# Multilinear Algebra

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# Contents


**1 From Bilinear Maps to Tensor Products** **3**

1.1 Multilinear Maps and Forms . . . 3
1.2 The Tensor Product . . . 10
1.3 Tensor Product as Functor . . . 16
1.4 Flatness . . . 23
1.5 Covariant Scalar Extension . . . 26
1.6 Composition Algebra . . . 31

**2 Tensorial Algebras** **33**

2.1 Tensor Algebra . . . 33
2.2 Mixed Tensor Algebra . . . 38
2.3 Exterior Algebra . . . 39
2.4 Determinant Identities . . . 44
2.5 Interior Product . . . 47
2.6 Mixed Exterior Algebra . . . 50
2.7 Algebraic Poincare Duality . . . 54
2.8 Pfaffian . . . 60
2.9 Symmetric Algebra . . . 65
2.10 Algebraic Differentiation . . . 68
2.11 Grassmann Algebra . . . 71
2.12 Graded Duality In Symmetric Algebras . . . 75
2.13 Plücker's Equations . . . 77

**3 Noncommutative Tensor Product** **82**

# 1 From Bilinear Maps to Tensor Products

## 1.1 Multilinear Maps and Forms

$$\texttt{Multilinear} :: \prod R \in \mathsf{ANN} \,.\, \left(\prod n \in \mathbb{N} \,.\, n \to R\text{-}\mathsf{MOD}\right) \to R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{MOD}$$
$$\texttt{Multilinear}\,(1 \mapsto V, M) = \mathcal{L}(V; M) := \mathcal{M}_{R\text{-}\mathsf{MOD}}(V, M)$$
$$\texttt{Multilinear}\,(V, M) = \mathcal{L}(V; M) := \mathcal{M}_{R\text{-}\mathsf{MOD}}\big(V_1, \mathcal{L}(V'; M)\big) \quad \texttt{where} \quad V' := \Lambda i \in n-1 \,.\, V_{i+1}$$

$$\texttt{multiEval} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V : n \to R\text{-}\mathsf{MOD} \,.\, \prod W \in R\text{-}\mathsf{MOD} \,.$$
$$.\, \mathcal{L}(V; W) \to \left(\prod i \in n \,.\, V_i\right) \to W$$
$$\texttt{multiEval}\,(T, v) = T(v) := T(v_1)(v') \quad \texttt{where} \quad v' := \Lambda i \in n-1 \,.\, v_{i+1}$$

$$\texttt{NForm} :: \prod R \in \mathsf{ANN} \,.\, \prod V, W \in R\text{-}\mathsf{MOD} \,.\, \mathbb{N} \to ?(V \to R)$$
$$F : \texttt{NForm} \iff \Lambda n \in \mathbb{N} \,.\, \exists T \in \mathcal{L}(i \mapsto V; R) : \forall v \in V \,.\, F(v) = L(i \mapsto v)$$

$$\texttt{coordinateTensor} :: \prod n \in \mathbb{N} \,.\, \prod V : n \to \texttt{FreeModule}(R) \,.\, \prod W \in \texttt{FreeModule}(R) \,.$$
$$.\, \mathcal{L}(V; W) \to \left(\prod i \in n \,.\, \texttt{Basis}(V_i)\right) \to \texttt{Basis}(W) \to R^{\left(\prod_{i=1}^{n} \operatorname{rank} V_i\right) \times W}$$
$$\texttt{coordinataTensor}\,(T, e, f) = T^{e;f} := \Lambda j : \prod i \in n \,.\, \operatorname{rank} V_i \,.\, \Lambda k \in \operatorname{rank} W \,.\, \alpha_k \quad \texttt{where} \quad \alpha f = L(i \mapsto e_{i, j_i})$$

$$\texttt{multiFromCoordinates} :: \prod n \in \mathbb{N} \,.\, \prod V : n \to \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$$
$$.\, \prod W \in \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.\, R^{\left(\prod_{i=1}^{n} \operatorname{rank} V_i\right) \times W} \to \left(\prod i \in n \,.\, \texttt{Basis}(V_i)\right) \to$$
$$\to \texttt{Basis}(W) \to \mathcal{L}(V; W)$$
$$\texttt{multiFromCoordinates}\,(A, e, f) = A_{e;f} := \Lambda \alpha e_1 \in V_1 \,.\, \sum_{i=1}^{\operatorname{rank} V_1} \alpha_i (A_i)_{e', f} \quad \texttt{where} \quad e' := \Lambda i \in n-1 \,.\, e_{i+1}$$

$$\texttt{Bilinear} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD}^3 \to R\text{-}\mathsf{MOD}$$
$$\texttt{Bililinear}\,(A, B, W) = \mathcal{L}(A, B; W) := \mathcal{L}(\lambda i \in 2 \,.\, \texttt{if } i == 1 \texttt{ then } A \texttt{ else } B; W)$$

$$\texttt{PermutationIsomorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to R\text{-}\mathsf{MOD} \,.\, \forall W \in R\text{-}\mathsf{MOD} \,.$$
$$\forall \sigma \in S_n \,.\, \mathcal{L}(V; W) \cong_{R\text{-}\mathsf{MOD}} \mathcal{L}(\sigma V; W)$$

$\texttt{Proof} \;=$

$\sigma^* T(v) := T(\sigma v)$ definetly acts as an isomorphism with the inverse provided by the $\sigma^{-1}$.

□

$$\texttt{QuadraticForm} := \Lambda R \in \mathsf{ANN} \,.\, \Lambda A, B \in R\text{-}\mathsf{MOD} \,.\, \texttt{NForm}(R, 2)\Big(\Lambda i \in 2 \,.\, \texttt{if } i == 1 \texttt{ then } A \texttt{ else } B\Big) :$$
$$: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD}^2 \to \texttt{Type};$$

$\texttt{MultiAdditive} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall V : n \to R\text{-}\mathsf{MOD} . \forall W \in R\text{-}\mathsf{MOD} . \forall T \in \mathcal{L}(V;W) .$
$\quad . \forall v, v', v'' . \forall i \in n . \forall a, b \in V_i . \forall [0] : v_i = a + b . \forall [00] : v'_i = a . \forall [000] : v''_i = b .$
$\quad . \forall [0000] : \forall j \in n . j \neq i \Rightarrow v_j = v'_j = v''_j . T(v) = T(v') + T(v'')$

$\texttt{Proof} =$

$♋ := \Lambda n \in \mathbb{N} . \forall m \in \mathbb{N} . m \leq n \Rightarrow \texttt{This}(R)(n) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume } [1] : n = 1,$

$[1.*] := \mathcal{I}\mathcal{L}(V;W)(T)[1]\mathcal{I}R\text{-}\mathsf{MOD}(V,W)(T)[0][00][000] : T(v) = T(a+b) = T(a) + T(b) = T(v') + T(v'');$

$\rightsquigarrow [1] := \mathcal{D}^{-1}♋ : ♋(1),$

$\texttt{Assume } m : \mathbb{N},$

$\texttt{Assume } [m.2] : ♋(m),$

$\texttt{Assume } [m.3] : n = m + 1,$

$\texttt{Assume } [m.4] : i = 1,$

$\hat{V} := \Lambda j \in m . V_{j+1} : m \to R\text{-}\mathsf{MOD},$

$\hat{v} := \Lambda j \in m . v_{j+1} : \prod_{j=1}^{m} \hat{V}_j,$

$[m.4.*] := \mathcal{I}\texttt{multiEval}(T, v)\mathcal{I}R\text{-}\mathsf{MOD}(V_1, \mathcal{L}(\hat{V};W))[0][00][000]\mathcal{D}\bar{v} :$
$\quad : T(v) = T(a+b)(\bar{v}) = T(a)(\bar{v}) + T(b)(\bar{v}) = T(v') + T(v'');$

$\rightsquigarrow [m.4] := I(\Rightarrow) : i = 1 \Rightarrow T(v) = T(v') + T(v''),$

$\texttt{Assume } [m.5] : i \neq 1,$

$\bar{V} := V_{|i-1} : (i-1) \to R\text{-}\mathsf{MOD},$

$\hat{V} := \Lambda j \in m + 2 - i . V_{i+j-1} : (n + 2 - i) \to R\text{-}\mathsf{MOD},$

$[m.6] := \texttt{NonegativeAdditionNondecrease}[m.5][m,3] + (i-2) : m + 2 - i \leq m,$

$\bar{v} := v_{|i-1} : \prod_{j=1}^{i-1} \hat{V}_j,$

$\hat{v} := \Lambda j \in m + 2 - i . v_{i+j-1} : \prod_{j=1}^{n+2-i} \hat{V}_j,$

$\hat{v}' := \Lambda j \in m + 2 - i . v'_{i+j-1} : \prod_{j=1}^{n+2-i} \hat{V}_j,$

$\hat{v}'' := \Lambda j \in m + 2 - i . v''_{i+j-1} : \prod_{j=1}^{n+2-i} \hat{V}_j,$

$[m.5.*] := \mathcal{D}^{-1}\bar{v}\mathcal{D}^{-1}\hat{v}[m.2](n + 2 - i, [m.6])[0][00][000][0000]\mathcal{D}\hat{v}''\mathcal{D}\hat{v}' :$
$\quad : T(v) = T(\bar{v})(\hat{v}) = T(\bar{v})(\hat{v}') + T(\bar{v})(\hat{v}'') = T(v') + T(v'');$

$\rightsquigarrow [m.5] := I(\Rightarrow) : i \neq 1 \Rightarrow T(v) = T(v') + T(v''),$

$[m.6] := \texttt{EqAlt}(\mathbb{N}, i, 1) : i = 1 | i \neq 1,$

$[m.*] := E(|)[m.4][m.5][m.6] : T(v) = T(v') + T(v'');$

$\rightsquigarrow [*] := \mathcal{I}\texttt{NaturalSet}(\mathbb{N})[1] : \texttt{This}(R),$

$\square$

$\texttt{MultiHomogen} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to R\text{-}\mathsf{MOD} \,.\, \forall W \in R\text{-}\mathsf{MOD} \,.\, \forall T \in \mathcal{L}(V;W) \,.$

$\quad . \forall v, v' \,.\, \forall \omega \in A \,.\, \forall i \in n \,.\, \forall a \in V_i \,.\, \forall [0] : v'_i = a \,.\, \forall [00] : v_i = \omega a \,.\, \forall [000] : \forall j \in n \,.\, j \neq i \Rightarrow v_j = v'_j \,.\, T(v) = \omega$

$\texttt{Proof} \ =$

$\text{♋} := \Lambda n \in \mathbb{N} \,.\, \forall m \in \mathbb{N} \,.\, m \leq n \Rightarrow \texttt{This}(R)(n) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ [1] : n = 1,$

$[1.*] := \daleth \mathcal{L}(V;W)(T)[1] \daleth R\text{-}\mathsf{MOD}(V,W)(T)[0][00][000] : T(v) = T(\omega a) = \omega T(a) = \omega T(v');$

$\rightsquigarrow [1] := \mathcal{D}^{-1}\text{♋} : \text{♋}(1),$

$\texttt{Assume}\ m : \mathbb{N},$

$\texttt{Assume}\ [m.2] : \text{♋}(m),$

$\texttt{Assume}\ [m.3] : n = m + 1,$

$\texttt{Assume}\ [m.4] : i = 1,$

$\hat{V} := \Lambda j \in m \,.\, V_{j+1} : m \to R\text{-}\mathsf{MOD},$

$$\hat{v} := \Lambda j \in m \,.\, v_{j+1} : \prod_{j=1}^{m} \hat{V}_j,$$

$[m.4.*] := \daleth \texttt{multiEval}(T, v) \daleth R\text{-}\mathsf{MOD}(V_1, \mathcal{L}(\hat{V}; W))[0][00][000] \mathcal{D} \bar{v} :$

$\quad : T(v) = T(\omega a)(\bar{v}) = \omega T(a)(\bar{v}) = \omega T(v');$

$\rightsquigarrow [m.4] := I(\Rightarrow) : i = 1 \Rightarrow T(v) = \omega T(v'),$

$\texttt{Assume}\ [m.5] : i \neq 1,$

$\bar{V} := V_{|i-1} : (i-1) \to R\text{-}\mathsf{MOD},$

$\hat{V} := \Lambda j \in m + 2 - i \,.\, V_{i+j-1} : (n + 2 - i) \to R\text{-}\mathsf{MOD},$

$[m.6] := \texttt{NonegativeAdditionNondecrease}[m.5][m,3] + (i-2) : m + 2 - i \leq m,$

$$\bar{v} := v_{|i-1} : \prod_{j=1}^{i-1} \hat{V}_j,$$

$$\hat{v} := \Lambda j \in m + 2 - i \,.\, v_{i+j-1} : \prod_{j=1}^{n+2-i} \hat{V}_j,$$

$$\hat{v}' := \Lambda j \in m + 2 - i \,.\, v'_{i+j-1} : \prod_{j=1}^{n+2-i} \hat{V}_j,$$

$[m.5.*] := \mathcal{D}^{-1} \bar{v} \mathcal{D}^{-1} \hat{v} [m.2](n + 2 - i, [m.6])[0][00][000][0000] \mathcal{D} \hat{v}'' \mathcal{D} \hat{v}' :$

$\quad : T(v) = T(\bar{v})(\hat{v}) = \omega T(\bar{v})(\hat{v}') = \omega T(v');$

$\rightsquigarrow [m.5] := I(\Rightarrow) : i \neq 1 \Rightarrow T(v) = \omega T(v'),$

$[m.6] := \texttt{EqAlt}(\mathbb{N}, i, 1) : i = 1 | i \neq 1,$

$[m.*] := E(|)[m.4][m.5][m.6] : T(v) = \omega T(v');$

$\rightsquigarrow [*] := \daleth \texttt{NaturalSet}(\mathbb{N})[1] : \texttt{This}(R),$

$\square$

$\texttt{NFormNHomogen} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : R\text{-}\mathsf{MOD} \,.\, \forall F : \texttt{NForm}(R, V, n) \,.$

$\quad . \forall v \in V \,.\, \forall \alpha \in R \,.\, F(\alpha v) = \alpha^n F(v)$

$\texttt{Proof} \ =$

$(T, [1]) := \daleth \texttt{NForm}(R, V, n) : \sum T \in \mathcal{L}(\Lambda i \in n \,.\, V; R) \,.\, \forall v \in V \,.\, F(v) = T(i \mapsto v),$

$[*] := [1] \texttt{MultiHomogen}^n(T)(\ldots)[1] : F(\alpha v) = T(i \mapsto \alpha v) = \alpha^n T(i \mapsto v) = \alpha^n F(v);$

$\square$

$\texttt{PolarLemma} :: \forall R \in \mathsf{ANN} \,.\, \forall V : R\text{-}\mathsf{MOD} \,.\, \forall F : \texttt{QuadraticForm}(V) \,.\, \forall v, w \in V \,.$
$\quad . \; F(v+w) + F(v-w) = 2F(v) + 2F(w)$

$\texttt{Proof} \; =$

$(T, [1]) := \mathcal{I}\texttt{NForm}(R, V, n) : \sum T \in \mathcal{L}(V, V; R) \,.\, \forall v \in V \,.\, F(v) = T(v, v),$

$[*] := [1]\texttt{MultiAdditive}^6(T)(\ldots)\texttt{MultiHomogen}^4(T)(\ldots)\texttt{SquareOfNegative}(-1)[1] :$

$$F(v+w) + F(v-w) = T(v+w, v+w) + T(v-w, v-w) =$$
$$= T(v,v) + T(v,w) + T(w,v) + T(w,w) - T(v,v) - T(v,w) - T(w,v) + T(v,v) =$$
$$= 2T(v,v) + 2T(w,w) = 2F(v) + 2F(w);$$

$\square$

$\texttt{multiNullset} :: \prod n \in \mathbb{N} \,.\, \prod V : n \to R\text{-}\mathsf{MOD} \,.\, \prod i \in n \,.\, \mathcal{L}(V; W) \to \texttt{Submodule}(V_i)$

$$\texttt{multiNullset}\,(T) = N_i(T) := \left\{ v \in V_i : \forall w : \prod j \in n \setminus \{i\} \,.\, V_j \,.\, T\Big(j \mapsto \texttt{if } j == i \texttt{ then } v \texttt{ else } w_j\Big) = 0 \right\}$$

$\texttt{multiReduction} :: \prod T \in \mathcal{L}(V; W) \,.\, \mathcal{L}\left(\dfrac{V}{N(T)}; W\right)$

$$\texttt{multiReduction}\,(T) = \tilde{T} := \texttt{reduce}(T) \texttt{ else } \texttt{Reduce}\, \Lambda v \in V_1 \,.\, \texttt{multiReduce}(T(v))$$

$\texttt{ReducedMultiIsReduced} :: \forall T \in \mathcal{L}(V; W) \,.\, \forall i \in n \,.\, N_i(\tilde{T}) = \{0\}$

$\texttt{Proof} \; =$

$\texttt{Assume}\; [v] : N_i(\tilde{T}),$

$[1] := \mathcal{I} N_i(\tilde{T}) : \forall [w] : \prod j \in n \setminus \{i\} \,.\, \dfrac{V_j}{N_j(T)} \,.\, \tilde{T}\Big(j \mapsto \texttt{if } j == i \texttt{ then } [v] \texttt{ else } [w_j]\Big) = 0,$

$\texttt{Assume}\; w : \prod j \in n \setminus \{i\} \,.\, V_j,$

$[w.1] := [1][w] : \tilde{T}\Big(j \mapsto \texttt{if } j == i \texttt{ then } [v] \texttt{ else } [w_j]\Big) = 0,$

$[w.*] := \mathcal{I}\tilde{T}[w.1] : T\Big(j \mapsto \texttt{if } j == i \texttt{ then } v \texttt{ else } w_j\Big) = 0;$

$\rightsquigarrow [v.*] := \mathcal{I}^{-1} N_i(T) : v \in N_i(T);$

$\rightsquigarrow [*] := \mathcal{I}\texttt{QuotientModule} : N_i(\tilde{T}) = \{0\};$

$\square$

$\texttt{NullSpaceInclusion} :: \forall T \in \mathcal{L}(V; W) \,.\, \forall A : W \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, \forall i \in n \,.\, N_i(T) \subset N_i(AT)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; v : N_i(T),$

$[1] := \mathcal{I} N_i(\tilde{T}) : \forall w : \prod j \in n \setminus \{i\} \,.\, V_j \,.\, \tilde{T}\Big(j \mapsto \texttt{if } j == i \texttt{ then } v \texttt{ else } w_j\Big) = 0,$

$\texttt{Assume}\; w : \prod j \in n \setminus \{i\} \,.\, V_j,$

$[2] := [1](w) : T\Big(j \mapsto \texttt{if } j == i \texttt{ then } v \texttt{ else } w_j\Big) = 0,$

$[w.*] := \mathcal{I} R\text{-}\mathsf{MOD}(W, M)(T)[2] : AT\Big(j \mapsto \texttt{if } j == i \texttt{ then } v \texttt{ else } w_j\Big) = 0;$

$\rightsquigarrow [v.*] := \mathcal{I}^{-1} N_i : v \in N_i(AT);$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Subset} : N_i \subset N_i(AT);$

$\square$

$\texttt{Alternating} :: \prod R \in \mathsf{ANN} \, . \, \prod V, W \in R\text{-}\mathsf{MOD} \, . \, \prod n \in \mathbb{N} \, . \, ?\mathcal{L}(\Lambda i \in n \, . \, V; W)$

$T : \texttt{Alternating} \iff \forall v \in V^n \, . \, \forall i \in (n-1) \, . \, v_i = v_{i+1} \Rightarrow T(v) = 0$

$\texttt{StrongAlternatingProperty} :: \forall R \in \mathsf{ANN} \, . \, \forall V, W \in R\text{-}\mathsf{MOD} \, . \, \forall n \in \mathbb{N} \, .$

$\forall T : \texttt{Alternating}(V, W, n) \, . \, \forall i, j \in n \, . \, \forall v \in V^n \, . \, \forall [0] : i < j \, . \, \forall [00] : v_i = v_j \, . \, T(v) = 0$

$\texttt{Proof} \; =$

$☿ := \Lambda k \in \mathbb{N} \, . \, \forall i, j \in n \, . \, \forall v \in V^n \, . \, i < j \;\&\; v_i = v_j \Rightarrow T(v) = 0 : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume } i, j : n,$

$\texttt{Assume } v : V^n,$

$\texttt{Assume } [1] : j - i = 1,$

$\texttt{Assume } [2] : v_i = v_j,$

$[1.*] := \mathit{\Gamma}\texttt{Alternating}(T)[1][2] : T(v) = 0;$

$\rightsquigarrow [1] := \mathcal{I}☿ : ☿(1),$

$\texttt{Assume } k : n - 2,$

$\texttt{Assume } [2] : ☿(k),$

$\texttt{Assume } i, j : n,$

$\texttt{Assume } v : V^n,$

$\texttt{Assume } [3] : j - i = k + 1,$

$\texttt{Assume } [4] : v_i = v_j,$

$[k.*] := \mathit{\Gamma}^k\texttt{Alternating}(T)\texttt{multiAdditive}^{k+1}(T)\mathit{\Gamma}\texttt{Alternating}(T)\texttt{MultAdditive}(T)\mathit{\Gamma}\texttt{Alternating}(T)$

$$[2](i, j-1, \ldots) : T(v) = T(v) + \sum_{L=1}^{k} T\left(\Lambda m \in n \, . \, \texttt{if } i \le m \le i + L \texttt{ then } \sum_{l=i}^{\min(m, i+L-1)} v_l \texttt{ else } v_m\right) =$$

$$= T\left(\Lambda m \in n \, . \, \texttt{if } i \le m \le i + k \texttt{ then } \sum_{l=i}^{m} v_l \texttt{ else } v_m\right) =$$

$$= T\left(\Lambda m \in n \, . \, \texttt{if } i \le m \le j \texttt{ then } \sum_{l=i}^{\min(m, j-1)} v_l \texttt{ else } v_m\right) -$$

$$- T\left(\Lambda m \in n \, . \, \texttt{if } i \le m < j \texttt{ then } \sum_{l=i}^{m} v_l \texttt{ else if } m = j \texttt{ then } \sum_{l=i+1}^{j-1} v_l \texttt{ else } v_m\right) =$$

$$= -T\left(\Lambda m \in n \, . \, \texttt{if } i \le m < j \texttt{ then } \sum_{l=i}^{m} v_l \texttt{ else if } m = j \texttt{ then } \sum_{l=i+1}^{j-1} v_l \texttt{ else } v_m\right) =$$

$$= -T\left(\Lambda m \in n \, . \, \texttt{if } i \le m < j-1 \texttt{ then } \sum_{l=i}^{m} v_l \texttt{ else if } j-1 \le m \le j \texttt{ then } \sum_{l=i+1}^{j-1} v_l \texttt{ else } v_m\right) -$$

$$- T\left(\Lambda m \in n \, . \, \texttt{if } i \le m \le i + k \texttt{ then } \sum_{l=i}^{\texttt{if } m<i+k \texttt{ then } m \texttt{ else } i} v_l \texttt{ else if } m = j-1 \texttt{ then } \sum_{l=i+1}^{j-1} v_l \texttt{ else } v_m\right) =$$

$$= 0;$$

$\rightsquigarrow [*] := \mathit{\Gamma}\texttt{InduciveSet}(\mathbb{N})\mathcal{I}☿ : \texttt{This};$

$\square$

$\texttt{LinearlyIndeprndentByAlternating} :: \forall k : \texttt{Field} . \forall V, W \in k\text{-VS} . \forall n \in \mathbb{N} .$

$. \forall T : \texttt{Alternating}(V, W, n) . \forall v : V^n . \forall [0] : T(v) \neq 0 . v : \texttt{LinearlyIndependent}(n, V)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : v \,!\, \texttt{LinearlyIndependent}(n, V),$

$(i, \alpha, [2]) := \mathcal{C}\texttt{LinearlyIndependent}(n, V)[1] : \sum i \in n . \sum \alpha \in R^{n \setminus \{i\}} : \alpha v_{|n \setminus \{i\}} = v_i,$

$[3] := [2]\texttt{MultiAdditive}^{n-1}(T)\texttt{MultiHomogen}^{n-1}(T)\texttt{StrongAlternatingProperty}^{n-1}(T, \ldots) :$

$: T(v) = \sum_{j \in n \setminus \{i\}} \alpha_j T(\Lambda k \in n . \texttt{ if } k = i \texttt{ then } v_j \texttt{ else } v_k) = 0,$

$[1.*] := [0][3] : \bot;$

$\rightsquigarrow [*] := E(\bot) : \Big(v : \texttt{LinearlyIndependent}(n, V)\Big);$

$\square$

$\texttt{Symmetric} :: \prod R \in \mathsf{ANN} . \prod V, W \in R\text{-}\mathsf{MOD} . \prod n \in \mathbb{N} . ?\mathcal{L}(\Lambda i \in n . V; W)$

$T : \texttt{Symmetric} \iff \forall \sigma \in S_n . \forall v \in V^n . T(\sigma^* v) = T(v)$

$\texttt{Antisymmetric} :: \prod R \in \mathsf{ANN} . \prod V, W \in R\text{-}\mathsf{MOD} . \prod n \in \mathbb{N} . ?\mathcal{L}(\Lambda i \in n . V; W)$

$T : \texttt{Antisymmetric} \iff \forall \sigma \in S_n . \forall v \in V^n . T(\sigma^* v) = (-1)^\sigma T(v)$

$\texttt{SymmetricNForm} :: \forall R \in \mathsf{ANN} . \forall V \in R\text{-}\mathsf{MOD} . \forall n \in \mathbb{N} . \forall F : \texttt{NForm}(V, n) .$

$\forall [0] : n! \in R^* . \exists S : \texttt{Symmetric}(V, R, n) : \forall v \in V . F(v) = S(i \mapsto v)$

$\texttt{Proof} \ =$

$\Big(T, [1]\Big) := \mathcal{C}\texttt{NForm}(V, n)(F) : \sum T : \mathcal{L}(\Lambda i \in n . V; R) . \forall v \in V . F(v) = T(i \mapsto v),$

$S := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{**} T : \texttt{Symmetric}(V, R, n),$

$\texttt{Assume}\ v : V,$

$[v.*] := [1][0]\texttt{NumberOfPermutations}(n)\mathcal{C}^{-1}\sigma^{**}\mathcal{D}^{-1}(S) :$

$: F(v) = T(i \mapsto v) = \frac{1}{n!} \sum_{\sigma \in S_n} T(i \mapsto v) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{**} T(i \mapsto v) = S(i \mapsto v);$

$\rightsquigarrow [*] := I(\forall) : \texttt{This};$

$\square$

$\texttt{MultilinearNullByIdealStructure} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \to \texttt{Ideal}(R) \,.\, \forall [0] : R = \sum_{i=1}^{n} I_i \,.$

$\,.\, \forall W \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}\left(\frac{R}{I}; W\right) = \{0\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ T : \mathcal{L}\left(\frac{R}{I}; W\right),$

$\texttt{Assume}\ [\alpha] : \prod_{i=1}^{n} \frac{R}{I_i},$

$\Big(\beta, [1]\Big) := [0]\alpha : \sum \beta : n \to \prod_{i=1}^{n} I_i \,.\, \forall i \in n \,.\, \alpha_i = \sum_{j=1}^{n} \beta_{i,j},$

$[2] := \texttt{MultiAdditive}(T)[1] : T[\alpha] = \sum_{j:n\to n} T[\beta_j],$

$\texttt{Assume}\ j : n \to n,$

$\texttt{Assume}\ [3] : T[\beta_j] \neq 0,$

$\beta' := \Lambda i \in n \,.\, \texttt{if}\ i = 1\ \texttt{then}\ [1]\ \texttt{else if}\ i = j_1\ \texttt{then}\ [\beta_1 \beta_{j_i}]\ \texttt{else}\ [\beta_{i,j_i}] : \prod_{i=1}^{n} \frac{R}{I_i},$

$[j.*] := \texttt{MultiHomogen}^2(T, \beta, \beta') Ɔ \beta' Ɑ \texttt{Ideal}(I_{j_1}) Ɑ \texttt{quotientRing} Ɔ \beta' \texttt{MultiHomogen} :$

$: T[\beta_j] = T[\beta'] = 0;$

$\rightsquigarrow [3] := I(\forall) : \forall j : n \to n \,.\, T[\beta_j] = 0,$

$\Big[[\alpha].*\Big] := [3][2] : T[\alpha] = 0;$

$\rightsquigarrow [T.*] := I(=, \to) : T = 0;$

$\rightsquigarrow [*] := Ɑ^{-1}\texttt{Subset}Ɑ^{-1}\{0\} : \mathcal{L}\left(\frac{R}{I}; W\right) = \{0\};$

$\square$

## 1.2 The Tensor Product

$$\texttt{TensorProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V : n \to R\text{-}\mathsf{MOD} \,.\, ?\sum W \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(V;W)$$

$$\left(W, \bigotimes\right) : \texttt{TensorProduct} \iff \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \mathcal{L}(V;M) \,.\, \exists! A : W \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, A\bigotimes = T$$

$$\texttt{TensorProductExists} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to R\text{-}\mathsf{MOD} \,.\, \exists \left(W, \bigotimes\right) : \texttt{TensorProduct}(R,n,V)$$

$\texttt{Proof} \;=$

$F := \texttt{FreeModule}\left(R, \prod_{i=1}^{n} V_i\right) : R\text{-}\mathsf{MOD},$

$f := \text{Ɔ}F\text{Ɑ}\texttt{FreeModulr} : \texttt{Basis}\left(\prod_{i=1}^{n} V_i\right),$

$U := \operatorname{span}\{f(v) - f(v') - f(v'')|\ldots\} \cup \{f(v) - \alpha f(v')\} : \texttt{Submodule}(F),$

$W := \dfrac{F}{U} : R\text{-}\mathsf{MOD},$

$T := \pi_U \circ f : \prod_{i=1}^{n} V_i \to W,$

$\text{♀} := \Lambda k \in n-1 \,.\, \forall v \in \prod_{i=1}^{n-k} V_i \,.\, T(v) \in \mathcal{L}(\Lambda i \in n-k \,.\, V_i) : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ v : \prod_{i=1}^{n-1} V_i,$

$[v.*] := \text{Ɔ}U\text{Ɔ}T(v)\text{Ɑ}^{-1}\mathcal{L}(V_n;W) : T(v) \in \mathcal{L}(V_n;W);$

$\leadsto [1] := \text{Ɔ}\text{♀} : \text{♀}(1),$

$\texttt{Assume}\ k : n-2,$

$\texttt{Assume}\ [2] : \text{♀}(k),$

$\texttt{Assume}\ v : \prod_{i=1}^{n-k-1} V_i,$

$[k.*] := \text{Ɔ}U\text{Ɔ}T(v)\text{Ɑ}^{-1}\mathcal{L}\left(\prod_{n-k-1}^{n} V; W\right) : T(v) \in \mathcal{L}(V_n;W);$

$\leadsto [2] := \text{Ɑ}^{-1}\mathcal{L}(V;W)\text{Ɑ}\texttt{InductiveSet}(n-1)\text{Ɔ}U\text{Ɔ}T : T \in \mathcal{L}(V;W),$

$\texttt{Assume}\ M : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ S : \mathcal{L}(V;M),$

$\left(A', [3]\right) := \text{Ɑ}\texttt{Adjoint}(\texttt{FreeModule})(S) : \sum A : F \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, A' \circ f = S,$

$[4] := \text{Ɑ}\mathcal{L}(V;M)[3] : U \subset \ker A,$

$(A, [5]) := \texttt{MorphismRestriction}[4] : \sum A : W \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, A \circ T = S,$

$\texttt{Assume}\ B : \mathcal{L}(V;M),$

$\texttt{Assume}\ [6] : B \circ T = S,$

$[7] := \texttt{GenSurjection}(f, \pi_U)\text{Ɔ}T : W = \operatorname{span}\operatorname{Im} T,$

$[M.*] := [7][6][5]\text{Ɑ}\texttt{span} : A = B;$

$\leadsto [*] := \text{Ɑ}^{-1}\texttt{TensorProduct} : \left((W,T) : \texttt{TensorProduct}(R,n,V)\right);$

$\square$

$\texttt{TensorProductsAreEq} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to R\text{-}\mathsf{MOD} \,.\, \forall (W,T),(U,S) : \texttt{TensorProduct}(R,n,V) \,.$

$\texttt{Proof} \;=$

$(A,[1]) := \mathcal{I}\texttt{TensorProduct}(R,n,V)(W,T)(S) : \sum A : W \xrightarrow{R\text{-}\mathsf{MOD}} U \,.\, S = A \circ T,$

$(B,[2]) := \mathcal{I}\texttt{TensorProduct}(R,n,V)(U,S)(T) : \sum A : U \xrightarrow{R\text{-}\mathsf{MOD}} W \,.\, T = B \circ S,$

$[3] := [1][2] : S = A \circ B \circ S,$

$[4] := [2][1] : T = B \circ A \circ T,$

$[5] := \mathcal{I}\texttt{TensorProduct}(R,n,V)[3] : A \circ B = \mathrm{id},$

$[6] := \mathcal{I}\texttt{TensorProduct}(R,n,V)[4] : B \circ A = \mathrm{id},$

$[*] := [5][6]\mathcal{I}^{-1}\texttt{Isomorphic} : W \cong_{R\text{-}\mathsf{MOD}} U;$

$\square$

$\texttt{tensorProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V : n \to R\text{-}\mathsf{MOD} \,.\, \texttt{TensorProduct}(R,n,V)$

$$\texttt{tensorProduct}\,() = \left(\bigotimes_{i=1}^{n} V_i, \bigotimes\right) := \texttt{TensorProductExists}(R,n,V)$$

$\texttt{tensorisation} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V : n \to R\text{-}\mathsf{MOD} \,.\, \prod W \in R\text{-}\mathsf{MOD} \,.$

$$.\, L(V;W) \to \bigotimes_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} W$$

$$\texttt{tensorisation}\,(T) = T^{\otimes} := \mathcal{I}\texttt{TensorProduct}(R,n,V)\left(\bigotimes_{i=1}^{n} V_i, \bigotimes\right)(T)$$

$\texttt{TensorProductOfFreeModules} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to \texttt{FreeModule}(R) \,.\, \bigotimes_{i=1}^{n} V_i : \texttt{FreeModule}(R)$

$\texttt{Proof} \;=$

$e := \texttt{FreeHasBasis}(V) : \prod i \in n \,.\, \texttt{Basis}(V_i),$

$$f := \Lambda j : \prod i \in n \,.\, \operatorname{rank} V_i \,.\, \bigoplus_{i=1}^{n} e_{i,j_i} : \left(\prod i \in n \,.\, \operatorname{rank} V_i\right) \to \bigoplus_{i=1}^{n} V_i,$$

$\texttt{Assume}\ v : \prod_{i=1}^{n} V_i,$

$$(\alpha,[1]) := \mathcal{I}\texttt{Basis}(v) : \sum \alpha : \left(\sum_{i=1}^{n} \operatorname{rank} V_i\right) \to R \,.\, \forall i \in n \,.\, v_i = \alpha_i e_i,$$

$$[*] := [1]\mathcal{I}\mathcal{L}(V;W)\bigoplus \mathcal{O}f : \bigoplus_{i=1}^{n} v_i = \sum_{j \in \prod_{i=1}^{n} \operatorname{rank} V_i} \prod_{i=1}^{n} \alpha_{i,j_i} f_j;$$

$$\rightsquigarrow [1] := \mathcal{I}\bigoplus_{i=1}^{n} V_i \mathcal{I}^{-1}\operatorname{span} : \bigoplus_{i=1}^{n} V_i = \operatorname{span}(f),$$

$\texttt{Assume}\ [2] : f \,!\, \texttt{LinearlyIndependent},$

$T := \delta_{e;\prod e} : \mathcal{L}\left(V; R^{\oplus \prod_{i=1}^{n} \operatorname{rank} V_i}\right),$

$(\alpha,[3]) := \mathcal{I}\texttt{LinearlyIndependent}[2] : \sum \alpha \in R^{\oplus \prod_{i=1}^{n} \operatorname{rank} V_i} \,.\, \alpha f = 0 \;\&\; \alpha \neq 0,$

$[4] := \mathcal{I}R\text{-}\mathsf{MOD}(T^{\otimes})[3]\mathcal{I}R\text{-}\mathsf{MOD}(T^{\otimes})\mathcal{O}(T) : 0 = T^{\otimes}(0) = T^{\otimes}(\alpha f) = \alpha T^{\otimes}(f) = \alpha,$

$[*] := [3][4] : \perp;$

$$\rightsquigarrow [2] := \text{Ɑ}^{-1}\mathtt{Basis}(\bot) : f : \mathtt{Basis}\left(\prod_{i=1}^n \operatorname{rank} V_i, \bigoplus_{i=1}^n V_i\right),$$

$$[*] := \mathtt{FreeByBasis}[2] : \left(\bigoplus_{i=1}^n V_i : \mathtt{FreeModule}(R)\right);$$

$\square$

$$\mathtt{RankOfTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to \mathtt{FreeModule}(R) \,.\, \operatorname{rank} \bigotimes_{i=1}^n V = \prod_{i=1}^n \operatorname{rank} V_i$$

$\mathtt{Proof}\ =$

$\ldots$

$\square$

$$\mathtt{TensorProductTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V : n \to R\text{-}\mathsf{MOD} \,.\, \forall k \in n-1 \,.$$

$$.\, \bigotimes_{i=1}^k V_i \otimes \bigotimes_{i=k+1}^n V_i \cong_{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^n V_i$$

$\mathtt{Proof}\ =$

$$T := \Lambda v : \prod_{i=1}^n V_i \,.\, \left(\bigotimes_{i=1}^k v_i\right) \otimes \left(\bigotimes_{i=k+1}^n v_i\right) : \mathcal{L}\left(V; \bigotimes_{i=1}^k V_i \otimes \bigotimes_{i=k+1}^n V_i\right),$$

$$S := \Lambda\left(\sum_{j=1} \bigotimes_{i=1}^k v_i^j, \sum_{j=1} \bigotimes_{i=k+1}^n w_i^j\right) : \bigotimes_{i=1}^k V_i \times \bigotimes_{i=k+1}^n V_i \,.\, \sum_{j=1} \sum_{l=1} \bigotimes_{i=1}^n \mathtt{if}\ i \le k \,.\, v_i^j\ \mathtt{else}\ w_i^l :$$

$$: \mathcal{L}\left(\bigotimes_{i=1}^k V_i, \bigotimes_{i=k+1}^n V_i; \bigotimes_{i=1}^n V_i\right),$$

$$[1] := \mathcal{I}T\mathcal{I}S : T^{\oplus}S^{\oplus} = \mathrm{id}\ \&\ S^{\oplus}S^{\oplus} = \mathrm{id},$$

$$[*] := \text{Ɑ}\mathtt{Isomorphic}[1] : \bigotimes_{i=1}^k V_i \otimes \bigotimes_{i=k+1}^n V_i \cong_{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^n V_i;$$

$\square$

$$\mathtt{AssociativeTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, C \in R\text{-}\mathsf{MOD} \,.\, (A \otimes B) \otimes C \cong_{R\text{-}\mathsf{MOD}} A \otimes (B \otimes C)$$

$\mathtt{Proof}\ =$

$[1] := \mathtt{TensorProductTensorProduct}(A, B, C, 1) : A \otimes (B \otimes C) \cong A \otimes B \otimes C,$

$[2] := \mathtt{TensorProdctTensorProduct}(A, B, C, 2) : (A \otimes B) \otimes C \cong A \otimes B \otimes C,$

$[3] := \text{Ɑ}\mathtt{Transitive}(\mathtt{Isomorphic})[1][2] : A \otimes (B \otimes C) \cong_{R\text{-}\mathsf{MOD}} (A \otimes B) \otimes A;$

$\square$

$\texttt{TensorProductPermutation} :: \forall R \in \mathsf{ANN} \, . \, \forall n \in \mathbb{N} \, . \, \forall n \to R\text{-}\mathsf{MOD} \, . \, \forall \sigma \in S_n \, . \, \bigoplus_{i=1}^{n} V_i \cong_{R\text{-}\mathsf{MOD}} \bigoplus_{i=1}^{n} V_{\sigma(i)}$

$\texttt{Proof} \ =$

$$T := \Lambda v \in \prod_{i=1}^{n} V_i \, . \, \bigotimes_{i=1}^{n} v_{\sigma(i)} : \mathcal{L}\left(V; \bigotimes_{i=1}^{n} V_i\right),$$

$$S := \Lambda v \in \prod_{i=1}^{n} V_{\sigma i} \, . \, \bigotimes_{i=1}^{n} v_{\sigma^{-1} i} : \mathcal{L}\left(\sigma^* V, \bigotimes_{i=1}^{n} V_i\right),$$

$[1] := \mathcal{D} T \mathcal{D} S : T^{\otimes} S^{\otimes} = \mathrm{id} \ \& \ S^{\otimes} T^{\otimes} = \mathrm{id},$

$$[*] := \mathcal{G}\,\texttt{Isomorphic}[1] : \bigotimes_{i=1}^{n} V_i \cong_{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^{n} V_{\sigma(i)};$$

□

$\texttt{TensorProductIdealQuotient} :: \forall R \in \mathsf{ANN} \, . \, \forall I : \texttt{Ideal}(R) \, . \, \forall n \in \mathbb{N} \, . \, \forall V : n \to R\text{-}\mathsf{MOD} \, .$

$$. \, \frac{\bigotimes_{i=1R}^{n} V_i}{I \bigotimes_{i=1R}^{n} V_i} \cong_{R\text{-}\mathsf{MOD}} \bigotimes_{i=1 \, \frac{R}{I}}^{n} \frac{V_i}{I V_i}$$

$\texttt{Proof} \ =$

$$T := \Lambda v \in \prod_{i=1}^{n} V_i \, . \, \bigotimes_{i=1}^{n} [v_i] : \mathcal{L}\left(V; \bigotimes_{i=1}^{n} \frac{V_i}{R V_i}\right),$$

$$[1] := \mathcal{G}^{-1} \ker \mathcal{G}\,\texttt{moduleQuotien}(\ldots)\texttt{MultiHomogen}(\ldots) : I \bigotimes_{i=1}^{n} V_i \subset \ker T^{\otimes},$$

$$\hat{T}^{\otimes} := \texttt{KerRestriction}[1] : \sum \hat{T}^{\otimes} : \frac{\bigotimes_{i=1R}^{n} V_i}{I \bigotimes_{i=1R}^{n} V_i} \xrightarrow{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^{n} \frac{V_i}{R V_i} \, . \, T^{\otimes} = \hat{T}^{\otimes},$$

$$S := \Lambda [v] \in \prod_{i=1}^{n} \frac{V_i}{S V_i} \, . \, \left[\bigoplus_{i=1}^{n}\right] : \mathcal{L}\left(\frac{V}{IV}; \frac{\bigoplus_{i=1}^{n} V_i}{I \bigoplus_{i=1}^{n} V_i}\right),$$

$\texttt{Assume} \ w : \prod_{i=1}^{n} I V_i,$

$$u := \Lambda L : n \to \{0,1\} \, . \, \Lambda i \in n \, . \, \texttt{if} \ L_i = 0 \ \texttt{then} \ v \ \texttt{else} \ w : (n \to \{1,0\}) \to \prod_{i=1}^{n} V_i,$$

$$[w.*] := \texttt{MultiAdditive}(S)\mathcal{G}\,\texttt{quotientModule}\mathcal{D} u : S\big([v+w]\big) = \sum_{L : n \to \{0,1\}} S[u] == S[v] + \sum_{L \neq 0} S[u] = 0;$$

$\leadsto [2] := \mathcal{G}\,\texttt{quotientModule} : S : \texttt{WellDefined},$

$[1] := \mathcal{D} T \mathcal{D} S : T^{\otimes} S^{\otimes} = \mathrm{id} \ \& \ S^{\otimes} T^{\otimes} = \mathrm{id},$

$$[*] := \mathcal{G}\,\texttt{Isomorphic}[1] : \frac{\bigotimes_{i=1}^{n} V_i}{\bigotimes_{i=1}^{n} V_i} \cong_{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^{n} \frac{V_i}{I V_i};$$

□

$\texttt{TrivialTensorProduct} :: \forall R \in \mathsf{ANN} \, . \, \forall V \in R\text{-}\mathsf{MOD} \, . \, R \otimes V \cong_{R\text{-}\mathsf{MOD}} V$

$\texttt{Proof} =$

$A := \Lambda^{\otimes}(\alpha, v) \in R \times V \, . \, \alpha v : R \otimes V \xrightarrow{R\text{-}\mathsf{MOD}} V,$

$B := \Lambda v \in V \, . \, 1 \otimes v : V \xrightarrow{R\text{-}\mathsf{MOD}} R \otimes V,$

$\texttt{Assume } v : V,$

$[v.1] := \eth B v \eth A : ABv = A(1 \otimes v) = v;$

$\rightsquigarrow [1] := I(=, \rightarrow) : AB = \mathrm{id},$

$\texttt{Assume } \alpha_i \otimes v_i : R \otimes V,$

$[.*] := \eth A \eth B \texttt{MultiHomogen}(\otimes) : BA(\alpha_i \otimes v_i) = B(\alpha_i v_i) = 1 \otimes \alpha_i v_i = \alpha_i \otimes v_i;$

$\rightsquigarrow [2] := I(=, \rightarrow) : BA = \mathrm{id},$

$[*] := \text{Ɑ}^{-1}\texttt{Isomorphic}[1][2] : R \otimes V \cong_{R\text{-}\mathsf{MOD}} V;$

$\square$

$\texttt{QuotientByTensorProduct} :: \forall R \in \mathsf{RING} \, . \, \forall I : \texttt{Ideal}(R) \, . \, \forall V \in R\text{-}\mathsf{MOD} \, . \, \frac{R}{I} \otimes V \cong_{R\text{-}\mathsf{MOD}} \frac{V}{IV}$

$\texttt{Proof} =$

$A := \Lambda\big([\alpha], v\big) \in \frac{R}{I} \times V \, . \, [\alpha][v] : \mathcal{L}\left(\frac{R}{I} \otimes V; \frac{V}{IV}\right),$

$B := \Lambda[v] \in \frac{V}{IV} \, . \, [1] \otimes v : \frac{V}{IV} \xrightarrow{R\text{-}\mathsf{MOD}} \frac{R}{I} \otimes V,$

$\texttt{Assume } w : IV,$

$(n, \alpha, u, [1]) :=: \sum n \in \mathbb{N} \, . \, \alpha : n \to I \, . \, \sum u : n \to V : w = \sum_{i=1}^{n} \alpha_i v_i,$

$[w.*] := \eth B \texttt{MultiAdditive}^{n+1}(\otimes)\texttt{MultiHomogen}^{n}(\otimes)\text{Ɑ}\texttt{quotientRingMultiHomogen}^{n}(\otimes)\eth^{-1}B :$

$: B[v + w] = [1] \otimes \left(v + \sum_{i=1}^{n} \alpha_i u_i\right) = [1] \otimes v + \sum_{i=1}^{n} \alpha_i [1] \otimes u_i = [1] \otimes v = B[v];$

$\rightsquigarrow [1] := \text{Ɑ}\texttt{QuotientModule} : (B : \texttt{WellDefined}),$

$\texttt{Assume } [v] : \frac{V}{VI},$

$[v.*] := \eth B \eth A : A^{\otimes} B[v] = A^{\otimes}([1] \otimes v) = [v];$

$\rightsquigarrow [2] := I(=, \rightarrow) : A^{\otimes} B = \mathrm{id},$

$\texttt{Assume } [\alpha] \otimes v : \frac{R}{I} \otimes V,$

$[.*] := \eth A \texttt{MultiHomogen}^{2}(\otimes)\text{Ɑ}\texttt{quotientRing}(R, I) : BA^{\otimes}([\alpha] \otimes v) = B[\alpha v] = [1] \otimes \alpha v = [\alpha] \otimes v;$

$\rightsquigarrow [3] := I(=, \rightarrow) : BA^{\otimes} = \mathrm{id},$

$[*] := \text{Ɑ}^{-1}\texttt{Isomorphic}[2][3] : \frac{R}{I} \otimes V \cong_{R\text{-}\mathsf{MOD}} \frac{V}{IV};$

$\square$

$\texttt{NakayamaTensorCondition} :: \forall R \in \mathsf{ANN} \,.\, \forall V \in R\text{-}\mathsf{MOD} \,.$

$$. \Big(\forall N \in \texttt{FinitelyGeneratedModule}(R) \,.\, N = \{0\} \iff N \otimes V = \{0\}\Big) \iff$$
$$\iff \forall I : \texttt{MaximalIdeal}(R) \,.\, IV \neq V$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \forall N \in \texttt{FinitelyGeneratedModule}(R) \,.\, N = \{0\} \iff N \otimes V = \{0\},$

$\texttt{Assume}\ I : \texttt{MaximalIdeal}(R),$

$[2] := \mathcal{C}\texttt{MaximalIdeal}(I) : \dfrac{R}{I} \neq \{0\},$

$[3] := \texttt{QuotientByTensorProduct}(V, I)[1][2] : \dfrac{V}{IV} \cong_{R\text{-}\mathsf{MOD}} V \otimes \dfrac{R}{I} \neq \{0\},$

$[*] := \mathcal{C}\texttt{quotientModule}[3] : V \neq IV;$

$\rightsquigarrow [1] := I(\Rightarrow)I(\forall) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \forall I : \texttt{MaximalIdeal}(R) \,.\, IV \neq V,$

$\texttt{Assume}\ N : \texttt{FinitelyGeneratedModule}(R),$

$\texttt{Assume}\ [3] : N \otimes V = \{0\},$

$\texttt{Assume}\ I : \texttt{maximalIdeal}(R),$

$[4] := \mathcal{C}\texttt{quotientModule}[3]\texttt{TensotProductIdealQuotient}(V, N; I) :$

$$: \{0\} = \frac{N \otimes_R V}{I(N \otimes_R V)} \cong_{R\text{-}\mathsf{MOD}} \frac{N}{IN} \otimes_{\frac{R}{I}} \frac{V}{IV},$$

$[5] := [2](I) : \dfrac{V}{IV} \neq 0,$

$[6] := \texttt{MaximallQuotientIsField}(I) : (\dfrac{R}{I} : \texttt{Field}),$

$[7] := [5][6] : \dfrac{N}{IN} = \{0\},$

$[I.*] := \mathcal{C}\texttt{quotientModule}[7] : N = IN;$

$\rightsquigarrow [4] := I(\forall) : \forall I : \texttt{MaximalIdeal}(R) \,.\, IN = N,$

$[2.*] := \texttt{NakayamaLemma}[4] : N = \{0\};$

$\rightsquigarrow [*] := I(\Rightarrow)\mathcal{C}\texttt{tensrProduct}I(\iff)I(\Rightarrow)I(\iff)[1] : \texttt{This},$

$\square$

$\texttt{tensorPower} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{MOD}$

$$\texttt{tensorPower}\,(V) = \mathbf{T}^n(V) := \bigoplus_{i=1}^{n} V$$

$\texttt{ZeroTensorInFGM} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall t \in A \otimes B \,.\, \forall [0] : t =_{A \otimes B} 0 \,.$

$$: \exists A' : \texttt{FinitelyGeneratedModule}(R)\ \&\ \texttt{Submodule}(R, A) \,.$$
$$: \exists B' : \texttt{FinitelyGeneratedModule}(R)\ \&\ \texttt{Submodule}(R, B) \,.\, t =_{A' \otimes B'} 0$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

### 1.3 Tensor Product as Functor

$$\texttt{tensorMap} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V, W : n \to R\text{-}\mathsf{MOD} \,.$$
$$.\left(\prod_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} W_i\right) \to \bigotimes_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} \bigotimes_{i=1}^{n} W_i$$
$$\texttt{tensorMap}\,(f) = \bigotimes_{i=1}^{n} f_i := \texttt{tensorisation}\ \Lambda v \in \prod_{i=1}^{n} V_i \,.\, \bigotimes_{i=1}^{n} f_i(v_i)$$

$$\texttt{TensorMapComposition} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod V, W, U : n \to R\text{-}\mathsf{MOD} \,.$$
$$.\,\forall f : \prod_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} W_i \,.\, \forall g : \prod_{i=1}^{n} W_i \xrightarrow{R\text{-}\mathsf{MOD}} U_i \,.\, \bigotimes_{i=1}^{n} g_i \circ \bigotimes_{i=1}^{n} f_i = \bigotimes_{i=1}^{n} g_i \circ f_i$$

$\texttt{Proof}\ =$

$\texttt{Assume}\ v : \prod_{i}^{n} V_i,$

$[v.*] := \text{Д}^2 \texttt{tensorMap}^{-1}(f,g)\text{Д}^{-1}\texttt{compose} :$

$$: \bigotimes_{i=1}^{n} g_i \circ \bigotimes_{i=1}^{n} f_i \bigotimes_{i=1}^{n} v_i = \bigotimes_{i=1}^{n} g_i \bigotimes_{i=1}^{n} f_i(v_i) = \bigotimes_{i=1}^{n} g_i\Big(f_i(v_i)\Big) = \bigotimes_{i=1}^{n} g_i \circ f_i(v_i);$$

$$\rightsquigarrow [*] := I(\forall)\text{Д}\texttt{tensorisation} : \bigotimes_{i=1}^{n} g_i \circ \bigotimes_{i=1}^{n} f_i = \bigotimes_{i=1}^{n} g_i \circ f_i;$$

$\square$

$$\texttt{tensorFunctor} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R\text{-}\mathsf{MOD}^n \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{MOD}$$
$$\texttt{tensorFunctor}\,() = \bigotimes_{i=1}^{n} := \left(\bigotimes_{i=1}^{n}, \bigotimes_{i=1}^{n}\right)$$

$$\texttt{TensorMapAdditive} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V, U \in R\text{-}\mathsf{MOD}^n \,.$$
$$.\,\forall f' : \prod_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} U_i \,.\, \forall i \in n \,.\, \forall g : V_i \xrightarrow{R\text{-}\mathsf{MOD}} U_i \,.\, \bigotimes_{i=1}^{n} f_i = \bigotimes_{i=1}^{n} f'_i + \bigotimes_{i=1}^{n} f''_i$$
$$\texttt{where}\quad f = \Lambda j \in n \,.\, \texttt{if}\ i == j\ \texttt{then}\ g + f'_i\ \texttt{else}\ f'_i,\ f'' = \Lambda j \in n \,.\, \texttt{if}\ i == j\ \texttt{then}\ g\ \texttt{else}\ f'_i$$

$\texttt{Proof}\ =$

...

$\square$

$$\texttt{TensorMapHomogen} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall V, U \in R\text{-}\mathsf{MOD}^n \,.$$
$$.\,\forall f : \prod_{i=1}^{n} V_i \xrightarrow{R\text{-}\mathsf{MOD}} U_i \,.\, \forall i \in n \,.\, \forall \alpha \in A \,.\, \bigotimes_{i=1}^{n} f'_i = \alpha \bigotimes_{i=1}^{n} f_i$$
$$\texttt{where}\quad f' = \Lambda j \in n \,.\, \texttt{if}\ i == j\ \texttt{then}\ \alpha f_i$$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{ExactTensorMapLemma1} :: \forall R \in \mathsf{ANN} \;.\; \forall n \in \mathbb{N} \;.\; \forall (V,f) : \texttt{RightShortExact}^n(R) \;.\; \bigotimes_{i=1}^{n} f_0^i : \bigotimes_{i=1}^{n} V_1^i \twoheadrightarrow \bigotimes_{i=1}^{n} V_0^i$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : \prod_{i=1}^{n} V_1^i,$

$\texttt{Assume}\; i : n,$

$[i.1] := \texttt{SurjByExact}\mathcal{I}\texttt{RightShortExact}(V^i, f^i) : (f_0^i : V_1^i \twoheadrightarrow V_0^1),$

$\Big(w_i, [i.*]\Big) := \mathcal{I}^{-1}\texttt{Surjective}(f^i)(v_i) : \sum w_i \in V_1^i \;.\; f_0^i(w_i) = v_i;$

$\rightsquigarrow (w, [v.1]) :=: \sum_{i=1}^{n} w_i \in V_1^i \;.\; f_0^i(w_i) = v_i,$

$[v.2] := \mathcal{I}\texttt{TensorFunc}(f_0)[v.1] : \bigotimes_{i=1}^{n} f_0^i(w_i) = \bigotimes_{i=1}^{n} v_i,$

$[v.*] := \mathcal{I}^{-1}\texttt{image}[v.2] : \bigotimes_{i=1}^{n} v_i \in \bigotimes_{i=1}^{n} f_0^i;$

$\rightsquigarrow [1] := I(\forall) : \forall v \in \prod_{i=1}^{n} V_1^i \;.\; \bigotimes_{i=1}^{n} v_i \in \bigotimes_{i=1}^{n} f_i,$

$[*] := \mathcal{I}\texttt{tensorProduct}[1]\mathcal{I}^{-1}\texttt{Surjective} : \left( f_)^i : \bigotimes_{i=1}^{n} V_1^i \twoheadrightarrow V_0^i \right);$

$\square$

$\texttt{ExactTensorMapLemma2} :: \forall R \in \mathsf{ANN} \;.\; \forall n \in \mathbb{N} \;.\; \forall (V,f) : \texttt{RightShortExact}^n(R) \;.$

$$\ker \bigotimes_{i=1}^{n} f_0^i = \sum_{i=1}^{n} N_i \quad \texttt{where} \quad N_i = \bigotimes_{j=1}^{n} \texttt{if}\; j == i \;\texttt{then}\; \operatorname{Im} f_1^i \;\texttt{else}\; V_0^i$$

$\texttt{Proof} \;=$

$\texttt{Assume}\; \sum_{i=1}^{n} t_i : \sum_{i=1}^{n} N_i,$

$\texttt{Assume}\; i : n,$

$\Big(K, v, [i.1]\Big) := \mathcal{I}\texttt{linearSum}\mathcal{O}(N_i)(t_i) :$

$$: \sum K \in \mathbb{N} \;.\; \sum v : K \to \sum_{j=1}^{n} \texttt{if}\; i == j \;\texttt{then}\; \operatorname{Im} f_1^i \;\texttt{else}\; V_1^i \;.\; t_i = \sum_{k=1}^{K} \bigotimes_{j=1}^{n} v_{k,j},$$

$\texttt{Assume}\; k : K,$

$[k.1] := \mathcal{I}\texttt{ChainComplex}(V^i, \varphi^i)(0) : \operatorname{Im} f_1^i \subset \ker f_i,$

$[k.2] := \mathcal{I} \ker[k.1](v_{j,i}) = 0 : f_0^i(v_{k,i}) = 0,$

$[k.*] := \mathcal{I}^{-1}\texttt{tensorMap}\mathcal{I}\texttt{TensorProduct}[k.2] : \bigotimes_{i=1}^{n} f_0^i \bigotimes_{i=1}^{n} v_{k,i} = \bigotimes_{i=1}^{n} f_0^i(v_{k,i}) = 0;$

$\rightsquigarrow [i.2] := I(\forall) : \forall k \in K \;.\; \bigotimes_{i=1}^{n} f_0^i \bigotimes_{i=1}^{n} v_{k,i} = 0,$

$[i.*] := \mathcal{I} R\text{-}\mathsf{MOD}\left( \bigotimes_{i=1}^{n} V_1^i, \bigotimes_{i=1}^{n} V_0^i \right)\left( \bigotimes_{i=1}^{n} f_0^i \right)[i.1][i.2] : \bigotimes_{j=1}^{n} f_0^j(t_i) = 0;$

$\rightsquigarrow [t.1] := I(\forall) : \forall i \in n \;.\; \bigotimes_{j=1}^{n} f_0^j(t_i) = 0,$

$$[t.*] := \daleth R\text{-MOD}\left(\bigotimes_{i=1}^{n} V_1^i, \bigotimes_{i=1}^{n} V_0^i\right)\left(\bigotimes_{i=1}^{n} f_0^i\right) : \bigotimes_{i=1}^{n} f_0^i \sum_{i=1}^{n} t_i = 0;$$

$$\rightsquigarrow [1] := \daleth^{-1}\texttt{Subset}\daleth^{-1}\ker : \sum_{i=1}^{n} N_i \subset \ker \bigotimes_{i=1}^{n} f_0^i,$$

$\texttt{Assume}\ i : n,$

$$\texttt{Assume}\ v, w : \prod_{i=1}^{n} V_1^i,$$

$\texttt{Assume}\ [i.1] : f_0^i(v_i) = f_0^i(w_i),$

$\texttt{Assume}\ [i.2] : \forall j \in n\ .\ j \neq i \Rightarrow v_j = w_j,$

$[i.3] := \daleth^{-1}\ker f_0^i[w.2] : v_i - w_i \in \ker f_0^i,$

$[i.4] := \daleth\texttt{RightShortExact}(R)(V^i, f^i)[w.3] : v_i - w_i \in \operatorname{Im} f_1^i,$

$$[i.*] := \eth N_i[w.4] : \bigotimes_{i=1}^{n} v_i - \bigotimes_{i=1}^{n} w_i \in N_i;$$

$$\rightsquigarrow [2] := I^4(\forall) : \forall i \in n\ .\ \forall v, w \in \prod_{i=1}^{n} V_1^i\ .\ \left(f_0^i(v_i) = f_0^i(w_i)\ \&\ \forall j \in n\ .\ j \neq i \Rightarrow\ .\ v_i = w_i\right) \Rightarrow$$

$$\Rightarrow \bigotimes_{i=1}^{n} v_i - \bigotimes_{i=1}^{n} w_i \in N_i,$$

$$\texttt{Assume}\ v, w : \prod_{i=1}^{n} V_1^i,$$

$\texttt{Assume}\ [w.1] : \forall i \in n\ .\ f_0^i(v_i) = f_0^i(w_i),$

$$[w.2] := \daleth R\text{-MOD} \bigotimes_{i=1}^{n} V_1^i : \bigotimes_{i=1}^{n} v_i - \bigotimes_{i=1}^{n} w_i =$$

$$= \sum_{i=0}^{n}\left(\bigotimes_{j=1}^{n} \texttt{if}\ j < i\ \texttt{then}\ w_i\ \texttt{else}\ v_i - \bigotimes_{j=1}^{n} \texttt{if}\ j \leq i\ \texttt{then}\ w_i\ \texttt{else}\ v_i\right),$$

$$[w.*] := \prod_{i=1}^{n}[2](i)[w.2] : \bigotimes_{i=1}^{n} v_i - \bigotimes_{i=1}^{n} w_i \in \sum_{i=1}^{n} N_i;$$

$$\rightsquigarrow [3] := I(\forall)I(\Rightarrow) : \forall v, w \in \prod_{i=1}^{n} V_1^i\ .\ \left(\forall i \in n\ .\ f_0^i(v_i) = f_0^i(w_i)\right) \Rightarrow \bigotimes_{i=1}^{n} v_i - \bigotimes_{i=1}^{n} w_i \in \sum_{i=1}^{n} N_i,$$

$$\texttt{Assume}\ v : \prod_{i=1}^{n} V_0^i,$$

$$[v.1] := \texttt{ExactTensorMap1}(v) : \sum w \in \prod_{i=1}^{n} V_1^i\ .\ \bigotimes_{i=1}^{n} f_1^i \bigotimes_{i=1}^{n} w_i = \bigotimes_{i=1}^{n} v_i,$$

$$G(v) := \left[\bigotimes_{i=1}^{n} w_i\right]_{\sum_{i=1}^{n} N_i} : \frac{\bigotimes_{i=1}^{n} V_1^i}{\sum_{i=1}^{n} N_i},$$

$$\texttt{Assume}\ u : \prod_{i=1}^{n} V_1^i,$$

$$\texttt{Assume}\ [u.1] : \bigotimes_{i=1}^{n} f_1^i \bigotimes_{i=1}^{n} u_i = \bigotimes_{i=1}^{n} v_i,$$

$$[u.*] := [3](w, u)[v.1][v.2] : \left[\bigotimes_{i=1}^{n} w_i\right] = \left[\bigotimes_{i=1}^{n} u_i\right];$$

$\rightsquigarrow [u.2] := \eth : (F : \texttt{WellDefined});$

$$\rightsquigarrow G := I(\to) : \mathcal{L}\left(V_0; \frac{\bigotimes_{i=1}^n V_1^i}{\sum_{i=1}^n N_i}\right),$$

$$g := \bigotimes_{i=1}^n f_0^i G^{\otimes} : \bigotimes_{i=1}^n V_1^i \xrightarrow{R\text{-MOD}} \frac{\bigotimes_{i=1}^n V_1^i}{\sum_{i=1}^n N_i},$$

$\texttt{Assume}\ t : \ker \bigotimes_{i=1}^n f_0^i,$

$[t.1] := \text{Ɑ} g(t) : t \in \ker g,$

$[t.2] := \text{Ↄ} g(t) : g(t) = [t],$

$[t.*] := [t.1][t.2] : t \in \sum_{i=1}^n N_i;$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{SetEq}[1]\text{Ɑ}^{-1}\texttt{Subset} : \ker \bigotimes_{i=1}^n f_0^i = \sum_{i=1}^n N_i;$

□

$\texttt{tensorWith} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{MOD} \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{MOD}$

$\texttt{tensorWith}\,(M) = T_M := \Big(\cdot \otimes M, \cdot \otimes \mathrm{id}_M\Big)$

$\texttt{ExactTensorTHM} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, T_M : \texttt{RightExact}\Big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\Big)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ A \xrightarrow{f} B \xrightarrow{g} C \to 0 : \texttt{RightShortExact}(R\text{-}\mathsf{MOD}),$

$[1] := \text{Ɑ}^{-1}\texttt{RightShortExact} : \Big(0 \to M \xrightarrow{\mathrm{id}_M} M \to 0 :\Big),$

$[2] := \texttt{ExactTensorLemma} : \ker g \otimes \mathrm{id}_M = \operatorname{Im} f \otimes M + B \otimes 0 = \operatorname{Im} f \otimes M,$

$[A.*] := \text{Ɑ}^{-1}\texttt{image}(\mathrm{id})\text{Ɑ}^{-1}\texttt{RightShortExact} : A \otimes M \xrightarrow{f \otimes \mathrm{id}} B \otimes M \xrightarrow{g \otimes \mathrm{id}} C \to 0 :$
$\quad : \texttt{RightExact}(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD});$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{RightExact} : \Big(T_M : \texttt{RightExact}\big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\big)\Big);$

□

$\texttt{TensorProductDistributive} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, M \in R\text{-}\mathsf{MOD} \,.\, M \otimes (A \oplus B) = (M \otimes A) \oplus (M \otimes B)$

$\texttt{Proof}\ =$

...

□

$\texttt{FreeTensoringDecomposition} :: \forall R \in \mathsf{ANN} \,.\, \forall F : \texttt{FreeModule}(R) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall E : \texttt{Basis}(F) \,.$
$\quad . \forall t \in F \otimes M \,.\, \exists a \in M^{\oplus E} \,.\, t = \sum_{e \in E} a \otimes e$

$\texttt{Proof}\ =$

...

□

$\texttt{FreeTensoringIsExact} :: \forall R \in \mathsf{ANN} \,.\, \forall F : \texttt{FreeModule}(R) \,.\, T_F : \texttt{Exact}\Big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ 0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0 : \texttt{ShortExact}(R),$

$[1] := \texttt{InjectiveByExact}(0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0) : (f : A \hookrightarrow B),$

$E := \texttt{FreeHasBasis}(F) : \texttt{Basis}(F),$

$\texttt{Assume}\ t : A \otimes F,$

$\texttt{Assume}\ [2] : f \otimes \mathrm{id}(t) = 0,$

$\big(a, [1]\big) := \texttt{TensorProductDistributive}(A, F)\text{Ɑ}\texttt{Basis}E : \sum a : A^{\oplus E} \,.\, t = \sum_{e \in E} a_e \otimes e,$

$[3] := [2]\text{Ɑ}\texttt{TensorMap}(f, t)[1] : 0 = f \otimes \mathrm{id}(t) = f(a_e) \otimes e,$

$[4] := \text{Ɑ}\texttt{Basis}(E)[3] : f(a) = 0,$

$[t.*] := \texttt{ZeroKernelTHM}[1][4] : a = 0;$

$\rightsquigarrow [2] := \texttt{ZeroKernelTHM} : f \otimes \mathrm{id} : A \otimes F \hookrightarrow B \otimes F,$

$[A.*] := \texttt{ExactTensorTHM}[2] : 0 \xrightarrow{0} A \otimes F \xrightarrow{f \otimes \mathrm{id}} B \otimes F \xrightarrow{g \otimes \mathrm{id}} C \otimes F \xrightarrow{0} 0 : \texttt{ShortExact}(R);$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{Exact}\text{Ɑ}^{-1}\texttt{TensorWith} : T_F : \texttt{Exact}\Big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\Big);$

$\square$

$\texttt{ProjectiveTensoringIsExact} :: \forall R \in \mathsf{ANN} \,.\, \forall P : \texttt{Projective}(R) \,.\, T_P : \texttt{Exact}\Big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\Big)$

$\texttt{Proof} \ =$

$\Big(Q, [1]\Big) := \text{Ɑ} : \sum Q \in R\text{-}\mathsf{MOD} \,.\, P \oplus Q : \texttt{FreeModule}(R),$

$\texttt{Assume}\ 0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} \xrightarrow{0} 0 : \texttt{ShortExact}(R),$

$[2] := \texttt{FreeTensoringIsExact}(P \oplus Q)[1] :$

$: 0 \xrightarrow{0} A \otimes (P \oplus Q) \xrightarrow{f \otimes \mathrm{id}} B \otimes (P \oplus Q) \xrightarrow{g \otimes \mathrm{id}} C \otimes (P \oplus Q) \xrightarrow{0} 0 : \texttt{ShortExact}(R\text{-}\mathsf{MOD}),$

$[3] := \texttt{InjectiveByExact}[2] : f \otimes \mathrm{id} : A \otimes (P \oplus Q) \hookrightarrow B \otimes (P \oplus Q),$

$[4] := \texttt{TensorProductDistributive}(P, Q)\text{Ɑ}^{-1}\texttt{image}(f \otimes \mathrm{id}) : \mathrm{Im}\, f \otimes \mathrm{id}_{|A \otimes P} = B \otimes P,$

$[5] := [4]\text{Ɑ}\texttt{Restrict}[4] : f \otimes \mathrm{id}_P = f \otimes \mathrm{id}^{B \otimes P}_{P \oplus Q | A \otimes P},$

$[6] := \texttt{RestrictionPreservesInj} : f \otimes \mathrm{id}_P : A \otimes P \hookrightarrow B \otimes P,$

$[A.*] := \texttt{ExactTensorTHM}[2] : 0 \xrightarrow{0} A \otimes P \xrightarrow{f \otimes \mathrm{id}} B \otimes P \xrightarrow{g \otimes \mathrm{id}} C \otimes P \xrightarrow{0} 0 : \texttt{ShortExact}(R);$

$\rightsquigarrow [*] := \text{Ɑ}^{-1}\texttt{Exact}\text{Ɑ}^{-1}\texttt{TensorWith} : T_P : \texttt{Exact}\Big(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{MOD}\Big);$

$\square$

$\texttt{KroneckerProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B : \texttt{FreeModule}\ \&\ \texttt{FinitelyGeneratedModule}(R) \,.$

$\,.\, \forall e : \texttt{Basis}(A) \,.\, \forall f : \texttt{Basis}(B) \,.\, \forall T : \mathrm{End}_{R\text{-}\mathsf{MOD}}(A) \,.\, \forall S ; \mathrm{End}_{R\text{-}\mathsf{MOD}}(B) \,.$

$(T \otimes S)^{e \otimes f, e \otimes f} = \texttt{fromBlocks}(\lambda i, j \in \mathrm{rank}\, A \,.\, T^{e,e}_{i,j} S^{f,f}, \mathrm{rank}\, A \times \mathrm{rank}\, B, \mathrm{rank}\, B, \mathrm{rank}\, B)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ImageOfTensor} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A, Bn \to R\text{-}\mathsf{MOD} \,.\, \forall T : \prod_{i=1}^{n} A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.\, \operatorname{Im} \bigotimes_{i=1}^{n} T_i = \bigotimes_{i=1}^{n} \operatorname{Im} T_i$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TensorMapRank} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{MOD} \,.$

$\forall B : n \to \texttt{FreeModule}(R) \,.\, \forall T : \prod_{i=1}^{n} : \prod_{i=1}^{n} A_i \xrightarrow{R\text{-}\mathsf{MOD}} B_i \,.\, \operatorname{rank} \bigotimes_{i=1}^{n} T_i = \prod_{i=1}^{n} \operatorname{rank} T_i$

$\texttt{Proof} \ =$

$[1] := \texttt{ImageOfTensor}(R, n, A, B, T) : \operatorname{Im} \bigotimes_{i=1}^{n} T_i = \bigotimes_{i=1}^{n} \operatorname{Im} T_i,$

$[2] := \texttt{RankOfTensorProduct}[1] : \operatorname{rank} \bigotimes_{i=1}^{n} T_i;$

$\square$

$\texttt{TensorMapTrace} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$. \, \forall T : \prod_{i=1}^{n} A_i \xrightarrow{R\text{-}\mathsf{MOD}} A_i \,.\, \operatorname{tr} \bigotimes_{i=1}^{n} T_i = \prod_{i=1}^{n} \operatorname{tr} T_i$

$\texttt{Proof} \ =$

$(m, e) := \texttt{FreeHasBase}(A) : \prod_{k=1}^{n} \sum m_k \in \mathbb{N} \,.\, e : \texttt{Basis}(m_k, A_k),$

$\texttt{Assume}\ I : \prod_{k=1}^{n} m_k,$

$[I.1] := \mathcal{I}\texttt{tensorMap}(T) \bigotimes_{i=1}^{n} \mathcal{I}^{-1}\texttt{matrixOfOperator}(T, e)\texttt{MultiAdditive}(\bigotimes)\texttt{MultiHomogen}(\bigotimes) :$

$\bigotimes_{i=1}^{n} T_i \bigotimes_{i=1}^{n} e_{i,I_i} = \bigotimes_{i=1}^{n} T_i(e_{i,I_i}) = \sum J \in \prod_{i=1}^{n} \,.\, \bigotimes_{i=1}^{n} T^{e_i,e_i}_{i;J_i,I_o} e_{J_i} = \sum J \in \prod_{i=1}^{n} \,.\, \prod_{i=1}^{n} T^{e_i,e_i}_{i;J_i,I_i} \bigotimes_{i=1}^{n} e_{J_i},$

$[I.*] := \texttt{BasisOfTensorProduct}[I.1](I) : \left( \bigotimes_{i=1}^{n} T_i \bigotimes_{i=1}^{n} e_{i,I_i} \right)_I = \prod_{i=1}^{n} T^{e_i,e_i}_{I_i,I_i};$

$\rightsquigarrow [1] := I(\forall) : \forall I \in \prod_{i=1}^{n} m_i \,.\, \left( \bigotimes_{i=1}^{n} T_i \bigotimes_{i=1}^{n} e_{i,I_i} \right)_I = \prod_{i=1}^{n} T^{e_i,e_i}_{i;I_i,I_i},$

$[*] := \mathcal{I}\texttt{trace}\mathcal{I}\mathsf{ANN}(R)\mathcal{I}^{-1}\texttt{trace} : \operatorname{tr} \bigotimes_{i=1}^{n} T_i = \sum J \in \prod_{i=1}^{n} m_i \,.\, \prod_{i=1}^{n} T^{e_i,e_i}_{i;I_i,I_i} = \prod_{i=1}^{n} \sum_{j=1}^{m_i} T^{e_i,e_i}_{i;j,j} = \prod_{i=1}^{n} \operatorname{tr} T_i;$

$\square$

$\texttt{DoubleTensorMapDet} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$$. \forall T : A \xrightarrow{R\text{-MOD}} A \,.\, \forall S : B \xrightarrow{R\text{-MOD}} B \,.\; \det T \otimes S = (\det T)^{\operatorname{rank} B} (\det S)^{\operatorname{rank} A}$$

$\texttt{Proof} \;=$

$e := \texttt{FreeHasBasis}(A) : \texttt{Basis}(\operatorname{rank} A, A),$

$f := \texttt{FreeHasBasis}(B) : \texttt{Basis}(\operatorname{rank} B, B),$

$\texttt{Assume}\ i : \operatorname{rank} A,$

$\texttt{Assume}\ j : \operatorname{rank} B,$

$[i.j.1] := \text{Ɑ}\texttt{tensorMap}(T, \mathrm{id})\text{Ɑ}^{-1}\texttt{matrixOfOperator} :$

$$: T \otimes \mathrm{id}_B(e_i \otimes f_j) = T(e_i) \otimes f_j = \sum_{a=1}^{\operatorname{rank} A} T^{e,e}_{i,i} e_a \otimes f_j = \sum_{a=1}^{\operatorname{rank} A} T^{e,e}_{i,i}(e_a \otimes f_j),$$

$\texttt{Assume}\ i' : \operatorname{rank} A,$

$\texttt{Assume}\ j' : \operatorname{rank} B,$

$[i.j.i'.j'.*] := \texttt{BasisOfTensorProduct}[i.j.1] : (T \otimes id_B(e_i \otimes f_j))_{(i',j')} = \delta^j_{j'} T^{e,e}_{i',i};$

$\rightsquigarrow [i.j.*] := I(\forall) : \forall i' \in \operatorname{rank} A \,.\, \forall j' \in \operatorname{rank} B \,.\; (T \otimes \mathrm{id}(e_i \otimes f_j))_{(i',j')} = \delta^j_{j'} T^{e,e}_{a,i};$

$\rightsquigarrow [1] := \text{Ɑ}^{-1}\texttt{matrixOfOperator}(T \otimes \mathrm{id}_B) : \forall i, i' \in \operatorname{rank} A \,.\, \forall j, j' \in \operatorname{rank} B \,.\, (T \otimes \mathrm{id}_B)^{e\otimes f, e\otimes f}_{(i',j'),(i,j)} = \delta^j_{j'} T^{e,e}_{i',i};$

$[2] := \text{Ɑ}^{-1}\texttt{BlockDiagonal}[1] : \Big( (T \otimes \mathrm{id}_B)^{e\otimes f.e\otimes f} : \texttt{BlockDaigonal}(\operatorname{rank} B, T^{e,e}) \Big),$

$[3] := \texttt{BlockDiagonalDet}[2] : \det T \otimes \mathrm{id}_B = (\det T)^{\operatorname{rank} B},$

$\texttt{Assume}\ i : \operatorname{rank} A,$

$\texttt{Assume}\ j : \operatorname{rank} B,$

$[i.j.1] := \text{Ɑ}\texttt{tensorMap}(\mathrm{id}, S)\text{Ɑ}^{-1}\texttt{matrixOfOperator} :$

$$: \mathrm{id}_A \otimes S(e_i \otimes f_j) = e_i \otimes S(f_j) = \sum_{b=1}^{\operatorname{rank} B} S^{f,f}_{b,j} e_i \otimes f_b = \sum_{b=1}^{\operatorname{rank} B} S^{f,f}_{b,j}(e_i \otimes f_b),$$

$\texttt{Assume}\ i' : \operatorname{rank} A,$

$\texttt{Assume}\ j' : \operatorname{rank} B,$

$[i.j.i'.j'.*] := \texttt{BasisOfTensorProduct}[i.j.1] : (\mathrm{id}_A \otimes S(e_i \otimes \mathrm{id}))_{(i',j')} = \delta^i_{i'} S^{f,f}_{j',j};$

$\rightsquigarrow [i.j.*] := I(\forall) : \forall i' \in \operatorname{rank} A \,.\, \forall j' \in \operatorname{rank} B \,.\; (\mathrm{id}_A \otimes S)(e_i \otimes f_j))_{(i',j')} = \delta^i_{i'} S^{f,f'}_{b,i};$

$\rightsquigarrow [4] := \text{Ɑ}^{-1}\texttt{matrixOfOperator}(\mathrm{id}_A \otimes S) : \forall i, i' \in \operatorname{rank} A \,.\, \forall j, j' \in \operatorname{rank} B \,.\, (\mathrm{id}_A \otimes S)^{e\otimes f, e\otimes f}_{(i',j'),(i,j)} = \delta^i_{i'} S^{f,f}_{j',j};$

$[5] := \text{Ɑ}^{-1}\texttt{BlockDiagonal}[1] : \Big( (\mathrm{id}_A \otimes T)^{e\otimes f.e\otimes f} : \texttt{BlockDaigonal}(\operatorname{rank} A, S^{f,f}) \Big),$

$[6] := \texttt{BlockDiagonalDet}[2] : \det \mathrm{id}_A \otimes S = (\det S)^{\operatorname{rank} A},$

$[*] := \texttt{DetProduct}[6][3] : \det T \otimes S = \det(T \otimes \underset{A}{\mathrm{id}}) \det(\underset{B}{\mathrm{id}} \otimes S) = (\det T)^{\operatorname{rank} B} (\det S)^{\operatorname{rank} A};$

$\square$

$\texttt{TensorMapDet} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to \texttt{FreeModule}(R) \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$$. \forall T : \prod_{i=1}^{n} A_i \xrightarrow{R\text{-MOD}} A_i \,.\; \det \bigotimes_{i=1}^{n} T_i = \prod_{i=1}^{n} \det(T_i)^{N_i} \quad \texttt{where} \quad N = \Lambda i \in n \,.\, \prod_{j \neq i}^{n} \operatorname{rank} A_j$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

## 1.4 Flatness

$\texttt{Flat} :: \prod R \in \mathsf{ANN} \,.\, ?R\text{-}\mathsf{MOD}$

$M : \texttt{Flat} \iff T_M : \texttt{Exact}$

$\texttt{ProjectiveIsFlat} :: \forall R \in \mathsf{ANN} \,.\, \forall P : \texttt{Projective}(R) \,.\, P : \texttt{Flat}(R)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{NonFlatQuotient} :: \forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{ProperIdeal}(R) \,.\, \forall a \in R^{\times} \,.\, \forall [0] : a \in I \,.\, \frac{R}{I} \,!\, \texttt{Flat}(R)$

$\texttt{Proof} \ =$

$[1] := [0]\text{Я}\texttt{ProperIdeal}[0] : a \notin R^*,$

$(b, [2]) := \text{Я}\texttt{ProperIdeal}(I) : \sum b \in R \,.\, b \notin I,$

$C := 0 \xrightarrow{0} R \xrightarrow{\cdot a} R \xrightarrow{\pi(a)} \frac{R}{(a)} \xrightarrow{0} 0 : \texttt{ShortExact},$

$[3] := \text{Я}\texttt{TensorProduct}[2] : [1] \otimes b \neq_{\frac{R}{I} \otimes R},$

$[4] := \text{Я}\texttt{TrivialModule} : R \otimes \frac{R}{I} \neq \{0\},$

$[5] := \text{Я}\texttt{TensorProduct}\text{Я}\texttt{quotientModule}\text{Я}\texttt{temsorMap} : a \otimes \mathrm{id}_{R \otimes \frac{R}{I}} = 0,$

$[6] := \text{Я}^{-1}\texttt{Exact}[5][4] : C \otimes \frac{R}{I} \,!\, \texttt{Exact},$

$[*] := \text{Я}^{-1}\texttt{Flat}[6] : \frac{R}{I} \,!\, \texttt{Flat};$

$\square$

$\texttt{FlatDirectSum} :: \forall R \in \mathsf{ANN} \,.\, \forall X \in \mathsf{SET} \,.\, \forall M : X \to \texttt{Flat}(R) \,.\, \bigoplus_{x \in X} M_x : \texttt{Flat}(R)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (V, f) : \texttt{ShortExact}(R\text{-}\mathsf{MOD}),$

$[1] := \text{Я}\texttt{Flat}(M, V) : \forall i \in n \,.\, M_i \otimes (V, f) : \texttt{Exact},$

$[2] := \texttt{ExactDirectSum}[1] : \left( \bigotimes_{i=1}^{n} (M_i \otimes (V, f)) : \texttt{Exact} \right),$

$[(V, f).*] := \texttt{TensorProductDistributive}[2] : \left( \left( \bigoplus_{i=1}^{n} M_i \right) \otimes (V, f) : \texttt{Exact} \right);$

$\square$

$\texttt{FlatTensorProduct} :: \forall R \in \mathsf{ANN} \, . \, \forall n \in \mathbb{N} \, . \, \forall M : n \to \texttt{Flat}(R) \, . \, \bigotimes_{i=1}^{n} M_i : \texttt{Flat}(R)$

$\texttt{Proof} \;=$

$\male := \Lambda n \in \mathbb{N} \, . \, \forall M : n \to \texttt{Flat}(R) \, . \, \bigotimes_{i=1}^{n} M_i : \texttt{Flat}(R) : \mathbb{N} \to \texttt{Type},$

$[1] := \mathcal{I}\texttt{tensorProduct}\mathcal{J}^{-1}\male : \male(1),$

$\texttt{Assume } n : \mathbb{N},$

$\texttt{Assume } [n.1] : \male(n),$

$\texttt{Assume } M : (n+1) \to \texttt{Flat}(R),$

$[M.1] := \mathcal{J}\male[n.1](M_{|n}) : (T_{\otimes_{i=1}^{n} M_i} : \texttt{Exact}),$

$[M.2] := \mathcal{I}\texttt{Flat}(M_{n+1}) : (T_{M_{n+1}} : \texttt{Exact}),$

$[M.*] := \texttt{ExactCompose}[M.1][M.2]\texttt{TensorProductAssoc}(M) :$

$\quad : T_{\bigotimes_{i=1}^{n+1} M_i} = T_{M_{n+1} \otimes \bigotimes_{i=1}^{n} M_i} = T_{M_{n+1}} T_{\bigotimes_{i=1}^{n} M_i} : \texttt{Exact};$

$\rightsquigarrow [n.*] := \mathcal{J}^{-1}\male\mathcal{I}^{-1}\texttt{Flat} : \male[n+1];$

$\rightsquigarrow [*] := \mathcal{I}\texttt{NaturalSet}(\mathbb{N})(n, M) : \left( \bigotimes_{i=1}^{n} M_i : \texttt{Flat}(R) \right);$

$\square$

$\texttt{FlatBySubmodules} :: \forall R \in \mathsf{ANN} \, . \, \forall M \in R\text{-}\mathsf{MOD} \, .$

$\quad . \, \forall [0] : \forall N : \texttt{FinitelyGeneratedModule}(R) \,\&\, \texttt{Submodule}(R, M) \, . \, N : \texttt{Flat}(R) \, . \, M : \texttt{Flat}(R)$

$\texttt{Proof} \;=$

$\texttt{Assume } (V, f) : \texttt{ShortExact}(R\text{-}\mathsf{MOD}),$

$\texttt{Assume } \sum_{i=1}^{k} m_i \otimes v_i : M \otimes V_2,$

$N := \operatorname{span}(m) : \texttt{FinitelyGeneratedModule}(R)\texttt{Submodule}(R, M),$

$[m.1] := [0](N) : (N : \texttt{Flat}(R)),$

$[m.*] := \texttt{ZeroKernelTHM}\,(m_i \otimes v_i)\, \texttt{InjectiveByExactness}(f_2 \otimes \mathrm{id}_N)\mathcal{I}\texttt{Flat}(R)(N) :$

$\quad : \sum_{i=1}^{n} m_i \otimes f_2(v_i) = 0 \iff \sum_{i=1}^{k} v_i \otimes m_i = 0;$

$\rightsquigarrow [1] := \texttt{ZeroKernelTHM}I(\forall) : (f_2 \otimes \mathrm{id}_M : V_2 \otimes M \hookrightarrow V_1),$

$\texttt{Assume } \sum_{i=1}^{k} v_i \otimes m_i : V_0 \otimes M,$

$N := \operatorname{span}(m) : \texttt{FinitelyGeneratedModule}(R)\texttt{Submodule}(R, M),$

$[m.1] := [0](N) : (N : \texttt{Flat}(R)),$

$[m.*] := \mathcal{I}\texttt{SurjectiveSurjectiveByExactness}\mathcal{I}\texttt{Flat}(R)(N)(V, f) :$

$\quad : \exists \sum_{i=1}^{k'} v'_i \otimes m'_i \in V_1 \otimes M \, . \, f_1 \otimes \mathrm{id}_N \left( \sum_{i=1}^{k'} v'_i \otimes m'_i \right) = \sum_{i=1}^{k} v_i \otimes m_i;$

$\rightsquigarrow [2] := \mathcal{I}^{-1}\texttt{Surjective} : f_1 \otimes \mathrm{id}_M : M \otimes V_1 \twoheadrightarrow M \otimes V_0,$

Assume $\sum_{i=1}^{k} v_i \otimes m_i : \ker f_1 \otimes \mathrm{id}_M$,

$N := \mathrm{span}(m) : \texttt{FinitelyGeneratedModule}(R)\texttt{Submodule}(R, M)$,

$[m.1] := [0](N) : (N : \texttt{Flat}(R))$,

$[m.*] := \reflectbox{$\mathcal{D}$}\texttt{Exact}\reflectbox{$\mathcal{D}$}\texttt{Flat}(R)(N)(V, f) : \exists \sum_{i=1}^{k'} v'_i \otimes m'_i \in V_2 \otimes M \ .\ f_2 \otimes \mathrm{id}_N \left( \sum_{i=1}^{k'} v'_i \otimes m'_i \right) = \sum_{i=1}^{k} v_i \otimes m_i$;

$\rightsquigarrow [V.*] := \reflectbox{$\mathcal{D}$}^{-1}\texttt{Exact}[1][2] : (M \otimes (V, f) : \texttt{Exact})$;

$\rightsquigarrow [*] := \reflectbox{$\mathcal{D}$}^{-1}\texttt{Flat}\reflectbox{$\mathcal{D}$}^{-1}\texttt{Exact}I(\forall) : (M : \texttt{Flat}(R))$;

$\square$

$\texttt{RatsAreFlatButNotProjective} :: \mathbb{Q} : \texttt{Flat}(\mathbb{Z})\ \&\ \mathbb{Q}\,!\,\texttt{Projective}(\mathbb{Z})$

Proof $=$

Assume $N : \texttt{FinitelyGeneratedModule}(\mathbb{Z})\ \&\ \texttt{Submodule}(\mathbb{Z}, \mathbb{Q})$,

$\left(n, \frac{a}{b}, [1]\right) := \reflectbox{$\mathcal{D}$} N : \sum n \in \mathbb{N} \ .\ \frac{a}{b} \in \mathbb{Q}^n \ .\ N = \mathrm{span}_{\mathbb{Z}} \left( \frac{a}{b} \right)$,

$[2] := [1]\reflectbox{$\mathcal{D}$}\mathbb{Q} : N \subset \frac{\mathbb{Z}}{\prod_{i=1}^{n} b_i}$,

$[3] := \texttt{CyclicSubsetIsCyclic}[2] : (N : \texttt{Cyclic})$,

$[4] := \texttt{InfiniteCyclic}[3] : N \cong_{\mathbb{Z}\text{-MOD}} \mathbb{Z}$,

$[N.*] := \texttt{FreeIsProjective}\ \texttt{ProjectiveIsFlat} : (N : \texttt{Flat}(\mathbb{Z}))$;

$\rightsquigarrow [1] := \texttt{FlatBySubmodules} : (\mathbb{Q} : \texttt{Flat}(\mathbb{Z}))$,

Assume $\varphi : \mathbb{Q} \xrightarrow{\mathbb{Z}\text{-MOD}} \mathbb{Z}$,

Assume $\frac{a}{b} : \mathbb{Q}$,

Assume $[2] : \varphi \left( \frac{a}{b} \right) \neq 0$,

$n := \varphi \left( \frac{a}{b} \right) : \mathbb{Z}^{\times}$,

$[3] := \reflectbox{$\mathcal{D}$}\mathbb{Z}\text{-MOD}(\mathbb{Q}, \mathbb{Z})(\varphi)\reflectbox{$\mathcal{C}$}^{-1}(n) : n\varphi \left( \frac{a}{nb} \right) = \varphi \left( \frac{a}{b} \right) = n$,

$[4] := \texttt{InjMult} : \varphi \left( \frac{a}{nb} \right) = 1$,

$[5] := \reflectbox{$\mathcal{D}$}\mathbb{Z}\text{-MOD} : 2\varphi \left( \frac{a}{2nb} \right) = \varphi \left( \frac{a}{nb} \right) = 1$,

$[6] := \texttt{DivisorsOfUnity}[5] : 2 = 1$,

$[\varphi.*] := I(\perp)[6] : \perp$;

$\rightsquigarrow [2] := I(\forall)I(=, \rightarrow)E(\perp) : \forall \varphi : \mathbb{Q} \xrightarrow{\mathbb{Z}\text{-MOD}} \mathbb{Z}$,

Assume $[4] : (\mathbb{Q} : \texttt{Projective}(\mathbb{Z}))$,

$(X, P, [5]) := \reflectbox{$\mathcal{D}$}\texttt{Projective}[4] : \sum X \in \mathsf{SET} \ .\ \sum P \in \mathbb{Z}\text{-MOD} \ .\ \mathbb{Q} \oplus P \cong X^{\mathbb{Z}}$,

Assume $x : X$,

$f := \iota_{\mathbb{Q}} \pi_x : \mathbb{Q} \xrightarrow{\mathbb{Z}\text{-MOD}} \mathbb{Z}$,

$[x.*] := [2](f) : f = 0$;

$\rightsquigarrow [5] := \reflectbox{$\mathcal{D}$}\texttt{Product} : \iota_{\mathbb{Q}} = 0$,

$[6] := \reflectbox{$\mathcal{D}$}\texttt{Injective}(\iota)\texttt{ZeroKernelTHM}[5] : \mathbb{Q} = \{0\}$,

$[4.*] := I(\perp)[6] : \perp$;

$\rightsquigarrow [*] := E(\perp) : \mathbb{Q}\,!\,\texttt{Projective}(R)$;

$\square$

## 1.5 Covariant Scalar Extension

$$\texttt{Bimodule} :: \prod R, S \in \mathsf{RING}\ .\ ?\sum M \in \mathsf{SET}\ .\ (M \times M \to M) \times (R \times M \to M) \times (S \times M \to M)$$

$$\Big(M, +, \odot_1, \odot_2\Big) : \texttt{Bimodule} \iff (M, +, \odot_1) \in R\text{-}\mathsf{MOD}\ \&\ (M, +, \odot_2) \in S\text{-}\mathsf{MOD}\ \&$$
$$\&\ \forall \alpha \in R\ .\ \forall \beta \in S\ .\ \forall a \in M\ .\ \beta \odot_2 (\alpha \odot_1 a) = \alpha \odot_1 (\beta \odot_2 a)$$

$$\texttt{bimoduleCategory} :: \mathsf{RING}^2 \to \mathsf{CAT}$$
$$\texttt{bimoduleCategory}\,(R, S) = (R, S)\text{-}\mathsf{MOD} := (\texttt{Bimodule}, R\text{-}\mathsf{MOD} \cap S\text{-}\mathsf{MOD}, \circ, \mathrm{id})$$

$$\texttt{leftTensorBimodule} :: \prod R, S \in \mathsf{ANN}\ .\ (R, S)\text{-}\mathsf{MOD} \to R\text{-}\mathsf{MOD} \to (R, S)\text{-}\mathsf{MOD}$$
$$\texttt{leftTensorBimodule}\,(V, M) = V \otimes_R M :=$$
$$:= \Big(V \otimes_R M, +, \cdot, \Lambda s \in S\ .\ \texttt{tensorization}(\Lambda v \in V\ .\ \Lambda m \in M\ .\ (sv) \otimes m)\Big)$$

$$\texttt{rightTensorBimodule} :: \prod R, S \in \mathsf{ANN}\ .\ (R, S)\text{-}\mathsf{MOD} \to R\text{-}\mathsf{MOD} \to (R, S)\text{-}\mathsf{MOD}$$
$$\texttt{rightTensorBimodule}\,(V, M) = M \otimes_R V :=$$
$$:= \Big(M \otimes_R V, +, \cdot, \Lambda s \in S\ .\ \texttt{tensorization}(\Lambda v \in V\ .\ \Lambda m \in M\ .\ m \otimes (sv))\Big)$$

$$\texttt{TensorCommutes} :: \forall R, S \in \mathsf{ANN}\ .\ \forall V \in (R, S)\text{-}\mathsf{MOD}\ .\ \forall M \in R\text{-}\mathsf{MOD}\ .\ M \otimes_R V \cong_{(R,S)\text{-}\mathsf{MOD}} V \otimes_R M$$

Proof =

...

□

$$\texttt{TensorAssociativityLaw} :: \forall R, S \in \mathsf{ANN}\ .\ \forall V \in (R, S)\text{-}\mathsf{MOD}\ .\ \forall A \in R\text{-}\mathsf{MOD}\ .\ \forall B \in S\text{-}\mathsf{MOD}\ .$$
$$(A \otimes_R V) \otimes_S B \cong_{(R,S)\text{-}\mathsf{MOD}} A \otimes_R (V \otimes_S B)$$

Proof =

...

□

$$\texttt{morphismExtension} :: \prod R, S \in \mathsf{ANN}\ .\ \Big(S \xrightarrow{\mathsf{ANN}} R\Big) \to R\text{-}\mathsf{MOD} \to (R, S)\text{-}\mathsf{MOD}$$
$$\texttt{morphismExtension}\,(\varphi, M) = M_\varphi := \Big(M, +, \cdot, \Lambda s \in S\ .\ \Lambda m \in M\ .\ \varphi(s)m\Big)$$

$\texttt{BasisOfCovariantExtension} :: \forall R, S \in \mathsf{ANN} \,.\, \forall \varphi : S \xrightarrow{\mathsf{ANN}} R \,.\, \forall F : \texttt{FreeModule}(R) \,.\, \forall E : \texttt{Basis}(F) \,.$

$\quad . \, E \otimes 1 : \texttt{Basis}(S, F \otimes_R S_\varphi)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ t : F \otimes_R S_\varphi,$

$\Big(s, [1]\Big) := \texttt{FreeTensotingDecomposition}(E, t) : \sum s : S^{\oplus E} \,.\, t = \sum_{e \in E} e \otimes s_e,$

$[t.*] := \mathcal{C}\!\!\!\!\!\;\texttt{rightTensorBimodule} : t = \sum_{e \in E} s_e e \otimes 1;$

$\rightsquigarrow [1] := \mathcal{C}\!\!\!\!\!\;^{-1}\texttt{span} : \big(F \otimes_R S = \mathrm{span}_S(E \otimes 1)\big),$

$\texttt{Assume}\ s : S^{\oplus E},$

$\texttt{Assume}\ [2] : sE \otimes 1 = 0,$

$\texttt{Assume}\ e : E,$

$[3] := [2](e) : e \otimes s_e = 0,$

$[e.*] := \mathcal{C}\!\!\!\!\!\;\texttt{Basis}(E)[3] : s_e = 0;$

$\rightsquigarrow [s.*] := I(=, \rightarrow) : s = 0;$

$\rightsquigarrow [*] := \mathcal{C}\!\!\!\!\!\;^{-1}\texttt{Basis}[1] : \big(E \otimes 1 : \texttt{Basis}(F \otimes S)\big);$

$\square$

$\texttt{FreeCovariantExtension} :: \forall R, S \in \mathsf{ANN} \,.\, \forall \varphi : S \xrightarrow{\mathsf{ANN}} R \,.\, \forall F : \texttt{FreeModule}(R) \,.\, F \otimes_R S_\varphi : \texttt{FreeModule}(S)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{FreeCovariantExtensionRank} :: \forall R, S \in \mathsf{ANN} \,.\, \forall \varphi : S \xrightarrow{\mathsf{ANN}} R \,.\, \forall F : \texttt{FreeModule}(R) \,.$

$\quad . \, \mathrm{rank}_S\, F \otimes_R S_\varphi = \mathrm{rank}_R\, F$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ProjectiveCovariantExtension} :: \forall R, S \in \mathsf{ANN} \,.\, \forall \varphi : S \xrightarrow{\mathsf{ANN}} R \,.\, \forall P : \texttt{Projective}(R) \,.$

$\quad . \, P \otimes RS_\varphi : \texttt{Projectve}(S)$

$\texttt{Proof} \;=$

$(Q, [1]) := \mathcal{C}\!\!\!\!\!\;\texttt{Projective}(P) : \sum Q \in R\text{-}\mathsf{MOD} \,.\, Q \oplus P : \texttt{FreeModule}(R),$

$[2] := \texttt{FreeCovariantExtension}(R, S, \varphi, Q \oplus P) : \Big(Q \oplus P \otimes_R S_\varphi : \texttt{FreeModule}(S)\Big),$

$[3] := \texttt{TensorProductDistributive}(S_\varphi, P, Q) : Q \otimes_R S_\varphi \oplus P \otimes_R S_\varphi \cong_{R\text{-}\mathsf{MOD}} Q \oplus P \otimes S_\varphi,$

$[4] := \mathcal{C}\!\!\!\!\!\;\texttt{leftBimodule}[3] : Q \otimes_R S_\varphi \oplus P \otimes_R S_\varphi \cong_{S\text{-}\mathsf{MOD}} Q \oplus P \otimes S_\varphi,$

$[*] := \mathcal{C}\!\!\!\!\!\;^{-1}\texttt{Projective} : \Big(P \otimes_R S_\varphi : \texttt{Projective}(S)\Big);$

$\square$

$\texttt{CovariantExtensionDistributive} :: \forall R, S \in \mathsf{ANN} . \forall \varphi : R \xrightarrow{\mathsf{ANN}} S . \forall A, B \in R\text{-}\mathsf{MOD} .$

$. (A \otimes_R S_\varphi) \otimes_S (B \otimes_R S_\varphi) \cong_{R\text{-}\mathsf{MOD}} (A \otimes_R B) \otimes_R S_\varphi$

$\texttt{Proof} =$

$$X := \texttt{tensorizes}_S\left(\Lambda \sum_{i=1}^{n} a_i \otimes \alpha_i \in A \otimes_R S_\varphi . \Lambda \sum_{i=1}^{m} b_i \otimes \beta_i \in B \otimes_R S_\varphi . \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \otimes b_j \otimes \alpha_i\beta_j\right) :$$

$$. (A \otimes_R S_\varphi) \otimes_S (A \otimes RS_\varphi) \xrightarrow{S\text{-}\mathsf{MOD}} (A \otimes_R B) \otimes_R S_\varphi,$$

$$Y := \Lambda \sum n \in \mathbb{N} . \sum_{i=1}^{n} a_i \otimes b_i \otimes s_i \in (A \otimes_R B) \otimes_R S_\varphi . \sum_{i=1}^{n} (a_i \otimes 1) \otimes (b_i \otimes s) :$$

$$: (A \otimes_R B) \otimes_R S_\varphi \xrightarrow{S\text{-}\mathsf{MOD}} (A \otimes_R S_\varphi) \otimes_S (B \otimes_R S_\varphi),$$

$\texttt{Assume}\ t : (A \otimes_R S_\varphi) \otimes_S (B \otimes_R S_\varphi),$

$\big(n, a, b, \alpha, \beta, [1]\big) := \mathcal{C}^3\texttt{tensorProduct}(t) :$

$$: \sum n \in \mathbb{N} . \sum a : n \to A . \sum b : n \to B . \sum \alpha, \beta : n \to S . t = \sum_{i=1}^{n} (a_i \otimes \alpha_i) \otimes (b_i \otimes \beta_i),$$

$[t.*] := [1]\mathcal{C} S\text{-}\mathsf{MOD}\mathcal{D}X\mathcal{D}Y\texttt{MultiHomogen}^{2n}(\dots, \alpha)[1] :$

$$: YX(t) = \sum_{i=1}^{n} YX\big((a_i \otimes \alpha_i) \otimes (b_i \otimes \beta_i)\big) = \sum_{i=1}^{n} Y(a_i \otimes b_i \otimes \alpha_i\beta_i) = \sum_{i=1}^{n} (a_i \otimes 1) \otimes (b_i \otimes \alpha_i\beta_i) =$$

$$= \sum_{i=1}^{n} (a_i \otimes \alpha_i) \otimes (b_i \otimes \beta_i) = t;$$

$\rightsquigarrow [1] := I(= . \to) : YX = \mathrm{id},$

$\texttt{Assume}\ t : A \otimes_R B \otimes_R S_\varphi,$

$\big(n, a, b, s, [1]\big) := \mathcal{C}\texttt{tensorProduct}(t) :$

$$: \sum n \in \mathbb{N} . \sum a : n \to A . \sum b : n \to B . \sum s : n \to S . t = \sum_{i=1}^{n} a_i \otimes b_i \otimes s_i,$$

$[t.*] := [1]\mathcal{C} S\text{-}\mathsf{MOD}(XY)\mathcal{D}Y\mathcal{D}X[1] :$

$$: XY(t) = \sum_{i=1}^{n} XY(a_i \otimes b_i \otimes s_i) = \sum_{i=1}^{n} X\big((a_i \otimes 1) \otimes (b_i \otimes s_i)\big) = \sum_{i=1}^{n} a_i \otimes b_i \otimes s_i = t;$$

$\rightsquigarrow [2] := I(=, \to) : XY = \mathrm{id},$

$[*] := \mathcal{C}^{-1}\texttt{Isomorphic}[2][3] : (A \otimes_R S_\varphi) \otimes_S (B \otimes_R S_\varphi) \cong_{R\text{-}\mathsf{MOD}} (A \otimes_R B) \otimes_R S_\varphi;$

$\square$

$\texttt{CovariantExtensionDistributive2} :: \forall R, S \in \mathsf{ANN} . \forall \varphi : R \xrightarrow{\mathsf{ANN}} S . \forall n \in \mathbb{N} . \forall A : n \to R\text{-}\mathsf{MOD} .$

$$. \bigotimes_{i=1}^{n} (A_i \otimes_R S_\varphi) \cong_{R\text{-}\mathsf{MOD}} \left(\bigotimes_{i=1}^{n} A_i\right) \otimes_R S_\varphi$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FlatCovariantExtension} :: \forall R, S \in \mathsf{ANN} \,.\, \forall \varphi : R \xrightarrow{\mathsf{ANN}} S \,.\, \forall M : \texttt{Flat}(R) \,.\, M \otimes_R S_\varphi : \texttt{Flat}(S)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (V, f) : \texttt{ShortExact}(S\text{-}\mathsf{MOD}),$

$[1] := \texttt{TensorAssociativityLaw}(R, S, \varphi, M, V) :$

$\quad : (M \otimes_R S_\varphi) \otimes_S (V, f) = M \otimes_R (S \otimes_S (V, f))_\varphi = M \otimes_R (V, f)_\varphi,$

$[2] := \texttt{ExactInAllStructures}((V, f), \varphi) : \Big((V, f)_\varphi : \texttt{Exact}(R\text{-}\mathsf{MOD})\Big),$

$[3] := \mathcal{J}\texttt{Flat}(M)[2] : \Big(M \otimes_R (V, f)_\varphi : \texttt{Exact}(R\text{-}\mathsf{MOD})\Big),$

$[.*] := \texttt{ExactInAllStructures}[3] : \Big(M \otimes_R (V, f)_\varphi : \texttt{Exact}(S\text{-}\mathsf{MOD})\Big);$

$\rightsquigarrow [*] := \mathcal{J}^{-1}\texttt{Flat}(S) : \Big(M \otimes_R S_\varphi : \texttt{Exact}(S)\Big);$

$\square$

$\texttt{FractionTensorZeroCondition} :: \forall R \in \mathsf{ANN} \,.\, \forall \Sigma : \texttt{MultiplicativeSubset}(R) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$

$\quad .\, \forall m \in M \,.\, \forall \sigma \in \Sigma \,.\, m \otimes \frac{1}{\sigma} = 0 \iff \exists \sigma' \in \Sigma : \sigma' m = 0$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : m \otimes \frac{1}{\sigma} = 0,$

$(M', S, [2]) := \texttt{ZeroTensorInFGM}[1] :$

$\quad : \sum M', S : \texttt{FinitelyGeneratedModule}(R) \,.\, M' \subset M \;\&\; S \subset \Sigma^{-1}R \;\&\;, m \otimes \frac{1}{\sigma} =_{M' \otimes S} 0,$

$(\varsigma, [3]) := \texttt{FGFractionSet}(S)[2] : \sum \varsigma \in \Sigma \,.\, S \subset \frac{R}{\sigma\varsigma},$

$\varphi := \Lambda r \in R \,.\, \frac{r}{\sigma\varsigma} : R \xrightarrow{R\text{-}\mathsf{MOD}} \frac{R}{\sigma\varsigma},$

$I := \ker \varphi : R\text{-}\mathsf{MOD},$

$C := I \hookrightarrow R \overset{\varphi}{\to} \frac{R}{\sigma\varsigma} : \texttt{ShortExact},$

$\texttt{Assume}\ a : I,$

$[a.1] := \mathcal{O}(a)(I) : \frac{a}{\sigma\varsigma} = 0,$

$(\alpha, [a.*]) := \mathcal{J}\Sigma^{-1}R[a.1] : \sum \alpha \in \Sigma \,.\, \alpha a = 0;$

$\rightsquigarrow (\alpha, [4]) :=: \sum \alpha I \to \Sigma \,.\, \forall a \in I \,.\, a\alpha_a = 0,$

$[5] := \texttt{TensorProductRightExact}(M, C) : (M \otimes C : \texttt{RightExact}),$

$[6] := \mathcal{J}\texttt{RightExact}(M \otimes C)[1] : m \otimes \varsigma \in M \otimes I,$

$(\varsigma', a, [7]) := [4][6] : \sum \varsigma' \in \Sigma \,.\, \varsigma\varsigma' m = 0,$

$[1.*] := \mathcal{J}\texttt{MultiplicativeSubset}(\Sigma)(\varsigma, \varsigma') : \varsigma\varsigma' \in \Sigma;$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ (\sigma', [2]) : \sum \sigma' \in \Sigma : \sigma' m = 0,$

$[2.*] := \texttt{MultiHomogen}(\sigma')[2]\texttt{MultiHomogen}(0) : m \otimes \frac{1}{\sigma} = \sigma' m \otimes \frac{1}{\sigma\sigma'} = 0 \otimes \frac{1}{\sigma\sigma'} = 0;$

$\rightsquigarrow [*] := I(\iff)[1]I(\Rightarrow) : \texttt{This},$

$\square$

$\texttt{FractionFormTensor} :: \forall R \in \mathsf{ANN} \,.\, \forall \Sigma : \texttt{MultiplicativeSubset}(R) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall t \in M \otimes \Sigma^{-1}R \,.$

$\quad . \, \exists m \in M : \exists \sigma \in \Sigma : t = m \otimes \frac{1}{\sigma}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{FractionsAreFlat} :: \forall R \in \mathsf{ANN} \,.\, \forall \Sigma : \texttt{MultiplicativeSubset}(R) \,.\, \Sigma^{-1}R : \texttt{Flat}(R)$

$\texttt{Proof} \; =$

$\texttt{Assume } A, B : R\text{-}\mathsf{MOD},$

$\texttt{Assume } f : A \xrightarrow{R\text{-}\mathsf{MOD}} B,$

$\texttt{Assume } m \otimes \frac{1}{\sigma} : \ker\left(f \otimes \mathrm{id}_{\Sigma^{-1}R}\right),$

$[1] := \mathcal{I} \ker \mathcal{I}\texttt{tensorMap} : 0 = f \otimes \mathrm{id}_{\Sigma^{-1}R}\left(m \otimes \frac{1}{\sigma}\right) = f(m) \otimes \frac{1}{\sigma},$

$(\sigma', [2]) := \texttt{FractionZeroTensorCondition} : \sum \sigma' \in \Sigma \,.\, \sigma' f(m) = 0,$

$[A.*] := \mathcal{I}^{-1}\texttt{kernel}\mathcal{I} : \sigma' m \in \ker f;$

$\rightsquigarrow [1] := I^3(\forall) : \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.\, \forall m \otimes \frac{1}{\sigma} \in \ker f \otimes \mathrm{id}_{\Sigma^{-1}R} \,.\, \exists \sigma' \in \Sigma : \sigma' m \in \ker f,$

$\texttt{Assume } (V, f) : \texttt{ShortExact}(R\text{-}\mathsf{MOD}),$

$[2] := [1](V_2, V_1, f_2) : \forall v \otimes \frac{1}{\sigma} \in \ker(f_2 \otimes \mathrm{id}_{\Sigma^{-1}R}) \,.\, \exists \sigma' \in \Sigma \,.\, \sigma' v = 0,$

$[3] := \texttt{ZeroFractionTensorCondition}[2] : \ker(f_2 \otimes \mathrm{id}_{\Sigma^{-1}R}) = \{0\},$

$[4] := \texttt{ZeroKernelTHM}[3] : f_3 \otimes \mathrm{id}_{\Sigma^{-1}R} : V_2 \otimes \Sigma^{-1}R \hookrightarrow V_1 \otimes \Sigma^{-1}R,$

$\texttt{Assume } v \otimes \frac{1}{\sigma} : \ker(f_1 \otimes \mathrm{id}_{\Sigma^{-1}R}),$

$(\sigma', [5]) := [1]\left(V_1, V_0, f_1, v \otimes \frac{1}{\sigma}\right) : \sum \sigma' \in \Sigma \,.\, f_1(\sigma' v) = 0,$

$(w, [6]) := \mathcal{I}\texttt{ShortExact}(f_1) : \sum w \in V_2 \,.\, f_2(w) = \sigma' v,$

$[v.*] := \mathcal{I}\texttt{tensorMap}[6]\texttt{MultiHomogeb} : f_2 \otimes \mathrm{id}_{\Sigma^{-1}R} w \otimes \frac{1}{\sigma\sigma'} = \sigma' v \otimes \frac{1}{\sigma\sigma'} = v \otimes \frac{1}{\sigma};$

$\rightsquigarrow [V.*] := \mathcal{I}^{-1}\texttt{Exact}[4] : ((V, f) \otimes \Sigma^{-1}R : \texttt{Exact});$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Flat} : \left(\Sigma^{-1}R : \texttt{Flat}(R)\right);$

$\square$

## 1.6 Composition Algebra

TensorBilinearProduct :: $\forall R \in \mathsf{ANN} . \forall V, W, U, X, Y, Z \in R\text{-}\mathsf{MOD} .$

$. \forall A \in \mathcal{L}(V, W; U) . \forall B \in \mathcal{L}(X, Y; Z) . \exists! C : \mathcal{L}(V \otimes X, W \otimes Y; U \otimes Z) :$

$: \forall v \in V . \forall x \in X . \forall w \in W . \forall y \in Y . C(v \otimes x, w \otimes y) = A(v, x) \otimes B(w, y)$

Proof =

...

□

TensorBilinearProduct2 :: $\forall R \in \mathsf{ANN} . \forall n, m \in \mathbb{N} . \forall W : n \to R\text{-}\mathsf{MOD} . \forall V : n \times m \to R\text{-}\mathsf{MOD} .$

$$. \forall A : \prod_{i=1}^{n} \mathcal{L}(V_i; W_i) . \exists! C : \mathcal{L}\left( \bigotimes_{i=1}^{n} V_n ; \bigotimes_{i=1}^{n} W_n \right) :$$

$$: \forall v \in \prod_{i,j=1}^{m \times n} V_{i,j} . C \left( \bigotimes_{j=1}^{n} v_{i,j} \right)_{i=1}^{m} = \bigotimes_{i=1}^{n} A_n(v_i)$$

Proof =

...

□

bilinearMapTensorProduct :: $\prod R \in \mathsf{ANN} . \prod n, m \in \mathbb{N} . \prod W : n \to R\text{-}\mathsf{MOD} .$

$$. \prod V : n \times m \to R\text{-}\mathsf{MOD} . \prod_{i=1}^{n} \mathcal{L}(V_n; W_n) \to \mathcal{L}\left( \bigotimes_{i=1}^{n} V_n ; \bigotimes_{i=1}^{n} W_n \right)$$

bilinearMapTensorProduct $(A) = \bigotimes_{i=1}^{n} A_i :=$ TensorBilinearProduct2

BilinearFuncTensorProduct :: $\forall R \in \mathsf{ANN} . \forall n, m \in \mathbb{N} . \forall V : n \times m \to R\text{-}\mathsf{MOD} .$

$$. \forall A : \prod_{i=1}^{n} \mathcal{L}(V_i; R) . \forall v \in \prod_{i,j=1}^{m \times n} V_{i,j} . \bigotimes_{i=1}^{n} A_i \left( \bigotimes_{j=1}^{n} v_{i,j} \right)_{i=1}^{m} = \prod_{i=1}^{n} A_i(v_i)$$

Proof =

...

□

NondegenerateNensorProductCondition :: $\forall k :$ Field $. \forall n, m \in \mathbb{N} . \forall V : n \times m \to k\text{-}\mathsf{VS} .$

$$. \forall A : \prod_{i=1}^{n} \mathcal{L}(V_i; k) . \bigotimes_{i=1}^{n} A_i : \mathtt{Nondegenerate}\left( k, \bigotimes_{i=1}^{n} V_i \right) \iff \forall i \in n . A_i : \mathtt{Nondegenerate}(k, V_i)$$

Proof =

...

□

$\texttt{DualTensorProduct} :: \forall k : \texttt{Field} . \forall V, W \in k\text{-VS} . \left(V \otimes W\right)^* \cong_{k\text{-VS}} V^* \otimes W^*$

$\texttt{Proof} =$

...

$\square$

$\texttt{SelfdualTensorProduct} :: \forall k : \texttt{Field} . \forall V \in k\text{-VS} . \left(V \otimes V^*\right)^* \cong_{k\text{-VS}} V^* \otimes V$

$\texttt{Proof} =$

...

$\square$

$\texttt{TensorProductReflexive} :: \forall k : \texttt{Field} . \forall V \in k\text{-VS} . \left(V \otimes V^*\right)^* \cong_{k\text{-VS}} V^* \otimes V$

$\texttt{Proof} =$

...

$\square$

$\texttt{DulMappingTensorProduct} :: \forall k : \texttt{Field} . \forall V, W, X, Y \in k\text{-VS} . \forall f : V \xrightarrow{k\text{-VS}} W . \forall g : X \xrightarrow{k\text{-VS}} Y .$
$. \left(f \otimes g\right)^* \cong_{k\text{-VS}} f^* \otimes g$

$\texttt{Proof} =$

...

$\square$

$\texttt{compositionAlgebra} :: \prod R \in \text{ANN} . R\text{-MOD} \to R\text{-ALG}$

$\texttt{compositionAlgebra}\,(V) = \text{CA}(V) :=$

$$:= \Big(V \otimes V^*, \cdot, +, \texttt{tensorisation}\Lambda v \otimes f, w \otimes g \in V \otimes V^* . f(w)(v \otimes g)\Big)$$

$\texttt{asOperator} :: \prod R \in \text{ANN} . \prod V \in R\text{-MOD} . \text{CA}(V) \xrightarrow{R\text{-ALG}} \mathrm{End}_{R\text{-MOD}}$

$$\texttt{asOperator}\left(\sum_{i=1}^{n} v_i \otimes f_i\right) := \Lambda w \in V . \sum_{i=1}^{n} f_i(w) v_i$$

$\texttt{InclusionOdCompositionAlgebras} :: \forall k : \texttt{Field} . \forall V : k\text{-VS} . \texttt{asOperator} : \text{CA}(V) \hookrightarrow \mathrm{End}_{R\text{-MOD}}$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompositionAlgebraIsOperators} :: \forall k : \texttt{Field} . \forall V : k\text{-VS} . \forall [0] : \dim V < \infty \texttt{asOperator} : \text{CA}(V) \xleftrightarrow{k\text{-ALG}} \mathrm{End}$

$\texttt{Proof} =$

...

$\square$

$\texttt{CompositionAlgebraIsNotOperators} :: \forall k : \texttt{Field} . \forall V : k\text{-VS} . \forall [0] : \dim V = \infty . \mathrm{id} \notin \texttt{asOperatoe}(\text{CA}(V))$

$\texttt{Proof} =$

...

$\square$

# 2 Tensorial Algebras

## 2.1 Tensor Algebra

$\texttt{TensorAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?\sum T : R\text{-}\mathsf{ALGE} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} T$

$(T, \iota) : \texttt{TensorAlgebra} \iff \forall A \in R\text{-}\mathsf{ALGE} \,.\, \forall \varphi : M R\text{-}\mathsf{MOD} A \,.\, \exists! f : T \xrightarrow{R\text{-}\mathsf{ALGE}} A : \varphi = \iota f$

$\texttt{IsomorphicTensorAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (T, \iota), (T', \iota') : \texttt{TensorAlgebra}(M) \,.$
$\quad .\, T \cong_{R\text{-}\mathsf{ALGE}} T'$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{TensorAlgebraUniverslInjective} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (T, \iota) : \texttt{TensorAlgebra}(M) \,.\, \iota : M \hookrightarrow T$

$\texttt{Proof} \ =$

$A := \texttt{leggedAlgebra}(M) : R\text{-}\mathsf{ALGE},$

$\varphi := \Lambda m \in M \,.\, (0, m) : M \xrightarrow{R\text{-}\mathsf{MOD}} A,$

$(f, [1]) := \text{Ɑ}\texttt{TensorAlgebra}(T, \iota) : \sum f : T \xrightarrow{R\text{-}\mathsf{ALGE}} A \,.\, \iota f = \varphi,$

$\texttt{Assume}\ m : M,$

$\texttt{Assume}\ [1] : \varphi(m) = 0,$

$[2.*] := \text{Ɔ}\varphi : m = 0;$

$\rightsquigarrow [2] := \texttt{ZeroKernelTHM} : (\varphi : M \hookrightarrow A),$

$[*] := \text{Ɑ}\texttt{MonoComp}[1][2] : (\iota : M \hookrightarrow T);$

$\square$

$\texttt{tensorAlgebra} :: R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{ALGE}(\mathbb{Z}_+)$

$\texttt{tensorAlgebra}\,(M) = M^\otimes := \left(\mathbb{Z}, \left(\bigoplus_{n=0}^{\infty} M^{\otimes n}, \otimes\right), \Lambda n \in \mathbb{Z}_+ \,.\, M^{\otimes n}\right)$

$\texttt{tensorImbedding} :: \prod M \in R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} M^\otimes$

$\texttt{tensorImbedding}\,(m) = \iota_{M^\otimes}(m) := \Lambda n \in \mathbb{Z}_+ \,.\, \texttt{if}\ n == 1\ \texttt{then}$

$\texttt{tensorAlgebraMap} :: \prod R \in \mathsf{ANN} \,.\, \prod X, Y \in R\text{-}\mathsf{MOD} \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y \to X^\otimes \xrightarrow{R\text{-}\mathsf{ALGE}} Y^\otimes$

$\texttt{tensorAlgebraMap}\,(f) = f^\otimes := \text{Ɑ}\texttt{TensorAlgebra}(M^\otimes, \iota_{M^\otimes})(f)$

$\texttt{tensorAlgebraFunctor} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Covariant}(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{ALGE})$

$\texttt{tensorAlgebraFunctor}\,() := (\texttt{tensorAlgebra}, \texttt{tensorAlgebraMap})$

$\texttt{TensorAlgebraTheorem} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, M^{\otimes} : \texttt{TensorAlgebra}(M)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : R\text{-}\mathsf{ALGE},$

$\texttt{Assume}\ \varphi : M \xrightarrow{R\text{-}\mathsf{MOD}} A,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ m : n \to M,$

$$f\left(\bigotimes_{i=1}^{n} m_i\right) := \prod_{i=1}^{n} \varphi(m_i) : M;$$

$\rightsquigarrow f := \mathcal{I} M^{\otimes} : M^{\otimes} \xrightarrow{R\text{-}\mathsf{MOD}} A,$

$[1] := \mathcal{I}\texttt{tensorProduct}\mathcal{O}f : (f : M^{\otimes} \xrightarrow{R\text{-}\mathsf{MOD}} A),$

$[2] := \mathcal{I}\texttt{tensorProduct}\mathcal{O}f : (f : M^{\otimes} \xrightarrow{R\text{-}\mathsf{ALGE}} A),$

$[3] := \mathcal{O}f\mathcal{I}\iota_{M^{\otimes}} : \iota_{M^{\otimes}} f = \varphi,$

$\texttt{Assume}\ f' : M^{\otimes} \xrightarrow{R\text{-}\mathsf{ALGE}} A,$

$\texttt{Assume}\ [4] : \varphi = \iota_{M^{\otimes}} f',$

$[f'.*] := \mathcal{O}f : f = f';$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{TensorProduct} : \Big(M^{\otimes} : \texttt{TensorAlgebra}\Big),$

$\square$

$\texttt{TensorAlgebraKer} :: \forall X, Y \in R\text{-}\mathsf{MOD} \,.\, \forall f : X \xrightarrow{R\text{-}\mathsf{MOD}} Y \,.\, \ker f^{\otimes} = \langle \ker f \rangle_{Y^{\otimes}}$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TensorMapSurjective} :: \forall X, Y \in R\text{-}\mathsf{MOD} \,.\, \forall f : X \xrightarrow{R\text{-}\mathsf{MOD}} Y \,.\, \big(f : X \twoheadrightarrow Y\big) \to (f^{\otimes} : X^{\otimes} \twoheadrightarrow Y^{\otimes})$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FromBasisToTensorMapExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall A \in R\text{-}\mathsf{ALGE} \,.\, \forall E : \texttt{Basis}(M) \,.\ .$

$\quad .\ \forall f : E \to A \,.\, \exists! f' : M^{\otimes} \xrightarrow{R\text{-}\mathsf{ALGE}} A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TensorAlgebraBasis} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall E : \texttt{Basis}(M) \,.$

$$.\ \left\{ \bigotimes_{i=1}^{n} e_i \middle| n \in \mathbb{N}, e : n \to E \right\} : \texttt{Basis}(M^{\otimes})$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FreeTensorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, M^{\otimes} : \texttt{FreeAssociativeAlgebra}(R)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{FlatTensorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{Flat}(R) \,.\, M^{\otimes} : \texttt{Flat}(R)$

$\texttt{Proof} \ =$

$\Big(Q, [1]\Big) := \mathcal{Q}\texttt{Flat}(R) : \sum Q \in R\text{-}\mathsf{MOD} \,.\, Q \oplus M : \texttt{FreeModule}(R),$

$[2] := \texttt{FreeTensorAlgebra}[1] : \Big((Q \oplus M)^{\otimes} : \texttt{FreeModule}(R)\Big),$

$\alpha := \Lambda m \in M \,.\, (0, m) : M \xrightarrow{R\text{-}\mathsf{MOD}} Q \oplus M,$

$\beta := \Lambda (q, m) \in Q \oplus M \,.\, m : Q \oplus M \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$[3] := \mathcal{O}\alpha\beta : \alpha^{\otimes}\beta^{\otimes} = \mathrm{id},$

$[4] := \texttt{IsomorphismDecompTHM}[3] : (Q \oplus M)^{\otimes} \cong_{R\text{-}\mathsf{MOD}} \ker \beta^{\otimes} \oplus M^{\otimes},$

$[*] := \mathcal{Q}^{-1}\texttt{Flat}[2][4] : \big(M^{\otimes} : \texttt{Flat}(R)\big);$

$\square$

$\texttt{CovariantExtensionOfTensorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall S \in \mathsf{ANN} \,.\, \forall \omega : R \xrightarrow{\mathsf{RING}} S \,.$

$\quad . \, \forall M \in R\text{-}\mathsf{MOD} \,.\, (M \otimes_{\omega} S)^{\otimes_S} \cong_{S\text{-}\mathsf{ALGE}} M^{\otimes_R} \otimes_{\omega} S$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{TensorAlgebraIsPolynomial} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, M^{\otimes} : \texttt{PolynomialGradedAlgebra}(R)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DerivationTensorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \in M^* \,.$

$\quad . \, \exists! D \in \mathcal{D}(M^{\otimes}) \,.\, D_{|M} = f$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SkewDerivationTensorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \in M^* \,.$

$\quad . \, \exists! D \in \widetilde{\mathcal{D}}(M^{\otimes}) \,.\, D_{|M} = f$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{PseudoCyclic} :: \prod R \in \mathsf{ANN} \,.\, ?R\text{-}\mathsf{MOD}$

$M : \texttt{PseudoCyclic} \iff \forall N \subset_{R\text{-}\mathsf{MOD}} M \,.\, N : \texttt{FinitelyGeneratedModule}(R) \Rightarrow \exists Z \subset_{R\text{-}\mathsf{MOD}} M \,.\, Z : \texttt{CyclicMo}$

$\texttt{CommutativeTensorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{PseudoCyclic}(R) \,.\, M^{\otimes} \in R\text{-}\mathsf{CALGE}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ n, m : \mathbb{Z}_+,$

$\texttt{Assume}\ x : n \to M,$

$\texttt{Assume}\ y : m \to M,$

$N := \Big\langle \{x_i | i \in n\} \cup \{y_i | i \in m\} \Big\rangle_M : \texttt{Submodule}(M),$

$[1] := \mathcal{O}N : \Big(N : \texttt{FinitelyGeneratedModule}(R)\Big),$

$\big(Z, [2]\big) := \mathcal{C}\texttt{PseusoCyclic}(M)(N, [1]) : \sum Z \subset_{R\text{-}\mathsf{MOD}} M \,.\, Z : \texttt{CyclicModule}(R) \ \&\ N \subset_{R\text{-}\mathsf{MOD}} Z,$

$\big(z, [3]\big) := \mathcal{C}\texttt{Cyclic}(Z) : \sum z \in M \,.\, Z = Rz,$

$\big(\alpha, \beta, [4]\big) := [3]\mathcal{O}N\mathcal{C}x\mathcal{C}y : \sum \alpha : n \to R \,.\, \sum \beta : m \to R \,.\, x = \alpha z \ \&\ y = \beta z,$

$[n.*] := [4]\mathcal{C}\texttt{MultiHomogen}^{2(n+m)}[4] :$

$$: \bigotimes_{i=1}^{n} x_i \otimes \bigotimes_{j=1}^{m} y_j = \bigotimes_{i=1}^{n} \alpha_i z \otimes \bigotimes_{j=1}^{m} \beta_j z = \prod_{i=1}^{n} \alpha_i \prod_{j=1}^{m} \beta_j \bigotimes_{i=1}^{n+m} z = \bigotimes_{j=1}^{m} \beta_j z \otimes \bigotimes_{i=1}^{n} \alpha_i z = \bigotimes_{j=1}^{m} y_j \otimes \bigotimes_{j=1}^{n} x_j;$$

$\rightsquigarrow [1] := \mathcal{C}M^{\otimes} : \forall t, s \in M^{\otimes} \,.\, ts = st,$

$[*] := \mathcal{C}R\text{-}\mathsf{CALGE}[1] : M^{\otimes} \in R\text{-}\mathsf{CALGE};$

$\square$

$\texttt{TensorAlgebraOfIntegralDomain} :: \forall R : \texttt{IntegralDomain} \,.\, \forall M : \texttt{FreeModule}(R) \,.\, M^{\otimes} : \texttt{IntegralDomain}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ n, m : \mathbb{Z}_+,$

$\texttt{Assume}\ x : M^{\otimes}_n,$

$\texttt{Assume}\ y : M^{\otimes}_m,$

$E := \texttt{FreeHasBasis}(M) : \texttt{Basis}(M),$

$$(\alpha, [1]) := \texttt{TensorAlgebraBasis}(M)(x) : \sum \alpha : (n \to E) \to R \,.\, x = \sum_{e : n \to E} \alpha_e \bigotimes_{i=1}^{n} e_i,$$

$$(\beta, [2]) := \texttt{TensorAlgebraBasis}(M)(y) : \sum \alpha : (m \to E) \to R \,.\, x = \sum_{e : m \to E} \beta_e \bigotimes_{i=1}^{m} e_i,$$

$$[3] := \mathcal{C}\texttt{tensorAlgebra}[1][2] : x \otimes y = \sum_{e : m+n \to E} \alpha_{e_{|n}} \beta_{e_{+n}} \bigotimes_{i=1}^{n+m} e_i,$$

$[n.*] := \mathcal{C}\texttt{IntegralDomain}(R)[3] : x \otimes y = 0 \Rightarrow x = 0 | y = 0;$

$\rightsquigarrow [4] := \mathcal{C}M^{\otimes} : \forall x, y \in M^{\otimes} \,.\, x \otimes y = 0 \Rightarrow x = 0 | y = 0,$

$[*] := \mathcal{C}^{-1}\texttt{IntegralDomain}[4] : \Big(M^{\otimes} : \texttt{IntegralDomain}\Big);$

$\square$

$\texttt{TensorAlgebraQuotient} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall I : \texttt{Ideal}(R) \,.\, \frac{M^{\otimes}}{(IM)^{\otimes}} \cong_{\frac{R}{I}\text{-}\mathsf{ALGE}} \left(\frac{M}{IM}\right)^{\otimes \frac{R}{I}}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [m] : \frac{M}{IM},$

$\varphi[m] := [m]_{\frac{M^{\otimes}}{(IM)^{\otimes}}} : \frac{M^{\otimes}}{(IM)^{\otimes}};$

$\rightsquigarrow \varphi := I(\rightarrow) : \frac{M}{IM} \xrightarrow{\frac{R}{I}\text{-}\mathsf{MOD}} \frac{M^{\otimes}}{(IM)^{\otimes}},$

$(f, [1]) := \mathcal{I}\texttt{TensorAlgebra}\left(\frac{M}{IM}\right)(\varphi) : \sum f : \left(\frac{M}{IM}\right)^{\otimes} \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} \frac{M^{\otimes}}{(IM)^{\otimes}} \,.\, \varphi = \iota f,$

$\psi := \phi^{|\left(\frac{M^{\otimes}}{(IM)^{\otimes}}\right)_1} : \frac{M}{IM} \xrightarrow{\frac{R}{I}\text{-}\mathsf{MOD}} \frac{M^{\otimes}}{(IM)^{\otimes}},$

$[2] := \mathcal{I}\psi\mathcal{I}\phi : \psi : \texttt{Surjective},$

$[3] := \texttt{SurjectiveTensorExtension}[2] : (f : \texttt{Surjective}),$

$\texttt{Assume}\ t : \texttt{Homogeneous}\left(\frac{M}{IM}\right)^{\otimes},$

$\texttt{Assume}\ [4] : f(t) = 0,$

$d := \deg t : \mathbb{Z}_+,$

$(k, m, [5]) := \mathcal{I}t : \sum k \in \mathbb{N} \,.\, m : k \to d \to M \,.t = \sum_{i=1}^{k} \bigotimes_{j=1}^{d} [m_{i,j}],$

$(m', [6]) := [4][5] : \sum m' : k \to d \to (IM)^{\otimes} \,.\, \sum_{i=1}^{k} \bigotimes_{j=1}^{d} \left(m_{i,j} + m'_{i,j}\right) \in (IM)^{\otimes},$

$[7] := \texttt{MultiAdditive}(\otimes)[6] : \forall i \in k \,.\, \forall j \in d \,.\, m_{i,j} \in IM,$

$[t.*] := [5][7] : t = 0;$

$\rightsquigarrow [*] := \mathcal{I}\texttt{Isomorphic}\mathcal{I}\texttt{Iso}[3]\texttt{ZeroKernelTHM} : \texttt{This};$

$\square$

$\texttt{DerivationTensorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.\, \forall f \in M \to M^{\otimes}_n \,.$

$\,.\, \exists! D \in \mathcal{D}^n(M^{\otimes}) \,.\, D_{|M} = f$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{SkewDerivationTensorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \in M^{\to} M^{\otimes}_n \,.$

$\,.\, \exists! D \in \widetilde{\mathcal{D}}^n(M^{\otimes}) \,.\, D_{|M} = f$

$\texttt{Proof} \ =$

$\ldots$

$\square$

## 2.2 Mixed Tensor Algebra

$$\texttt{mixedTensorAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{ALGE}(\mathbb{Z}_+^2)$$
$$\texttt{mixedTensorAlgebra}\,(M) = M^{\otimes,*} := M^{\otimes} \otimes \left(M^*\right)^{\otimes}$$

$$\texttt{totalDegree} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \texttt{Homogeneous}(M^{\otimes,*}) \to \mathbb{Z}_+$$
$$\texttt{totalDegree}\,(h) = \overline{\deg} h := i + j \quad \texttt{where} \quad (i,j) = \deg h$$

$$\texttt{DecomposableTensor} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?M^{\otimes,*}$$
$$t : \texttt{DecomposableTensor} \iff \exists p, q \in \mathbb{Z}_+ : \exists m : M^p$$

$$\texttt{contraction} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \prod p, q \in \mathbb{N} \,.\, p \to q \to M^{\otimes,*}_{p,q} \to M^{\otimes,*}_{p-1,q-1}$$
$$\texttt{contraction}\,(k,l,t) = \mathrm{tr}_{k,l}\, t := \texttt{tensorize}\left(\Lambda v \in M^p \,.\, \Lambda f \in (M^*)^q \,.\, f_j(v_i) \bigotimes_{i=1}^{p-1} \widehat{v}_{k,i} \otimes \bigotimes_{j=1}^{1-1} \widehat{f}_{l,j}\right)$$

$$\texttt{BasisContraction} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall E : \texttt{Basis}(M) \,.\, \forall p, q \in \mathbb{N} \,.$$
$$.\, \forall \sum_{e \in E^p} \sum_{f \in (E^*)^q} \alpha_{e,f} \bigotimes_{i=1}^{p} e_i \otimes \bigotimes_{i=1}^{q} f_i \,.\, \forall i \in p \,.\, \forall j \in q \,.$$
$$.\, \mathrm{tr}_{i,j}\, \alpha_{t,s} \bigotimes_{i=1}^{p} e_{t_i} \otimes \bigotimes_{i=1}^{q} e^*_{s_i} = \sum_{e' \in E^{p-1}} \sum_{f' \in (E^*)^{q-1}} \left(\sum e \in E_p \,.\, \sum f \in E^*_q \,.\, e^*_i = f_j \;\&\; \hat{e}_i = e' \;\&\; \hat{f}_j = f' \,.\, \alpha_{e,f}\right)$$
$$\bigotimes_{i=1}^{p} e'_i \otimes \bigotimes_{i=1}^{q} f'_i$$

Proof $=$

...

□

$$\texttt{mixedTensorMap} :: \prod R \in \mathsf{ANN} \,.\, \prod M, N \in R\text{-}\mathsf{MOD} \,.\, \left(M \xleftrightarrow{R\text{-}\mathsf{MOD}} N\right) \to \left(M^{\otimes,*} \xrightarrow{R\text{-}\mathsf{ALGE}} N^{\otimes,*}\right)$$
$$\texttt{mixedTensorMap}\,(T) = T^{\otimes,*} := T^{\otimes} \otimes \left(T_*^{-1}\right)^{\otimes}$$

$$\texttt{MixedTensorFunctor} :: \forall R \in \mathsf{ANN} \,.\, (\texttt{mixedTensorAlgebra}, \texttt{mixedTensorMap}) :$$
$$: \texttt{Covariant}\Big(\texttt{groupoid}(R\text{-}\mathsf{MOD}), R\text{-}\mathsf{ALGE}\Big)(\mathbb{Z}_+^2)$$

Proof $=$

...

□

$$\texttt{Tensorial} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \prod a, b, p, q \in \mathbb{Z}_+ \,.\, ?M^{\otimes,*}_{a,b} \xrightarrow{R\text{-}\mathsf{MOD}} M^{\otimes,*}_{p,q}$$
$$T : \texttt{Tensorial} \iff \forall A \in \mathrm{Aut}_{R\text{-}\mathsf{MOD}}(M) \,.\, TM^{\otimes,*} = M^{\otimes,*}T$$

## 2.3 Exterior Algebra

$$\texttt{alternator} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M^{\otimes} \xrightarrow{R\text{-}\mathsf{MOD}} M^{\otimes}$$

$$\texttt{alternator}\,(t) = \wedge(t) := \text{Ɑ}\texttt{TensorAlgebra}\Lambda k \in \mathbb{N} \,.\, \lambda m \in M^k \,.\, \sum_{\sigma \in S^n} (-1)^{\sigma} \bigotimes_{i=1}^{k} m_{\sigma(i)}$$

$$\texttt{exteriorPower} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to \mathbb{Z}_+ \to R\text{-}\mathsf{MOD}$$

$$\texttt{exteriorPower}\,(M, n) = M^{\wedge n} := \frac{M_n^{\otimes}}{\ker \wedge}$$

$$\texttt{exteriorAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to \texttt{SkewAlgebra}R(\mathbb{Z}_+)$$

$$\texttt{exteriorAlgebra}\,(M) = M^{\wedge} := \frac{M^{\otimes}}{M^{\otimes}\{a \otimes a\}M^{\otimes}}$$

$$\texttt{ExteriorAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?\sum E : \texttt{SkewAlgebra}(R) \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} E$$

$$(E, \iota) : \texttt{ExtoriarAlgebra} \iff \forall A : \texttt{SkewAlgebra}(R) \,.\, \forall \varphi : M R\text{-}\mathsf{MOD} A \,.\, \exists! f : T \xrightarrow{R\text{-}\mathsf{ALGE}} A : \varphi = \iota f$$

$$\texttt{IsomorphicExteriorAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (E, \iota), (E', \iota') : \texttt{ExteriorAlgebra}(M) \,.$$
$$.\, T \cong_{R\text{-}\mathsf{ALGE}} T'$$

Proof $=$

...

$\square$

$$\texttt{ExteriorAlgebraUniversalInjective} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (E, \iota) : \texttt{TensorAlgebra}(M) \,.$$
$$.\, \iota : M \hookrightarrow T$$

Proof $=$

...

$\square$

$$\texttt{exteriorImbedding} :: \prod M \in R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} M^{\wedge}$$

$$\texttt{exteriorImbedding}\,(m) = \iota_{M^{\wedge}}(m) := \Lambda n \in \mathbb{Z}_+ \,.\, \textbf{if } n == 1 \textbf{ then } m \textbf{ else } 0$$

$$\texttt{ExteriorAlgebraTheorem} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, M^{\wedge} : \texttt{ExteriorAlgebra}(M)$$

Proof $=$

...

$\square$

$$\texttt{exteriorAlgebraMap} :: \prod R \in \mathsf{ANN} \,.\, \prod X, Y \in R\text{-}\mathsf{MOD} \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y \to X^{\wedge} \xrightarrow{R\text{-}\mathsf{ALGE}} Y^{\wedge}$$

$$\texttt{exteriorAlgebraMap}\,(f) = f^{\wedge} := \text{Ɑ}\texttt{ExteriorAlgebra}(M^{\wedge}, \iota_{M^{\wedge}})(f)$$

$$\texttt{tensorAlgebraFunctor} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Covariant}(R\text{-}\mathsf{MOD}, R\text{-}\mathsf{ALGE})$$

$$\texttt{tensorAlgebraFunctor}\,() := (\texttt{exteriorAlgebra}, \texttt{exteriorAlgebraMap})$$

$\texttt{exteriorProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}\Big(M^\wedge, M^\wedge\Big)$

$\texttt{exteriorProduct}\,(t,s) = t \wedge s := \wedge(t \otimes s)$

$\texttt{TensorAlgebraKer} :: \forall X, Y \in R\text{-}\mathsf{MOD} \,.\, \forall f : X \xrightarrow{R\text{-}\mathsf{MOD}} Y \,.\, \ker f^\wedge = \langle \ker f \rangle_{Y^\wedge}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{TensorMapSurjective} :: \forall X, Y \in R\text{-}\mathsf{MOD} \,.\, \forall f : X \xrightarrow{R\text{-}\mathsf{MOD}} Y \,.\, \big(f : X \twoheadrightarrow Y\big) \to (f^\otimes : X^\wedge \twoheadrightarrow Y^\wedge)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ExteriorAlgebraBasis} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall E : \texttt{Basis}(M) \,.$

$\quad . \left\{ \bigwedge_{i=1}^{n} e_i \middle| n \in \mathbb{N}, e : \texttt{Injective} \,\&\, \texttt{Ascending}\Big(n, (E, o)\Big) \right\} : \texttt{Basis}(M^\otimes)$

$\quad \texttt{where} \quad o = \daleth\texttt{WellOrderingPrinciple}(E)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{FreeExteriorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, M^\wedge : \texttt{FreeModule}(R)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ExteriorHPAlgebraRank} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, \forall n, k \in \mathbb{Z}_+ \,.$

$\quad . \,\forall [0] : \operatorname{rank} M = n \,.\, \operatorname{rank} M^\wedge_k = C^k_n$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ExteriorAlgebraRank} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, \forall n \in \mathbb{Z}_+ \,.$

$\quad . \,\forall [0] : \operatorname{rank} M = n \,.\, \operatorname{rank} M^\wedge = 2^n$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ExteriorProductDirectSum} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall M : n \to R\text{-}\mathsf{MOD} \,.$

$$\left(\bigoplus_{i=1}^{n} M_i\right)^{\wedge} \cong_{R\text{-}\mathsf{ALGE}(\mathbb{Z}_+)} \widetilde{\bigotimes}_{i=1}^{n} M_i^{\wedge}$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A : R\text{-}\mathsf{MOD},$

$\texttt{Assume}\ B : B\text{-}\mathsf{MOD},$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ b : B,$

$\varphi(a,b) := a \otimes 1 + 1 \otimes b : A^{\wedge}\widetilde{\otimes}B^{\wedge};$

$\rightsquigarrow \varphi := \mathcal{I}\texttt{ExteriorAlgebra}(A \oplus B) : (A \oplus B)^{\wedge} \xrightarrow{R\text{-}\mathsf{ALGE}} A^{\wedge}\widetilde{\otimes}B^{\wedge},$

$T := \pi_A^{\wedge} \wedge \pi_B^{\wedge} : \mathcal{L}\left(A^{\wedge}, B^{\wedge}; (A \oplus B)^{\wedge}\right),$

$\psi := \texttt{tensorize}(T) : A^{\wedge}\widetilde{\otimes}B^{\wedge} \xrightarrow{R\text{-}\mathsf{MOD}} (A \oplus B)^{\wedge},$

$\texttt{Assume}\ n, m : \mathbb{Z}_+,$

$\texttt{Assume}\ t : n \to \texttt{Homogeneous}(A^{\wedge}),$

$\texttt{Assume}\ t' : m \to \texttt{Homogeneous}(A^{\wedge}),$

$\texttt{Assume}\ s : n \to \texttt{Homogeneous}(B^{\wedge}),$

$\texttt{Assume}\ s' : m \to \texttt{Homogeneous}(B^{\wedge}),$

$k := i, j \mapsto \deg s_i \deg t'_j : n \times m \to \mathbb{Z},$

$[1] := \mathcal{I}\texttt{skewTensorProduct}\mathcal{O}\psi : \psi\Big((t_i \otimes s_i)(t'_j \otimes s'_j)\Big) = \psi((-1)^{k_{i,j}} t_i \wedge t'_j \otimes s_i \wedge s'_j) =$

$$= (-1)^{k_{i,j}} \bigwedge_{l=1}^{\deg t_i} (t_{i,l}, 0) \wedge \bigwedge_{l=1}^{\deg t'_j} (t'_{j,l}, 0) \wedge \bigwedge_{l=1}^{\deg s_i} (0, s_{i,l}) \wedge \bigwedge_{l=1}^{\deg s'_j} (0, s'_{j,l}),$$

$[2] := \mathcal{O}\psi\mathcal{I}\texttt{exteriorProduct} : \psi(t_i \otimes s_i)\psi(t'_j \otimes s'_j) = \bigwedge_{l=1}^{\deg t_i} (t_{i,l}, 0) \wedge \bigwedge_{l=1}^{\deg s_i} (0, s_{i,l}) \wedge \bigwedge_{l=1}^{\deg t'_j} (t'_{j,l}, 0) \wedge \bigwedge_{l=1}^{\deg s'_j} (0, s'_{j,l}) =$

$$= (-1)^{k_{i,j}} \bigwedge_{l=1}^{\deg t_i} (t_{i,l}, 0) \wedge \bigwedge_{l=1}^{\deg t'_j} (t'_{j,l}, 0) \wedge \bigwedge_{l=1}^{\deg s_i} (0, s_{i,l}) \wedge \bigwedge_{l=1}^{\deg s'_j} (0, s'_{j,l}),$$

$[*] := [1][2] : \psi\Big((t_i \otimes s_i)(t_j \otimes s_j)\Big) = \psi(t_i \otimes s_i)\psi(t'_j \otimes s'_j);$

$\rightsquigarrow [1] := \mathcal{I}R\text{-}\mathsf{ALGE} : \psi : A^{\wedge}\widetilde{\otimes}B^{\wedge} \xrightarrow{R\text{-}\mathsf{ALGE}} (A \oplus B)^{\wedge},$

$\texttt{Assume}\ (a,b) : A \oplus B,$

$[(a,b).*] := \mathcal{O}\varphi\mathcal{O}\psi\mathcal{I}\texttt{directSum} : \psi\varphi(a,b) = \psi(a \otimes 1 + 1 \otimes b) = (a,0) + (0,b) = (a,b),$

$(a,b).* := \mathcal{O}\psi\mathcal{O}\varphi\mathcal{I}\texttt{tensorProduct} : \varphi\psi(a \otimes 1) = \varphi(a,0) = a \otimes 1 + 1 \otimes 0 = a \otimes 1,$

$(a,b), * := \mathcal{O}\psi\mathcal{O}\varphi\mathcal{I}\texttt{tensorProduct} : \varphi\psi(1 \otimes b) = \varphi(0,b) = 0 \otimes a + 1 \otimes b = 1 \otimes b;$

$\rightsquigarrow [2] := \mathcal{I}\texttt{exteriorAlgebra}\mathcal{I}\texttt{tensorProduct} : \varphi\psi = \mathrm{id}\ \ \&\ \psi\varphi = \mathrm{id},$

$[*] := \mathcal{I}\texttt{Isomotphic}[2] : \texttt{This};$

$\square$

$\texttt{CovariantExtensionOfExteriorAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall S \in \mathsf{ANN} \,.\, \forall \omega : R \xrightarrow{\mathsf{RING}} S \,.$

$\quad . \forall M \in R\text{-}\mathsf{MOD} \,.\, (M \otimes_\omega S)^{\wedge_S} \cong_{S\text{-}\mathsf{ALGE}} M^{\wedge_R} \otimes_\omega S$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ExteriorAlgebraIsPolynomial} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, M^\wedge : \texttt{PolynomialGradedAlgebra}(R)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{SkewDerivationExteriorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \in M^* \,.$

$\quad . \exists! D \in \widetilde{\mathcal{D}}(M^\wedge) \,.\, D_{|M} = f$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{skewExtension} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M^* \xrightarrow{R\text{-}\mathsf{MOD}} \widetilde{D}(M^\wedge)$

$\texttt{skewExtension}\,(f) = D_f := \text{\textit{Ɑ}}\texttt{SkewDerivarionExteriorAlgebraExtension}(f)$

$\texttt{skewBilinearExtension} :: \prod R \in \mathsf{ANN} \,.\, \prod M, U \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(U, M; R) \to U \to \widetilde{D}(M^\wedge)$

$\texttt{skewBilinearExtension}\,(T, u) = D_{T,u} := D_f \quad \texttt{where} \quad f = \Lambda m \in M \,.\, T(u, m)$

$\texttt{skewBilinearExteriorAsComp} :: \forall R \in \mathsf{ANN} \,.\, \forall M, U \in R\text{-}\mathsf{MOD} \,.\, \forall T \in \mathcal{L}(U, M; R) \,.\, \forall n \in \mathbb{N} \,.\, \forall u : n \to U \,.$

$$. D_T^\wedge \bigwedge_{i=1}^n u_i = \prod_{i=0}^{n-1} D_{T, u_{n-i}}$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{skewBilinearExteriorAppByDet} :: \forall R \in \mathsf{ANN} \,.\, \forall M, U \in R\text{-}\mathsf{MOD} \,.\, \forall T \in \mathcal{L}(U, M; R) \,.\, \forall n \in \mathbb{N} \,.\, \forall m \in n \,.$

$$. \forall u : m \to U \,.\, \forall v : n \to M \,.\, D_T^\wedge \left( \bigwedge_{i=1}^m u_i \right) \left( \bigwedge_{i=1}^n v_i \right) = (-1)^m \sum_{k : m \to n} (-1)^{|k|} \det \Lambda i, j \in m \,.\, T(u_i, v_{k_i})$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{skewBilinearExteriorAppByDet2} :: \forall R \in \mathsf{ANN} \,.\, \forall M, U \in R\text{-}\mathsf{MOD} \,.\, \forall T \in \mathcal{L}(U, M; R) \,.\, \forall n \in \mathbb{N} \,.$

$$. \forall u : n \to U \,.\, \forall v : n \to M \,.\, D_T^\wedge \left( \bigwedge_{i=1}^n u_i \right) \left( \bigwedge_{i=1}^n v_i \right) = (-1)^{\frac{n(n+3)}{2}} \det \Lambda i, j \in n \,.\, T(u_i, v_i)$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{ExteriorAlgebraQuotient} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall I : \texttt{Ideal}(R) \,.\, \dfrac{M^\wedge}{(IM)^\wedge} \cong_{\frac{R}{I}\text{-}\mathsf{ALGE}} \left(\dfrac{M}{IM}\right)^{\wedge \frac{R}{I}}$

$\texttt{Proof} =$

$\texttt{SkewDerivationExteriorAlgebraExtension} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.\, \forall f \in M^{\wedge n *} \,.$

$\,.\, \exists! D \in \widetilde{\mathcal{D}}^n(M^\wedge) \,.\, D_{|M^{\wedge n}} = f$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Decomposable} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?M^\wedge$

$$x : \texttt{Decomposable} \iff \exists p \in \mathbb{N} : \exists m : p \to M \,.\, x = \bigwedge_{i=1}^{p} m_i$$

## 2.4 Determinant Identities

$\texttt{DeterminantTHM} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS}\, \forall T \in \mathrm{End}_{k\text{-VS}}(V) . \forall e : \texttt{Basis}(V, \dim V) .$

$$. T^\wedge\left(\bigwedge_{i=1}^{n} e_i\right) = \det T \bigwedge_{i=1}^{n} e_i$$

$\texttt{Proof} =$

...

$\square$

$\texttt{complementaryIncreaingSequance} :: \prod n \in \mathbb{N} . \texttt{Increasing}(n, 2n) \to \texttt{Increasing}(n, 2n)$

$\texttt{complementatyIncreasingSequance}(I) = I^{\complement} := \Lambda i \in n . \texttt{ if } i = 1 \texttt{ then } u(1) \texttt{ else } u\Big(I^{\complement}(n-1)+1\Big)$

$\quad \texttt{where} \quad u = \Lambda i \in 2n . \texttt{ if } i \in \mathrm{Im}\, I \texttt{ then } u(i+1) \texttt{ else } i$

$\texttt{IndependentIncreasing} :: \prod n \in \mathbb{N} . ?\texttt{Increasing}^2(n, 2n)$

$I, J : \texttt{IndependentIncreasing} \iff I \perp J \iff \mathrm{Im}\, I \cap \mathrm{Im}\, J = \emptyset$

$\texttt{independentAsPermutation} :: \prod n \in \mathbb{N} . \texttt{IndependentIncreasing}(n) \to S_{2n}$

$\texttt{independentAsPermutation}(I, J) = I \oplus J := \Lambda i \in 2n . \texttt{ if } i \le n \texttt{ then } I(n) \texttt{ else } J(i-n)$

$\texttt{IndependentComplements} :: \forall n \in \mathbb{N} . \forall I : n \uparrow 2n . (I, I^{\complement}) : \texttt{IndependentIncreasing}(n)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LaplaceDeterminantIdentity} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall A \in R^{2n \times n} .$

$$. \sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} \det(A_{I_i, j})_{i,j=1}^{n} \det(A_{I^{\complement}_i, j})_{i,j=1}^{n} = 0$$

$\texttt{Proof} =$

$$K := \sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} : \mathbb{Z},$$

$[1] := \sigma R\text{-MOD}(M^\wedge) \sigma M^\wedge\ \texttt{determinantTHM} \sigma^{-1} \texttt{ExteriorAlgebraFunctor}$

$\quad \sigma R\text{-ALGE}(M^\wedge, M^\wedge)(A \oplus A)^\wedge_e \sigma R\text{-MOD}(M^\wedge, M^\wedge) \mathcal{O}^{-1} K\ \texttt{DeterminantTHM SingularDeterminantTHM} :$

$$: \left(\sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} \det(A_{I_i, j})_{i,j=1}^{n} \det(A_{I^{\complement}_i, j})_{i,j=1}^{n}\right) \bigwedge_{i=1}^{2n} e_i = \sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} \det(A_{I_i, j})_{i,j=1}^{n} \det(A_{I^{\complement}_i, j})_{i,j=1}^{n} \bigwedge_{i=1}^{2n} e_i =$$

$$= \sum_{I : n \uparrow 2n} \det(A_{I_i, j})_{i,j=1}^{n} \bigwedge_{i=1}^{n} e_{I_i} \wedge \det(A_{I^{\complement}_i, j})_{i,j=1}^{n} \bigwedge_{i=1}^{n} e_{I^{\complement}_i} = \sum_{I : n \uparrow 2n} A^\wedge_{e_I} \bigwedge_{i=1}^{n} e_{I_i} \wedge A^\wedge_{e_{I^{\complement}}} \bigwedge_{i=1}^{n} e_{I^{\complement}_i} =$$

$$= \sum_{I : n \uparrow 2n} (A \oplus A)^\wedge_e \bigwedge_{i=1}^{n} e_{I_i} \wedge (A \oplus A)^\wedge_e \bigwedge_{i=1}^{n} e_{I^{\complement}_i} = \sum_{I : n \uparrow 2n} (A \oplus A)^\wedge_e \left(\bigwedge_{i=1}^{n} e_{I_i} \wedge \bigwedge_{i=1}^{n} e_{I^{\complement}_i}\right) =$$

$$= (A \oplus A)^\wedge_e \left(K \bigwedge_{i=1}^{2n} e_i\right) = \det(A \oplus A) K \bigwedge_{i=1}^{2n} e_i = 0,$$

$$[*] := \sigma R\text{-MOD}(M^\wedge)[1] : \sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} \det(A_{I_i, j})_{i,j=1}^{n} \det(A_{I^{\complement}_i, j})_{i,j=1}^{n} = 0;$$

$\square$

$\texttt{IncreasingSwapLemma} :: \forall n : \texttt{Even} \,.\, \forall I : n \uparrow 2n \,.\, (-1)^{I \oplus I^{\complement}} = (-1)^{I^{\complement} \oplus I}$

$\texttt{Proof} \; =$

$\sigma := \prod_{k=1}^{n} (k, n-k+1) : S_{2n},$

$[1] := \mathcal{D}\sigma\mathcal{I}\,\texttt{independentAsPermutation} : I \oplus I^{\complement}\sigma = I^{\complement} \oplus I,$

$[2] := \mathcal{I}\texttt{Even}\mathcal{I}\texttt{SignByTransposition}(\sigma) : (-1)^{\sigma} = 1,$

$[*] := \texttt{SignIsHomo}[1][2] : (-1)^{I^{\complement} \oplus I} = (-1)^{\sigma I \oplus I^{\complement}} = (-1)^{\sigma}(-1)^{I \oplus I^{\complement}} = (-1)^{I \oplus I^{\complement}};$

$\square$

$\texttt{SpecialLaplaceDeterminantIdentity} :: \forall R \in \mathsf{ANN} \,.\, \forall n : \texttt{Even} \,.\, \forall k \in n \,.\, \forall A \in R^{2n \times n} \,.$

$$. \sum_{I : n \uparrow 2n : k \in \operatorname{Im} I} (-1)^{I \oplus I^{\complement}} \det(A_{I_i,j})_{i,j=1}^{n} \det(A_{I_i^{\complement},j})_{i,j=1}^{n} = 0$$

$\texttt{Proof} \; =$

$(s, I) := \texttt{enumerate}\{I : n \uparrow 2n : k \in \operatorname{Im} I\} : \sum s \in \mathbb{N} \,.\, s \to n \uparrow 2n,$

$[1] := \texttt{LaplaceDeterminantIdentity}(R, n, A)\mathcal{D}^{-1}I\mathcal{I}I^{\complement}\texttt{IncreasingSwapLemma}(n, I) :$

$$: 0 = \sum_{I : n \uparrow 2n} (-1)^{I \oplus I^{\complement}} \det(A_{I_i,j})_{i,j=1}^{n} \det(A_{I_i^{\complement},j})_{i,j=1}^{n} =$$

$$= \sum_{t=1}^{s} (-1)^{I_t \oplus I_t^{\complement}} \det(A_{I_{t,i},j})_{i,j=1}^{n} \det(A_{I_{t,i}^{\complement},j})_{i,j=1}^{n} - \sum_{t=1}^{s} (-1)^{I_t^{\complement} \oplus I_t} \det(A_{I_{t,i},j})_{i,j=1}^{n} \det(A_{I_{t,i}^{\complement},j})_{i,j=1}^{n} =$$

$$= 2 \sum_{t=1}^{s} (-1)^{I_t \oplus I_t^{\complement}} \det(A_{I_{t,i},j})_{i,j=1}^{n} \det(A_{I_{t,i}^{\complement},j})_{i,j=1}^{n},$$

$[*] :=$ Argue for the ring $\mathbb{Z}[\mathbb{Z}_{+}^{n \times n}]$ and apply homomorphism $x_{i,j} \mapsto A_{i,j}[1] : \texttt{This};$

$\square$

$\texttt{DeterminantPermutation} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall I : n \uparrow 2n \,.\, \forall A : R^{2n \times 2n} \,.\; \det A_{I \oplus I^{\complement},\cdot} = (-1)^{I \oplus I^{\complement}} \det A$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{Antiminor} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{2n \times 2n} \to (n \uparrow 2n)^2 \to R$

$\texttt{Antiminor}\,(A, I, J) = \Gamma_{I,J}(A) := (-1)^{|I|+|J|} \det A_{I^{\complement},J^{\complement}}$

$\texttt{AntiminorSummationTHM} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A \in R^{2n \times 2n} \,.\, \forall k \in n \,.\, \forall I, J : (k \uparrow n) \,.$

$$. \sum_{K : n-k \uparrow n} \Gamma_{I,K}(A) \det A_{J,K} = \texttt{if } I == J \texttt{ then } \det A \texttt{ else } 0$$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{exteriorPowerOfTheMatrix} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod k \in n \,.\, R^{n \times n} \to R^{\frac{n!}{k!(n-k)!} \times \frac{n!}{k!(n-k)!}}$

$\texttt{exteriorPowerOfTheMatrix}\,(A) = A^{\wedge k} := \left(A_{e,e}^{\wedge k}\right)^{e^{\wedge k}, e^{\wedge k}}$

$\texttt{ExteriorDeterminantMult} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall A \in R^{n\times n} . \forall k \in n . \exists U \in R^{\frac{n!}{k!(n-k)!} \times \frac{n!}{k!(n-k)!}} .$

$. \; UA^{\wedge k} = \det AI = A^{\wedge k}U$

$\texttt{Proof} \; =$

$\texttt{Assume} \; I : k \uparrow n,$

$$[I.*] := \text{Ɑ}\texttt{ExteriorAlgebraFunctor} : A^{\wedge k}_{e,e} \bigwedge_{i=1}^{k} e_{I_i} = \sum_{J:k\to 2n} \det A_{I,J} \bigwedge_{i=1}^{k} e_{J_i};$$

$\rightsquigarrow [1] := \text{Ɑ}\texttt{exteriorPowerOfTheMatrix} : \forall I, J : k \uparrow n \; . \; A^{\wedge k}_{I,J} = \det A_{I,J},$

$U := \Lambda I, J : k \uparrow n \; . \; \Gamma_{I,J}(A) : R^{\frac{n!}{k!(n-k)!} \times \frac{n!}{k!(n-k)!}},$

$[*] := \texttt{AntiminorSummationTHM}\text{Ɔ}U[1] : \texttt{This};$

$\square$

$\texttt{IrreducibleDeterminant} :: \forall n \in \mathsf{ANN} \; . \; \det(x_{i,j})_{i,j=1}^{n} : \texttt{Irreducible} \; \mathbb{Z}\left[\mathbb{Z}_{+}^{n\times n}\right]$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{ExteriorPowerDeterminant} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall A \in R^{n\times n} . \forall k \in n .$

$. \; \det A^{\wedge k} = (\det A)^{\frac{(n-1)!}{(k-1)!(n-k)!}}$

$\texttt{Proof} \; =$

$X(x) := (x_{i,j})_{i,j=1}^{n} : \left(\mathbb{Z}\left[\mathbb{Z}_{+}^{n\times n}\right]\right)^{n\times n},$

$d(x) := \det X(x) : \mathbb{Z}\left[\mathbb{Z}_{+}^{n\times n}\right],$

$(U, [1]) := \texttt{ExteriorDeterminantMult}\Big(X(x)\Big)^{\wedge k} :$

$: \sum U(x) \in \left(\mathbb{Z}\left[\mathbb{Z}_{i=1}^{n\times n}\right]\right)^{\frac{n!}{(n-k)!k!} \times \frac{n!}{(n-k)!k!}} . \; U(x)\Big(X(x)\Big)^{\wedge k} = \det X(x)I,$

$[2] := \texttt{DetHomo}[1] : \det U(x) \det \Big(X(x)\Big)^{\wedge k} = \Big(\det X(x)\Big)^{\frac{n!}{k!(n-k)!}} = \Big(d(x)\Big)^{\frac{n!}{k!(n-k!)}},$

$[3] := \texttt{IrreducibleDeterminant}(n) : (d(x) : \texttt{Irreducible}),$

$(p, q, s, [4]) := [2][3] : \sum p, q \in \mathbb{Z}_{+} \; . \; \sum s \in \{1, -1\} \; .$

$$. \; \det U(x) = sd^{p}(x) \;\&\; \det X^{\wedge k}(x) = sd^{q}(x) \;\&\; p + q = \frac{n!}{k!(n-k!)},$$

$$[5] := \text{Use special dummy matrices } e_1 \mapsto \alpha e_1 \text{ and } E : p = \frac{(n-1)!}{(n-k)!(k-1)!} \;\&\; s = 1,$$

$[*] := \text{Map dummy variable to the enties of } A[4][5] : \texttt{This};$

$\square$

## 2.5 Interior Product

$\texttt{leftExteriorMult} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} M^{\wedge} \xrightarrow{R\text{-}\mathsf{MOD}} M^{\wedge}$

$\texttt{leftExteriorMult}\,(m,t) = L_m(t) := m \wedge t$

$\texttt{dualExteriorApp} :: \prod R \in \mathrm{Ann} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(M^{*\wedge}, M^{\wedge}; M^{\wedge})$

$\texttt{dualExteriorApp}\,(f,v) = f(v) := \daleth M^{\wedge} \daleth M^{*\wedge} \Lambda f : \texttt{Decomposable}(M^{*\wedge}) \,.\, \Lambda v : \texttt{Decomposable}(M^{\wedge}) \,.$

$$\cdot \sum_{I:\deg f \uparrow \deg v} \sum_{\sigma \in S_{\deg f}} (-1)^{I^{*}\sigma} \prod_{i=1}^{n} f_i(v_{I_{\sigma(i)}}) \bigwedge_{i=1}^{\deg v - \deg f} v_{I_i^{\complement}}$$

$\texttt{interiorProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M^{\wedge} \to ?\Big(M^{*\wedge} \xrightarrow{E\text{-}\mathsf{ALGE}} M^{*\wedge}\Big)$

$\texttt{interiorProduct}\,(a,f,v) = \mathbf{i}_a(f)(v) := f(a \wedge v)$

$\texttt{InteriorProductComposition} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall a,b \in M^{\wedge} \,.\, \mathbf{i}_{a\wedge b} = \mathbf{i}_b \circ \mathbf{i}_a$

$\texttt{Proof} \;=$

$\texttt{Assume}\; f : M^{*\wedge},$

$\texttt{Assume}\; v : M^{\wedge},$

$[f.*] := \daleth \mathbf{i} : \mathbf{i}_{a\wedge b}(f)(v) = f(a \wedge b \wedge v) = \mathbf{i}_a(f)(b \wedge v) = \mathbf{i}_b\Big(\mathbf{i}_a(f)\Big)(v);$

$\rightsquigarrow [*] := I(\to, =) : \mathbf{i}_a \mathbf{i}_b = \mathbf{i}_{a\wedge b};$

$\square$

$\texttt{NonAnnihilatingInteriorProductExists} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{VS} \,.\, \forall t \in V^{\star\wedge 2} \,.\, \exists x \in V : \mathbf{i}_x(t) \neq 0$

$\texttt{Proof} \;=$

$E := \texttt{FreeHasBasis}(V^{*}) : \texttt{Basis}(V^{*}),$

$o := \texttt{WellOrderingTHM}(E) : \texttt{WellOrderingTHM}(E),$

$(\alpha, [1]) := \texttt{ExteriorAlgebraBasis}(E)(t) : \alpha \in k^{\oplus E \times E} \,.\, t = \sum_{f \in E} \sum_{g >_o f} \alpha_{f,g} f \wedge g,$

$f := \min_o \{f \in E : \exists g \in E \,.\, \alpha_{f,g} \neq 0\} : E,$

$(e, [2]) := \texttt{CanonicalIsoTHM}(f) : \sum e \in V \,.\, e^{**} = f^{*},$

$[3] := \daleth \mathbf{i}_e(t)[1][2] : \mathbf{i}_e(t) = \sum_{g \in E} \alpha_{f,g} g,$

$[*] := \daleth \texttt{Basis} \beth f[3] : \mathbf{i}_e(t) \neq 0;$

$\square$

$$\mathtt{DecomposableByAnnihilator} :: \forall k : \mathtt{Field} \,.\, \forall V \in k\text{-}\mathsf{VS} \,.\, \forall t \in V^{\wedge 2} \,.\, \exists x \in V \,.$$

$$.\ \forall [0] x \neq 0 \,.\, \forall [00] v \wedge x = 0 \,.\, v : \mathtt{Decomposable}(V^{\wedge})$$

$\mathtt{Proof}\ =$

$$(E, [1]) := \mathtt{BasisExtension}(\{x\})[0] : \sum E \subset V \,.\, \{x\} \cap E : \mathtt{Basis}(V),$$

$$o := \mathtt{WellOrderingTHM}(E) : \mathtt{WellOrderingTHM}(E),$$

$$(\alpha, [2]) := \mathtt{ExteriorAlgebraBasis}(E)(t) : \alpha \in k^{\oplus \{x\} \cup E \times \{x\} \cup E} \,.\, t = \sum_{f \in E} \alpha_{x,g} x \wedge f + \sum_{g >_o f} \alpha_{f,g} f \wedge g,$$

$$[3] := [00][1][2] : 0 = v \wedge x = \sum_{f \in E} \sum_{g >_o f} \alpha_{f,g} f \wedge g \wedge x,$$

$$[4] := \mathtt{ExteriorAlgebraBasis} \daleth \mathtt{Basis} : \forall f, g \in E \,.\, \alpha_{f,g} = 0,$$

$$[5] := [4][3] : t = x \wedge \sum_{e \in E} \alpha_{x,e} e,$$

$$[i] := \daleth \mathtt{Decomposable}[5] : \mathtt{This};$$

$\square$

$$\mathtt{InteriorProductAntiderivation} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall a \in M \,.\, \mathbf{i}_a \in \widetilde{\mathcal{D}}(M^{*\wedge})$$

$\mathtt{Proof}\ =$

$$\mathtt{Assume}\ f, g : \mathtt{Decomposable}(M^{*\wedge}),$$

$$p := \deg f : \mathbb{Z}_+,$$

$$q := \deg g : \mathbb{Z}_+,$$

$$N := p + q : \mathbb{Z}_+,$$

$$\mathtt{Assume}\ v : \mathtt{Decomposable}(M^{\wedge}),$$

$$m := \deg v : \mathbb{Z}_+,$$

$$M := m + 1 : \mathbb{Z}_+,$$

$$[v, *] := \daleth \mathbf{i}_a(f \wedge g)(v) \daleth \mathtt{leftExteriorMult}(f \wedge g, a \wedge v) \daleth (-1)^{I^* \sigma} \daleth :$$

$$: \mathbf{i}_a(f \wedge g)(v) = (f \wedge g)(a \wedge v) =$$

$$= \sum_{I : N \to M} \sum_{\sigma \in S_N} (-1)^{I^* \sigma} f_{(\sigma I)^{-1}(1)}(a) \prod_{i \in (\sigma I)^{-1}([1]+n) \cap p} f_i(v_{I\sigma(i)})$$

$$\prod_{i \in (\sigma I)^{-1}([1]+n) \cap [q]+p} g_i(v_{I\sigma(i)}) \bigwedge_{i \in (I^{C})^{-1}\{1\}} a \wedge \bigwedge_{i \in I^{\complement} \cap [m]+n} v_i +$$

$$+ \sum_{I : N \to M} \sum_{\sigma \in S_N} (-1)^{I^* \sigma} g_{(\sigma I)^{-1}(1)}(a) \prod_{i \in (\sigma I)^{-1}([1]+n) \cap p} f_i(v_{I\sigma(i)})$$

$$\prod_{i \in (\sigma I)^{-1}([1]+n) \cap [q]+p} g_i(v_{I\sigma(i)}) \bigwedge_{i \in (I^{C})^{-1}\{1\}} a \wedge \bigwedge_{i \in I^{\complement} \cap [m]+n} v_i =$$

$$\sum_{I : N \to M} \sum_{\sigma \in S_N} (-1)^{I^* \sigma} f_{(\sigma I)^{-1}(1)}(a) \prod_{i \in (\sigma I)^{-1}([1]+n) \cap p} f_i(v_{I\sigma(i)})$$

$$\prod_{i \in (\sigma I)^{-1}([1]+n) \cap [q]+p} g_i(v_{I\sigma(i)}) \bigwedge_{i \in (I^{\complement})^{-1}\{1\}} a \wedge \bigwedge_{i \in I^{\complement} \cap [m]+n} v_i +$$

$$+ (-1)^p \sum_{I : N \to M} \sum_{\sigma \in S_N} (-1)^{I^* \sigma} g_{(\sigma I)^{-1}(1)}(a) \prod_{i \in (\sigma I)^{-1}([1]+n) \cap q} f_i(v_{I\sigma(i)})$$

$$\prod_{i \in (\sigma I)^{-1}([1]+n) \cap [p]+q} g_i(v_{I\sigma(i)}) \bigwedge_{i \in (I^{C})^{-1}\{1\}} a \wedge \bigwedge_{i \in I^{\complement} \cap [m]+n} v_i =$$

$$\mathbf{i}_a(f)(v) + (-1)^p \mathbf{i}_a(g)(v);$$

$$\rightsquigarrow [(f, g).*] := \daleth M^{\wedge} : \mathbf{i}_a(f \wedge g) = \mathbf{i}_a(f) + (-1)^p \mathbf{i}_a(g);$$

$$\rightsquigarrow [*] := \daleth \widetilde{\mathcal{D}}(M^{\star\wedge}) : \iota_a \in \widetilde{\mathcal{D}}(M^{\star} \wedge);$$

$\square$

$\texttt{ExtDecomposableProperty} :: \forall k : \texttt{NonBinary} \,.\, \forall V : k\text{-VS} \,.\, \forall x \in V^{\wedge 2} \,.\, x \wedge x = 0 \iff x : \texttt{Decomoposable}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : x \wedge x = 0,$

$\texttt{Assume}\ [2] : x \neq 0,$

$(f, [3]) := \texttt{NonAnnihilatingInteriorProductExists}(x, [2])\texttt{CanonicalIsomorphismTHM} :$

$\quad : \sum f \in V^* \,.\, \mathbf{i}_f(x) \neq 0,$

$[4] := \mathcal{A} k\text{-VS}(V^\wedge, V^\wedge)(\mathbf{i}_f)[1]\mathcal{A}\widetilde{\mathcal{D}}(V^\wedge)\mathcal{A}V^{wedge} : 0 = \mathbf{i}_f(0) = \mathbf{i}_f(x \wedge x) = \mathbf{i}_f(x) \wedge x + x \wedge \mathbf{i}_f(x) = 2\mathbf{i}_f(x) \wedge x,$

$[2.*] := \texttt{DecomposableByAnnihalator}[2] : (x : \texttt{Decomposable}(V));$

$\rightsquigarrow [2] := I(\Rightarrow) : x \neq 0 \Rightarrow x : \texttt{Decomposable}(V),$

$[3] := \mathcal{A}V^\wedge : x = 0 \Rightarrow x : \texttt{Decomposable}(V),$

$[1.*] := E(|)\texttt{LEM}[2][3] : \Big(x : \texttt{Decomposable}(V)\Big);$

$\rightsquigarrow [1] := I(\Rightarrow) : x \wedge x = 0 \Rightarrow x : \texttt{Decompasble}(V),$

$[4] := \mathcal{A}\texttt{Decomposable}(V)\mathcal{A}V^\wedge I(\Rightarrow) : (x : \texttt{Decomposable}(V)) \Rightarrow x \wedge x = 0,$

$[*] := [3][4] : \texttt{This};$

$\square$

$\texttt{DecomposableByMatrix} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.\, \forall t \in V_2^\wedge \,.$

$\quad .\, \forall \alpha : \dim V \times \dim V \to k \,.\, \forall [0] : t = \alpha_{i \wedge j} e_i \wedge e_j \,.\; \operatorname{rank} \alpha = 1 \Rightarrow t : \texttt{Decomposable}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \operatorname{rank} \alpha = 1,$

$(C, \beta, [2]) := \mathcal{A} \operatorname{rank} \alpha[1] : \sum C \in \mathrm{GL}(k, \dim V) \,.\, ☼\beta \in k \dim V \,.\, C^{-1}\alpha C = \Lambda i, j \in \dim V \,.\, \delta_1^j \beta_i,$

$f := Ce : \texttt{Basis}(V),$

$$[3] := \mathcal{O} f[2]\mathcal{A}V^\wedge : t = \sum_{i=1}^{\dim V} \beta_i f_1 \wedge f_i = f_1 \wedge \sum_{i=1}^{\dim V} \beta_i f_i,$$

$[1.*] := \mathcal{A}^{-1}\texttt{Decomposable} : \Big(t : \texttt{Decomposable}(V)\Big);$

$\rightsquigarrow [8] := I(\Rightarrow) : \operatorname{rank} \alpha = 1 \Rightarrow t : \texttt{Decomposable}(V);$

$\square$

$\texttt{InterorProductMapping} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall \varphi : A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.\, \forall a \in A^\wedge \,.\, \varphi^{*\wedge}\mathbf{i}_a = \mathbf{i}_{\varphi^\wedge(a)}\varphi^{*\wedge}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : B^{\star\wedge},$

$\texttt{Assume}\ v : A^\wedge,$

$[f.*] := \ldots : \varphi^{*\wedge}\mathbf{i}_a(f)(v) = \varphi^{*\wedge}(f)(a \wedge v) = f\Big(\varphi^\wedge(a \wedge v)\Big) =$

$\quad = f\Big(\varphi^\wedge(a) \wedge \varphi^\wedge(v)\Big) = \mathbf{i}_{\varphi^\wedge(a)}(f)\Big(\varphi^\wedge(v)\Big) = \varphi^{*\wedge}\mathbf{i}_{\varphi^\wedge(a)}(f)(v);$

$\rightsquigarrow [*] := I(=, \to) : \texttt{This};$

$\square$

$\texttt{InteriorProductDerivations} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall a \in M^\wedge \,.\, \forall f \in M^* \,.\, \widetilde{D}_f\mathbf{i}_a = \mathbf{i}_{D_f(a)} + \mathbf{i}_a\widetilde{D}_f$

$\texttt{Proof} \;=$

$\ldots$

$\square$

## 2.6 Mixed Exterior Algebra

$\texttt{mixedExteriorAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{ALGE}$

$\texttt{mixedExteriorAlgebra}\,(M) = M^{\wedge,*} := M^{\wedge} \otimes M^{*\wedge}$

$\texttt{exteriorPower} :: \prod k : \texttt{NumberField} \,.\, \prod V \in R\text{-}\mathsf{VS} \,.\, V^{\wedge,*} \to \mathbb{Z}_+ \to V^{\wedge,*}$

$\texttt{exteriorPower}\,(t, 0) = t^0 := 1 \otimes 1$

$\texttt{exteriorPower}\,(t, n) = t^n := \frac{1}{n!} \prod_{i=1}^{n} t$

$\texttt{AnticommutativeMixedProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall p, p', q, q' \in \mathbb{Z}_+ \,.\, \forall t \in M^{\wedge,*}_{(p,q)} \,.\, \forall s \in M^{\wedge,*}_{(p',q')} \,.$

$\quad .\, t \wedge s = (-1)^{pp'+qq'} s \wedge t$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{MixedExteriorBinomialFormula} :: \forall k : \texttt{NumberField} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall k \in \mathbb{Z}_+ \,.\, \forall x, y \in M^{\wedge,*} \,.$

$\quad .\, (x+y)^k = \sum_{n+m=k} x^n y^m$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{dualExteriorInnerProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(M^{*\wedge}, M^{\wedge}; R)$

$\texttt{dualExteriorInnerProduct}\,(f, v) = \langle f, v\rangle := \text{Ɑ} M^{\wedge} \text{Ɑ} M^{*\wedge}$

$\quad \Lambda f : \texttt{Decomposable}(M^*) \,.\, \Lambda v : \texttt{Decomposable}(M) \,.\, \texttt{if}\ \deg f = \deg v \,.\ \texttt{then}\ \det(f_i(v_j))_{i,j=1}^{\deg f}\ \texttt{else}\ 0$

$\texttt{mixedExteriorInnerProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \texttt{InnerProduct}(M^{\wedge,*})$

$\texttt{mixedExteriorInnerProduct}\,(f, v) = \langle f, v\rangle := \text{Ɑ} M^{\wedge,*} \Lambda f \in M^{*\wedge} \,.\, \Lambda v \in M^{\wedge} \,.\, \Lambda g \in M^{*\wedge} \,.\, \Lambda w \in M^{\wedge} \,.$

$\quad .\, \langle f, w\rangle \langle g, v\rangle$

$\texttt{MixedInteriorProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, M^{\wedge,*} \to M^{\wedge,*} \to M^{\wedge,*}$

$T : \texttt{MixedInteriorProduct} \iff \forall a, b, c \in M^{\wedge,*} \,.\, \langle T(a)(b), c\rangle = \langle b, ac\rangle$

$\texttt{MixedInteriorProducUnique} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T, S : \texttt{MixedIneriorProduct}(M, R) \,.\, T = S$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{mixedInterioreProduct} :: \prod k : \texttt{Field} \,.\, \prod V : k\text{-}\mathsf{VS} \,.\, \texttt{MixedInteriorProduct}(V)$

$\texttt{mixedInteriorProduct}\,(a, b) = \mathbf{i}_a(b) := L^{\star}_a(b)$

$\texttt{DegreeOfmixedInteriorProcuct} :: \forall k : \texttt{Field} . \forall V \in k\text{-VS} . \forall a : \texttt{Homogeneous}\left(M^{\wedge,*}\right) . \forall n, m \in \mathbb{Z}_+ .$

$. \forall[0] : \deg a = (n, m) . \deg \mathbf{i}_a = (-n, -m)$

Proof $=$

...

$\square$

$\texttt{MixedInteriorProductComposition} :: \forall k : \texttt{Field} . \forall V \in k\text{-VS} . \forall a, b \in M^{\wedge,*} . \mathbf{i}_a \mathbf{i}_b = \mathbf{i}_{ab}$

Proof $=$

...

$\square$

$\texttt{diagonalSubalgebra} :: \prod R \in \text{ANN} . R\text{-MOD} \to R\text{-ALGE}$

$$\texttt{diagonalSubalgebra}\,(M) = M^\Delta := \bigoplus_{n=0}^{\infty} M^{\wedge,\star}_{(n,n)}$$

$\texttt{mixedExteriorMap} :: \forall R \in \text{ANN} . \forall A, B \in R\text{-MOD} . (A \xrightarrow{R\text{-MOD}} B) \times (B \xrightarrow{R\text{-MOD}} A) \to A^{\wedge,*} \xrightarrow{R\text{-ALGE}} B^{\wedge,*}$

$\texttt{mixedExteriorMap}\,(f, g) = (f, g)^{\wedge,*} := f^\wedge \otimes g^{*\wedge}$

$\texttt{MixedExteriorMapTHM} :: \forall R \in \text{ANN} . \forall A, B \in R\text{-MOD} . \forall \phi : A \xrightarrow{R\text{-MOD}} B . \forall \psi : B \xrightarrow{R\text{-MOD}} A .$

$. \forall a \in A^{\wedge,*} . (\psi, \phi)^{\wedge,*} \mathbf{i}_a = \mathbf{i}_{(\phi,\psi)^{\wedge,*} a} (\psi, \phi)^{\wedge,*}$

Proof $=$

...

$\square$

$\texttt{asExteriorLinearMap} :: \prod R \in \text{ANN} . \prod M \in R\text{-MOD} . M^{wedge,*} \xrightarrow{R\text{-MOD}} \mathcal{L}(M^\wedge, M^\wedge)$

$\texttt{asExteriorLinearMap}\,(x) = T_x := \Lambda t \in M^\wedge . f_i(t) v_i \quad \textbf{where} \quad x = f_i \otimes v_i$

$\texttt{boxproduct} :: \prod R \in \text{ANN} . \prod M \in R\text{-MOD} . \prod n \in \mathbb{N} . (n \to M \xrightarrow{R\text{-MOD}} M) . M^\wedge_n \xrightarrow{R\text{-MOD}} M^\wedge_n$

$$\texttt{boxProducut}\,(\phi) = \boxtimes_{i=1}^{n} \phi_i := \text{Ꭵ} M^\wedge \Lambda \bigwedge_{i=1}^{n} v_i \in M^\wedge_n . \sum_{\sigma \in S_n} (-1)^\sigma \bigwedge_{i=1}^{n} \phi_i(v_{\sigma(i)})$$

$\texttt{PermutationPreservesBoxProduct} :: \forall R \in \text{ANN} . \forall M \in R\text{-MOD} . \forall n \in \mathbb{N} . \forall \phi : n \to M \xrightarrow{R\text{-MOD}} M .$

$. \forall \sigma \in S_n . \boxtimes_{i=1}^{n} \phi_{\sigma(i)} = \boxtimes_{i=1}^{n} \phi_i$

Proof $=$

...

$\square$

$\texttt{BoxProductDistributivity} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.\, \forall x : n \to M \otimes M^* \,.$

$$T_{\prod_{i=1}^n x_i} = \boxtimes_{i=1}^n T_{x_i}$$

$\texttt{Proof} =$

$\texttt{Assume}\ f : n \to M^*,$

$\texttt{Assume}\ u : n \to M,$

$\texttt{Assume}\ [1] : x = u \otimes f,$

$\texttt{Assume}\ \bigwedge_{i=1}^n v_i : \texttt{Decomposable}(M),$

$[1.*] := [1] \,\mathcal{I}\, \texttt{mixedExteriorAlgebra} \,\mathcal{I}\, \texttt{asExteriorLinearMap} \,\mathcal{I}\, \texttt{dualExteriorInnerProduct} \,\mathcal{I}\, \det \,\mathcal{I}\, M^\wedge$
$\mathcal{I}^{-1}\texttt{asExteriorLinearMap}\,\mathcal{I}^{-1}\texttt{bixProduct} :$

$$: T\left(\prod_{i=1}^n x_i\right)\left(\bigwedge_{i=1}^n v_i\right) = T\left(\prod_{i=1}^n u_i \otimes f_i\right)\left(\bigwedge_{i=1}^n v_i\right) = T\left(\bigwedge_{i=1}^n u_i \otimes \bigwedge_{i=1}^n f_i\right)\left(\bigwedge_{i=1}^n v_i\right) =$$

$$= \bigwedge_{i=1}^n f_i\left(\bigwedge_{i=1}^n v_i\right)\bigwedge_{i=1}^n u_i = \det\left(f_i(v_j)\right)_{i,j=1}^n \bigwedge_{i=1}^n u_i = \sum_{\sigma\in S_n}(-1)^\sigma\left(\prod_{i=1}^n f_i(v_{\sigma(i)})\right)\bigwedge_{i=1}^n u_i =$$

$$= \sum_{\sigma\in S_n}(-1)^\sigma\bigwedge_{i=1}^n f_i(v_{\sigma(i)})u_i = \sum_{\sigma\in S_n}(-1)^\sigma\bigwedge_{i=1}^n T_{x_i}(v_{\sigma(i)}) = \boxtimes_{i=1}^n T_{x_i}\left(\bigwedge_{i=1}^n v_i\right);$$

$\rightsquigarrow [*] := \mathcal{I}\, M^\wedge : T_{\prod_{i=1}^n x_i} = \boxtimes_{i=1}^n T_{x_i};$

$\square$

$\texttt{exteriorCompositionProduct} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}\Big(M^{\wedge,*}, M^{\wedge,*}; M^{\wedge,*}\Big)$

$\texttt{exteriorCompositionProduct}\,(f \otimes v, g \otimes u) = f \otimes v \circ g \otimes u := (g_i(v_i))f_i \otimes u_i$

$\texttt{compositionProductProperty} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \otimes v, g \otimes u\, \mathrm{Im}\, M^{\wedge,*} \,.\, T_{f\otimes v \circ g\otimes u} = T_{f\otimes v} \circ T_{g\otimes u}$

$\texttt{Proof} =$

$\ldots$

$\square$

InteriorProductOfMixedProductFormula :: $\forall R \in \mathsf{ANN} . \forall M \in R\text{-}\mathsf{MOD} . \forall x \in M \otimes M^* .$

$$\forall n \in \mathbb{N} . \forall y : n \to M \otimes M^* . \mathbf{i}_x \prod_{i=1}^{n} y_i = \sum_{i=1}^{n} \langle x, y_i \rangle \prod_{j=1}^{n-1} \hat{y}_{i,j} - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (y_i \circ x \circ y_j + y_j \circ x \circ y_i) \prod_{k=1}^{n-2} \hat{y}_{(i,j),k}$$

Proof $=$

Assume $f : M^*$,

Assume $v : M$,

Assume $g : n \to M^*$,

Assume $u : n \to M$,

Assume $[1] : x = v \otimes f$,

Assume $[2] : y = u \otimes g$,

$[\ldots *] := [1][2] \mathcal{I} \mathbf{i} \mathcal{I} \mathsf{ABEL}(M^{\wedge,\star}) \mathcal{I} M^{\wedge} \mathcal{I} M^{\star\wedge}[1][2] :$

$$: \mathbf{i}_x \prod_{i=1}^{n} y_i = \mathbf{i}_{v \otimes f} \bigwedge_{i=1}^{n} u_i \otimes \bigwedge_{i=1}^{n} f_i = \sum_{i,j=1}^{n} (-1)^{i+j} f(u_i) g_i(v) \bigwedge_{k=1}^{n-1} \hat{u}_{i,k} \otimes \bigwedge_{k=1}^{n-1} f_i =$$

$$\sum_{i=1}^{n} \langle v \otimes f, u_i \otimes g_i \langle \bigwedge_{k=1}^{n-1} \hat{u}_{i,k} \otimes \bigwedge_{k=1}^{n-1} \hat{f}_{j,k} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (-1)^{2i+2j+1} f(u_i) g_j(v) v_j \otimes g_i \bigwedge_{k=1}^{n-1} \hat{u}_{i,k} \otimes \bigwedge_{k=1}^{n-w} f_{(i,j),k} +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{2i+2j} f(u_i) g_j(v) v_j \otimes g_i \bigwedge_{k=1}^{n-2} \hat{u}_{(i,j),k} \otimes \bigwedge_{k=1}^{n-2} \hat{f}_{(i,j),k} =$$

$$= \sum_{i=1}^{n} \langle x, y_i \rangle \prod_{j=1}^{n-1} \hat{y}_{i,j} - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (y_i \circ x \circ y_j + y_j \circ x \circ y_i) \prod_{k=1}^{n-2} \hat{y}_{(i,j),k};$$

$\rightsquigarrow [*] := \mathcal{I} M \otimes M^* :$ This;

$\square$

InteriorProductOfMixedProductFormula :: $\forall R \in \mathsf{ANN} . \forall M \in R\text{-}\mathsf{MOD} . \forall x \in M \otimes M^* .$

$$\forall n \in \mathbb{N} . \forall y \in M \otimes M^* . \mathbf{i}_x y^n = \langle x, y \rangle y^{n-1} - (y \circ x \circ y) y^{n-2}$$

Proof $=$

$\ldots$

$\square$

## 2.7 Algebraic Poincare Duality

$\texttt{asLinearMapInDegrees} :: \prod R \in \mathsf{ANN} \,.\, \prod V \in R\text{-}\mathsf{MOD} \,.\, \prod p, q \in \mathbb{Z}_+ \,.\, M^{\wedge,*}_{(p,q)} \xleftrightarrow{R\text{-}\mathsf{MOD}} \mathcal{L}(V^\wedge_p, V^\wedge_q)$

$\texttt{asLinearMapInDegrees}\,(v) = T^{V^{\wedge q}}_{v|V^{\wedge p}} := T^{p,q}_v$

$\texttt{TraceInnerProduct} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall p, q \in \mathbb{Z}_+ \,.\, \forall x \in V^{\wedge,*}_{(p,q)} \,.\, \forall y \in V^{\wedge,*}_{(q,p)} \,.$

$\quad \operatorname{tr} T^{p,q}_x \circ T^{q,p}_y = \langle x, y\rangle$

Proof $=$

...

$\square$

$\texttt{CompositionIsomorphism} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall p \in \mathbb{Z}_+ \,.\, T^{p,p} : (V^\Delta_p, \circ) \xleftrightarrow{k\text{-}\mathsf{ALGE}} \mathcal{L}(V^{\wedge p}; V^{\wedge p})$

Proof $=$

...

$\square$

$\texttt{unitTensor} :: \prod k : \texttt{Field} \,.\, \prod V \in k\text{-}\mathsf{FDVS} \,.\, V^{\wedge,*}$

$\texttt{unitTensor}\,() = \text{☿} := T^{-1}(\mathrm{id}_{V^\wedge})$

$\texttt{unitTensorOfDegree} :: \prod k : \texttt{Field} \,.\, \prod V \in k\text{-}\mathsf{FDVS} \,.\, \prod p \in \dim V \,.\, V^\Delta_p$

$\texttt{unitTensorOfDegree}\,() = \text{☿}_p := (T^{p,p})^{-1}(\mathrm{id}_{V^\wedge_p})$

$\texttt{UnitTensorDecomposition} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \text{☿} = \sum_{i=0}^{\dim V} \text{☿}_i$

Proof $=$

...

$\square$

$\texttt{UnitTensorPower} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall p \in \dim V \,.\, \text{☿}_p = \text{☿}^p_1$

Proof $=$

...

$\square$

$\texttt{UnitTensorTraceRelation} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall p \in \dim V \,.\, \forall w \in V^\Delta_p \,.\, \langle \text{☿}_p, w\rangle = \operatorname{tr} T^{p,p}_w$

Proof $=$

...

$\square$

$\texttt{UnitTensorMult} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall p, q \in \dim V \,.\, \text{☿}_p \text{☿}_q = C^p_{p+q} \text{☿}_{p+q}$

Proof $=$

...

$\square$

$\texttt{UnitTensorIneriorMult} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall p, q \in \dim V . \mathbf{i}_{\maltese_p} \maltese_q = C^p_{\dim V - q + p} \maltese_{q-p}$

$\texttt{Proof} =$

$\texttt{Assume } [1] : p = 1,$

$[5.*] := \texttt{UnitTensorPower}(q)\texttt{InteriorProductOfMixedPower}(\maltese_1, \maltese_1, q)\texttt{UnitTensorTraceRelation}\mathcal{I}$

$\quad \mathrm{id}\,\mathcal{I}^{-1}\texttt{binomialCoeficient} :$

$$: \mathbf{i}_{\maltese_1}(\maltese_q) = \mathbf{i}_{\maltese_1}(\maltese_1^q) = \left(\langle \maltese_1, \maltese_1 \rangle \maltese_1^{1-1} - \maltese_1^{\circ 3} \maltese_1^{q-2}\right) = \left((\dim V)\maltese_1^{q-1} - (p-1)\maltese_1^{q-1}\right) =$$

$$(\dim V - q + 1)\maltese_{q-1} = C^1_{\dim V - q + 1}\maltese_{q-1};$$

$\rightsquigarrow [1] := I(\Rightarrow) : p = 1 \Rightarrow \mathbf{i}_{\maltese_p} \maltese_q = C^p_{\dim v - q + p} \maltese_{q-p},$

$\texttt{Assume } n : \mathbb{N},$

$\texttt{Assume } [2] : \forall k \in n . p = n \Rightarrow \mathbf{i}_{\maltese_p} \maltese_q = C^p_{\dim v - q + p} \maltese_{q-p},$

$\texttt{Assume } [3] : p = n + 1,$

$[n.*] := \texttt{UnitTensorMultMixedExtrioorProductComp}[1][2][3]\mathcal{I}\texttt{BinomialCoefficient} :$

$$: \mathbf{i}_{\maltese_p} \maltese_q = \frac{1}{p}\mathbf{i}_{\maltese_1 \maltese_n} \maltese_q = \frac{1}{p}\mathbf{i}_{\maltese_1}\mathbf{i}_{\maltese_n} \maltese_q = \frac{1}{p} C^n_{\dim V - q + n}\mathbf{i}_{\maltese_1} \maltese_{q-n} =$$

$$= \frac{1}{p} C^n_{\dim V - q + n}(\dim V - q + p)\maltese_{q-p} = \frac{(\dim V - q + p)!}{(\dim V - q)!p!}\maltese_{q-p} = C^{p!}_{(\dim V - q + p)!}\maltese_{q-p};$$

$\rightsquigarrow [*] := \mathcal{I}\texttt{NaturalSet}(\dim V)[1]I(\Rightarrow) : \texttt{This};$

$\square$

$\texttt{flatPoincareIsomorphism} :: \prod k : \texttt{Field} . \prod V : k\text{-FDVS} . \prod e : \texttt{Basis}(V, \dim V) . V^\wedge \xleftrightarrow{k\text{-ALGE}} V^{*\wedge}$

$$\texttt{flatPoincareIsomorphism}(t) = D_{\flat e} t := \mathbf{i}_t \bigwedge_{i=1}^{\dim V} e_i^*$$

$\texttt{sharpPoincareIsomorphism} :: \prod k : \texttt{Field} . \prod V : k\text{-FDVS} . \prod e : \texttt{Basis}(V, \dim V) . V^{*\wedge} \xleftrightarrow{k\text{-ALGE}} V^\wedge$

$$\texttt{sharpPoincareIsomorphism}(t) = D_{\sharp e} t := \mathbf{i}_t \bigwedge_{i=1}^{\dim V} e_i$$

$\texttt{flatPoincareIsomorphismScalarMult} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall \alpha \in k^* .$

$\quad . D_{\flat \alpha e} = \alpha^{-\dim V} D_{\flat e}$

...

$\square$

$\texttt{flatPoincareIsomorphismScalarMult} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall \alpha \in k^* .$

$\quad . D_{\sharp \alpha e} = \alpha^{\dim V} D_{\sharp e}$

...

$\square$

`flatPoincareIsomorphismExteriorMult` :: $\forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.\, \forall x, y \in V^{\wedge} \,.$

$\quad . \; D_{\flat e}(x \wedge y) = \mathbf{i}_y D_{\flat e}(x)$

...

□

`flatPoincareIsomorphismScalarMult` :: $\forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.\, \forall f, g \in V^{*\wedge} \,.$

$\quad . \; D_{\sharp e}(f \wedge g) = \mathbf{i}_g D_{\sharp e}(x)$

...

□

`PoincareIsometry` :: $\forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.$

$\quad . \; \forall v \in V^{\wedge} \,.\, \forall f \in V^{*\wedge} \,.\, \langle D_{\sharp e} f, D_{\flat e} v \rangle = \langle f, v \rangle$

`Proof` $=$

`Assume` $[1] : (v : \texttt{Homogeneous}(V^{\wedge}))$,

`Assume` $[2] : (f : \texttt{Homogeneous}(V^{*\wedge}))$,

$p := \deg v : \mathbb{Z}_+$,

`Assume` $[3] : \deg g = p$,

$[4] := \daleth D_\sharp \daleth D_\flat \daleth^{-1}\texttt{mixedExteriorInnerProduct}\daleth^{-1}\texttt{mixedInteriorProduct}\daleth\texttt{MixedInteriorProduct}$
$\quad \daleth M^{\Delta} \daleth\texttt{MixedIneriorProductUnitTensorInteriorMult}\daleth\texttt{mixedExteriorInnerProduct} :$

$$\langle D_{\sharp e} f, D_{\flat e} v \rangle = \left\langle \mathbf{i}_f \bigwedge_{i=1}^{n} e_i, \mathbf{i}_v \bigwedge_{i=1}^{n} e^*_i \right\rangle = \left\langle \mercury_{n-p}, \mathbf{i}_f \bigwedge_{i=1}^{n} e_i \otimes \mathbf{i}_v \bigwedge_{i=1}^{n} e^*_i \right\rangle = \langle \mercury_{n-p}, \mathbf{i}_{f \otimes v} \mercury_n \rangle =$$
$$= \langle (f \otimes v) \mercury_{n-p}, \mercury_n \rangle = \langle \mercury_{n-p}(f \otimes v), \mercury_n \rangle = \langle f \otimes v, \mathbf{i}_{\mercury_{n-p}} \mercury_n \rangle = \langle f \otimes v, \mercury_p \rangle = \langle f, v \rangle,$$

$\rightsquigarrow [*] := \daleth V^{\wedge} \daleth V^{*\wedge} : \texttt{This}$;

□

`FlatPoincareDuality` :: $\forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.\, \forall p \in \dim V \,.$

$\quad . \; D^*_{\flat e | V^{\wedge}_p} = (-1)^{p(n-p)} D_{\flat e | V^{\wedge}_{n-p}}$

`Proof` $=$

`Assume` $v : V^{\wedge}_p$,

`Assume` $u : V^{\wedge}_{n-p}$,

$[v.*] := \daleth D_\flat \daleth\texttt{InteriorProduct}\daleth\texttt{exteriorProduct}\daleth\texttt{InteriorProduct}\daleth^{-1} D_\flat :$

$$: \langle D_{\flat e} v, u \rangle = \left\langle \mathbf{i}_v \bigwedge_{i=1}^{n} e^*_i, u \right\rangle = \left\langle \bigwedge_{i=1}^{n} e^*_i, v \wedge u \right\rangle = (-1)^{p(n-p)} \left\langle \bigwedge_{i=1}^{n} e^*_i, u \wedge v \right\rangle =$$
$$= (-1)^{p(n-p)} \left\langle \mathbf{i}_v \bigwedge_{i=1}^{n} e^*_i, v \right\rangle = (-1)^{p(n-p)} \langle D_{\flat e} u, v \rangle = \langle v, (-1)^{p(n-p)} D_{\flat e} u \rangle;$$

$\rightsquigarrow [*] := \daleth\texttt{DualMap} : D^*_{\flat e | V^{\wedge}_p} = (-1)^{p(n-p)} D_{\flat e | V^{\wedge}_{n-p}}$,

□

`SharpPoincareDuality` :: $\forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V, \dim V) \,.\, \forall p \in \dim V \,.$

$\quad . \; D^*_{\sharp e | V^{*\wedge}_p} = (-1)^{p(n-p)} D_{\sharp e | V^{*\wedge}_{n-p}}$

`Proof` $=$

...

□

FlatPoincareSemiinversion :: $\forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall p \in \dim V .$

$. D_{\flat e|V_p^\wedge} D_{\sharp e|V_{n-p}^\wedge} = (-1)^{p(n-p)} \mathrm{id}$

Proof $=$

...

$\square$

SharpPoincareSemiinversion :: $\forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall p \in \dim V .$

$. D_{\sharp e|V_p^\wedge} D_{\flat e|V_{n-p}^\wedge} = (-1)^{p(n-p)} \mathrm{id}$

Proof $=$

...

$\square$

FlatPoincareNaturality :: $\forall k : \texttt{Field} . \forall V, U \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall \varphi : V \xleftrightarrow{k\text{-VS}} U .$

$. \varphi^\wedge D_{\flat\varphi(e)} = D_{\flat e} \varphi^{-1*\wedge}$

Proof $=$

...

$\square$

SharpPoincareNaturality :: $\forall k : \texttt{Field} . \forall V, U \in k\text{-FDVS} . \forall e : \texttt{Basis}(V, \dim V) . \forall \varphi : V \xleftrightarrow{k\text{-VS}} U .$

$. \varphi^{-1*\wedge} D_{\sharp e} = D_{\sharp\varphi(e)} \varphi$

Proof $=$

...

$\square$

NaturalPoincareIsomorphism :: $\prod k : \texttt{Field} . \prod V : k\text{-FDVS} . V^{\wedge,*} \xleftrightarrow{R\text{-ALGE}} V^{\wedge,*}$

NaturalPoincareIsomorphism $(t) = D_\natural(t) := \mathbf{i}_t(☿)$

NaturalPIDecomposition :: $\forall k : \texttt{Field} . \forall V : k\text{-FDVS} . \forall e : \texttt{Basis}k . \forall v \in V^\wedge . \forall f \in V^{*\wedge} .$

$. D_\natural v \otimes f = D_{\flat e}(v) \otimes D_{\sharp e}(f)$

Proof $=$

...

$\square$

NaturalPIMult :: $\forall k : \texttt{Field} . \forall V : k\text{-FDVS} . \forall t, s \in V^{*,\wedge} . D_\natural(t \cdot s) = \mathbf{i}_t D_\natural(s)$

Proof $=$

...

$\square$

PoincareInvolution :: $\prod k : \texttt{Field} . \prod V : k\text{-FDVS} . V^{*,\wedge} \to V^{*,\wedge}$

PoincareInvolution $() = \omega_\natural := ⅁ V^{\wedge,*} \Lambda p, q \in \mathbb{Z}_+ . \Lambda v \in V^{\wedge p} . \Lambda f \in V^{*\wedge q} . (-1)^{q(n-p)+p(n-q)} v \otimes f$

$n = \dim V$

$\texttt{BalancedPoincareIsometry} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . \forall t, s \in V^{\wedge,*} . \langle D_\natural t, D_\natural s \rangle = \langle t, s \rangle$

Proof $=$

...

$\square$

$\texttt{BalancedPoincareDuality} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} .$

$. D_\natural^* = \omega_\natural \circ D_\natural$

Proof $=$

...

$\square$

$\texttt{FlatPoincareSemiinversion} :: \forall k : \texttt{Field} . \forall V \in k\text{-FDVS} . D_\natural^{\circ 2} = \omega_\natural$

Proof $=$

...

$\square$

$\texttt{FlatPoincareNaturality} :: \forall k : \texttt{Field} . \forall V, U \in k\text{-FDVS} . \forall \varphi : V \xleftrightarrow{k\text{-VS}} U .$

$. \varphi^{\wedge,*} D_\natural = D_\natural \varphi^{\wedge,*}$

Proof $=$

...

$\square$

$\texttt{intersectionProduct} :: \prod k : \texttt{Field} . \prod V : k\text{-FDVS} . \prod e : \texttt{Basis}(V) . V^\wedge \times V^\wedge \to V^\wedge$

$\texttt{intersectionProduct}(t, s) = t \cap_e s := D_{\sharp e}(D_{\sharp e}^{-1} t \wedge D_{\sharp e}^{-1} s)$

$\texttt{IntersectionProductAnticommute} :: \forall k : \texttt{Field} . \forall V : k\text{-FDVS} . \forall e : \texttt{Basis}(V) . \forall t, s : \texttt{Homogeneous}(V^\wedge) .$

$. \forall p \in \mathbb{Z}_+ . \forall q \in \mathbb{Z}_+ . \forall [0] : \deg t = p . \forall [00] : \deg s = q . t \cap_e s = (-1)^{(n-p)(n-q)} s \cap_e t \quad \texttt{where} \quad n = \dim V$

Proof $=$

...

$\square$

$\texttt{IntersectionProductWithBasis} :: \forall k : \texttt{Field} . \forall V : k\text{-FDVS} . \forall e : \texttt{Basis}(V) . \forall t : V^\wedge .$

$. t \cap_e \bigwedge_{i=1}^{n} e_i = t$

Proof $=$

...

$\square$

$\texttt{PoincareAlgebraHomo} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V) \,.\, \forall t, s \in V^\wedge \,.$

$\quad . \; D_{\flat e}(t \cap_e s) = D_{\flat e}(t) \wedge D_{\flat e}(s)$

$\texttt{Proof} \;=$

$n := \dim V : \mathbb{Z}_+,$

$[*] := \mathcal{A}\,\texttt{intersectionProduct}(V, e)\texttt{FlatPoincarePseudoInverse}^2(V, e)\mathcal{A}\,\mathcal{L}(V^\wedge, L^\wedge; L^\wedge)(\wedge)\mathcal{A}(-1) :$

$$D_{\flat e}(t \cap_e s) = D_{\flat e} D_{\sharp e}\Big(D^{-1}_{\sharp e}(t) \wedge D^{-1}_{\sharp e}(s)\Big) = \sum_{p,q=0}^{n} (-1)^{(2n-p-q)(p+q-n)} D^{-1}_{\sharp e}(t_p) \wedge D^{-1}_{\sharp e}(s_q) =$$

$$= \sum_{p,q=0}^{n} (-1)^{(2n-p-q)(p+q-n)} \Big((-1)^{p(n-p)} D_{\flat e} t_p \wedge (-1)^{q(n-q)} s_q\Big) = \sum_{p,q=0}^{n} (-1)^{2np+2nq-2p^2-2q^2} D_{\flat e} t_p \wedge D_{\flat e} s_q =$$

$$= D_{\flat e} t \wedge D_{\flat e} s;$$

$\square$

$\texttt{PoincareAlgebraHomo2} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V) \,.\, D_{\flat e} : (V^\wedge, \cap_e) \xrightarrow{k\text{-}\mathsf{ALGE}} (V^\wedge, \wedge)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{externalProduct} :: \prod k : \texttt{Field} \,.\, \prod V : k\text{-}\mathsf{FDVS} \,.\, \prod e : \texttt{Basis}(V) \,.\, V^{(\dim V)-1} \to V$

$$\texttt{externalProduct}\,(v) = [v]_e := D_{\flat e} \bigwedge_{i=1}^{\dim V - 1} v_i$$

$\texttt{ExternalProductOrthogonality} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V) \,.$

$\quad . \; \forall v : (\dim V - 1) \to V \,.\, \forall i \in \dim V - 1 \,.\, \big\langle [v]_e, v_i \big\rangle = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{LagrandgeIdentity} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-}\mathsf{FDVS} \,.\, \forall e : \texttt{Basis}(V) \,.\, \forall v : (n-1) \to V \,.\, \forall f : (n-1) \to V^* \,.$

$\quad . \; \Big\langle [f]_{e^*}, [v]_e \Big\rangle = \det \Big(f_i(v_j)\Big)_{i,j=1}^{n-1} \quad \texttt{where} \quad n = \dim V$

$\texttt{Proof} \;=$

$\ldots$

$\square$

## 2.8 Pfaffian

$\texttt{leftDiffeomult} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \texttt{Alternating}(M,R) \to M \xrightarrow{R\text{-}\mathsf{MOD}} M^\wedge \xrightarrow{R\text{-}\mathsf{MOD}} M^\wedge$

$\texttt{leftDiffeomult}\,(T,m,t) = \Lambda_{T,m}(t) := L_m(t) + D_{T,m}(t)$

$\texttt{AlternatingDiffeomult} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \texttt{Alternating}(M,R) \,.\, \forall m \in M \,.\, \Lambda^2_{T,m} = 0$

$\texttt{Proof} \ =$

$\texttt{Assume}\ t : \texttt{Alternationg}(M,R),$

$[t.*] := \mathcal{I}\Lambda_{T,m}\mathcal{I}\,\texttt{exteriorAlgebra}(M)\mathcal{I}D_{T,m}\mathcal{I}\,\texttt{Alernating}(M,r)(T)\mathcal{I}\,\mathsf{ABEL}(M^\wedge) :$

$\quad : \Lambda^2_{T,m}(t) = \Lambda_{T,m}\left(L_m(t) + D_{T,m}(t)\right) = L^2_m(t) + L_m D_{T,m}(t) + D_{T,m}L_m(t) + D^2_{T,m}(t) =$

$\quad = m \wedge m \wedge t + m \wedge D_{T,m}(t) + D_{T,m}(m \wedge t) = m \wedge D_{T,m}(t) + T(m,m) \wedge t - m \wedge D_{T,m}(t) = 0;$

$\rightsquigarrow [*] := I(=,\to) : \Lambda^2_{T,m} = 0,$

$\square$

$\texttt{DiffeomultExteriorAsComp} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \texttt{Alternating}(M,R) \,.\, \forall n \in \mathbb{N} \,.\, \forall u : n \to M \,.$

$$.\ \Lambda^\wedge_T \bigwedge_{i=1}^{n} u_i = \prod_{i=0}^{n-1} \Lambda_{T,u_{n-i}}$$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{higherDiffeomult} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \texttt{Alternating}(M,R) \to M^\wedge \xrightarrow{R\text{-}\mathsf{MOD}} M^\wedge$

$\texttt{higherDiffeomult}\,(T,t) = \Omega_T(t) := \Lambda^\wedge_T(t)(1)$

$\texttt{higherAntidiffeomult} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, \texttt{Alternating}(M,R) \to M^\wedge \xrightarrow{R\text{-}\mathsf{MOD}} M^\wedge$

$\texttt{higherAntidiffeomult}\,(T,t) = \overline{\Omega}_T(t) := \Omega_{-T}(t)$

$\texttt{DiffeomultDecomp} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \texttt{Alternating}(M,R) \,.\, \forall t \in M^\wedge \,.\, \forall m \in M \,.$

$\quad .\ \Omega_T(m \wedge t) = \Lambda_{T,m}\Omega_T(t)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DiffeomultAndAntiderivationCommute} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \texttt{Alternating}(M, R) \,.$

$\quad . \forall D \in \widetilde{\mathcal{D}}(M) \,.\, D\Omega_T = \Omega_T D$

$\texttt{Proof} \ =$

$\texttt{Assume}\ t : \texttt{Decomposable}(M^\wedge),$

$\texttt{Assume}\ [0] : \deg t = 0,$

$[1] := \mathcal{I}\Omega_T : \Omega_T(t) \in R,$

$[t.*] := \mathcal{I}\texttt{MapOfDegree}(M^\wedge, -1)(D)[1] : D\omega_T(t) = 0 = \omega_T D(t);$

$\rightsquigarrow [0] := I(\forall) : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \deg t = 0 \Rightarrow D\omega_T(t) = \omega_T D(t),$

$\texttt{Assume}\ n : \mathbb{Z}_+,$

$\texttt{Assume}\ [1] : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \deg t \le n \Rightarrow D\omega_T(t) = \omega_T D(t),$

$\texttt{Assume}\ t : \texttt{Decomposable}(M^\wedge),$

$\texttt{Assume}\ [2] : \deg t = n + 1,$

$\big(m, s, [3]\big) := \mathcal{I}\texttt{Decomposable}(t) : \sum m \in M \,.\, \sum s : \texttt{Decomposable}(M^\wedge) \,.\, t = m \wedge s,$

$[4] := [2][3] : \deg s = n,$

$[5] := [3]\mathcal{I}\texttt{SkewDerivation}(D)\mathcal{I}R\text{-}\mathsf{MOD}(M^\wedge, M^\wedge)(\Omega_T)\texttt{DiffeomultComp}(T)\mathcal{I}\Lambda_{T,m} :$

$\quad : \Omega_T D(t) = \Omega_T D(m \wedge s) = \Omega_T\Big(D(m)s - m \wedge D(s)\Big) =$

$\quad = D(m)\Omega_T(s) - \Lambda_{T,m}\Omega_T(s) = D(m)\Omega_T(s) - m \wedge \Omega_T(D(s)) - D_{T,m}\Omega_T(D(s)),$

$[6] := [3]\texttt{DiffeomultDecomp}\mathcal{I}\Lambda_{T,m}\mathcal{I}\texttt{skewDerivation}(D)\texttt{SkewDerivationAnticommute}(D_{T,m}, D)[1]\big(s, [4]\big) :$

$\quad : D\Omega_T(t) = D\Omega_T(m \wedge s) = D\Big(m \wedge \Omega_T(s) + D_{T,m}\Omega_T(s)\Big) =$

$\quad = D(m)\Omega_T(s) - m \wedge D\Omega_T(s) + DD_{T,m}\Omega_T(s) = D(m)\Omega_T(s) - m \wedge D\Omega_T(s) - D_{T,m}D\Omega_T(s) =$

$\quad = D(m)\Omega_T(s) - m \wedge \Omega_T(D(s)) - D_{T,m}\Omega_T(D(s)),$

$[1.*] := [5][6] : \Omega_T D(t) = D\Omega_T(t);$

$\rightsquigarrow [1] := \mathcal{I}\texttt{NaturalSet}(\mathbb{Z}_+)[0][1] : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \Omega_T D(t) = D\Omega_T(t),$

$[*] := \mathcal{I}M^\wedge[1] : \Omega_T D = D\Omega_T;$

$\square$

$\texttt{DiffeomultInverse} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall T : \texttt{Alternating}(M, R) \,.\, \Omega_T^{-1} = \overline{\Omega}_T$

$\texttt{Proof} \ =$

$\texttt{Assume}\ t : \texttt{Decomposable}(M^\wedge),$

$\texttt{Assume}\ [0] : \deg t = 0,$

$[1] := \mathcal{I}\Omega_T \mathcal{I}\overline{\Omega}_T(t) : \Omega_T(t) = t = \overline{\Omega}_T(t),$

$[t.*] := [1]^2 : \Omega_T\overline{\Omega}_T(t) = t \ \&\ \overline{\Omega}_T\Omega_T(t) = t;$

$\rightsquigarrow [0] := I(\forall) : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \deg t = 0 \Rightarrow \Omega_T\overline{\Omega}_T(t) = \overline{\Omega}_T\Omega_T(t) = t,$

$\texttt{Assume}\ n : \mathbb{Z}_+,$

$\texttt{Assume}\ [1] : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \deg t \le n \Rightarrow \Omega_T\overline{\Omega}_T(t) = \overline{\Omega}_T\Omega_T(t) = t,$

$\texttt{Assume}\ t : \texttt{Decomposable}(M^\wedge),$

$\texttt{Assume}\ [2] : \deg t = n + 1,$

$\big(m, s, [3]\big) := \mathcal{I}\texttt{Decomposable}(t) : \sum m \in M \,.\, \sum s : \texttt{Decomposable}(M^\wedge) \,.\, t = m \wedge s,$

$[4] := [2][3] : \deg s = n,$

$[5] := [3]\texttt{DiffeomultComp}^2(T)\mathcal{I}\overline{\Omega}_T\mathcal{I}^2\Lambda_{T,m}\texttt{DiffeomultAndAntiderivativeCommute}\mathcal{I}\,\mathsf{ABEL}(M^\wedge) :$

$\quad : \Omega_T\overline{\Omega}_T(t) = \Omega_T\overline{\Omega}_T(m \wedge s) = \Omega_T\Big(m \wedge \overline{\Omega}_T(s) - D_{T,m}\overline{\Omega}_T(s)\Big) =$

$\quad = m \wedge \Omega_T\overline{\Omega}_T(s) + D_{T,m}\Omega_T\overline{\Omega}_T\overline{\Omega}_T(s) - \Omega_T D_{T,m}\overline{\Omega}_T(s) = m \wedge s + D_{T,m}\Omega_T\overline{\Omega}_T(s) - D_{T,m}\Omega_T\overline{\Omega}_T(s) = t,$

$[6] := [3]\texttt{DiffeomultComp}^2(T)\mathcal{I}\overline{\Omega}_T\mathcal{I}^2\Lambda_{T,m}\texttt{DiffeomultAndAntiderivativeCommute}\mathcal{I}\,\mathsf{ABEL}(M^\wedge) :$

$\quad : \overline{\Omega}_T\Omega_T(t) = \overline{\Omega}_T\Omega_T(m \wedge s) = \overline{\Omega}_T\Big(m \wedge \Omega_T(s) + D_{T,m}\Omega_T(s)\Big) =$

$\quad = m \wedge \overline{\Omega}_T\Omega_T(s) - D_{T,m}\overline{\Omega}_T\Omega_T(s) - \overline{\Omega}_T D_{T,m}\Omega_T(s) = m \wedge s + D_{T,m}\overline{\Omega}_T\Omega_T(s) - D_{T,m}\overline{\Omega}_T\Omega_T(s) = t,$

$[1.*] := [5][6] : \Omega_T\overline{\Omega}_T(t) = t \ \&\ \overline{\Omega}_T\Omega_T(t) = t;$

$\rightsquigarrow [1] := \mathcal{I}\texttt{NaturalSet}(\mathbb{Z}_+)[0][1] : \forall t : \texttt{Decomposable}(M^\wedge) \,.\, \Omega_T\overline{\Omega}_T(t) = t \ \&\ \overline{\Omega}_T\Omega_T(t) = t,$

$[*] := \mathcal{I}M^\wedge[1] : \Omega_T\overline{\Omega}_T = \mathrm{id} \ \&\ \overline{\Omega}_T\Omega_T = \mathrm{id};$

$\square$

$\texttt{PfaffTHM} :: \forall R \in \mathsf{ANN} \, . \, \forall M \in R\text{-}\mathsf{MOD} \, . \, \forall T : \texttt{Alternating}(M, R) \, . \, \forall n \in \mathbb{N} \, . \, \forall m : n \to M \, .$

$$. \det \Lambda i, j \in n \, . \, T(m_i, m_j) = \pi_0^2 \Omega_T \bigwedge_{i=1}^{n} m_i$$

$\texttt{Proof} \;=$

$$x := \bigwedge_{i=1}^{n} m_i : M^\wedge,$$

$[1] := \texttt{SkewExtertiotAppByDet2}(T)\texttt{SkewExtensionExteriorComp}\mathcal{I}^{-1}\pi_0\texttt{DiffeomultExteriorAsComp}(T) :$

$$: \det \Lambda i, j \in n \, . \, T(m_i, m_j) = (-1)^{n(n-3)/2} D^\wedge_{T,x}(x) = (-1)^{n(n-3)/2} \prod_{i=0}^{n-1} D_{T, m_{n-i}} \bigwedge_{i=1}^{n} m_i =$$

$$= (-1)^{n(n-3)/2} \pi_0 \prod_{i=0}^{n-1} (L_{m_{n-i}} + D_{T, m_{n-i}}) \bigwedge_{i=1}^{n} m_i = (-1)^{n(n-3)/2} \pi_0 \Lambda^\wedge_{T,x}(x),$$

$\bar{x} := \overline{\Omega}_T(x) : M^\wedge,$

$[2] := \texttt{DiffeomultInverse}(T)\mathcal{J}(\bar{X})\mathcal{I}\Omega_T\texttt{DiffeomultExteriorComp}\mathcal{I}^{-1}\Omega_T :$

$$: \Lambda^\wedge_{T,x}(x) = \Lambda^\wedge_{T,x}\big(\Omega_T(\bar{x})\big) = \Lambda^\wedge_{T,x}\Lambda^\wedge_{T,\bar{x}}(1) = \Lambda^\wedge_{T, x\wedge\bar{x}}(1) = \Omega_T(x \wedge \bar{x}),$$

$[3] := \mathcal{I}\overline{\Omega}_T\bar{\mathcal{J}}(x)\mathcal{I}^{-1}\texttt{genAlgebra} : \bar{x} \in \Big\langle \{m_i | i \in n\} \Big\rangle,$

$[4] := \mathcal{I}\texttt{exteriorAlgebra}[3] : x \wedge \bar{x} = \pi_0(\bar{x})x,$

$[6] := [1][2][4] : \det \Lambda i, j \in n \, . \, T(m_i, m_j) = (-1)^{n(n-3)/2}\pi_0(\Omega_T(x))\pi_0(\overline{\Omega}_T(x)),$

$\Big(F, [7]\Big) := \texttt{DiffeomultExteriorComp}\mathcal{I}^{-1}\texttt{MapOfDegree} : \sum n \in \mathbb{N} \, . \, \prod i \in n \, .$

$$. \, F_i : \texttt{MapOfDegree}(M^\wedge, n - 2i) \, . \, \Lambda^\wedge_x = \sum_{i=1}^{n} F_i,$$

$$[8] := \mathcal{I}\bar{x}[7] : \Lambda^\wedge_{\bar{x}} = \sum_{i=1}^{n} (-1)^i F_i,$$

$[9] := [8][7]\mathcal{I}^{-1}\pi_0 : \pi_0(\overline{\Omega}_T(x)) = (-1)^{n/2}\pi_0(\Omega_T(x)),$

$[10] := \mathcal{I}(-1) : n : \texttt{Even} \Rightarrow (-1)^{n/2} = (-1)^{k(k-2)/3},$

$[11] := [8][7] : n : \texttt{Odd} \Rightarrow \pi_0(\Omega_T(x)) = 0 = \pi_0(\overline{\Omega}_T(x)),$

$[12] := \texttt{OddOrEven}[10][11] : \pi_0\Omega_T(x) = (-1)^{n(n+3)/2}\pi_0(\overline{\Omega}_T(x)),$

$[*] := [12][6] : \det \Lambda i, j \in n \, . \, T(m_i, m_j) = \pi_0^2(\Omega_T(x));$

$\square$

$\texttt{pfaffian} :: \prod R \in \mathsf{ANN} \, . \, \prod M : \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \, .$

$$. \prod e : \texttt{Basis}(M) \, . \, \texttt{Alternating}(M, R) \to R$$

$$\texttt{pfaffian}\,(T) = \mathrm{pf}_e\, T := \pi_0 \left( \Omega_T \bigwedge_{i=1}^{n} e_i \right)$$

$\texttt{PfaffianProperty} :: \forall R \in \mathsf{ANN} \, . \, \forall M : \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \, .$

$$. \forall e : \texttt{Basis}(M) \, . \, \forall T : \texttt{Alternating}(M, R) \, . \; \mathrm{pf}^2_e\, T = \det T^e$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{PfaffianChangeOfBasis} :: \forall R \in \mathsf{ANN} . \forall M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) .$

$. \forall e, e' : \texttt{Basis}(M) . \ \mathrm{pf}_{e'} T = \det C_{e \to e'} \, \mathrm{pf}_e T$

$\texttt{Proof} \ =$

$[*] := \mathcal{A}\, \mathrm{pf}_{e'} T \mathcal{A}^{-1} C_{e \to e'} \mathcal{A}^{-1} \texttt{exteriorAlgebraFunctorDeterminantTHM} \mathcal{A}\, R\text{-}\mathsf{MOD}(M^\wedge, R)(\Omega \pi_0) \mathcal{A}^{-1} \mathrm{pf}_e T :$

$$: \mathrm{pf}_{e'} T = \pi_0 \Omega \left( \bigwedge_{i=1}^{p} e'_i \right) = \pi_0 \Omega \left( \bigwedge_{i=1}^{p} C_{e \to e'} e_i \right) = \pi_0 \Omega \left( C^\wedge_{e \to e'} \bigwedge_{i=1}^{p} e_i \right) = \pi_0 \Omega \left( (\det C_{e \to e'}) \bigwedge_{i=1}^{p} e_i \right) =$$

$$= (\det C_{e \to e'}) \pi_0 \Omega \left( \bigwedge_{i=1}^{p} e_i \right) = \det C_{e \to e'} \, \mathrm{pf}_e T;$$

$\square$

$\texttt{matrixPfaffian} :: \prod R \in \mathsf{ANN} . \prod n \in \mathbb{N} . \texttt{AlternatingMatrix}(R, n) \to R$

$\texttt{matrixPfaffian}\,(A) = \mathrm{pf}\, A := \mathrm{pf}_e A_{e,e}$

$\texttt{BlockDiagonalPfaffian1} :: \forall R \in \mathsf{ANN} . \forall n, m \in \mathbb{N} . \forall A : \texttt{AlternatingMatrix}(R, n) .$

$. \forall B : \texttt{AlternatingMatix}(R, m) . \ \mathrm{pf}\, A \oplus B = \mathrm{pf}\, A \, \mathrm{pf}\, B$

$\texttt{Proof} \ =$

$[1] := \mathcal{A} \texttt{matrixPfaffian}(A \oplus B) \mathcal{A} \texttt{pfaffian} \mathcal{A} \Omega \mathcal{A}\, A \oplus B :$

$$\mathrm{pf}(A \oplus B) = \pi_0 \Big( \Omega_{(A \oplus B)_{e,e}} \bigwedge_{i=1}^{n+m} e_i \Big) = \pi_0 \prod_{i=1}^{n+m} (L_{e_i} + T_{(A \oplus B)_{e,e,e_i}}) \bigwedge_{i=1}^{n+m} e_i =$$

$$= \pi_0 \prod_{i=1}^{n} (L_{e_i} + T_{(A \oplus B)_{e,e,e_i}}) \prod_{i=n+1}^{n+m} (L_{e_i} + T_{B_{e,e,e_i}}) \bigwedge_{i=1}^{n+m} e_i,$$

$$\Big(x, [2]\Big) := \mathcal{A}\, \mathrm{pf}\, B[1] : \sum x : m \to R^{n+m\wedge} . \ \mathrm{pf}(A \oplus B) = \pi_0 \prod_{i=1}^{n} (L_{e_i} + T_{(A \oplus B)_{e,e,e_i}}) \left( \mathrm{pf}\, B \bigwedge_{i=1}^{n} e_i + \sum_{i=1}^{m} e_{i+1} \wedge x_i \right),$$

$$\Big(y, [*]\Big) := \mathcal{A} T \mathcal{A}^{-1} \mathrm{pf}\, A \mathcal{A} \pi_0 [2] : \sum y : m \to R^{n+m\wedge} . \ \mathrm{pf}(A \oplus B) = \pi_0 \left( \mathrm{pf}\, A \, \mathrm{pf}\, B + \sum_{i=1}^{m} e_{i+1} \wedge y_i \right) = \mathrm{pf}\, A \, \mathrm{pf}\, B,$$

$\square$

$\texttt{doublyReducedMatrix} :: \prod X \in \mathsf{SET} . \prod n \in \mathbb{N} . X^{n \times n} \to n \times n \to X^{(n-2) \times (n-2)}$

$\texttt{doublyReducedMatrix}\,(A, (i, j)) = \widehat{A}_{((i,j))} := \widehat{\left( \widehat{A}_{(i,j)} \right)}_{(j,i)}$

$\texttt{PfaffianFormula} :: \forall R \in \mathsf{ANN} . \forall n \in \mathbb{N} . \forall A : \texttt{AlternatingMatrix}(R, n) .$

$$. \ \mathrm{pf}\, A = \sum_{i=1}^{n} (-1)^i A_{1,i} \, \mathrm{pf}\, \widehat{A}_{((1,i))}$$

$\texttt{Proof} \ =$

$\dots$

$\square$

## 2.9 Symmetric Algebra

$$\texttt{SymmetricAlgebra} :: \prod R \in \mathsf{ANN} \;.\; \prod M \in R\text{-}\mathsf{MOD} \;.\; ?\sum S : R\text{-}\mathsf{CALGE} \;.\; M \xrightarrow{R\text{-}\mathsf{MOD}} S$$
$$(S,\iota) : \texttt{SymmetricAlgebra} \iff \forall A \in R\text{-}\mathsf{CALGE} \;.\; \forall \varphi : M \xrightarrow{R\text{-}\mathsf{MOD}} A \;.\; \exists! f : S \xrightarrow{R\text{-}\mathsf{ALGE}} A : \varphi = \iota f$$

$$\texttt{IsomorphicTensorAlgebras} :: \forall R \in \mathsf{ANN} \;.\; \forall M \in R\text{-}\mathsf{MOD} \;.\; \forall (S,\iota),(S',\iota') : \texttt{SymmetricAlgebra}(M) \;.$$
$$.\; T \cong_{R\text{-}\mathsf{CALGE}} T'$$
$\texttt{Proof} \; =$

$\ldots$

$\square$

$$\texttt{SymmetricAlgebraUniversalInjective} :: \forall R \in \mathsf{ANN} \;.\; \forall M \in R\text{-}\mathsf{MOD} \;.$$
$$.\; \forall (S,\iota) : \texttt{SymmetricAlgebra}(M) \;.\; \iota : M \hookrightarrow T$$
$\texttt{Proof} \; =$

$\ldots$

$\square$

$$\texttt{symmetricAlgebra} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{CALGE}$$
$$\texttt{symmetricAlgebra}\,(M) = M^{\vee} := \frac{M^{\otimes}}{\Big\langle \{x \otimes y - y \otimes x | x, y \in M\} \Big\rangle}$$

$$\texttt{symmetricProduct} :: \prod R \in \mathsf{ANN} \;.\; \prod M \in R\text{-}\mathsf{MOD} \;.\; \mathcal{L}(M^{\vee}, M^{\vee}; M^{\vee})$$
$$\texttt{symmetricProduct}\,([t],[s]) = [t] \vee [s] := [t \otimes s]$$

$$\texttt{symmetricEmbedding} :: \prod R \in \mathsf{ANN} \;.\; \prod M \in R\text{-}\mathsf{MOD} \;.\; M \xrightarrow{R\text{-}\mathsf{MOD}} M^{\vee}$$
$$\texttt{symmetricEmbedding}\,(m) = \iota^{\vee}_M(m) := \Big[\iota^{\otimes}_M(m)\Big]$$

$$\texttt{SymmetricAlgebraTHM} :: \forall R \in \mathsf{ANN} \;.\; \forall M \in R\text{-}\mathsf{MOD} \;.\; (M^{\vee}, \iota^{\vee}_M) : \texttt{SymmetricAlgebra}(R, M)$$
$\texttt{Proof} \; =$

$\ldots$

$\square$

$$\texttt{symmetricMapping} :: \prod R \in \mathsf{ANN} \;.\; \prod M, N \in R\text{-}\mathsf{MOD} \;.\; (M \xrightarrow{R\text{-}\mathsf{MOD}} N) \to (M^{\vee} \xrightarrow{R\text{-}\mathsf{CALGE}} N^{\vee})$$
$$\texttt{symmetricMapping}\,(f) = f^{\vee} := \text{Ɑ}\texttt{SymmetricAlgebra}(R, M)(M^{\vee})(f\iota^{\vee}_N)$$

$$\texttt{symmetricFunctor} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{MOD} \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{CALGE}$$
$$\texttt{symmetricFunctor}\,() := \Big(\texttt{symmetricAlgebra}, \texttt{symmetricMap}\Big)$$

$\texttt{BasisOfSymmetricAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, \forall E : \texttt{Basis}(R) \,.$

$$. \left\{ \bigvee_{i=1}^{n} e_i | e : \texttt{Nondecreasing}\Big(n, (E, o)\Big) \right\} : \texttt{Basis}(M^\vee) \quad \texttt{where} \quad o = \texttt{wellOrderingTHM}(E)$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{FreeSymmetricAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, M^\vee : \texttt{FreeModule}(R)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{SymmetricAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in \texttt{FreeModule}(R) \,.\, M^\vee : \texttt{FreeModule}(R)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{SymmetricAlgebraDirectSum} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \forall M : n \to R\text{-}\mathsf{MOD} \,.\, \forall \left( \bigoplus_{i=1}^{n} M_i \right)^\vee \cong_{R\text{-}\mathsf{CALGE}} \bigotimes_{i=1}^{n} M_i^\vee$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{SymmetricAlgebraPoincareSeries} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{FreeModule} \;\&\; \texttt{FinitelyGeneratedModule}(R) \,.$

$$. \, P(M^\vee)(x) = \frac{1}{(1-x)^n} \quad \texttt{where} \quad n = \operatorname{rank} M$$

$\texttt{Proof} \;=$

$[1] := \texttt{SymmetricAlgebraBasis} : \forall n \in \mathbb{N} \,.\, \dim R_n^\vee = 1,$

$[2] := \mathcal{I}\texttt{seriesOfPoincare}[1] : P(R^\vee)(x) = \dfrac{1}{1-x},$

$[3] := \texttt{FreeAsSum}(M) : M = \bigoplus_{i=1}^{n} R,$

$[4] := \texttt{SymmetrcAlgebraDirectSum} : M^\wedge \cong_{R\text{-}\mathsf{CALGE}} \bigotimes_{i=1}^{n} R^\wedge,$

$[*] := \texttt{PoincareSeriesProduct}[2][4] : P(M^\vee)(x) = \dfrac{1}{(1-x)^n};$

$\square$

SymmetricAlgebraDimension :: $\forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) \,.$

$$. \forall p \in \mathbb{Z}_+ \,.\, \dim M_p^\vee = \binom{n+p-1}{p} \quad \texttt{where} \quad n = \operatorname{rank} M$$

Proof =

$[1] :=$ SymmetricAlgebraPoincareSeries$(M)$FractionDiff GeometricSeries SeriesDiff$\mathcal{I}^{-1}$binom :

$$P(M^\wedge)(x) = \frac{1}{(1-x)^n} = \frac{d^{(n-1)}}{dx^{n-1}} \frac{1}{(n-1)!(1-x)} = \frac{d^{(n-1)}}{dx^{n-1}} \sum_{p=0}^{\infty} \frac{x^p}{(n-1)!} = \sum_{p=0}^{\infty} \frac{d^{n-1}}{dx^{n-1}} \frac{x^p}{(n-1)!} =$$

$$= \sum_{p=0}^{\infty} \frac{(n+p-1)!}{(n-1)!p!} x^p = \sum_{p=0}^{\infty} \binom{n+p-1}{p} x^p,$$

$[*] := \mathcal{I}$PoincareSeries$[1] : \forall p \in \mathbb{Z}_+ \,.\, \dim M_p^\vee = \binom{n+p-1}{p};$

□

ProjectiveSymmetricAlgebra :: $\forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{Projective}(R) \,.\, M^\vee : \texttt{Projective}$

Proof =

...

□

SymmetricCovariantExtension :: $\forall A, B \in \mathsf{ANN} \,.\, \forall \omega A \xrightarrow{\mathsf{RING}} B \,.\, \forall M \in A\text{-}\mathsf{MOD} \,.$

$. M^\vee \otimes_\omega B \cong_{B\text{-}\mathsf{ALGE}} (M \otimes_\omega B)^\vee$

Proof =

...

□

SymmetricDerivationExtension :: $\forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall f \in M^* \,.\, \exists! D \in \mathcal{D}(M^\vee) : D_{|M_1^\vee} = f$

Proof =

...

□

SymmetricGeneralisedDerivationExtension :: $\forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.$

$. \forall f \in \left(M_n^\wedge\right)^* \,.\, \exists! D \in \mathcal{D}^n(M^\vee) : D_{M_n^\vee} = f$

Proof =

...

□

SymmetricAlgebraQuotient :: $\forall R \in \mathsf{ANN} \,.\, \forall I : \texttt{Ideal}(R) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \left(\frac{M}{IM}\right)^\vee \cong_{\frac{R}{I}\text{-}\mathsf{ALGE}} \frac{M^\vee}{IM^\vee}$

Proof =

...

□

## 2.10 Algebraic Differentiation

$$\texttt{DifferentialOperator} :: \prod R \in \mathsf{ANN} \ . \ \prod M \in R\text{-}\mathsf{MOD} \ . \ \prod n \in \mathbb{N} \ . \ \texttt{MapOfDegree}(M^\vee, n)$$

$$F : \texttt{DifferentialOperator} \iff F \in \nabla^n M \iff \exists! f \in (M_n^\vee)^* : \forall k : \texttt{After}(n) \ .$$

$$. \ \forall m : k \to M \ . \ F\left(\bigvee_{i=1}^{k} m_i\right) = \sum_{I:n\uparrow k} f\left(\bigvee_{i=1}^{m} m_{I_i}\right) \bigvee_{i=1}^{k-n} m_{I_i^\complement}$$

$$\texttt{DifferentialOperatorComposition} :: \forall R \in \mathsf{ANN} \ . \ \forall M \in R\text{-}\mathsf{MOD} \ . \ \forall n, m \in \mathbb{N} \ . \ \forall A \in \nabla^n(M) \ .$$
$$. \ \forall B \in \nabla^m(M) \ . \ AB \in \nabla^{n+m}(M)$$

$\texttt{Proof} \ =$

$$(a, [1]) := \mathcal{A}\nabla^n(M)(A) : \sum a : M_n^\vee \xrightarrow{R\text{-}\mathsf{MOD}} R \ . \ \dots,$$

$$(b, [2]) := \mathcal{A}\nabla^m(M)(A) : \sum b : M_m^\vee \xrightarrow{R\text{-}\mathsf{MOD}} R \ . \ \dots,$$

$\texttt{Assume}$ $K : \mathbb{N}$,

$\texttt{Assume}$ $[3] : K \geq n + m$,

$\texttt{Assume}$ $x : K \to M$,

$$[K.*.1] := [1](K, x)\mathcal{A}R\text{-}\mathsf{MOD}(M^\wedge)(b)[3][2](K - n, \dots)\texttt{RearangeArange}(\dots) :$$

$$: AB\left(\bigvee_{i=1}^{K} x_i\right) = B\left(\sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right) \bigvee_{i=1}^{K-n} x_{I_i^\complement}\right) = \sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right) B \bigvee_{i=1}^{K-n} x_{I_i^\complement} =$$

$$= \sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right) \sum_{J:m\uparrow K-n} b\left(\bigvee_{i=1}^{m} x_{I^\complement_{J_i}}\right) \bigvee_{i=1}^{K-n-m} x_{I^\complement_{J_i^\complement}} =$$

$$= \sum_{L:n+m\uparrow K} \left(\sum_{H:n\uparrow n+m} a\left(\bigvee_{i=1}^{n} x_{L_{H_i}}\right) b\left(\bigvee_{i=1}^{m} x_{L_{H_i^\complement}}\right)\right) \bigvee_{i=1}^{K-n-m} x_{L_i^\complement},$$

$$f := \mathcal{A}M_{n+m}^\vee \Lambda x : (n+m) \to M \ . \ \sum_{H:n\uparrow n+m} a\left(\bigvee_{i=1}^{n} x_{H_i}\right) b\left(\bigvee_{i=1}^{m} x_{H_i^\complement}\right) : M_{n+m}^\vee \to R,$$

$$[K.*.2] := \mathcal{I} f \mathcal{A}R\text{-}\mathsf{MOD}(M_a^\wedge, R)(a)\mathcal{A}R\text{-}\mathsf{MOD}(M_m^\wedge, R)(b) : \left(f : M_{n+m}^\wedge \xrightarrow{R\text{-}\mathsf{MOD}} R\right);$$

$$\rightsquigarrow [*] := \mathcal{A}\nabla^{n+m}(M) : AB \in \nabla^{n+m}(M);$$

$\square$

$\texttt{DifferentialOperatorsCommute} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n, m \in \mathbb{N} \,.$

$\quad . \forall A \in \nabla^n(M) \,.\, \forall B \in \nabla^m(M) \,.\, AB = BA$

$\texttt{Proof} \ =$

$(a, [1]) := \mathcal{A}\nabla^n(M)(A) : \sum a : M_n^\vee \xrightarrow{R\text{-}\mathsf{MOD}} R \,.\, \dots,$

$(b, [2]) := \mathcal{A}\nabla^m(M)(A) : \sum b : M_m^\vee \xrightarrow{R\text{-}\mathsf{MOD}} R \,.\, \dots,$

$\texttt{Assume}\ K : \mathbb{N},$

$\texttt{Assume}\ [3] : K \geq n + m,$

$\texttt{Assume}\ x : K \to M,$

$[K.*] := [1](K, x)\mathcal{A}R\text{-}\mathsf{MOD}(M^\wedge)(b)[3][2](K - n, \dots)\texttt{RearangeArange}(\dots) :$

$$: AB\left(\bigvee_{i=1}^{K} x_i\right) = B\left(\sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right)\bigvee_{i=1}^{K-n} x_{I_i^\complement}\right) = \sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right) B \bigvee_{i=1}^{K-n} x_{I_i^\complement} =$$

$$= \sum_{I:n\uparrow K} a\left(\bigvee_{i=1}^{n} x_{I_i}\right) \sum_{J:m\uparrow K-n} b\left(\bigvee_{i=1}^{m} x_{I^\complement_{J_i}}\right) \bigvee_{i=1}^{K-n-m} x_{I^\complement_{J_i^\complement}} =$$

$$= \sum_{L:n+m\uparrow K}\left(\sum_{H:n\uparrow n+m} a\left(\bigvee_{i=1}^{n} x_{L_{H_i}}\right) b\left(\bigvee_{i=1}^{m} x_{L_{H_i^\complement}}\right)\right)\bigvee_{i=1}^{K-n-m} x_{L_i^\complement} =$$

$$= \sum_{L:n+m\uparrow K}\left(\sum_{H:m\uparrow n+m} b\left(\bigvee_{i=1}^{m} x_{L_{H_i^\complement}}\right) a\left(\bigvee_{i=1}^{n} x_{L_{H_i}}\right)\right)\bigvee_{i=1}^{K-n-m} x_{L_i^\complement} =$$

$$= BA\left(\bigvee_{i=1}^{K} x_i\right);$$

$\rightsquigarrow [*] := \mathcal{A}M^\vee I(=, \to) : AB = BA;$

$\square$

$\texttt{ExtensionToDifferentialOperator} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.$

$\quad . \forall f \in M_n^\vee \xrightarrow{R\text{-}\mathsf{MOD}} R \,.\, \exists! A \in \nabla^n(M) \,.\, A_{|M_n^\vee} = f$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DifferentialOperatorsAsFunctional} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.$

$\quad . \left(M_n^\vee\right)^* \cong_{R\text{-}\mathsf{MOD}} \nabla^n(M)$

$\texttt{Proof} \ =$

$\texttt{algebraOfDifferentialOperators} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{CALGE}(\mathbb{Z})$

$$\texttt{algebraOfDifferentialOperators}(M) = \nabla M := \left(\bigoplus_{n=0}^{\infty} \nabla^n M, \Lambda n \in \mathbb{Z} \,.\, \texttt{if}\ n \geq 0\ \texttt{then}\ \nabla^n M\ \texttt{else}\ 0\right)$$

$\texttt{partialDifferentiation} :: \prod R \in \mathsf{ANN} \,.\, \prod X \in \mathsf{SET} \,.\, X \to \nabla^1 R^X$

$\texttt{partialDifferential}\,(\alpha) = \dfrac{\partial}{\partial x_\alpha} := \text{\char"A3}\texttt{ExtensionToDifferentialOperators}(e^*_\alpha)$

$\texttt{higherPartialDifferentiation} :: \prod R \in \mathsf{ANN} \,.\, \prod X \in \mathsf{SET} \,.\, \prod n \in \mathbb{N} \,.\, \left(n \hookrightarrow X \,\&\, n \to \mathbb{N}\right) \to \nabla R^X$

$$\texttt{higherPartialDifferential}\,(\alpha, m) = \frac{\partial^{\sum_{i=1}^n m_i}}{\prod_{i=1}^n \partial x_{\alpha_i}^{m_i}} := \prod_{i=1}^n \left(\frac{\partial}{\partial x_{\alpha_i}}\right)^{m_i}$$

$\texttt{HigherPolynomialDifferentiation} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall \alpha : n \to \mathbb{Z}_+ \,.\, \forall \beta : n \to \mathbb{Z}_+ \,.$

$$.\ \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i} \bigwedge_{i=1}^n e_i^{\vee \beta_i} = \prod_{i=1}^n \frac{\beta_i!}{(\beta_i - \alpha_i)!} \bigwedge_{i=1}^n e_i^{\vee \beta_i - \alpha_i}$$

Proof $=$

...

$\square$

$\texttt{standartDifferentialOperator} :: \prod R \in \mathsf{ANN} \,.\, \prod X \in \mathsf{SET} \,.\, \prod n \in \mathbb{N} \,.\, \left(n \hookrightarrow X \,\&\, n \to \mathbb{N}\right) \to \nabla R^X$

$$\texttt{standartDifferentialOperator}\,(\alpha, m) = \mathrm{D}^{\alpha, m} := \prod_{i=1}^n \frac{1}{\alpha_i!} \frac{\partial^{\sum_{i=1}^n \alpha_i}}{\prod_{i=1}^n \partial x_i}$$

$$\texttt{MultinomialComposition} :: \forall R \in \mathsf{ANN} \,.\, \forall X \in \mathsf{SET} \,.\, \forall n \in \mathbb{N} \,.\, \forall \alpha, \beta : n \to \mathbb{Z}_+ \,.\, \mathrm{D}^\alpha \mathrm{D}^\beta = \binom{\alpha + \beta}{\alpha} D^{\alpha+\beta}$$

Proof $=$

...

$\square$

$\texttt{permanent} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, R^{n \times n} \to R$

$$\texttt{permanent}\,(A) = \mathrm{perm}(A) := \sum_{\sigma \in S_n} A_{i, \sigma(i)}$$

$\texttt{billinearAsDifferential} :: \prod R \in \mathsf{ANN} \,.\, \prod A, B \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(A, B; R) \to A \to \nabla B$

$\texttt{billinearAsDifferential}\,(\gamma, a) = D^\gamma_a := \texttt{ExtensionToDifferentialOperators}(\Lambda b \in B \,.\, \gamma(a, b))$

$\texttt{PermanentComposition} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.\, \forall a : n \to A \,.\, \forall b : n \to B \,.$

$$.\ \prod_{i=1}^n D^\gamma_{a_i} \bigwedge_{i=1}^n b_i = \mathrm{perm}(\gamma(a_i, b_i))_{i,j=1}^n$$

Proof $=$

...

$\square$

## 2.11 Grassmann Algebra

$\texttt{exteriorComultiplication} :: \prod R \in \mathsf{ANN} \, . \, \prod A \in R\text{-}\mathsf{MOD} \, . \, A^\wedge \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} A^\wedge \widetilde{\otimes} A^\wedge$

$\texttt{exteriorComultiplication}\,() = \Delta := \varGamma A^\wedge \Lambda a \in A \, . \, a \otimes 1 + 1 \otimes a$

$\texttt{Assume}\ a, b : A,$

$[1] := \varGamma \Delta \varGamma \texttt{twistedTensorProduct} :$

$: \Delta\Big(a \wedge b\Big)(a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) = (a \wedge b) \otimes 1 + a \otimes b - b \otimes a + 1 \otimes (a \wedge b),$

$[2] := \varGamma \Delta \varGamma \texttt{twistedtensorProduct} \varGamma A^\wedge :$

$: \Delta\Big(b \wedge a\Big)(b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) = (b \wedge a) \otimes 1 + b \otimes a - a \otimes b + 1 \otimes (b \wedge a) =$

$= -(a \wedge b) \otimes 1 - a \otimes b + b \otimes a - 1 \otimes (a \wedge b),$

$[a, b.*] := \varGamma \Delta [1][2] : \Delta(a \wedge b + b \wedge a) = \Delta(a \wedge b) + \Delta(b \wedge a) = 0;$

$\rightsquigarrow [*] := \varGamma \Delta \varGamma E^\wedge : \texttt{WellDefinied}(\Delta);$

$\square$

$\texttt{exteriorCounit} :: \prod R \in \mathsf{ANN} \, . \, \prod A \in R\text{-}\mathsf{MOD} \, . \, A^\wedge \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} R$

$\texttt{exteriorCounit}\,() = \eta := \varGamma R\text{-}\mathsf{ALGE}(A^\wedge, R)(0)$

$\texttt{exteriorAntipode} :: \prod R \in \mathsf{ANN} \, . \, \prod A \in R\text{-}\mathsf{MOD} \, . \, A^\wedge \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} A^\wedge$

$\texttt{exteriorAntipode}\,() = \sigma := \varGamma R\text{-}\mathsf{ALGE}(A^\wedge, R)(-\mathrm{id}_A)$

$\texttt{ExteriorAlgebraIsASkewCoalgebra} :: \forall R \in \mathsf{ANN} \, . \, \forall A \in R\text{-}\mathsf{MOD}(A^\wedge, \Delta, \eta) \in R\text{-}\mathsf{SCOALG}(\mathbb{Z})$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{ExteriorAlgebraIsATwistedHopfAlgebra} :: \forall R \in \mathsf{ANN} \, . \, \forall A \in R\text{-}\mathsf{MOD} \, . \, A^\wedge \in \widetilde{R\text{-}\mathsf{HOPF}}$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{ExteriorAlgebraMapIsAHopfMorphism} :: \forall R \in \mathsf{ANN} \, . \, \forall A, B \in R\text{-}\mathsf{MOD} \, . \, \forall f : A \xrightarrow{R\text{-}\mathsf{MOD}} B \, .$

$. \, f^\wedge : A^\wedge \xrightarrow{\widetilde{R\text{-}\mathsf{HOPF}}} B^\wedge$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{disjointSequenceSum} :: \prod A \in \mathsf{ABEL} \prod n \in \mathbb{Z}_+ .$

$$. \left( \sum k, l \in \mathbb{Z}_+ . \sum [0] : k + l = 0 . \left( \left( k \uparrow [n]_\mathbb{N} \right) \times \left( l \uparrow [n]_\mathbb{N} \right) \right) \to A \right) \to A$$

$$\texttt{disjointSequenceSum}(F) = \sum_{I \sqcup J \uparrow n} F(I, J) :=$$

$$:= \sum (I, J) \in \left\{ (I, J) \in \left( k \uparrow [n]_\mathbb{N} \right) \times \left( l \uparrow [n]_\mathbb{N} \right) : \operatorname{Im} I \cap \operatorname{Im} J = \emptyset \middle| k, l \in \mathbb{Z}_+ : k + l = n \right\} . F(I, J)$$

$\texttt{ComultiplicationOfExteriorProduct} :: \forall R \in \mathsf{ANN} . \forall A \in R\text{-}\mathsf{MOD} . \forall n \in \mathbb{N} . \forall a : n \to A .$

$$. \Delta \left( \bigwedge_{i=1}^{n} a_i \right) = \sum_{I \sqcup J \uparrow n} (-1)^{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right)$$

$\texttt{Proof} =$

$$\male := \Lambda n \in \mathbb{N} . \forall m \in n . \forall a : m \to A . \Delta \left( \bigwedge_{i=1}^{n} a_i \right) = \sum_{I \sqcup J \uparrow n} (-1)^{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right) : \mathbb{N} \to \texttt{Type},$$

$[1] := \mathcal{C} \Delta \mathcal{C}^{-1} (-1)^{I,J} \mathcal{C}^{-1} \texttt{disjointSequenceSum} : \male(1),$

$\texttt{Assume } m : \mathbb{N},$

$\texttt{Assume } [2] : \male(m),$

$\texttt{Assume } a : (m + 1) \to A,$

$$\mathcal{I} := \left\{ (I, J) \in \left( k \uparrow [n]_\mathbb{N} \right) \times \left( l \uparrow [n]_\mathbb{N} \right) : \operatorname{Im} I \cap \operatorname{Im} J = \emptyset \middle| k, l \in \mathbb{Z}_+ : k + l = m \right\} : \mathsf{SET},$$

$$\mathcal{I}_+ := \left\{ (I, J) \in \left( k \uparrow [n]_\mathbb{N} \right) \times \left( l \uparrow [n]_\mathbb{N} \right) : \operatorname{Im} I \cap \operatorname{Im} J = \emptyset \middle| k, l \in \mathbb{Z}_+ : k + l = m + 1 \right\} : \mathsf{SET},$$

$$\left( s, [3] \right) := \mathcal{C} R\text{-}\mathsf{ALGE}(A^\wedge, A^\wedge \otimes A^\wedge) \mathcal{C} \Delta \mathcal{C} R\text{-}\mathsf{ALGE}(A^\wedge) \mathcal{C}^{-1} \mathcal{C}^{-1} \texttt{disjointSequenceSum} :$$

$$: \sum s : \mathcal{I}_+ \to \{-1, 1\} .$$

$$. \left( \bigwedge_{i=1}^{m+1} a_i \right) = \bigwedge_{i=1}^{n} \Delta(a_i) = \bigwedge_{i=1}^{n} (a_i \otimes 1 + 1 \otimes a_i) = \sum_{I \sqcup J \uparrow (m+1)} s_{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right);$$

$\square$

$$[4] := [2](a_{|m}) : \Delta \left( \bigwedge_{i=1}^{m} a_i \right) = \sum_{I \sqcup J \uparrow m} (-1)^{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right),$$

$[5] := [1] \mathcal{C} R\text{-}\mathsf{ALGE}(A^\wedge, A^\wedge \widetilde{\otimes} A^\wedge)[2] :$

$$: \sum_{I \sqcup J \uparrow (m+1)} s_{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right) = \Delta \left( \bigwedge_{i=1}^{m+1} a_i \right) = \Delta \left( \bigwedge_{i=1}^{m} a_i \right) \Delta(a_{m+1}) =$$

$$= \left( \sum_{I \sqcup J \uparrow m} (-1)^{I,J} \left( \bigwedge_{i \in \operatorname{dom} I} a_{I_i} \right) \otimes \left( \bigwedge_{j \in \operatorname{dom} J} a_{J_j} \right) \right) \left( (a_{m+1} \otimes 1) + (1 \otimes a_{m+1}) \right),$$

Assume $(I,J):\mathcal{I}$,

$$[6]:=\mathcal{I}\texttt{twistedTensorProduct}:(-1)^{I,J}\left(\bigwedge_{i\in\operatorname{dom}I}a_{I_i}\right)\otimes\left(\bigwedge_{j\in\operatorname{dom}J}a_{J_j}\right)(a_{m+1}\otimes 1)=$$

$$=(-1)^{I,J}(-1)^{|J|}\left(\bigwedge_{i\in\operatorname{dom}\left(I\sqcup(m+1)\right)}a_{(I\sqcup(m+1))_i}\right)\otimes\left(\bigwedge_{j\in J}a_{J_j}\right),$$

$$[7]:=\mathcal{I}\texttt{twistedTensorProduct}:(-1)^{I,J}\left(\bigwedge_{i\in\operatorname{dom}I}a_{I_i}\right)\otimes\left(\bigwedge_{j\in\operatorname{dom}J}a_{J_j}\right)(1\otimes a_{m+1})=$$

$$=(-1)^{I,J}\left(\bigwedge_{i\in\operatorname{dom}\left(I\sqcup(m+1)\right)}a_{(I\sqcup(m+1))_i}\right)\otimes\left(\bigwedge_{j\in\operatorname{dom}(J\sqcup(m+1))}a_{(J_j\sqcup(m+1))_j}\right),$$

$[8]:=\mathcal{I}\texttt{permutatioSign}\mathcal{I}\texttt{doubleIncreasingAsPermutation}:(-1)^{I,J}(-1)^{|}J|=(-1)^{I\sqcup(m+1),J}$,

$[9]:=\mathcal{I}\texttt{permutationSign}\mathcal{I}\texttt{doubleIncreasignAsPermutation}:(-1)^{I,J}=(-1)^{I,J\sqcup(m+1)}$,

$\Big[(I,J).*\Big]:=[3][6][7][8][9]:s_{I\sqcap(m+1),J}=(-1)^{I\sqcap(m+1),J}\ \&\ s_{I,J\sqcap(m+1)}=(-1)^{I,J\sqcap(m+1)}$;

$\rightsquigarrow[m.*]:=I(\forall)[3]\mathcal{I}^{-1}♂:♂(m+1)$;

$\rightsquigarrow[*]:=\mathcal{I}\mathbb{N}\mathcal{I}♂:$ This;

$\square$

$\texttt{exteriotDualProduct}::\prod R\in\mathsf{ANN}\,.\prod A\in R\text{-}\mathsf{MOD}A^{\wedge *}\otimes A^{\wedge *}\to A^{\wedge *}$

$\texttt{exteriorDualProduct}\,(f,g)=f\wedge g:=\Delta_A(f\otimes g)\mu_R$

$\texttt{ExteriorDualProductAction}::\forall R\in\mathsf{ANN}\,.\,\forall A\in R\text{-}\mathsf{MOD}\,.\,\forall n\in\mathbb{N}\,.\,\forall a:n\to A\,.$

$$.\,\forall f,g\in A^{\wedge *}(f\wedge g)\left(\bigwedge_{i=1}^{n}a_i\right)=\sum_{I\sqcup J\uparrow n}(-1)^{I,J}f\left(\bigwedge_{i\in\operatorname{dom}I}a_{I_i}\right)g\left(\bigwedge_{j\in\operatorname{dom}J}a_{J_j}\right)$$

Proof $=$

...

$\square$

$\texttt{GrassmannAlgebra}::\prod R\in\mathsf{ANN}\,.\,R\text{-}\mathsf{MOD}\xrightarrow{\mathsf{CAT}}\widetilde{R\text{-}\mathsf{HOPF}}$

$\texttt{GrassmannAlgebra}\,(A)=\mathfrak{G}(A):=\mathfrak{D}(A^{\wedge})$

$\texttt{GrassmannIsomorphism}::\forall R\in R\text{-}\mathsf{MOD}\,.\,\forall A:\texttt{FinitelyGeneratedModule}(R)\,.\,\mathfrak{G}(A)\cong_{R\text{-}\mathsf{MOD}(\mathbb{Z})}A^{\wedge}$

Proof $=$

...

$\square$

$\texttt{StrongGrassmannIsomorphism}::\forall R\in R\text{-}\mathsf{MOD}\,.\,\forall A:\texttt{FinitelyGeneratedModule}(R)\,.\,\mathfrak{G}(A)\cong_{\widetilde{R\text{-}\mathsf{HOPF}}}A^{*\wedge}$

Proof $=$

...

$\square$

ExteriorDualProductAction :: $\forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{MOD} \,.\, \forall n \in \mathbb{N} \,.\, \forall a : n \to A \,.$

$$. \forall f : m \to A^* \left(\bigwedge_{i=1}^n f_i\right)\left(\bigwedge_{i=1}^n a_i\right) = \det\Big(f_i(m_j)\Big)_{i,j=1}^n$$

Proof $=$

$\female := \Lambda n \in \mathbb{Z}_+ \,.\, \forall m \in n \,.\, \forall a : m \to A \,.\, \forall f : m \to A^* \,.\, \left(\bigwedge_{i=1}^n f_i\right)\left(\bigwedge_{i=1}^n a_i\right) = \det(f_i(a_j))_{i,j=1}^n : \mathbb{Z}_+ \to \mathtt{Type},$

$[1] := \mathcal{I}^{-1} \det \mathcal{I}^{-1} \female : \female(0) \,\&\, \female(1),$

Assume $m : \mathbb{N},$

Assume $[2] : \female(m),$

Assume $a : (m+1) \to A,$

Assume $f : (m+1) \to A^*,$

$g := \bigwedge_{j=1}^m \widehat{f}_{1,j} : (m+1) \to A^{\wedge *},$

$[m.*] := \mathcal{O}^{-1}(g)$ExteriorDualProductAction$\mathcal{O}(g)\mathcal{I}\female[2]$DeterminantDecomposition :

$$: \left(\bigwedge_{i=1}^{m+1} f_i\right)\left(\bigwedge_{i=1}^{m+1} a_i\right) = (f_1 \wedge g)\left(\bigwedge_{i=1}^{m+1} a_i\right) = \sum_{i=1}^n (-1)^i f_1(a_i) g\left(\bigwedge_{j=1}^n \widehat{a}_{i,j}\right) =$$

$$= \sum_{i=1}^n (-1)^i f_1(a_i) \det\Big(\hat{f}_{1,j}(\hat{a}_{i,l})\Big)_{j,l}^m = \det(f_i(a_j))_{i,j=1}^n;$$

$\rightsquigarrow [*] := [1]\mathcal{I}\mathbb{Z}_+ :$ This;

$\square$

ExterirorIntegrals :: $\forall k : \mathtt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall n \in \mathbb{N} \,.\, \forall [0] : \dim V = n \,.\, \int_{V^\wedge}^l = \int_{V^\wedge}^r = V_n^\wedge$

Proof $=$

. . .

$\square$

GrassmannIntegrals :: $\forall k : \mathtt{Field} \,.\, \forall V \in k\text{-}\mathsf{FDVS} \,.\, \forall n \in \mathbb{N} \,.\, \forall [0] : \dim V = n \,.\, \int_{\mathfrak{G}(V)}^l = \int_{\mathfrak{G}(V)}^r = \mathfrak{G}_n(V)$

Proof $=$

. . .

$\square$

integralOfBerezin :: $\prod k : \mathtt{Field} \,.\, \prod V \in k\text{-}\mathsf{FDVS} \,.\, \prod n \in \mathbb{N} \,.\, \prod e : \mathtt{Basis}(n, V) \,.\, \int_{\mathfrak{G}(V)}^l$

integralOfBerezin $\left(\sum_{I \subset n} \alpha_I \bigwedge_{i \in I} e_i\right) = \mathbf{B}_e \sum_{I \subset n} \alpha_I \bigwedge_{i \in I} := \alpha_n$

## 2.12 Graded Duality In Symmetric Algebras

$\texttt{symmetricComultiplication} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{MOD} \,.\, A^\vee \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} A^\vee \widetilde{\otimes} A^\vee$

$\texttt{symmetricComultiplication}\,() = \Delta := \text{Ꟗ} A^\vee \Lambda a \in A \,.\, a \otimes 1 + 1 \otimes a$

$\texttt{Assume}\ a, b : A,$

$[1] := \text{Ꟗ}\Delta\text{Ꟗ}\texttt{TensorProduct} :$

$\quad : \Delta\Big(a \vee b\Big)(a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) = (a \vee b) \otimes 1 + a \otimes b + b \otimes a + 1 \otimes (a \vee b),$

$[2] := \text{Ꟗ}\Delta\text{Ꟗ}\texttt{twistedtensorProduct}\text{Ꟗ} A^\wedge :$

$\quad : \Delta\Big(b \vee a\Big)(b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) = (b \vee a) \otimes 1 + b \otimes a + a \otimes b + 1 \otimes (b \vee a) =$

$\quad = (a \vee b) \otimes 1 + a \otimes b + b \otimes a + 1 \otimes (a \vee b) =,$

$[a, b.*] := \text{Ꟗ}\Delta[1][2] : \Delta(a \vee b - b \vee a) = \Delta(a \vee b) - \Delta(b \vee a) = 0;$

$\rightsquigarrow [*] := \text{Ꟗ}\Delta\text{Ꟗ} E^\wedge : \texttt{WellDefinied}(\Delta);$

$\square$

$\texttt{SymmetricCounit} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{MOD} \,.\, A^\vee \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} R$

$\texttt{exteriorCounit}\,() = \eta := \text{Ꟗ} R\text{-}\mathsf{ALGE}(A^\vee, R)(0)$

$\texttt{exteriorAntipode} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{MOD} \,.\, A^\vee \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} A^\vee$

$\texttt{symmetricAntipode}\,() = \sigma := \text{Ꟗ} R\text{-}\mathsf{ALGE}(A^\vee, R)(-\mathrm{id}_A)$

$\texttt{SymmetricAlgebraIsACocommutativeCoalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{MOD}(A^\vee, \Delta, \eta) \in R\text{-}\mathsf{CCOALG}(\mathbb{Z})$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{SemmetricAlgebraIsAHopfAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{MOD} \,.\, A^\vee \in R\text{-}\mathsf{HOPF}(\mathbb{Z})$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{SymmetricAlgebraMapIsAHopfMorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.$

$\quad .\, f^\vee : A^\vee \xrightarrow{R\text{-}\mathsf{HOPF}(\mathbb{Z})} B^\vee$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{DifferentialOperatorsAsGradedDuals} :: \forall R \in \mathsf{ANN} . \forall A : \texttt{FreeModule} \,\&\, \texttt{FinitelyGeneratedModule}(R) .$
$\nabla A \cong_{R\text{-}\mathsf{ALGE}(\mathbb{Z})} \mathfrak{D}(A)$

Proof $=$

...

□

$\texttt{symmetricDualEmbedding} :: \prod R \in \mathsf{ANN} . \prod A \in R\text{-}\mathsf{MOD} . A^{*\vee} \xrightarrow{R\text{-}\mathsf{HOPF}(\mathbb{Z})} \mathfrak{D}(A^\vee)$

$\texttt{symmeticDualEmbedding}\,(f) = \lambda(f) := \mathit{\Pi}\, R\text{-}\mathsf{HOPF}(\mathbb{Z}) \mathit{\Pi}\, \mathfrak{D}(A)(f)$

$\texttt{SymmetricDualAction} :: \forall R \in \mathsf{ANN} . \forall A \in R\text{-}\mathsf{MOD} . \forall n \in \mathbb{Z}_+ . \forall a : n \to A . \forall f : n \to A^* .$

$$. \left(\bigvee_{i=1}^n f_i\right)\left(\bigvee_{i=1}^n a_i\right) = \operatorname{perm}(f_i(a_j))_{i,j=1}^n$$

Proof $=$

...

□

$\texttt{SymmetricDualBasisAction1} :: \forall R \in \mathsf{ANN} . \forall A \in R\text{-}\mathsf{MOD} . \forall n \in \mathbb{N} . \forall x : \texttt{Bais}(n, A) .$

$$. \forall p : n \to \mathbb{Z}_+ . \left(\bigvee_{i=1}^n \left(\mathrm{d}x_i\right)^{p_i}\right)\left(\bigvee_{i=1}^n x_i^{p_i}\right) = \prod_{i=1}^n p_i!$$

Proof $=$

...

□

$\texttt{SymmetricDualBasisAction2} :: \forall R \in \mathsf{ANN} . \forall A \in R\text{-}\mathsf{MOD} . \forall n \in \mathbb{N} . \forall x : \texttt{Bais}(n, A) .$

$$. \forall p, q : n \to \mathbb{Z}_+ . \forall [0] : p \neq q . \left(\bigvee_{i=1}^n \left(\mathrm{d}x_i\right)^{p_i}\right)\left(\bigvee_{i=1}^n x_i^{q_i}\right) = 0$$

Proof $=$

...

□

## 2.13 Plücker's Equations

$\texttt{DecomposableByHighDegree} :: \forall k : \texttt{Numeric} \,.\, \forall V \in k\text{-FDVS} \,.\, \forall t \in V^{\wedge(n-1)} \,.\, t : \texttt{Decomposable}(V)$

$\quad n = \dim V$

$\texttt{Proof} =$

$\mathsf{P} := \Lambda m \in \mathbb{N} \,.\, \forall V \in k\text{-FDVS} \,.\, \dim V = m \Rightarrow \forall t \in V^{\wedge(m-1)} \,.\, t : \texttt{Decomposable}(V) : \mathbb{N} \to \texttt{Type},$

$[0] := \mathcal{I}\mathsf{P}\mathcal{A}\,\texttt{exteriorPower}(0) : \mathsf{P}(0),$

$\texttt{Assume}\ n-1 : \mathbb{N},$

$\texttt{Assume}\ [1] : \mathsf{P}(n-1),$

$\texttt{Assume}\ V : k\text{-FDVS},$

$\texttt{Assume}\ [01] : \dim V = n,$

$\texttt{Assume}\ t : t \in V^{\wedge(n-1)},$

$e := \texttt{FreeHasBasis}(V) : \texttt{Basis}(V, n),$

$\Big(\alpha, [2]\Big) := \texttt{ExteriorAlgebraBasis}(V, e)\mathcal{A}\texttt{Basis} : \sum \alpha : k^n \,.\, t = \sum_{i=1}^{n} \alpha_i \bigwedge_{j \neq i} e_i,$

$U := \operatorname{span}\{e_i\}_{i=2}^{n} : \texttt{VectorSubspace}(V),$

$\texttt{Assume}\ [00] : \exists i \in \overline{2n} \,.\, \alpha_i \neq 0,$

$\Big(u, [3]\Big) := [1](U)\mathcal{A}\texttt{Decomposable}[00] : \sum u : \texttt{LinearlyIndependent}(U, n-2) \,.\, \sum_{i=2}^{n} \alpha_i \bigwedge_{j\neq i,1}^{n} e_j = \bigwedge_{i=0}^{n-1} u_i,$

$\Big(w, [4]\Big) := \texttt{BasisExtension}(U, u) : \sum w \in U \,.\, w \oplus u : \texttt{Basis}(U, n-1),$

$\Big(\beta, [5]\Big) := \mathcal{A}U^{\wedge n-1}\texttt{ExteriorAlgebraBasis}(U, w \oplus u) : \alpha_1 \bigwedge_{i=2}^{n} e_i = \beta w \wedge \bigwedge_{i=1}^{n-1} u_i,$

$[6] := [2]\mathcal{A}V^{\wedge}[3][5]\mathcal{A}V^{\wedge} :$

$$: t = \alpha_n \sum_{i=1}^{n} \bigwedge_{j\neq i} e_i = e_1 \wedge \left(\sum_{i=2}^{n} \alpha_i \bigwedge_{j\neq 1,i}^{n} e_j\right) + \alpha_1 \bigwedge_{i\neq 1}^{n} e_i = e_1 \wedge \bigwedge_{i=1}^{n-1} u_i + \beta w \wedge \bigwedge_{i=1}^{n-1} u_i = (e_1 + \beta w) \wedge \bigwedge_{i=1}^{n-1} u_i,$$

$[00.*] := \mathcal{A}^{-1}\texttt{Decomposable}[6] : \Big(t : \texttt{Decomposable}(V)\Big);$

$\rightsquigarrow [00] := I(\Rightarrow) : \Big(\exists i \in \overline{2n} \,.\, \alpha_i \neq 0\Big) \Rightarrow t : \texttt{Decomposable}(V),$

$[10] := [2]\mathcal{A}^{-1}\texttt{Decomposable}I(\Rightarrow) : \Big(\forall i \in \overline{2n} \,.\, \alpha_i = 0\Big) \Rightarrow t : \texttt{Decomposable}(V),$

$n.* := \texttt{LEM}[00][10] : \Big(t : \texttt{Decomposable}(V)\Big);$

$\rightsquigarrow [*] := \mathcal{A}\mathbb{N}\mathcal{I}\mathsf{P} : \texttt{This};$

$\square$

$\texttt{tensorRank} :: \prod k : \texttt{Numeric} \,.\, \prod V : k\text{-FDVS} \,.\, V^{\wedge} \to \mathbb{Z}_+$

$\texttt{tensorRank}\,(t) = \operatorname{rank} t := \min\{\dim U : U \subset_{k\text{-VS}} V \;\&\; t \in U^{\wedge}\}$

$\texttt{tensorAnnihilator} :: \prod k : \texttt{Numeric} \,.\, \prod V : k\text{-FDVS} \,.\, V^{\wedge} \to \texttt{VectorSubspace}(V^*)$

$\texttt{tensorAnnihilator}\,(t) = \operatorname{Ann} t := \{f \in V^* \,.\, \mathbf{i}(f)(t) = 0\}$

$\texttt{DecomposableByRank} :: \forall k : \texttt{Numeric} \,.\, \forall V : k\text{-FDVS} \,.\, \forall n \in \dim V \,.\, \forall t \in V^{\wedge n} \,.\, \forall [0] : t \neq 0$

$\quad . \, t : \texttt{Decomposable}(V) \iff \operatorname{rank} t = n$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \Big(t : \texttt{Decomposable}(V)\Big),$

$$\Big(v, [2]\Big) := \mathcal{I}\texttt{Decomposable}[1] : \sum v : n \to V \,.\, t = \bigwedge_{i=1}^{n} v_i,$$

$[3] := [2][0] : \Big(v : \texttt{LinearlyIndependent}(V, n)\Big),$

$U := \operatorname{span}\{v_i\}_{i=1}^{n} : \texttt{VectorSubspace}(V),$

$[4] := [2]\mathcal{O} : t \in U^{\wedge},$

$[5] := [3]\mathcal{O} : \dim U = n,$

$[1.*] := \texttt{ExteriorPowerRank}[4][5][0] : \operatorname{rank} t = n;$

$\rightsquigarrow [1] := I(\Rightarrow) : t : \texttt{Decomposable}(V) \Rightarrow \operatorname{rank} t = n,$

$\texttt{Assume}\ [2] : \operatorname{rank} t = n,$

$\Big(U, [3]\Big) := \mathcal{I}\texttt{tensorRank}(t)[2] : \sum U \subset_{k\text{-VS}} V \,.\, t \in U^{\wedge} \ \&\ \dim U = n,$

$[2.*] := \texttt{ExteriorPowerRank}[3]\mathcal{I}^{-1}\texttt{Decomposable} : \Big(t : \texttt{Decomosable}\Big);$

$\rightsquigarrow [*] := I(\Rightarrow)I(\iff)[1] : \Big(t : \texttt{Decomposable}(V) \iff \operatorname{rank} t = n\Big);$

$\square$

$\texttt{RankAnnTHM} :: \forall k : \texttt{numeric} \,.\, \forall V : k\text{-FDVS} \,.\, \forall t \in V^{\wedge} \,.\ \operatorname{Ann}(t) + \operatorname{rank}(t) = \dim V$

$\texttt{Proof} \ =$

$n := \operatorname{rank} t : \mathbb{N},$

$\Big(U, [1]\Big) := \mathcal{I}\texttt{tensorRank}[t] : \sum U \subset_{k\text{-VS}} V \,.\, t \in U^{\wedge} \ \&\ \dim U = n,$

$u := \texttt{FreeHasBasis}[1] : \texttt{Basis}(U),$

$\Big(w, [2]\Big) := \texttt{BasisExtension}(u) : \sum w : \texttt{LinearlyIndependent}(V, n) \,.\, u \oplus w : \texttt{Basis}(V),$

$e := u \oplus w : \texttt{Basis}(V),$

$d := \deg t : \mathbb{Z}_+,$

$\texttt{Assume}\ \beta : n \to k,$

$\texttt{Assume}\ [4] : \beta \neq 0,$

$$f := \sum_{i=1}^{n} \beta_i u_i^* : V^*,$$

$$v := \sum_{f=1}^{n} \beta_i u_i^1 : V,$$

$[5] := \mathcal{O}(v)[4] : v \neq 0,$

$\Big(u', [6]\Big) := \texttt{OrthogonalBasisExtension}(v)[5] : \sum u' : (n-1) \to V \,.\, v \oplus u' : \texttt{Basis}(V)\ \&$

$\quad \&\ \forall i \in (n-1) \,.\, f(u'_i) = 0,$

$[7] := \mathcal{O} f \mathcal{O} v : f(v) \neq 0,$

$\Big(\alpha,\gamma,[3]\Big) := \texttt{ExteriorAlgebraBasis}(t)[1] : \sum \alpha : \texttt{Skew}(d-1,n,k) \;.\; \sum \gamma : \texttt{Skew}(d,n,k) \;.$

$$. \; t = \sum_{m=1}^{d-1} \sum_{I:m\uparrow n-1} \alpha_I v \wedge \bigwedge_{i=1}^{m} u'_{I_i} + \sum_{m=1}^{d-1} \sum_{I:m\uparrow n-1} \gamma_I \bigwedge_{i=1}^{m} u'_{I_i},$$

$[8] := [3]\mathcal{O}\beta\mathcal{C}\,\texttt{interiorProduct}[6] :$

$$: \mathbf{i}(f)(t) = \sum_{m=1}^{d-1} \sum_{I:m\uparrow n-1} \alpha_I \mathbf{i}(f) v \wedge \bigwedge_{i=1}^{m} u'_{I_i} + \sum_{m=1}^{d-1} \sum_{I:m\uparrow n-1} \gamma_I \mathbf{i}(f) \bigwedge_{i=1}^{m} u'_{I_i} = \sum_{m=1}^{d-1} \sum_{I:m\uparrow n-1} f(v)\alpha_I \bigwedge_{i=1}^{m} u'_{I_i},$$

$U' := \operatorname{span}\{u'_i\}_{i=1}^{n} : \texttt{VectorSubspace}(U),$

$\texttt{Assume}\ [9] : \mathbf{i}(f)(t) = 0,$

$[10] := [9][8] : \alpha = 0,$

$[11] := [3][10] : t \in U'^{\wedge},$

$[12] := \mathcal{O}U'[11] : \operatorname{rank} t = n-1,$

$[9.*] := [12]\mathcal{O}(n)\texttt{PrivIneq} : \bot;$

$\rightsquigarrow [\beta.*] := E(\bot) : \mathbf{i}(f)(t) \neq 0;$

$\rightsquigarrow [5] := I(\forall)\mathcal{C}^{-1}U^*\mathcal{C}^{-1}\operatorname{Ann} t : U^* \bot \operatorname{Ann} t = \{0\},$

$W := \operatorname{span}\{w_i\}_{i=1}^{n} : \texttt{VectorSubspace}(V),$

$[6] := \mathcal{O}(W)\mathcal{C}^{-1}\operatorname{Ann}(t) : \forall f \in W^* \;.\; f \in \operatorname{Ann} t,$

$[7] := \mathcal{C}\dim[6] : \dim \operatorname{Ann} t \geq \dim V - n,$

$[8] := [5]\texttt{NonitersectDim} : \dim \operatorname{Ann}(t) + \dim U^* \leq \dim V,$

$[*] := \mathcal{O}U[8][7]\texttt{DoubleIneq} : \dim \operatorname{Ann}(t) + \operatorname{rank}(t) = \dim V;$

$\square$

$\texttt{SkewMatrixRank} :: \forall k : \texttt{Numeric} \;.\; \forall n \in \mathbb{N} \;.\; \forall \alpha : \texttt{SkewMatrix}(k,n) \;.\; \operatorname{rank} \alpha = \operatorname{rank} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha_{i,j} e_i \wedge e_j$

$\texttt{Proof} \;=$

$$t := \sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha_{i,j} e_i \wedge e_j : k^{n\wedge},$$

$\texttt{Assume}\ \beta : \ker \alpha,$

$$f := \sum_{i=1}^{n} \beta_i e_i^* : k^{n*},$$

$[1] := \mathcal{C}\ker\alpha(\beta)\mathcal{C}\texttt{matrixMultiplication}\mathcal{C}\texttt{SkewMatrix}(\alpha)\mathcal{C}^{-1}\texttt{exteriorMultiplication} :$

$$: 0 = \alpha\beta = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \beta_j \alpha_{i,j} \right) e_i = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \alpha_{i,j}(\beta_i e_j - \beta_j e_i) = \mathbf{i}(f)(t),$$

$[f.*] := \mathcal{C}^{-1}\operatorname{Ann} f : f \in \operatorname{Ann} t;$

$\rightsquigarrow [1] := \texttt{IsoDim} : \dim \ker \alpha = \dim \operatorname{Ann} t,$

$[*] := \texttt{RankKerTHM}(\alpha)\texttt{RankAnnTHM}(t)[1] : \operatorname{rank} t = \operatorname{rank} \alpha;$

$\square$

$\texttt{BivectorRank} :: \forall k : \texttt{Numeric} \;.\; \forall V \in k\text{-}\mathsf{VS} \;.\; \forall t \in V^{\wedge 2} \;.\; \operatorname{rank} t : \texttt{Even}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ProductTensorRank} :: \forall k : \texttt{Numeric} \,.\, \forall V \in k\text{-VS} \,.\, \forall U, W \subset_{k\text{-VS}} V \,.\, \forall [0] : U \cap W = \{0\} \,.\, \forall n, m \in \mathbb{Z}_+ \,.$
$\quad .\, \forall t \in U^{\wedge n} \,.\, \forall s \in W^{\wedge m} \,.\, \forall [00] : t \neq 0 \neq s \,.\; \operatorname{rank} t \wedge s = \operatorname{rank} t + \operatorname{rank} s$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SumTensorRank} :: \forall k : \texttt{Numeric} \,.\, \forall V \in k\text{-VS} \,.\, \forall U, W \subset_{k\text{-VS}} V \,.\, \forall [0] : U \cap W = \{0\} \,.\, \forall n, m : \mathbb{N} \,.$
$\quad .\, \forall t \in U^{\wedge n} \,.\, \forall s \in W^{\wedge m} \,.\, \forall [00] : (n, m) \neq (1, 1) \,.\; \operatorname{rank}(t + s) = \operatorname{rank} t + \operatorname{rank} s$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{mapOfPlücker} :: \prod k : \texttt{Numeric} \,.\, \prod V \in k\text{-FDVS} \,.\, \prod n \in \mathbb{N} \,.\, V^{\wedge n} \xrightarrow{k\text{-VS}} V^* \xrightarrow{k\text{-VS}} V^{\wedge(n-1)}$

$\texttt{mapOfPlücker}\,(t, f) = p_t(f) := \mathbf{i}(f)(t)$

$\texttt{dualMapOfPlücker} :: \prod k : \texttt{Numeric} \,.\, \prod V \in k\text{-FDVS} \,.\, \prod n \in \mathbb{N} \,.\, V^{\wedge n} \xrightarrow{k\text{-VS}} V^{*\wedge(n-1)} \xrightarrow{k\text{-VS}} V$

$\texttt{dualMapOfPlücker}\,(t, s) = b_t(s) := \mathbf{i}(s)(t)$

$\texttt{PlückerDuality} :: \forall k : \texttt{Numeric} \,.\, \forall V \in k\text{-FDVS} \,.\, \forall n \in \mathbb{N} \,.\, \forall t \in V^{\wedge n} \,.\, \exists \sigma \in \{-1, +1\} \,.\, p_t^* = s b_t$

$\texttt{Proof} \;=$

$\texttt{Assume}\; F : V^{\wedge(n-1)*},$

$(m, f, [1]) := \texttt{FiniteExteriorDuality}(F) : \sum m \in \mathbb{N} \,.\, \sum f : m \to (n-1) \to V^* \,.\, F = \mathbf{i}\left(\sum_{i=1}^{m} \bigwedge_{j=1}^{n-1} f_{i,j}\right),$

$s := \sum_{i=1}^{m} \bigwedge_{j=1}^{n-1} f_{i,j} : V^{*\wedge(n-1)},$

$\texttt{Assume}\; g : V^*,$

$[g.*] := \text{ꓷ}\,\texttt{dualMap}\,\text{ꓷ}\, p_t \,\text{ꓷ}\, k\text{-ALGE}\mathbf{i}\,\text{ꓷ}\, V^{\wedge} :$

$$p_t^* F(f) = F\Big(p_t g\Big) = \mathbf{i}\left(\sum_{i=1}^{m} \bigwedge_{j=1}^{n-1} f_{i,j}\right) \mathbf{i}(g)(t) = \mathbf{i}\left(\sum_{i=1}^{m} g \wedge \bigwedge_{j=1}^{n-1} f_{i,j}\right)(t) =$$
$$\mathbf{i}\left((-1)^{n-1} \sum_{i=1}^{m} \left(\bigwedge_{j=1}^{n-1} f_{i,j}\right) \wedge g\right)(t) = \mathbf{i}(g)\left((-1)^{n-1} \sum_{i=1}^{m} \bigwedge_{j=1}^{n-1} f_{i,j}\right)(t) = (-1)^{n-1}\mathbf{i}(g)\mathbf{i}(s)(t) =$$
$$= (-1)^{n-1} g\Big(b_t(s)\Big) = (-1)^{n-1} \epsilon(b_t(s))(g);$$

$\rightsquigarrow [F.*] := I(=, \to)\texttt{NaturalIsomorphism}(V) : p_t^*(F) = (-1)^{n-1} b_t(s);$

$\rightsquigarrow [*] := I(=, \to)\texttt{ExteriorDuality}(V) : p_t^* = b_t;$

$\square$

$\texttt{PlückerRankLemma} :: \forall k : \texttt{Numeric} \,.\, \forall V \in k\text{-FDVS} \,.\, \forall n \in \mathbb{N} \,.\, \forall t \in V^{\wedge n} \,.\; \operatorname{rank} p_t = \operatorname{rank} t$

$\texttt{Proof} \;=$

$[1] := \text{ꓷ}\, \operatorname{Ann} t \,\text{ꓷ}\, \ker p_t : \operatorname{Ann} t = \ker p_t,$

$[*] := \texttt{RankAnnTHM}(t)\texttt{RankkerTHM}(p_t) : \operatorname{rank} p_t = \operatorname{rank} t;$

$\square$

$\texttt{DualPlückerRankLemma} :: \forall k : \texttt{Numeric} . \forall V \in k\text{-FDVS} . \forall n \in \mathbb{N} . \forall t \in V^{\wedge n} . \operatorname{rank} b_t = \operatorname{rank} t$

$\texttt{Proof} =$

$[1] := \texttt{PlückerRankLemma}(k, V, n, t) : \operatorname{rank} p_t = \operatorname{rank} t,$

$[*] := \texttt{DualRankTheorem}[1]\texttt{PlückerDuality}(k, V, n, t) : \operatorname{rank} b_t = \operatorname{rank} t;$

$\square$

$\texttt{PlückerEquations} :: \forall k : \texttt{Numeric} . \forall V \in k\text{-FDVS} . \forall n \in \mathbb{N} . \forall t \in V^{\wedge n} .$

$$\Big(t : \texttt{Decomposable}(V)\Big) \iff \forall \xi \in V^{*\wedge(n-1)} . b_t(\xi) \wedge t = 0$$

$\texttt{Proof} =$

$\texttt{Assume } [1] : \Big(t : \texttt{Decomposable}(V)\Big),$

$\Big(v, [2]\Big) := \mathit{I}\!\!\!\!C\texttt{Decomposable}[1] : \forall v : n \to V . t = \bigwedge_{i=1}^{n} v_i,$

$[*] := \mathit{I}\!\!\!\!C b_t[2]\mathit{I}\!\!\!\!C V^{\wedge} : \forall \xi \in V^{*\wedge(n-1)} . b_t(\xi) \wedge t = 0;$

$\rightsquigarrow [1] := I(\Rightarrow) : \Big(t : \texttt{Decomposable}\Big)(V) \Rightarrow \forall \xi \in V^{*\wedge(n-1)} . b_t(\xi) \wedge t = 0,$

$\texttt{Assume } [2] : \forall \xi \in V^{*\wedge(n-1)} . b_t(\xi) \wedge t = 0,$

$N := \operatorname{rank} t : \mathbb{Z}_+,$

$\Big(U, [3]\Big) := \mathit{I}\!\!\!\!C \operatorname{rank} t : \sum U \subset_{k\text{-VS}} V . t \in U^{\wedge} \ \&\ \dim U = N,$

$[4] := \texttt{DualPlückerRankLemma}(t)\mathcal{O}N : \operatorname{rank} b_t = N,$

$[5] := \mathit{I}\!\!\!\!C b_t[3] : \operatorname{Im} b_t \subset U,$

$[6] := [3][4][5] : \operatorname{Im} b_t = U,$

$u := \texttt{VectorSpaceIsFree}(U)\texttt{FreeHasBasis}(U) : \texttt{Basis}(U),$

$\Big(\alpha, [7]\Big) := \texttt{ExteriorBasis}(u)(t)[3] : \sum \alpha : (n \uparrow N) \to k . t = \sum_{I:(n\uparrow N)} \alpha_I \bigwedge_{i=1}^{n} u_{I_i},$

$\texttt{Assume } i : N,$

$\Big(f, [8]\Big) := \mathit{I}\!\!\!\!C\texttt{image}[6](u_i) : \sum f \in V^{*\wedge(n-1)} . b_t(f) = u_i,$

$[9] := [8][7][2](f) : \sum_{I:(n\uparrow N)} u_i \wedge \alpha_I \bigwedge_{i=1}^{n} u_{I_i} = u_i \wedge t = b_t(f) \wedge t = 0,$

$[*.1] := [9]\mathit{I}\!\!\!\!C V^{\wedge} : \forall I : n \uparrow N . i \notin \operatorname{Im} I \Rightarrow \alpha_I = 0;$

$\rightsquigarrow [8] := I(\forall) : \forall i \in N . \forall I : n \uparrow N . (\alpha_I \neq 0) \Rightarrow (i \in \operatorname{Im} I),$

$\texttt{Assume } [9] : t \neq 0,$

$[10] := [9][3] : \alpha \neq 0,$

$[11] := [10][8] : N = n,$

$[9.*] := \mathcal{O}N\texttt{DecomposableByRank} : \Big(t : \texttt{Decompasble}\Big)(V);$

$\rightsquigarrow [9] := I(\Rightarrow) : t \neq 0 \Rightarrow t : \texttt{Decomposable}(V),$

$[10] := \mathit{I}\!\!\!\!C^{-1}\texttt{Decomposable}\mathit{I}\!\!\!\!C V^{\wedge} : t \Longrightarrow t : \texttt{Decomposable}(V),$

$[2.*] := E(|)\texttt{LEM}(t = 0)[9][10] : \Big(t : \texttt{Decomposable}(V)\Big);$

$\rightsquigarrow [*] := I(\Rightarrow)[1]I(\iff) : \texttt{This};$

$\square$

# 3 Noncommutative Tensor Product