Category Theory

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1 Basic Concepts of Category Theory

1.1 Definition

$$\texttt{Category} :: ? \sum \mathcal{O} : \texttt{Kind} \;.\; \sum \mathcal{M} : \mathcal{O} \times \mathcal{O} \to \texttt{Kind} \;.$$
$$.\; \sum c : \prod A, B, C \in \mathcal{O} \;.\; \mathcal{M}(A,B) \times \mathcal{M}(B,C) \to \mathcal{M}(A,C) \;.\; \sum I : \prod A \in \mathcal{O} \;.\; \mathcal{M}(A,A)$$
$$\mathcal{C} : \texttt{Category} \iff \forall A, B, C, D \in \mathcal{O} \;.\; \forall f \in \mathcal{M}(A,B) \;.\; \forall g \in \mathcal{M}(B,C) \;.\; \forall h \in \mathcal{M}(C,D) \;.$$
$$.\; c\Big(f, I(A)\Big) = f = c\Big(I(A), f\Big) \;\&\; c(c(f,g),h) = c(f, c(g,h))$$

$$\texttt{objects} :: \texttt{Category} \to \texttt{Kind}$$
$$\texttt{objects}\,(\mathcal{C}) = \mathcal{O}_{\mathcal{C}} := \mathcal{O} \quad \texttt{where} \quad \mathcal{C} = (\mathcal{O}, \mathcal{M}, c, I)$$

$$\texttt{morphisms} :: \prod \mathcal{C} : \texttt{Category} \;.\; \mathcal{O}_{\mathcal{C}} \times \mathcal{O}_{\mathcal{C}} \to \texttt{Kind}$$
$$\texttt{morphisms}\,(A,B) = \mathcal{M}_{\mathcal{C}}(A,B) := \mathcal{M}(A,B) \quad \texttt{where} \quad \mathcal{C} = (\mathcal{O}, \mathcal{M}, c, I)$$

$$\texttt{compositionLaw} :: \prod \mathcal{C} : \texttt{Category} \;.\; \Big(\prod A, B, C \in \mathcal{O}_{\mathcal{C}} \;.\; \mathcal{M}_{\mathcal{C}}(A,B) \times \mathcal{M}_{\mathcal{C}}(B,C) \to \mathcal{M}_{\mathcal{C}}(A,C)\Big)$$
$$\texttt{compositionLaw}\,(\mathcal{C}) := c \quad \texttt{where} \quad \mathcal{C} = (\mathcal{O}, \mathcal{M}, c, i)$$

$$\texttt{composeInCat} :: \prod \mathcal{C} : \texttt{Category} \;.\; \prod A, B, C \in \mathcal{O}_{\mathcal{C}} \;.\; \mathcal{M}_{\mathcal{C}}(A,B) \times \mathcal{M}_{\mathcal{C}}(B,C) \to \mathcal{M}_{\mathcal{C}}(A,C)$$
$$\texttt{composeInCat}\,(f,g) = fg := \texttt{compostionLae}(\mathcal{C})(f,g)$$

$$\texttt{idMorphism} :: \prod \mathcal{C} : \texttt{Category} \;.\; \prod A \in \mathcal{O}_{\mathcal{C}} \;.\; \mathcal{M}(A,A)$$
$$\texttt{idMorphism}\,(A) = \underset{A}{\mathrm{id}} := I(A) \quad \texttt{where} \quad \mathcal{C} = (\mathcal{O}, \mathcal{M}, c, I)$$

$$\texttt{InCat} :: \texttt{Category} \to \texttt{Type}$$
$$a : \texttt{InCat} \iff \Lambda\mathcal{C} : \texttt{Category} \;.\; a \in \mathcal{C} \iff \Lambda\mathcal{C} : \texttt{Category} \;.\; a \in \mathcal{O}_{\mathcal{C}}$$

$$\texttt{CategoriesAsKinds} :: \forall \mathcal{C} : \texttt{Category} \;.\; \Big(\mathcal{C}, \texttt{InCat}(\mathcal{C})\Big) : \texttt{Kind}$$

Proof =

...

□

$$\texttt{RealInCat} :: \texttt{Category} \to \texttt{Type}$$
$$f : \texttt{realInCat} \iff \Lambda\mathcal{C} : \texttt{Category} \;.\; f \in \vec{\mathcal{C}} \iff \Lambda\mathcal{C} : \texttt{Category} \;.\; \exists A, B \in \mathcal{O}_{\mathcal{C}} : f \in \mathcal{M}_{\mathcal{C}}(A,B)$$

$$\texttt{CorrectCategoriesAsKinds} :: \forall C : \texttt{Category} \;.\; \Big(\mathcal{C}, \texttt{RealInCat}(\mathcal{C})\Big) : \texttt{Kind}$$

Proof =

...

□

$\texttt{domain} :: \prod \mathcal{C} : \texttt{Category} \, . \, \overrightarrow{\mathcal{C}} \to \texttt{Set}(\mathcal{C})$

$\texttt{domain}\,(f) = \operatorname{dom} f := \{A \in \mathcal{C} : \exists B \in \mathcal{C} . f \in \mathcal{M}_{\mathcal{C}}(A, B)\}$

$\texttt{codomain} :: \prod \mathcal{C} : \texttt{Category} \, . \, \overrightarrow{\mathcal{C}} \to \texttt{Set}(\mathcal{C})$

$\texttt{codomain}\,(f) = \operatorname{codom} f := \{B \in \mathcal{C} : \exists B \in \mathcal{C} \, . \, f \in \mathcal{M}_{\mathcal{C}}(B, A)\}$

$\texttt{Arrow} :: \prod C : \texttt{Category} \, . \, ?(\mathcal{C} \times \mathcal{C} \times \overrightarrow{\mathcal{C}})$

$(A, B, f) : \texttt{Arrow} \iff f : A \xrightarrow{\mathcal{C}} B \iff f \in \mathcal{M}_{\mathcal{C}}(A, B)$

$\texttt{Small} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{Small} \iff \exists O, M : \texttt{Set} \, . \, \mathcal{O}_{\mathcal{C}} = O \,\&\, \overrightarrow{\mathcal{C}} = M$

$\texttt{LocallySmall} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{LocallySmall} \iff \forall A, B \in \mathcal{C} \, . \, \exists M : \texttt{Set} \, . \, \mathcal{M}_{\mathcal{C}}(A, B) = M$

$\texttt{Preorder} :: ?\texttt{LocallySmall}$

$\mathcal{C} : \texttt{Preorder} \iff \forall A, B \in \mathcal{C} \, . \, |\mathcal{M}_{\mathcal{C}}(A, B)| \leq 1$

$\texttt{Identity} :: \prod \mathcal{C} \in \texttt{Category} \, . \, \prod A \in \mathcal{C} \, . \, ?\mathcal{M}_{\mathcal{C}}(A, A)$

$e : \texttt{Identity} \iff \Big(\forall X \in \mathcal{C} \, . \, \forall f \in \mathcal{M}_{\mathcal{C}}(X, A) \, . \, fe = f\Big) \,\&\, \Big(\forall Y \in \mathcal{C} \, . \, \forall f \in \mathcal{M}_{\mathcal{C}}(A, Y) \, . \, ef = f\Big)$

$\texttt{IdentityIsUnique} :: \forall \mathcal{C} \in \texttt{Category} \, . \, \forall A \in \mathcal{C} \, . \, \exists! e : \texttt{Identity}(\mathcal{C})(A)$

$\texttt{Proof} =$

Use the fact that an identity element is unique in monoids.

$\square$

$\texttt{Connected} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{Connected} \iff \forall A, B \in \mathcal{C} \, . \, \exists n \in \mathbb{N} : \exists X : n \to \mathcal{C} \, . \, X_1 = A \,\&\, X_n = B \,\&$

$\quad \& \, \forall i \in (n-1) \, . \, (\exists X_i \xrightarrow{\mathcal{C}} X_{i+1}) | (\exists X_{i+1} \xrightarrow{\mathcal{C}} X_i)$

$\texttt{Discrete} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{Discrete} \iff \forall A, B \in \mathcal{C} \, . \, A = B \Rightarrow \mathcal{M}_{\mathcal{C}}(A, B) = \{\mathrm{id}_A\} \,\&\, A \neq B \Rightarrow \mathcal{M}_{\mathcal{C}}(A, B) = \emptyset$

$\texttt{Antidiscret} :: \texttt{LocallySmall}$

$\mathcal{C} : \texttt{Antidiscrete} \iff \forall A, B \in \mathcal{C} \, . \, \#\mathcal{M}_{\mathcal{C}}(A, B) = 1$

$\texttt{unitCategory} :: \texttt{Category}$

$\texttt{unitCategory}\,() = \mathbf{1} := \Big(\{1\}, 1 \mapsto \{1\}, (1, 1) \mapsto 1, 1 \mapsto 1\Big)$

1.2 Types of Morphisms

$\texttt{Inverse} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, (A \overset{\mathcal{C}}{\leftrightarrow} B) \to ?(B \overset{\mathcal{C}}{\leftrightarrow} A)$

$g : \texttt{Inverse} \iff \Lambda f : A \overset{\mathcal{C}}{\leftrightarrow} B \,.\, fg = \underset{A}{\mathrm{id}} \;\&\; gf = \underset{B}{\mathrm{id}}$

$\texttt{Isomorphism} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, ?\Big(A \overset{\mathcal{C}}{\to} B\Big)$

$f : \texttt{Isomorphism} \iff f : A \overset{\mathcal{C}}{\leftrightarrow} B \iff \exists g : B \overset{\mathcal{C}}{\to} A \,.\, g : \texttt{Inverse}(f)$

$\texttt{Isomorphic} :: \prod \mathcal{C} : \texttt{Category} \,.\, ?(\mathcal{C} \times \mathcal{C})$

$(A, B) : \texttt{Isomorphic} \iff A \cong B \iff \exists f : A \overset{\mathcal{C}}{\leftrightarrow} B$

$\texttt{Endomorphism}(A) := \mathrm{End}_{\mathcal{C}}(A) = \mathcal{M}_{\mathcal{C}}(A, B) : \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Kind};$

$\texttt{Automorphism}(A) := \mathrm{Aut}_{\mathcal{C}}(A) = A \overset{\mathcal{C}}{\leftrightarrow} A : \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Kind};$

$\texttt{Groupoid} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{Groupoid} \iff \forall A, B \in \mathcal{C} \,.\, \forall f : A \overset{\mathcal{C}}{\to} B \,.\, f : A \overset{\mathcal{C}}{\leftrightarrow} B$

$\texttt{Subcategory} :: \texttt{Small} \to ?\texttt{Small}$

$\mathcal{S} : \texttt{Subcategry} \iff \Lambda \mathcal{C} : \texttt{Category} \,.\, \mathcal{S} \subset \mathcal{C} \iff \mathcal{O}_{\mathcal{S}} \subset \mathcal{C} \;\&\; \forall A, B \in \mathcal{S} \,.\, \mathcal{M}_{\mathcal{S}}(A, B) \subset \mathcal{M}_{\mathcal{C}}(A, B)$

$\texttt{maximalGroupoid} :: \texttt{Category} \to \texttt{Groupoid}$

$\texttt{maximalGroupoid}((\mathcal{O}, \mathcal{M}, c, I)) := \Big(\mathcal{O}, \texttt{Isomorphism}(\mathcal{C}), c, I\Big)$

$\texttt{InverseIsUnique} :: \forall f : \texttt{Isomorphism}(\mathcal{C})(A, B) \,.\, \exists! \texttt{Inverse}(f)$

$\texttt{Proof} =$

$\texttt{Assume}\ g, h : \texttt{Inverse}(f),$

$(1) := \eth\texttt{Inverse}(f)(g) : fg = \mathrm{id},$

$(2) := h(1) : hfg = h,$

$(3) := \eth\texttt{Inverse}(h)(f) : hf = \mathrm{id},$

$(4) := (2)(3) : g = h;$

$\rightsquigarrow (*) := I(\exists!)\eth\texttt{Isomorphism}(\mathcal{C})(A, B)(f) : \exists! \texttt{Inverse}(f);$

$\square$

$\texttt{inverse} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, \prod f : A \overset{\mathcal{C}}{\leftrightarrow} B \,.\, \texttt{Inverse}(f)$

$\texttt{inverse}\,() = f^{-1} := \texttt{InverseIsUniq}(f)$

$$\texttt{CatSliceUnder} :: \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Category}$$

$$\texttt{CatSliceUnder}\,(A) = \frac{\mathcal{C}}{A} := \left(\sum X \in C \,.\, \mathcal{M}_{\mathcal{C}}(A, X), \Big((X, f), (Y, g)\Big) \mapsto \Big\{h : X \xrightarrow{\mathcal{C}} Y | fh = g\Big\}, \right.$$
$$\left. , \texttt{compositionLaw}(\mathcal{C}), (X, f) \mapsto \underset{X}{\mathrm{id}} \right)$$

$$\texttt{CatSliceOver} :: \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Category}$$

$$\texttt{CatSliceOver}\,(A) = \frac{A}{\mathcal{C}} := \left(\sum X \in C \,.\, \mathcal{M}_{\mathcal{C}}(X, A), \Big((X, f), (Y, g)\Big) \mapsto \Big\{h : X \xrightarrow{\mathcal{C}} Y | hf = g\Big\}, \right.$$
$$\left. , \texttt{compositionLaw}(\mathcal{C}), (X, f) \mapsto \underset{X}{\mathrm{id}} \right)$$

$$\texttt{Monic} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, ?(A \xrightarrow{\mathcal{C}} B)$$
$$f : \texttt{Monic} \iff f : A \overset{\mathcal{C}}{\hookrightarrow} B \iff \forall X \in \mathcal{C} \,.\, \forall g, h : X \xrightarrow{\mathcal{C}} A \,.\, gf = hf \Rightarrow g = h$$

$$\texttt{Epic} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, ?(A \xrightarrow{\mathcal{C}} B)$$
$$f : \texttt{Epic} \iff f : A \overset{\mathcal{C}}{\twoheadrightarrow} B \iff \forall X \in \mathcal{C} \,.\, \forall g, h : B \xrightarrow{\mathcal{C}} X \,.\, fg = fh \Rightarrow g = h$$

$$\texttt{Section} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, (A \xrightarrow{\mathcal{C}} B) \to ?(B \xrightarrow{\mathcal{C}} A)$$
$$g : \texttt{Section} \iff \Lambda f : A \xrightarrow{\mathcal{C}} B \,.\, gf = \underset{B}{\mathrm{id}}$$

$$\texttt{Retraction} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, (A \xrightarrow{\mathcal{C}} B) \to ?(B \xrightarrow{\mathcal{C}} A)$$
$$g : \texttt{Retraction} \iff \Lambda f : A \xrightarrow{\mathcal{C}} B \,.\, fg = \underset{A}{\mathrm{id}}$$

$$\texttt{SplitMono} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, ?(A \xrightarrow{\mathcal{C}} B)$$
$$f : \texttt{SplitMono} \iff \exists \texttt{Retraction}(f)$$

$$\texttt{SplitEpic} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, ?(A \xrightarrow{\mathcal{C}} B)$$
$$f : \texttt{SplitEpic} \iff \exists \texttt{Section}(f)$$

$$\texttt{idempotent} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A \in \mathcal{C} \,.\, \mathrm{End}_{\mathcal{C}}(A)$$
$$f : \texttt{idempotent} \iff ff = f$$

$\texttt{SplitMonoIsMono} :: \forall f : \texttt{SplitMono}(\mathcal{C})(A,B) \,.\, f : A \xhookrightarrow{\mathcal{C}} B$

$\texttt{Proof} \;=$

$r := \eth\texttt{SplitMono}(f) : \texttt{Retraction}(r),$

$(1) := \eth\texttt{Retraction}(f)(r) : fr = \underset{A}{\mathrm{id}},$

$\texttt{Assume}\ X : \mathcal{C},$

$\texttt{Assume}\ g,h : X \xrightarrow{\mathcal{C}} A,$

$\texttt{Assume}\ (2) : gf = hf,$

$() := (1)((2)r)(1) : g = gfr = hfr = h;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Mono}(\mathcal{C})(A,B) : \left(f : A \xhookrightarrow{\mathcal{C}} B\right);$

$\square$

$\texttt{SplitEpicIsEpic} :: \forall f : \texttt{SplitEpic}(\mathcal{C})(A,B) \,.\, f : A \xrightarrow{\mathcal{C}}\!\!\!\!\rightarrow B$

$\texttt{Proof} \;=$

$s := \eth\texttt{SplitEpic}(f) : \texttt{Section}(r),$

$(1) := \eth\texttt{Section}(f)(s) : sf = \underset{A}{\mathrm{id}},$

$\texttt{Assume}\ Y : \mathcal{C},$

$\texttt{Assume}\ g,h : B \xrightarrow{\mathcal{C}} Y,$

$\texttt{Assume}\ (2) : fg = fh,$

$() := (1)(s(2))(1) : g = sfg = sfh = h;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Epic}(\mathcal{C})(A,B) : \left(f : A \xrightarrow{\mathcal{C}}\!\!\!\!\rightarrow B\right);$

$\square$

$\texttt{LeftRightInverse} :: \forall f : A \xrightarrow{\mathcal{C}} B \,.\, \forall s : \texttt{Section}(f) \,.\, \forall r : \texttt{Retraction}(f) \,.\, s = r$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Retraction}(f)(r) : fr = \mathrm{id},$

$(2) := s(1) : sfr = s,$

$(3) := \eth\texttt{Section}(f)(s) : sf = \mathrm{id},$

$(*) := (2)(3) : s = r;$

$\square$

$\texttt{IsoBySplit} :: \forall f : \texttt{SplitMono} \,\&\, \texttt{SplitEpic}(\mathcal{C})(A,B) \,.\, f : A \xleftrightarrow{\mathcal{C}} B$

$\texttt{Proof} \;=$

$r := \eth\texttt{SplitMono}(f) : \texttt{Retraction}(f),$

$s := \eth\texttt{SplitEpic}(f) : \texttt{Section}(f),$

$(1) := \texttt{LeftRightInverse}(f,s,r) : s = r,$

$(2) := \eth^{-1}\texttt{Inverse}(f)\eth\texttt{Retraction}(f)(r)(1)\eth\texttt{Section}(f)(s) : \left[r : \texttt{Inverse}(f)\right],$

$(*) := \eth^{-1}\texttt{Iso}(2) : \left[f : A \xleftrightarrow{\mathcal{C}} B\right];$

$\square$

1.3 Functors

$$\texttt{Covariant} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, ?\sum F : \mathcal{A} \to \mathcal{B} \,.$$
$$.\ \prod X, Y \in \mathcal{A} \,.\, (X \xrightarrow{\mathcal{A}} Y) \to \Big(F(X) \xrightarrow{\mathcal{B}} F(Y)\Big)$$
$$(F, F') : \texttt{Covariant} \iff \Big(\forall X, Y, Z \in \mathcal{A} \,.\, \forall f : X \xrightarrow{\mathcal{A}} Y \,.\, \forall g : Y \xrightarrow{\mathcal{A}} Z \,.$$
$$.\ F'_{X,Z}(fg) = F'_{X,Y}(f)F'_{Y,Z}(g)\Big) \ \&\ \Big(\forall A \in \mathcal{A} \,.\, F'_{A,A} \underset{A}{\mathrm{id}} = \underset{F(A)}{\mathrm{id}}\Big)$$

$$\texttt{Contravariant} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, ?\sum F : \mathcal{A} \to \mathcal{B} \,.$$
$$.\ \prod X, Y \in \mathcal{A} \,.\, (X \xrightarrow{\mathcal{A}} Y) \to \Big(F(Y) \xrightarrow{\mathcal{B}} F(X)\Big)$$
$$(F, F') : \texttt{Contravariant} \iff \Big(\forall X, Y, Z \in \mathcal{A} \,.\, \forall f : X \xrightarrow{\mathcal{A}} Y \,.\, \forall g : Y \xrightarrow{\mathcal{A}} Z \,.$$
$$.\ F'_{X,Z}(fg) = F'_{Y,Z}(g)F'_{X,Y}(f)\Big) \ \&\ \Big(\forall A \in \mathcal{A} \,.\, F'_{A,A} \underset{A}{\mathrm{id}} = \underset{F(A)}{\mathrm{id}}\Big)$$

$$\texttt{Functor} := \texttt{Covariant}|\texttt{Contravariant} : \texttt{Category} \times \texttt{Category} \to \texttt{Type};$$

$$\texttt{actOnObjects} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, \texttt{Functor}(\mathcal{A}, \mathcal{B}) \to \mathcal{A} \to \mathcal{B}$$
$$\texttt{actOnObjects}\,(F, X) = F(X) := F''(X) \quad \textbf{where} \quad F = (F', F'')$$

$$\texttt{actOnMorphism} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, \texttt{Functor}(\mathcal{A}, \mathcal{B}) \to \vec{\mathcal{A}} \to \vec{\mathcal{B}}$$
$$\texttt{actOnMotphism}\,(F, f) = F(f) := F''(f) \quad \textbf{where} \quad F = (F', F'')$$

$$\texttt{functorCompose} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C} : \texttt{Category} \,.\, \texttt{Functor}\,(\mathcal{A}, \mathcal{B}) \times \texttt{Functor}\,(\mathcal{B}, \mathcal{C}) \to \texttt{Functor}\,(\mathcal{A}, \mathcal{C})$$
$$\texttt{functorCompose}\,(F, G) = FG = G \circ F := \Big(X \mapsto G\big(F(X)\big), f \mapsto G\big(F(f)\big)\Big)$$

$$\texttt{Full} :: ?\texttt{Functor}\,(\mathcal{A}, \mathcal{B})$$
$$(F, F') : \texttt{Full} \iff \forall X, Y \in \mathcal{A} \,.\, F'_{X,Y} : \texttt{Surjective}$$

$$\texttt{Faithful} :: ?\texttt{Functor}\,(\mathcal{A}, \mathcal{B})$$
$$(F, F') : \texttt{Faithful} \iff \forall X, Y \in \mathcal{A} \,.\, F'_{X,Y} : \texttt{Injective}$$

$$\texttt{FullyFaithful} :: ?\texttt{Functor}\,(\mathcal{A}, \mathcal{B})$$
$$(F, F') : \texttt{FullyFaithful} \iff \forall X, Y \in \mathcal{A} \,.\, F'_{X,Y} : \texttt{Bijective}$$

$$\texttt{EmbeddingFunctor} :: ?\texttt{FullyFaithful}(\mathcal{A}, \mathcal{B})$$
$$(F, F') : \texttt{EmbeddingFunctor} \iff F : \texttt{Injective}(\mathcal{A}, \mathcal{B})$$

$\texttt{CovariantPreservesIso} :: \forall F : \texttt{Covariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : X \overset{\mathcal{A}}{\leftrightarrow} Y \,.\, F(f) : F(X) \overset{\mathcal{B}}{\leftrightarrow} F(Y)$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F \ \underset{X}{\mathrm{id}})\eth^{-1} f f^{-1} \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F) :$

$\quad : \underset{F(X)}{\mathrm{id}} = F \ \underset{X}{\mathrm{id}} = F\left(f f^{-1}\right) = F(f) F\left(f^{-1}\right),$

$(2) := \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F \ \underset{X}{\mathrm{id}})\eth^{-1} f^{-1} f \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F) :$

$\quad : \underset{F(X)}{\mathrm{id}} = F \ \underset{X}{\mathrm{id}} = F\left(f^{-1} f\right) = F\left(f^{-1}\right) F(f),$

$(3) := \eth^{-1}\texttt{Inverse}(1,2) : \left(F(f)\right)^{-1} = F\left(f^{-1}\right),$

$(*) := \eth^{-1}\texttt{Iso}(3) : \left[F(f) : F(X) \overset{\mathcal{B}}{\leftrightarrow} F(Y)\right];$

$\square$

$\texttt{CovntravariantPreservesIso} :: \forall F : \texttt{Contravariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : X \overset{\mathcal{A}}{\leftrightarrow} Y \,.\, F(f) : F(Y) \overset{\mathcal{B}}{\leftrightarrow} F(X)$

$\texttt{Proof} \ =$

$(1) := \eth\texttt{Contravariant}(\mathcal{A},\mathcal{B})(F \ \underset{X}{\mathrm{id}})\eth^{-1} f^{-1} f \eth\texttt{Contravariant}(\mathcal{A},\mathcal{B})(F) :$

$\quad : \underset{F(X)}{\mathrm{id}} = F \ \underset{X}{\mathrm{id}} = F\left(f^{-1} f\right) = F(f) F\left(f^{-1}\right),$

$(2) := \eth\texttt{Contravariant}(\mathcal{A},\mathcal{B})(F \ \underset{X}{\mathrm{id}})\eth^{-1} f f^{-1} \eth\texttt{Contravariant}(\mathcal{A},\mathcal{B})(F) :$

$\quad : \underset{F(X)}{\mathrm{id}} = F \ \underset{X}{\mathrm{id}} = F\left(f f^{-1}\right) = F\left(f^{-1}\right) F(f),$

$(3) := \eth^{-1}\texttt{Inverse}(1,2) : \left(F(f)\right)^{-1} = F\left(f^{-1}\right),$

$(*) := \eth^{-1}\texttt{Iso}(3) : \left[F(f) : F(Y) \overset{\mathcal{B}}{\leftrightarrow} F(X)\right];$

$\square$

$\texttt{FunctorPreservesIso} :: \forall F : \texttt{Functor}\,(\mathcal{A},\mathcal{B}) \,.\, \forall f : \texttt{Iso}(\mathcal{A}) \,.\, F(f) : \texttt{Iso}(\mathcal{B})$

$\texttt{Proof} \ =$

Combine two last statements.

$\square$

$\texttt{CovariantMapsSplitMonoToSplitMono} :: \forall F : \texttt{Covariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : \texttt{SplitMono}\mathcal{A}(X,Y) \,.$

$\quad . \, F(f) : \texttt{SplitMono}(\mathcal{B})\left(F(X), F(Y)\right)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{CovariantMapsEpiEpiToSplitEpi} :: \forall F : \texttt{Covariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : \texttt{SplitEpi}\mathcal{A}(X,Y) \,.$

$\quad . \, F(f) : \texttt{SplitEp}(\mathcal{B})\left(F(X), F(Y)\right)$

$\texttt{Proof} \ =$

...

$\square$

$\texttt{ContravriantMapsSplitMonoToSplitEpi} :: \forall F : \texttt{Contravariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : \texttt{SplitMono}\mathcal{A}(X,Y) \,.$

$\quad .\, F(f) : \texttt{SplitEpi}(\mathcal{B})\Big(F(Y), F(X)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ContravariantMapsSplitRpiToSplitMono} :: \forall F : \texttt{Contravariant}(\mathcal{A},\mathcal{B}) \,.\, \forall f : \texttt{SplitEpi}\mathcal{A}(X,Y) \,.$

$\quad .\, F(f) : \texttt{SplitMono}(\mathcal{B})\Big(F(Y), F(X)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{sign} :: \texttt{Functor}\,(\mathcal{A},\mathcal{B}) \to \texttt{Sign}$

$\texttt{sign}\,(\texttt{Covariant}, F) := +1$

$\texttt{sign}\,(\texttt{Contravariant}, F) := -1$

$\texttt{InverseFunctoriality} :: \forall F : \texttt{Covariant}\ \&\ \texttt{FullyFaithful}(\mathcal{A},\mathcal{B}) \,.\, \forall X, Y, Z \in \mathcal{A} \,.$

$\quad .\, \forall f : F(X) \xrightarrow{\mathcal{B}} F(Y) \,.\, \forall g : F(Y) \xrightarrow{\mathcal{B}} F(X) \,.\, F^{-1}_{X,Y}(f) F^{-1}_{Y,Z}(g) = F^{-1}_{X,Z}(fg)$

$\texttt{Proof} \;=$

$f' := F^{-1}_{X,Y}(f) : X \xrightarrow{\mathcal{A}} Y,$

$g' := F^{-1}_{X,Y}(g) : X \xrightarrow{\mathcal{A}} Y,$

$(1) := \eth\texttt{Covariant}(F)\eth f'g' : F_{X,Z}(f'g') = F_{X,Y}(f')F_{Y,Z}F(g') = fg,$

$(*) := F^{-1}_{X,Z}(1) : f'g' = F^{-1}_{X,Z}(fg);$

$\square$

$\texttt{FullyFaithfulReflectsIso} :: \forall F : \texttt{Covariant}\ \&\ \texttt{FullyFaithful}(\mathcal{A},\mathcal{B}) \,.\, \forall X, Y \in \mathcal{A} \,.\, \forall f : X \xrightarrow{\mathcal{A}} Y \,.$

$\quad .\, \Big[F(f) : F(X) \overset{\mathcal{B}}{\leftrightarrow} F(Y)\Big] \Rightarrow \Big[f : X \overset{\mathcal{A}}{\leftrightarrow} Y\Big]$

$\texttt{Proof} \;=$

$(g, 1) := \eth\texttt{Iso}(F(f))\eth\texttt{FullyFaithful} : \sum g : Y \xrightarrow{\mathcal{A}} X \,.\, F(f)F(g) = \mathrm{id}_{F(X)}\ \&\ F(g)F(f) = \mathrm{id}_{F(Y)},$

$(2) := \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F)(1)\eth\texttt{FullyFaithful}(F) : fg = \mathrm{id}_A\ \&\ gf = \mathrm{id}_B,$

$(*) := \eth^{-1}\texttt{Iso}(2) : \Big[f : X \overset{\mathcal{A}}{\leftrightarrow} Y\Big];$

$\square$

$\texttt{FaithfulReflectsMono} :: \forall \mathcal{A}, \mathcal{B} : \texttt{Category} . \forall F : \texttt{Faithful}(\mathcal{A}, \mathcal{B}) . \forall f : A \xrightarrow{\mathcal{A}} B .$

$. \left[F(f) : F(A) \overset{\mathcal{B}}{\hookrightarrow} F(B)\right] \Rightarrow \left[f : A \overset{\mathcal{A}}{\hookrightarrow} B\right]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ C : \mathcal{A},$

$\texttt{Assume}\ h, g : C \xrightarrow{\mathcal{A}} A,$

$\texttt{Assume}\ (1) : hf = gf,$

$(2) := \eth\texttt{Covariant}(F)F(1)\eth\texttt{Covariant}(F) : F(h)F(f) = F(hf) = F(gf) = F(g)F(f),$

$(3) := \eth\texttt{Monic}\big(F(f)\big)(2) : F(h) = F(f),$

$() := \eth\texttt{Faithful}(F)(3) : h = f;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Monic} : [f : A \overset{\mathcal{A}}{\hookrightarrow} B],$

$\square$

$\texttt{FaithfulReflectsEpi} :: \forall \mathcal{A}, \mathcal{B} : \texttt{Category} . \forall F : \texttt{Faithful}(\mathcal{A}, \mathcal{B}) . \forall f : A \xrightarrow{\mathcal{A}} B .$

$. \left[F(f) : F(A) \overset{\mathcal{B}}{\twoheadrightarrow} F(B)\right] \Rightarrow \left[f : A \overset{\mathcal{A}}{\twoheadrightarrow} B\right]$

$\texttt{Proof} \ =$

Apply dual trick to previous theorem.

$\square$

1.4 Duality through Opposition

$\texttt{oppositeCategory} :: \texttt{Category} \to \texttt{Category}$
$\texttt{oppositeCategory}\,(\mathcal{C}) = \mathcal{C}^{\mathrm{op}} := (O, M \circ \texttt{swap}, c \circ \texttt{swap}, I) \quad \texttt{where} \quad \mathcal{C} = (O, M, c, I)$

$\texttt{oppose} :: \prod \mathcal{C} : \texttt{Category} \,.\, \texttt{Contravariant}(\mathcal{C}, \mathcal{C}^{\mathrm{op}})$
$\texttt{oppose}\,(X) = X := X$
$\texttt{oppose}\,(f) = f^{\mathrm{op}} := f$

$\texttt{ReflexiveOpposition} :: \forall \mathcal{C} \in \texttt{Category} \,.\, ((\mathcal{C})^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{dualStatement} :: (\texttt{Category} \to \texttt{Type}) \to (\texttt{Category} \to \texttt{Type})$
$\texttt{dualStatement}\,(\mathbb{T}) = \mathbb{T}^* := \Lambda \mathcal{C} \in \texttt{Category} \,.\, \mathbb{T}(\mathcal{C}^{\mathrm{op}})$

$\texttt{DualStatementIsReflexive} :: \forall \mathbb{T} : \texttt{Category} \to \texttt{Type} \,.\, \mathbb{T}^{**} = \mathbb{T}$
$\texttt{Proof} \;=$
$\ldots$
$\square$

$\texttt{DualTrick} :: \forall \mathbb{L} : \texttt{Category} \to \texttt{Logical} \,.\, \Big(\forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}(\mathcal{C})\Big) \iff \Big(\forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}^*(\mathcal{C})\Big)$
$\texttt{Proof} \;=$
$\texttt{Assume}\ (1) : \forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}(\mathcal{C}),$
$\texttt{Assume}\ \mathcal{C} : \texttt{Category},$
$(2) := (1)(\mathcal{C}^{\mathrm{op}}) : \mathbb{L}(\mathcal{C}^{\mathrm{op}}),$
$() := \eth^{-1}\texttt{dualStatement}(2) : \mathbb{L}^*(\mathcal{C});$
$\rightsquigarrow (3) := I(\forall)) : \forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}^*(\mathcal{C});$
$\rightsquigarrow (1) := I(\Rightarrow) : \Big(\forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}(\mathcal{C})\Big) \Rightarrow \Big(\forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}^*(\mathcal{C})\Big),$
$\texttt{Assume}\ (2) : \forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}^*(\mathcal{C}),$
$(3) := (1)(2) : \forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}^{**}(\mathcal{C}),$
$(4) := \texttt{DualStatementIsReflexive}(\mathbb{L}) : \mathbb{L}^{**} = \mathbb{L},$
$() := E(=)(4)(3) : \forall \mathcal{C} \in \texttt{Category} \,.\, \mathbb{L}(\mathcal{C});$
$\rightsquigarrow (*) := I(\iff)I(\Rightarrow)(1) : \texttt{This};$
$\square$

$\texttt{OppositeOfMonoIsEpi} :: \forall f : A \overset{\mathcal{C}}{\hookrightarrow} B \;.\; f^{\mathrm{op}} : B \xrightarrow{\mathcal{C}^{\mathrm{op}}} \!\!\!\rightarrow A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{OppositeOfEpiIsMono} :: \forall f : A \xrightarrow{\mathcal{C}} \!\!\!\rightarrow \;.\; f^{\mathrm{op}} : B \overset{\mathcal{C}^{\mathrm{op}}}{\hookrightarrow} A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{reverse} :: \texttt{Functor}\,(\mathcal{A}, \mathcal{B}) \to \texttt{Functor}\,(\mathcal{A}^{\mathrm{op}}, \mathcal{B})$

$\texttt{reverse}\,(F) = -F := F$

$\texttt{reverse}\,(F_{X,Y}) = -F_{X,Y} := F_{X,Y} \circ \texttt{swap}$

$\texttt{orientatedAlong} :: \prod \mathcal{A}, \mathcal{B} \in \texttt{Category} \;.\; \texttt{Functor}\,(\mathcal{A}, \mathcal{B}) \to \texttt{Category}$

$\texttt{orientatedAlong}\,(F) = |\mathcal{A}|_F := \texttt{if}\ \texttt{Functor}\,(F, =) - 1\ \texttt{then}\ \mathcal{C}^{\mathrm{op}}\ \texttt{else}\ \mathcal{C}$

$\texttt{wellOrientatedFunctor} :: \prod \mathcal{A}, \mathcal{B} \in \texttt{Category} \;.\; \texttt{Functor}\,(\mathcal{A}, \mathcal{B}) \to \texttt{Covariant}(|\mathcal{A}|_F, \mathcal{B})$

$\texttt{wellOrientatedFunctor}\,(F) = |F| := \texttt{sign}(F)F$

1.5 Representation of Functors

$\texttt{covariantRepresentedByObject} :: \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Covariant}(\mathcal{C}, \mathsf{SET})$

$\texttt{covariantRepresrntedByObject}\,(A) = \mathcal{M}_{\mathcal{C}}(A, \_\_) :=$

$:= \Big(X \mapsto \mathcal{M}_{\mathcal{C}}(A, X), \Lambda f \in \mathcal{M}_{\mathcal{C}}(X, Y) \,.\, \Lambda g \in \mathcal{M}_{\mathcal{C}}(A, X) \,.\, gf\Big)$

$\texttt{PushForward} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, X, Y \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(X, Y) \to \Big(\mathcal{M}_{\mathcal{C}}(A, X) \to \mathcal{M}_{\mathcal{C}}(A, Y)\Big)$

$\texttt{PushForward}\,(f) = (A) \quad f_* := \mathcal{M}_{\mathcal{C}}(A, \_\_)(f)$

$\texttt{contravariantRepresentedByObject} :: \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Contravariant}(\mathcal{C}, \mathsf{SET})$

$\texttt{contravariantRepresentedByObject}\,(A) = \mathcal{M}_{\mathcal{C}}(\_\_, A) :=$

$:= \Big(X \mapsto \mathcal{M}_{\mathcal{C}}(X, A), \Lambda f \in \mathcal{M}_{\mathcal{C}}(X, Y) \,.\, \Lambda g \in \mathcal{M}_{\mathcal{C}}(Y, A) \,.\, fg\Big)$

$\texttt{PullBack} :: \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, X, Y \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(X, Y) \to \Big(\mathcal{M}_{\mathcal{C}}(Y, A) \to \mathcal{M}_{\mathcal{C}}(X, A)\Big)$

$\texttt{PushForward}\,(f) = (A) \quad f^* := \mathcal{M}_{\mathcal{C}}(\_\_, A)(f)$

$\texttt{PushForwardOfIsoIsBijection} :: \forall \mathcal{C} \in \texttt{Category} \,.\, \forall A, X, Y \in \mathcal{C} \,.\, \forall f : X \overset{\mathcal{C}}{\leftrightarrow} Y \,.\, (A) \quad f_* : \texttt{Bijection}$

$\texttt{Proof} \;=$

By $\texttt{FunctorPreservesIso}$.

$\square$

$\texttt{PullBackOfIsoIsBijection} :: \forall \mathcal{C} \in \texttt{Category} \,.\, \forall A, X, Y \in \mathcal{C} \,.\, \forall f : X \overset{\mathcal{C}}{\leftrightarrow} Y \,.\, (A) \quad f^* : \texttt{Bijection}$

$\texttt{Proof} \;=$

By $\texttt{FunctorPreservesIso}$.

$\square$

$\texttt{PushForwardOfMonoIsInjection} :: \forall \mathcal{C} \in \texttt{Category} \,.\, \forall A, X, Y \in \mathcal{C} \,.\, \forall f : X \overset{\mathcal{C}}{\hookrightarrow} Y \,.\, (A) f_* : \texttt{Injection}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; g, h : \mathcal{M}_{\mathcal{C}}(A, X),$

$\texttt{Assume}\; (1) : f_*(g) = f_*(h),$

$(2) := ð((A) \quad f_*)(1) : gf = hf,$

$() := ð\texttt{Monic}(f)(2) : g = h;$

$\rightsquigarrow (*) := ð^{-1}\texttt{Injection} : \Big(f_* : \texttt{Injection}\Big),$

$\square$

$\texttt{PullBackOfEpiIsSurjection} :: \forall \mathcal{C} \in \texttt{Category} \,.\, \forall A, X, Y \in \mathcal{C} \,.\, \forall f : X \overset{\mathcal{C}}{\hookrightarrow} Y \,.\, (A) f^* : \texttt{Surjection}$

$\texttt{Proof} \;=$

Apply dual trick to precious theorem.

$\square$

1.6 Bifunctors

$\texttt{ProductOfCats} :: \texttt{Category} \times \texttt{Category} \to \texttt{Category}$

$\texttt{ProductOfCats}\,(\mathcal{A}, \mathcal{B}) = \mathcal{A} \times \mathcal{B} := \Big(\mathcal{O}_{\mathcal{A}} \times \mathcal{O}_{\mathcal{B}}, (A, B), (X, Y) \mapsto \mathcal{M}_{\mathcal{A}}(A, X) \times \mathcal{M}_{\mathcal{B}}(B, Y), \cdot \times \cdot, \mathrm{id} \times \mathrm{id}\Big)$

$\texttt{Bifunctor} := \Lambda \mathcal{C}, \mathcal{X} : \texttt{Category} \,.\, \texttt{Covariant}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{X}) : \texttt{Category} \to \texttt{Category} \to \texttt{Type};$

$\texttt{TwoSidedRepresented} :: \prod \mathcal{C} : \texttt{LocallySmall} \,.\, \texttt{Bifunctor}(\mathcal{C}, \mathsf{SET})$

$\texttt{TwoSidesRepresented}\,() = \mathcal{M}_{\mathcal{C}}(\_\_, \_\_) :=$

$:= \Big(\Lambda(X, Y) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(X, Y),$

$, \Lambda(A, B), (X, Y) \in C^{\mathrm{op}} \times C \,.\, \Lambda(f, g) : \mathcal{M}_{\mathcal{C}}(X, A) \times \mathcal{M}_{\mathcal{C}}(B, Y) \,.\, \Lambda h \in \mathcal{M}_{\mathcal{C}}(A, B) \,.\, fhg\Big)$

1.7 Categories of Categories

$$\texttt{identityFunctor} :: \prod \mathcal{C} \; . \; \texttt{Functor}\,(\mathcal{C}, \mathcal{C})$$

$$\texttt{identityFunctor}\,() = \mathrm{Id}_{\mathcal{C}} := (\Lambda X \in \mathcal{C} \; . \; X, \Lambda X, Y \in \mathcal{C} \; . \; \Lambda f : X \xrightarrow{\mathcal{C}} Y \; . \; f)$$

$$\texttt{Categories} :: \texttt{Category}$$

$$\texttt{Categories}\,() = \mathsf{LSCAT} := (\texttt{LocallySmall}, \texttt{Functor}, \circ, \mathrm{Id})$$

$$\texttt{SmallCategories} :: \texttt{Category}$$

$$\texttt{SmallCategories}\,() = \mathsf{SCAT} := (\texttt{Small}, \texttt{Functor}, \circ, \mathrm{Id})$$

1.8 Comma Categories

$\texttt{commaCategory} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C}, : \texttt{Category} \, . \, \texttt{Covariant}(\mathcal{A}, \mathcal{C}) \times \texttt{Covariant}(\mathcal{B}, \mathcal{C}) \to \texttt{Category}$

$\texttt{commaCategory}\,(F, G) = F \downarrow G := \Big(\sum A \in \mathcal{A} \, . \, B \in \mathcal{B} \, . \, FA \xrightarrow{\mathcal{C}} GB,$

$\Lambda(A, B, f), (X, Y, g) \, . \, \sum (\alpha, \beta) \in \mathcal{M}_{\mathcal{A}}(A, X) \times \mathcal{M}_{\mathcal{B}}(B, Y) \, . \, fG(\beta) = F(\alpha)g,$

$\texttt{compositionRule}(\mathcal{A}) \times \texttt{compositionRule}(\mathcal{B}), \mathrm{id} \times \mathrm{id} \Big)$

$\texttt{Assume}\ (A, B, f), (C, D, g), (X, Y, h) : F \downarrow G,$

$\texttt{Assume}\ (\alpha, \beta) : (A, B, f) \xrightarrow{F \downarrow G} (C, D, g),$

$\texttt{Assume}\ (\gamma, \delta) : (C, D, g) \xrightarrow{F \downarrow G} (X, Y, h),$

$(1) := \eth(\alpha, \beta) : fG(\beta) = F(\alpha)g,$

$(2) := \eth(\gamma, \delta) : gG(\delta) = F(\gamma)h,$

$(3) := \eth\texttt{Covariant}(G(\beta\delta))(1)(2)\eth\texttt{Covariant}(F(\alpha)F(\gamma)) :$

$: fG(\beta\delta) = fG(\beta)G(\delta) = F(\alpha)gG(\delta) = F(\alpha)F(\gamma)h = F(\alpha\gamma)h,$

$(*) := \eth^{-1} F \downarrow G(3) : (\alpha\gamma, \beta\delta) : (A, B, f) \xrightarrow{F \downarrow G} (C, D, g);$

$\square$

$\texttt{leftProjFunctor} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C} : \texttt{Category} \, . \, \prod F : \texttt{Covariant}(\mathcal{A}, \mathcal{C}) \, . \, \prod G : \texttt{Covariant}(\mathcal{B}, \mathcal{C}) \, .$

$. \, \texttt{Covariant}(F \downarrow G, \mathcal{A})$

$\texttt{leftProjFunctor}\,(A, B, f) = \Pi_{\mathcal{A}}(A, B, f) := A$

$\texttt{leftProjFunctor}\,(\alpha, \beta) = \Pi_{\mathcal{A}}(\alpha, \beta) := \alpha$

$\texttt{rightProjFunctor} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C} : \texttt{Category} \, . \, \prod F : \texttt{Covariant}(\mathcal{A}, \mathcal{C}) \, . \, \prod G : \texttt{Covariant}(\mathcal{B}, \mathcal{C}) \, .$

$. \, \texttt{Covariant}(F \downarrow G, \mathcal{B})$

$\texttt{rightProjFunctor}\,(A, B, f) = \Pi_{\mathcal{B}}(A, B, f) := B$

$\texttt{rightProjFunctor}\,(\alpha, \beta) = \Pi_{\mathcal{B}}(\alpha, \beta) := \beta$

$\texttt{SliceUnderAsComma} :: \forall \mathcal{C} : \texttt{Category} \, . \, \forall A \in \mathcal{C} \, . \, \dfrac{\mathcal{C}}{A} = I \downarrow \mathrm{Id}_{\mathcal{C}}$

where

$I = (1 \mapsto A, \underset{1}{\mathrm{id}} \mapsto \underset{A}{\mathrm{id}}) : \texttt{Covariant}(\mathbf{1}, \mathcal{C})$

Proof $=$

...

$\square$

$\texttt{SliceOverAsComma} :: \forall \mathcal{C} : \texttt{Category} \, . \, \forall A \in \mathcal{C} \, . \, \dfrac{A}{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}} \downarrow I$

where

$I = (1 \mapsto A, \underset{1}{\mathrm{id}} \mapsto \underset{A}{\mathrm{id}}) : \texttt{Covariant}(\mathbf{1}, \mathcal{C})$

Proof $=$

...

$\square$

1.9 Natural Transformations

$\texttt{NaturalTransform} :: \prod \mathcal{C}\mathcal{D} \in \texttt{Category} \;.\; F, G : \texttt{Covariant}(\mathcal{C}, \mathcal{D}) \;.\; \prod X \in \mathcal{C} \;.\; F(X) \xrightarrow{\mathcal{D}} G(X)$

$\alpha : \texttt{NaturalTransform} \iff (F \Rightarrow G) \iff \forall f : A \xrightarrow{\mathcal{C}} B \;.\; \alpha(A)G(f) = F(f)\alpha(B)$

$\texttt{NaturalIso} :: ?\texttt{NaturalTransform}(\mathcal{C}, \mathcal{D}, F, G)$

$\alpha : \texttt{NaturalIso} \iff (F \iff G) \iff \forall X \in \mathcal{C} \;.\; \alpha(X) : F(X) \xleftrightarrow{\mathcal{D}} G(X)$

$\texttt{invert} :: \texttt{NaturalIso}(\mathcal{C}, \mathcal{D}, F, G) \to \texttt{natuatalIso}(\mathcal{D}, \mathcal{C}, F, G)$
$\texttt{invert}\,(\alpha) = \alpha^{-1} := \Lambda X \in \mathcal{C} \;.\; \alpha_X^{-1}$

1.10 Equivalence of Categories

$\texttt{EquivalentCategories} :: ?(\texttt{Category} \times \texttt{Category})$

$(\mathcal{A}, \mathcal{B}) : \texttt{EquivalentCategories} \iff \mathcal{A} \simeq \mathcal{B} \iff \exists F : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) : \exists G : \texttt{Covariant}(\mathcal{B}, \mathcal{A}) :$
$: \exists(\mathrm{Id}_{\mathcal{A}} \iff FG) \times (\mathrm{Id}_{\mathcal{B}} \iff GF)$

$\texttt{CategoryEqIsEq} :: \Big[\texttt{EquivalentCategries} : \texttt{Equivalence}(\texttt{Category})\Big]$

$\texttt{Proof} =$

$\texttt{Assume}\ \mathcal{A}, \mathcal{B}, \mathcal{C} : \texttt{Category},$

$\texttt{Assume}\ (1) : \mathcal{A} \simeq \mathcal{B},$

$\texttt{Assume}\ (2) : \mathcal{B} \simeq \mathcal{C},$

$(F, G, \alpha, \beta) := \eth\texttt{EquivalentCategories}(1) : \sum F : \texttt{Covariant}(\mathcal{A}, \mathcal{B})\ .$
$.\ \sum G : \texttt{Covariant}(\mathcal{B}, \mathcal{A})\ .\ (\mathrm{Id}_{\mathcal{A}} \iff FG)\ \&\ (\mathrm{Id}_{\mathcal{B}} \iff GF),$

$(F', G', \alpha', \beta') := \eth\texttt{EquivalentCategories}(1) : \sum F' : \texttt{Covariant}(\mathcal{B}, \mathcal{C})\ .$
$.\ \sum G' : \texttt{Covariant}(\mathcal{C}, \mathcal{B})\ .\ (\mathrm{Id}_{\mathcal{B}} \iff F'G')\ \&\ (\mathrm{Id}_{\mathcal{C}} \iff G'F'),$

$\texttt{Assume}\ X : \mathcal{A},$

$\alpha''(X) := \alpha(X)G\Big(\alpha'\big(F(X)\big)\Big) : X \overset{\mathcal{A}}{\leftrightarrow} FF'G'G(X),$

$\texttt{Assume}\ Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$(3) := \eth\alpha''(X)FF'G'G(f)\eth\texttt{Covariant}(\mathcal{B}.\mathcal{A})(G)\eth\texttt{NaturalTransform}(\alpha')\big(F(X)\big)\big(F(f)\big)$
$\eth\texttt{Covariant}(\mathcal{B}, \mathcal{A})(G)\eth\texttt{NaturalTransform}(\alpha)(X)(f)\eth^{-1}\alpha''(Y) :$
$: \alpha''(X)FF'G'G(f) = \alpha(X)G\Big(\alpha'\big(F(X)\big)G(F'G(F(f))\Big) = \alpha(X)G\Big(\alpha'\big(F(X)\big)F'G'\big(F(f)\big)\Big) =$
$= \alpha(X)G\Big(F(f)\alpha'\big(F(Y)\big)\Big) = \alpha(X)FG(f)G\Big(\alpha'\big(F(Y)\big)\Big) = f\alpha(Y)G\Big(\alpha'\big(F(Y)\big)\Big) = f\alpha''(Y);$

$\rightsquigarrow \alpha'' := \eth^{-1}\texttt{NaturalIso} : \mathrm{Id}_{\mathcal{A}} \iff FF'G'G,$

$\texttt{Assume}\ X : \mathcal{C},$

$\beta''(X) := \beta'(X)F'\Big(\beta\big(G'(X)\big)\Big) : X \overset{\mathcal{A}}{\leftrightarrow} FF'G'G(X),$

$\texttt{Assume}\ Y : \mathcal{C},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{C}} Y,$

$(3) := \eth\beta''(X)G'GFF'(f)\eth\texttt{Covariant}(\mathcal{B}.\mathcal{C})(F')\eth\texttt{NaturalTransform}(\beta)\big(G'(X)\big)\big(G'(f)\big)$
$\eth\texttt{Covariant}(\mathcal{B}, \mathcal{C})(F')\eth\texttt{NaturalTransform}(\beta')(X)(f)\eth^{-1}\beta''(Y) :$
$: \beta''(X)G'GFF'(f) = \beta'(X)F'\Big(\beta\big(G'(X)\big)F'(GF(G'(f))\Big) = \beta'(X)F'\Big(\beta\big(G'(X)\big)GF\big(G'(f)\big)\Big) =$
$= \beta'(X)F'\Big(G'(f)\beta\big(G'(Y)\big)\Big) = \beta'(X)F'G'(f)F'\Big(\beta\big(G'(Y)\big)\Big) = f\beta'(Y)G\Big(\alpha'\big(F(Y)\big)\Big) = f\beta''(Y);$

$\rightsquigarrow \beta'' := \eth^{-1}\texttt{NaturalIso} : \mathrm{Id}_{\mathcal{C}} \iff G'GFF',$

$() := \eth^{-1}\texttt{EquivalentCategory}(FF', G'G, \alpha'', \beta'') : \mathcal{A} \simeq \mathcal{C};$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Transitive} : \Big[\texttt{EquivalentCategory} : \texttt{Transitive}\Big],$

$(*) := \ldots : \texttt{This};$

$\square$

$\texttt{ProvidesEquivalence} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, ?\texttt{Covariant}(\mathcal{A}, \mathcal{B})$

$F : \texttt{ProvidesEquivalence} \iff \exists G : \texttt{Covariant}(\mathcal{B}, \mathcal{A}) : \exists(\mathrm{Id}_{\mathcal{A}} \iff FG) \times (\mathrm{Id}_{\mathcal{B}} \iff GF)$

$\texttt{EquivalenceProvison} :: \forall \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, \forall F : \texttt{ProvidesEquivalence}(\mathcal{A}, \mathcal{B}) \,.\, A \simeq \mathcal{B}$

$\texttt{Proof} \;=$

$\square$

$\texttt{isoClass} :: \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \to \texttt{Kind}$

$\texttt{isoClass}\,(A) := \left\{ B \in \mathcal{C} : \exists f : A \overset{\mathcal{C}}{\leftrightarrow} B \right\}$

$\texttt{IsoClass} :: \texttt{Category} \to \texttt{Kind}$

$A : \texttt{IsoClass} \iff \Lambda\mathcal{C} : \texttt{Category} \,.\, \exists a \in \mathcal{C} : A = \texttt{isoclass}(a)$

$\texttt{Embedding} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, ?\texttt{Faithful}(\mathcal{A}, \mathcal{B})$

$(E, E') : \texttt{Embedding} \iff \left[E : \texttt{Injective}(\mathcal{A}, \mathcal{B}) \right]$

$\texttt{Subcategory} := \prod \mathcal{C} : \texttt{Category} \,.\, \sum \mathcal{A} : \texttt{Category} \,.\, \texttt{Embedding}(\mathcal{A}, \mathcal{C}) : \texttt{Category} \to \texttt{Type};$

$\texttt{synecdoche} :: \prod \mathcal{C} : \texttt{Category} \,.\, \texttt{Subcategory}(\mathcal{C}) \to \texttt{Category}$

$\texttt{synecdoche}\,(\mathcal{A}, E) := \mathcal{A}$

$\texttt{EssSubcat} :: ?\texttt{Subcategory}(\mathcal{C})$

$(\mathcal{A}, E) : \texttt{EssSubcat} \iff \forall C : \texttt{Isoclass}(\mathcal{C}) \,.\, \exists A \in \mathcal{A} : \exists c \in C : F(A) = c$

$\texttt{Essentially} :: \prod \mathbb{T} : \prod A, B : \texttt{Kind} \,.\, ?(A \to B) \,.\, \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, ?\texttt{Covariant}(\mathcal{A}, \mathcal{B})$

$(F, F') : \texttt{Essentially} \iff \exists \mathcal{A}' : \texttt{EssSubcat}(\mathcal{A}) : \exists \mathcal{B}' : \texttt{EssSubcat}(\mathcal{B}) : F(\mathcal{A}') \subset \mathcal{B}' \;\&\; F_{|\mathcal{A}'} : \mathbb{T}(\mathcal{A}', \mathcal{B}')$

$\texttt{Isofunctor} := \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, \texttt{FullyFaithful} \;\&\; \texttt{Essentially Bijective}(\mathcal{A}, \mathcal{B}) :$

$\quad : \texttt{Category} \times \texttt{Category} \to \texttt{Type};$

$\texttt{IsomorphismLemma} :: \forall \mathcal{C} : \texttt{Category} \,.\, \forall A, B, A', B' \in \mathcal{C} \,.\, \forall f : A \overset{\mathcal{C}}{\to} B \,.\, \forall \varphi : A \overset{\mathcal{C}}{\leftrightarrow} A' \,.\, \forall \psi : B \overset{\mathcal{C}}{\leftrightarrow} B' \,.$

$\quad . \exists! : f' : A' \overset{\mathcal{C}}{\to} B' : f\psi = \varphi f' \;\&\; \varphi^{-1} f \psi = f' \;\&\; f = \varphi f \psi^{-1} \;\&\; \varphi^{-1} f = f' \psi^{-1}$

$\texttt{Proof} \;=$

$f' := \varphi^{-1} f \psi : A' \overset{\mathcal{C}}{\to} B',$

$(1) := \varphi \eth f' : f\psi = \varphi f',$

$(2) := (1)\psi : f = \varphi f' \psi^{-1},$

$(*) := \varphi^{-1}(2) : \varphi^{-1} f = f' \psi^{-1};$

$\square$

$\texttt{EquivalenceProviderIsIsofunctor} :: \forall F : \texttt{EquivalenceProvider}(\mathcal{A}, \mathcal{B}) \, . \, \Big[F : \texttt{Isofunctor}(\mathcal{A}, \mathcal{B}) \Big]$

$\texttt{Proof} \; =$

$\texttt{Assume} \; \mathcal{A}, \mathcal{B} : \texttt{Category},$

$\texttt{Assume} \; F : \texttt{EquivalencProvider}(\mathcal{A}, \mathcal{B}),$

$(G, \alpha, \beta) := \eth F : \sum G : \texttt{Covariant}(\mathcal{B}, \mathcal{A}) \, . \, (\mathrm{Id}_{\mathcal{A}} \iff FG) \times (\mathrm{Id}_{\mathcal{B}} \iff GF),$

$\texttt{Assume} \; X, Y : \mathcal{A},$

$\texttt{Assume} \; f : X \xrightarrow{\mathcal{A}} Y,$

$(1) := \eth\alpha(f) : f\alpha(Y) = \alpha(X)FG(f),$

$() := \alpha^{-1}(X)(f) : f = \alpha(X)FG(f)\alpha^{-1}(Y);$

$\rightsquigarrow (1) := I(\forall) : \forall f : X \xrightarrow{\mathcal{A}} Y \, . \, f = \alpha^{-1}(X)FG(f)\alpha(Y),$

$(2) := \texttt{InjectiveByComposition}(F, 1) : \Big[F_{X,Y} : \mathcal{M}_{\mathcal{A}}(X, Y) \hookrightarrow \mathcal{M}_{\mathcal{B}}\big(F(X), F(Y)\big) \Big];$

$\rightsquigarrow (1) := I(\forall)\eth^{-1}\texttt{Faithful}I(\forall) : \forall F : \texttt{EquivalenceProvider}(\mathcal{A}, \mathcal{B}) \, . \, \Big[F : \texttt{Faithful}(\mathcal{A}, \mathcal{B}) \Big],$

$(G, \alpha, \beta) := \eth F : \sum G : \texttt{Covariant}(\mathcal{B}, \mathcal{A}) \, . \, (\mathrm{Id}_{\mathcal{A}} \iff FG) \times (\mathrm{Id}_{\mathcal{B}} \iff GF),$

$(2) := (1)(G) : \Big(G : \texttt{Faithful}(\mathcal{B}, \mathcal{A}) \Big),$

$\texttt{Assume} \; X, Y : \mathcal{A},$

$\texttt{Assume} \; g : F(X) \xrightarrow{\mathcal{B}} F(Y),$

$(f, 3) := \texttt{IsomorphismLemma}\Big(G(g), \alpha(X), \alpha(Y) \Big) : \sum f : X \xrightarrow{\mathcal{A}} Y \, . \, f = \alpha(X)G(g)\alpha^{-1}(Y),$

$(4) := \big((3)\alpha(Y)\big)\eth\alpha(f) : \alpha(X)G(g) = f\alpha(Y) = \alpha(X)FG(f),$

$(5) := \alpha^{-1}(X)(4) : G(g) = FG(f),$

$() := \texttt{InjectionIsRightInvertible}\eth\texttt{Faithful}(G) : g = F(f);$

$\rightsquigarrow () := \eth^{-1}\texttt{Bijective}\Big((1)(F)\eth\texttt{Faithful} \Big)\eth^{-1}\texttt{Surjective} : \Big[F_{X,Y} : \mathcal{M}_{\mathcal{A}}(X, Y) \leftrightarrow \mathcal{M}_{\mathcal{B}}\big(F(X), G(X)\big) \Big];$

$\rightsquigarrow (3) := \eth^{-1}\texttt{FullyFaithful} : \big[F : \texttt{FullyFaithul} \big],$

$\texttt{Assume} \; b : \mathcal{B},$

$(4) := \eth^{-1}\texttt{Isomorphic}\eth\beta(b) : b \cong_{\mathcal{B}} GF(b),$

$() := \eth\texttt{Image}(F)\Big(GF(b) \Big) : GF(b) \in \mathrm{Im}\, F;$

$\rightsquigarrow (4) := \eth^{-1} \ldots : \Big[F : \texttt{Essentially Bijective}(\mathcal{A}, \mathcal{B}) \Big],$

$() := \eth^{-1}\texttt{Isofuncctor}(3, 4) : \Big[F : \texttt{Isofuncot}(\mathcal{A}, \mathcal{B}) \Big];$

$\square$

$\texttt{IsofunctorProvidesEquivalence} :: \forall F : \texttt{IsoFunctor}(\mathcal{A},\mathcal{B}) \;.\; \forall (0) : \texttt{Choice} \;.$

$\quad . \Big[F : \texttt{EquivalenceProvider}(\mathcal{A},\mathcal{B})\Big]$

$\texttt{Proof} \;=$

$C := \Lambda S : \texttt{Isoclass}(\mathcal{B}) \;.\; F^{-1}(S) : \texttt{Isoclass}(\mathcal{B}) \to ?\mathcal{A},$

$(1) := \eth\texttt{Isofunctor}(F)\eth C : \forall S : \texttt{Isoclass}(\mathcal{B}) \;.\; C(S) \neq \emptyset,$

$(A,2) := (0)(C) : \sum A : \texttt{Isoclass}(\mathcal{B}) \to \mathcal{A} \;.\; \forall S : \texttt{Isoclass}(\mathcal{B}) \;.\; F(A(S)) \in S,$

$G' := \Lambda B \in \mathcal{B} \;.\; A(\texttt{isoclass}(B)) : \mathcal{B} \to \mathcal{A},$

$\texttt{Assume}\; B : \mathcal{B},$

$(3) := \eth\texttt{Isoclass}(2)\eth G'(B') : F\big(G'(B)\big) \cong_{\mathcal{B}} B,$

$\beta(B) := \eth\texttt{Isomorphic}(3) : B \overset{\mathcal{B}}{\leftrightarrow} F\big(G'(B)\big);$

$\rightsquigarrow \beta := I\left(\prod\right) : \prod B \in \mathcal{B} \;.\; B \overset{\mathcal{B}}{\leftrightarrow} F\big(G'(B)\big),$

$\texttt{Assume}\; A : \mathcal{A},$

$(3) := \eth G'(1)G'(F(A)) : F\Big(G'\big(F(A)\big)\Big) \cong_{\mathcal{B}} F(A),$

$\alpha(A) := F^{-1}_{A,G'FA}\Big(\beta\big(F(A)\big)\Big) : A \overset{\mathcal{A}}{\leftrightarrow} G'\big(F(A)\big);$

$\texttt{Assume}\; B, B' : \mathcal{B},$

$\texttt{Assume}\; f : B \overset{\mathcal{B}}{\to} B',$

$f' := \beta^{-1}(B)f\beta(B') : F\big(G'(B)\big) \overset{\mathcal{B}}{\to} F\big(G'(B')\big),$

$G''_{B,B'}(f) := F^{-1}_{G'(B),G'(B')}(f') : G'(B) \overset{\mathcal{A}}{\to} G''(B);$

$\rightsquigarrow G'' := I\left(\prod\right) I(\to) : \prod B, B' \in \mathcal{B} \;.\; (B \overset{\mathcal{B}}{\to} B') \to \big(G'(B) \overset{\mathcal{A}}{\to} G'(B')\big),$

$\texttt{Assume}\; B, B', B'' : \mathcal{B},$

$\texttt{Assume}\; f : B \overset{\mathcal{B}}{\to} B',$

$\texttt{Assume}\; f' : B' \overset{\mathcal{B}}{\to} B'',$

$() := \eth G''_{B,B'}(f)G''_{B',B''}(f')\texttt{InverseFunctoriality}(F)\eth\texttt{Inverse}\big(\beta(B')\big)\eth^{-1}G''_{B,B''}(ff') :$

$\quad : G''_{B,B'}(f)G''_{B',B'''}(f') = F^{-1}_{G'(B),G'(B')}\big(\beta^{-1}(B)f\beta(B')\big)F^{-1}_{G'(B'),G'(B'')}\big(\beta^{-1}(B')f'\beta(B'')\big) =$

$\quad = F^{-1}_{G'(B),G'(B'')}\big(\beta^{-1}(B)ff'\beta(B'')\big) = G''_{B,B''}(ff');$

$\rightsquigarrow (3) := I^3(\forall) : \forall B, B', B'' \in \mathcal{B} \;.\; \forall f : B \overset{\mathcal{B}}{\to} B' \;.\; \forall f' : B' \overset{\mathcal{B}}{\to} B'' \;.\; G''_{B,B'}(f)G'_{B',B''}(f') = G''_{B,B''}(ff'),$

$\texttt{Assume}\; B : \mathcal{B},$

$() := \eth G''_{B,B}\eth\texttt{Identity}(B)\eth\texttt{Inverse}\big(\beta(B)\big)\eth F :$

$\quad : G''_{B,B}(\mathrm{id}_B) = F^{-1}_{G'(B),G'(B)}(\beta^{-1}(B)\mathrm{id}_B\beta(B)) = F^{-1}_{G'(B),G'(B)}(\mathrm{id}_{FG'(B)}) = \mathrm{id}_{G'(B)};$

$\rightsquigarrow (4) := I(\forall) : \forall B \in \mathcal{B} \;.\; G''_{B,B}(\mathrm{id}_B) = \mathrm{id}_{G'(B)},$

$G := \big(G', G'', (3), (4)\big) : \texttt{Covariant}(\mathcal{B},\mathcal{A}),$

$\texttt{Assume}\ X, Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$() := \eth G(\alpha(X)FG(f))\texttt{InverseFunctoriality}(F)\eth^{-1}\alpha\eth\texttt{Inverse}(\alpha(X)) :$

$: \alpha(X)FG(f) = \alpha(X)F^{-1}_{FG(X),FG(Y)}(\beta^{-1}(F(X))F(f)\beta(F(Y))) = \alpha(X)\alpha^{-1}(X)f\alpha(Y) = f\alpha(Y);$

$\rightsquigarrow (5) := \eth^{-1}\texttt{NaturalIso} : [\alpha : \mathrm{Id}_A \iff FG],$

$\texttt{Assume}\ X, Y : \mathcal{B},$

$\texttt{Assume}\ f : X \xrightarrow{B} Y,$

$() := \eth G(\beta(X)GF(f))\eth\texttt{Inverse}(F)\eth\texttt{Inverse}\big(\beta(X)\big) :$

$: \beta(X)GF(f) = \beta(X)F\Big(F^{-1}\big(\beta^{-1}(X)f\beta(Y)\big)\Big) = f\beta(Y);$

$\rightsquigarrow (6) := \eth^{-1}\texttt{naturalIso} : [\beta : \mathrm{Id}_B \iff GF],$

$(*) := \eth^{-1}\texttt{EquivalenceProvider}(F)(G, \alpha, \beta) : \Big[F : \texttt{EquivalenceProvider}(\mathcal{A}, \mathcal{B})\Big];$

$\square$

$\texttt{IsofunctorComposition} :: \forall F : \texttt{Isofunctor}(\mathcal{A}, \mathcal{B}) \ .\ \forall G : \texttt{Isofunctor}(\mathcal{B}, \mathcal{C}) \ .\ FG : \texttt{isofunctor}(\mathcal{A}, \mathcal{C})$

$\texttt{Proof}\ =$

$\square$

$\texttt{Skeletal} :: ?\texttt{Category}$

$\mathcal{C} : \texttt{Skeletal} \iff \forall A \in \mathcal{C} \ .\ \#\texttt{Isoclass}(A) = 1$

$\texttt{Skeleton} :: \prod \mathcal{C} : \texttt{Category} \ .\ ?\texttt{Skeletal}$

$\mathcal{A} : \texttt{Skeleton} \iff A \simeq \mathcal{C}$

$\texttt{Essentially} :: ?\texttt{Category} \to ?\texttt{Category}$

$\mathcal{C} : \texttt{Essentially} \iff \Lambda\mathbb{T} :?\texttt{Category} \ .\ \exists\mathcal{A} : \mathbb{T} : \mathcal{A} \simeq \mathcal{C}$

1.11 Commutative Diagrams

$\texttt{Diagram} := \prod \mathcal{C} : \texttt{Category} \,.\, \sum \mathcal{I} : \texttt{Small} \,.\, \texttt{Covariant}(\mathcal{I}, \mathcal{C}) : \texttt{Category} \to \texttt{Type};$

$\texttt{index} :: \texttt{Diagram} \to \texttt{Small}$
$\texttt{index}\,(\mathcal{I}, D) := \mathcal{I}$

$\texttt{MorphismChain} := \prod \mathcal{C} : \texttt{Category} \,.\, \prod A, B \in \mathcal{C} \,.\, \sum n \in \mathbb{N} \,.\, \sum X : (n+1) \to \mathcal{C} \,.$
$\,.\, \sum f : \prod i \in n \,.\, X_i \xrightarrow{\mathcal{C}} X_{i+1} \,.\, X_1 = A \,\&\, X_{n+1} = B : \prod \mathcal{C} : \texttt{Category} \,.\, \mathcal{C} \times \mathcal{C} \to \texttt{Type};$

$\texttt{Commutative} :: ?\texttt{Diagram}$

$$(\mathcal{I}, D) : \texttt{Commutative} \iff \forall (n, X, f), (m, Y, g) : \texttt{MorphismChain}(\mathcal{I})(I, J) \,.\, \prod_{i=1}^{n} D(f_i) = \prod_{i=1}^{m} D(g_i)$$

$\texttt{FunctorPreservesCommutativity} :: \forall (\mathcal{I}, D) : \texttt{Commutative}(\mathcal{A}) \,.\, \forall F : \texttt{Functor}(\mathcal{A}, \mathcal{B}) \,.$
$\,.\, (\mathcal{I}, DF) : \texttt{Commutative}(\mathcal{B})$

Proof =

...

□

$\texttt{Initial} :: \prod \mathcal{C} : \texttt{Category} \,.\, ?\mathcal{C}$

$I : \texttt{Initial} \iff \forall A \in \mathcal{C} \,.\, \exists! f : I \xrightarrow{\mathcal{C}} A$

$\texttt{Terminal} :: \prod \mathcal{C} : \texttt{Category} \,.\, ?\mathcal{C}$

$T : \texttt{Terminal} \iff \forall A \in \mathcal{C} \,.\, \exists! f : A \xrightarrow{\mathcal{C}} T$

$\texttt{Zero} := \texttt{Initial} \,\&\, \texttt{Terminal} : \texttt{Category} \to \texttt{Type};$

$\texttt{CommutativityOfChainsAtZero} :: \forall (n, X, f), (m, Y, g) : \texttt{MorphismChain}(\mathcal{C})(A, B) \,.$

$$.\, \Big(A : \texttt{Initial}(\mathcal{C}) \Big| B : \texttt{Terminal}(\mathcal{C})\Big) \Rightarrow \prod_{i=1}^{n} f_i = \prod_{i=1}^{m} g_i$$

Proof =

...

□

$\texttt{Concrete} := \sum \mathcal{C} : \texttt{Category} \,.\, \texttt{Faithful}(\mathcal{C}, \mathsf{SET}) : \texttt{Type};$

$\texttt{synecdoche} :: \texttt{Concrete} \to \texttt{Category}$
$\texttt{synecdoche}\,(\mathcal{C}, F) := \mathcal{C}$

$\texttt{FaithfulReflectsCommutative} :: \forall(\mathcal{I}, D) : \texttt{Diagramm}(\mathcal{A}) \;.\; \forall F : \texttt{Faithful}(\mathcal{A}, \mathcal{B}) \;.$

$$. \left[(\mathcal{I}, FD) : \texttt{Commutative}(\mathcal{B})\right] \Rightarrow \left[(\mathcal{I}, D) : \texttt{Commutative}(\mathcal{A})\right]$$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{ZeroCondition} :: \forall I : \texttt{Initial}(\mathcal{C}) \;.\; \forall T : \texttt{Terminal}(\mathcal{C}) \;.\; \forall f : T \xrightarrow{\mathcal{C}} I \;.\; I \cong T$

$\texttt{Proof} \;=$

$\dots$

$\square$

1.12 Coalgebras

$$\texttt{Coalgebra} := \prod T : \texttt{Covariant}(\mathcal{C},\mathcal{C}) \;.\; ?\sum X \in \mathcal{C} \;.\; X \xrightarrow{\mathcal{C}} TX : \prod \mathcal{C} : \texttt{Category} \;.\; \texttt{Covariant}(\mathcal{C},\mathcal{C}) \to \texttt{Type};$$

$$\texttt{CoalgebraMorphism} :: \prod (A,\alpha),(B,\beta) : \texttt{Coalgebra}(\mathcal{C})(T) \;.\; ?(A \xrightarrow{\mathcal{C}} B)$$
$$f : \texttt{CoalgebraMorphism} \iff \alpha T f = f\beta$$

$$\texttt{coalgebraCategory} :: \prod \mathcal{C} : \texttt{Category} \;.\; \texttt{Covariant}(\mathcal{C},\mathcal{C}) \to \texttt{Category}$$
$$\texttt{coalgebraCategory}\,(T) = \mathsf{COALG}(T) :=$$
$$:= \Big(\texttt{Coalgebra}(T), \texttt{CoalgebraMorphism}, \texttt{compositionLaw}(\mathcal{C}), \texttt{idMorphism}(\mathcal{C})\Big)$$

$$\texttt{TerminalCoalgebra} :: \forall (X,\gamma) : \texttt{Terminal}\Big(\mathsf{COALG}(\mathcal{C})(T)\Big) \;.\; \gamma : X \overset{\mathcal{C}}{\leftrightarrow} TX$$

Proof $=$

$\gamma' := T\gamma : TX \overset{\mathcal{C}}{\leftrightarrow} T^2X,$

$(1) := \eth\texttt{Coalgebra}(TX,\gamma') : \Big[(TX,\gamma') : \texttt{Coalgebra}(T)\Big],$

$(2) := \eth\gamma'(\gamma\gamma') : \gamma\gamma' = \gamma T\gamma,$

$(3) := \eth^{-1}\texttt{CoalgMorphism}(2) : \Big[\gamma : (X,\gamma) \xrightarrow{\mathsf{COALG}(T)} (TX,\gamma')\Big],$

$f := \eth\texttt{Terminal}(\mathsf{COALG}(T))(X,\gamma) : (TX,\gamma') \xrightarrow{\mathsf{COALG}(T)} (X,\gamma),$

$(4) := \eth\texttt{Terminal}(\mathsf{COALG}(T))(X,\gamma)(\gamma f) : \gamma f = \mathrm{id}_{(X,\gamma)},$

$(5) := \eth\texttt{Covariant}(T)(4)\eth\texttt{CoalgebraMorphism}(f)\eth\texttt{Covariant}(T) :$

$\quad : \mathrm{id}_{(TX,T\gamma)} = T\Big(\mathrm{id}_{(X,\gamma)}\Big) = T(\gamma f) = T\gamma T f = f\gamma,$

$(6) := \eth^{-1}\texttt{Inverse} : f = \gamma^{-1},$

$(*) := \eth\mathsf{COALG}(\mathcal{C})(T)(6) : \Big[\gamma : X \overset{\mathcal{C}}{\leftrightarrow} TX\Big];$

$\square$

1.13 Functor Category

$\texttt{verticalCompositonOfNT} :: \prod \mathcal{A}, \mathcal{B} : \texttt{Category} \,.\, F, G, H : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) \,.$

$\quad . \,(F \Rightarrow G) \times (G \Rightarrow H) \to (F \Rightarrow H)$

$\texttt{verticalCompositionOfNT}\,(\alpha, \beta) = \alpha\beta := \prod X \in \mathcal{A} \,.\, \alpha(X)\beta(X)$

$\texttt{Assume}\ X, Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$(1) := \eth\alpha(f) : \alpha(X)G(f) = F(f)\alpha(Y),$

$(2) := \eth\beta(f) : \beta(X)H(f) = G(f)\beta(Y),$

$() := \eth\alpha\beta\Big(\alpha\beta(X)H(f)\Big)(2)(1)\eth^{-1}\alpha\beta :$

$\quad : \alpha\beta(X)H(f) = \alpha(X)\beta(X)H(f) = \alpha(X)G(f)\beta(Y) = F(f)\alpha(Y)\beta(Y) = F(f)\alpha\beta(Y);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturalTransform}(\mathcal{A}, \mathcal{B}) : [\alpha\beta : F \Rightarrow H];$

$\square$

$\texttt{functorCategory} :: \texttt{Category} \times \texttt{Category} \to \texttt{Category}$

$\texttt{functorCategory}\,(\mathcal{A}, \mathcal{B}) = \mathcal{B}^{\mathcal{A}} :=$

$\quad := \Big(\texttt{Covariant}(\mathcal{A}, \mathcal{B}), (F, G) \mapsto F \Rightarrow G, \texttt{verticalCompositionOfNT}, F \mapsto \big(X \mapsto \mathrm{id}_{F(X)}\big)\Big)$

1.14 2-Category of All Categories

$\texttt{horisontalComposition} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C} \in \texttt{Category} . \forall F, G : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) . \forall H, E : \texttt{Covariant}(\mathcal{B}, \mathcal{C}) .$
$. (F \Rightarrow G) \times (H \Rightarrow E) \to (FH \Rightarrow GE)$

$\texttt{horisontalComposition}(\alpha, \beta) = \alpha * \beta := \prod X \in \mathcal{A} . H\big(\alpha(X)\big)\beta\big(G(X)\big)$

$\texttt{Assume } X, Y : \mathcal{A},$

$\texttt{Assume } f : X \xrightarrow{\mathcal{A}} Y,$

$(1) := \eth\beta(\alpha(X)) : \beta(F(X))E(\alpha(X)) = H(\alpha(X))\beta(G(X)),$

$(2) := \eth\alpha(f) : \alpha(X)G(f) = F(f)\beta(X),$

$(3) := \eth\beta\Big(F(f)\alpha(Y)\Big) : \beta(F(X))E(F(f)\alpha(Y)) = H(F(f)\alpha(Y))\beta(G(Y)),$

$(*) := \eth\alpha * \beta(\alpha * \beta(X)GE(f))(1)\eth\texttt{Covariant}E(2)(3)\eth\texttt{Covariant}(H)\eth^{-1}\alpha * \beta :$

$: \alpha * \beta(X)GE(f) = H\big(\alpha(X)\big)\beta\big(G(X)\big)GE(f) = \beta(F(X))E(\alpha(X))GE(f) = \beta(F(X))E(\alpha(X)G(f)) =$

$= \beta(F(X))E(F(f)\alpha(Y)) = H(F(f)\alpha(Y))\beta(G(Y)) = FH(f)H(\alpha(Y))\beta(G(Y)) = FH(f)\alpha * \beta(Y);$

$\leadsto (*) := \eth^{-1}\texttt{NaturalTransform} : [\alpha * \beta : FH \Rightarrow GE];$

□

$\texttt{FourInterchangeLemma} :: \forall \mathcal{A}, \mathcal{B}, \mathcal{C} : \texttt{Category} . \forall F, G, H : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) . \forall K, L, M : \texttt{Covariant}(\mathcal{B}, \mathcal{C}) .$
$. \forall \alpha : F \Rightarrow G . \forall \beta : G \Rightarrow H . \forall \gamma : K \Rightarrow L . \forall \delta : L \Rightarrow M . (\alpha\beta) * (\gamma\delta) = (\alpha * \gamma)(\beta * \delta)$

$\texttt{Proof} =$

$\texttt{Assume } X : \mathcal{A},$

$(*) := \eth\texttt{horisontalComposition}\eth\texttt{verticalComposition}\eth\gamma(\beta(X))\eth\texttt{Covariant}(K)$
$\eth^{-1}\texttt{horisontalComposition}\eth^{-1}\texttt{verticalComposition} :$

$: (\alpha\beta) * (\gamma\delta)(X) = K(\alpha\beta(X))\gamma\delta(H(X)) = K(\alpha(X)\beta(X))\gamma(H(X))\delta(H(X)) =$

$= K(\alpha(X))K(\beta(X))\gamma(H(X))\delta(H(X)) = K(\alpha(X))\gamma(G(X))L(\beta(X))\delta(H(X)) = (\alpha * \gamma)(\beta * \delta)(X);$

□

$\texttt{TwoCategory} :: ?\sum \mathcal{C} : \texttt{Category} . \sum \mathcal{F} : \prod A, B \in \mathcal{C} . \texttt{Category} . \prod A, B, C \in \mathcal{C} .$
$. \forall A, B \in \mathcal{C} . \mathcal{M}_{\mathcal{C}}(A, B) = \mathcal{F}(A, B)\ \&$
$\& \prod F, G \in \mathcal{F}(A, B) . \prod H, E \in \mathcal{F}(B, C) . \mathcal{M}_{\mathcal{F}(A,B)}(F, G) \times \mathcal{M}_{\mathcal{F}(B,C)}(H, E) \to \mathcal{M}_{\mathcal{F}(A,C)}(FH, GE)$

$(\mathcal{C}, \mathcal{F}, \mathrm{h}) : \texttt{TwoCategory} \iff \text{2-}\texttt{Category} \iff \forall A, B, C \in \mathcal{C} . \forall f, g, h : A \xrightarrow{\mathcal{C}} B . \forall x, y, z : B \xrightarrow{\mathcal{C}} C .$
$\forall \alpha : f \xrightarrow{\mathcal{M}_{\mathcal{C}}(A,B)} g . \forall \beta : g \xrightarrow{\mathcal{M}_{\mathcal{C}}(A,B)} h . \forall \gamma : x \xrightarrow{\mathcal{M}_{\mathcal{C}}(B,C)} y . \forall \delta : y \xrightarrow{\mathcal{M}_{\mathcal{C}}(B,C)} z . \mathrm{h}(\alpha\beta, \gamma\delta) = \mathrm{h}(\alpha, \gamma)\mathrm{h}(\beta, \delta)\ \&$
$\& \Big(\mathcal{C}, (A, B) \mapsto \sum f, g : A \xrightarrow{\mathcal{C}} B . f \xrightarrow{\mathcal{M}_{\mathcal{C}}(A,B)} g, \mathrm{h}, A \mapsto (\mathrm{id}_A, \mathrm{id}_A, \mathrm{id}_{\mathrm{id}_A})\Big) : \texttt{Category}$

$\texttt{catCat} :: \text{2-}\texttt{Category}$

$\texttt{catCat}() = \mathsf{CAT} := \Big((\texttt{Category}, \texttt{Covariant}, \circ, \mathrm{Id}), (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{B}^{\mathcal{A}}, \texttt{verticalComposition}\Big)$

$\texttt{synecdoche} :: \text{2-}\texttt{Category} \to \texttt{Category}$

$\texttt{synecdoche}(\mathcal{C}, \mathcal{F}, \mathrm{h}) := \mathcal{C}$

$\texttt{arrow2} :: \prod \mathcal{C} : 2\text{-}\texttt{Category} \;.\; \prod A, B \in \mathcal{C} \;.\; (A \xrightarrow{\mathcal{C}} B) \to \texttt{Kind}$

$\texttt{arrow2}\,(f, g) = f \overset{\mathcal{C}}{\Rightarrow} g := \mathcal{M}_{\mathcal{F}(A,B)}(f, g) \quad \texttt{where} \quad \mathcal{C} = (\mathcal{C}, \mathcal{F}, \mathrm{h})$

$\texttt{composition2} :: \prod \mathcal{C} : 2\text{-}\texttt{Category} \;.\; \prod A, B, C \in \mathcal{C} \;.\; \prod f, g : A \xrightarrow{\mathcal{C}} B \;.\; \prod x, y : B \xrightarrow{\mathcal{C}} C \;.$

$\quad .\; (f \overset{\mathcal{C}}{\Rightarrow} g) \times (x \overset{\mathcal{C}}{\Rightarrow} y) \to (fx \overset{\mathcal{C}}{\Rightarrow} gy)$

$\texttt{Composition2}\,(\alpha, \beta) = \alpha * \beta := \mathrm{h}(\alpha, \beta) \quad \texttt{where} \quad \mathcal{C} = (\mathcal{C}, \mathcal{F}, \mathrm{h})$

2 Yoneda's Theory

2.1 Representation

$\texttt{toSetFunctor} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathsf{SET}$

$\texttt{toSetFunctor}\,(A) = \{\cdot\}(A) := \{A\}$

$\texttt{toSetFunctor}\,(f) = \{\cdot\}_{A,B}(f) := A \mapsto B$

$\texttt{InitialRepresentation} :: \forall \mathcal{C} \in \mathsf{CAT} \,.\, \forall I \in \mathcal{C} \,.\, \Big[I : \texttt{Initial}(\mathcal{C}) \Big] \iff \mathcal{M}_{\mathcal{C}}(I, \cdot) \cong \{\cdot\}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : \Big[I : \texttt{Initial} \Big],$

$\texttt{Assume}\ X : \mathcal{C},$

$(2) := \eth\texttt{Initial}(\mathcal{C})(I)(X) : \Big[\mathcal{M}_{\mathcal{C}}(I, X) : \texttt{Singleton}\big(\mathcal{M}_{\mathcal{C}}(I, X)\big) \Big],$

$(f, 3) := \eth\texttt{Singleton}(2) : \sum f : I \xrightarrow{\mathcal{C}} X \,.\, \mathcal{M}_{\mathcal{C}}(I, X) = \{f\},$

$\alpha(X)(f) := X : \{f\} \xleftrightarrow{\mathsf{SET}} \{X\};$

$\rightsquigarrow \alpha := I\left(\prod\right) : \prod X \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(I, X) \iff \{X\},$

$\texttt{Assume}\ X, Y : \mathcal{C},$

$\texttt{Assume}\ \varphi : X \xrightarrow{\mathcal{C}} Y,$

$() := \eth\texttt{Singleton} : \varphi\alpha(Y)(f) = Y = \alpha(X)\{\cdot\}(f);$

$\rightsquigarrow (2) := \eth^{-1}\texttt{NaturalIso} : [\alpha : \mathcal{M}_{\mathcal{C}}(I, \cdot) \iff \{\cdot\}],$

$(3) := \eth^{-1}\texttt{Isomorphic}(2) : \mathcal{M}_{\mathcal{C}}(I, \cdot) \cong \{\cdot\};$

$\rightsquigarrow (1) := I(\Rightarrow) : \Big[I : \texttt{Initial}(\mathcal{C}) \Big] \Rightarrow \mathcal{M}_{\mathcal{C}}(I, \cdot) \cong \{\cdot\},$

$\texttt{Assume}\ (2) : \mathcal{M}_{\mathcal{C}}(I, \cdot) \cong \{\cdot\},$

$\alpha := \eth\texttt{Isomorphic}(2) : \mathcal{M}_{\mathcal{C}}(I, \cdot) \iff \{\cdot\},$

$\texttt{Assume}\ X : \mathcal{C},$

$(3) := \eth\alpha(X) : \mathcal{M}_{\mathcal{C}}(I, X) \cong_{\mathsf{SET}} \{X\},$

$(4) := \eth^{-1}\texttt{Card}(3) : \Big| \mathcal{M}_{\mathcal{C}}(I, X) \Big| \cong \Big| \{X\} \Big|;$

$\rightsquigarrow () := \eth^{-1}\texttt{Initial} : \Big[I : \texttt{Initial}(\mathcal{C}) \Big];$

$(*) := I(\iff)(1)I(\Rightarrow) : \texttt{This};$

$\square$

$\texttt{TerminalRepresentation} :: \forall \mathcal{C} \in \texttt{Category} \,.\, \forall T \in \mathcal{C} \,.\, \Big[T : \texttt{Terminal}(\mathcal{C}) \Big] \iff \mathcal{M}_{\mathcal{C}^{\mathrm{op}}}(\cdot, T) \cong \{\cdot\}$

$\texttt{Proof} \;=$

Apply dual trick.

$\square$

$\texttt{Representable} :: \prod \mathcal{C} \in \mathsf{LSCAT} \ . \ ?\texttt{Functor}\,(\mathcal{C}, \mathsf{SET})$

$F : \texttt{Representable} \iff \exists R \in \mathcal{C} \ . \ |F| \cong \mathcal{M}_{|\mathcal{C}|_F}(R, \cdot)$

$\texttt{RepresentedBy} :: \prod \mathcal{C} \in \mathsf{LSCAT} \ . \ \mathcal{C} \to \texttt{Functor}\,(\mathcal{C}, \mathsf{SET})$

$F : \texttt{RepresentedBy} \iff \Lambda R \in \mathcal{C} \ . \ |F| \cong \mathcal{M}_{|\mathcal{C}|_F}(R, \cdot)$

$\texttt{Representing} :: \prod \mathcal{C} \in \mathsf{LSCAT} \ . \ \texttt{Functor}\,(\mathcal{C}, \mathsf{SET}) \to ?\mathcal{C}$

$F : \texttt{Representing} \iff \Lambda F \in \texttt{Functor}\,(\mathcal{C}, \mathsf{SET}) \ . \ |F| \cong \mathcal{M}_{|\mathcal{C}|_F}(R, \cdot)$

$\texttt{RepresentablePresevesMono} :: \forall \mathcal{C} \in \mathsf{LSCAT} \ . \ \forall F : \texttt{Representable}(\mathcal{C}) \ \& \ \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \ .$

$\quad . \ \forall A, B \in \mathcal{C} \ . \ \forall f : A \overset{\mathcal{C}}{\hookrightarrow} B \ . \ F_{A,B}(f) : F(A) \xhookrightarrow{\mathsf{SET}} F(B)$

$\texttt{Proof} \ =$

$(X, 1) := \eth\texttt{Representable}(\mathcal{C})(f) : \sum X \in \mathcal{C} \ . \ F \cong \mathcal{M}_{\mathcal{C}}(X, \cdot),$

$(2) := (1)\eth\mathcal{M}_{\mathcal{C}}(X, \cdot)(f) : F(f) \cong f_*,$

$(*) := (2)\texttt{PushForwardOfMonoIsInjection}(X, A, B, f) : \Big[F(f) : \texttt{Injection}\Big];$

$\square$

2.2 Yoneda's Lemma

$\texttt{YonedasLemma} :: \forall \mathcal{C} \in \mathsf{LSCAT} \;.\; \forall F : \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \;.\; \forall X \in \mathcal{C} \;.\; F(X) \cong_{\mathsf{SET}} \mathcal{M}_{\mathsf{SET}^{\mathcal{C}}}\Big(\mathcal{M}_{\mathcal{C}}(X, \cdot), F\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : F(X),$

$\texttt{Assume}\; A : \mathcal{C},$

$\phi(x)(A) := \Lambda f \in \mathcal{M}_{\mathcal{C}}(X, A) \;.\; F(f)(x) : \mathcal{M}_{\mathcal{C}}(X, A) \to F(A);$

$\rightsquigarrow \phi(x) := I\left(\prod\right) : \prod A \in \mathcal{C} \;.\; \mathcal{M}_{\mathcal{C}}(X, A) \to F(A),$

$\texttt{Assume}\; A, B : \mathcal{C},$

$\texttt{Assume}\; f : A \xrightarrow{\mathcal{C}} B,$

$\texttt{Assume}\; g : X \xrightarrow{\mathcal{C}} A,$

$() := \eth f_* \eth \phi(x) B \eth \texttt{Covariant}(\mathcal{C}, \mathsf{SET})(F) \eth^{-1} \phi(x)(A) :$

$\quad : f_*\phi(x)(B)(g) = \phi(x)(B)(gf) = F(gf)(x) = F(g)F(f)(x) = F(f)\Big(\phi(x)(A)(g)\Big);$

$\rightsquigarrow () := I(=, \to) : f_*\phi(x)(B) = \phi(x)(A)F(f);$

$\rightsquigarrow () := \eth^{-1}\texttt{NaturalTransform} : \phi(x) : \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F;$

$\rightsquigarrow \phi := I(\to) : F(X) \to \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F,$

$\psi := \Lambda \alpha : \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F \;.\; \alpha(X)(\mathrm{id}_X) : \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F \to F(X),$

$(1) := \eth \phi \psi \eth \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \eth^{-1} \mathrm{id}_{F(X)} : \phi\psi = \Lambda x \in F(x) \;.\; F(\mathrm{id}_X)(x) = \Lambda x \in F(x) \;.\; x = \mathrm{id}_{F(X)},$

$(2) := \eth \psi \phi \Lambda \alpha \in \mathcal{M}_{\mathcal{C}}(X, \cdot) \;.\; \eth \texttt{NaturalTransform}(\alpha) \eth^{-1}\, \mathrm{id} :$

$\quad : \psi\phi = \Lambda \alpha \in \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F \;.\; \Lambda A \in \mathcal{C} \;.\; \Lambda f : X \xrightarrow{\mathcal{C}} A \;.\; F(f)\big(\alpha(X)(\mathrm{id}_X)\big) =$

$\quad = \Lambda \alpha \in \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F \;.\; \Lambda A \in \mathcal{C} \;.\; \Lambda f : X \xrightarrow{\mathcal{C}} A \;.\; \alpha(A)(f) = \underset{\mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F}{\mathrm{id}},$

$(*) := \eth^{-1}\texttt{Isomorphic}(1)(2) : F(X) \cong \mathcal{M}_{\mathsf{SET}^{\mathcal{C}}}\Big(\mathcal{M}_{\mathcal{C}}(X, \cdot), F\Big);$

$\square$

$\texttt{functorOfYoneda} :: \prod \mathcal{C} \in \mathsf{LSCAT} \;.\; \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \to \texttt{Covariant}(\mathcal{C}, \mathsf{SET})$

$\texttt{functorOfYoneda}\,(F, X) = \mathbb{Y}^F(X) := \mathcal{M}_{\mathsf{SET}^{\mathcal{C}}}\Big(\mathcal{M}_{\mathcal{C}}(X, \cdot), F\Big)$

$\texttt{functorOfYoneda}\,(F, X, Y, f) = \mathbb{Y}^F_{X,Y}(f) := \Lambda \alpha \in \mathbb{Y}^F(X) \;.\; \Lambda g \in \mathcal{M}_{\mathcal{C}}(Y, A) \;.\; \alpha(A)(fg)$

$\texttt{mapOfYoneda} :: \prod \mathcal{C} \in \texttt{Category} \;.\; \prod F : \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \;.\; \mathbb{Y}^F \iff F$

$\texttt{mapOfYoneda}\,(X, \alpha) = Y^F(X)(\alpha) := \alpha(X)(\mathrm{id}_X)$

$\texttt{Assume}\; A, B : \mathcal{C},$

$\texttt{Assume}\; f : A \xrightarrow{\mathcal{C}} B,$

$() := \eth Y^F(A) \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; \eth \alpha \eth f_* \eth^{-1} \mathbb{Y}^F_{A,B} \eth^{-1} Y^f :$

$\quad : Y^F(A) F_{A,B}(f) = \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; F_{A,B}(f)\Big(\alpha(A)(\mathrm{id}_A)\Big) =$

$\quad = \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; \alpha(A) F_{A,B}(f)(\mathrm{id}_A) = \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; f_*\alpha(B)(\mathrm{id}_A) = \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; \alpha(B)(f) =$

$\quad = \Lambda \alpha \in \mathbb{Y}^F(A) \;.\; \mathbb{Y}^F(f)(\alpha)(\mathrm{id}_B) = \mathbb{Y}^F_{A,B}(f) Y^F(B);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturalIso} : \Big[Y^F : \mathbb{Y}^F(f) \iff F\Big];$

$\square$

$\texttt{functorOfYoneda2} :: \prod \mathcal{C} \in \mathsf{LSCAT} \,.\, \mathcal{C} \to \texttt{Covariant}\left(\mathsf{SET}^{\mathcal{C}}_{\mathrm{Cov}}, \mathsf{SET}\right)$

$\texttt{functorOfYoneda2}\,(X, F) = \mathbb{Y}^X(F) := \mathcal{M}_{\mathsf{SET}^{\mathcal{C}}}\left(\mathcal{M}_{\mathcal{C}}(X, \cdot), F\right)$

$\texttt{functorOfYoneda2}\,(X, F, G, \alpha) = \mathbb{Y}^X_{F,G}(\alpha) := \Lambda\beta : \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F \,.\, \beta\alpha$

$\texttt{evaluationFunctor} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \mathcal{C} \to \texttt{Covariant}(\mathsf{SET}^{\mathcal{C}}, \mathcal{C})$

$\texttt{evaluetionFuncor}\,(X, F) = \mathrm{Ev}^X(F) := F(X)$

$\texttt{evaluationFunctor}\,(X, F, G, \alpha) = \mathrm{Ev}^X_{F,G}(\alpha) := \alpha(X)$

$\texttt{mapOfYoneda2} :: \prod \mathcal{C} \in \mathsf{LSCAT} \,.\, \prod X \in \mathcal{C} \,.\, \mathbb{Y}^X \iff \mathrm{Ev}^X$

$\texttt{mapOfYoneda2}\,(F, \alpha) = Y^X(F)(\alpha) := \alpha(X)(\mathrm{id}_X)$

$\texttt{Assume}\ F, G : \texttt{Covariant}(\mathcal{C}, \mathsf{SET}),$

$\texttt{Assume}\ \alpha : F \Rightarrow G,$

$() := \eth \mathrm{Ev}^X_{F,G} \eth Y^X(F) \eth^{-1} \texttt{verticalComposition}(\alpha, \beta) \eth^{-1} \mathbb{Y}^X_{F,G} \eth^{-1} Y^X(G) :$

$: Y^X(F)\mathrm{Ev}^X_{F,G}(\alpha) = Y^X(F)\alpha(X) = \Lambda\beta \in \mathbb{Y}^X(F) \,.\, \alpha(X)\Big(\beta(X)(\mathrm{id}_X)\Big) =$

$= \Lambda\beta \in \mathbb{Y}^X(F) \,.\, \beta\alpha(X)(\mathrm{id}_X) = \mathbb{Y}^X_{F,G}(\alpha) Y^X(G);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{naturalIso} : \left[Y^X : \mathbb{Y}^X \iff \mathrm{Ev}^X\right];$

$\square$

$\texttt{DualYonedasLemma} :: \forall \mathcal{C} \in \mathsf{LSCAT} \,.\, \forall F : \texttt{Contravariant}(\mathcal{C}, \mathsf{SET}) \,.\, \forall X \in \mathcal{C} \,.$

$.\, \mathcal{M}_{\mathsf{SET}^{\mathcal{C}}}\Big(\mathcal{M}(\cdot, X), F\Big) \cong_{\mathsf{SET}} F(X)$

$\texttt{Proof} =$

Apply dual trick to Yoneda's Lemma

$\square$

2.3 Yoneda's Embedding

$$\texttt{embeddingOfYoneda} :: \prod \mathcal{C} \in \mathsf{LSCAT} \,.\, \texttt{FullyFaithful} \,\&\, \texttt{Embedding}(\mathcal{C}, \mathsf{SET}^{\mathcal{C}^{\mathrm{op}}})$$
$$\texttt{embeddingOfYoneda}\,(X) = \mathrm{y}(X) := \mathcal{M}_{\mathcal{C}}(\cdot, X)$$
$$\texttt{embeddingOfYoneda}\,(X, Y, f) = \mathrm{y}_{X,Y}(f) := f_*$$

$$\texttt{embeddingOfYoneda2} :: \prod \mathcal{C} \in \mathsf{LSCAT} \,.\, \texttt{FullyFaithful} \,\&\, \texttt{Embedding}(\mathcal{C}^{\mathrm{op}}, \mathsf{SET}^{\mathcal{C}})$$
$$\texttt{embeddingOfYoneda2}\,(X) = \mathrm{y}(X) := \mathcal{M}_{\mathcal{C}}(X, \cdot)$$
$$\texttt{embeddingOfYoneda2}\,(X, Y, f) = \mathrm{y}_{X,Y}(f) := f^*$$

2.4 Universal Property

$$\texttt{UniversallyEq} :: \prod \mathcal{C} \in \mathsf{LSCAT} \;.\; ?\Big(\mathcal{C} \times \mathcal{C}\Big)$$
$$A, B : \texttt{UniversallyEq} \iff \mathrm{y}(A) \cong \mathrm{y}(B)$$

$$\texttt{UniversallyEqAreIsomorphic} :: \prod \mathcal{C} \in \mathsf{LSCAT} \;.\; \forall A, B \in \mathcal{C}\big[(A, B) : \texttt{UniversallyEq}(\mathcal{C})\big] \iff A \cong_{\mathcal{C}} B$$

Proof $=$

Use the fact that fully faithful functor both creates and preserves isomorphism

□

$$\texttt{UnversalMappingProperty} := \Lambda \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \;.\; \Lambda F : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) \;.\; \Lambda B \in \mathcal{B} \;.$$
$$.\; \texttt{Initial}\Big(\mathrm{Const}(B) \downarrow F\Big) : \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \;.\; \texttt{Covariant}(\mathcal{A}, \mathcal{B}) \to \mathcal{B} \to \texttt{Type};$$

$$\texttt{mediatingMorphismMap} :: \prod (1, A, f) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) \;.\; \prod X \in \mathcal{A} \;.$$
$$.\; \mathcal{M}_{\mathcal{B}}(B, F(X)) \to \mathcal{M}_{\mathcal{A}}(A, X)$$
$$\texttt{mediatingMorphismMap}\,(g) = \tau_{B,X}(A, f)(g) := \Big(\eth\texttt{Initial}(\mathrm{Const}(B) \downarrow F)(1, A, f)(1, X, g)\Big)_2$$

$$\texttt{MediatingInverse} :: \forall (1, A, f) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) \;.\; \forall X \in \mathcal{A} \;.$$
$$.\; \forall h : A \xrightarrow{\mathcal{B}} X \;.\; \tau_{B,X}(A, f)\Big(fF(h)\Big) = h$$

Proof $=$

$g := fF(h) : B \xrightarrow{F} (X),$

$(1) := \eth g : fF(h) = g,$

$(2) := \eth \mathrm{Const}(B) \downarrow F : \Big[(\mathrm{id}, h) : (1, A, f) \xrightarrow{\mathrm{Const}(B)\downarrow F} (1, X, g)\Big],$

$(*) := E\texttt{Unique}\eth\texttt{Initial}(\mathrm{Const}(B) \downarrow F)(1, A, f) : \tau_{B,X}(A, f)\Big(fF(h)\Big) = h;$

□

$$\texttt{MediatingInverseFormula} :: \forall (1, A, f) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) \;.\; \forall X \in \mathcal{A} \;.$$
$$\tau^{-1}_{B,X}(A, f) = \Lambda h : X \xrightarrow{\mathcal{A}} A \;.\; fF(h)$$

Proof $=$

...

□

$$\texttt{MediatingFormula} :: \forall (1, A, f) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) \;.\; \forall X \in \mathcal{A} \;.$$
$$.\; \forall g : B \xrightarrow{\mathcal{B}} F(X) \;.\; fF\big(\tau_{B,X}(A, f)(g)\big) = g$$

Proof $=$

$(1) := \texttt{MediatingInverse}(1, A, f)(X)\Big(\tau_{B,X}(A, f)(g)\Big) : \tau_{B,X}(A, f)\Big(fF\big(\tau_{B,X}(A, f)(g)\big)\Big) = \tau_{B,X}(A, f)(g),$

$(*) := \tau^{-1}_{B,X}(A, f)(1) : fF\big(\tau_{B,X}(A, f)(g)\big);$

□

$\texttt{MediatingMorphismsAreNatural} :: \forall (1, A, f) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) .$

$\tau_{B,\cdot}(A, f) : F\mathcal{M}_{\mathcal{B}}(B, \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, \cdot)$

$\texttt{Proof} =$

$\texttt{Assume } X, Y : \mathcal{A},$

$\texttt{Assume } y : X \xrightarrow{\mathcal{A}} Y,$

$\texttt{Assume } g : B \xrightarrow{\mathcal{B}} F(X),$

$() := \eth\texttt{Covariant}(F)\texttt{MediatingFormula}(A, f)(X)(g) : fF(\tau_{B,X}(A, f)(g)y) = fF(\tau_{B,X}(A, f)g)F(y) = gF(y);$

$\rightsquigarrow (1) := \eth\mathrm{Const}(B) \downarrow F : \left[\tau_{B,X}(A, f)(g)y : (1, A, f) \xrightarrow{\mathrm{Const}(B)\downarrow F} (1, Y, gF(y))\right],$

$() := \eth\big(F(y)\big) * \eth\tau_{B,Y}(A, f)\eth\texttt{UnversalMappingProperty}(1, A, f)(1) :$

$: \big(F(y)\big)_* \tau_{B,Y}(A, f) = \Lambda g : B \xrightarrow{\mathcal{B}} F(X) . \tau_{B,Y}(A, f)(gF(y)) =$

$= \Lambda g : B \xrightarrow{\mathcal{B}} F(X) . \tau_{B,X}(A, f)(g)y = \tau_{B,X}(A, f)y_*;$

$\rightsquigarrow () := \eth\texttt{NaturalIso} : \tau_{B,\cdot}(A, f) : F\mathcal{M}_{\mathcal{B}}(B, \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, \cdot);$

□

$\texttt{UniversalPropertyIsUnique} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{CAT} . \forall F : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) . \forall B \in \mathcal{B} .$

$. \forall (T, f), (S, g) : \texttt{UnversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) . \exists \alpha : T \overset{\mathcal{A}}{\leftrightarrow} S : g = fF(\alpha)$

$\texttt{Proof} =$

Use properties of initial objects.

□

$\texttt{CouniversalMappingProperty} := \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} . \prod F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} . \prod B \in \mathcal{B} .$

$. \texttt{Terminal}\big(F \downarrow \mathrm{Const}(B)\big) : \prod \mathcal{A}, \mathcal{B} \in \texttt{Category} . \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \to \mathcal{B} \to \texttt{Type};$

$\texttt{comediatingMorphismMap} :: \prod (A, 1, f) : \texttt{CouniversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) . \prod X \in \mathcal{A} .$

$. \mathcal{M}_{\mathcal{B}}(F(X), A) \to \mathcal{M}_{\mathcal{A}}(X, A)$

$\texttt{comediatingMorphismMap}(g) = \mu_{B,X}(A, f)(g) := \Big(\eth\texttt{Terminal}(F \downarrow \mathrm{Const}(B))(A, 1, f)(X, 1, g)\Big)_2$

$\texttt{ComediatingInverseFormula} :: \forall (A, 1, f) : \texttt{CouniversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) . \forall X \in \mathcal{A} .$

$\mu^{-1}_{B,X}(A, f) = \Lambda h : X \xrightarrow{\mathcal{A}} A . F(h)f$

$\texttt{Proof} =$

...

□

$\texttt{comediatingFormula} :: \forall (A, 1, f) : \texttt{CouniversalMappingProperty}(\mathcal{A}, \mathcal{B}, F, B) . \forall X \in \mathcal{A} .$

$. \forall g : F(X) \xrightarrow{\mathcal{B}} B . F\big(\mu_{B,X}(A, f)(g)\big)f = g$

$\texttt{Proof} =$

...

□

comediatingMorphismsAreNatural :: $\forall (A, f)$: UnversalMappingProperty$(\mathcal{A}, \mathcal{B}, F, B)$.
$\tau_{B,\cdot}(A, f) : F\mathcal{M}_{\mathcal{B}}(\cdot, B) \iff \mathcal{M}_{\mathcal{A}}(\cdot, A)$

Proof $=$

$\ldots$

$\square$

2.5 Category of Universal Elements

$$\texttt{categoryOfElements} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Covariant}\Big(\mathcal{C}, \mathsf{SET}\Big) \to \mathsf{CAT}$$

$$\texttt{categoryOfElements}\,(F) = \int F :=$$

$$:= \Big(\sum X \in \mathcal{C} \,.\, F(X), \big((X,x),(Y,y)\big) \mapsto \{f : X \xrightarrow{\mathcal{C}} Y : F(f)(x) = y\}, \cdot_{\mathcal{C}}, \mathrm{id}\Big)$$

$$\texttt{categoryOfElements2} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Contravariant}\Big(\mathcal{C}, \mathsf{SET}\Big) \to \mathsf{CAT}$$

$$\texttt{categoryOfElements2}\,(F) = \int F :=$$

$$:= \Big(\sum C \in \ \,.\, F(X), \big((X,x),(Y,y)\big) \mapsto \{f : X \xrightarrow{\mathcal{C}} Y : F(f)(x) = y\}, \cdot_{\mathcal{C}}, \mathrm{id}\Big)$$

$$\texttt{projectionFunctor} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \prod F : \texttt{Covariant}\Big(\mathcal{C}, \mathsf{SET}\Big) \,.\, \texttt{Covariant}\left(\int F, \mathcal{C}\right)$$

$$\texttt{projectionFunctor}\,(X,x) = \Pi(X,x) := X$$

$$\texttt{projectionFunctoe}\,((X,x),(Y,y),f) = \Pi(f) := f$$

$$\texttt{ElementsAsComma} :: \forall \mathcal{C} \in \mathsf{CAT} \,.\, \forall F : \texttt{Contravariant}(\mathcal{C}, \mathsf{SET}) \,.\, \int F \simeq \mathrm{y} \downarrow \mathrm{Const}(F)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (X,x) : \int F,$

$\texttt{Assume}\ A : \mathcal{C},$

$\alpha(A) := \Lambda f \in \mathcal{M}_{\mathcal{C}}(A,X) \,.\, F(f)(x) : \mathcal{M}_{\mathcal{C}}(A,X) \to F(A);$

$\rightsquigarrow \alpha := I\left(\prod\right) I(\to) : \prod A \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(A,X) \to F(A),$

$\texttt{Assume}\ A, B : \mathcal{C},$

$\texttt{Assume}\ f : B \xrightarrow{\mathcal{C}} A,$

$() := \eth f^* \,.\, \eth\alpha(B)\eth\texttt{Contravariant}(\mathcal{C}, \mathsf{SET})(F)\eth\eth^{-1}\alpha(A) :$

$\quad : f^*\alpha(B) = \Lambda g \in \mathcal{M}_{\mathcal{C}}(A,X) \,.\, \alpha(B)(fg) = \Lambda g \in \mathcal{M}_{\mathcal{C}}(A,X) \,.\, F(fg)(x) =$

$\quad = \Lambda g \in \mathcal{M}_{\mathcal{C}}(A,X) \,.\, F(g)F(f)(x) = \alpha(A)F(f);$

$\rightsquigarrow () := \eth^{-1}\texttt{NaturalTransform} : \big[\alpha : \mathcal{M}_{\mathcal{C}}(\cdot, X) \Rightarrow F\big],$

$G'(X,x) := (X, 1, \alpha) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$\rightsquigarrow G' := I(\to) : \int F \to \mathrm{y} \downarrow \mathrm{Const}(F),$

$\texttt{Assume}\ f : (X,x) \xrightarrow{\int F} (Y,y),$

$G''\Big((X,x),(Y,y)\Big)(f) := (f, \mathrm{id}_1) : \Big(X \xrightarrow{\mathcal{C}} Y\Big) \times \Big(1 \xrightarrow{\mathbf{1}} 1\Big),$

$(X, 1, \alpha) := G'(X,x) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$(Y, 1, \beta) := G'(Y,y) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$$() := \eth\mathtt{Covariant}\left(\mathbf{1}, \mathsf{SET}^{\mathcal{C}^{\mathrm{op}}}\right)\eth\alpha\eth^{-1}\mathtt{categoryOfElements}(f)\eth\mathtt{Contravariant}(\mathcal{C}, \mathsf{SET})(F)\eth^{-1}\beta\eth^{-1}f_* :$$

$$: \alpha\mathrm{Const}(F)(\mathrm{id}_1) = \alpha\mathrm{id}_F = \Lambda A \in \mathcal{C} \ . \ \Lambda g \in \mathcal{M}_{\mathcal{C}}(A, X) \ . \ F(g)(x) =$$

$$= \Lambda A \in \mathcal{C} \ . \ \Lambda g \in \mathcal{M}_{\mathcal{C}}(A, X) \ . \ F(f)F(g)(y) = \Lambda A \in \mathcal{C} \ . \ \Lambda g \in \mathcal{M}_{\mathcal{C}}(A, X) \ . \ F(gf)(y) = f_*\beta;$$

$$\rightsquigarrow G'' := I\left(\prod\right) I(\rightarrow) : \prod(X, x), (Y, Y) \in \int F \ . \ (X, x) \xrightarrow{\int F} (Y, y) \to G'(X, x) \xrightarrow{\mathrm{y}\downarrow\mathrm{Const}(F)} G'(Y, y),$$

$$G := (G', G'') : \mathtt{Covariant}\left(\int F, \mathrm{y} \downarrow \mathrm{Const}(F)\right),$$

$\mathtt{Assume}\ (X, 1, \alpha) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$x := \alpha(X)(\mathrm{id}_X) : F(X),$

$$H'(X, 1, \alpha) := (X, x) : \int F;$$

$$\rightsquigarrow H' := I(\rightarrow) : \mathrm{y} \downarrow \mathrm{Const}(F) \to \int F,$$

$\mathtt{Assume}\ (X, 1, \alpha), (Y, 1, \beta) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$\mathtt{Assume}\ (f, \mathrm{id}) : (X, 1, \alpha) \xrightarrow{\mathrm{y}\downarrow\mathrm{Const}(F)} (Y, 1, \beta),$

$H''(f, \mathrm{id}) := f : X \xrightarrow{\mathcal{C}^{\mathrm{op}}} Y,$

$(1) := \eth\mathtt{commaCategory}(f) : \forall A \in \mathcal{C} \ . \ \alpha(A) = f_*\beta(A),$

$$(2) := (1)(X)\eth^{-1}f^*\eth\mathtt{NaturalTransform}(\beta)f : \alpha(X)(\mathrm{id}_X) = f_*\beta(X)(\mathrm{id}_X) = f^*\beta(X)(\mathrm{id}_Y) =$$

$$= F(f)\big(\beta(Y)(\mathrm{id}_Y)\big);$$

$$\rightsquigarrow H'' := I\left(\prod\right) I(\rightarrow) : \prod(X, 1, \alpha), (Y, 1, \beta) \in \mathrm{y} \downarrow \mathrm{Const}(F) \ .$$

$$. \ (X, 1, \alpha) \xrightarrow{\mathrm{y}\downarrow\mathrm{Const}(F)} (Y, 1, \beta) \to H'(X, 1, \alpha) \xrightarrow{\int F} H'(Y, 1, \beta)),$$

$$H := (H', H'') : \mathtt{Covariant}\left(\mathrm{y} \downarrow \mathrm{Const}(F), \int F\right),$$

$$\mathtt{Assume}\ (X, x) : \int F,$$

$$() := \eth G \eth H : GH(X, x) = H\Big(X, 1, \Lambda A \in \mathcal{C} \ . \ \Lambda f : A \xrightarrow{\mathcal{C}} X \ . \ F(f)(x)\Big) = \big(X, F(\mathrm{id}_X)(x)\big) = (X, X);$$

$$\rightsquigarrow (1) := I(\forall) : \forall (X, x) \in \int F \ . \ GH(X, x) = (X, x),$$

$\mathtt{Assume}\ (X, 1, \alpha) : \mathrm{y} \downarrow \mathrm{Const}(F),$

$$() := \eth H \eth G \eth\mathtt{NaturalTransform}(\alpha)\eth^{-1}\alpha : HG(X, 1, \alpha) = G\big(X, \alpha(X)(\mathrm{id}_X)\big) =$$

$$= \Big(X, 1, \Lambda A \in \mathcal{C} \ . \ \Lambda f \in \mathcal{M}(A, X) \ . \ F(f)(\alpha(X)(\mathrm{id}_X))\Big) =$$

$$= \Big(X, 1, \Lambda A \in \mathcal{C} \ . \ \Lambda f \in \mathcal{M}(A, X) \ . \ \alpha(A)(f)\Big) = (X, 1, \alpha);$$

$$\rightsquigarrow (2) := I(\forall) : \forall (X, 1, \alpha) \in \mathrm{y} \downarrow \mathrm{Const}(F) \ . \ HG(X, 1, \alpha) = (X, 1, \alpha),$$

$(3) := (1)(2) : HG = \mathrm{Id}GH = \mathrm{Id},$

$$(*) := \eth^{-1}\mathtt{Isomorphic}(\mathsf{CAT})(3) : \int F \cong_{\mathsf{CAT}} \mathrm{y} \downarrow \mathrm{Const}(F);$$

$\square$

$\texttt{RepresentableIffInitialElement} :: \forall \mathcal{C} \in \mathsf{LSCAT} \,.\, \forall F : \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \,.$

$$. \left[F : \texttt{Representable}(\mathcal{C})\right] \iff \exists \texttt{Initial}\left(\int F\right)$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : \left[F : \texttt{Representable}(\mathcal{C})\right],$

$(X, \alpha) := \eth\texttt{Representable}(\mathcal{C})(F) : \prod X \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(X, \cdot) \iff F,$

$\texttt{Assume}\ (A, \alpha(f)) : \int F,$

$() := \eth \int F \eth^{-1} \mathcal{M}_{\int F}(X, \alpha_X(\mathrm{id}_X))(A, \alpha_A(f)) : \mathcal{M}_{\int F}\Big((X, \alpha_X(\mathrm{id}_X)), (A, \alpha_A(f))\Big) = \{f\};$

$\rightsquigarrow () := \eth^{-1}\texttt{Initial} : \left[(X, \alpha_X(\mathrm{id}_X)) : \texttt{Initial}\left(\int F\right)\right];$

$\rightsquigarrow (1) := I(\Rightarrow) : \left[F : \texttt{Representable}\right] \Rightarrow \exists\texttt{initial}\left(\int F\right),$

$\texttt{Assume}\ (X, x) : \texttt{Initial}\left(\int F\right),$

$\texttt{Assume}\ A : \mathcal{C},$
$\texttt{Assume}\ f : \mathcal{M}_{\mathcal{C}}(X, A),$
$\alpha(A)(f) := F(f)(x) : F(A);$
$\rightsquigarrow \alpha := I\left(\prod\right) I(\rightarrow) : \mathcal{M}_{\mathcal{C}}(X, \cdot) \Rightarrow F,$
$\texttt{Assume}\ A : \mathcal{C},$
$\texttt{Assume}\ a : F(A),$

$(f, 2) := \eth\texttt{Initial}\left(\int F\right)(X, x) : \sum f : (X, x) \xrightarrow{\int F} (A, a) \,.\, \mathcal{M}_{\int F}\Big((X, x), (A, a)\Big) = \{f\},$

$(3) := \eth \int F(f) : F(f)(x) = a,$

$\beta(A)(a) := f : X \xrightarrow{\mathcal{C}} A;$

$\rightsquigarrow \beta := I\left(\prod\right) I(\rightarrow) : \prod A \in \mathcal{C} \,.\, F(A) \rightarrow X \xrightarrow{\mathcal{C}} A,$

$\texttt{Assume}\ A, B : \mathcal{C},$

$\texttt{Assume}\ \phi : A \xrightarrow{\mathcal{C}} B,$

$\texttt{Assume}\ a : F(A),$
$(3) := \eth\beta(A)(a)(x) : \beta(A)(a)(x) = a,$
$(4) := F(\phi)(3) : \beta(A)(a)F(\phi)(x) = F(\phi)(a),$
$(5) := \eth\beta(B)(F(\phi)(a))(x) : \beta(B)(F(\phi)(a))(x) = F(\phi)(a),$

$(6) := \eth\beta\eth\texttt{Initial}\left(\int F\right)(X, x) : \beta(B)(F(\phi))(a) = \beta(A)(a),$

$() := \eth\phi_*(6) : \beta(A)\phi_*(a) = \beta(A)(a)\phi = \beta(B)\big(F(\phi)(a)\big) = F(\phi)\beta(B)(a);$
$\rightsquigarrow (3) := \eth^{-1}\texttt{NaturalTransform} : [\beta : F \Rightarrow \mathcal{M}_C(X, \cdot)],$

$\texttt{Assume}\ A : \mathcal{C},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{C}} A,$

$() := \eth\alpha(A)\eth\beta(A)\eth\texttt{Initial}\left(\int F\right)(X,x) : \alpha(A)\beta(A)(f) = \beta(A)\big(F(f)(x)\big) = f;$

$\rightsquigarrow (4)^* := I(=,\to) : \alpha(A)\beta(A) = \mathrm{id},$

$\texttt{Assume}\ a : F(A),$

$() := \eth\beta(A)\eth\alpha(A)\eth\texttt{Initial}\left(\int F\right)(X,x) : \beta(A)\alpha(A)(a) = a;$

$\rightsquigarrow () := I(=,\to) : \beta(A)\alpha(A) = \mathrm{id};$

$\rightsquigarrow (4) := \eth\texttt{Inverse} : \beta = \alpha^{-1},$

$(5) := \eth^{-1}\texttt{NaturalIso} : \big[\alpha : \mathcal{M}(X,\cdot) \iff F\big],$

$() := \eth^{-1}\texttt{Representable} : \Big[F : \texttt{Representable}(\mathcal{C})\Big];$

$\rightsquigarrow () := I(\iff)(1)I(\Rightarrow) : \texttt{This};$

□

$\texttt{RepresentableIffTerminalElements} :: \forall \mathcal{C} \in \mathsf{CAT}\ .\ \forall F : \texttt{Contra}(\mathcal{C}, \mathsf{SET})\ .$

$.\ \Big[F : \texttt{Reprsentable}(\mathcal{C})\Big] \iff \exists\texttt{Terminal}\left(\int F\right)$

$\texttt{Proof}\ =$

Apply dual trick to previous theorem.

□

$\texttt{ContractibleGroupoid} :: ?\mathsf{CAT}$

$F : \texttt{ContractibleGoupoid} \iff \forall A, B \in \mathcal{C}\ .\ \exists f : A \overset{\mathcal{C}}{\leftrightarrow} B\ .\ \mathcal{M}_{\mathcal{C}}(A,B) = \{f\}$

$\texttt{RepresentationSpanCG} :: \forall \mathcal{C} \in \mathsf{CAT}\ .\ \forall F : \texttt{Representable}(\mathcal{C})\ .$

$.\ \texttt{cat}\left(\int F, \texttt{Representation}(F)\right) : \texttt{ContractibleGroupoid}$

$\texttt{Proof}\ =$

Use the fact that every representation corresponds to an initial object and all initial objects are isomotphic.

□

$\texttt{ElementsAsComma2} :: \forall \mathcal{C} \in \mathsf{LSCAT} \,.\, \forall F : \texttt{Covariant}(\mathcal{C}, \mathsf{SET}) \,.\, \int F \cong_{\texttt{Category}} \mathrm{Const}\big(\{1\}\big) \downarrow F$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (X, x) : \int F,$

$G'(X, x) := (1, X, x) : F \downarrow \mathrm{Const}\big(\{1\}\big);$

$\rightsquigarrow G' := I(\to) : \int F \to F \downarrow \mathrm{Const}\big(\{1\}\big),$

$\texttt{Assume}\ (X, x), (Y, y) : \int F,$

$\texttt{Assume}\ f : (X, x) \xrightarrow{\int F} (Y, y),$

$G''(f) := (\mathrm{id}, f) : (1 \xrightarrow{\mathbf{1}} 1) \times (X \xrightarrow{\mathcal{C}} Y),$

$() := \eth f : y = f(x);$

$\rightsquigarrow G'' := I\left(\prod\right) I(\to) : \prod (X, x), (Y, y) \in \int F \,.\, \left((X, x) \xrightarrow{\int F} (Y, y)\right) \to \left((1, X, x) \xrightarrow{\mathrm{Const}(\{1\})\downarrow F} (1, Y, y)\right),$

$G := (G', G'') : \texttt{Covariant}\left(\int F, \mathrm{Const}\big(\{1\}\big) \downarrow F\right),$

$\texttt{Assume}\ (1, X, x) : \mathrm{Const}\big(\{1\}\big) \downarrow F,$

$H'(1, X, x) := (X, x) : \int F;$

$\rightsquigarrow H' := I(\to) : \left(\mathrm{Const}\big(\{1\}\big) \downarrow F\right) \to \int F,$

$\texttt{Assume}\ (1, X, x), (1, Y, y) : \mathrm{Const}\big(\{1\}\big) \downarrow F,$

$\texttt{Assume}\ (\mathrm{id}, f) : (1, X, x) \to (1, Y, y),$

$H''(\mathrm{id}, f) := f : X \xrightarrow{\mathcal{C}} Y,$

$() := \eth f : f(x) = y;$

$\rightsquigarrow H'' := I\left(\prod\right) I(\to) : \prod (1, X, x), (1, Y, y) \in \mathrm{Const}\big(\{1\}\big) \downarrow F \,.$

$\quad .\, \left((1, X, x) \to (1, Y, y)\right) \to ((X, x) \to (Y, y)),$

$H := (H', H'') : \texttt{Covariant}\left(\mathrm{Const}(\{1\}), \int F\right),$

$(1) := \eth H \eth G : H = G^{-1},$

$(*) := \eth\texttt{Isomorphic}(\mathsf{CAT}) : \int F \cong_{\mathsf{CAT}} \mathrm{Const}\big(\{1\}\big) \downarrow F;$

$\square$

$\texttt{twistedArrows} :: \mathsf{LSCAT} \to \mathsf{CAT}$

$\texttt{twistedArrows}\,(\mathcal{C}) = \mathcal{C}^{\rightsquigarrow} := \int \mathcal{M}_{\mathcal{C}}$

$\texttt{elemants} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Covariant}\left(\mathsf{SET}^{\mathcal{C}}, \frac{\mathsf{CAT}}{\mathcal{C}}\right)$

$\texttt{elements}\,(F) = \int F := \left(\int F, \Pi\right)$

$\texttt{elements}\,(F, G, \alpha) = \int_F^G \alpha := \left(\Lambda(X, x) \in \int F \,.\, (X, \alpha(X)(x)), \mathrm{id}\right)$

3 Limits and Colimits

3.1 From Cones and Cocones to Limits and Colimits

$$\texttt{ConstantFunctor} :: \prod \mathcal{C}, \mathcal{I} \in \mathsf{CAT} \,.\, \mathcal{C} \to \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C}$$
$$\texttt{ConstantFunctor}\,(\mathcal{I}, A) = \mathrm{Const}_{\mathcal{I}}(A) := (X \mapsto A, f \mapsto \mathrm{id}_X)$$

$$\texttt{FunctorEmbedding} :: \prod \mathcal{C}, \mathcal{I} \in \texttt{Category} \,.\, \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{C}^{\mathcal{I}}$$
$$\texttt{FunctorEmbeddibg}\,(A) = \Delta_{\mathcal{I}}(A) := \Big(X \mapsto \mathrm{Const}_{\mathcal{I}}(A), f \mapsto \Lambda i \in \mathcal{I} \,.\, f\Big)$$

$$\texttt{Cone} := \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\sum(\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \sum A \in \mathcal{C} \,.\, \Delta_{\mathcal{I}}(A) \Rightarrow D : \mathsf{CAT} \to \texttt{Type};$$

$$\texttt{summit} :: \texttt{Cone}(\mathcal{C}) \to \mathcal{C}$$
$$\texttt{summit}\,((\mathcal{I}, D), A, \lambda) := A$$

$$\texttt{synecdoche} :: \texttt{Cone}(\mathcal{C}) \to \mathsf{CAT}$$
$$\texttt{synecdoche}\,((\mathcal{I}, D), A, \lambda) := \mathcal{I}$$

$$\texttt{synecdoche} :: \prod \big((\mathcal{I}, D), A, \lambda\big) : \texttt{Cone}(\mathcal{C}) \,.\, \mathcal{I}\mathsf{CAT}\mathcal{C}$$
$$\texttt{sunecdoche}\,() := D$$

$$\texttt{legs} :: \prod C : \texttt{Cone}(\mathcal{C}) \,.\, \Delta_C\big(\texttt{summit}(C)\big) \Rightarrow C$$
$$\texttt{legs}\,() = \lambda^C := \lambda \quad \texttt{where} \quad \big((\mathcal{I}, D), A, \lambda\big) = C$$

$$\texttt{Cocone} := \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\sum(\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \sum A \in \mathcal{C} \,.\, D \Rightarrow \Delta_{\mathcal{I}}(A) : \mathsf{CAT} \to \texttt{Type};$$

$$\texttt{nadir} :: \texttt{Cocone}(\mathcal{C}) \to \mathcal{C}$$
$$\texttt{nadir}\,((\mathcal{I}, D), A, \lambda) := A$$

$$\texttt{synecdoche} :: \texttt{Cocone}(\mathcal{C}) \to \mathsf{CAT}$$
$$\texttt{synecdoche}\,((\mathcal{I}, D), A, \lambda) := \mathcal{I}$$

$$\texttt{synecdoche} :: \prod \big((\mathcal{I}, D), A, \lambda\big) : \texttt{Cocone}(\mathcal{C}) \,.\, \mathcal{C}\mathsf{CAT}\mathcal{I}$$
$$\texttt{sunecdoche}\,() := D$$

$$\texttt{legs} :: \prod C : \texttt{Cocone}(\mathcal{C}) \,.\, C \Rightarrow \Delta_C\big(\texttt{summit}(C)\big)$$
$$\texttt{legs}\,() = \lambda^C := \lambda \quad \texttt{where} \quad \big((\mathcal{I}, D), A, \lambda\big) = C$$

$$\texttt{coneCategory} :: \texttt{Diagram}(\mathcal{C}) \to \mathsf{CAT}$$
$$\texttt{coneCategory}\,((\mathcal{I}, D)) = \mathsf{CONE}_{\mathcal{C}}(\mathcal{I}, D) :=$$
$$:= \Big(\{C : \texttt{Cone}(\mathcal{C}) : (\mathcal{I}, D) = (C, C)\}, A, B \mapsto \big\{f : \texttt{summit}(A) \xrightarrow{\mathcal{C}} \texttt{summit}(B) : \forall i \in \mathcal{I} \,.\, f\lambda_i^B = \lambda_i^A\big\}, \cdot, \mathrm{id}\Big)$$

$$\texttt{coconeCategory} :: \texttt{Diagram}(\mathcal{C}) \to \mathsf{CAT}$$
$$\texttt{coconeCategory}\,((\mathcal{I}, D)) = \mathsf{CONE}_{\mathcal{C}}(\mathcal{I}, D) :=$$
$$:= \Big(\{C : \texttt{Cocone}(\mathcal{C}) : (\mathcal{I}, D) = (C, C)\}, A, B \mapsto \big\{f : \texttt{nadir}(A) \xrightarrow{\mathcal{C}} \texttt{nadir}(B) : \forall i \in \mathcal{I} \,.\, \lambda_i^A f = \lambda_i^B\big\}, \cdot, \mathrm{id}\Big)$$

$$\texttt{cone} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Diagram}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathsf{SET}$$
$$\texttt{cone}\,(\mathcal{I}, D) = \mathrm{Cone}_{\mathcal{I}}(\cdot, D) :=$$
$$:= \Big(\Lambda X \in \mathcal{C} \,.\, \big\{C \in \mathsf{CONE}_{\mathcal{C}}(\mathcal{I}, F) : \texttt{summit}(C) = X\big\}, f \mapsto f^*\Big)$$

$$\texttt{Limit} := \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Terminal}\left(\int \mathrm{Cone}_{\mathcal{I}}(\cdot, D)\right) : \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Diagram}(\mathcal{C}) \to \texttt{Type};$$

$$\texttt{limit} :: \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Limit}(\mathcal{I}, D) \to \mathcal{C}$$
$$\texttt{limit}\,(L, \lambda) = \lim D \quad (L, \lambda) := L$$

$$\texttt{universalCone} :: \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Limit}(\mathcal{I}, D) \to \texttt{Cone}(\mathcal{C})$$
$$\texttt{universalCone}\,(L, \lambda) := \lambda$$

$$\texttt{cocone} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Diagram}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathsf{SET}$$
$$\texttt{cocone}\,(\mathcal{I}, D) = \mathrm{Cocone}_{\mathcal{I}}(\cdot, D) :=$$
$$:= \Big(\Lambda X \in \mathcal{C} \,.\, \big\{C \in \mathsf{COCONE}_{\mathcal{C}}(\mathcal{I}, F) : \texttt{nadir}(C) = X\big\}, f \mapsto f_*\Big)$$

$$\texttt{Colimit} := \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Initial}\left(\int \mathrm{Cocone}_{\mathcal{I}}(\cdot, D)\right) : \prod \mathcal{C} \in \mathsf{CAT} \,.\, \texttt{Diagram}(\mathcal{C}) \to \texttt{Type};$$

$$\texttt{limit} :: \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Colimit}(\mathcal{I}, D) \to \mathcal{C}$$
$$\texttt{limit}\,(L, C) = \mathrm{colim}\, D \quad (L, C) := C$$

$$\texttt{universalCocone} :: \prod (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) \,.\, \texttt{Colimit}(\mathcal{I}, D) \to \texttt{Cocone}(\mathcal{C})$$
$$\texttt{universalCocone}\,(L, C) := C$$

$$\texttt{discreteCat} :: \mathsf{SET} \to \texttt{Discrete}$$

$$\texttt{discreteCat}\,(X) := \Big(X, x, y \mapsto \texttt{if}\ x == y\ \texttt{then}\ \{1\}\ \texttt{else}\ \emptyset, (1,1) \mapsto 1, x \mapsto 1\Big)$$

$$\texttt{Product} := \prod \mathcal{C} \in \texttt{Category}\ .\ \prod \mathcal{I} : \texttt{Discrete}\ .\ \prod X : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Limit}(\mathcal{I}, X) :$$

$$: \prod \mathcal{C} \in \texttt{Category}\ .\ \prod \mathcal{I} : \texttt{Discrete}\ .\ \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$$

$$(P, p) : \texttt{Product} \iff (P, \pi) = \prod_{i \in \mathcal{I}} X_i$$

$$\texttt{synecdoche} :: \texttt{Product}(\mathcal{C})(\mathcal{I}, X) \to \mathcal{C}$$

$$\texttt{synecdoche}\,(P, p) := P$$

$$\texttt{projections} :: \texttt{Product}(\mathcal{C})(\mathcal{I}, X) \to \texttt{Cone}(\mathcal{C})$$

$$\texttt{projections}\,(P, p) = (P) \quad \pi := \lambda^p$$

$$\texttt{Coproduct} := \prod \mathcal{C} \in \texttt{Category}\ .\ \prod \mathcal{I} : \texttt{Discrete}\ .\ \prod X : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Colimit}(\mathcal{I}, X) :$$

$$: \prod \mathcal{C} \in \texttt{Category}\ .\ \prod \mathcal{I} : \texttt{Discrete}\ .\ \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$$

$$(S, s) : \texttt{Coproduct} \iff (S, \iota) = \coprod_{i \in \mathcal{I}} X_i$$

$$\texttt{synecdoche} :: \texttt{Coproduct}(\mathcal{C})(\mathcal{I}, X) \to \mathcal{C}$$

$$\texttt{synecdoche}\,(S, s) := S$$

$$\texttt{inclusions} :: \texttt{Coproduct}(\mathcal{C})(\mathcal{I}, X) \to \texttt{Cone}(\mathcal{C})$$

$$\texttt{inclusions}\,(S, s) = (S) \quad \iota := \lambda^s$$

$$\texttt{parallelPair} :: \texttt{Small}$$

$$\texttt{parallelPair}\,() = \bullet \rightrightarrows \bullet := \Big(\{1, 2\},$$

$$, \big\{((1,1), \{1\}), ((2,2), \{1\}), ((1,2), \{1,2\}), ((2,1), \emptyset)\big\}, (f, 1) \mapsto f|(1, f) \mapsto 1, x \mapsto 1\Big)$$

$$\texttt{Equalizer} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \bullet \rightrightarrows \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Limit}(\bullet \rightrightarrows \bullet, X) :$$

$$: \prod \mathcal{C} \in \mathsf{CAT}\ .\ \bullet \rightrightarrows \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$$

$$\texttt{Coequalizer} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \bullet \rightrightarrows \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Colimit}(\bullet \rightrightarrows \bullet, X) :$$

$$: \prod \mathcal{C} \in \mathsf{CAT}\ .\ \bullet \rightrightarrows \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$$

$\texttt{wedgeCategory} :: \texttt{Small}$

$\texttt{wedgeCategory}\,() = \bullet \to \bullet \leftarrow \bullet := \Big(\{1,2,3\},$
$\quad , \{((1,1),1),((2,2),\{1\}),((3,3),\{1\}),((1,2),\{1\}),((1,3),\emptyset),((2,3),\emptyset),((3,2),\{1\}),((3,1),\emptyset),((2,1),\emptyset)\},$
$\quad , 1 \mapsto 1, x \mapsto 1\Big)$

$\texttt{Pullback} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \bullet \to \bullet \leftarrow \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Limit}(\bullet \to \bullet \leftarrow \bullet, X) :$
$\quad : \prod \mathcal{C} \in \mathsf{CAT}\ .\ \bullet \to \bullet \leftarrow \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$

$\texttt{veeCategory} :: \texttt{Small}$
$\texttt{veeCategory}\,() = \bullet \leftarrow \bullet \to \bullet := (\bullet \to \bullet \leftarrow \bullet)^{\mathrm{op}}$

$\texttt{Pushout} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \bullet \leftarrow \bullet \to \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .\ \texttt{Colimit}(\bullet \leftarrow \bullet \to \bullet, X) :$
$\quad : \prod \mathcal{C} \in \mathsf{CAT}\ .\ \bullet \leftarrow \bullet \to \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$

$\texttt{sequenceCategory} :: \texttt{Small}$
$\texttt{sequenceCategory}\,() = \mathsf{NAT} := (\mathbb{N}, \Lambda n, m \in \mathbb{N}\ .\ \texttt{if}\ n \geq m\ \texttt{then}\ \{1\}\ \texttt{else}\ \emptyset, (1,1) \mapsto 1, n \mapsto 1)$

$\texttt{InverseLimit} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \mathsf{NAT} \xrightarrow{\mathsf{CAT}}\ .\ \texttt{Limit}(\mathsf{NAT}, X) :$
$\quad : \prod \mathcal{C} \in \mathsf{CAT}\ .\ \mathsf{NAT} \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$
$(L, C) : \texttt{InverseLimit} \iff (L, C) = \varprojlim X$

$\texttt{synecdoche} :: \texttt{InverseLimit}(\mathcal{C}, X) \to \mathcal{C}$
$\texttt{synecdoche}\,(L, C) := L$

$\texttt{DirectLimit} := \prod \mathcal{C} \in \mathsf{CAT}\ .\ \prod X : \mathsf{NAT}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}}\ .\ \texttt{Colimit}(\mathsf{NAT}^{\mathrm{op}}, X) :$
$\quad : \prod \mathcal{C} \in \mathsf{CAT}\ .\ \mathsf{NAT} \xrightarrow{\mathsf{CAT}} \mathcal{C} \to \texttt{Type};$
$(L, C) : \texttt{DirectLimit} \iff (L, C) = \varinjlim X$

$\texttt{synecdoche} :: \texttt{DirectLimit}(\mathcal{C}, X) \to \mathcal{C}$
$\texttt{synecdoche}\,(L, C) := L$

3.2 Categories with Limits

$\texttt{WithLimit} :: \prod \mathbb{T} : \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{C}) \,.\, ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithLimit} \iff \forall D : \mathbb{T}(\mathcal{C}) \,.\, \exists \texttt{Limit}(D)$

$\texttt{Complete} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{Complete} \iff \forall D : \texttt{Diagram}(\mathcal{C}) \,.\, \exists \texttt{Limit}(D)$

$\texttt{SetIsComplete} :: [\mathsf{SET} : \texttt{Complete}]$

$\texttt{Proof} =$

$\texttt{Assume}\ (\mathcal{I}, X) : \texttt{Diagram}(\mathsf{SET}),$

$$L := \left\{ x \in \prod_{i \in \mathcal{I}} X_i : \forall i, j \in \mathcal{I} \,.\, \forall f \in i \xrightarrow{\mathcal{I}} j \,.\, X_{i,j}(f)(x_i) = x_j \right\} : \texttt{Set},$$

$\pi := \Lambda i \in \mathcal{I} \,.\, \Lambda x \in L \,.\, x_i : \prod i \in \mathcal{I} \,.\, L \to X_i,$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ f : i \xrightarrow{\mathcal{I}} j,$

$() := \eth \pi_i \eth L \eth^{-1} \pi_j : \pi_i X_{i,j}(f) = \Lambda x \in L \,.\, X_{i,j}(f)(x_i) = \Lambda x \in L \,.\, x_j = \pi_j;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : \Big[\big((\mathcal{I}, X), L, \pi\big) : \texttt{Cone}(\mathcal{I}, X) \Big],$

$\texttt{Assume}\ (C, c) : \displaystyle\int \mathrm{Cone}(\cdot, X),$

$\lambda := \lambda^c : \Delta_I(C) \Rightarrow X,$

$\phi := \Lambda x \in C \,.\, \Lambda i \in \mathcal{I} \,.\, \lambda_i(x) : C \to \displaystyle\prod_{i \in \mathcal{I}} X_i,$

$() := \eth \mathsf{CONE}(\mathcal{I}, X)(C)(\phi) \eth^{-1} L : \mathrm{Im}\, \phi \subset L;$

$\rightsquigarrow () := \eth^{-1}\texttt{Limit} : (L, \pi) : \texttt{Limit}(\mathcal{I}, X);$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Complete} : \big[X : \texttt{Complete}(\mathsf{SET}) \big];$

$\square$

$\texttt{WithColimit} :: \prod \mathbb{T} : \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{C}) \,.\, ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithColimit} \iff \forall D : \mathbb{T}(\mathcal{C}) \,.\, \exists \texttt{Colimit}(D)$

$\texttt{Cocomplete} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{Cocomplete} \iff \forall D : \texttt{Diagram}(\mathcal{C}) \,.\, \exists \texttt{Colimit}(D)$

$\texttt{SetIsCocomplete} :: [\mathsf{SET} : \texttt{Cocomplete}]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ (\mathcal{I}, X) : \texttt{Diagram}(\mathsf{SET}),$

$E := \texttt{spanEq}\left\{\big((i,x),(j,X_{i,j}(f)(x))\big) \in \coprod_{i\in\mathcal{I}} X_i \times \coprod_{i\in\mathcal{I}} X_i \middle| i,j \in \mathcal{I}, f : i \xrightarrow{\mathcal{I}} j\right\} : \texttt{Equivalence} \coprod_{i\in\mathcal{I}} X_i,$

$L := \frac{\coprod_{i\in\mathcal{I}} X_i}{E} : \texttt{Set},$

$\iota := \Lambda i \in \mathcal{I} \,.\, \Lambda x \in X_i \,.\, \big[(i,x)\big]_E : \prod i \in \mathcal{I} \,.\, X_i \to L,$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ f : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\iota_j\eth\texttt{quetient}(E)\eth^{-1}\iota_i : X_{i,j}(f)\iota_j = \Lambda x \in X_i \,.\, \Big[\big((j, X_{i,j}(f)(x)\big)\Big]_E = [(i,x)]_E = \iota_i;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cocone} : \Big[\big((\mathcal{I},X),L,\iota\big) : \texttt{Cocone}(\mathcal{I},X)\big)\Big],$

$\texttt{Assume}\ (C,c) : \int \mathsf{Cone}_{\mathcal{I}}(\cdot, X),$

$\lambda := \lambda^c : X \Rightarrow \Delta_{\mathcal{I}}(C),$

$\texttt{Assume}\ y : L,$

$\texttt{Assume}\ (i,x,2),(j,x',3) : \sum i \in \mathcal{I} \,.\, \sum x \in X \,.\, (i,x_i) \in y,$

$(k,f,g,4) := \eth y \eth L(3) : \sum k \in \mathcal{I} \,.\, \sum z \in X_k \,.\, \sum f : k \xrightarrow{\mathcal{I}} i \,.\, \sum g : k \xrightarrow{\mathcal{I}} j \,.$

$\quad .\, X_{k,i}(f)(z) = x \,\&\, X_{k,j}(g)(z) = y,$

$() := \eth\texttt{Cocone}(C,c)\eth(\lambda)(4) : \lambda_i(x) = \lambda_j(y);$

$\rightsquigarrow (2) := I(\forall) : \forall y \in L \,.\, \forall (i,x,2),(j,x',3) : \sum i \in \mathcal{I} \,.\, \sum x \in X_i : (i,x') \in y \,.\, \lambda_i(x) = \lambda_j(x'),$

$\phi := \Lambda y \in L \,.\, \eth\texttt{Singleton}\left\{\lambda_i(x) | (i,x,\cdot) : \sum i \in \mathcal{I} : \sum x \in X_i : (i,x_i) \in y\right\} : L \to C,$

$() := \eth\phi : \forall i,j \in \mathcal{I} \,.\, \forall f : i \xrightarrow{\mathcal{I}} j \,.\, \iota_i\phi = \lambda_j;$

$\rightsquigarrow () := \eth^{-1}\texttt{Colimit} : \Big[(L,\iota) : \texttt{Limit}(\mathcal{I},X)\Big];$

$(*) := \eth\texttt{Cocomplete} : [\mathsf{SET} : \texttt{Cocomplete}];$

$\square$

3.3 Limits under Functors

$$\texttt{mapDiagram} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \to \texttt{Diagram}(\mathcal{A}) \to \texttt{Diagram}(\mathcal{B})$$

$$\texttt{mapDiagram}\,(F, (\mathcal{I}, D)) = F(\mathcal{I}, D) := (\mathcal{I}, DF)$$

$$\texttt{mapDiagramType} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \to ?\texttt{Diagram}(\mathcal{A}) \to ?\texttt{Diagram}(\mathcal{B})$$

$$\texttt{mapDiagramType}\,(F, \mathbb{T}) = F\mathbb{T} := \left\{ FD : D \in \mathbb{T} \right\}$$

$$\texttt{PreservesLimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{PreservesLimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Limit}(\mathcal{A})(D) \,.$$
$$.\, \Big[FL : \texttt{Limit}(\mathcal{B})(D) \Big]$$

$$\texttt{ReflectsLimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{ReflectsLimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Cone}(\mathcal{A})(D) \,.$$
$$.\, \Big[FL : \texttt{Limit}(\mathcal{B}))(D) \Big] \Rightarrow \Big[L : \texttt{Limit}(\mathcal{A})(D) \Big]$$

$$\texttt{CreatesLimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{CreatesLimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Limit}(\mathcal{B})(D) \,.$$
$$.\, \exists L' : \texttt{Limit}(\mathcal{A})(D) : FL' = L$$

$$\texttt{PreservesColimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{PreservesColimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Colimit}(\mathcal{A})(D) \,.$$
$$.\, \Big[FL : \texttt{Colimit}(\mathcal{B})(FD) \Big]$$

$$\texttt{ReflectsColimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{ReflectsColimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Cocone}(\mathcal{A})(D) \,.$$
$$.\, \Big[FL : \texttt{Colimit}(\mathcal{B}))(FD) \Big] \Rightarrow \Big[L : \texttt{Colimit}(\mathcal{A})(D) \Big]$$

$$\texttt{CreatesLimits} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?\texttt{Diagram}(\mathcal{A}) \to ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$

$$F : \texttt{CreatesLimits} \iff \Lambda\mathbb{T} : ?\texttt{Diagram}(\mathcal{A}) \,.\, \forall D \in \mathbb{T} \,.\, \forall L : \texttt{Colimit}(\mathcal{B})(FD) \,.$$
$$.\, \exists L' : \texttt{Colimit}(\mathcal{A})(D) : FL' = L$$

$\texttt{ExistanceOfLimitsByFunctor} :: \forall \mathcal{A},\mathcal{B} \in \mathsf{CAT} \,.\, \forall \mathbb{T} :?\texttt{Diagram}(\mathcal{A}) \,.\, \forall F : \texttt{CreatesLimit}(\mathcal{A},\mathcal{B})(\mathbb{T}) \,.$
$\quad .\, \forall (1) : \Big[\mathcal{B} : \texttt{WithLimit}(F\mathbb{T})\Big] \,.\, \Big[\mathcal{A} : \texttt{WithLimit}(\mathbb{T})\Big] \;\&\; \Big[F : \texttt{PreservesLimit}(\mathbb{T})\Big]$

$\texttt{Proof} \;=$

$\texttt{Assume}\ D : \mathbb{T},$

$L := (1)(FD) : \texttt{Limit}(\mathcal{B})(FD),$

$(L', 2) := \eth\texttt{CreatesLimit}((\mathcal{A},\mathcal{B})F) : \sum L' : \texttt{Limit}(D) \,.\, FL' = L;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{WithLimit} : \Big[\mathcal{A} : \texttt{WithLimit}\mathbb{T}\Big],$

$\texttt{Assume}\ D : \mathbb{T},$

$\texttt{Assume}\ L : \texttt{Limit}(D),$

$M := (1)(FD) : \texttt{Limit}(\mathcal{B})(FD),$

$(M', 3) := \eth\texttt{CreatesLimit}((\mathcal{A},\mathcal{B})F) : \sum M' : \texttt{Limit}(D) \,.\, FM' = M,$

$(4) := \texttt{IsomorphicTerminal}\eth\texttt{Limit}(D)(L, M') : L \cong_{\mathcal{A}} M',$

$(5) := \texttt{FunctorPreservesIso}(4)(3) : FL \cong M,$

$(6) := \eth^{-1}\texttt{LimitIsomorphicTerminal}(5) : \Big[FL : \texttt{Limit}(\mathcal{B})\Big];$

$\square$

$\texttt{FullyFaithfulReflectsLimits} :: \forall \mathcal{A} : \texttt{WithLimit}(\mathbb{T}) \,.\, \forall \mathcal{B} \in \mathsf{CAT} \,.\, \forall F : \texttt{FullyFaithul}(\mathcal{A},\mathcal{B}) \,.$
$\quad .\, \Big[F : \texttt{ReflectsLimits}(\mathcal{A},\mathcal{B})(\mathbb{T})\Big]$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mathcal{I}, D : \mathbb{T},$

$\texttt{Assume}\ (C,\lambda) : \texttt{Cone}(\mathcal{I}, D),$

$\texttt{Assume}\ (1) : \Big[(FC, F\lambda) : \texttt{Limit}F(\mathcal{I}, D)\Big],$

$(L,\mu) := \eth\texttt{WithLimit}(\mathbb{T})(\mathcal{A})(\eth)(\mathcal{I}, D) : \texttt{Limit}(\mathcal{I}, D),$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ f : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F)\eth\texttt{Cone}(L,\mu) : F_{L,D_i}(\mu_i)DF_{i,j}(f) = F_{L,D_j}\Big(\mu_i D_{i,j}(f)\Big) = F_{L.D_j}(\mu_j);$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : \Big[(FL, F\mu)\Big] : \texttt{Cone}F(\mathcal{I}, D),$

$\phi := \eth\texttt{Limit}(L,\mu)(C,\lambda) : C \xrightarrow{\int \mathrm{Cone}_{\mathcal{I}}(\cdot, D)} L,$

$\psi := \eth\texttt{Limit}(FC, F\lambda)(FL, F\mu) : FL \xrightarrow{\int \mathrm{Cone}_{\mathcal{I}}(\cdot, FD)} FC,$

$\texttt{Assume}\ i : \mathcal{I},$

$() := \eth\phi\eth\texttt{Covariant}(\mathcal{A},\mathcal{B})(F) : F(\lambda_i) = F(\phi\mu_i) = F(\phi)F(\mu_i);$

$\rightsquigarrow (1) := \eth \displaystyle\int \mathrm{Cone}_{\mathcal{I}}(\cdot, FD) : \Big[F(\phi) : FC \xrightarrow{\int \mathrm{Cone}_{\mathcal{I}}(\cdot, FD)}\Big],$

$(1) := \eth\texttt{Terminal}(FC,\lambda) : \psi F(\phi) = e,$

$(2) := \eth\texttt{FullyFaithul}(F)(1) : (C,\lambda) \cong_{\int \mathrm{Cone}(\cdot, D)} (L,\mu),$

$() := \eth^{-1}\texttt{Limit}(\mathcal{I}, D)(2) : \Big[(C,\lambda) : \texttt{Limit}(\mathcal{I}, D)\Big];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{ReflectsLimitis} : \Big[F : \texttt{ReflectsLimits}(\mathcal{A},\mathcal{B})(\mathbb{T})\Big];$

$\square$

$\texttt{StrictlyCreatesL} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \; . \; \prod \mathbb{T} :?\texttt{Diagram}(\mathcal{A}) \; . \; ?\texttt{CreatesLimits}(\mathcal{A}, \mathcal{B})$

$F : \texttt{StrictlyCreatesL} \iff \forall D \in \mathbb{T} \; . \; \forall C : \texttt{Limit}(FD) \; . \; \exists! C' : \texttt{Limit}(D) \; . \; C = FC'$

$\texttt{Connected} :: ?\texttt{Diagram}(\mathcal{C})$

$(\mathcal{I}, D) : \texttt{Connected} \iff [\mathcal{I} : \texttt{Connected}]$

$\texttt{forgetfulDemorph} :: \prod \mathcal{C} \in \mathsf{CAT} \; . \; \prod X \in \mathcal{C} \; . \; \frac{\mathcal{C}}{X} \xrightarrow{\mathsf{CAT}} \mathcal{C}$

$\texttt{forgetfulDemorph}\,(A, f) = \Pi^X(A, f) := A$

$\texttt{forgetfulDemorph}\,((A, f), (B, g), (h)) = \Pi^X_{(A,f),(B,g)}(h) := h$

$\texttt{ForgetfulDemorphStrictlyCreatesL} :: \forall \mathcal{C} \in \mathsf{CAT} \; . \; \forall Y \in \mathcal{C} \; .$

$$\left[\Pi^Y : \texttt{StrictlyCreatesL}\left(\texttt{Connected}\left(\frac{\mathcal{C}}{Y}\right)\right)\right]$$

$\texttt{Proof} \; =$

$\texttt{Assume}\; \Big(\mathcal{I}, \big((X, f), \phi\big)\Big) : \texttt{Connected}\left(\frac{\mathcal{C}}{Y}\right),$

$D := \Big(\mathcal{I}, \big((X, f), \phi\big)\Big) : \texttt{Connected}\left(\frac{\mathcal{C}}{Y}\right),$

$(1) := \eth D : \forall i, j \in \mathcal{I} \; . \; \mathcal{M}_{\mathcal{I}}(i, j) \neq \emptyset \Rightarrow \phi_{i,j} f_j = f_i,$

$:\texttt{Assume}\; (L, \lambda) : \texttt{Limit}\big(\Pi^Y D\big),$

$(2) := \eth\texttt{Cone}(C, \lambda) : \forall i, j \in \mathcal{I} \; . \; \mathcal{M}_{\mathcal{I}}(i, j) \neq \emptyset \Rightarrow \lambda_i \phi_{i,j} = \lambda_j,$

$\texttt{Assume}\; i, j : \mathcal{I},$

$(n, k, h) := (00)(D)\eth\texttt{Connected}(D) : \texttt{MorphismChain}(i, j),$

$\texttt{Assume}\; l : n,$

$() := (1)(2)(l, l+1) : \lambda_{k_l} f_{k_l} = \lambda_i \phi_{k_l, k_{l+1}}(h) f_{k_{l+1}} = \lambda_{k_{l+1}} f_{k_{l+1}};$

$\leadsto (1) := I(\forall) : \forall l \in n \; . \; \lambda_{k_l} f_{k_l} = \lambda_{k_{l+1}} f_{k_{l+1}},$

$() := \texttt{EqChain}(1)\eth\texttt{MorphismChain}(i, j)(n, k, h) : \lambda_i f_i = \lambda_j f_j;$

$\leadsto (1) := I(\forall) : \forall i, j \in \mathcal{I} \; . \; \lambda_i f_i = \lambda_j f_j,$

$g := \eth\texttt{Singleton}\{\lambda_i f_i | i \in \mathcal{I}\} : L \xrightarrow{\mathcal{C}} Y,$

$(2) := \eth g \eth(L.\lambda)\eth^{-1}\texttt{Cone}(D) : \Big[\big((L, g), \lambda\big) : \texttt{Cone}(D)\Big],$

$\texttt{Assume}\; \big((C, h), \mu\big) : \texttt{Cone}(D),$

$(3) := \eth^{-1}\texttt{Cone}(\Pi^Y F) : \Big[(C, \mu) : \texttt{Cone}\big(\Pi^Y D\big)\Big],$

$(\psi, 4) := \eth^{-1}\texttt{Limit}\big(\Pi^Y D\big)(L, \lambda) : \sum \psi : C \xrightarrow{\mathcal{C}} L \; . \; \forall i \in \mathcal{I} \; . \; \mu_i = \psi \lambda_i,$

$(5) := \forall i \in I \; . \; \eth\texttt{Cone}(C, \mu)(i)(4)\eth\texttt{Cone}(L, \lambda)(i) : \forall i \in I \; . \; h = \mu_i f_i = \psi \lambda_i f_i = \psi g,$

$(6) := \eth\texttt{Connected}(D)\eth\texttt{NonEmpty}(\mathcal{I})(5) : h = \psi g,$

$(7) := \eth\left(\frac{\mathcal{C}}{Y}\right) : \left[\psi : (C, h) \xrightarrow{\frac{\mathcal{C}}{Y}} (L, g)\right];$

$\leadsto () := \eth^{-1}\texttt{Limit} : \Big[\big((L, g), \lambda\big) : \texttt{Limit}(D)\Big];$

$\leadsto (*) := \eth^{-1}\texttt{StrictlyCreates} : \Big[\Pi^Y : \texttt{StrictlyCreates}(\mathbb{T})\Big];$

$\square$

$\texttt{StrictlyCreatesC} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, \prod \mathbb{T} :?\texttt{Diagram}(\mathcal{A}) \,.\, ?\texttt{CreatesColimits}(\mathcal{A}, \mathcal{B})$

$F : \texttt{StrictlyCreatesC} \iff \forall D \in \mathbb{T} \,.\, \forall C : \texttt{Colimit}(FD) \,.\, \exists! C' : \texttt{Colimit}(D) \,.\, C = FC'$

$\texttt{ForgetfulDemorphStriclyCreatesC} :: \forall \mathcal{C} i \in \mathsf{CAT} \,.\, \forall Y \in \mathsf{CAT} \,.$

$\quad . \left[\Pi^Y : \texttt{StrictlyCreatesC}\left(\texttt{Diagram}\left(\frac{\mathcal{C}}{Y}\right)\right)\right]$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \Big(\mathcal{I}, \big((X, f), \phi\big)\Big) : \texttt{Diagram}\left(\frac{\mathcal{C}}{Y}\right),$

$D := \Big(\mathcal{I}, \big((X, f), \phi\big)\Big) : \texttt{Diagram}\left(\frac{\mathcal{C}}{Y}\right),$

$(1) := \eth\frac{\mathcal{C}}{Y}(D) : \forall i, j \in \mathcal{I} \,.\, \forall h : i \xrightarrow{\mathcal{I}} j \,.\, \phi_{i,j}(h) f_j = f_i,$

$(2) := \eth^{-1}\texttt{Cocone}(1) : (Y, f) \in \int \mathrm{Cocone}_{\mathcal{I}}\Big(\cdot, (X, \phi)\Big),$

$\texttt{Assume}\ (L, \lambda) : \texttt{Colimit}\Big(\Pi^Y D\Big),$

$(\psi, 3) := \eth\texttt{Colimit}\Big(\Pi^Y D\Big)(L.\lambda)(Y, f) : \sum \psi : L \xrightarrow{\mathcal{C}} Y \,.\, \forall i \in \mathcal{I} \,.\, \lambda_i \psi = f_i,$

$(4) := \eth^{-1}\texttt{Cocone}(D)(3) : \Big[\big((L, \psi), \lambda\big) : \texttt{Cocone}(D)\Big],$

$\texttt{Assume}\ \Big((C, h), \mu\Big) : \texttt{Cocone}(D),$

$(5) := \eth\texttt{Cocone}\Big(\Pi^Y D\Big)(C, \mu) : \Big[(C, \mu) : \texttt{Cocone}(D)\Big],$

$(\chi, 6) := \eth\texttt{Colimit}\Big(\Pi^Y D\Big) : \sum \chi : L \xrightarrow{\mathcal{C}} C \,.\, \forall i \in \mathcal{I} \,.\, \lambda_i \chi = \mu_i,$

$() := \eth\texttt{Unique}\eth\texttt{Colimit}\eth\texttt{Initial}\eth\psi : \chi h = \psi;$

$\rightsquigarrow () := \eth^{-1}\texttt{Colimit}\eth\texttt{Unique}\eth\texttt{Colimit}\eth\texttt{Initial}\, I(\exists) : \Big[\big((L, \psi), \lambda\big) : \texttt{Colimit}(D)\Big];$

$(*) := \eth^{-1}\texttt{StrictlyCreatesC}\eth\Pi_Y\eth\texttt{Unique}\eth\texttt{Colimit}\eth\texttt{Initial} : \Big[\Pi^Y : \texttt{StrictlyCreatesC}\Big];$

$\square$

$\texttt{objectCategory} :: \mathsf{CAT} \to \mathsf{CAT}$

$\texttt{objectCategory}\,(\mathcal{C}) = \mathcal{C}^{\mathrm{obj}} := \Big(\mathcal{O}(\mathcal{C}), \Lambda X, Y \in \mathcal{C} \,.\, \texttt{if}\ X == Y\ \texttt{then}\ \{\mathrm{id}_X\}\ \texttt{else}\ \emptyset, \cdot_{\mathcal{C}}, \underset{\mathcal{C}}{\mathrm{id}}\Big)$

$\texttt{functorRelaxation} :: \prod \mathcal{C}, \mathcal{A} \in \mathsf{CAT} \,.\, \mathcal{C}^{\mathcal{A}} \xrightarrow{\mathsf{CAT}} \mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}$

$\texttt{functorRelaxation}\,(F) = \mathrm{R}\ F := F$

$\texttt{functorRelaxation}\,(F, G, \alpha) = \mathrm{R}_{F,G}\ \alpha := \alpha$

$\texttt{evaluateFunctorAt} :: \prod \mathcal{C}, \mathcal{A} \in \mathsf{CAT} \,.\, \mathcal{A} \to \mathcal{C}^{\mathcal{A}} \xrightarrow{\mathsf{CAT}} \mathcal{C}$

$\texttt{evaluateFunctorAt}\,(F) = \mathrm{Ev}_A\ F := F(A)$

$\texttt{evaluateFunctorAt}\,(F, G, \alpha) = (\mathrm{Ev}_A)_{F,G}\ \alpha := \alpha(A)$

$\texttt{EvaluationPreservesLimits} :: \forall \mathcal{A}, \mathcal{C} : \mathsf{CAT} \,.\, \forall A \in \mathcal{A} \,.\, \left[\mathrm{Ev}_A : \texttt{PreservesLimits}\left(\mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}, \mathcal{A}\right)\right]$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \big(\mathcal{I}, (F, \alpha)\big) : \texttt{Diagram}(\mathcal{C}^{\mathcal{A}}),$

$D := \big(\mathcal{I}, (F, \alpha)\big) : \texttt{Diagram}(\mathcal{C}^{\mathcal{A}}),$

$\texttt{Assume}\ (L, \lambda) : \texttt{Limit}(D),$

$\dots$

$\square$

$\texttt{FunctorLimitFibration} :: \forall \mathcal{C} \in \mathsf{CAT} \,.\, \forall \mathcal{A} : \texttt{Small} \,.\, \forall \mathbb{T} :?\texttt{Diagrm}(\mathcal{C}^{\mathcal{A}}) \,.$

$\quad . \forall (0) : \forall A \in \mathcal{A} \,.\, \left[\mathcal{C} : \texttt{WithLimit}(\mathrm{Ev}_A \mathbb{T})\right] \,.\, \mathrm{R} : \texttt{StrictlyCreatesL}\Big(\mathbb{T}\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \Big(\mathcal{I}, (F, \alpha)\Big) : \mathbb{T},$

$D := \Big(\mathcal{I}, (F, \alpha)\Big) : \mathbb{T},$

$\texttt{Assume}\ A : \mathcal{A},$

$(1) := \mathrm{Ev}_A\, \eth D : \left[\mathrm{Ev}_A\ D : \mathrm{Ev}_A\ \mathbb{T}\right],$

$\Big(L'(A), \lambda'(A)\Big) := (0)(\mathrm{Ev}_A\ D) : \texttt{Limit}(\mathrm{Ev}_A\ D);$

$\rightsquigarrow (L', \lambda') := I\left(\prod\right) : \prod A \in \mathcal{A} \,.\, \texttt{Limit}\Big(\mathrm{Ev}_A\ D\Big),$

$\texttt{Assume}\ X, Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$\mu := \Lambda i \in \mathcal{I} \,.\, \lambda_i(X)(F_i)_{X,Y}(f) : \prod i \in \mathcal{I} \,.\, L(X) \xrightarrow{\mathcal{C}} F_i(Y),$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ h : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\mu_i \eth\texttt{NaturalTransform}(F_i, F_j)\alpha_{i,j}(h)\eth\texttt{Cone}(\mathrm{Ev}_X\ D)(L(X), \lambda(X))\eth^{-1}\mu_j :$

$\quad : \mu_i\alpha_{i,j}(h)(Y) = \lambda_i(X)(F_i)_{X,Y}(f)\alpha_{i,j}(h)(Y) = \lambda_i(X)\alpha_{i,j}(h)(X)(F_j)_{X,Y}(f) = \lambda_j(X)(F_j)_{X,Y}(f) = \mu_j;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone}(\mathrm{Ev}_Y\ D) : \Big[\big(L'(X), \mu\big) : \texttt{Cone}(\mathrm{Ev}_Y\ D)\Big],$

$\big(L''_{X,Y}(f), 2\big) := \eth\texttt{Limit}(\mathrm{Ev}_Y\ D)(L'(Y), \lambda'(Y)) : \sum L''_{X,Y}(f) : L'(X) \xrightarrow{L}{}' (Y) \,.\, \forall i \in \mathcal{I} \,.\, L''_{X,Y}(f)\lambda'_i(Y) = \mu_i;$

$\rightsquigarrow (L'', 1) := I\left(\prod\right) : \prod X, Y \in \mathcal{A} \,.\, \prod f : X \xrightarrow{\mathcal{A}} Y \,.\, \sum L'_{X,Y}(f) : L(X) \xrightarrow{\mathcal{C}} L(Y) \,.$

$\quad . \forall i \in \mathcal{I} \,.\, L''_{X,Y}(f)\lambda'_i(Y) = \lambda'_i(X)(F_i)_{X,Y}(f),$

$\texttt{Assume}\ X, Y, Z : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$\texttt{Assume}\ g : Y \xrightarrow{\mathcal{B}} Z,$

$\texttt{Assume}\ i : \mathcal{I},$

$() := (1)(Y, Z, g, i)(1)(X, Y, f, i)\eth\texttt{Covariant}(\mathcal{A}, \mathcal{C})(F_i) :$

$\quad : L''_{X,Y}(f)L''_{Y,Z}(g)\lambda'_i(Z) = L''_{X,Y}(f)\lambda'_i(Y)(F_i)_{X,Y}(f) = \lambda'_i(X)(F_i)_{X,Y}(f)(F_i)_{Y,Z}(g) = \lambda'_i(X)(F_i)_{X,Z}(fg);$

$\rightsquigarrow (2) := I(\forall) : \forall i \in \mathcal{I} \,.\, L''_{X,Y}(f)L''_{Y,Z}(g)\lambda'_i(Z) = \lambda'_i(X)(F_i)_{X,Z}(fg),$

$() := \eth\texttt{Unique}\eth L''(2) : L''_{X,Y}(f)L''_{Y,Z}(g) = L''_{X,Z}(fg);$

$\rightsquigarrow (2) := \eth^{-1}\texttt{Covariant} : \Big[(L', L'') : \texttt{Covariant}(\mathcal{A}, \mathcal{C})\Big],$

$L := (L', L'') : \texttt{Covariant}(\mathcal{A}, \mathcal{C}),$

$\lambda := \Lambda i \in \mathcal{I} \,.\, \Lambda A \in \mathcal{A} \,.\, \lambda'_i(A) : \prod i \in \mathcal{I} \,.\, L \Rightarrow F_i,$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ h : i \xrightarrow{\mathcal{I}} j,$

$\texttt{Assume}\ A : \mathcal{A},$

$() := \eth\texttt{Cone}(L(A), \lambda(A)) : \alpha_{i,j}(h)(A)\lambda_j(A) = \lambda_i(A);$

$\rightsquigarrow () := I(\to, =) : \alpha_{i,j}(h)\lambda_j = \lambda_i;$

$(3) := \eth^{-1}\texttt{Cone} : \Big[(L, \lambda) : \texttt{Cone}(D)\Big];$

$\texttt{Assume}\ (M, \mu) : \texttt{Cone}(\mathrm{R}D),$

$\texttt{Assume}\ A : \mathcal{A},$

$(5) := \texttt{EvaluationPreservesLimits}(A)(4) : \Big[(M(A), \mu(A)) : \texttt{Limit}(\mathrm{Ev}_A D)\Big],$

$() := \texttt{Terminalisomprphic}\eth L'(5) : (M(A), \mu(A)) \cong_{\int_{\mathcal{C}} \mathrm{Cone}(x, \mathrm{Ev}_A D)\mathrm{d}x} (L(A), \mu(A));$

$\rightsquigarrow (6) := \eth\mathcal{A}^{\mathrm{obj}}\eth L : (M, \mu) \cong_{\int_{\mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}} \mathrm{Cone}(x, \mathrm{R}D)\mathrm{d}x} (L, \lambda),$

$(7) := \eth^{-1}\texttt{Limit}\ \texttt{TerminalIsomorphic}(6) : \Big[(L, \lambda) : \texttt{Limit}(\mathrm{R}\ D)\Big],$

$\phi := \eth^{-1}\texttt{Isomorphic}(6) : \sum \phi : L \xleftrightarrow{\mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}} M\ .\ \forall i \in \mathcal{I}\ .\ \phi\mu_i = \lambda_i,$

$M^* := (M, \Lambda X, Y \in \mathcal{A}\ .\ \Lambda f : X \xrightarrow{\mathcal{A}} Y\ .\ \phi^{-1}(X)L_{X,Y}(f)\phi(Y)) : \texttt{Covariant}(\mathcal{A}, \mathcal{C}),$

$(8) := \eth\phi\eth M^* : (M^*, \mu) \cong_{\int_{\mathcal{C}^{\mathcal{A}}} \mathrm{Cov}(x, D)\mathrm{d}x} (L, \lambda),$

$\texttt{Assume}\ (C, \beta) : \texttt{Cone}(D),$

$(\psi, 9) := \eth\texttt{Limit}(L, \lambda)\mathrm{R}(C, \beta) : \sum\ .\ C \xrightarrow{\mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}} L\ .\ \forall i \in \mathcal{I}\ .\ \psi\lambda_i = \beta_i,$

$\texttt{Assume}\ X, Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$\rho := \beta(X)F_{X,Y}(f) : C(X) \Rightarrow F(Y),$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ h : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\rho_i\eth\alpha\eth\texttt{Cone}(D)(C, \beta)\eth^{-1}\rho_j : \rho_i\alpha_{i,j}(h)(Y) = \beta_i(X)(F_i)_{X,Y}(f)\alpha_{i,j}(h)(Y) = \beta_i(X)\alpha_{i,j}(h)(X)(F_j)_{X,Y}(f) = \beta_j$

$\rightsquigarrow (10) := \eth^{-1}\texttt{Cone} : \Big[(C(X), \rho) : \texttt{Cone}(\mathrm{Ev}_Y\ D)\Big],$

$\texttt{Assume}\ i : \mathcal{I},$

$()_1 := \eth\rho_i\eth\beta(9) : \rho_i = \beta_i(X)(F_i)_{X,Y}(f) = C_{X,Y}(f)\beta_i = C_{X,Y}(f)\psi\lambda_i(Y),$

$()_2 := \eth\rho_i(9)\eth\lambda : \rho_i = \beta_i(X)(F_i)_{X,Y}(f) = \psi\lambda_i(X)(F_i)_{X,Y}(f) = \psi L_{X,Y}(f)\lambda_i(Y);$

$\rightsquigarrow (11) := \eth \int_{\mathcal{C}} \mathrm{Cone}(x, \mathrm{Ev}_Y D)\mathrm{d}x : C_{X,Y}(f)\psi(Y), \psi(X)L_{X,Y}(f) : (C, \rho) \xrightarrow{\int_{\mathcal{C}} \mathrm{Cone}(x, \mathrm{Ev}_Y D)\mathrm{d}x} (L, \lambda),$

$() := \eth\texttt{Limit}(\mathrm{R}D)(L, \lambda) : C_{X,Y}(f)\psi(Y) = \psi(X)L_{X,Y}(f);$

$\rightsquigarrow () := \eth\mathcal{C}^{\mathcal{A}} : \Big[\psi : (C, \beta) \xrightarrow{\mathcal{C}^{\mathcal{A}}} (L, \lambda)\Big];$

$\rightsquigarrow (9) := \eth^{-1}\texttt{Limit} : \Big[(L, \lambda) : \texttt{Limit}(D)\Big],$

$() := \texttt{TerminalIso}(8, 9) : \Big[(M^*, \mu) : \texttt{Limit}(D)\Big];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{StrictlyCreatesL} : \Big[R : \texttt{StrictlyCreatesL}(\mathbb{T})\Big];$

$\square$

$\texttt{ExistanceOfColimitsByFunctor} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \ . \ \forall \mathbb{T} :?\texttt{Diagram}(\mathsf{CAT}) \ . \ \forall F : \texttt{CreatesColimits}(\mathbb{T}) \ .$

$. \ \forall (1) : \Big[\mathcal{B} : \texttt{WithColimit}(F\mathbb{T})\Big] \ . \ \Big[\mathcal{A} : \texttt{Colimit}(F\mathbb{T})\Big] \ \& \ \Big[F : \texttt{CreatesLimits}(F\mathbb{T})\Big]$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FullyFaithfulReflectsColimits} :: \forall \mathcal{A} : \texttt{WithColimit}(\mathbb{T}) \ . \ \forall \mathcal{B} \in \mathsf{CAT} \ . \ \forall F : \texttt{FullyFaithful}(\mathcal{A}, \mathcal{B}) \ .$

$\Big[F : \texttt{ReflectsColimits}(\mathbb{T})\Big]$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{forgetfufulDemorph2} :: \prod \mathcal{C} \in \mathsf{CAT} \ . \ \prod X \in \mathcal{C} \ . \ \dfrac{X}{\mathcal{C}} \xrightarrow{\mathsf{CAT}} \mathcal{C}$

$\texttt{forgetfulDemorph2}\,(A, f) = \Pi_X(A, f) := A$

$\texttt{forgetfulDemorph2}\,((A, f), (B, g), h) = (\Pi_X)_{(A,f),(B,g)}(h) := h$

$\texttt{ForgetfulDemorph2StrictlyCreatesC} :: \forall \mathcal{C} \in \mathsf{CAT} \ . \ \forall X \in \mathcal{C} \ .$

$. \ \left[\Pi_X : \texttt{StrictlyCreatesC}\left(\texttt{Connected}\left(\dfrac{X}{\mathcal{C}}\right)\right)\right]$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{ForgetfulDemorph2StrictlyCreatesL} :: \forall \mathcal{C} \in \mathsf{CAT} \ . \ \forall X \in \mathcal{C} \ .$

$. \ \left[\Pi_X : \texttt{StrictlyCreatesL}\left(\texttt{Diagram}\left(\dfrac{X}{\mathcal{C}}\right)\right)\right]$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{EvaluationPreservesColimits} :: \forall C, A \in \mathsf{CAT} \ . \ \Big[\mathrm{Ev}_A : \texttt{PreservesColimits}\big(\mathcal{C}^{\mathcal{A}^{\mathrm{obj}}}, \mathcal{C}\big)\Big]$

$\texttt{Proof} \ =$

$\texttt{FunctorColimitFibration} :: \forall C \in \mathsf{CAT} \ . \ \forall \mathcal{A} : \texttt{Small} \ . \ \forall \mathbb{T} :?\texttt{Diagram}(\mathcal{C}^{\mathcal{A}}) \ .$

$. \ \forall (0) : \forall A \in \mathcal{A} \ . \ \Big[\mathcal{C} : \texttt{WithColimit}(\mathrm{Ev}_A\mathbb{T})\Big] \ . \ \mathrm{R} : \texttt{StrictlyCreatesC}(\mathbb{T}) \ .$

$\texttt{Proof} \ =$

$\ldots$

$\square$

3.4 Representation and Limits

$\texttt{ConeRepresentation} :: \forall \mathcal{C} : \texttt{LocallySmall} \,.\, \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{A}) \,.\, \lim_{i \in \mathcal{I}} \mathcal{M}_{\mathcal{C}}(\cdot, X_i) \iff \mathrm{Cone}_{\mathcal{I}}(\cdot, X)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ A : \mathcal{C},$

$(L, \lambda) := \lim_{i \in \mathcal{I}} \mathcal{M}_{\mathcal{C}}(A, X_i) : \sum L : \texttt{Set} \,.\, \lambda : L \Rightarrow \mathcal{M}_{\mathcal{C}}(A, X),$

$\texttt{Assume}\ x : L,$

$\alpha(A)(x) := (A, \lambda(x)) : \sum A \in \mathcal{C} \,.\, \prod i \in \mathcal{I} \,.\, A \xrightarrow{\mathcal{C}} X_i,$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ f : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\texttt{NaturalTransform}(\lambda) : \lambda_i(x) X_{i,j}(f) = \lambda_j(x);$

$\rightsquigarrow () := \eth^{-1}\texttt{Cone} : \Big[\alpha(A)(x) \in \mathrm{Cone}_{\mathcal{I}}(A, X)\Big];$

$\rightsquigarrow \alpha(A) := I(\to) : L \to \mathrm{Cone}_{\mathcal{I}}(A, X),$

$\texttt{Assume}\ (A, \mu) : \mathrm{Cone}_{\mathcal{I}}(A, X),$

$(1) := \eth\texttt{Cone}(\mathcal{I}, X)(A, \mu) : \Big[\Big(\mathrm{End}_{\mathcal{C}}(A), \mu^*\Big) : \texttt{Cone}(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(A, X))\Big],$

$(\phi, 2) := \eth\texttt{Limit}(L)(C, \mu) : \sum \phi : \mathrm{End}_{\mathcal{C}}(A) \to L \,.\, \forall i \in \mathcal{I} \,.\, \phi\lambda_i = \mu_i^*,$

$\beta(A)(A, \mu) := \phi(\mathrm{id}_A) : L;$

$\rightsquigarrow \beta(A) := I(\to) : \mathrm{Cone}_{\mathcal{I}}(A, X) \to L,$

$(1) := \eth\texttt{Cone}(L, \lambda)\eth\frac{L}{\lambda}\eth^{-1}\texttt{Cone} : \left[\left(\frac{L}{\lambda}, \lambda\right) : \texttt{Cone}\Big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(A, X)\Big)\right],$

$(2) := \eth\texttt{Limit}(L, \lambda) : \exists! \psi : \frac{L}{\lambda} \to L : \forall i \in \mathcal{I} \,.\, \psi\lambda_i = \lambda_i,$

$(3) := \eth^{-1}\texttt{Injection}\eth\frac{L}{\lambda} : \left[\lambda : L \hookrightarrow \prod i \in \mathcal{I} \,.\, A \xrightarrow{\mathcal{C}} X_i\right],$

$()_1 := \eth\alpha(A)\eth\beta(A)\eth\texttt{Limit}(L, \lambda)\texttt{InjectionRetracts}(3)\eth^{-1}\mathrm{id}_L :$

$: \alpha(A)\beta(A) = \Lambda x \in L \,.\, \beta(A)(A, \lambda(x)) = \Lambda x \in L \,.\, \phi_{\lambda(x)}(\mathrm{id}_A) = \Lambda x \in L \,.\, x = \mathrm{id}_L,$

$()_2 := \eth\beta(A)\eth\alpha(A) :$

$: \beta(A)\alpha(A) = \Lambda(A, \mu) \in \mathrm{Cone}_{\mathcal{I}}(A, X) \,.\, \Big(A, \lambda\big(\phi_\mu(\mathrm{id}_A)\big)\Big) = \Lambda(A, \mu) \in \mathrm{Cone}_{\mathcal{I}}(A, X) \,.\, (A, \mu) = \mathrm{id};$

$\rightsquigarrow \alpha := I\left(\prod\right) : \prod A \in \mathcal{C} \,.\, \lim_{\mathcal{I}} \mathcal{M}_{\mathcal{C}}(A, X_i) \leftrightarrow \mathrm{Cone}_{\mathcal{I}}(A, X),$

$\texttt{Assume}\ A, B : \mathcal{C},$

$\texttt{Assume}\ f : B \xrightarrow{\mathcal{C}} A,$

$(L^A, \lambda^A) := \lim_{i \in \mathcal{I}} \mathcal{M}_{\mathcal{C}}(A, X_i) : \texttt{Limit}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(A, X)\big),$

$(L^B, \lambda^B) := \lim_{i \in \mathcal{I}} \mathcal{M}_{\mathcal{C}}(B, X_i) : \texttt{Limit}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(B, X)\big),$

$\mu := f_* \lambda^A : \prod i \in \mathcal{I} \,.\, L^A \to \mathcal{M}_{\mathcal{C}}(B, X),$

Assume $i, j : \mathcal{I}$,

Assume $h : i \xrightarrow{\mathcal{I}} j$,

$() := \eth \mu_i \eth \Lambda_A \eth^{-1} \mu_j : \mu_i X^*_{i,j}(h) = f_* \lambda^A_i X^*_{i,j}(h) = f_* \lambda^A_j = \mu_j$;

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : \Big[(L^A, \mu) : \texttt{Cone}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(B, X)\big)\Big]$,

$(\psi, 2) := \eth\texttt{Limit}(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(A, X))(1) : \sum \psi : L^A \to L^B \;.\; \psi \lambda^B = f_* \lambda^A$,

$() := \eth\alpha(A)(2)\eth^{-1}\alpha(B) :$

$: \alpha(A)\mathrm{Cone}_{\mathcal{I};A,B}(f, X) = \Lambda x \in \lim_{i\in\mathcal{I}} \mathcal{M}_{\mathcal{C}}(A, X_i) \;.\; \big(B, f(\lambda^A(x))\big) =$

$= \Lambda x \in \lim_{i\in\mathcal{I}} \mathcal{M}_{\mathcal{C}}(A, X_i) \;.\; (B, \lambda^B(\psi(x))) = \lim_{i\in I;A,B} \mathcal{M}_{\mathcal{C}}(f, X_i)\alpha(B)$;

$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturalTransform} : \Big[\alpha : \lim_{i\in\mathcal{I}} \mathcal{M}_C(\cdot, X) \iff \mathrm{Cone}_{\mathcal{I}}(\cdot, X)\Big]$;

$\square$

RepresentationCommutesWithLimit $::$ $\forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C})$.

$. \forall (0) : \Big[\mathcal{C} : \texttt{WithLimit}\{(\mathcal{I}, X)\}\Big] \;.\; \mathcal{M}_{\mathcal{C}}(\cdot, \lim_{i\in\mathcal{I}} X_i) \iff \lim_{i\in\mathcal{I}} \mathcal{M}_{\mathcal{C}}(\cdot, X_i)$

Proof $=$

$(L.\lambda) := \eth(0) : \texttt{Limit}(\mathcal{I}, X)$,

Assume $A : \mathcal{C}$,

Assume $f : A \xrightarrow{\mathcal{C}} L$,

$\mu := f\lambda : \prod i \in \mathcal{I} \;.\; A \xrightarrow{\mathcal{C}} X_i$,

Assume $i, j : \mathcal{I}$,

Assume $h : i \xrightarrow{\mathcal{I}} j$,

$() := \eth\mu_i \eth\lambda \eth^{-1}\mu_j : \mu_i X_{i,j}(h) = f\lambda_i X_{i,j}(h) = \lambda_j X_{i,j}(h)$;

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : \Big[(A, \mu) : \texttt{Cone}\Big]$,

$\alpha(A)(f) := \mathcal{I}(\to : \mathcal{M}_{\mathcal{C}}(A, L) \to \mathrm{Cone}_{\mathcal{I}}(A, X)$;

Assume $(A, \mu) : \mathrm{Cone}_{\mathcal{I}}(A, X)$,

$(\phi, 2) := \eth\texttt{Limit}(L, \lambda)(A, \mu) : \sum \phi : A \xrightarrow{\mathcal{C}} L \;.\; \forall i \in \mathcal{I} \;.\; \phi\lambda_i = \mu_i$,

$\beta(A)(A, \mu) := \phi : A \xrightarrow{\mathcal{C}} L$;

$\rightsquigarrow \beta(A) := I(\to) : \mathrm{Cone}_{\mathcal{I}}(A, X) \to A \xrightarrow{\mathcal{C}} L$,

$()_1 := \eth\alpha(A)\eth\beta(A)\eth\texttt{Limit}\eth\texttt{unique} : \alpha(A)\beta(A) = \Lambda f : A \xrightarrow{\mathcal{C}} L \;.\; \beta(A)(A, f\lambda) = \Lambda f : A \xrightarrow{\mathcal{C}} L \;.\; f = \mathrm{id}$,

$()_2 := \eth\beta(A)\alpha(A)\eth\texttt{Limit} : \beta(A)\alpha(A) = \Lambda(A, \mu) : \mathrm{Cone}_{\mathcal{I}}(A, X) \;.\; (A, \phi_\mu\lambda) = \Lambda(A, \mu) : \mathrm{Cone}_{\mathcal{I}}(A, X) \;.\; (A, \mu) = \mathrm{id}$

$\rightsquigarrow \alpha := I\left(\prod\right) : \prod A \in \mathcal{C} \;.\; \alpha(A) : \mathcal{M}_{\mathcal{C}}(A, L) \leftrightarrow \mathrm{Cone}_{\mathcal{I}}(A, X)$,

Assume $A, B : \mathcal{C}$,

Assume $f : B \xrightarrow{\mathcal{C}} A$,

$:= \eth\alpha(A)\eth f^* \eth^{-1} f^8 \eth^{-1} : \alpha(A)f^* = \Lambda g : A \xrightarrow{\mathcal{C}} L \;.\; (B, fg\lambda) = f_*\alpha(B)$;

$\rightsquigarrow (1) := \eth\texttt{NaturalTransform} : \Big[\alpha : \mathcal{M}(\cdot, L) \iff \mathrm{Cone}_{\mathcal{I}}(\cdot, L)\Big]$,

$(*) := (1)\texttt{ConeRepresentation}\eth L : \mathcal{M}_{\mathcal{C}}(\cdot, \lim_{i\in\mathcal{I}} X_i) \iff \lim_{i\in\mathcal{I}} \mathcal{M}_{\mathcal{C}}(\cdot, X_i)$;

$\square$

$\texttt{LimRepInterpretation1} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall A \in \mathcal{C} \;.\; \left[\mathcal{M}_{\mathcal{C}}(A, \cdot) : \texttt{PreservesLimits}(\mathcal{C}, \mathsf{SET})\right]$

$\texttt{Proof} =$

Trivially follows from previous therorem.

□

$\texttt{LimRepInterpretarion2} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \;.\; \mathcal{M}_{\mathcal{C}}(\cdot, \lim_{i \in \mathcal{I}} X) = \lim_{i \in \mathcal{I}} X_i \mathrm{y}$

$\texttt{Proof} =$

Trivially follows from previous therorem.

□

$\texttt{YonedasEmbeddingOnRepresentation} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.$

$\;.\; \mathrm{y} : \texttt{PreservesLimits} \,\&\, \texttt{ReflectsLimits}(\mathcal{C}, \mathsf{SET}^{\mathcal{C}^{\mathrm{op}}})$

$\texttt{Proof} =$

Use the fact that Yoneda's embedding is fully faithful embedding and apply the to second interpretation.

□

$\texttt{ColimitAndCorepresentationCocommute} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall (\mathcal{I}, X) : \texttt{Diagram} \;.$

$\;.\; \forall (0) : \left[\mathcal{C} : \texttt{WithColimit}\{(\mathcal{I}, X)\}\right] \;.\; \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathcal{M}_{\mathcal{C}}(X_i, \cdot) \iff \mathcal{M}_{\mathcal{C}}(\operatorname{colim}_{i \in \mathcal{I}} X_i, \cdot)$

$\texttt{Proof} =$

Use dual tricks in the proofs simmilar to one in the begining of this chapters.

□

$\texttt{ColimRepInterpretation1} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall A \in \mathcal{C} \;.\; \left[\mathcal{M}_{\mathcal{C}}(\cdot, A) : \texttt{PreservesLimits}(\mathcal{C}, \mathsf{SET})\right]$

$\texttt{Proof} =$

Trivially follows from previous therorem.

□

$\texttt{ColimRepInterpretarion2} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.\; \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \;.\; \mathcal{M}_{\mathcal{C}}(\operatorname{colim}_{i \in \mathcal{I}} X, \cdot) = \lim_{i \in \mathcal{I}^{\mathrm{op}}} X_i \mathrm{y}$

$\texttt{Proof} =$

Trivially follows from previous therorem.

□

$\texttt{YonedasEmbeddingOnRepresentation2} :: \forall \mathcal{C} : \texttt{LocallySmall} \;.$

$\;.\; \mathrm{y} : \texttt{PreservesLimits} \,\&\, \texttt{ReflectsLimits}(\mathcal{C}^{\mathrm{op}}, \mathsf{SET}^{\mathcal{C}})$

$\texttt{Proof} =$

Use the fact that Yoneda's embedding is fully faithful embedding and apply the to second interpretation.

□

$\texttt{WithProducts} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithProducts} \iff \forall I \in \mathsf{SET} \;.\; \forall X : I \to \mathcal{C} \;.\; \exists \texttt{Limit}(\texttt{discrete}(X), \mathrm{Id})$

$\texttt{WithCoproducts} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithCoproducts} \iff \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathcal{C} \,.\, \exists \texttt{Limit}(\texttt{discrete}(X), \mathrm{Id})$

$\texttt{WithEqualizers} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithEqualizers} \iff \forall A, B \in \mathcal{C} \,.\, \forall f, g : A \xrightarrow{\mathcal{C}} B \,.\, \exists \texttt{Equalizer}(A, B, f, g)$

$\texttt{WithCoequalizer} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithCoequalizers} \iff \forall A, B \in \mathcal{C} \,.\, \forall f, g : A \xrightarrow{\mathcal{C}} B \,.\, \exists \texttt{Coequalizer}(A, B, f, g)$

$\texttt{WithProductsAndEqualizersIsComplete} :: \forall \mathcal{C} : \texttt{WithProducts} \,\&\, \texttt{WithEqualizers} \,\&\, \texttt{LocallySmall} \,.$

$\quad . \, \mathcal{C} : \texttt{Complete}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}),$

$A := \prod_{i \in \mathcal{I}} X_i : \mathcal{C},$

$B := \prod_{i,j \in \mathcal{I}} \prod_{h \in \mathcal{M}_{\mathcal{I}}(i,j)} X_j : \mathcal{C},$

$f := \prod_{i,j \in \mathcal{I}} \prod_{h \in \mathcal{M}_{\mathcal{I}}(i,j)} \pi_j : A \xrightarrow{\mathcal{C}} B,$

$g := \prod_{i,j \in \mathcal{I}} \prod_{h \in \mathcal{M}_{\mathcal{I}}(i,j)} \pi_i X_{i,j}(h) : A \xrightarrow{\mathcal{C}} B,$

$C := \eth\texttt{WithEqualizer}(\mathcal{C})(A, B, f, g) : \texttt{Equalizer}(A, B, f, g),$

$\texttt{Assume} \ Y : \mathcal{C},$

$(1) := \mathrm{Ev}_X \mathrm{y}\texttt{EvaluationPreserveslimits}\eth C : \Big[\mathcal{M}_{\mathcal{C}}(X, C) : \texttt{Equalizer}(\mathcal{M}_{\mathcal{C}}(Y, A), \mathcal{M}_{\mathcal{C}}(Y, B), f_*, g_*)\Big],$

$(2) := \texttt{RepresentationCommutesWithLimit}(1) :$

$$: \left[\mathcal{M}_{\mathcal{C}}(Y, C) : \texttt{Equalizer}\left(\prod_{i \in \mathcal{I}} \mathcal{M}_{\mathcal{C}}(Y, X_i), \prod_{i,j \in \mathcal{I}} \prod_{f \in \mathcal{M}_{\mathcal{I}}(i,j)} \mathcal{M}_{C}(Y, X_i)\right), f_*, g_*\right],$$

$(3) := \eth\texttt{Complete}(\mathsf{SET})(2) : \Big[\mathcal{M}_{\mathcal{C}}(Y, X) : \texttt{Limit}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(Y, X)\big)\Big];$

$\rightsquigarrow (1) := I(\forall) : \forall Y \in \mathcal{C} \,.\, \mathcal{M}_{\mathcal{C}}(Y, C) : \texttt{Limit}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(Y, X)\big),$

$(2) := \texttt{LimitFibration}(1) : \Big[\mathcal{M}_{\mathcal{C}}(C, \cdot) : \texttt{Limit}\big(\mathcal{I}, \mathcal{M}_{\mathcal{C}}(Y, X)\big)\Big],$

$() := \texttt{YonedasEmbeddingRepresentation}(2) : \Big[C : \texttt{Limit}(\mathcal{I}, X)\Big];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{Complete} : \Big[\mathcal{C} : \texttt{Complete}\Big];$

$\square$

$\texttt{WithCoproductsAndCoequalizersIsCocomplete} ::$

$\quad :: \forall \mathcal{C} : \texttt{WithCoproducts} \,\&\, \texttt{WithCoequalizer} \,\&\, \texttt{LocallySmall} \,.$

$\quad . \, \mathcal{C} : \texttt{Cocomplete}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FinetlyComplete} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{FinetlyComplete} \iff \forall(\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \ . \ \left|\sum i, j \in \mathcal{I} \ . \ i \xrightarrow{\mathcal{I}} j\right| < \infty \Rightarrow \exists\texttt{Limit}(I.X)$

$\texttt{FinetlyCocomplete} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{FinetlyCocomplete} \iff \forall(\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \ . \ \left|\sum i, j \in \mathcal{I} \ . \ i \xrightarrow{\mathcal{I}} j\right| < \infty \Rightarrow \exists\texttt{Colimit}(I.X)$

$\texttt{WithPullbacks} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithPullbacks} \iff \forall X : \bullet \to \bullet \leftarrow \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \ . \ \exists\texttt{PullBack}(X)$

$\texttt{WithTerminal} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithTerminal} \iff \exists X : \texttt{Terminal}(\mathcal{C})$

$\texttt{terminal} :: \prod \mathcal{C} : \texttt{WithTerminal}$

$\texttt{terminal}\,() = 0_{\mathcal{C}} := \eth\texttt{WithTerminal}$

$\texttt{WithFiniteProducts} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithFiniteProducts} \iff \forall I : \texttt{Finite} \ . \ \forall X : I \to \mathcal{C} \ . \ \exists\texttt{Product}(X)$

$\texttt{WithPushouts} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithPushouts} \iff \forall X : \bullet \leftarrow \bullet \to \bullet \xrightarrow{\mathsf{CAT}} \mathcal{C} \ . \ \exists\texttt{Pushout}(X) \ .$

$\texttt{WithInitial} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{With} \iff \exists X : \texttt{Initial}(\mathcal{C})$

$\texttt{initial} :: \prod \mathcal{C} : \texttt{WithInitial} \ . \ \texttt{Initial}(\mathcal{C})$

$\texttt{initial}\,() = 0_{\mathcal{C}} := \eth\texttt{Withinitial}$

$\texttt{WithFiniteCoproducts} :: ?\mathsf{CAT}$

$\mathcal{C} : \texttt{WithFiniteCoroducts} \iff \forall I : \texttt{Finite} \ . \ \forall X : I \to \mathcal{C} \ . \ \exists\texttt{Coproduct}(X)$

$\texttt{ProductAsPullBack} :: \forall \mathcal{C} : \texttt{WithPullbacks} \ \& \ \texttt{WithTerminal} \ . \ \forall A, B \in \mathcal{C} \ .$

$\quad . \ \exists A \times B : \texttt{Product}(A, B) : \Big[A \times B : \texttt{PullBack}(B \to 1 \leftarrow A)\Big]$

$\texttt{Proof} \ =$

$(A \times B, \pi_A, \pi_B) := \eth\texttt{WithPullback}(A, B)(B \to 1 \leftarrow A) : \texttt{Pullback}(B \to 1 \leftarrow A),$

$(1) := \eth^{-1}\texttt{Cone}(A \times B, \pi_A, \pi_B)\eth\texttt{Pushout} : \Big[(A \times B, \pi_A, \pi_B) : \texttt{Cone}(A, B)\Big],$

$\texttt{Assume} \ (C, \lambda_A, \lambda_B) : \texttt{Cone}(A, B),$

$f := \eth\texttt{Terminal}(1) : C \xrightarrow{\mathcal{C}} 1,$

$(2) := \eth^{-1}\texttt{Cone}\eth\texttt{Teminal}(1)\eth\texttt{Cone}(C, \lambda_A, \lambda_B) : \Big[(C, \lambda_A, \lambda_B, f) : \texttt{Cone}(A \to 1 \leftarrow B)\Big],$

$(\phi, 3) := \eth\texttt{Pullback}(A \times B)(2) : \sum \phi : C \to A \times B \ . \ \phi\pi_A = \lambda_A \ \& \ \phi p_B = \lambda_B;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{ProductTerminal}(1) : \Big[A \times B : \texttt{Product}(A.B)\Big];$

$\square$

$\texttt{EqualizerAsPullBack}$ :: $\forall \mathcal{C} : \texttt{WithPullbacks}$ & $\texttt{WithTerminal}$. $\forall A, B \in \mathcal{C}$. $\forall f, g : A \xrightarrow{\mathcal{C}}$.

$$. \exists E : \texttt{Equalizer}(f, g) : \left[E : \texttt{Pullback}(A \xrightarrow[(f,g)]{} B \times B \xleftarrow[(\mathrm{id},\mathrm{id})]{} B)\right]$$

Proof =

For any wedge Cone $(C, \lambda_A, \lambda_B, \lambda_{A\times B})$ it holds that $\lambda_A f = \lambda_A g = \lambda_B$,
Hence it is also a parralel cone for f and g.
For any parallel Cone $(C, \lambda_A, \lambda_B)$ it is possible to define $\lambda_{B\times B} = (\lambda_B, \lambda_B)$ making it into wedge-cone.
Hence the limits for these cone agree a limitis for pullbacks exists in the category.

□

$\texttt{TerminalAndPullbacksGiveFiniteProduct}$ :: $\forall \mathcal{C} : \texttt{WithPullbacks}$ & $\texttt{WithTerminal}$.

. $[\mathcal{C} : \texttt{WithFiniteProducts}]$

Proof =

...

□

$\texttt{TerminalAndPullbacksFinitelyComplete}$ :: $\forall \mathcal{C} : \texttt{WithPullbacks}$ & $\texttt{WithTerminal}$ &

& $\texttt{LocallyFinite}(\mathcal{C})$. $[\mathcal{C} : \texttt{FinitelyComplete}]$

Proof =

...

□

$\texttt{CoproductAsPushout}$ :: $\forall \mathcal{C} : \texttt{WithPushouts}$ & $\texttt{WithInitial}$. $\forall A, B \in \mathcal{C}$.

$$. \exists A \sqcup B : \texttt{Coproduct}(A, B) : \left[A \sqcup B : \texttt{Pushout}(B \leftarrow 0 \rightarrow A)\right]$$

Proof =

...

□

$\texttt{CoequalizerAsPushout}$ :: $\forall \mathcal{C} : \texttt{WithPushout}$ & $\texttt{WithInitial}$. $\forall A, B \in \mathcal{C}$. $\forall f, g : A \xrightarrow{\mathcal{C}}$.

$$. \exists E : \texttt{Coequalizer}(f, g) : \left[E : \texttt{Pushout}(A \xleftarrow[f|g]{} B \times B \xrightarrow[\mathrm{id}\,|\,\mathrm{id}]{} B)\right]$$

Proof =

...

□

$\texttt{IsnitalAndPushoutssGiveFiniteCoroduct}$ :: $\forall \mathcal{C} : \texttt{WithPushouts}$ & $\texttt{WithInitial}$.

. $[\mathcal{C} : \texttt{WithFiniteCoproducts}]$

Proof =

...

□

$\texttt{TerminalAndPushoutsFinitelyCocomplete}$ :: $\forall \mathcal{C} : \texttt{WithPushoutss}$ & $\texttt{WithInitial}$ &

& $\texttt{LocallyFinite}(\mathcal{C})$. $[\mathcal{C} : \texttt{FinitelyCocomplete}]$

Proof =

...

□

3.5 Working with Complete Categories

$$\texttt{Continuous} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$
$$F : \texttt{Continuous} \iff F : \texttt{PreserveLimits}\Big(\texttt{Diagram}(\mathcal{A})\Big)$$

$$\texttt{Cocontinuous} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \,.\, ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$$
$$F : \texttt{Cocontinuous} \iff F : \texttt{PreserveColimits}\Big(\texttt{Diagram}(\mathcal{A})\Big)$$

$$\texttt{CompleteSlice} :: \forall \mathcal{C} : \texttt{Complete} \,.\, \forall X \in \mathcal{C} \,.\, \frac{X}{\mathcal{C}} : \texttt{Complete}$$

$\texttt{Proof} =$

$\texttt{Assume}\ \Big(\mathcal{I}, \big((Y, f), \phi\big)\Big) : \texttt{Diagram}\ \dfrac{X}{\mathcal{C}},$
$O := \mathcal{O}(\mathcal{I}) \cup \{X\} : \mathsf{SET},$
$M(X, X) := \{\mathrm{id}_X\} : \mathsf{SET},$
$M := \Lambda a, b \in O \,.\, \texttt{if } b == X! = a \texttt{ then } \emptyset \texttt{ else if } a == X \texttt{ then } \{f_b\} \texttt{ else } \mathcal{M}_{\mathcal{I}}(a, b) : O \times O \to \mathsf{SET},$
$C(X, X, \cdot)(\mathrm{id}_X, \cdot) := \Lambda b \in O \,.\, \Lambda g \in M(X, b) \,.\, g : \prod b \in O \,.\, M(X, b) \to M(X, b),$
$C(X, \cdot, \cdot) := \Lambda a, b \in \mathcal{O}(\mathcal{I}) \,.\, \Lambda g \in M(X, a) \,.\, \Lambda h \in M(a, b) \,.\, f_b : \prod a, b \in O . M(X, a) \times M(a, b) \to M(X, b),$
$C := \Lambda a, b, c \in \mathcal{O}(\mathcal{I}) \,.\, \Lambda g \in M(a, b) \,.\, \Lambda h \in M(b, c) \,.\, gh : \prod a, b, c \in O \,.\, M(a, b) \times M(b, c) \to M(a, c),$
$I(X) := \mathrm{id}_X : M(X, X),$
$I := \Lambda a \in \mathcal{O}(\mathcal{I}) \,.\, \mathrm{id}_a : \prod a \in O \,.\, M(a, a),$
$\mathcal{I}' := (O, M, C, I) : \texttt{Small},$
$F'(X) := X : \mathcal{C},$
$F' := \Lambda i \in \mathcal{I} \,.\, Y_i : \mathcal{I}' \to \mathcal{C},$
$F''(X, \cdot) := \Lambda a \in \mathcal{I}' \,.\, \Lambda g : X \xrightarrow{\mathcal{I}'} a \,.\, g : \prod a \in \mathcal{I}' \,.\, X \xrightarrow{\mathcal{I}'} a \to X \xrightarrow{\mathcal{C}} Y_a,$
$F'' := \Lambda a, b \in \mathcal{I} \,.\, \Lambda h \in a \xrightarrow{\mathcal{I}} b \,.\, \phi_{a,b}(h) : \prod a, b \in \mathcal{I}' \,.\, a \xrightarrow{\mathcal{I}'} b \to Y_a \xrightarrow{\mathcal{C}} Y_b,$
$F := (F', F'') : \texttt{Covariant}(\mathcal{I}', \mathcal{C}),$
$(L, \lambda) := \eth\texttt{Complete} : \texttt{Limit}(\mathcal{C})(\mathcal{I}', F),$
$(1) := \eth^{-1}\texttt{Cone}\dfrac{X}{\mathcal{C}}\eth(L, \lambda)\eth(\mathcal{I}, F) : \Big[\big((L, \lambda_X), \lambda_{|\mathcal{I}}\big) : \texttt{Cone}\Big(\mathcal{I}, \big((Y, f), \phi\big)\Big)\Big],$
$\texttt{Assume}\ \big((C, g), \mu\big) : \texttt{Cone}\Big(\mathcal{I}, \big((Y, f), \phi\big)\Big),$
$(2) := \eth^{-1}\texttt{Cone}\eth\big((C, g), \mu\big) : \big(C, (\mu_i)_{i \in \mathcal{I}} \oplus (g)_{i = X}\big) : \texttt{Cone}(\mathcal{I}', F),$
$(\varphi, 3) := \eth\texttt{Limit}(L, \lambda)(2) : \sum \varphi : C \xrightarrow{\mathcal{C}} L \,.\, \Big(\forall i \in \mathcal{I} \,.\, \mu_i = \varphi\lambda_i\Big) \ \&\ g = \varphi\lambda_X,$
$() := \eth^{-1} \displaystyle\int \big((Y, f), \phi\big)(3) : \Big[\varphi : \big((C, g), \mu\big) \xrightarrow{\int((Y,f),\phi)} \big((L, \lambda_X), \lambda_{|\mathcal{I}}\big)\Big];$
$\rightsquigarrow () := \eth^{-1}\texttt{Limit} : \Big[\big((L, \lambda_X), \lambda_{|\mathcal{I}}\big) : \texttt{Limit}\Big(\mathcal{I}, \big((Y, f), \phi\big)\Big)\Big];$
$\rightsquigarrow (*) := \eth^{-1}\texttt{Complete} : \Big[\dfrac{X}{\mathcal{C}} : \texttt{Complete}\Big];$

$\square$

$\texttt{CocompleteSlice} :: \forall \mathcal{C} : \texttt{LocallySmall} \,.\, \forall X \in \mathsf{CAT} \,.\, \frac{X}{\mathcal{C}} : \texttt{Cocomplete}$

$\texttt{Proof} \;=$

$(1) := \texttt{ForgetfulDemorph2StrictlyCreatesC}\eth\Pi_X : \left[\frac{\mathcal{C}}{X} : \texttt{WithLimits}(\texttt{Connected})\right],$

$(2) := \eth\frac{X}{\mathcal{C}}\eth\mathrm{id}_X\eth^{-1}\texttt{Initial} : \left[(X, \mathrm{id}_X) : \texttt{Initial}\ \frac{C}{\mathcal{C}}\right],$

$(3) := \texttt{CoproductsAsPushouts}^{\omega}(1)(2) : \left[\frac{X}{\mathcal{C}} : \texttt{WithCoproducts}\right],$

$(*) := \texttt{WithCoproductsAndCoequalizersIsCocomplete}(1,3) : \left[\frac{X}{\mathcal{C}} : \texttt{Cocomplete}\right];$

$\square$

$\texttt{CompleteCoslice} :: \forall \mathcal{C} : \texttt{LocallySmall} \,.\, \forall X \in \mathsf{CAT} \,.\, \frac{\mathcal{C}}{X} : \texttt{Complete}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CocompleteCoslice} :: \forall \mathcal{C} \in \mathsf{CAT} \,.\, \forall X \in \mathsf{CAT} \,.\, \frac{\mathcal{C}}{X} : \texttt{Cocomplete}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{Subobject} := \prod \mathcal{C} \in \mathsf{CAT} \,.\, \prod A \in \mathcal{C} \,.\, \sum X \in \mathcal{C} \,.\, A \overset{\mathcal{C}}{\hookrightarrow} X : \prod \mathcal{C} \in \mathsf{CAT} \,.\, \prod A \in \mathcal{C} \,.\, \texttt{Type};$

$\texttt{KernelPair} := \prod \mathcal{C} : \texttt{WithPullbacks} \,.\, \prod X, Y \in \mathcal{C} \,.\, \prod f : X \xrightarrow{\mathcal{C}} Y \,.\, \texttt{Pullback}(X \underset{f}{\overset{\mathcal{C}}{\rightarrow}} Y \underset{f}{\overset{\mathcal{C}}{\leftarrow}} X) :$

$: \prod \mathcal{C} : \texttt{WithPullbacks} \,.\, \prod X, Y \in \mathcal{C} \,.\, X \xrightarrow{\mathcal{C}} Y \to \texttt{Type};$

$\texttt{EquivalenceObject} :: \prod \mathcal{C} : \texttt{FinetlyComplete} \,.\, \prod X \in \mathcal{C} \,.\, ?\texttt{Subobject}(X \times X)$

$(R, \phi) : \texttt{EquivaenceObject} \iff \exists \rho : X \xrightarrow{\mathcal{C}} R : \rho\phi\pi_1 = \rho\phi\pi_2 = \mathrm{id}_X\ \&$

$\quad \&\ \exists \sigma : R \xrightarrow{\mathcal{C}} R : \phi\pi_1 = \sigma\phi\pi_2\ \&\ \phi\pi_2 = \sigma\phi\pi_1\ \&$

$\quad \&\ \exists \tau : R \times R \xrightarrow{\mathcal{C}} R : \pi_1\phi\pi_1 = \tau\phi\pi_1\ \&\ \pi_2\phi\pi_2 = \tau\phi\pi_2$

$\texttt{quetientObject} :: \texttt{EquivalenceObject}(\mathcal{C}, X) \to \mathcal{C}$

$\texttt{quetientObject}\,(R, \phi) = \frac{X}{R} := \eth\texttt{FinetlyComplete}(\mathcal{C})(R \underset{\phi\pi_1}{\overset{\mathcal{C}}{\longrightarrow}} X \underset{\phi\pi_2}{\overset{\mathcal{C}}{\longleftarrow}} R)$

3.6 Functoriality of Limits

$\mathtt{LimitFunctor} :: \prod \mathcal{C} : \mathtt{Complete} \,.\, \prod \mathcal{I} : \mathtt{Small} \,.\, ?(\mathcal{C}^{\mathcal{I}} \xrightarrow{\mathsf{CAT}} \mathcal{C})$

$L : \mathtt{LimitFunctor} \iff L = \lim_{\mathcal{I}} \iff \forall X \in \mathcal{C}^{\mathcal{I}} \,.\, \exists \lambda : \mathrm{Const}_{\mathcal{I}}(L(X)) \Rightarrow X : \Big[(L(X), \lambda) : \mathtt{Limit}(\mathcal{I}, X)\Big]$

$\mathtt{LimitFunctorExists} :: \forall \mathcal{C} : \mathtt{Complete} \,.\, \forall \mathcal{I} : \mathtt{Small} \,.\, \exists \mathtt{LimitFunctor}(\mathcal{C}, \mathcal{I})$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ X : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C},$

$(L, \lambda) := \eth\mathtt{Complete}(\mathcal{C})(\mathcal{I}, X) : \mathtt{Limit}(\mathcal{I}, X),$

$(F'(X), \lambda^X) := (L, \lambda) : \mathtt{Limit}(\mathcal{I}, X),$

$(F', \lambda) := I\left(\prod\right) : \prod X : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C} \,.\, \mathtt{Limit}(\mathcal{I}, X);$

$\mathtt{Assume}\ X, Y : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C},$

$\mathtt{Assume}\ \alpha : X \Rightarrow Y,$

$\mu := \lambda^X \alpha : \prod i \in \mathcal{I} \,.\, F(X) \to Y_i,$

$\mathtt{Assume}\ i, j : \mathcal{I},$

$\mathtt{Assume}\ h : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\mu_i \eth\mathtt{NaturalTransform}(\alpha)\eth\mathtt{Cone}(F(X), \lambda^X)\eth^{-1}\mu_j :$

$\quad : \mu_i Y_{i,j}(h) = \lambda^X_i \alpha_i Y_{i,j}(h) = \lambda^X_i X_{i,j}(h)\alpha_j = \lambda^X_j \alpha_j = \mu_j;$

$\rightsquigarrow (1) := \eth^{-1}\mathtt{Cone} : \Big[\big(F'(X), \mu\big) : \mathtt{Cone}(\mathcal{I}, Y)\Big],$

$\big(F_{X,Y}(\alpha), (2)\big) := \eth\mathtt{Limit}(F('Y), \lambda^Y)(1) : \sum F''_{X,Y}(\alpha) : F'(X) \xrightarrow{\mathcal{C}} F'(Y) \,.\, \lambda^X \alpha = F''_{X,Y}(\alpha)\lambda^Y;$

$\rightsquigarrow F'' := I\left(\prod\right) :$

$\quad : \prod X, Y : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C} \,.\, \sum F''_{X,Y} : (X \Rightarrow Y) \to (F'(X) \xrightarrow{\mathcal{C}} F'(Y)) \,.\, \forall \alpha : X \Rightarrow Y \,.\, \lambda^X \alpha = F''_{X,Y}(\alpha)\lambda^Y,$

$\mathtt{Assume}\ X, Y, Z : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C},$

$\mathtt{Assume}\ \alpha : X \Rightarrow Y,$

$\mathtt{Assume}\ \beta : Y \Rightarrow Z,$

$(1) := \eth F'' : \lambda^X \alpha\beta = F''_{X,Y}(\alpha)\lambda^Y \beta = F''_{X,Y}(\alpha)F''_{Y,Z}(\beta)\lambda_X,$

$() := \eth F'' \eth\mathtt{Limit}(1) : F''_{X,Y}(\alpha)F''_{Y,Z}(\beta) = F''_{X,Z}(\alpha\beta);$

$\rightsquigarrow (*) := \eth^{-1}\mathtt{LimitFunctor}\eth(F', F'')\eth^{-1}\mathtt{Covariant} : \Big[(F', F'') : \mathtt{LimitFunctor}(\mathcal{C}, \mathcal{I})\Big];$

$\square$

$\mathtt{ColimitFunctor} :: \prod \mathcal{C} : \mathtt{Cocomplete} \,.\, \prod \mathcal{I} : \mathtt{Small} \,.\, ?(\mathcal{C}^{\mathcal{I}} \xrightarrow{\mathsf{CAT}} \mathcal{C})$

$L : \mathtt{LimitFunctor} \iff L = \mathrm{colim}_{\mathcal{I}} \iff \forall X \in \mathcal{C}^{\mathcal{I}} \,.\, \exists \lambda : \mathrm{Const}_{\mathcal{I}}(L(X)) \Leftarrow X : \Big[(L(X), \lambda) : \mathtt{Colimit}(\mathcal{I}, X)\Big]$

$\mathtt{ColimitFunctorExists} :: \forall \mathcal{C} : \mathtt{Cocomplete} \,.\, \forall \mathcal{I} : \mathtt{Small} \,.\, \exists \mathtt{ColimitFunctor}(\mathcal{C}, \mathcal{I})$

$\mathtt{Proof}\ =$

$\dots$

$\square$

`AssociativeProducts` :: $\forall \mathcal{C}$: `WithFiniteProducts` . $\forall X, Y, Z \in \mathcal{C}$. $(X \times Y) \times Z \cong X \times (Y \times Z)$

`Proof` $=$

$(X \times Y) \times Z \cong X \times Y \times Z \cong X \times (Y \times Z)$

$\square$

3.7 Freyd Theorem

$\texttt{SelfLimitIsInitial} :: \forall \mathcal{I} : \texttt{Small} \,.\, \forall (L, \lambda) = \lim_{\mathcal{I}} \mathrm{Id} \,.\, L : \texttt{Initial}$

$\texttt{Proof} \;=$
$(1) := \eth\texttt{Cone}(\mathcal{I}, \mathrm{Id})(\lambda_L) : \forall X \in \mathcal{I} \,.\,\,.\, \lambda_L \lambda_X = \lambda_X,$
$(2) := \eth \int_{\mathcal{I}} \mathrm{Id}(1) : \lambda_L : (L, \lambda) \xrightarrow{\int_{\mathcal{I}} \mathrm{Id}} (L, \lambda),$
$(3) := \eth\texttt{Limit}(L, \lambda)(2) : \lambda_L = \mathrm{id}_L,$
$\texttt{Assume}\ X : \mathcal{I},$
$\texttt{Assume}\ f : L \xrightarrow{\mathcal{I}} X,$
$() := \eth\texttt{Cone}(L, \lambda)(X, f)(3) : \lambda_X = \lambda_L f = f;$
$\rightsquigarrow (*) := \eth^{-1}\texttt{Initial}\eth\lambda : [X : \texttt{Initial}(\mathcal{I})];$
$\square$

$\texttt{SelfColimitIsTerminal} :: \forall \mathcal{I} : \texttt{Small} \,.\, \forall (L, \lambda) = \operatorname{colim}_{\mathcal{I}} \mathrm{id} \,.\, L : \texttt{Terminal}$

$\texttt{Proof} \;=$

$\texttt{WeaklyInitial} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\mathcal{C}$

$A : \texttt{WeaklyInitial} \iff \forall X \in \mathcal{C} \,.\, \exists A \xrightarrow{\mathcal{C}} X$

$\texttt{WeaklyTerminal} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\mathcal{C}$

$A : \texttt{WeaklyTerminal} \iff \forall X \in \mathcal{C} \,.\, \exists X \xrightarrow{\mathcal{C}} A$

$\texttt{KComplete} :: \mathsf{CARD} \to ?\mathsf{CAT}$
$\mathcal{C} : \texttt{KComplete} \iff \mathcal{C} : \kappa\text{-}\texttt{Complete} \iff \Lambda\kappa \in \mathsf{CARD} \,.\, \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \,.$
$\quad .\, \left| \sum i, j \in \mathcal{I} \,.\, i \xrightarrow{\mathcal{I}} j \right| \le \kappa \Rightarrow \exists\texttt{Limit}(\mathcal{I}, X)$

$\texttt{KCocomplete} :: \mathsf{CARD} \to ?\mathsf{CAT}$
$\mathcal{C} : \texttt{KCocomplete} \iff \mathcal{C} : \kappa\text{-}\texttt{Cocomplete} \iff \Lambda\kappa \in \mathsf{CARD} \,.\, \forall (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{C}) \,.$
$\quad .\, \left| \sum i, j \in \mathcal{I} \,.\, i \xrightarrow{\mathcal{I}} j \right| \le \kappa \Rightarrow \exists\texttt{Colimit}(\mathcal{I}, X)$

$\texttt{KSmall} :: \mathsf{CARD} \to ?\mathsf{CAT}$

$\mathcal{C} : \texttt{KSmall} \iff \mathcal{C} : \kappa\text{-}\texttt{Small} \iff \left| \sum X, Y \in \mathcal{C} \,.\, X \xrightarrow{\mathcal{I}} Y \right| \le \kappa$

$\texttt{KPower} :: \prod \kappa \in \mathsf{CARD} \,.\, \prod \mathcal{C} : \kappa\text{-}\texttt{Complete} \,.\, \mathcal{C} \to \kappa \to \mathcal{C}$

$\texttt{KPower}\,(X, \omega) = X^\omega := \prod_{i \in \omega} X$

$\texttt{KTensor} :: \prod \kappa \in \mathsf{CARD} \,.\, \prod \mathcal{C} : \kappa\text{-}\texttt{Cocomplete} \,.\, \mathcal{C} \to \kappa \to \mathcal{C}$

$\texttt{KTensor}\,(X, \omega) = \omega X := \coprod_{i \in \omega} X$

$\texttt{FreydTheorem} :: \forall \kappa \in \mathsf{CARD} \;.\; \forall \mathcal{C} : \kappa\text{-}\texttt{Small} \;\&\; \kappa\text{-}\texttt{Completete} \;.\; \mathcal{C} : \texttt{Preorder}$

$\texttt{Proof} \;=$

$\omega := \left| \sum X, Y \in \mathcal{C} \;.\; X \xrightarrow{\mathcal{C}} Y \right| : \mathsf{CARD},$

$(1) := \eth\kappa\text{-}\texttt{Small}(\mathcal{C})\eth\omega : \omega \leq \kappa,$

$\texttt{Assume}\; X, Y : \mathcal{C},$

$\texttt{Assume}\; f, g : X \xrightarrow{\mathcal{C}} Y,$

$\texttt{Assume}\; (2) : f \neq g,$

$(3) := \eth\texttt{Product}\eth Y^{\omega}\eth\kappa\text{-}\texttt{Complete}(\mathcal{C})(1) : \left| X \xrightarrow{\mathcal{C}} Y^{\omega} \right| \leq 2^{\omega},$

$(4) := \eth^{-1}\texttt{CardLess}(\eth\omega)(2)\texttt{CantorTHM}(\omega) : \omega < 2^{\omega} \leq \omega,$

$(5) := \eth\texttt{CardGreater}(4) : \omega \neq \omega,$

$(6) := I(\bot)(5) : \bot;$

$\rightsquigarrow (*) := E(\bot)\eth\texttt{Preorder} : [\mathcal{C} : \texttt{Preorder}];$

$\square$

$\texttt{FreydTheorem2} :: \forall \kappa \in \mathsf{CARD} \;.\; \forall \mathcal{C} : \kappa\text{-}\texttt{Small} \;\&\; \kappa\text{-}\texttt{Cocompletete} \;.\; \mathcal{C} : \texttt{Preorder}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

3.8 Interaction of Limits and Colimits

$$\texttt{DoubleLimit} :: \forall \mathcal{I}, \mathcal{J} : \texttt{Small} \,.\, \forall \mathcal{C} : \texttt{Complete} \,\&\, \texttt{LocallySmall} \,.\, \forall F : \mathcal{I} \times \mathcal{J} \xrightarrow{\mathsf{CAT}} \mathcal{C} \,.$$

$$.\, \forall X : \prod i \in \mathcal{I} \,.\, \texttt{Limit}\big(\mathcal{J}, F(i, \cdot)\big) \,.\, \forall Y : \prod j \in \mathcal{J} \,.\, \texttt{Limit}\big(\mathcal{I}, F(\cdot, j)\big) \,.$$

$$\lim_{i \in \mathcal{I}} X_i \cong_{\mathcal{C}} \lim_{(i,j) \in \mathcal{I} \times \mathcal{J}} F(i,j) \cong_{\mathcal{C}} \lim_{j \in \mathcal{J}} Y_j$$

$\texttt{Proof} =$

$\texttt{Assume}\ A : \mathcal{C},$

$H := \Lambda i, j \in \mathcal{I} \,.\, \mathcal{M}_{\mathcal{C}}\big(A, F(i,j)\big) : \mathcal{I} \times \mathcal{J} \xrightarrow{\mathsf{CAT}} \mathsf{SET},$

$(1) := \eth H\texttt{LimCommutes}(F) : \lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} H(i,j) \cong_{\mathsf{SET}} \lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} \mathcal{M}_{\mathcal{C}}\big(A, F(i,j)\big) = \mathcal{M}_{\mathcal{C}}\big(A, \lim_{i \in \mathcal{I}} \lim_{i \in \mathcal{J}} F(i,j)\big),$

$(2) := \texttt{LimCommutes}(F) : \lim_{(i,j) \in \mathcal{I} \times \mathcal{J}} H(i,j) \cong_{\mathsf{SET}} \mathcal{M}_{\mathcal{C}}\big(A, \lim_{(i,j) \in \mathcal{I} \times \mathcal{J}} F(i,j)\big),$

$(3) := \texttt{LimitRepresentation}(H) : \lim_{(i,j) \in \mathcal{I} \times \mathcal{J}} H(i,j) \cong_{\mathsf{SET}} \mathrm{Cone}_H(1),$

$(4) := \texttt{LimitReprsentatuin}(\lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{I}} H(i,j)) : \lim_{i \in \mathcal{I}} \lim_{j \in \mathcal{I}} H(i,j) \cong_{\mathsf{SET}} \mathrm{Cone}_{\lim_{j \in \mathcal{J}} H(\cdot, j)}(1),$

$\texttt{Assume}\ (1, \lambda) : \mathrm{Cone}_H(1),$

$\texttt{Assume}\ i : \mathcal{I},$

$(5) := \eth\texttt{Cone}(1, \lambda) : \Big[(1, \lambda_{|\{i\} \times \mathcal{J}} i) : \texttt{Cone}(\mathcal{J}, H(i, \cdot))\Big],$

$\mu_i := \eth\texttt{Limit}(\lim_{j \in \mathcal{J}} H(i,j)) : 1 \to \lim_{j \in \mathcal{J}} H(i,j);$

$\rightsquigarrow \mu := I\left(\prod\right) : \prod i \in \mathcal{I} \,.\, \mu_i : 1 \to \lim_{j \in \mathcal{J}} H(i,j),$

$\texttt{Assume}\ i, i' : \mathcal{I},$

$\eta := \texttt{legs}(\lim_{j \in \mathcal{J}} H(i,j)) : \prod j \in \mathcal{J} \,.\, \lim_{k \in \mathcal{J}} H(i,k) \to H(i,j),$

$\eta' := \texttt{legs}(\lim_{j \in \mathcal{J}} H(i',j)) : \prod j \in \mathcal{J} \,.\, \lim_{k \in \mathcal{J}} H(i',k) \to H(i',j),$

$\texttt{Assume}\ h : i \xrightarrow{\mathcal{I}} i',$

$\texttt{Assume}\ j : \mathcal{J},$

$(5) := \eth \lim_{j \in \mathcal{J}} H\big((i,i'), j\big)(h)\eth\mu : \mu_i \lim_{j \in \mathcal{J}} H\big((i,i'), j\big)(h)\eta'_j = \mu_i \eta_j = \lambda_{i,j},$

$\rightsquigarrow () := \eth\texttt{Limit}\eth\mu(5)(i,i') : \mu_i \lim_{j \in \mathcal{J}} H\big((i,i'), j\big)(h) = \mu_{i'};$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Cone} : \Big[(1, \mu) : \mathrm{Cone}_{\lim_{j \in \mathcal{J}} H(\cdot, j)}(1)\Big],$

$\varphi(1, \lambda) := (1, \mu) : \mathrm{Cone}_{\lim_{j \in \mathcal{J}} H(\cdot, j)}(1);$

$\rightsquigarrow \varphi := I(\to) : \mathrm{Cone}_H(1) \to \mathrm{Cone}_{\lim_{j \in \mathcal{J}} H(\cdot, j)}(1),$

$\texttt{Assume}\ (1, \lambda) : \mathrm{Cone}_{\lim_{j \in \mathcal{J}} H(\cdot, j)}(1),$

$\texttt{Assume}\ i : \mathcal{I},$

$\eta := \texttt{legs}(\lim_{j \in \mathcal{J}} H(i,j)) : \prod j \in \mathcal{J} \,.\, \lim_{k \in \mathcal{J}} H(i,k) \to H(i,j),$

$\texttt{Assume}\ j : \mathcal{J},$

$\mu_{i,j} := \lambda_i \eta_j : 1 \to H(i,j);$

$\rightsquigarrow \mu := I\left(\prod\right) : \prod (i,j) \in \mathcal{I} \times \mathcal{J} \,.\, 1 \to H(i,j),$

$\texttt{Assume}\ i, i' : \mathcal{I},$

$\texttt{Assume}\ j, j' : \mathcal{J},$

$\eta := \texttt{legs}(\lim_{j\in\mathcal{J}} H(i,j)) : \prod j \in \mathcal{J}\ .\ \lim_{k\in\mathcal{J}} H(i,k) \to H(i,j),$

$\eta' := \texttt{legs}(\lim_{j\in\mathcal{J}} H(i',j)) : \prod j \in \mathcal{J}\ .\ \lim_{k\in\mathcal{J}} H(i',k) \to H(i',j),$

$\texttt{Assume}\ h : (i,i') \xrightarrow{\mathcal{I}\times\mathcal{J}} (j,j'),$

$() := \eth\mu_{i,j}\eth\texttt{Covariant}(\mathcal{I}\times\mathcal{J},\mathsf{SET})(H)\eth\mathcal{I}\times\mathcal{J}\eth\eta\eth^{-1}\lim_{k\in\mathcal{J}} H\big((i,i'),k\big)(\pi_1 h)\eth\lambda\eth^{-1}\mu_{i',j'} :$

$: \mu_{i,j}H_{(i,j),(i',j')}(h) = \lambda_i\eta_j H_{(i,j),(i,j')}(\mathrm{id}\times\pi_2 h)H_{(i,j'),(i',j')}(\pi_1 h\times\mathrm{id}) =$

$= \lambda_i\eta_{j'}H_{(i,j'),(i',j')}(\pi_1 h\times\mathrm{id}) = \lambda_i\lim_{k\in\mathcal{J}} H\big((i,i'),k\big)(\pi_1 h)\eta'_{j'} = \lambda_{i'}\eta'_{j'} = \mu_{i',j'};$

$\rightsquigarrow (5) := \eth^{-1}\texttt{Cone} : \Big[(1,\mu) : \mathrm{Cone}_H(1)\Big],$

$\psi(1,\lambda) := (1,\mu) : \mathrm{Cone}_H(1);$

$\rightsquigarrow \psi := I(\to) : \mathrm{Cone}_H(1) \to \mathrm{Cone}_{\lim_{j\in\mathcal{J}} H(\cdot,j)}(1),$

$(5) := \eth\varphi\eth\texttt{Limit}\eth^{-1}\mathrm{id} : \varphi\psi = \Lambda(1,\lambda) : \mathrm{Cone}_H(1)\ .\ (1,\lambda) = \mathrm{id},$

$(6) := \eth\psi : \psi\varphi = \mathrm{id},$

$(7) := \eth\texttt{Inverse}\eth\texttt{Isomorphis}(\mathsf{SET}) : \mathrm{Cone}_H(1) \cong \mathrm{Cone}_{\lim_{j\in\mathcal{J}} H(\cdot,j)}(1),$

$(*) := \texttt{YonedasLemma}(1)(2)(3)(4)(7) : \texttt{This};$

$\square$

$\texttt{DoubleColimit} :: \forall \mathcal{I},\mathcal{J} : \texttt{Small}\ .\ \forall\mathcal{C} : \texttt{Cocomplete}\ \&\ \texttt{LocallySmall}\ .\ \forall F : \mathcal{I}\times\mathcal{J} \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .$

$.\ \forall X : \prod i \in \mathcal{I}\ .\ \texttt{Colimit}\big(\mathcal{J}, F(i,\cdot)\big)\ .\ \forall Y : \prod j \in \mathcal{J}\ .\ \texttt{Colimit}\big(\mathcal{I}, F(\cdot,j)\big)\ .$

$\operatorname{colim}_{i\in\mathcal{I}} X_i \cong_{\mathcal{C}} \operatorname{colim}_{(i,j)\in\mathcal{I}\times\mathcal{J}} F(i,j) \cong_{\mathcal{C}} \operatorname{colim}_{j\in\mathcal{J}} Y_j$

$\texttt{Proof}\ =$

$\dots$

$\square$

$\texttt{ColimLimTHM} :: \forall \mathcal{I},\mathcal{J} : \texttt{Small}\ .\ \forall\mathcal{C} : \texttt{Complete}\ \&\ \texttt{Cocomplete}\ .\ \forall F : \texttt{Bifunctor}(\mathcal{I},\mathcal{J},\mathcal{C})\ .$

$.\ \exists k : \operatorname{colim}_{i\in\mathcal{I}}\lim_{j\in\mathcal{J}} F(i,j) \xrightarrow{\mathcal{C}} \lim_{j\in\mathcal{J}}\operatorname{colim}_{i\in\mathcal{I}} F(i,j)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ i : \mathcal{I},$

$\lambda := \texttt{legs}(\lim_{k\in\mathcal{J}} F(i,k)) : \prod k \in \mathcal{J}\ .\ \lim_{l\in\mathcal{J}} F(i,l) \xrightarrow{\mathcal{C}} F(i,k),$

$\texttt{Assume}\ j : \mathcal{J},$

$\mu := \texttt{legs}(\lim_{k\in\mathcal{I}} F(k,j)) : \prod k \in \mathcal{I}\ .\ F(k,j) \xrightarrow{\mathcal{C}} \operatorname{colim}_{l\in\mathcal{I}} F(l,j),$

$\eta_j := \lambda_j\mu_i : \lim_{k\in\mathcal{J}} F(i,k) \xrightarrow{\mathcal{C}} \operatorname{colim}_{k\in\mathcal{I}} F(k,j);$

$\rightsquigarrow \eta := I\left(\prod\right) : \prod j \in \mathcal{J}\ .\ \lim_{k\in\mathcal{J}} F(i,k) \xrightarrow{\mathcal{C}} \operatorname{colim}_{k\in\mathcal{I}} F(k,j),$

$\texttt{Assume}\ j, j' : \mathcal{J},$

$\texttt{Assume}\ h : j \xrightarrow{\mathcal{J}} j',$

$() := \eth\eta_j\eth\operatorname{colim}\eth\lambda\eth^{-1}\eta_{j'} :$

$: \eta_j\operatorname{colim}_{k\in\mathcal{I}} F\big(k,(j,j')\big)(h) = \lambda_j\mu_i^j\operatorname{colim}_{k\in\mathcal{I}} F\big(k,(j,j')\big)(h) = \lambda_j F\big(i,(j,j')\big)(h)\mu_i^{j'} = \lambda_{j'}\mu_i^{j'} = \eta_{j'};$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cone} : \left[\left(\lim_{j\in\mathcal{J}} F(i,j), \eta\right) : \texttt{Cone}\left(\mathcal{J}, \operatorname{colim}_{i\in\mathcal{I}} F(i,\cdot)\right)\right],$

$(\phi_i, 2) := \eth^{-1}\texttt{Limit} : \sum \phi_i : \lim_{j\in\mathcal{J}} F(i,j) \xrightarrow{\mathcal{C}} \lim_{j\in\mathcal{J}} \operatorname{colim}_{i\in\mathcal{I}} F(i,j) \;.\; \forall j \in \mathcal{J} \;.\; \phi_i\nu_j = \eta_j;$

$\rightsquigarrow \phi := I\left(\prod\right) : \prod i \in \mathcal{I} \;.\; \sum \phi_i : \lim_{j\in\mathcal{J}} F(i,j) \xrightarrow{\mathcal{C}} \lim_{j\in\mathcal{J}} \operatorname{colim}_{i\in\mathcal{I}} F(i,j) \;.\; \forall j \in \mathcal{J} \;.\; \phi_i\nu_j = \eta_j,$

$\texttt{Assume}\ i, i' : \mathcal{I},$

$\texttt{Assume}\ h : i \xrightarrow{\mathcal{I}} i',$

$\texttt{Assume}\ j : \mathcal{J},$

$() := \eth\phi\eth\eta : \lim_{j\in\mathcal{J}} F\Big((i,i'),j\Big)(h)\phi_{i'}\nu_j = \lim_{j\in\mathcal{J}} F\Big((i,i'),j\Big)(h)\eta_j^{i'} = \eta_j^i;$

$\rightsquigarrow (1) := \eth \int \mathrm{Cone} : \left[\lim_{j\in\mathcal{J}} F\big((i,i'),j\big)\phi_i : \lim_{j\in\mathcal{J}} F(i',j) \xrightarrow{\int \mathrm{Cone}} \lim_{j\in\mathcal{J}} F(i,j)\right],$

$() := \eth\texttt{Limit}\eth\phi : \lim_{j\in\mathcal{J}} F\big((i,i'),j\big)(h)\phi_i = \phi_{i'};$

$\rightsquigarrow (1) := \eth^{-1}\texttt{Cocone} : \left[\left(\lim_{j\in\mathcal{J}} \operatorname{colim}_{i\in\mathcal{I}} F(i,j), \phi\right) : \texttt{Cocone}(\mathcal{I}, \lim_{j\in\mathcal{J}} F(\cdot, J))\right],$

$\psi := \eth\texttt{Colimit}(1) : \operatorname{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \xrightarrow{\mathcal{C}} \lim_{j\in\mathcal{J}} \operatorname{colim}_{i\in\mathcal{I}} F(i,j);$

$\square$

$\texttt{KFiltered} :: \mathsf{CARD} \to ?\mathsf{CAT}$

$\mathcal{I} : \texttt{KFilterded} \iff \mathcal{I} : \kappa\text{-}\texttt{Filtered} \iff \forall \mathcal{J} \in \texttt{Category} \;.\; \forall () : |\mathcal{J}^{\to}| < \kappa \;.\; \forall I : \mathcal{J} \xrightarrow{\mathsf{CAT}} \mathcal{I} \;.$
$\;.\; \exists \texttt{Cocone}(\mathcal{J}, I)$

$\texttt{FilteredColimitStructure} :: \forall \mathcal{I} : \texttt{Small}\ \&\ \texttt{Filtered} \;.\; \forall X : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathsf{SET} \;.\; \operatorname{colim}_{i\in\mathcal{I}} X_i = \frac{\coprod_{i\in\mathcal{I}} X_i}{R}$

$\texttt{where}$

$$R = \left\{\Big((i,x),(j,y)\Big) \in \coprod_{i\in\mathcal{I}} X_i \times \coprod_{i\in\mathcal{I}} X_i : \exists t \in \mathcal{I} : \exists f : i \xrightarrow{\mathcal{I}} t : \exists g : j \xrightarrow{\mathcal{I}} t : X_{i,t}(f)(x) = X_{j,t}(g)(y)\right\}$$

$\texttt{Proof} =$

Inspect the definition of equivalence relation $(\sim)$ in $\texttt{SetIsCocomplete}$.
$(\sim) = \texttt{eqclosure}(T)$ end it is easy to see that $T \subset R \subset (\sim)$,
and hence, as R is equivalence by property of $\mathcal{I}$ of being filtered, $R = (\sim)$.

$\square$

$\mathtt{FilterdColimCommutesWithLim} :: \forall \mathcal{I} : \aleph_0\text{-}\mathtt{Filtered}\ \&\ \mathtt{Small} \,.\, \forall n \in \mathbb{N} \,.\, \forall \mathcal{J} : n\text{-}\mathtt{Small} \,.$

$\quad . \forall F : \mathcal{I} \times \mathcal{J} \xrightarrow{\mathsf{CAT}} \mathsf{SET} \,.\, \operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \cong_{\mathsf{SET}} \lim_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(j,i)$

$\mathtt{Proof}\ =$

$\kappa := \mathtt{ColimLimTHM}(F) : \operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \to \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} F(i,j),$

$(1) := \mathtt{LimRepresentation}(\operatorname*{colim}_{i\in\mathcal{I}} F(i,\cdot)) : \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} F(i,j) \cong \mathrm{Cone}_{\operatorname{colim}_{i\in\mathcal{I}} F(i,\cdot)}(1),$

$\mathtt{Assume}\ (1,\lambda) : \mathrm{Cone}_{\operatorname{colim}_{i\in\mathcal{I}} F(i,\cdot)}(1),$

$\mathtt{Assume}\ j : \mathcal{J},$

$((t_j,x_j),2) := \mathtt{SetIsCocomplete}\lambda_j(1) : \sum (t_j,x_j) \in \coprod_{i\in\mathcal{I}} F(i,j) \,.\, [t_j,x_j] = \lambda_j(1);$

$\rightsquigarrow (t,x,2) := I\left(\prod\right) : \prod j \in \mathcal{J} \,.\, \sum t_j \in \mathcal{I} \,.\, \sum x \in F(i,j) \,.\, [x] = \lambda_j(1),$

$\mathtt{Assume}\ j,j' : \mathcal{J},$

$\mathtt{Assume}\ h : j \xrightarrow{\mathcal{J}} j',$

$f := \operatorname*{colim}_{i\in\mathcal{I}} F(i,(j,j'))(h) : \operatorname*{colim}_{i\in\mathcal{I}} F(i,j) \to \operatorname*{colim}_{i\in\mathcal{I}} F(i,j'),$

$(3) := \eth\mathtt{Cone}(1,\lambda) : f[t_j,x_j] = [t_{j'},x_{j'}],$

$(4) := \eth \operatorname*{colim}_{i\in\mathcal{I}} F(i,j)\eth f : \forall i \in \mathcal{I} \,.\, \Big[F(i,(j,j'))(h)\Big] = f[\cdot],$

$(5) := (4)(3) : \big[F(t_j,(j,j'))(h)(x_j)\big] = [t_{j'},x_{j'}],$

$(\tau,g,g',6) := \mathtt{FilteredColimitStructure}(5) :$

$\quad : \sum \tau \in \mathcal{I} \,.\, \sum g : t_j \to \tau \,.\, \sum g' : t_{j'} \to \tau \,.\, F(t_j,(j,j'))(h)F((t_j,\tau),j')(g)(x_j) = F((t_{j'},\tau),j')(g')(x'_j),$

$(t_j,x_j) := \Big(\tau, F\big((t_j,\tau),j\big)(g)(x)\Big) : \sum t_j \in \mathcal{I} \,.\, H(\tau,j) \quad !\mathtt{Redefine}!,$

$(t_{j'},x_{j'}) := \Big(\tau, F\big((t_{j'},\tau),j'\big)(g)(x)\Big) : \sum t_{j'} \in \mathcal{I} \,.\, H(\tau,j') \quad !\mathtt{Redefine}!;$

$\rightsquigarrow (3) := I(\forall) : \forall j,j' \in \mathcal{J} \,.\, \forall h : j \xrightarrow{\mathcal{J}} j' \,.\, t_j = t_{j'},$

$(s,\mu) := \eth\mathtt{Filtered}(\mathcal{I})(t) : \mathtt{Cocone}(\mathcal{J},(t,\mathrm{id})),$

$\lambda' := \Lambda j \in \mathcal{J} \,.\, \Lambda 1 \in 1 \,.\, F((t_j,s),j)(\mu_j)(x_j) : \prod j \in \mathcal{J} \,.\, 1 \to F(s,j),$

$(4) := \eth^{-1}\mathtt{Cone}\eth\lambda'\eth x : \Big[(1,\lambda') : \mathtt{Cone}\big(\mathcal{J},F(s,\cdot)\big)\Big],$

$() := \eth\kappa : \kappa[1,\lambda'] = (1,\lambda);$

$\rightsquigarrow (2) := \eth^{-1}\mathtt{Surjective} : \Big[\kappa : \operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \twoheadrightarrow \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} F(i,j)\Big],$

$\mathtt{Assume}\ \big[i,(1,\alpha)\big], \big[i',1,\beta\big] : \operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j),$

$\mathtt{Assume}\ (3) : \kappa\big[i,(1,\alpha)\big] = \kappa\big[i',(1,\beta)\big],$

$(t,h,h',4) := \eth\kappa(3) : \prod j \in \mathcal{J} \,.\, \sum t_j \in \mathcal{I} \,.\, \sum h_j : i \xrightarrow{\mathcal{I}} t_j \,.\, \sum h'_j : i' \xrightarrow{\mathcal{I}} t_j \,.\, F((i,t_j),j)(h_j)(\alpha_j(1)) = F((i,t_j)$

$(s,\mu) := \eth\mathtt{Filtered}\mathcal{I}(t,h,h') : \mathtt{Cocone}(\mathcal{J}+2, t+(i,i')),$

$(5) := \eth\mathtt{Cocone}(s,\mu)(4) : \forall j \in \mathcal{J} \,.\, F((i,s),j)(h_j\mu_j)(\alpha_j(1)) = F((i',s),j)(h'_j\mu_j)(\beta_j(1)),$

$(6) := \mathtt{FilteredColimitStructure}(5) : \big[i,(1,\alpha)\big] = \big[i',(1,\beta)\big];$

$\rightsquigarrow (3) := \eth^{-1}\mathtt{Bijective}\eth^{-1}\mathtt{Surjective} : \Big[\kappa : \operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \leftrightarrow \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} F(i,j)\Big],$

$(*) := \eth\mathtt{Isomorphic}(\mathsf{SET})(3) : \Big[\operatorname*{colim}_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \cong_{\mathsf{SET}} \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} F(i,j)\Big];$

$\square$

3.9 Exponentiation

$$\texttt{Exponent} :: \prod \mathcal{C} : \texttt{WithFiniteProduct} \ . \ \prod A, B \in \mathcal{C} \ . \ \sum A^B \in \mathcal{C} \ . \ A^B \times B \xrightarrow{\mathcal{C}} A$$

$$(A^B, \epsilon) : \texttt{Exponent} \iff \forall X \in \mathcal{C} \ . \ \forall f : X \times B \xrightarrow{\mathcal{C}} A \ . \ \exists! \tau : X \xrightarrow{\mathcal{C}} A^B : (\tau \times \mathrm{id}_{\mathcal{B}})\epsilon = f$$

$$\texttt{implicit} :: \texttt{Exponent}(\mathcal{C}, A, B) \to \mathcal{C}$$

$$\texttt{implicit}\left(A^B, \epsilon\right) := A^B$$

$$\texttt{evaluation} :: \prod E : \texttt{Exponent}(\mathcal{C}, A, B) \ . \ E \times B \xrightarrow{\mathcal{C}} A$$

$$\texttt{evaluation}() = \mathrm{ev}_E := \epsilon \quad \texttt{where} \quad (A^B, \epsilon) = E$$

$$\texttt{curry} :: \prod E : \texttt{Exponent}(\mathcal{C}, A, B) \ . \ \prod X \in \mathcal{C} \ . \ (X \times B \xrightarrow{\mathcal{C}} A) \to X \xrightarrow{\mathcal{C}} E$$

$$\texttt{curry}() = \lambda^E := \eth\texttt{Exponent}(\mathcal{C}, A, B)(E)$$

$$\texttt{WithExponentiation} :: ?\mathsf{CAT}$$

$$\mathcal{C} : \texttt{WithExponentiation} \iff \forall A, B \in \mathcal{C} \ . \ \exists \texttt{Exponentiation}(A, B, \mathcal{C})$$

$$\texttt{exponent} :: \prod \mathcal{C} : \texttt{WithExponent} \ . \ \prod A, B \in \mathcal{C} \ . \ \texttt{Exponent}(\mathcal{C}, A, B)$$

$$\texttt{exponent}() = A^B := \eth\texttt{WithExponent}(\mathcal{C})(A, B)$$

4 Adjunctions

4.1 Adjoint Functors

$$\texttt{Adjoint} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, ?\Big((\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B}) \times (\mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A})\Big)$$

$$(F, G) : \texttt{Adjoint} \iff F \dashv G \iff \exists \alpha : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \Big(\mathcal{M}_{\mathcal{B}}\Big(F(A), B\Big) \leftrightarrow \mathcal{M}_{\mathcal{A}}\Big(A, G(B)\Big)\Big) :$$

$$: \Big(\forall B \in \mathcal{B} \,.\, \alpha : \mathcal{M}_{\mathcal{B}}\Big(F(\cdot), B\Big) \iff \mathcal{M}_{\mathcal{A}}\Big(\cdot, G(B)\Big)\Big) \,\&$$

$$\& \Big(\forall A \in \mathcal{A} \,.\, \alpha : \mathcal{M}_{\mathcal{B}}\Big(F(A), \cdot\Big) \iff \mathcal{M}_{\mathcal{A}}\Big(A, G(\cdot)\Big)\Big)$$

$$\texttt{transpose} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod \sum F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \sum G : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, F \dashv G \,.$$

$$.\, \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \mathcal{M}_{\mathcal{B}}(F(A), B) \to \mathcal{M}_{\mathcal{A}}(A, G(B))$$

$$\texttt{transpose}\,(f) = f^{\top_{F,G}} := \eth\texttt{Adjoint}(F, G)(f)$$

$$\texttt{antitranspose} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod \sum F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \sum G : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, F \dashv G \,.$$

$$.\, \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \mathcal{M}_{\mathcal{A}}(A, G(B)) \to \mathcal{M}_{\mathcal{B}}(F(A), B)$$

$$\texttt{antitranspose}\,(f) = f^{\perp_{F,G}} := \eth\texttt{Adjoint}(F, G)(f)$$

$$\texttt{AdjointFunctorsChar} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \forall G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \,.$$

$$\forall \alpha : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, (\mathcal{M}_{\mathcal{B}}(F(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, G(B))) \,.\, (F \dashv G, \alpha) \iff$$

$$\iff : \Big(\forall A, A' \in \mathcal{A} \,.\, \forall B, B' \in \mathcal{B} \,.\, \forall f : F(A) \xrightarrow{\mathcal{B}} B \,.\, \forall g : F(A') \xrightarrow{\mathcal{B}} B' \,.\, \forall h : A \xrightarrow{\mathcal{A}} A' \,.\, \forall k : B \xrightarrow{\mathcal{B}} B' \,.$$

$$.\, fk = F(h)g \iff \alpha(A, B)(f)G(k) = h\alpha(A', B')(g)\Big)$$

$\texttt{Proof} =$

$\texttt{Assume}\ (1) : (F \dashv G, \alpha),$

$\texttt{Assume}\ A, A' : \mathcal{A},$

$\texttt{Assume}\ B, B' : \mathcal{B},$

$\texttt{Assume}\ f : F(A) \xrightarrow{\mathcal{B}} B,$

$\texttt{Assume}\ g : F(A') \xrightarrow{\mathcal{B}} B',$

$\texttt{Assume}\ h : A \xrightarrow{\mathcal{A}} A',$

$\texttt{Assume}\ k : B \xrightarrow{\mathcal{B}} B',$

$$(2) := \eth^{-1}k^*\eth^{-1}\mathcal{M}_{\mathcal{B}}\Big((F(A), F(A), (B, B'))\Big)\eth\texttt{NaturalTransform}(\alpha)\eth^{-1}G(k) :$$

$$: \alpha(A, B')fk = \alpha(A, B')k^*f = \alpha(A, B')\mathcal{M}_{\mathcal{B}}\Big((F(A), F(A)), (B, B')\Big)(\mathrm{id} \times k)(f) =$$

$$= \mathcal{M}_{\mathcal{A}}\Big((A, A), (G(B), G(B'))\Big)(\mathrm{id} \times k)\alpha(A, B)(f) = \alpha(A, B)(f)G(k),$$

$$(3) := \eth^{-1}F_*(h)\eth^{-1}\mathcal{M}_{\mathcal{B}}\Big((F(A), F(A)), (B, B')\Big)\eth\texttt{NaturalTransform}(\alpha)\eth^{-1}h_*\eth h_* :$$

$$: \alpha(A, B')F(h)g = \alpha(A, B')F(h)_*g = \alpha(A, B')\mathcal{M}_{\mathcal{B}}\Big((F(A), F(A')), (B', B'))(h)(g) =$$

$$= \mathcal{M}_{\mathcal{A}}\Big((A, A), (G(B'), G(B'))\Big)(h \times \mathrm{id})\alpha(A', B')(g) = h\alpha(A', B')(g),$$

$() := \eth\texttt{Bijection}\alpha(A, B')(1)(2) : fk = F(h)g \iff \alpha(A, B)(f)G(k) = h\alpha(A', B')(g);$
$\rightsquigarrow (2) := I(\Rightarrow)I(\forall) : \texttt{Left} \Rightarrow \texttt{Right},$
$\texttt{Assume } R : \texttt{Right},$
$\texttt{Assume } B : \mathcal{B},$
$\texttt{Assume } A, A' : \mathcal{A},$
$\texttt{Assume } h : A \xrightarrow{\mathcal{A}^{\mathrm{op}}} A',$
$\texttt{Assume } z : F(A) \to B,$
$k := \mathrm{id}_B : B \xrightarrow{\mathcal{B}} B,$
$f := F(h)z : F(A') \xrightarrow{\mathcal{B}} B,$
$g := z : F(A) \xrightarrow{\mathcal{B}} B,$
$(3) := \eth F(h)g\eth fk : fk = F(h)g,$
$() := R(3) : h\alpha(A, B)(z) = h\alpha(A, B)(g) = \alpha(A', B)(g)G(k) = \alpha(A', B)(F(h)z);$
$\rightsquigarrow (3) := I(\forall)\eth^{-1}\texttt{NaturalTransform} : \forall B \in \mathcal{B} \;.\; \alpha(\cdot, B) : \mathcal{M}_{\mathcal{B}}(F(\cdot), B) \iff \mathcal{M}_{\mathcal{A}}(\cdot, G(B)),$
$\texttt{Assume } A : \mathcal{A},$
$\texttt{Assume } B, B' : \mathcal{B},$
$\texttt{Assume } k : B \xrightarrow{\mathcal{A}^{\mathrm{op}}} B',$
$\texttt{Assume } z : F(A) \to B,$
$h := \mathrm{id}_A : A \xrightarrow{\mathcal{A}} A,$
$f := F(h)z : F(A) \xrightarrow{\mathcal{B}} B,$
$g := zk : F(A) \xrightarrow{\mathcal{B}} B',$
$(4) := \eth F(h)g\eth fk : fk = F(h)g,$
$() := R(3) : \alpha(A, B)(z)G(k) = h\alpha(A, B)(g) = \alpha(A', B)(g)G(k) = \alpha(A', B)(zk);$
$\rightsquigarrow (4) := I(\forall)\eth^{-1}\texttt{NaturalTransform} : \forall A \in \mathcal{A} \;.\; \alpha(A, \cdot) : \mathcal{M}_{\mathcal{B}}(F(A), \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, G(\cdot)),$
$() := \eth^{-1}\texttt{Adjoint}(3, 4) : (F \dashv G, \alpha);$
$\rightsquigarrow (*) := I(\Leftarrow)I(\iff) : \texttt{This};$
$\square$

$\texttt{AdjointTriple} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \;.\; \forall R, S : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \;.\; \forall U : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \;.\; R \dashv U \dashv S \Rightarrow RU \dashv SU$
$\texttt{Proof} =$
$\alpha := \eth R \dashv U : \prod A \in \mathcal{A} \;.\; \prod B \in \mathcal{B} \;.\; \mathcal{M}_{\mathcal{B}}(R(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, U(B)),$
$\beta := \eth U \dashv S : \prod B \in \mathcal{B} \;.\; \prod A \in \mathcal{A} \;.\; \mathcal{M}_{\mathcal{A}}(U(B), A) \leftrightarrow \mathcal{M}_{\mathcal{B}}(B, S(A)),$
$\texttt{Assume } X, Y : \mathcal{A},$
$\omega(X, Y) := \beta(R(X), Y)\alpha(X, S(Y)) : \mathcal{M}_{\mathcal{A}}(RU(X), Y) \leftrightarrow \mathcal{M}_{\mathcal{B}}(X, SU(Y));$
$\rightsquigarrow \omega := I\left(\prod\right) : \prod X \in \mathcal{A} \;.\; \prod Y \in \mathcal{B} \;.\; \mathcal{M}_{\mathcal{A}}(RU(X), Y) \leftrightarrow \mathcal{M}_{\mathcal{B}}(X, SU(Y)),$
$(*) := \eth\omega\eth R \dashv U\eth S \dashv U : RU \dashv SU;$
$\square$

$\texttt{CommaIsomorphismByAdjunction} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \ . \ \forall (F, G) : \texttt{Asjoint}(\mathcal{A}, \mathcal{B}) \ .$

$. \ \exists T : F \downarrow \mathrm{id}_{\mathcal{B}} \xleftrightarrow{\mathsf{CAT}} \mathrm{id}_{\mathcal{A}} \downarrow G : \Pi_1 = T\Pi_2$

$\texttt{where}$

$\Pi_1 = (\Lambda(X, Y, f) \in F \downarrow \mathrm{id}_{\mathcal{B}} \ . \ (X, Y), \mathrm{id})$

$\Pi_2 = (\Lambda(X, Y.f) \in \mathrm{id}_{\mathcal{A}} \downarrow G \ . \ (X, Y), \mathrm{id})$

$\texttt{Proof} \ =$

$T := \Lambda(X, Y, f) \in F \downarrow \mathrm{id}_{\mathcal{B}} \ . \ (X, Y, f^{\top}) : F \downarrow \mathrm{id}_{\mathcal{B}} \leftrightarrow \mathrm{id}_{\mathcal{A}} \downarrow G,$

$(*) := \texttt{AdjointFunctorsChar}\eth T : \left[T : F \downarrow \mathrm{id}_{\mathcal{B}} \xleftrightarrow{\mathsf{CAT}} \mathrm{id}_{\mathcal{A}} \downarrow G \right];$

$\square$

4.2 Category of Fractions[!]

$$\texttt{Quiver} := \sum A, B : \texttt{Kind} \, . \, \sum O : A \, . \, O \times O \to B : \texttt{Type};$$

$$\texttt{implicit} :: \mathsf{CAT} \to \texttt{Quiver}$$
$$\texttt{implicit}\,(\mathcal{C}) := (\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$$

$$\texttt{Cogrounded} :: ?(\mathsf{LSCAT} \times \mathsf{LSCAT})$$
$$(\mathcal{A}, \mathcal{B}) : \texttt{Cogrounded} \iff \mathcal{O}_{\mathcal{A}} = \mathcal{O}_{\mathcal{B}}$$

$$\texttt{catUnion} :: \texttt{Cogrounded} \to \texttt{Quiver}$$
$$\texttt{catUnion}\,(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cup \mathcal{B} := \Big(\mathcal{O}_{\mathcal{A}}, \Lambda A, B \in \mathcal{O}_{\mathcal{A}} \, . \, \mathcal{M}_{\mathcal{A}}(A, B) \cup \mathcal{M}_{\mathcal{B}}(A, B)\Big)$$

$$\texttt{Chain} :: \prod (O, M) : \texttt{Quiver} \, . \, \prod A, B \in O \, . \, ?\sum n \in \mathbb{N} \, . \, n \to \sum X, Y \in O \, . \, M(X, Y)$$
$$(l, (X, Y, f)) : \texttt{Chain} \iff X_1 = A \; \& \; Y_l = B \; \& \; \forall i \in (l-1) \, . \, Y_i = X_{i+1}$$

$$\texttt{Pseudochain} := \Lambda(O, M) : \texttt{Quiver} \, . \, \Lambda A, B \in O \, .$$
$$\textbf{if } A == B \textbf{ then } \texttt{Chain}((O.M), A, A) | [\text{empty}]_{\text{atom}} \texttt{ else } \texttt{Chain}((O, M), A, B) :$$
$$: \prod (O, M) : \texttt{Quiver} \, . \, O^2 \to \texttt{Type};$$

$$\texttt{categoryOfFractions} :: \mathsf{LSCAT} \to \mathsf{LSGROUPOID}$$
$$\texttt{categoryOfFractions}\,(\mathcal{C}) := \left(\mathcal{O}_{\mathcal{C}}, \Lambda A, B \in \mathcal{C} \, . \, \frac{\mathcal{M}_{\mathcal{C}_+}(A, B)}{R(A, B)}, \Big([\alpha], [\beta]\Big) \mapsto [\alpha\beta], A \mapsto [(\mathrm{id}_A)_{i=1}^{1}]\right)$$

where

$$\mathcal{C}_+ = \Big(\mathcal{O}_{\mathcal{C}}, \texttt{Chain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, B), (a, b) \mapsto a \oplus b, A \mapsto (\mathrm{id}_A)\Big)$$
$$R(A, B) = \texttt{eqclosure}(C_1(A, B) \cup C_2(A, B) \cup O_1(A, B) \cup O_2(A, B) \cup I(A, B))$$
$$C_1(A, B) = \Big\{\Big(\alpha(X, Y, f)(Y, Z, g)\beta, \alpha(X, Z, fg)\beta\Big) \Big| X, Y, Z \in \mathcal{C}, f : X \xrightarrow{\mathcal{C}} Y, g : G \xrightarrow{\mathcal{C}} Z,$$
$$, \alpha : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, X), \beta : \texttt{Pesudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, Z, B)\Big\}$$
$$C_2(A, B) = \Big\{\Big(\alpha(X, Y, f^{\mathrm{op}})(Y, Z, g^{\mathrm{op}})\beta, \alpha(X, Z, (gf)^{\mathrm{op}})\beta\Big) \Big| X, Y, Z \in \mathcal{C}, f : X \xrightarrow{\mathcal{C}} Y, g : G \xrightarrow{\mathcal{C}} Z,$$
$$, \alpha : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, X), \beta : \texttt{Pesudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, Z, B)\Big\}$$
$$O_1(A, B) = \Big\{\Big(\alpha(X, Y, f)(Y, X, f^{\mathrm{op}})\beta, \alpha(X, X, \mathrm{id}_X)\beta\Big) \Big| X, Y \in \mathcal{C}, f : X \xrightarrow{\mathcal{C}} Y$$
$$, \alpha : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, X), \beta : \texttt{Pesudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, X, B)\Big\}$$
$$O_2(A, B) = \Big\{\Big(\alpha(X, Y, f^{\mathrm{op}})(Y, X, f)\beta, \alpha(X, X, \mathrm{id}_X)\beta\Big) \Big| X, Y \in \mathcal{C}, f : X \xrightarrow{\mathcal{C}} Y$$
$$, \alpha : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, X), \beta : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, X, B)\Big\}$$
$$I(A, B) = \Big\{\Big(\alpha(X, Y, f^{-1})\beta, \alpha(X, Y, f^{\mathrm{op}})\beta\Big) \Big| X, Y \in \mathcal{C},$$
$$, \alpha : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, A, X), \beta : \texttt{Pseudochain}(\mathcal{C} \cup \mathcal{C}^{\mathrm{op}}, Y, B)\Big\}$$

`fracCatFunctor` :: $\mathsf{LSCAT} \xrightarrow{\mathsf{CAT}} \mathsf{LSGroupoid}$

`fracCatFunctor` $(\mathcal{C}) = \mathrm{Frac}(\mathcal{C}) :=$ `categoryOfFractions`$(\mathcal{C})$

`fracCatFunctor` $(\mathcal{A}, \mathcal{B}, F) = \mathrm{Frac}_{\mathcal{A},\mathcal{B}}(F) :=$

$$:= \left(\Lambda A \in \mathrm{Frac}(\mathcal{A}) \;.\; F(A), \Lambda[f_i]_{i=1}^{n} : X \xrightarrow{\mathrm{Frac}(\mathcal{A})} \mathrm{Frac}(\mathcal{B}) \;.\; \Big[F(f_i)\Big]_{i=1}^{n}\right)$$

`GroupoidEmbeddingAjoint` :: $\mathrm{MG} \dashv \mathrm{E} \dashv \mathrm{Frac}$

`Proof` $=$

$\ldots$

$\square$

4.3 Unit and Counit

$\texttt{unit} :: \prod(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \,.\, \prod A \in \mathcal{A} \,.\, A \xrightarrow{\mathcal{A}} FG(A)$

$\texttt{unit}\,() = \eta_A^{F,G} := \mathrm{id}_{F(A)}^{\top}$

$\texttt{UnitIsNatural} :: \forall(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \,.\, \eta^{F,G} : \mathrm{id}_{\mathcal{A}} \Rightarrow FG$

$\texttt{Proof} \;=$

$\texttt{Assume}\; X, Y : \mathcal{A},$

$\texttt{Assume}\; f : X \Rightarrow \mathcal{A} Y,$

$(1) := I(=)F(f) : F(f) = F(f),$

$() := \texttt{AdjointFunctorChar}(F,G)(\mathrm{id},\mathrm{id},F(f),F(f))(1) : \eta_X FG(f) = f\eta_Y;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturalTransform} : \eta^{F,G} : \mathrm{id}_{\mathcal{A}} \Rightarrow FG;$

$\square$

$\texttt{counit} :: \prod(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \,.\, \prod B \in \mathcal{B} \,.\, GF(B) \xrightarrow{\mathcal{A}} B$

$\texttt{counit}\,() = \epsilon_B^{F,G} := \mathrm{id}_{G(B)}^{\perp}$

$\texttt{CounitIsNatural} :: \forall(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \,.\, \epsilon^{F,G} : GF \Rightarrow \mathrm{id}_{\mathcal{B}}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; X, Y : \mathcal{B},$

$\texttt{Assume}\; f : X \Rightarrow \mathcal{A} Y,$

$(1) := I(=)G(f) : G(f) = G(f),$

$() := \texttt{AdjointFunctorChar}(F,G)(\mathrm{id},\mathrm{id},G(f),G(f))(1) : \epsilon_X f = FG(f)\epsilon_Y;$

$\rightsquigarrow (*) := \eth^{-1}\texttt{NaturalTransform} : \epsilon^{F,G} : GF \Rightarrow \mathrm{id}_{\mathcal{B}};$

$\square$

$\texttt{TriangleId} :: \prod F : \mathcal{A} \xrightarrow{\mathcal{C}} \mathcal{B} \,.\, \prod G : \mathcal{B} \xrightarrow{\mathcal{C}} \mathcal{A} \,.\, ?(\mathrm{id}_{\mathcal{A}} \Rightarrow FG \times GF \Rightarrow \mathrm{id}_{\mathcal{B}})$

$(\alpha,\beta) : \texttt{TriangleId} \iff F(\alpha(\cdot))\beta(F(\cdot)) = \mathrm{id}_F \;\&\; \alpha(G(\cdot))G(\beta(\cdot)) = \mathrm{id}_G$

$\mathtt{AdjointFunctorsChar2} :: \forall F : \mathcal{A} \xrightarrow{\mathcal{C}} \mathcal{B} \,.\, \forall G : \mathcal{B} \xrightarrow{\mathcal{C}} \mathcal{A} \,.\, F \dashv G \iff \exists \mathtt{TriangleId}(F, G)$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ (1) : F \dashv G,$

$\mathtt{Assume}\ A : \mathcal{A},$

$\mathtt{Assume}\ B : \mathcal{B},$

$()_1 := \eth\eta(A)\eth\epsilon(F(A))\eth\mathtt{NaturalTransform}(\mathtt{antitranspose})(1) :$

$\quad : F(\eta(A))\epsilon(F(A)) = F\Big(\mathrm{id}^{\top}_{F(A)}\Big)\mathrm{id}^{\perp}_{FG(A)} = \Big(\mathrm{id}^{\top}_{F(A)}\Big)^{\perp} = \mathrm{id}_{F(A)},$

$()_2 := \eth\eta(A)\eth\epsilon(F(A))\eth\mathtt{NaturalTransform}(\mathtt{transpose})(1) :$

$\quad : \eta(G(B))G(\epsilon(B)) = \mathrm{id}^{\top}_{GF(B)}G(\mathrm{id}^{\perp}_{G(B)}) = \Big(\mathrm{id}^{\perp}_{G(B)}\Big)^{\top} = \mathrm{id}_{G(B)};$

$\rightsquigarrow () := \eth^{-1}\mathtt{TriangleId} : \Big[(\eta, \epsilon) : \mathtt{TriangleId}(F, G)\Big];$

$\rightsquigarrow (1) := I(\Rightarrow) : \mathtt{Left} \Rightarrow \mathtt{Right},$

$\mathtt{Assume}\ (\mu, \nu) : \mathtt{TriangleId}(F, G),$

$\mathtt{Assume}\ A : \mathcal{A},$

$\mathtt{Assume}\ B : \mathcal{B},$

$\mathtt{Assume}\ f : F(A) \xrightarrow{\mathcal{B}} B,$

$\mathtt{Assume}\ g : A \xrightarrow{\mathcal{A}} G(B),$

$\alpha(A, B)(f) := \mu(A)G(f) : A \xrightarrow{\mathcal{A}} G(B),$

$\beta(A, B)(g) := F(g)\nu(B) : F(A) \xrightarrow{\mathcal{B}} B,$

$()_1 := \eth\mathtt{Covariant}(F)\eth\mathtt{NaturalTransform}\nu\eth\mathtt{Triangleid}(\mu, \nu) : F\Big(\mu(A)G(f)\Big)\nu(B) = F(\mu(A))GF(f)\nu(B) =$

$()_2 := \eth\mathtt{Covariant}(G)\eth\mathtt{NaturalTransform}\mu\eth\mathtt{TriangleId}(\mu, \nu) : \mu(A)G\Big(F(g)\nu(B)\Big) = \mu(A)FG(g)F(\nu(B)) = g$

$\rightsquigarrow \alpha := I\left(\prod\right) : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \mathcal{M}_{\mathcal{B}}(F(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, G(B)),$

$\mathtt{Assume}\ A, A' : \mathcal{A},$

$\mathtt{Assume}\ B, B' : \mathcal{B},$

$\mathtt{Assume}\ f : F(A) \xrightarrow{\mathcal{B}} B,$

$\mathtt{Assume}\ g : F(A') \xrightarrow{\mathcal{B}} B',$

$\mathtt{Assume}\ h : A \xrightarrow{\mathcal{A}} A',$

$\mathtt{Assume}\ k : B \xrightarrow{\mathcal{B}} B',$

$(2) := \eth\mathtt{Covariant}(G)\eth^{-1}\alpha :$

$\quad : \mu(A)G(fk) = \mu(A)G(f)G(k) = \alpha(A, B)(f)G(k),$

$(3) := \eth\mathtt{Covariant}(G)\eth\mathtt{NaturalTransform}\mu\eth^{-1}\alpha :$

$\quad : \mu(A)G(F(h)g) = \mu(A)FG(h)G(g) = h\mu(A')G(g) = h\alpha(A', B')(g),$

$(4) := F(2)\nu(B')\eth\mathtt{Covariant}(F)\eth\mathtt{NaturalTransform}(\nu)\eth\mathtt{TriangleId}(\mu, \nu) :$

$\quad : F\Big(\alpha(A, B)(f)G(k)\Big)\nu(B') = F(\mu(A))GF(fk)\nu(B') = F(\mu(A))\nu(F(A))fk = fk,$

$(5) := F(3)\nu(B')\eth\mathtt{Covariant}(F)\eth\mathtt{NaturalTransform}\nu\eth\mathtt{TriangleId}(\mu, \nu) :$

$\quad : F\Big(h\alpha(A', B')(g)\Big)\nu(B') = F\Big(\mu(A)G(F(h)g)\Big)\nu(B') = F\Big(\mu(A)\Big)GF(F(h)g)\nu(B') =$

$\quad = F\Big(\mu(A)\Big)\nu(F(A))F(h)g = F(h)g,$

$() := \texttt{MapEq}(2,3,4,5) : fk = F(h)g \iff \alpha(A,B)(f)G(k) = h\alpha(A',B')(g);$

$\rightsquigarrow () := \texttt{AdjointFunctorsChar} : F \dashv G;$

$\rightsquigarrow (*) := I(\Leftarrow)I(\iff) : \texttt{This};$

$\square$

$\texttt{unitalSubcategory} :: \prod \mathcal{C} \in \mathsf{SCAT} \; . \; \prod F : \mathrm{End}_{\mathsf{CAT}}(\mathcal{C}) \; . \; (\underset{\mathcal{C}}{\mathrm{id}} \Rightarrow F) \to \mathsf{CAT}$

$\texttt{unitalSubcategory}\,(\alpha) = \mathcal{C}^\alpha := \left(\{X \in \mathcal{O}_\mathcal{C}(\mathcal{C}) : \alpha(X) : X \overset{\mathcal{C}}{\leftrightarrow} F(X)\}, \mathcal{M}_\mathcal{C}, \cdot_\mathcal{C}, \underset{\mathcal{C}}{\mathrm{id}}\right)$

$\texttt{EqSubcategoriesFromAdjunction} :: \forall (F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \; . \; \mathcal{A}^\eta \simeq \mathcal{B}^\epsilon$

$\texttt{Proof} \; =$

$\texttt{Assume}\; A : \mathcal{A}^\eta,$

$(1) := \texttt{AdjointFunctorsChar2}(F,G)\eth\texttt{TriangleId}(\nu,\epsilon) : F(\eta(A))\epsilon(F(A)) = \mathrm{id}_{F(A)},$

$(2) := F(\eta^{-1}(A))(1)F(\eta) : \epsilon(F(A))F(\eta(A)) = \mathrm{id}_{F(A)},$

$() := \eth^{-1}\texttt{Inverse}(\epsilon(F(A)))(F(\eta(A)))(1)(2) : F(A) \in \mathcal{B}^\epsilon;$

$\rightsquigarrow (1) := I(\forall) : \forall A \in \mathcal{A}^\eta \; . \; F(A) \in \mathcal{B}^\epsilon,$

$\texttt{Assume}\; B : \mathcal{B}^\epsilon,$

$(2) := \texttt{AdjointFunctorsChar2}(F,G)\eth\texttt{TriangleId}(\nu,\epsilon) : \eta(G(B))G(\epsilon(B)) = \mathrm{id}_{F(A)},$

$(3) := G(\epsilon^{-1}(A))(1)G(\epsilon) : G(\epsilon(B))\eta(G(B)) = \mathrm{id}_{F(A)},$

$() := \eth^{-1}\texttt{Inverse}(\eta(G(B)))(G(\epsilon(B)))(2)(3) : G(B) \in \mathcal{A}^\eta;$

$\rightsquigarrow (2) := I(\forall) : \forall B \in \mathcal{B}^\epsilon \; . \; G(B) \in \mathcal{A}^\eta,$

$(3) := \texttt{AdjointFunctorsChar2}(F,G)\eth\mathcal{A}^\eta\eth\mathcal{B}^\epsilon : \eta_{|A^\eta} : \mathrm{id}_{\mathcal{A}^\eta} \iff F_{|\mathcal{A}^\epsilon}G_{|\mathcal{B}^\epsilon},$

$(4) := \texttt{AdjointFunctorsChar2}(F,G)\eth\mathcal{B}^\epsilon\eth\mathcal{A}^\eta : \epsilon_{|B^\epsilon} : G_{|\mathcal{B}^\epsilon}F_{\mathcal{A}^\eta} \iff \mathrm{id}_{\mathcal{B}^\epsilon},$

$(*) := \eth^{-1}\texttt{CatEq}(3,4) : \mathcal{A}^\eta \simeq \mathcal{B}^\epsilon;$

$\square$

4.4 Morphisms of Adjunctions

$\texttt{MorphismOfAdjoints} :: \prod(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \;.\; \prod(F',G') : \texttt{Adjoint}(\mathcal{A}',\mathcal{B}') \;.$

$\quad .\; ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{A}' \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{B}')$

$(H,K) : \texttt{MorphismOfAdjoints} \iff \forall A \in \mathcal{A} \;.\; \forall B \in \mathcal{B} \;.\; HG = G'K \;\&\; KF = F'H \;\&$

$\quad \Big(K_{F(A),B}(\cdot)\Big)^{\top} = H_{A,G(B)}\Big(\cdot\Big)^{\top}$

$\texttt{MorphismOfAdjointsChar1} :: \forall(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \;.\; \forall(F',G') : \texttt{Adjoint}(\mathcal{A}',\mathcal{B}') \;.$

$\quad .\; \forall H : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{A}' \;.\; \forall K : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{B}' \;.\; \forall(0) : HG = G'K \;\&\; kF = F'H \;.$

$\quad .\; (H,K) : \texttt{MorphismOfAdjoints}(\mathcal{A},\mathcal{B}) \iff H(\eta) = \eta'(H)$

$\quad\quad \texttt{where}$

$\quad\quad\quad \eta = \eta^{F,G}$

$\quad\quad\quad \eta' = \eta^{F',G'}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (1) : \Big[(H,K) : \texttt{MorphismOfAdjoint}\Big],$

$\texttt{Assume}\ A : \mathcal{A},$

$() := \eth\eta(A)\eth\texttt{MorphismOfAdjoint}(H,K)\eth\texttt{Covariant}(K)\eth\texttt{MorphismOfAsjoints}(H,K)\eth^{-1}\eta' :$

$\quad : H\Big(\eta(A)\Big) = H_{A,FG(A)}\Big(\mathrm{id}^{\top}_{F(A)}\Big) = \Big(K_{F(A),F(A)}(\mathrm{id}_{F(A)})\Big)^{\top} = \mathrm{id}^{\top}_{KF(A)} = \mathrm{id}^{\top}_{F'H(A)} = \eta'(H(A));$

$\rightsquigarrow (1) := I(\forall)I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ (2) : H\eta = \eta' H,$

$\texttt{Assume}\ A : \mathcal{A},$

$\texttt{Assume}\ B : \mathcal{B},$

$\texttt{Assume}\ f : F(A) \xrightarrow{\mathcal{B}} B,$

$(3) := I(=)(f) : f = f,$

$(4) := \texttt{AdjointFunctorsChar}(\mathrm{id}, f, \mathrm{id}, f)(3) : \eta(A)G(f) = f^{\top},$

$(5) := I(=)(K(f)) : K(f) = K(f),$

$(6) := \texttt{AdjointFunctorsChar}(\mathrm{id}, K(f), K(f), \mathrm{id}) : \eta'(H(A))G'K(f) = (Kf)^{\top},$

$() := (6)(0)(2)\eth\texttt{Covariant}H(4) :$

$\quad : \Big(K_{F(A),B}(f)\Big)^{\top} = \eta'(H(A))G'K(f) = \eta'(H(A))HG(f) = H(\eta(A))HG(f) = H(\eta(A)G(f)) = H(f^{\top});$

$\rightsquigarrow (*) := I(\iff)I(\Leftarrow)\eth^{-1}\texttt{MorphismOfAdjunctions} : \texttt{This};$

$\square$

$\texttt{MorphismOfAdjointsChar2} :: \forall(F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) \;.\; \forall(F',G') : \texttt{Adjoint}(\mathcal{A}',\mathcal{B}') \;.$

$\quad .\; \forall H : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{A}' \;.\; \forall K : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{B}' \;.\; \forall(0) : HG = G'K \;\&\; kF = F'H \;.$

$\quad .\; (H,K) : \texttt{MorphismOfAdjoints}(\mathcal{A},\mathcal{B}) \iff K(\epsilon) = \epsilon'(K)$

$\quad\quad \texttt{where}$

$\quad\quad\quad \epsilon = \epsilon^{F,G}$

$\quad\quad\quad \epsilon' = \epsilon^{F',G'}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

4.5 Contravariant Adjunctions

$$\texttt{LeftAdjoint} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, ?\Big((\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B}) \times (\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A})\Big)$$

$$(F, G) : \texttt{LeftAdjoint} \iff \exists \alpha : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \mathcal{M}_{\mathcal{B}}(F(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(G(B), A) :$$

$$: \forall A \in \mathcal{A} \,.\, \Lambda B \in \mathcal{B} \,.\, \alpha(A, B) : \mathcal{M}_{\mathcal{B}}(F(A), \cdot) \Rightarrow \mathcal{M}_{\mathcal{A}}(G(\cdot), \cdot) \,\&$$

$$\& \,\forall B \in \mathcal{B} \,.\, \Lambda A \in \mathcal{A} \,.\, \alpha(A, B) : \mathcal{M}_{\mathcal{B}}(F(\cdot), B) \Rightarrow \mathcal{M}_{\mathcal{A}}(G(B), \cdot)$$

$$\texttt{RightAdjoint} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, ?\Big((\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B}) \times (\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A})\Big)$$

$$(F, G) : \texttt{RightAdjoint} \iff \exists \alpha : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, \mathcal{M}_{\mathcal{B}}(B, F(A)) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, G(B)) :$$

$$: \forall A \in \mathcal{A} \,.\, \Lambda B \in \mathcal{B} \,.\, \alpha(A, B) : \mathcal{M}_{\mathcal{B}}(\cdot, F(A)) \Rightarrow \mathcal{M}_{\mathcal{A}}(A, G(\cdot)) \,\&$$

$$\& \,\forall B \in \mathcal{B} \,.\, \Lambda A \in \mathcal{A} \,.\, \alpha(A, B) : \mathcal{M}_{\mathcal{B}}(B, F(\cdot)) \Rightarrow \mathcal{M}_{\mathcal{A}}(\cdot, G(B))$$

$$\texttt{transposeLeft} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod (F, G) : \texttt{LeftAdjoint}(\mathcal{A}, \mathcal{B}) \,.\, \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.$$

$$.\, \mathcal{M}_{\mathcal{B}}(F(A), B) \to \mathcal{M}_{\mathcal{A}}(G(B), A)$$

$$\texttt{transpose}\,(f) = f^{\top_{F,G}} := \eth\texttt{LeftAdjoint}(F, G)(f)$$

$$\texttt{antitransposeLeft} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod (F, G) : \texttt{LeftAdjoint} \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.$$

$$.\, \mathcal{M}_{\mathcal{A}}(G(B).A) \to \mathcal{M}_{\mathcal{B}}(F(A), B)$$

$$\texttt{antitransposeLeft}\,(f) = f^{\perp_{F,G}} := \eth\texttt{LeftAdjoint}(F, G)(f)$$

$$\texttt{transposeRight} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod (F, G) : \texttt{RightAdjoint} \,.\, \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.$$

$$\mathcal{M}_{\mathcal{B}}(B, F(A)) \to \mathcal{M}_{\mathcal{A}}(A, G(B))$$

$$\texttt{transposeRight}\,(f) = f^{\top_{F,G}} := \eth\texttt{RightAdjoint}(F, G)(f)$$

$$\texttt{antitransposeRight} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \prod (F, G) : \texttt{RightAdjoint} \,.\, \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.$$

$$\mathcal{M}_{\mathcal{A}}(A, G(B)) \to \mathcal{M}_{\mathcal{B}}(B, F(A))$$

$$\texttt{antitransposeRight}\,(f) = f^{\perp_{F,G}} := \eth\texttt{RightAdjoint}(F, G)(f)$$

$$\texttt{LeftAdjointFunctorsChar1} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \forall F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \forall G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A} \,.$$

$$\forall \alpha : \prod A \in \mathcal{A} \,.\, \prod B \in \mathcal{B} \,.\, (\mathcal{M}_{\mathcal{B}}(F(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(G(B), A)) \,.\, (F, G, \alpha) : \texttt{LeftAdjoint} \iff$$

$$\iff : \Big(\forall A, A' \in \mathcal{A} \,.\, \forall B, B' \in \mathcal{B} \,.\, \forall f : F(A') \xrightarrow{\mathcal{B}} B \,.\, \forall g : F(A) \xrightarrow{\mathcal{B}} B' \,.\, \forall h : A \xrightarrow{\mathcal{A}} A' \,.\, \forall k : B \xrightarrow{\mathcal{B}} B' \,.$$

$$.\, fk = F(h)g \iff G(k)\alpha(A', B)(f) = \alpha(A, B')(g)h\Big)$$

$\texttt{Proof}$ $=$

...

□

$\texttt{RightAdjointFunctorsChar1} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} . \forall F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B} . \forall G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A} .$

$\forall \alpha : \prod A \in \mathcal{A} . \prod B \in \mathcal{B} . (\mathcal{M}_{\mathcal{B}}(B, F(A)) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, G(B))) . (F, G, \alpha) : \texttt{RightAdjoint} \iff$

$\iff : \Big(\forall A, A' \in \mathcal{A} . \forall B, B' \in \mathcal{B} . \forall f : B \xrightarrow{\mathcal{B}} F(A') . \forall g : B' \xrightarrow{\mathcal{B}} F(A) . \forall h : A \xrightarrow{\mathcal{A}} A' . \forall k : B \xrightarrow{\mathcal{B}} B' .$

$. fF(h) = kg \iff h\alpha(A', B)(f) = \alpha(A, B')(g)G(k)\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{counit1} :: \prod (F, G) : \texttt{LeftAdjoint}(\mathcal{A}, \mathcal{B}) . GF \Rightarrow \mathrm{id}_{\mathcal{B}}$

$\texttt{counit1}\,() = \epsilon_1^{F,G} := \mathrm{id}^{\perp}_{G(B)}$

$\texttt{counit2} :: \prod (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) . \ . FG \Rightarrow \mathrm{id}_{\mathcal{A}}$

$\texttt{counit2}\,() = \epsilon_2^{F,G} := \mathrm{id}^{\perp}_{F(A)}$

$\texttt{unit1} :: \prod (F, G) : \texttt{RightAdjoint}(\mathcal{A}, \mathcal{B}) . \prod A \in \mathcal{A} . \mathrm{id}_{\mathcal{A}} \Rightarrow FG$

$\texttt{unit1}\,() = \eta_1^{F,G} := \mathrm{id}^{\top}_{F(A)}$

$\texttt{unit2} :: \prod (F, G) : \texttt{RightAdjoint}(\mathcal{A}, \mathcal{B}) . \mathrm{id}_{\mathcal{B}} \Rightarrow GF$

$\texttt{unit2}\,() = \eta_2^{F,G} := \mathrm{id}^{\top}_{G(B)}$

$\texttt{LeftTriangleId} :: \prod F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B} . \prod G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A} . ?(GF \Rightarrow \mathrm{id}_{\mathcal{B}} \times FG \Rightarrow \mathrm{id}_{\mathcal{A}})$

$(\alpha, \beta) : \texttt{LeftTriangleId} \iff F\alpha\beta F = \mathrm{id}_F \ \&\ G\beta\alpha G = \mathrm{id}_G$

$\texttt{RightTriangleId} :: \prod F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{B} . \prod G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} \mathcal{A} . ?(\mathrm{id}_{\mathcal{B}} \Rightarrow GF \times \mathrm{id}_{\mathcal{A}} \Rightarrow FG)$

$(\alpha, \beta) : \texttt{LeftTriangleId} \iff \beta FF\alpha = \mathrm{id}_F \ \&\ \alpha GG\beta = \mathrm{id}_G$

$\texttt{LeftAdjointFunctorsChar2} :: \forall F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{B} . \forall G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{A} . (F, G) : \texttt{LeftAdjoint}(\mathcal{A}, \mathcal{B}) \iff$

$\iff \exists \texttt{LeftTriangleId}(F, G)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{RightAdjointFunctorsChar2} :: \forall F : \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{B} . \forall G : \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{A} . (F, G) : \texttt{RightAdjoint}(\mathcal{A}, \mathcal{B}) \iff$

$\iff \exists \texttt{RightTriangleId}(F, G)$

$\texttt{Proof} =$

$\ldots$

$\square$

4.6 Extension to Adjoint Functor

FunctorAdjointExtension :: $\forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} . \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} . \forall G : \mathcal{B} \to \mathcal{A}$.

$. \forall \alpha : \prod A \in \mathcal{A} . \prod B \in \mathcal{B} . \mathcal{M}_{\mathcal{B}}(F(A), B) \leftrightarrow \mathcal{M}_{\mathcal{A}}(A, G(B))$.

$. \forall (0) : \forall B \in \mathcal{B} . \Lambda A \in \mathcal{A} . \alpha(A, B) : \mathcal{M}_{\mathcal{B}}(F(\cdot), B) \iff \mathcal{M}_{\mathcal{A}}(\cdot, G(B))$.

$. \exists ! G^\star : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} : G^\star_1 = G \ \& \ \forall A \in \mathcal{A} . \Lambda B \in \mathcal{B} . \alpha(A, B) : \mathcal{M}(F(A), \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, G(\cdot))$

Proof $=$

Assume $B, B' : \mathcal{B}$,

Assume $f : B \xrightarrow{\mathcal{B}} B'$,

$(1) := \eth f_* : \left[f_* : \mathcal{M}_{\mathcal{B}}(F(\cdot), B) \xrightarrow{\mathsf{SET}^{\mathcal{A}}} \mathcal{M}_{\mathcal{B}}(F(\cdot), B') \right]$,

$\gamma(B')(f) := \alpha^{-1}(\cdot, B) f_* \alpha(\cdot, B') : \mathcal{M}_{\mathcal{A}}(\cdot, G(B)) \xrightarrow{\mathsf{SET}^{\mathcal{A}}} \mathcal{M}_{\mathcal{A}}(\cdot, G(B'))$,

$G'_{B,B'}(f) :=$ ContravariantYonedaLemma$(\mathcal{A})(G(B))(\gamma) : G(B) \xrightarrow{\mathcal{C}} G(B')$;

$\rightsquigarrow (G', 1) := I \left(\prod \right) : \prod B, B' \in \mathcal{B} . \sum G_{B,B'} : (B \xrightarrow{\mathcal{B}} B') \to (G(B) \xrightarrow{\mathcal{B}} G(B'))$.

$. \forall f : B \xrightarrow{B'} B' . G^*_{B,B',*}(f) = \alpha^{-1}(B)(f_*)\alpha(B')$,

$G^\star := \Big(G, (G', (1))$PushforwardReflectsEquality$) \Big) : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A}$,

Assume $A : \mathcal{A}$,

Assume $B, B' : \mathcal{B}$,

Assume $f : B \xrightarrow{\mathcal{B}} B'$,

$() := \alpha(B)(1)(B, B', f)\eth$Inverse$\alpha(B) : \alpha(B) G^\star_{B,B',*}(f) = \alpha(B)\alpha^{-1}(B) f_* \alpha(B') = f_* \alpha(B')$;

$\rightsquigarrow (2) := I(\forall)\eth^{-1}$NaturalTransform $: \forall A \in \mathcal{A} . \Lambda B \in \mathcal{B} . \alpha(A, B) : \mathcal{M}(F(A), \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, G(\cdot))$,

Assume $H : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A}$,

Assume $(3) : H_1 = G$,

Assume $(4) : \forall A \in \mathcal{A} . \Lambda B \in \mathcal{B} . \alpha(A, B) : \mathcal{M}(F(A), \cdot) \iff \mathcal{M}_{\mathcal{A}}(A, H(\cdot))$,

Assume $B, B' : \mathcal{B}$,

Assume $f : B \xrightarrow{\mathcal{B}} B'$,

$(5) := (2)\eth$NaturalTransform$(\alpha)(3)\eth$NaturalTransform$(\alpha)(4) : \alpha(B) G^\star_{B,B',*}(f) = f_* \alpha(B) = \alpha(B) H_{B,B',*}(f)$,

$() := \alpha^{-1}(B)(5) : G^\star_{B,B',*}(f) = H_{B,B',*}(f)$;

$\rightsquigarrow () := I \left(=, \prod \right) : G^\star = H$;

$\rightsquigarrow (*) := \eth^{-1}$Unique : This;

$\square$

MultivariableFunctorAdjointExtensionRight :: $\forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{LSCAT}\ .\ \forall F : \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .$

$.\ \forall G : \prod A \in \mathcal{A}\ .\ \sum G_A : \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B}\ .\ F_A \dashv G_A\ .$

$.\ \exists! G^\star : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B} : \forall A \in \mathcal{A}\ .\ \forall C \in \mathcal{C}\ .\ G^\star(A, C) = G_A(C)\ \&$

$\&\ \exists \alpha : \prod A \in \mathcal{A}\ .\ \prod B \in \mathcal{B}\ .\ \prod C \in \mathcal{C}\ .\ \mathcal{M}_{\mathcal{C}}\big(F(A, B), C\big) \leftrightarrow \mathcal{M}_{\mathcal{B}}\big(B, G^\star(A, C)\big) :$

$: \Big(\forall A \in \mathcal{A}\ .\ \forall B \in \mathcal{B}\ .\ \Lambda C \in \mathcal{C}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(A, B), \cdot\big) \iff \mathcal{M}_{\mathcal{C}}\big(B, G^\star(A, \cdot)\big)\Big)\ \&$

$\&\ \Big(\forall C \in \mathcal{C}\ .\ \forall B \in \mathcal{B}\ .\ \Lambda A \in \mathcal{A}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(\cdot, B), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(B, G^\star(\cdot, C)\big)\Big)\ \&$

$\&\ \Big(\forall A \in \mathcal{A}\ .\ \forall C \in \mathcal{B}\ .\ \Lambda B \in \mathcal{B}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(A, \cdot), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(\cdot, G^\star(A, C)\big)\Big)$

Proof $=$

Assume $A, A' : \mathcal{A}$,

Assume $C, C' : \mathcal{C}$,

Assume $f : A' \xrightarrow{\mathcal{A}} A$,

Assume $h : C \xrightarrow{\mathcal{C}} C'$,

$k := h_* F^*_{A',A}(f \times \mathrm{id}) : \mathcal{M}_{\mathcal{C}}(F(A, \cdot), C) \xrightarrow{\mathsf{SET}^{\mathcal{B}}} \mathcal{M}_{\mathcal{C}}(F(A', \cdot), C')$,

$\gamma := \Lambda B \in \mathcal{B}\ .$ antitranspose$(F_A, G_A, B, C) k$ transpose$(F_{A'}, G_{A'}, B, C') : \mathcal{M}(\cdot, G_A(C)) \xrightarrow{\mathsf{SET}^{\mathcal{B}}} \mathcal{M}(\cdot, G_{A'}(C))$,

$G'_{(A,B),(A',B')}(f \times h) :=$ ControvariantYonedaLemma$(\mathcal{B}, G_A(C), \gamma) : G_A(C) \xrightarrow{\mathcal{B}} G_{A'}(C')$;

$\rightsquigarrow G' := I\left(\prod\right) : \prod A, A' \in \mathcal{A}\ .\ \prod C, C' \in \mathcal{C}\ .\ \left((A, C) \xrightarrow{\mathcal{A}^{\mathrm{op}} \times \mathcal{C}} (A', C')\right) \to (G_A(C) \xrightarrow{\mathcal{B}} G_{A'}(C'))$,

$G^\star := (G, G') : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B}$,

$\dots$

$\square$

MultivariableFunctorAdjointExtensionLeft :: $\forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{LSCAT}\ .\ \forall F : \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{C}\ .$

$.\ \forall H : \prod B \in \mathcal{B}\ .\ \sum H_B : \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{A}\ .\ F_B \dashv H_B\ .$

$.\ \exists! H^\star : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B} : \forall B \in \mathcal{B}\ .\ \forall C \in \mathcal{C}\ .\ H^\star(B, C) = H_B(C)\ \&$

$\&\ \alpha : \prod A \in \mathcal{A}\ .\ \prod B \in \mathcal{B}\ .\ \prod C \in \mathcal{C}\ .\ \mathcal{M}_{\mathcal{C}}\big(F(A, B), C\big) \leftrightarrow \mathcal{M}_{\mathcal{B}}\big(A, H^\star(B, C)\big) :$

$: \Big(\forall A \in \mathcal{A}\ .\ \forall B \in \mathcal{B}\ .\ \Lambda C \in \mathcal{C}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(A, B), \cdot\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, H^\star(B, \cdot)\big)\Big)\ \&$

$\&\ \Big(\forall C \in \mathcal{C}\ .\ \forall B \in \mathcal{B}\ .\ \Lambda A \in \mathcal{A}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(\cdot, B), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(\cdot, H^\star(B, C)\big)\Big)\ \&$

$\&\ \Big(\forall A \in \mathcal{A}\ .\ \forall C \in \mathcal{B}\ .\ \Lambda B \in \mathcal{B}\ .\ \alpha(A, B, C) : \mathcal{M}_{\mathcal{C}}\big(F(A, \cdot), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, H^\star(\cdot, C)\big)\Big)$

Proof $=$

$\dots$

$\square$

$\texttt{rightClosure1} :: \prod F : \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{LSCAT}} \mathcal{C} \; . \; \left(\prod A \in \mathcal{A} \; . \; \sum G_A : \mathcal{C} \xrightarrow{\mathsf{LSCAT}} \mathcal{B} \; . \; F_A \dashv G_A\right) \to$

$\quad \to \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B}$

$\texttt{rightClosure1}\,(G) := \texttt{MultivariableFunctorAdjointExtensionRight}(F, G)$

$\texttt{leftClosure1} :: \prod F : \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{LSCAT}} \mathcal{C} \; . \; \left(\prod B \in \mathcal{B} \; . \; \sum G_B : \mathcal{C} \xrightarrow{\mathsf{LSCAT}} \mathcal{A} \; . \; F_B \dashv G_B\right) \to$

$\quad \to \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{A}$

$\texttt{leftClosure1}\,(G) := \texttt{MultivariableFunctorAjointExtensionRight}(F, G)$

$\texttt{ClosuresRightAdjoint} :: \forall \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{LSCAT} \; . \; \forall F : \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{C} \; .$

$\quad \forall g : \prod A \in \mathcal{A} \; . \; g_A : \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B} \; . \; F_A \dashv g_A \; .$

$\quad \forall h : \prod B \in \mathcal{B} \; . \; h_B : \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{A} \; . \; F_B \dashv g_B \; .$

$\quad \forall C \in \mathcal{C} \; . \; (G_C, H_C) : \texttt{RightAdjoint}$

$\quad\quad \texttt{where}$

$\quad\quad\quad G = \texttt{rightClosure1}(F, g)$

$\quad\quad\quad H = \texttt{leftClosure1}(F, h)$

$\texttt{Proof} \; =$

Use composition of natural isomorphisms

$\square$

4.7 Multivariable Adjunctions

$$\texttt{Biadjoint} :: \prod \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathsf{LSCAT} \, . \, ?\Big((\mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{C}) \times (\mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B}) \times (\mathcal{B}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{A})\Big)$$

$$(F, G, H) : \texttt{Biadjoint} \iff$$

$$\iff \Bigg(\exists \alpha : \prod A \in \mathcal{A} \, . \, \prod B \in \mathcal{B} \, . \, \prod C \in \mathcal{C} \, . \, \mathcal{M}_{\mathcal{C}}\big(F(A,B), C\big) \leftrightarrow \mathcal{M}_{\mathcal{B}}\big(B, G(A,C)\big) :$$

$$: \Big(\forall A \in \mathcal{A} \, . \, \forall B \in \mathcal{B} \, . \, \Lambda C \in \mathcal{C} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(F(A,B), \cdot\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, G(B,\cdot)\big)\Big) \, \&$$

$$\& \Big(\forall C \in \mathcal{C} \, . \, \forall B \in \mathcal{B} \, . \, \Lambda A \in \mathcal{A} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(F(\cdot,B), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(\cdot, G(B,C)\big)\Big) \, \&$$

$$\& \Big(\forall A \in \mathcal{A} \, . \, \forall C \in \mathcal{B} \, . \, \Lambda B \in \mathcal{B} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(F(A,\cdot), C\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, G(\cdot,C)\big)\Big)\Bigg) \, \&$$

$$\& \Bigg(\exists \beta : \prod A \in \mathcal{A} \, . \, \prod B \in \mathcal{B} \, . \, \prod C \in \mathcal{C} \, . \, \mathcal{M}_{\mathcal{B}}\big(B, G(A,C)\big) \leftrightarrow \mathcal{M}_{\mathcal{A}}\big(A, H(B,C)\big) :$$

$$: \Big(\forall A \in \mathcal{A} \, . \, \forall B \in \mathcal{B} \, . \, \Lambda C \in \mathcal{C} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(B, G(A,\cdot)\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, H(B,\cdot)\big)\Big) \, \&$$

$$\& \Big(\forall C \in \mathcal{C} \, . \, \forall B \in \mathcal{B} \, . \, \Lambda A \in \mathcal{A} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(B, G(\cdot,C)\big) \iff \mathcal{M}_{\mathcal{C}}\big(\cdot, H(B,C)\big)\Big) \, \&$$

$$\& \Big(\forall A \in \mathcal{A} \, . \, \forall C \in \mathcal{C} \, . \, \Lambda B \in \mathcal{B} \, . \, \alpha(A,B,C) : \mathcal{M}_{\mathcal{C}}\big(\cdot, G(A,C)\big) \iff \mathcal{M}_{\mathcal{C}}\big(A, H(\cdot,C)\big)\Big)\Bigg)$$

$$\texttt{synecdoche} :: \texttt{Biadjoint}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to \mathcal{A} \times \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{C}$$

$$\texttt{synecdoche}\,(F, G, H) := F$$

$$\texttt{leftClosure} :: \texttt{Biadjoint}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{A}$$

$$\texttt{leftClosure}\,(F, G, H) := G$$

$$\texttt{rightClosure} :: \texttt{Bidjoint}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{B}$$

$$\texttt{rightClosure}\,(F, G, H) := H$$

$$\texttt{Closed} :: \prod \mathcal{C} \in \mathsf{LSCAT} \, . \, ?(\mathcal{C} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{C})$$

$$F : \texttt{Closed} \iff \exists (F, G, H) : \texttt{Biadjoint}(\mathcal{C}, \mathcal{C}, \mathcal{C}) : G \cong H$$

$$\texttt{productBifunctor} :: \prod \mathcal{C} : \texttt{WithFiniteProducts} \, . \, \mathcal{C} \times \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{C}$$

$$\texttt{productBifunctor}\,(A, B) = A \times B := \eth\texttt{WithFiniteProducts}(A, B)$$

$$\texttt{productBifunctor}\,((A, B), (A', B'), (f, g)) = f \times g := \eth\texttt{Limit}(A' \times B')(\pi_1 f, \pi_2 g)$$

$$\texttt{CartesianClosed} :: ?(\mathsf{LSCAT} \, \& \, \texttt{WithFiniteProducts})$$

$$\mathcal{C} : \texttt{CartesianClosed} \iff \texttt{productBifunctor}(\mathcal{C}) : \texttt{Closed}(\mathcal{C})$$

$$\texttt{MultivariableAdjunction} :: \prod I : \mathsf{SET} \,.\, \prod \mathcal{X} : I \to \mathsf{LSCAT} \,.\, \prod \mathcal{Y} \,.$$

$$.\, ?\left(\prod i \in \mathcal{I} \,.\, \texttt{Biadjoint}\left(\mathcal{X}_i, \prod_{j \in \mathcal{I} : j \neq i} \mathcal{X}_j, \mathcal{Y}\right)\right)$$

$$F : \texttt{MultivariableAdjunction} \iff \forall i, j \in \mathcal{I} \,.\, F_i =_{\prod_{i \in \mathcal{I}} \mathsf{X}_j \xrightarrow{\mathsf{CAT}} \mathsf{Y}_j} F_j$$

4.8 Calculus of Adjunction

TwoLeftAdjointsAreIso $:: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \;.\; \forall F, F' : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \;.\; \forall G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \;.\; \forall (0) : F \dashv G \;\&\; F' \dashv G \;.$

$\quad . \; \exists \theta : F \iff F' : \eta G(\theta) = \eta' \;\&\; (\theta G)\epsilon' = \epsilon$

where

$\qquad \eta' = \eta^{F',G} \quad \eta = \eta^{F,G} \quad \epsilon' = \epsilon^{F',G} \quad \epsilon = \epsilon^{F,G}$

Proof $=$

Assume $A : \mathcal{A}$,

$\theta_A := \left(\mathrm{id}_{F(A)}^{\top_{F,G}}\right)^{\perp_{F',G}} : F'(A) \xrightarrow{\mathcal{B}} F(A),$

$\theta'_A := \left(\mathrm{id}_{F'(A)}^{\top_{F',G}}\right)^{\perp_{F,G}} : F(A) \xrightarrow{\mathcal{B}} F'(A),$

$(1) := \eth\theta_A\theta'_A$AdjointFunctorsChar2$(F,G)(F',G)\eth$Covariant$(G)\eth$NaturalTransform$(\eta')\eth$TriangleId$(\eta',\epsilon')\eth$

$\quad : (\theta_A\theta'_A)^{\top_{F',G}} = \left(\left(\mathrm{id}_{F(A)}^{\top_{F,G}}\right)^{\perp_{F',G}} \left(\mathrm{id}_{F'(A)}^{\top_{F',G}}\right)^{\perp_{F,G}}\right)^{\top_{F',G}} = \eta'_A G\Big(F'(\eta_A)\epsilon'_{F(A)} F(\eta'_A)\epsilon_{F'(A)}\Big) =$

$\quad = \eta'_A F'G(\eta_A) G(\epsilon'_{F(A)}) FG(\eta'_A) G(\epsilon_{F'(A)}) = \eta_A \eta'_{FG(A)} G(\epsilon'_{F(A)}) FG(\eta'_A) G(\epsilon_{F'(A)}) = \eta_A FG(\eta'_A) G(\epsilon_{F'(A)}) =$

$\quad = \eta'_A \eta_{F'G(A)} G(\epsilon_{F'(A)}) = \eta'_A,$

$(2) := (1)^{\perp_{F',G}} : \theta_A\theta'_A = \mathrm{id},$

$(3) := \eth\theta'_A\theta_A$AdjointFunctorsChar2$(F,G)(F',G)\eth$Covariant$(G)\eth$NaturalTransform$(\eta)\eth$TriangleId$(\eta,\epsilon)\eth$Na

$\quad : (\theta'_A\theta_A)^{\top_{F,G}} = \left(\left(\mathrm{id}_{F'(A)}^{\top_{F',G}}\right)^{\perp_{F,G}} \left(\mathrm{id}_{F(A)}^{\top_{F,G}}\right)^{\perp_{F',G}}\right)^{\top_{F,G}} = \eta_A G\Big(F(\eta'_A)\epsilon_{F'(A)} F'(\eta_A)\epsilon'_{F(A)}\Big) =$

$\quad = \eta_A FG(\eta'_A) G(\epsilon_{F'(A)}) F'G(\eta_A) G(\epsilon'_{F(A)}) = \eta'_A \eta_{F'G(A)} G(\epsilon_{F'(A)}) F'G(\eta_A) G(\epsilon'_{F(A)}) = \eta'_A F'G(\eta_A) G(\epsilon'_{F(A)}) =$

$\quad = \eta_A \eta'_{FG(A)} G(\epsilon'_{F(A)}) = \eta_A,$

$() := (3)^{\perp_{F,G}} : \theta'_A\theta_A = \mathrm{id};$

$\rightsquigarrow \theta := I\left(\prod\right) : \prod A \in \mathcal{A} \;.\; F(A) \xleftrightarrow{\mathcal{B}} F'(A),$

Assume $X, Y : \mathcal{A}$,

Assume $f : X \xrightarrow{\mathcal{A}} Y$,

$() := \eth\theta(Y)\eth$Covariant$(F)\eth$NaturalTransform$(\eta')\eth$Covariant$(F)\eth$NaturalTransform$(\epsilon)\eth^{-1}\theta :$

$\quad : F(f)\theta(Y) = F(f)F(\eta'_Y)\epsilon_{F'(Y)} = F(f\eta'_Y)\epsilon_{F'(Y)} = F(\eta'_X F'G(f))\epsilon_{F'(Y)} = F(\eta'_X)F'GF(f)\epsilon_{F'(Y)} =$

$\quad = F(\eta'_X)\epsilon_{F'(X)}F'(f) = \theta(X)F'(f);$

$\rightsquigarrow (1) := \eth^{-1}$NaturalTransform $: [\theta : F \iff F'],$

Assume $A : \mathcal{A}$,

$()_1 := \eth\theta(A)\eth$Covariant$(G)\eth$NaturalTransform$(\eta)\eth$TriangleId$(\eta,\epsilon)$:

$\quad : \eta_A G(\theta(A)) = \eta_A G(F(\eta'_A)\epsilon_{F'(A)}) = \eta_A FG(\eta'_A) G(\epsilon_{F'(A)}) = \eta'_A \eta_{F'G(A)} G(\epsilon_{F'(A)}) = \eta'_A,$

$()_2 := \eth\theta(G(A))\eth$Covariant$(F)\eth$NaturalTransform$(\epsilon)\eth$TriangleId :

$\quad : \theta(G(A))\epsilon'_A = F(\eta'_G(A))\epsilon_{GF'(A)}\epsilon'_A = F(\eta'_G(A))GF(\epsilon'_A)\epsilon_A = \epsilon_A;$

$\rightsquigarrow (*) := I(=, \to)^2 :$ This;

$\square$

$\texttt{CompositionOfAdjoints} :: \forall \mathcal{A},\mathcal{B},\mathcal{C} \in \mathsf{LSCAT} . \forall (F,G) : \texttt{Adjoint}(\mathcal{A},\mathcal{B}) . \forall (F',G') : \texttt{Adjoint}(\mathcal{B},\mathcal{C}) .$
$\quad . (FF', G'G) : \texttt{Adjoint}(\mathcal{A},\mathcal{C})$

$\texttt{Proof} =$

$\eta' := \eta^{F',G'} : \mathrm{id}_{\mathcal{B}} \Rightarrow F'G',$

$\eta := \eta^{F,G} : \mathrm{id}_{\mathcal{A}} \Rightarrow FG,$

$\epsilon' := \epsilon^{F',G'} : G'F' \Rightarrow \mathrm{id}_{\mathcal{C}},$

$\epsilon := \epsilon^{F,G} : GF \Rightarrow \mathrm{id}_{\mathcal{B}},$

$\bar\eta := \eta G \eta' F : \mathrm{id}_{\mathcal{A}} \Rightarrow FF'G'G,$

$\bar\epsilon := F'\epsilon G'\epsilon' : G'GFF' \Rightarrow \mathrm{id}_{\mathcal{C}},$

$\texttt{Assume } A : \mathcal{A},$

$() := \eth\bar\eta\eth\bar\epsilon\eth\texttt{Covariant}(FF')\eth\texttt{Covariant}(F')\eth\texttt{NaturalTransform}(\epsilon)\eth\texttt{Covariant}(F')$
$\quad \eth\texttt{TriangleId}^2(\eta,\epsilon)(\eta',\epsilon')\eth\texttt{Covariant}(F')\eth\texttt{Identity}(\mathrm{id}_{FF'A}) :$
$\quad : FF'(\bar\eta_A)\bar\epsilon_{FF'A} = FF'(\eta_A G(\eta'_{FA}))F'(\epsilon_{FF'G'A})\epsilon'_{FF'A} = FF'(\eta_A)GFF'(\eta'_{FA})F'(\epsilon_{FF'G'A})\epsilon'_{FF'A} =$
$\quad = FF'(\eta_A)F'\Big(GF(\eta'_{FA})\epsilon_{FF'G'A}\Big)\epsilon'_{FF'A} = FF'(\eta_A)F'(\epsilon_{FA}\eta'_{FA})\epsilon'_{FF'A} = F'\Big(F(\eta_A)\epsilon_{FA}\Big)F'(\eta_{FA})\epsilon'_{FF'A} =$
$\quad = F'(\mathrm{id}_{F(A)})\mathrm{id}_{FF'A} = \mathrm{id}_{FF'A};$

$\rightsquigarrow (1) := I(=,\to) : FF'(\eta)\bar\epsilon FF' = \mathrm{id}_{FF'},$

$\texttt{Assume } C : \mathcal{C},$

$() := \eth\bar\eta\eth\bar\epsilon\eth\texttt{Covariant}(G'G)\eth\texttt{Covariant}(G)\eth\texttt{NaturalTransform}(\eta')\eth\texttt{Covariant}(G)$
$\quad \eth\texttt{TriangleId}^2(\eta,\epsilon)(\eta',\epsilon')\eth\texttt{Covariant}(G)\eth\texttt{Identity}(\mathrm{id}_{GG'A}) :$
$\quad : \bar\eta_{G'GC}G'G(\bar\epsilon_C) = \eta_{G'GC}G(\eta'_{G'GFC})G'G\Big(F'(\epsilon_{G'C})\epsilon'_C\Big) = \eta_{G'GC}G(\eta'_{G'GFC})F'G'G(\epsilon_{G'C})G'G(\epsilon'_C) =$
$\quad = \eta_{G'GC}G(\eta'_{G'GFC}F'G'(\epsilon_{G'C}))G'G(\epsilon'_C) = \eta_{G'GC}G(\epsilon_{G'C}\eta'_{G'C})G'G(\epsilon'_C) = \eta_{G'GC}G(\epsilon_{G'C}\eta'_{G'C})G\Big(\eta'_{GC}G'(\epsilon'_C)\Big) =$
$\quad = \mathrm{id}_{G'GC}G(\mathrm{id}_{G'C});$

$\rightsquigarrow (2) := \eth^{-1}\texttt{TriangleId} : \Big[(\bar\eta,\bar\epsilon) : \texttt{TriangleId}(\mathcal{A},\mathcal{B})\Big],$

$(*) := \texttt{AdjointFunctorsChar2}(2) : FF' \dashv GG';$

$\square$

$\texttt{AdjointEquivalence} :: ?\texttt{Adjoint}(\mathcal{A},\mathcal{B})$

$(F,G) : \texttt{AdjointEquivalence} \iff \Big(\eta^{F,G} : \mathrm{id}_{\mathcal{A}} \iff FG\Big) \,\&\, \Big(\epsilon^{F,G} : GF \iff \mathrm{id}_{\mathcal{B}}\Big)$

$\texttt{EveryEquivalenceIsAdjoint} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \forall (0) : \mathcal{A} \simeq \mathcal{B} \,.\, \exists \texttt{AdjointEquivalence}(\mathcal{A}, \mathcal{B})$

$\texttt{Proof} \;=$

$(F, G, \eta, \epsilon) := \eth\texttt{EqCat}(1) : \sum F : \texttt{FullyFaithful}(\mathcal{A}, \mathcal{B}) \,.\, \sum G : \texttt{FullyFaithful}(\mathcal{B}, \mathcal{A}) \,.$

$\quad . \,(\mathrm{id}_{\mathcal{A}} \iff FG) \times (GF \iff \mathrm{id}_{\mathcal{B}}),$

$\gamma := \eta G(G\epsilon) : G \iff G,$

$\epsilon' := (F\gamma^{-1})\epsilon : GF \iff \mathrm{id}_{\mathcal{B}},$

$\texttt{Assume}\ B : \mathcal{B},$

$() := \eth\epsilon'\eth\texttt{NaturalTransform}(\eta)\eth^{-1}\gamma\eth\texttt{Inverse} :$

$\quad : \eta_{GB} G(\epsilon'_B) = \eta_{GB} GF\gamma_B^{-1} G\epsilon_B = \gamma_B^{-1}\eta_{GB} G\epsilon_B = \gamma_B^{-1}\gamma_B = \mathrm{id}_{GB};$

$\rightsquigarrow (1) := I(=, \rightarrow) I(\forall) : \eta G G\epsilon = \mathrm{id}_G,$

$\texttt{Assume}\ A : \mathcal{A},$

$(2) := \eth^2\texttt{NaturalTransform}(\epsilon')\eth\texttt{NaturalTransform}(\eta) :$

$\quad : F(\eta_A)\epsilon'_{FA} F(\eta_A)\epsilon'_{FA} = F(\eta_A) FGF(\eta_A)\epsilon'_{FGFA}\epsilon'_{FA} = F(\eta_A GF(\eta_A)) GF\epsilon'_{FA}\epsilon'_{FA} =$

$\quad = F(\eta_A\eta_{FGA}) GF\epsilon'_{FA}\epsilon'_{FA} = F(\eta_A) F(\eta_{FGA}) GF\epsilon'_{FA}\epsilon'_{FA} = F(\eta_A)\epsilon'_{FA},$

$(3) := \eth^{-1}\texttt{Idempotent}(2) : \Big[F(\eta_A)\epsilon'_{FA} : \texttt{Idempotent}(FA)\Big],$

$() := \texttt{InvertibleIdempotentIsId}(3) : F(\eta_A)\epsilon'_{FA} = \mathrm{id}_{F(A)};$

$\rightsquigarrow (2) := \eth^{-1}\texttt{TriangleId}(1) : \Big[(\eta, \epsilon') : \texttt{TriangleId}\Big],$

$(*) := \texttt{AdjointFunctorsChar2}(2) : F \dashv G;$

$\square$

$\texttt{AdjointExponentiationI} :: \forall \mathcal{A}, \mathcal{B} : \mathsf{LSCAT} \,.\, \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \,.\, \forall \mathcal{I} : \mathsf{SCAT} \,.$

$\quad . \,(F_*, G_*) : \texttt{Adjoint}(\mathcal{A}^{\mathcal{I}}, \mathcal{B}^{\mathcal{I}})$

$\texttt{Proof} \;=$

$\eta' := \Lambda H \in \mathcal{A}^{\mathcal{I}} \,.\, \eta H : \prod H \in \mathcal{A}^{\mathcal{I}} \,.\, H \Rightarrow HFG,$

$\epsilon' := \Lambda H \in \mathcal{B}^{\mathcal{I}} \,.\, \epsilon H : \prod H \in \mathcal{A}^{\mathcal{I}} \,.\, HGF \Rightarrow H,$

$\texttt{Assume}\ X, Y : \mathcal{A}^{\mathcal{I}},$

$\texttt{Assume}\ \alpha : X \Rightarrow Y,$

$() := \eth\eta'_Y\eth\texttt{NaturalTransform}\eta\eth^{-1}\eta'_X : \alpha\eta'_Y = \alpha\eta Y = (\eta X) FG\alpha = \eta'_X FG\alpha;$

$\rightsquigarrow (1) := \eth^{-1}\texttt{NaturalTransform} : [\eta' : \mathrm{id}_{\mathcal{A}^{\mathcal{I}}} \Rightarrow F_* G_*],$

$\texttt{Assume}\ X, Y : \mathcal{B}^{\mathcal{I}},$

$\texttt{Assume}\ \alpha : X \Rightarrow Y,$

$() := \eth\epsilon'_Y\eth\texttt{NaturalTransform}\epsilon\eth^{-1}\epsilon'_X : GF\alpha\epsilon'_Y = GF\alpha\epsilon Y = (\epsilon X)\alpha = \epsilon'_X FG\alpha;$

$\rightsquigarrow (2) := \eth^{-1}\texttt{NaturalTransform} : [\epsilon' : G_* F_* \Rightarrow \underset{\mathcal{B}}{\mathrm{id}}],$

$\texttt{Assume}\ X : \mathcal{A}^{\mathcal{I}},$

$() := \eth\eta'_X\eth\epsilon'\eth\texttt{TriangleId}(\eta, \epsilon) : F_*(\eta'_X)\epsilon'_{F_* X} = F\eta_X\epsilon_{XF} = \mathrm{id}_{F_* X};$

$\rightsquigarrow (3) := I(=, \rightarrow) : F_*\eta\epsilon F_* = \mathrm{id}_{F_*},$

$\texttt{Assume}\ X : \mathcal{B}^{\mathcal{I}},$

$() := \eth\eta'_X\eth\epsilon'\eth\texttt{TriangleId}(\eta, \epsilon) : (\eta'_{G_* X}) G_*\epsilon'_X = \eta_{XG} G\epsilon_X = \mathrm{id}_{G_* X};$

$\rightsquigarrow (4) := \eth^{-1}\texttt{TriangleId} : \Big[(\eta', \epsilon') : \texttt{TriangleId}(F_*, G_*)\Big],$

$(*) := \texttt{AdjointFunctorsChar2}(4) : F_* \dashv G_*;$

$\square$

$\texttt{AdjointExponentiationII} :: \forall \mathcal{A}, \mathcal{B} : \mathsf{LSCAT} \; . \; \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \; . \; \forall \mathcal{C} : \mathsf{LSCAT} \; .$
$\quad . \; (F^*, G^*) : \texttt{Adjoint}(\mathcal{C}^{\mathcal{B}}, \mathcal{C}^{\mathcal{A}})$

$\texttt{Proof} \; =$

$\ldots$

$\square$

4.9 Limits and Adjunctions

$\mathtt{WithLimitsByAdjuction} :: \forall \mathcal{C} \in \mathsf{LSCAT} \,.\, \forall \mathcal{I} \in \mathsf{SCAT} \,.\, \mathcal{C} : \mathtt{WithLimit}(\mathcal{I}) \iff \mathrm{Const}_{\mathcal{I}} \dashv \lim_{i \in \mathcal{I}}$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ (1) : \mathcal{C} : \mathtt{WithLimit}(\mathcal{I}),$

$\mathtt{Assume}\ X : \mathcal{C},$

$\mathtt{Assume}\ F : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C},$

$\lambda := \mathtt{legs}(\lim_{i \in \mathcal{I}} F_i) : \prod j \in \mathcal{I} \,.\, \lim_{i \in \mathcal{I}} F_i \xrightarrow{\mathcal{C}} F_j,$

$\mathtt{Assume}\ \alpha : \mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F,$

$\mathtt{Assume}\ i, j : \mathcal{I},$

$\mathtt{Assume}\ f : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\mathtt{NaturalTransform}\alpha : \alpha_i F_{i,j}(f) = \alpha_j;$

$\rightsquigarrow (2) := \eth^{-1}\mathtt{Cone} : (X, \alpha) : \mathtt{Cone}(F),$

$\tau_{X,F}(f) := \eth\mathtt{Limit}(1) : X \xrightarrow{\int_{\mathcal{C}} \mathrm{Cone}_F} \lim_{i \in \mathcal{I}} F_i;$

$\rightsquigarrow \tau_{X,F} := I(\rightarrow) : \Big(\mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F\Big) \rightarrow \Big(X \xrightarrow{\mathcal{C}} \lim_{i \in \mathcal{I}} F_i\Big),$

$\mathtt{Assume}\ f : X \xrightarrow{\mathcal{C}} \lim_{i \in \mathcal{I}} F_i,$

$\sigma(f) := f\lambda : \prod j \in \mathcal{I} \,.\, X \xrightarrow{\mathcal{C}} F_j,$

$\mathtt{Assume}\ i, j : \mathcal{I},$

$\mathtt{Assume}\ g : i \xrightarrow{\mathcal{I}} j,$

$() := \eth\sigma_i(f)\eth\mathtt{Cone}(\lim_{i \in \mathcal{I}} f_i, \lambda)\eth^{-1}\sigma_j(f) : \sigma_i(f)F_{i,j}(g) = f\lambda_i F_{i,j}(g) = f\lambda_j = \sigma_j(f);$

$\rightsquigarrow () := \eth^{-1}\mathtt{NaturalTransform} : \Big[\sigma : \mathrm{Const}(X) \Rightarrow F\Big];$

$\sigma := I(\rightarrow) : \Big(X \xrightarrow{\mathcal{C}} \lim_{i \in \mathcal{I}} F_i\Big) \rightarrow \Big(\mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F\Big),$

$(2) := \eth\sigma\eth \int_{\mathcal{C}} \mathrm{Cone}_F(x)\, \mathrm{d}x(\tau_{X,F}(\alpha))\eth^{-1}\mathrm{id}_{\mathcal{C}^{\mathcal{I}}} :$

$\quad : \tau_{X,F}\sigma = \Lambda\alpha : \mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F \,.\, \tau_{X,F}(\alpha)\lambda = \Lambda\alpha : \mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F \,.\, \alpha = \mathrm{id}_{\mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F},$

$(3) := \eth\mathtt{Limit}(F) : \sigma\tau_{X,F} = \mathrm{id}_{X \to \lim_{i \in \mathcal{I}} F_i},$

$() := \eth^{-1}\mathtt{Inverse}(2)(3) : \sigma = \tau_{X,F}^{-1};$

$\rightsquigarrow \tau := I\Big(\prod\Big) : \prod X \in \mathcal{C} \,.\, \prod F : \mathcal{I} \xrightarrow{\mathsf{CAT}} \mathcal{C} \,.\, \Big(\mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F\Big) \leftrightarrow \Big(X \xrightarrow{\mathcal{C}} \lim_{i \in \mathcal{I}} F_i\Big),$

$\mathtt{Assume}\ F, G : \mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F,$

$\mathtt{Assume}\ X, Y : \mathcal{C},$

$\mathtt{Assume}\ f : X \xrightarrow{\mathcal{C}} Y,$

$\mathtt{Assume}\ \beta : F \Rightarrow G,$

$()_1 := \eth f^*\eth\mathtt{Limit}(F)\eth^{-1}f^* : f^*\tau_{Y,F} = \Lambda\alpha : \mathrm{Const}(X) \Rightarrow F \,.\, \tau_{Y,F}(f\alpha) = \Lambda\alpha : \mathrm{Const}(X) \Rightarrow F \,.\, f\tau_{X,F}(\alpha) = \tau_{X,F}\, f$

$()_2 := \eth\beta_*\eth\mathtt{Limit}\eth^{-1}\beta_* : \beta_*\tau_{X,G} = \Lambda\alpha : \mathrm{Const}(X) \Rightarrow F \,.\, \tau_{X,G}(\alpha\beta) = \Lambda\alpha : \mathrm{Const}(X) \Rightarrow F \,.\, \tau_{X,F}(\alpha) \lim_{i \in \mathcal{I}} \beta_i = \tau_{X,F}$

$\rightsquigarrow (2) := \eth^{-1}\mathtt{NaturalTransform} : \tau : \Big(\mathrm{Const}_{\mathcal{I}}(X) \Rightarrow F\Big) \iff \Big(X \xrightarrow{\mathcal{C}} \lim_{i \in \mathcal{I}} F_i\Big),$

$(*) := \eth^{-1}\mathtt{Adjoint}(\mathcal{C}, \mathcal{C}^{\mathcal{I}})(\tau) : \mathrm{Const}_{\mathcal{I}} \dashv \lim_{i \in \mathcal{I}};$

$\square$

$\texttt{ColimitsByAdjunction} :: \forall \mathcal{C} : \texttt{LSCAT} \,.\, \forall \mathcal{J} : \texttt{SCAT} \,.\, \mathcal{C} : \texttt{WithLimits} \iff \operatorname*{colim}_{i \in \mathcal{I}} \dashv \mathrm{Const}_{\mathcal{I}}$

$\texttt{Proof} =$

...

$\square$

$\texttt{RAPL} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \forall G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \,.\, \forall (0) : G \dashv F \,.\, F : \texttt{PreservesLimits}$

$\texttt{Proof} =$

$\texttt{Assume}\ (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{A}),$

$\texttt{Assume}\ (L, \lambda) : \texttt{Limit}(X),$

$\texttt{Assume}\ (C, \mu) : \texttt{Cone}(XF),$

$(1) := \eth\texttt{Covariant}(F)\eth^{-1}\texttt{Cone} : \Big[(GC, G\mu) : \texttt{Cone}(XFG)\Big],$

$(2) := \eth^{-1}\texttt{Cone}\eth^{-1}\epsilon : \Big[(GC, G\mu\epsilon_X) : \texttt{Cone}(X)\Big],$

$(\psi, 3) := \eth\texttt{Limit}(L, \lambda)(2) : \sum \psi : GC \xrightarrow{\mathcal{A}} L \,.\, \forall i \in \mathcal{I} \,.\, \psi\lambda_i = G(\mu_i)\epsilon_{X_i},$

$\phi := \eta_C F(\psi) : C \xrightarrow{\mathcal{B}} FL,$

$\texttt{Assume}\ i : \mathcal{I},$

$() := \eth\phi\eth\texttt{Covariant}F(3)\eth\texttt{Covariant}G\eth\texttt{NaturalTransform}(\eta)\eth\texttt{TriangleId}(\epsilon, \eta) :$

$: \phi F\lambda_i = \eta_C F(\psi)F(\lambda_i) = \eta_C F(\psi\lambda_i) = \eta_C F(G(\mu_i)\epsilon_{X_i}) = \eta_C FG(\mu_i)F(\epsilon_{X_i}) = \mu_i\eta_{FX_i}F(\epsilon_{X_i}) = \mu_i;$

$\rightsquigarrow (4) := \eth^{-1}\int_{\mathcal{B}} \mathrm{Cone}_{FX} : \Big[\phi : C \xrightarrow{\int_{\mathcal{B}} \mathrm{Cone}_{FX}} FL\Big],$

$\texttt{Assume}\ \phi' : C \xrightarrow{\int_{\mathcal{B}} \mathrm{Cone}_{FX}} FL,$

$(5) := \eth\texttt{antitranspose} : [\phi^\perp, (\phi')^\perp : GC \xrightarrow{\mathcal{A}} L],$

$\texttt{Assume}\ i : \mathcal{I},$

$() := \eth\texttt{NaturalTransformantitranspose}\eth\int_{\mathcal{B}} \mathrm{Cone}\eth\texttt{NaturalTransformantitranspose}\eth^{-1}\epsilon : \phi^\perp\lambda_i = (\phi F\lambda_i)$

$\rightsquigarrow (6) := \eth^{-1}\int_{\mathcal{A}} \mathrm{Cone}_{X_i} : \Big[\phi^\perp, (\phi')^\perp : GC \xrightarrow{\int_{\mathcal{A}} \mathrm{Cone}_X} L\Big],$

$(7) := \eth\texttt{Limit}(X)(L, \lambda)(6) : \phi^\perp = (\phi')^\perp,$

$() := (7)^\top : \phi = \phi';$

$\rightsquigarrow () := \eth^{-1}\texttt{Limit} : \Big[(FL, F\lambda) : \texttt{Limit}(FX)\Big];$

$\rightsquigarrow (*) := \eth^{-1}\texttt{PreservesLimits} : [F : \texttt{PreservesLimits}];$

$\square$

$\texttt{LAPC} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \,.\, \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} \,.\, \forall G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \,.\, \forall (0) : F \dashv G \,.\, F : \texttt{PreservesColimits}$

$\texttt{Proof} =$

$G \ldots$

$\square$

4.10 Reflective Subcategory

$$\mathtt{ReflectiveSubcat} :: \prod \mathcal{C} \in \mathsf{LSCAT} \;.\; ?\mathtt{FullSubcat}(\mathcal{C})$$

$$(\mathcal{D}, I) : \mathtt{ReflectiveSubcat} \iff \exists L : \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{D} : L \dashv I$$

$$\mathtt{localization} :: \prod \mathcal{D} : \mathtt{ReflectiveSubcat} \;.\; \mathcal{C} \xrightarrow{\mathsf{CAT}} \mathcal{D}$$

$$\mathtt{localization}\,() = L_{\mathcal{D}} := \eth\mathtt{ReflectiveSubcal}$$

$$\mathtt{RightAdjointEpimorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \;.\; \forall (F, G) : \mathtt{Adjoint}(\mathcal{A}, \mathcal{B}) \;.$$

$$.\; G : \mathtt{Faithfull}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \;.\; \mathtt{Epic}(GFB, B)$$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ (1) : [G : \mathtt{Faithful}(\mathcal{B}, \mathcal{A})],$

$\mathtt{Assume}\ B : \mathcal{B},$

$\mathtt{Assume}\ Y : \mathcal{B},$

$\mathtt{Assume}\ f, g : B \xrightarrow{\mathcal{B}} Y,$

$\mathtt{Assume}\ (2) : \epsilon_B f = \epsilon_B g,$

$(3) := \eth\epsilon_B\eth\mathtt{NaturalTransformtranspose} : (Gf)^\top = (\underset{GB}{\mathrm{id}})^\top f = \epsilon_B f = \epsilon_B g = (\underset{GB}{\mathrm{id}})^\top g = (Gg)^\top,$

$(4) := (3)^\perp : Gf = Gg,$

$() := \eth^{-1}\mathtt{Faithful}(G)(1)(4) : f = g;$

$\rightsquigarrow (1) := I(\Rightarrow) : \mathtt{Left} \Rightarrow \mathtt{Right},$

$\mathtt{Assume}\ (2) : \forall B \in \mathcal{B} \;.\; \epsilon_B : \mathtt{Epic}(GFB, B),$

$\mathtt{Assume}\ X, Y : \mathcal{B},$

$\mathtt{Assume}\ f, g : X \xrightarrow{\mathcal{B}} Y,$

$\mathtt{Assume}\ (3) : Gf = Gg,$

$(4) := (3)^\top : (Gf)^\top = (Gg)^\top,$

$(5) := \eth^{-1}\epsilon_B\eth\mathtt{NaturalTransformtranspose}(4) : \epsilon_B f = \epsilon_B g,$

$(6) := \eth\mathtt{Epic}(\epsilon_B)(5) : f = g;$

$\rightsquigarrow (*) := I(\iff)(1) : \mathtt{This};$

$\square$

$\texttt{RightAdjointSplitMonomorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \ . \ \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \ .$

$\quad . \ G : \texttt{Full}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \ . \ \texttt{SplitMonic}(GFB, B)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ (1) : [G : \texttt{Full}(\mathcal{B}, \mathcal{A})],$

$\texttt{Assume} \ B : \mathcal{B},$

$\gamma := G^{-1}\eta_{GB} : \Big[B \mapsto GFB\Big],$

$() := \eth\epsilon_B\eth\texttt{NaturalTransformtranspose}\eth\eta_{GB}\eth\texttt{Inverse} :$

$\quad : \epsilon_B G^{-1}\eta_{GB} = (\mathrm{id}_{GB})^\top G^{-1}\alpha_{GB} = (\alpha_{GB})^\top = \mathrm{id}_{GFB};$

$\rightsquigarrow (1) := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume} \ (2) : \forall B \in \mathcal{B} \ . \ \epsilon_B : \texttt{SplitMonic}(GFB, B),$

$\texttt{Assume} \ X, Y : \mathcal{B},$

$\texttt{Assume} \ y : GX \xrightarrow{\mathcal{A}} GY,$

$x := \epsilon_X^{-1} y^\perp : X \xrightarrow{\mathcal{B}} Y,$

$(3) := \texttt{AdjointFunctorsChar2}(F, G)(Gx)\eth\texttt{NaturalTransform}\epsilon\eth\texttt{Retraction}(e_X) :$

$\quad : (Gx)^\perp = FG\Big(\epsilon_X^{-1} y^\perp\Big)\epsilon_Y = \epsilon_X \epsilon_X^{-1} y^\perp = y^\perp,$

$(4) := (3)^\top : Gx = y;$

$\rightsquigarrow () := \eth^{-1}\texttt{Surjective} : \Big[G_{X,Y} : (X \xrightarrow{\mathcal{B}} Y) \twoheadrightarrow (GX \xrightarrow{\mathcal{A}} GY)\Big];$

$\rightsquigarrow (*) := I(\iff)(1) : \texttt{This};$

$\square$

$\texttt{RightAdjointIsomorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \ . \ \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \ .$

$\quad . \ G : \texttt{FullyFaithful}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \ . \ \texttt{Iso}(GFB, B)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LeftAdjointMonomorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \ . \ \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \ .$

$\quad . \ F : \texttt{Faithful}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \ . \ \texttt{Monic}(GFB, B)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LeftAdjointSolitEpimorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \ . \ \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \ .$

$\quad . \ F : \texttt{Full}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \ . \ \texttt{SplitEpic}(GFB, B)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{LeftAdjointIsomorphism} :: \forall \mathcal{A}, \mathcal{B} \in \mathsf{LSCAT} \;.\; \forall (F, G) : \texttt{Adjoint}(\mathcal{A}, \mathcal{B}) \;.$

$\quad . \; F : \texttt{Faithful}(\mathcal{B}, \mathcal{A}) \iff \epsilon : \prod B \in \mathcal{B} \;.\; \texttt{Iso}(GFB, B)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{ReflectiveInclusionCreatesLimits} :: \forall \mathcal{D} : \texttt{Reflective} \;.\; I_{\mathcal{D}} : \texttt{CreatesLimits}$

$\texttt{Proof} \; =$

$G\texttt{Assume}\ (\mathcal{I}, X) : \texttt{Diagram}(\mathcal{D}),$

$\texttt{Assume}\ (Y, \lambda) : \texttt{Limit}(I_{\mathcal{D}} X),$

$(A, \mu) := (L_{\mathcal{D}} Y, L_{\mathcal{D}} \lambda) : \sum A \in \mathcal{D} \;.\; \prod i \in \mathcal{I} \;.\; A \xrightarrow{\mathcal{D}} X_i,$

$\texttt{Assume}\ i, j : \mathcal{I},$

$\texttt{Assume}\ f : i \xrightarrow{f} j,$

$() := \eth \mu_i \eth \texttt{Reflective} \eth \eth \texttt{Cone}(Y, \lambda) \eth^{-1} \mu_j :$

$\quad : \mu_i X_{i,j}(f) = \Big(L_{\mathcal{D}} \lambda_i\Big) X_{i,j}(f) = L_{\mathcal{D}} \lambda_i I_{\mathcal{D}} L_{\mathcal{D}} X_{i,j}(f) = L_{\mathcal{D}}(\lambda_i I_{\mathcal{D}} X_{i,j}(f)) = L_{\mathcal{D}}(\lambda_j) = \mu_j;$

$\rightsquigarrow (1) := \eth^{-1} \texttt{Cone} : \Big[(A, \mu) : \texttt{Cone}(\mathcal{I}, X)\Big],$

$\texttt{Assume}\ (C, \alpha) : \texttt{Cone}(\mathcal{I}, X),$

$(2) := \eth^{-1} \texttt{Cone} \eth \texttt{Covariant}(I_{\mathcal{D}}) \eth \texttt{Cone}(C, \alpha) : \Big[(I_{\mathcal{D}} C, I_{\mathcal{D}} \alpha) : \texttt{Cone}(\mathcal{I}, I_{\mathcal{D}} \alpha)\Big],$

$\phi := \eth \texttt{Limit}(Y, \lambda) : I_{\mathcal{D}} C \xrightarrow{\int \mathrm{Cone}_{I_{\mathcal{D}} X}} Y,$

$\psi := L_{\mathcal{D}} \phi : C \xrightarrow{\mathcal{D}} A,$

$\texttt{Assume}\ i : \mathcal{I},$

$() := \eth \psi \eth \mu_i \eth \texttt{Covariant} L_{\mathcal{D}} \eth \phi \eth \texttt{Reflective}(\mathcal{D}) : \psi \mu_i = L_D(\phi \lambda_i) = L_{\mathcal{D}}(I_{\mathcal{D}} \alpha_i) = \alpha_i;$

$\rightsquigarrow (3) := \eth^{-1} \int_{\mathcal{D}} \mathrm{Cone}_X(x)\, \mathrm{d}x : \Big[\psi : C \xrightarrow{\int \mathrm{Cone}_X} A\Big],$

$\texttt{Assume}\ \psi' : (C, \alpha) \xrightarrow{\int \mathrm{Cone}_X} (A, \mu),$

$(4) := \eth \texttt{Covariant} I_{\mathcal{D}} : \Big[I_{\mathcal{D}} \psi' : (I_{\mathcal{D}} C, I_{\mathcal{D}} \alpha) \xrightarrow{\int \mathrm{Cone}_{I_{\mathcal{D}} X}} (I_{\mathcal{D}} A, I_{\mathcal{D}} \mu_i)\Big],$

$\varphi := \eth \texttt{Limit}(Y, \lambda) : \Big[(I_{\mathcal{D}} A, I_{\mathcal{D}} \mu_i) \xrightarrow{\int \mathrm{Cone}_{I_{\mathcal{D}} X}} (Y, \lambda)\Big],$

$\phi' := I_{\mathcal{D}} \psi' \varphi : (I_{\mathcal{D}} C, I_{\mathcal{D}} \alpha) \xrightarrow{\int \mathrm{Cone}_{I_{\mathcal{D}} X}} (Y, \lambda),$

$(5) := \eth \texttt{Limit}(Y, \lambda)(\phi', \phi) : \phi = \phi' \varphi,$

$:=: (L_{\mathcal{D}} \varphi)^{\top} : I_{\mathcal{D}} A \xrightarrow{\mathcal{C}} I_{\mathcal{D}} A,$

$() :=:$

$\quad : \psi = L_{\mathcal{D}} \phi' = I_{\mathcal{D}} L_{\mathcal{D}} \psi' L_{\mathcal{D}} \varphi = \psi' L_{\mathcal{D}} \varphi = \psi';$

$\dots$

$\square$

$\texttt{ReflectiveInclusionCreatesColimits} :: \forall \mathcal{D} : \texttt{Reflective} \;.\; I_{\mathcal{D}} : \texttt{CreatesColimits}$

$\texttt{Proof} \; =$

$\dots$

$\square$

4.11 Existance of Adjoint Functors

$\texttt{LeftAdjointExistsByComma} :: \forall \mathcal{A}, \mathcal{B} : \texttt{LSCAT} \,.\, \forall U : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} \,.$

$\quad .\, \exists F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B} : F \dashv U \iff \forall A \in \mathcal{A} \,.\, \exists \texttt{Initial}(\mathrm{Const}(A) \downarrow U)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ F : \mathcal{A} \xrightarrow{\mathsf{CAT}} B,$

$\texttt{Assume}\ (1) : F \dashv U,$

$\texttt{Assume}\ A : \mathcal{A},$

$\texttt{Assume}\ (X, f) : \mathrm{Const}(A) \downarrow U,$

$(2) := \eth \mathrm{Const}(A) \downarrow U(X, f) : [f : A \xrightarrow{\mathcal{A}} U(X)],$

$(3) := \texttt{AdjointFunctorsChar2}(F, U)\eth\texttt{Inverse} : \eta_A U(f^\perp) = (f^\perp)^\top = f,$

$(4) := \eth^{-1}\eth \mathrm{Const}(A) \downarrow U : \left[f^\perp : (FA, \eta_A) \xrightarrow{\mathrm{Const}(A)\downarrow U} (X, U)\right],$

$\texttt{Assume}\ g : (FA, \eta_A) \xrightarrow{\mathrm{Const}(A)\downarrow U} (X, U),$

$(5) := \eth^\perp \mathrm{Const}(A) : f = \eta_A U g = g^\top,$

$() := (5)^\perp : f^\perp = g;$

$\rightsquigarrow () := \eth^{-1}\texttt{Initial} : (FA, \eta_A) : \texttt{Initial}(\mathrm{Const}(A) \downarrow U);$

$\rightsquigarrow (1) := I(\Rightarrow) : \texttt{Right} \Rightarrow \texttt{Left},$

$\texttt{Assume}\ (2) : \forall A \in \mathcal{A} \,.\, \exists \texttt{Initial}(\mathrm{Const}(A) \downarrow U),$

$\texttt{Assume}\ A : \mathcal{A},$

$(X, f) := (2)(A) : \texttt{Initial}(\mathrm{Const}(A) \downarrow U),$

$F'(A) := X : \mathcal{B};$

$\texttt{Assume}\ X, Y : \mathcal{A},$

$\texttt{Assume}\ f : X \xrightarrow{\mathcal{A}} Y,$

$(F'(X), g) := (2)(X) : \texttt{Initial}(\mathrm{Const}(X) \downarrow U),$

$(F'(Y), h) := (2)(X) : \texttt{Initial}(\mathrm{Const}(Y) \downarrow U),$

$(3) := (2)(X) : (F'(Y), fh) \in \mathrm{Const}(X) \downarrow U,$

$F''(f) := \eth\texttt{Initial}(F'(X), g))(3) : F'(X) \xrightarrow{\mathcal{B}} F'(Y);$

$\rightsquigarrow F'' := I\left(\prod\right) : \prod X, Y \in \mathcal{A} \,.\, (X \xrightarrow{\mathcal{A}} Y) \to (F'(X) \xrightarrow{\mathcal{B}} F'(Y)),$

$F := (F', F'') : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B},$

$\texttt{Assume}\ A : \mathcal{A},$

$\texttt{Assume}\ B : \mathcal{B},$

$\texttt{Assume}\ f : F(A) \xrightarrow{\mathcal{B}} B,$

$(F(A), \eta_A) := (2)(A) : \texttt{Initial}(\mathrm{Const}(A) \downarrow U),$

$\tau(f) := \eta_A U(f) : A \xrightarrow{\mathcal{B}} U(f);$

$\rightsquigarrow \tau := I(\to) : \mathcal{M}_\mathcal{B}(F(\cdot), \cdot) \Rightarrow \mathcal{M}_\mathcal{A}(\cdot, U(\cdot)),$

$\ldots$

$\square$

$\texttt{SolutionSetCondition} :: \prod \mathcal{A}, \mathcal{B} \in \mathsf{CAT} \;.\; ?\texttt{Continuous}(\mathcal{A}, \mathcal{B})$

$U : \texttt{SolutionSetCondition} \iff \forall B \in \mathcal{B} \;.\; \exists \sum I \in \mathsf{SET} \;.\; \sum A : I \to \mathcal{A} \;.$
$\;.\; \sum f : \prod i \in I \;.\; B \to UA_i \;.\; \forall X \in \mathcal{A} \;.\; \forall \phi : B \xrightarrow{\mathcal{B}} UA \;.\; \exists i \in I : \exists \alpha_i : A_i \xrightarrow{\mathcal{A}} X : \phi = f_i U \alpha_i$

$\texttt{JointlyWeaklyInitial} :: \prod \mathcal{C} \in \mathsf{CAT} \;.\; ?\sum I \in \mathsf{SET} \;.\; I \to \mathcal{C}$

$C : \texttt{JointlyWeaklyInitial} \iff \forall X \in \mathcal{C} \;.\; \exists i \in I : \exists C_i \xrightarrow{\mathcal{C}} X$

$\texttt{JWILemma} :: \forall \mathcal{C} : \texttt{Complete} \;.\; \forall (I, C) : \texttt{JointlyWeaklyInitial}(\mathcal{C}) \;.\; \exists \texttt{Initial}(\mathcal{C})$

$\texttt{Proof} =$

$\mathcal{D} := \texttt{fullSubcat}(\mathcal{C}, (I, C)) : \mathsf{SCAT},$

$(L, \lambda) := \lim_{d \in \mathcal{D}} d : \sum L \in \mathcal{C} \;.\; \prod i \in \mathcal{I} \;.\; L \xrightarrow{\mathcal{C}} C_i,$

$\texttt{Assume}\ X : \mathcal{C},$

$\texttt{Assume}\ (1) : \exists i \in I \;.\; X = C_i,$

$f_X := \lambda_i : L \xrightarrow{\mathcal{C}} X;$

$\leadsto (1) := I(\texttt{if}\ ) : \texttt{if}\ X = C_i\ \texttt{then}\ f_X = \lambda_i,$

$\texttt{Assume}\ (2) : \forall i \in I \;.\; X \neq C_i,$

$(i, \phi) := \eth \texttt{JointlyWeaklyInitial}(C)(X) : \sum i \in I \;.\; C_i \xrightarrow{\mathcal{C}} X,$

$f_X := \lambda_i \phi : L \xrightarrow{X};$

$\leadsto f := I\left(\prod\right) : \prod X \in \mathcal{C} \;.\; L \xrightarrow{\mathcal{C}} X,$

$\texttt{Assume}\ X, Y : \mathcal{C},$

$\texttt{Assume}\ g : X \xrightarrow{\mathcal{C}} Y,$

$(i, \phi_i) := \eth \texttt{JointlyWeaklyInitial}(C)(X) : \sum i \in I \;.\; C_i \xrightarrow{\mathcal{C}} X,$

$(j, \phi_j) := \eth \texttt{JointlyWeaklyInitial}(C)(X) : \sum j \in I \;.\; C_i \xrightarrow{\mathcal{C}} X,$

$(P, \pi_i, \pi_j) := \texttt{pullback}(\phi_j, \phi_i g) : \sum P \in \mathcal{C} \;.\; (P \xrightarrow{\mathcal{A}} C_i) \times (P \xrightarrow{\mathcal{A}} C_j),$

$(k, \phi_k) := \eth \texttt{JointlyWeaklyInitial}(C)(P) : \sum k \in I \;.\; C_k \xrightarrow{\mathcal{C}} X,$

$() := \eth f_X \eth \texttt{Cone}(L, \lambda)(\phi_k \pi_i) \eth \texttt{Pullback}(P, \pi_i, \pi_j) \eth \texttt{Cone}(L, \lambda)(\phi_k \pi_j) \eth^{-1} f_Y :$
$\quad f_X g = \lambda_i \phi_i g = \lambda_k \phi_k \pi_i \phi_i g = \lambda_k \phi_k \pi_j \phi_j = \lambda_j \phi_j = f_Y;$

$\leadsto (1) := \eth^{-1} \texttt{Cone} : \Big[(L, f) : \texttt{Cone}(\mathcal{C}, \mathrm{id})\Big],$

$(2) := \eth \texttt{Identity}(1) : f_L = \mathrm{id}_L,$

$\texttt{Assume}\ (K, \mu) : \texttt{Cone}(\mathcal{C}, \mathrm{id}),$

$(3) := \eth(K, \mu) : \mu_L : K \xrightarrow{\mathcal{C}} L,$

$(4) := \eth(K, \mu)(2)(1) : \mu_L = \lambda_K^{-1};$

$(3) := \eth^{-1} \texttt{Limit} : \Big[(L, \lambda) : \texttt{Limit}(\mathcal{C}, \mathrm{id})\Big],$

$(*) := \texttt{SelfLimitIsInitial} : \Big[L : \texttt{Initial}(\mathcal{C}, \mathrm{id})\Big];$

$\square$

$\texttt{GeneralAdjointFunctorTheorem} :: \forall \mathcal{A} \in \mathsf{LSCAT} \,.\, \forall \mathcal{B} : \mathsf{LSCAT} \,\&\, \texttt{Complete} \,.$
$\quad .\, \forall U : \texttt{SolutionSetCondition}(\mathcal{B}, \mathcal{A}) \,.\, \exists F : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) : F \dashv U$

$\texttt{Proof} \; =$

$\texttt{Assume } A : \mathcal{A},$

$(I, B, \phi, 1) := \eth\texttt{SolutionSetCondition} :$

$\quad : \sum I \in \mathsf{SET} \,.\, \sum B : I \to \mathcal{B} \,.\, \sum \phi : \prod i \in \mathcal{I} \,.\, A \xrightarrow{\mathcal{A}} UB_i \,.$

$\quad .\, \forall X \in \mathcal{B} \,.\, \forall f : A \xrightarrow{\mathcal{A}} UX \,.\, \exists i \in I : \exists \beta : B_i \xrightarrow{\mathcal{B}} B : f = \phi_i U\beta,$

$(2) := \eth^{-1}\texttt{JointlyWeaklyInitial}(1) : \Big[\big(I, (B, \phi)\big) : \texttt{JointlyWeaklyInitial}(\mathrm{Const}(A) \downarrow U)\Big],$

$(3) := \texttt{ContinuousCommaComplete}(A, U) : \Big[\mathrm{Const}(A) \downarrow U : \texttt{Complete}\Big],$

$L := \texttt{JWILemma}(2, 3)(I, (B, \phi)) : \texttt{Initial}(\mathrm{Const}(A) \downarrow U);$

$\rightsquigarrow (1) := I(\forall)I(\exists) : \forall A \in \mathcal{A} \,.\, \exists\texttt{Initial}(\mathrm{Const}(A) \downarrow U),$

$(*) := \texttt{LeftAdjointExistsByComma} : \texttt{This};$

$\square$

$\texttt{GeneratingSet} :: \prod \mathcal{C} \in \mathsf{CAT} \,.\, ?\left(\sum I \in \mathsf{SET} \,.\, \sum C : I \to \mathcal{C} \,.\, \prod X \in \mathcal{C} \,.\, \sum i \in I \,.\, C_i \xrightarrow{\mathcal{C}_i} X\right)$

$(I, C, h) : \texttt{GeneratingSet} \iff \forall X, Y \in \mathcal{C} \,.\, \forall f, g : X \xrightarrow{\mathcal{C}} Y \,.\, f \neq g \Rightarrow h_X f \neq h_X g$

$\texttt{CogeneratingSet}(\mathcal{C}) := \texttt{GeneratingSet}(\mathcal{C}^{\mathrm{op}}) : \mathsf{CAT} \to \texttt{Type};$

$\texttt{Intersection} := \prod \mathcal{C} : \mathsf{CAT} \,.\, \prod X \in \mathcal{C} \,.\, \prod J \in \mathsf{SET} \,.\, \prod (C, I) : J \to \texttt{Subobject}(X) \,.\, \texttt{Limit}(I) : \texttt{Type};$

$\texttt{Intersectable} :: ?\texttt{Complete}$

$\mathcal{C} : \texttt{Intersectable} \iff \forall X \in \mathcal{C} \,.\, \exists J \in \texttt{Set} : \exists (C, I) : J \to \texttt{Subobject}(X) \,.\, \forall (S, i) : \texttt{Subobject}(X) \,.$
$\quad .\, \exists j \in J \,.\, (C_j, I_j) \cong (S, i)$

$\texttt{IntersectableLemma} :: \forall \mathcal{C} : \texttt{Intersectable} \,\&\, \mathsf{LSCAT} \,.\, \forall (I, C, h) : \texttt{CogeneratingSet}(\mathcal{C}) \,.\, \exists\texttt{Initial}\mathcal{C}$

$\texttt{Proof} \; =$

$P := \prod_{i \in I} C_i : \mathcal{C},$

$(J, S, \iota) := \eth\texttt{Intersectable}(J) : \sum J \in \mathsf{SET} \,.\, \sum S : J \to \mathcal{C} \,.\, \prod j \in J \,.\, S \xhookrightarrow{\mathcal{C}} P,$

$T := \lim_{j \in J}(S, \iota_j) : \mathcal{C},$

$\texttt{Assume } X : \mathcal{C},$

$P' := \prod_{i \in \mathcal{I}} \prod_{f \in \mathcal{M}_{\mathcal{C}}(X, C_i)} C_i : \mathcal{C},$

$\phi := \prod_{i \in \mathcal{I}} \prod_{f \in \mathcal{M}_{\mathcal{C}}(X, C_i)} f : X \xrightarrow{\mathcal{C}} P',$

$(1) := \eth\phi\eth\texttt{CogeneratingSet}(I, C, h) : [\phi : X \xhookrightarrow{\mathcal{C}} P'],$

$\psi := \prod_{i \in \mathcal{I}} \prod_{f \in \mathcal{M}_{\mathcal{C}}(X, C_i)} \pi_i \mathrm{id}_{C_i} : P \xrightarrow{P} {}',$

$(P'', \theta_1, \theta_2) := \texttt{pullback}(\phi, \psi) : \sum P'' \in \mathcal{C} \,.\, (P'' \xrightarrow{\mathcal{C}} P') \times (P'' \xrightarrow{\mathcal{C}} P)$,

$(2) := \eth\texttt{Pullback}(P'')(1) : \left[\theta_2 : X \overset{\mathcal{C}}{\hookrightarrow} P\right]$,

$(3) := \eth^{-1}\texttt{Subobject} : \left[(P'', \theta^2) : \texttt{Subobject}(P)\right]$,

$\xi := \eth T(3) : T \xrightarrow{\mathcal{C}} P''$,

$f := \xi\theta_1 : T \xrightarrow{\mathcal{C}} X$,

$\texttt{Assume}\ g : T \xrightarrow{\mathcal{C}} X$,

$\texttt{Assume}\ (4) : f \neq g$,

$(T', \nu) := \texttt{coequalizer}(f, g) : \sum T' \in \mathcal{C} \,.\, T' \xrightarrow{\mathcal{C}} X$,

$(5) := \eth^{-1}\texttt{Cone} : \left[(T', \nu\mu) : \texttt{Cone}(S, \iota)\right]$,

$(6) := \eth T'(5)(4) : T \,!\, \texttt{Limit}(S, \iota)$,

$() := \eth T(6) : \perp$;

$\rightsquigarrow (*) := \eth^{-1}\texttt{Initial} : \left[T : \texttt{Initial}(\mathcal{C})\right]$,

$\square$

$\texttt{SpecialAdjointFunctorTHM} :: \forall \mathcal{A} : \mathsf{LSCAT} \,.\, \forall \mathcal{B} : \mathsf{LSCAT}\ \&\ \texttt{Intersectable} \,.$

$\quad .\, \forall U : \texttt{Continuous}(\mathcal{B}, \mathcal{A}) \,.\, \forall (I, C, h) : \texttt{Cogenerating}(\mathcal{B}) \,.\, \exists F : \texttt{Covariant}(\mathcal{A}, \mathcal{B}) \,.\, F \dashv U$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{CocompletenesByCogenerating} :: \forall \mathcal{C} : \mathsf{LSCAT}\ \&\ \texttt{Intersectable} \,.$

$\quad .\, \forall (I, C, h) : \texttt{Cogenerating}(\mathcal{C}) \,.\, \mathcal{C} : \texttt{Cocomplete}$

$\texttt{Proof}\ =$

$\ldots$

$\square$

4.12 Locally Presentable Categories

ContinuousIsRepresentable :: $\forall \mathcal{A} : \mathsf{LSCAT}\ \&\ \texttt{Intersectable}\ .\ \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathsf{SET}\ .$
$.\ \forall (I, C, h) : \texttt{Cogenerating}(\mathcal{C})\ .\ F : \texttt{Representable}$

Proof =

...

□

FreydRepresentabilityTheorem :: $\forall \mathcal{A} : \mathsf{LSCAT}\ \&\ \texttt{Complete}\ .\ \forall F : \texttt{SolutionSetCondition}(\mathcal{A}, \mathsf{SET})\ .$
$.\ F : \texttt{Representable}$

Proof =

...

□

KPresentable :: $\prod \kappa : \texttt{RegularCardinal}\ .\ ?(\mathsf{LSCAT}\ \&\ \texttt{Cocomplete})$

$\mathcal{C} : \texttt{KPresentable} \iff \mathcal{C} : \kappa\text{-}\texttt{Presenable} \iff \exists S :?\mathcal{C} : \forall C \in \mathcal{C}\ .\ \exists (\mathcal{I}, D) : \texttt{Diagram}(\mathcal{C}) :$
$: \operatorname{Im} D \in S\ \&\ C = \operatorname*{colim}_{i \in \mathcal{I}} D_i\ \&\ \forall s \in S\ .\ \mathcal{M}_{\mathcal{C}}(s, \cdot) : \texttt{PreservesLimits}(\kappa\text{-}\texttt{Filtered})$

LocallyPresentable :: $?(\mathsf{LSCAT}\ \&\ \texttt{Cocomplete})$

$\mathcal{C} : \texttt{LocallyPresentable} \iff \exists \kappa : \texttt{RegularCardinal}\ .\ \mathcal{C} : \kappa\text{-}\texttt{Presentable}$

Accessible :: $\prod \mathcal{A}, \mathcal{B} : \kappa\text{-}\texttt{Presentable}\ .\ ?(\mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B})$

$F : \texttt{Accessible} \iff F : \texttt{PresereveslLimits}(\kappa\text{-}\texttt{Filtered})$

FunctorBetweenLPCAdmitsRA :: $\forall \mathcal{A}, \mathcal{B} : \texttt{LocallyPresentable}\ .\ \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B}\ .$
$\exists G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A} : F \dashv G \iff F : \texttt{Cocontinuous}$

Proof =

...

□

FunctorBetweenLPCAdmitsLA :: $\forall \mathcal{A}, \mathcal{B} : \texttt{LocallYPresentable}\ .\ \forall F : \mathcal{A} \xrightarrow{\mathsf{CAT}} \mathcal{B}\ .$
$\exists G : \mathcal{B} \xrightarrow{\mathsf{CAT}} \mathcal{A}\ .\ G \dashv F \iff F : \texttt{Continuous}\ \&\ \texttt{Accessible}$

Proof =

...

□

5 Monads and Monoids

6 Kan Extension