Algebras

Uncultured Tramp

December 15, 2019

Contents

1 Associative Algebras over Commutative Rings

1.1 Categories Of Algebras

$$\texttt{Algebra} := \prod R \in \mathsf{ANN} \;.\; \sum X \in R\text{-}\mathsf{MOD} \;.\; X \otimes X \to X : \mathsf{ANN} \to \texttt{Type};$$

$\texttt{multiplication} :: \prod (A, \odot) : \texttt{Algebra}(R) \;.\; A \otimes A \to A$

$\texttt{multiplication}\,(A) = (\cdot_A) := (\odot)$

$\texttt{AlgebraModule} :: \prod (A, \odot) : \texttt{Algebra}(R) \;.\; L\text{-}\mathsf{MOD}$

$\texttt{RingGroup}\,(A) = A := A$

$\texttt{UnitalAlgebra} :: \prod R \in \mathsf{ANN} ? \texttt{Algebra}(R)$

$A : \texttt{UnitalAlgebra} \iff \exists e \in A : \texttt{Identity}(\odot)$

$\texttt{identity} :: \prod A : \texttt{UnitalAlgebra} \;.\; A$

$\texttt{identity}\,(R) = 1_R := Ɑ\texttt{UnitalAlgebra}(A)$

$\texttt{CommutativeAlgebra} :: ?\texttt{Algebra}(R)$

$A : \texttt{CommutativeAlgebra} \iff (\cdot_A) : \texttt{Commutative}(A)$

$\texttt{DivisionAlgebra} :: ?\texttt{Algebra}(R)$

$(R, +, \cdot) : \texttt{DivisioniAlgebra} \iff (\cdot) : \texttt{Invertible}(A \setminus 0)$

$\texttt{AlgebraHomo} :: \prod A, B : \texttt{Algebra}(R) \;.\; ?(A \xrightarrow{R\text{-}\mathsf{MOD}} B)$

$f : \texttt{AlgebraHomo} \iff \forall x, y \in A \;.\; f[x, y] = \Big[f(x), f(y)\Big]$

$\texttt{UnitalHomo} :: \prod A, B : \texttt{UnitalAlgebra}(R) \;.\; ?\texttt{AlgebraHomo}(A, B)$

$f : \texttt{UnitalHomo} \iff f(e) = e$

$\texttt{IdIsHomo} :: \forall A : \texttt{Algebra}(R) \;.\; \mathrm{id}_A : \texttt{RingHomo}$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b : A,$

$(*) := Ɑ\,\mathrm{id} : \mathrm{id}[a, b] = [a, b] = [\mathrm{id}(a), \mathrm{id}(b)];$

$\square$

$\texttt{IdIsUnital} :: \forall A : \texttt{UnitalAlgebra}(R) \;.\; \mathrm{id}_A : \texttt{UnitalHomo}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{structuralHomomrphism} :: \forall A : \texttt{UnitalAlgebra}(R) \;.\; R \xrightarrow{R\text{-}\mathsf{MOD}} A$

$\texttt{structuralHomomrphis}\,(\alpha) = \epsilon(\alpha) := \alpha e$

$\texttt{AlgebraHomoCompos} :: \forall A, B, C : \texttt{Algebra}(R) \;.\; \forall f : \texttt{AlgebraHomo}(A, B) \;.\; \forall g : \texttt{AlgebraHomo}(B, C) \;.$

$\quad .\; g \circ f : \texttt{UnitalAlgebraHomo}(A, C)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UnitalAlgebraHomoCompos} :: \forall A, B, C : \texttt{UnitalAlgebra}(R) \;.\; \forall f : \texttt{UnitalAlgebraHomo}(A, B) \;.$

$\quad .\; \forall g : \texttt{UnitalAlgebraHomo}(B, C) \;.\; g \circ f : \texttt{UnitalAlgebraHomo}(A, C)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{AlgebraCat} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{AlgebraCat}\,(R) = R\text{-}\mathsf{LG} := \Big(\texttt{Algebra}(R), \texttt{AlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{CommAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{CommAlgebraCat}\,(R) = R\text{-}\mathsf{CLG} := \Big(\texttt{CommutativeAlgebra}(R), \texttt{AlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{assAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{assAlgebraCat}\,(R) = R\text{-}\mathsf{ALG} := \Big(\texttt{AssociativeAlgebra}(R), \texttt{AlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{commAssAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{commAssAlgebraCat}\,(R) = R\text{-}\mathsf{CALG} :=$

$\quad := \Big(\texttt{AssociativeAlgebra} \,\&\, \texttt{CommutativeAlgebra}(R), \texttt{AlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{unitalAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{unitalAlgebraCat}\,(R) = R\text{-}\mathsf{LGE} := \Big(\texttt{UnitalAlgebra}(R), \texttt{UnitalAlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{unitalCommAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{unitalCommAlgebraCat}\,(R) = R\text{-}\mathsf{CLGE} :=$

$\quad := \Big(\texttt{CommutativeAlgebra} \,\&\, \texttt{UnitalAlgebra}(R), \texttt{UnitalAlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{unitalAssAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{unitalAssAlgebraCat}\,(R) = R\text{-}\mathsf{ALGE} :=$

$\quad := \Big(\texttt{UnitalAlgebra} \,\&\, \texttt{AssociativeAlgebra}(R), \texttt{UnitalAlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{unitalAssCommAlgebraCat} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{unitalAssCommAlgebraCat}\,(R) = R\text{-}\mathsf{CALGE} :=$

$\quad := \Big(\texttt{CommutativeAlgebra} \,\&\, \texttt{UnitalAlgebra} \,\&\, \texttt{AssociativeAlgebra}(R), \texttt{UnitalAlgebraHomo}, \circ, \mathrm{id}\Big)$

$\texttt{Subalgebra} :: \prod R \in \mathsf{RING} \,.\, \prod A \in R\text{-}\mathsf{LG}??A$

$B : \texttt{Subalgebra} \iff B \subset_{R\text{-}\mathsf{ALG}} A \iff \left((B, \odot_{A|B}) : \texttt{Algebra}(R)\right)$

$\texttt{UnitalSubalgebra} :: \prod R \in \mathsf{RING} \,.\, \prod A \in R\text{-}\mathsf{LG}??A$

$B : \texttt{UnitalSubalgebra} \iff B \subset_{R\text{-}\mathsf{ALGE}} A \iff \left((B, \odot_{A|B}) : \texttt{UnitalAlgebra}(R)\right)$

$\texttt{TrivialRing} :: \mathsf{ANN}$

$\texttt{TrivialRing}\,() = \star := \Big(\{\star\}, (\star, \star) \mapsto \star, (\star, \star) \mapsto \star\Big)$

$\texttt{MultZero} :: \forall A \in \texttt{Algebra}(R) \,.\, \forall a \in A \,.\, [0, a] = [a, 0] = 0$

$\texttt{Proof} \;=$

$[0] := \mathcal{I}\texttt{Algebra}(R) : \Big(R \oplus A, \Lambda(\alpha, a), (\beta, b) \in (R \oplus A) \otimes (R \oplus A) \,.\, \alpha\beta + \beta a + \alpha b + [a, b]\Big) : R\text{-}\mathsf{LGE},$

$[1] := \mathcal{I}\texttt{Identity}(1)\mathcal{I}\texttt{Distrivutive}(R, +, \cdot)\mathcal{I}\texttt{Identity}(0)\mathcal{I}\texttt{Identity}(1) : [0, a] + a = [0 + 1, a] = [1, a] = a,$

$[2] := \mathcal{I}\texttt{Identity}(1)\mathcal{I}\texttt{Distrivutive}(R, +, \cdot)\mathcal{I}\texttt{Identity}(0)\mathcal{I}\texttt{Identity}(1) : [a, 0] + a = [a, 0 + 1] = [a, 1] = a,$

$(*) := \texttt{IdentityIsUnique}(1)(2) : [a, 0] = 0 = [0, a];$

$\square$

$\texttt{MultNeg} :: \forall R \in \mathsf{RING} \,.\, \forall A \in R\text{-}\mathsf{LGE} \,.\, \forall a \in A \,.\, [-e, a] = -a = [a, -e]$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}\texttt{Identity}\mathcal{I}\texttt{Distributive}(R)\mathcal{I}\texttt{Inverse}(1) : a + [-e, a] = [e - e, a] = [0, a] = 0,$

$[2] := \mathcal{I}\texttt{Identity}\mathcal{I}\texttt{Distributive}(R)\mathcal{I}\texttt{Inverse}(1) : a + [a, e] = [a, e - e] = 0,$

$(*) := \texttt{InverseIsUnique}(1)(2) : [-1, a] = -a = [a, -1];$

$\square$

$\texttt{SubalgebraImage} :: \forall R \in \mathsf{RING} \,.\, \forall A, B \in R\text{-}\mathsf{LG} \,.\, \forall S : \texttt{Subalgebra}(A) \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, f(S) \subset_{R\text{-}\mathsf{LG}} B$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SubringPreimage} :: \forall R \in \mathsf{RING} \,.\, \forall A, B \in \mathsf{RING} \,.\, \forall S : \texttt{Subalgebra}(B) \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, f^{-1}(S) \subset_{\mathsf{RING}} A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{AlgebraOfFunctions} :: \forall X \in \mathsf{SET} \,.\, \forall R \in \mathsf{ANN} \,.\, \Big(\mathcal{M}_{\mathsf{SET}}(X, R), +, \cdot\Big) \in R\text{-}\mathsf{ALG}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{productAlgebra} :: \prod I \in \mathsf{SET} \;.\; \prod R \in \mathsf{ANN} \;.\; (I \to R\text{-}\mathsf{LG}) \to R\text{-}\mathsf{LG}$

$\texttt{productAlgebra}\,(A) = \prod_{i \in I} A_i := \left(\prod i \in I \;.\; A_i, a, b \mapsto \Lambda i \in I \;.\; a_i b_i \right)$

$\texttt{projection} :: \prod I \in \mathsf{SET} \;.\; \prod R \in \mathsf{ANN} \;.\; \prod R : I \to \mathsf{RING} \;.\; \prod i \in I \;.\; \prod_{i \in I} R_i \xrightarrow{R\text{-}\mathsf{LG}} R_i$

$\texttt{projection}\,(a) = \pi_i(a) := a_i$

$\texttt{rightMultiplication} :: \prod R \in \mathsf{ANN} \;.\; \prod R \in R\text{-}\mathsf{LG} \;.\; A \xrightarrow{R\text{-}\mathsf{LG}} \mathrm{End}_{R\text{-}\mathsf{LG}}(A)$

$\texttt{rightMultiplication}\,(a) = \rho_a := \Lambda b \in R \;.\; ab$

$\texttt{leftMultiplication} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{LG} \;.\; A \xrightarrow{R\text{-}\mathsf{LG}} \mathrm{End}_{R\text{-}\mathsf{LG}}(A)$

$\texttt{leftMultiplication}\,(a) = \lambda_a := \Lambda b \in A \;.\; ba$

$\texttt{AssociativeAlgebrasAreRings} :: \forall R \in \mathsf{ANN} \;.\; \forall A \in R\text{-}\mathsf{ALGE} \;.\; \left(A, [\cdot, \cdot]\right) \in \mathsf{RING}$

$\texttt{Proof} =$

...

$\square$

$\texttt{RingsAreAssociativeAlgebras} :: \mathsf{RING} \cong_{\mathsf{CAT}} \mathbb{Z}\text{-}\mathsf{ALGE}$

$\texttt{Proof} =$

...

$\square$

$\texttt{LeftUnit} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{LGE} \;.\; ?A$

$u : \texttt{LeftUnit} \iff \exists a \in A : au = e$

$\texttt{RightUnit} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{LGE} \;.\; ?A$

$u : \texttt{RightUnit} \iff \exists a \in A : ua = e$

$\texttt{LeftZeroDivisor} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{LG} \;.\; ?A$

$x : \texttt{LeftZeroDivizor} \iff \exists a \in A \;.\; xa = 0 \;\&\; x \neq 0$

$\texttt{RightZeroDivisor} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{LG} \;.\; ?A$

$x : \texttt{RightZeroDivizor} \iff \exists a \in R \;.\; ax = 0 \;\&\; x \neq 0$

$\texttt{ZeroDivisor} := \Lambda R \in \mathsf{RING} \;.\; \Lambda A \in R\text{-}\mathsf{LG} \;.\; \texttt{RightZeroDivisor}|\texttt{LeftZeroDivisor}(A) : \prod R \in \mathsf{RING} \;.\; R\text{-}\mathsf{LG} \to$

$\texttt{Regular} := \Lambda R \in \mathsf{RING} \;.\; \Lambda A \in R\text{-}\mathsf{LG} \;.\; !\texttt{ZeroDivisor}(A) : \prod R \in \mathsf{RING} \;.\; R\text{-}\mathsf{LG} \to \texttt{Type};$

$\texttt{Unit} := \Lambda R \in \mathsf{RING} \;.\; \Lambda A \in R\text{-}\mathsf{LGE} \;.\; \texttt{LeftUnit} \;\&\; \texttt{RightUnit}(A) : \prod R \in \mathsf{RING} \;.\; R\text{-}\mathsf{LGE} \to \texttt{Type};$

$\texttt{UnitsAreRegular} :: \forall R \in \mathsf{RING} \,.\, \forall A \in R\text{-}\mathsf{ALGE} \,.\, \forall u : \texttt{Unit}(A) \,.\, u : \texttt{Regular}(A)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a : R,$

$\texttt{Assume}\ (1) : [u, a] = 0,$

$\texttt{Assume}\ (2) : a \neq 0,$

$(3, v) := \mathcal{I}\texttt{LeftUnit}(u) : \sum v \in A \,.\, [v, u] = e,$

$(4) := \mathcal{I}\texttt{Identity}(1)(a)(3)(vua)(1)\texttt{ZeroMult}(v) : a = [e, a] = \Big[[v, u], a\Big] = \Big[v, [u, a]\Big] = 0,$

$() := (2)(4) : \perp;$

$\rightsquigarrow (1) := \mathcal{I}^{-1}\texttt{RightZeroDivisor}E(\perp) : [u \,!\, \texttt{RightZeroDivisor}(R)],$

$\texttt{Assume}\ a : R,$

$\texttt{Assume}\ (2) : [a, u] = 0,$

$\texttt{Assume}\ (3) : a \neq 0,$

$(4, v) := \mathcal{I}\texttt{LeftUnit}(u) : \sum v \in R \,.\, [u, v] = e,$

$(4) := \mathcal{I}\texttt{Identity}(1)(a)(3)(auv)(1)\texttt{ZeroMult}(v) : a = [a, e] = \Big[a, [u, v]\Big] = [0, v] = 0,$

$() := (2)(4) : \perp;$

$\rightsquigarrow (2) := \mathcal{I}^{-1}\texttt{LeftZeroDivisor}E(\perp) : [u \,!\, \texttt{LeftZeroDivisor}(R)],$

$(3) := \mathcal{I}^{-1}\texttt{Regualar}(1)(2) : [u : \texttt{Regular}];$

$\square$

$\texttt{groupOfUnits} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{ALGE} \to \mathsf{GRP}$

$\texttt{groupOfUnits}\,(R) = R^* := (\texttt{Unit}(R), \cdot_R)$

$\texttt{Nillpotent} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?A$

$a : \texttt{Nillpotent} \iff \exists n \in \mathbb{N} : a^n = 0$

$\texttt{Unipotent} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LGE}?A$

$a : \texttt{Unipotent} \iff a - e : \texttt{Nillpotent}(R)$

$\texttt{Idempotent} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?A$

$a : \texttt{Idempotent} \iff a^2 = a$

$\texttt{Involution} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LGE} \,.\, ?A$

$a : \texttt{Involution} \iff a^2 = e$

$\texttt{NillpotentProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALG} \,.\, \forall a : \texttt{Nillpotent}(A) \,.$
$\quad .\, \forall b : \texttt{Commutes}(A, \cdot_R)(a) \,.\, [a, b] : \texttt{Nillpotent}(R)$

$\texttt{Proof} \ =$

$(1, n) := \mathcal{I}\texttt{Nillpotent}(a) : \sum n \in \mathbb{N} \,.\, a^n = 0,$

$(2) := \mathcal{I}\texttt{Commutes}(b)(ab)^n(1)\texttt{ZeroMult}(R)(b^n) : (ab)^n = a^n b^n = 0b^n = 0,$

$() := \mathcal{I}^{-1}\texttt{Nillpotent}(2) : [ab : \texttt{NillPotent}(R)];$

$\square$

$\texttt{NillpotentSum} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALG} \,.\, \forall a, b : \texttt{Nillpotent}(A) \,.\, \texttt{Commutes}(A, \cdot_A)(a, b) \Rightarrow a + b : \texttt{Nillpote}$

$\texttt{Proof} \ =$

$(1, n) := \mathcal{A}\texttt{Nillpotent}(a) : \sum n \in \mathbb{N} \,.\, a^n = 0,$

$(2, m) := \mathcal{A}\texttt{Nillpotent}(b) : \sum m \in \mathbb{N} \,.\, b^m = 0,$

$$(3) := \texttt{BinomialSum}(b, m, n+m)(1)(2) : (a+b)^{n+m} = \sum_{i=1}^{n+m} C^i_{n+m} a^i b^{n+m-i} = 0,$$

$() := \mathcal{A}^{-1}\texttt{Nillpotent}(3) : [a + b : \texttt{NillPotent}(R)];$

$\square$

$\texttt{UnitDiff} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALGE} \,.\, \forall a \in A^* \,.\, \forall b : \texttt{Nillpotent}(A) \,.\, \texttt{Commutes}(A, \cdot_R)(a, b) \Rightarrow a - b \in A^*$

$\texttt{Proof} \ =$

$(n, 1) := \mathcal{A}\texttt{Nillpotent}(b) : \sum n \in \mathbb{N} \,.\, b^n = 0,$

$$(2) := \texttt{SumOfPowers}(a, b, n)(1)\mathcal{A}\texttt{Inverse} : (a-b)\left(\sum_{i=0}^{n-1} a^i b^{n-1-i}\right) a^{-n} = (a^n - b^n)a^{-n} = a^n a^{-n} = 1,$$

$(*) := \mathcal{A}^{-1}A^*(2) : a - b \in R^*;$

$\square$

$\texttt{LeftIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?\texttt{Subgroup}(A)$

$I : \texttt{LeftIdeal} \iff \forall a \in I \,.\, \forall b \in A \,.\, ba \in I$

$\texttt{RightIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?\texttt{Subgroup}(A)$

$I : \texttt{RightIdeal} \iff \forall b \in I \,.\, \forall b \in A \,.\, ab \in I$

$\texttt{TwoSidedIdeal} := \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, \texttt{LeftIdeal}(R) \;\&\; \texttt{RightIdeal}(R) : \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{LG} \to \texttt{Ty}$

$\texttt{CommutativeIdeal} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{CLG} \,.\, \forall I : \texttt{LeftIdeal}(R) \,.\, I : \texttt{TwoSidedIdeal}(R)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{Ideal} := \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{CLG} \,.\, \texttt{LeftIdeal}(R) : \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{CLG} \to \texttt{Type};$

$\texttt{quatMult} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \prod I : \texttt{TwoSidedIdeal} \,.\, \frac{R}{I} \to \frac{R}{I} \to \frac{R}{I}$

$\texttt{quatMult}\,([a], [b]) = [a][b] := [ab]$

$\texttt{Assume}\ x, y : I,$

$(1) := \mathcal{A}\texttt{RightIdeal}(a, y) : ay \in I,$

$(2) := \mathcal{A}\texttt{LeftIdeal}(b, x) : xb \in I,$

$(3) := \mathcal{A}\texttt{RightIdeal}(x, y) : xy \in I,$

$(*) := \ldots : [a + x][b + y] = [ab + xb + ay + xy] = [ab];$

$\square$

$\texttt{quotientAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \texttt{TwoSidedIdeal} \to \mathsf{GRP}$

$\texttt{quotientAlgebra}\,(I) = \dfrac{R}{I} := \left(\dfrac{R}{I}, +, \texttt{quatMult}\right)$

$\texttt{LeftIdealPreimage} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{LG} \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, \forall I : \texttt{LeftIdeal}(B) \,.\, f^{-1}(I) : \texttt{LeftIdeal}($

Proof $=$

. . .

□

$\texttt{RightIdealPreimage} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{LG} \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, \forall I : \texttt{RightIdeal}(B) \,.\, f^{-1}(I) : \texttt{RightIdea}$

Proof $=$

. . .

□

$\texttt{TwoSidedIdealPreimage} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{LG} \,.\, \forall f : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, \forall I : \texttt{TwoSidedIdeal}(B) \,.$
$\quad .\, f^{-1}(I) : \texttt{TwoSidedIdeal}(A)$

Proof $=$

. . .

□

$\texttt{IdealPreimage} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{CLG} \,.\, \forall f : A \xrightarrow{\mathsf{RING}} B \,.\, \forall I : \texttt{Ideal}(B) \,.\, f^{-1}(I) : \texttt{Ideal}(A)$

Proof $=$

. . .

□

$\texttt{LeftIdealIntersection} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{LeftIdeal}(A) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{LeftIdea}$

Proof $=$

. . .

□

$\texttt{RightIdealIntersection} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{RightIdeal}(R) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{RightI}$

Proof $=$

. . .

□

$\texttt{TwoSidedtIdealIntersection} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{TwoSidedIdeal}(A) \,.$
$\quad .\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{TwoSidedIdeal}(A)$

Proof $=$

. . .

□

$\texttt{IdealIntersection} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{CLG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{Ideal}(A) \,.\, \bigcap_{\alpha \in \mathcal{A}} I_\alpha : \texttt{Ideal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfLeftIdeals} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{CLG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{LeftIdeal}(A) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{LeftIdeal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfRightIdeals} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{RightIdeal}(A) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{RightIdeal}(A$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfTwoSidedIdeals} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{TwoSidedIdeal}(A) \,.$
$\sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{TwoSidedIdeal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{SumOfIdeals} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{CLG} \,.\, \forall \mathcal{A} \in \mathsf{SET} \,.\, \forall I : \mathcal{A} \to \texttt{Ideal}(A) \,.\, \sum_{\alpha \in \mathcal{A}} I_\alpha : \texttt{Ideal}(A)$

$\texttt{Proof} =$

...

$\square$

$$\texttt{compositeIdeal} :: \prod R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \texttt{LeftIdeal}(A) \times \texttt{RightIdeal}(A) \to \texttt{TwoSidedIdeal}(A)$$

$$\texttt{compositeIdeal}\,(I, J) = IJ := \left\{ \sum_{\alpha=1}^{n} a_\alpha b_\alpha | n \in \mathbb{N}, a : n \to I, b : n \to J \right\}$$

$$\texttt{compositeIdeal2} :: \prod R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{CALG} \,.\, \prod n \in \mathbb{N} \,.\, n \to \texttt{Ideal}(A) \to \texttt{Ideal}(A)$$

$$\texttt{compositeIdeal2}\,(I) = \prod_{\alpha=1}^{n} I_\alpha := \left\{ \sum_{\beta=1}^{m} \prod_{\alpha=1}^{n} a_{\alpha,\beta} | m \in \mathbb{N}, a : \prod \alpha \in n \,.\, m \to I_\alpha \right\}$$

$\texttt{genLeftIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?A \to \texttt{LeftIdeal}(A)$

$\texttt{genLeftIdeal}\,(S) := \bigcap\{I : \texttt{LeftIdeal}(A) : S \subset A\}$

$\texttt{genRightIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?A \to \texttt{RightIdeal}(A)$

$\texttt{genRightIdeal}\,(S) := \bigcap\{I : \texttt{RightIdeal}(A) : S \subset A\}$

$\texttt{genTwoSidedIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{LG} \,.\, ?A \to \texttt{TwoSidedIdeal}(A)$

$\texttt{genTwoSidedIdeal}\,(S) := \bigcap\{I : \texttt{TwoSidedIdeal}(A) : S \subset A\}$

$\texttt{genIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{CLG} \,.\, ?A \to \texttt{Ideal}(A)$

$\texttt{genIdeal}\,(S) := \bigcap\{I : \texttt{Ideal}(A) : S \subset A\}$

$\texttt{kernelIdeal} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{LG} \,.\, \forall \varphi : A \xrightarrow{R\text{-}\mathsf{LG}} B \,.\, \ker \varphi : \texttt{TwoSidedIdeal}(A)$

Proof $=$

...

$\square$

$\texttt{IdealProjectionIsAlgebraHomo} :: \forall R \in \mathsf{RING} \,.\, \forall I : \texttt{TwoSidedIdeal}(R) \,.\, \pi_I : R \xrightarrow{\mathsf{RING}} \dfrac{R}{I}$

Proof $=$

$(1) := \mathcal{I}\pi_I(1) : \pi_I(1) = [1],$

Assume $a, b : R,$

$() := \mathcal{I}\pi_I(ab)\mathcal{I}\,\texttt{quotMult}([a],[b])\mathcal{I}^{-1}\pi_I(a)\mathcal{I}^{-1}\pi_I : \pi_I(ab) = [ab] = [a][b] = \pi_I(a)\pi_I(b);$

$\square$

$\texttt{EveryIdealIsRHKernel} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{LG} \,.\, \forall I : \texttt{TwoSidedIdeal}(R) \,.\, I = \ker \pi_I$

Proof $=$

...

$\square$

$\texttt{freeCAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Covariant}(\mathsf{SET}, R\text{-}\mathsf{CALGE})$

$\texttt{freeCAlgebra}\,(X) = F_{R\text{-}\mathsf{CALGE}}(X) := R\left[\mathbb{Z}_+^X\right]$

$\texttt{freeCAlgebra}\,(X, Y, f) = F_{R\text{-}\mathsf{CALGE},X,Y}(f) := \Lambda \sum_{p:X\to\mathbb{Z}_+} \alpha_p \prod_{x\in X} x^{p_x} \,.\, \sum_{p:X\to\mathbb{Z}_+} \alpha_p \prod_{x\in X} f(x)^{p_x}$

$\texttt{FinitelyGeneratedCommutativeAlgebra} :: \prod R \in \mathsf{ANN} \,.\, ?R\text{-}\mathsf{CALGE}$

$A : \texttt{FinitelyGeneratedCommutativeAlgebra} \iff \exists X \in \mathsf{SET} \,.\, \exists I : \texttt{Ideal}\Big(F_{R\text{-}\mathsf{CALGE}}(X)\Big) \,.\, A = \dfrac{F_{R\text{-}\mathsf{CALGE}}(X)}{I}$

$\texttt{freeAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Covariant}(\mathsf{SET}, R\text{-}\mathsf{ALGE})$

$\texttt{freeAlgebra}\,(X) = F_{R\text{-}\mathsf{ALGE}}(X) := R^{\oplus \texttt{String}(X)}$

$\texttt{freeAlgebra}\,(X, Y, f) = F_{R\text{-}\mathsf{ALGE},X,Y}(f) := \Lambda \sum_{x\in\texttt{String}(X)\to\mathbb{Z}_+} \alpha_x \prod_{i=1}^{|x|} x_i \,.\, \sum_{x\in\texttt{String}(X)} \alpha_x \prod_{i=1}^{|x|} f(x_i)$

1.2 Tensor Product Of Algebras

$$\texttt{tensorProductOfAlgebras} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, n \to R\text{-}\mathsf{ALG} \to R\text{-}\mathsf{ALG}$$

$$\texttt{tensorProductOfAlgebras}\,(A) = \bigotimes_{i=1}^{n} A_i :=$$

$$:= \left(\bigotimes_{i=1}^{n} A_i, \texttt{tensorize}\Lambda \sum_{i=1}^{m} \bigotimes_{j=1}^{n} a_{i,j}, \sum_{i=1}^{m'} \bigotimes_{j=1}^{n} b_{i,j} \,.\, \sum_{i=1}^{m} \sum^{m'} \bigotimes_{j=1}^{n} a_{i,j} b_{i',j} \right)$$

$\texttt{TensorProductOfUnitalAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{ALGE} \,.\, \bigotimes_{i=1}^{n} A_i \in R\text{-}\mathsf{ALGE}$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{TensorProductOfCommutativeAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{CALG} \,.\, \bigotimes_{i=1}^{n} A_i \in R\text{-}\mathsf{CALG}$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{AssociativeTensorProductOfAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, C \in R\text{-}\mathsf{ALG} \,.$
$(A \otimes B) \otimes C \cong_{R\text{-}\mathsf{ALG}} A \otimes (B \otimes C)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{TensorProductOfAlgebrasPermutation} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{ALG} \,.\, \forall \sigma \in S_n \,.$
$.\ \bigotimes_{i=1}^{n} A_i \cong_{R\text{-}\mathsf{ALG}} \bigotimes_{i=1}^{n} A_{\sigma(i)}$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{TrivialTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALG} \,.\, R \otimes A \cong A$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{TensorProductOfFractionAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall \Sigma_1, \Sigma_2 \in \texttt{MultiplicativeSet}(R) \,.$
$\quad .\, \Sigma_1^{-1} R \otimes \Sigma_2^{-1} R \cong_{R\text{-}\mathsf{ALGE}} (\Sigma_1 \Sigma_2)^{-1} R$

$\texttt{Proof} \;=$

$\varphi := \texttt{tensorize}\left(\Lambda \frac{a}{\sigma} \in \Sigma_1^{-1} \,.\, \Lambda \frac{b}{\sigma'} \,.\, \frac{ab}{\sigma\sigma'}\right) : \Sigma_1^{-1} R \otimes \Sigma_2^{-1} R \xrightarrow{R\text{-}\mathsf{MOD}} (\Sigma_1 \Sigma_2)^{-1} R,$

$[1] := \mathcal{I}\varphi : \varphi(1 \otimes 1) = 1,$

$[2] := \mathcal{I}\mathsf{ANN}(R)\mathcal{O}\varphi[2] : \left(\varphi : \Sigma_1^{-1} R \otimes \Sigma_2^{-1} \xrightarrow{R\text{-}\mathsf{ALGE}} (\Sigma_1 \Sigma_2)^{-1} R\right),$

$\texttt{Assume}\ \frac{a}{\sigma} : (\Sigma_1 \Sigma_2)^{-1} R,$

$(\alpha, \beta, [1]) := \mathcal{I}\Sigma_1\Sigma_2(\sigma) : \sum \alpha \in \Sigma_1 \sum \beta \in \Sigma_2 \,.\, \sigma = \alpha\beta,$

$[\ldots *] := I(\varphi) : \varphi\left(a\frac{1}{\alpha} \otimes \frac{1}{\beta}\right) = \frac{a}{\alpha\beta} = \frac{a}{\sigma};$

$\rightsquigarrow [3] := \mathcal{I}^{-1}\texttt{Surjective} : \left(\varphi : \Sigma_1^{-1} R \otimes \Sigma_2^{-1} \twoheadrightarrow (\Sigma_1 \Sigma_2)^{-1} R\right),$

$\texttt{Assume}\ t : \Sigma_1^{-1} R \otimes \Sigma_2^{-1} R,$

$\texttt{Assume}\ [4] : \varphi(t) = 0,$

$(r, \alpha, \beta, [5]) := \mathcal{I}t : \sum r \in R \,.\, \sum \alpha \in \Sigma_1 \,.\, \sum \beta \in \sigma_2 \,.\, t = r\frac{1}{\alpha} \otimes \frac{1}{\beta},$

$[6] := [4][5]\mathcal{O}\varphi : 0 = \varphi(t) = \frac{r}{\alpha\beta},$

$[7] := \mathcal{I}\texttt{MultiplicativeSet}(\Sigma_1, \Sigma_2)[6] : r = 0,$

$[t.4.*] := [5][7] : t = 0;$

$\rightsquigarrow [4] := \texttt{ZeroKernelTHM}[3] : \left(\varphi : \Sigma_1^{-1} R \otimes \Sigma_2^{-1} \xleftrightarrow{R\text{-}\mathsf{ALGE}} (\Sigma_1 \Sigma_2)^{-1} R\right),$

$[5] := \mathcal{I}^{-1}\texttt{Isomotphic}[4] : \texttt{This};$

$\square$

1.3 Graded Algebras

$\texttt{GradedAlgebra} :: \prod R \in \mathsf{ANN} \,.\, ?\sum \Delta : \texttt{CommutativeMonoid} \,.\, \sum A \in R\text{-}\mathsf{ALG} \,.\, \Delta \to \texttt{Submodule}(R, A)$

$(\Delta, A, H) : \texttt{GradedAlgebra} \iff A = \bigoplus_{\delta \in \Delta} H_i \;\&\; \forall \alpha, \beta \in \Delta \,.\, \forall a \in H_\alpha \,.\, \forall b \in H_\beta \,.\, a + b \in H_{\alpha+\beta}$

$\texttt{Homogeneous} :: \prod R \in \mathsf{ANN} \,.\, \prod (\Delta, A, H) : \texttt{GradedAlgebra}(A) \,.\, ?A$

$a : \texttt{Homogeneous} \iff \exists \delta \in \Delta : a \in H_\delta$

$\texttt{homogeneousElement} :: \prod R \in \mathsf{ANN} \,.\, \prod (\Delta, A, H) : \texttt{GradedAlgebra}(R) \,.\, A \to \Delta \to A$

$\texttt{homogeneousElement}\,(a, \delta) = a_\delta := b_\delta$

where

$$(b, [\ldots]) = \textit{C\!\!I}\,\texttt{GradedAlgebra}(\Delta, A, H) \,.\, \sum b : \prod_{\delta \in \Delta} H_\delta \,.\, a = \sum_{\delta \in \Delta} b_\delta$$

$\texttt{ZerothHomogeneousSubalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall (\Delta, A, H) : \texttt{GradedAlgebra}(R) \,.\, H_0 \subset_{R\text{-}\mathsf{ALG}} A$

Proof $=$

$\ldots$

$\square$

$\texttt{ZerothHomogeneousUnitalSubalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall (\Delta, A, H) : \texttt{GradedAlgebra}(R) \,.\, A \in R\text{-}\mathsf{ALGE} \Rightarrow H_0 \subset$

Proof $=$

$\ldots$

$\square$

$\texttt{PolynomialGradedAlgebra} :: ?\texttt{GradedAlgebra}(R)$

$(\mathbb{Z}, A, H) : \texttt{PolynomialGradedAlgebra} \iff A = \langle H_1 \rangle_{R\text{-}\mathsf{ALGE}}$

$\texttt{FreeCoefficientLemma} :: \forall R \in \mathsf{ANN} \,.\, \forall (\mathbb{Z}, A, H) : \texttt{PolynomialGradedAlgebra}(R) \,.\, \forall [0] : A \in R\text{-}\mathsf{ALGE} \,.$
$\quad .\, H_0 = \langle e \rangle$

Proof $=$

Assume $a : H_0$,

$(b, [1]) := \textit{C\!\!I}\,\texttt{GradedAlgebra}(\mathbb{Z}, A, H)(a) : \sum b : \prod_{\delta \in \mathbb{Z}} H_\delta \,.\, a = \sum_{\delta \in \Delta} b_\delta,$

$(c, [2]) := [00](b) : \sum c : \prod_{n \in \mathbb{N}} n \to H_1 \,.\, b_0 \in Re \;\&\; \forall n \in \mathbb{N} \,.\, b_n = \prod_{i=1}^{n} c_{n,i},$

$[3] := \textit{C\!\!I}\, a[1] : \forall n \in \mathbb{N} \,.\, b_n = 0,$

$[a.*] := [00][1][3][2] : a \in Re;$

$\rightsquigarrow [*] := \texttt{ZerothHomogeneousUnitalSubalgebra} : H_0 = \langle e \rangle,$

$\square$

$\texttt{HomogeneousIdeal} :: \prod R \in \mathsf{ANN} \,.\, \prod (\Delta, A, H) : \texttt{GradedSubalgebra}(k) \,.\, ?\texttt{TwoSidedIdeal}(A)$

$I : \texttt{HomogeneousIdeal} \iff \forall a \in I \,.\, \forall \delta \in \Delta \,.\, a_\delta \in I$

$\texttt{HomogeneousIdealLemma} :: \forall R \in \mathsf{ANN} \,.\, \forall (\Delta, A, H) : \texttt{GradedSubalgebra}(k) \,.\, \forall I : \texttt{TwoSidedIdeal}(A) \,.$
$I : \texttt{HomogeneousIdeal}(A) \iff \exists X :? \texttt{Homogeneous}(A) : I = \langle X \rangle$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{Assume}\ R : \mathsf{ANN},$

$\texttt{Assume}\ (\Delta, A, H) : \texttt{GradedAlgebra}(R),$

$\texttt{Assume}\ [0] : A \in R\text{-}\mathsf{ALGE},$

$\texttt{HomogeneousIdealAsGradedModule} :: \forall I : \texttt{HomogeneousIdeal}(\Delta, A, H) \,.\, (I, I \cap H) : \texttt{GradedModule}(\Delta, A, H)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{HomogeneousQuotient} :: \forall I : \texttt{HomogeneousIdeal}(\Delta, A, H) \,.\, \left(\Delta, \frac{A}{I}, \frac{I+H}{I}\right) : \texttt{GradedAlgebra}(R)$

$\texttt{Proof} =$

$\texttt{Assume}\ [a] : \prod_{\delta \in \Delta} \frac{I + H_\delta}{I},$

$\texttt{Assume}\ [1] : \sum_{\delta \in \Delta} [a_\delta] = 0,$

$[2] := \mathcal{I}\texttt{quatientModule}[1] : \sum_{\delta \in \Delta} a_\delta \in H,$

$[3] := \mathcal{I}\texttt{HomogeneousIdeal}[2] : \forall \delta \in \Delta \,.\, a_\delta \in H,$

$[a.*] := \mathcal{I}\texttt{quatientModule}[3] : \forall \delta \in \Delta \,.\, [a_\delta] = 0;$

$\rightsquigarrow [1] := \mathcal{I}^{-1}\texttt{DirectSum} : \frac{I}{H} = \bigoplus_{\delta \in \Delta} \frac{I + H_\delta}{I},$

$\texttt{Assume}\ \alpha, \beta : \Delta,$

$\texttt{Assume}\ [a] : \frac{I + H_\alpha}{I},$

$\texttt{Assume}\ [b] : \frac{I + H_\beta}{I},$

$[2] := \mathcal{I}\texttt{GradedAlgebra}(\Delta, A, H)(a, b) : ab \in H_{\alpha+\beta},$

$[a.*] := \mathcal{I}\texttt{quotientAlgebra}[1][2] : [a][b] = [ab] \in \frac{H_{\alpha+\beta} + I}{I};$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{GradedAlgebra} : \left(\left(\Delta, \frac{A}{I}, \frac{I+H}{I}\right) : \texttt{GradedAlgebra}(R)\right),$

$\square$

$\texttt{GradedTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall (\Delta, A, H) : n \to \texttt{GradedAlgebra}(R) \,.$

$$. \left(\prod_{i=1}^{n} \Delta_i, \bigotimes_{i=1}^{n} A_i, \bigotimes_{i=1}^{n} H_i \right) : \texttt{GradedAlgebra}(R)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{IntegralGradedTensorProduct} :: \forall n \in \mathbb{N} \,.\, \forall (\mathbb{Z}, A, H) : n \to \texttt{GradedAlgebra}(R) \,.$

$$. \left(\mathbb{Z}, \bigotimes_{i=1}^{n} A_i, \Lambda m \in \mathbb{Z} \,.\, \bigoplus \sum i : n \to \mathbb{Z} \,.\, \sum_{k \in n} i_k = m \,.\, \bigotimes_{j=1}^{n} H_{j,i_j} \right) : \texttt{GradedAlgebra}(R)$$

$\texttt{Proof} =$

...

$\square$

$\texttt{GradedAlgHomo} :: \prod (\Delta, A, H), (\Delta, B, H') : \texttt{GradedAlgebra}(R) \,.\, ?A \xrightarrow{R\text{-}\mathsf{ALGE}} B$

$f : \texttt{GradedeAlgHomo} \iff \forall \delta \in \Delta \,.\, f^{-1} H'_\delta = H_\delta$

$\texttt{categoryOfGradedAlgebras} :: \mathsf{ANN} \to \texttt{commutativeMonoid} \to \mathsf{CAT}$

$\texttt{categoryOfGradedAlgebras}\,(R, \Delta) = R\text{-}\mathsf{ALGE}(\Delta) := (\texttt{GradedAlgebra}(R), \texttt{GradedAlgHomo}, \circ, \mathrm{id})$

$\texttt{AssociativeTensorProductOfAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, C \in R\text{-}\mathsf{ALG}(\Delta) \,.$

$(A \otimes B) \otimes C \cong_{R\text{-}\mathsf{ALG}(\Delta)} A \otimes (B \otimes C)$

$\texttt{Proof} =$

...

$\square$

$\texttt{TensorProductOfGradedAlgebrasPermutation} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{ALG}(\Delta) \,.\, \forall \sigma \in S_n \,.$

$$. \bigotimes_{i=1}^{n} A_i \cong_{R\text{-}\mathsf{ALG}(\Delta)} \bigotimes_{i=1}^{n} A_{\sigma(i)}$$

$\texttt{Proof} =$

...

$\square$

$\texttt{TrivialTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALG}(\Delta) \,.\, R \otimes A \cong_{R\text{-}\mathsf{ALG}(\Delta)} A$

$\texttt{Proof} =$

...

$\square$

$\texttt{TensorProductOfGradedHomo} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A, B : n \to R\text{-}\mathsf{ALG}(\Delta) \,.$

$$. \forall f : \prod_{i=1}^{n} A_i \xrightarrow{R\text{-}\mathsf{ALG}(\Delta)} B_i \,.\, \bigotimes_{i=1}^{n} f : \bigotimes_{i=1}^{n} A_i \xrightarrow{R\text{-}\mathsf{ALG}(\Delta)} \bigotimes_{i=1}^{n} B_i$$

$\texttt{Proof} =$

...

$\square$

$\texttt{CentaralIdemppotentHasDegreeZero} :: \forall R \in \mathsf{ANN} \,.\, \forall (\mathbb{Z}_+, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}_+) \,.$
$\quad .\, \forall a \in Z(A) \,.\, \forall [0] : a^2 = a \,.\, a \in H_0$

$\texttt{Proof} \; =$

$b := a - a_0 : A,$

$[1] := \mathcal{D} b : b_0 = 0,$

$[2] := [0] \mathcal{C} a_0 : a_0^2 = a_0,$

$[3] := [2] \mathcal{C} Z(A) \texttt{BinomialExpansion}(2) : (1 - a_0)^2 = 1 - 2a_0 + a_0 = 1 - a_0,$

$[4] := \mathcal{D} b[0] : (1 - a_0)a = (1 - a_0)(b + a_0) = (1 - a_0)b,$

$[5] := [4][3] : \Big((1 - a_0)b : \texttt{Idempotent}(Z(A))\Big),$

$[6] := [5][1] : (1 - a_0)b = 0,$

$[7] := \mathcal{C} R\text{-}\mathsf{ALGE}(A)[6] : a_0 b = b,$

$[8] := \mathcal{D} b[0] \mathcal{D} b[2][7] : a_0 + b = a = a^2 = a_0^2 + 2ba_0 + b^2 = a_0 + 2b + b^2,$

$[9] := \mathcal{C} R\text{-}\mathsf{ALGE}(A) : b^2 = -b,$

$[10] := [9][1] : b = 0,$

$[*] := \mathcal{D} b \mathcal{C} a_0 : a \in H_0;$

$\square$

$\texttt{leggedAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{MOD} \to R\text{-}\mathsf{ALGE}(\mathbb{Z}_+)$

$\texttt{leggedAlgebra}\,(M) := \Big(\mathbb{Z}, \big(R \times M, \Lambda(\alpha, m), (\beta, n) \in R \times M \,.\, (\alpha, \beta, \beta n + \alpha m\big),$
$\quad , \Lambda k \in \mathbb{Z}_+ \,.\, \texttt{if } k == 0 \texttt{ then } R \times \{0\} \texttt{ else if } k == 1 \texttt{ then } \{0\} \times M \texttt{ else } \{0\}\Big)$

$\texttt{LeggedAlgebraIsCommutative} :: \forall R \in \mathsf{ANN} \,.\, \forall (\mathbb{Z}_+, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}_+) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$
$\quad (\mathbb{Z}_+, A, H) = \texttt{leggedAlgebra}(M) \Rightarrow A \in R\text{-}\mathsf{CALGE}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{LeggedAlgebraIsCommutative} :: \forall R \in \mathsf{ANN} \,.\, \forall (\mathbb{Z}_+, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}_+) \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$
$\quad (\mathbb{Z}_+, A, H) = \texttt{leggedAlgebra}(M) \Rightarrow A \in R\text{-}\mathsf{CALGE}$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{LeggedAlgebraIsPolynomial} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, M : \texttt{PolynomialGradedAlgebra}(R)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{PoincareGradedAlgebra} :: \prod k : \texttt{Field} \; . \; ?k\text{-ALGE}(\mathbb{Z})$

$(\mathbb{Z}, A, H) : \texttt{PoincareGradedAlgebra} \iff \forall n \in \mathbb{Z} \; . \; \dim H_n < \infty$

$\texttt{seriesOfPoincare} :: \texttt{PoincareGradedAlgebra}(k) \to \mathbb{Z}[[\mathbb{Z}]]$

$$\texttt{seriesOfPoincare}\,(\mathbb{Z}, A, H) = P(\mathbb{Z}, A, H)(x) := \sum_{n \in \mathbb{Z}} (\dim H_n) x^n$$

$\texttt{LorantGradedAlgebra} :: \prod k : \texttt{Field} \; . \; ?\texttt{PoincareGradedAlgebra}$

$(\mathbb{Z}, A, H) : \texttt{LorantGradedAlgebra} \iff \exists N \in \mathbb{Z} \; . \; \forall n : \texttt{Before}(N) \; . \; \dim H_n = 0$

$\texttt{PoincareSeriesProduct} :: \forall k : \texttt{Field} \; . \; \forall n \in \mathbb{N} \; . \; \forall A : n \to \texttt{LorantGradedAlgebra}(k) \; . \; .$

$$. \; P\left(\bigotimes_{i=1}^{n} A_i\right)(x) = \prod_{i=1}^{n} P(A_i)(x)$$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{PositiveHomogeneous} :: \prod R \in \text{ANN} \; . \; \prod A \in R\text{-ALGE}(\mathbb{Z}) \; . \; ?\texttt{Homogeneous}(A)$

$a : \texttt{PositiveHomogeneous} \iff \deg a > 0$

$\texttt{HilbertModule} :: \prod k : \texttt{Field} \; . \; \prod A \in k\text{-ALGE}(\mathbb{Z}) \; . \; ?A\text{-MOD}(\mathbb{Z}_+)$

$(M, H) : \texttt{HilbertModule} \iff \forall n \in \mathbb{Z}_+ \; . \; \dim_k H_n < \infty \; \& \; M : \texttt{Noetherian}(A)$

$\texttt{seriesOfHilbert} :: \prod k : \texttt{Field} \; . \; \prod A \in k\text{-ALGE} \; . \; \texttt{HilbertModule}(A) \to \mathbb{Z}[[\mathbb{Z}_+]]$

$$\texttt{seriesOfHilbert}\,(M, O) = H(M, O)(x) := \sum_{n=0}^{\infty} (\dim_k O_n) x^n$$

$\texttt{HilbertSeriesTheorem} :: \forall k : \texttt{Field} \; . \; \forall A \in k\text{-ALGE}(\mathbb{Z}) \; . \; \forall M : \texttt{HilbertModule}(A) \; . \; \forall n \in \mathbb{Z}_+ \; .$

$$. \; \forall a : n \to \texttt{PositiveHomogeneous}(A) \; . \; \forall [0] : A = \Big\langle \{a_n | n \in \mathbb{N}\} \Big\rangle_{R\text{-ALGE}} \; . \; \forall [00] : A = Z(A)$$

$$\exists! Q \in \mathbb{Z}[\mathbb{Z}_+] \; . \; H(A)(x) = \frac{Q(M)}{\prod_{i=1}^{n} (1 - x^{\deg a_i})}$$

$\texttt{Proof} \; =$

$(!)$

$\texttt{structuralPolynomial} :: \texttt{HilbertModule}(A) \to \sum n \in \mathbb{Z}_+ \,.\, \sum a : n \to \texttt{PositiveHomogneous}(A) \,.\, A = \langle\{a_i$

$\texttt{structuralPolynomial}\,(M, (n, a, \star)) = Q(M, n, a) := \texttt{HilbertSeriesTheorem}(M, n, a, \star)$

$\texttt{HilbertAlgebra} :: \prod k : \texttt{Field} \,.\, ?\texttt{PolynomialGradedAlgebra}(k)$

$(\mathbb{Z}, A, H) : \texttt{HilbertAlgebra} \iff Z(A) = A \;\&\; A : \texttt{FinitelyGeneratedAlgebra}(k)$

$\texttt{HilbertPolynomial} :: \prod k : \texttt{Field} \,.\, \prod A \in k\text{-}\mathsf{ALGE} \,.\, \prod M : \texttt{HilbertModule}(k) \,.\, ?\mathbb{Q}[\mathbb{Z}_+]$

$$h : \texttt{HilbertPolynomial} \iff \forall n \in \mathbb{Z}_+ \,.\, H(M)(x) = \sum_{n=0}^{\infty} h(n)x^n$$

$\texttt{HibertPolynomialTheorem} :: \forall k : \texttt{Field} \,.\, \forall A : \texttt{HilbertAlgebra}(k) \,.\, \forall M : \texttt{HilbertModule}(A) \,.$

$\quad .\, \exists! h : \texttt{HilbertPolynomial}(M)$

$\texttt{Proof} =$

(!)

$\texttt{polynomialOfHilbert} :: \prod A : \texttt{HilbertAlgebra}(k) \,.\, \texttt{HilbertModule}(k) \to \mathbb{Q}[\mathbb{Z}_+]$

$\texttt{polynomialOfHilbert}\,(M) = h(M)(x) := \texttt{HilbertPolynomialTHM}$

1.4 Skew Tensor Product and Skew Algebras

$$\texttt{doubleMultiindexSign} :: \prod n \in \mathbb{N} . \mathbb{Z}^n \times \mathbb{Z}^n \to \{1, -1\}$$

$$\texttt{doubleMultiindexSign}\,(I, J) = (-1)^{I,J} := \texttt{if isEven} \left(\sum_{k=1}^{n} \sum_{l=k+1}^{n} I_l J_k \right) \texttt{then}\ 1\ \texttt{else}\ -1$$

$$\texttt{skewTensorProduct} :: \prod n \in \mathbb{N} . n \to R\text{-ALG}(\mathbb{Z}) \to R\text{-ALG}(\mathbb{Z})$$

$$\texttt{skewTensorProduct}\,((\mathbb{Z}, A, H)) = \widetilde{\bigotimes}_{i=1}^{n} (\mathbb{Z}, A_i, H_i) :=$$

$$:= \left(\mathbb{Z}, \left(\bigotimes_{i=1}^{n} A_i, \text{Д} R\text{-ALGE}(\Delta) \Lambda \sum I, J \in \mathbb{Z}^n . \left(x : \prod_{k=1}^{n} H_{I_k}, y : \prod_{k=1}^{n} H_{J_k} \right) : \bigotimes_{i=1}^{n} A_i \times \bigotimes_{i=1}^{n} A_i . \right.\right.$$
$$\left.\left. . (-1)^{I,J} \bigotimes_{i=1}^{n} x_i y_i \right), \Lambda N \in \mathbb{Z} . \bigoplus \sum I \in \mathbb{Z}^n . \sum_{k=1}^{n} I_k = N . \prod_{k=1}^{n} H_{I_k} \right)$$

$\texttt{AssociativeSkewTensorProductOfAlgebras} :: \forall R \in \text{ANN} . \forall A, B, C \in R\text{-ALG}(\mathbb{Z}) .$
$(A \widetilde{\otimes} B) \widetilde{\otimes} C \cong_{R\text{-ALG}(\mathbb{Z})} A \widetilde{\otimes} (B \widetilde{\otimes} C)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SkewAlgebra} :: ?R\text{-ALG}(\mathbb{Z})$

$(\mathbb{Z}, A, H) : \texttt{SkewAlgebra} \iff \forall a, b : \texttt{Homogeneous}(A) . ab = (-1)^{ij} ba \quad \texttt{where} \quad a \in H_i \ \& \ b \in H_j$

$\texttt{AlternatingAlgebraTHM} :: \forall R \in \text{ANN} . \forall (\mathbb{Z}, A, H) : \texttt{PolynomialGradedAlgebra}(R) .$

$\forall [0] : \forall a \in A . 2a = 0 \Rightarrow a = 0 . \left(\forall a \in H_1 . a^2 = 0 \right) \iff A : \texttt{SkewAlgebra}(R)$

$\texttt{Proof} =$

$\texttt{Assume}\ L : \forall a \in H_1 . a^2 = 0,$

$[1] := \texttt{AlternateIsSkew}(L) : \forall a, b \in H_1 . ab = -ba,$

$\texttt{Assume}\ n, m : \mathbb{N},$

$\texttt{Assume}\ a : H_1^n,$

$\texttt{Assume}\ b : H_1^m,$

$[\ldots *] := [1]^{nm} : \prod_{i=1}^{n} a_i \prod_{i=1}^{m} b = (-1)^{n+m} \prod_{i=1}^{m} b_i \prod_{i=1}^{n} a_i;$

$\rightsquigarrow [L.*] := [0] \text{Д}^{-1} \texttt{SkewAlgebra} : ((\mathbb{Z}, A, H) : \texttt{SkewAlgebra}(R)));$

$\rightsquigarrow [1] := I(\rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ R : ((\mathbb{Z}, A, H) : \texttt{SkewAlgebra}(R)),$

$\texttt{Assume}\ a : H_1,$

$[2] := \text{Д}\texttt{SkewAlgebra}(R) : a^2 = -a^2,$

$[a.*] := [00][2] : a^2 = 0;$

$\rightsquigarrow [a.*] := I(\iff)[1] I(\Rightarrow) I(\forall) : \texttt{This};$

$\square$

$\texttt{SkewTensorProductTheorem} :: \forall n \in \mathbb{N} \,.\, \forall(\mathbb{Z}, A, H) : n \to \texttt{SkewAlgebra}(R) \,.$

$\quad . \widetilde{\bigotimes}^{n}_{i=1}(\mathbb{Z}, A_i, H_i) : \texttt{SkewAlgebra}(R)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ (\mathbb{Z}, A, H), (\mathbb{Z}, B, H') : \texttt{SkewAlgebra}(R),$

$\texttt{Assume}\ i, i', j, j' : \mathbb{Z},$

$\texttt{Assume}\ a : H_i,$

$\texttt{Assume}\ x : H_j,$

$\texttt{Assume}\ b : H'_{i'},$

$\texttt{Assume}\ y : H'_{j'},$

$[1] := \mathcal{I}\texttt{SkewTensorProduct} : (a\widetilde{\otimes}b)(x\widetilde{\otimes}y) = (-1)^{ij'}ax\widetilde{\otimes}by,$

$[\ldots *] := \mathcal{I}\texttt{SkewTensorProduct}\mathcal{I}\texttt{SkewAlgevra}(R)(A, B)[1]\mathcal{I}(-1)\mathcal{I}\mathsf{RING}(\mathbb{Z}) :$

$\quad : (x\widetilde{\otimes}y)(a\widetilde{\otimes}b) = (-1)^{ji'}xa\widehat{\otimes}yb = (-1)^{i'j}\Big((-1)^{ij}ax\Big)\widetilde{\otimes}\Big((-1)^{i'j'}by\Big) = (-1)^{ij+i'j'+i'j-ij'}(a\widetilde{\otimes}b)(x\widetilde{\otimes}y) =$

$\quad = (-1)^{ij+i'j'+i'j'+ij'}(a\widetilde{\otimes}b)(y\widetilde{\otimes}x) = (-1)^{(i+i')(j+j')}(a\widetilde{\otimes}b)(y\widetilde{\otimes}x);$

$\rightsquigarrow [\ldots *] := \mathcal{I}\texttt{SkewTensorProduct}\mathcal{I}^{-1}\texttt{SkewAlgebera} : \Big((\mathbb{Z}, A, H)\widetilde{\otimes}(\mathbb{Z}, B, H') : \texttt{SkewAlgebra}(R)\Big);$

$\rightsquigarrow [*] := \texttt{AssociateveSkewTensorProductOfAlgebras} : \texttt{This};$

$\square$

$\texttt{twistingIsomorphism} :: \prod A, B \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \,.\, A \otimes B \xrightarrow{R\text{-}\mathsf{MOD}} B \otimes A$

$\texttt{twistingIsmorphism}\,() = \tau_{A,B} := \mathcal{I}R\text{-}\mathsf{ALGE}(\mathbb{Z})\Lambda n, m \in \mathbb{Z} \,.\, \Lambda a \in A_n \,.\, \Lambda b \in B_m \,.\, (-1)^{mn} b \otimes a$

$\texttt{TwistingIsomorphismTheorem} :: \forall A, B \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \,.\, \tau_{A,B} : A\widetilde{\otimes}B \xleftrightarrow{R\text{-}\mathsf{ALGE}} B\widetilde{\otimes}A$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}\tau_{A,B} : \tau_{A,B}(e_A \otimes e_B) = e_B \otimes e_A,$

$\texttt{Assume}\ n, n', m,' m' : \mathbb{Z},$

$\texttt{Assume}\ a : A_n,$

$\texttt{Assume}\ a' : A_{n'},$

$\texttt{Assume}\ b : B_m,$

$\texttt{Assume}\ b' : B_{m'},$

$[1] := \mathcal{I}\texttt{SkewTensorProduct}\mathcal{I}\texttt{twistingIsomorphism}\mathcal{I}(-1) :$

$\quad \tau_{A,B}\Big(a \otimes b \cdot a' \otimes b'\Big) = (-1)^{n'm}\tau_{A,B}\Big(aa' \otimes bb'\Big) = (-1)^{nm'+nm+n'm+2n'm}(bb' \otimes aa') =$

$\quad = (-1)^{nm+n'm+nm'}(bb' \otimes aa'),$

$[2] := \mathcal{I}\texttt{twistingIsomorphism}\mathcal{I}\texttt{SkewTensorProduct} :$

$\quad \tau_{A,B}(a \otimes b) \cdot \tau_{A,B}(a' \otimes b') = (-1)^{nm+n'm'+nm'}\Big(b \otimes a \cdot b' \otimes a'\Big) = (-1)^{nm+n'm'+nm'}bb' \otimes aa',$

$[\ldots *] := [1][2] : \tau_{A,B}(a \otimes b)\tau_{A,B}(a' \otimes b') = \tau_{A,B}(a \otimes b \cdot a' \otimes b');$

$\rightsquigarrow [*] := [1]\mathcal{I}\texttt{SkewTensorProduct} : \texttt{This};$

$\square$

$\texttt{SkewTensorProductPermutation} :: \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{ALGE}(\mathbb{Z}) \,.\, \forall \sigma \in S_n \,.\, \widetilde{\bigotimes}^{n}_{i=1} A_i \cong_{R\text{-}\mathsf{ALGE}} \widetilde{\bigotimes}^{n}_{i=1} A_{\sigma(i)}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SkewMultiplicationMorphism} :: \forall A : \texttt{SkewAlgebra}(R) \;.\; \mu_A : A\widetilde{\otimes}A :\xrightarrow{R\text{-ALGE}} A$

$\texttt{Proof} \;\; =$

$[1] := \reflectbox{$D$}\mu_A : \mu_A(e \otimes e) = e,$

$\texttt{Assume}\; n, n', n,' m' : \mathbb{Z},$

$\texttt{Assume}\; a : A_n,$

$\texttt{Assume}\; a' : A_{n'},$

$\texttt{Assume}\; b : A_m,$

$\texttt{Assume}\; b' : A_{m'},$

$[1] := \reflectbox{$D$}\texttt{skewTensorProduct}\reflectbox{D}\mu : \mu(a \otimes b \cdot a' \otimes b') = (-1)^{n'm}\mu(aa' \otimes bb') = (-1)^{n'm}aa'bb',$

$[2] := \reflectbox{$D$}\mu\reflectbox{$D$}\texttt{SkewAlgebra} : \mu(a \otimes b)\mu(a' \otimes b') = aba'b' = (-1)^{n'm}aa'bb',$

$\ldots * := [1][2] : \mu\Big(a \otimes b \cdot a' \otimes b'\Big) = \mu(a \otimes b)\mu(a' \otimes b');$

$\rightsquigarrow [*] := [1]\reflectbox{D}\texttt{SkewTensorProduct} : \texttt{This};$

$\square$

$\texttt{doublingDegrees} :: \prod R \in \mathsf{ANN} \;.\; R\text{-ALGE}(\mathbb{Z}) \to R\text{-ALGE}(\mathbb{Z})$

$\texttt{doublingDegrees}\,(\mathbb{Z}, A, H) = (\mathbb{Z}, A, H)^{\mathrm{dd}} := \Big(\mathbb{Z}, A, \Lambda n \in \mathbb{Z} \;.\; \texttt{if isOdd}(n) \;\texttt{then}\; \{0\} \;\texttt{else}\; H_{\frac{n}{2}}\Big)$

$\texttt{DoublingDegreesTensorDistributive} :: \forall A, B \in R\text{-ALGE}(\mathbb{Z}) \;.\; A^{\mathrm{dd}} \otimes B^{\mathrm{dd}} = (A \otimes B)^{\mathrm{dd}}$

$\texttt{Proof} \;\; =$

$X := \mathcal{A}^{\mathrm{dd}} \otimes \mathcal{B}^{\mathrm{dd}} : R\text{-ALGE}(\mathbb{Z}),$

$Y := \Big(A \otimes B\Big)^{\mathrm{dd}} : R\text{-ALGE}(\mathbb{Z}),$

$\texttt{Assume}\; n : \mathbb{Z},$

$\texttt{Assume}\; [1] : (n : \texttt{Odd}),$

$[2] := \reflectbox{$D$}\texttt{integralTensorProduct} : X_n = \bigoplus \sum k, l \in \mathbb{Z} \;.\; k + l = n \;.\; A_k^{\mathrm{dd}} \otimes B_l^{\mathrm{dd}},$

$\texttt{Assume}\; k, l : \mathbb{Z},$

$\texttt{Assume}\; [3] : k + l = n,$

$[4] := \texttt{OddSum}[3] : \Big(k : \texttt{Odd}\Big| l : \texttt{Odd}\Big),$

$[\ldots *] := \reflectbox{D}\texttt{doublingDegrees}[4]\reflectbox{D}\texttt{TensorProduct} : A_k^{\mathrm{dd}} \otimes B_l^{\mathrm{dd}} = \{0\};$

$\rightsquigarrow [3] := \reflectbox{D}\texttt{directSum}[2] : X_n = 0,$

$[1.*] := \reflectbox{$D$}\texttt{doublingDegrees}[3] : Y_n = X_n;$

$\rightsquigarrow [1] := I(\Rightarrow) : n : \texttt{Odd} \Rightarrow Y_n = X_n,$

$\texttt{Assume}\; [2] : (n : \texttt{Even}),$

$[3] := \reflectbox{$D$}\texttt{integralTensorProduct} : X_n = \bigoplus \sum k, l \in \mathbb{Z} \;.\; k + l = n \;.\; A_k^{\mathrm{dd}} \otimes B_l^{\mathrm{dd}},$

$\texttt{Assume}\; k, l : \mathbb{Z},$

$\texttt{Assume}\; [4] : k + l = n,$

$\texttt{Assume}\; [5] : A_k^{\mathrm{dd}} \otimes B_l^{\mathrm{dd}} \neq \{0\},$

$[6] := [5]\reflectbox{$D$}\texttt{doublingDegrees} : (k, l : \texttt{Even}),$

$[\ldots *] := \reflectbox{D}\texttt{doublingDegrees}[6] : A_k^{\mathrm{dd}} \otimes B_l^{\mathrm{dd}} = A_{\frac{k}{2}} \otimes B_{\frac{k}{2}};$

$\rightsquigarrow [2.*] := [2]\reflectbox{$\mathcal{C}$}Y : X_n = Y_n;$

$\rightsquigarrow [2] := I(\Rightarrow) : n : \texttt{Even} \;.\; \Rightarrow Y_n = X_n,$

$[n.*] := E(|)\texttt{EvenOrOdd}[1][2] : Y_n = X_n;$

$\rightsquigarrow [1] := I(\forall) : \forall n \in \mathbb{Z} \;.\; Y_n = X_n,$

$[*] := \reflectbox{$\mathcal{C}$}X\reflectbox{$\mathcal{C}$}Y[1] : X = Y;$

$\square$

1.5 Derivations on Algebras

$\texttt{MapOfDegree} :: \prod(\Delta, A, H) : \texttt{GradedAlgebra}(R) \;.\; ?(A \to A)$

$f : \texttt{MapOfDegree} \iff \exists! \delta \in \Delta : \forall \alpha \in \Delta \;.\; f(H_\alpha) \subset H_{\alpha+\delta}$

$\texttt{mapDegree} :: \texttt{MapOfDegree}(\Delta, A, H) \to \Delta$

$\texttt{mapDegree}\,(f) = \deg f := \mathcal{C}\!\texttt{MapOfDegree}(\Delta, A, H)(f)$

$\texttt{Derivation} :: \prod A \in R\text{-}\mathsf{LG} \;.\; ?\mathrm{End}_{R\text{-}\mathsf{MOD}}(A)$

$D : \texttt{Derivation} \iff \forall a, b \in A \;.\; D[a, b] = [Da, b] + [a, Db]$

$\texttt{GradedDerivation} :: \prod(\mathbb{Z}, A, H) \;.\; ?\texttt{Derivation}(A)$

$D : \texttt{GradedDerivation} \iff D \in \mathcal{D}(A, H) \iff D : \texttt{MapOfDegree}(\mathbb{Z}, A, H) \;\&\; \deg D = -1$

$\texttt{DerivationOfProduct} :: \forall A \in R\text{-}\mathsf{ALG} \;.\; \forall D : \texttt{Derivation}(A) \;.\; \forall n \in \mathbb{N} \;.\; \forall a : n \to A \;.$

$$D \prod_{i=1}^{n} a_i = \sum_{k=1}^{n} \prod_{i=1}^{k-1} a_i D a_k \prod_{i=k+1}^{n} a_i$$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ModuleOfDerivations} :: \forall A \in R\text{-}\mathsf{LG} \;.\; \texttt{Derivatopn}(A) \in R\text{-}\mathsf{MOD}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ModuleOfDerivations2} :: \forall(\Delta, A, H) : \texttt{GradedAlgebra}(A) \;.\; \mathcal{D}(A, H) \in R\text{-}\mathsf{MOD}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{NeutralDerivation} :: \forall A \in R\text{-}\mathsf{LGE} \;.\; \forall D : \texttt{Derivation}(A) \;.\; De = 0$

$\texttt{Proof} =$

$[1] := \mathcal{C}\!\texttt{Neutral}(e)\mathcal{C}\!\texttt{Derivation}(D)\mathcal{C}\!\texttt{Neutral}(e) : De = D[e, e] = [De, e] + [e, De] = 2De,$

$[*] := \mathcal{C}\mathsf{ABEL}(A)[1] : De = 0;$

$\square$

$\texttt{PolynomialDerivationsCommute} :: \forall (A,H) : \texttt{PolynomialGradedAlgebra}(R) \,.\, \forall D, D' \in \mathcal{D}(A,H) \,.$
$\quad .\, DD' = D'D$

$\texttt{Proof} \ =$

$[1] := \texttt{FreeCoeffiecientLemma}(A) : H_0 = Re,$

$\texttt{Assume}\ a : H_1,$

$(\alpha, [2]) := \mathcal{I}\texttt{mapDegree}\mathcal{I}\mathcal{D}(A,H)(D)[1] : \sum \alpha \in R \,.\, Da = \alpha e,$

$(\beta, [3]) := \mathcal{I}\texttt{mapDegree}\mathcal{I}\mathcal{D}(A,H)(D')[1] : \sum \beta \in R \,.\, D'a = \beta e,$

$[a.*] := [2]\mathcal{I}R\text{-}\mathsf{MOD}(A,A)(D')[1]\mathcal{I}R\text{-}\mathsf{MOD}(A,A)(D')[3] : D'Da = D'\alpha e = \alpha D'e = 0 = \beta De = D\beta e = DD'a;$

$\rightsquigarrow [2] := I(\forall) : \forall a \in H_1 \,.\, D'Da = D'Da,$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ b : H_n,$

$\big(m, a, [3]\big) := \mathcal{I}\texttt{PolynomialGradedAlgebra}(R)(A)(b) : \sum m \in \mathbb{N} \,.\, \sum a : H_1^{m\times n} \,.\, b = \sum_{i=1}^{m}\prod_{j=1}^{n} a_{i,j},$

$[n.b.*] := [3](D'Da)\texttt{DerivationProduct}^2(D)(D')[2](a)\texttt{DerivationProduct}^2(D')(D)[3] :$

$$D'Db = D'D\sum_{i=1}^{m}\prod_{j=1}^{n} a_{i,j} = D'\sum_{i=1}^{m}\sum_{k=1}^{n}\prod_{j=1}^{k-1} a_{i,j} D a_{i,k} \prod_{j=k+1}^{n} a_{i,j} =$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{k\neq l=1}^{n}\prod_{j=1}^{\min(k-1,l-1)} a_{i,j}\Big([k<l]Da_{i,k} + [l<k]D'a_{i,l}\Big)\prod_{j=\min(k+1,l+1)}^{\max(k-1,l-1)} a_{i,j}\Big([k>l]Da_{i,k} + [l>k]D'a_{i,l}\Big)$$
$$\prod_{j=\max(k+1,l+1)}^{n} a_{i,j} + \sum_{k=1}^{n}\prod_{i=1}^{k-1} a_i D'Da_k \prod_{i=k+1}^{n} a_i =$$
$$= \sum_{i=1}^{m}\sum_{l=1}^{n}\sum_{l\neq k=1}^{n}\prod_{j=1}^{\min(k-1,l-1)} a_{i,j}\Big([k<l]Da_{i,k} + [l<k]D'a_{i,l}\Big)\prod_{j=\min(k+1,l+1)}^{\max(k-1,l-1)} a_{i,j}\Big([k>l]Da_{i,k} + [l>k]D'a_{i,l}\Big)$$
$$\prod_{j=\max(k+1,l+1)}^{n} a_{i,j} + \sum_{k=1}^{n}\prod_{i=1}^{k-1} a_i DD'a_k \prod_{i=k+1}^{n} a_i = D\sum_{i=1}^{m}\sum_{l=1}^{n}\prod_{j=1}^{l-1} a_{i,j} D'a_{i,l}\prod_{j=l+1}^{n} a_{i,j} = DD'\sum_{i=1}^{m}\prod_{j=1}^{n} a_{i,j} = DD'b;$$

$\rightsquigarrow [3] := I^2(\forall) : \forall n \in \mathbb{N} \,.\, \forall b \in H_n \,.\, D'Db = DD'b,$

$[*] := \mathcal{I}\texttt{GradedAlgebra}(\mathbb{Z}, A, H)[3] : D'D = DD';$

$\square$

$\texttt{PolynomialDerivationsAgree} :: \forall (A,H) : \texttt{PolynomialGradedAlgebra}(R) \,.\, \forall D, D' \in \mathcal{D}(A,H) \,.$
$\quad .\, \forall [0] : \forall a \in H_1 \,.\, Da = D'a \,.\, D = D'$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{SkewDerivation} :: \prod(\mathbb{Z}, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \,.\, ?\texttt{MapOfDegree}(\mathbb{Z}, A, H)$

$D : \texttt{SkewDerivation} \iff D \in \widetilde{\mathcal{D}}(A,H) \iff \deg D = -1 \ \&$
$\quad \& \ \forall n \in \mathbb{N} \,.\, \forall a \in A \,.\, \forall h \in H_n D(ha) = (Dh)a + (-1)^n h Da$

`SkewDerivationOfProduct` $:: \forall (\mathbb{Z}, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \;.\; \forall D \in \widetilde{\mathcal{D}}(A, H) \;.\; \forall n \in \mathbb{N} \;.\; \forall a : n \to H_1 \;.$

$$D \prod_{i=1}^{n} a_i = \sum_{k=1}^{n} (-1)^{k+1} \prod_{i=1}^{k-1} a_i D a_k \prod_{i=k+1}^{n} a_i$$

`Proof` $=$

$\ldots$

$\square$

`NeutralSkewDerivation` $:: \forall (\mathbb{Z}, A, H) \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \;.\; \forall D :$ `Derivation`$(A, H) \;.\; De = 0$

`Proof` $=$

$[1] :=$ `UnitDegree`$(A, H) : \deg e = 0,$

$[2] := \daleth$`Neutral`$(e)\daleth$`SkewDerivation`$(D)\daleth$`Neutral`$(e) : De = De^2 = (De)e + e(De) = 2De,$

$[*] := \daleth\,\mathsf{ABEL}(A)[1] : De = 0;$

$\square$

`PolynomialSkewDerivationsAnticommute` $:: \forall (A, H) :$ `PolynomialGradedAlgebra`$(R) \;.\; \forall D, D' \in \mathcal{D}(A, H) \;.$
$\;.\; DD' + D'D = 0$

`Proof` $=$

$[1] :=$ `FreeCoeffiecientLemma`$(A) : H_0 = Re,$

`Assume` $a : H_1,$

$(\alpha, [2]) := \daleth$`mapDegree`$\daleth\mathcal{D}(A, H)(D)[1] : \sum \alpha \in R \;.\; Da = \alpha e,$

$(\beta, [3]) := \daleth$`mapDegree`$\daleth\mathcal{D}(A, H)(D')[1] : \sum \beta \in R \;.\; D'a = \beta e,$

$[a.*] := [2][3]\daleth R\text{-}\mathsf{MOD}(A, A)(D)(D')$`NeutralSkewDerivation`$(D)(D') : (DD' + D'D)a = \beta De + \alpha D'e = 0;$

$\rightsquigarrow [2] := I(\forall) : \forall a \in H_1 \;.\; (DD' + D'D)a = 0,$

`Assume` $n : \mathbb{N},$

`Assume` $[3] : \forall a \in H_n \;.\; (DD' + D'D)(a) = 0,$

`Assume` $a : H_1,$

`Assume` $b : H_n,$

$[a.b.*] := \daleth$`SkewDerivation`$(D)(D')\daleth R\text{-}\mathsf{ALGE}(A)\daleth R\text{-}\mathsf{MOD}(A, A)(D)(D')[2][3] : (DD' + D'D)(ab) = (DD'a)b$

$\rightsquigarrow [n.3.*] := \daleth$`PolynomialGradedAlgebra`$(A, H) : \forall a \in H_{n+1} \;.\; (DD' + D'D)a = 0;$

$\rightsquigarrow [3] := \daleth\mathbb{N}[2] : \forall n \in \mathbb{N} \;.\; \forall a \in H_n \;.\; (DD' + D'D)a = 0,$

$[*] := \daleth(\text{-}\mathsf{ALGE}R)(\mathbb{Z})[3] : DD' + D'D = 0;$

$\square$

`PolynomialSkewDerivationsZeroSquare` $:: \forall (A, H) :$ `PolynomialGradedAlgebra`$(R) \;.\; \forall D \in \mathcal{D}(A, H) \;.$
$\;.\; D^2 = 0$

`Proof` $=$

$\ldots$

$\square$

`PolynomialSkewDerivationsAgree` $:: \forall (A, H) :$ `PolynomialGradedAlgebra`$(R) \;.\; \forall D, D' \in \widetilde{\mathcal{D}}(A, H) \;.$
$\;.\; \forall [0] : \forall a \in H_1 \;.\; Da = D'a \;.\; D = D'$

`Proof` $=$

$\ldots$

$\square$

$\texttt{DerivationOfDegree} :: \prod A : R\text{-}\mathsf{ALGE}(\mathbb{Z})\ .\ \prod n \in \mathbb{Z}\ .\ ?\texttt{MapOfDegree}(R)$

$D : \texttt{DerivationOfDegree} \iff D \in \mathcal{D}^n(A) \iff \deg D = n\ \&\ D : \texttt{Derivation}(A)$

$\texttt{GeneralisedDerivationTHM} :: \forall A \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \forall n, m \in \mathbb{Z}\ .\ \forall D \in \mathcal{D}^n(A)\ .\ \forall D' \in \mathcal{D}^m(A)\ .$
$\quad .\ DD' - D'D \in \mathcal{D}^{n+m}(A)$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b : A,$

$[a, b.*] := \mathcal{A}\texttt{Derivation}(D)\mathcal{A}\texttt{Derivation}(D')\mathcal{A}R\text{-}\mathsf{MOD}(A, A)(DD')(D'D) :$

$\quad : \big(DD' - D'D\big)(ab) = D\Big((D'a)b + a(D'b)\Big) - D'\Big((Da)b + a(Db)\Big) =$

$\quad (DD'a)b + (D'a)(Db) + (Da)(D'b) + a(DD'b) - (D'Da)b - (Da)(D'b) - (D'a)(Db) - a(D'Db) =$

$\quad \Big(\big(DD' - D'D\big)a\Big) + a\Big(\big(DD' - D'D\big)b\Big);$

$\rightsquigarrow [*] := \mathcal{A}\mathcal{D}^{n+m}(A) : DD' - D'D \in \mathcal{D}^{n+m}(A);$

$\square$

$\texttt{mainInvolution} :: \prod A \in R\text{-}\mathsf{ALGE}(\mathbb{Z})\ .\ \mathrm{Aut}_{R\text{-}\mathsf{ALGE}(\mathbb{Z})}(A)$

$\texttt{mainInvolution}\,() = J_A := \mathcal{A}R\text{-}\mathsf{ALGE}(\mathbb{Z})(A)\Lambda n \in \mathbb{Z}\ .\ \Lambda a \in A_n\ .\ J_A(a) = (-1)^n a$

$\texttt{SkewDerivationOfDegree} :: \prod A : R\text{-}\mathsf{ALGE}(\mathbb{Z})\ .\ \prod n \in \mathbb{Z}\ .\ ?\texttt{MapOfDegree}(R)$

$D : \texttt{SkewDerivationOfDegree} \iff D \in \widetilde{\mathcal{D}}^n(A) \iff \deg D = n\ \&$
$\quad \&\ \forall a, b \in A\ .\ D(ab) = (Da)b + J_A^n(a)Db$

$\texttt{GeneralisedSkewDerivationTHM} :: \forall A \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \forall n, m \in \mathbb{Z}\ .\ \forall D \in \mathcal{D}^n(A)\ .\ \forall D' \in \mathcal{D}^m(A)\ .$
$\quad .\ DD' - (-1)^{nm}D'D \in \mathcal{D}^{n+m}(A)$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b : A,$

$[a, b.*] := \mathcal{A}\texttt{SkewDerivationOfDegree}(D)\mathcal{A}\texttt{SkewDerivationOfDegree}(D')\mathcal{A}R\text{-}\mathsf{MOD}(A, A)(DD')(D'D) :$

$\quad : \big(DD' - (-1)^{nm}D'D\big)(ab) = D\Big((D'a)b + J^m(a)(D'b)\Big) - (-1)^{nm}D'\Big((Da)b + J^n a(Db)\Big) =$

$\quad = (DD'a)b + J^n(D'a)(Db) + DJ^m(a)(D'b) + J^{m+n}(a)(DD'b)-$

$\quad - (-1)^{nm}\Big((D'Da)b - J^m(Da)(D'b) - D'J^n(a)(Db) - J^{m+n}(a)(D'Db)\Big) =$

$\quad (DD'a)b + J^n(D'a)(Db) + (-1)^{nm}J^m(Da)(D'b) + J^{m+n}(a)(DD'b)-$

$\quad - (-1)^{nm}\Big((D'Da)b - J^m(Da)(D'b) - (-1)^{nm}J^n(D'a)(Db) - J^{m+n}(a)(D'Db)\Big) =$

$\quad \Big(\big(DD' - (-1)^{nm}D'D\big)a\Big) + J^{m+n}(a)\Big(\big(DD' - (-1)^{nm}D'D\big)b\Big);$

$\rightsquigarrow [*] := \mathcal{A}\mathcal{D}^{n+m}(A) : DD' - D'D \in \mathcal{D}^{n+m}(A);$

$\square$

1.6 Finite-Dimensional Associative Algebras over Fields

$\texttt{AlgebraRepresentation} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.$
$\quad .\, ?\Big(A \xrightarrow{R\text{-}\mathsf{ALGE}} \mathrm{End}_{R\text{-}\mathsf{MOD}}(M)\Big)$

$\texttt{Faithful} :: ?\texttt{AlgebraRepresentation}(R, A, M)$
$\rho : \texttt{Faithful} \iff \rho : A \hookrightarrow \mathrm{End}_{R\text{-}\mathsf{MOD}}(M)$

$\texttt{lefttRegularRepresentation} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \texttt{Faithul}(R; A, A)$
$\texttt{leftRegularRepresentation}\,(a) = L_A(a) := \Lambda b \in A \,.\, ab$

$\texttt{leftRegularMatrixRepresentation} :: \prod k : \texttt{Field} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,\&\, R\text{-}\mathsf{FDVS} \,.$
$\quad .\, \texttt{Basis}(A) \to \texttt{Faithul}(R, A, A^{\dim A \times \dim A})$
$\texttt{leftRegularMatrixRepresentation}\,(e, a) = L_{A,e}(a) := L_A(a)^{e,e}$

$\texttt{FiniteRankIdealProperty} :: \forall k : \texttt{Field} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \Big\{a \in A |\, \mathrm{rank}\, L_{A,e}(a) \le \infty\Big\} : \texttt{Ideal}(A)$
$\texttt{Proof} \;=$
$\dots$
$\square$

$\texttt{finiteRankIdeal} :: \prod k : \texttt{Field} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \texttt{Ideal}(A)$
$\texttt{finiteRankIdeal}\,() = I_{\mathrm{rank}<\infty}(A) :=$

$\texttt{FiniteRankIdealTHM} :: \forall k : \texttt{Field} \,.\, \prod V \in R\text{-}\mathsf{VS} \,.\, \forall I : \texttt{Ideal}(\mathrm{End}_{k\text{-}\mathsf{VS}}(V)) \,.$
$\quad .\, \forall [0] : I \ne 0 \,.\, I_{\mathrm{rank}<\infty}(\mathrm{End}_{k\text{-}\mathsf{VS}}(V)) \subset I$
$\texttt{Proof} \;=$
$\Big(B, [3]\Big) := \mathcal{Q} I[0] : \sum B \in I : B \ne 0,$
$\texttt{Assume}\; A : I_{\mathrm{rank}<\infty}(\mathrm{End}_{k\text{-}\mathsf{VS}}(V)),$
$\Big(F, [1]\Big) := \mathcal{Q} I_{\mathrm{rank}<\infty}(A) \mathcal{Q}\, \mathrm{rank} : \sum F : \mathrm{rank}\, A \to \mathrm{End}_{k\text{-}\mathsf{VS}} \,.\, A = \sum^{\mathrm{rank}\, A} \forall i \in \mathrm{rank}\, A \,.\; \mathrm{rank}\, F = 1,$
$\texttt{Assume}\; i : \mathrm{rank}\, f,$
$\Big(v, u, [2]\Big) := \texttt{Rank1Reperesentation}(F_i)[1] : \sum u, v \in V \,.\, F_1(u) = v \,\&\, \ker F_1 \oplus \mathrm{span}(u) = V,$
$\Big(x, [4]\Big) := \mathcal{Q} 0[3] : \sum x \in V \,.\, Bx \ne 0,$
$[4] := \mathcal{Q} T_{B(x),v}, T_{u,x}[2] : F_i = T_{u,x} B T_{B(x),v},$
$[i.*] := \mathcal{Q}\,\texttt{Ideal}[4] : F_i \in I;$
$\rightsquigarrow [A.*] := \mathcal{Q}\,\texttt{Ideal}[1] : A \in I;$
$\rightsquigarrow [*] := \mathcal{Q}\,\texttt{Subset} : I_{\mathrm{rank}<\infty}(\mathrm{End}_{k\text{-}\mathsf{VS}(V)}) \subset I;$
$\square$

$\texttt{Algebraic} :: \prod k : \texttt{Field} \,.\, \prod A \in k\text{-}\mathsf{ALGE} \,.\, ?A$

$a : \texttt{Algebraic} \iff \exists f \in k[x] \,.\, f(a) = 0$

$\texttt{minimalPolynomial} :: \prod k : \texttt{Field} \,.\, \prod A \in k\text{-}\mathsf{ALGE} \,.\, \texttt{Algebraic}(A) \to k[x]$

$\texttt{minimalPolynomial}\,(a) = M_a := Ɑ\texttt{PrincipleIdealDomain}\Big(k[x]\Big)Ɑ\texttt{Algebraic}(a)$

$\texttt{AlgebraicSubalgeraStructure} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{ALGE} \,.\, \forall a : \texttt{Algebraic}(A) \,.$

$\quad .\ k[a] \cong_{k\text{-}\mathsf{ALGE}} \dfrac{k[x]}{M_a}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{FiniteDimensionalIsAlgebraic} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{ALGE} \,.$

$\quad .\ \dim A < \infty \Rightarrow \forall a \in A \,.\, a : \texttt{Algebraic}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{AlgebraicInvertibility} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{ALGE} \,.\, \forall a : \texttt{Algebraic}(A) \,.$

$\quad .\ a \in A^* \iff a \in A^\times$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{MinimalAlgebraicRoots} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{ALGE} \,.\, \forall a : \texttt{Algebra}(A) \,.\, \forall \rho \in k \,.$

$\quad .\ \rho \in \texttt{roots}(k, m_a(x)) \iff a - \rho e \notin A^\times$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{spectreOfElement} :: \prod k : \texttt{Field} \,.\, \prod A \in k\text{-}\mathsf{ALGE} \,.\, \texttt{Algebraic}(A) \to \texttt{Measure}(k, 2^k)$

$\texttt{spectreOfElement}\,(a) = \sigma(a) := \Lambda K \subset k \,.\, \sum_{\alpha \in K} \max \Big\{ t \in \mathbb{Z}_+ : (x - \alpha)^t | m_a(x) \Big\}$

CommutativeSpectre :: $\forall k : \texttt{AlgebraicallyClosedField} \,.\, \forall A : k\text{-}\mathsf{ALGE} \,.\, \forall a, b : \texttt{Algebraic}(A) \,.$

$\quad \operatorname{supp} \sigma(ab) = \operatorname{supp} \sigma(ba)$

Proof $=$

Assume $\rho : A$,

Assume $[1] : \rho \neq 0$,

Assume $[2] : \sigma(ab)\{\rho\} = 0$,

$[3] :=$ MinimalAlgebraicRoots AlgebraicInvertability$[2] : ab - \rho e \in A^*$,

$[4] := \daleth k\text{-}\mathsf{ALGE}(A) \daleth A^*[3] \daleth \mathsf{ABEL}(A) : (ba - \rho e)\Big(b(ab - \rho e)^{-1}a - e\Big) = b(ab - \rho e)(ab - \rho e)^{-1}a - ba + \rho e = ba$

$[5] := [1]$AlgebraicInvertability$[4] : ba - \rho e \in A^*$,

$[\rho.*] :=$ MinimalAlgebraicRoots$[5] : \sigma(ba)\{\rho\} = 0$;

$\rightsquigarrow [1] := \daleth \subset : \operatorname{supp} \sigma(ba) \subset \operatorname{supp} \sigma(ab)$,

Assume $[2] : \sigma(ab)\{0\} = 0$,

$[3] :=$ AlgebraicInvertability MinimalAlgebraicRoots $: ab \in A^*$,

$[4] := \daleth k\text{-}\mathsf{ALGE}(A, \mathrm{End}_{k\text{-}\mathsf{VS}}(A))(L_A)[3] : L_A(a)L_A(b) = L_A(ab) \in \mathrm{Aut}_{k\text{-}\mathsf{VS}}(A)$,

$[5] :=$ InvertibleProduct$[4] : L_A(a), L_A(b) \in \mathrm{Aut}_{k\text{-}\mathsf{VS}}(A)$,

$[6] := \daleth k\text{-}\mathsf{ALGE}(A, \mathrm{End}_{k\text{-}\mathsf{VS}}(A))(L_A)[5] : a, b \in A^\times$,

$[7] :=$ AlgebraicInvertability $: a, b \in A^*$,

$[8] := \daleth R^*[7] : ba \in A^*$,

$[9] :=$ AlgebraicInvertability MinimalAlgebraicRoots $: \sigma(ba)\{0\} = 0$;

$\rightsquigarrow [10] :=$ SymmetricArgument$\daleth$Subset $: \operatorname{supp}(ab) = \operatorname{supp}(ba)$,

$\square$

quaternions :: $\mathbb{R}\text{-}\mathsf{ALGE}$

quaternions $() = \mathbb{H} := \dfrac{\mathrm{Free}_{\mathbb{R}\text{-}\mathsf{ALGE}}\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}}{\Big(\mathrm{i}^2 + 1, \mathrm{j}^2 + 1, \mathrm{k}^2 + 1, \mathrm{ij} - \mathrm{k}\Big)}$

quaternionicIdentities :: $\mathrm{ik} = -\mathrm{j} \;\&\; \mathrm{kj} = -\mathrm{i}$

Proof $=$

...

$\square$

tripleQuaternionicIdentity :: $\mathrm{jik} = 1$

Proof $=$

...

$\square$

ReversedQuaternionicIdentities :: $\mathrm{ij} = -k \;\&\; \mathrm{ki} = \mathrm{j} \;\&\; \mathrm{jk} = \mathrm{i}$

Proof $=$

...

$\square$

DimensionOfQuaternions :: $\dim \mathbb{H} = 4$

Proof $=$

...

$\square$

$\texttt{QuaternionicBasis} :: \{1, \mathrm{i}, \mathrm{j}, \mathrm{k} : \texttt{Basis}(\mathbb{H})\}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{InvetibleQuaternions} :: \mathbb{H} : \texttt{DivisionAlgebra}(\mathbb{R})$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{AlgebraiclyClosedDivision} :: \forall k : \texttt{AlgebraicallyClosedField} \ . \ \forall A : \texttt{DivisionAlgebra}(k) \ . \ A : \texttt{Field}$

$\texttt{Proof} \ =$

$\texttt{WidderburnsTheorem} :: \forall q : \texttt{PrimePower} \ . \ \forall A : \texttt{DivisionAlgebra}(\mathbb{F}_q) \ .$

$\quad . \ \forall [0] : |A| < \infty \ . \ A : \texttt{Field}$

$\texttt{Proof} \ =$

$(p, k, [00]) := \varGamma \texttt{PrimePower} : \sum p : \texttt{Prime}(\mathbb{Z}) \ . \ \sum k \in \mathbb{N} \ . \ q = p^k,$

$[1] := \texttt{CommutativeByConjugation} \varGamma \texttt{centre}[0] : |A^*| = |Z^*(A)| + \sum_{\gamma \in C(A,*) : |\gamma| \neq 1} |\gamma|,$

$[2] := \varGamma \texttt{Field}(Z(A)) : \Big(Z(A) \in \texttt{Field}\Big),$

$[3] := \varGamma \mathbb{F}_q\text{-ALGE}(A)[2] : A \in Z(A)\text{-VS},$

$n := |Z(A)| : \mathbb{N},$

$(5, t) := [4][0] \varGamma \dim_{Z(A)} A : \sum t \in \mathbb{N} |A| = n^t,$

$(\alpha, [6]) := \texttt{ClassEquation}(A) : \sum \alpha \subset A \ . \ \sum_{\gamma \in C(A,*) : |\gamma| \neq 1} |\gamma| = \sum_{a \in \alpha} \frac{|A^*|}{|Z_A^*(a)|},$

$\texttt{Assume} \ a : \alpha,$

$[7] := \varGamma Z(A)\text{-VS} \varGamma (Z_A(a)) : \Big(Z_A(a) : Z(A)\text{-VS}\Big),$

$(s(a), a.*) := \varGamma \dim_{Z(A)} Z_A(a) : \sum s(a) \in \mathbb{N} \ . \ |Z_A(a)| = n^{s(a)};$

$\rightsquigarrow (s, [7]) := I\left(\sum\right) : \sum s : \alpha \to \mathbb{N} \ . \ \forall a \in \alpha \ . \ |Z_A(a)| = n^{s(a)},$

$[8] := [5][6][7] : n^t - 1 = n - 1 + \sum_{a \in \alpha} \frac{n^t - 1}{n^{s(a)} - 1},$

$[9] := \varGamma \mathbb{Z} : n(n^{t-1} - 1) = \sum_{a \in \alpha} \frac{n^t - 1}{n^{s(a)} - 1},$

$[10] := \texttt{SubgroupOrder}(A, Z_A(a))[5][7] : \forall a \in \alpha \ . \ n^{s(a)} - 1 | n^t - 1,$

$[11] := \texttt{CyclicDivisibility}[10] : s(a) | t,$

$[12] := \texttt{CyclotomicDivision}[11][8] : Q_t(n) | n - 1,$

$[13] := \texttt{ComplexDifferenceEstimates}(n, 1, \texttt{PrimitiveRootsOfUnity}(\mathbb{C}, t))$

$\quad \texttt{IncreasingProduct} : t > 1 \Rightarrow |Q_t(n)| > n - 1,$

$[14] := \texttt{NaturalDivisorsAreLess}[12][13] : t = 1,$

$[*] := \varGamma \texttt{Field}[14][6][1] : (A : \texttt{Field});$

$\square$

$\texttt{FrobeneusTheorem} :: \forall A : \texttt{DivisionAlgebra}(\mathbb{R}) \,.\, \forall [0] : \dim A < \infty \,.\, A = \mathbb{R} | A = \mathbb{C} | A = \mathbb{H}$

$\texttt{Proof} \ =$

$D := \{a \in A : \exists \alpha \in \mathbb{R}_- \,.\, a^2 = \alpha e\} :?A,$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ [1] : a \neq 0,$

$[2] := \Game\texttt{DivisionAlgebra}(A)(A)\texttt{MinimalAlgebraicRootsAlgebraicInvertability} :$

$\quad : \Big(m_a(x) : \texttt{Irreducible}(\mathbb{R})\Big),$

$\texttt{Assume}\ [3] : (\deg m_a(x) = 1),$

$[3.*] := \Game m_a(x)[3] : \exists \alpha \in \mathbb{R} \,.\, a = \alpha e;$

$\leadsto [3] := I(\Rightarrow) : \deg m_a(x) = 1 \Rightarrow a \in \mathbb{R},$

$\texttt{Assume}\ [4] : \deg m_a(x) = 2,$

$\Big(\alpha, \beta, [5]\Big) := \texttt{RealIrreducibleQuadric}[2][4] :: \sum \alpha, \beta \in \mathbb{R} \,.\, m_a(x) = x^2 + \alpha x + \beta \ \&\ \alpha^2 < 4\beta,$

$[6] := \Game m_a(x)\texttt{BinomialEquation} :: 0 = m_a(a) = a^2 + \alpha a + \beta e = \left(a + \frac{\alpha e}{2}\right)^2 + \beta e - \frac{\alpha^2}{4}e,$

$[a.*] := [6][5] : \left(a + \frac{\alpha e}{2}\right)^2 \in \mathbb{R}_{--}e;$

$\leadsto [1] := \Game DE(|) : A = D + \mathbb{R}e,$

$\texttt{Assume}\ u, v : D,$

$[2] := \texttt{RealSquaresPositive}(u, v) : u, v \notin \mathbb{R}e,$

$(\lambda, \mu, [3]) := \Game D(u, v) : \sum \lambda, \mu \in \mathbb{R}^+ + \ \,.\, u^2 = -\lambda e \ \&\ v^2 = -\mu e,$

$\texttt{Assume}\ [5] : \Big(\{u, b\} : \texttt{LinearlyIndependent}(\mathbb{R}, A)\Big),$

$(\alpha, x, \beta, y, [4]) := [1](u, v) : \sum x, y \in D \,.\, \sum \alpha, \beta \in \mathbb{R} \,.\, u + v = x + \alpha e \ \&\ u - v = y + \beta e,$

$\texttt{Assume}\ [6] : \Big(\{y, x, e\} : \texttt{LinearlyDependent}(A)\Big),$

$\Big(\alpha, \beta, [7]\Big) := [6] : \sum \alpha, \beta \in \mathbb{R} \,.\, x = \alpha y + \beta e,$

$[8] := \Game\mathbb{R}\text{-}\mathsf{ALGE}(A) : x^2 = \alpha^2 y^2 + 2\beta\alpha v + \beta^2 e,$

$[6.*] := [2][8] : \perp;$

$\leadsto [6] := E(\perp) : \Big(\{x, y, e\} : \texttt{LinearlyIndependent}(\mathbb{R}, a)\Big),$

$[7] := \Game\mathbb{R}\text{-}\mathsf{ALGE}[3][4] : -2(\lambda + \mu)e = (u + v)^2 + (u - v)^2 =$

$\quad = (x + \alpha e)^2 + (y + \beta e)^2 = x^2 + y^2 + 2\alpha x + 2\beta y + (\alpha^2 + \beta^2)e,$

$[8] := [7] - \ldots : 2\alpha x + 2\beta y = x^2 + y^2 + (2\lambda + 2\mu + \alpha^2 + \beta^2)e,$

$[9] := [8][6] : \alpha = 0 \ \&\ \beta = 0,$

$[(u, v).*] := [4][9] : u + v \in D;$

$\leadsto [2] := \Game\mathbb{R}\text{-}\mathsf{VS}\Game\texttt{InnerSum} : A = D \oplus \mathbb{R}e,$

$[1.1] := [2] \dim D = 0 : \dim D = 0 \Rightarrow A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{R},$

$[2.2] := [2] \dim D = 1 : \dim D = 1 \Rightarrow A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{C},$

$\texttt{Assume}\ [3] : \dim D > 1,$

$(\mathrm{i}, [4]) := \Game \dim D\Game D\texttt{PositiveRealSquareRoot} : \sum \mathrm{i} \in D \,.\, \mathrm{i}^2 = -e,$

$p := \Lambda u, v \in D \,.\, -uv - vu : \mathcal{L}(D, D; D),$

$\texttt{Assume}\ u, v : D,$

$[5] := \Game\mathbb{R}\text{-}\mathsf{ALGE}(A)\Game D : p^2(u, v)(-uv - vu)^2 = u^2v^2 + vu^2v + uv^2u_v^2u^2 = 4v^2u^2 \in \mathbb{R}_+e,$

$(u, v).* := \texttt{MinimalAlgebraicRoots}[5] : p(u, v) \in \mathbb{R}e;$

$\leadsto [5] := \Game^{-1}\texttt{InnerProduct}(D) : \Big(p : \texttt{InnerProduct}(D)\Big),$

$(S,[6]) := [0]\texttt{OrthogonalDecompositionExists}(\mathbb{R}\mathrm{i}) : \sum S \subset_{\mathbb{R}\text{-}\mathsf{VS}} D \,.\, D = \mathbb{R}\mathrm{i}\perp S,$

$(\mathrm{j},[8]) := [3][6]\text{Ɑ}D : \sum \mathrm{j} \in S \,.\, \mathrm{j}^2 = -e,$

$\mathrm{k} := \mathrm{ij} : A,$

$[9] := \text{Ɑ}\mathbb{R}\text{-}\mathsf{ALGE}(A)\text{Ɑ}\mathrm{k} : 0 = (\mathrm{k}-\mathrm{k})^2 = (\mathrm{ij}+\mathrm{ji})^2 = 2\mathrm{k}^2 + 2,$

$[10] := \text{Ɑ}D[9] : \mathrm{k} \in D,$

$[11] := \text{Ɑ}\mathrm{k}\text{Ɑ}\mathrm{j} : p(\mathrm{i},\mathrm{k}) = \mathrm{i}p(\mathrm{i},\mathrm{j}) = 0,$

$[12] := \text{Ɑ}\mathrm{k}\text{Ɑ}\mathrm{j} : p(\mathrm{j},\mathrm{k}) = p(\mathrm{i},\mathrm{j})\mathrm{j} = 0,$

$(Z,[13]) := \texttt{OrthogonalDecompositioExists} : \sum Z \subset_{\mathbb{R}\text{-}\mathsf{VS}} S \,.\, D = \mathbb{R}\mathrm{i}\perp\mathbb{R}\mathrm{j}\perp\mathbb{R}\mathrm{k}\perp Z,$

$\texttt{Assume}\ z : Z,$

$\texttt{Assume}\ [14] : z^2 = -e,$

$[15] := \text{Ɑ}\texttt{Orthogonal}\text{Ɑ}p[13] : \mathrm{i}z = -z\mathrm{i},$

$[16] := \text{Ɑ}\texttt{Orthogonal}\text{Ɑ}p[13] : \mathrm{j}z = -z\mathrm{j},$

$[17] := \text{Ɑ}\texttt{Orthogonal}\text{Ɑ}p[13] : \mathrm{k}z = -z\mathrm{k},$

$[18] := [17]\text{Ɑ}\mathrm{k}[15][16] : \mathrm{ij}z = -z\mathrm{ij} = \mathrm{i}z\mathrm{j} = -\mathrm{ij}z,$

$[19] := [19]\text{Ɑ}\mathsf{ABEL}(A) : \mathrm{ij}z = 0,$

$[z.*] := \text{Ɑ}\texttt{DivisionAlgebra}(A)[14][8][4] : \perp;$

$\rightsquigarrow [14] := E(\perp) : Z = \{0\},$

$[3.*] := \text{Ɑ}\mathbb{H}[14] : A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{H};$

$\rightsquigarrow [3.3] := I(\Rightarrow) : \dim D > 1 \Rightarrow A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{H},$

$[*] := \text{Ɑ}\mathbb{Z}_+E(|)[1.1][2.2][3.3] : A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{R} | A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{C} | A \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathbb{H};$

$\square$

1.7 Widderburn Representation Theorems

$\texttt{RepresentationInvariantMaps} :: \prod R \in \mathsf{ANN} \,.\, \prod A, B : R\text{-}\mathsf{MOD} \,.$

$\quad .\, \texttt{Representayion}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \to ?R\text{-}\mathsf{MOD}(A, B)$

$T : \texttt{RepresentationInvariantMap} \iff \Lambda\rho : \texttt{Representation}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \,.\, T \in \mathcal{L}_\rho(A; B) \iff$

$\quad \iff \Lambda\rho : \texttt{Representation}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \,.\, \forall f \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(A) \,.\, fT = T\rho(f)$

$\texttt{RepresentationInvariantOperators} :: \prod R \in \mathsf{ANN} \,.\, \prod A, B : R\text{-}\mathsf{MOD} \,.$

$\quad .\, \texttt{Representayion}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \to ?\mathrm{End}_{R\text{-}\mathsf{MOD}}(B)$

$T : \texttt{RepresentationInvariantMap} \iff \Lambda\rho : \texttt{Representation}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \,.\, T \in \mathcal{L}_\rho(B) \iff$

$\quad \iff \Lambda\rho : \texttt{Representation}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \,.\, \forall f \in \mathrm{End}_{R\text{-}\mathsf{MOD}}(A) \,.\, \rho(f)T = T\rho(f)$

$\texttt{tensorEvaluation} :: \prod R \in \mathsf{ANN} \,.\, \prod A, B \in R\text{-}\mathsf{MOD} \,.\, \mathcal{L}(A; B) \otimes A \xrightarrow{R\text{-}\mathsf{MOD}} B$

$\texttt{tensorEvaluation}\,(T \otimes a) = \mathcal{E}(T \otimes a) := T(a)$

$\texttt{InvariantEvaluation} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{MOD} \,.\, \forall \rho : \texttt{Representation}(\mathrm{End}_{R\text{-}\mathsf{MOD}}(A), B) \,.$

$\quad .\, \forall f : A \xrightarrow{R\text{-}\mathsf{MOD}} B \,.\, \Big(\mathcal{E}\rho(f)\Big)_{|\mathcal{L}_\rho(A,B)\otimes A} = \Big((\mathrm{id} \otimes f)\mathcal{E}\Big)_{|\mathcal{L}_\rho(A,B)\otimes A}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T : \mathcal{L}_\rho(A, B),$

$\texttt{Assume}\; a : A,$

$[T.*] := \text{Ɑ}\mathcal{E}\text{Ɑ}L_\rho(A, B)(T)\text{Ɑ}^{-1}\mathcal{E}\text{Ɑ}^{-1}(\mathrm{id} \otimes f) :$

$\quad : (T \otimes a)\Big(\mathcal{E}\rho(f)\Big) = a\; T\; \rho(f) = a\; f\; T = \Big(T \otimes (a\; f)\Big)\mathcal{E} = (T \otimes a)\Big((\mathrm{id} \otimes f)\mathcal{E}\Big);$

$\rightsquigarrow [*] := \text{Ɑ}\texttt{tensorProduc} : \texttt{This};$

$\square$

$\texttt{WidderburnEvaluationTheorem} :: \forall k : \texttt{Field} \,.\, \forall V, W : k\text{-}\mathsf{FDVS} \,.\, \forall \rho : \texttt{Representation}(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), W) \,.$

$\quad .\, \mathcal{E} : \mathcal{L}_\rho(V, W) \otimes V \xleftrightarrow{k\text{-}\mathsf{VS}} W$

$\texttt{Proof} \;=$

$(n, e) := \text{Ɑ}k\text{-}\mathsf{FDVS} : \sum n \in \mathbb{N} \,.\, \sum e : \texttt{Basis}(n, V),$

$\texttt{Assume}\; i, j : n,$

$T(e_i \otimes e_j^*) := \Lambda v \in V \,.\, \alpha_j e_j^*(v) e_i : \mathrm{End}_{k\text{-}\mathsf{VS}}(V);$

$\rightsquigarrow T := \texttt{TensorProductBasis}(V, V^*)\text{Ɑ}\texttt{Basis}(n, V)(e)\text{Ɑ}e^* : V \otimes V^* \xleftrightarrow{k\text{-}\mathsf{VS}} \mathrm{End}_{k\text{-}\mathsf{VS}}(V),$

$F := \Lambda i \in n \,.\, \Lambda v \in V \,.\, \Lambda w \in W \,.\, \sum_{j=1}^{n} e_j^*(x)\rho\Big(T(e_j \otimes e_i^*)\Big)(y) : n \to \mathcal{L}(V, W; W),$

`Assume` $i : n$,

`Assume` $S : \mathcal{L}_\rho(V;W)$,

`Assume` $v, u : V$,

$[u,*] := \text{Ↄ}F_i\text{Ɑ}\mathcal{L}_\rho(V;W)(Sy)\text{Ↄ}T\texttt{OperatorByBasis}(S) :$

$$: F_i(u)(Sv) = \sum_{j=1}^{n} e_j^*(u)\rho\Big(T(e_j \otimes e_i^*)\Big)(Sv) =$$

$$= \sum_{j=1}^{n} e_j^*(u)(v\ T(e_j \otimes e_i^*)\ S) = \sum_{j=1}^{n} e_j^*(u)v^iSe_j = v^iS(u);$$

$\rightsquigarrow [i.*] := I(=,\rightarrow) : F_i(Sv) = v^iS;$

$\rightsquigarrow [1] := I^3(\forall) : \forall i \in n\ .\ \forall S \in \mathcal{L}_\rho(V;W)\ .\ \forall v \in V\ .\ F_i(Sv) = v^iS,$

`Assume` $w : W$,

$[w.*] := \text{Ↄ}F\text{Ɑ}e^*\text{Ɑ}k\text{-}\mathsf{VS}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(\rho)\text{Ↄ}T\text{Ɑ}k\text{-}\mathsf{ALGE}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(\rho) :$

$$: \sum_{i=1}^{n} F_i(e_i, w) = \sum_{i=1}^{n}\sum_{j}^{n} e_j^*(e_i)\rho\Big(T(e_j \otimes e_i^*)\Big)(w) = \sum_{i=1}^{n}\rho\Big(T(e_i \otimes e_i^*)\Big)(w) =$$

$$= \rho\left(\sum_{i=1}^{n} T(e_i \otimes e_i^*)\right)(w) = \rho(\mathrm{id}_V)(w) = \mathrm{id}_W(w) = w;$$

$$\rightsquigarrow [2] := I(\forall) : \forall w \in W\ .\ \sum_{i=1}^{n} F_i(e_i, w) = w,$$

`Assume` $w : W$,

`Assume` $i : n$,

`Assume` $S : \mathrm{End}_{k\text{-}\mathsf{VS}}(V)$,

`Assume` $v : V$,

$[v.*] := \text{Ↄ}F_i\texttt{OperatorInCoordinates}(S)\text{Ɑ}k\text{-}\mathsf{VS}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(\rho)$

$\quad \text{Ɑ}k\text{-}\mathsf{VS}\Big(V \otimes V^*, \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(T)\text{Ↄ}T\text{Ɑ}k\text{-}\mathsf{ALGE}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(\rho)\text{Ↄ}^{-1}F_i :$

$$: v\ S\ F_i(w) = F_i\big(v\ S, w\big) = \sum_{j=1}^{n} e_j^*(v\ S)\rho(T(e_j \otimes e_i^*))(w) = \sum_{j=1}^{n}\sum_{t=1}^{n} v^tS_{j,t}\rho(T(e_j \otimes e_i^*))(w) =$$

$$= \sum_{t=1}^{n} v^t\rho\left(\sum_{j=1}^{n} T(S_{j,t}e_j \otimes e_i^*)\right)(w) = \sum_{t=1}^{n} v^t\rho(T(e_t \otimes e_i^*)S)(w) = w\left(\sum_{t=1}^{n} e_t^*(v)\rho\Big(T(e_t \otimes e_i^*)\Big)\right)\rho(S) =$$

$$= F_i(v, w)\rho(S) = v\ F_i(w)\ \rho(S);$$

$\rightsquigarrow [S.*] := I(=,\rightarrow) : SF_i(w) = F_i(w)\rho(S);$

$\rightsquigarrow [w.*] := I(\forall)\text{Ɑ}\mathcal{L}_\rho(V;W) : F_i(w) \in \mathcal{L}_\rho(V;W);$

$\rightsquigarrow [3] := I^2(\forall)\text{Ɑ}^{-1}\texttt{Subset}\text{Ɑ}^{-1}\texttt{image} : F_i(W) \subset \mathcal{L}_\rho(V;W),$

$$\mathcal{A} := \Lambda w \in W\ .\ \sum_{i=1}^{n} F_i(w) \otimes e_i : W \xrightarrow{k\text{-}\mathsf{VS}} \mathcal{L}_\rho(V, W) \otimes V,$$

`Assume` $S : \mathcal{L}_\rho(V, W,$

`Assume` $v : V$,

$[S.*] := \text{Ɑ}\mathcal{E}\text{Ↄ}\mathcal{A}[1]\text{Ɑ}\mathcal{L}\Big(\mathcal{L}_\rho(V;W), V, \mathcal{L}_\rho(V;W)\Big)(\otimes)\text{Ɑ}\texttt{coordinates}(e, v) :$

$$: (S \otimes v)\mathcal{E}\mathcal{A} = v\ S\mathcal{A} = \sum_{i=1}^{n} F_i(v\ S) \otimes e_i = \sum_{i=1}^{n} v^iS \otimes e_i = S \otimes \left(\sum_{i=1}^{n} v^ie_i\right) = S \otimes v;$$

$\rightsquigarrow [4] := \text{Ɑ}\texttt{tensorProduct} : \mathcal{E}\mathcal{A} = \mathrm{id},$

Assume $w : W$,

$[w.*] := \mathcal{I}\mathcal{A}\mathcal{I}\mathcal{E}[2] : w\mathcal{A}\mathcal{E}\Big(\sum_{i=1}^{n} F_i(w) \otimes e_i\Big)\mathcal{E} = \sum_{i=1}^{n} F_i(e_i, w) = w;$

$\rightsquigarrow [5] := I(=, \to) : \mathcal{A}\mathcal{E} = \mathrm{id},$

$[6] := \mathcal{I}^{-1}\mathtt{Inverse}[4][5] : \mathcal{E}^{-1} = \mathcal{A},$

$[*] := \mathcal{I}^{-1}\mathtt{Iso} : \mathtt{This};$

$\square$

$\mathtt{RepresentationInvariantDimension} :: \forall k : \mathtt{Field} \,.\, \forall V, W \in k\text{-}\mathsf{FDVS} \,.$

$\quad .\, \forall \rho : \mathrm{End}_{k\text{-}\mathsf{VS}}(V) \xrightarrow{k\text{-}\mathsf{ALGE}} \mathrm{End}_{k\text{-}\mathsf{VS}}(W) \,.\, \dim V \dim \mathcal{L}_\rho(V; W) = \dim W$

$\mathtt{Proof} \;=$

...

$\square$

$\mathtt{tensorComposition} :: \prod R \in \mathsf{ANN} \,.\, \prod A, B \in R\text{-}\mathsf{MOD} \,.$

$\quad .\, R\text{-}\mathsf{MOD}(A, B) \otimes \mathrm{End}_{R\text{-}\mathsf{MOD}(B)} \xrightarrow{R\text{-}\mathsf{MOD}} R\text{-}\mathsf{MOD}(A, B)$

$\mathtt{tensorComposition}\,(T \otimes S) = \mathcal{C}(T \otimes S) := TS$

$\mathtt{WidderburnCompositionTheorem} :: \forall k : \mathtt{Field} \,.\, \forall V, W : k\text{-}\mathsf{FDVS} \,.\, \forall \rho : \mathtt{Representation}(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), W) \,.$

$\quad .\, \mathcal{C} : L_\rho(W) \xleftrightarrow{k\text{-}\mathsf{ALGE}} \mathrm{End}_{k\text{-}\mathsf{VS}}\Big(\mathcal{L}_\rho(V, W)\Big)$

$\mathtt{Proof} \;=$

$\mathcal{B} := \Lambda\Omega \in \mathrm{End}_{k\text{-}\mathsf{VS}}\Big(\mathcal{L}_\rho(V, W)\Big) \,.\, \mathcal{E}^{-1}(\Omega \otimes \mathrm{id})\mathcal{E} : \mathrm{End}_{k\text{-}\mathsf{VS}}\Big(\mathcal{L}_\rho(V, W)\Big) \xrightarrow{k\text{-}\mathsf{VS}} \mathrm{End}_{k\text{-}\mathsf{VS}}(W),$

Assume $\Omega : \mathrm{End}_{k\text{-}\mathsf{VS}}\Big(\mathcal{L}_\rho(V, W)\Big),$

Assume $S : \mathrm{End}_{k\text{-}\mathsf{VS}}(V),$

Assume $w : W$,

$[w.*] := \mathcal{I}\mathcal{B}\mathcal{I}\mathcal{E}^{-1}\mathcal{I}\mathtt{tensorMap}\mathcal{I}\mathcal{E}\mathcal{I}F_i\mathcal{I}k\text{-}\mathsf{ALGE}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)(\rho)\mathcal{I}T$

$\quad \mathcal{I}^{-1}\mathtt{operatorMatrix}(S, e)\mathcal{I}k\text{-}\mathsf{VS}\Big(\mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(V)\Big)\rho\Omega\mathcal{I}\mathtt{operatorMatrix}(S, e)\mathcal{I}^{-1}F$

$\quad \mathcal{I}\mathcal{L}_\rho(V, W)(wF_t\Omega)\mathcal{I}^{-1}\mathcal{B} :$

$$: w\rho(S)\mathcal{B}(\Omega) = w\rho(S)\mathcal{E}^{-1}(\Omega \otimes \mathrm{id})\mathcal{E} = \left(\sum_{i=1}^{n} F_i\Big(w\rho(S)\Big) \otimes e_i\right)(\Omega \otimes \mathrm{id})\mathcal{E} = \sum_{i=1}^{n} (w\;\rho(S)\;F_i\Omega)(e_i) =$$

$$= \sum_{i,j=1}^{n} \Omega\Big(e_j^*\rho\big(T(e_j \otimes e_i^*)\big)\rho(S)(w)\Big)(e_i) = \sum_{i,j=1}^{n} \Omega\Big(e_j^*\rho\big(ST(e_j \otimes e_i^*)\big)(w)\Big)(e_i) =$$

$$= \sum_{i,j=1}^{n} \Omega\left(e_j^*\rho\left(\sum_{t=1}^{n} S_{i,t}T(e_j \otimes e_t^*)\right)(w)\right)(e_i) = \sum_{i,j,t=1}^{n} \Omega\left(e_j^*\rho\left(\sum_{t=1}^{n} T(e_j \otimes e_t^*)\right)(w)\right)(S_{i,t}e_i) =$$

$$= \sum_{j,t=1}^{n} \Omega\left(e_j^*\rho\left(\sum_{t=1}^{n} T(e_j \otimes e_t^*)\right)(w)\right)(Se_t) = \sum_{t=1}^{n} (wF_t\Omega)(Se_t) = \left(\sum_{t=1}^{n} (wF_t\Omega)(e_t)\right)\rho(S) = w\rho(S)\mathcal{B}(\Omega);$$

$\rightsquigarrow [S.*] := I(=, \to) : \rho(S)\mathcal{B}(\Omega) = \mathcal{B}(\Omega)\rho(S);$

$\rightsquigarrow [\Omega.*] := \mathcal{I}\mathcal{L}_\rho(W) : \mathcal{B}(\Omega) \in \mathcal{L}_\rho(W);$

$\rightsquigarrow [7] := \mathcal{I}^{-1}\mathtt{Subset}\mathcal{I}^{-1}\mathtt{image} : \mathrm{Im}\,\mathcal{B} \subset \mathcal{L}_R(W),$

$\texttt{Assume}\ \Omega : \mathrm{End}_{k\text{-VS}}\Big(L_\rho(V,W)\Big),$

$\texttt{Assume}\ S : \mathcal{L}_\rho(V,W),$

$\texttt{Assume}\ v : V,$

$[v.*] := \mathcal{O}\mathcal{B}\mathcal{I}\mathcal{C}\mathcal{O}\mathcal{E}^{-1}![1]\mathcal{I}\mathcal{L}\Big(\mathcal{L}_\rho(V,W), V; \mathcal{L}_\rho(V,W)\otimes V\Big)(\otimes)\mathcal{I}\,\texttt{coordinates}(v)\mathcal{I}\,\texttt{tensorMap}(\Omega,\mathrm{id})\mathcal{I}\mathcal{E} :$

$$: v\Big(S(\Omega\mathcal{B}\mathcal{C})\Big) = v\Big(S\big(\mathcal{E}^{-1}(\Omega\otimes\mathrm{id})\mathcal{E}\mathcal{C}\big)\Big) = (v\ S)\Big(\mathcal{E}^{-1}(\Omega\otimes\mathrm{id})\mathcal{E}\Big) = \left(\sum_{i=1}^n F_i\Big(vS\Big)\otimes e_i\right)(\Omega\otimes\mathrm{id})\mathcal{E} =$$

$$= \left(\sum_{i=1}^n v^i S\otimes e_i\right)(\Omega\otimes\mathrm{id})\mathcal{E} = (S\otimes v)(\Omega\otimes\mathrm{id})\mathcal{E} = v\ \Omega(S);$$

$\rightsquigarrow [S.*] := I(=,\to) : S(\Omega\mathcal{B}\mathcal{C}) = S\ \Omega;$

$\rightsquigarrow [\Omega.*] := I(=,\to) : \Omega\ \mathcal{B}\mathcal{C} = \Omega;$

$\rightsquigarrow [8] := I(=,\to) : \mathcal{B}\mathcal{C} = \mathrm{id},$

$\texttt{Assume}\ S : \mathcal{L}_\rho(W),$

$\texttt{Assume}\ w : W,$

$[w.*] := \mathcal{O}\mathcal{B}\mathcal{O}\mathcal{E}^{-1}\mathcal{I}\,\texttt{tensorMap}\,\mathcal{I}\mathcal{C}\mathcal{O}F_i\mathcal{I}\mathcal{E}![2] :$

$$: w(S\mathcal{C}\mathcal{B}) = w\Big(\mathcal{E}^{-1}(S\mathcal{C}\otimes\mathrm{id})\mathcal{E}\Big) = \sum_{i=1}^n (F_i(w)\otimes e_i)(S\mathcal{C}\otimes\mathrm{id})\mathcal{E} = \sum_{i=1}^n (F_i(w)S\otimes e_i)\mathcal{E} = \sum_{i=1}^n F_i(e_i,w)S = w\ S;$$

$\rightsquigarrow [S.*] := I(=,\to) : S\ \mathcal{C}\mathcal{B} = S;$

$\rightsquigarrow [9] := I(=,\to) : \mathcal{C}\mathcal{B} = \mathrm{id},$

$[10] := \mathcal{I}^{-1}\texttt{Inverse}[8][9] : \mathcal{C}\mathcal{B} = \mathrm{id},$

$[*] := \mathcal{I}\,\texttt{Iso}[10] : \texttt{This};$

$\square$

$\texttt{EquevalentAlgebraRepresentation} :: \prod R\in\mathsf{ANN}\ .\ \prod A\in R\text{-}\mathsf{ALGE}\ .\ \prod X,Y\in R\text{-}\mathsf{MOD}\ .$

$.\ ?\Big(\texttt{Representation}(A,X)\times\texttt{Representation}(A,Y)\Big)$

$(\rho,\rho') : \texttt{EquivalentAlgebraRepresentation} \iff \rho\cong\rho' \iff \exists\varphi : X\xrightarrow{R\text{-}\mathsf{MOD}} Y : \forall a\in A\ .\ \rho(a)\varphi = \rho'(a)\varphi$

$\texttt{TensorRepresentationEquivalence} :: \forall k : \texttt{Field}\ .\ \forall V,W\in k\text{-}\mathsf{FDVS}\ .\ \forall A\in k\text{-}\mathsf{ALGE}\ .$

$\forall R : A\otimes\mathrm{End}_{k\text{-VS}}(V)\xrightarrow{k\text{-}\mathsf{ALGE}}\mathrm{End}_{k\text{-VS}}(W)\ .\ \exists\rho' : A\xrightarrow{k\text{-}\mathsf{ALGE}}\mathrm{End}_{R\text{-VS}}\Big(\mathcal{L}_\rho(V,W)\Big) : R\cong\rho'\otimes\mathrm{id}$

$\texttt{where}\ \rho = \Lambda T\in\mathrm{End}_{kG\text{-VS}}(V)\ .\ R(e_A\otimes T)$

$\texttt{Proof}\ =$

$p := \Lambda a\in A\ .\ R(a\otimes\mathrm{id}) : A\xrightarrow{k\text{-}\mathsf{ALGE}}\mathrm{End}_{k\text{-VS}}(W),$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ S : \mathrm{End}_{k\text{-VS}}(V),$

$[S.*] := \mathcal{O}\rho\mathcal{O}p\mathcal{I}^2 k\text{-}\mathsf{ALGE}\Big(A\otimes\mathrm{End}_{k\text{-VS}}(V),\mathrm{End}_{k\text{-VS}}(W)\Big)R\mathcal{O}^{-1}\rho\mathcal{O}^{-1}p :$

$$\rho(S)p(a) = R(e\otimes S)R(a\otimes\mathrm{id}) = R(a\otimes S) = R(a\otimes\mathrm{id})R(e\otimes S) = p(a)\rho(S);$$

$\rightsquigarrow [a.*] := \mathcal{I}\mathcal{L}_\rho(W) : p(a)\in\mathcal{L}_\rho(W);$

$\rightsquigarrow [11] := \mathcal{I}^{-1}\texttt{Subset}\mathcal{I}^{-1}\texttt{Image} : \mathrm{Im}\,p\subset\mathcal{L}_\rho(W),$

$\rho' := p\mathcal{C} : A\xrightarrow{k\text{-}\mathsf{ALGE}}\mathrm{End}_{k\text{-VS}}\Big(\mathcal{L}_\rho(V,W)\Big),$

$\texttt{Assume}\ a : A$,

$\texttt{Assume}\ f : \mathrm{End}_{k\text{-}\mathsf{VS}}(V)$,

$\texttt{Assume}\ S : \mathcal{L}_\rho(V, W)$,

$\texttt{Assume}\ v : W$,

$[S.*] := \llparenthesis\texttt{tensorMap}\rrparenthesis\rho'\llparenthesis\mathcal{E}\rrparenthesis p\llparenthesis\mathcal{C}\llparenthesis\mathcal{L}_\rho(V, W)(S)\rrparenthesis\rho\llparenthesis k\text{-}\mathsf{ALGE}\Big(A \otimes \mathrm{End}_{k\text{-}\mathsf{VS}}(V), \mathrm{End}_{k\text{-}\mathsf{VS}}(W)\Big)R\llparenthesis^{-1}\mathcal{E} :$

$\quad : (S \otimes v)(\rho'(a) \otimes f)\mathcal{E} = (S\ \rho'(a)) \otimes (v\ f)\mathcal{E} = (v\ f)\Big(S\ (ap\mathcal{C})\Big) = (v\ f)\Big(S\ R(a \otimes \mathrm{id})\mathcal{C}\Big) =$

$\quad = v\ f\ S\ R(a \otimes \mathrm{id}) = v\ S\ \rho(f)\ R(a \otimes \mathrm{id}) = v\ S\ R(e \otimes f)R(a \otimes \mathrm{id}) = v\ S\ R(a \otimes f) = (S \otimes v)\mathcal{E}R(a \otimes f)$;

$\leadsto [a.*] := I(=, \to) : (\rho'(a) \otimes f)\mathcal{E} = \mathcal{E}R(a \otimes f)$;

$[*] := \llparenthesis\texttt{tensorProduct}\llparenthesis^{-1}\texttt{EquivalentAlgebraRepresentation} : \texttt{This}$;

$\square$

2 Coalgebras and Comodules

2.1 Coalgebras

$$\texttt{Coalgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod A : R\text{-}\mathsf{MOD} \,.\, \left(A \otimes A \xrightarrow{R\text{-}\mathsf{MOD}} A\right) \times AR\text{-}\mathsf{MOD} \xrightarrow{R\text{-}\mathsf{MOD}} R$$
$$(A, \Delta, \eta) : \texttt{A} \iff \Delta(\mathrm{id} \otimes \Delta) = \Delta(\Delta \otimes \mathrm{id}) \;\&\; \Delta(\mathrm{id} \otimes \eta) = \mathrm{id} \otimes 1 \;\&\; \Delta(\eta \otimes \mathrm{id}) = 1 \otimes \mathrm{id}$$

$$\texttt{comultiplication} :: \prod A : \texttt{Coalgebra} \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes A$$
$$\texttt{comultiplication}\,((A, \Delta, \eta, [1], [2], [3])) = \Delta_A := \Delta$$

$$\texttt{comultiplication} :: \prod A : \texttt{Coalgebra} \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes A$$
$$\texttt{comultiplication}\,((A, \Delta, \eta, [1], [2], [3])) = \Delta_A := \Delta$$

$$\texttt{counit} :: \prod A : \texttt{Coalgebra} \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}} R$$
$$\texttt{counit}\,((A, \Delta, \eta, [1], [2], [3])) = \eta_A := \eta$$

$$\texttt{comultiplicationProperty} :: \prod A : \texttt{Coalgebra} \,.\, \texttt{Type}$$
$$\texttt{comultiplicationProperty}\,(A, \Delta, \eta, [1], [2], [3]) := [1]$$

$$\texttt{rightCounitProperty} :: \prod A : \texttt{Coalgebra} \,.\, \texttt{Type}$$
$$\texttt{rightCounitProperty}\,(A, \Delta, \eta, [1], [2], [3]) := [2]$$

$$\texttt{leftCounitProperty} :: \prod A : \texttt{Coalgebra} \,.\, \texttt{Type}$$
$$\texttt{leftCounitProperty}\,(A, \Delta, \eta, [1], [2], [3]) := [3]$$

$$\texttt{Cocomutative} :: ?\texttt{Coalgebra}(A)$$
$$A : \texttt{Cocomutative} \iff \texttt{swap}\ \Delta_A = \Delta_A$$

$$\texttt{SweedlerSum} :: \prod A : \texttt{Coalgebra}(R) \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes A$$
$$\texttt{SweedlerSum}\,(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} := \Delta_A(a)$$

$$\texttt{trivialCoalgebra} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Coalgebra}(R)$$
$$\texttt{trivialCoalgebra}\,() := \Big(R, \Lambda r \in R \,.\, r(e \otimes e), \mathrm{id}\Big)$$

$$\texttt{dividedPowerCoalgebra} :: \prod R \in \mathsf{ANN} \,.\, \texttt{Coalebra(R)}$$
$$\texttt{dividedPowerCoalgebra}\,() := \left(R[x], Ɑ\texttt{Free}(R[x])\Lambda n \in \mathbb{Z}_+ \,.\, \sum_{i=0}^{n} C_n^i x^i \otimes x^{n-i}, Ɑ\texttt{Free}(R[x])\Lambda n \in \mathbb{Z}_+ \,.\, \delta_0^n\right)$$

$\texttt{Coideal} :: \prod A : \texttt{Coalgebra}(R) \,.\, \texttt{Submodule}(A)$

$I : \texttt{Coideal} \iff \forall a \in I \,.\, \Delta(a) \subset I \otimes A + A \otimes I \;\&\; \eta(a) = 0$

$\texttt{CoidealQuotient} :: \forall A : \texttt{Coalgebra}(R) \,.\, \forall I : \texttt{Coideal}(A) \,.\, \left(\frac{A}{I}, \Delta\Big([\cdot]_I \otimes [\cdot]_I\Big), \eta\right) : \texttt{Coalgebra}(I)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ h : I,$

$[1] := \mathit{\Pi}\texttt{Coideal}(I)(h) : \Delta(h) \subset A \otimes I + A \otimes I,$

$[h.*] := \mathit{\Pi}\texttt{SweedlerSum}\mathit{\Pi}\texttt{quotient}[1] : \sum_{(h)} [h_1] \otimes [h_2] = 0;$

$\rightsquigarrow [1] := \mathit{\Pi}\texttt{Subset} : I \subset \ker \Delta[\cdot]_I \otimes [\cdot]_I,$

$\Big(\phi, [2]\Big) := \texttt{QuotientMapTHM}[1] : \sum \phi : \frac{A}{I} \xrightarrow{R\text{-MOD}} \frac{A}{I} \otimes \frac{A}{I} \,.\, \pi_I \phi = \Delta(\pi_I \otimes \pi_I),$

$\Big(\eta', [3]\Big) := \texttt{QuotientMapTHM}\mathit{\Pi}\texttt{Coideal}(A)(I) : \sum \eta' : \frac{A}{I} \xrightarrow{R\text{-MOD}} R \,.\, \pi_I \eta' = \eta,$

$[*] := \mathit{\Pi}^{-1}\texttt{Coalgebra}(R)[2][3] : \texttt{This};$

$\square$

$\texttt{quotientCoalgebra} :: \prod A : \texttt{Coalgebra}(R) \,.\, \texttt{Coideal}(I) \to \texttt{Coalgebra}(A)$

$\texttt{quotientCoalgebra}\,(I) = \frac{A}{I} := \texttt{CoidealQuotient}$

$\texttt{Grouplike} :: \prod A : \texttt{Coalgebra}(R) \,.\, ?A$

$g : \texttt{Grouplike} \iff \Delta(g) = g \otimes g \;\&\; g \neq 0$

$\texttt{GrouplikeCounit} :: \forall A : \texttt{Coalgebra} \;\&\; \texttt{TorsionFree}(R) \,.\, \forall g : \texttt{Grouplike}(A) \,.\, \eta(g) = e$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{GrouplikeOfDividedPower} :: \forall R \in \texttt{IntegralDomain} \,.\, \texttt{Grouplike}\ \texttt{dividedPowerCoalgebra}(R) = \{1\}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a_i x^i : \texttt{Grouplike}\ \texttt{dividedPowerCoalgebra},$

$n := \deg a_i x^i : \mathbb{Z}_+,$

$[1] := \mathit{\Pi}\texttt{Groupllike}(a_i x^i)\mathit{\Pi}\texttt{dividedPowerCoalgebra} : a_i x^i \otimes a_i x^i = \Delta(a_i x^i) = a_i C_i^j x^j \otimes x^{i-j},$

$\texttt{Assume}\ [0] : n > 0,$

$[2] := \texttt{TensorProductBasis}[1] : a_n^2 = 0,$

$[0.*] := \mathit{\Pi}\ \deg[2] : \bot;$

$\rightsquigarrow [2] := E(\bot)\mathit{\Pi}\ \deg : a_i x^i = a_0,$

$[\ldots *] := \texttt{GrouplikeCounit}\mathit{\Pi}\texttt{dividedPowerCoalgebra}[2] : a_i x^i = 1;$

$\rightsquigarrow [*] := \mathit{\Pi}^{-1}\texttt{Singleton} : \texttt{This};$

$\square$

$\texttt{GrouplikeLinearlyIndependent} :: \forall R : \texttt{IntegralDomain}\ .$
$\quad . \forall A : \texttt{Coalgebra}(R)\ \&\ \texttt{TorsionFree}(R)\ .\ \texttt{Grouplike}(A) : \texttt{LinearlyIndependentSet}(A)$

$\texttt{Proof} \ =$

$G := \texttt{Grouplike}(A) :?A,$

$\texttt{Assume}\ a, b : G,$

$\texttt{Assume}\ \alpha : R,$

$\texttt{Assume}\ [1] : \alpha a = b,$

$[2] := \mathcal{O}\texttt{Grouplike}(a, b) : \alpha a \otimes a = \Delta(\alpha a) = \Delta(b) = b \otimes b = \alpha^2 a \otimes a,$

$[(a, b)*] := \mathcal{O}\texttt{TorsionFree}(R)(A)\mathcal{O}\texttt{IntegralDomain}(R) : \alpha = 1;$

$\rightsquigarrow [0] := I(\forall) : \forall a, b \in G\ .\ \forall \alpha \in R\ .\ \alpha a = b \Rightarrow a = b,$

$\texttt{Assume}\ \alpha : R^{\oplus G},$

$\texttt{Assume}\ [1] : \alpha_g g = 0,$

$\texttt{Assume}\ [2] : \alpha \neq 0,$

$(g, [3]) := E(\#, \rightarrow)[2] : \sum g \in G\ .\ \alpha_g \neq 0,$

$k := \mathrm{Frak}(R) : \texttt{Field},$

$V := A \otimes_R k : k\text{-}\mathsf{VS},$

$[4] := [3][1] : g =_V \dfrac{\alpha_h}{\alpha_g} h,$

$I := \Big\{h \in G \setminus \{g\} : \alpha_h \neq 0\Big\} : \texttt{Finite}(G),$

$\texttt{Assume}\ [5] : (I : \texttt{LinearlyIndependentSet}(V)),$

$[6] := \mathcal{O}\texttt{Grouplike}(g)[4] : \dfrac{\alpha_h \alpha_f}{\alpha_f} h \otimes f = g \otimes g = \Delta(g) = \dfrac{\alpha_h}{\alpha_g} h \otimes h,$

$[7] := \texttt{TensorProductBasis}\mathcal{O}\texttt{LinearlyIndependentSet}(V)(I) : \forall h, f \in I\ .\ \alpha_h \alpha_f = 0,$

$[8] := \mathcal{O}\texttt{IntegralDomain}(R)[7] : \alpha_I = 0,$

$[\alpha.*] := \mathcal{D}I[8] : I = \emptyset;$

$\rightsquigarrow [1] := I(\Rightarrow) : G : \texttt{LineatlyDependent}(A) \Rightarrow \forall I \subset G\ .\ |I| > 1 \Rightarrow I : \texttt{LinearlyDependent}(A),$

$[*] := [1][0] : (G : \texttt{LinearlyIndependentSet}(A));$

$\square$

$\texttt{CoalgebraMorphism} :: \prod A, B : \texttt{Coalgebra}(R)\ .\ ?(A \xrightarrow{R\text{-}\mathsf{MOD}} B)$

$f : \texttt{CoalgebraMorphism} \iff \forall x \in A\ .\ \Delta_A(f \otimes f) = f\Delta_B\ \&\ \eta_A = f\eta_B$

$\texttt{coalgebraCategory} :: \mathsf{RING} \to \mathsf{CAT}$

$\texttt{coalgebraCategory}\,(R) = R\text{-}\mathsf{COALG} := \Big(\texttt{Coalgebra}(R), \texttt{CoalgebraMorphism}(R), \circ, \mathrm{id}\Big)$

$\texttt{CounitMorphism} :: \forall A \in \texttt{Coalgebra}(R)\ .\ \eta_A : A \xrightarrow{R\text{-}\mathsf{COALG}} R$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{HomoPreservesGrouplike} :: \forall A, B \in \texttt{Coalgrbra}(R)\ .\ \forall f : A \xrightarrow{R\text{-}\mathsf{COALG}} B\ .$
$\quad . \forall g : \texttt{Grouplike}(A)\ .\ f(g) : \texttt{Grouplike}(B)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{tensorProductOfCoalgebras} :: R\text{-COALG} \times R\text{-COALG} \to R\text{-COALG}$

$$\texttt{tensorProductOfCoalgebras}\,(A,B) = A \otimes B := \Big(A \otimes B,$$
$$,\mathcal{C}\texttt{TensorProduct}\Lambda a \in A\,.\,\Lambda b \in B\,.\,\sum_{(a),(b)} (a_1 \otimes b_1) \otimes (a_2 \otimes b_2),$$
$$,\mathcal{C}\texttt{TensorProduct}\Lambda a \in A\,.\,\Lambda b \in B\,.\,\eta_A(a)\eta_B(b)\Big)$$

$\texttt{freeCoalgebra} :: \Big[k\text{-VS}\Big]_e \xrightarrow{\text{CAT}} k\text{-COALG}$

$\texttt{freeCoalgebra}\,(V,E) = F_{R\text{-COALG}}(V,E) := (M, \mathcal{C}\texttt{Basis}(V,E)\Lambda e \in E\,.\,e \otimes e, \mathcal{C}\texttt{Basis}(V,E)\Lambda e \in E\,.\,1)$

$\texttt{LeftCoideal} :: \prod A : \texttt{Coideal}(R)\,.\,?\texttt{Submodule}(A)$

$I : \texttt{LeftCoideal} \iff \eta(A) = 0 \,\&\, \Delta(I) \subset I \otimes A$

$\texttt{RightCoideal} :: \prod A : \texttt{Coideal}(R)\,.\,?\texttt{Submodule}(A)$

$I : \texttt{RightCoideal} \iff \eta(A) = 0 \,\&\, \Delta(I) \subset I \otimes A$

$\texttt{RightCoidealIsCoideal} :: \forall I : \texttt{RightCoideal}(A)\,.\,I : \texttt{Coideal}(A)$

Proof $=$

$\ldots$

$\square$

$\texttt{LeftCoidealIsCoideal} :: \forall I : \texttt{RightCoideal}(A)\,.\,I : \texttt{Coideal}(A)$

Proof $=$

$\ldots$

$\square$

$\texttt{SumOfCoideals} :: \forall I, J : \texttt{Coideal}(A)\,.\,I + J : \texttt{Coideal}(A)$

Proof $=$

$\ldots$

$\square$

2.2 Algebra-Coalgebra Duality

$$\texttt{dualAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{COALG}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{ALGE}$$

$$\texttt{dualAlgebra}\,(A) = A^* := \left(A^*, \Lambda f, g \in A^* \,.\, \Lambda a \in A \,.\, \sum_{(a)} f(a_1)g(a_2), \eta\right)$$

$$\texttt{Cofinite} :: \prod R \in \mathsf{ANN} \,.\, \prod A : R\text{-}\mathsf{ALGE} \,.\, ?\texttt{Ideal}(A)$$

$$I : \texttt{Cofinite} \iff \exists F : \texttt{Finite}\left(\frac{A}{I}\right) \,.\, \frac{A}{I} = \langle F \rangle_{E\text{-}\mathsf{MOD}}$$

$$\texttt{finiteDual} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{ALGE} \to R\text{-}\mathsf{ALGE}$$

$$\texttt{finiteDual}\,(A) = A^\circ := \Big\{ f \in M^* : \exists I : \texttt{Cofinite}(A) : f(I) = \{0\} \Big\}$$

$$\texttt{FiniteDualWhitness} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALGE} \,.\, \forall f \in A^\circ \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall [0] : f(I) = \{0\} \,.$$
$$\exists \overline{f} \in \left(\frac{A}{I}\right)^* \,.\, \pi_I^* \overline{f} = f$$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{FiniteDualTensorProduct} :: \prod R : \texttt{Field} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \mu_A^*(A^\circ) \subset A^\circ \otimes A^\circ$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : A^\circ,$

$\Big(I, [1]\Big) := \mathcal{I} A^\circ(f) : \sum I : \texttt{cofinite}(A) \,.\, f(I) = \{0\},$

$\Big(F, [2]\Big) := \mathcal{I}\texttt{Cofinite}(A)(I) : \sum F : \texttt{Finite}\left(\frac{A}{I}\right) \,.\, \frac{A}{I} = \langle F \rangle_{R\text{-}\mathsf{MOD}},$

$\Big(\overline{f}, [3]\Big) := \texttt{FiniteDualWhitness}(f, I) : \sum \overline{f} \in \left(\frac{A}{I}\right)^* \,.\, \pi_I^* \overline{f} = f,$

$\Big(\phi, [4]\Big) := \texttt{FGDualTensorBasis}\Big(\Big(\mu^*_{\frac{A}{I}}\Big)\overline{f}\Big) : \sum \phi : F^2 \to R \,.\, \Big(\mu^*_{\frac{A}{I}}\Big)\overline{f} = \sum_{a,b \in F} \phi_{a,b} a^* \otimes b^*,$

$\texttt{Assume}\ \sum_{i=1}^n x_i \otimes y_i : A \otimes A,$

$$\dots * := \mathcal{I}\mu_A^*[3]\mathcal{I}^{-1}\mu^*_{\frac{A}{I}}\mathcal{I}\texttt{Ideal}(A)(I)[4]\mathcal{I}^{-1}\pi_I^* : (\mu_A^* f)\sum_{i=1}^n x_i \otimes y_i = \sum_{i=1}^n f(x_i y_i) = \sum_{i=1}^n \pi_I^* \overline{f}(x_i y_i) =$$
$$= \Big(\mu^*_{\frac{A}{I}}\Big)\overline{f}\sum_{i=1}^n [x_i] \otimes [y_i] = \sum_{a,b \in F} \phi_{a,b}(a^* \otimes b^*)\sum_{i=1}^n [x_i] \otimes [y_i] = \sum_{a,b \in F} \phi_{a,b}(\pi_I^* a^* \otimes \pi_I^* b^*)\sum_{i=1}^n x_i \otimes y_i;$$

$\rightsquigarrow [5] := I(=, \to) : \mu_A^* f = \sum_{a,b \in F} \phi_{a,b}(\pi_I^* a^* \otimes \pi_I^* b^*),$

$[6] := \mathcal{I}\pi_I^*\mathcal{I}\pi_I : \forall a \in F \,.\, I \subset \ker \pi_I^* a,$

$[7] := \mathcal{I}\texttt{Cofinite}(I) : \forall a \in F \,.\, \pi_I^* a \in A^\circ,$

$[*] := [5][7] : \mu_A * (A^\circ) \subset A^\circ \otimes A^\circ;$

$\square$

$\texttt{FiniteDualIsCoalgebra} :: \forall k : \texttt{Field} . \forall A \in k\text{-ALGE} . (A^\circ, \mu_A^*, e_A^*) \in k\text{-COALG}$

$\texttt{Proof} =$

$\texttt{Assume}\ f : A^*,$

$\texttt{Assume}\ a, b, c : A,$

$[(a,b,c)\ldots] := \mathcal{Q}\mu_A^* : \mu_A^*(\mu_A^* \otimes \mathrm{id})(f)(a \otimes b \otimes c) = f(abc) = \mu_A^*(\mathrm{id} \otimes \mu_A^*)(f)(a \otimes b \otimes c);$

$\rightsquigarrow f.(*) := \mathcal{Q}\texttt{TensorProduct}I(=,\to) : \mu_A^*(\mu_A^* \otimes \mathrm{id})(f) = (\mu_A^*(\mathrm{id} \otimes \mu_A^*(f);$

$\rightsquigarrow [1] := I(=,\to) : \mu_A^*(\mu_A^* \otimes \mathrm{id}) = \mu_A^*(\mu_A^* \otimes \mu_A^*),$

$\texttt{Assume}\ f : A^*,$

$\texttt{Assume}\ a : A,$

$[f.*] := \texttt{TrivialTensorProduct}\mathcal{Q}e_A^*\mathcal{Q}e : \mu_A^*(\mathrm{id} \otimes e_A^*)(f)(a) = (\mathrm{id} \otimes e_A^*)(f \circ \mu_A^*)(a \otimes 1) =$

$= f(ae) = f(a) = f(ea) = \mu_A^*(e_A^* \otimes \mathrm{id});$

$\rightsquigarrow [2] := I(=,\to) : \mu_A^*(\mathrm{id} \otimes e_A^*) = \mathrm{id} = \mu_A^*(e_A^* \otimes \mathrm{id}),$

$[*] := \mathcal{Q}k\text{-COALG} : A^\circ \in k\text{-COALG};$

$\square$

$\texttt{finiteDualCoalgebra} :: \prod k : \texttt{Field} . k\text{-ALGE}^{\mathrm{op}} \xrightarrow{\mathsf{CAT}} k\text{-COALG}$

$\texttt{finiteDualCoalgebra}(A) = A^\circ := \texttt{FiniteDualIsCoalgebra}$

$\texttt{finiteDualCoalgebra}(A, B, \varphi) = \varphi^\circ := f^*_{|B^\circ}$

$\texttt{Assume}\ f : B^\circ,$

$\texttt{Assume}\ a \otimes a' : A \otimes A,$

$\Big(a \otimes a'.*\Big) := \mathcal{Q}\mu^*B\mathcal{Q}\varphi^\circ\mathcal{Q}k\text{-ALGE}(A,B)(\varphi)\mathcal{Q}^{-1}\varphi^\circ : \mu_B^*(\varphi^\circ \otimes \varphi^\circ)(f)(a \otimes a') = f\Big(\varphi(a)\varphi(a')\Big) = f\Big(\varphi(aa')\Big) =$

$= \varphi^\circ f(aa') = \varphi^\circ \mu_A^*(f)(a \otimes a');$

$\rightsquigarrow f.* := \mathcal{Q}\texttt{TensorProduct}I(=,\to) : \mu_B^*(\varphi^\circ \otimes \varphi^\circ)(f) = \varphi^\circ \mu_A^*(f);$

$\rightsquigarrow [1] := I(=,\to) : \mu_B^*(\varphi^\circ \otimes \varphi^\circ) = \varphi^\circ \mu_A^*,$

$\texttt{Assume}\ f : B^\circ,$

$[f.*] := \mathcal{Q}\varphi^\circ\mathcal{Q}e_A^*\mathcal{Q}k\text{-ALGE}(A,B)(\varphi) : \varphi^\circ e_A^*(f) = f\Big(\varphi(e_A)\Big) = f(e_B) = e_B^* f;$

$\rightsquigarrow [2] := I(=,\to) : \varphi^\circ e_A^* = e_B^*,$

$[*] := \mathcal{Q}k\text{-COALG}(B,A) : \varphi^\circ \in k\text{-COALG}(B,A);$

$\square$

$\texttt{FiniteMonoidAlgebraDual} :: \forall M : \texttt{FiniteMonoid} . \forall k : \texttt{Field} . \forall m \in M .$

$$. \Delta_{k^\circ[M]}(\mathrm{d}x_m) = \sum_{a,b \in M : ab = m} \mathrm{d}x^a \otimes \mathrm{d}x^b$$

$\texttt{Proof} =$

$\texttt{Assume}\ P, Q : k[M],$

$\Big[(P,Q).*\Big] := \mathcal{Q}\texttt{finiteDualCoalgebra}\big(k[M]\big)\mathcal{Q}\mu_{k[M]}^*\mathcal{Q}k[M]\mathcal{Q}\mathrm{d}x^m\mathcal{Q}^{-1}\mathrm{d}x \otimes \mathrm{d}x\mathcal{Q}^{-1}P(x) \otimes Q(x) :$

$$= \Delta_{k^\circ[M]}(\mathrm{d}x^m)\Big(P(x) \otimes Q(x)\Big) = \mu_{k[M]}^*(\mathrm{d}x^m)\Big(P(x) \otimes Q(x)\Big) = \mathrm{d}x^m\Big(P(x)Q(x)\Big) = \mathrm{d}x^m \sum_{a,b \in M} P_a Q_b x^{ab} =$$

$$= \sum_{a,b \in M : ab = m} P_a Q_b = \sum_{a,b \in M : ab = m} \mathrm{d}x^a \otimes \mathrm{d}x^b \sum_{a,b \in M} P a Q_b (x^a \otimes x^b) = \sum_{a,b \in M : ab = m} \mathrm{d}x^a \otimes \mathrm{d}x^b \Big(P(x) \otimes Q(x)\Big);$$

$\rightsquigarrow [*] := \mathcal{Q}\texttt{TensorProduct}I(=,\to) : \texttt{This};$

$\square$

$\texttt{CommutativeDualCoalg} :: \forall A \in k\text{-CALGE} \,.\, A^\circ : \texttt{Cocommutative}(k)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : A^\circ,$

$\texttt{Assume}\ a \otimes a' : A \otimes A,$

$[f.*] := \texttt{FunctionalSwap}\mathcal{I}\mu_A^*\mathcal{I}k\text{-CALGE}\mathcal{I}^{-1}\mu_A^* : \Big(\mu_A^*\texttt{swap}(f)\Big)(a \otimes b) = \mu_A^*(f)(b \otimes a) =$

$\quad = f(ba) = f(ab) = \mu_A^*(f)(a \otimes b);$

$\rightsquigarrow [1] := \mathcal{I}\texttt{TensorProduct}I^2(=, \to) : \mu_A^*\texttt{swap} = \mu_A^*,$

$[*] := \mathcal{I}^{-1}\texttt{Cocomutative}[1] : \texttt{This};$

$\square$

$\texttt{CocommutativeDualAlg} :: \forall A \in R\text{-COALG} \,.\, \forall [0] : (A : \texttt{Cocommutative}(R)) \,.\, A^* \in R\text{-CALGE}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f, g : A^*,$

$\texttt{Assume}\ a : A,$

$[(f,g).*] := \mathcal{I}\mu_{A*}\mathcal{I}\texttt{Cocomutative}(A)\texttt{FunctionalSwap}\mathcal{I}^{-1}\mu_{A^*} : fg(a) = (f \otimes g)\Delta(a) = (f \otimes g)\Big(\Delta\ \texttt{swap}(a)\Big) =$

$\quad = (g \otimes f)\Delta(a) = gf(a);$

$\rightsquigarrow [1] := I(\forall)I(=, \to) : \forall f, g \in At^* \,.\, fg = gf,$

$[*] := \mathcal{I}R\text{-CALGE} : \texttt{This};$

$\square$

$\texttt{linearlyRecursiveSequances} :: \prod k : \texttt{Field} \,.\, k\text{-VS}$

$$\texttt{linearlyRecursiveSequances}\,() = \mathrm{LR}(k) := \left\{ s \in K^{\mathbb{Z}_+} : \exists! P(x) \in k[x] : \forall n \in \mathbb{Z}_+ \,.\, s_{n+\deg P+1} = \sum_{i=0}^{\deg P} P_i s_{n+i} \right\}$$

$\texttt{linearlyRecursiveDegree} :: \prod k : \texttt{Field} \,.\, \mathrm{LR}(k) \to \mathbb{N}$

$\texttt{linearlyRecursiveDegree}\,(s) = \deg s := \deg P + 1\ \texttt{where}\ P = \mathcal{I}\mathrm{LR}(k)(s)$

$\texttt{characteristicPolynomial} :: \prod k : \texttt{Field} \,.\, \mathrm{LR}(k) \to \texttt{Monic}(k)$

$\texttt{characteristicPolynomial}\,(s) = \chi_s(x) := x^{\deg s} - P(x)\ \texttt{where}\ P(x) = \mathcal{I}\mathrm{LR}(k)(s)$

$\texttt{dualLRPolynomialEmbedding} :: \prod k : \texttt{Field} \,.\, \mathrm{LR}(k) \xrightarrow{k\text{-VS}} \Big(k[x]\Big)^*$

$$\texttt{dualLRPolynomialEmbedding}\,(s) = f_s := \sum_{n=0}^{\infty} s_i \mathrm{d}x^n$$

$\texttt{linearRecursion} :: \prod k : \texttt{Field} \,.\, \prod n \in \mathbb{N} \,.\, k^n \to k^n \to \mathrm{LR}(k)$

$$\texttt{linearRecursion}\,(a, v) = s := \Lambda i \in \mathbb{Z}_+ \,.\, \texttt{if}\ i < n\ \texttt{then}\ v_{i+1}\ \texttt{else}\ \sum_{j=0}^{n} a_j s_{i-n+j}$$

$\texttt{LRIsomorphism} :: \forall k : \texttt{Field} \,.\, f : \mathrm{LR}(k) \xleftrightarrow{k\text{-VS}} \Big(k[x]\Big)^\circ$

$\texttt{Proof} \;=$

$\texttt{Assume}\; s : \mathrm{LR}(k),$

$n := \deg s : \mathbb{N},$

$P := \mathit{\Pi}\,\mathrm{LR}(k)(s) : k[x],$

$\texttt{Assume}\; m : \mathbb{Z}_+,$

$[m.*] := \mathit{\Pi}\, f_s \mathit{\Pi}\, \chi_s(x) \mathit{\Pi}\, \mathrm{d}x \mathit{\Pi}\, P : f_s\chi_s(x)x^m = \sum_{i=1}^{\infty} s_i \mathrm{d}x^i\big(x^{n+m} - P(x)x^m\big) = s_{n+m} - \sum_{i=0}^{n-1} s_{i+m}P_i = 0;$

$\rightsquigarrow [1] := \mathit{\Pi}\, k[x] \mathit{\Pi}^{-1}\texttt{proncipleIdeal}(\chi_s(x))\mathit{\Pi}\,\texttt{Subset}\mathit{\Pi}\, \ker f_s : \Big(\chi_s(x)\Big) \subset \ker f_s,$

$[2] := \texttt{PrincipleQuotientDim}\big(\chi_s(x)\big)\mathit{\Pi}\, \chi_s(x) : \dim \dfrac{k[x]}{\Big(\chi_s(x)\Big)} = \deg s,$

$[3] := \mathit{\Pi}^{-1}\texttt{Cofinite}[2] : \Big(\big(\chi_s(x)\big) : \texttt{Cofinite}\big(k[x]\big)\Big),$

$(s.*) := \mathit{\Pi}\Big(k[x]\Big)^\circ [3][1] : f_s \in \Big(k[x]\Big)^\circ;$

$\rightsquigarrow [1] := \mathit{\Pi}\,\texttt{image} : \mathrm{Im}\, f \subset \Big(k[x]\Big)^\circ,$

$\texttt{Assume}\; g : \big(k[x]\big)^\circ,$

$\Big(I, [2]\Big) := \mathit{\Pi}\big(k[x]\big)^\circ(g) : \sum I : \texttt{Ideal}\big(k[x]\big) \,.\, \dim \dfrac{k[x]}{I} < \infty \;\&\; g(I) = \{0\},$

$\Big(Q(x), [3]\Big) := \mathit{\Pi}\,\texttt{PrincipleIdealDomain}\big(k[x]\big)(I)[2] : \sum Q(x) : \texttt{Monic}(k) \,.\, I = \Big(Q(x)\Big),$

$n := \deg Q(x) : \mathbb{Z}_+,$

$\texttt{Assume}\; [4] : n = 0,$

$[5] := [2][4] : g = 0,$

$[4.*] := \mathit{\Pi}\, f[5] : g = f_0;$

$\rightsquigarrow [4] := I(\Rightarrow) : n = 0 \Rightarrow \exists s \in \mathrm{LR}(k) : g = f_s,$

$\texttt{Assume}\; [5] : n \in \mathbb{N},$

$P(x) := x^n - Q(x) : k[x],$

$v := \lambda i \in n \,.\, g(x^{i-1}) : k^n,$

$s := \texttt{linearRecursion}(P, v) : \mathrm{LR}(k),$

$[5.*] := \mathit{\Omega}\, s \mathit{\Omega}\, v[2] : f_s = g;$

$\rightsquigarrow [5] := I(\Rightarrow) : n \in \mathbb{N} \Rightarrow \exists s \in \mathrm{LR}(k) : g = f_s,$

$[g.*] := E(|)\mathit{\Pi}(\mathbb{Z}_+)[4][5] : \exists s \in \mathrm{LR}(k) : g = f_s;$

$\rightsquigarrow [2] := \mathit{\Pi}^{-1}\texttt{Surjective} : \Big(f : \mathrm{LR}(k) \twoheadrightarrow \big(k[x]\big)^\circ\Big),$

$[*] := \mathit{\Pi}\, f[2] : \Big(f : \mathrm{LR}(k) \xleftrightarrow{k\text{-VS}} \big(k[x]\big)^\circ\Big);$

$\square$

$\texttt{linearlyRecursiveCoalgebra} :: \forall k : \texttt{Field} \,.\, k\text{-COALG}$

$\texttt{linearlyRecursiveCoalgebra}\,() = \mathrm{LR}(k) := \Big(\mathrm{LR}(k), f^*\Delta_{\left(k[x]\right)^\circ}(f^{-1} \otimes f^{-1}), f^*\eta_{\left(k[x]\right)^\circ}\Big)$

$\texttt{hitAction} :: \prod A : R\text{-ALGE} \,.\, A \xrightarrow{R\text{-MOD}} (A^* R\text{-MOD} A^*)$

$\texttt{hitAction}\,(a, f) = a \rightharpoonup f := \Lambda b \in A \,.\, f(ab)$

$\texttt{hitByAction} :: \prod A : R\text{-ALGE} \,.\, A \xrightarrow{R\text{-MOD}} (A^* R\text{-MOD} A^*)$

$\texttt{hitByAction}\,(a, f) = f \leftharpoonup a := \Lambda b \in A \,.\, f(ba)$

$\texttt{FiniteHitAction} :: \forall R : \texttt{Field} \,.\, \forall A : R\text{-ALGE} \,.\, \forall f \in A^* \,.\, f \in A^\circ \iff \dim(A \rightharpoonup f) < \infty$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : f \in A^\circ,$

$\Big(I, [2], [3]\Big) := \mathcal{I}\texttt{finiteDual}[1] : \sum I : \texttt{Ideal}(A) \,.\, I \subset \ker f \;\&\; \dim \dfrac{A}{I} < \infty,$

$\Big(\overline{f}, [4]\Big) := \texttt{IsomorphismTHM}(A, I, [2]) : \sum \overline{f} : \dfrac{A}{I} \xrightarrow{R\text{-MOD}} k \,.\, f = \pi_I \overline{f},$

$\texttt{Assume}\ a, b : A,$

$[(a, b)*] := \mathcal{I}\texttt{hitAction}(a, f)[4] : (a \rightharpoonup f)(b) = f(ab) = \overline{f}[ab];$

$\rightsquigarrow [1] := \mathcal{I}^{-1}\texttt{Injective InjectiveDim ImageDim}(\cdot \rightharpoonup \overline{f})[3] : \dim(A \rightharpoonup f) \le \dim \left(\dfrac{A}{I} \rightharpoonup \overline{f}\right) < \infty;$

$\rightsquigarrow [1] := I(\Rightarrow) : f \in A^\circ \Rightarrow \dim(A \rightharpoonup f) < \infty,$

$\texttt{Assume}\ [2] : \dim(A \rightharpoonup f) < \infty,$

$K := \{a \in A : \forall b, c \in A \,.\, f(bac)\} : \texttt{Submodule}(A),$

$[3] := \mathcal{I} \ker f \mathcal{I} K : K \subset \ker f,$

$[4] := \mathcal{I}^{-1}\mathcal{I}\texttt{Ideal}\mathcal{I} \ker f : \Big(K : \texttt{Ideal}(A)\Big),$

$[5] := \texttt{KerImTHM SubsetDim EndDim}[2] : \dim \dfrac{A}{I} = \dim \Big(A \rightharpoonup (A \rightharpoonup f)\Big) \le \dim_R \mathrm{End}_{R\text{-VS}}(A \rightharpoonup f) < \infty,$

$[2.*] := \mathcal{I} A^\circ [5] : f \in A^\circ;$

$\rightsquigarrow [2] := I(\Rightarrow) : \dim(A \rightharpoonup f) < \infty \Rightarrow f \in A^\circ,$

$[*] := I(\iff)[1][2] : f \in A^\circ \iff \dim(A \rightharpoonup f) < \infty;$

$\square$

$\texttt{FiniteHitByAction} :: \forall R : \texttt{Field} \,.\, \forall A : R\text{-ALGE} \,.\, \forall f \in A^* \,.\, f \in A^\circ \iff \dim(f \leftharpoonup A) < \infty$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FiniteDualGrouplike} :: \forall A : R\text{-ALGE} \,.\, \forall f \in \mathcal{A}^\circ \,.\, f : \texttt{Grouplike}(A^\circ) \iff f : A \xrightarrow{R\text{-ALGE}} R$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{expEvaluation} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, R^G \xrightarrow{R\text{-MOD}} (k[G])^\circ$

$\texttt{expEvaluation}\,(f) = \phi(f) := \Lambda \alpha_i x^g \,.\, \alpha f(g)$

$\texttt{representativeCoalgebra} :: \prod k : \texttt{Field} \,.\, \texttt{Monoid} \to k\text{-COALG}$

$\texttt{representativeCoalgebra}\,(G) = \mathcal{R}_k(G) := \phi^{-1}\Big((k[G])^\circ\Big)$

$\texttt{FiniteCanonicalInjection} :: \forall k : \texttt{Field} . \forall A : k\text{-COALG} . \forall a \in A . \epsilon(a) \in A^{*\circ}$

$\quad \texttt{where} \quad \epsilon = \texttt{canonicalInjection}(A)$

$\texttt{Proof} =$

$\texttt{Assume } f, g : A^*,$

$[(f,g).*] := \mathcal{I}\texttt{hitAction}\mathcal{I}\texttt{canonicalInjection}\mathcal{I}\texttt{dualAlgebra}\mathcal{I}^{-1}\texttt{CanonicalInjection} :$

$$: \Big(f \rightharpoonup \epsilon(a)\Big)(g) = \epsilon(a)(fg) = fg(a) = \sum_{(a)} f(a_1)g(a_2) = \sum_{(a)} f(a_1)\epsilon(a_2)(g);$$

$\rightsquigarrow [1] := I(\forall)I(=,\to) : \forall f \in A^* . \big(f \rightharpoonup \epsilon(a)\big) = \sum_{(a)} f(a_1)\epsilon(a_2),$

$[2] := [1]\mathcal{I}^{-1}\texttt{span} : A^* \rightharpoonup \epsilon(a) \subset \operatorname{span}\{\epsilon(a_2)\}_{(a)},$

$[3] := \mathcal{I}k\text{-COALG}\mathcal{I}^{-1}\texttt{dimension}[2] : \dim(A^* \rightharpoonup \epsilon(a)) < \infty,$

$[*] := \texttt{FiniteHitAction}[3] : \epsilon(a) \in A^{*\circ};$

$\square$

$\texttt{CanonicalInjectionCoalgHomo} :: \forall k : \texttt{Field} . \forall A : k\text{-COALG} . \epsilon : A \xrightarrow{k\text{-COALG}} A^{*\circ} .$

$\quad \texttt{where} \quad \epsilon = \texttt{canonicalInjection}(A)$

$\texttt{Proof} =$

$\texttt{Assume } a : A,$

$[a.*] := \mathcal{I}\texttt{finiteDualCoalgebra}\mathcal{I}\texttt{dualAlgebra}\mathcal{I}\texttt{canonicalInjection} :$

$$: \eta_{A^{*\circ}}\big(\epsilon(a)\big) = \epsilon(a)(e_{A^*}) = \epsilon(a)(\eta_A) = \eta_A(a);$$

$\rightsquigarrow [1] := I(=,\to) : \varepsilon\eta_{A^{*\circ}} = \eta_A,$

$\texttt{Assume } a : A,$

$\texttt{Assume } f, g : A^*,$

$[a.*] := \mathcal{I}\texttt{dualFiniteCoalg}\mathcal{I}\texttt{canonicalInjection}\mathcal{I}\texttt{dualAlgebra}\mathcal{I}^{-1}\texttt{canonicalInjection}\mathcal{I}\texttt{SweedlerSum} :$

$$: \Delta\big(\epsilon(a)\big)(f \otimes g) = \epsilon(a)(fg) = fg(a) = \sum_{(a)} f(a_1)g(a_2) = \sum_{(a)} \Big(\epsilon(a_1) \otimes \epsilon(a_2)\Big)(f \otimes g) = (\epsilon \otimes \epsilon)(\Delta\, a)(f \otimes g);$$

$\rightsquigarrow [2] := I(=,\to) : \Delta\epsilon = (\epsilon \otimes \epsilon)\Delta,$

$[3] := \mathcal{I}k\text{-COALG}(A, A^{*\circ})[1][2] : \texttt{This};$

$\square$

$\texttt{Coreflexive} :: \prod k : \texttt{Field} . ?k\text{-COALG}$

$A : \texttt{Coreflexive} \iff \epsilon : A \xleftrightarrow{k\text{-COALG}} A^{*\circ}$

$\quad \texttt{where} \quad \epsilon = \texttt{canonicalInjection}(A)$

$\texttt{TopologicalCoreflexivityCriterion} :: \forall k : \texttt{Field} . \forall A \in k\text{-COALG} . \forall A : \texttt{Coreflexive} \iff$

$$\iff \forall I : \texttt{Ideal}(A^*) . \dim \frac{A^*}{I} < \infty \Rightarrow I : \texttt{Closed}\Big(A^*, \mathcal{F}(A,k)\Big)$$

$\texttt{Proof} =$

$\texttt{Assume } [1] : A : \texttt{Coreflexive},$

$\texttt{Assume } I : \texttt{Ideal}(A^*),$

$\texttt{Assume } [2] : \dim \frac{A^*}{I} < \infty,$

$V := \epsilon^{-1}(I^\perp \cap A^{*\circ}) : \texttt{VectorSubspace}(A),$

$[3] := \texttt{ComplementDim}[2] : \dim I^\perp = \operatorname{codim} I < \infty,$

$\texttt{Assume } h : I^\perp,$

$[4] := \mathcal{G}\texttt{Orthogonal}(I, h) : I \subset \ker h,$

$[h.*] := \mathcal{G}\texttt{finiteComplement}[4] : h \in A^{*\circ};$

$\leadsto [4] := \mathcal{G}\texttt{Subset}\mathcal{G}V[3] : \dim V = \operatorname{codim} I < \infty,$

$[5] := \texttt{OrthogonalIsomorphism}(V) : V^{\perp\perp} \cong_{k\text{-VS}} V,$

$[6] := \mathcal{G}V\texttt{DoubleOrthogonalTheorem}(I) : \overline{I} = V^\perp,$

$[7] := \texttt{ComplementDim}[4][6][2] : \dim V^{\perp\perp} = \operatorname{codim} \overline{I} = \dim V = \operatorname{codim} I,$

$[8] := \mathcal{G}\texttt{closureEqualByCodimmension}[7] : I = \overline{I},$

$[I.*] := \mathcal{G}\texttt{closure}[8] : \Big(I : \texttt{Closed}\big(A^*, \mathcal{F}(A,k)\big)\Big);$

$\leadsto LR := I(\Rightarrow)I(\forall)I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume } R : \texttt{Right},$

$\texttt{Assume } F : A^{*\circ}\circ,$

$\mathcal{I} := \{I : \texttt{Ideal}(A^*) . I \subset \ker F \ \&\ \operatorname{codim} I < \infty\} :?\texttt{Ideal}(I),$

$[2] := \mathcal{G}\texttt{finiteDual}(A^*)(F) : \mathcal{I} \neq \emptyset,$

$\texttt{Assume } I : \mathcal{I},$

$[3] := \mathcal{D}\mathcal{I}(I) : I \subset \ker F \ \&\ \dim \frac{A^*}{I} < \infty,$

$[4] := R[3] : \Big(I : \texttt{Closed}\big(A^*, \mathcal{F}(A,k)\big)\Big),$

$\Big(V, [5]\Big) := \texttt{ClosedSubspaceIsOrtgogonal} : \sum V \subset_{k\text{-VS}} A . V^\perp = I,$

$[6] := \texttt{ClosedOrthogonalIsomorphism} : \epsilon_{|V} : V \overset{k\text{-VS}}{\longleftrightarrow} I^\perp,$

$[*.I] := \mathcal{G}\texttt{Surjective} : \exists a \in A : F = \epsilon(a);$

$\leadsto [2.*] := I(\forall)[2] : \exists a \in A : F = \epsilon(a);$

$\leadsto [*] := \mathcal{G}^{-1}\texttt{Coreflexive}I(\Rightarrow)I(\iff) : \texttt{Left} \iff \texttt{Right};$

$\square$

$\texttt{CoalgebraAsRepresentative} :: \forall k \in \texttt{Field} . \forall A \in k\text{-COALG} .$

$. \exists M : \texttt{Monoid} : \exists R \subset_{k\text{-COALG}} \mathcal{R}_k(M) : R \cong_{k\text{-COALG}} A$

$\texttt{Proof} =$

$M := (A^*, \mu) : \texttt{Monoid},$

$\varphi := \Lambda a \in A . Ɑ k[M] \Lambda f \in A^* . f(a) : A \xrightarrow{k\text{-VS}} M^k,$

$\texttt{Assume } a : A,$

$[1] := Ɑ \ker \mathcal{J}\varphi : \ker \varphi(a) = \left\langle \{a\}^\perp \right\rangle,$

$[a.*] := \texttt{FiniteCanonicalInjection}(A) : \varphi(a) \in A^*;$

$\rightsquigarrow [1] := Ɑ \mathcal{R}_k(M) : \varphi : A \xrightarrow{VSk} \mathcal{R}_k(M),$

$[2] := Ɑ M : k\big[M\big] \cong_{K\text{-ALGE}} A^*,$

$[*] := Ɑ \varphi Ɑ \texttt{finiteCanonicalInjection} : A \cong_{k\text{-COALG}} \varphi(A);$

$\square$

$\texttt{CanonicalInjectionAlgHomo} :: \forall k : \texttt{Field} . \forall A : k\text{-ALGE} . \epsilon : A \xrightarrow{k\text{-ALGE}} A^{\circ *} .$

$\texttt{where} \quad \epsilon = \texttt{canonicalInjection}(A)$

$\texttt{Proof} =$

$\texttt{Assume } F : A^{\circ *},$

$\texttt{Assume } f : A^\circ,$

$[a.*.1] := Ɑ \texttt{dualAlgebracanonicalInjection} Ɑ \texttt{finiteDualCoalg} Ɑ k\text{-VS}(A,k)(f_2) Ɑ k\text{-COALG}(A^\circ) :$

$$: \epsilon(e)F(f) = \sum_f \epsilon(e)(f_1)F(f_2) = \sum_f f_1(e)F(f_2) = F\left(\sum_f \eta(f_1) f_2\right) = F(f),$$

$[a.*.2] := Ɑ \texttt{dualAlgebracanonicalInjection} Ɑ \texttt{finiteDualCoalg} Ɑ k\text{-VS}(A,k)(f_1) Ɑ k\text{-COALG}(A^\circ) :$

$$: F\epsilon(e)(f) = \sum_f F(f_1)\epsilon(e_1) = \sum_f F(f_1) f_2(e) = F\left(\sum_f \eta(f_2) f_1\right) = F(f),$$

$\rightsquigarrow [1] := I(=, \rightarrow) : \epsilon(e) = e,$

$\texttt{Assume } a, b : A,$

$\texttt{Assume } f : A^\circ,$

$[a.*] := Ɑ \texttt{dualAlgebra} Ɑ \texttt{canonicalInjection} Ɑ \texttt{finiteDualCoalg} Ɑ^{-1} \texttt{canonicalInjection} :$

$$: \epsilon(a)\epsilon(b)(f) = \sum_f \epsilon(a)(f_1)\epsilon(b)(f_2) = \sum_f f_1(a) f_2(b) = f(ab) = \epsilon(ab)(f);$$

$\rightsquigarrow [2] := I(=, \rightarrow) : \mu\epsilon = (\epsilon \otimes \epsilon)\mu,$

$[3] := Ɑ k\text{-ALGE}(A, A^{\circ *})[1][2] : \texttt{This};$

$\square$

$\texttt{Proper} :: \prod k : \texttt{Field} . ?k\text{-ALGE}$

$A : \texttt{Proper} \iff \epsilon_{|A^\circ} : \texttt{Injective}(A, A^{\circ *})$

$\texttt{WeaklyReflexive} :: \prod k : \texttt{Field} . ?k\text{-ALGE}$

$A : \texttt{WeaklyReflexive} \iff \epsilon_{|A^\circ} : \texttt{Surjective}(A, A^{\circ *})$

$\texttt{Reflexive} :: \prod k : \texttt{Field} . ?k\text{-ALGE}$

$A : \texttt{Reflexive} \iff \epsilon_{|A^\circ} : \texttt{Bijective}(A, A^{\circ *})$

$\texttt{TopologicalPropernesCriterion} :: \forall k : \texttt{Field} \, . \, \forall A : k\text{-}\mathsf{ALGE} \, . \, A : \texttt{Proper}(k) \iff A^\circ : \texttt{Dense}\Big(A^*, \mathcal{F}(A,k)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ [1] : (A : \texttt{Proper}(k))\,,$

$\texttt{Assume} \ f : A^*,$

$\texttt{Assume} \ U : O \in \mathcal{U}(f),$

$\texttt{Assume} \ [2] : O \cap A^\circ = \emptyset,$

$\Big(n, a, \alpha, [3]\Big) := \mathcal{C}\mathcal{F}(A,k)[2] : \sum n \in \mathbb{N} \, . \, \sum a : \texttt{LinearlyIndependent}(n, A) \, . \, \sum \alpha : n \to A \, .$

$\quad . \, \forall f \in A^\circ \, . \, \exists i \in n \, . \, f(a_i) \neq \alpha_i,$

$\Big(i, [4]\Big) := [3](0) : \sum i \in n \, . \, \alpha_i \neq 0,$

$[5] := \mathcal{C} k\text{-}\mathsf{VS}(A^*)[3][4] : \forall f \in A^* \, . \, f(a_i) = 0,$

$[6] := \mathcal{C}\texttt{LinearlyIndependent}(n, A)(a)(a_i) : a_i \neq 0,$

$[7] := \mathcal{C}\texttt{Proper}(k)(A)\mathcal{C}\texttt{Injectivive}(A, A^{\circ *})[6] : \epsilon_{|A^\circ}(a) \neq 0,$

$[8] := \mathcal{C}^{-1}\epsilon_{|A^\circ}[5] : \epsilon_{|A^\circ}(a) = 0,$

$[1.*] := [7][8] : \perp;$

$\rightsquigarrow LR := E(\perp)\mathcal{C}^{-1}I^2(\forall)\texttt{Dense}\Big(A^*, \mathcal{F}(A,k)\Big)I(\Rightarrow) : \Big(A : \texttt{Proper}(k) \Rightarrow A^\circ : \texttt{Dense}\big(A^*, \mathcal{F}(A,k)\big)\Big),$

$\texttt{Assume} \ [1] : \Big(A^\circ : \texttt{Dense}\big(A^*, \mathcal{F}(A,k)\big)\Big),$

$\texttt{Assume} \ a, b : A,$

$\texttt{Assume} \ [2] : a \neq b,$

$\Big(f, [3]\Big) := \mathcal{C}\texttt{Injective}(\epsilon)\big(a, b, [2]\big)\mathcal{C}\epsilon : \sum f \in A^* \, . \, f(a) \neq f(b),$

$U := \{g \in A^* : g(a) = f(a) \ \& \ g(b) = f(b)\} : \mathcal{F}(A,k),$

$\Big(g, [4]\Big) := \mathcal{C}\texttt{Dense}\Big(A, \mathcal{F}(A,k)\Big)(A^\circ)(U) : \sum g \in A^\circ \, . \, g \in U,$

$[5] := \mathcal{O}(U)[4][3] : g(a) = f(a) \neq f(b) = g(b),$

$[1.*] := \mathcal{C}^{-1}\varepsilon_{|A^\circ}[5] : \varepsilon_{|A^\circ}(a) \neq \varepsilon_{|A^\circ}(b);$

$\rightsquigarrow [*] := I(\Rightarrow)I(\forall)\mathcal{C}^{-1}\texttt{Injective}\mathcal{C}^{-1}\texttt{Reflexive}I(\Rightarrow)I(\iff)(LR) : \texttt{This};$

$\square$

$\texttt{IdealPropernesCriterion} :: \forall k : \texttt{Field} . \forall A : k\text{-}\mathsf{ALGE} .$

$$. \, A : \texttt{Proper}(k) \iff \bigcap\{I : \texttt{Ideal}(A) : \operatorname{codim} I < \infty\} = \{0\}$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : (A : \texttt{Proper}(k))\,,$

$\texttt{Assume}\ a : \bigcap\{I : \texttt{Ideal}(A) : \operatorname{codim} I < \infty\} = \{0\},$

$[2] := \mathcal{I} A^\circ \mathcal{D}(a) : \forall f \in A^\circ \ . \ f(a) = 0,$

$[3] := \mathcal{I}^{-1} \epsilon_{|A^\circ}[2] : \epsilon_{|A^\circ}(a) = 0,$

$[1.*] := \mathcal{I}\texttt{Proper}(k)(A)\mathcal{I}\texttt{Injective}[3] : a = 0;$

$\leadsto [LR] := I(\forall)\mathcal{I}^{-1}\texttt{Singleton}I(\Rightarrow) : A : \texttt{Proper}(k) \Rightarrow \bigcap\{I : \texttt{Ideal}(A) : \operatorname{codim} I < \infty\} = \{0\},$

$\texttt{Assume}\ [1] : \bigcap\{I : \texttt{Ideal}(A) : \operatorname{codim} I < \infty\} = \{0\},$

$\texttt{Assume}\ a : A,$

$\texttt{Assume}\ [2] : a \neq 0,$

$\Big(I, [3]\Big) := [1][2] : \sum I : \texttt{Ideal}(A) \ . \ \operatorname{codim} I < \infty \ \&\ a \notin I,$

$\Big(f, [4]\Big) := \texttt{FunctionalConstruction}[3] : \sum f \in A^* \ . \ I \subset \ker f \ \&\ f(a) = 1,$

$[5] := \mathcal{I} A^\circ[4] : f \in A^\circ,$

$[6] := [5][4] : \epsilon_{|A^\circ}(a);$

$\leadsto [*] := I(\Rightarrow)I(\forall)\mathcal{I}^{-1}\texttt{Injective}\mathcal{I}^{-1}\texttt{Reflexive}I(\Rightarrow)I(\iff)(LR) : \texttt{This};$

$\square$

$\texttt{DualAlgebraLeftAdjoint} :: \forall k : \texttt{Field} .$

$$. \, \Big(\texttt{finiteDualCoalgebra}(k), \texttt{dualAlgebra}(k)\Big) : \texttt{LeftAdjoint}(k\text{-}\mathsf{ALGE}, k\text{-}\mathsf{COALG})$$

$\texttt{Proof} \ =$

$\dots$

$\square$

2.3 Main Theorem of Coalgebras

$\texttt{Subcoalgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{COALG} \,.\, ??A$

$B : \texttt{Subcoalgebra} \iff B \subset_{R\text{-}\mathsf{COALG}} A \iff (B, \Delta_A, \eta_A) \in R\text{-}\mathsf{COALG}$

$\texttt{IdealsSubcoalgebrasDuality} :: \forall k : \texttt{Field} \,.\, \forall A : k\text{-}\mathsf{COALG} \,.\, \forall I : \texttt{Ideal}(A^*) \,.\, \epsilon^{-1}(I^\perp) : \texttt{Subcoalgebra}(A)$

$\texttt{Proof} \;=$

$B := \epsilon^{-1}\, I^\perp : \texttt{VectorSubspace}(A),$

$\texttt{Assume}\; b : B,$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; v, u : \texttt{LinearlyIndependent}(n, A),$

$\texttt{Assume}\; [5] : \Delta(b) = \sum_{i=1}^{n} v_i \otimes u_i,$

$\texttt{Assume}\; i : n,$

$\texttt{Assume}\; [6] : v_i \notin B,$

$\Big(f, [7]\Big) := \mathit{\Pi} B[6] : \sum f \in I \,.\, f(v_i) \neq 0,$

$\Big(g, [8]\Big) := \texttt{AlgebraicReizReprezentationTHM}(u_i, 1, \widehat{u}_i) : \sum g \in A^* \,.\, g(u_i) = 1 \;\&\; \forall j \in (n-1) \,.\, h(u_j) = 0,$

$[9] := \mathit{\Pi}\texttt{dualAlgebra}(A)[8][7] : fg(b) = \sum_{i=1}^{n} f(v_i) g(u_i) = f(v_i) \neq 0,$

$[10] := \mathit{\Pi}\texttt{Ideal}(I)(f, g) : fg \in I,$

$[11] := \mathit{\Pi} B[10] : fg(b) = 0,$

$[6.*] := [9][11] : \perp;$

$\leadsto [b.*.1] := E(\perp) : v_i \in B,$

$\texttt{Assume}\; [6] : u_i \notin B,$

$\Big(f, [7]\Big) := \mathit{\Pi} B[6] : \sum f \in I \,.\, f(u_i) \neq 0,$

$\Big(g, [8]\Big) := \texttt{AlgebraicReizReprezentationTHM}(v_i, 1, \widehat{v}_i) : \sum g \in A^* \,.\, g(v_i) = 1 \;\&\; \forall j \in (n-1) \,.\, h(v_j) = 0,$

$[9] := \mathit{\Pi}\texttt{dualAlgebra}(A)[8][7] : gf(b) = \sum_{i=1}^{n} g(v_i) f(u_i) = f(u_i) \neq 0,$

$[10] := \mathit{\Pi}\texttt{Ideal}(I)(f, g) : gf \in I,$

$[11] := \mathit{\Pi} B[10] : gf(b) = 0,$

$[6.*] := [9][11] : \perp;$

$\leadsto [b.*.2] := E(\perp) : u_i \in B;$

$\leadsto [5] := \mathit{\Pi} k\text{-}\mathsf{COALG}\mathit{\Pi}\texttt{Subcoalgebra} : \Big(B : \texttt{Subcoalgebra}(V)\Big);$

$\square$

$\texttt{SubcoalgebrasIdealsDuality} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall B :\subset_{k\text{-}\mathsf{COALG}} A \,.\, B^\perp : \texttt{Ideal}(A^*)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{QuotientDuality} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-COALG} \,.\, \forall B :\subset_{k\text{-COALG}} A \,.\, B^* \cong_{k\text{-ALGE}} \dfrac{A^*}{B^\perp}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MainTheoremOfCoalgebras} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-COALG} \,.\, \forall a \in A \,.\, \exists B \subset_{k\text{-COALG}} A : a \in B \;\&\; \dim B < \infty$

$\texttt{Proof} \;=$

$[1] := \texttt{FiniteCanonicalInjection}(a) : \epsilon(a) \in A^{*\circ},$

$\Big([2], I\Big) := \mathcal{G}\texttt{finiteDual}[1] : \sum I : \texttt{Ideal}(A^*) \,.\, \operatorname{codim} I < \infty \;\&\; I \subset \ker \epsilon(a),$

$B := \epsilon^{-1}\, I^\perp : \texttt{VectorSubspace}(A),$

$[3] := \texttt{InjectionDim}(\epsilon)\texttt{OrthogonalDim}(I)[2] : \dim B \le \dim I^\perp = \operatorname{codim} I < \infty,$

$[4] := \mathcal{G}\texttt{preimage}\mathcal{G}\texttt{kernel}\mathcal{D}B[2] : a \in B,$

$[5] := \texttt{IdealsSubcoalgebraDuality}(A)(I)\mathcal{D}(B) : \Big(B : \texttt{Subcoalgebra}(A)\Big),$

$[*] := I(\;\&\;)[3][4][5] : \texttt{This};$

$\square$

$\texttt{IdealsSubcoalgebrasDuality2} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-ALGE} \,.\, \forall I : \texttt{Ideal}(A) \,.\, I^\perp \cap A^\circ : \texttt{Subcoalgebra}(A^\circ)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SubcoalgebrasIdealsDuality2} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-ALGE} \,.\, \forall B :\subset_{k\text{-COALG}} A^\circ \,.\, \epsilon^{-1}\Big(B^\perp\Big) : \texttt{Ideal}(A)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CoidealsSubalgebrasDuality} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-COALG} \,.\, \forall I : \texttt{Coideal}(A) \,.\, I^\perp : \texttt{Subalgebra}(A^*)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SubalgebrasCoidealsDuality} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-COALG} \,.\, \forall B \subset_{k\text{-ALGE}} A^* \,.\, \epsilon^{-1}\Big(B^\perp\Big) : \texttt{Coideal}(A)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SubalgebrasXoidealsDuality2} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-ALGE} \,.\, \forall B \subset_{k\text{-ALGE}} A \,.\, B^\perp \cap A^\circ : \texttt{Coideal}(A^\circ)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CoidealsSubalgebraDuality2} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{ALGE} . \forall I : \texttt{Coideal}(A^\circ) . \epsilon^{-1}\left(I^\perp\right) : \texttt{Subalgebra}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LeftIdealsCoidealsDuality} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{ALGE} . \forall I : \texttt{LeftIdeal}(A) . I^\perp \cap A^\circ : \texttt{LeftCoideal}(A^\circ)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LeftCoidealIdealsDuality} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{ALGE} . \forall I : \texttt{LeftCoideal}(A^\circ) . \epsilon^{-1}\left(I^\perp\right) : \texttt{LeftIdeal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LeftCoidealIdealsDuality2} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{COALG} . \forall I : \texttt{LeftCoideal}(A) . I^\perp : \texttt{LeftIdeal}(A^*)$

$\texttt{Proof} =$

...

$\square$

$\texttt{LeftIdealsCoidealsDuality} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{COALG} . \forall I : \texttt{LeftIdeal}(A^*) .$
$. \epsilon^{-1}\left(I^\perp\right) : \texttt{LeftCoideal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{RightIdealsCoidealsDuality} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{ALGE} . \forall I : \texttt{RightIdeal}(A) .$
$. I^\perp \cap A^\circ : \texttt{RightCoideal}(A^\circ)$

$\texttt{Proof} =$

...

$\square$

$\texttt{RightCoidealIdealsDuality} :: \forall k : \texttt{Field} . \forall A \in k\text{-}\mathsf{ALGE} . \forall I : \texttt{RightCoideal}(A^\circ) .$
$. \epsilon^{-1}\left(I^\perp\right) : \texttt{RightIdeal}(A)$

$\texttt{Proof} =$

...

$\square$

$\texttt{RightCoidealIdealsDuality2} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall I : \texttt{RightCoideal}(A) \,.\, I^{\perp} : \texttt{RightIdeal}(A^*)$

Proof $=$

...

$\square$

$\texttt{RightIdealsCoidealsDuality} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall I : \texttt{RightIdeal}(A^*) \,.$

$\,.\, \epsilon^{-1}\left(I^{\perp}\right) : \texttt{RightCoideal}(A)$

Proof $=$

...

$\square$

$\texttt{SubcoalgebraIntersection} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall X \in \mathsf{SET} \,.\, \forall I : X \to \texttt{Subcoalgebra}(A) \,.$

$\,.\, \bigcap_{x \in X} I_x : \texttt{Subcoalgebra}(A)$

Proof $=$

...

$\square$

$\texttt{LeftCoidealIntersection} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall X \in \mathsf{SET} \,.\, \forall I : X \to \texttt{LeftCoideal}(A) \,.$

$\,.\, \bigcap_{x \in X} I_x : \texttt{LeftCoideal}(A)$

Proof $=$

...

$\square$

$\texttt{RightCoidealIntersection} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{COALG} \,.\, \forall X \in \mathsf{SET} \,.\, \forall I : X \to \texttt{RightCoideal}(A) \,.$

$\,.\, \bigcap_{x \in X} I_x : \texttt{RightCoideal}(A)$

Proof $=$

...

$\square$

2.4 Tensor Products of Coalgebras

$\texttt{GradedCoalgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod I : \texttt{Monoid} \,.\, \sum M : R\text{-}\mathsf{MOD}(I) \,.$

$\quad . \, M \xrightarrow{R\text{-}\mathsf{MOD}(I)} M \otimes M \times M \xrightarrow{R\text{-}\mathsf{MOD}(I)} k \,.$

$(M, \Delta, \eta) : \texttt{GradedCoalgebra} \iff (M, \Delta, \eta) \in R\text{-}\mathsf{COALG}$

$\texttt{GradedCoalgebraHomo} :: \prod R \in \mathsf{ANN} \,.\, \prod I : \texttt{Monoid} \,.\, \prod A, B : \texttt{GradedCoalgebra}(R, I) \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}(I)} B$

$f : \texttt{GradedCoalgebraHomo} \iff f : A \xrightarrow{R\text{-}\mathsf{COALG}} B \;\&\; f : A \xrightarrow{R\text{-}\mathsf{MOD}(\mathcal{I})} B$

$\texttt{categoryOfGradedCoalgebras} :: \mathsf{ANN} \to \texttt{Monoid} \to \mathsf{CAT}$

$\texttt{categoryOfGradedCoalgebra}\,(R, M) = R\text{-}\mathsf{COALG}(M) := \Big(\texttt{GradedCoalgebra}, \texttt{GradedCoalgebraHomo}, \mathrm{id}, \circ\Big)$

$\texttt{TensorProductOfCoalgebraHomo} :: \forall R \in \mathsf{ANN} \,.\, \forall X, X', Y, Y' : R\text{-}\mathsf{COALG} \,.$

$\quad . \, \forall \varphi : X \xrightarrow{R\text{-}\mathsf{COALG}} Y \,.\, \forall \psi : X' \xrightarrow{R\text{-}\mathsf{COALG}} Y' \,.\, \varphi \otimes \psi : X \otimes X' \xrightarrow{R\text{-}\mathsf{COALG}} Y \otimes Y'$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : X,$

$\texttt{Assume}\; x' : X',$

$[x'.*.1] := \mathcal{I}\texttt{homoTensorProduct}\mathcal{I} R\text{-}\mathsf{COALG}(X, X')(\varphi)\mathcal{I} R\text{-}\mathsf{COALG}(Y, Y')(\psi)\mathcal{I}\texttt{coalgebraTensorProduct} :$

$\quad : \Delta\Big((\varphi \otimes \psi)(x \otimes x')\Big) = \Delta\Big(\varphi(x) \otimes \psi(x')\Big) = \sum_{x, x'} (\varphi(x_1) \otimes \psi(x_1') \otimes (\varphi(x_1) \otimes \psi(x_1')) =$

$\quad = (\varphi \otimes \psi) \otimes (\varphi \otimes \psi)\Delta(x \otimes x');$

$[x'.*.2] := \mathcal{I}\texttt{homoTensorProduct}\mathcal{I} R\text{-}\mathsf{COALG}(X, X')(\varphi)\mathcal{I} R\text{-}\mathsf{COALG}(Y, Y')(\psi)\mathcal{I}\texttt{coalgebraTensorProduct} :$

$\quad : \eta\Big((\varphi \otimes \psi)(x \otimes x')\Big) = \eta\Big(\varphi(x) \otimes \psi(x')\Big) = \eta\Big(\varphi(x)\Big)\Big(\psi(x)\Big) = \eta(x)\eta'(x') = \eta(x \otimes x');$

$\rightsquigarrow [*] := I(\forall)\mathcal{I}\texttt{TensorProduct} : \varphi \otimes \psi : X \otimes X' \xrightarrow{R\text{-}\mathsf{COALG}} Y \otimes Y';$

$\square$

$\texttt{CoalgTensorProductAssociativty} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, C \in R\text{-}\mathsf{COALG} \,.$

$\quad . \, (A \otimes B) \otimes C \cong_{R\text{-}\mathsf{COALG}} A \otimes (B \otimes C)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CoalgTensorProductPermutation} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{COALG} \,.\, \forall \sigma \in S_n \,.$

$\quad . \, \bigotimes_{i=1}^{n} A_i \cong_{R\text{-}\mathsf{COALG}} \bigotimes_{i=1}^{n} A_{\sigma(i)}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CoalgTrivialTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall A : n \to R\text{-}\mathsf{COALG} \,.\, R \otimes A \cong_{R\text{-}\mathsf{COALG}} A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TensorProductOfGradedCoalgebraHomo} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{Monoid} \forall X, X', Y, Y' : R\text{-}\mathsf{COALG}(M) \,.$

$\quad .\, \forall \varphi : X \xrightarrow{R\text{-}\mathsf{COALG}(M)} Y \,.\, \forall \psi : X' \xrightarrow{R\text{-}\mathsf{COALG}(M)} Y' \,.\, \varphi \otimes \psi : X \otimes X' \xrightarrow{R\text{-}\mathsf{COALG}(M)} Y \otimes Y'$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CoalgTensorProductAssociativty} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{Monoid} \,.\, \forall A, B, C \in R\text{-}\mathsf{COALG}(M) \,.$

$\quad .\, (A \otimes B) \otimes C \cong_{R\text{-}\mathsf{COALG}(M)} A \otimes (B \otimes C)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{GradedCoalgTensorProductPermutation} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{Monoid} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{COALG}(M) \,.$

$$.\, \forall \sigma \in S_n \,.\, \bigotimes_{i=1}^{n} A_i \cong_{R\text{-}\mathsf{COALG}(M)} \bigotimes_{i=1}^{n} A_{\sigma(i)}$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CoalgTrivialTensorProduct} :: \forall R \in \mathsf{ANN} \,.\, \forall M : \texttt{Monoid} \,.\, \forall A : n \to R\text{-}\mathsf{COALG}(M) \,.\, R \otimes A \cong_{R\text{-}\mathsf{COALG}(M)} A$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$$\texttt{skewTensorProductOfCoalgebras} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, n \to R\text{-}\mathsf{COALG}(\mathbb{Z}) \to R\text{-}\mathsf{COALG}(\mathbb{Z})$$

$$\texttt{skewTensorProductOfCoalgebras}\,(A) = \widetilde{\bigotimes}_{i=1}^{n} A_i :=$$

$$:= \Bigg(A \otimes B, \text{Д}\texttt{TensorProduct}\text{Д}\texttt{GradedAlgebra} \,.\, \Lambda a \in \prod_{i=1}^{n} \texttt{Homogeneous} A_i \,.$$

$$.\, \sum_{a} (-1)^{I,J} \bigotimes_{i=1}^{n} a_{i,1} \otimes \bigotimes_{i=1}^{n} a_{i,2} \quad \texttt{where} \quad I = (\deg a_{i,1})_{i=1}^{n}, J = (\deg a_{i,2})_{i=1}^{n}; \eta_{A \otimes B} \Bigg)$$

$\texttt{SkewTensorProductOfGradedHomo} :: \forall R \in \mathsf{ANN} \,.\, \forall n : \mathbb{N} \to R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.$

$$.\, \forall X, Y, : n \to R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.\, \forall \varphi : \prod_{i=1}^{n} X_i \xrightarrow{R\text{-}\mathsf{COALG}(\mathbb{Z})} Y_i \,.\, \bigotimes_{i=1}^{n} \varphi_i : \widetilde{\bigotimes}_{i=1}^{n} X_i \xrightarrow{R\text{-}\mathsf{COALG}(\mathbb{Z})} \widetilde{\bigotimes}_{i=1}^{n} Y_i$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CoalgSkewTensorProductAssociativty1} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A \in n \to R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.$

$\quad .\, A_1 \widetilde{\otimes} \widetilde{\bigotimes}_{i=2}^{n} A_i \cong_{R\text{-}\mathsf{COALG}(\mathbb{Z})} \widetilde{\bigotimes}_{i=1}^{n} A_i$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CoalgSkewTensorProductAssociativty2} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A \in n \to R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.$

$\quad .\, \left(\widetilde{\bigotimes}_{i=1}^{n-1} A_i \right) \widetilde{\otimes} A_n \cong_{R\text{-}\mathsf{COALG}(\mathbb{Z})} \widetilde{\bigotimes}_{i=1}^{n} A_i$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CoalgSkewTensorProductAssociativty} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B, C \in R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.$

$\quad .\, (A \widetilde{\otimes} B) \widetilde{\otimes} C \cong_{R\text{-}\mathsf{COALG}(\mathbb{Z})} A \widetilde{\otimes} (B \widetilde{\otimes} A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{TwistingCoalgebraHomomorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{COALG}(\mathbb{Z}) \,.\, \tau_{A,B} : A \widetilde{\otimes} B \xleftrightarrow{R\text{-}\mathsf{COALG}(\mathbb{Z})} B \widetilde{\otimes} A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{categoryOfCocommutativeCoalgebras} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{categoryOfCocommutativeCoalgebras}\,(R) = R\text{-}\mathsf{CCOALG} := \Big(\texttt{Cocommutaive}, \texttt{CoalgebraHomo}, \circ, \mathrm{id} \Big)$

$\texttt{CoskewCoalgebra} :: \prod R \in \mathsf{ANN} \,.\, ?R\text{-}\mathsf{COALG}$

$A : \texttt{CoskewCoalgebra} \iff \Delta_A\, T_{A,A} = \Delta_A$

$\texttt{categoryOfCoskewCoalgebras} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{categoryOfCoskewCoalgebras}\,(R) = R\text{-}\mathsf{SCOALG} := \Big(\texttt{CoskewCoalgebra}, \texttt{CoalgebraHomo}, \circ, \mathrm{id} \Big)$

$\texttt{TensorProductsPreserveCocommutativity} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{CCOALG} \,.\, A \otimes B \in R\text{-}\mathsf{CCOALG}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SkewTensorProductsPreserveSkewCocommutativity} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in R\text{-}\mathsf{SCOALG} \,.$

$\quad .\, A \widetilde{\otimes} B \in R\text{-}\mathsf{SCOALG}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{counitalProjection} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, \prod A : n \to A\text{-}\mathsf{COALG} \,.\, \prod_{i=1}^{n} \bigotimes_{j=1}^{n} A_j \xrightarrow{R\text{-}\mathsf{COALG}} A_i$

$\texttt{counitalProjection}\,(t) = \pi_i(t) := \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j}(t)$

$\texttt{Assume}\ a : \prod_{i=1}^{n} A_i,$

$a.*.1 := \mathcal{I}^{-1}\texttt{SweedlerNotation}\mathcal{I}\pi_i\mathcal{I}\texttt{TensorFunc}\mathcal{I}R\text{-}\mathsf{MOD}(A_j,R)(\eta_j)\mathcal{I}R\text{-}\mathsf{COALG}(A_j)$
$\quad \mathcal{I}\texttt{SweedlerNotation}\mathcal{I}^{-1}\pi_i :$

$$: \bigotimes_{j=1}^{n} a_j \Delta\pi_i \otimes \pi_i = \sum_a \bigotimes_{j=1}^{n} a_{j,1} \otimes \bigotimes a_{j,2} \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j}(t) \otimes \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j}(t) =$$

$$= \sum_a \prod_{j=1, j\neq i}^{n} \eta(a_{j,1})\eta(a_{j,2}) a_{i,1} \otimes a_{i,2} = \prod_{j=1,j\neq i}^{n} \eta\left(\sum_{a_j} \eta(a_{j,2})a_{j,1}\right) \sum_{a_i} a_{i,1} \otimes a_{i,2} = \prod_{j=1,j\neq i}^{n} \eta(a_j) \sum_{a_i} a_{i,1} \otimes a_{i,2} =$$

$$= \prod_{j=1,j\neq i}^{n} \eta(a_j) a_i \Delta = \bigotimes_{j=1}^{n} a_j\ \pi_i\ \Delta,$$

$a.*.2 := \mathcal{I}\pi_i\mathcal{I}R\text{-}\mathsf{MOD}(A_i,R)(\eta_i)\mathcal{I}^{-1}\eta : \bigotimes_{j=1}^{n} a_j\pi_i\eta = \prod_{j=1,j\neq 1}^{n} \eta(a_j)a_i\eta_{A_i} = \prod_{j=1}^{n} \eta(a_j) = \bigotimes_{j=1}^{n} a_j\eta;$

$\rightsquigarrow [*] := \mathcal{I}\texttt{TensorProduct} : \texttt{This};$

$\square$

$\texttt{TensorProductIsCCOALGProduct} :: \forall R \in \mathsf{ANN} \,.\, \Big(\texttt{tensorProduct}, \pi\Big) : \texttt{FiniteProduct}(k\text{-}\mathsf{CCOALG})$

$\texttt{Proof}\ =$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ A : n \to R\text{-}\mathsf{CCOALG},$

$\texttt{Assume}\ B : R\text{-}\mathsf{CCOALG},$

$\texttt{Assume}\ \varphi : \prod_{i=1}^{n} B \xrightarrow{R\text{-}\mathsf{COALG}} A_i,$

$\psi := \Delta^n \bigotimes_{i=1}^{n} \varphi_i : B \xrightarrow{R\text{-}\mathsf{COALG}} \bigotimes_{i=1}^{n} A_i,$

$\texttt{Assume}\ i : n,$

$\texttt{Assume}\ b : B,$

$i.* := \mathcal{D}\psi\mathcal{D}\pi_i\mathcal{I}^{-1}\texttt{SweedlersNotation}\mathcal{I}\texttt{tensorMap}(\varphi)\mathcal{I}\texttt{tensorMap}(\eta_A)$
$\quad \mathcal{I}R\text{-}\mathsf{MOD}(B,A_i)(\varphi_i)\mathcal{I}R\text{-}\mathsf{COALG}(B,A_j)(\varphi_j)\mathcal{I}R\text{-}\mathsf{COALG}(B) :$

$$: b\ \psi\ \pi_i = b\ \Delta^n \bigotimes_{j=1}^{n} \varphi_j \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j} = \sum_b \bigotimes_{j=1}^{n} b_j \bigotimes_{j=1}^{n} \varphi_j \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j} =$$

$$= \sum_b \bigotimes_{j=1}^{n} \varphi_j(b_j) \bigotimes_{j=1}^{i-1} \eta_{A_j} \otimes \mathrm{id}_{A_i} \otimes \bigotimes_{j=i+1}^{n} \eta_{A_j} = \sum_b \prod_{j=1,j\neq i}^{n} \eta\Big(\varphi_j(b_j)\Big)\varphi_i(b_i) = \varphi_i\left(\sum_b \prod_{j=1,j\neq i}^{n} \eta(b_j)b_i\right) = \varphi_i(b);$$

$\rightsquigarrow [1] := I(=,\to)I(\forall) : \forall i \in n \,.\, \psi\pi_i = \varphi_i,$

$\texttt{Assume}\ \psi' : B \xrightarrow{R\text{-}\mathsf{COALG}} \bigotimes_{i=1}^{n} A_i,$

$\texttt{Assume}\ [2] : \forall i \in n \,.\, \psi'\pi_i = \varphi_i,$

$\texttt{Assume}\ b : B,$

$$n.* := \Game\psi[2]\daleth^{-1}\texttt{SweedlerSum}\daleth\pi_i\daleth\texttt{tensorMap}(\eta_A)\daleth\mathcal{L}\left(A;\bigotimes_{i=1}^{n}A_i\right)(\otimes)\daleth R\text{-CCOALG}(B)$$

$$\daleth\texttt{coalgebraTensorProduct}(A)\daleth R\text{-COALG}\left(\bigotimes_{i=1}^{n}\right) : b\ \psi = b\ \Delta^n\ \otimes_{i=1}^{n}\varphi_i = \sum_b \bigotimes_{i=1}^{n} b_i\ \otimes_{i=1}^{n}\psi'\pi_i =$$

$$= \sum_b \bigotimes_{i=1}^{n}\left(\psi'(b_i)\bigotimes_{j=1}^{i-1}\eta_{A_j}\otimes \operatorname*{id}_{A_i}\otimes\bigotimes_{j=i+1}^{n}\eta_{A_j}\right) = \sum_b\bigotimes_{i=1}^{n}\sum_{a_i=\psi'(b_i)}\prod_{j=1,j\neq i}^{n}\eta(a_i^j)a_i^i =$$

$$\sum_b\sum_{a_i=\psi'(b_i)}\prod_{i\neq j}\eta(a_i^j)\bigotimes_{i=1}^{n}a_i^i = \sum_b\sum_{a_i=\psi'(b_i)}\prod_{i=1}\prod_{j=2}\eta(a_i^j)\bigotimes_{i=1}^{n}a_1^i = \sum_b\prod_{i\neq 1}\eta\Big(\psi'(b_i)\Big)\psi'(b_1) = b\ \psi';$$

$\rightsquigarrow [*] := \daleth^{-1}\texttt{FiniteProduct} : \texttt{This};$

$\square$

$\texttt{TensorProductIsSCOALGProduct} :: \forall R \in \text{ANN} . \Big(\texttt{SkewTensorProduct}, \pi\Big) : \texttt{FiniteProduct}(k\text{-SCOALG})$

$\texttt{Proof} =$

$\texttt{Assume}\ n : \mathbb{N},$

$\texttt{Assume}\ A : n \to R\text{-SCOALG},$

$\texttt{Assume}\ B : R\text{-SCOALG},$

$$\texttt{Assume}\ \varphi : \prod_{i=1}^{n} B \xrightarrow{R\text{-COALG}} A_i,$$

$$\psi := \Delta^n\bigotimes_{i=1}^{n}\varphi_i : B \xrightarrow{R\text{-COALG}} \bigotimes_{i=1}^{n}A_i,$$

$\texttt{Assume}\ i : n,$

$\texttt{Assume}\ b : B,$

$$i.* := \Game\psi\Game\pi_i\daleth^{-1}\texttt{SweedlersNotation}\daleth\texttt{tensorMap}(\varphi)\daleth\texttt{tensorMap}(\eta_A)$$

$$\daleth R\text{-MOD}(B, A_i)(\varphi_i)\daleth R\text{-COALG}(B, A_j)(\varphi_j)\daleth R\text{-COALG}(B) :$$

$$: b\ \psi\ \pi_i = b\ \Delta^n\bigotimes_{j=1}^{n}\varphi_j\bigotimes_{j=1}^{i-1}\eta_{A_j}\otimes\operatorname*{id}_{A_i}\otimes\bigotimes_{j=i+1}^{n}\eta_{A_j} = \sum_b\bigotimes_{j=1}^{n}b_j\bigotimes_{j=1}^{n}\varphi_j\bigotimes_{j=1}^{i-1}\eta_{A_j}\otimes\operatorname*{id}_{A_i}\otimes\bigotimes_{j=i+1}^{n}\eta_{A_j} =$$

$$= \sum_b\bigotimes_{j=1}^{n}\varphi_j(b_j)\bigotimes_{j=1}^{i-1}\eta_{A_j}\otimes\operatorname*{id}_{A_i}\otimes\bigotimes_{j=i+1}^{n}\eta_{A_j} = \sum_b\prod_{j=1,j\neq i}^{n}\eta\Big(\varphi_j(b_j)\Big)\varphi_i(b_i) = \varphi_i\left(\sum_b\prod_{j=1,j\neq i}^{n}\eta(b_j)b_i\right) = \varphi_i(b);$$

$\rightsquigarrow [1] := I(=,\to)I(\forall) : \forall i \in n\ .\ \psi\pi_i = \varphi_i,$

$$\texttt{Assume}\ \psi' : B \xrightarrow{R\text{-COALG}} \bigotimes_{i=1}^{n}A_i,$$

$\texttt{Assume}\ [2] : \forall i \in n\ .\ \psi'\pi_i = \varphi_i,$

$\texttt{Assume}\ b : B,$

$$n.* := \Game\psi[2]\daleth^{-1}\texttt{SweedlerSum}\daleth\pi_i\daleth\texttt{tensorMap}(\eta_A)\daleth\mathcal{L}\left(A;\bigotimes_{i=1}^{n}A_i\right)(\otimes)\daleth R\text{-SCOALG}(B)$$

$$\daleth\texttt{coalgebraTensorProduct}(A)\daleth R\text{-COALG}\left(\bigotimes_{i=1}^{n}\right) : b\ \psi = b\ \Delta^n\ \otimes_{i=1}^{n}\varphi_i = \sum_b \bigotimes_{i=1}^{n} b_i\ \otimes_{i=1}^{n}\psi'\pi_i =$$

$$= \sum_b \bigotimes_{i=1}^{n}\left(\psi'(b_i)\bigotimes_{j=1}^{i-1}\eta_{A_j}\otimes \operatorname*{id}_{A_i}\otimes\bigotimes_{j=i+1}^{n}\eta_{A_j}\right) = \sum_b\bigotimes_{i=1}^{n}\sum_{a_i=\psi'(b_i)}\prod_{j=1,j\neq i}^{n}\eta(a_i^j)a_i^i =$$

$$\sum_b\sum_{a_i=\psi'(b_i)}\prod_{i\neq j}\eta(a_i^j)\bigotimes_{i=1}^{n}a_i^i = \sum_b\sum_{a_i=\psi'(b_i)}\prod_{i=1}\prod_{j=2}\eta(a_i^j)\bigotimes_{i=1}^{n}a_1^i = \sum_b\prod_{i\neq 1}\eta\Big(\psi'(b_i)\Big)\psi'(b_1) = b\ \psi';$$

$\rightsquigarrow [*] := \daleth^{-1}\texttt{FiniteProduct} : \texttt{This};$

$\square$

2.5 Cofreedom

$\texttt{CofreeCoalgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod M \in R\text{-}\mathsf{MOD} \,.\, ?\sum A \in R\text{-}\mathsf{COALG} \,.\, A \xrightarrow{R\text{-}\mathsf{MOD}} M$

$(A,\pi) : \texttt{CofreeCoalgebra} \iff \forall B : R\text{-}\mathsf{COALG} \,.\, \forall \varphi : B \xrightarrow{R\text{-}\mathsf{MOD}} M \,.\, \exists! \psi : B \xrightarrow{R\text{-}\mathsf{COALG}} A \,.\, \psi\pi = \varphi$

$\texttt{CofreeCoalgebraSurjectivity} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, \forall (A,\pi) : \texttt{CofreeCoalgebra}(M) \,.\, \pi : A \twoheadrightarrow M$

$\texttt{Proof} \ =$

$\texttt{Assume}\ m : M,$

$\mu := \Lambda t \in R \,.\, tm : R \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$\Big(\psi, [1]\Big) := \text{Д}\,\texttt{CofreeCoalgebra}(A,\pi)(\nu) : \sum \psi : R \xrightarrow{R\text{-}\mathsf{COALG}} A \,.\, \psi\pi = \mu,$

$[2] := [1]\text{Э}(\mu) : \psi\pi(e) = \mu(e) = m,$

$[m.*] := \text{Д}\,\texttt{image}[2] : m \in \operatorname{Im}\pi;$

$\rightsquigarrow [*] := I(\forall)\text{Д}^{-1}\texttt{Surjective} : (\pi : A \twoheadrightarrow M);$

$\square$

$\texttt{IsomorphicCofreeCoalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$

$\quad . \,\forall (A,\pi),(B,\pi') : \texttt{CofreeCoalgebra}(M) \,.\, A \cong_{R\text{-}\mathsf{COALG}} B$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DoubleDualCofreeCoalgebra} :: \forall k : \texttt{Field} \,.\, \forall V \in k\text{-}\mathsf{VS} \,.\, \exists \texttt{CofreeCoalgebra}(V^{**})$

$\texttt{Proof} \ =$

$\pi := \Lambda f \in V^{*\otimes\circ} \,.\, f_{|V_1^{*\otimes}} : V^{*\otimes\circ} \xrightarrow{k\text{-}\mathsf{VS}} V^{**},$

$\texttt{Assume}\ A : R\text{-}\mathsf{COALG},$

$\Big(\phi, [1]\Big) := \texttt{DualAdjucntion}(V,A) : \sum \phi : (A \xrightarrow{k\text{-}\mathsf{VS}} V^{**}) \xleftrightarrow{k\text{-}\mathsf{VS}} (V^* \xrightarrow{k\text{-}\mathsf{VS}} A^*) \,.$

$\quad . \,\forall T : A \xrightarrow{k\text{-}\mathsf{VS}} V^* * \ . \,\forall f \in V^* \,.\, \forall a \in A \,.\, \phi(T)(f)(a) = T(a)(f),$

$\phi' := (\cdot)^{\otimes} : (V^* \xrightarrow{k\text{-}\mathsf{VS}} A^*) \xleftrightarrow{k\text{-}\mathsf{VS}} (V^{*\otimes} \xrightarrow{k\text{-}\mathsf{ALGE}} A^*),$

$\phi'' := \texttt{DualAlgebraLeftAdjoint} : (V^{*\otimes} \xrightarrow{k\text{-}\mathsf{ALGE}} A^*) \xleftrightarrow{\mathsf{SET}} (A \xrightarrow{k\text{-}\mathsf{COALG}} V^{*\otimes\circ}),$

$\texttt{Assume}\ \varphi : A \xrightarrow{k\text{-}\mathsf{VS}} V^{**},$

$\psi := \phi\phi'\phi''(\varphi) : A \xrightarrow{k\text{-}\mathsf{COALG}} V^{*\otimes\circ},$

$\texttt{Assume}\ a : A,$

$[a.*] := \text{Э}\psi\text{Э}\phi\text{Э}\phi'\text{Э}\phi''\text{Д}\pi I(\rightarrow)(\varphi(a)) :$

$$\psi\pi(a) = \phi\phi'\phi''(\varphi)\pi(a) = \pi\Big(\phi'\phi''\big(\Lambda f \in V^* \,.\, \Lambda a \in A \,.\, \varphi(a)(f)\big)(a)\Big) =$$

$$= \pi\left(\phi''\left(\Lambda \sum_{i=1}^{n}\bigotimes_{j=1}^{i} f_{i,j} \in V^{*\otimes} \,.\, \Lambda a \in A \,.\, \sum_{i=1}^{n}\prod_{j=1}^{i}\varphi(a_{i,j})(f_{i,j})\right)(a)\right) =$$

$$= \pi\left(\Lambda a \in A \,.\, \Lambda \sum_{i=1}^{n}\bigotimes_{j=1}^{i} f_{i,j} \in V^{*\otimes} \,.\, \sum_{i=1}^{n}\prod_{j=1}^{i}\varphi(a_{i,j})(f_{i,j})(a)\right) =$$

$$\pi\left(\Lambda \sum_{i=1}^{n}\bigotimes_{j=1}^{i} f_{i,j} \in V^{*\otimes} \,.\, \sum_{i=1}^{n}\prod_{j=1}^{i}\varphi(a_{i,j})(f_{i,j})\right) = \Lambda f \in V^* \,.\, \varphi(a)(f) = \varphi(a);$$

$\rightsquigarrow [1] := E(=,\rightarrow) : \psi\pi = \varphi,$

$\texttt{Assume}\ \psi' : A \xrightarrow{k\text{-COALG}} V^{*\otimes\circ},$

$\texttt{Assume}\ [2] : \psi'\pi = \varphi,$

$\Big(\varphi', [3]\Big) := \mathit{\Game}\texttt{Bijection}(\phi\phi'\phi'') : \sum \varphi : A \xrightarrow{k\text{-VS}} V^{**} \ .\ \phi_1\phi_2\phi_3(\varphi') = \psi',$

$[4] := \dots [3][2] : \varphi = \varphi',$

$[A.*] := E(=,\rightarrow)[4]\mathcal{I}\psi[3] : \psi = \psi';$

$\rightsquigarrow [*] := I(\exists!)I^2(\forall)\mathit{\Game}^{-1}\texttt{CofreeCoalgebra} : \Big((V^{*\otimes\circ}, \pi) : \texttt{CofreeCoalgebra}(V)\Big);$

$\square$

$\texttt{InheritingCofreeCoalgebra} :: \forall k : \texttt{Field} \ .\ \forall V \in k\text{-VS} \ .\ \forall U \subset_{k\text{-VS}} V \ .\ \forall (A, \pi) : \texttt{CofreeCoalgebra}(V) \ .$
$\ .\ \exists \texttt{CofreeCoalgebra}(U)$

$\texttt{Proof}\ =$

$B := \sum \{E \subset_{k\text{-COALG}} A : \pi(E) \subset U\} : k\text{-COALG},$

$\texttt{Assume}\ C : k\text{-COALG},$

$\texttt{Assume}\ \varphi : C \xrightarrow{k\text{-VS}} U,$

$\Big(\psi, [1]\Big) := \mathit{\Game}\texttt{CofreeCoalgebra}(V)(A,\pi)(C,\varphi) : \sum \psi : C \xrightarrow{k\text{-COALG}} A \ .\ \psi\pi = \varphi,$

$\texttt{Assume}\ a : \operatorname{Im}\psi,$

$Z := \langle a \rangle_{k\text{-COALG}} : \texttt{Subcoalgebra}(A),$

$\Big(c, [2]\Big) := \mathit{\Game}\texttt{image}(\psi)(a) : \sum c \in C \ .\ a = \psi(c),$

$\texttt{Assume}\ z : Z,$

$\Big(n, m, i, j, y, [3]\Big) := \mathit{\Game}\texttt{spawnedCoalgebra}(a)(z) : \sum n, m \in \mathbb{N} \ .\ \sum i \in n \ .\ \sum j \in m \ .\ \sum y : m \to n \to A \ .$
$\ .\ \Delta^n(a) = \sum_{i=1}^{m} \bigotimes_{j=1}^{n} y_{i,j} \ \&\ y_{i,j} = z,$

$\Big(m', x, [4]\Big) := \mathit{\Game}k\text{-COALG}(C)(c)(n) : \sum m' \in \mathbb{N} \sum x : m' \to n \to A \ .\ \Delta^n(c) = \sum_{i=1}^{m'} \bigotimes_{i=1}^{n} x_{i,j},$

$[5] := [4]\mathit{\Game}k\text{-COALG}(C, A)(\psi)[3] : \sum_{i=1}^{m'} \bigotimes_{i=1}^{n} \psi(x_{i,j}) = \psi^{\otimes n}\Big(\Delta^n(c)\Big) = \Delta^n\psi(c) = \Delta^n(a) = \sum_{i=1}^{m} \bigotimes_{j=1}^{n} y_{i,j},$

$[6] := [3][5] : z \in \operatorname{Im}\psi,$

$[z.*] := [1][6] : \pi(z) \in U;$

$\rightsquigarrow [3] := I(\forall)\mathit{\Game}^{-1}\texttt{Subset} : \pi(Z) \subset U,$

$[a.*] := \mathcal{I}B\mathcal{I}Z[3] : a \in B;$

$\rightsquigarrow [2] := I(\forall)\mathit{\Game}^{-1}\texttt{Subset} : \operatorname{Im}\psi \subset B,$

$[3] := [2][1] : \psi^{|B}\pi_{|B} = \varphi,$

$\texttt{Assume}\ \psi' : C \xrightarrow{k\text{-COALG}} B,$

$\texttt{Assume}\ [4] : \psi'\pi_{|B} = \psi,$

$[C.*] := \mathit{\Game}\texttt{CofreeCoalgebra}(V)(A,\pi)(\psi')\mathit{\Game}\texttt{Unique} : \psi = \psi';$

$\rightsquigarrow [*] := I(\exists!)I^2(\forall)\mathit{\Game}^{-1}\texttt{CofreeCoalgebra} : \Big((B, \pi_{|B}) : \texttt{FreeCoalgebra}(U)\Big);$

$\square$

$\texttt{CofreeCoalgebraExists} :: \forall k : \texttt{Field} \,.\, \forall V : k\text{-}\mathsf{VS} \,.\, \exists \texttt{CofreeCoalgebra}(V)$

$\texttt{Proof} \;=$

$(A, \pi) := \texttt{DoubleDualCofreeCoalgebra}(V) : \texttt{CofreCoalgebra}(V^{**}),$

$(A', \pi') := \texttt{InheretingCofreeCoalgebra}\Big(V^{**}, \epsilon\, V, (A, \pi)\Big) : \texttt{CofreeCoalgebra}(\epsilon\, V),$

$[*] := \mathcal{A}\,\texttt{Isomorphism}(\epsilon)\mathcal{A}\,\texttt{CofreeCoalgebra}(\epsilon\, V)(A', \pi') : \Big((A', \pi'\epsilon^{-1}) : \texttt{CofreeCoalgebra}(V)\Big);$

$\square$

$\texttt{cofreeCoalgebraFunctor} :: \prod k : \texttt{Field} \,.\, k\text{-}\mathsf{VS} \xrightarrow{\mathsf{CAT}} k\text{-}\mathsf{COALG}$

$\texttt{cofreeCoalgebraFunctor}\,(V) = \mathrm{CF}(V) := \texttt{CofreeCoalgebraExists}(V)$

$\texttt{cofreeCoalgebraFunctor}\,(V, W, T) = \mathrm{CF}_{V,W}(T) := \mathcal{A}\,\texttt{CofreeCoalgebra}(\mathrm{CF}(W), \pi')(\pi T)$

$\quad \texttt{where} \quad (\mathrm{CF}(V), \pi) = \texttt{CofreeCoalgebraExists}(V)$

$\qquad\qquad\quad (\mathrm{CF}(W), \pi') = \texttt{CofreeCoalgebraExists}(W)$

$\texttt{CoalgebrasForgetfulFunctorAdjoint} :: \forall k : \texttt{Field} \,.\, (\mathrm{CF}, U_{k\text{-}\mathsf{COALG}, k\text{-}\mathsf{VS}}) : \texttt{RightAdjoint}(k\text{-}\mathsf{COALG}, k\text{-}\mathsf{VS})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CocommutativeCofreeCoalgebra} :: \prod k : \texttt{Field} \,.\, \prod V : k\text{-}\mathsf{VS} \,.\, ?\sum A : k\text{-}\mathsf{CCOALG} \,.\, A \xrightarrow{k\text{-}\mathsf{VS}} V$

$(A, \pi) : \texttt{CocommutiveCofreeCoalgebra} \iff \forall B : R\text{-}\mathsf{COALG} \,.\, \forall \varphi : B \xrightarrow{R\text{-}\mathsf{MOD}} M \,.$

$\quad .\, \exists! \psi : B \xrightarrow{R\text{-}\mathsf{COALG}} A \,.\, \psi\pi = \varphi$

$\texttt{CofreeCoalgebraSurjectivity} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$

$\quad .\, \forall (A, \pi) : \texttt{CocommutativeCofreeCoalgebra}(M) \,.\, \pi : A \twoheadrightarrow M$

$\texttt{Proof} \;=$

$\texttt{Assume}\; m : M,$

$\mu := \Lambda t \in R \,.\, tm : R \xrightarrow{R\text{-}\mathsf{MOD}} M,$

$\Big(\psi, [1]\Big) := \mathcal{A}\,\texttt{CocommutativeCofreeCoalgebra}(A, \pi)(\nu) : \sum \psi : R \xrightarrow{R\text{-}\mathsf{CCOALG}} A \,.\, \psi\pi = \mu,$

$[2] := [1]\mathcal{O}(\mu) : \psi\pi(e) = \mu(e) = m,$

$[m.*] := \mathcal{A}\,\texttt{image}[2] : m \in \operatorname{Im} \pi;$

$\leadsto [*] := I(\forall)\mathcal{A}^{-1}\texttt{Surjective} : (\pi : A \twoheadrightarrow M);$

$\square$

$\texttt{IsomorphicCofreeCoalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.$

$\quad .\, \forall (A, \pi), (B, \pi') : \texttt{CocommutativeCofreeCoalgebra}(M) \,.\, A \cong_{R\text{-}\mathsf{COALG}} B$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CocommutativeCofreeCoalgebraExists} :: \forall k : \texttt{Field} . \forall V : k\text{-VS} . \exists \texttt{CocommutativeCofreeCoalgebra}(M) .$

$\texttt{Proof} \ =$

$(A, \pi) := \texttt{CofreeCoalgebraExists}(V) : \texttt{CofreeCoalgebra}(V),$

$B := \sum\{E \subset A : (E, \delta, \eta) \in k\text{-CCOALG}\} : k\text{-CCOALG},$

$\texttt{Assume}\ C : k\text{-CCOALG},$

$\texttt{Assume}\ \varphi : C \xrightarrow{k\text{-VS}} V,$

$\Big(\psi, [1]\Big) := \mathcal{I}\texttt{CofreeCoalgebra}(V)(A, \pi) : \sum \psi : C \xrightarrow{k\text{-COALG}} A \ . \ \psi\pi = \varphi,$

$\texttt{Assume}\ a : \operatorname{Im}\psi,$

$\Big(c, [2]\Big) := \mathcal{I}\texttt{image}\mathcal{I}a : \sum c \in C \ . \ \psi(c) = a,$

$[2] := \mathcal{I}k\text{-COALG}(C, A)(\psi)[2] : \langle a \rangle_{z\text{-COALG}} \in \Im\psi,$

$[3] := [2]\mathcal{I}k\text{-COALG}(C, A)(\psi)\mathcal{I}\texttt{swap}\mathcal{I}\texttt{Cocommutative}(C)\mathcal{I}k\text{-COALG}(C, A)[2] :$

$\quad : a\ \Delta_A\ \texttt{swap}\ = c\ \psi\ \Delta_A\ \texttt{swap} = c\ \Delta_C\ (\psi \otimes \psi)\ \texttt{swap} = c\ \Delta_C\ \texttt{swap}\ (\psi \otimes \psi) = c\ \Delta_C\ (\psi \otimes \psi) =$

$\quad = c\ \psi\ \Delta_A = a\ \Delta_A,$

$[a.*] := \mathcal{D}B[2][3] : a \in B;$

$\rightsquigarrow [2] := \mathcal{I}\texttt{Subset} : \operatorname{Im}\psi \subset B,$

$[3] := [1][2] : \psi^{|B}\pi_{|B} = \varphi,$

$\texttt{Assume}\ \psi' : C \xrightarrow{k\text{-CCOALG}} B,$

$\texttt{Assume}\ [4] : \psi'\pi_{|B} = \varphi,$

$[C.*] := \mathcal{I}\texttt{CofreeCoalgebra}(V)(A, \pi)(\psi')\mathcal{I}\texttt{Unique} : \psi = \psi';$

$\rightsquigarrow [*] := I(\exists!)I^2(\forall)\mathcal{I}^{-1}\texttt{CocommutativeCofreeCoalgebra} : \Big((B, \pi_{|B}) : \texttt{CocommutativeFreeCoalgebra}(U)\Big);$

$\square$

$\texttt{CocommutativeCoalgebraFunctor} :: \prod k : \texttt{Field} . \ k\text{-VS} \xrightarrow{\text{CAT}} k\text{-CCOALG}$

$\texttt{cofreeCoalgebraFunctor}\ (V) = \mathrm{CCF}(V) := \texttt{CocommutativeCofreeCoalgebraExists}(V)$

$\texttt{cofreeCoalgebraFunctor}\ (V, W, T) = \mathrm{CCF}_{V,W}(T) := \mathcal{I}\texttt{CocommutativeCofreeCoalgebra}(\mathrm{CF}(W), \pi')(\pi T)$

$\quad \texttt{where} \quad (\mathrm{CCF}(V), \pi) = \texttt{CocommutativeCofreeCoalgebraExists}(V)$

$\qquad\qquad (\mathrm{CCF}(W), \pi') = \texttt{CocommutativeCofreeCoalgebraExists}(W)$

$\texttt{CommutativeCoalgebrasForgetfulFunctorAdjoint} ::$

$\quad :: \forall k : \texttt{Field} . \ (\mathrm{CCF}, U_{k\text{-CCOALG}, k\text{-VS}}) : \texttt{RightAdjoint}(k\text{-COALG}, k\text{-VS})$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{CocommutativeCofreeCoalgebraOfSum} :: \forall k : \texttt{Field} . \forall V, V' \in l\text{-}\mathsf{VS} .$

$. \left(A \otimes A', \pi\right) : \texttt{CocmmutativeCofreeCoalgebra}(V \oplus V')$

$\texttt{where} \quad (A, \nu) = \texttt{CocommutativeCofreeCoalgebraExists}(V)$

$(A', \nu') = \texttt{CocommutativeCofreeCoalgebraExists}(V')$

$\pi = Ɑ\texttt{TensorProduct}(A, A')\Lambda a \in A . a' \in A' . \left(\eta'(a')\nu(a), \eta(a)\nu'(a')\right)$

$\texttt{Proof} =$

$\texttt{Assume}\ B : k\text{-}\mathsf{CCOALG},$

$\texttt{Assume}\ \varphi : B \xrightarrow{k\text{-}\mathsf{VS}} V \oplus V',$

$(\psi, [1]) := Ɑ\texttt{CocommutativeCofreeCoalgebra}(V)(A, \nu)(\varphi\pi_1) : \sum \psi : B \xrightarrow{k\text{-}\mathsf{CCOALG}} A . \psi\nu = \varphi\pi_1,$

$(\psi', [2]) := Ɑ\texttt{CocommutativeCofreeCoalgebra}(V')(A', \nu')(\varphi\pi_2) : \sum \psi' : B \xrightarrow{k\text{-}\mathsf{CCOALG}} A' . \psi'\nu' = \varphi\pi_2,$

$\texttt{Assume}\ b : B,$

$[b.*] := \texttt{SweedlerNotation}(b)Ɔ\pi Ɑ k\text{-}\mathsf{COALG}(B, A)(\psi)Ɑ k\text{-}\mathsf{COALG}(B, A')(\psi')[1][2]Ɑ k\text{-}\mathsf{VS}(\varphi)(B, V)$

$$Ɑ k\text{-}\mathsf{COALG}(B)Ɑ\texttt{directSum}(V, V') : b\,\Delta\,(\psi \otimes \psi')\,\pi = \left(\sum_b b_1 \otimes b_2\right)\,(\psi \otimes \psi')\,\pi =$$

$$= \left(\sum_b \eta'\big(\psi'(b_2)\big)\nu\big(\psi(b_1)\big), \sum_b \eta\big(\psi(b_1)\big)\nu'\big(\psi'(b_2)\big)\right) = \left(\sum_b \eta(b_2)\varphi(b_1), \sum_b \eta(b_1)\varphi(b_2)\right) =$$

$$= \left(\pi_1\varphi\left(\sum_b \eta(b_2)b_1\right), \pi_2\varphi\left(\sum_b \eta(b_1)b_2\right)\right) = \big(\pi_1\ \varphi(b), \pi_2\ \varphi(b)\ \big) = \eta(b)\varphi(b);$$

$\rightsquigarrow [1] := I(=, \to) : \Delta(\psi \otimes \psi')\ \pi = \mathrm{id},$

$\texttt{Assume}\ \widehat{\psi} : B \xrightarrow{k\text{-}\mathsf{CCOALG}} A \otimes A',$

$\texttt{Assume}\ [2] : \widehat{\psi}\pi = \varphi,$

$\overline{\psi} := \widehat{\psi}(\mathrm{id} \otimes \eta') : B \xrightarrow{k\text{-}\mathsf{CCOALG}} A,$

$\overline{\psi}' := \widehat{\psi}(\eta \otimes \mathrm{id}) : B \xrightarrow{k\text{-}\mathsf{CCOALG}} A',$

$[3] := Ɔ\overline{\psi}Ɑ k\text{-}\mathsf{VS}(B, A)(\nu)Ɔ^{-1}\pi : \overline{\psi}\ \nu = \widehat{\psi}\ (\mathrm{id} \otimes \eta')\ \nu = \widehat{\psi}\ (\nu \otimes \eta') = \widehat{\psi}\pi\pi_1,$

$[4] := Ɑ\texttt{CocommutativeFreeCoalgebra}(V)(A, \nu))[3] : \overline{\psi} = \psi,$

$[5] := Ɔ\overline{\psi}'Ɑ k\text{-}\mathsf{VS}(B, A')(\nu')Ɔ^{-1}\pi : \overline{\psi}'\ \nu' = \widehat{\psi}\ (\eta \otimes \mathrm{id})\ \nu' = \widehat{\psi}\ (\eta \otimes \nu') = \widehat{\psi}\pi\pi_2,$

$[6] := Ɑ\texttt{CocommutativeFreeCoalgebra}(V)(A', \nu')[5] : \overline{\psi}' = \psi',$

$\texttt{Assume}\ b : B,$

$[B.*] := [5][6]Ɔ\widehat{\psi}Ɔ\widehat{\psi}'Ɑ\texttt{sweedlerNotation}\ Ɑ\texttt{tensorFunction}Ɑ\mathcal{L}(A, A'; A \otimes A')(\texttt{tensorproduct})$

$Ɑ k\text{-}\mathsf{CCOALG}(B, A \otimes A')(\widehat{\psi})Ɑ\texttt{tensorProductCoalgebra}Ɑ^{-1}\widehat{\psi}Ɑ k\text{-}\mathsf{CCOALG}(B, A \otimes A')(\psi)$

$$Ɑ k\text{-}\mathsf{COALG}(B) : b\,\Delta\,(\psi \otimes \psi') = b\,\Delta\,(\overline{\psi} \otimes \overline{\psi}') = \sum_b \left(b_1\ \widehat{\psi}\ (\mathrm{id} \otimes \eta') \otimes b_2\ \widehat{\psi}\ (\eta \otimes \mathrm{id})\right) =$$

$$= \sum_b \left(\sum_{a_1 = \widehat{\psi}(b_1)} \eta'(a_1^2)a_1^1\right) \otimes \left(\sum_{a_2 = \widehat{\psi}(b_2)} \eta(a_2^1)a_2^2\right) = \sum_b \sum_{a_1 = \widehat{\psi}(b_1)} \sum_{a_2 = \widehat{\psi}(b_2)} \eta(a_2^1)\eta'(a_1^2)a_1^1 \otimes a_2^2 =$$

$$= \sum_b \sum_{a_1 = \widehat{\psi}(b_1)} \sum_{a_2 = \widehat{\psi}(b_2)} \eta(a_2^1)\eta'(a_2^2)a_1^1 \otimes a_1^2 = \sum_b \eta(\widehat{\psi}(b_2))\widehat{\psi}(b_1) = \sum_b \eta(b_2)\widehat{\psi}(b_1) = \widehat{\psi}(b);$$

$\rightsquigarrow [*] := I(\exists!)I^2(\forall)Ɑ^{-1}\texttt{CocommutativeCofreeCoalgebra} :$

$: \Big((A \otimes A', \pi) : \texttt{CocommutativeFreeCoalgebra}(U)\Big);$

$\square$

2.6 Comodules

$\texttt{LeftAlgebraModule} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \sum M : R\text{-}\mathsf{MOD} \,.\, A \otimes M \xrightarrow{R\text{-}\mathsf{MOD}} M$

$(M, \mu) : \texttt{LeftAlgebraModule} \iff (\mathrm{id} \otimes \mu)\mu = (\mu_A \otimes \mathrm{id})\mu \;\&\; (e_A \otimes \mathrm{id})\mu = \cdot$

$\texttt{RightAlgebraModule} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.\, \sum M : R\text{-}\mathsf{MOD} \,.\, M \otimes A \xrightarrow{R\text{-}\mathsf{MOD}} M$

$(M, \mu) : \texttt{RightAlgebraModule} \iff (\mu \otimes \mathrm{id})\mu = (\mathrm{id} \otimes \mu_A)\mu \;\&\; (\mathrm{id} \otimes e_A)\mu = \cdot$

$\texttt{LeftComodule} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{COALG} \,.\, \sum M : R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes M$

$(M, \rho) : \texttt{LeftComodule} \iff \rho(\mathrm{id} \otimes \rho) = \rho(\Delta \otimes \mathrm{id}) \;\&\; \rho(\eta_A \otimes \mathrm{id}) = \mathrm{id}$

$\texttt{RightAlgebraComodule} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{COALG} \,.\, \sum M : R\text{-}\mathsf{MOD} \,.\, M \xrightarrow{R\text{-}\mathsf{MOD}} M \otimes A$

$(M, \rho) : \texttt{RightComodule} \iff \rho(\rho \otimes \mathrm{id}) = \rho(\mathrm{id} \otimes \Delta) \;\&\; \rho(\mathrm{id} \otimes \eta_A) = \mathrm{id}$

$\texttt{LeftAlgebraModuleMorphism} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.$

$\quad . \prod X, Y : \texttt{LeftAlgebraModule}(A) \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y$

$\varphi : \texttt{LeftAlgebraModuleMorphism} \iff (\mathrm{id} \otimes \varphi)\mu_Y = \mu_X \varphi$

$\texttt{RightAlgebraModuleMorphism} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{ALGE} \,.$

$\quad . \prod X, Y : \texttt{RightAlgebraModule}(A) \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y$

$\varphi : \texttt{RightAlgebraModuleMorphism} \iff (\varphi \otimes \mathrm{id})\mu_Y = \mu_X \varphi$

$\texttt{LeftComoduleMorphism} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{COALG} \,.\, \prod X, Y : \texttt{LeftComodule}(A) \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y$

$\varphi : \texttt{LeftComoduleMorphism} \iff \rho_X(\mathrm{id} \otimes \varphi) = \varphi \rho_Y$

$\texttt{RightComoduleMorphism} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{COALG} \,.\, \prod X, Y : \texttt{RightComodule}(A) \,.\, X \xrightarrow{R\text{-}\mathsf{MOD}} Y$

$\varphi : \texttt{RightComoduleMorphism} \iff \rho_X(\varphi \otimes \mathrm{id}) = \varphi \rho_Y$

$\texttt{leftAlgebraModuleCategory} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{ALGE} \to \mathsf{CAT}$

$\texttt{leftAlgebraModuleCategory}\,(A) = {}_A\mathsf{MOD} :=$

$\quad := \Big(\texttt{LeftAlgebraModule}(A), \texttt{LeftAlgebraiModuleMorphism}(A), \circ, \mathrm{id}\Big)$

$\texttt{rightAlgebraModuleCategory} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{ALGE} \to \mathsf{CAT}$

$\texttt{rightAlgebraModuleCategory}\,(A) = \mathsf{MOD}_A :=$

$\quad := \Big(\texttt{RightAlgebraModule}(A), \texttt{RightAlgebraModuleMorphism}(A), \circ, \mathrm{id}\Big)$

$\texttt{leftComoduleCategory} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{COALG} \to \mathsf{CAT}$

$\texttt{leftComoduleCategory}\,(A) = {}^A\mathsf{MOD} := \Big(\texttt{LeftAlgebraComodule}(A), \texttt{LeftComoduleMorphism}(A), \circ, \mathrm{id}\Big)$

$\texttt{rightComoduleCategory} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{COALG} \to \mathsf{CAT}$

$\texttt{rightComoduleCategory}\,(A) = \mathsf{MOD}^A := \Big(\texttt{RightComodule}(A), \texttt{RightComoduleMorphism}(A), \circ, \mathrm{id}\Big)$

$\texttt{CoalgebraAsComodule} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{COALG} \,.\, (A, \Delta) \in \mathsf{MOD}^A$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{ConstructedComoduleStructure} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{COALG} \,.\, \forall M \in R\text{-}\mathsf{MOD} \,.\, (M \otimes A, \mathrm{id} \otimes \Delta) \in \mathsf{MOD}^A$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{setComodule} :: \prod R \in \mathsf{ANN} \,.\, \prod X : \mathsf{SET} \,.\, (X \to R\text{-}\mathsf{MOD}) \to \mathsf{MOD}^{\mathrm{F}(X)}$

$$\texttt{setComodule}\,(M) := \left(\bigoplus_{x \in X} M_x, \mathcal{I}\texttt{DirectSum}(X, M) \,.\, \Lambda x \in X \,.\, \Lambda m \in M \,.\, m \otimes e_x\right)$$

$\texttt{FundamentalTheoremOfComodules} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{COALG} \,.\, \forall M \in \mathsf{MOD}^A \,.\, \forall m \in M \,.$

$\quad .\, \exists N \subset_{\mathsf{MOD}^A} M : m \in N \;\&\; \dim N < \infty$

$\texttt{Proof} \; =$

$\dots$

$\square$

2.7 Rationality

2.8 Bicomodules

2.9 Cotensor Products

2.10 Simplicity and Injectivity

2.11 Torsion Theories

2.12 Cosemisimplicity

2.13 Semiperfectnes

2.14 Duals of Frobeneus Theories

3 Theory of Hopf Algebras

3.1 Bialgebras

$$\texttt{Bialgebra} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{MOD} \;.$$
$$.\;(A \otimes A \xrightarrow{R\text{-}\mathsf{MOD}} A) \times (R \xrightarrow{R\text{-}\mathsf{MOD}}) \times (A \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes A) \times (A \xrightarrow{R\text{-}\mathsf{MOD}} R)$$
$$(A, \mu, e, \Delta, \eta) : \texttt{Bialgebra} \iff (A, \Delta, \eta) \in R\text{-}\mathsf{COALG} \;\&\; (A, \mu, e) \in R\text{-}\mathsf{ALGE} \;\&$$
$$\&\; \mu : A \otimes A \xrightarrow{R\text{-}\mathsf{COALG}} A \;\&\; e : R \xrightarrow{R\text{-}\mathsf{COALG}} A \;\&\; \Delta : A \xrightarrow{R\text{-}\mathsf{ALGE}} A \otimes A \;\&\; \eta : A \xrightarrow{R\text{-}\mathsf{ALGE}} R$$

$$\texttt{BialgebraMorphism} :: \prod R \in \mathsf{ANN} \;.\; \prod A, B : \texttt{Bialgebra}(R) \;.\; A \xrightarrow{R\text{-}\mathsf{ALGE}} B$$
$$f : \texttt{BialgebraMorphism} \iff f : A \xrightarrow{R\text{-}\mathsf{COALG}} B$$

$$\texttt{bialgebraCategory} :: \mathsf{ANN} \to \texttt{Category}$$
$$\texttt{bialgebraCategory}\,(R) = R\text{-}\mathsf{BIALG} := \Big(\texttt{Bialgebra}, \texttt{BialgebraMorphism}, \mathrm{id}, \circ\Big)$$

$$\texttt{Primitive} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{BIALG} \;.\; ?A$$
$$a : \texttt{Primitive} \iff \Delta(a) = a \otimes e + e \otimes a$$

$$\texttt{monoidBialgebra} :: \prod R \in \mathsf{ANN} \;.\; \texttt{Monoid} \to R\text{-}\mathsf{BIALG}$$
$$\texttt{monoidBialgebra}\,(M) = RM := \Big(R^{\oplus M}, Ɑ\,\texttt{directPower}\,\Lambda a, b \in M \;.\; ab, \Lambda \alpha \in R \;.\; \alpha e_M,$$
$$, Ɑ\,\texttt{directPower}\,\Lambda a \in M \;.\; a \otimes a, Ɑ\,\texttt{directPower}\,\Lambda a \in M \;.\; e_A\Big)$$

$$\texttt{grouplike} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{BIALG}$$
$$\texttt{grouplike}\,() := \Big(R[x], Ɑ\,R[x](x \otimes x), Ɑ\,R[x](1)\Big)$$

$$\texttt{primitive} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{BIALG}$$
$$\texttt{primitive}\,() := \Big(R[x], Ɑ\,R[x](x \otimes 1 + 1 \otimes x), Ɑ\,R[x](0)\Big)$$

$$\texttt{Biideal} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{BIALG} \;.\; ?A$$
$$I : \texttt{Biideal} \iff I : \texttt{Ideal} \;\&\; \texttt{Coideal}(A)$$

$$\texttt{BialgebraQuotient} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{BIALG} \;.\; \texttt{Biideal}(A) \to R\text{-}\mathsf{BIALG}$$
$$\texttt{BialgebraQuotient}\,(I) = \frac{A}{I} := \frac{A}{I}$$

$\texttt{PolynomialBialgebraClassification} :: \forall R \in \mathsf{ANN} \,.\, \forall \Delta : R[x] \xrightarrow{R\text{-}\mathsf{ALGE}} R[x] \otimes R[x] \,.\, \forall \eta : R[x] \xrightarrow{R\text{-}\mathsf{ALGE}} R \,.$

$\quad . \left(R[x], \Delta, \eta\right) : R\text{-}\mathsf{BIALG} \Rightarrow \left(R[x], \Delta, \eta\right) \cong \texttt{grouplike}(R) \Big| \left(R[x], \Delta, \eta\right) \cong \texttt{primitive}(R)$

$\texttt{Proof} \; =$

$A := \left(R[x], \Delta, \eta\right) : R\text{-}\mathsf{BIALG},$

$[1] := \daleth k\text{-}\mathsf{BIALG}(A) : \Delta(1) = 1 \otimes 1 \,\&\, \eta(1) = e_k,$

$\left(n, m, \beta[2]\right) := \texttt{TensorProductBasis}(A, A)(\Delta(x)) : \sum n, m \in \mathbb{Z}_+ \,.\, \beta : n \to m \to R \,.\, \Delta(x) = \beta_{i,j} x^i \otimes x^j,$

$[3] := \daleth A \daleth A\text{-}\mathsf{COALG}[2] : \{(i,j) \in n \times m \,.\, \beta_{imj} \neq 0\} \neq \emptyset,$

$I := \max\{i \in n : \exists j \in m : \beta_{i,j} \neq 0\} : n,$

$J := \max\{j \in n : \beta_{I,j} \neq 0\} : m,$

$[4] := \daleth \texttt{multideg} \daleth R\text{-}\mathsf{BIALG}(A) \daleth R\text{-}\mathsf{COALG}(A) \daleth \texttt{multideg}(\ldots) :$

$\quad (I^2, IJ, J) = \operatorname{multideg}\, (\Delta \otimes \mathrm{id})\Delta(x) = \operatorname{multideg}\, (\mathrm{id} \otimes \Delta)\Delta(x) = (I, IJ, J^2),$

$[5] := \texttt{IdempotentIntegers}[4] : (I, J) \in \{0, 1\}^2,$

$J' := \max\{j \in m : \exists i \in n : \beta_{i,j} \neq 0\} : m,$

$I' := \max\{j \in n : \beta_{I,j} \neq 0\} : n,$

$[6] := \daleth \texttt{multideg} \daleth R\text{-}\mathsf{BIALG}(A) \daleth R\text{-}\mathsf{COALG}(A) \daleth \texttt{multideg}(\ldots) :$

$\quad (I'^2, I'J', J') = \operatorname{multideg}\, (\Delta \otimes \mathrm{id})\Delta(x) = \operatorname{multideg}\, (\mathrm{id} \otimes \Delta)\Delta(x) = (I', I'J', J'^2),$

$[7] := \texttt{IdempotentIntegers}[4] : (I', J') \in \{0, 1\}^2,$

$\texttt{Assume}\ [8] : (I = 0,$

$[9] := [8][2] : \Delta(x) = \beta_{0,0} 1 \otimes 1 + \beta_{0,1} 1 \otimes x,$

$[10] := \daleth R\text{-}\mathsf{COALG}(A)[9][1] : x = (\mathrm{id} \otimes \eta) \circ \Delta(x) = \beta_{0,0} + \beta_{0,1}\eta(x),$

$[8.*] := \daleth R[x] : \perp;$

$\rightsquigarrow [8] := E(\perp) : I \neq 0,$

$\texttt{Assume}\ [9] : (I, J) = (1, 0),$

$[10] := [9][2][7] : \Delta(x) = \beta_{1,0} x \otimes 1 + \beta_{0,0} 1 \otimes 1 + \beta_{0,1} 1 \otimes x,$

$[11] := \daleth R\text{-}\mathsf{COALG}(A)[10][1] : x = (\eta \otimes \mathrm{id}) \circ \Delta(x) = \beta_{1,0}\eta x + \beta_{0,0} + \beta_{0,1} x,$

$[12] := \daleth R\text{-}\mathsf{COALG}(A)[10][1] : x = (\mathrm{id} \otimes \eta) \circ \Delta(x) = \beta_{0,1}\eta x + \beta_{0,0} + \beta_{1,0} x,$

$[13] := \daleth R[x][10][11] : \beta_{1,0} = \beta_{0,1} = 0, \beta_{0,0} = -\eta(x),$

$\varphi := \daleth R[x](x - \beta_{0,0}) : A \xleftrightarrow{R\text{-}\mathsf{BIALG}} A,$

$[9.*] := [13](A) : A \cong_{R\text{-}\mathsf{BIALG}} \texttt{primitive}(R);$

$\rightsquigarrow [9] := I(\Rightarrow) : (I, J) = (1, 0) \Rightarrow A \cong_{R\text{-}\mathsf{BIALG}} \texttt{primitive}(R),$

$\texttt{Assume}\ [10] : (I, J) = (1, 1),$

$[11] := [10][2][7] : \Delta(x) = \beta_{0,0} 1 \otimes 1 + \beta_{1,0} x \otimes 1 + \beta_{0,1} 1 \otimes x + \beta_{1,1} x \otimes x,$

$[12] := \daleth R\text{-}\mathsf{COALG}(A)[11] : x = (\mathrm{id} \otimes \eta)\Delta(x) = (\beta_{0,0} + \beta_{0,1}\eta(x)) + (\beta_{1,0} + \beta_{1,1}\eta(x))x,$

$[13] := \daleth R\text{-}\mathsf{COALG}(A)[11] : x = (\eta \otimes \mathrm{id})\Delta(x) = (\beta_{0,0} + \beta_{1,0}\eta(x)) + (\beta_{0,1} + \beta_{1,1}\eta(x))x,$

$y := \daleth R[x](x - \beta_{0,0}) : A \xleftrightarrow{R\text{-}\mathsf{BIALG}} R[y],$

$[14] := \daleth y[12][13] : \Delta(y) = 1 \otimes y + y \otimes 1 + \beta_{1,1} y \otimes y,$

$[15] := \Game(I, J)[10] : \beta_{1,1} \neq 1,$

$z := \beta_{1,1} y + 1 : R[y] \xleftrightarrow{R\text{-}\mathsf{BIALG}} R[z],$

$[16] := \Game z \daleth \mathcal{L}\Big(R[y], R[y]; R[y] \otimes R[y]\Big) \texttt{tensorProduct} \Game z : \Delta(z) = \Delta(\beta_{1,1} y + 1) = 1 \otimes 1 + \beta_{1,1}(1 \otimes y) + \beta_{1,1}(y \otimes$

$[10.*] := \Game z \Game y \daleth^{-1} \texttt{grouplike}[16] : A \cong_{R\text{-}\mathsf{BIALG}} \texttt{grouplike}(A);$

$\rightsquigarrow [10] := I(\Rightarrow) : (I, J) = (1, 1) \Rightarrow A \cong_{R\text{-}\mathsf{BIALG}} \texttt{grouplike}(A),$

$[*] := [5][8][9][10] : A \cong_{R\text{-}\mathsf{BIALG}} \texttt{primitive}(A) \Big| A \cong_{R\text{-}\mathsf{BIALG}} \texttt{grouplike}(A);$

$\square$

$\texttt{BialgebraModuleAlgebra} :: \prod R \in \mathsf{ANN} \;.\; \prod B : R\text{-}\mathsf{BIALG} \;.\; ?\; {}_B\mathsf{MOD} \,\&\, R\text{-}\mathsf{ALGE}$

$A : \texttt{BialgebraModuleAlgebra} \iff (\mathrm{id} \otimes \mu_A)\mu_{B,A} = (\Delta_B \otimes \mathrm{id}_A \otimes \mathrm{id}_A)\mu^{\otimes 2}_{B,A}\mu_A \,\&\, (\mathrm{id} \otimes e_A)\mu_{B,A} = \eta_B e_A$

$\texttt{categoryOfBialgebraModuleAlgebras} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{BIALG} \to \mathsf{CAT}$

$\texttt{categoryOfBialgebraModuleAlgebras}\,(B) = {}_B\mathsf{ALGE} :=$

$:= \Big(\texttt{BialgebraModuleAlgebra},\ {}_B\mathsf{MOD} \,\&\, R\text{-}\mathsf{ALGE}, \mathrm{id}, \circ\Big)$

$\texttt{BialgebraModuleCoalgebra} :: \prod R \in \mathsf{ANN} \;.\; \prod B : R\text{-}\mathsf{BIALG} \;.\; ?\mathsf{MOD}_B \,\&\, R\text{-}\mathsf{COALG}$

$A : \texttt{BialgebraModuleCoalgebra} \iff \mu_{A,B}\Delta_A = (\Delta_A \otimes \Delta_B)\mu^{\otimes 2}_{A,B} \,\&\, \mu_{A,B}\eta_A = (\eta_A \otimes \eta_B)\mu_R$

$\texttt{categoryOfBialgebraModuleCoalgebras} :: \prod R \in \mathsf{ANN} \;.\; R\text{-}\mathsf{BIALG} \to \mathsf{CAT}$

$\texttt{categoryOfBialgebraModuleCoalgebra}\,(B) = \mathsf{COALG}_B :=$

$:= \Big(\texttt{BialgebraModuleAlgebra},\ {}_B\mathsf{MOD} \,\&\, R\text{-}\mathsf{ALGE}, \mathrm{id}, \circ\Big)$

$\texttt{rightBimoduleDualAlgebra} :: \prod R \in \mathsf{ANN} \;.\; \prod B \in R\text{-}\mathsf{BIALG} \;.\; {}_B\mathsf{ALGE}$

$\texttt{rightBimoduleDualAlgebra}\,() = B^* := (B^*, \Lambda b \in B \;.\; \Lambda f \in B^* \;.\; a \rightharpoonup f)$

$\texttt{Assume}\ f, g : B^\circ,$

$\texttt{Assume}\ b, x : B,$

$\Big[(f,g).*.1\Big] := \daleth B^* \daleth \texttt{hitAction} \daleth \texttt{dualAlgebra} \daleth R\text{-}\mathsf{BIALG}(B) \daleth^{-1} \texttt{hitAction} \daleth^{-1} B^* :$

$\Big(b(fg)\Big)(x) = (b \rightharpoonup fg)(x) = fg(bx) = \sum_{y=bx} f(y_1)g(y_2) = \sum_b \sum_x f(b_1x_1)g(b_2x_2) =$

$= \sum_b \sum_x (b_1 \rightharpoonup f)(x_1)(b_2 \rightharpoonup g)(x_2) \sum_b (b_1 \rightharpoonup f)(b_2 \rightharpoonup g)(x) \sum_b (b_1 f)(b_2 f)(x),$

$\Big[(f,g).*.1\Big] := \daleth B^* \daleth \texttt{hitAction} \daleth R\text{-}\mathsf{ALGE}(B,R)(\eta) \daleth^{-1} B^* :$

$be_{B^*}(x) = (b \rightharpoonup \eta)(x) = \eta(bx) = \eta(b)\eta(x) = \eta(b)e_{B^*}(x);$

$\rightsquigarrow [*] := \daleth\ {}_B\mathsf{ALGE} : B^* \in {}_B\mathsf{ALGE};$

$\square$

$\texttt{FiniteDualBialgebra} :: \forall R : \texttt{Field} \,.\, \forall B \in R\text{-BIALG} \,.\, B^\circ \in R\text{-BIALG}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f, g : B^\circ,$

$\texttt{Assume}\ b, x : B,$

$[(b, x).*] := Л\texttt{hitBy}Л\texttt{dualAlgebra}ЛR\text{-BIALG}Л^{-1}\texttt{hitBy}Л^{-1}\texttt{dualAlgebra} :$

$$: (fg \leftharpoonup b)(x) = fg(xb) = \sum_{y=xb} f(y_1)g(y_2) = \sum_x \sum_a f(x_1b_1)g(x_1b_2) =$$

$$= \sum_x \sum_a (f \leftharpoonup b_1)(x_1)(f \leftharpoonup b_2)(x_2) = \sum_a (f \leftharpoonup b_1)(f \leftharpoonup b_2)(x);$$

$\rightsquigarrow [1] := Л^{-1}\texttt{Subset} : (fg \leftharpoonup B) \subset (f \leftharpoonup B)(g \leftharpoonup B),$

$[2] := \texttt{FiniteHitByAction}(f) : \dim(f \leftharpoonup B) < \infty,$

$[3] := \texttt{FiniteHitByAction}(g) : \dim(g \leftharpoonup B) < \infty,$

$[4] := \texttt{SubsetDimension}[1]\texttt{ProductDimension}[2][3] : \dim(fg \leftharpoonup B) < \dim(f \leftharpoonup B)(g \leftharpoonup B) M \infty,$

$[(f, g).*] := \texttt{FiniteHitByAction}[4] : fg \in B^\circ;$

$\rightsquigarrow [1] := I(\forall) : \forall f, g \in B^\circ \,.\, fg \in B^\circ,$

$[2] := Л\texttt{kernel}ЛR\text{-BIALG}Л^{-1}\texttt{Ideal} : \Big(\ker e_{B^*} : \texttt{Ideal}(B)\Big),$

$[3] := ЛB^\circ[2] : e_{B^*} \in B^\circ,$

$\texttt{Assume}\ f, g : B^*,$

$\texttt{Assume}\ x, y : B,$

$\texttt{Assume}\ \alpha : R,$

$\Big[(f, g).*.1\Big] := Л\texttt{finiteDualCoalgebra}Л\texttt{dualAlgebra}ЛR\text{-BIALG}(B)$

$Л^{-1}\texttt{finiteDualCoalgebr}Л^{-1}\texttt{dualAlgebra}Л^{-1}\texttt{tensorProductAlgebra} : \Delta(fg)(x \otimes y) = fg(xy) =$

$$= \sum_{z=xy} f(z_1)g(z_2) = \sum_x \sum_y f(x_1y_1)g(x_2y_2) = \sum_x \sum_y \Delta(f)(x_1 \otimes y_1)\Delta(g)(x_2 \otimes y_2) = \Delta(f)\Delta(g)(x \otimes y),$$

$\Big[(f, g).*.2\Big] := Л\texttt{finiteDualCoalgebra}Л\texttt{dualAlgebra}ЛR\text{-BIALG}(B)$

$Л^{-1}\texttt{finiteDualCoalgebra}Л\texttt{tensorProductCoalgebra} : \eta(fg) = fg(e_B) = f(e_B)g(e_B) = \eta(f)\eta(g) =$

$= \eta(f \otimes g),$

$\Big[(f, g).*.3\Big] := Л\texttt{finiteDualCoalgebra}Л\texttt{dualAlgebra}ЛR\text{-BIALG}(B)$

$Л^{-1}\texttt{finiteDualCoalgebra}Л\texttt{tensorProductCoalgebra} : \Delta\Big(e_{B^\circ}(\alpha)\Big)(x \otimes y) = e_{B^\circ}(\alpha)(xy) = \alpha\eta_B(xy) =$

$= \alpha\eta_B(x)\eta_B(y) = e_{B^\circ} \otimes e_{B^\circ}\Big(\Delta(\alpha)\Big)(x \otimes y),$

$\Big[(f, g).*.4\Big] := Л\texttt{finiteDualCoalgebra}Л\texttt{dualAlgebra}ЛR\text{-BIALG}(B)$

$Л^{-1}\texttt{finiteDualCoalgebra}Л\texttt{tensorProductCoalgebra} : \eta_{B^\circ}(e_{B^\circ}(\alpha)) = e_{B^\circ}(\alpha)(e_B) = \alpha\eta_B(e_B) = \alpha =$

$= \eta_R(\alpha);$

$\rightsquigarrow [*] := ЛR\text{-BIALG} : B^\circ \in R\text{-BIALG};$

$\square$

$\texttt{productOfHadamard} :: \prod k : \texttt{Field} \, . \, \mathrm{LR}(k) \otimes \mathrm{LR}(k) \xrightarrow{k\text{-COALG}} \mathrm{LR}(k)$

$\texttt{productOfHadamard}\,() = \odot_H := \texttt{multiplication}\Big(\texttt{grouplike}(A)\Big)^{\circ}$

$\texttt{HadamardProductFormula} :: \forall k : \texttt{Field} \, . \, \forall s, t \in \mathrm{LR}(k) \, . \, s \odot_H t = \Lambda n \in \mathbb{Z}_+ \, . \, s_n t_n$

$\texttt{Proof} \;=$

$\texttt{Assume}\; p : k[x],$

$n := \deg p : \mathbb{Z}_+,$

$[p.*] := \text{Ɑ} k[x](p) \text{Ɑ} \texttt{dualAlgera}(\texttt{grouplike}(k)) \text{Ɑ} \texttt{tensorMap}(s,t) \text{Ɑ} \mathrm{LR}(k) \text{Ɑ}^{-1} \text{Ɑ} \mathrm{LR}(k) \text{Ɑ} k[x](p) :$

$$: p\ (s \odot_H t) = \sum_{i=0}^{n} p_i x^i\ (s \odot_H t) = \sum_{i=1}^{n} p_i x^i \otimes x^i\ (s \otimes t) \sum_{i=0}^{n} p_i s(x^i) t(x^i) = \sum_{i=0}^{n} p_i s_i t_i$$

$$= p\ \Lambda i \in \mathbb{Z}_+ \, . \, s_i t_i;$$

$\rightsquigarrow [*] := I(=, \to) : \texttt{This};$

$\square$

$\texttt{HadamrdProductCharacteristicPolynomial} :: \forall k : \texttt{NumericField} \, . \, \forall s, t \in \mathrm{LR}(k) \, . \, \forall n, m \in \mathbb{N} \, .$

$$. \, \forall \alpha : n \hookrightarrow \widehat{k} \, . \, \forall \beta : m \hookrightarrow \widehat{k} \, . \, \forall (0.1) : \chi_s(x) = \prod_{i=1}^{n} (x - \alpha_i) \, . \, \forall (0.2) : \chi_t(x) = \prod_{i=1}^{n} (x - \beta_i) \, .$$

$$. \, \chi_{t \odot_H s}(x) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - \alpha_i \beta_j)$$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{productOfHurwitz} :: \prod k : \texttt{Field} \, . \, \mathrm{LR}(k) \otimes \mathrm{LR}(k) \xrightarrow{k\text{-COALG}} \mathrm{LR}(k)$

$\texttt{productOfHurwitz}\,() = *_H := \texttt{multiplication}\Big(\texttt{primitive}(A)\Big)^{\circ}$

$\texttt{HurwitzProductFormula} :: \forall k : \texttt{Field} \, . \, \forall s, t \in \mathrm{LR}(k) \, . \, s \odot_H t = \Lambda n \in \mathbb{Z}_+ \, . \, \sum_{i=0}^{n} C_n^i s_{n-i} t_i$

$\texttt{Proof} \;=$

$\texttt{Assume}\; p : k[x],$

$n := \deg p : \mathbb{Z}_+,$

$[p.*] := \text{Ɑ} k[x](p) \text{Ɑ} \texttt{dualAlgera}(\texttt{primitive}(k)) \text{Ɑ} \texttt{tensorMap}(s,t) \text{Ɑ} \mathrm{LR}(k) \text{Ɑ}^{-1} \text{Ɑ} \mathrm{LR}(k) \text{Ɑ} k[x](p) :$

$$: p\ (s *_H t) = \sum_{m=0}^{n} p_m x^i\ (s *_H t) = \sum_{m=0}^{n} p_m \sum_{i=0}^{m} C_m^i x^{m-i} \otimes x^i\ (s \otimes t) =$$

$$= \sum_{m=0}^{n} p_m \sum_{i=0}^{m} C_m^i s(x^{m-i}) t(x^i) = \sum_{m=0}^{n} p_m \sum_{i=0}^{m} C_m^i s_{m-i} t_i$$

$$= p\ \Lambda m \in \mathbb{Z}_+ \, . \, \sum_{i=0}^{m} s_{m-i} t_i;$$

$\rightsquigarrow [*] := I(=, \to) : \texttt{This};$

$\square$

HurwitzProductCharacteristicPolynomial :: $\forall k :$ NumericField . $\forall s, t \in \mathrm{LR}(k)$. $\forall n, m \in \mathbb{N}$.

$$. \forall \alpha : n \hookrightarrow \widehat{k} \;.\; \forall \beta : m \hookrightarrow \widehat{k} \;.\; \forall (0.1) : \chi_s(x) = \prod_{i=1}^{n} (x - \alpha_i) \;.\; \forall (0.2) : \chi_t(x) = \prod_{i=1}^{n} (x - \beta_i) \;.$$

$$. \chi_{t *_H s}(x) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x - \alpha_i - \beta_j)$$

Proof =

...

$\square$

3.2 Algebraic Myhill-Nerode Theorem

$\texttt{AlgebraicFiniteIndexLemma} :: \forall \Sigma : \texttt{Finite} \,.\, \forall L \in \mathcal{L}(\Sigma) \,.\, \forall k : \texttt{Field} \,.$

$\quad . \, L : \texttt{FiniteIndex}(\Sigma) \iff \left|\sigma^* \rightharpoonup_k \chi_L\right| < \infty$

$\texttt{Proof} \;=$

$\mathcal{F} := (\omega \rightharpoonup_k \chi_L) :?k\Sigma^{**},$

$\texttt{Assume}\ f : \mathcal{F},$

$\Big(\alpha, [1]\Big) := \mathcal{O}(\mathcal{F}) : \sum \alpha \in \Sigma^* \,.\, f = \alpha \rightharpoonup \chi_L,$

$\texttt{Assume}\ \beta : \Sigma^*,$

$\texttt{Assume}\ [2] : \beta \rightharpoonup \chi_L \neq f,$

$\Big(\omega, [3]\Big) := \mathcal{C}\texttt{hitAction} : \sum \omega \in \Sigma^* \,.\, \chi_L(\alpha\omega) \neq \chi_L(\beta\omega),$

$[\beta.*] := \mathcal{C}\texttt{characteristicFunction}[3] : \alpha \not\sim_L \beta;$

$\rightsquigarrow [2] := I(\forall)I(\Rightarrow) : \forall \beta \in \Sigma^* \,.\, \beta \rightharpoonup \chi_L \neq f \Rightarrow \alpha \not\sim_L \beta;$

$\rightsquigarrow [1] := I(\forall) : \forall f \in \mathcal{F} \,.\, \forall \alpha, \beta \in \Sigma^* \,.\, \Big((\alpha \rightharpoonup \chi_L) = f \ \&\ (\beta \rightharpoonup \chi_L) \neq f\Big) \Rightarrow \alpha \not\sim_L \beta,$

$[*] := \mathcal{C}\texttt{FiniteIndex}[1] : \texttt{This};$

$\square$

$\texttt{AlgebraicMyhillNerodeTheorem} :: \forall M : \texttt{Monoid} \,.\, \forall k : \texttt{Field} \,.\, \forall f \in kM^* \,.\, \Big|(M \rightharpoonup f)\Big| < \infty \iff$

$\quad \iff \Big(\exists B : k\text{-BIALG} : \exists \psi : kM \xrightarrow{k\text{-BIALG}} B : \exists p \in B^\circ : \dim B < \infty \ \&\ f = \psi\, p\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : |M \rightharpoonup f| < \infty,$

$\mathcal{F} := M \rightharpoonup f :?kM^*,$

$R := \{\Lambda g \in \mathcal{F} \,.\, m \rightharpoonup g | m \in M\} :?(\mathcal{F} \to \mathcal{F}),$

$[2] := \texttt{PowerSetCardinality}\mathcal{O}(R)[1] : |R| < \infty,$

$\texttt{Assume}\ A, B : R,$

$(a, b, [3]) := \mathcal{O}(R) : \sum a, b \in M \,.\, A = (a \rightharpoonup \cdot) \ \&\ B = (b \rightharpoonup \cdot),$

$AB := (a \rightharpoonup \cdot) : R;$

$\rightsquigarrow (\cdot) := I(\to) : (R \times R) \to R,$

$[3] := \mathcal{O}(R, \cdot)\mathcal{C}\texttt{Monoid}(M) : \Big((R, \cdot) : \texttt{Monoid}\Big),$

$B := kR : k\text{-BIALG},$

$[4] := \mathcal{C}kR[2] : \dim B < \infty,$

$\psi := \Lambda p \in kM \,.\, \sum_{m \in M} p_m(m \rightharpoonup \cdot) : kM \xrightarrow{k\text{-BIALG}} B,$

$\texttt{Assume}\ A : R,$

$(a, [5]) := \mathcal{O}(R)[5] : \sum a \in M \,.\, A = a \rightharpoonup \cdot,$

$p(A) := f(a) : k,$

$\texttt{Assume}\ b : M,$

$\texttt{Assume}\ [6] : A = b \rightharpoonup \cdot,$

$[A.*] := \mathcal{O}R(A)[6][5] : f(b) = (Af)(e) = f(a);$

$\rightsquigarrow p := \mathcal{C}\texttt{monoidBialgebra} : B^\circ,$

$[*] := \mathcal{O}p : f = \psi\, p;$

$\rightsquigarrow [1] := I(\Rightarrow) : \texttt{Left} \Rightarrow \texttt{Right},$

$\texttt{Assume}\ [2] : \texttt{Right},$

$R := \texttt{Grouplike}(B) :?B,$

$[3] := \texttt{LinearlyIndependentGrouplike}(R) : \Big(R : \texttt{LinearlyIndependent}(B)\Big),$

$[4] := \daleth\texttt{dimension}[2] : |B| < \infty,$

$\mu_{B,kM} := \Lambda b \in B \,.\, \Lambda a \in kM \,.\, b\psi(a) : B \otimes kM \xrightarrow{k\text{-VS}} B,$

$\texttt{Assume}\ b : B,$

$\texttt{Assume}\ a : kM,$

$$[b.*] := \daleth\mu_{B,kM}\daleth k\text{-BIALG}(B)\daleth k\text{-BIALG}(kM,B)(\psi)\daleth\texttt{SweedlerNotation}(b,a)$$
$$\daleth\texttt{tensorProductAlgebra}(B,kM)\daleth^{-1}\mu_{B,kM} : \Delta(ba) = \Delta\Big(b\psi(a)\Big) = \Delta(b)\Delta\Big(\psi(a)\Big) =$$
$$= \Delta(b)(\psi\otimes\psi)\Delta(a) = \sum_{b,a} b_1\psi(a_1)\otimes b_2\psi(a_2) = \sum_{b,a} b_1a_1 \otimes b_2a_2;$$

$\rightsquigarrow [5] := \daleth\mathsf{COALG}_{kM} : (B, \mu_{B,kM}) \in \mathsf{COALG}_{kM},$

$\texttt{Assume}\ r : R,$

$\texttt{Assume}\ a : M,$

$[6] := \daleth\mathsf{COALG}_{kM}[5]\daleth\texttt{monoidBialgebra}(k,M) : \Delta(ra) = ra \otimes ra,$

$[r.*] := \daleth R[6] : ra \in R;$

$\rightsquigarrow [6] := \daleth\texttt{Subset} : RM \subset M,$

$\texttt{Assume}\ a, x : M,$

$$[7] := \daleth\texttt{hitAction}[2]\daleth k\text{-BIALG}(kM,B)(\psi)\daleth k\text{-BIALG}(B)\daleth^{-1}\mu_{B,kM} :$$
$$: (a \rightharpoonup f)(x) = f(ax) = p\Big(\psi(ax)\Big) = p\Big(\psi(a)\psi(x)\Big) = p\Big(e\psi(a)\psi(x)\Big) = p\Big(eax\Big),$$

$\Big[(a,x).*\Big] := [6][7] : \exists r \in R \,.\, (a \rightharpoonup f)(x) = (r \rightharpoonup p)\psi(x);$

$\rightsquigarrow [7] := I(\forall)I(=,\to) : \forall a \in M \,.\, \exists r \in R \,.\, (a \rightharpoonup f) = (r \rightharpoonup p)\psi,$

$[2.*] := [4][7] : |M \rightharpoonup f| < \infty;$

$\rightsquigarrow [*] := I(\Rightarrow)I(\iff)[1] : \texttt{Left} \iff \texttt{Right};$

$\square$

$\texttt{MyhillNerodeAlgebra} :: \prod k : \texttt{Field} \,.\, \prod M : \texttt{Monoid} \,.\, \prod f \in kM^* \,.\, ?k\text{-BIALG}$

$B : \texttt{MyhillNerodeAlgebra} \iff \exists\psi : kM \xrightarrow{k\text{-BIALG}} B : \exists p \in B^\circ : \dim B < \infty \;\&\; f = \psi\, p$

$$\texttt{algebraicFiniteAutomaton} :: \prod \Sigma : \texttt{Finite} \,.\, \prod L \in \mathcal{L}(\Sigma) \,.\, \prod k : \texttt{Field} \,.$$
$$.\, \texttt{MyhillNerodeAlgebra}(k, \Sigma^*, \chi_L) \to \sum A : \texttt{FiniteAutomoton} \,.\, \texttt{language}(A) = L$$

$\texttt{algebraicFiniteAutomoton}\,((B,\psi,p,[0])) := \Big(\Sigma, \texttt{Grouplike}(B), \mu_{B,k\Sigma^*}, e_B, e_BL\Big)$

$\texttt{Assume}\ \omega, \omega' : \sigma^*,$

$\texttt{Assume}\ [1] : \omega \in L,$

$\texttt{Assume}\ [2] : \omega \notin L,$

$[3] := \omega\daleth\chi_L[1][0]\daleth k\text{-ALGE}(B)\daleth\mathsf{COALG}_{k\Sigma^*}(B) : 1 = \chi_L(\omega) = p\Big(\psi(\omega)\Big) = p\Big(e_B\psi(\omega)\Big) = p(e_B\omega),$

$[4] := \omega'\daleth\chi_L[1][0]\daleth k\text{-ALGE}(B)\daleth\mathsf{COALG}_{k\Sigma^*}(B) : 0 = \chi_L(\omega') = p\Big(\psi(\omega')\Big) = p\Big(e_B\psi(\omega')\Big) = p(e_B\omega'),$

$\Big[[(\omega,\omega').*]\Big] := I(\to,\#) : e_B\omega \neq e_B\omega';$

$\rightsquigarrow [*] := \mathcal{O}(A)\daleth\texttt{language} : \texttt{language}(A) = L;$

$\square$

3.3 Regular Sequences

$\texttt{RegularSequence} :: \prod k : \texttt{Field} . ?(\mathbb{N} \to k)$

$x : \texttt{RegularSequence} \iff \exists M : \texttt{Monoid} : \exists m : \mathbb{N} \leftrightarrow M : \exists f \in kM^* : x = f(m) \ \&\ |M \rightharpoonup f| < \infty$

$\texttt{RegularSequenceCharacterization} :: \forall k : \texttt{Field} . \forall M : \texttt{Monoid} . \forall m : \mathbb{N} \leftrightarrow M . \forall x : \mathbb{N} \to k .$
$\quad x : \texttt{RegularSequenc}(k) \iff \exists f \in kM^* . x = f(m) \ \&\ \dim(kM \rightharpoonup f) < \infty \ \&\ |f(M)| < \infty$

$\texttt{Proof} \ =$

$\texttt{Assume}\ f : kM^*,$

$\texttt{Assume}\ [1] : x = f(M),$

$\texttt{Assume}\ [2] : \dim(kM \rightharpoonup f) < \infty,$

$\texttt{Assume}\ [3] : |f(M)| < \infty,$

$n := \dim(kM \rightharpoonup f) : \mathbb{Z}_+,$

$\Big(g, p, [4]\Big) := \texttt{BasisWithSpecialSupportTHM} : \sum g : \texttt{Basis}(kM \rightharpoonup f) . \sum p : n \to M .$
$\quad . \forall i, j \in n . g_i(p_j) = \delta^i_j,$

$\texttt{Assume}\ a : M,$

$\Big(\alpha, [5]\Big) := \mathcal{O}\texttt{Basis}(kM \rightharpoonup f)(g)(a \rightharpoonup f) : \sum \alpha \in k^n . a \rightharpoonup f = \alpha g,$

$[6] := [4][5] : \forall i \in n . \alpha_i = f(a_i) \in f(M),$

$[*] := \mathcal{O}\texttt{SetImage}[5][6] : (a \rightharpoonup f) \in f(M)\{g_i\}_{i=1}^n;$

$\rightsquigarrow [5] := \mathcal{O}\texttt{Subset} : (M \rightharpoonup f) \subset f(M)\{g_i\}_{i=1}^n,$

$[6] := \texttt{FiniteProduct}[5][3] : |M \rightharpoonup f| < \infty,$

$[*] := \mathcal{O}^{-1}\texttt{RegularSequence}[1][6] : \Big(x : \texttt{RegularSequence}(k)\Big);$

$\square$

$\texttt{FiniteFieldLinearlyRecursiveIsRegular} :: \forall p : \texttt{Prime}(\mathbb{Z}) . \forall n \in \mathbb{N} . \forall x \in \mathrm{LR}(\mathbb{F}_{p^n}) .$
$\quad . x : \texttt{RegularSequence}(\mathbb{F}_{p^n})$

$\texttt{Proof} \ =$

$q := p^n : \mathbb{N},$

$k := \mathbb{F}_q : \texttt{Field},$

$M := \mathbb{Z}_+ : \texttt{Monoid},$

$[1] := \mathcal{D}M\mathcal{D} : kM \cong_{k\text{-BIALG}} k[x],$

$f := \mathcal{O}k[x]\Lambda i \in \mathbb{Z}_+ . f(x^i) = s_i : \Big(k[x]\Big)^\circ,$

$[2] := \texttt{FiniteHitAction}\mathcal{D}\mathcal{O}s : \dim\Big(k[x] \rightharpoonup f\Big) < \infty,$

$[3] := \mathcal{O}\mathbb{F}_q : |f(M)| < \infty,$

$[*] := \texttt{RegularSequenceCharactrization}\Big([1], [2]\Big)[3] : \Big(s : \texttt{RegularSequence}(\mathbb{F}_{p^n})\Big);$

$\square$

3.4 Hopf Algebras

$\texttt{HopfAlgebra} :: \prod R \in \mathsf{ANN}\ .\ \prod B : R\text{-}\mathsf{BIALG}\ .\ B \xrightarrow{R\text{-}\mathsf{MOD}} B$

$(B, \sigma) : \texttt{HopfAlgebra} \iff \Delta(\mathrm{id} \otimes \sigma)\mu = \eta e = \Delta(\sigma \otimes \mathrm{id})\mu$

$\texttt{antipode} :: \prod R \in \mathsf{ANN}\ .\ \prod (B, \sigma) : \texttt{HopfAlgebra}(R)\ .\ B \xrightarrow{R\text{-}\mathsf{MOD}} B$

$\texttt{antipode}\,() := \sigma$

$\texttt{categoryOfHopfAlgebra} :: \mathsf{ANN} \to \mathsf{CAT}$

$\texttt{categoryOfHopfAlgebra}\,(R) = R\text{-}\mathsf{HOPF} := \Big(\texttt{HopfAlgebra}(R), R\text{-}\mathsf{BIALG}, \circ, \mathrm{id}\Big)$

$\texttt{groupHopfAlgebra} :: \prod R \in \mathsf{ANN}\ .\ \to \mathsf{GRP}\texttt{HopfAlgebra}(R)$

$\texttt{groupHopfAlgebra}\,(G) = RG := \Big(RG, \text{Ɑ}\, RG \Lambda g \in G\ .\ g^{-1}\Big)$

$\texttt{QuantumGroup} :: \prod R \in \mathsf{ANN}\ .\ ?R\text{-}\mathsf{HOPF}$

$A : \texttt{QuantumGroup} \iff A\,!\,\texttt{Commutative}(R)\ \&\ A\,!\,\texttt{Cocommutative}(R)$

$\texttt{convolutionProduct} :: \prod R \in \mathsf{ANN}\ .\ \prod A \in R\text{-}\mathsf{COALG}\ .\ \prod B \in R\text{-}\mathsf{ALGE}\ .$

$\quad .\ R\text{-}\mathsf{MOD}(A, B) \otimes R\text{-}\mathsf{MOD}(A, B) \xrightarrow{R\text{-}\mathsf{MOD}} R\text{-}\mathsf{MOD}(A, B)$

$\texttt{convolutionProduduct}\,(\varphi \otimes \psi) = \varphi * \psi := \mu_B(\varphi \otimes \psi)\Delta_A$

$\texttt{ConvolutionMonoid} :: \forall R \in \mathsf{ANN}\ .\ \forall A \in R\text{-}\mathsf{COALG}\ .\ \forall B \in R\text{-}\mathsf{ALGE}\ .\ \Big(R\text{-}\mathsf{MOD}(A, B), *\Big) : \texttt{Monoid}$

$\texttt{Proof}\ =$

$\texttt{Assume}\ \phi, \phi', \phi'' : R\text{-}\mathsf{MOD}(A, B),$

$\texttt{Assume}\ a : A,$

$[*] := \text{Ɑ}\, R\text{-}\mathsf{ALGE}(B)\,\text{Ɑ}\, R\text{-}\mathsf{COALG}(A)\,\text{Ɑ}\,\texttt{SweedlerNotation} :$

$\quad : (\phi * \phi') * \phi''(a) = \sum_a \phi(a_1)\phi'(a)\phi''(a) = \phi * (\phi' * \phi'')(a);$

$\rightsquigarrow [1] := I(=, \to)\text{Ɑ}^{-1}\texttt{Associtaive} : \Big((*) : \texttt{Associative}\Big),$

$\texttt{Assume}\ \phi : R\text{-}\mathsf{MOD}(A, B),$

$\texttt{Assume}\ a : A,$

$[\phi. * .1] := \text{Ɑ}\,\texttt{convolution}\,\text{Ɑ}\, e\,\text{Ɑ}\,\mathsf{GRP}(A, B)(\phi)\,\text{Ɑ}\, R\text{-}\mathsf{COALG}(A) :$

$\quad : (\eta e * \phi)(a) = \sum_a \eta(a_1)\phi(a_2) = \phi\left(\sum_a \eta(a_1)a_2\right) = \phi(a),$

$[\phi. * .2] := \text{Ɑ}\,\texttt{convolution}\,\text{Ɑ}\, e\,\text{Ɑ}\,\mathsf{GRP}(A, B)(\phi)\,\text{Ɑ}\, R\text{-}\mathsf{COALG}(A) :$

$\quad : (\phi * \eta a)(a) = \sum_a \eta(a_2)\phi(a_1) = \phi\left(\sum_a \eta(a_2)a_1\right) = \phi(a);$

$\rightsquigarrow [2] := \text{Ɑ}\,\texttt{Neutral} : \eta_A e_B : \texttt{Neutral}(*),$

$[*] := \text{Ɑ}^{-1}\texttt{Monoid}[1][2] : \texttt{This};$

$\square$

$\texttt{AntipodIsInverseOfIdentity} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \sigma_A * \mathrm{id}_A = \eta_A e_A = \mathrm{id}_A * \sigma_A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{AntipodeAntihomo} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \forall a, b \in A \,.\, \sigma(ab) = \sigma(b)\sigma(a)$

$\texttt{Proof} \ =$

$\varphi := \mu(\sigma \otimes \sigma)\texttt{swap} : A \otimes A \xrightarrow{R\text{-}\mathsf{MOD}} A,$

$\texttt{Assume}\ a, b : A,$

$\Big[(a,b).*.1\Big] := \mathcal{I}\,\texttt{convolution}\,\mathcal{I}\,\texttt{tensorProductCoalgebra}\,\mathcal{J}\varphi\mathcal{I}\,\texttt{tensorMap}\,\mathcal{I}^2(\text{-}\mathsf{HOPF}R)(A) :$

$$: (a \otimes b)(\mu * \varphi) = (a \otimes b)\Delta(\mu \otimes \varphi)\mu = \sum_a \sum_b (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)(\mu \otimes \varphi)\mu =$$

$$= \sum_a \sum_b a_1 b_1 \sigma(b_2)\sigma(a_2) = \eta(b) \sum_a a_1 \sigma(a_2) = \eta(b)\eta(a)e,$$

$\Big[(a,b).*.2\Big] := \mathcal{I}\,\texttt{convolution}\,\mathcal{I}\,\texttt{tensorProductCoalgebra}\,\mathcal{J}\varphi\mathcal{I}\,\texttt{tensorMap}\,\mathcal{I}^2(\text{-}\mathsf{HOPF}R)(A) :$

$$: (a \otimes b)(\varphi * \mu) = (a \otimes b)\Delta(\varphi \otimes \mu)\mu = \sum_a \sum_b (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)(\varphi \otimes \mu)\mu =$$

$$= \sum_a \sum_b \sigma(b_1)\sigma(a_1)a_2 b_2 = \eta(a) \sum_b \sigma(b_1) b_2 = \eta(b)\eta(a)e;$$

$\rightsquigarrow [1] := \mathcal{I}^{-1}\texttt{Inverse} : \varphi = \mu^{-1},$

$\texttt{Assume}\ a, b : A,$

$\Big[(a,b).*.1\Big] := \mathcal{I}\,\texttt{convolution}\,\mathcal{I}\,\texttt{tensorProductCoalgebra}\,\mathcal{I}\,\texttt{tensorMap}\,\mathcal{I}R\text{-}\mathsf{BIALG}(A)\mathcal{I}R\text{-}\mathsf{HOPF}(A)$

$$\mathcal{I}R\text{-}\mathsf{BIALG}(A)\mathcal{I}\mathsf{ANN}(R) : (a \otimes b)(\mu\sigma * \mu) = (a \otimes b)\Delta(\mu\sigma * \mu)\mu = \sum_a \sum_b (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)(\mu\sigma * \mu) =$$

$$= \sum_a \sum b\sigma(a_1 b_1)a_2 b2 = \sum_{c=ab} \sigma(c_1)c_2 = \eta(ab)e = \eta(b)\eta(a)e,$$

$\Big[(a,b).*.2\Big] := \mathcal{I}\,\texttt{convolution}\,\mathcal{I}\,\texttt{tensorProductCoalgebra}\,\mathcal{I}\,\texttt{tensorMap}\,\mathcal{I}R\text{-}\mathsf{BIALG}(A)\mathcal{I}R\text{-}\mathsf{HOPF}(A)$

$$\mathcal{I}R\text{-}\mathsf{BIALG}(A)\mathcal{I}\mathsf{ANN}(R) : (a \otimes b)(\mu * \mu\sigma) = (a \otimes b)\Delta(\mu * \mu\sigma)\mu = \sum_a \sum_b (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)(\mu * \mu\sigma) =$$

$$= \sum_a \sum b a_1 b_1 \sigma(a_2 b2) = \sum_{c=ab} c_1 \sigma(c_2) = \eta(ab)e = \eta(b)\eta(a)e,$$

$\rightsquigarrow [2] := \mathcal{I}^{-1}\texttt{Inverse}(*) : \mu\sigma = \mu^{-1},$

$[3] := [2][1] : \mu\sigma = \varphi,$

$[*] := \mathcal{I}\varphi[3] : \texttt{This};$

$\square$

$\texttt{UnityAntipode} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, e\sigma = e$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{InvolutionAntipode} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,\&\, \texttt{Cocommutative} \,.\, \sigma^2 = \mathrm{id}$

$\texttt{Proof} \; =$

$\texttt{Assume}\ a : A,$

$[a.*.1] := \Game\,\texttt{convolution}\Game\,\texttt{SweedlerNotation}\Game\,\texttt{tensorMap}\texttt{AntipodeAntihomo}(A)\Big(a_1, \sigma(a_2)\Big)$

$\quad \Game\,\texttt{Cocommutative}(A)\Game R\text{-}\mathsf{MOD}(A,A)(\sigma)\Game R\text{-}\mathsf{HOPF}(A)\Game R\text{-}\mathsf{MOD}(A,A)(\sigma)\texttt{UnityAntipode}(A) :$

$$: a\ (\sigma * \sigma^2) = a\Delta\ (\sigma \otimes \sigma^2)\ \mu = \sum_a a_1 \otimes a_2\ (\sigma \otimes \sigma^2)\ \mu = \sum_a \sigma(a_1)\sigma^2(a_2) = \sum_a \sigma\Big(\sigma(a_2)a_1\Big) =$$

$$= \sum_a \sigma\Big(a_1\sigma(a_2)\Big) = \sigma\left(\sum_a a_1\sigma(a_2)\right) = \sigma(\eta(a)e) = \eta(a),$$

$[a.*.2] := \Game\,\texttt{convolution}\Game\,\texttt{SweedlerNotation}\Game\,\texttt{tensorMap}\texttt{AntipodeAntihomo}(A)\Big(a_1, \sigma(a_2)\Big)$

$\quad \Game\,\texttt{Cocommutative}(A)\Game R\text{-}\mathsf{MOD}(A,A)(\sigma)\Game R\text{-}\mathsf{HOPF}(A)\Game R\text{-}\mathsf{MOD}(A,A)(\sigma)\texttt{UnityAntipode}(A) :$

$$: a\ (\sigma^2 * \sigma) = a\Delta\ (\sigma^2 \otimes \sigma)\ \mu = \sum_a a_1 \otimes a_2\ (\sigma^2 \otimes \sigma)\ \mu = \sum_a \sigma^2(a_1)\sigma(a_2) = \sum_a \sigma\Big(a_2\sigma(a_1)\Big) =$$

$$= \sum_a \sigma\Big(\sigma(a_1)a_2\Big) = \sigma\left(\sum_a \sigma(a_1)a_2\right) = \sigma(\eta(a)e) = \eta(a);$$

$\rightsquigarrow [1] := \Game^{-1}\texttt{Invers}(*) : \sigma^2 = \sigma^{-1},$

$[*] := \texttt{AntipodeIsInverseOfIdentity}[1] : \sigma^2 = \mathrm{id};$

$\square$

$\texttt{ComultiplicationOfAntipode} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \sigma\ \Delta = \Delta\ (\sigma \otimes \sigma)\ \texttt{swap}$

$\texttt{Proof} \; =$

$\varphi := \Delta\ (\sigma \otimes \sigma)\ \texttt{swap} : A \xrightarrow{R\text{-}\mathsf{MOD}} A \otimes A,$

$\texttt{Assume}\ a : A,$

$[a.*.1] := \Game\,\texttt{convolution}\Game\,\texttt{SweedlerNotation}\Game\,\texttt{tensorMap}\Game\,\texttt{tensorProductAlgebra}$

$\quad \Game R\text{-}\mathsf{COALG}(A)\Game R\text{-}\mathsf{HOPF}(A)\Game\,\texttt{tensorProduct}\Game R\text{-}\mathsf{COALG}(A)\Game R\text{-}\mathsf{HOPF}(A) :$

$$: a\ \Delta * \varphi = a\ \Delta(\Delta \otimes \varphi)\mu = \sum_a (a_1 \otimes a_2)\ (\Delta \otimes \varphi)\mu = \sum_a (a_1 \otimes a_2) \otimes \Big(\sigma(a_4) \otimes \sigma(a_3)\Big)\mu =$$

$$= \sum_a a_1\sigma(a_4) \otimes a_2\sigma(a_3) = \sum_a a_1\sigma(a_3) \otimes \eta(a_2)e = \left(\sum_a a_1\eta(a_2)\sigma(a_3)\right) \otimes e = \left(\sum_a a_1\sigma(a_2)\right) \otimes e =$$

$$= \eta(a)e \otimes e,$$

$[a.*.2] := \Game\,\texttt{convolution}\Game\,\texttt{SweedlerNotation}\Game\,\texttt{tensorMap}\Game\,\texttt{tensorProductAlgebra}$

$\quad \Game R\text{-}\mathsf{COALG}(A)\Game R\text{-}\mathsf{HOPF}(A)\Game\,\texttt{tensorProduct}\Game R\text{-}\mathsf{COALG}(A)\Game R\text{-}\mathsf{HOPF}(A) :$

$$: a\ \varphi * \Delta = a\ \Delta(\varphi \otimes \Delta)\mu = \sum_a (a_1 \otimes a_2)\ (\varphi \otimes \Delta)\mu = \sum_a \Big(\sigma(a_2) \otimes \sigma(a_1)\Big) \otimes (a_3 \otimes a_4)\mu =$$

$$= \sum_a \sigma(a_2)a_3 \otimes \sigma(a_1)a_4 = \sum_a \eta(a_2)e \otimes \sigma(a_1)a_4 = e \otimes \left(\sum_a a_1\eta(a_2)\sigma(a_3)\right) = e \otimes \left(\sum_a a_1\sigma(a_2)\right) =$$

$$= \eta(a)e \otimes e;$$

$\rightsquigarrow [1] := \Game^{-1}\texttt{Inverse}(*) : \varphi = \Delta^{-1},$

$\texttt{Assume}\ a : A,$

$$\left[a.*.1\right] := \daleth\,\texttt{convolution}\,\daleth\,\texttt{SweedlerNotation}\,\daleth\,\texttt{tensorMap}\,\daleth\,R\text{-}\mathsf{BIALG}(A)\daleth\,R\text{-}\mathsf{MOD}(A, A\otimes A)$$

$$\daleth\,R\text{-}\mathsf{HOPF}(A)\daleth\,R\text{-}\mathsf{BIALG}(A) : a\ \sigma\Delta * \Delta = a\ \Delta\Big(\sigma\Delta\otimes\Delta\Big)\mu = \sum_a a_1\otimes a_2\ \Big(\sigma\Delta\otimes\Delta\Big)\mu =$$

$$= \sum_a \left(\sum_{b=\sigma(a_1)} b_1\otimes b_2\right)\otimes(a_2\otimes a_3)\mu = \sum_a\sum_{b=\sigma(a_1)} b_1a_2\otimes b_2a_3 = \sum_a \Delta(\sigma(a_1)a_2) = \Delta\left(\sum_a \sigma(a_1)a_2\right) =$$

$$= \Delta(\eta(a)e) = \eta(a)e\otimes e,$$

$$\left[a.*.1\right] := \daleth\,\texttt{convolution}\,\daleth\,\texttt{SweedlerNotation}\,\daleth\,\texttt{tensorMap}\,\daleth\,R\text{-}\mathsf{BIALG}(A)\daleth\,R\text{-}\mathsf{MOD}(A, A\otimes A)$$

$$\daleth\,R\text{-}\mathsf{HOPF}(A)\daleth\,R\text{-}\mathsf{BIALG}(A) : a\ \Delta * \sigma\Delta = a\ \Delta\Big(\Delta\otimes\sigma\Delta\Big)\mu = \sum_a a_1\otimes a_2\ \Big(\Delta\otimes\sigma\Delta\Big)\mu =$$

$$= \sum_a (a_1\otimes a_2)\otimes\left(\sum_{b=\sigma(a_3)} b_1\otimes b_2\right)\mu = \sum_a\sum_{b=\sigma(a_3)} a_1b_1\otimes a_2b_2 = \sum_a \Delta(a_1\sigma(a_2)) = \Delta\left(\sum_a a_1\sigma(a_2)\right) =$$

$$= \Delta(\eta(a)e) = \eta(a)e\otimes e,$$

$\rightsquigarrow [2] := \daleth^{-1}\texttt{Inverse}(*) : \sigma\Delta = \Delta^{-1},$

$[*] := [1][2]\mathcal{I}\varphi : \texttt{This};$

□

$\texttt{CounitOfAntipode} :: \forall R\in\mathsf{ANN}\ .\ \forall A\in R\text{-}\mathsf{HOPF}\ .\ \sigma\ \eta = \eta$

$\texttt{Proof}\ =$

$\texttt{Assume}\ a : A,$

$[a.*] := \daleth\,R\text{-}\mathsf{BIALG}(A)\daleth\,R\text{-}\mathsf{HOPF}(A)\daleth\,R\text{-}\mathsf{BIALG}(A)\daleth\,R\text{-}\mathsf{MOD}(A,A)(\sigma\eta)\daleth\,R\text{-}\mathsf{COALG}(A) :$

$: a\ \eta = a\ \eta\ e\ \eta = a\ \Delta\ (\sigma\otimes\mathrm{id})\ \mu\ \eta = a\ \Delta\ (\sigma\otimes\mathrm{id})\ (\eta\otimes\eta)\ \mu = a\ \Delta(\eta\otimes\mathrm{id})\ \mu\ \sigma\ \eta = a\ \sigma\ \eta;$

$\rightsquigarrow [*] := I(=,\rightarrow) : \eta = \sigma\ \eta;$

□

$\texttt{HopfIdeal} :: \prod R\in\mathsf{ANN}\ .\ \prod A\in R\text{-}\mathsf{HOPF}\ .\ ?\texttt{Biideal}(A)$

$I : \texttt{HopfIdeal} \iff \sigma(I)\subset I$

$\texttt{HopfQuotient} :: \forall R\in\mathsf{ANN}\ .\ \forall A\in R\text{-}\mathsf{HOPF}\ .\ \forall I : \texttt{HopfIdeal}(A)\ .\ \left(\frac{A}{I}, \widehat{\sigma}_I\right)\in R\text{-}\mathsf{HOPF}$

$\texttt{Proof}\ =$

...

□

HopfDuality $::$ $\forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \forall [0] : \dim A < \infty \,.\, A^\circ \in R\text{-}\mathsf{HOPF}$

Proof $=$

Assume $f : A^\circ$,

Assume $a : A$,

$[f. * .1] := \text{Ɑ}\,\texttt{finiteDualBialgebra}\,\text{Ɑ}\,\mathsf{ABEL}(A, R)(f)\,\text{Ɑ}\,R\text{-}\mathsf{HOPF}(A) :$

$$: a\ f\ \Delta\ (\mathrm{id} \otimes \sigma^*)\ \mu = \sum_a f\big(a_1 \sigma(a_2)\big) = f\left(\sum_a a_1 \sigma(a_2)\right) = f(a),$$

$[f. * .1] := \text{Ɑ}\,\texttt{finiteDualBialgebra}\,\text{Ɑ}\,\mathsf{ABEL}(A, R)(f)\,\text{Ɑ}\,R\text{-}\mathsf{HOPF}(A) :$

$$: a\ f\ \Delta\ (\sigma^* \otimes \mathrm{id})\ \mu = \sum_a f\big(\sigma(a_1) a_2\big) = f\left(\sum_a \sigma(a_1) a_2\right) = f(a);$$

$\rightsquigarrow [*] := \text{Ɑ}\,R\text{-}\mathsf{HOPF} : A^\circ \in R\text{-}\mathsf{HOPF};$

$\square$

3.5 Integrals of Hopf Algebras

$$\texttt{LeftIntegral} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{HOPF} \;.\; ?A$$

$$a : \texttt{LeftIntegral} \iff a \in \int_A^l \iff \forall x \in A \;.\; xa = \eta(x)a$$

$$\texttt{RightIntegral} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in R\text{-}\mathsf{HOPF} \;.\; ?A$$

$$a : \texttt{RightIntegral} \iff a \in \int_A^r \iff \forall x \in A \;.\; ax = \eta(x)a$$

$$\texttt{IntegralsAreSubMod} :: \forall R \in \mathsf{ANN} \;.\; \forall A \in R\text{-}\mathsf{HOPF} \;.\; \int_A^l, \int_A^r \subset_{R\text{-}\mathsf{MOD}} A$$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{IntegralsAreIdeal} :: \forall R \in \mathsf{ANN} \;.\; \forall A \in R\text{-}\mathsf{HOPF} \;.\; \int_A^l, \int_A^r : \texttt{Ideal}(A)$$

$\texttt{Proof} =$

$\texttt{Assume}\ a : \int_A^l,$

$\texttt{Assume}\ x, y : A,$

$[a.*.1] := \text{Ꟈ} \int_A^l : yax = \eta(y)ax,$

$[a.*.2] := \text{Ꟈ} \int_A^l \text{Ꟈ} R\text{-}\mathsf{BIALG}(A) \text{Ꟈ} \int_A^l : yxa = \eta(yx)a = \eta(y)\eta(x)a = \eta(y)xa;$

$\rightsquigarrow [1] := \text{Ꟈ} \int_A^l \text{Ꟈ}^{-1}\texttt{Ideal}(A) : \int_A^l \in \mathsf{Spec}(A),$

$\texttt{Assume}\ a : \int_A^r,$

$\texttt{Assume}\ x, y : A,$

$[a.*.1] := \text{Ꟈ} \int_A^r \text{Ꟈ} R\text{-}\mathsf{BIALG}(A) \text{Ꟈ} \mathsf{ANN}(R) \text{Ꟈ} R\text{-}\mathsf{MOD}(A) \text{Ꟈ} \int_a^r :$

$\quad : axy = \eta(xy)a = \eta(x)\eta(y)a = \eta(y)\eta(x)a = \eta(y)ax,$

$[a.*.2] := \text{Ꟈ} \int_A^r : xay = xa\eta(y);$

$\rightsquigarrow [2] := \text{Ꟈ} \int_A^r \text{Ꟈ}^{-1}\texttt{Ideal}(A) : \int_A^l \in \mathsf{Spec}(A),$

$[*] := [1][2] : \texttt{This};$

$\square$

$$\texttt{Unimodular} :: \prod_{R \in \mathsf{ANN}} ?R\text{-}\mathsf{HOPF}$$

$$A : \texttt{Unimodular} \iff \int_A^l = \int_A^r$$

$\texttt{IntegralsOfFiniteGroupAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \int_{RG}^{l} = \int_{RG}^{r} = R \sum_{g \in G} g$

$\texttt{Proof} \; =$

$\texttt{Assume}\ \alpha, \beta : R,$

$\texttt{Assume}\ h : G,$

$$\Big[(\alpha, \beta). * .1\Big] := \daleth\,\mathsf{GRP}(G)\daleth RG : (\beta h)\alpha \sum_{g \in G} g = (\beta\alpha) \sum_{g \in G} g = \eta(\beta h)\alpha \sum_{g \in G} g,$$

$$\Big[(\alpha, \beta). * .2\Big] := \daleth\,\mathsf{GRP}(G)\daleth RG : \left(\alpha \sum_{g \in G} g\right)(\beta h) = (\beta\alpha) \sum_{g \in G} g = \eta(\beta h)\alpha \sum_{g \in G} g,$$

$$\leadsto [1] := \daleth \int_{A}^{l} \daleth \int_{A}^{r} \daleth^{-2}\texttt{Subset} : R \sum_{g \in G} g \subset \int_{A}^{r} \cap \int_{A}^{l},$$

$\texttt{Assume}\ v : RG,$

$\texttt{Assume}\ [2] : v \notin R \sum_{g \in G} g,$

$$\Big(g, h, [3]\Big) := \daleth RG[2] : \sum_{g,h \in G} v_g \neq v_h,$$

$$[4] := \daleth RG[3] : (gh^{-1}v)_h \neq v_h \;\&\; (vg^{-1}h)_g \neq v_g,$$

$$[v.*] := \daleth RG \daleth \int_{RG}^{r} \daleth \int_{RG}^{l} : v \notin \int_{RG}^{r} \;\&\; v \notin \int_{RG}^{l};$$

$$\leadsto [*] := \daleth\texttt{SetEq}[1] : R \sum_{g} G = \int_{RG}^{r} = \int_{RG}^{l};$$

$\square$

$\texttt{IntegralsOfFiniteGroupDualAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall G : \texttt{FiniteGroup} \,.\, \int_{RG^*}^{l} = \int_{RG^*}^{r} = R\,\mathrm{de}$

$\texttt{Proof} \; =$

$\texttt{Assume}\ \alpha, \beta : R,$

$\texttt{Assume}\ g, h : h,$

$$\Big[(g, h).*\Big] := \daleth\texttt{finiteDualAlgebra}\daleth\texttt{differential}\daleth\texttt{finiteDualAlgebra} : (\beta \mathrm{d}g)(\alpha \mathrm{de})(h) = \alpha\beta \mathrm{d}g(h)\mathrm{de}(h) = $$

$$\leadsto [1] := \daleth\texttt{Subset} : R\mathrm{de} \subset \int_{GR^*}^{l},$$

$\texttt{Assume}\ f : \int_{GR^*}^{l},$

$$[2] := \daleth \int_{GR^*}^{l} \daleth RG* : (\mathrm{de}\ f) = \eta(\mathrm{de})f = f,$$

$$[3] := \daleth\texttt{differential}\daleth RG^* : (\mathrm{de}\ f)(v) = f^e \mathrm{de},$$

$$[f.*] := [2][3] : f \in R\mathrm{de};$$

$$\leadsto [*] := \daleth\texttt{Subset}[1]\daleth\texttt{SetEq} : \int_{RG^*}^{l} = \int_{RG^*}^{r} = R\ \mathrm{de};$$

$\square$

$$\texttt{dualComodule} :: \prod k : \texttt{Field} \,.\, \prod A : k\text{-}\mathsf{BIALG} \,.\, \prod n \in \mathbb{N} \,.\, \texttt{Basis}(n, A) \to \mathsf{MOD}^A$$

$$\texttt{dualComodule}\,(e) = A^*_e := \left(A^*, \Lambda f \in A^* \,.\, \sum_{i=1}^n e^i f \otimes e_i\right)$$

$$\Big(\alpha, [1]\Big) := \mathcal{I}\texttt{Basis}(n, A)(e)\Delta : \sum \alpha : n^3 \to k \,.\, \forall i \in n \,.\, \Delta(e_i) = \alpha_{i,j,l} e_j \otimes e_l,$$

$$\Big(\beta, [2]\Big) := \mathcal{I}\texttt{Basis}(n, A)(e)\Delta^2 : \sum \alpha : n^4 \to k \,.\, \forall i \in n \,.\, \Delta^2(e_i) = \beta_{i,j,l,t} e_i \otimes e_l \otimes e_t,$$

$$[3] := \mathcal{I} k\text{-}\mathsf{COALG}(A)[1][2] : \forall i, t \in n \,.\, \sum_{j,l=1}^n \beta_{i,j,l,t} e_j \otimes e_l = \sum_{j=1}^n \alpha_{i,j,t}\Delta(e_j),$$

Assume $f : A^*$,

Assume $a : A$,

$[f.*] := \mathcal{I}\rho\mathcal{I}\texttt{tensorMap}\mathcal{I}\rho\mathcal{I}\texttt{SweedlerNotation}\mathcal{I}\texttt{dualBasis}[1][3][2]\mathcal{I}^{-1}\texttt{tensorMap}\mathcal{I}^{-1}\rho :$

$$: a(f\ \rho(\rho \otimes \mathrm{id})) = a\left(\left(\sum_{i=1}^n e^i f \otimes e_i\right)(\rho \otimes \mathrm{id})\right) = a \sum_{i,j=1}^n e^j e^i f \otimes e_j \otimes e_i = \sum_a \sum_{i,j=1}^n a_1^j a_2^i f(a_3) e_j \otimes e_i =$$

$$= \sum_{i,j,l,t=1}^n f_l a^t \beta_{t,i,j,l} \alpha e_j \otimes e_i = \sum_{t,l,i=1}^n a^t f_l \alpha_{t,i,l} \Delta(e_i) = a\left(\sum_{i=1}^n e^i f \otimes \Delta(e_i)\right) =$$

$$= a\left(\left(\sum_{i=1}^n e^i f \otimes e_i\right) \mathrm{id} \otimes \Delta\right) = a(f\ \rho\ \mathrm{id} \otimes \Delta);$$

$\rightsquigarrow [4] := I(=, \to) : \rho(\mathrm{id} \otimes \Delta) = \rho(\Delta \otimes \mathrm{id}),$

Assume $f : A^*$,

Assume $a : A$,

$[f.*] := \mathcal{I}\rho\mathcal{I}\texttt{tensorMap}\mathcal{I}\texttt{SweedlerNotation}[1]\mathcal{I}k\text{-}\mathsf{MOD}(A, k)(f)\mathcal{I}k\text{-}\mathsf{COALG}(A)\mathcal{I}\texttt{Basis}(n, A) :$

$$: a(f\ \rho(\mathrm{id} \otimes \eta)) = a\left(\left(\sum_{i=1}^n e^i f \otimes e_i\right)(\mathrm{id} \otimes \eta)\right) = \sum_a a_1^i f(a_2)\eta(e_i) = \sum_{i,j,l=1}^n a^i f_l \alpha_{i,j,l} \eta(e_j) =$$

$$= \sum_{i=1}^n a^i f\left(\sum_{j,l=1}^n \alpha_{i,j,l}\eta(e_j) e_l\right) = \sum_{i=1}^n a^i f(e_i) = \sum_{i=1}^n a^i f_i = f(a);$$

$\rightsquigarrow [5] := I(=, \to) : \rho(\mathrm{id} \otimes \eta) = \mathrm{id},$

$[*] := \mathcal{I}\mathsf{MOD}^A[5][4] : A^* \in \mathsf{MOD}^A;$

$\square$

$$\texttt{HopfModule} :: \prod R \in \mathsf{ANN} \,.\, \prod A \in R\text{-}\mathsf{HOPF} \,.\, ?(\mathsf{MOD}^A \,\&\, \mathsf{MOD}_A)$$

$$M : \texttt{HopfModule} \iff \rho_M : M \xrightarrow{\mathsf{MOD}_A} M \otimes A$$

$$\texttt{categoryOfHopfModule} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{HOPF} \to \mathsf{CAT}$$

$$\texttt{categoryOfHopfModule}\,(A) = \mathsf{MOD}^A_A := \Big(\texttt{HopfModule}, \mathsf{MOD}^A \,\&\, \mathsf{MOD}_A, \circ, \mathrm{id}\Big)$$

$$\texttt{MainTheoremOfHopfModules} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \forall M \in \mathsf{MOD}^A_A \,.\, M \cong_{\mathsf{MOD}^A_A} W \otimes A$$

where $\quad W = \{m \in M : \rho(m) = m \otimes e_A\}$

Proof $=$

$\ldots$

$\square$

$\texttt{dualModuleOfHopfAlg} :: \prod k : \texttt{Field} \,.\, \prod A : k\text{-HOPF} \,.\, \mathsf{MOD}_A$

$\texttt{dualModuleOfHopfAlg}\,() = A^* := \Big(A^*, \Lambda f \in A^* \,.\, \Lambda a \in A \,.\, f \leftharpoondown \sigma(a)\Big)$

$\texttt{AntiRightDualAlgebra} :: \prod k : \texttt{Field} \,.\, \prod A : k\text{-HOPF} \,.\, \forall f, g \in A^* \,.\, \forall a \in A \,.\, (fg)a = \sum_a (fa_2)(ga_1)$

Proof $=$

Assume $x : A$,

$[x.*] := \mathcal{I}\,\texttt{dualModuleOfHopfAlgebra}\,\mathcal{I}\,\texttt{HitByAction}\,\mathcal{I}\,R\text{-BIALG}(A)$

$\texttt{ComultiplicationOfAntipode}\,\mathcal{I}\,\texttt{dualAlgebra} :$

$$: x\ (fg)a = x\big(fg \leftharpoondown \sigma(a)\big) = \sum_{y = x\sigma(a)} f(y_1)g(y_2) = \sum_x \sum_{z=\sigma(a)} f(x_1 z_1)g(x_2 z_2) =$$

$$= \sum_x \sum_a f\big(x_1\sigma(a_1)\big)g\big(x_2\sigma(a_2)\big) = x \sum_a (fa_2)(ga_1);$$

$\rightsquigarrow [*] := I(=, \rightarrow) : (fg)a = \sum_a (fa_2)(ga_1);$

$\square$

$\texttt{dualAsHopfModule} :: \forall k : \texttt{Field} \,.\, \forall A : k\text{-HOPF} \,.\, \forall n \in \mathbb{N} \,.\, \forall e : \texttt{Basis}(n, A) \,.\, A^*_e \in \mathsf{MOD}^A_A$

Proof $=$

Assume $f, g : A^*$,

Assume $a, x : A$,

$\Big[(f,g).*\Big] := \mathcal{I}\,k\text{-COALG}(A)\,\mathcal{I}\,k\text{-VS}(A^* \otimes A^*; A^*)(\mu_{A^*_e})\,\mathcal{I}\,k\text{-ALGE}(A)\,\mathcal{I}\,k\text{-HOPF}(A)\,\mathcal{I}\,\texttt{dualAlgebra}(A)$

$\mathcal{I}\,\texttt{dualModuleOfHoplfAlgebra}(A)\texttt{AntipodeAntihomo}(A)\texttt{AntipodeComultiplication}(A)\,\mathcal{I}\,k\text{-HOPF}(A)$

$\texttt{AntipodeCounit}(A)\,\mathcal{I}\,k\text{-HOPF}(A)\,\mathcal{I}\,\texttt{finiteDualAlgebra}(A)$

$\mathcal{I}^{-1}\texttt{dualModuleOfHoplfAlgebra}(A)\texttt{AntiRightDualAlgebra}\,\mathcal{I}\,\texttt{Basis}(n, A)(e)\,\mathcal{I}\,k\text{-VS}(A, A^*)\texttt{eval}$

$\mathcal{I}\,\texttt{finiteDualAlgebra}(A) :$

$$x\ f(ga) = x\ f\left(g\sum_a a_1\eta(a_2)\right) = x \sum_a \big(f\eta(a_2)e_A\big)(ga_1) = x \sum_a \big(f\sigma(a_2)a_3\big)(ga_1) =$$

$$\sum_x\sum_a \Big(x_1\ f\sigma(a_2)a_3\Big)\big(x_2\ ga_1\big) = \sum_x\sum_a f\Big(x_1\sigma\big(\sigma(a_2)a_3\big)\Big)g\big(x_2\sigma(a_1)\big) =$$

$$= \sum_x\sum_a f\big(x_1\sigma(a_3)\sigma^2(a_2)\big)g\big(x_2\sigma(a_1)\big) = \sum_x\sum_a f\big(x_1\sigma\eta(a_2)e\big)g\big(x_2\sigma(a_1)\big) =$$

$$\sum_x\sum_a f\big(x_1\eta(a_2)e\big)g\big(x_2\sigma(a_1)\big) = \sum_x\sum_a f\big(x_1\sigma(a_2)a_3 g\big(x_2\sigma(a_1)\big) =$$

$$= \sum_x\sum_a\sum_f f_1\big(x_1\sigma(a_2)\big)f_2(a_3)g\big(x_2\sigma(a_1)\big) = \sum_x\sum_a\sum_f f_2(a_3)(f_1a_2)(x_1)(ga_1)(x_2) =$$

$$= \sum_a\sum_f f_2(a_3)\Big(x\ (f_1a_2)(ga_1)\Big) = \sum_a\sum_f f_2(a_2)\big(x\ (f_1g)a_1\big) =$$

$$= \sum_a\sum_f f_2(a_2)\left(x\ \left(\sum_{i=1}^n f_1(e_i)e^i g\right) a_1\right) = \sum_a\sum_f\sum_{i=1}^n f_1(e_i)f_2(a_2)(x\ e^i ga_1) =$$

$$= \sum_a\sum_{i=1}^n f(e_ia_2)(x\ e^i ga_1);$$

$\rightsquigarrow [1] := I(\forall)I(=,\to) : \forall f, g \in A^* \,.\, \forall a \in A \,.\, f(ga) = \sum_a \sum_{i=1}^n f(e_i a_2)(e^i g a_1),$

Assume $g : A^*,$

Assume $a : A,$

Assume $i : n,$

$\left[.*\right] := [1](e^i, g, a) \mathcal{I}\,$tensorProduct $:$

$$: e^i(ga) \otimes e_i = \left(\sum_a \sum_{j=1}^n e^i(e_j a_2)(e^j g a_1)\right) \otimes e_i = \sum_a \sum_{j=1}^n (e^j g a_1) \otimes e^i(e_j a_2) e_i =;$$

$\rightsquigarrow [2] := I(\forall) : \forall g \in A^* \,.\, \forall a \in A \,.\, \forall i \in n \,.\, e^i(ga) \otimes e_i = \sum_a \sum_{j=1}^n (e^j g a_1) \otimes e^i(e_j a_2) e_i,$

Assume $f : A^*,$

Assume $a : A,$

$[f.*] := \mathcal{I}\rho\Big(\forall i \in n \,.\, [2](f,a,i)\Big)\mathcal{I}\,$dualBasis$\,\mathcal{I}\,$tensorProduct$\,\mathcal{I}\,\mathsf{MOD}_A(A^* \otimes A)\mathcal{I}^{-1}\rho :$

$$: \rho(fa) = \sum_{i=1}^n e^i(fa) \otimes e_i = \sum_{i,j=1}^n \sum_a (e^j f a_1) \otimes e^i(e_j a_2) e_i = \sum_{j=1}^n \sum_a (e^j f a_1) \otimes e_j a_2 = \left(\sum_{j=1}^n e^j f \otimes e_j\right) a =$$

$$= \rho(f)a;$$

$\rightsquigarrow [*] := \mathcal{I}\,\mathsf{MOD}_A^A : A_e^* \in \mathsf{MOD}_A^A;$

$\square$

IntergralOfHopfDual $::$ $\forall k :$ Field $.\ \forall A \in k\text{-}\mathsf{HOPF} \,.\, \forall [0] : \dim A < \infty \,.\, \int_{A^*}^l = \{f \in A^* : \rho(f) = f \otimes e\}$

Proof $=$

$n := \dim A : \mathbb{N},$

$e :=$ BasisExists $:$ Basis$(n, A),$

$W := \{f \in A^* : \Delta(f) = f \otimes \eta_A\} :?A^*,$

Assume $f : \int_{A^*}^l,$

$[1] := \mathcal{I} \int_{A^*}^l \mathcal{I}\,$dualCoalgebra$(A)\mathcal{I}\,$Unimodul$(A^*) : \forall g \in A^* \,.\, gf = \eta(g) f = g(e) f = fg,$

$[2] := \mathcal{I}\rho[1]\mathcal{I}\,$dualBasis$\,\mathcal{I}\,$tensorProduct $: \rho(f) = \sum_{i=1}^n e^i f \otimes e_i = \sum_{i=1}^n e^i(e) f \otimes e_i = \sum_{i=1}^n f \otimes e^i(e) e_i = f \otimes e,$

$[f.*] := \mathcal{O}W[2] : f \in W;$

$\rightsquigarrow [1] := \mathcal{I}\,$Subset $: \int_{A^*}^l \subset W,$

Assume $f : W,$

Assume $g : A^*,$

$[g.*] := \mathcal{I}\,$dualBasis$\,\mathcal{I}^{-1}$tensorMap$\,\mathcal{I}^{-1}\rho\mathcal{O}W\mathcal{I}\,$tensorMap$\,\mathcal{I}\,$finiteDualAlgebra $:$

$$: gf = \sum_{i=1}^n g(e_i) e^i f = (\sum_{i=1}^n e^i f \otimes e_i)(\mathrm{id} \otimes g)\mu = \rho(f)(\mathrm{id} \otimes g)\mu = (f \otimes e)(\mathrm{id} \otimes g)\mu = g(e) f = \eta(g) f;$$

$\rightsquigarrow [f.*] := \mathcal{I} \int_{A^*}^l : f \in \int_{A^*}^l;$

$\rightsquigarrow [*] := [1]\mathcal{I}\,$SetEq $: \int_{A^*}^l = W;$

$\square$

$\texttt{antipodalAutoconvolution} :: \prod R \in \mathsf{ANN} \;.\; \prod A \in k\text{-}\mathsf{HOPF} \;.\; \prod M \in \mathsf{MOD}_A^A \;.\; M \xrightarrow{k\text{-}\mathsf{VS}} M$

$$\texttt{antipodalAutoconvolution}\,(m) = S(m) := \sum_m m_0\sigma(a_1)$$

$\texttt{AutoconvolutionMultiplication} :: \forall R \in \mathsf{ANN} \;.\; \forall A \in R\text{-}\mathsf{HOPF} \;.\; \forall M \in \mathsf{MOD}_A^A \;.\; \forall f \in M \;.\; \forall a \in A \;.$
$\quad .\; S(fa) = \eta(a)S(f)$

$\texttt{Proof} \;=$

$[*] := \text{ᗡ} S \text{ᗡ} \mathsf{MOD}_A^A(M)\texttt{AntipodeAntihomo}(A)\text{ᗡ} R\text{-}\mathsf{HOPF}(A)\text{ᗡ} R\text{-}\mathsf{ALGE}(A)\text{ᗡ}^{-1}S :$

$$: S(fa) = \sum_{g=fa} g_0\sigma(g_1) = \sum_{f,a} f_0a_1\sigma(f_1a_2) = \sum_{f,a} f_0a_1\sigma(a_2)\sigma(f_1) = \sum_f f_0\eta(a)e\sigma(f_1) = \eta(a)\sum_f f_0\sigma(f_1) =$$
$$= \eta(a)S(f);$$

$\square$

$\texttt{AutoconvolutionComultiplication} :: \forall R \in \mathsf{ANN} \;.\; \forall A \in R\text{-}\mathsf{HOPF} \;.\; \forall M \in \mathsf{MOD}_A^A \;.\; \forall f \in M \;.\; \rho(S(f)) = S(f)$

$\texttt{Proof} \;=$

$[*] := \text{ᗡ} S \text{ᗡ} R\text{-}\mathsf{MOD}(M, M \otimes A)(\rho)\text{ᗡ}\mathsf{MOD}_A^A(M)\text{ᗡ}\texttt{SweedlerNotation}\text{ᗡ}\mathsf{MOD}_A^A(M \otimes A)\text{ᗡ} R\text{-}\mathsf{HOPF}(A)$
$\quad \text{ᗡ}\texttt{tensorProduct}\text{ᗡ}\mathsf{MOD}^A(M)\text{ᗡ}^{-1}S :$

$$: \rho(S(f)) = \rho\left(\sum_f f_0\sigma(f_1)\right) = \sum_f \rho\big(f_0\sigma(f_1)\big) = \sum_f \rho(f_0)\sigma(f_1) = \sum_f (f_0 \otimes f_1)\sigma(f_2) =$$
$$= \sum_f f_0\sigma(f_3) \otimes f_1\sigma(f_2) = \sum_f f_0\sigma(f_2) \otimes \eta(f_1)e = \left(\sum_f f_0\eta(f_1)\sigma(f_2)\right) \otimes e = \left(\sum_f f_0\sigma(f_1)\right) \otimes e =$$
$$= S(f) \otimes e;$$

$\square$

$\texttt{integralHopfModule} :: \prod k : \texttt{Field} \;.\; \prod A \in k\text{-}\mathsf{HOPF} \;.\; \prod n \in \mathbb{N} \;.\; \prod e : \texttt{Basis}(n, A) \;.\; \mathsf{MOD}_A^A$

$$\texttt{integralHopfModule}\,() = \int_{A^*} := \left(\int_{A^*}, \Lambda f \in A^* \;.\; \lambda a \in A \;.\; \eta(a)f\right)$$

$\texttt{SweedlerLarsonTHM} :: \forall k : \texttt{Field} \;.\; \forall A \in k\text{-}\mathsf{HOPF} \;.\; \forall n \in \mathbb{N} \;.\; \forall e : \texttt{Basis}(n, A) \;.\; A_e^* \cong_{\mathsf{MOD}_A^A} \int_{A^*} \otimes A$

$\texttt{Proof} \;=$

$\varphi := \text{ᗡ}\texttt{tensorProduct}\Lambda f \in \int_{A^*} \;.\; \Lambda a \in A \;.\; f \cdot_{A^*} a : \int_{A^*} \otimes A \xrightarrow{\mathsf{MOD}_A^A} A^*,$

$\psi := \rho(S \otimes \mathrm{id}) : A^* \xrightarrow{\mathsf{MOD}_A^A} \int_{A^*} \otimes A,$

$\texttt{Assume}\; f : \int_{A^*},$

$\texttt{Assume}\; a : A,$

$[f.*] := \text{Ɔ}\varphi\text{Ɔ}\psi\text{ᗡ}\mathsf{MOD}_A^A(A_e^*)\texttt{IntegralOfHopfDual}(f)\text{ᗡ}\texttt{integralHopfModule}\text{ᗡ}\texttt{tensorMap}$
$\quad \text{ᗡ} S\texttt{IntegralOfHopfDualAntipodeUnit} :$

$$: (f \otimes a)\varphi\psi = \left(f \cdot_{A^*} a\right)\rho(S \otimes \mathrm{id}) = \big(\rho(f)a\big)(S \otimes \mathrm{id}) = (f \otimes a)(S \otimes \mathrm{id}) = S(f) \otimes a = f\sigma(e) \otimes a = f \otimes a;$$

$\rightsquigarrow [1] := I(=, \rightarrow)) : \varphi\psi = \mathrm{id},$

$\texttt{Assume}\ f : A^*$,

$[f.*] := \mathcal{D}\psi\mathcal{D}\varphi\mathcal{A}\,\texttt{SweedlerNotation}\,\mathcal{A}\,\texttt{tensorMap}\,\mathcal{A}\,S\mathcal{A}\,k\text{-}\mathsf{HOPF}(A)\,\texttt{AntipodeUnit}\,\mathcal{A}\,\mathsf{MOD}^A(A_e^*) :$

$: f\psi\varphi = f\rho(S\otimes \mathrm{id})\mu = \sum_f f_0 \otimes f_1(S\otimes\mathrm{id})\mu = \sum_f f_0\sigma(f_1)f_2 = \sum_f f_0\eta(f_1)e = \sum_f \eta(f_1)f_0 = f;$

$\rightsquigarrow [2] := I(=,\to) : \psi\varphi = \mathrm{id},$

$[3] := \mathcal{A}^{-1}\texttt{Inverse}[3][2] : \psi = \varphi^{-1},$

$[4] := \mathcal{A}^{-1}\texttt{Isomorphic} : \int_{A_e^*} \otimes A \cong_{\mathsf{MOD}_A^A} A_e^*;$

$\square$

$\texttt{IntegralsAreLines} :: \forall k : \texttt{Field} \,.\, \forall A \in k\text{-}\mathsf{HOPF} \,.\, \forall n \in \mathbb{N} \,.\, \forall e : \texttt{Basis}(n, A) \,.\, \dim \int_A = 1$

$\texttt{Proof} =$

$[1] := \texttt{DualDimendion} : \dim A = \dim A^*,$

$[2] := \texttt{SweedlerLarsonTHM}(A^*)\texttt{FinitDimReflexive} : A \cong_{\mathsf{MOD}_A^A} \int_A \otimes A^*,$

$[3] := \texttt{TensorProductDimension} : \dim A^* = \dim \int_A \dim A^*,$

$[4] := \texttt{NeutralNaturalNumberisOne}[3] : \dim \int_A = 1;$

$\square$

$\texttt{GeneratingIntegralsExists} :: \forall R : \texttt{PrincipleIdealDomain} \,.\, \forall A \in R\text{-}\mathsf{HOPF} \,.\, \forall [0] : \mathrm{rank}\, A < \infty \,.$

$. \exists \Lambda \in \int_A : \int_A = R\Lambda$

$\texttt{Proof} =$

$\ldots$

$\square$

3.6 Hopf Orders [!!]

$$\texttt{Order} :: \prod R : \texttt{IntegralDomain} \;.\; \prod G \in \mathsf{GRP} \;.\; \texttt{UnitalSubalgebra}\Big(\mathrm{Frac}(R)G\Big)$$

$$O : \texttt{Order} \iff O : \texttt{Projective} \;\&\; \texttt{FiniteGroup}(R) \;\&\; \exists E : \texttt{Basis}\Big(\mathrm{Frac}(R)G\Big) : E \subset O$$

$\texttt{OrderRoots} :: \forall R : \texttt{IntegralDomain} \;.\; \forall G \in \mathsf{GRP} \;.\; \forall O : \texttt{Order}(R, G) \;.$

$\quad .\; \forall o \in O \;.\; \exists f(x) \in R[x] \;.\; f(o) = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{FreeOrder} :: \forall R : \texttt{IntegralDomain} \;\&\; \texttt{LocalIntegrallyClosed} \;.\; \forall G \in \mathsf{GRP} \;.$

$\quad .\; \forall O : \texttt{Order}(R, G) \;.\; O \cong_{R\text{-}\mathsf{MOD}} R^{|G|}$

$\texttt{Proof} \;=$

$$\texttt{HopfOrder} :: \prod R : \texttt{IntegralDomain} \;.\; \prod G \in \mathsf{GRP} \;.\; ?\texttt{Order}(R, G)$$

$$O : \texttt{HopfOrder} \iff \Delta(O) \subset O \times O$$

3.7 Graded Duality

$$\texttt{gradedDual} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, R\text{-}\mathsf{MOD}(G) \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{MOD}^o(G)$$

$$\texttt{gradedDual}\,(M) = \mathfrak{D}(M) := \bigoplus_{g \in G} M_g^*$$

$$\texttt{gradedDual}\,(A, B, \varphi) = \mathfrak{D}_{A,B}(\varphi) := \bigoplus_{g \in G} \varphi^*_{|M_g}$$

$$\texttt{gradedDualAction} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, \prod M : R\text{-}\mathsf{MOD}(G) \,.\, \mathfrak{D}(M) \xrightarrow{R\text{-}\mathsf{MOD}} M^*$$

$$\texttt{gradedDualAction}\,(f, m) = f(m) := \sum_{g \in G} f_g(m_g)$$

$$\texttt{GradedDualAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall G : \texttt{Monoid} \,.\, \forall A \in R\text{-}\mathsf{COALG}(G)\, (\mathfrak{D}(M), \mathfrak{D}(\Delta), \mathfrak{D}(\eta)) \in R\text{-}\mathsf{ALGE}(G)$$

Proof $=$

. . .

□

$$\texttt{gradedDualAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, R\text{-}\mathsf{COALG}(G) \xrightarrow{\mathsf{CAT}} R\text{-}\mathsf{ALGE}^o(G)$$

$$\texttt{gradedDualAlgebra}\,(A) = \mathfrak{D}(A) := \Big(\mathfrak{D}(A), \mathfrak{D}(\Delta), \mathfrak{D}(\eta)\Big)$$

$$\texttt{gradedDualAlgebra}\,(A, B, \varphi) = \mathfrak{D}_{A,B}(\varphi) := \mathfrak{D}(\varphi)$$

$$\texttt{CoskewDualAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{SCOALG} \,.\, \mathfrak{D}(A) : \texttt{SkewAlgebra}(A)$$

Proof $=$

. . .

□

$$\texttt{GradedHopfAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, ?\Big(R\text{-}\mathsf{HOPF} \,\&\, \big(R\text{-}\mathsf{ALGE} \,\&\, R\text{-}\mathsf{COALG}\big)(G)\Big)$$

$$A : \texttt{GradedHopfAlgebra} \iff \sigma_A : A \xrightarrow{R\text{-}\mathsf{MOD}(G)} A$$

$$\texttt{categoryOfGradedHopfAlgebras} :: \mathsf{ANN} \to \texttt{Monoid} \to \mathsf{CAT}$$

$$\texttt{categoryOfGradedHopfAlgebras}\,(R, G) = R\text{-}\mathsf{HOPF}(G) :=$$

$$:= \Big(\texttt{GradedHopfAlgebra}, R\text{-}\mathsf{HOPF} \cap R\text{-}\mathsf{MOD}(G), \circ, \mathrm{id}\Big)$$

$$\texttt{TwistedHopfAlgebra} :: \prod R \in \mathsf{ANN} \,.\, ?\Big(R\text{-}\mathsf{BIALG} \,\&\, \big(R\text{-}\mathsf{ALGE} \,\&\, R\text{-}\mathsf{COALG}\big)(\mathbb{Z})\Big)$$

$$A : \texttt{TwistedHopfAlgebra} \iff \Delta_A : A \xrightarrow{R\text{-}\mathsf{ALGE}(\mathbb{Z})} \Big(A \tilde{\otimes} A\Big) \,\&\, \mu_A : \Big(A \tilde{\otimes} A\Big) \xrightarrow{R\text{-}\mathsf{COALG}(\mathbb{Z})} A$$

$$\texttt{categoryOfTwistedHopfAlgebras} :: \mathsf{ANN} \to \mathsf{CAT}$$

$$\texttt{categoryOfTwistedHopfAlgebras}\,(R) = \widetilde{R\text{-}\mathsf{HOPF}} :=$$

$$:= \Big(\texttt{TwistedHopfAlgebra}, R\text{-}\mathsf{BIALG} \cap R\text{-}\mathsf{MOD}(G), \circ, \mathrm{id}\Big)$$

$\texttt{TensorProductOfHopfAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall A : n \to R\text{-}\mathsf{HOPF} \left(\bigotimes_{i=1}^n A_i, \bigotimes_{i=1}^n \sigma_i \right) \in R\text{-}\mathsf{HOPF}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a : \prod_{i=1}^n A_i,$

$[a.*.1] := \mathcal{I}\texttt{tensorProductCoalgebra}(n,A)\mathcal{I}\texttt{tensorMap}(n,\sigma)\mathcal{I}\texttt{tensorProductAlgebra}(n,A)$
$\mathcal{I}\texttt{tensorProduct}(n,A)\mathcal{I}R\text{-}\mathsf{HOPF}(A)\mathcal{I}\texttt{tensorProductAlgebra}(n,A)\mathcal{I}\texttt{tensorProductCoalgebra}(n,A) :$

$$: \bigotimes_{i=1}^n a_i \;\Delta \left(\mathrm{id} \otimes \bigotimes_{i=1}^n \sigma_{A_i} \right) \mu = \sum_a \bigotimes_{i=1}^n a_{i,1} \otimes \bigotimes_{i=1}^n a_{i,2} \left(\mathrm{id} \otimes \bigotimes_{i=1}^n \sigma_{A_i} \right) \mu = \sum_a \bigotimes_{i=1}^n a_{i,1}\sigma_{A_i}(a_{i,2}) =$$

$$= \bigotimes_{i=1}^n \eta_{A_i}(a_i)e_{A_i} = \prod_{i=1}^n \eta_{A_i}(a_i) \otimes_{i=1}^n e_{A_i} = \eta \left(\bigotimes_{i=1}^n a_i \right) e,$$

$[a.*.2] := \mathcal{I}\texttt{tensorProductCoalgebra}(n,A)\mathcal{I}\texttt{tensorMap}(n,\sigma)\mathcal{I}\texttt{tensorProductAlgebra}(n,A)$
$\mathcal{I}\texttt{tensorProduct}(n,A)\mathcal{I}R\text{-}\mathsf{HOPF}(A)\mathcal{I}\texttt{tensorProductAlgebra}(n,A)\mathcal{I}\texttt{tensorProductCoalgebra}(n,A) :$

$$: \bigotimes_{i=1}^n a_i \;\Delta \left(\bigotimes_{i=1}^n \sigma_{A_i} \otimes \mathrm{id} \right) \mu = \sum_a \bigotimes_{i=1}^n a_{i,1} \otimes \bigotimes_{i=1}^n a_{i,2} \left(\bigotimes_{i=1}^n \sigma_{A_i} \otimes \mathrm{id} \right) \mu = \sum_a \bigotimes_{i=1}^n \sigma_{A_i}(a_{i,1}) =$$

$$= \bigotimes_{i=1}^n \eta_{A_i}(a_i)e_{A_i} = \prod_{i=1}^n \eta_{A_i}(a_i) \otimes_{i=1}^n e_{A_i} = \eta \left(\bigotimes_{i=1}^n a_i \right) e,$$

$\rightsquigarrow [*] := \mathcal{I}\texttt{tensorProduct}\mathcal{I}R\text{-}\mathsf{HOPF} : \texttt{This},$

$\square$

$\texttt{tensorProductOfHopfAlgebras} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, (n \to R\text{-}\mathsf{HOPF}) \to R\text{-}\mathsf{HOPF}$

$$\texttt{tensorProductOfHopfAlgebras}\,(A) = \bigotimes_{i=1}^n A_i := \left(\bigotimes_{i=1}^n A_i, \bigotimes_{i=1}^n \sigma_i \right)$$

$\texttt{TwistedTensorProductOfTwistedHopfAlgebras} :: \forall R \in \mathsf{ANN} \,.\, \forall A, B \in \widetilde{R\text{-}\mathsf{HOPF}} \,.\, A \widetilde{\otimes} B \in \widetilde{R\text{-}\mathsf{HOPF}}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{twistedTensorProductOfHopfAlgebras} :: \prod R \in \mathsf{ANN} \,.\, \prod n \in \mathbb{N} \,.\, (n \to \widetilde{R\text{-}\mathsf{HOPF}}) \to \widetilde{R\text{-}\mathsf{HOPF}}$

$$\texttt{twistedTensorProductOfHopfAlgebras}\,(A) = \widetilde{\bigotimes}_{i=1}^n A_i := \widetilde{\bigotimes}_{i=1}^n A_i$$

$\texttt{FiniteFreeComponents} :: \prod R \in \mathsf{ANN} \,.\, \prod G : \texttt{Monoid} \,.\, ?R\text{-}\mathsf{MOD}(G)$

$M : \texttt{FiniteFreeComponents} \iff \forall g \in G \,.\, \exists n \in \mathbb{N} \,.\, M_g \cong_{R\text{-}\mathsf{MOD}} R^n$

$\texttt{GradedNaturalIsomorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall G : \texttt{Monoid} \,.\, \forall M : \texttt{FiniteFreeComponents}(R, G) \,.$

$\quad . \, \epsilon_M : M \xleftrightarrow{R\text{-}\mathsf{MOD}(G)} \mathfrak{D}^2(M)$

Proof $=$

Assume $m : M$,

Assume $g, h : G$,

Assume $f : M_h^*$,

Assume $[1] : \epsilon(m)(f) \neq 0$,

$[2] := [1] \daleth \epsilon : 0 \neq \epsilon(m_g)(f) = f(m_g)$,

$[3] := \daleth \texttt{gradedDualAction} : g = h$,

$[m.*] := \daleth M_g^{**}[3] : m_g \in M_g^{**}$;

$\rightsquigarrow [1] := \daleth R\text{-}\mathsf{MOD} : \left(\epsilon : M \xrightarrow{R\text{-}\mathsf{MOD}(G)} \mathfrak{D}^2(M)\right),$

Assume $f : \mathfrak{D}^2(M)$,

$(m, [2]) := \forall \texttt{NaturalIsomorphisTheorem}(f_g) : \sum m : \prod_{g \in G} M_g \,.\, \forall g \in G \,.\, f_g = \epsilon(m_g),$

$[3] := \daleth \mathfrak{D}^2(M)[2] : m \in M,$

$[f.*] := [2][3] : f = \epsilon(m);$

$\rightsquigarrow [*] := \daleth^{-1}\texttt{Surjective}\daleth^{-1}\texttt{Bijective} : \texttt{This};$

$\square$

$\texttt{TensorProductGradedDuality} :: \forall R \in \mathsf{ANN} \,.\, \forall n \in \mathbb{N} \,.\, \forall M : n \to \texttt{FiniteFreeComponents}(R, \mathbb{Z}) \,.$

$\quad . \, \mathfrak{D}\left(\bigotimes_{i=1}^{n} M_i\right) = \bigotimes_{i=1}^{n} \mathfrak{D}(M_i)$

Proof $=$

...

$\square$

$\texttt{GradedDualAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{COALG}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \,.\, \left(\mathfrak{D}(A), \mathfrak{D}(\Delta), \mathfrak{D}(\eta)\right) :$

Proof $=$

...

$\square$

$\texttt{gradedDualAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{COALG}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \to R\text{-}\mathsf{ALGE}(\mathbb{Z})$

$\texttt{gradedDualCoalgebra}\,(A) := \left(\mathfrak{D}(A), \mathfrak{D}(\Delta), \mathfrak{D}(\eta)\right)$

$\texttt{GradedDualCoalgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{ALGE}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \,.\, \left(\mathfrak{D}(A), \mathfrak{D}(\mu), \mathfrak{D}(e)\right) :$

Proof $=$

...

$\square$

$\texttt{gradedDualAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{ALGE}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \to R\text{-}\mathsf{COALG}(\mathbb{Z})$

$\texttt{gradedDualCoalgebra}\,(A) := \left(\mathfrak{D}(A), \mathfrak{D}(\mu), \mathfrak{D}(e)\right)$

$\texttt{GradedNaturalAlgebraIsomorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall A : R\text{-}\mathsf{ALGE}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents} \,.$

$\quad .\, \epsilon_A : A \xleftrightarrow{R\text{-}\mathsf{ALGE}(G)} \mathfrak{D}^2(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{GradedNaturalCoalgebraIsomorphism} :: \forall R \in \mathsf{ANN} \,.\, \forall A : R\text{-}\mathsf{COALG}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents} \,.$

$\quad .\, \epsilon_A : A \xleftrightarrow{R\text{-}\mathsf{COALG}(G)} \mathfrak{D}^2(A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{GradedDualHopfAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in R\text{-}\mathsf{HOPF}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \,.$

$\quad .\, \mathfrak{D}(A) : R\text{-}\mathsf{HOPF}(\mathbb{Z})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{gradedDualHopfAlgebra} :: \prod R \in \mathsf{ANN} \,.\, R\text{-}\mathsf{HOPF}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \to R\text{-}\mathsf{HOPF}(\mathbb{Z})$

$\texttt{gradedDualHopfAlgebra}\,(A) := \mathfrak{D}(A)$

$\texttt{GradedDualTwistedHopfAlgebra} :: \forall R \in \mathsf{ANN} \,.\, \forall A \in \widetilde{R\text{-}\mathsf{HOPF}}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \,.$

$\quad .\, \mathfrak{D}(A) : \widetilde{R\text{-}\mathsf{HOPF}}(\mathbb{Z})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{gradedDualAlgebra} :: \prod R \in \mathsf{ANN} \,.\, \widetilde{R\text{-}\mathsf{HOPF}}(\mathbb{Z}) \;\&\; \texttt{FiniteFreeComponents}(R) \to \widetilde{R\text{-}\mathsf{HOPF}}(\mathbb{Z})$

$\texttt{gradedDualCoalgebra}\,(A) := \mathfrak{D}(A)$

4 Classical Clifford Algebras

4.1 Clifford Structure

$\texttt{CliffordMap} :: \prod k : \texttt{Field} . \prod A \in k\text{-ALGE} . \prod V : \texttt{OrthogonalVectorSpace}(k) . ?(V \xrightarrow{k\text{-VS}} A)$

$\varphi : \texttt{CliffordMap} \iff \forall x \in A . (x\ \varphi)^2 = \langle x, x \rangle e$

$\texttt{CliffordMapProduct} :: i\forall k : \texttt{Field} . \forall A \in k\text{-ALGE} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$

$. \forall \varphi : \texttt{CliffordMap} . \ \forall x, y \in A . \varphi(x)\varphi(y) + \varphi(y)\varphi(x) = 2\langle x, y \rangle e$

$\texttt{Proof} =$

$[1] := \mathcal{I} R\text{-ALGE}[1] : \varphi^2(x) + \varphi(x)\varphi(y) + \varphi(y)\varphi(x) + \varphi^2(y) = \varphi^2(x+y) = \langle x+y, x+y \rangle e =$

$= \langle x, x \rangle e + 2\langle x, y \rangle e + \langle y, y \rangle e = \varphi^2(x) + 2\langle x, y \rangle e + \varphi^2(y),$

$[*] := [1] - \varphi^2(x) - \varphi^2(y) : \varphi(x)\varphi(y) + \varphi(y)\varphi(x) = 2\langle x, y \rangle e;$

$\square$

$\texttt{CliffordAlgebra} :: \prod k : \texttt{Field} . \sum A \in k\text{-ALGE} . \sum V : \texttt{OrthogonalVectorSpace}(k) . \texttt{CliffordMap}(A, V$

$(A, V, \mathbf{i}) : \texttt{CliffordAlgebra} \iff \Big\langle \mathbf{i}(V) \Big\rangle_{k\text{-ALGE}} = A \ \&\ \forall B \in k\text{-ALGE} . \varphi : \texttt{CliffordMap}(B, V) .$

$. \exists f : A \xrightarrow{k\text{-ALGE}} B . \mathbf{i}f = \varphi$

$\texttt{categoryOfClifford} :: \texttt{Field} \to \mathsf{CAT}$

$\texttt{categoryOfClifford}(k) = k\text{-CLIF} := \Big(\texttt{CliffordAlgebra}, (k\text{-VS}, k\text{-ALGE}), (\circ, \circ^o), (\mathrm{id}, \mathrm{id})\Big)$

$\texttt{complexCliffordAlgebra} :: \mathbb{R}\text{-CLIF}$

$\texttt{complexCliffordAlgebra}() = \mathbb{C}_{\mathbb{R}} := \Big((\mathbb{R}, \Lambda a, b \in \mathbb{R} . \ -ab), \mathbb{C}, \Lambda a \in \mathbb{R} . ai\Big)$

$\texttt{Assume}\ a : \mathbb{R},$

$[a.*] := \mathcal{I}\mathbf{i}\mathcal{I}\mathbb{C} : \mathbf{i}^2(a) = (a\mathrm{i})^2 = -a^2 = \langle a, a \rangle 1;$

$\rightsquigarrow [1] := \mathcal{I}^{-1}\texttt{CliffordMap} : \Big(\mathbf{i} : \texttt{CliffordMap}(\mathbb{R}; \mathbb{R}, \mathbb{C})\Big),$

$\texttt{Assume}\ A : \mathbb{R}\text{-ALGE},$

$\texttt{Assume}\ T : \texttt{CliffordMap}(\mathbb{R}; \mathbb{R}; A),$

$\psi := \lambda a + b\mathrm{i} \in \mathbb{C} . ae_A + T(b) : \mathbb{C} \xrightarrow{k\text{-VS}} A,$

$\texttt{Assume}\ a + b\mathrm{i}, a' + b'\mathrm{i} : \mathbb{C},$

$[T.*] := \mathcal{I}\mathbb{C}\mathcal{O}\psi\mathcal{I}\mathbb{R}\text{-VS}(T)\mathcal{I}\texttt{CliffordMap}(\mathbb{R})(\mathbb{R}, A)(T)\mathcal{O}^{-1}\psi :$

$: \psi\Big((a + b\mathrm{i})(a' + b'\mathrm{i})\Big) = \psi\Big(aa' - bb' + (a'b + ab')\mathrm{i}\Big) = (aa' - bb')e_A + T(a'b + ab') =$

$= aa'e_A + bb'T^2(1) + a'T(b) + aT(b') = aa'e_A + T(b)T(b') + a'T(b) + aT(b') = (ae_A + T(b))(a'e_A + T(b')) =$

$= \psi\Big(a + b\mathrm{i}\Big)\psi(a' + b'\mathrm{i});$

$\rightsquigarrow [*] := \mathcal{I}\mathbb{R}\text{-CLIF} : \mathbb{C}_{\mathbb{R}} \in \mathbb{R}\text{-CLIF};$

$\square$

$\texttt{realQuaternionCliffordAlgebra} :: \mathbb{R}\text{-CLIF}$

$\texttt{realQueternionCliffordAlgebra}() = \mathbb{H}_{\mathbb{R}} :=$

$$\Bigg(\Bigg(\mathbb{R}^2, \texttt{quadraticByMatrix}\Big((e_1, e_1) \mapsto -1, (e_2, e_2) \mapsto -1, (e_1, e_2) \mapsto 0, (e_2, e_1) \mapsto 0\Big)\Bigg), \mathbb{H},$$

$$\mathcal{I}\texttt{Basis}(2, \mathbb{R}^2)(e)\Big(e_1 \mapsto \mathrm{i}, e_2 \mapsto \mathrm{j}\Big)\Bigg)$$

$\texttt{Assume}\ a, b : \mathbb{R},$

$\Big[(a, b). *\Big] := \mathcal{I}\mathbf{i}\mathcal{I}\mathbb{H} : \mathbf{i}^2(a, b) = (a\mathrm{i} + b\mathrm{j})^2 = -a^2 - b^2 + ab\mathrm{k} - ab\mathrm{k} = \Big\langle (a, b), (a, b)\Big\rangle 1;$

$\leadsto [1] := \mathcal{I}\texttt{CliffordMap} : \Big(\mathbf{i} : \texttt{CliffordMap}(\mathbb{R})(\mathbb{R}^2, \mathbb{H})\Big),$

$\texttt{Assume}\ A : \mathbb{R}\text{-ALGE},$

$\texttt{Assume}\ T : \texttt{CliffordMap}(\mathbb{R}^2, A),$

$\psi := \Lambda a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \in \mathbb{H}\ .\ ae_A + bT(e_1) + cT(e_2) + dT(e_1)T(e_2) : \mathbb{H} \xrightarrow{\mathbb{R}\text{-VS}} \mathbb{R}^2,$

$\texttt{Assume}\ a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}, a' + b'\mathrm{i} + c'\mathrm{j} + d'\mathrm{k} : \mathbb{H},$

$[1] := \texttt{CliffordMapProduct}(\mathbf{i})(e_1, e_2) : T(e_1)T(e_2) = -T(e_2)T(e_1),$

$[A.*] := \mathcal{I}\mathbb{H}\mathcal{D}\psi[1]\mathcal{I}\texttt{CliffoeMap}(T)\mathcal{I}R\text{-ALGE}(A)\mathcal{D}^{-1}\psi :$

$$= \psi\Big((a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k})(a' + b'\mathrm{i} + c'\mathrm{j} + d'\mathrm{k})\Big) =$$

$$= \psi\Big((aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')\mathrm{i} + (ac' + ca' - bd' + db')\mathrm{j} + (ad' + da' + bc' - cb')\mathrm{k}\Big) =$$

$$= (aa' - bb' - cc' - dd')e_A + (ab' + ba' - cd' + dc')T(e_1) +$$
$$+ (ac' + ca' - bd' + db')T(e_2) + (ad' + da' + bc' - cb')T(e_1)T(e_2) =$$

$$= aa' + bb'T^2(e_1) + cc'T^2(e_1) + dd'\big(T(e_1)T(e_2)\big)^2 + (ab' + ba')T(e_1) + cd'T(e_2)T(e_1)T(e_2) +$$
$$+ dc'T(e_1)T^2(e_2) + (ac' + ca')T(e_2) + bd'T^2(e_1)T(e_3) + db'T(e_1)T(e_2)T(e_1) + (ad' + da')T(e_1)T(e_2) +$$
$$+ bc'T(e_1)T(e_2) + cb'T(e_1)T(e_2) =$$

$$= \Big(a + bT(e_1) + cT(e_2) + dT(e_1)T(e_2)\Big)\Big(a' + b'T(e_1) + c'T(e_2) + d'T(e_1)T(e_2)\Big) =$$

$$= \psi\Big(a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}\Big)\psi(a' + b'\mathrm{i} + c'\mathrm{j} + d'\mathrm{k});$$

$\leadsto [*] := \mathcal{I}\mathbb{R}\text{-CLIF} : \mathbb{H}_{\mathbb{R}} \in \mathbb{R}\text{-CLIF};$

$\square$

$\texttt{CliffordUniversalProperty} :: \forall k : \texttt{Field}\ .\ \forall (V, C, \mathbf{i}) \in k\text{-CLIF}\ .\ \forall A \in k\text{-ALGE}\ .$

$.\ \forall T : \texttt{CliffordMap}(k)(V, A)\ .\ \exists! f : C \xrightarrow{k\text{-ALGE}} A : \mathbf{i}f = T$

$\texttt{Proof}\ =$

$\texttt{Assume}\ g : C \xrightarrow{k\text{-ALGE}} A,$

$\texttt{Assume}\ [1] : \mathbf{i}g = T,$

$\texttt{Assume}\ y : C,$

$\Big(n, m, x, [2]\Big) := \mathcal{I}k\text{-CLIF}(V, C, \mathbf{i}) : \sum n \in \mathbb{N}\ .\ \sum m : n \to \mathbb{N}\ .\ \sum x : \prod_{i=1}^{n} V^{m_i}\ .\ y = \sum_{i=1}^{n}\prod_{j=1}^{m_i}\mathbf{i}(x_{i,j}),$

$[y.*] := [2]\mathcal{I}k\text{-ALGE}(g)[1]\mathcal{I}k\text{-CLIF}\mathcal{I}k\text{-ALGE}(f)[1] :$

$$: g(y) = g\left(\sum_{i=1}^{n}\prod_{j=1}^{m_i}\mathbf{i}(x_{i,j})\right) = \sum_{i=1}^{n}\prod_{j=1}^{m_i}\mathbf{i}g(x_{i,j}) = \sum_{i=1}^{n}\prod_{j=1}^{m_i}T(x_{i,j}) = \sum_{i=1}^{n}\prod_{j=1}^{m_i}\mathbf{i}f(x_{i,j}) = f\left(\sum_{i=1}^{n}\prod_{j=1}^{m_i}\mathbf{i}(x_{i,j})\right) = f(y);$$

$\leadsto [g.*] := I(=, \to) : f = g;$

$\leadsto [*] := \mathcal{I}^{-1}\texttt{Unique} : \texttt{This};$

$\square$

$\texttt{CliffordAlgebraIsUnique} :: \forall V : \texttt{OrthogonalVectorSpace}k \; . \; \forall A, B \in k\text{-}\mathsf{ALGE} \; . \; \forall f : \texttt{CliffordMap}(V, A) \; .$
$\; . \; \forall g : \texttt{CliffordMap}(V, B) \; . \; (V, A, g), (V, B, f) \in k\text{-}\mathsf{CLIF} \Rightarrow (V, A, g) \cong_{k\text{-}\mathsf{CLIF}} (V, B, F)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{algebraOfClifford} :: \prod k : \texttt{Field} \; . \; \texttt{OrthogonalVectorSpace}(k) \to k\text{-}\mathsf{CLIF}$

$$\texttt{algebraOfClifford}\,(V) = \mathrm{CL}(V) := \left(V, \frac{V^{\otimes}}{I}, \iota_{\otimes}\pi_i\right) \quad \texttt{where} \quad I = \texttt{ideal}\{x \otimes x - \langle x, x \rangle | x \in V\}$$

$\texttt{CliffordAlgebraOfDegenerateSpace} :: \forall k : \texttt{Field} \; . \; \forall V \in k\text{-}\mathsf{VS} \; . \; \mathrm{CL}(V, 0) = V^{\wedge}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{InjectiveCliffordMap} :: \forall k : \texttt{Field} \; . \; \forall (V, C, \mathbf{i}) \in k\text{-}\mathsf{CLIF} \; . \; \mathbf{i} : V \hookrightarrow C$

$\texttt{Proof} \; =$

$U := \ker\langle \circ, \circ \rangle : \texttt{VectorSubspace}(V),$

$(W, [1]) := \texttt{OrthogonalStructure}(V) : \sum W : \texttt{VectorSubspace}(V) \; . \; V = W \bot U,$

$(V, A, \mathbf{i}_W) := \mathcal{CL}(W) : k\text{-}\mathsf{CLIF},$

$(V, U^{\wedge}, \mathbf{i}_U) := \mathcal{CL}(U) : k\text{-}\mathsf{CLIF},$

$\phi := \mathbf{i}_W \otimes 1 + e_A \otimes \mathbf{i}_U : V \xrightarrow{k\text{-}\mathsf{VS}} A \widetilde{\otimes} U^{\wedge},$

$\texttt{Assume} \; v : V,$

$\Big(u, w, [2]\Big) := [1](v) : \sum u \in U \; . \; \sum w \in W \; . \; w + u = v,$

$[*] := [2] \mathcal{I} A \widetilde{\otimes} U^{\wedge} \mathcal{I} k\text{-}\mathsf{VS}(V, A \otimes U^{\wedge}) \mathcal{O} \phi \mathcal{I} \texttt{OrthogonalVectorSpace}k[1][2]\text{-}\mathsf{ALGE} : \phi^2(v) = \phi^2(u + w) = \phi^2(u) + \phi$

$\rightsquigarrow [2] := \mathcal{I}^{-1} \texttt{CliffordMap} : \Big(\phi : \texttt{CliffordMap}(k)(V, A \otimes U^{\wedge})\Big),$

$\Big(f, [3]\Big) := \mathcal{I} \texttt{CliffordAlgebra}(k)(V, C, \mathbf{i}) : \sum f : C \xrightarrow{A} \widetilde{\otimes} U^{\wedge} \; . \; \sqsupset f = \phi,$

$[4] := \mathcal{O} \varphi \texttt{CliffordAlgebraOfDegenerateSpace}(U) : \Big(\phi : C \hookrightarrow A \widetilde{\otimes} U^{\wedge}\Big),$

$[*] := \texttt{InjectiveByComposition}[4] : \texttt{This};$

$\square$

$\texttt{functorOfClifford} :: \prod k : \texttt{Field} \; . \; k\text{-}\mathsf{OVS} \xrightarrow{\mathsf{CAT}} k\text{-}\mathsf{CLIF}$

$\texttt{functorOfClifford}\,(V) = \mathrm{CL}(V) := \mathrm{CL}(V)$

$\texttt{functorOfClifford}\,(V, W, T) = \mathrm{CL}_{V,W}(T) := \Big(T, \mathcal{I} k\text{-}\mathsf{CLIF}\big(\mathrm{CL}(V)\big)(T\mathbf{i}_W)\Big)$

4.2 Natural Involutions

$\texttt{dwgreeInvolution} :: \prod k : \texttt{Field} \,.\, \prod V : \texttt{OrthogonalVectorSpace}k\mathrm{CL}(V) \xrightarrow{k\text{-CLIF}} \mathrm{CL}(V)$

$\texttt{degreeInvolution}\,() = \omega_V := \mathrm{CL}_{V,V}(-\mathrm{id})$

$\texttt{DegreeInvolutionIsInvolution} :: \forall k : \texttt{Field} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,.\, \omega_V^2 = \mathrm{id}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{partZero} :: \prod k : \texttt{Field} \,.\, \prod V : \texttt{OrthogonalVectorSpace}k \,.\, \texttt{VectorSubspace}(\mathrm{CL}(V))$

$\texttt{partZero}\,() = \mathrm{CL}_0(V) := \ker(\omega_V - \mathrm{id})$

$\texttt{partOne} :: \prod k : \texttt{Field} \,.\, \prod V : \texttt{OrthogonalVectorSpace}k \,.\, \texttt{VectorSubspace}(\mathrm{CL}(V))$

$\texttt{partOne}\,() = \mathrm{CL}_1(V) := \ker(\omega_V + \mathrm{id})$

$\texttt{InvolutionaryDecomposition} :: \forall k : \texttt{Type}NonBinary \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,.\, \mathrm{CL}(V) = \mathrm{CL}_0(V) \oplus$

$\texttt{Proof} \ =$

$\texttt{Assume}\ y : \mathrm{Im}(\omega_V - \mathrm{id}),$

$\big(x, [2]\big) := \mathcal{I}\,\texttt{image} : \sum x \in V \,.\, y = (\omega_V - \mathrm{id})x,$

$[y.*] := [2]\mathcal{I}\omega_V[2] : (\omega_V - \mathrm{id})y = (\omega_V - \mathrm{id})^2 x = 2(\mathrm{id} - \omega)x = -2y;$

$\rightsquigarrow [2] := \mathcal{I}\ker : \ker(\omega_V - \mathrm{id}) \cap \mathrm{Im}(\omega_V - \mathrm{id}) = \{0\},$

$[3] := \texttt{DegreeInvolutionIsInvolution}(V) : (\mathrm{id} + \omega_V)(\mathrm{id} - \omega_V) = \mathrm{id} - \mathrm{id} = 0,$

$[4] := \mathcal{I}^{-1}\ker \mathcal{I}^{-1}\mathrm{Im}[3] : \mathrm{Im}(\omega_V + \mathrm{id}) \subset \ker(\omega_V - \mathrm{id}),$

$[5] := [4][2] : \mathrm{Im}(\omega V + \mathrm{id}) \cap \mathrm{Im}(\omega V - \mathrm{id}),$

$\texttt{Assume}\ x : \ker(\omega_V - \mathrm{id}),$

$[x.*] := \mathcal{I}k\text{-VS}(\mathrm{CL}(x))\mathcal{I}x\mathcal{I}k\text{-VS}(\omega_V + \mathrm{id}) : x = \frac{1}{2}(\omega_V + \mathrm{id})x - \frac{1}{2}(\omega_V - \mathrm{id})x = \frac{1}{2}(\omega_V + \mathrm{id})x = (\omega_V + \mathrm{id})\left(\frac{1}{2}x\right)$

$\rightsquigarrow [6] := I(\forall)\mathcal{I}^{-1}\texttt{image}\mathcal{I}^{-1}\texttt{Subset} : \mathrm{Im}(\omega_V + \mathrm{id}) = \ker(\omega_V - \mathrm{id}),$

$\texttt{Assume}\ x : \ker(\omega_V + \mathrm{id}),$

$$[x.*] := \mathcal{I}k\text{-VS}(\mathrm{CL}(x))\mathcal{I}x\mathcal{I}k\text{-VS}(\omega_V + \mathrm{id}) : x = \frac{1}{2}(\omega_V + \mathrm{id})x - \frac{1}{2}(\omega_V - \mathrm{id})x = \frac{1}{2}(\omega_V + \mathrm{id})x = \\ = (\omega_V + \mathrm{id})\left(-\frac{1}{2}x\right);$$

$\rightsquigarrow [7] := 2[4] - x : \mathrm{Im}(\omega_V - \mathrm{id}) = \ker(\omega_V - \mathrm{id}),$

$[*] := \mathcal{I}\texttt{DirectSum}[5][6][7] : \mathrm{CL}(V) = \mathrm{CL}_0(V) \oplus \mathrm{CL}_1(V);$

$\square$

$\texttt{ZeroPartProduct} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \mathrm{CL}_0(V)\mathrm{CL}_0(V) \subset \mathrm{CL}_0(V)$

$\texttt{Proof} =$

$\texttt{Assume}\ x, y : \mathrm{CL}_0(V),$

$[1] := \mathcal{I}\mathrm{CL}_0(V)\mathcal{I}(x, y) : x, y \in \ker(\omega_V - \mathrm{id}),$

$[2] := \mathcal{I}k\text{-}\mathsf{VS}\Big(\mathrm{CL}(V)\Big)\mathcal{I}k\text{-}\mathsf{ALGE}(\mathrm{CL}(V))[1] :$

$: (\omega_V - \mathrm{id})(xy) = \omega_V(xy) - xy = \omega_V(x)\omega_V(y) - x(\omega_V(y) - y) + (\omega_V(x) - x)y - xy = \omega_V(x)\omega_V(y) - x\omega_V(y)$

$[3] := \mathcal{I} \ker[2] : xy \in \ker(\omega_V - \mathrm{id}),$

$\Big[(x, y). *\Big] := \mathcal{I}\mathrm{CL}_0(V)[3] : xy \in \mathrm{CL}_0(V);$

$\rightsquigarrow [*] := I(\forall)I\texttt{Subset} : \mathrm{CL}_0(V)\,\mathrm{CL}_0(V) \subset \mathrm{CL}_0(V);$

$\square$

$\texttt{ZeroPartOnePartProduct} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \mathrm{CL}_1(V)\mathrm{CL}_0(V) \subset \mathrm{CL}_1(V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{OnePartZeroPartProduct} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \mathrm{CL}_0(V)\mathrm{CL}_1(V) \subset \mathrm{CL}_1(V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{OnePartProduct} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \mathrm{CL}_1(V)\mathrm{CL}_1(V) \subset \mathrm{CL}_0(V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{DegreeGradingOfCliffordAlgebra} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$

$. \Big(\mathrm{CL}(V), \mathbf{F}_2, (0 \mapsto \mathrm{CL}_0(V), 1 \mapsto \mathrm{CL}_0(V))\Big) \in k\text{-}\mathsf{ALGE}(\mathbf{F}_2)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ZeroPartStructure} :: \forall k : \texttt{NonBinary} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$

$$. \ \mathrm{CL}_0(V) = \mathrm{span}\left\{ \prod_{i=1}^{2n} v_i \middle| n \in \mathbb{Z}_+, v : 2n \to V \right\}$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{OnePartStructure} :: \forall k : \texttt{NonBinary} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$

$$. \ \mathrm{CL}_0(V) = \mathrm{span}\left\{ \prod_{i=1}^{2n+1} v_i \middle| n \in \mathbb{Z}_+, v : (2n+1) \to V \right\}$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CliffordAlgebraDirectDecomposition} :: \forall k : \texttt{NonBinary} . \forall A, B : \texttt{OrthogonalVectorSpace}(k) .$

$. \ \mathrm{CL}(A \oplus B) \cong_{k\text{-ALGE}(\mathbb{F}_2)} \mathrm{CL}(A) \tilde{\otimes} \mathrm{CL}(B)$

$\texttt{Proof} =$

$\varphi := \mathcal{I}\,\texttt{tensorProduct}\Lambda a \in \mathrm{CL}(A) . \Lambda b \in \mathrm{CL}(B)\, \mathrm{CL}_{A,A\oplus B}(\iota_A)(a)\, \mathrm{CL}_{B,A\oplus B}(b) :$

$: \mathrm{CL}(A) \tilde{\otimes} \mathrm{CL}(B) \xrightarrow{k\text{-VS}} \mathrm{CL}(A \oplus B),$

$\texttt{Assume } a : A,$

$\texttt{Assume } b : B,$

$[1] := \mathcal{I} k\text{-CLIF}\,\texttt{CliffordMapProduct}\,\mathcal{I}\,\texttt{sumInnerProduct} :$

$: \mathrm{CL}_{A,A\oplus B}(\iota_A)(\mathbf{i}_A(a))\, \mathrm{CL}_{B,A\oplus B}(\iota_B)(\mathbf{i}_B(b)) + \mathrm{CL}_{B,A\oplus B}(\iota_B)(\mathbf{i}_B(b))\, \mathrm{CL}_{B,A\oplus A}(\iota_A)(\mathbf{i}_A(a)) =$

$= (a\iota_A \mathbf{i}_{A\oplus B})(b\iota_B \mathbf{i}_{A\oplus B}) + (b\iota_B \mathbf{i}_{A\oplus B})(a\iota_A \mathbf{i}_{A\oplus B}) = 2\langle (a,0), (0,b)\rangle e = 0,$

$\Big[(a,b). *\Big] := \mathcal{I} k\text{-ALGE}\, \mathrm{CL}(A \oplus B)[1] :$

$: \Big(a\mathbf{i}_A\, \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big)\Big(b\mathbf{i}_B\, \mathrm{CL}_{B,A\oplus B}(\iota_B)\Big) = -\Big(b\mathbf{i}_B\, \mathrm{CL}_{B,A\oplus B}(\iota_B)\Big)\Big(a\mathbf{i}_A\, \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big);$

$\rightsquigarrow [1] := I^2(\forall) : \forall a \in A . \forall b \in B .$

$. \ \Big(a\mathbf{i}_A\, \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big)\Big(b\mathbf{i}_B\, \mathrm{CL}_{B,A\oplus B}(\iota_B)\Big) = -\Big(b\mathbf{i}_B\, \mathrm{CL}_{B,A\oplus B}(\iota_B)\Big)\Big(a\mathbf{i}_A\, \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big),$

$\texttt{Assume } n, n', m, m' : \mathbb{Z}_+,$

$\texttt{Assume } a : n \to A,$

$\texttt{Assume } b : m \to B,$

$\texttt{Assume } a' : n' \to A,$

$\texttt{Assume } b' : m' \to B,$

$[\ldots *] := \mathcal{I}\,\texttt{skewTensorProduct}\,\mathcal{O}\varphi\mathcal{I} k\text{-ALGE}(\mathrm{CL}(A), \mathrm{CL}(A\oplus B))\, \mathrm{CL}(\iota_A)$

$\mathcal{I} k\text{-ALGE}(\mathrm{CL}(B), \mathrm{CL}(A\oplus B))\, \mathrm{CL}(\iota_B)[1]\mathcal{O}^{-1}\varphi :$

$$: \varphi\left(\left(\prod_{i=1}^{n} a_i\, \mathbf{i}_A \otimes \prod_{i=1}^{m} b_i\, \mathbf{i}_B\right)\left(\prod_{i=1}^{n'} a'_i\, \mathbf{i}_A \otimes \prod_{i=1}^{m'} b'_i\, \mathbf{i}_B\right)\right) = (-1)^{mn'}\varphi\left(\prod_{i=1}^{n} a_i\, \mathbf{i}_A \prod_{i=1}^{n'} a'_i\, \mathbf{i}_A \otimes \prod_{i=1}^{m} b_i\, \mathbf{i}_B \prod_{i=1}^{m'} b'_i\, \mathbf{i}_B\right) =$$

$$= (-1)^{mn'}\, \mathrm{CL}_{A,A\oplus B}(\iota_A)\left(\prod_{i=1}^{n} a_i\, \mathbf{i}_A \prod_{i=1}^{n'} a'_i\, \mathbf{i}_A\right) \mathrm{CL}_{B,A\oplus B}(\iota_B)\left(\prod_{i=1}^{m} b_i\, \mathbf{i}_B \prod_{i=1}^{m'} b'_i\, \mathbf{i}_B\right) =$$

$$= (-1)^{n'm} \prod_{i=1}^{n} a_i\, \mathbf{i}_A\ \mathrm{CL}_{A,A\oplus B}(\iota_A) \prod_{i=1}^{n'} a'_i\, \mathbf{i}_A\ \mathrm{CL}_{A,A\oplus B}(\iota_A) \prod_{i=1}^{m} b_i\, \mathbf{i}_B\ \mathrm{CL}_{B,A\oplus B}(\iota_B) \prod_{i=1}^{m'} b'_i\, \mathbf{i}_B\ \mathrm{CL}_{B,A\oplus B}(\iota_B) =$$

$$= \prod_{i=1}^{n} a_i\, \mathbf{i}_A\ \mathrm{CL}_{A,A\oplus B}(\iota_A) \prod_{i=1}^{m} b_i\, \mathbf{i}_B\ \mathrm{CL}_{B,A\oplus B}(\iota_B) \prod_{i=1}^{n'} a'_i\, \mathbf{i}_A\ \mathrm{CL}_{A,A\oplus B}(\iota_A) \prod_{i=1}^{m'} b'_i\, \mathbf{i}_B\ \mathrm{CL}_{B,A\oplus B}(\iota_B) =$$

$$= \varphi\left(\prod_{i=1}^{n} a_i \mathbf{i}_A \otimes \prod_{i=1}^{m} b_i \mathbf{i}_B\right) \varphi\left(\prod_{i=1}^{n'} a'_i \mathbf{i}_A \otimes \prod_{i=1}^{m'} b'_i \mathbf{i}_B\right);$$

$\rightsquigarrow [2] := \mathcal{I} k\text{-CLIF} : \Big(\varphi : \mathrm{CL}(A)\tilde{\otimes}\mathrm{CL}(B) \xrightarrow{k\text{-ALGE}(\mathbb{F}_2)} \mathrm{CL}(A \oplus B)\Big),$

$\psi := \mathcal{I} k\text{-ALGE}\Lambda a \in A . \Lambda b \in B . (a\, \mathbf{i}_A) \otimes e_{\mathrm{CL}(B)} + e_{\mathrm{CL}(A)} \otimes (b\, \mathbf{i}_B) : \mathrm{CL}(A \oplus B) \xrightarrow{R\text{-ALGE}(\mathbb{F}_0)} \mathrm{CL}(A)\tilde{\otimes}\mathrm{CL}(B),$

$\texttt{Assume } a : A,$

$\texttt{Assume } b : B,$

$[a. * .1] := \mathcal{O}\psi\mathcal{O}\varphi\mathcal{I}\,\texttt{functorOfClifford}\,\mathcal{I} k\text{-ALGE}(\mathrm{CL}(A\oplus B))\mathcal{I}\,\texttt{directSum} :$

$: (a,b)\, \mathbf{i}_{A\oplus B}\ \psi\ \varphi = \Big((a\, \mathbf{i}_A) \otimes e_B + e_A \otimes (b\, \mathbf{i}_B)\Big)\ \varphi =$

$= \Big(a\, \mathbf{i}_A\, \mathrm{CL}_{A,A\oplus B}(\iota_A)(e_{\mathrm{CL}(B)}\, \mathrm{CL}(B, A\oplus B)(\iota_B)\Big) + \Big(e_{\mathrm{CL}(A)}\, \mathrm{CL}(A, A\oplus B)(\iota_B)(b\, \mathbf{i}_B\, \mathrm{CL}_{B,A\oplus B}(\iota_B)\Big) =$

$= (a\ \iota_A\ \mathbf{i}_{A\oplus B})e_{\mathrm{CL}(A\oplus B} + e_{\mathrm{CL}(A\oplus B)}(b\ \iota_B\ \mathbf{i}_{A\oplus B}) = (a,b)\ \mathbf{i}_{A\oplus B},$

$[a.*.2] := \mathcal{D}\varphi\mathcal{A}\texttt{functorOfClifford}\mathcal{D}\psi :$

$: (a\ \mathbf{i}_A) \otimes e_{\mathrm{CL}(B)}\ \varphi\ \psi = \Big(a\ \mathbf{i}_A\ \ \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big)e_{\mathrm{CL}(A\oplus B)}\ \varphi = (a,0)\ \mathbf{i}_{A\oplus B}\ \varphi = (a\ \mathbf{i}_A) \otimes e_{\mathrm{CL}(B)},$

$[a.*.3] := \mathcal{D}\varphi\mathcal{A}\texttt{functorOfClifford}\mathcal{D}\psi :$

$: e_{\mathrm{CL}(A)} \otimes (b\ \mathbf{i}_B)\ \varphi\ \psi = e_{\mathrm{CL}(A\oplus B)}\Big(b\ \mathbf{i}_A\ \ \mathrm{CL}_{A,A\oplus B}(\iota_A)\Big)\ \varphi = (0,b)\ \mathbf{i}_{A\oplus B}\ \varphi = e_{\mathrm{CL}(A)} \otimes (b\ \mathbf{i}_B);$

$\rightsquigarrow [*] := \mathcal{A}\texttt{Generating}\mathcal{A}^{-1}\texttt{Inverse}\mathcal{A}^{-1}\texttt{Isomorphic} : \texttt{This};$

$\square$

$\texttt{CliffordsFunctorPreservesMonomorphisms} :: \forall k : \texttt{NonBinary} . \forall V, W : \texttt{OrthogonalVectorSpace}k .$

$. \forall T : \texttt{Isometry}(V, W) . T : V \hookrightarrow W \Rightarrow \mathrm{CL}_{V,W}(T) : \mathrm{CL}(V) \hookrightarrow \mathrm{CL}(W)$

$\texttt{Proof} =$

$U := T(V) : \texttt{VectorSubspace},$

$\Big(H, [1]\Big) := \texttt{OrthogonalDecomposition}(W, U) : \sum H \subset_{k\text{-}\mathsf{VS}} W . W = U \perp H,$

$[2] := \texttt{CliffordAlgebraDirectDecomposition}[1] : \mathrm{CL}(W) \cong_{k\text{-}\mathsf{ALGE}} \mathrm{CL}(U)\widetilde{\otimes}\,\mathrm{CL}(H),$

$\varphi := \mathcal{A}\texttt{Isomorphic}[2] : \mathrm{CL}(U)\widetilde{\otimes}\,\mathrm{CL}(H) \xleftrightarrow{k\text{-}\mathsf{ALGE}} \mathrm{CL}(W),$

$[2] := \mathcal{D}\varphi : T(\mathbf{i}_U \otimes e_H)\varphi = \mathrm{CL}_{V,W}(T),$

$[4] := \texttt{InjectiveCompositon}[4] : \Big(\mathrm{CL}_{V,W}(T) : \mathrm{CL}(V) \hookrightarrow \mathrm{CL}(W)\Big);$

$\square$

$\texttt{CliffordsFunctorPreservesEpimorphisms} :: \forall k : \texttt{NonBinary} . \forall V, W : \texttt{OrthogonalVectorSpace}(k) .$

$. \forall T : \texttt{Isometry}(V, W) . T : V \twoheadrightarrow W \Rightarrow \mathrm{CL}_{V,W}(T) : \mathrm{CL}(V) \twoheadrightarrow \mathrm{CL}(W)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{semiconjugation} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}(k) . \mathrm{CL}(V) \xrightarrow{k\text{-}\mathsf{ALGE}(\mathbb{F}_0)} \mathrm{CL}^{\mathrm{op}}(V)$

$\texttt{semiconjugation}\,() = S_V := \mathcal{A}k\text{-}\mathsf{CLIFi}^{\mathrm{CL}^{\mathrm{op}}(V)}$

$\texttt{SemiconjugationIsInvolution} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}k . S_V^2 = \mathrm{id}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SemiconjugationPreservesCliffordMap} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \mathbf{i}_V\ S_V = \mathbf{i}_V$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{SemiconjugationPreservesCommutesWithDegreeInvolutiom} :: \forall k : \texttt{Field} .$

$. \forall V : \texttt{OrthogonalVectorSpace}(k) . \omega_V\ S_V = S_V\ \omega_V$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{conjugation} :: \prod k : \texttt{Field} \, . \, \prod V : \texttt{OrthogonalVectorSpace}(k) \, . \, \mathrm{CL}(V) \xrightarrow{k\text{-}\mathsf{VS}} \mathrm{CL}(V)$

$\texttt{conjugataion}\,(x) = \overline{x} := x\,\omega_V\,S_V$

$\texttt{CliffordMapConjugation} :: \forall k : \texttt{Field} \, . \, \forall V : \texttt{OrthogonalVectorSpace}(k) \, . \, \forall v \in V \, . \, \overline{v \sqsupset_V} = -(v \sqsupset_V)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ProductConjugation} :: \forall k : \texttt{Field} \, . \, \forall V : \texttt{OrthogonalVectorSpace}(k) \, . \, \forall a, b \in \mathrm{CL}(V) \overline{ab} = \overline{b}\,\overline{a}$

$\texttt{Proof} =$

...

$\square$

4.3 Clifford Algebras over Finite-Dimensional Vector Spaces

$\texttt{CliffordAlgebraDimension1} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$
$\quad \dim V = 1 \Rightarrow \dim \mathrm{CL}(V) = 2$

$\texttt{Proof} =$

$\Big(v, [1]\Big) := \mathcal{I} \dim V : \sum v \in V . v \neq 0,$
$A := \mathrm{span}(e, v\ \mathbf{i}) : k\text{-VS},$
$[1] := \mathcal{I}\texttt{CliffordMap}(\mathbf{i})\mathcal{I}A : A \in k\text{-ALGE},$
$[2] := \mathcal{D}A\mathcal{I}\texttt{span}\mathcal{D}\texttt{functorOfClifford} : \dim A = 2,$
$\Big(\varphi, [3]\Big) := \mathcal{I}A\text{-CLIF} : \sum \varphi : \mathrm{CL}(V) \xrightarrow{k\text{-ALGE}} A . \beth_A\varphi = \iota_A,$
$[4] := \mathcal{I}\texttt{Monomorphism}[3](\mathbf{i}_A) : \Big(\varphi : \mathrm{CL}(A) \hookrightarrow A\Big),$
$[5] := \mathcal{D}A\mathcal{I}k\text{-ALGE}(\varphi) : \Big(\varphi : \mathrm{CL}(A) \twoheadrightarrow A\Big),$
$[6] := \mathcal{I}\texttt{Isomorphic}\mathcal{I}\texttt{Isomorphism}[4][5] : A \cong_{k\text{-ALGE}} \mathrm{CL}(A),$
$[7] := [2][6] : \dim \mathrm{CL}(V) = 2;$
$\square$

$\texttt{CliffordAlgebraDimension} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) . \forall n \in \mathbb{N} .$
$\quad \dim V = n \Rightarrow \dim \mathrm{CL}(V) = 2^n$

$\texttt{Proof} =$

$\Big[e, [1]\Big] := \texttt{OrthogonalBasisExists}(V) : \sum e : \texttt{Basis}(n, V) . \forall i, j \in n . i \neq j \Rightarrow \langle e_i, e_j \rangle = 0,$
$U := \Lambda i \in n . \mathrm{span}(e_i) : \sum_{i=1}^{n} \texttt{VectorSubspace}(V),$
$[2] := \texttt{CliffordAlgebraDirectDecomposition}(U) : \mathrm{CL}(V) \cong_{k\text{-ALGE}} \widetilde{\bigotimes}_{i=1}^{n} \mathrm{CL}(U_i),$
$[3] := \mathcal{I}\texttt{Span}\mathcal{I}^{-1} \dim : \forall i \in n . \dim U_i = 1,$
$[4] := \texttt{CliffordAlgebraDimension1}[3] : \forall i \in n . \dim \mathrm{CL}(U_i) = 2,$
$[5] := [2][4] : \dim \mathrm{CL}(V) = 2^n;$
$\square$

$\texttt{CliffordAlgebraBasis} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}k . \forall n \in \mathbb{N} . \forall x : \texttt{Basis}(n, V) .$
$\quad . \left(\prod_{i=1}^{n} v_{i,\alpha_i}\right)_{\alpha : n \to \mathbb{B}} \quad \texttt{where} \quad v : \Lambda i \in n . \Lambda b \in \mathbb{B} . \texttt{if } b == 0 \texttt{ then } x_i \texttt{ else } e$

$\texttt{Proof} =$

$\ldots$
$\square$

$\texttt{alternatingIsomorphism} :: \prod k : \texttt{Numeric} . \prod V : \texttt{OrthogonalVectorSpace}(k)\ \&\ k\text{-FDVS} .$
$\quad . V^\wedge \xleftrightarrow{k\text{-VS}} \mathrm{CL}(V)$

$\texttt{alternatingIsomorphism}() = \xi_V := \mathcal{I}\texttt{alternatingAlgebra}\Lambda n \in \mathbb{N} . \Lambda v : n \to V . \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} x_{\sigma(i)}\ \mathbf{i}_V$

$\texttt{determinantElement} :: \prod k : \texttt{Numeric} \,.\, \prod V : \texttt{OrthogonalVectorSpace}(k) \,\&\, k\text{-FDVS} \,.$

$\quad .\, \prod x : \texttt{OrthogonalBasis}(V) \,.\, \mathrm{CL}(V)$

$$\texttt{determinantElement}\,() = x_\Delta := \prod_{i=1}^{\dim V} x_i \,\mathbf{i}_V$$

$\texttt{determinantScalar} :: \prod k : \texttt{Numeric} \,.\, \prod V : \texttt{OrthogonalVectorSpace}(k) \,\&\, k\text{-FDVS} \,.$

$\quad .\, \prod x : \texttt{OrthogonalBasis}(V) \,.\, \mathrm{CL}(V)$

$$\texttt{determinantScalar}\,() = \Delta(x) := \prod_{i=1}^{\dim V} \langle x_i, x_j \rangle$$

$\texttt{determinantProduct} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k) \,\&\, k\text{-FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, x_\Delta^2 = (-1)^{\frac{n(n-1)}{2}} \Delta(x) e \quad \texttt{where} \quad n = \dim V$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{NondegenerateByDeterminantElement} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,\&\, k\text{-FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) V : \texttt{Nondegenerate}(k) \iff x_\Delta : \texttt{Invertible}\Big(\mathrm{CL}(V)\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{DegenerateDeterminantElement} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,\&\, k\text{-FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, V : \texttt{Degenerate}(k) \Rightarrow x_\Delta^2 = 0$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{DeterminantElementTransposition1} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,\&\, k\text{-FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, \forall v \in V \,.\, x_\Delta(v\,\mathbf{i}_V) = (-1)^{1-\dim V}(v\,\mathbf{i}_V) x_\Delta$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{DeterminantElementTransposition2} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \,\&\, k\text{-FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, \forall a \in \mathrm{CL}(V) \,.\, x_\Delta a = \omega^{(\dim V)-1}(a) x_\Delta$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CenterIsGradedSubalgebra} :: \forall k : \texttt{NonBinary} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \;\&\; k\text{-}\mathsf{FDVS} \,.$

$\quad .\, Z\Big(\mathrm{CL}(V)\Big) \in k\text{-}\mathsf{ALGE}(\mathbb{F}_2)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{NondegenerateByDeterminantElement} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}k \;\&\; k\text{-}\mathsf{FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, \dim V : \texttt{Odd} \iff x_\Delta \in Z\Big(\mathrm{CL}(V)\Big)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{TrivialAnticentre} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate} \;\&\; k\text{-}\mathsf{FDVS} \,.\, \mathrm{AZ}_1\Big(\mathrm{CL}(V)\Big) = \{0\}$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{LinearCentre} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate} \;\&\; k\text{-}\mathsf{FDVS} \,.\, \mathrm{Z}_0\Big(\mathrm{CL}(V)\Big) = ke$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OddDimensionalCentreStructure} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \;\&\; k\text{-}\mathsf{FDVS} \,.$

$\quad .\, \forall x : \texttt{OrthogonalBasis}(V) \,.\, \dim V : \texttt{Odd} \Rightarrow Z\Big(\mathrm{CL}(V)\Big) = ke + kx_\Delta$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OddDimensionalAnticentreStructure} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \;\&\; k\text{-}\mathsf{FDVS} \,.$

$\quad .\, \dim V : \texttt{Odd} \Rightarrow AZ\Big(\mathrm{CL}(V)\Big) = 0$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{EvenDimensionalCentreStructure} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \;\&\; k\text{-}\mathsf{FDVS} \,.$

$\quad .\, \dim V : \texttt{Odd} \Rightarrow Z\Big(\mathrm{CL}(V)\Big) = ke$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{EvenDimensionalAnticentreStructure} :: \forall k : \texttt{Numeric} . \forall V : \texttt{Nondegenerate}(k) \,\&\, k\text{-FDVS} .$

$. \forall x : \texttt{OrthogonalBasis}(V) . \dim V : \texttt{Odd} \Rightarrow AZ\Big(\mathrm{CL}(V)\Big) = kx_\Delta$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{inverseCliffordAlgebra} :: \prod k : \texttt{Field} . \texttt{OrthogonalVectorSpace}(k) \to k\text{-ALGE}$

$\texttt{inverseCliffordAlgebra}\,(V) = \mathrm{CL}(-V) := \mathrm{CL}\Big(\big(V, -\langle\cdot,\cdot\rangle_V\big)\Big)$

$\texttt{inverseDeterminantScalar} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}k \,\&\, k\text{-FDVS} .$

$. \texttt{OrthogonalBasis}(V) \to k$

$\texttt{inverseDeterminantScalar}\,(x) = \Delta^-(x) := (-1)^{\dim V}\Delta(x)$

$\texttt{InverseCliffordAlgebraIsomorphism} :: \forall k : \texttt{Numeric} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$

$. \dim V : \texttt{Even} \Rightarrow \mathrm{CL}(V) \cong_{k\text{-ALGE}(\mathbb{F}_2)} \mathrm{CL}(-V)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{crossDualSpace} :: \prod k : \texttt{Numeric} . k\text{-VS} \to \texttt{OrthogonalVectorSpace}(k)$

$\texttt{crossDualSpace}\,(V) = V^{*,*} := \left(V \oplus V^*, \Lambda(v,f), (w,g) \in V^* . \frac{1}{2}\big(g(v) + f(w)\big)\right)$

$\texttt{CrossDualSpaceIsExteriorOperators} :: \forall k : \texttt{Numeric} . \forall V : k\text{-FDVS} . V^{*,*} \cong_{k\text{-ALGE}} k\text{-VS}(V^\wedge, V^\wedge)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{GeneratorsOfExteriorOperators} :: \forall k : \texttt{Field} . \forall V : k\text{-FDVS} .$

$. k\text{-VS}(V^\wedge, V^\wedge) = \Big\langle \{\rho_v | v \in V\} \,\&\, \{\sigma_v | f \in V^*\}\Big\rangle_{k\text{-ALGE}}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{CliffordExteriorOperatorsIsomorphismCriterion} :: \forall k : \texttt{Field} . \forall V : k\text{-FDVS} .$

$. \forall n \in \mathbb{N} . \forall [0] : \dim V = 2n . \forall \omega : \texttt{involution}(V) . \forall [00] . \omega^\top = -\omega .$

$. \mathrm{CL}(V) \cong_{k\text{-ALGE}} k\text{-VS}\Big(\ker^\wedge(\omega - \mathrm{id}), \ker^\wedge(\omega - \mathrm{id})\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{naturalProjection} :: \prod k \in \texttt{Field} \,.\, \prod V \in \texttt{OrthogonalVectorSpace}(k) \,.\, \mathrm{CL}(V) \xrightarrow{k\text{-VS}} k$

$\texttt{naturalProjection}() = \pi_V := \xi_V^{-1}\pi_0$

$\texttt{naturalCliffordForm} :: \prod k \in \texttt{Field} \,.\, \prod V \in \texttt{OrthogonalVectorSpace}(k) \,.\, \mathcal{L}\Big(\mathrm{CL}(V), \mathrm{CL}(V); k)$

$\texttt{naturalCliffordForm}(a, b) = Q_V(a, b) := \pi_V(ab)$

$\texttt{specialCategoryOfOrthogonalVectorSpaces} :: \texttt{Numperic} \to \mathsf{CAT}$

$\texttt{specialCategoryOfOrthogonalVectorSpaces}(k) = k\text{-}\mathsf{SOVS} :=$

$:= \Bigg(\sum V : \texttt{OrthogonalVectorSpace}(k) \,.\, \texttt{VectorSubspaces}(k), \Lambda(V, A), (W, B) \in k\text{-}\mathsf{SOVS} \,.\,,$

$, \sum f : \texttt{Isometry}(V, W) \,.\, f(V) \subset B, \circ, \mathrm{id}\Bigg)$

$\texttt{forgetfulCliffordFunctor} :: \prod k : \texttt{Numeric} \,.\, k\text{-}\mathsf{CLIF} \xrightarrow{\mathsf{CAT}} k\text{-}\mathsf{SOVS}$

$\texttt{forgetfulCliffordFunctor}(V, A, \mathbf{i}) = U^{k\text{-}\mathsf{CLIF}}(V, A, \mathbf{i}) := (A, Q_V)$

$\texttt{forgetfulCliffordFunctor}((V, A, \mathbf{i}), (W, B, \mathbf{j}), (T, \varphi)) = U^{k\text{-}\mathsf{CLIF}}_{(V,A,\mathbf{i}),(W,B,\mathbf{j})}(T, \varphi) := \varphi\, \xi_W^{-1}\, \pi_1$

$\texttt{Assume } x, a : A,$

$[1] := \text{Д}\xi_V \text{Д}\texttt{Isometry}(T)\text{Д}\varphi : (xa)\, \xi_V^{-1}\, T^\wedge = (xa)\, \varphi\, \xi_W^{-1},$

$\Big[(x, a). *\Big] := \text{Д}Q_V \text{Д}\pi_V \text{Д}\texttt{exteriotMap}(T)[1]\text{Д}^{-1}\pi_W \text{Д}^{-1} Q_W :$

$: Q_V(x, a)(xa)\, \pi_V = 0(xa)\, \xi_V^{-1}\pi_0 = (xa)\, \xi_V^{-1}\, T^\wedge \pi_0 = (xa)\, \varphi\, \xi_W^{-1}\pi_0 = \varphi(x)\varphi(a)\, \pi_W = Q_W\Big(\varphi(x), \varphi(a)\Big);$

$\rightsquigarrow [*] := \text{Д}^{-1}\texttt{Isometrty} : \Bigg(\varphi : \texttt{Isometry}\Big((A, Q_V), (B, Q_W)\Big)\Bigg);$

$\square$

$\mathtt{CliffordAdjoint} :: \forall k : \mathtt{Numeric} \,.\, \left(\mathrm{CL}, U^{k\text{-}\mathsf{CLIF}}\right) : \mathtt{Adjoint}(\mathtt{OrthogonalVectorSpace}(k), k\text{-}\mathsf{CLIF})$

$\mathtt{Proof} \; =$

$\mathtt{Assume}\ V : \mathtt{OrthogonalVectorSpace}(k),$

$\mathtt{Assume}\ \Big(W, A, \mathbf{i}\Big) : k\text{-}\mathsf{CLIF},$

$\mathtt{Assume}\ (T, \varphi) : \mathrm{CL}(V) \xrightarrow{k\text{-}\mathsf{CLIF}} (W, A, \mathbf{i}),$

$F(T, \varphi) := T\mathbf{i} : V \xrightarrow{k\text{-}\mathsf{VS}} A,$

$\mathtt{Assume}\ v, v' : V,$

$\Big[(v, v').\, *\Big] := \supset F(T, \varphi)\mathtt{VectorElementNaturalCliffordMap}\mathit{\Lambda}\mathtt{Isometry}(V, W)(T) : Q_W\Big(v\ F(T, \varphi), v'\ F(T, \varphi)$

$\rightsquigarrow \Big[(T, \varphi).\, *\Big] := \mathit{\Lambda}^{-1}\mathtt{Isometry} : F(T, \varphi) : \mathtt{Isometry}\Big(V, (A, Q_W)\Big);$

$\rightsquigarrow F := I(\rightarrow) : k\text{-}\mathsf{CLIF}\Big(\mathrm{CL}(V), (A, Q_W)\Big) \to \mathtt{Isometry}\Big(V, (A, Q_W)\Big),$

$\mathtt{Assume}\ T : \mathtt{Isometry}\Big(V, (A, Q_W)\Big),$

$S := T\xi_W^{-1}\pi_1 : V \xrightarrow{k\text{-}\mathsf{VS}} W,$

$\mathtt{Assume}\ x, y : V,$

$$:=: \langle Sx, Sy\rangle_W = \langle x\ T\ \xi_W^{-1}\ \pi_1, y\ T\ \xi_W^{-1}\ \pi_1\rangle_W = Q_V\Big(x\ T\ \xi_W^{-1}\ \pi_1\ \mathbf{i}_W, y\ T\ \xi_W^{-1}\ \pi_1\ \mathbf{i}_W\Big) =$$

$$= Q_V\Big(x\ T\ \xi_W^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1, y\ T\ \xi_W^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1\Big) = \bigg(\Big(x\ T\xi_W^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1\Big)\Big(y\ T\xi_W^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1\Big)\bigg)\pi_W =$$

$$= \bigg(\Big(x\ T\xi_V^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1\Big)\Big(y\ T\xi_V^{-1}\ \mathbf{i}_W^{\wedge}\ \pi_1\Big)\bigg)\xi_W^{-1}\pi_0 = \Big((w_i\mathbf{i}_W)(w'_j\mathbf{i}_W)\Big)\xi_W^{-1}\pi_0 = \langle e_k, e_k\rangle w_{i,k}w'_{j,k} = \langle x, y\rangle;$$

$\ldots$

$\square$

4.4 Towards Low-Dimensional Classification

$\texttt{AllFDComplexCliffordAlgebrasAreIsomorphic} :: \forall A, B \in \mathbb{C}\text{-CLIF} . \forall [0] : \dim A < \infty .$
$. \forall [00] : \dim A = \dim B . A \cong_{\mathbb{C}\text{-CLIF}} B$

$\texttt{Proof} =$

...

$\square$

$\texttt{signatureCliffordAlgebra} :: (\mathbb{Z}_+ \times \mathbb{Z}_+) \to \mathbb{R}\text{-CLIF}$

$\texttt{signatureCliffordAlgebra}\,(p, q) = \mathrm{CL}(p, q) := \mathrm{CL}\left(\mathbb{R}^p \oplus \mathbb{R}^q, Q_e(I) \oplus Q_e(-I)\right)$

$\texttt{positiveCliffordAlgebra} :: \mathbb{Z}_+ \to \mathbb{R}\text{-CLIF}$

$\texttt{positiveCliffordAlgebra}\,(n) = \mathrm{CL}_n(+) := \mathrm{CL}(n, 0)$

$\texttt{negativeCliffordAlgebra} :: \mathbb{Z}_+ \to \mathbb{R}\text{-CLIF}$

$\texttt{negativeCliffordAlgebra}\,(n) = \mathrm{CL}_n(-) := \mathrm{CL}(0, n)$

$\texttt{PositiveDoubleStepTheorem} :: \forall p, q \in \mathbb{Z}_+ . \ \mathrm{CL}(p, q) \otimes \mathrm{CL}_2(+) \cong_{\mathbb{R}\text{-CLIF}} \mathrm{CL}(p + 2, q)$

$\texttt{Proof} =$

...

$\square$

$\texttt{PositiveDoubleStepTheorem} :: \forall p, q \in \mathbb{Z}_+ . \ \mathrm{CL}(p, q) \otimes \mathrm{CL}_2(-) \cong_{\mathbb{R}\text{-CLIF}} \mathrm{CL}(p, q + 2)$

$\texttt{Proof} =$

...

$\square$

$\texttt{QuarticEquivalence} :: \forall p, q \in \mathbb{Z}_+ . p - q =_{Z_4} 0 \Rightarrow \mathrm{CL}(p, q) \cong_{\mathbb{R}\text{-CLIF}} \mathrm{CL}(q, p)$

$\texttt{Proof} =$

...

$\square$

$\texttt{ZeroSignatureStructure} :: \forall p \in \mathbb{Z}_+ . \ \mathrm{CL}(p, p) \cong_{\mathbb{R}\text{-ALGE}} \mathbb{R}\text{-VS}\left(\mathbb{R}^{p\wedge}, \mathbb{R}^{p\wedge}\right)$

$\texttt{Proof} =$

...

$\square$

$\texttt{quaternionicIsomorphism4} :: \mathbb{H} \otimes \mathbb{H} \xrightarrow{\mathbb{R}\text{-VS}} \mathbb{R}\text{-VS}\left(\mathbb{H}, \mathbb{H}\right)$

$\texttt{quaternionicIsomorphism}_4\,() = \Lambda t \in \mathbb{H} \otimes \mathbb{H} . T_t := \mathit{Ɑ}\,\texttt{tensorProduct}\,\Lambda a, b, x \in \mathbb{H} . ax\overline{b}$

$\texttt{QuaternionicIsomorphis} :: T : \mathrm{CL}_4(-) \xleftrightarrow{\mathbb{R}\text{-ALGE}} \mathbb{R}\text{-VS}(\mathbb{R}^4, \mathbb{R}^4)$

$\texttt{Proof} =$

...

$\square$

4.5 Representation of Clifford Algebras

$\texttt{NondegenerateRepresentationIsInjective} :: \forall k : \texttt{Field} . \forall V : \texttt{Nondegenerate}(k) .$

$. \forall W \in k\text{-FDVS} . \forall\Big(\mathrm{CL}(V), W, \rho\Big) : \mathsf{AR}(k) . \rho : \mathrm{CL}(V) \hookrightarrow \mathcal{L}(W; W)$

Proof $=$

$\dots$

$\square$

$\texttt{Orthogonal} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}(k) . \prod W : \texttt{InnerProductSpace}(k) .$

$. ?\texttt{Representation}\Big(\mathrm{CL}(V), W\Big)$

$\rho : \texttt{Orthogonal} \iff \exists \sigma \in \{-1, +1\} : \forall x \in V . \forall a, b \in W . \Big\langle \rho(x)a, \rho(x)b \Big\rangle = \sigma\langle x, x\rangle\langle a, b\rangle$

$\texttt{signOfOrthogonal} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}(k) . \prod W : \texttt{InnerProductSpace}(k) .$

$\texttt{Orthogonal}(V, W) \to \{-1, +1\}$

$\texttt{signOfOrthogonal}\,(\rho) = \sigma(\rho) := \mathcal{I}\texttt{Orthogonal}$

$\texttt{PositiveOrthogonal} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}(k) . \prod W : \texttt{InnerProductSpace}(k) .$

$. ?\texttt{Orthogonal}(V, W)$

$\rho : \texttt{PositiveOrthogonal} \iff \sigma(\rho) = 1$

$\texttt{NegativeOrthogonal} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace}(k) . \prod W : \texttt{InnerProductSpace}(k) .$

$. ?\texttt{Orthogonal}(V, W)$

$\rho : \texttt{NegativeOrthogonal} \iff \sigma(\rho) = -1$

$\texttt{PositivelyOrhognallyRepresentedAreSymmetric} :: \forall k : \texttt{Field} . \forall V : \texttt{NonDegenerate}(k) .$

$. \forall W \in \texttt{InnerProductSpace}(k) . \forall \rho : \texttt{PositiveOrthogonal}(V, W) . \forall x \in V . \rho(x) : \texttt{Symmetric}(W)$

Proof $=$

Assume $v, w : W$,

$\Big[(v, w).*.1\Big] := \mathcal{I}\texttt{PositiveOrthogonal}(V, W)(\rho)(v, w)\mathcal{I}\texttt{AdjointOperator} :$

$: \langle x, x\rangle\langle v, w\rangle = \langle v\ \rho(x), w\ \rho(x)\rangle = \langle v\ \rho(x)\ \rho^*(x), w\rangle,$

$\Big[(v, w).*.2\Big] :=$

$: \mathcal{I}k\text{-ALGE}\Big(\mathrm{CL}(V), k\text{-VS}(W, W)\Big)\mathcal{I}k\text{-CLIF}\Big(\mathrm{CL}(V)\Big)\mathcal{I}k\text{-ALGE}\Big(\mathrm{CL}(V), k\text{-VS}(W, W)\Big) :$

$: \langle v\ \rho^2(x), w\rangle = \langle v\ \rho(x^2), w\rangle = \Big\langle v\ \langle x, x\rangle\rho(e), w\Big\rangle = \langle x, x\rangle\langle v, w\rangle,$

$\rightsquigarrow \Big[1\Big] := \texttt{NonDegenerateDefines} : \rho(x)\ \rho^*(x) = \rho^2(x),$

$[*] := \mathcal{I}^{-1}\texttt{Symmetric}[1] : \Big(\rho(x) : \texttt{Symmetric}(W)\Big);$

$\square$

$\texttt{NegativelyOrhognallyRepresentedAreSkew} :: \forall k : \texttt{Field} . \forall V : \texttt{OrthogonalVectorSpace}(k) .$
$. \forall W \in \texttt{InnerProductSpace}(k) . \forall \rho : \texttt{NegativeOrthogonal}(V, W) . \forall x \in \mathrm{CL}(V) . \rho(x) : \texttt{Skew}$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{PositiveRepresentationClassification} :: \forall n \in \mathbb{N} . \forall V : \mathbb{R}\text{-}\mathsf{VS} . \forall \rho : \texttt{Representation}(\mathbb{R}, \mathrm{CL}_n(+), V) .$
$. \exists W : \texttt{InnerProductSpace}(\mathbb{R}) : \exists \rho' : \texttt{PositiveOrthogonal}(\mathbb{R}^n, W) . \rho \sim \rho'$
$\texttt{Proof} =$
$[1] := \mathcal{I} e \mathcal{I}\, \mathrm{CL}_n(+) : \forall i, j \in n . (e_i\ \mathbf{i})(e_j\ \mathbf{i}) + (e_j\ \mathbf{i})(e_i\ \mathbf{i}) = 2\delta_{i,j} e,$
$[2] := \mathcal{I}\texttt{Automorphism}[1] : \forall i \in n . \rho(e_i) \in \mathrm{Aut}_{\mathbb{R}\text{-}\mathsf{VS}}(V),$
$G := \Big\langle \rho(e_i) \Big\rangle_{\mathrm{Aut}_{\mathbb{R}\text{-}\mathsf{VS}}(V)} : \texttt{Subgroup}\Big(\mathrm{Aut}_{\mathbb{R}\text{-}\mathsf{VS}}(V)\Big),$
$[3] := \mathcal{I}^{-1}\texttt{FiniteGroup}[1] : \Big(G : \texttt{FiniteGroup}\Big),$
$Q := \lambda v, w \in V . \sum_{g \in G} \langle v\ \rho(g), w\ \rho(g) \rangle : \texttt{SymmetricForm}(V),$
$W := (V, Q) : \texttt{OrthogonalVectorSpace}(\mathbb{R}),$
$\texttt{Assume}\ v, w : W,$
$\texttt{Assume}\ g : G,$
$\Big[(v, w). *\Big] := \mathcal{D} W \mathcal{I} \mathbb{R}\text{-}\mathsf{ALGE}\Big(\mathrm{CL}_n(+), \mathbb{R}\text{-}\mathsf{VS}(V, V)\Big)(\rho)\texttt{GroupCycle}(G)(g)\mathcal{D}^{-1} W :$
$\Big\langle v\ \rho(g), w\ \rho(g) \Big\rangle_W = \sum_{f \in G} \Big\langle v; \rho(g)\rho(f), w\ \rho(g)\rho(f) \Big\rangle_V = \sum_{f \in G} \Big\langle v; \rho(gf), w\ \rho(gf) \Big\rangle_V =$
$= \sum_{f \in G} \Big\langle v; \rho(f), w\ \rho(f) \Big\rangle_V = \langle v, w \rangle_W;$
$\rightsquigarrow [4] := I(\forall) : \forall v, w \in V . \forall g \in G . \langle v\ \rho(g), w\ \rho(g) \rangle_W = \langle v, w \rangle_W,$
$[5] := [4]\mathcal{I}^{-1}\texttt{PositiveOrthogonal} : \Big(\rho : \texttt{PositiveOrthogonal}(\mathbb{R}^n, W)\Big),$
$[*] := \mathcal{I}\texttt{Reflexivity}(\texttt{EquivalentAlgRepr})(\rho) : \rho \sim \rho;$
$\square$

$\texttt{NegativeRepresentationClassification} :: \forall n \in \mathbb{N} . \forall V : \texttt{InnerProductSpace}(\mathbb{R}) .$
$. \forall \rho : \texttt{Representation}(\mathbb{R}, \mathrm{CL}_n(-), V) . \exists W : \texttt{InnerProductSpace}(\mathbb{R}) :$
$: \exists \rho' : \texttt{NegativeOrthogonal}(\mathbb{R}^n, W) . \rho \sim \rho'$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{twistedAdjointRepresentation} :: \prod k : \texttt{Field} . \prod V : \texttt{OrthogonalVectorSpace} k .$
$. \texttt{Representation}\Big(\mathrm{CL}^*(V), \mathrm{CL}(V)\Big)$
$\texttt{twistedAdjointRepresentation}\,(x) = \widetilde{\mathrm{ad}}\, x := \Lambda a \in \mathrm{CL}(V) . \omega_V(x) a x^{-1}$

$\texttt{TwistedAdjointRepresentationKernel} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{NonDegenerate}(k) \,.$

$\quad .\ \ker \widetilde{\mathrm{ad}}_V = ke_{\mathrm{CL}(V)}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \lambda : k,$

$\texttt{Assume}\ a : \mathrm{CL}(V),$

$[a.*] := \text{\unicode{x0421}}\ \widetilde{\mathrm{ad}}(\lambda e) \text{\unicode{x0421}}\, k\text{-}\mathsf{ALGE}\texttt{UnitityInverse} \text{\unicode{x0421}}\, e_{\mathrm{CL}(V)} \text{\unicode{x0421}}\, \texttt{inverse} :$

$\quad : \widetilde{\mathrm{ad}}(\lambda e)a = \omega_V(\lambda e)a(\lambda e)^{-1}\lambda ea(\lambda^{-1}e) = \lambda\lambda^{-1}a = a;$

$\rightsquigarrow [1] := I(\forall) : \forall a \in \mathrm{CL}(V) \,.\, \widetilde{\mathrm{ad}}(\lambda e)a = a,$

$[\lambda.*] := \texttt{UniqueIdentity}[1] : \widetilde{\mathrm{ad}}(\lambda e) = \mathrm{id};$

$\rightsquigarrow [1] := \text{\unicode{x0421}} \ker \text{\unicode{x0421}}\, \texttt{Subset} : ke \subset \ker \widetilde{\mathrm{ad}}_V,$

$\square$

4.6 Clifford Group

$\texttt{groupOfClifford} :: \prod k : \texttt{Field} \,.\, \texttt{OrthogonalVectorSpace}(k) \to \mathsf{GRP}$

$\texttt{groupOfClifford}\,(V) = \Gamma(V) := \mathrm{Stab}\Big(\mathrm{CL}^*(V), V\,\mathbf{i}\Big)\Big(\widetilde{\mathrm{ad}}\Big)$

$\texttt{NondegenerateVectorsInCliffordGroup} :: \forall k : \texttt{Field} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k) \,.\, \forall v \in V \,.$
$\quad .\, \forall [0] : \langle v, v\rangle \neq 0 \,.\, v\,\mathbf{i}_V \in \Gamma(V)$

$\texttt{Proof} \;=$

$[1] := \mathcal{I}\, k\text{-}\mathsf{CLIF}\Big(\mathrm{CL}(V)\Big)\mathcal{I}^{-1}\texttt{inverse} : (v\,\mathbf{i}_V)^{-1} = \dfrac{v\,\mathbf{i}_V}{\langle v, v\rangle},$

$\texttt{Assume}\; w : V,$

$[2] := \mathcal{I}\,\widetilde{\mathrm{ad}}[1]\mathcal{I}\, k\text{-}\mathsf{CLIF}\Big(\mathrm{CL}(V)\Big) :$

$$: \widetilde{\mathrm{ad}}(v\,\mathbf{i}_V)(w\,\mathbf{i}_V) = \omega_V(v\,\mathbf{i}_V)(w\,\mathbf{i}_V)(v\,\mathbf{i}_V) = -(v\,\mathbf{i}_V)(w\,\mathbf{i}_V)\frac{v\,\mathbf{i}_V}{\langle v, v\rangle} = \frac{1}{\langle v, v\rangle}(w\,\mathbf{i}_V)(v\,\mathbf{i}_V)^2 + 2\frac{\langle w, v\rangle}{\langle v, v\rangle}(v\,\mathbf{i}_V) =$$
$$= (w\,\mathbf{i}_V) + 2\frac{\langle w, v\rangle}{\langle v, v\rangle}(v\,\mathbf{i}_V) \in \mathbf{i}(V),$$

$\rightsquigarrow [*] := \mathcal{I}\Gamma(V) : v\,\mathbf{i} \in \Gamma(V);$

$\square$

$\texttt{CliffordGroupDegreeInvolution} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, \forall x \in \Gamma(V) \,.$
$\quad .\, \omega_V(x) \in \Gamma(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : V,$

$[*] := \mathcal{I}\,\widetilde{\mathrm{ad}}\,\mathcal{I}\omega_V\mathcal{I}\, k\text{-}\mathsf{ALGE}\Big(\mathrm{CL}(V), \mathrm{CL}(V)\Big)(\omega_V)\mathcal{I}\Gamma(V)(x)\mathcal{I}\omega_V\mathcal{I}\Gamma(V)(x) :$

$$: \widetilde{\mathrm{ad}}\Big(x\,\omega_V\Big)(v\,\mathbf{i}) = (x\,\omega_V^2)(v\,\mathbf{i})(x\,\omega_V)^{-1} = -(x\,\omega_V)(v\,\mathbf{i})(x^{-1})\,\omega_V = (x\,\omega_V)(v\,\mathbf{i})(x^{-1}) \in V\,\mathbf{i};$$

$\rightsquigarrow [*] := \mathcal{I}\Gamma(V) : x \in \Gamma(V);$

$\square$

$\texttt{CliffordGroupSemiconjugation} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, \forall x \in \Gamma(V) \,.$
$\quad .\, S_V(x) \in \Gamma(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : V,$

$[*] := \mathcal{I}\,\widetilde{\mathrm{ad}}\,\mathcal{I} S_V \texttt{SemiconjugationPreseresCliffordMap}(v)$
$\quad \texttt{SemiconjugationCommutedWithDegreeInvolutin}\mathcal{I}^{-1}\,\widetilde{\mathrm{ad}}\,\mathcal{I}\Gamma(V)(x\omega(V))^{-1} :$

$$: \widetilde{\mathrm{ad}}\Big(x\,S_V\Big)(v\,\mathbf{i}) = (x\,\omega_V S_V)(v\,\mathbf{i})(x\,S_V)^{-1} = (x^{-1})(v\,\mathbf{i})(x\,\omega_V)\,S_V = \widetilde{\mathrm{ad}}\Big(x\omega_V\Big)^{-1}(v) \in V\,\mathbf{i};$$

$\rightsquigarrow [*] := \mathcal{I}\Gamma(V) : x \in \Gamma(V);$

$\square$

$\texttt{CliffordGroupConjugation} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, \forall x \in \Gamma(V) \,.$
$\quad .\, \overline{x} \in \Gamma(V)$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{CliffordGroupConjugateSquare} :: \forall k : \texttt{Numeric} . \forall V : \texttt{Nondegenerate}(k) .$

$. \exists \lambda : \Gamma(V) \xrightarrow{\mathsf{GRP}} k^* : \forall x \in \Gamma(V) . x\overline{x} = \lambda(x)e$

$\texttt{Proof} =$

$\texttt{Assume } x : \Gamma(V),$

$\texttt{Assume } v : V,$

$w := \widetilde{\mathrm{ad}}(\overline{x})(v\mathbf{i}) : V,$

$[1] := \texttt{SemiconjugationPreservesCliffordMap}(w) : S_V(w\mathbf{i}) = w\mathbf{i},$

$[2] := Ɑ^{-1}\,\widetilde{\mathrm{ad}}\,Ɔ w Ɑ S_V[1] : \omega(\overline{x})(v\mathbf{i})\overline{x}^{-1} = \widetilde{\mathrm{ad}}(\overline{x})(v\mathbf{i}) = w = w\ S_V = (\overline{x}\ S_V)^{-1}(v\mathbf{i})(\overline{x}\ \omega_V\ S_V),$

$[3] := Ɑ k\text{-}\mathsf{ALGE}\omega_V\Big(\mathrm{CL}(V), \mathrm{CL}(V)\Big)Ɑ\texttt{conjugation}(\overline{x}\ S_V)[2]\overline{x}Ɑ^{-1}\texttt{conjugation} :$

$: \omega(x\overline{x})(v\mathbf{i})(\overline{x}S_V)\omega(\overline{(x)})(v\mathbf{i}) = (v\mathbf{i})(x\ \omega_V\ S_V)\overline{x} = (v\mathbf{i})x\overline{x},$

$[v.*] := Ɑ\,\widetilde{\mathrm{ad}}[3] : \widetilde{\mathrm{ad}}\left(x\overline{x}\right)(v\ \mathbf{i}_V) = \omega(x\overline{x})(v\ \mathbf{i}_V)(x\overline{x})^{-1} = (v\ \mathbf{i}_V)x\overline{x}(x\overline{x})^{-1} = (v\ \mathbf{i}_V);$

$\rightsquigarrow [1] := I(=, \to) : \widetilde{\mathrm{ad}}(x\overline{x})_{|V\mathbf{i}} = \mathrm{id},$

$[2] := [1]Ɑ\,\widetilde{\mathrm{ad}}\,Ɑ^{-1}Z_0\ \mathrm{CL}(V) : (x\overline{x})_0 \in Z_0\ \mathrm{CL}(V)\ \&\ (x\overline{x}_1)_1 \in AZ_1\ \mathrm{CL}(V),$

$[3] := \texttt{TrivialAnicentre}\ \&\ \texttt{LinearCentre}[2] : (x\overline{x})_0 \in ke\ \&\ (x\overline{x})_1 = 0,$

$\Big(\lambda(x), [1]\Big) := Ɑ ke[3] : \sum \lambda(x) \in k . x\overline{x} = \lambda(x)e,$

$[x.*] := Ɑ\mathsf{GRP}\Gamma(V)(x)Ɑ k\text{-}\mathsf{ALGE}\Big(\mathrm{CL}(V)\Big) : \lambda(x) \in k^*;$

$\rightsquigarrow \lambda := I\left(\sum\right) I\left(\sum\right) : \prod x \in \Gamma(V) . \sum \lambda(x) \in k^* . x\overline{x} = \lambda e,$

$\texttt{Assume } x, y : \Gamma(V),$

$[1] := Ɑ_1^{-1}\lambda(xy)\texttt{ConjugationAntihomo}(xy)Ɑ_1\lambda(y)Ɑ k\text{-}\mathsf{ALGE}\,\mathrm{CL}(V)Ɑ_1\lambda(x)Ɑ\mathsf{ANN}(k) :$

$: \lambda(xy)e = xy\overline{xy} = x\ y\ \overline{y}\ \overline{x} = x\lambda(y)e\overline{x} = \lambda(y)x\overline{x} = \lambda(y)\lambda(x)e = \lambda(x)\lambda(y)e,$

$\Big[(x, y). *\Big] := Ɑ\texttt{Field}Ɑ k\text{-}\mathsf{ALGE}\,\mathrm{CL}(V) : \lambda(xy) = \lambda(x)\lambda(y);$

$\rightsquigarrow [*] := Ɑ\mathsf{GRP} : \lambda : \Gamma(V) \xrightarrow{k\text{-}\mathsf{VS}} k^*;$

$\square$

$\texttt{conjugationSquare} :: \prod k : \texttt{Numeric}(V) . \prod V : \texttt{NonDegenerate}(k) . \Gamma(V) \xrightarrow{\mathsf{GRP}} k^*$

$\texttt{conjugationSquareMap}\,(x) = \lambda_V(x) := \texttt{CliffordGroupConjugationSquare}$

$\texttt{degreeinvolutionpreservesconjugatesquare} :: \forall k : \texttt{Type}numeric . \forall v : \texttt{Type}nondegenerate(k) .$

$. \forall x \in \Gamma(v) . \omega_v(x)\overline{\omega_v(x)} = x\overline{x}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{DegreeInvolutionPreservesConjugateSquareMap} :: \forall k : \texttt{Numeric} . \forall V : \texttt{Nondegenerate}(k) .$

$. \omega_V\lambda_V = \lambda_V$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{TwistedAdjugationPreservesConjugateSquareMap} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.$

$\quad . \,\forall a \in \Gamma(V) \,.\, \widetilde{\mathrm{ad}}(a)\lambda_V = \lambda_V$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{twistedAdjugationIsoquadric} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.$

$\quad . \,\forall a \in \Gamma(V) \,.\, \widetilde{\mathrm{ad}}(a)_{|V} : \texttt{Isoquadric}(V, V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; v : V,$

$\texttt{Assume}\; [0] : \langle v, v\rangle \neq 0,$

$[1] := \mathcal{I}\,\texttt{conjugation}\mathcal{I}\,k\text{-}\mathsf{CLIF}\;\;\mathrm{CL}(V) : (v\mathbf{i})\overline{v\mathbf{i}} = -(v\mathbf{i})^2 = -\langle v, v\rangle e,$

$[2] := \mathcal{I}^{-1}\lambda_V[1][0] : \lambda(v\mathbf{i}) = -\langle v, v\rangle,$

$[0.*] := [2]\texttt{TwistedAdjugationPreservesConjugateSquareMap}[2] :$

$\quad : -\langle \widetilde{\mathrm{ad}}(a)v, \widetilde{\mathrm{ad}}(a)v\lambda_V\Big(\widetilde{\mathrm{ad}}(a)v\mathbf{i}\Big) = \lambda_V(v\mathbf{i}) = -\langle v, v\rangle;$

$\rightsquigarrow [1] := I(\Rightarrow) : \langle v, v\rangle \neq 0 \Rightarrow \langle \widetilde{\mathrm{ad}}(a)v, \widetilde{\mathrm{ad}}(a)v\rangle = \langle v, v\rangle,$

$\texttt{Assume}\; [0] : \langle v, v\rangle = 0,$

$[0.*] := \mathcal{I}\,\widetilde{\mathrm{ad}}\,a\mathcal{I}\,k\text{-}\mathsf{CLIF}(V)\mathcal{I}\,\mathsf{GRP}\Gamma(V) : \langle \widetilde{\mathrm{ad}}\,av, \widetilde{\mathrm{ad}}\,av\rangle = 0 = \langle v, v\rangle;$

$\rightsquigarrow [2] := I(\Rightarrow) : \langle v, v\rangle = 0 \Rightarrow \langle \widetilde{\mathrm{ad}}(a)v, \widetilde{\mathrm{ad}}(a)v\rangle = \langle v, v\rangle,$

$[v.*] := \texttt{LEM}(\langle v, v\rangle = 0)[1][2]E(|) : \langle \widetilde{\mathrm{ad}}(a)v, \widetilde{\mathrm{ad}}(a)v\rangle = \langle v, v\rangle;$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{Isoquadric} : \Big(\widetilde{\mathrm{ad}}(a) : \texttt{Isoquadric}(V, V)\Big);$

$\square$

$\texttt{asOrthogonalTransform} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.$

$\quad . \,\Gamma(V) \xrightarrow{\mathsf{GRP}} \mathbf{O}(V)$

$\texttt{asOrthogonalTransform}\,(a) = O(a) := \widetilde{\mathrm{ad}}\,a_{|V}$

$\texttt{VectorsProduceReflections} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.$

$\quad . \,\langle v, v\rangle \neq 0 \Rightarrow O(v\mathbf{i}) = \sigma_v$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CliffordGroupSpawnsOrthogonalGroup} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, O_V : \Gamma(V) \twoheadrightarrow \mathbf{O}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ T : \mathbf{O}(V),$

$$\Big(n, S, [1]\Big) := \texttt{OrthogonalGroupStructure}(T) : \sum n \in \mathbb{Z}_+ \,.\, \sum S : n \to \texttt{Symmetry}(V) \,.\, T = \prod_{i=1}^{n} S_i,$$

$$\Big(v, [2]\Big) := \mathcal{A}\texttt{Symmetry}(S) : \sum v : n \to V \,.\, \forall i \in n \,.\, \langle v, v\rangle \neq 0 \;\&\; S_i = \sigma_{v_i},$$

$$[T.*] := \mathcal{A}\mathsf{GRP}\Big(\Gamma(V), \mathbf{O}(V)\Big)(O_V)\forall i \in n \,.\, \texttt{VectorsProduceReflections}(v_i)[2][1] :$$

$$: O\left(\prod_{i=1}^{n} v_i\mathbf{i}\right) = \prod_{i=1}^{n} O(v_i\mathbf{i}) = \prod_{i=1}^{n} \sigma_{v_i} = \prod_{i=1}^{n} S_i = T;$$

$$\rightsquigarrow [*] := \mathcal{A}^{-1}\texttt{Surjection} : \Big(O_V : \Gamma(V) \twoheadrightarrow \mathbf{O}(V)\Big);$$

$\square$

$\texttt{CliffordGroupScalarCriterion} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, \forall x \in \Gamma(V) \,.$

$$\Big(\forall v \in V \,.\, \omega_V(x)(v\mathbf{i}) = (v\mathbf{i})x\Big) \Rightarrow x \in ke_{\mathrm{CL}(V)}$$

$\texttt{Proof} \;=$

$\texttt{CliffordGroupStructure} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.$

$$.\ \Gamma(V) = \left\langle \Big\{ (v\mathbf{i}) \Big| v \in V : \langle v, v\rangle \neq 0 \Big\} \right\rangle$$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \Gamma(V),$

$T := O(x) : \mathbf{O}(V),$

$$\Big(n, v, [1]\Big) := \texttt{CliffordGroupSpawnsOrthogonalGroup}(T) : \sum n \in \mathbb{N} \,.\, \sum v : n \to V \,.\, T = O\left(\prod_{i=1}^{n} v_i\mathbf{i}\right),$$

$$a := x^{-}1 \prod_{i=1}^{n} (v_i\mathbf{i}) : \Gamma(V),$$

$$[2] := \mathcal{D}a\mathcal{A}\mathsf{GRP}\Big(\Gamma(V), \mathbf{O}(V)\Big)(O)\mathcal{D}^{-1}T[1]\mathcal{A}\texttt{inverse} : O(a) = O\left(x^{-1}\prod_{i=1}^{n} v_i\mathbf{i}\right) = O^{-1}(x)O\left(\prod_{i=1}^{n} v_i\mathbf{i}\right) =$$

$$= T^{-1}T = \mathrm{id},$$

$$[3] := \mathcal{A}O[2] : \forall v \in V \,.\, \omega(a)(v\mathbf{i}) = (v\mathbf{i})a^{-1}_{,}$$

$$[4] := \texttt{CliffordGroupScalarCriterion}[3] : a \in ke_{\mathrm{CL}(V)},$$

$$\Big(\lambda, [x.*]\Big) := [4]\mathcal{D}a : \sum \lambda \in k^* \,.\, x = \left((\lambda v_1\mathbf{i})\prod_{i=2}^{n}(v_i\mathbf{i})\right)^{-1};$$

$$\rightsquigarrow [*] := \mathcal{A}\texttt{generateGroup} : \texttt{This};$$

$\square$

$\texttt{DegreeInvolutionByDeterminant} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{Nondegenerate}(k) \,.\, \forall x \in \Gamma(V) \,.$

$\omega_V(x) = \big(\det O_V(x)\big)x$

$\texttt{Proof} \;=$

$\varphi := \Lambda x \in \Gamma(V) \,.\, \big(\det O_V(x)\big)x : \Gamma(V) \xrightarrow{\mathsf{GRP}} \Gamma(V),$

$\texttt{Assume}\ v : V,$

$\texttt{Assume}\ [0] : \langle v, v \rangle \neq 0,$

$[1] := \texttt{VectorsProduceReflections}(v)\texttt{ReflectionDeterminant} : \det O(v) = -1,$

$[v.*] := \mathcal{C}\omega_V[1]\mathcal{D}\varphi : \omega_V(v) = \varphi(v);$

$\rightsquigarrow [1] := I(\forall) : \forall v \in V \,.\, \omega_V(v\mathbf{i}) = \varphi(v\mathbf{i}),$

$\Big(n, v, [2]\Big) := \texttt{CliffordGroupStructure}(x) : \sum n \in \mathbb{N} \,.\, \sum v : n \to V \,.\, x = \prod_{i=1}^{n} (v_i\ \mathbf{i}) \,.\, ,$

$[*] := [1][2]\mathcal{C}\,\mathsf{GRP}\Big(\Gamma(V), \Gamma(V)\Big)(V) : \omega_V(x) = \varphi(x);$

$\square$

4.7 Spin Group and Representation

$$\texttt{pinGroup} :: \prod k : \texttt{Numeric} \,.\, \texttt{Nondegenerate}(k) \to \mathsf{GRP}$$

$$\texttt{pinGroup}\,(V) = \mathbf{PIN}(V) := \left\{ x \in \Gamma(V) \middle| \lambda_V(x) \in \{-1, +1\} \right\}$$

$$\texttt{spinGroup} :: \prod k : \texttt{Numeric} \,.\, \texttt{Nondegenerate}(k) \to \mathsf{GRP}$$

$$\texttt{SpinGroup}\,(V) = \mathbf{SPIN}(V) := \left\{ x \in \Gamma(V) \middle| \lambda_V(x) = 1 \right\}$$

$\texttt{PinGroupSpawnsOrthogonalGroup} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \,.\, \mathbf{PIN}(V)\; O = \mathbf{O}(V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; T : \mathbf{O}(V),$

$\Big(n, v, [1]\Big) := \texttt{CliffordGroupSpawnsOrthogonalGroup}(T) : \sum n \in \mathbb{N} \,.\, \sum v : n \to V \,.$

$$T = O\left(\prod_{i=1}^{n} v_i \mathbf{i}\right) \;\&\; \forall i \in n \,.\, \langle v_i, v_i \rangle \neq 0,$$

$\texttt{Assume}\; i : n,$

$$u_i := \frac{v_i}{\sqrt{|\langle v_i, v_i \rangle|}} : V,$$

$$[2] := \lbag u_i \rbag \lambda_V \rbag \texttt{absValue} : \lambda_V(u_i) = -\frac{\langle v_i, v_i \rangle}{|\langle v_i, v_i \rangle|} \in \{-1, +1\},$$

$[i.*] := \rbag \mathbf{PIN}(V) : u_i \in \mathbf{PIN}(V);$

$\rightsquigarrow u := I(\rightarrow) : n \to V,$

$$[T.*] := \rbag O \rbag \widetilde{\mathrm{ad}} : O\left(\prod_{i=1}^{n} u_i\right) = T;$$

$\rightsquigarrow [*] := I(\forall) : \texttt{This},$

$\square$

$\texttt{PinKernel} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \,.\, \ker O_{V|\mathbf{PIN}(V)} = \mathbb{S}^0$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SpinGroupSpawnsSpecialOrthogonalGroup} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \,.\, \mathbf{SPIN}(V)\; O = \mathbf{SO}(V)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{SPinKernel} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \,.\, \ker O_{V|\mathbf{SPIN}(V)} = \mathbb{S}^0$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{MetricComplexStructure} :: \prod V : \texttt{NonDegenerate}(\mathbb{R}) \;.\; ?\texttt{ComplexStructure}(V)$

$J : \texttt{MetricComplexStructure} \iff J : \texttt{Isoquadric}(V)$

$\texttt{MetricComplexStructureAdjoint} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.\; J^\star = -J$

$\texttt{Proof} \;=$

$[1] := \mathit{\Pi}\texttt{Isoquadric}(V)(J)\mathit{\Pi}\texttt{Nondenerate}(\mathbb{R})(V) : J^\star = J^{-1},$

$[*] := \mathit{\Pi}\texttt{ComplexStructure}(V)(J)[1] : J^\star = -J;$

$\square$

$\texttt{MetricComplexStructureIsSkew} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.$

$\quad .\; J : \texttt{Skew}(V)$

$\texttt{Proof} \;=$

$\texttt{complexInvolution} :: \prod V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \texttt{MetricComplexStructure}(V) \to \mathbb{C}\otimes V \xrightarrow{\mathbb{C}\text{-VS}} \mathbb{C}\otimes V$

$\texttt{complexInvolution}\,(J) = \omega_J := \mathit{\Pi}\texttt{tensorProduct}\Lambda z \in \mathbb{C} \;.\; \Lambda v \in V \;.\; \mathrm{i}z \otimes v\; J$

$\texttt{ComplexInvolutionIsInvolution} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.\; \omega_J^2 = \mathrm{id}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; z : \mathbb{C},$

$\texttt{Assume}\; v : V,$

$[z.*] := \mathit{\Pi}\omega_J\mathit{\Pi}\mathrm{i}\mathit{\Pi}\texttt{ComplexStructure}(V)(J)\mathit{\Pi}\mathcal{L}(\mathbb{C}, V; \mathbb{C}\otimes V)(\otimes)\texttt{NegativeSquare}(\mathbb{R}) :$

$\quad : \omega_J^2(z\otimes v) = \mathrm{i}^2 z\otimes v\; J^2 = -z\otimes -v = (-1)^2 z\otimes v = z\otimes v;$

$\leadsto [*] := \mathit{\Pi}\texttt{tensorProduct}I(=,\to) : \omega_J^2 = \mathrm{id};$

$\square$

$\texttt{ComplexInvolutionIsSkew} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.$

$\quad .\; \omega_J : \texttt{Skew}(\mathbb{C}\otimes V)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; t : \mathbb{C}\otimes V,$

$\Big(v, w, [1]\Big) := \texttt{TensorProductBasis}(t) : \sum v, w \in V \;.\; t = 1\otimes v + \mathrm{i}\otimes w,$

$[t.*] := [1]\mathit{\Pi}\mathcal{L}(\mathbb{C}\otimes V, \mathbb{C}\otimes V; \mathbb{C})(\mathbb{C}\otimes V)\mathit{\Pi}\omega_J\mathit{\Pi}\texttt{innerProductTensorProduct}\mathit{\Pi}\texttt{Skew}(V)(J)\mathit{\Pi}\texttt{adjOp}(J)$

$\quad \texttt{MetricComplexStructuteAdjoint}(J)\mathit{\Pi}\mathcal{L}(\mathbb{C}\otimes V, \mathbb{C}\otimes V; \mathbb{C})V\mathit{\Pi}\texttt{inverse} :$

$\quad : \langle t\,\omega_J, t\rangle = \langle 1\otimes v\,\omega_J, 1\otimes v\rangle + \langle 1\otimes v\,\omega_J, \mathrm{i}\otimes w\rangle + \langle \mathrm{i}\otimes w\,\omega_J, \mathrm{i}\otimes w\rangle + \langle \mathrm{i}\otimes w\,\omega_J, 1\otimes v\rangle =$

$\quad \langle \mathrm{i}\otimes v\; J, 1\otimes v\rangle + \langle \mathrm{i}\otimes v\; J, \mathrm{i}\otimes w\rangle + \langle -1\otimes w\; J, \mathrm{i}\otimes w\rangle + \langle -1\otimes w\; J, 1\otimes v\rangle =$

$\quad = \mathrm{i}\langle v\; J, v\rangle - \langle v\; J, w\rangle + -\mathrm{i}\langle w\; J, w\rangle - \langle w\; J, v\rangle = -\langle v\; J, w\rangle - \langle v\; J^\star, w\rangle = -\langle v\; J, w\rangle + \langle v\; J, w\rangle = 0;$

$\leadsto [*] := \mathit{\Pi}^{-1}\texttt{Skew} : \Big(\omega_J : \texttt{Skew}(\mathbb{C}\otimes V)\Big);$

$\square$

$\texttt{exteriorComplexIso} :: \prod V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \prod J : \texttt{MetricComplexStructure}(V) \;.$

$\quad .\; \mathrm{CL}(\mathbb{C}\otimes V) \xleftrightarrow{\mathbb{C}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)$

$\texttt{exteriorComplexIso}\,() = R_J := \texttt{CliffordExteriorOperatorsIsomorphismCriterion}(\omega_J)$

$$\texttt{complexCliffordEmbedding} :: \prod V : \texttt{OrthogonalVectorSpace}(\mathbb{R}) \;.\; \mathrm{CL}(V) \hookrightarrow \mathrm{CL}(\mathbb{C} \otimes V)$$

$$\texttt{complexCliffordEmbedding}\,(x) = 1 \otimes x := \mathrm{CL}(\iota)(x) \;\texttt{where}\; \iota = \Lambda v \in V \;.\; 1 \otimes v$$

$$\texttt{spinRepresentation} :: \prod V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \prod J : \texttt{MetricComplexStructure}(V) \;.$$
$$.\; \mathrm{CL}(V) \xrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}\Big(\ker^{\wedge}(\mathrm{id} - \omega_J)\Big)$$

$$\texttt{spinRepresentation}\,(x) = S_J(x) := R_J(1 \otimes x)$$

$$\texttt{ComplexIrreducible} :: \prod A : \mathbb{R}\text{-ALGE} \;.\; \prod U : \mathbb{C}\text{-VS} \;.\; ?A \xrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}(U)$$

$$\rho : \texttt{ComplexIrreducible} \iff \texttt{Invariant}(\mathbb{C})(\rho(A)) = \Big\{\{0\}, U\Big\}$$

$$\texttt{SpinRepresentationIsIrreducible} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.$$
$$.\; S_J : \texttt{ComplexIrreducible}\Big(\mathrm{CL}(V), \ker^{\wedge}(\mathrm{id} - \omega_J)\Big)$$

$\texttt{Proof} =$

$\texttt{Assume}\; U : \texttt{Invariant}\Big(S_J(\mathrm{CL}(V))\Big),$

$[1] := \mathcal{I}\,\texttt{Invariant}(U) : \forall x \in \mathrm{CL}(V) \;.\; S_J(x)(U) \subset U,$

$\texttt{Assume}\; t : \mathrm{CL}(\mathbb{C} \otimes V),$

$\Big(z, x, [2]\Big) := \texttt{CliffordAlgebraScalarExtension}(t) : \sum z \in \mathbb{C} \;.\; \sum x \in \mathrm{CL}(V) \;.\; t = z \otimes x,$

$[t.*] := [2]\mathcal{I}\,\mathbb{C}\text{-VS}\Big(\mathrm{CL}(\mathbb{C} \otimes V), \mathrm{End}_{\mathbb{C}\text{-VS}}\big(\ker(\mathrm{id} - \omega_J)\big)^{\wedge}(R_J)\mathcal{I}^{-1}(S_J)[1](x)\mathcal{I}\,\texttt{VectorSubspace}(\mathbb{C})(U) :$

$\quad R_J(t)(U) = R_J(z \otimes x)(U) = zR_J(1 \otimes x)(U) = zS_J(x)(U) \subset U;$

$\rightsquigarrow [2] := \mathcal{I}^{-1}\texttt{Invariant} : \Bigg(U : \texttt{Invariant}\Big(R_J\big(\mathrm{CL}(\mathbb{C} \otimes V)\big)\Big)\Bigg),$

$[U.*] := \texttt{IsomorphismIsIrreducible}(R_J)\mathcal{I}\,\texttt{Irreducible}[2] : U = \{0\}\Big|U = \ker^{\wedge}(\mathrm{id} - \omega_J);$

$\rightsquigarrow [*] := \mathcal{I}^{-1}\texttt{ComplexIrreducible} : \texttt{This};$

$\square$

$$\texttt{HermitianSubstitution} :: \forall V : \texttt{Nondegenerate}(\mathbb{R}) \;.\; \forall J : \texttt{MetricComplexStructure}(V) \;.\; \forall t \in \mathbb{C} \otimes V \;.$$
$$.\; t\sigma_H = \bar{t}\,\sigma$$

$\texttt{Proof} =$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; x : n \to \ker(\mathrm{id} - \omega_J),$

$\texttt{Assume}\; z : (n-1) \to \ker(\mathrm{id} - \omega_J),$

$[n.*] := \mathcal{I}\,\texttt{hermitianSubstitution}\mathcal{I}\,\texttt{hermitianProduct}\mathcal{I}^{-1}\texttt{substitution}\mathcal{I}^{-1}\texttt{hermitianProduct} :$

$$: \left\langle (t\;\sigma_H) \bigwedge_{i=1}^{n} x_i, \bigwedge_{i=1}^{n-1} y_i \right\rangle_H = \left\langle \bigwedge_{i=1}^{n} x_i, t \wedge \bigwedge_{i=1}^{n-1} y_i \right\rangle_H = \left\langle \bigwedge_{i=1}^{n} x_i, \bar{t} \wedge \bigwedge_{i=1}^{n-1} \overline{y}_i \right\rangle = \left\langle (\bar{t}\;\sigma) \bigwedge_{i=1}^{n} x_i, \bigwedge_{i=1}^{n-1} \overline{y}_i \right\rangle =$$
$$= \left\langle (\bar{t}\;\sigma) \bigwedge_{i=1}^{n} x_i, \bigwedge_{i=1}^{n-1} y_i \right\rangle;$$

$\rightsquigarrow [*] := \mathcal{I}\,\texttt{HermitianProduct}\mathcal{I}\,\texttt{Nondegenerate}(V)\mathcal{I}\,\texttt{exteriorAlgebra} : \texttt{This},$

$\square$

$\texttt{SpinRepresentationOfVectorsIsHermitianSymmetric} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .$

$.\ \forall J : \texttt{MetricComplexStructure}(V)\ .\ \forall v \in V\ .\ S_J(v\ \mathbf{i}) : \texttt{Symmetric}\Big(\ker^\wedge(\mathrm{id} - \omega_J), \langle\cdot,\cdot\rangle_H\Big)$

$\texttt{Proof}\ =$

$a := \dfrac{1}{2}\Big(1 \otimes v + \mathrm{i} \otimes (v\ J)\Big) : \mathbb{C} \otimes V,$

$b := \dfrac{1}{2}\Big(1 \otimes v - \mathrm{i} \otimes (v\ J)\Big) : \mathbb{C} \otimes V,$

$[1] := \mathcal{D}a\mathcal{D}b : \overline{a} = b,$

$\texttt{Assume}\ t, s : \ker^\wedge(\mathrm{id} - \omega_J),$

$\Big[(t,s).*\Big] := \mathcal{D}S_J \mathcal{C}\mathcal{L}\Big((\mathbb{C}\otimes V)^\wedge, (\mathbb{C}\otimes V)^\wedge; \mathbb{C}\Big)\Big(\langle\cdot,\cdot\rangle_H\Big)\mathcal{D}^{-1}a\mathcal{D}^{-1}b\mathcal{C}^{-1}\sigma_H(a)$

$: \texttt{HermitianSubstitution}^2(b)(a)[1]\mathcal{C}\sigma_H(a)\mathcal{C}\mathcal{L}\Big((\mathbb{C}\otimes V)^\wedge, (\mathbb{C}\otimes V)^\wedge; \mathbb{C}\Big)\Big(\langle\cdot,\cdot\rangle_H\Big)\mathcal{D}^{-1}S_J :$

$: \langle S_J(v\ \mathbf{i})t, s\rangle_H = \langle a \wedge t, s\rangle_H + \langle \sigma(b)t, s\rangle_H = \langle t, \sigma_H(a)s\rangle_H + \langle \sigma_H(a)t, s\rangle_H = \langle t, \sigma(b)s\rangle_H + \langle t, a\wedge s\rangle_H =$

$= \langle t, S_J(v\ \mathbf{i})s\rangle_H;$

$\rightsquigarrow [*] := \mathcal{C}^{-1}\texttt{Symmetric} : \texttt{This},$

$\square$

$\texttt{SpinRepresentationOfVectorsIsQuasiquadric} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .$

$.\ \forall J : \texttt{MetricComplexStructure}(V)\ .\ \forall v \in V\ .\ \forall s, t \in \ker^\wedge(\mathrm{id} - \omega_J)\ .$

$.\ \Big\langle S_J(v\mathbf{i})(t), S_J(v\mathbf{i})(s)\Big\rangle_H = \langle v, v\rangle \langle t, s\rangle_H$

$\texttt{Proof}\ =$

$[1] := \mathcal{C}\mathbb{R}\text{-}\mathsf{ALGE}\Big(\mathrm{CL}(V), \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}\big(\ker^\wedge(\mathrm{id} - \omega_V)\big)\Big)(S_J)\mathcal{C}(\text{-}\mathsf{CLIF}\mathbb{R})\Big(\mathrm{CL}(V)\Big)$

$\mathcal{C}\mathbb{R}\text{-}\mathsf{ALGE}\Big(\mathrm{CL}(V), \mathrm{End}_{\mathbb{C}\text{-}\mathsf{VS}}\big(\ker^\wedge(\mathrm{id} - \omega_V)\big)\Big)(S_J) : S_J^2(v\mathbf{i}) = S_J(v\mathbf{i})^2 = S_J\Big(\langle v, v\rangle e_{\mathrm{CL}(V)}\Big) = \langle v, v\rangle\ \mathrm{id},$

$[*] := \texttt{SpinRepresentationOfVectorsIsHermitianSymmetric}(V, J, v)[1]$

$: \mathcal{C}\mathcal{L}\Big((\mathbb{C}\otimes V)^\wedge, (\mathbb{C}\otimes V)^\wedge; \mathbb{C}\Big)\Big(\langle\cdot,\cdot\rangle_H\Big) :$

$: \Big\langle S_J(v\mathbf{i})(t), S_J(v\mathbf{i})(s)\Big\rangle_H = \Big\langle S_J^2(v\mathbf{i})(t), s\Big\rangle_H = \Big\langle \langle v, v\rangle t, s\Big\rangle_H = \langle v, v\rangle\langle t, s\rangle_H;$

$\square$

$\texttt{SphereSpinRepresentationIsUnitary} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .\ \forall J : \texttt{MetricComplexStructure}(V)\ .$

$.\ S_J\Big(\mathbb{S}_V\Big) \subset \mathbf{U}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{evenSpinSpace} :: \prod V : \texttt{Nondegenerate}(\mathbb{R})\ .\ \prod J : \texttt{MetricComplexStructure}(V)\ .$

$.\ \texttt{VectorSubspace}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)$

$$\texttt{evenSpinSpace}\,() = V_J^0 := \sum_{n=0}^{\infty}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)_{2n}$$

$\texttt{oddSpinSpace} :: \prod V : \texttt{Nondegenerate}(\mathbb{R})\ .\ \prod J : \texttt{MetricComplexStructure}(V)\ .$

$.\ \texttt{VectorSubspace}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)$

$$\texttt{evenSpinSpace}\,() = V_J^1 := \sum_{n=0}^{\infty}\Big(\ker^\wedge(\mathrm{id} - \omega_J)\Big)_{2n+1}$$

$\texttt{EvenCliffordElementsPreservesSpinSpaces} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .$

$\quad .\ \forall J : \texttt{MetricComplexStructure}(V)\ .\ V_J^0, V_J^1 : \texttt{Invariant}\Big(S_J\big(\mathrm{CL}_0(V)\big)\Big)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{evenHalfSpinRepresentation} :: \prod V : \texttt{Nondegenerate}(\mathbb{R})\ .\ \prod J : \texttt{MetricComplexStructure}(V)\ .$

$\quad .\ \mathrm{CL}_0(V) \xrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}(V_J^0)$

$\texttt{evenHalfSpinRepesentation}\,(x) = S_J^0(x) := \Big(S_J(x)\Big)_{V_J^0}$

$\texttt{oddHalfSpinRepresentation} :: \prod V : \texttt{Nondegenerate}(\mathbb{R})\ .\ \prod J : \texttt{MetricComplexStructure}(V)\ .$

$\quad .\ \mathrm{CL}_0(V) \xrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}(V_J^1)$

$\texttt{oddHalfSpinRepesentation}\,(x) = S_J^1(x) := \Big(S_J(x)\Big)_{V_J^1}$

$\texttt{EvenHalfSpinRepresentationIsIso} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .$

$\quad .\ \forall J : \texttt{MetricComplexStructure}(V)\ .\ S_J^0 : \mathrm{CL}_0(V) \xleftrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}(V_J^0)$

$\texttt{Proof}\ =$

...

$\square$

$\texttt{OddHalfSpinRepresentationIsIso} :: \forall V : \texttt{Nondegenerate}(\mathbb{R})\ .$

$\quad .\ \forall J : \texttt{MetricComplexStructure}(V)\ .\ S_J^1 : \mathrm{CL}_0(V) \xleftrightarrow{\mathbb{R}\text{-ALGE}} \mathrm{End}_{\mathbb{C}\text{-VS}}(V_J^1)$

$\texttt{Proof}\ =$

...

$\square$

4.8 Radon-Hurwitz Number

$\texttt{RadonHurwitzOrthogonalSystem} :: \forall n, k \in \mathbb{N} \,.\, \forall \rho : \texttt{OrthogonalRepresentation}\Big(\mathrm{CL}_k(-), \mathrm{End}_{\mathbb{R}\text{-VS}}(\mathbb{R}^n)\Big) \,.$

$\quad .\, \forall e : \texttt{OrthogonalBasis}(\mathbb{R}^k) \,.\, \forall a :\in \mathbb{S}^{n-1} \,.\, a \oplus \rho(e\ \mathbf{i})(a) : \texttt{Orthonormal}(\mathbb{R}^n)$

$\texttt{Proof} \;=$

$\sigma := \rho(e\ \mathbf{i}) : k \to \mathrm{End}_{\mathbb{R}\text{-VS}}(\mathbb{R}^n),$

$[1] := \mathcal{A}\mathbb{R}\text{-ALGE}\Big(\mathrm{CL}_k(-), \mathrm{End}_{\mathbb{R}\text{-VS}}(\mathbb{R}^n)\Big)\mathcal{A}\texttt{OrhogonalBasis}(\mathbb{R}^k)(e)\mathcal{A}\mathbb{R}\text{-CLIF}\Big(\mathrm{CL}_k\Big) :$

$\quad : \forall i, j \in k \,.\, \sigma_i\sigma_j + \sigma_j\sigma_i = -2\delta^i_j,$

$v := \sigma(a) : k \to \mathbb{R}^n,$

$\texttt{Assume}\ i : k,$

$[2] := \mathcal{D}v_i\mathcal{D}\sigma_i\mathcal{A}\texttt{OrthogonalRepresentation}(\rho)\texttt{NegativelyOrthogonalRepresented}\mathcal{D}v_i\mathcal{D} :$

$\quad : \langle v_i, a\rangle = \langle \sigma_i(a), a\rangle = \langle \rho(e_i\ \mathbf{i})(a), a\rangle = -\langle a, \rho(e_i\ \mathbf{i})(a)\rangle = -\langle a, \sigma_i(a)\rangle = -\langle v_i, a\rangle,$

$[i.*] := [2] - [2] : \langle v_i, a\rangle = 0;$

$\rightsquigarrow [2] := I(\forall) : \forall i \in n \,.\, \langle v_i, a\rangle = 0,$

$\texttt{Assume}\ i, j : k,$

$\texttt{Assume}\ [3] : i \neq j,$

$[4] := \mathcal{D}v\mathcal{D}\sigma\texttt{NegativelyOrthogonalRepresented}\mathcal{A}\mathbb{R}\text{-ALGE}\Big(\mathrm{CL}_k(-), \mathrm{End}_{\mathbb{R}\text{-VS}}(\mathbb{R}^n)\Big)[1]$

$\quad \texttt{NegativelyOrthogonalRepresented}\mathcal{D}^{-1}v\mathcal{A}\texttt{Symmetric} :$

$\quad : \langle v_i, v_j\rangle = \Big\langle \sigma_i(a), \sigma_j(a)\Big\rangle = \Big\langle \rho(e_i\mathbf{i})(a), \rho(e_j\mathbf{i})(a)\Big\rangle = -\Big\langle \rho(e_j\mathbf{i})\rho(e_i\mathbf{i})(a), a\Big\rangle = -\Big\langle \rho\big((e_i\mathbf{i})(e_j\mathbf{i})a\big), a\Big\rangle =$

$\quad = \Big\langle \rho\big((e_j\mathbf{i})(e_i\mathbf{i})a\big), a\Big\rangle = -\Big\langle \rho(e_j\mathbf{i})a, \rho(e_i\mathbf{i})a\Big\rangle = -\Big\langle v_j, v_i\Big\rangle = -\Big\langle v_i, v_j\Big\rangle,$

$[3.*] := \dfrac{[4] - [4]}{2} : \Big\langle v_i, v_j\Big\rangle = 0;$

$\rightsquigarrow \Big[(i,j).*.1\Big] := I(\Rightarrow) : (i \neq j) \Rightarrow \langle v_i, v_j\rangle = 0,$

$\Big[(i,j).*.1\Big] := \mathcal{D}v\mathcal{D}\sigma_i\texttt{NegativelyOrthogonalRepresented}\mathcal{A}\mathbb{R}\text{-ALGE}\Big(\mathrm{CL}_k(-), \mathbb{R}^n\Big)[1]\mathcal{A}\mathbb{S}^n(a) :$

$\quad : \langle v_i, v_i\rangle = \Big\langle \sigma_i(a), \sigma_i(a)\Big\rangle = \Big\langle \rho(e_i\ \mathrm{i})(a), \rho(e_i\ \mathrm{i})(a)\Big\rangle = -\Big\langle \rho^2(e_i\ \mathrm{i})(a), a\Big\rangle = -\Big\langle \rho\Big((e_i\ \mathrm{i})^2\Big)(a), a\Big\rangle = \langle a, a\rangle = 1;$

$\rightsquigarrow [*] := [2]\mathcal{A}\texttt{Orthonormal} : \texttt{This};$

$\square$

$\texttt{RadonHurwitzDimensionBound} :: \forall k \in \mathbb{N} \,.\, \forall V \in \mathbb{R}\text{-VS} \,.$

$\quad .\, \forall \rho : \texttt{OrthogonalRepresentation}\Big(\mathrm{CL}_k(-), V\Big) \,.\, \dim V > k$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{numberOfRadonHurwitz} :: \mathbb{Z}_+ \to \mathbb{Z}_+$

$\texttt{numberofRadonHurwitz}\,(n) = K(n) :=$

$\quad := \max\Big\{k \in \mathbb{N} : \exists \rho : \texttt{OrthogonalRepresentation}\big(\mathrm{CL}_k(-), End_{\mathbb{R}\text{-VS}}(V)\big)\Big\}$

$\texttt{RadonHurwitzBound} :: \forall n \in \mathbb{N} \,.\, K(n) < n$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{RadonHurwitzRecurrentRelation} :: \forall n \in \mathbb{N} \,.\, K(16n) = K(n) + 8$

$\texttt{Proof} \ =$

$k := K(n) : \mathbb{N},$

$\rho := \reflectbox{$D$} K(n) \mathcal{J} k : \texttt{OrthogonalRepresentation}\Big(\mathrm{CL}_{16k}(-), \mathbb{R}^n\Big),$

$[1] := \texttt{BottPeriodicity}(k) : \mathrm{CL}_{k+8}(-) \cong_{\mathbb{R}\text{-}\mathsf{ALGE}} \mathrm{CL}_k(-) \otimes \mathrm{End}_{\mathbb{R}\text{-}\mathsf{VS}}(\mathbb{R}^{16}),$

$\varphi := \reflectbox{$D$} \texttt{Isomorphic} : \mathrm{CL}_{k+8}(-) \xrightarrow{\mathbb{R}\text{-}\mathsf{ALGE}} \mathrm{CL}_k(-) \otimes \mathrm{End}_{\mathbb{R}\text{-}\mathsf{VS}}(\mathbb{R}^{16}),$

$R := \varphi(\mathrm{id} \otimes \rho) : \texttt{OrthogonalRepresentation}\Big(\mathrm{CL}_{k+8}, \mathbb{R}^{16n}\Big),$

$[2] := \reflectbox{$D$} k(16n)(R) : k(16n) \geq k(n) + 8,$

$\texttt{Assume}\ t : \mathbb{N},$

$\texttt{Assume}\ [3] : t > 8,$

$\texttt{Assume}\ R : \texttt{OrthogonalRepresentation}(\mathrm{CL}_t(-), \mathbb{R}^{16n}),$

$\Big(\rho', [4]\Big) := \texttt{BottPeriodicity}(t-8)\texttt{TensorRepresentationEquivalens}(R) :$

$\quad : \sum \rho' : \texttt{OrthogonalRepresentation}\Big(\mathrm{CL}_{t-8}(-), \mathbb{R}^n\Big) \,.\, \rho' \otimes \mathrm{id} \cong R,$

$[t.*] := \reflectbox{$D$} k[4] : t \leq k + 8;$

$\rightsquigarrow [*] := \texttt{DoubleIneq}[2] : k(16n) = k(n) + 8;$

$\square$

$\texttt{RadonHurwitzNumberLittleBound} :: \forall b \in 3 \,.\, \forall c : \texttt{Odd} \,.\, k(2^b c) < 8$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : k(2^b c) \geq 8,$

$[2] := \reflectbox{$D$} \mathbb{Z}_+[1] : k(2^b c) - 8 \in \mathbb{Z}_+,$

$\Big(n, [3]\Big) := \texttt{RadonHurwitzRecurrentRelation}[2] : n * 16 = 2^b c,$

$[1.*] := [3]\reflectbox{$D$} b \reflectbox{$D$} c \texttt{MainTheoremOfArithmetics} : \perp;$

$\rightsquigarrow [*] := E(\perp)\reflectbox{D}\texttt{GreaterOrEqual} : k(2^b c) < 8;$

$\texttt{RadonHurwitzNumberExpression} :: \forall a \in \mathbb{N} \,.\, \forall b \in \ 3 \,.\, \forall c : \texttt{Odd} \,.\, K(16^a 2^b c) = 8a + 2^b - 1$

$\texttt{Proof} \ =$

$\ldots$

$\square$

4.9 Towards Enumeration of Orthonormal Frames

$$\texttt{OrthogonalFamily} :: \prod k : \texttt{Field} \,.\, \prod V : \texttt{OrthogonalVectorSpace}(k) \,.\, \prod n \in \mathbb{N} \,.\, ?\Big(n \to \texttt{Skew}(V)\Big)$$

$$\sigma : \texttt{OrthogonalFamily} \iff \forall i, j \in n \,.\, \sigma_i\sigma_j + \sigma_j\sigma_i = -2\delta^i_j \operatorname{id}$$

$\texttt{OrhogonalFamilyInnerProduct} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k) \,.\, \forall n \in \mathbb{N} \,.$

$\quad . \forall \sigma : \texttt{OrthogonalFamily}(V, n) \,.\, \forall v \in V \,.\, \forall i, j \in n \,.\, \langle \sigma_i v, \sigma_j v\rangle = \delta^i_j \langle v, v\rangle$

Proof $=$

...

$\square$

$\texttt{orthogonalMultiplication} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k)\ \&\ k\text{-FDVS} \,.$

$\quad . \texttt{Orthonormal}(V) \to \texttt{OrthogonalFamily}(\dim V - 1, V) \to \mathcal{L}(V, V; V)$

$$\texttt{orthogonalMultiplication}\,(e, \sigma, v, u) = v \odot_{e,\sigma} u := v_1 u + \sum_{i=2}^{n} v_i \sigma_{i-1}(u)$$

$\texttt{OrthogonalMultiplication} :: \prod k : \texttt{Numeric} \,.\, \prod V : \texttt{OrthogonalVectorSpace}(k)\ \&\ k\text{-FDVS}$

$\quad . ?\mathcal{L}(V, V; V)$

$$\mu : \texttt{OrthogonalMultiplication} \iff \forall v, u \in V \,.\, \Big\langle \mu(v, u), \mu(v, u)\Big\rangle = \langle v, v\rangle\langle u, u\rangle\ \&\ \exists v \in \mathbb{S}_V : \mu(v, \cdot) = \operatorname{id}$$

$\texttt{OrthogonalMultiplicationProperty} :: \forall k : \texttt{Numeric} \,.\, \forall V : \texttt{OrthogonalVectorSpace}(k)\ \&\ k\text{-FDVS} \,.$

$\quad . \forall e : \texttt{Orthonormal}(V) \,.\, \forall \sigma : \texttt{OrthogonalFamily}(\dim V - 1, V) \,.\, \odot_{e,\sigma} : \texttt{OrthogonalMultiplication}$

Proof $=$

...

$\square$

$\texttt{OrthogonalMultiplicationConstruction} :: \forall n \in \mathbb{N} \,.\, \forall \mu : \texttt{OrthogonalMultiplication}(\mathbb{R}^{n+1}) \,.$

$\quad . \exists e : \texttt{Orthonormal}(V) \,.\, \exists \sigma : \texttt{OrthogonalFamily}(\dim V - 1, V) \,.\, \odot_{e,\sigma} = \mu$

Proof $=$

...

$\square$

$\texttt{DimensionByOrthogonalMultiplication} :: \forall n \in \mathbb{N} \,.\, \forall \mu : \texttt{orthogonalMultiplication}(\mathbb{R}^n) \,.$

$\quad . \dim n \in \{1, 2, 4, 8\}$

Proof $=$

...

$\square$