LieGroups.Know

Uncultured Tramp

July 9, 2016

Prereqs: Manifolds, Group theory.

1 Lie Groups

1.1 Definitions

$\texttt{LieGroup} :: ?\texttt{Group}\&\texttt{SManifold}$
$G : \texttt{LieGroup} \iff (\cdot_G) \in C^\infty(G \times G, G) \wedge (\cdot)_G^{-1} \in C^\infty(G, G)$

$(\mathbb{R}^n, +), (\mathbb{C}^n, +), U(1) = (S^1, \cdot_{\mathbb{C}}) : \texttt{LieGroup}$

$\mathrm{GL}(n, \mathbb{R}) :: \texttt{LieGroup},$
$\mathrm{GL}(m, \mathbb{R}) := \{M \in \mathcal{M}^n(\mathbb{R}) : \det M \neq 0\}$

$\texttt{DijointCosets} :: \forall G : \texttt{Group} \,.\, \forall H : \texttt{Subgroup}(G) \,.\, G = \bigsqcup GH$
$\texttt{Proof} =$
$\texttt{Assume}\ G : \texttt{Group},$
$\texttt{Assume}\ H : \texttt{Subgroup}(G),$
$\texttt{Subgroup}(G)(H) \to e \in H\ \texttt{as}\ (1),$
$\texttt{Assume}\ g \in G,$
$(1) \to g \in gH;$
$G = \bigcup GH\ \texttt{as}\ (2),$
$\texttt{Assume}\ a, b \in G,$
$\texttt{Assume}\ g \in aH \cap bH,$
$g \in aH \to \exists x \in H : g = ax\ \texttt{Extract},$
$g \in bH \to \exists x \in H : g = bx\ \texttt{Extract as}\ y,$
$\texttt{Assume}\ c \in aH,$
$c \in bH \to \exists x \in H : c = ax\ \texttt{Extract as}\ z,$
$by = g, ax = g \to by = ax \to b = axy^{-1} \in aH \to$
$\quad \to c = az = axy^{-1}yx^{-1}z = byx^{-1}z \in bH;$
$aH \subset bH\ \texttt{as}\ (3),$
$\texttt{SymmetricArgument}(3) \to bH \subset aH\ \texttt{as}\ (4),$
$(3, 4) \to aH = bH;;$
$G = \bigsqcup GH;;\square$

$\texttt{OpenCosets} :: \forall G : \texttt{LieGroup} \,.\, \forall H : \texttt{Open\&Subgroup}(G) \,.\, g \in G \,.\, gH : \texttt{Open}(G)$

$\texttt{Proof} =$

$\texttt{Assume}\ G : \texttt{LieGroup},$

$\texttt{Assume}\ H : \texttt{Open\&Subgroup}(G),$

$\texttt{Assume}\ g \in G,$

$\mu := \Lambda x \in G \,.\, g^{-1}x : C^\infty(G, G),$

$gH = \mu^{-1}(H) \to gH : \texttt{Open}(G);;; \square$

$\texttt{DisconnectedSubgroup} :: \forall G : \texttt{LieGroup} \,.\, \forall H : \texttt{Open\&Subgroup}(G) \,.\, H : \texttt{Closed}(H)$

$\texttt{Proof} =$

$\texttt{Assume}\ G : \texttt{LieGroup},$

$\texttt{Assume}\ H : \texttt{Open\&Subgroup}(G),$

$g \in G,$

$\texttt{OpenCosets}(G, H, g) \to gH : \texttt{Open}(G);$

$\forall g \in G \,.\, gH : \texttt{Open}(G)\ \texttt{as}\ (1),$

$\texttt{DijointCosets}(G, H) \to G = \bigsqcup GH\ \texttt{as}\ (2),$

$(1, 2) \to G \setminus H : \texttt{Open}(G) \to H : \texttt{Closed}(G);; \square$

$\texttt{NeighbourhoodOfUnity} :: \forall G : \texttt{LieGroup\&Connected} \,.\, \forall U \in \mathcal{U}(e_G) \,.\, \texttt{genGroup}(U) = G$

$\texttt{Proof} =$

$\texttt{Assume}\ G : \texttt{LieGroup\&Connected},$

$\texttt{Assume}\ U \in \mathcal{U}(e),$

$V := U \cap U^{-1} :?U\&\texttt{Open}(G),$

$U \in \mathcal{U}(e_G) \to e \in U\ \texttt{as}\ (1),$

$e^{-1} = e \to_{(1)} V \neq \emptyset,$

$V \subset U \to \texttt{genGroup}(V) \subset \texttt{genGroup}(U),$

$\texttt{Assume}\ k \in \mathbb{N},$

$S_k := \texttt{if}\ k == 1\ \texttt{then}\ V\ \texttt{else}\ VS_{k-1},$

$V : \texttt{Open}(G), S_k : \texttt{Open}(G) \to S_{k+1} = VS_k = \bigcup_{v \in V} vS_k : \texttt{Open}(G)$

$H := \bigcup_{k \in \mathbb{N}} S_k : \texttt{Subgroup\&Open}(G),$

$\texttt{DisconnectedSubgroup}(G, H) \to H : \texttt{Closed}(G)\ \texttt{as}\ (2),$

$V \neq \emptyset \to H \neq \emptyset\ \texttt{as}\ (3),$

$G : \texttt{Connected} \to_{(1,3)} H = G,$

$H \subset \texttt{genGroup}(U) \subset G \to \texttt{genGroup}(U) = G;; \square$

$\texttt{IdComponent} :: \prod G : \texttt{LieGroup} \;.\; ?\texttt{CC}(G)$

$H : \texttt{IdComponent} \iff e \in H$

$\texttt{LieSubgroup} :: \prod G : \texttt{LieGroup} \;.\; ?\texttt{Subgroup}(G)$

$H : \texttt{LieSubgroup} \iff i_H : \texttt{Immersion}(H, G)$

$\texttt{RegularLieSubgroup} :: \forall G : \texttt{LieGroup} \;.\; \forall H : \texttt{Subgroup\&Regular}(G) \;.$
$\quad .\; H : \texttt{LieSubgroup\&Closed}(H)$

$\texttt{Proof} =$

$\texttt{Assume}\ G : \texttt{LieGroup},$

$\texttt{Assume}\ H : \texttt{Subgroup\&Regular}(G),$

$H : \texttt{Regular}(G) \to i_H : \texttt{Immersion} \to H : \texttt{LieSubgroup}(M)\ \texttt{as}\ (1),$

$(U, x) := \texttt{Regular}(G, H, e),$

$V := \texttt{Separable3}(U, e) \to e \in V \subset \overline{V} \subset U,$

$\delta := \Lambda(a, b) \in G \times G \;.\; a^{-1}b : C^\infty(G \times G, G)$

$\Theta := \delta^{-1}(V) : \texttt{Open}(G \times G) \to \exists O : \texttt{Open}(G) : O \times O \subset \Theta : e \in O,$

$\texttt{Assume}\ X \in \overline{H},$

$X \in \overline{H} \to \exists x \in \texttt{ConvergentFrom}(G, H) : \lim_{n\to\infty} x_n = X\ \texttt{Extract},$

$\lim_{n\to\infty} \lim_{k\to\infty} \delta(x_n, x_k) = \lim_{n\to\infty} \lim_{k\to\infty} x_n^{-1} x_k = \left(\lim_{n\to\infty} x_n^{-1}\right)\left(\lim_{k\to\infty} x_k\right) = X^{-1}X = e \to$
$\quad \to \exists N \in \mathbb{N} : \forall n, k > N \;.\; \delta(x_n, x_k) \in V\ \texttt{Extract},$

$(U, x) : \texttt{SliceChart}(G, H) \to H \cap U : \texttt{Closed}(G),$

$H \cap U : \texttt{Closed}(G) \to H \cap U \cap \overline{V} = H \cap \overline{V} : \texttt{Closed}(G),$

$\texttt{Assume}\ n, k \in \mathbb{N} : n, k > N,$

$x : \texttt{ConvergentFrom}(G, H) \to \delta(x_n, x_k) = x_n^{-1} x_k \in H,$

$\texttt{def}(N) \to \delta(x_n, x_k) \in V \to \delta(x_n, x_k) \in H \cap V;$

$: \forall n, k \in \mathbb{N} : n, k > N \;.\; \delta(x_n, x_k) \in H \cap V\ \texttt{as}\ (2),$

$\texttt{Assume}\ n \in \mathbb{N} : n > N,$

$\lim_{k\to\infty} \delta(x_n, x_k) = x_n^{-1}\left(\lim_{k\to\infty} x_k\right) = x_n^{-1} X \in_{(2)} \overline{H \cap V} = H \cap \overline{V} \to$
$\quad \to x^{-1} X \in H \to X \in xH = H; ;$

$: \forall X \in \overline{H} \;.\; X \in H \to H : \texttt{Closed}(G)\ \texttt{as}\ (3),$

$(1, 3) \to \texttt{LieSubgroup\&Closed}(G); ; \square$

$\texttt{CLieSubgroup}(G) := \texttt{LieSubgroup\&Closed}(G)$

$H : \texttt{CLieSubgroup}(G) \iff H \subset_{LG} G$

1.2 Linear Lie Groups

$\texttt{GeneralLinearGroup} :: \mathsf{VS}(\mathbb{F}) \to \texttt{Group}$

$\texttt{GeneralLinearGroup}(V) = \mathrm{GL}(V) := \Big(\{T : \mathcal{L}(V, V) : T : \texttt{Invertible}\}, \circ\Big)$

$\texttt{GeneralMatrixGroup} :: \texttt{Field} \to \mathbb{N} \to \texttt{LieGroup}$

$\texttt{GeneralMatrixGroup}(\mathbb{F}, n) = \mathrm{GL}(\mathbb{F}, n) := \Big(\{M \in \mathcal{M}^n(\mathbb{R}) : \det M \neq 0\}, \cdot\Big)$

$\texttt{SpecialLinearGroup} :: \mathsf{FVS}(\mathbb{F}) \to \texttt{LieGroup}$

$\texttt{SpecialLinearGroup}(V) = \mathrm{SL}(V) := \{T \in \mathrm{GL}(V) : \det T = 1\}$

$\texttt{Nondegenerate} :: \prod V : \mathsf{VS}(\mathbb{F}) \ . \ ?\mathcal{L}_2(V, V; \mathbb{F})$

$\beta : \texttt{Nondegenerate} \iff \forall \sigma \in S(2) \ . \ \Lambda v \in V \ . \ \Lambda w \in W \ . \ (v, w)\sigma\beta : \texttt{Iso}_{\mathsf{VS}(\mathbb{F})}(V, V^*)$

$\texttt{ScalarProduct}(V) := \texttt{Nondegenerate}\&\texttt{Symmetric}(V)$

$\texttt{ScalarProductSpace} := \sum V : \mathsf{VS}(\mathbb{F}) \ . \ \texttt{ScalarProduct}(V)$

$\texttt{OrthogonalLinearGroup} :: \mathsf{IPVS}(\mathbb{F}) \to \texttt{Group}$

$\texttt{OrthogonalLinearGroup}(V) = \mathrm{Aut}(V) := \big\{A \in \mathrm{GL}(V) : \forall v, w \in V \ . \ \langle Av, Aw\rangle = \langle v, w\rangle\big\}$

$\texttt{SLAsClosedLieSubgroup} :: \forall V : \mathsf{FVS}(\mathbb{F}) \ . \ \mathrm{SL}(V) \subset_{LG} \mathrm{GL}(V)$
$\texttt{Proof} =$
$\texttt{Assume}\ V : \mathsf{FVS}(\mathbb{F}),$
$\texttt{Assume}\ T \in \mathrm{SL}(V) \to T \in \mathrm{GL}(V),$
$(U, x) := \texttt{chartCentredAt}(\mathrm{GL}(V), T),$

$x' := \Lambda A \in \mathrm{GL}(V) \ . \ (\det A - 1) \oplus \bigoplus_{i=2}^{n} x^i(A),$

$(U.x') : \texttt{SliceChart}(\mathrm{GL}(V), \mathrm{SL}(V), T);$
$\mathrm{SL}(V) : \texttt{Regular}(\mathrm{GL}(V)),$
$\texttt{RegularLieSubgroup}(\mathrm{GL}(V), \mathrm{SL}(V)) \to \mathrm{SL}(V) \subset_{LG} \mathrm{GL}(V); \square$

$\texttt{AutAsClosedLieSubgroup} :: \forall V : \mathsf{FIPVS}(\mathbb{F}) \; . \; \mathrm{SL}(V) \subset_{LG} \mathrm{GL}(V)$

$\texttt{Proof} =$

$\texttt{Assume}\ V : \mathsf{FIPVS}(\mathbb{F}),$

$\texttt{Assume}\ v, w \in V,$

$a := \langle v, w \rangle,$

$F_{v,w} := \big\{ A \in \mathrm{GL}(V) : \langle Av, Aw \rangle = a \big\},$

$\texttt{Assume}\ T \in F_{v,w} \to T \in \mathrm{GL}(V),$

$(U, x) := \texttt{chartCentredAt}(\mathrm{GL}(V), T),$

$$x' := \Lambda A \in \mathrm{GL}(V) \; . \; (\langle Av, Aw \rangle - a) \oplus \bigoplus_{i=2}^{n} x^i(A),$$

$(U.x') : \texttt{SliceChart}(\mathrm{GL}(V), F_{v,w}, T);$

$F_{v,w} : \texttt{Regular}(\mathrm{GL}(V));$

$$\mathrm{Aut}(V) = \bigcap_{v,w \in V} F_{v,w} : \texttt{Regular}(\texttt{GL}(V)),$$

$\texttt{RegularLieSubgroup}(\mathrm{GL}(V), \mathrm{Aut}(V)) \to \mathrm{Aut}(V) \subset_{LG} \mathrm{GL}(V); \square$

1.3 Simplectic Forms

$\texttt{SimplecticProduct} :: \prod V : \mathsf{FVS}(K) \ . \ (V \times V) \to (V \times V) \to K$

$\texttt{SimplecticProduct}(v, w)(a, b) = ((v, w), (a, b))_\nabla := \sum_{i=1}^{n} v_i b_i - \sum_{i=1}^{n} w_i a_i$

$\texttt{SimplecticGroup} :: \mathsf{FVS}(K) \to \texttt{LieGroup}$

$\texttt{SimplecticGroup}(V) = \mathrm{Sp}(V) := \left\{T \in \mathrm{GL}(V \oplus V) : \forall v, w \in V \oplus V \ . \ (Tv, Tw)_\nabla = (v, w)_\nabla\right\}$

$\texttt{Quaternion} :: \texttt{DivisionRing}$

$\texttt{Quaternion} = \mathbb{H} := (\mathbb{R}^4, +, \Lambda a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}, x + y\mathrm{i} + z\mathrm{j} + u\mathrm{k} \in \mathbb{H} \ .$

$\quad . \ (ax - by - cz - du) + (ay + bx + cu - dz)\mathrm{i} + (az + cx - bu + dy)\mathrm{j} + (au + dx + bz - cy)]\mathrm{k}$

$\quad)$

where

$(a, b, c, d) \in \mathbb{H} \iff (a, b, c, d) = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$

$\texttt{Real} :: ?\mathbb{H}$

$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : \texttt{Real} \iff b = c = d = 0$

$\texttt{Imagenary} :: ?\mathbb{H}$

$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : \texttt{Imagenary} \iff a = 0$

$\texttt{conjugate} :: \mathbb{H} \to \mathbb{H}$

$\texttt{conjugate}(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{j}) := a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$

$\texttt{conjugate}(x) := \overline{x}$

$\texttt{valuate} :: \mathbb{H} \to \mathbb{R}$

$\texttt{valuate}(x) = |x| := \sqrt{\overline{x}x}$

$\texttt{inverse} :: \mathbb{H} \setminus \{0\} \to \mathbb{H}$

$\texttt{inverse}(x) = x^{-1} = \dfrac{\overline{x}}{|x|^2}$

$\texttt{QVectors} :: \mathbb{N} \to \texttt{RightModule}(\mathbb{H})$

$\texttt{QVectors}(n) = \mathbb{H}^n := \left(\mathbb{H}^n, +, \Lambda v \in \mathbb{H}^n \ . \ \Lambda a \in \mathbb{H}^n \ . \ [v_i a]_i^n\right)$

$\texttt{quaternify} :: \mathbb{C} \to \mathbb{H}$

$\texttt{quaternify}(a + b\mathrm{i}) = a + b\mathrm{i} + 0\mathrm{j} + 0\mathrm{k}$

$\texttt{ToQuaternion} :: \texttt{ISO}_{\mathsf{VS}(\mathbb{R})}(\mathbb{C}^2.\mathbb{H})$

$\texttt{ToQuaternion}(x, y) = \nu(x, y) := x + y\mathrm{j}$

$\texttt{ComplexDecomposition} :: \forall M \in \mathcal{M}^{n\times m}(\mathbb{H})\ .\ \exists A, B \in \mathcal{M}^{n\times m}(\mathbb{C}) :$

$: M = A + B\mathrm{j}$

$\texttt{Scatch}$

$M = [q_{i,j}]_{i,j=1}^{n,m} = [a_{i,j} + b_{i,j}\mathrm{j}]_{i,j=1}^{n,m} = [a_{i,j}]_{i,j=1}^{n,m} + [b_{i,j}]_{i,j=1}^{n,m}\mathrm{j} = A + B\mathrm{j}$

$\texttt{ToComplexMatrix} :: \mathcal{M}_{\mathsf{VS}(\mathbb{R})}\Big(\mathcal{M}^{n\times m}(\mathbb{H}), \mathcal{M}^{2n\times 2k}(\mathbb{C})\Big)$

$\texttt{ToComplexMatrix}(M) = \vartheta(M) := \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}$

$\texttt{where}(A, B) := \texttt{ComplexDecomposition}(M)$

$\texttt{MatrixRepresentation} :: \vartheta : \texttt{Iso}_{\mathsf{GRP}}(U(1, \mathbb{H}), SU(2))$

$\texttt{Scatch}:$

$\vartheta((a + b\mathrm{j})(x + y\mathrm{j})) = \vartheta(ax + ay\mathrm{j} + b\overline{x}\mathrm{j} - b\overline{y}) =$

$= \begin{bmatrix} ax - b\overline{y} & ay + b\overline{x} \\ -\overline{ay} + \overline{b}x & \overline{ax} - \overline{b}y \end{bmatrix} = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix} = \vartheta(a + b\mathrm{j})\vartheta(x + y\mathrm{j})$

$|a + b\mathrm{j}| = 1 \to |a|^2 + |b|^2 = 1 \to \det \vartheta(a + b\mathrm{j}) = 1$

$\vartheta(a + b\mathrm{j})^* \vartheta(a + b\mathrm{j}) = I$

$|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1,$

$\overline{a}c + \overline{b}d = a\overline{c} + b\overline{d} = 0,$

$ad - bc = 1$

$\to c = \overline{a}$

$\to d = -\overline{b}$

$\texttt{GenLinMark} :: \forall M \in \mathcal{M}^n(\mathbb{H})\ .\ M \in \mathrm{GL}(n, \mathbb{H}) \iff \vartheta M \in \mathrm{GL}(2n, \mathbb{C})$

$\texttt{Scatch}:$

$\vartheta M = \nu^{-1} M \nu, \mathrm{null}\ \nu = \{0\} \rightsquigarrow \Box$

$\texttt{scalarProductH} :: \mathbb{H}^n \to \mathbb{H}^n \to \mathbb{H}$

$\texttt{scalarproductH}(a, b) = \langle a, b\rangle := \sum_{i=1}^{n} \overline{a}b$