# 1 Problem about tensor products

## Question a)

$\mathbb{Q}(\sqrt[3]{2})\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{2})$ will be a field as $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt{2})$ have no common proper subfields. Moreover,

$$\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$$

as they have the same basis over $\mathbb{Q}$.

## Question b)

$\mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ is not a field as $\mathbb{Q}(\sqrt[4]{2})$ contains $\mathbb{Q}(\sqrt{2})$ as a subfield, so it has alements

$$a = \sqrt{2} \otimes_{\mathbb{Q}} \sqrt{2} - 2 \otimes_{\mathbb{Q}} 1, b = \sqrt{2} \otimes_{\mathbb{Q}} \sqrt{2} + 2 \otimes_{\mathbb{Q}} 1$$

and

$$ab = (\sqrt{2} \otimes_{\mathbb{Q}} \sqrt{2})^2 - (2 \otimes_{\mathbb{Q}} 1)^2 = 2 \otimes_{\mathbb{Q}} 2 - 4 \otimes_{\mathbb{Q}} 1 = 4(1 \otimes_{\mathbb{Q}} 1 - 1 \otimes_{\mathbb{Q}} 1) = 0$$

and this can not happen in the field.

## Question c)

We assume that $T$ is transcendental over $\mathbb{F}_2$. Note that minimal polynomial of $\mathbb{F}_2(\sqrt{T})$ over $\mathbb{F}_2(T)$ is

$$X^2 - T = (X - \sqrt{T})^2,$$

so it is inseparable. This means that thet tensor product contain nilpotent elements, say

$$\sqrt{T} \otimes_{\mathbb{F}_2(T)} \sqrt{T} + T \otimes_{\mathbb{F}_2(T)} 1.$$

So

$$\mathbb{F}_2(\sqrt{T}) \otimes_{\mathbb{F}_2(T)} \mathbb{F}_2(\sqrt{T})$$

is not a field.

## Question d)

Here minimal polynomial is separable:

$$X^3 + T = (X + \sqrt[3]{T})(X + a\sqrt[3]{T})(X + a\sqrt[3]{T})(X + a\sqrt[3]{T} + \sqrt[3]{T}),$$

where $a$ is a primitive element of $\mathbb{F}_4$ over $\mathbb{F}_2$. However, the tensor product still not a field. Consider elements

$$A = \sqrt[3]{T^2} \otimes_{\mathbb{F}_4(T)} \sqrt[3]{T} + T \otimes_{\mathbb{F}_4(T)} 1$$

$$B = T(\sqrt[3]{T} \otimes_{\mathbb{F}_4(T)} \sqrt[3]{T^2}) + T(\sqrt[3]{T^2} \otimes_{\mathbb{F}_4(T)} \sqrt[3]{T}) + T^2 \otimes_{\mathbb{F}_4(T)} 1.$$

Then

$$AB = T^2 \otimes_{\mathbb{F}_4(T)} T + T^3 \otimes_{\mathbb{F}_4(T)} 1 = 0.$$