . V = \bigperp_{x \in X} H_x$

$\texttt{NonSingularCompletion} :: \prod V : \texttt{MetricVectorSpace}(k) . \texttt{VectorSubspace}(V) \to ?\texttt{VectorSubspace}(V)$

$U : \texttt{NonSingularCompletion} \iff \Lambda S \subset_{k\text{-VS}} . U \in \min\left\{W \subset_{k\text{-VS}} V : (W : \texttt{Nonsingular}(k))\right\}$

$\texttt{Assume} \ k : \texttt{NonBinaryField},$

$\texttt{HyperbolicPlaneOfIsotropic} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)$.

$. \forall v : \texttt{IsotropicVector}(V) . \forall S \subset_{k\text{-vs}} V . \forall [0] : kv \perp S \subset_{k\text{-vs}} V . \exists H : \texttt{HyperbolicPlane}(V) :$

$: kv \perp S \subset_{k\text{-vs}} H \perp S$

$\texttt{Proof} =$

$[1] := \texttt{DoubleOrthogonalComplement}(S) : S = S^{\perp\perp},$

$[2] := [0][1] : v \notin S^{\perp\perp},$

$\Big(u, [3]\Big) := \mathcal{C}\texttt{OrthogonalComplement}[2] : \sum u \in S^\perp . \langle v, u\rangle \neq 0,$

$[4] := \mathcal{C}\texttt{MetricVectorSpace}(k)(V) : \Big(V : \texttt{OrthogonalVectorSpace}(k) | V : \texttt{SymplecticVectorSpace}(k)\Big),$

$\texttt{Assume } [5] : \Big(V : \texttt{SymplecticVectorSpace}(k)\Big),$

$[6] := \mathcal{C}\texttt{SymplecticVectorSpace}(k)(u) : \Big(u : \texttt{IsotropicVector}(V)\Big),$

$w := \dfrac{u}{\langle v, u\rangle} : S^\perp,$

$H := \operatorname{span}(v, w) : \texttt{HyperbolicPlane}(V),$

$[5.*] := \mathcal{C}^{-1}\texttt{OrthogonalDirectSum}[1] : \texttt{This};$

$\rightsquigarrow [5] := I(\Rightarrow) : V : \texttt{SymplecticVectorSpace}(k) \Rightarrow \texttt{This},$

$\texttt{Assume } [6] : \Big(V : \texttt{OrthogonalVectorSpace}(k)\Big),$

$' := w : \langle u, v\rangle u - \dfrac{\langle u, u\rangle}{2} v,$

$S^\perp[7] := \mathcal{D} w' \texttt{MultiAdditive}\Big(\langle\cdot,\cdot\Big)\texttt{MuliHomogen}\Big(\langle\cdot,\cdot\rangle\Big)\mathcal{C}\texttt{OrthogonalVectorSpace}(k)(V)\mathcal{C}\texttt{IsotropicVector}(V$

$: \langle w', w'\rangle = \left\langle \langle u, v\rangle u - \dfrac{\langle u, u\rangle}{2} v, \langle u, v\rangle u - \dfrac{\langle u, u\rangle}{2} v \right\rangle = \langle u, v\rangle^2\langle u, u\rangle - \langle u, v\rangle^2 \langle u, u\rangle = 0,$

$[8] := \mathcal{C}^{-1}\texttt{IsotropicVector}[7] : \Big(w : \texttt{IsotropicVector}(V)\Big),$

$w := \dfrac{w'}{\langle v, w'\rangle} : S^\perp,$

$H := \operatorname{span}(v, w) : \texttt{HyperbolicPlane}(V),$

$[9.*] := \mathcal{C}^{-1}\texttt{OrthogonalDirectSum}[1] : \texttt{This};$

$\rightsquigarrow [6] := I(\Rightarrow) : V : \texttt{OrthogonalVectorSpace}(k) \Rightarrow \texttt{This},$

$[*] := E(|)[4][5][6] : \texttt{This};$

$\square$

$\texttt{HyperbolicExtensionTHM} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)$.

$. \forall n \in \mathbb{N} . \forall v : \texttt{LinearlyIndependent}(n, V) . \forall W : \texttt{Nonsingular}(k) \,\&\, \texttt{VectorSubspace}(k)$.

$. \forall [0] : \operatorname{span}\{v_i | i \in n\} \perp W \subset_{k\text{-vs}} V . \forall [00] : \forall i \in n . v_i \in \sqrt{\operatorname{span}\{v_i | i \in\} \oplus W}$.

$. \exists H : \texttt{HyperbolicSpace}(k) \,\&\, \texttt{VectorSubspace}(V) : \operatorname{span}\{v_n | n \in \mathbb{N}\} \perp W \subset_{k\text{-vs}} H \perp W$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{hyperbolicExtension} :: \prod V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k) \;.\; \texttt{VectorSubspac}$

$\texttt{hyperbolicExtension}\,(U) = \overline{U} := \texttt{HyperbolicExtension}(V, v, W, [1], \mathcal{D}v[1])$

$\quad \texttt{where} \quad (W, [1]) = \texttt{SingularDecomposition}(U) \,\&\, v = \texttt{FreeHasBasis}(\sqrt{U})$

$\texttt{NonsingularCompletionTHM} :: \forall V : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k) \;.$

$\quad . \forall U \subset_{k\text{-vs}} V \;.\; \overline{U} : \texttt{NonsingularCompletion}(V)$

$\texttt{Proof} \;=$

$W, [1] := \texttt{RadicalDecomposition}(U) : \sum W : \texttt{Nonsingular}(k) \;.\; U = W \perp \sqrt{U},$

$v := \texttt{FreeHasBasis}(\sqrt{U}) : \texttt{Basis}(\sqrt{U}),$

$m := \dim \sqrt{U} : \mathbb{N},$

$(u, H, [2]) := \mathcal{I}\overline{U} : \sum u : n \to V \;.\; \sum H : \texttt{HYperbolicSpace}(k) \;.\; H = \operatorname{span}(v, u) \,\&$

$\quad \& \,\forall i \in n \;.\; (v_i, u_i) : \texttt{HyperbolicPair}(V) \,\&\, \overline{U} = H \perp W,$

$\texttt{Assume}\; x : \overline{U} \setminus 0,$

$(h, w, [3]) := [2](x) : \sum h \in H \;.\; \sum w \in W \;.\; x = h + w,$

$\texttt{Assume}\; [4] : h \neq 0,$

$(\alpha, \beta, [5]) := [4][3] : \sum \alpha, \beta \in k^m : \alpha \oplus \beta \neq 0 \,\&\, h = \alpha v + \beta u,$

$\texttt{Assume}\; [6] : \alpha \neq 0,$

$(i, [7]) := \mathcal{I}k^m[6] : \sum i \in n \;.\; \alpha_i \neq 0,$

$[6.*] := \mathcal{I}\texttt{HyperbolicPair}(v_i, u_i)[7] : \langle u_i, x\rangle = \alpha_i \neq 0;$

$\rightsquigarrow [6] := I(\Rightarrow) : \alpha \neq 0 \Rightarrow x \,!\, \texttt{Singular}(\overline{(}U)),$

$\texttt{Assume}\; [7] : \beta \neq 0,$

$(i, [8]) := \mathcal{I}k^m[7] : \sum i \in n \;.\; \beta_i \neq 0,$

$[7.*] := \mathcal{I}\texttt{HyperbolicPair}(v_i, u_i)[8] : \langle v_i, x\rangle = \beta_i \neq 0;$

$\rightsquigarrow [7] := I(\Rightarrow) : \beta \neq 0 \Rightarrow x \,!\, \texttt{Singular}(\overline{U}),$

$[4.*] := E(|)[5][6][7] : x \,!\, \texttt{Singular}(\overline{U});$

$\rightsquigarrow [4] := I(\Rightarrow) : h \neq 0 \Rightarrow x \,!\, \texttt{Singular}(\overline{U}),$

$\texttt{Assume}\; [5] : w \neq 0,$

$\rightsquigarrow (y, [6]) := \mathcal{I}\texttt{Nonsingular}(U)(w) : \sum y \in W \;.\; \langle w, y\rangle \neq 0,$

$[5.*] := [6][3] : \langle x, y\rangle \neq 0;$

$\rightsquigarrow [5] := I(\Rightarrow) : w \neq 0 \Rightarrow x \,!\, \texttt{Singular}(\overline{U}),$

$[x.*] := E(|)[3][4][5] : x \,!\, \texttt{Singular}(\overline{U});$

$\rightsquigarrow [3] := \mathcal{I}^{-1}\texttt{NonSingular}(k) : (\overline{U} : \texttt{Nonsingular}(k)),$

$\texttt{Assume}\; X : \texttt{StrictVectorSubspace}(\overline{U}),$

$\texttt{Assume}\; [4] : U \subset_{k\text{-vs}} X,$

$[X.*] := \texttt{DimSumTHM} : X^\perp \cap \overline{U} \neq \{0\};$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{SingularCompletion} : \left(\overline{U} : \texttt{SingularCompletion}\right);$

$\square$

$\texttt{HyperbolicDimension} :: \forall V : \texttt{FiniteDimensionalMetricVectorSpace} \ \& \ \texttt{Nonsingular}(k) \ .$

$\quad . \ \forall U : \texttt{VectorSubspace}(V) \ . \ \dim \overline{U} = \dim U + \dim \sqrt{U}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{NonsingularExtensionIsHyperbolic} :: \forall V : \texttt{FiniteDimensionalMetricVectorSpace} \ \& \ \texttt{Nonsingular}(k) \ .$

$\quad . \ \forall U \subset_{k\text{-}\mathsf{vs}} V \ . \ \forall X : \texttt{SingularCompletion}(U) \ . \ X \cong \overline{U}$

$\texttt{Proof} \ =$

$\texttt{NonsingularExtension} :: \forall V, W : \texttt{FiniteDimensionalMetricVectorSpace} \ \& \ \texttt{Nonsinglar}(k) \ .$

$\quad . \ \forall U :\subset_{k\text{-}\mathsf{vs}} V \ . \ \forall T : \texttt{Isoquadric}(U, W) \ . \ \exists T' : \texttt{Isoquadric}(\overline{U}, W) : T'_{|U} = T$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{IsoquadricSpaces} :: ?\texttt{QuadraticSpace}^2(k)$

$V, W : \texttt{IsoquadricSpaces} \iff A \approx B \iff \exists \texttt{Isoquadric} \ \& \ \texttt{Bijection}(V, W)$

3.3 Witt Theory

$\texttt{WittExtensionProperty} :: ?\texttt{QuadraticSpace}^2(k)$

$V, W : \texttt{WittExtensionProperty} \iff \forall X \subset_{k\text{-vs}} V \,.\, \forall Y \subset_{k\text{-vs}} W \,.\, \forall T : \texttt{Isoquadric} \,\&\, \texttt{Bijection}(X, Y) \,.$
$\quad .\, \exists T' : \texttt{Isoqadric} \,\&\, \texttt{Bojection}(V, W) : T'_{|X=T}$

$\texttt{WittCancelationProperty} :: ?\texttt{QuadraticSpace}^2(k)$

$V, W : \texttt{WittCancelationProperty} \iff \forall X \subset_{k\text{-vs}} V \,.\, \forall Y \subset_{k\text{-vs}} W \,.$
$\quad .\, \forall D : V = X \perp X^{\perp} \,\&\, W = Y \perp Y^{\perp} \,.\, X \approx Y \Rightarrow Y \approx Y^{\perp}$

$\texttt{WittMetatheorem} :: \forall V, W : \texttt{FiniteDimensionalMetricVectorSpace} \,\&\, \texttt{Nonsingular}(k)$
$\quad .\, (V, W) : \texttt{WittExtensionProperty}(k) \iff (V, W) : \texttt{WittCancelationProperty}(k)$

$\texttt{Proof} =$

$\texttt{Assume } L : \Big((V, W) : \texttt{WittExtensionProperty}(k)\Big),$

$\texttt{Assume } X : \texttt{VectorSubspace}(X),$

$\texttt{Assume } Y : \texttt{VectorSubspace}(Y),$

$\texttt{Assume } D : V = X \perp X^{\perp} \,\&\, W = Y \perp Y^{\perp},$

$\texttt{Assume } [1] : X \approx Y,$

$T := \mathcal{I}\texttt{IsoquadricSpaces}[1] : \texttt{Isoquadric} \,\&\, \texttt{Bijection}(X, Y),$

$(T', [2]) := \mathcal{I}\texttt{WittExtensionProperty}(L, T) : \sum T' : \texttt{Isoquadric} \,\&\, \texttt{Bijection}(V, W) : T'_{|X} = T,$

$[3] := \texttt{IsoquadricComplementTranslation}(T')[2] : T'X^{\perp} = Y^{\perp},$

$[L.*] := \mathcal{I}^{-1}\texttt{IsoquadricSpaces} : X^{\perp} \approx Y^{\perp};$

$\leadsto L := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume } R : \Big((V, W) : \texttt{WittCancelationProperty}(k)\Big),$

$\texttt{Assume } X : \texttt{vectorSubspace}(V),$

$\texttt{Assume } Y : \texttt{vectorSubspace}(W),$

$\texttt{Assume } T : \texttt{Isoquadric} \,\&\, \texttt{Bijection}(X, Y),$

$\Big(T'.[1]\Big) := \texttt{NonSingularExtension}(T) : \sum T' : \texttt{Isoaquadric}(\overline{X}, \overline{(}Y)) \,.\, T'_{|X} = X,$

$[2] := \mathcal{I}\texttt{Nonsinguler}(\overline{X})\texttt{DimSumTHM} : V = \overline{X} \perp \overline{X}^{\perp},$

$[3] := \mathcal{I}\texttt{Nonsinguler}(\overline{T})\texttt{DimSumTHM} : W = \overline{Y} \perp \overline{Y}^{\perp},$

$[4] := \mathcal{I}\texttt{WittCanceleatioProperty}[2][3][1] : \overline{X}^{\perp} \approx \overline{Y}^{\perp},$

$S' := \mathcal{I}\texttt{IsoquadreicSpaces}[4] : \texttt{Ioquadric} \,\&\, \texttt{Bijection}\Big(\overline{X}^{\perp}, \overline{Y}^{\perp}\Big),$

$T'' := T' \oplus S' : V \xrightarrow{\text{1-VS}} W,$

$[5] := \mathcal{O}T''[2][3] : (T'' : \texttt{Isoquadric}(V) \,\&\, \texttt{Bijection}(V, W)),$

$[R.*] := \mathcal{O}T''[1][2] : T''|_X = T;$

$\leadsto [*] := I(\iff)I(\Rightarrow)(L) : \texttt{This},$

$\square$

3.4 Classification of Symplectic Spaces

$\texttt{NonsingularSymplecticIsHyperbolic} :: \forall V : \texttt{SymplecticVectorSpace} \ \& \ \texttt{Nonsingular}(k) .$
$\quad . \forall [0] : \dim V < \infty . V : \texttt{HyperbolicSpace}(k)$

$\texttt{Proof} \ =$

$\mathcal{H} := \{H \subset_{k\text{-VS}} V : (H : \texttt{HyperbolicSpace}(V))\} :?\texttt{VectorSubspace}(V),$

$v := \mathcal{C}\texttt{SymplecticVectorSpace}(k)(V) : (v : \text{`}IsotropicVector(V)),$

$(w, [1]) := \mathcal{C}\texttt{Nonsingular}(k)(V)(v) : \sum w \in V : \langle v, w\rangle \neq 0,$

$[2] := \mathcal{C}^{-1}\texttt{HyperbolicPlane}(V)[1][2] : \Big(\operatorname{span}\{v, w\} : \texttt{HyperbolicPlane}(k)\Big),$

$[3] := \mathcal{O}\mathcal{H}[2] : \mathcal{H} \neq \emptyset,$

$[4] := [3][0] : \max \mathcal{H} \neq \emptyset,$

$\texttt{Assume} \ H : \max \mathcal{H},$

$\texttt{Assume} \ [5] : V \neq H,$

$[6] := \mathcal{C}\texttt{Nonsingular}(H)\texttt{DimSumTHM} : V = H \perp H^{\perp},$

$\big(u, [7]\big) := [5][6] : \sum u \in H^{\perp} . u \neq 0,$

$[8] := [6]\texttt{HyperbolicOfIsotropic}(V, u) : \exists H' : \texttt{Hyperbolic}(V) . H \perp H' H \cap H' = \{0\},$

$[9] := \mathcal{C}^{-1}\texttt{HyperbolicSpace}[8] : H \oplus H' : \texttt{HyperbolicSpace}(k),$

$[H.*] := \mathcal{C} \max[9] : \perp;$

$\rightsquigarrow [5] := E(\perp) : \max \mathcal{H} = \{V\},$

$[*] := \mathcal{O}\mathcal{H}[5] : \Big(V : \texttt{HyperbolicSpace}(k)\Big);$

$\square$

$\texttt{SymplecticClassification} :: \forall V : \texttt{SymplecticVectorSpace} \ \& \ \texttt{Nonsingular}(k) .$
$\quad . \forall [0] : \dim V < \infty . \exists H : \texttt{HyperbolicSpace}(k) : V = \sqrt{V} \perp H$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CanonicalAlternatingMatrix} :: \forall V \in k\text{-FDVS} . \forall p : \texttt{Alternating}(V) . \exists e : \texttt{Basis}(V) :$

$$: p^e = \texttt{blockDiagonal}\left(\Lambda i \in m . \texttt{if } i < m \texttt{ then } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \texttt{ else } 0\right)$$

$$\texttt{where} \quad m = \frac{\operatorname{rank} \phi(p)}{2} + 1$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$$\begin{bmatrix} & 1 & & & & & \\ -1 & & & & & & \\ & & & 1 & & & \\ & & -1 & & & & \\ & & & & & 1 & \\ & & & & -1 & & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

3.5 Witt Theorems for Symplectic Spaces

$\texttt{SVSWittExtensionTheorem} :: \forall V, W : \texttt{SymplecticVectorSpace} \ \& \ \texttt{Nonsingular}(k) \ . \ \forall [0] : \dim V < \infty \ .$

$\quad . \ (V, W) : \texttt{WittExtensionPropertrty}(k)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ X : \texttt{vectorSubspace}(V),$

$\texttt{Assume} \ Y : \texttt{vectorSubspace}(w),$

$\texttt{Assume} \ T : \texttt{Isoquadric} \ \& \ \texttt{Bijection}(X, Y),$

$\Big(T'.[1]\Big) := \texttt{NonSingularExtension}(T) : \sum T' : \texttt{Isoaquadric}(\overline{X}, \overline{(}Y)) \ . \ T'_{|X} = X,$

$[2] := \mathit{Ɑ}\texttt{Nonsinguler}(\overline{X})\texttt{DimSumTHM} : V = \overline{X} \perp \overline{X}^{\perp},$

$[3] := \mathit{Ɑ}\texttt{Nonsinguler}(\overline{Y})\texttt{DimSumTHM} : W = \overline{Y} \perp \overline{Y}^{\perp},$

$[4] := \mathit{Ɑ}\texttt{Nonsingular}(V, W)[2][3] : \Big(\overline{X}^{\perp}, \overline{Y}^{\perp} : \texttt{Nonsingular}(k)\Big),$

$[5] := \texttt{NonsingularSymplecticIsHyperbolic}\Big(\overline{(}X)^{\perp}, \overline{(}Y)^{\perp}\Big) : \Big(\overline{X}^{\perp}, \overline{Y}^{\perp} : \texttt{HyperbolicSpace}(k)\Big),$

$m := \dfrac{\dim \overline{X}^{\perp}}{2} : \mathbb{N},$

$(x, \hat{x}, [6]) := \mathit{Ɑ}\texttt{HyperbolicSpace}(k)\Big(\overline{X}^{\perp}\Big)[5] : \sum (x, \hat{x}) : m \to \texttt{HyperbolicPair}(V) \ .$

$\quad . \ \overline{X}^{\perp} = \bigperp_{i=1}^{m} \operatorname{span}\{x_i, \hat{x}_i\},$

$(y, \hat{y}, [7]) := \mathit{Ɑ}\texttt{HyperbolicSpace}(k)\Big(\overline{Y}^{\perp}\Big)[5] : \sum (y, \hat{y}) : m \to \texttt{HyperbolicPair}(V) \ .$

$\quad . \ \overline{Y}^{\perp} = \bigperp_{i=1}^{m} \operatorname{span}\{y_i, \hat{y}_i\},$

$S' := \Lambda \alpha x + \beta \hat{x} \ . \ \alpha y + \beta \hat{y} : \overline{X}^{\perp} \xrightarrow{k\text{-VS}} \overline{Y}^{\perp},$

$T'' := T' \oplus S' : V \xrightarrow{1\text{-VS}} W,$

$[5] := \mathcal{D} T''[2][3] : (T'' : \texttt{Isoquadric}(V) \ \& \ \texttt{Bijection}(V, W)),$

$[R.*] := \mathcal{D} T''[1][2] : T''|_X = T;$

$\rightsquigarrow [*] := \mathit{Ɑ}^{-1}\texttt{WittExtensionProperty} : \texttt{This},$

$\square$

$\texttt{SVSWittExtensionTheorem} :: \forall V, W : \texttt{SymplecticVectorSpace} \ \& \ \texttt{Nonsingular}(k) \ . \ \forall [0] : \dim V < \infty \ .$

$\quad . \ (V, W) : \texttt{WittCancelationPropertrty}(k)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

3.6 Symplectic Group

$\texttt{transvection} :: \prod V : \texttt{QuadraticSpace}(k) \,.\, (V \setminus 0) \to k \to \mathrm{End}_{k\text{-VS}}(V)$

$\texttt{transvection}\,(v, \alpha) = \tau_{v,\alpha} := \Lambda u \in v \,.\, u + \alpha\langle u, v\rangle v$

$\texttt{SymplecticTransvection} :: \forall V : \texttt{SymplecticVectorSpace}(k) \,.\, \forall v \in V \setminus \{0\} \,.\, \forall \alpha \in k \,.\, \tau_{v,\alpha} \in \mathbf{Sp}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x, y : V,$

$[(x, y).*] := \text{Д}\,\texttt{transvection}(v, \alpha)\texttt{MultiAdditive}^3\Big(\langle\cdot,\cdot\rangle\Big)\texttt{MultiHomogen}^4\Big(\langle\cdot,\cdot\rangle\Big)$

$\quad \text{Д}\,\texttt{SymplecticVectorSpace}(k)\text{Д}\,\texttt{Inverse}(k, +) : \langle \tau_{v,\alpha}x, \tau_{v,\alpha}y\rangle = \Big\langle x + \alpha\langle x, v\rangle v, y + \alpha\langle y, v\rangle v\Big\rangle =$

$\quad = \langle x, y\rangle + \alpha\langle x, v\rangle\langle v, y\rangle + \alpha\langle y, v\rangle\langle x, v\rangle + \alpha^2\langle x, v\rangle\langle y, v\rangle\langle v, v\rangle = \langle x, y\rangle + \alpha\langle x, v\rangle\langle v, y\rangle - \alpha\langle x, v\rangle\langle y, v\rangle = 0;$

$\rightsquigarrow [*] := \text{Д}\,\mathbf{Sp}(V) : \tau_{v,\alpha} \in \mathbf{Sp}(V);$

$\square$

$\texttt{TransvectionIdentity} :: \forall V : \texttt{SymplecticVectorSpace}\,\&\,\texttt{Nonsingular}(k) \,.\, \forall v \in V \setminus \{0\} \,.\, \forall \alpha \in k \,.\, \tau_{v,\alpha} = \mathrm{id}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OrthogonalTransvection} :: \forall V : \texttt{SymplecticVectorSpace}\,\&\,\texttt{Nonsingular}(k) \,.\, \forall v \in V \setminus \{0\} \,.$

$\quad .\, \forall \alpha \in k^* \,.\, \forall w \in V \,.\, v \perp w \iff \tau_{v,\alpha}(w) = w$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TransvectionAddition} :: \forall V : \texttt{SymplecticVectorSpace}(k) \,.\, \forall v \in V \setminus \{0\} \,.\, \forall \alpha, \beta \in k \,.\, \tau_{v,\alpha+\beta} = \tau_{v,\alpha}\tau_{v,\beta}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TransvectionInverse} :: \forall V : \texttt{SymplecticVectorSpace}(k) \,.\, \forall v \in V \setminus \{0\} \,.\, \forall \alpha \in k \,.\, \tau_{v,\alpha}^{-1} = \tau_{v,-\alpha}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TransvectionConjugation} :: \forall V : \texttt{SymplecticVectorSpace}(k) \,.\, \forall v \in V \setminus \{0\} \,.\, \forall \alpha \in k \,.\, \forall \sigma \in \mathbf{Sp}(V) \,.$

$\quad .\, \sigma\tau_{v,\alpha}\sigma^{-1} = \tau_{\sigma v,\alpha}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; w : V,$

$[(w.*)] := \text{Д}\,\texttt{transvection}(v, \alpha)\text{Д}\,\texttt{Inverse}\big(\mathbf{Sp}(V)\big)\text{Д}\,\mathbf{Sp}(V)\text{Д}^{-1}\texttt{transvection}(\sigma\; v, \alpha) :$

$\quad : \sigma\tau_{v,\alpha}\sigma^{-1}(w) = \sigma\Big(\sigma^{-1}(w) + \alpha\langle\sigma^{-1}(w), v\rangle v\Big) = w + \alpha\langle w, \sigma v\rangle\sigma v = \tau_{\sigma\; v,\alpha}(w);$

$\rightsquigarrow [*] := I(=, \to) : \sigma\tau_{v,\alpha}\sigma^{-1} = \tau_{\sigma v,\alpha};$

$\square$

$\texttt{TransvectionScalarMult} :: \forall V : \texttt{SymplecticVectorSpace}(k)\, \forall \alpha \in k \;.\; \forall \beta \in k^* \;.\; \tau_{\beta v,\alpha} = \tau_{v.\alpha^2\beta}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{Transvection} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?\mathbf{Sp}(V)$

$T : \texttt{Transvection} \iff \exists v \in V \setminus \{0\} : \exists \alpha \in k \;.\; T = \tau_{v,\alpha}$

$\texttt{ConnectedHyperbolicPairs} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?\texttt{HyperbolicPair}^2(V)$

$\Big((v,w),(x,y)\Big) : \texttt{ConnectedHyperbolicPairs} \iff \exists n \in \mathbb{N} : \exists T : n \to \texttt{Transvection}(V) \;.$

$\quad . \prod_{i=1}^{n} T_i v = x \;\&\; \prod_{i=1}^{n} T_i w = y$

$\texttt{SymplecticIsConnected} :: \forall V : \texttt{SymplecticVectorSpace} \;\&\; \texttt{Nonsingular}(k) \;.\; \forall [0] : \dim V < \infty \;.$

$\quad . \forall (v,w),(x,y) : \texttt{HyperbolicPair}(V) \;.\; \Big((v,w),(x,y)\Big) : \texttt{ConnectedHyperbolicPairs}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; (v,w) : \texttt{HyperbolicPair}(V),$

$\texttt{Assume}\; u : V,$

$\texttt{Assume}\; [1] : \langle v,u \rangle \neq 0,$

$\alpha := \dfrac{1}{\langle v,u \rangle} : k^*,$

$[(v,w).*] := \mathcal{I}\texttt{transvection}\mathcal{I}\texttt{SymplecticVectorSpace}(V)\mathcal{D}\alpha : \tau_{u-v,\alpha}(v) = v - \alpha\langle v, u-v\rangle(u-v) = u;$

$\rightsquigarrow [1] := I(\forall)I(\forall)I(\Rightarrow) : \forall (v,w) : \texttt{HyperbolicPair}(V) \;.\; \forall u \in V \;.\; \langle v,u \rangle \neq 0 \Rightarrow$

$\quad \Rightarrow \exists x \in V : \Big((v,w),(u,x)\Big) : \texttt{ConnectedHyperbolicPair}(V),$

$\texttt{Assume}\; v,u,w : V,$

$\texttt{Assume}\; [2] : \Big((v,w),(u,w) : \texttt{HyperbolicPair}(V)\Big),$

$\texttt{Assume}\; [3] : \langle v,u \rangle \neq 0,$

$[4] := \texttt{MultiAddive}\Big(\langle\cdot,\cdot\rangle\Big)\mathcal{I}\texttt{HyperbolicPair}(V)(v,w)(u,w)\mathcal{I}\texttt{Inverse}(k,+)(1) :$

$\quad : \langle u-v, w \rangle = \langle u,w \rangle - \langle v,w \rangle = 1 - 1 = 0,$

$[5] := \mathcal{I}^{-1}\texttt{Orthogonal}[4] : u - v \perp w,$

$\alpha := \dfrac{1}{\langle v,u \rangle} : k^*,$

$[6] := \texttt{orthogonalTransvection}(V, v-u, \alpha) : \tau_{u-v,\alpha}(w) = w,$

$[(v,u,w).*] := [1][6] : \bigg(\Big((v,w),(u,w)\Big) : \texttt{ConnectedHyperbolicPair}(V)\bigg);$

$\rightsquigarrow [2] := I(\forall)I(\Rightarrow) : \forall v,u,w \in V \;.\; (v,w),(u,w) : \texttt{HyperbolicPair}(V) \;\&\; \langle v,u \rangle \neq 0 \Rightarrow$

$\quad \Rightarrow \Big((v,w),(u,w) : \texttt{ConnectedHyperbolicPair}(V)\Big),$

$\texttt{Assume}\ v, u, w : V,$

$\texttt{Assume}\ [3] : \Big((v,w),(u,w) : \texttt{HyperbolicPair}(V)\Big),$

$[4] := \mathcal{C}\texttt{SymplecticVectorSpace}(k)(V)\mathcal{C}\texttt{HyperbolicPair}(v,w) :$

$: \Big(\{v,w\} : \texttt{LinearlyIndependentSet}(V)\Big),$

$\texttt{Assume}\ [6] : u \in \texttt{LinearlyIndependent}(v,w),$

$[6.*] := \mathcal{C}\texttt{NonBinary}(k)[6] : \exists f \in V^\star : f(w) = 1\ \&\ f(v) \neq 0\ \&\ f(u) \neq 0;$

$\rightsquigarrow [6] := I(\Rightarrow) : \dots,$

$\texttt{Assume}\ [7] : u \notin \texttt{LinearlyIndependent}(v,w),$

$[7.*] := \mathcal{C}\texttt{LinearlyIndependent}(v,w,u)[6] : \exists f \in V^\star : f(w) = 1\ \&\ f(v) \neq 0\ \&\ f(u) \neq 0;$

$\rightsquigarrow [7] := I(\Rightarrow) : \dots,$

$(f,[8]) := E(|)\texttt{LEM}[6][7] : \sum \in V^\star : f(w) = 1\ \&\ f(v) \neq 0\ \&\ f(u) \neq 0,$

$(x,[9]) := \texttt{ReiszRepresentationTheorem2}(V,[8]) : \sum x \in V\ .\ \langle x,w\rangle = 1\ \&\ \langle x,v\rangle \neq 0\ \&\ \langle x,u\rangle \neq 0,$

$[(v,u,w).*] := [2](v,w,x)[9][2](u,w,x)[9] : \Big(\big((v,w),(u,w)\big) : \texttt{ConnectedHyperbolicPair}\Big);$

$\rightsquigarrow [3] := I(\forall)I(\Rightarrow) : \forall v,u,w \in V\ .\ (v,w),(u,w) : \texttt{HyperbolicPair}(V) \Rightarrow$

$\Rightarrow \Big((v,w),(u,w) : \texttt{ConnectedHyperbolicPair}(V)\Big),$

$(z,[4]) := [1](v,w,x) : \sum z \in V\ .\ \big((v,w),(x,z)\big) : \texttt{ConnectedHyperbolicPair}(V),$

$[5] := [3](x,y,z) : \Big(\big((x,z),(x,y)\big) : \texttt{ConnectedHyperbolicPair}(V)\Big),$

$[*] := [4][5] : \Big(\big((v,w),(x,y)\big) : \texttt{ConnectedHyperbolicPair}(V)\Big);$

$\square$

$\texttt{SymplecticTOperatorStructure} :: \forall V : \texttt{SymplecticVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, \forall [0] : \dim V < \infty \,.$

$$. \forall S \in \mathbf{Sp}(V) \,.\, \exists n \in \mathbb{N} : \exists T : n \to \texttt{Transvection}(V) \,.\, S = \prod_{i=1}^{n} T_i$$

$\texttt{Proof} \;=$

$$♂ := \Lambda n \in \mathbb{N} \,.\, \dim V \leq n \Rightarrow \exists n \in \mathbb{N} : \exists T : n \to \texttt{Transvection}(V) \,.\, S = \prod_{i=1}^{n} T_i : \mathbb{N} \to \texttt{Type},$$

$\texttt{Assume}\ [1] : \dim V = 1,$

$[2] := \texttt{SymplecticClassification} : (V : \texttt{HyperbolicPlane}(V)),$

$[1.*] := \mathcal{G}\texttt{ConnectedHyperbolicPairSymplecticIsConnected}[2] :$

$$. \exists n \in \mathbb{N} : \exists T : n \to \texttt{Transcection}(V) \,.\, S = \prod_{i=1}^{n} T_i;$$

$\rightsquigarrow [1] := \mathcal{D}♂ : ♂(1),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [3] : ♂(n),$

$(W, H, [4]) := \texttt{SymplecticClassification}[3] :$

$$. \sum H : \texttt{HyperbolicPlane}(k) \,.\, \sum W : \texttt{SymplecticSpace}(k) \,.\, V = H \perp W,$$

$\texttt{Assume}\ [5] : \dim V = n + 1,$

$[6] := \mathcal{G} \dim[4] : \dim W = \dim V - 2 \geq n,$

$[7] := \mathcal{G}\mathbf{Sp}(V)\mathcal{G}\texttt{OrthofonalDirectSum}[4] : S = H \boxplus W,$

$$(n, T, [8]) := [1][7] : \sum n \in \mathbb{N} \,.\, \sum T : n \to \texttt{Transvection}(H) \,.\, S_{|H} = \prod_{i=1}^{n} T_i,$$

$$(m, T[9]) := [3][7][6] : \sum m \in \mathbb{N} \,.\, \sum T' : m \to \texttt{Transvection}(W) \,.\, S_{|W} = \prod_{i=1}^{m} T'_i,$$

$$(10) := \texttt{TransvectionOrthogonal}(T)[4] : \Big(T \oplus \mathrm{id} : n \to \texttt{Transvection}(V)\Big),$$

$$(11) := \texttt{TransvectionOrthogonal}(T')[4] : \Big(\mathrm{id} \oplus T' : m \to \texttt{Transvection}(V)\Big),$$

$$(n.*) := [8][9] : T = \prod_{i=1}^{n} T \oplus \mathrm{id} \prod_{i=1}^{m} T';$$

$\rightsquigarrow [2] := \mathcal{G}\texttt{NaturalSet}(\mathbb{N}) : \forall n \in \mathbb{N} \,.\, ♂(n),$

$[*] := ♂(\dim V) : \texttt{This};$

$\square$

3.7 Classification of Orthogonal Spaces

$\texttt{OrthonormalBasis} :: \prod V : \texttt{QuadraicSpace}(k) \,.\, ?\texttt{Basis}(V)$

$E : \texttt{OrthonormalBasis} \iff \forall e, e' \in E \,.\, \Big(\langle e, e' \rangle = 0 \iff e \neq e'\Big)$

$\texttt{OrthogonalSymplectic} :: \forall V : \texttt{SymplecticVectorSpace} \,\&\, \texttt{OrthogonalVectorSpace}(k) \,.$
$\quad .\, \forall [0] : (k : \texttt{MomBinary}) \,.\, \langle \cdot, \cdot \rangle_V = 0$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrthogonalStructure} :: \forall V : \texttt{OrthogonalVectorSpace}(k) \,.\, \forall [0] : \dim V < \infty \,.\, \exists U, W \subset_{k\text{-VS}} V :$
$\quad : \Big(\exists! \texttt{OrthonormalBasis}(U)\Big) \,\&\, W : \texttt{SymplecticVectorSpace}(k) \,\&\, V = U \perp W$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrthogonalBasisTheorem} :: \forall k : \texttt{NonBinary} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k) \,.$
$\quad .\, \forall [0] : \dim V < \infty \,.\, \exists \texttt{OrthonormalBasis}(V)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrthogonalForm} :: \forall k : \texttt{NonBinary} \,.\, \forall V : k\text{-FDVS} \,.\, \forall p : \texttt{Symmetric}(V) \,.$
$\quad \exists e : \texttt{Basis}(V) : \exists \alpha : (\operatorname{rank} p^e) \to k^* : p^e = \operatorname{diag}(\alpha \oplus 0)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrthogonalClassificationForACF} :: \forall k : \texttt{AlgebraicallyClosedField} \,.\, \forall V : k\text{-FDVS} \,.$
$\quad .\, \forall p : \texttt{Symmetric}(V) \,.\, \exists e : \texttt{Basis}(V) : p^e = \operatorname{diag}(1_{\operatorname{rank} p^e} \oplus 0) \,.$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SylvesterLawOfInertia} :: \forall V : \mathbb{R}\text{-FDVS} \,.$
$\quad .\, \forall p : \texttt{Symmetric}(V) \,.\, \exists e : \texttt{Basis}(V) : \exists n \in \mathbb{N} \,.\, \exists m \in \mathbb{N} \,.\, p^e = \operatorname{diag}(1_n \oplus -1_m \oplus 0)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SylvesterMatrix} :: \prod n \in \mathbb{N}.?\mathbb{R}^{n\times n}$

$A : \texttt{SylvesterMatrix} \iff \exists n \in \mathbb{N} \,.\, \exists m \in \mathbb{N} \,.\, A = \mathrm{diag}(1_n \oplus 1_m \oplus 0)$

$\texttt{SylvesterTHM} :: \forall n \in \mathbb{N} \,.\, \forall A : \texttt{Symmetric}(\mathbb{R}, n) \,.\, \exists! S : \texttt{SylvesterMatrix}(n) : A \cong S$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{signature} :: \prod n \in \mathbb{N} \,.\, \texttt{Symmetric}(\mathbb{R}, n) \to \mathbb{Z}$

$\texttt{signature}\,(A) := \mathrm{tr}\, S \quad \texttt{where} \quad S = \texttt{SylvesterTHM}(n, A)$

$\texttt{inertia} :: \prod n \in \mathbb{N} \,.\, \texttt{Symmetric}(\mathbb{R}, n) \to \texttt{partition}(n)$

$\texttt{inertia}\,(A) := (k, l, n - j - l) \quad \texttt{where} \quad S = \texttt{SylvesterTHM}(n, A), (k, l) = Ɑ\texttt{SylvesterMatrix}(S)$

$\texttt{UniversalForm} :: \prod V \in k\text{-}\mathsf{VS} \,.\, ?\mathcal{L}(V, V; k)$

$p : \texttt{UniversalForm} \iff \forall \alpha \in k \,.\, \exists v \in V : \;.\, p(v, v) = \alpha$

$\texttt{UniversalFormTHM} :: \forall q : \texttt{PrimePower} \,.\, \forall V : \texttt{OrthogonalVectorSpace}\mathbb{F}_q \,.\, \forall U \subset_{\mathbb{F}_q\text{-}\mathsf{VS}} V \,.$
$\quad .\, \forall[0] : \dim U \geq 2 \,.\, \forall[00] : (U : \texttt{Nonsingular}) \,.\, \forall[000] : q \neq 2 \,.\, \langle\cdot,\cdot\rangle : \texttt{Universal}(V)$

$\texttt{Proof} \;=$

$(v, w, \alpha, \beta, [1]) := \texttt{OrthogonalForm}(U) : \sum v, w \in U \,.\, \sum \alpha, \beta \in \mathbb{F}_q^* \,.\, \langle v, v\rangle = \alpha \;\&\; \langle w, w\rangle = \beta \;\&\; \langle v, w\rangle = 0,$

$\texttt{Assume}\ \gamma : k,$

$[2] := Ɑ\texttt{SquaresCardinality}(\dots) : \left|\{\alpha\mu^2 | \mu \in k\}\right| = \dfrac{q+1}{2},$

$[3] := Ɑ\texttt{SquaresCardinality}(\dots) : \left|\{\gamma - \beta\mu^2 | \mu \in k\}\right| = \dfrac{q+1}{2},$

$(\mu, \nu, [4]) := Ɑ\texttt{FiniteFieldCardinality}[2][3] : \sum \mu \in k \,.\, \mu^2\alpha + \nu^2\beta = \gamma,$

$[\gamma.*] := Ɑ\texttt{MultiAdditive}(\langle\cdot,\cdot\rangle)\texttt{MultiHimogen}(\langle\cdot,\cdot\rangle)[1][4] : \langle \alpha v + \nu w, \alpha v + \nu w\rangle = \mu^2\alpha + \nu^2\beta = \gamma;$

$\rightsquigarrow [*] := Ɑ^{-1}\texttt{UniversalForms} : \Big(\langle *, * \rangle : \texttt{UniversalForm}(V)\Big),$

$\square$

$\texttt{OrthogonalClassificationForFiniteField} :: \forall q : \texttt{PrimePower} \,.\, \forall V : \mathbb{F}_q\text{-}\mathsf{FDVS} \,.$
$\quad .\, \forall p : \texttt{Symmetric}(V) \,.\, \exists e : \texttt{Basis}(V) : \exists \alpha \in \mathbb{F}_q^* : p^e = \mathrm{diag}(1_{\mathrm{rank}\, p^e - 1} \oplus \alpha \oplus 0)$

$\texttt{Proof} \;=$

$\dots$

$\square$

3.8 Orthogonal Group

$\texttt{SpecialOrthogonalGroup} :: \prod V : \texttt{OrthogonalVectorSpace}k \,.\, ?\mathbf{O}(V)$

$T : \texttt{SpecialOrthogogonalGroup} \iff T \in \mathbf{SO}(V) \iff \det T = 1$

$\texttt{NotIsotropic} :: \prod V : \texttt{QuadraticSpace}(k) \,.\, ?V$

$v : \texttt{NotIsotropic} \iff \langle v, v \rangle \neq 0$

$\texttt{symmetry} :: \prod V : \texttt{quadraticSpace} \,.\, \texttt{NotIsotropic}(V) \to \mathrm{End}_{k\text{-}\mathsf{VS}}(V)$

$\texttt{symmetry}\,(v) = \sigma_v := \Lambda x \in V \,.\, x - \dfrac{2\langle v, x\rangle}{\langle v, v\rangle} v$

$\texttt{OrthogonalSymmetry} :: \forall V : \texttt{OrthogonalVectorSpace}k \,.\, \forall v : \texttt{NotIsotropic}(V) \,.\, \sigma_v \in \mathbf{O}(V)$

$\texttt{Proof} =$

$[1] := \mathit{\Pi}\texttt{NotIsotropic}(V)(v)\mathit{\Pi}^{-1}\texttt{Nonsingular} : \Big(kv : \texttt{Nonsingular}(k)\Big),$

$[2] := \texttt{DimSumTHM}[1] : V = kv \perp (kv)^{\perp},$

$\texttt{Assume}\ x, y : V,$

$\Big(\alpha, \beta, w, u, [3]\Big) := [2](x, y) : \sum \alpha, \beta \in k \,.\, w, u \in (kv)^{\perp} \,.\, x = \alpha v + w \,\&\, y = \beta v + u,$

$[\ldots *] := \mathit{\Pi}\sigma_v[3]\texttt{MultAdditive}\Big(\langle\cdot,\cdot\rangle\Big)\texttt{MultiHomogen}\Big(\langle\cdot,\cdot\rangle\Big)\mathit{\Pi}(kv)^{\perp}\mathit{\Pi}\texttt{Inverse}(k, +) :$

$\quad : \langle \sigma_v x, \sigma_v y\rangle = \left\langle \alpha v + w - 2\dfrac{\langle v, \alpha v + w\rangle v}{\langle v, v\rangle} v, \beta v + u - 2\dfrac{\langle v, \alpha v + w\rangle}{\langle v, v\rangle} v \right\rangle = \langle x, y\rangle - 2\alpha\beta - 2\alpha\beta + 4\alpha\beta = \langle x, y\rangle;$

$\rightsquigarrow [*] := \mathit{\Pi}^{-1}\mathbf{O}(V) : \sigma_v \in \mathbf{O}(V),$

$\square$

$\texttt{Symmetry} :: \prod V : \texttt{OrthogonalVectorSpace}k \,.\, ?\mathrm{End}_{k\text{-}\mathsf{VS}}(V)$

$S : \texttt{Symmetry} \iff \exists v : \texttt{NotIsotropic}(V) \,.\, S = \sigma_v$

$\texttt{ReflectionAlongSymmetry} :: \forall V : \texttt{OrthogonalVectorSpace}k \,.\, \forall v : \texttt{NotIsotropic}(V) \,.\, \sigma_v(v) = -v$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ReflectionAlongSymmetry2} :: \forall V : \texttt{OrthogonalVectorSpace}k \,.\, \forall v : \texttt{NotIsotropic}(V) \,.$

$\quad .\, \forall u : (kv)^{\perp} \,.\, \sigma_v(u) = u$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{OrthogonalCpnnection} :: \forall V : \texttt{Nonsingular} \ \& \ \texttt{OrthogonalVectorSpace}k \ . \ \forall v, u : \texttt{nonIsotropic} \ .$

$. \ \forall [0] : \langle v, v\rangle = \langle u, u\rangle \ . \ \exists w : \texttt{NonIsotropic}(V) \ . \ \sigma_w(v) = u$

$\texttt{Proof} \ =$

$[1] := \mathcal{I}^{-1}\texttt{NonIsotropic}[0] : v + u : \texttt{NonIsotropic}(V) | v - u : \texttt{NonIsotropic}(V),$

$[2] := \texttt{MultiAdditive}[0]\mathcal{I}^{-1}\texttt{Orthogonal} : v + u \perp v - u,$

$\texttt{Assume} \ [3] : \Big(v + u : \texttt{NonIsotropic}(V)\Big),$

$[4] := \mathcal{I}\sigma_{v+u}(v + u) : \sigma_{v+u}(v + u) = -v - u,$

$[5] := \mathcal{I}\sigma_{v+u}(v - u)[2] : \sigma_{v+u}(v - u) = v - u,$

$[3.*] := \mathcal{I}k\text{-}\mathsf{VS}(V, V)(\sigma_{v+u})[3][4] : \sigma_{v+u}(v) = u;$

$\rightsquigarrow [3] := I(\Rightarrow) : v + u : \texttt{NonIsotropic}(V) \Rightarrow \texttt{This},$

$\texttt{Assume} \ [4] : \Big(v - u : \texttt{NonIsotropic}(V)\Big),$

$[5] := \mathcal{I}\sigma_{v-u}(v + u) : \sigma_{v+u}(v + u) = v + u,$

$[4] := \mathcal{I}\sigma_{v+u}(v - u)[4] : \sigma_{v+u}(v - u) = -v + u,$

$[4.*] := \mathcal{I}k\text{-}\mathsf{VS}(V, V)(\sigma_{v+u})[5][6] : \sigma_{v+u}(v) = u;$

$\rightsquigarrow [4] := I(\Rightarrow) : v + u : \texttt{NonIsotropic}(V) \Rightarrow \texttt{This},$

$[*] := E(|)[1][3][4] : \texttt{This};$

$\square$

$\texttt{StructureOfOrthogonalOperator} :: \forall V : \texttt{Nonsingular} \ \& \ \texttt{OrthogonalVectorSpace}(k) \ . \ \forall [0] : \dim V < \infty \ .$

$$. \ \forall T \in \mathbf{O}(V) \ . \ \exists n \in \mathbb{N} \ . \ \exists S : n \to \texttt{Symmetry}(V) \ . \ T = \prod_{i=1}^{n} S_i$$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : \dim V = 1,$

$v := \texttt{OrthogonalSymplectic}(V) : \texttt{NonIsotropic}(V),$

$\Big(\alpha, [2]\Big) := \mathcal{I}k\text{-}\mathsf{VS}(V, V)(k)[1] : \sum \alpha \in k \ . \ Tv = \alpha v,$

$[3] := \mathcal{I}\texttt{NonIsotropic}(V)(v)\mathcal{I}\mathbf{O}(V)(T)(v)[1]\texttt{MultiHomogen}(V) : 0 \neq \langle v, v\rangle = \langle Tv, Tv\rangle = \langle \alpha v, \alpha v\rangle = \alpha^2\langle v, v\rangle,$

$[4] := \texttt{RootsCardinality}(k,)[3] : \alpha = \pm 1,$

$\texttt{Assume} \ [5] : \alpha = 1,$

$[6] := [1][2][5] : T = \mathrm{id},$

$[5.*] := \texttt{ReflectionAlongSymmetry2}[1][6] : T = \sigma_v\sigma_v;$

$\rightsquigarrow [5] := I(\Rightarrow) : \alpha = 1 \Rightarrow \texttt{This},$

$\texttt{Assume} \ [6] : \alpha = -1,$

$[7] := [1][2][6] : T = -\mathrm{id},$

$[5.*] := \texttt{ReflectionAlongSymmetry2}[1][7] : T = \sigma_v\sigma_v;$

$\rightsquigarrow [6] := I(\Rightarrow) : \alpha = -1 \Rightarrow \texttt{This},$

$[7] := E(|)[4][5][6] : \texttt{This};$

$\rightsquigarrow [1] := I(\Rightarrow) : \dim V = 1 \Rightarrow \texttt{This},$

$d := \dim V : \mathbb{N},$

$\texttt{Assume}$ $[2] : \forall W : \texttt{Nonsingular} \ \& \ \texttt{OrthogonalVectorSpace}(k) \ . \ \dim W < d \Rightarrow \forall T \in \mathbf{O}(W) \ .$

$$. \ \exists n \in \mathbb{N} \ . \ \exists S : n \to \texttt{Symmetry}(V) \ . \ T = \prod_{i=1}^{n} S_i,$$

$v := \texttt{OrthogonalSymplectic}(V) : \texttt{NonIsotropic}(V),$

$[3] := \texttt{DimSumTHM}(v) : V = kv \perp (kv)^\perp,$

$[4] := \mathcal{I}\mathbf{O}(V)[3] : T = kv \boxplus (kv)^\perp,$

$$(n, S, [5]) := [2](kv, [3], T_{|kv},) : \sum n \in \mathbb{N} \ . \ \sum S : n \to \texttt{Symmetry}(kv) \ . \ T_{|kv} = \prod_{i=1}^{n} S_i,$$

$$(n', S', [6]) := [2]\left((kv)^\perp, [3], T_{|(kv)^\perp}\right) : \sum n' \in \mathbb{N} \ . \ \sum S' : n' \to \texttt{Symmetry}(kv)^\perp \ . \ T_{|(kv)^\perp} = \prod_{i=1}^{n'} S'_i,$$

$$[7] := \texttt{ReflectionAlongSymmetry}[3][5][6] : T = \prod_{i=1}^{n} S_i \prod_{i=1}^{n'} S'_i;$$

$\leadsto [*] := \mathcal{I}\mathbb{N}[1] : \texttt{This};$

$\square$

3.9 Witt Theorems for Orthogoanl Spaces

$\texttt{OVSWittCancelationTranslation} :: \forall n \in \mathbb{N}\ .$

$. \forall [0] : \Big(\forall V : \texttt{OrthogonalVectorSpace} \ \& \ \texttt{Nonsingular}(k)\ .\ \dim V = n \Rightarrow$

$\Rightarrow (V, V) : \texttt{WittCancelationProperty}(k)\Big)\ .$

$. \Big(\forall V, U : \texttt{OrthogonalVectorSpace} \ \& \ \texttt{Nonsingular}(k)\ .\ \forall [00] : \dim V = \dim U = n \ \& \ (V, U) : \texttt{Isometric}(k)$

$\Rightarrow (V, U) : \texttt{WittCancelationProperty}(k)\Big)$

$\texttt{Proof} \ =$

$A := \mathcal{C}\texttt{Isometric}[00] : \texttt{Isometry} \ \& \ \texttt{Bijection}(V, W),$

$\texttt{Assume}\ X : \texttt{Vectorsubspace}(V),$

$\texttt{Assume}\ Y : \texttt{Vectorsubspace}(W),$

$\texttt{Assume}\ T : \texttt{Isometry} \ \& \ \texttt{Bijection}(X, Y),$

$[1] := \mathcal{C}^{-1}\texttt{IsometricIsometryComposition} : \Big((X, A^{-1}Y) : \texttt{Isometry}\Big),$

$[2] := \mathcal{C}\texttt{WittCancelationProperty}[0][1] : \Big((X^\perp, (A^{-1}Y)^\perp) : \texttt{Isometric}(k)\Big),$

$[3] := \texttt{QuadraticOrhogonalTranslation}(A, A^{-1}Y) : A(A^{-1}Y)^\perp = Y^\perp,$

$[\ldots *] := [2][3] : \Big((X^\perp, T^\perp) : \texttt{Isometric}(k)\Big);$

$\rightsquigarrow [*] := \mathcal{C}^{-1}\texttt{isometric} : \texttt{This},$

$\square$

$\texttt{OVSWittCancelationPreTHM} :: \forall V : \texttt{OrthogonalVectorSpace} \ \& \ \texttt{Nonsingular}(k)\ .$

$. (V, V) : \texttt{WittCancelationProperty}(k)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ X, Y : \texttt{VectorSubspace}(V),$

$\texttt{Assume}\ T : \texttt{Isometry} \ \& \ \texttt{Bijection}(X, Y),$

$[1] := \texttt{DimSumTHM} : V = X \perp X^\perp,$

$[2] := \texttt{DimSumTHM} : W = Y \perp Y^\perp,$

$\texttt{Assume}\ [3] : \dim X = 1,$

$(x, [4]) := \mathcal{C}\dim[1] : \sum x \in X\ .\ X = kx,$

$(s, v, [5]) := \texttt{OrthogonalConnection}(Tx) : \sum s = \pm 1\ .\ \sum v : \texttt{NonIsotropic}(V)\ .\ s\sigma_v(x) = Tx,$

$[3.*] := \texttt{QuadraticOrthogonalTranslation}(s\sigma_v(x)) : s\sigma_v\ X^\perp = Y^\perp;$

$\rightsquigarrow [3] := I(\Rightarrow)I(\exists) : \dim X = 1 \Rightarrow (X^\perp, Y^\perp) : \texttt{Isometric}(k),$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ [4] : \forall X', Y' \subset_{k\text{-VS}} V\ .\ \dim X' < n \ \& \ (X', Y')i : \texttt{Isometric}(V) \Rightarrow (X'^\perp, Y'^\perp) : \texttt{Isometric}(V),$

$\texttt{Assume}\ [5] : \dim X = n,$

$x := \texttt{OrthogonalSymplectic}(X) : \texttt{NonIsotropic}(X),$

$(X', [6]) := \texttt{DimSumTHM}(x) : X = kx \perp X',$

$(Y', [7]) := \texttt{DimSumTHM}(x) : Y = kTx \perp Y',$

$[8] := [3][6][7] : \Big((X', Y') : \texttt{Isometric}(k)\Big),$

$[9] := \texttt{OSVCancelationTranslation}[4][8] : \Big((X^\perp, Y^\perp) : \texttt{Isometric}(k)\Big);$

$\rightsquigarrow [*] := \text{Ɑ}\mathbb{N} : \texttt{This};$

$\square$

$\texttt{OVSWittCancelationTHM} :: \forall V, W : \texttt{OrthogonalVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, \forall [0] : \dim V = \dim W < \infty$

$\quad . \, (V, W) : \texttt{WittCancelationProperty}(k)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{OVSWittExtensionTHM} :: \forall V, W : \texttt{OrthogonalVectorSpace} \,\&\, \texttt{Nonsingular}(k) \,.\, \dim V = \dim W < \infty$

$\quad . \, (V, W) : \texttt{WittExtensionProperty}(k)$

$\texttt{Proof} \ =$

$\dots$

$\square$

3.10 Maximal Hyperbolic Subspaces

$\texttt{DegenerateSpace} :: ?\texttt{QuadraticSpace}(k)$
$V : \texttt{DegenerateSpace} \iff \forall v, u \in V \;.\; \langle v, u \rangle = 0$

$\texttt{MaximalDegenerateSubspace} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?\texttt{VectorSubspace}(V)$
$U : \texttt{MaximalDegenerateSubspace} \iff U : \texttt{DegenerateSpace}(k) \;\&\; \forall W \subset_{k\text{-VS}} V \;.\; U \subsetneq W \Rightarrow$
$\quad \Rightarrow W \;!\; \texttt{DegenerateSpace}(k)$

$\texttt{WittIndexTHM} :: \forall V : \texttt{OrthogonalVectorSpace} \;\&\; \texttt{Nonsingular}(k) \;.\; \forall U, U' : \texttt{MaximalDegenerateSybspace}(V$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \dim U < \dim U',$

$(W, T) := \texttt{DimensionIsomorphism}(U, U')[1] : \sum W \subset_{k\text{-VS}} U' \;.\; \sum T : UToIso(\text{-VS}k)W,$

$[2] := \Game\texttt{DegenerateSpace}(V)(T) : \Big(T : \texttt{Isometry}(V, W)\Big),$

$(T', [3]) := \texttt{OSWWittExtensionTHM}(V)\Game\texttt{WittExtensionProperty}[2] : \sum T' : \texttt{Isometry}(V, V) \;.\; T'_{|U} = T,$

$[4] := \Game\texttt{Isometry} : \Big(T'^{-1}U' : \texttt{DegenerateSpace}(k)\Big),$

$[5] := \Game\texttt{Preimage}[3][1] : U \subset T'^{-1}U',$

$[6] := \Game\texttt{MaximalDegenerateSubspace} : \bot;$

$\rightsquigarrow [*] := \texttt{SymmetricProof}\Game\texttt{Antisymmetric} : \dim U = \dim U';$

$\square$

$\texttt{DegenerateSpacExist} :: \forall V : \texttt{QuadraticSpace}(k) \;.\; \exists U \subset_{k\text{-VS}} V : U : \texttt{DegenerateSpace}(k)$

$\texttt{Proof} =$

$[*] := \Game^{-1}\texttt{DegenerateSpace} : \Big(\{0\} : \texttt{DegenerateSpace}(k)\Big);$

$\texttt{indexOfWitt} :: \texttt{OrthogonalVectorSpace} \;\&\; \texttt{NonSingular} \;\&\; k\text{-FDVS} \to \mathbb{N}$
$\texttt{indexOfWitt}\,(V) = \mathrm{WI}(V) := \Game\texttt{Singleton}\{\dim U | U : \texttt{DegenerateSpace}(k)\}$

$\texttt{MaximalHyperbolicSubspace} :: \prod V : \texttt{QuadraticSpace}(k) \;.\; ?\texttt{VectorsSubspace}(V)$
$U : \texttt{MaximalHyperbolicSubspace} \iff U : \texttt{HypervolicSpace}(k) \;\&\; \forall W \subset_{k\text{-VS}} V \;.\; U \subsetneq W \Rightarrow W \;!\; \texttt{HyperbolicS}$

$\texttt{MaximalHyperbolicTHM} :: \forall V : \texttt{OrthogonalVectorSpace} \;\&\; \texttt{Nonsingular}(k) \;.$
$\quad \forall U : \texttt{MaximalHyperbolicSubspace}(V) \;.\; \dim U = 2\,\mathrm{WI}(U)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{AnisotropicDecompositionTHM} :: \forall V : \texttt{OrthogonalVectorSpace}(k) \;.\; \forall [0] : \dim V < \infty \;.$
$\quad \exists H : \texttt{HyperbolicSpace}(k) \;.\; \exists U : \texttt{AnIsotropic}(k) \;.\; V = H \bot U \bot \sqrt{V}$

$\texttt{Proof} =$

$\ldots$

$\square$