Lp.Know

Uncultured Tramp

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Contents

1 L^p Spaces

1.1 Basic Definition

$\texttt{ComplexValuedBorelMeasurableFunction} :: \prod X \in \texttt{BOR} \,.\, ?(X \to \mathbb{C})$

$f : \texttt{ComplexValuedBorelMeasurableFunction} \iff \Im f : \texttt{Measurable}(X) \;\&\; \Re f : \texttt{Measurable}(X)$

$\texttt{Integrable} :: \prod X \in \texttt{MEAS} \,.\, ?(X \to \mathbb{C})$

$f : \texttt{Integrable} \iff \Im f : \texttt{Integrable}(X) \;\&\; \Re f : \texttt{Integrable}(X)$

$\texttt{Integrate} :: \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \to \mathbb{C}$

$\texttt{Integrate}(f) := \int_\Omega f \mathrm{d}\mu = \int_\Omega \Re f \mathrm{d}\mu + \mathrm{i} \int_\Omega \Im f \mathrm{d}\mu,$

$\texttt{AbsValIntegralInequality} :: \forall f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu) \,.\, \left| \int_O f \mathrm{d}\mu \right| \le \int_\Omega |f| \mathrm{d}\mu$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu),$

$r\mathrm{e}^{\mathrm{i}\theta} := \int_\Omega f \mathrm{d}\mu \in \mathbb{C},$

$\eth(r, \theta) \leadsto E_1 : \left| \int_O f \mathrm{d}\mu \right| = r = \mathrm{e}^{-\mathrm{i}\theta} r\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{-\mathrm{i}\theta} \int_\Omega f \mathrm{d}\mu = \int_\Omega \mathrm{e}^{-\mathrm{i}\theta} f \mathrm{d}\mu,$

$\rho\mathrm{e}^{\mathrm{i}\phi} := f : \texttt{Integrable}(\Omega, \mathcal{F}, \mu),$

$E_1 \leadsto I_1 : \int_\Omega \mathrm{e}^{-\mathrm{i}\theta} f \mathrm{d}\mu = \int_\Omega \rho \cos \circ \phi \mathrm{d}\mu + \mathrm{i} \int_\Omega \rho \sin \circ \phi \mathrm{d}\mu = \int_\Omega \rho \cos \circ \phi \mathrm{d}\mu \le \int_\Omega \rho \mathrm{d}\mu = \int_\Omega |f| \mathrm{d}\mu;$

$\square$

$\texttt{Lp} :: \prod X \in \texttt{MEAS} \,.\, \mathbb{R}_{++} \to \texttt{Set}(X \to \mathbb{C})$

$f \in \texttt{Lp}(p) \iff f \in L^p \iff |f|^p : \texttt{Integrable}(X)$

1.2 Inequalities

$\texttt{Ineq1} :: \forall a, b, x, y \in \mathbb{R}_{++} : x + y = 1 \ . \ a^x b^y \leq xa + yb$

$\texttt{Proof} \ =$

$\texttt{Assume} \ a, b, x, y \in \mathbb{R}_{++} : x + y = 1$

$I_1 := \eth\texttt{Convex}(-\log, x, y) : -\log(xa + yb) \leq -x\log(a) - y\log(b),$

$\texttt{InverseMonotoneActIneq}(I_1, \exp \circ - \mathrm{id}) :: \exp(x\log(a) + y\log(b)) =$

$= \exp(\log(a^x b^y)) = a^x b^y \leq xa + xb = \exp\log(xa + xb); \ \square$

$\texttt{Ineq2} :: \forall a, b, x, y \in \mathbb{R}_{++} : (1/x) + (1/y) = 1 \ . \ ab \leq \frac{a^x}{x} + \frac{b^y}{y}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ a, b, x, y \in \mathbb{R}_{++} : (1/x) + (1/y) = 1,$

$\texttt{Ineq1}(a, b, 1/x, 1/y)) :: ab \leq \frac{a^x}{x} + \frac{b^y}{y}; \square$

$\texttt{HölderInequality} :: \forall p, q \in (1, \infty) : (1/p) + (1/q) = 1 \ .$

$. \ \forall f \in L^p(\Omega, \mathcal{F}, \mu) \ . \ \forall g \in L^q(\Omega, \mathcal{F}, \mu) \ . \ \int_\Omega |fg| \mathrm{d}\mu \leq \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} \sqrt[q]{\int_\Omega |g|^q \mathrm{d}\mu}$

$\texttt{Proof} \ =$

$\texttt{Assume} \ p, q \in (1, \infty) : (1/p) + (1/q) = 1,$

$\texttt{Assume} \ f \in L^p(\Omega, \mathcal{F}, \mu),$

$\texttt{Assume} \ g \in L^q(\Omega, \mathcal{F}, \mu),$

$a := \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} \in \mathit{Reals}_+,$

$b := \sqrt[q]{\int_\Omega |g|^q \mathrm{d}\mu} \in \mathbb{R}_+,$

$\texttt{Assume Alternative} \ A : a = 0 | b = 0,$

$I_1 := \texttt{LowerBound}(\mathbb{R}_+, A) : ab = 0 = \int_\Omega |fg| \mathrm{d}\mu;$

$\texttt{Close Alternative} \ A : a \neq 0 \ \& \ b \neq 0,$

$\texttt{Assume} \ \omega \in \Omega,$

$\texttt{Ineq1}(|f(\omega)|/a, |g(\omega)|/b, p, q) : \frac{|f(\omega)||g(\omega)|}{ab} \leq \frac{|f(\omega)|^p}{pa^p} + \frac{|g(\omega)|^q}{qb^q};$

$R_1 : \forall \omega \in \Omega \ . \ \frac{|f(\omega)||g(\omega)|}{ab} \leq \frac{|f(\omega)|^p}{pa^p} + \frac{|g(\omega)|^q}{qb^q},$

$\texttt{IntegralIneq}(R_1) : \frac{1}{ab} \int_\Omega |fg| \mathrm{d}\mu = \int_\Omega \frac{|f||g|}{ab} \mathrm{d}\mu \leq \int_\Omega \frac{|f(\omega)|^p}{pa^p} + \frac{|g(\omega)|^q}{qb^q} \mathrm{d}\mu =$

$$= \frac{1}{pa^p}\int_\Omega |f(\omega)|^p \mathrm{d}\mu + \frac{1}{qb^q}\int_\Omega |g|^q \mathrm{d}\mu = \frac{1}{p} + \frac{1}{q} = 1 \rightsquigarrow \int_\Omega |fg|\mathrm{d}\mu \le ab; ; ; ; \square$$

Ineq3 $:: \forall a, b \in \mathbb{R}_+ . \forall p \in [1, \infty) . (a+b)^p \le 2^{p-1}(a^p + b^p)$

Proof $=$

Assume $a, b \in \mathbb{R}_+$,

$p \in [1, \infty)$,

$f := \Lambda x \in \mathbb{R}_+ . (a+x)^p - 2^{p-1}(a^p + x^p) : \mathbb{R}_+ \to \mathbb{R}$,

$E_1 :=$ ðDerivative $: f' = \Lambda x \in \mathbb{R}_+ . p(a+x)^{p-1} - 2^{p-1}px^{p-1}$,

$E_1 \rightsquigarrow f'(a) = 0 \rightsquigarrow a :$ Extremum(f),

Assume $x \in \mathbb{R}_+ : x > a$,

$f'(x) > 0$;

Assume $x \in \mathbb{R}_+ : x < a$,

$f'(x) < 0$;

$a :$ Maximum(f),

$f(a) = 0 \rightsquigarrow (a+b)^p \le 2^{p-1}(a^p + b^p); ; \square$

MinkowskiInequality $:: \forall p \in [1, \infty) . \forall f, g \in L^p(\Omega, \mathcal{F}, \mu) . f + g \in L^p$ &

$$\& \sqrt[p]{\int_\Omega |f+g|^p \mathrm{d}\mu} \le \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} + \sqrt[p]{\int_\Omega |g|^p \mathrm{d}\mu}$$

Proof $=$

Assume $p \in [1, \infty)$,

Assume $f, g \in L^p$,

Assume $\omega \in \Omega$,

$I_1 :=$ Ineq3$(|f(\omega)|, |g(\omega)|, p) : (|f(\omega)| + |g(\omega)|)^p \le 2^{p-1}(|f(\omega)|^p + |g(\omega)|^p)$,

$I_2 :=$ MonotoneActIneq(AbsSumIneq$(f(\omega), g(\omega)), \mathrm{id}^p) : |f(\omega) + g(\omega)|^p \le (|f(\omega)| + |g(\omega)|)^p$,

$I_2 I_1 : |f(\omega) + g(\omega)|^p \le 2^{p-1}(|f(\omega)|^p + |g(\omega)|^p)$;

$R_1 : \forall \omega \in \Omega : |f(\omega) + g(\omega)|^p \le 2^{p-1}(|f(\omega)|^p + |g(\omega)|^p)$,

$$I_1 := \texttt{IntegralIneq}(R_1) : \int_\Omega |f+g|^p \mathrm{d}\mu \le 2^{p-1}\int_\Omega |f|^p \mathrm{d}\mu + 2^{p-1}\int_\Omega |g|^p \mathrm{d}\mu < \infty \rightsquigarrow$$

$\rightsquigarrow f + g \in L^p$,

Assume Alternative $p = 1$,

$$I_1 \rightsquigarrow \int_\Omega |f+g|\mathrm{d}\mu \le \int_\Omega |f|\mathrm{d}\mu + \int_\Omega |g|\mathrm{d}\mu;$$

Close Alternative $p > 1$,

$q := p/(p-1) \in (1, +\infty)$,

Assume $\omega \in \Omega$,

$\texttt{MonotoneActIneq}\left(\texttt{AbsSumIneq}(f(\omega), g(\omega)), \mathrm{id}|f(\omega)+g(\omega)|^{(p-1)}\right) : |f(\omega)+g(\omega)|^p \leq$
$\leq |f(\omega)+g(\omega)||f(\omega)+g(\omega)|^{(p-1)} \leq |f(\omega)||f(\omega)+g(\omega)|^{(p-1)} + |g(\omega)||f(\omega)+g(\omega)|^{(p-1)};$
$A_2 : \forall \omega \in \Omega \;.\; |f(\omega)+g(\omega)|^p \leq |f(\omega)||f(\omega)+g(\omega)|^{(p-1)} + |g(\omega)||f(\omega)+g(\omega)|^{(p-1)};$
$\int_\Omega (|f+g|^{p-1})^q \mathrm{d}\mu = \int_\Omega |f+g|^p \mathrm{d}\mu \rightsquigarrow |f+g|^{p-1} \in L^q(\Omega, \mathcal{F}, \mu);$
$E_1 := \eth(1/q) + 1/p : \dfrac{1}{q} + \dfrac{1}{p} = \dfrac{1}{p} + \dfrac{p-1}{p} = 1,$
$I_2 := \texttt{HölderInequality}\Big(|f|, |f+g|^{p-1}, p, q\Big) : \int_\Omega |f||f+g|^{p-1} \mathrm{d}\mu \leq$
$\leq \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} \sqrt[q]{\int_\Omega |f+g|^p \mathrm{d}\mu},$
$I_3 := \texttt{HölderInequality}\Big(|g|, |f+g|^{p-1}, p, q\Big) : \int_\Omega |g||f+g|^{p-1} \mathrm{d}\mu \leq$
$\leq \sqrt[p]{\int_\Omega |g|^p \mathrm{d}\mu} \sqrt[q]{\int_\Omega |f+g|^p \mathrm{d}\mu},$
$(I_2, I_3, A_2) : \int_\Omega |f+g|^p \mathrm{d}\mu \leq \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} \sqrt[q]{\int_\Omega |f+g|^p \mathrm{d}} + \sqrt[p]{\int_\Omega |g|^p \mathrm{d}\mu} \sqrt[q]{\int_\Omega |f+g|^p \mathrm{d}\mu} =$
$= \left(\sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} + \sqrt[p]{\int_\Omega |g|^p \mathrm{d}\mu}\right) \sqrt[q]{\int_\Omega |f+g|^p \mathrm{d}\mu} \rightsquigarrow_{E_1}$
$\rightsquigarrow_{E_1} \sqrt[p]{\int_\Omega |f+g|^p \mathrm{d}p} \leq \left(\sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu} + \sqrt[p]{\int_\Omega |g|^p \mathrm{d}\mu}\right);;; \square$

$\texttt{LpIsVS} :: \forall p \in [1, \infty) \;.\; L^p : \mathsf{VS}(\mathbb{C})$

1.3 L^p as Topological Vector Space

$\texttt{LpSeminorm} :: L^p(\Omega, \mathcal{F}, \mu) \to \mathbb{R}_+$

$$\texttt{LpSeminorm}(f) = \|f\|_p := \sqrt[p]{\int_\Omega |f|^p \mathrm{d}\mu}$$

$\texttt{LpSpace} :: [1, \infty) \to \mathsf{MAES} \to \mathsf{NVS}(\mathbb{C})$

$$\texttt{LpSpace}(p)(X) = \mathbf{L}^p(X) = \left(\frac{L^p(X)}{\{(f,g) \in (L^p(L))^2 : \|f - g\|_p = 0\}}, \|\cdot\|_p \right)$$

$\texttt{ChebyshevIneq} :: \forall f \in L^p(\Omega, \mathcal{F}, \mu) : f > 0 \,.\, \forall t \in \mathbb{R}_{++} \,.\, \mu\{\omega \in O : f(\omega) > t\} \le \dfrac{1}{t^p} \displaystyle\int_\Omega f^p \mathrm{d}\mu$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f \in L^p(\Omega, \mathcal{F}, \mu) : f > 0,$

$\texttt{Assume}\ t \in \mathbb{R}_{++},$

$A := \mu\{\omega \in O : f(\omega) > t\} \in \mathcal{F},$

$$\int_\Omega f^p \mathrm{d}\mu \ge \int_A f^p \mathrm{d}\mu \ge \int_A t^p \mathrm{d}\mu = t^p \int_A \mathrm{d}\mu = t^p \mu(A) \rightsquigarrow \mu(A) \le \frac{1}{t_p} \int_\Omega f^p \mathrm{d}\mu \,\square$$

$\texttt{LpConvergenceLemma} :: \forall f : \mathbb{N} \to L^p(\Omega, \mathcal{F}, \mu) : \forall k \in \mathbb{N} \,.\, \|f_k - f_{k+1}\|_p \le (1/4)^k \,.$

$\quad .\, f : \texttt{Converge}(\mathbb{C}) \quad \mathrm{a\,.\,e\,.}\ [\mu]$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \mathbb{N} \to L^p(\Omega, \mathcal{F}, \mu) : \forall k \in \mathbb{N} \,.\, \|f_k - f_{k+1}\|_p \le (1/4)^k,$

$\texttt{Assume}\ k \in \mathbb{N},$

$A_k := \left\{\omega \in \Omega : |f_k(\omega) - f_{k+1}(\omega)| \le 2^{-k}\right\} \in \mathcal{F},$

$$\texttt{ChebyshevIneq}(|f_k - f_{k-1}|, 2^{-k}) : \mu(A) \le 2^k \int_\Omega |f_k(\omega) - f_{k+1}(\omega)| \mathrm{d}\mu = \frac{1}{2^k};$$

$$a \rightsquigarrow \mu\left(\limsup_n A_n\right) = 0 \rightsquigarrow f : \texttt{Cauchy}(\mathbb{C}) \quad \mathrm{a\,.\,e\,.}\ [\mu] \rightsquigarrow f : \texttt{Convergent}(\mathbb{C}) \square$$

$\texttt{LpIsComplete} :: \mathbf{L}^p(X) : \texttt{Complete}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f : \texttt{Cauchy}(\mathbf{L}^p(X)),$

$\texttt{Assume}\ k \in \mathbb{N},$

$N_k := \eth\texttt{Cauchy}(\mathbf{L}^p(X))(f) \in \mathbb{N} : \forall n, m \in \mathbb{N} : n \ge N : m \ge N : d(f_n, f_m) \le 4^{-k};$

$N : \texttt{Subseqer},$

$g := \texttt{subseq}(f, N) : \mathbb{N} \to \mathbf{L}^p(X),$

$\texttt{LpConvergenceLemma}(g) : \big(g : \texttt{Convergent}\ \mathrm{a\,.\,e\,.}\ [\mu_X]\big)$

$\phi := \lim_{n\to\infty} g_n : \mathbf{L}^p(X),$

$\texttt{Assume}\ \epsilon \in \mathbb{R}_{++},$

$M := \eth\texttt{Cauchy}(\mathbf{L}^p(X))(f) \in \mathbb{N} : \forall n, m \in \mathbb{N} : n > M : m > M \ .\ d(f_n, f_m) < \epsilon,$

$\texttt{Assume}\ n \in \mathbb{N} : n \geq M,$

$N' := [N_k : N_k \geq M] : \texttt{Subseqer},$

$h := f \circ N' : \mathbb{N} \to \mathbf{L}^p(X),$

$$I_1 := \eth\epsilon : \epsilon^p > \liminf_{m\to\infty} \mathrm{d}^p(f_n, h_m) = \liminf_{n\to\infty} \int_X |f_m - h_n|^p \mathrm{d}\mu_X \geq$$
$$\geq \int_X \liminf_{m\to\infty} |f_m - h_n|^p \mathrm{d}\mu_X = \int_X |f_n - \phi|^p \mathrm{d}\mu_X = d^p(f_m, \phi);$$

$\exists n \in \mathbb{N} \ .\ d(f_m, \phi) \leq \epsilon;$

$a_2 : \lim_{n\to\infty} f_n = \phi \rightsquigarrow f : \texttt{Convergent}(\mathbf{L}^p(X));$

$\mathbf{L}^p(X) : \texttt{Complete}\ \square$

$\texttt{LpSeq} :: [1, \infty) \to \texttt{Set} \to \mathsf{NVS}(\mathbb{C}),$

$\texttt{LpSeq}(p)(X) = l^p(X) := \mathbf{L}^p(X, 2^X, \#)$

$\texttt{SimpleAreDense} :: \forall X \in \mathsf{MEAS}\ .\ \texttt{Simple}(X) : \texttt{Dense}(\mathbf{L}^p(X))$

$\texttt{ContAreDense} :: \left[C\left(\mathbb{R}^d\right)\right] : \texttt{Dense}(\mathbf{L}^p(X))$

1.4 L^∞ Space

$\texttt{EssentialSupremum} :: \texttt{Measurable}(\Omega, \mathcal{F}, \mu) \rightarrow \overset{\infty}{\mathbb{R}}$

$\texttt{EssentialSupremum}(f) = \operatorname{ess\,sup} f := \inf \left\{ c \in \overset{\infty}{\mathbb{R}} : \mu\{\omega \in \Omega : f(\omega) > c\} = 0 \right\}$

$\texttt{LInftySeminorm} :: \texttt{ComplexValuedBorelMeasurableFunction} \rightarrow \overset{\infty}{\mathbb{R}}_+$

$\texttt{LInftySeminorm}(f) = \|f\|_\infty = \operatorname{ess\,sup} |f|$

Extend $L^{\cdot}(X)$ on $\{\infty\}$

$L^\infty(X) = \{f : \texttt{ComplexValuedBorelMeasurableFunction} : \|f\|_\infty < \infty\}$

Extend $\mathbf{L}^{\cdot}(X)$ on $\{\infty\}$

Extend $l^{\cdot}(X)$ on $\{\infty\}$