Boolean Algebra

Uncultured Tramp

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Contents

1 Boolean Rings and Algebras

1.1 Stone Theory

1.1.1 Definition And Examples

$\texttt{BooleanRing} :: ?\mathsf{RNG}$
$B : \texttt{BooleanRing} \iff \forall b \in B \,.\, b^2 = b$

$\texttt{BooleanAlgebra} :: ?\mathsf{RING}$
$B : \texttt{BooleanAlgebra} \iff \forall b \in B \,.\, b^2 = b$

$\texttt{AlgebraOfSubsets} :: \forall X \in \mathsf{SET} \,.\, \texttt{BooleanAlgebra}(?X, \cap, \triangle)$
$\texttt{Proof} =$
$[1] := \mathsf{IRNG}\Big(\Lambda A, B, C : ?X \,.\, \texttt{CheckingTruthTables}\big(A \cap (B \triangle c)), (A \cap B) \triangle (A \cap C)\big)\Big) : (?X, \cap, \triangle) \in \mathsf{RNG},$
$[2] := \mathsf{IRING}\Big([1], \Lambda A : ?X \,.\, \texttt{CheckingTruthTables}\big(A \cap X, A\big)\Big) : (?X, \cap, \triangle) \in \mathsf{RING},$
$[*] := \texttt{IBooleanAlgebra}\Big([2], \texttt{CheckingTruthTables}\big(A \cap A, A\big)\Big) : \texttt{BooleanAlgebra}(X^2, \cap, \triangle);$
$\square$

$\texttt{SetTheoreticAlgebra} :: \prod_{X \in \mathsf{SET}} ?^3 X$
$\mathcal{A} : \texttt{SetTheoreticAlgebra} \iff \texttt{Algebra}(X) \iff$
$\iff (\emptyset, X \in \mathcal{A}) \;\&\; (\forall A, A' \in \mathcal{A} \,.\, A \triangle A' \in \mathcal{A}) \;\&\; (\forall A, A' \in \mathcal{A} \,.\, A \cap A' \in \mathcal{A})$

$\texttt{SetTheoreticAlgebraIsBoolean} :: \forall X \in \mathsf{SET} \,.\, \forall \mathcal{A} : \texttt{Algebra}(X) \,.\, \texttt{BooleanAlgebra}(\mathcal{A}, \cap, \triangle)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{Bool} := \mathbb{B} = \top | \top : \texttt{Type};$

$\texttt{BooleanAdd} :: \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
$\texttt{BooleanAdd}\,(a, b) = a + b := a \oplus b$

$\texttt{BooleanMult} :: \mathbb{B} \times \mathbb{B} \to \mathbb{B}$
$\texttt{BooleanMult}\,(a, b) = ab := a \wedge b$

$\texttt{BoolIsASet} :: \mathbb{B} \in \mathsf{SET}$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{BoolIsBooleanAlgbra} :: \texttt{BooleanAlgebra}(\mathbb{B}, \cdot, +)$
$\texttt{Proof} =$
$\dots$
$\square$

$\texttt{BoolIsAField} :: \texttt{Field}(\mathbb{B}, \cdot, +)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{BooleanRingHasChar2} :: \forall A : \mathsf{CRNG} \,.\, \forall a \in A \,.\, a + a = 0$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a \in A,$

$[2] := \texttt{EBooleanRing}(A)[1](a+a)\texttt{BinomialExpansion}(A,a,a,2)\texttt{EBooleanRing}(A)[1](a) :$

$\quad : a + a = (a+a)^2 = a^2 + 2a^2 + a^2 = (a+a) + (a+a),$

$[*] := [2] - (a+a) : a + a) = 0;$

$\rightsquigarrow [*] := \texttt{I}\forall : \forall a \in A \,.\, a + a = 0;$

$\texttt{BooleanRingIsCommutative} :: \forall A : \texttt{BooleanRing} \,.\, A \in \mathsf{CRNG}$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a, b : A,$

$[1] := \texttt{EBooleanRing}(A)(a+b)\texttt{BinomialExpansion}(A,a,b,2)\texttt{EBooleanRing}(A)(a)\texttt{EBooleanRing}(A)(a) :$

$\quad : a + b = (a+b)^2 a^2 + ab + ba + b^2 = a + ab + ba + b,$

$[2] := [1] - a - b - ab : -ab = ba,$

$\Big[(a,b).*] := \texttt{BooleanRingHasChar2}[2] : ab = ba;$

$\rightsquigarrow [*] := \texttt{I}\mathsf{CRNG} : A \in \mathsf{CRNG},$

$\dots$

$\square$

$\texttt{BooleanSumByParity} :: \forall n \in \mathbb{N} \,.\, \forall b : [1,\dots,n] \to \mathbb{B} \,.\, \bigoplus_{i=1}^{n} b_i = 1 \iff \texttt{Odd}\left|b^{-1}(1)\right|$

$\texttt{Proof} \;=$

$\triangle := \Lambda n \in \mathbb{N} \,.\, \forall b : [1,\dots,n] \to \mathbb{B} \,.\, \bigoplus_{i=1}^{n} b_i = 1 \iff \texttt{Odd}\left|b^{-1}(1)\right| : \mathbb{N} \to \texttt{Type},$

$\texttt{Assume}\ b : [1,\dots,1] \to \mathbb{B},$

$\texttt{Assume}\ [1] : \bigoplus_{i=1}^{1} b_i = 1,$

$[2] := [1]\texttt{EiteratedOperator}(\mathbb{B}, b, 1) : 1 = \bigoplus_{i=1}^{1} b_i = b_1,$

$[3] := \texttt{SingletonPreimage}[2] : b^{-1}(1) = \{1\},$

$[4] := \Big|[3]\Big| : \left|b^{-1}(1)\right| = 1,$

$[1.*] := \texttt{OnnIsOdd}[4] : \texttt{Odd}\left|b^{-1}(1)\right|;$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : \bigoplus_{i=1}^{1} b_i \Rightarrow \texttt{Odd}\left|b^{-1}(1)\right|,$

$\texttt{Assume}\ [2] : \texttt{Odd}\left|b^{-1}(1)\right|,$

$[3] := \texttt{SubsetCardinality}\Big([1,\dots,1], b^{-1}(1)\Big)\texttt{SingletonCardinality}\Big([1,\dots,1]\Big) : \left|b^{-1}(1)\right| \leq \left|[1,\dots,1]\right| = 1,$

$[4] := \texttt{EOdd}[2][3] : \left|b^{-1}(1)\right| = 1,$

$[5] := \mathsf{ECARD}[4] : b^{-1}(1) = \{1\},$

$[6] := \texttt{SingletonPreimage}[5] : b(1) = 1,$

$$[7] := \texttt{EIteratedOperatort}[6] : \bigoplus_{i=1}^{1} b_i = 1,$$

$$\rightsquigarrow [2] := \texttt{I} \Rightarrow : \texttt{Odd}\left|b^{-1}(1)\right| \Rightarrow \bigoplus_{i=1}^{1} b_i = 1,$$

$$[b.*] := \texttt{I} \iff [2][3] : \bigoplus_{i=1}^{1} b_i = 1 \iff \texttt{Odd}\left|b^{-1}(1)\right|;$$

$\rightsquigarrow [1] := \texttt{I}\forall\texttt{I}\underline{\frown} : \underline{\frown}(1),$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ [2] : \underline{\frown}(n),$

$\texttt{Assume}\ b : [1,\dots,n+1] \to \mathbb{B},$

$$\texttt{Assume}\ [3] : \bigoplus_{i=1}^{n+1} b_i = 1,$$

$$[4] := \texttt{EiteratedOperator}(\oplus)[3] : 1 = \bigoplus_{i=1}^{n+1} b_i = b_{n+1} \oplus \bigoplus_{i=1}^{n} b_i,$$

$$[5] := \texttt{E}\bigoplus[4] : b_{n+1} = 0\ \&\ \bigoplus_{i=1}^{n} b_i = 1 \Bigg| b_{n+1} = 1\ \&\ \bigoplus_{i=1}^{n} b_i = 0,$$

$[3.*] := \texttt{E}\underline{\frown}[2][5] : \texttt{Odd}\left|b^{-1}(1)\right|;$

$$\rightsquigarrow [3] := \Rightarrow : \bigoplus_{i=1}^{n+1} b_i = 1 \Rightarrow \texttt{Odd}\left|b^{-1}(1)\right|,$$

$$[4] := \texttt{PigeonholePrinciple} : \texttt{Odd}\left|b^{-1}(1)\right| \Rightarrow \bigoplus_{i=1}^{n+1} b_i = 1,$$

$$[n.*] := \texttt{I}(\iff)[3][4] : \bigoplus_{i=1}^{n+1} b_i = 1 \Rightarrow \texttt{Odd}\left|b^{-1}(1)\right|;$$

$\rightsquigarrow [n.*] := \texttt{E}\mathbb{N} : \forall n \in \mathbb{N} . \underline{\frown}(n),$

$\square$

$$\texttt{SymmetricSumByParity} :: \forall X \in \mathsf{SET} . \forall n \in \mathbb{N} . \forall A : n \to ?X . \sum_{i=1}^{n} A_n = \Big\{x \in X : \texttt{Odd}\big\{i \in [1,\dots,n] : x \in A_i\big\}$$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{FiniteBooleanRingCard} :: \forall A : \texttt{BooleanRing} \,.\, |A| < \infty \Rightarrow \exists k \in \mathbb{Z}_+ \,.\, |A| = 2^k$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{FiniteBooleanRingIsAlgebra} :: \forall A : \texttt{BooleanRing} \,.\, |A| < \infty \Rightarrow \texttt{BooleanAlgebra}(A)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.1.2 First form of Stone's Theorem

$\texttt{FirstStoneLemma} :: \forall A : \texttt{BooleanRing} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall a \in A \setminus I \,.\, \exists A \xrightarrow{\phi} \mathbb{B} : \mathsf{RNG} \,.\, \phi(a) = 1 \;\&\; \phi(I) = \{0\}$

$\texttt{Proof} \;=$

$\Big(J, [1]\Big) := \texttt{MaximalIdealExists}(A, I, a) : \sum J : \texttt{MaximaIdeal}(A, a) \,.\, I \subset J,$

$K := \Lambda b \in A \,.\, \{c \in A : cb \in J\} : A \to \texttt{Ideal}(A),$

$[2] := \mathtt{E}K[1]\mathtt{E}\texttt{MaximalIdeal}(A, a, J) : \forall b \in A \,.\, a \notin K_b \Rightarrow K_b = J,$

$[3] := \mathtt{E}\texttt{BooleanRing}(A)[2] : K_a = J,$

$[4] := [3][2] : \forall b \in J^{\complement} \,.\, K_b = J,$

$[5] := [4][4] : \forall b, c \in J^{\complement} \,.\, bc \in J^{\complement},$

$\texttt{Assume}\ b, c \in J^{\complement},$

$[6] := [5](bc) : bc \in J^{\complement},$

$[7] := [4][6] : K_{bc} = J,$

$[8] := \mathtt{E}\mathsf{RNG}(A, bc, b, c)\mathtt{E}\texttt{BooleanRing}(A)(b)\mathtt{E}\texttt{BooleanRing}(A)(c)\texttt{BooleanRingHasChar2} :$

$\quad : bc(b + c) = b^2c + bc^2 = bc + bc = 0,$

$[9] := \mathtt{E}K_{bc}[8] : b + c \in K_{ab},$

$[*] := [9][7] : b + c \in J;$

$\rightsquigarrow [6] := \mathtt{I}\forall : \forall b, c \in J^{\complement} \,.\, a + b \in J,$

$\phi := \Lambda b \in A \,.\, \bigwedge_{j \in J} j \mathrel{!=} b : A \to \mathbb{B},$

$[7] := \mathtt{E}\phi(a)[1] : \phi(a) = 1,$

$[8] := \mathtt{E}\phi(I)[1] : \phi(I) = \{0\},$

$\texttt{Assume}\ c, b \in A,$

$\texttt{Assume}\ [9] : \phi(c + b) = 0,$

$[10] := \mathtt{E}\phi[9] : c + b \in J,$

$[11] := \mathtt{E}\texttt{Ideal}(A, j)[10] : c, b \in J \Big| c, b \in J^{\complement},$

$[9.*] := \mathtt{E}\phi[3][6]\mathtt{E}\mathbb{B} : \phi(c) + \phi(b) = \phi(c + b);$

$\rightsquigarrow [9] := \mathtt{I} \Rightarrow : \phi(c + b) = 0 \Rightarrow \phi(c + b) = \phi(c) + \phi(b),$

$\texttt{Assume}\ [10] : \phi(c + b) = 1,$

$[10] := \mathtt{E}\phi[9] : c + b \notin J,$

$[11] := [10][6] : (c, b) \in J \times J^{\complement} \Big| (c, b) \in J^{\complement} \times J,$

$[10.*] := \mathtt{E}\phi[11]\mathtt{E}\mathbb{B} : \phi(c) + \phi(b) = \phi(c + b);$

$\rightsquigarrow [10] := \mathtt{I} \Rightarrow : \phi(c + b) = 1 \Rightarrow \phi(c + b) = \phi(c) + \phi(b),$

$\Big([c, b].*\Big) := \mathtt{E}|\texttt{BooleanAlternative}\Big(\phi(c + b)\Big)[4][5] : \phi(c + b) = \phi(c) + \phi(b),$

$\texttt{Assume}\ [11] : \phi(cb) = 1,$

$[12] := \mathtt{E}\phi[11] : cb \notin J,$

$[13] := \mathtt{E}\texttt{Ideal}[12] : c, b \notin J,$

$[11.*] := [13]\mathtt{E}\phi\mathtt{E}\mathbb{B} : \phi(c)\phi(b) = \phi(cb);$

$\rightsquigarrow [11] := \mathtt{I} \Rightarrow : \phi(cb) = 1 \Rightarrow \phi(cb) = \phi(c)\phi(b),$

$\texttt{Assume}\ [12] : \phi(cb) = 0,$

$[13] := \mathtt{E}\phi[11] : cb \in J,$

$[14] := [5][14] : c \in J | b \in J,$

$[12.*] := [14]\mathtt{E}\phi\mathtt{E}\mathbb{B} : \phi(c)\phi(b) = \phi(cb);$

$\rightsquigarrow [12] := \mathtt{I} \Rightarrow: \phi(cb) = 0 \Rightarrow \phi(cb) = \phi(c)\phi(b),$

$\Big([c, b]. *\Big) := \mathtt{E}|\mathtt{BooleanAlternative}\Big(\phi(cb)\Big)[4][5] : \phi(cb) = \phi(c)\phi(b);$

$\rightsquigarrow [*] := \mathtt{I}\mathsf{RNG} : \phi \in \mathsf{RNG}(A, \mathbb{B});$

$\square$

$\mathtt{StoneSpace} :: \mathtt{BooleanRing} \to \mathsf{SET}$

$\mathtt{StoneSpace}\,(A) = Z_A := \left\{A \xrightarrow{\phi} \mathbb{B} : \mathsf{RNG} : \phi \neq 0\right\}$

$\mathtt{StoneRepresentation} :: \prod A : \mathtt{BooleanRing}\,.\, A \to ?Z_A$

$\mathtt{StoneRepresentation}\,(a) = S_A(a) := \{\phi \in Z_A : \phi(a) = 1\}$

$\mathtt{StoneTHM1stForm} :: \forall A : \mathtt{BooleanRing}\,.\, \mathtt{Injective}\ \&\ \mathsf{RNG}(A, ?Z_A, S_A)$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ a, b : A,$

$\mathtt{Assume}\ [1] : a \neq b,$

$\mathtt{Assume}\ [2] : a \in \langle b\rangle, b \in \langle a\rangle,$

$\Big(c, d, [3]\Big) := \mathtt{EIdeal}[2] : b = ac\ \&\ a = bd,$

$[4] := [3]^3\mathtt{EBooleanRing}(A) : a = acd = bcd^2 = bcd\ \&\ b = bcd,$

$[5] := [4][1] : \perp;$

$\rightsquigarrow [2] := \mathtt{E}\perp : a \notin \langle b\rangle \Big| b \notin \langle a\rangle,$

$\mathtt{Assume}\ [3] : a \notin \langle b\rangle,$

$\Big(\phi, [4]\Big) := \mathtt{FirstStoneLemma}\Big(A, \langle b\rangle, a\Big) : \sum A \xrightarrow{\phi} \mathbb{B} : \phi(a) = 1\ \&\ \phi\langle b\rangle = \{0\},$

$[5] := \mathtt{Eimage}[4] : \phi(b) = 0,$

$[3.*] := \mathtt{E}S_A[3][4] : S_A(a) \neq S_A(b);$

$\rightsquigarrow [3] := \mathtt{I} \Rightarrow: a \notin \langle b\rangle \Rightarrow S_A(a) \neq S_A(b),$

$\mathtt{Assume}\ [4] : b \notin \langle a\rangle,$

$\Big(\phi, [5]\Big) := \mathtt{FirstStoneLemma}\Big(A, \langle a\rangle, b\Big) : \sum A \xrightarrow{\phi} \mathbb{B} : \phi(b) = 1\ \&\ \phi\langle a\rangle = \{0\},$

$[6] := \mathtt{Eimage}[5] : \phi(a) = 0,$

$[4.*] := \mathtt{E}S_A[3][4] : S_A(a) \neq S_A(b);$

$\rightsquigarrow [4] := \mathtt{I} \Rightarrow: b \notin \langle a\rangle \Rightarrow S_A(a) \neq S_A(b),$

$\Big[(a, b). *\Big] := \mathtt{E}(|)[2][3][4] : S_A(a) \neq S_A(b);$

$\rightsquigarrow [*.1] := \mathtt{IInjective} : \mathtt{Injective}\Big(A, ?Z_A, S_A\Big),$

$[*.2] := \mathtt{E}Z_A\mathtt{E}\mathsf{RNG}\mathtt{E}\mathbb{B}\mathtt{I}\mathsf{RNG} : S_Z \in \mathsf{RNG}(A, ?Z_A);$

$\square$

1.1.3 Translating set theoretic notions

$$\texttt{andOperator} :: \prod A : \texttt{BooleanRing}(X) \ . \ A^2 \to A$$
$$\texttt{andOperator}\,(a,b) = a \cap b := ab$$

$$\texttt{andOperator} :: \prod A : \texttt{BooleanRing}(X) \ . \ A^2 \to A$$
$$\texttt{andOperator}\,(a,b) = a \cup b := a + b + ab$$

$$\texttt{symmetricDifferenceOperator} :: \prod A : \texttt{BooleanRing}(X) \ . \ A^2 \to A$$
$$\texttt{symmetricDifferenceOperator}\,(a,b) = a \bigtriangleup b := a + b$$

$$\texttt{setMinusOpera} :: \prod A : \texttt{BooleanRing}(X) \ . \ A^2 \to A$$
$$\texttt{andOperator}\,(a,b) = a \setminus b := (a+b)a$$

$$\texttt{Disjoint} :: \prod A : \texttt{BooleanRing}(X) \ . \ ?A^2$$
$$(a,b) : \texttt{Disjoint} \iff a \perp b \iff ab = 0$$

$$\texttt{PairwiseDisjointElements} :: \prod A : \texttt{BooleanRing}(X) \ . \ ??A$$
$$P : \texttt{PairwiseDisjointElements} \iff \forall a, b \in A \ . \ a \perp b$$

$$\texttt{PartitionOfUnity} :: \prod A : \texttt{BooleanRing}(X) \ . \ ?\texttt{PairwiseDisjointElements}(A)$$
$$P : \texttt{PartitionOfUnity} \iff \forall c \in A \ . \ c \neq 0 \Rightarrow \exists a \in P : ac \neq 0$$

$$\texttt{PartitionOfUnityIsMaximalDisjoint} :: \forall A : \texttt{BooleanRing} \ . \ \forall P : \texttt{PairwiseDisjointElements}(A) \ .$$
$$. \ \texttt{PartitionOfUnity}(A, P) \iff P \in \max \texttt{PairwiseDisjointElements}(A)$$

Proof =

. . .

$\square$

$$\texttt{DisjointHasPartitionOfUnity} :: \forall A : \texttt{BooleanRing} \ . \ \forall B : \texttt{PairwiseDisjointElements}(A) \ .$$
$$. \ \exists P : \texttt{PartitionOfUnity}(A) \ . \ P \subset B$$

Proof =

. . .

$\square$

$$\texttt{Refinement} :: \prod A : \texttt{BooleanRing} \ . \ \texttt{PartitionOfUnity}(A) \to ?\texttt{PartitionOfUnity}(A)$$
$$Q : \texttt{Refinement} \iff \Lambda P : \texttt{PartitionOfUnity}(A) \ . \ \forall p \in P \ . \ \exists q \in Q : pq = q$$

1.1.4 Order of Boolean Ring

$\texttt{BooleanOrder} :: \prod A : \texttt{BooleanRing} \, . \, ?(A \times A)$

$a, b : \texttt{BooleanOrder} \iff a \leq b \iff ab = a$

$\texttt{BooleanOrderByStoneRepresentation} :: \forall A : \texttt{BooleanRing} \, . \, \forall a, b \in A \, . \, a \leq b \iff S_A(a) \subset S_A(b)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{BooleanOrderIsPartialOrder} :: \forall A : \texttt{BooleanRing} \, . \, \texttt{PartialOrder}\Big(A, \leq\Big)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{booleanRingAsPoset} :: \texttt{BooleanRing} \to \mathsf{POSET}$

$\texttt{booleaRingAsPoset}\,(A) = A := (A, \leq)$

$\texttt{MinimalElementInBooleanRing} :: \forall A : \texttt{BooleanRing} \, . \, \min A = 0$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{MaximalElementInBooleanAlgebra} :: \forall A : \texttt{BooleanAlgebra} \, . \, \max A = e_A$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{BooleanRingIsLattice} :: \forall A : \texttt{BooleanRing} \, . \, (A, \cap, \cup) \in \mathsf{LAT}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{booleanRingAsLattice} :: \texttt{BooleanRing} \to \mathsf{LAT}$

$\texttt{booleaRingAsLattice}\,(A) = A := (A, \cap, \cup)$

1.1.5 Topology of Stone Space

$$\texttt{StoneTopology} :: \prod A : \texttt{BooleanRing} \,.\, ??Z_A$$

$$\texttt{StoneTopology}\,() = \mathcal{T} := \Big\{ U \subset Z_A : \forall u \in U \,.\, \exists a \in A : u \in S_A(a) \subset U \Big\}$$

$\texttt{StoneTopologyIsTopology} :: \forall A : \texttt{BooleanRing} \,.\, \texttt{Topology}(Z_A, \mathcal{T}_A)$

$\texttt{Proof} =$

$[1] := \texttt{E}\mathcal{T}_A : \emptyset \in \mathcal{T}_A,$

$\texttt{Assume}\ f \in Z_A,$

$\Big(a, [2]\Big) := \texttt{E}Z_A(f) : \sum a \in A \,.\, f(a) = 1,$

$[3] := \texttt{E}S_A[2] : f \in S_A(a),$

$[f.*] := \texttt{E}S_A(a) : S_A(a) \subset Z_A;$

$\rightsquigarrow [1] := \texttt{E}\mathcal{T}_Z : Z_A \in \mathcal{T}_Z,$

$\texttt{Assume}\ \mathcal{U} :?\mathcal{T}_Z,$

$\texttt{Assume}\ u \in \bigcup \mathcal{U},$

$\Big(U, [2]\Big) := \texttt{Eunion}\mathcal{U} : \sum U \in \mathcal{U} \,.\, u \in U,$

$[3] := \texttt{ESubset}(\mathcal{U}, U) : U \in \mathcal{T}_A,$

$\Big(a, [4]\Big) := \texttt{E}\mathcal{T}_Z(U, u) : \sum a \in A \,.\, u \in S_A(a) \subset U,$

$[u.*] := \texttt{SubsetOfUnion}[4](\mathcal{U}, U) : u \in S_A(a) \subset \bigcup \mathcal{U};$

$\rightsquigarrow [\mathcal{U}.*] := \texttt{E}\mathcal{T}_A : \bigcup \mathcal{U} \in \mathcal{T}_A;$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall \mathcal{U} \in ?\mathcal{T}_A \,.\, \bigcup \mathcal{U} \in \mathcal{T}_A,$

$\texttt{Assume}\ n \in \mathbb{N},$

$\texttt{Assume}\ U : [1, \ldots, n] \to \mathcal{T}_A,$

$\texttt{Assume}\ u \in \bigcap_{i=1}^{n} U_n,$

$[3] := \texttt{Eintersect}(U, u) : \forall n \in \mathbb{N} \,.\, u \in U_n,$

$\Big(a, [4]\Big) := \texttt{E}\mathcal{T}_A[3] : \sum a : [1, \ldots, n] \to A : \forall i \in [1, \ldots, n] \,.\, u \in S_A(a_n) \subset U_n,$

$b := \prod_{i=1}^{n} a_i : A,$

$[5] := \texttt{E}S_A[4] : \forall i \in [1, \ldots, n] \,.\, u(a_n) = 1,$

$[6] := \texttt{E}b\texttt{E}\mathsf{RNG}(A, \mathbb{B})[5]\texttt{E}\mathbb{B} : u(b) = u\left(\prod_{i=1}^{n} a_i\right) = \prod_{i=1}^{n} u(a_i) = \prod_{i=1}^{n} 1 = 1,$

$[u.*] := \texttt{E}S_A(b)[6]\texttt{E}b\texttt{E}S_A(a)\texttt{IntersectOfSubsets}(S_A(a), U)[4] : u \in S_A(b) \subset \bigcap_{i=1}^{n} S_A(a_n) \subset \bigcap_{i=1}^{n} U_n;$

$\rightsquigarrow [n.*] := \texttt{E}\mathcal{T}_A : \bigcap_{i=1}^{n} U_n \in \mathcal{T}_A;$

$\rightsquigarrow [3] := \texttt{I}\forall\texttt{I}\forall : \forall n \in \mathbb{N} \,.\, \forall U : [1, \ldots, n] \to \mathcal{T}_A \,.\, \bigcap_{i=1}^{n} U_n \in \mathcal{T}_A,$

$[*] := \texttt{ITopology}[1][2][3] : \texttt{Topology}(Z_A, \mathcal{T}_A);$

$\square$

$\texttt{StoneSpace} :: \texttt{BooleanRing} \to \mathsf{TOP}$

$\texttt{StoneSpace}\,(A) = Z_A := (Z_A, \mathcal{T}_A)$

$\texttt{StoneRepresentationIsOpen} :: \forall a \in A \,.\, S_A(a) \in \mathcal{T}(Z_A)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; f \in S_A(a),$

$[f.*] := \texttt{E}f\texttt{SelfSubset}(S_A(a)) : f \in \S_A(a) \subset \S_A(a);$

$\leadsto [*] := \texttt{E}Z_A\texttt{E}\mathcal{T}_A : S_A(a) \in \mathcal{T}(Z_A);$

$\square$

$\texttt{StoneSpaceIsHausdorff} :: \forall A : \texttt{BooleanRing} \,.\, \texttt{T2}(Z_A)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; u, v \in Z_A,$

$\texttt{Assume}\; [1] : u \neq v,$

$\Big(a, [2]\Big) := \texttt{E}Z_A[1] : \sum a \in A \,.\, u(a) = 1 \;\&\; v(a) = 0 | u(a) = 0 \;\&\; v(a) = 1,$

$\texttt{Assume}\; [3] : u(a) = 1 \;\&\; v(a) = 0,$

$\Big(b, [4]) := \texttt{E}Z_A(v) : \sum b \in A \,.\, v(b) = 1,$

$b' := b + ba \in A,$

$[5] := \texttt{E}b' : u(b') = 0 \;\&\; v(b') = 1,$

$[6] := \texttt{I}S_A[3] : u \in S_A(a) \;\&\; v \notin S_A(a),$

$[7] := \texttt{I}S_A[5] : v \in S_A(b') \;\&\; u \notin S_A(b'),$

$[8] := \texttt{E}b'\texttt{E}S_A : S_A(a) \cap S_A(b') = \emptyset;$

$\leadsto [*] := \texttt{StoneRepresentationIsOpenIT2} : \texttt{T2}(Z_A);$

$\square$

$\texttt{StoneRepresentationIsClopen} :: \forall A : \texttt{BooleanRing}(A) \,.\, \forall a \in A \,.\, \forall \texttt{Clopen}\big(Z_A, S_A(a)\big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ f \in S_A^{\complement}(a),$

$[1] := \mathtt{E}S_A(a, f) : f(a) = 0,$

$\Big(b, [2]\Big) := \mathtt{E}Z_A(f) : \sum_{b \in A} f(b) = 1,$

$[3] := [1][2]\mathtt{E}\mathsf{RNG}(A, \mathbb{B}, f) : f(b + ba) = 1,$

$[4] := \mathtt{E}S_A(b + ba)[3] : f \in S_A(b + ba),$

$[f.*] := \texttt{UnionMembership}[4] : f \in \bigcup_{b \in A} S_A(b + ba);$

$\rightsquigarrow [1] := \mathtt{I} \subset : S_A^{\complement}(a) \subset \bigcup_{b \in A} S_A(b + ba),$

$\texttt{Assume}\ f \in \bigcup_{b \in A} S_A,$

$\Big(b, [2]\Big) := \mathtt{E}\cup(f) : \sum b \in A \,.\, f \in S_A(b + ba),$

$[3] := \mathtt{E}S_A[2] : f(b + ba) = 1,$

$[4] := \mathtt{E}\mathsf{RNG}(A, \mathbb{B}, f)[3] : f(a) = 0,$

$[f.*] := \mathtt{E}S_A[4] : f \in S_A^{\complement}(a);$

$\rightsquigarrow [2] := \mathtt{I} \subset : \bigcup_{b \in A} S_A(b + ba) \subset S_A^{\complement}(a),$

$[3] := \texttt{ISetEq}[1][2] : S_A^{\complement}(a) = \bigcup_{b \in A} S_A(b + ba),$

$[4] := \texttt{StoneRepresentationIsOpen}(A)[3]\texttt{ETopology}(Z_A, \mathcal{T}_A) : S_A^{\complement}(a) \in \mathcal{T}(Z_A),$

$[5] := \texttt{IClosed}(Z_A)[4] : \texttt{Closed}\Big(Z_A, S_A(a)\Big),$

$[*] := \texttt{SoneRepresentationIsOpen}(A, a)[5]\texttt{IClopen} : \texttt{Clopen}\Big(Z_a, S_A(a)\Big);$

$\square$

$\texttt{StoneSpaceIsZeroDimensional} :: \forall A : \texttt{BooleanRing} \,.\, \dim_{\mathsf{TOP}} Z_A = 0$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{StoneRepresentationIsCompact} :: \forall A : \texttt{BooleanRing} \,.\, \forall a \in A \,.\, \texttt{CompactSubset}(Z_A, S_A(a))$

$\texttt{Proof} \;=$

$\texttt{Assume}\ \mathcal{F} : \texttt{Ultrafilter}\ \mathcal{T}\Big(S_A(a)\Big),$

$\texttt{Assume}\ a \in A,$

$[1] := \texttt{E}\mathcal{T}_A\texttt{EUltrafilter}(\mathcal{F}) : \texttt{ConvergentFilter}(\mathbb{B}, \mathcal{F}(a)),$

$f(a) := \lim \mathcal{F}(a) : \mathbb{B};$

$\leadsto f := \texttt{I} \to : A \to \mathbb{B},$

$[2] := \texttt{E}f : \forall b \in A \,.\, f(b) = \lim \mathcal{F}(b),$

$[3] := \texttt{E}\lim \mathcal{F}(A) : \forall b \in A \,.\, \forall U \in \mathcal{U}(f(b)) \,.\, U \in \mathcal{F}(b),$

$[4] := \texttt{E}S_A(a)\texttt{E}\mathcal{F}\texttt{E}f : f(a) = 1,$

$\texttt{Assume}\ b, c \in A,$

$\Big(U, [5]\Big) := \texttt{E}f(b+c) : \sum U \in \mathcal{F} \,.\, \forall u \in U \,.\, u(b+c) = f(b+c),$

$\Big(V, [6]\Big) := \texttt{E}f(b) : \sum V \in \mathcal{F} \,.\, \forall v \in V \,.\, v(b) = f(b),$

$W := U \cap V \in W,$

$[7] := \texttt{E}W[5][6] : \forall w \in W \,.\, w(c) = f(b+c) + f(b),$

$\Big[(b,c).*\Big] := \texttt{E}f[7] : f(b+c) = f(b) + f(c);$

$\leadsto [5] := \texttt{I}\forall : \forall b, c \in A \,.\, f(b+c) = f(b) + f(c),$

$\texttt{Assume}\ b, c \in A,$

$\Big(U, [6]\Big) := \texttt{E}f(bc) : \sum U \in \mathcal{F} \,.\, \forall u \in U \,.\, u(bc) = f(bc)\ \&\ u(b) = f(b)\ \&\ u(c) = f(c),$

$\Big[7.*\Big] := \texttt{E}f[6] : f(bc) = f(b)f(c);$

$\leadsto [6] := \texttt{I}\forall : \forall b, c \in A \,.\, f(bc) = f(b)f(c),$

$[7] := \texttt{E}S_a(A)[4][5][6] : f \in S_a(A),$

$\texttt{Assume}\ U \in \mathcal{U}_(f),$

$\Big(b, [8]\Big) := \texttt{E}\mathcal{T}_A(U, f) : \sum b \in A \,.\, f \in S_A(b) \subset U,$

$[9] := \texttt{E}S_A(b)[8] : \forall s \in S_A(a) \,.\, s \in S_A(b) \iff s(b) = f(b) = 1,$

$\Big(V, [10]\Big) := \texttt{E}f[9][8] : \sum V \subset S_A(b) \subset U \,.\, V \in \mathcal{F},$

$[U.*] := \texttt{EFilter}\Big(\mathcal{T}\Big(S_A(a)\Big), \mathcal{F}\Big)[10] : U \in \mathcal{F};$

$\leadsto [\mathcal{F}.*] := \texttt{EFilterLimit} : f = \lim \mathcal{F};$

$\leadsto [1] := \texttt{CompactByUltrafilters} : \texttt{Compact}\big(S_A(a)\big),$

$[*] := \texttt{CompactAsSubset}[1] : \texttt{CompactSubset}\big(Z_A, S_A(a)\big);$

$\square$

$\texttt{StoneSpaceIsLocallyCompact} :: \forall A : \texttt{BooleanRing} \,.\, \texttt{LocallyCompact}(Z_A)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CompactOpenAreStoneRepresentation} ::$

$:: \forall A : \texttt{BooleanRing} \,.\, \forall U \in \mathcal{T}(Z_A) \,.\, \texttt{CompactSubset}(Z_A, U) \Rightarrow \exists a \in A \,.\, U = S_A(a)$

$\texttt{Proof} \ =$

$[1] := \mathtt{E}\mathcal{T}_A(U) : \forall f \in U \,.\, \exists a \in A : f \in S_A(a) \subset U,$

$\Big(n, a, [2]\Big) := \texttt{ECompactSubset}(Z_A, U)[1] : \sum_{n=1}^{\infty} \sum_{a:[1,\dots,n]\to A} U = \bigcup_{i=1}^{n} S_A(a_i),$

$b := \bigcup_{n=1}^{\infty} a_i \in A,$

$\texttt{Assume}\ f \in U,$

$\Big(i, [3]\Big) := \texttt{Eunion}[2] : f \in S_A(a_i),$

$[4] := \mathtt{E}S_A(a_i)(f) : f(a_i) = 1,$

$[5] := \mathtt{E}b\mathtt{E}\mathsf{RNG}(A, \mathbb{B}, f)[4] : f(b) = 1,$

$[f.*] := \mathtt{E}S_A(b)[5] : f \in S_A(b);$

$\rightsquigarrow [3] := \mathtt{I} \subset : U \subset S_A(b),$

$\texttt{Assume}\ f \in S_A(b),$

$[4] := \mathtt{E}S_A : f(b) = 1,$

$\Big(i, [5]\Big) := \mathtt{E}\mathsf{RNG}(A, \mathbb{B}, f)\mathtt{E}b[4] : \sum_{i=1}^{n} f(a_i) = 1,$

$[6] := \mathtt{E}S_A[5] : f \in S_A(a_i),$

$[f.*] := \texttt{SubsetUnion}(S_A(a_i), S_A(a))[3]\mathtt{E} \subset : f \in U;$

$\rightsquigarrow [4] := \mathtt{I} \subset : U \subset S_A(b),$

$[*] := \texttt{ISetEq}[3][4] : U = S_A(b);$

$\square$

$\texttt{BooleanAlgebraByCompactness} :: \forall A : \texttt{BooleanRing} \,.\, \texttt{BooleanAlgebra}(A) \iff \texttt{Compact}(Z_A)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{StoneSpaceAsCantorSubset} :: \forall A : \texttt{BooleanRing} \,.\, Z_A \subset_{\mathsf{TOP}} \mathbb{B}^A$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{StoneSpace} := \texttt{T2} \,\&\, \texttt{LocallyCompact} \,\&\, \texttt{OneDimensional} :?\mathsf{TOP};$

$\mathcal{TK} := \Lambda X \in \mathsf{TOP} \,.\, \texttt{CompactSubset} \,\&\, \texttt{Open}(X) : \prod_{X \in \mathsf{TOP}} \texttt{Algebra}(X);$

$\texttt{StoneHomeomorphism} :: \forall X : \texttt{StoneSpace} \,.\, Z_{\mathcal{TK}(X)} \cong_{\mathsf{TOP}} X$

$\texttt{Proof} \; =$

$\texttt{Assume}\ f \in Z_{\mathcal{TK}(X)},$

$\Big(U, [1]\Big) := \mathtt{E}Z_{\mathcal{TK}(X)} : \sum U \in \mathcal{TK}(X) \,.\, f(U) = 1,$

$\mathcal{A} := \{A \in \mathcal{TK}(X) : f(A) = 1\} :?\mathcal{TK}(X),$

$[2] := \mathtt{E}\mathcal{A}[1] : \mathcal{A} \neq \emptyset,$

$[3] := \mathtt{E}Z_{\mathcal{TK}(X)}\mathtt{E}\mathcal{A}\mathtt{E}\mathsf{RNG}(\mathcal{TK}(X), \mathbb{B}) : \forall A, B \in \mathcal{A} \,.\, A \cap B \neq \emptyset,$

$[4] := \texttt{CantorIntersectionTHM}[2][3] : \bigcap \mathcal{A} \neq \emptyset,$

$\texttt{Assume}\ a \in \bigcap \mathcal{A},$

$[5] := \mathtt{E}\mathcal{A}[1] : a \in U,$

$\texttt{Assume}\ V : \mathcal{TK}(\mathcal{A}),$

$\texttt{Assume}\ [6] : a \in V,$

$[7] := \mathtt{I}\texttt{Intersect}[5][6] : a \in V \cap U,$

$[8] := \mathtt{E}\mathcal{A}(a)[7] : f(V \setminus U) = 0 = f(U \setminus C),$

$[9] := \mathtt{E}Z_{\mathcal{TK}(X)}[1]\mathtt{E}\mathsf{RNG}\Big(\mathcal{TK}(X), \mathbb{B}, f\Big)[8] : 1 = f(V \cup U) = f(V \setminus U) + f(U \cap V) + f(U \setminus V) = f(U \cap V),$

$[10] := \mathtt{E}Z_{\mathcal{TK}(X)}[9] : f(V) = 1,$

$[a.*] := \mathtt{E}\mathcal{A}[10] : V \in \mathcal{A};$

$\rightsquigarrow [5] := \mathtt{I} \iff \mathtt{E}\mathcal{A} : \forall a \in \bigcap \mathcal{A} \,.\, \forall V \in \mathcal{TK}(X) \,.\, a \in V \iff f(V) = 1,$

$\texttt{Assume}\ a, b : \bigcup \mathcal{A},$

$\Big(V, [6]\Big) := \mathtt{E}\texttt{StoneSpace}(X)[5](a, b)) : \sum V \in \mathcal{TK}(X) \,.\, a \in V \,\&\, b \notin V,$

$[7] := [5][6] : 1 = f(V) = 0,$

$\Big[(a, b).*\Big] := [7][7] : \bot;$

$\rightsquigarrow [6] := \mathtt{I}\mathsf{CARD} : \Big|\bigcap \mathcal{A}\Big| \leq 1,$

$[7] := [4][6] : \Big|\bigcap \mathcal{A}\Big| = 1,$

$\Big(\varphi(f), [8]\Big) := \mathtt{E}\texttt{Singleton}[7] : \sum \varphi(f) \in X \,.\, \bigcap \mathcal{A} = \Big\{\varphi(X)\Big\},$

$[f.*] := [5]\big(\varphi(f), [8]\big) : \forall W \in \mathcal{TK}(X) \,.\, \varphi(f) \in W \iff f(W) = 1;$

$\rightsquigarrow \Big(\varphi, [1]\Big) := \mathtt{I}\sum : \sum \varphi : Z_{\mathcal{TK}(X)} \to X \,.\, \forall f \in Z_{\mathcal{TK}(X)} \,.\, \forall W \in \mathcal{TK}(X) \,.\, \varphi(f) \in W \iff f(W) = 1,$

$[2] := \mathtt{E}Z_{\mathcal{TK}(X)}[1]\mathtt{I}\varphi : \texttt{Bijection}\Big(Z_{\mathcal{TK}(X)}, X, \varphi\Big),$

$\texttt{Assume}\ U \in \mathcal{T}(X),$

$[3] := \mathtt{E}\varphi^{-1}(U) : \varphi^{-1}(U) = \bigcup \Big\{S_{\mathcal{TK}(X)}(V) \Big| V \in \mathcal{TK}(X) \,\&\, V \subset U\Big\},$

$[U.*] := \mathtt{E}\texttt{Topology}(Z_{\mathcal{TK}(X)}, \mathcal{T}_{\mathcal{TK}(X)})[3] : \varphi^{-1}(U) \in \mathcal{T}_{\mathcal{TK}(X)};$

$\rightsquigarrow [3] := \mathtt{E}\mathsf{TOP} : \varphi \in \mathsf{TOP}(Z_{\mathcal{TK}(X)}, X),$

$[4] := \mathtt{E}\varphi : \forall V \in \mathcal{TK}(X) \,.\, \varphi\Big(S_{\mathcal{TK}(X)}(V)\Big) = V,$

$[*] := \mathtt{E}\texttt{Base}\Big(S_{\mathcal{TK}(X)}\Big)[4][3] : \texttt{Homeo}\Big(Z_{\mathcal{TK}(X)}, X, \varphi\Big);$

$\square$

1.1.6 Identifying Lattices as Boolean Algebras

$\texttt{BooleanLattice} :: ?\Big(\texttt{DistributiveLattice} \,\&\, \texttt{ComplimentaryLattice}\Big)$

$L : \texttt{BooleanLattice} \iff \exists 0 \in L : 0 = \min L \,\&\, \forall a \in L \,.\, a \wedge \neg a = 0$

$\texttt{BooleanAlgebraIsBooleanLattice} :: \forall A : \texttt{BooleanAlgebra} \,.\, \texttt{BooleanLattice}(A)$

$\texttt{Proof} =$

$\texttt{Assume}\ a, b, c \in A,$

$\Big[(a,b,c).*\big] := \texttt{E}(\cap,\cup)\texttt{E}\mathsf{RNG}(A)\texttt{EBooleanAlgebra}(A)\texttt{I}(\cap,\cup) :$

$\quad : a \cap (b \cup c) = a(b + c + bc) = ab + ac + abc = ab + ac + a^2bc = (a \cap b) \cup (a \cap c);$

$\rightsquigarrow [1] := \texttt{IDistributiveLattice} : \texttt{DistributiviveLattice}(A),$

$n := \Lambda a \in A \,.\, e + a : A \to A,$

$\texttt{Assume}\ a \in A,$

$\Big[a.*\big] := \texttt{E}n\texttt{BooleanRingHasChar2}(A) : n^2(a) = e + e + a = a;$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall a \in A \,.\, n^2(a) = a,$

$\texttt{Assume}\ a, b \in A,$

$\texttt{Assume}\ [3] : a \le b,$

$[4] := \texttt{EBooleanOrder}[3] : ab = a,$

$[5] := \texttt{E}(n(a)n(b))[4]\texttt{BooleanRingHasChar2}(A)\texttt{I}n(b) :$

$\quad : n(a)n(b) = (a + e)(b + e) = ab + b + a + e = b + e = n(b),$

$\Big[(a,b).*\Big] := \texttt{IBooleanOrder} : n(b) \le n(a);$

$\rightsquigarrow [3] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall a, b \in A \,.\, a \le b \Rightarrow n(b) \le n(a),$

$[4] := \texttt{IComplement}[2][3] : \texttt{Complement}(A, n),$

$[5] := \texttt{EBooleanOrderI} \min \texttt{I}0 : 0 = \min A,$

$\texttt{Assume}\ a \in A,$

$[a.*] := \texttt{E}\cap\texttt{E}n(a)\texttt{E}\mathsf{RNG}(A)\texttt{EBooleanRing}(A)\texttt{BooleanRingHasChar2} : a \cap n(a) = a(a + e) = a^2 + a = a + a = 0$

$\rightsquigarrow [6] := \texttt{I}\forall : \forall a \in A \,.\, a \cap n(a) = 0,$

$[*] := \texttt{IBooleanLattice}[1][4][5][6] : \texttt{BooleanLattice}(A);$

$\square$

$\texttt{DeMorganaLaw1} :: \forall L : \texttt{BooleanLattice} \,.\, \forall a, b \in L \,.\, \neg(a \vee b) = \neg a \wedge \neg b$

$\texttt{Proof} \; =$

$[1] := \texttt{E}(a \vee b, b) : a \vee b \geq a,$

$[2] := \texttt{E}(a \wedge b, a) : a \vee b \geq b,$

$[3] := \texttt{EComplimentaryLattice}(L)[1] : \neg(a \vee b) \leq \neg a,$

$[4] := \texttt{EComplimentaryLattice}(L)[2] : \neg(a \vee b) \leq \neg b,$

$[5] := \texttt{I} \wedge [3][4] : \neg(a \vee b) \leq \neg a \wedge \neg b,$

$\texttt{Assume}\ c \in L,$

$\texttt{Assume}\ [6] : c \leq \neg a \wedge \neg b,$

$[7] := \texttt{E} \wedge [6] : c \leq \neg a \;\&\; c \leq \neg b,$

$[8] := \neg[6] : \neg c \geq a \;\&\; \neg c \geq b,$

$[9] := \texttt{I}\vee : \neg c \geq a \vee b,$

$[c.*] := \neg[9] : c \leq \neg(a \vee b);$

$\rightsquigarrow [6] := \texttt{I}\forall : \forall c \in L \,.\, c \leq \neg a \wedge \neg b \Rightarrow c \leq \neg(a \vee b),$

$[*] := \texttt{E}\mathsf{LAT}(L)[1][6] : \neg a \wedge \neg b = \neg(a \vee b);$

$\square$

$\texttt{DeMoraganaLaw} :: \forall L : \texttt{BooleanLattice} \,.\, \forall a, b \in L \,.\, \neg a \vee \neg b = \neg(a \wedge b)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{BooleanAlgebraIdentification} :: \forall L : \texttt{BooleanLattice} \,.\, \exists \Delta : L \times L \to L \,.$

$\quad .\, \texttt{BooleanAlgebra}(L, \wedge, \Delta) \;\&\; \texttt{order}(L) = \texttt{order}(L, \wedge, \Delta)$

$\texttt{Proof} \; =$

$\oplus := \Lambda a, b \in L \,.\, (a \wedge \neg b) \vee (\neg a \wedge b) : L \times L \to Ł,$

$\texttt{Assume}\ a, b, c \in L,$

$\Big[(a, b, c).*\Big] := \texttt{E}(a \oplus b)\texttt{E}\oplus(c)\texttt{EDestributiveLattice}(L)\texttt{DeMorganaLaw1}(L)\texttt{DeMorganaLaw2}(L)$

$\quad \texttt{EDestributiveLattice}(L)\texttt{DeMorganaLaw1}(L)\texttt{DeMorganaLaw2}(L)\texttt{I}(b \oplus c)\texttt{I}\oplus(a) :$

$$: (a \oplus b) \oplus c = \Big((a \wedge \neg b) \vee (\neg a \wedge b)\Big) \oplus c =$$

$$= \bigg(\Big((a \wedge \neg b) \vee (\neg a \wedge b)\Big) \vee \neg c\bigg) \vee \bigg(\neg\Big((a \wedge \neg b) \vee (\neg a \wedge b)\Big) \vee c\bigg) =$$

$$= (a \wedge \neg b \neg c) \vee (\neg a \wedge b \neg c) \vee \neg a \wedge \neg b \wedge c \vee a \wedge b \wedge \neg c =$$

$$= \bigg(a \vee \neg\Big((b \wedge \neg c) \vee (\neg b \wedge c)\Big)\bigg) \vee \bigg(\neg a \vee \Big((b \wedge \neg c) \vee (\neg b \wedge c)\Big)\bigg) = a \oplus \Big((b \wedge \neg c) \vee (\neg b \wedge c)\Big) = a \oplus (b \oplus c);$$

$\rightsquigarrow [1] := \texttt{IAssociative} : \texttt{Associative}(L, \oplus),$

$\texttt{Assume}\ a \in L,$

$[a.*] := \texttt{E}(a \oplus 0)\texttt{EBooleanLattice}(L)\texttt{E}\mathsf{LAT}(L) : a \oplus 0 = (a \wedge \neg 0) \vee (\neg a \wedge 0) = a \vee 0 = a;$

$\rightsquigarrow [2] := \texttt{INeutral} : \texttt{Neutral}(L, \oplus, 0),$

$\texttt{Assume}\ a \in L,$

$[3] := \texttt{E}(a \oplus a)\texttt{EBooleanLattice}(L)\texttt{E}\mathsf{LAT}(L) : a \oplus a = (a \wedge \neg a) \vee (\neg a \wedge a) = 0 \vee 0 = 0,$

$[*] := \texttt{IInvertible}[1][2] : \texttt{Invertible}\Big(L, \oplus, a\Big);$

$\rightsquigarrow [3] := \texttt{I}\forall : \forall a \in L \,.\, \texttt{Invertible}(L, \oplus, a),$

$[4] := \texttt{I}\mathsf{GRP}[1][2][3] : (L, \oplus) \in \mathsf{GRP},$

$\texttt{Assume } a, b, c : L,$

$\Big[(a,b,c).*\Big] := \texttt{E}(a \oplus b)\texttt{EDestributiveLattice}(L)\texttt{EBooleanLattice}(L)\texttt{EDestributiveLattice}(L)\texttt{I}(\oplus) :$

$: (a \oplus b) \wedge c = \Big((a \wedge \neg b) \vee (\neg a \wedge b)\Big) \wedge c = (a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) =$

$= (a \wedge c) \wedge \neg(b \wedge c) \vee \neg(a \wedge c) \wedge (b \wedge c) = (a \wedge c) \oplus (b \wedge c);$

$\rightsquigarrow [5] := \texttt{I}\mathsf{RNG} : (L, \oplus, \wedge) \in \mathsf{RNG},$

$[6] := \texttt{EBooleanLattice}(L)[5]\texttt{I}\mathsf{RING} : (L, \oplus, \wedge) \in \mathsf{RING},$

$[7] := \texttt{E}\mathsf{LAT}(L)[6] : \texttt{BooleanAlgebra}(L, \oplus, \wedge),$

$[*] := \texttt{E}\mathsf{LAT}(L)\texttt{Iorder} : \texttt{order}(L) = \texttt{order}(L, \wedge, \Delta);$

$\square$

1.1.7 Extension of boolean rings to algebras

$$\texttt{doubleAddition} :: \prod A : \texttt{BooleanRing} \,.\, \Big((A \sqcup A) \times (A \sqcup A)\Big) \to (A \sqcup A)$$

$$\texttt{doubleAddition}\,((0,a),(0,b)) = (0,a) +' (0,b) := (0, a+b)$$
$$\texttt{doubleAddition}\,((0,a),(1,b)) = (0,a) +' (1,b) := (1, a+b)$$
$$\texttt{doubleAddition}\,((1,a),(0,b)) = (1,a) +' (0,b) := (1, a+b)$$
$$\texttt{doubleAddition}\,((1,a),(1,b)) = (1,a) +' (1,b) := (0, a+b)$$

$$\texttt{doubleMult} :: \prod A : \texttt{BooleanRing} \,.\, \Big((A \sqcup A) \times (A \sqcup A)\Big) \to (A \sqcup A)$$

$$\texttt{doubleMult}\,((0,a),(0,b)) = (0,a) \cdot' (0,b) := (0, ab)$$
$$\texttt{doubleMult}\,((0,a),(1,b)) = (0,a) \cdot' (1,b) := (0, a+ab)$$
$$\texttt{doubleMult}\,((1,a),(0,b)) = (1,a) \cdot' (0,b) := (0, b+ab)$$
$$\texttt{doubleMult}\,((1,a),(1,b)) = (1,a) \cdot' (1,b) := (1, a+b+ab)$$

$$\texttt{complementationEmbedding} :: \prod A : \texttt{BooleanRing} \,.\, A \to (A \sqcup A)$$

$$\texttt{complementationEmbedding}\,(a) = a^{\complement} := (1, a)$$

$$\texttt{doubleExtenstion} :: \texttt{BooleanRing} \to \texttt{BooleanAlgebra}$$

$$\texttt{doublextension}\,(A) = A' := \Big(A \sqcup A, +', \cot\Big)$$

$[1] := \texttt{E}A' : (A', +) \cong_{\mathsf{GRP}} \mathbb{B} \oplus A,$

$\texttt{Assume}\ (x,a),(y,b),(z,c) \in A',$

$\Big[\big((x,a),(y,b),(z,c)\big).*\Big] := \texttt{E}^2(\cdot')\texttt{E}\mathsf{RNG}(\mathbb{B}\ \&\ A)\texttt{I}(\cdot') :$

$: \Big((x,a)(y,b)\Big)(z,c) = (xy, xb + ya + ab)(z,c) = (xyz, xyc + xzb + xbc + yac + yza + zab + abc) =$
$(x,a)(yz, yc + zb + bc) = (x,a)\Big((y,b),(z,c)\Big);$

$\leadsto [2] := \texttt{I}\forall : \forall a,b,c \in A' \,.\, a(bc) = (ab)c,$

$\texttt{Assume}\ (x,a),(y,b),(z,c) \in A',$

$\Big[\big((x,a),(y,b),(z,c)\big).*\Big] := \texttt{E}(+')\texttt{E}(\cdot')\texttt{E}\mathsf{RNG}(\mathbb{B}\ \&\ A)\texttt{I}(+')\texttt{I}(\cdot') :$

$: (x,a)\Big((y,b)+(z,c)\Big) = (x,a)(y+z, b+c) = \Big(x(y+z), a(b+c) + x(b+c) + (y+z)a\Big) =$
$= (xy + xz, ab + ac + xb + xc + ya + za) = (xy, ab + xb + ya) + (xz, ac + xc + za) =$
$= (x,a)(y,b) + (x,a)(y,b);$

$\leadsto [3] := \texttt{I}\forall : \forall a,b,c \in A' \,.\, a(b+c) = ab + ac,$

$[4] := \texttt{I}\mathsf{RNG}[2][3] : A' \in \mathsf{RNG},$

$\texttt{Assume}\ (x,a) : A',$

$\Big[(x,a).*\Big] := \texttt{E}(\cdot)\texttt{EBooleanRing}(A\ \&\ \mathbb{B}) : (x,a)^2 = (x^2, ax + ax + a^2) = (x,a);$

$\leadsto [5] := \texttt{IBooleanRing} : \texttt{BooleanRing}(A'),$

$\texttt{Assume}\ (x,a) : A',$

$\Big[(x,a).*\Big] := \texttt{E}(\cdot')\texttt{ZeroMult}(A) : (x,a)(1,0) = (x, a + x0 + 0a) = (x,a);$

$\leadsto [*] := \texttt{IBooleanAlgebra} : \texttt{BooleanAlgebra}(A);$

$\square$

$\texttt{IdealInExtension} :: \forall A : \texttt{BooleanRing} . \texttt{Ideal}(A', A)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{StoneSpaceOfExtensionIsOnePointCompactification} :: \forall A : \texttt{BooleanRing} . Z_{A'} \cong_{\mathsf{TOP}} Z_A^*$

$\texttt{Proof} =$

$\texttt{Assume}\ f \in Z_A \sqcup \{0\},$

$\varphi(f) := \Lambda(x, a) \in A' . x + f(a) : A' \to \mathbb{B},$

$\texttt{Assume}\ (x, a), (y, b) : A',$

$$\Big[\big((x, a), (y, b)\big). *\Big] := \mathtt{E}(+')\mathtt{E}\varphi(f)\mathtt{E}\mathsf{GRP}(A', \mathbb{B})(f)\mathtt{I}\varphi(f) :$$

$$: \varphi(f)\Big((x, a) + (y, b)\Big) = \varphi(f)(x + y, a + b)x + y + f(a + b) = x + y + f(a) + f(b) =$$

$$= \varphi(f)(x, a) + \varphi(f)(y, b);$$

$\rightsquigarrow [1] := \mathtt{I}\mathsf{GRP} : \varphi(f) \in \mathsf{GRP}(A', \mathbb{B}),$

$\texttt{Assume}\ (x, a), (y, b) : A',$

$$\Big[\big((x, a), (y, b)\big). *\Big] := \mathtt{E}(\cdot')\mathtt{E}\varphi(f)\mathtt{E}\mathsf{RNG}(A, \mathbb{B}, f)\mathtt{I}(\cdot') :$$

$$: \varphi(f)\Big((x, a)(y, b)\Big) = \varphi(f)(xy, ya + xb + ab) = xy + f(ya + xb + ab) = xy + yf(a) + xf(b) + f(a)f(b) =$$

$$= \Big(x + f(a)\Big)\Big(y + f(b)\Big) = \varphi(f)(x, a)\varphi(f)(y, b);$$

$\rightsquigarrow [f.*] := \mathtt{I}\mathsf{RNG} : \varphi(f) \in \mathsf{RNG}(A', \mathbb{B});$

$\rightsquigarrow \varphi := \mathtt{I}(\to) : Z_A \sqcup \{0\} \to Z_{A'},$

$\texttt{Assume}\ f \in Z_{A'},$

$g := f_{|A} \in Z_A \sqcup \{0\},$

$\texttt{Assume}\ (x, a) \in A',$

$$\Big[(x, a). *\Big] := \mathtt{E}\varphi\mathtt{E}g\mathtt{E}Z_{A'}(f)\mathtt{E}\mathsf{RNG}(A', \mathbb{B}, f) : \varphi(g)(x, a) = x + g(a) = f(x, 0) + f(0, a) = f(x, a);$$

$\rightsquigarrow [f.*] := \mathtt{I}(\to, =) : f = \varphi(g);$

$\rightsquigarrow [1] := \texttt{ISurjective} : \texttt{Surjective}\Big(Z_A \sqcup \{0\}, Z_{A'}, \varphi\Big),$

$[2] := [1]\texttt{InjectiveCardinlaity}[1] : \Big|Z_{A'} \setminus \varphi(Z_A)\Big| = 1,$

$\texttt{Assume}\ (x, a) \in A',$

$[3] := \mathtt{E}S\mathtt{E}\varphi : \varphi^{-1}\Big(S_{A'}(x, a)\Big) = \texttt{if}\ x\ \texttt{then}\ S_A^{\mathcal{C}}(a)\ \texttt{else}\ S_A(a),$

$\Big[(x, a). *\Big] := \texttt{StoneRepresentationIsClopen}[3] : \varphi^{-1}\Big(S_{A'}(x, a)\Big) \in \mathcal{T}(Z_A);$

$\rightsquigarrow [3] := \mathtt{I}\mathsf{TOP} : \varphi \in \mathsf{TOP}(Z_A, Z_{A'}),$

$\texttt{Assume}\ a \in A,$

$[4] := \mathtt{E}S\mathtt{E}\varphi : \varphi\Big(S_A(a)\Big) = \varphi\Big(S_{A'}(0, a)\Big),$

$\Big[(x, a). *\Big] := \texttt{StoneRepresentationIsClopen}[4] : \varphi\Big(S_A(a)\Big) \in \mathcal{T}(Z_{A'});$

$\rightsquigarrow [4] := \texttt{IHomeo} : \varphi : \texttt{Homeo}\Big(Z_A, \varphi(Z_A)\Big),$

$[*] := [4]\mathtt{E}\mathcal{T}_{A'} : Z_{A'} \cong Z_A^*;$

$\square$

1.2 Exploiting the ring structure

1.2.1 Subalgebras

$\texttt{categoryOfBooleanRings} :: \mathsf{CAT}$

$\texttt{categoryOfBooleanRings}\,() = \mathsf{BOL} := \Big(\texttt{BooleanRing}, \mathsf{RNG}, \circ, \mathrm{id}\Big)$

$\texttt{categoryOfBooleanRings} :: \mathsf{CAT}$

$\texttt{categoryOfBooleanRings}\,() = \mathsf{BOOL} := \Big(\texttt{BooleanAlgebra}, \mathsf{RING}, \circ, \mathrm{id}\Big)$

$\texttt{complement} :: \prod A : \texttt{BooleanAlgebra} \,.\, A \to A$

$\texttt{complement}\,(a) = a^{\complement} := a + e$

$\texttt{LawOfExcludedMiddle} :: \forall A : \texttt{BooleanAlgebra} \,.\, \forall a \in A \,.\, a \cap a^{\complement} = 0$

$\texttt{Proof} \;=$

$[*] := \texttt{E}(\cap)\texttt{E}\complement\texttt{E}\mathsf{RING}(A)\texttt{EBooleanAlgebra}(A)\texttt{BooleanRingHasChar2} :$

$\quad : a \cap a^{\complement} = aa^{\complement} = a(a+e) = a^2 + a = a + a = 0;$

$\square$

$\texttt{BooleanSubalgebraCriterion1} :: \forall A \in \mathbb{B} \,.\, \forall B \subset A \,.\, B \subset_{\mathbb{B}} A \iff$

$\quad \iff 0 \in B \;\&\; \forall a, b \in B \,.\, a \cup b \in B \;\&\; \forall a \in B \,.\, a^{\complement} \in B$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1.1] : 0 \in B,$

$\texttt{Assume}\ [1.2] : \forall a, b \in B \,.\, a \cup B \in B,$

$\texttt{Assume}\ [1.3] : \forall a \in B \,.\, a^{\complement} \in B,$

$[2] := [1.1][1.3] : e \in B,$

$[3] := [1.3][1.2][1.3] : \forall a, b \in B \,.\, ab = (a^{\complement} \cup b^{\complement})^{\complement} \in B,$

$[4] := [1.3][3][1.2] : \forall a, b \in B \,.\, a + b = (a \cap b^{\complement}) \cup (a^{\complement} \cap b) \in B,$

$[*] := \texttt{I}\mathsf{BOOL}[2][3][4] : B \subset_{\mathsf{BOOL}} A;$

$\square$

$\texttt{BooleanSubalgebraCriterion2} :: \forall A \in \mathbb{B} \,.\, \forall B \subset A \,.\, B \subset_{\mathbb{B}} A \iff$

$\quad \iff B \neq \emptyset \;\&\; \forall a, b \in B \,.\, a \cap b \in B \;\&\; \forall a \in B \,.\, a^{\complement} \in B$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1.1] : B \neq \emptyset,$

$\texttt{Assume}\ [1.2] : \forall a, b \in B \,.\, a \cap B \in B,$

$\texttt{Assume}\ [1.3] : \forall a \in B \,.\, a^{\complement} \in B,$

$a := \texttt{ENonEmpty}(B) \in B,$

$[2] := \texttt{LawOfExluedeMiddle}(A, a)[1.3][1.2] : 0 = a \cap a^{\complement} \in B,$

$[3] := [1.3][1.2][1.3] : \forall a, b \in B \,.\, a \cup b = (a^{\complement} \cap b^{\complement})^{\complement} \in B,$

$[*] := \texttt{BooleanSubalgebraCriterion1}[2][3][1.3] : B \subset_{\mathbb{B}} A;$

$\square$

$\texttt{SubalgebraGenratedByAdditionalElement} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, \{(b \cap a) \cup (c \setminus a) | b, c \in B\}$

$\texttt{Proof} \ =$

$C := \{(b \cap a) \cup (c \setminus a) | b, c \in B\} :?A,$

$[1] := \texttt{E}(0)\texttt{E}(\cup) : (0 \cap a) \cup (0 \setminus a) = 0 \cup 0 = 0,$

$[2] := \texttt{ESubring}(A, B)\texttt{E}C[1] : 0 \in C,$

$\texttt{Assume}\ d \in C,$

$\Big(b, c, [3]\Big) := \texttt{E}C(d) : \sum b, c \in B \,.\, d = (b \cap a) \cup (c \setminus a),$

$[d.*] := [3]\texttt{CheckingTruthTablesE}C :$

$: d^{\complement} = \Big((b \cap a) \cup (c \setminus a)\Big)^{\complement} = (b^{\complement} \cup a^{\complement}) \cap (c^{\complement} \cup a) = (b^{\complement} \cap a) \cup (c^{\complement} \setminus a) \in C;$

$\rightsquigarrow [3] := \texttt{I}\forall : \forall d \in C \,.\, d^{\complement} \in C,$

$\texttt{Assume}\ d, d' : C,$

$\Big(b, c, [4]\Big) := \texttt{E}C(d) : \sum b, c \in B \,.\, d = (b \cap a) \cup (c \setminus a),$

$\Big(b', c', [5]\Big) := \texttt{E}C(d') : \sum b', c' \in B \,.\, d' = (b' \cap a) \cup (c' \setminus a),$

$\Big[(d, d').*\Big] := [4][5]\texttt{CheckingTruthTablesE}C :$

$: d \cup d' = (b \cap a) \cup (c \setminus a) \cup (b' \cap a) \cup (c' \setminus a) = \Big((b \cup b') \cap a\Big) \cup \Big((b \cup b') \setminus a\Big) \in C;$

$\rightsquigarrow [4] := \texttt{I}\forall : \forall d, d' \in C \,.\, d \cup d' \in C,$

$[*] := \texttt{BooleanSubalgebraCriterion1}[4] : C \subset_{\mathsf{BOOL}} A;$

$\square$

$\texttt{oneElementSubalgebraExtension} :: \prod_{A \in \mathsf{BOOL}} \texttt{Subalgebra}(A) \to A \to \texttt{Subalgebra}(A)$

$\texttt{oneElementSubalgebraExtension}\,(B, c) = B_c := \{(b \cap a) \cup (x \setminus a) | b, c \in B\}$

$\texttt{OneElementExtensionProperty} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, B \subset_{\mathsf{BOOL}} B_a \ \&\ a \in B_a$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.2.2 Ideals

$\texttt{IdealCriterion} :: \forall A \in \mathsf{BOOL} \,.\, \forall I \subset A \,.\, \texttt{Ideal}(A, I) \iff$

$\iff 0 \in I \;\&\; \forall a, b \in I \,.\, a \cup b \in I \;\&\; \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{Ideal}(A, I),$

$[*.1] := \texttt{EIdeal}(A, I : 0 \in I,$

$\texttt{Assume}\ a, b \in I,$

$[2] := \texttt{EIdeal}(A, I)(a, b) : ab \in I,$

$\Big[(a, b).*\Big] := \texttt{E}(a \cup b)\texttt{ESubgroup}(A, I)[2] : a \cup b = ab + a + b \in I;$

$\leadsto [*.2] := \texttt{I}\forall : \forall a, b \in I \,.\, a \cup b \in I,$

$\texttt{Assume}\ a \in I,$

$\texttt{Assume}\ b \in A,$

$\texttt{Assume}\ [2] : b \leq a,$

$[3] := \texttt{EBooleanOrder} : b = ab,$

$[a.*] := \texttt{EIdeal}(A, I)[3] : b \in I;$

$\leadsto [*.3] := \texttt{I} \Rightarrow \texttt{I}^2\forall : \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I;$

$\leadsto [1] := \texttt{I} \Rightarrow: \texttt{Ideal}(A, I) \Rightarrow 0 \in I \;\&\; \forall a, b \in I \,.\, a \cup b \in I \;\&\; \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I,$

$\texttt{Assume}\ [2.1] : 0 \in I,$

$\texttt{Assume}\ [2.2] : \forall a, b \in I \,.\, a \cup b \in I,$

$\texttt{Assume}\ [2.3] : \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ i \in I,$

$[3] := \texttt{EBooleanAlgebra}(A) : ai^2 = ai,$

$[4] := \texttt{EBooleanOrder}(A)[3] : ai \leq i,$

$[a.*] := [2.3][4] : ai \in I;$

$\leadsto [3] := \texttt{I}^2\forall : \forall a \in A \,.\, \forall i \in I \,.\, ai \in I,$

$\texttt{Assume}\ a, b \in A,$

$[4] := [3](b^{\complement}, a) : ab^{\complement} \in I,$

$[5] := [3](a^{\complement}, a) : a^{\complement}b \in I,$

$\Big[(a, b).*\Big] := \texttt{E}\oplus [2.2] : a + b = ab^{\complement} \cup a^{\complement}b \in I;$

$\leadsto [4] := \texttt{I}\forall : \forall a, b \in I \,.\, a + b \in I,$

$[2.*] := \texttt{IIdeal}(A)[2.1][3][4] : \texttt{Ideal}(A, I);$

$\leadsto [2] := \texttt{I} \Rightarrow: 0 \in I \;\&\; \forall a, b \in I \,.\, a \cup b \in I \;\&\; \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I \Rightarrow \texttt{Ideal}(A, I),$

$[*] := \texttt{I} \iff [1][2] : \texttt{Ideal}(A, I) \iff 0 \in I \;\&\; \forall a, b \in I \,.\, a \cup b \in I \;\&\; \forall a \in I \,.\, \forall b \in A \,.\, b \leq a \Rightarrow b \in I;$

$\square$

$\texttt{PrincipleIdealStructure} :: \forall A \in \mathsf{BOOL} \,.\, \forall a \in A \,.\, \langle a \rangle = \{b \in A : b \le a\}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ b \in \langle a \rangle,$

$\big(c, [1]\big) := \texttt{E}\langle a \rangle(b) : b = ca,$

$[2] := [1]\texttt{EBooleanAlgebra}(A)[1] : ab = ca^2 = ca = b,$

$[3] := \texttt{EBooleanOrder}[2] : b \le a;$

$\rightsquigarrow [1] := \texttt{I} \subset : \langle a \rangle \subset \{b \in A : b \le a\},$

$\texttt{Assume}\ b \in A,$

$\texttt{Assume}\ [2] : b \le a,$

$[3] := \texttt{E}(\le)[2] : ba = b,$

$[b.*] := \texttt{E}\langle a \rangle[3] : b \in \langle a \rangle;$

$\rightsquigarrow [2] := \texttt{I} \subset : \{b \in A : b \le a\} \subset \langle a \rangle,$

$[*] := \texttt{ISetEq}[1][2] : \{b \in A : b \le a\} = \langle a \rangle;$

$\square$

$\texttt{PrincipleIdealIsAlgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall a \in A \,.\, \langle a \rangle \in \mathsf{BOOL}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.2.3 Morphisms

$\texttt{MorphismPreservesOrer} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall A \xrightarrow{f} B : \mathsf{BOOL} \,.\, \forall x, y \in A \,.\, x \leq y \Rightarrow f(x) \leq f(y)$

$\texttt{Proof} \;=$

$[1] := \texttt{E}(\leq)[0] : xy = x,$

$[2] := \texttt{E}\mathsf{BOOL}(A, B, f)[1] : f(x)f(y) = f(xy) = f(x),$

$[*] := \texttt{I}(\leq)[2] : f(x) \leq f(y);$

$\square$

$\texttt{IntersectionIsSurjectiveHomo} :: \forall A \in \mathsf{BOOL} \,.\, \forall a \in A \,.\, \lambda_a : \mathsf{BOOL} \;\&\; \texttt{Surjective}\Big(A, \langle a \rangle\Big)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$$\texttt{fixedPointAlgebra} :: \prod_{A \in \mathsf{BOOL}} \mathrm{End}_{\mathsf{BOOL}}(A) \to \mathsf{BOOL}$$

$$\texttt{fixedPointAlgebra}\,(f) = \mathrm{Fix}(f) := \{a \in A : f(a) = a\}$$

$\texttt{BooleanPosetIsomorphismIsBooleanIsomorphism} :: \forall A, B \in \mathbb{B} \,.\, \forall A \overset{f}{\longleftrightarrow} B : \mathsf{POSET} \,.\, A \overset{f}{\longleftrightarrow} B : \mathsf{BOOL}$

$\texttt{Proof} \;=$

$[1] := \texttt{PosetIsomorphismPreservesMin}(A) : f(0) = 0,$

$[2] := \texttt{PosetIsomorphismPreservesMax}(A) : f(e) = e,$

$[3] := \texttt{PosetIsomorphismPresevesLatticeStructure}(A) :$

$\quad : \forall a, b \in A \,.\, f(a \cap b) = f(a) \cap f(b) \;\&\; f(a \cup b) = f(a) \cup f(b),$

$[4] := [3.2]\texttt{LawOfRxcludedMiddle}(A)[2] : \forall a \in A \,.\, f(a) \cup f(a^{\complement}) = f(a \cup a^{\complement}) = f(e) = e,$

$[5] := [3.1]\texttt{LawOfRxcludedMiddle}(A)[1] : \forall a \in A \,.\, f(a) \cap f(a^{\complement}) = f(a \cap a^{\complement}) = f(0) = 0,$

$[6] := \texttt{UniqueComplementatioTheorem}[4][5] : \forall a \in A \,.\, f(a^{\complement}) = f^{\complement}(a),$

$[*] := \texttt{I}(\oplus)[6][3.2][3.1] : \texttt{Isomorphism}(\mathsf{BOOL}, A, B, f);$

$\square$

$\mathtt{HomomorphismExtension} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall A' \subset_{\mathsf{BOOL}} A \,.\, \forall A' \xrightarrow{f} B : \mathsf{BOOL} \,.\, \forall c \in A \,.\, \forall v \in B \,.$

$: \forall [0] : \forall a, b \in A' \,.\, a \le c \le b \iff f(a) \le v \le f(b) \,.\, \exists A'_c \xrightarrow{f'} B : \mathsf{BOOL} : f'(c) = v \;\&\; f'_{|A'} = f$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ d \in A'_c,$

$\Big(a, b, [1]\Big) := \mathtt{E}A'_c(d) : \sum a, b \in A' : d = (a \cap c) \cup (b \setminus c),$

$\mathtt{Assume}\ a', b' \in A',$

$\mathtt{Assume}\ [2] : d = (a' \cap c) \cup (b' \setminus c),$

$[3] := [1][2] \cap c : a \cap c = d \cap c = a' \cap c,$

$[4] := \mathtt{I}(\triangle)[3] : (a \triangle a') \cap c = 0,$

$[5] := \mathtt{I}(\setminus) a : c \subset (a \triangle a')^{\complement},$

$[6] := [0][5] : v \subset (f(a) \triangle f(a'))^{\complement},$

$[7] := \mathtt{I}(\cap)[6] : f(a) \cap v = f(a') \cap v,$

$[8] := [1][2] \setminus c : b \setminus c = d \setminus c = b' \setminus c,$

$[9] := \mathtt{I} \triangle \ : (b \triangle b') \setminus c = 0,$

$[10] := \mathtt{E}(\setminus)[9] : b \triangle b' \subset c,$

$[11] := [0][10] : f(b) \triangle f(b') \subset v,$

$[12] := \mathtt{I}(\setminus)[11] : \Big(f(b) \triangle f(b')\Big) \setminus v = 0,$

$[13] := \mathtt{E} \triangle [12] : f(b) \setminus v = f(b') \setminus v,$

$\Big[(a', b').*\Big] := [7] \cup [13] : (f(a) \cap v) \cup (f(b) \setminus v) = (f(a') \cap v) \cup (f(b') \setminus v);$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall a', b' \in A' \,.\, d = (a' \cap c) \cup (b' \setminus c) \Rightarrow (f(a) \cap v) \cup (f(b') \setminus v),$

$f'(d) := (f(a) \cap v) \cup (f(b) \setminus v) : B;$

$\rightsquigarrow f' := \mathtt{I} \to : A'_c \to B,$

$[*.1] := \mathtt{E}A'_c \mathtt{E} f' \mathtt{E}\mathbb{B}(A', B, f) \mathtt{E}\mathsf{RING}(B) \mathtt{E}(\cup) :$

$: f'(c) = f'\Big((e \cap c) \cup (0 \setminus c)\Big) = f(e) \cap v \cup (f(0) \setminus v) = e \cap v \cup (0 \setminus v) = v \cup 0 = v,$

$\mathtt{Assume}\ a \in A',$

$[a.*] := \mathtt{IntersectDifferenceDecomposition}(a, c) \mathtt{E} f' \mathtt{IntersectDifferenceDecomposition}\Big(f(a), v\Big) : f'(a) =$

$\rightsquigarrow [*.2] := \mathtt{I}\forall : \forall a \in A' \,.\, f'(a) = f(a),$

$\mathtt{Assume}\ d \in A'_c,$

$\Big(a, b, [1]\Big) := \mathtt{E}A'_c(d) : \sum a, b \in A' : d = (a \cap c) \cup (b \setminus c),$

$[d.*] := [1] \mathtt{CheckingTruthTableE} f' \mathtt{E}\mathsf{BOOL}(A, B, f) \mathtt{CheckingTruthTableI} f' :$

$: f'(d^{\complement}) = f'\Big((a \cap c) \cup (b \setminus c)\Big)^{\complement} = f'\Big((a^{\complement} \cap c) \cup (b^{\complement} \setminus c) = (f(a^{\complement}) \cap v) \cup (f(b^{\complement}) \setminus v) =$

$= (f^{\complement}(a) \cap v) \cup (f^{\complement}(b) \setminus v) = \Big(\big(f(a) \cap v\big) \cup \big(f(b) \setminus v\big)\Big)^{\complement} = \Big(f'(d)\Big)^{\complement};$

$\rightsquigarrow [1] := \mathtt{I}\forall : \forall d \in A'_c \,.\, f'(d^{\complement}) = \Big(f'(d)\Big)^{\complement},$

$\texttt{Assume}\ d, d' \in A'_c,$

$\Big(a, b, [2]\Big) := \texttt{E}A'_c(d) : \sum a, b \in A' : d = (a \cap c) \cup (b \setminus c),$

$\Big(a', b', [3]\Big) := \texttt{E}A'_c(d) : \sum a', b' \in A' : d' = (a' \cap c) \cup (b' \setminus c),$

$[4.*] := [2][3]\texttt{CheckingTruthTablesE}f'\texttt{E}\mathsf{BOOL}(A', B, f)\texttt{CheckingTruthTablesI}f' :$

$: f'(d \cup d') = f'\Big((a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c)\Big) = f'\Big(\big((a \cup a') \cap c\big) \cup \big((b \cup b') \setminus c\big)\Big) =$

$= \Big(f(a \cup a') \cap v\Big) \cup \Big(f(b \cup b') \setminus v\Big) = f(a) \cap v \cup f(b) \setminus v \cup f(a') \cap v \cup f(b') \setminus v = f'(d) \cup f'(d');$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall d, d' \in A'_c \,.\, f'(d \cup d') = f'(d) \cup f'(d'),$

$[*.3] := \texttt{I}\mathsf{BOOL}[1][2] : f' \in \mathsf{BOOL}(A'_c, B);$

$\square$

1.2.4 Quotitent Algebras

$\texttt{BooleanQuotientAlgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \frac{A}{I} \in \mathsf{BOOL}$

$\texttt{Proof} \; =$

$\texttt{Assume}\ [a] \in \frac{A}{I},$

$\Big[[a].*\Big] := \texttt{E}\frac{A}{I}\texttt{EBooleanAlgebra}(A) : [a]^2 = [a^2] = [a];$

$\leadsto [*] := \texttt{IBooleanAlgebra} : \texttt{BooleanAlgebra}\left(\frac{A}{I}\right),$

$\square$

$\texttt{QuotientOrder} :: \forall A \in \mathsf{BOOL} \,.\, \forall I : \texttt{Ideal}(A) \,.\, \forall [a],[b] \in \frac{A}{I} \,.\, [a] \le [b] \Rightarrow a \setminus b \in I$

$\texttt{Proof} \; =$

$[1] := \texttt{E}(\le)[0] : [a][b] = [a],$

$[2] := \texttt{E}(\setminus)[1]\texttt{BooleanRingHasChar2}\,() : [a] \setminus [b] = [a] + [a][b] = [a] + [a] = 0,$

$[*] := \texttt{E}\mathsf{BOOL}\left(A, \frac{A}{I}, [\cdot]\right) : a \setminus b \in I;$

$\square$

1.2.5 Stone Functor

$\texttt{setOfIdeals} :: \texttt{Contravariant}(\mathsf{RING}, \mathsf{POSET})$
$\texttt{setOfIdeals}\,(R) = \mathcal{I}(R) := \texttt{Ideal}(R)$
$\texttt{setOfIdeals}\,(R, S, f) = \mathcal{I}_{R,S}(f) := \Lambda I \in \mathcal{I}(S) \;.\; f^{-1}(S)$

$\texttt{StoneTopologyRingIdealsCorrespondance} :: \forall B \in \mathsf{BOOL} \;.\; \mathcal{T}(Z_B) \cong_{\mathsf{POSET}} \mathcal{I}(B)$
$\texttt{Proof} \;=$
$\texttt{Assume}\; U \in \mathcal{T}(Z_B),$
$F(U) := \{b \in B : S_B(b) \subset U\} :?B,$
$\texttt{Assume}\; b \in F(U),$
$[1] := \texttt{E}F(U)(b) : S_B(b) \subset U,$
$\texttt{Assume}\; a \in B,$
$[2] := \texttt{E}\texttt{BooleanOrder} : ab \le b,$
$[3] := \texttt{BooleanOrderBy}[2][1] : S_B(ab) \subset S_B(b) \subset U,$
$[b.*] := \texttt{E}F(U)[3] : ab \in F(U);$
$\rightsquigarrow [U.*] := \texttt{I}\mathcal{I} : F(U) \in \mathcal{I}(B);$
$\rightsquigarrow F := \texttt{I}(\to) : \mathcal{T}(Z_B) \to \mathcal{I}(B),$
$G := \Lambda I \in \mathcal{I}(B) \;.\; \bigcup_{b \in \mathcal{I}} S_B(b) \in \texttt{Poset}\Big(\mathcal{I}(B), \mathcal{T}(Z_B)\Big),$
$[1] := \texttt{E}F\texttt{E}G : FG = \mathrm{id} \;\&\; GF = \mathrm{id},$
$[*] := \texttt{I} \cong [1] : \mathcal{T}(Z_B) \cong_{\mathsf{POSET}} \mathcal{I}(B);$
$\square$

$\texttt{StoneHomoAndCCorespondance} :: \forall A, B \in \mathsf{BOOL} \;.\; \exists \mathsf{BOOL}(A, B) \overset{\gamma}{\leftrightarrow} \mathsf{TOP}(Z_B, Z_A) \;.$
$\;.\; \forall \varphi \in \mathsf{BOOL}(A, B) \;.\; \forall a \in A \;.\; S_B\Big(\varphi(a)\Big) = \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big)$
$\texttt{Proof} \;=$
$\texttt{Assume}\; f \in \mathsf{TOP}(Z_B, Z_A),$
$\texttt{Assume}\; a \in A,$
$[1] := \texttt{StoneRepresentationsAreClopen}(A, a) : \texttt{Clopen}(Z_A, S_A(a)),$
$[2] := \texttt{ClopenCPreimage}[1] : \texttt{Clopen}\bigg(Z_B, f^{-1}\Big(S_A(a)\Big)\bigg),$
$[3] := \texttt{ClosedSubsetOfCompactIsCompact}[2] : \texttt{CompactSubset}\bigg(Z_B, f^{-1}\Big(S_A(a)\Big)\bigg),$
$\Big(b, [4]\Big) := \texttt{CompactOpenIsStoneRepresentation}[2][3] : \sum_{b \in B} f^{-1}\Big(S_A(a)\Big) = S_B(b),$
$\delta(f)(a) := b : B;$
$\rightsquigarrow \delta(f)(a) := \texttt{I}(\to) : A \to B,$
$[f.*] := \texttt{E}S_B\texttt{E}\delta(f) : \delta(f) \in \mathsf{BOOL}(A, B);$
$\rightsquigarrow \delta := \texttt{I}(\to) : \mathsf{TOP}(Z_A, Z_B) \to \mathsf{BOOL}(A, B),$

$\texttt{Assume } \varphi \in \mathsf{BOOL}(A,B),$
$\texttt{Assume } f \in Z_B,$
$\gamma(\varphi)(f) := \varphi f : Z_A;$
$\rightsquigarrow \gamma(\varphi) := \texttt{I}(\to) : Z_B \to Z_A,$
$\texttt{Assume } a \in A,$
$\texttt{Assume } f \in \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big),$
$[1] := \texttt{Epreimage} : \gamma(\varphi)(f) \in S_A(a),$
$[2] := \texttt{I}\gamma\Big(\varphi(a)\Big)[1]\texttt{E}S_A(a) : f\Big(\varphi(a)\Big) = \bigg(\Big(\gamma(\varphi)\Big)(f)\bigg)(a) = 1,$
$[f.*] := \texttt{E}S_B\Big(\varphi(a)\Big)[2] : f \in S_B\Big(\varphi(a)\Big);$
$\rightsquigarrow [1] := \texttt{I} \subset : \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big) \subset S_B\Big(\varphi(a)\Big),$
$\texttt{Assume } f \in S_B(\varphi(a)),$
$[2] := \texttt{E}\gamma\Big(\varphi(a)\Big)[1]\texttt{E}S_B\Big(\varphi(a)\Big) : \bigg(\Big(\gamma(\varphi)\Big)(f)\bigg)(a) = f\Big(\varphi(a)\Big) = 1,$
$[*] := \texttt{E}Z_A(a) : f \in \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big);$
$\rightsquigarrow [2] := \texttt{I} \subset : S_B\Big(\varphi(a)\Big) \subset \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big),$
$[a.*] := \texttt{I}(=)[1][2] : \Big(\gamma(\varphi)\Big)^{-1}\Big(S_A(a)\Big) = S_B\Big(\varphi(a)\Big);$
$\rightsquigarrow [\varphi.*] := \texttt{E}\mathcal{T}_A : \gamma(\varphi) \in \mathsf{TOP}(Z_B, Z_A);$
$\rightsquigarrow \gamma := \texttt{I}(\to) : \mathsf{BOOL}(A,B) \to \mathsf{TOP}(Z_B, Z_A),$
$\texttt{Assume } \varphi \in \mathsf{TOP}(Z_B, Z_A),$
$\texttt{Assume } f \in Z_B,$
$[f.*] := \texttt{E}\delta\texttt{E}\gamma\texttt{E}S\texttt{E}\Lambda :$

$$: \delta\gamma(\varphi)(f) = \delta\Big(\Lambda a \in A \,.\, \varphi^{-1}S_B^{-1}\big(S_A(a)\big)\Big)(f) = \Lambda a \in A \,.\, f\Big(\varphi^{-1}S_B^{-1}\big(S_A(a)\big)\Big) = \Lambda a \in A \,.\, \varphi(f)(a) = \varphi(f);$$

$\rightsquigarrow [1] := \texttt{I}(=,\to) : \delta\gamma = \mathrm{id},$
$\texttt{Assume } \varphi \in \mathsf{BOOL}(A,B),$
$\texttt{Assume } a \in A,$
$[a.*] := \texttt{E}\gamma\texttt{E}\delta\texttt{E}S :$

$$: \gamma\delta(\varphi)(a) = S_B^{-1}\Big(\delta(\varphi)\Big)^{-1}\big(S_A(a)\big) = \varphi(a);$$

$\rightsquigarrow [2] := \texttt{I}(=,\to) : \gamma\delta = \mathrm{id},$
$* := [1][2] : \mathsf{BOOL}(A,B) \overset{\gamma}{\leftrightarrow} \mathsf{TOP}(Z_B, Z_A);$
$\square$

$\texttt{functorOfStone} :: \texttt{Contravariant}(\mathbb{B}, \mathsf{HC})$
$\texttt{functorOfStone}\,(B) = \mathsf{Z}(B) := A$
$\texttt{functorOfStone}\,(f) = \mathsf{Z}_{A,B}(f) := \texttt{StoneHomoAndCCorrespondance}$

$\texttt{StoneFunctorMirrorsInjection} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall A \xrightarrow{f} B : \mathsf{BOOL} \,.$

$\texttt{Injective}(A, B, f) \iff \texttt{Surjective}\Big(\mathsf{Z}(A), \mathsf{Z}(B), \mathsf{Z}_{A,B}(f)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StoneFunctorMirrorsSurjection} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall A \xrightarrow{f} B : \mathsf{BOOL} \,.$

$\texttt{Surjective}(A, B, f) \iff \texttt{Injective}\Big(\mathsf{Z}(A), \mathsf{Z}(B), \mathsf{Z}_{A,B}(f)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{principalIdealProjection} :: \prod_{A \in \mathsf{BOOL}} \prod_{a \in A} \mathsf{BOOL}\Big(A, \langle a \rangle\Big)$

$\texttt{principalIdealProjection}\,(b) = \pi_a(b) := ab$

$\texttt{StonePrincipalIdealEmbedding} :: \forall A \in \mathsf{BOOL} \,.\, \forall a \in A \,.\, \texttt{TopologicalEmbedding}\Big(\mathsf{Z}\langle a \rangle, \mathsf{Z}(A), \mathsf{Z}_{A,\langle a \rangle}(\pi_a)\Big)$

$\texttt{Proof} =$

...

$\square$

$\texttt{StonePrincipalIdealEmbeddingIsStoneRepresentation} ::$

$:: \forall A, B \in \mathsf{BOOL} \,.\, \forall a \in A \,.\, \mathsf{Z}_{A,\langle a \rangle}(\pi_a)\Big(\mathsf{Z}\langle a \rangle\Big) = S_a(A)$

$\texttt{Proof} =$

...

$\square$

1.3 Order Continuity

1.3.1 Inf and Sup

`SupremumComplementation` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, \forall c \in A \,.\, \forall b = \sup B \,.\, c \setminus b = \inf(c \setminus B)$

`Proof` $=$

$\ldots$

$\square$

`InfimumComplementation` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, \forall c \in A \,.\, \forall b = \inf B \,.\, c \setminus b = \sup(c \setminus B)$

`Proof` $=$

$\ldots$

$\square$

`SupremumMult` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, \forall c \in A \,.\, \forall b \in \sup B \,.\, bc \in \sup Bc$

`Proof` $=$

$\ldots$

$\square$

`SupremumMult2` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B, C \subset A \,.\, \forall b \in \sup B \,.\, \forall c \in \sup C \,.\, bc = \sup BC$

`Proof` $=$

$\ldots$

$\square$

`InfinumMult` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, \forall c \in A \,.\, \forall b \in \inf B \,.\, bc = \inf Bc$

`Proof` $=$

$\ldots$

$\square$

`InfimumMult2` $:: \forall A \in \mathsf{BOOL} \,.\, \forall B, C \subset A \,.\, \forall b \in \inf B \,.\, \forall c \in \inf C \,.\, bc = \inf BC$

`Proof` $=$

$\ldots$

$\square$

$\texttt{SupremumAsUnion} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, b = \sup B \iff S_A(b) = \overline{\bigcup_{c \in B} S_A(c)}$

$\texttt{Proof} \ =$

$\texttt{Assume}\ b \in A,$

$\texttt{Assume}\ [1] : b = \sup B,$

$[2] := \texttt{E}_1 \sup B[1] : \forall c \in B \,.\, b \geq c,$

$[3] := [2]\texttt{BooleanOrderByStoneRepresentationI}\cup : \bigcup_{c \in B} S_A(c) \subset S_A(b),$

$[4] := \texttt{E}_2 \sup B : \forall a \in A \,.\, a \geq B \Rightarrow a \geq c,$

$[5] := [4]\texttt{BoleanOrderByStoneRepresentationI}\inf : S_A(b) = \min\left\{U \in \mathcal{TK}\,\mathsf{Z}(A) : \bigcup_{c \in B} S_A(c)\right\},$

$[6] := \texttt{Iclosure}[2]\texttt{StoneRepresentationIsClopen}(A, b) : \overline{\bigcup_{c \in A} S_A(c)} \subset S_A(b),$

$\texttt{Assume}\ f \in S_A(b) \setminus \overline{\bigcup_{c \in A} S_A(c)},$

$\Big(U, [7]\Big) := \texttt{StoneRepresentationIsOpen}(A, b)\texttt{EZeroDimensional}\Big(A\Big) :$

$\quad : \sum U : \texttt{Clopen}\Big(\mathsf{Z}(A)\Big) f \in U \ \&\ U \cap \overline{\bigcup_{c \in A} S_A(c)} = \emptyset,$

$V := S_A(b) \setminus U : \texttt{Clopen}\Big(\mathsf{Z}(A)\Big),$

$[8] := \texttt{E}V[6][7] : \overline{\bigcup_{c \in A} S_A(c)} \subset V,$

$[9] := \texttt{E}V\texttt{E}(\setminus)[7] : V \subsetneq S_A(b),$

$[10] := [9][5] : \bot;$

$\leadsto [1.*] := \texttt{E}\bot[6] : S_A(b) = \overline{\bigcup_{c \in B} S_A(c)};$

$\leadsto [1] := \texttt{I} \Rightarrow : b = \inf B \Rightarrow S_A(b) = \overline{\bigcap_{c \in B} S_A},$

$\texttt{Assume}\ [2] : S_A(b) = \overline{\bigcap_{b \in B} S_A(c)},$

$[3] := \texttt{Eclosure}[2] : \bigcap_{b \in B} S_A(c) \subset S_A(b),$

$[4] := \texttt{BooleanOrderByStoneRepresentation}[3] : B \leq b,$

$[5] := \texttt{Eclosure}\texttt{BooleanOrderByStoneRepresentation}^2[3] : \forall a \in A \,.\, a \geq B \Rightarrow a \geq b,$

$[2.*] := \texttt{I}\sup[4][5] : b = \sup B;$

$\leadsto [2] := \texttt{I} \Rightarrow : S_A(b) = \overline{\bigcap_{c \in B} S_A(c)} \Rightarrow b = \sup B,$

$[*] := \texttt{I} \iff [1][2] : b = \sup B \iff S_A(b) = \overline{\bigcap_{c \in B} S_A(c)};$

$\square$

$\texttt{InfimumAsIntersect} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, b = \inf B \iff S_A(b) = \operatorname{int} \bigcap_{c \in B} S_A(c)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{ZeroInfimumCriterion} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset A \,.\, b = \inf B \iff \texttt{NowhereDense}\left(\mathsf{Z}(A), \bigcap_{c \in B} S_A(c)\right)$

$\texttt{Proof} =$

$\ldots$

$\square$

1.3.2 Sigma Algebras and Ideals

$\texttt{OrderClosed} :: ?\mathsf{POSET}$

$X : \texttt{OrderClosed} \iff \left(\forall D : \texttt{Directed}(X) \,.\, \exists \sup A\right) \,\&\, \left(\forall D : \texttt{Directed}(X^{\mathrm{op}}) \,.\, \exists \inf A\right)$

$\texttt{SequentiallyOrderClosed} :: ?\mathsf{POSET}$

$X : \texttt{SequentiallyOrderClosed} \iff \left(\forall x : \mathbb{N} \uparrow X \,.\, \exists \sup_{n=1} x_n\right) \,\&\, \left(\forall x : \mathbb{N} \downarrow X \,.\, \exists \inf_{n=1} x_n\right)$

$\texttt{SigmaAlgebra} = \sigma\texttt{-Algebra} := \mathsf{BOOL} \,\&\, \texttt{SequentiallyOrderClosed} : \texttt{Type};$

$\texttt{SigmaSubalgebra} = \sigma\texttt{-Subalgebra} = ?^{\sigma}_{\mathsf{BOOL}} := \Lambda A \in \mathsf{BOOL} \,.\, \texttt{Subring}(A) \,\&\, \texttt{SequentiallyOrderClosed} :$
$: \mathsf{BOOL} \to \mathsf{SET};$

$\texttt{SigmaIdeal} = \sigma\texttt{-Ideal} = \mathcal{I}^{\sigma} := \Lambda A \in \mathsf{BOOL} \,.\, \texttt{Ideal}(A) \,\&\, \texttt{SequentiallyOrderClosed} : \mathsf{BOOL} \to \mathsf{SET};$

$\texttt{SigmaAlgebraBySuprema} :: \forall A \in \mathsf{BOOL} \,.\, \sigma\texttt{-Algebra}(A) \iff \forall x : \mathbb{N} \uparrow A \,.\, \exists \sup_{n=1} x_n \in A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SigmaAlgebraByInfima} :: \forall A \in \mathsf{BOOL} \,.\, \sigma\texttt{-Algebra}(A) \iff \forall x : \mathbb{N} \uparrow A \,.\, \exists \inf_{n=1} x_n \in A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SigmaAlgebraHasAllSuprema} :: \forall A \in \mathsf{BOOL} \,.\, \sigma\texttt{-Algebra}(A) \Rightarrow \forall x : \mathbb{N} \to A \,.\, \exists \sup_{n=1} x_n \in A$

$\texttt{Proof} =$

$a := \Lambda n \in \mathbb{N} \,.\, \bigvee_{i=1}^{n} x_i : \mathbb{N} \uparrow A,$

$s := \sup_{n=1} a_n \in A,$

$[1] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}\bigvee(x_{|[1,\dots,n]})\mathtt{I}a\mathtt{E}s\mathtt{E}_1 \sup : \forall n \in \mathbb{N} \,.\, x_n \le \bigvee_{i=1}^{n} x_n = a_n \le s,$

$\texttt{Assume}\ b \in A,$
$\texttt{Assume}\ [2] : \forall n \in \mathbb{N} \,.\, x_n \le b,$

$[3] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}a_n\mathtt{E} \vee [2] : \forall n \in \mathbb{N} \,.\, a_n = \bigvee_{i=1}^{n} x_n \le b,$

$[4] := \mathtt{E}s\mathtt{E}_2 s[3] : s \le b;$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall b \in A \,.\, \left(\forall n \in \mathbb{N} \,.\, x_n \le b\right) \Rightarrow s \le b,$

$[*] := \mathtt{I}\sup[1][2] : s = \sup_{n=1} x_n;$

$\square$

$\texttt{SigmaAlgebraHasAllInfima} :: \forall A \in \mathsf{BOOL} \,.\, \sigma\texttt{-Algebra}(A) \Rightarrow \forall x : \mathbb{N} \to A \,.\, \exists \sup_{n=1} x_n \in A$

$\texttt{Proof} =$

...

$\square$

$\texttt{SigmaIdealCriterion} :: \forall A \in \mathsf{BOOL} \,.\, \forall I \in \mathcal{I}(A) \,.\, I \in \mathcal{I}_\sigma(A) \iff \forall x : \mathbb{N} \to I \,.\, \exists \sup_{n=1} x_n \in I$

$\texttt{Proof} =$

...

$\square$

$\texttt{sigmaClosure} :: \prod A : \sigma\texttt{-Algebra} \,.\, ?A \to \sigma\texttt{-Subalgebra}(A)$

$\texttt{sigmaClosure}\,(B) = \sigma(B) := \bigcap \{C : \sigma\texttt{-Subalgebra}(A) : B \subset C\}$

$\texttt{DifferenceClass} :: \prod_{A \in \mathsf{BOOL}} ??^{\sigma}_{\mathsf{BOOL}} A$

$C : \texttt{DifferenceClass} \iff \forall a, b \in A \,.\, a \setminus b \in A$

$\texttt{DifferenceClassLemma} :: \forall A : \sigma\texttt{-Algebra} \forall I \subset_{\mathsf{MONO}} (A, \wedge) \,.\, \forall C : \texttt{DifferenceClass}(A) \,.\, I \subset C \Rightarrow \sigma(I) \subset C$

$\texttt{Proof} \; =$

$\mathcal{I} := \Big\{ I \subset J \subset C : \forall a, b \in J \,.\, ab \in J \Big\} :?\texttt{Subobjectc}\Big(\mathsf{MONO}, A\Big),$

$J := \texttt{ZornLemma}(\mathcal{I}) : \max \mathcal{I},$

$B := \{a \in A : \forall j \in J \,.\, aj \in C\} :?A,$

$\texttt{Assume}\; a \in B,$

$[2] := \texttt{E}J\texttt{E}\mathcal{I}\texttt{I}e : e \in J,$

$[a.*] := \texttt{E}B(a, e)[2] : a \in C;$

$\rightsquigarrow [*] := \texttt{I} \subset : B \subset C,$

$[3] := \texttt{E}B\texttt{E}J\texttt{E}\mathcal{I} : J \subset B,$

$\texttt{Assume}\; c \in A \setminus J,$

$K := J \cup \{cb | b \in J\} :?A,$

$[4] := \texttt{E}K(e) : c \in K,$

$[5] := [4]\texttt{E}K : B \subsetneq K,$

$[6] := \texttt{E}J[3][5] : K \notin \mathcal{J},$

$[7] := \texttt{EE}\mathcal{I}\texttt{E}K[3] : \forall a, b \in J \,.\, ab \in K,$

$\texttt{Assume}\; a, b \in J,$

$\Big[(a, b).*\Big] := \texttt{ECommutative}(\wedge)\texttt{E}K : (ca)b = (ca)(cb) = c(ab) \in K;$

$\rightsquigarrow [8] := \texttt{I}\forall : \forall a, b \in J \,.\, (ca)b = (ca)(cb) \in K,$

$[9] := \texttt{E}K[7][8] : \forall k, k' \in K \,.\, kk' \in K,$

$[10] := \texttt{E}\mathcal{I}[6][9] : K \not\subset C,$

$\Big(b, [11]\Big) := \texttt{E}K[10] : \sum b \in J \,.\, cb \notin C,$

$[c.*] := \texttt{I}B : c \notin B;$

$\rightsquigarrow [4] := \texttt{ISetEq}[3] : J = B,$

$\texttt{Assume}\; a, b \in J,$

$\texttt{Assume}\; c \in J,$

$[c.*] := \texttt{DiffIntersercDistributivity}(a, c, c)\texttt{ESemigroup}(A, J)\texttt{EDifferenceClass}(A, C) :$

$\quad : (a \setminus b)c = (ac) \setminus b \in C;$

$\rightsquigarrow [5] := \texttt{I}\forall : \forall c \in J \,.\, (a \setminus b)c \in C,$

$[6] := \texttt{E}B[5] : a \setminus b \in B,$

$\Big[(a, b).*\Big] := \texttt{E}\Big(=, [3], [6]\Big) : a \setminus b \in J;$

$\rightsquigarrow [5] := \texttt{I}\forall : \forall a, b \in J \,.\, a \setminus b \in J,$

$[6] := \texttt{BooleanSubAlgebraCriterion}[5] : J \subset_{\mathsf{BOOL}} A,$

$[*] := \texttt{EDifferenceClass}(C)[6]\texttt{I}\sigma : \sigma(I) \subset \sigma(J) \subset C;$

$\square$

1.3.3 Sigma-continuity

$\texttt{OrderContinuous} :: \prod X, Y \in \mathsf{LAT} \;.\; ?\mathsf{LAT}(X, Y)$

$f : \texttt{OrderContinuous} \iff \left(\forall x : \mathbb{N} \uparrow X \;.\; \sup_{n=1} f(x_n) = f\big(\sup_{n=1} x_n\big)\right) \;\&\; \left(\forall x : \mathbb{N} \downarrow X \;.\; \inf_{n=1} f(x_n) = f\big(\inf_{n=1} x_n\big)\right)$

$\texttt{OrderContinuousByPreimage} :: \forall X, Y \in \mathsf{LAT} \;.\; \forall f \in \mathsf{LAT} \;.\; \sigma\texttt{-Continuous}(X, Y, f) \iff$

$\iff \forall A : \texttt{SequentiallyOrderClosed}(Y) \;.\; \texttt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \sigma\texttt{-Continuous}(X, Y, f),$

$\texttt{Assume}\ A : \texttt{SequentiallyOrderClosed}(Y),$

$\texttt{Assume}\ x : \mathbb{N} \uparrow f^{-1}(A),$

$\texttt{Assume}\ s \in X,$

$\texttt{Assume}\ [2] : s = \sup_{n=1} x_n,$

$[3] := \texttt{E}\sigma\texttt{-Continuous}(X, Y, f) : f(s) = \sup_{n=1} f(x_n),$

$[4] := \texttt{ESequentiallyOrderClosed}(Y, A)[3]\texttt{Epreimage}(x) : f(s) \in A,$

$[s.*] := \texttt{Ipreimage}[4] : s \in f^{-1}(A);$

$\rightsquigarrow [A.*] := \texttt{ISequentiallyOrderClosed} : \texttt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big);$

$\rightsquigarrow [1.*] := \texttt{I}\forall\texttt{I} \Rightarrow : \sigma\texttt{-Continuous}(X, Y, f) \Rightarrow$

$\Rightarrow \forall A : \texttt{SequentiallyOrderClosed}(Y) \;.\; \texttt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big);$

$\texttt{Assume}\ [2] : \forall A : \texttt{SequentiallyOrderClosed}(Y) \;.\; \texttt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big),$

$\texttt{Assume}\ x : \mathbb{N} \uparrow X,$

$\texttt{Assume}\ s \in X,$

$\texttt{Assume}\ [3] : s = \sup x_n,$

$[4] := \texttt{E}\sup x_n \texttt{E}\mathsf{POSET}(X, Y, f) : f(x) \le f(s),$

$\texttt{Assume}\ y \in Y,$

$\texttt{Assume}\ [5] : f(x) \le y,$

$A := \{z \in Y : z \le y\} :?Y,$

$[6] := \texttt{E}A\texttt{ISequentiallyOrderClosed} : \texttt{SequentiallyOrderClosed}(Y, A),$

$[7] := [2][6] : \texttt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big),$

$[8] := \texttt{E}A\texttt{I}f^{-1} : \forall n \in \mathbb{N} \;.\; x_n \in f^{-1}(A),$

$[9] := \texttt{E}[7][8][3] : s \in f^{-1}(A),$

$[10] := \texttt{Epreimage}[9] : f(s) \in A,$

$[y.*] := \texttt{E}A[10] : f(s) \le y;$

$\rightsquigarrow [5] := \texttt{I}\forall\texttt{I} \Rightarrow : \forall y \in Y \;.\; y \ge f(x) \Rightarrow y \ge f(s),$

$[x.*] := \texttt{I}\sup : \sup_{n=1} f(x_n) = f(s);$

$\rightsquigarrow [2.*] := \texttt{I}\sigma\texttt{-Continuous} : \sigma\texttt{-Continuous}(X, Y, f);$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow: \forall A : \mathtt{SequentiallyOrderClosed}(Y) \ . \ \mathtt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big) \Rightarrow$

$\Rightarrow \sigma\text{-}\mathtt{Continuous}(X, Y, f),$

$[*] := \mathtt{I}(\iff)[1][2] : \sigma\text{-}\mathtt{Continuous}(X, Y, f) \iff$

$\iff \forall A : \mathtt{SequentiallyOrderClosed}(Y) \ . \ \mathtt{SequentiallyOrderClosed}\Big(X, f^{-1}(A)\Big);$

$\square$

$\mathtt{OCByOCAtZero} :: \forall A, B \in \mathsf{BOOL} \ . \ \forall f \in \mathsf{BOOL}(A, B) \ . \ \Big(\forall x : \mathbb{N} \downarrow A \ . \ \inf_{n=1} x_n = 0 \Rightarrow \inf_{n=1} f(x_n) = 0\Big) \Rightarrow$

$\Rightarrow \sigma\text{-}\mathtt{Continuous}(A, B, f)$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ x : \mathbb{N} \downarrow A,$

$\mathtt{Assume}\ a \in A,$

$\mathtt{Assume}\ [1] : \inf_{n=1} x_n = a,$

$y := \Lambda n \in \mathbb{N} \ . \ (x_n \setminus a) : \mathbb{N} \downarrow A,$

$[2] := \Lambda n \in \mathbb{N} \ . \ \mathtt{ZeroIsMinimum}(A, y_n) : \forall n \in \mathbb{N} \ . \ y_n \geq 0,$

$\mathtt{Assume}\ b \in A,$

$\mathtt{Assume}\ [3] : \forall n \in \mathbb{N} \ . \ y_n \geq b,$

$[4] := \mathtt{E}y[3] : ba = 0,$

$[5] := \mathtt{E}y\mathtt{E}\inf[3][1] : a \leq b \cup a \leq a,$

$[6] := \mathtt{DoubleIne}[5] : b \cup a = a,$

$[b.*] := [6][4] : b = 0;$

$\rightsquigarrow [3] := \mathtt{I}\forall : \forall b \in A \ . \ \Big(\forall n \in \mathbb{N} \ . \ y_n \geq b) \Rightarrow b = 0,$

$[4] := \mathtt{I}\inf[2][3] : \inf_{n=1} y_n = 0,$

$[5] := [0][4] : \inf_{n=1} f(y_n) = 0,$

$[6] := \mathtt{E}_1 \inf[1] : \forall n \in \mathbb{N} \ . \ x_n \geq a,$

$[7] := \mathtt{E}y[y] : x = y \cup a,$

$[8] := [7]\mathtt{E}\mathsf{BOOL}(A, B, f)\mathtt{E}\cup : f(x) = f(y \cup a) = f(y) \cup f(a) \geq f(a),$

$\mathtt{Assume}\ b \in B,$

$\mathtt{Assume}\ [9] : f(x) \geq b,$

$[10] := \mathtt{E}\mathsf{BOOL}(A, B, f)\mathtt{I}y : f(y) \geq \big(b \setminus f(a)\big),$

$[11] := \mathtt{E}_2[5][10]\mathtt{ZeroIsMin}(B) : b \setminus f(a) = 0,$

$[b.*] := \mathtt{E}(\setminus)[11]\mathtt{IBooleanOrder}(B) : f(a) \geq b;$

$\rightsquigarrow [9] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall b \in B \ . \ f(x) \geq b \Rightarrow f(a) \geq b,$

$[x.*] := \mathtt{I}\inf[8][9] : \inf f(x_n) = f(a);$

$\rightsquigarrow [1] := \mathtt{I}\forall : \forall x : \mathbb{N} \downarrow A \ . \ \inf_{n=1} f(x_n) = f\big(\inf_{n=1} x_n\big),$

$[2] := [1]^{\complement} : \forall x : \mathbb{N} \uparrow A \ . \ \sup f(x_n) = f\big(\sup_{n=1} x_n\big),$

$[*] := \mathtt{I}\sigma\text{-}\mathtt{Continuous} : \sigma\text{-}\mathtt{Continuous}(A, B, f);$

$\square$

$\texttt{BooleanOrderContinuitySup} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\texttt{Continuous}(A, B) \,.$
$\quad .\, \forall X : \texttt{Countable}(A) \,.\, f(\sup X) = \sup f(X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [1] : a = \sup X,$

$x := \texttt{enumerate}(X) : \mathbb{N} \leftrightarrow X,$

$y := \Lambda n \in \mathbb{N} \,.\, \bigvee_{i=1}^{n} x_i : \mathbb{N} \uparrow A,$

$[2] := \texttt{E}y\texttt{E}_1 \sup[1] : \forall n \in \mathbb{N} \,.\, y_n \le a,$

$\texttt{Assume}\ a' \in A,$

$\texttt{Assume}\ [3] : \forall n \in \mathbb{N} \,.\, y_n \le a',$

$[4] := \texttt{E}y[3] : \forall n \in \mathbb{N} \,.\, x_n \le a',$

$[5] := \texttt{E}x[4] : \forall z \in X \,.\, z \le a',$

$[a'.*] := \texttt{E}\sup[1][5] : a \le a';$

$\rightsquigarrow [3] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall a' \in A \,.\, y \le a' \Rightarrow a \le a',$

$[4] := \texttt{I}\sup[2][3] : \sup_{n=1} y_n = a,$

$[5] := \texttt{E}\sigma\text{-}\texttt{Continuous}(A, B, f)[4] : \sup_{n=1} f(y_n) = f(a),$

$[6] := \texttt{E}_1 \sup[5]\texttt{E}y\texttt{BooleanMorphismIsOrderPreserving}(A, B, f) : \forall z \in X \,.\, f(z) \le f(a),$

$\texttt{Assume}\ b \in B,$

$\texttt{Assume}\ [7] : \forall z \in X \,.\, f(z) \le b,$

$[8] := [7]\texttt{I}y\texttt{BooleanMorphismIsOrderPreserving} : \forall n \in \mathbb{N} \,.\, f(y_n) \le b,$

$[b.*] := \texttt{E}_2 \sup[5][8] : f(a) \le b;$

$\rightsquigarrow [7] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall b \in B \,.\, f(X) \le b \Rightarrow f(a) \le b,$

$[*] := \texttt{I}\sup[6][7][1] : \sup f(X) = f(\sup X);$

$\square$

$\texttt{BooleanOrderContinuityInf} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\texttt{Continuous}(A, B) \,.$
$\quad .\, \forall X : \texttt{Countable}(A) \,.\, f(\inf X) = \inf f(X)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{OrderContinuousByPoUImages} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \mathsf{BOOL}(A, B) \,.\, \sigma\texttt{-Continuous}(A, B, f) \iff$

$\iff \forall P : \texttt{PartitionOfUnity}(A) \,.\, |P| \leq \aleph_0 \Rightarrow \texttt{PartitionOfUnity}\Big(B, f(P)\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \sigma\texttt{-Continuous}(A, B, f),$

$\texttt{Assume}\ P : \texttt{PartitionOfUnity}(A),$

$\texttt{Assume}\ [2] : |P| \leq \aleph_0,$

$[3] := \texttt{EPartitionOfUnity}(A)\texttt{I}\sup : \sup P = e,$

$[4] := \texttt{E}\sigma\texttt{-Continuous}(A, B, f)[3]\texttt{E}\mathsf{BOOL}(A, B, f) : \sup f(P) = f(e) = e,$

$\texttt{Assume}\ x, y \in f(P),$

$\texttt{Assume}\ [5] : x \neq y,$

$\Big(a, b, [6]\Big) := \texttt{Eimage}[4] : \sum a, b \in P \,.\, f(a) = x \ \&\ f(b) = y,$

$[7] := \texttt{E}(=, \to)[5][6] : a \neq b,$

$\Big[(x, y).*\Big] := [6]\texttt{E}\mathsf{RNG}(A, B, f)\texttt{EPartitionOfUnity}(A, P)\texttt{E}\mathsf{RNG}(A, B, f) : xy = f(a)f(b) = f(ab) = f(0) = 0;$

$\leadsto [1.*] := \texttt{IPartitionOfUnity}[4] : \texttt{PartitionOfUnity}\Big(B, f(P)\Big);$

$\leadsto [1] := \texttt{I} \Rightarrow \texttt{I}\forall\texttt{I} \Rightarrow:$

$\sigma\texttt{-Continuous}(A, B, f) \Rightarrow \forall P : \texttt{PartitionOfUnity}(A) \,.\, \Big(|P| \leq \aleph_0 \Rightarrow \texttt{PartitionOfUnity}\big(B, f(P)\big)\Big),$

$\texttt{Assume}\ [2] : \forall P : \texttt{PartitionOfUnity}(A) \,.\, \Big(|P| \leq \aleph_0 \Rightarrow \texttt{PartitionOfUnity}\big(B, f(P)\big)\Big),$

$\texttt{Assume}\ x : \mathbb{N} \uparrow A,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [3] : \sup_{n=1} x_n = a,$

$P := \Im X \cup \{a^{\complement}\} : \texttt{Countable}(A),$

$[4] := \texttt{E}P\texttt{E}\sup[3]\texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}(A),$

$[5] := [2](P) : \texttt{PartitionOfUnity}\big(B, f(P)\big),$

$[6] := \texttt{EPartitionOfUnity}\big(B, f(P)\big)\texttt{I}\sup : \sup f(P) = e,$

$[x.*] := \texttt{E}P\texttt{E}\mathsf{BOOL}(A, B, f)\texttt{EPartitionOfUnity}\big(B, f(P)\big)[6] :$

$: \sup_n f(x_n) = \sup_n f(P) \setminus \{f(a^{\complement})\} = \sup_n f(P) \setminus \{f^{\complement}(a)\} = f(a);$

$\leadsto [2.*] := \texttt{OCByOCAtZero} : \sigma\texttt{-Continuous}(A, B, f);$

$\leadsto [2] := \texttt{I} \Rightarrow:$

$: \forall P : \texttt{PartitionOfUnity}(A) \,.\, \Big(|P| \leq \aleph_0 \Rightarrow \texttt{PartitionOfUnity}\big(B, f(P)\big)\Big) \Rightarrow \sigma\texttt{-Continuous}(A, B, f),$

$[*] := \texttt{I} \iff [1][2] :$

$: \forall P : \texttt{PartitionOfUnity}(A) \,.\, \Big(|P| \leq \aleph_0 \Rightarrow \texttt{PartitionOfUnity}\big(B, f(P)\big)\Big) \iff$

$\iff \sigma\texttt{-Continuous}(A, B, f);$

$\square$

$\texttt{BooleanIsomorphismIsOrderContinuous} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall A \overset{f}{\leftrightarrow} B : \mathsf{BOOL} \,.\, \sigma\texttt{-Continuous}(A, B, f)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{OrderContinuousPreimagingOrderSubalgebra} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\texttt{Continuous}(A, B) \,.$
$\quad .\, \forall B' \subset^{\sigma}_{\mathsf{BOOL}} B \,.\, f^{-1}(B') \subset^{\sigma}_{\mathsf{BOOL}} A$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{OrderContinuousOrderSubalgebraImage} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\texttt{Continuous}(A, B) \,.\, \forall C \subset A \,.$
$\quad .\, f\ \sigma\ C = \sigma\ f\ C$

$\texttt{Proof} =$

$[1] := \mathrm{E}\sigma\texttt{OrderContinuousPreimagingOrderSubalgebra} : \sigma\ C \subset f^{-1}\ \sigma f\ C,$

$[*] := f[1] : f\sigma\ C \subset \sigma f\ C;$

$\square$

$\texttt{OrderContinuousOrderSubalgebraImage} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\texttt{Continuous}\ \&\ \texttt{Surjective}(A, B) \,.$
$\quad .\, \forall C \subset A \,.\, \sigma C = A \Rightarrow B = f\ \sigma\ C = \sigma\ f\ C$

$\texttt{Proof} =$

$[1] := \texttt{OrderContinuousSubalgebraImage}(A, B) : f\ \sigma\ C \subset \sigma\ f\ C,$

$[2] := [0]\texttt{ESurjective}(f) : f\ \sigma C = fA = B,$

$[3] := \texttt{UniversumSubset}\Big(B, \sigma\ f\ C\Big)[2] : \sigma\ f\ C \subset B = f\ \sigma\ C,$

$[*] := \texttt{ISetEq}[1][3] : \sigma\ f\ C = f\ \sigma\ C;$

$\square$

1.3.4 Order-density

$$\mathtt{OrderDense} :: \prod_{A \in \mathsf{BOOL}} ?A$$

$$D : \mathtt{OrderDense} \iff \forall a \in A \,.\, \exists d \in D : d \neq 0 \,\&\, d \leq a$$

$\mathtt{DensitySupTHM} :: \forall A \in \mathsf{BOOL} \,.\, \forall D \in \mathtt{OrderDense}(A) \,.\, \forall a \in A \,.\, \exists C : \mathtt{PairwiseDisjointElements}(A) \,.$
$\quad .\, C \subset D \,\&\, a = \sup D$

$\mathtt{Proof} \; =$

$D' := \{d \in D : d \leq a\} :?D,$

$\mathcal{C} := \{C : \mathtt{PairwiseDisjointElements}(A) : C \subset D'\} :?\mathtt{PairwiseDisjointElements}(A),$

$C := \mathtt{ZornLemma}(\mathcal{C}) : \max \mathcal{C},$

$[1] := \mathtt{E}C\mathtt{E}\mathcal{C}\mathtt{E}D' : C \leq a,$

$\mathtt{Assume}\; b \in A,$

$\mathtt{Assume}\; [0] : C \leq b,$

$\mathtt{Assume}\; [2] : a \setminus b \neq 0,$

$\Big(d, [3]\Big) := \mathtt{EOrderDense}(D)(a \setminus b) : \sum d \in D \,.\, d \neq 0 \,\&\, d \leq (a \setminus b),$

$[4] := \mathtt{E}a[3]\mathtt{SetDifferenceOrder}(a, b) : d \leq (a \setminus b) \leq a,$

$[5] := \mathtt{E}D'[4] : d \in D',$

$\Big(c, [5]\Big) := \mathtt{E}\mathcal{C}\mathtt{E}C : \sum c \in C \,.\, cd \neq 0,$

$[6] := [0](c) : c \leq b,$

$[7] := \mathtt{EBooleanOrder}[6] : cb = c,$

$[8] := [5][3.2]\mathtt{E}(\setminus)\mathtt{E}\mathsf{RNG}(A)[7]\mathtt{BooleanRingHasChar2}(A) :$
$\quad 0 \neq cd \leq c(a \setminus b) = c(a + ab) = ca + cab = ca + ca = 0,$

$[2.*] := \mathtt{ZeroIsMinimal}(A)[8] : \perp;$

$\leadsto [2] := \mathtt{E}(\perp) : a \setminus b = 0,$

$[b.*] := \mathtt{E}(\setminus)[2]\mathtt{IBooleanOrder} : a \leq b;$

$\leadsto [2] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall b \in A \,.\, C \leq b \Rightarrow a \leq b,$

$[*] := \mathtt{I}\sup[1][2] : \sup C = a;$

$\square$

$\texttt{OrderDenseContainsPartitionOfUnity} :: \forall A \in \mathsf{BOOL} \,.\, \forall D : \texttt{OrderDense} \,.$

$\quad .\, \exists P : \texttt{PartitionOfUnity}(A) : P \subset D$

$\texttt{Proof} \;=$

$\Big(P, [1]\Big) := \texttt{DensitySupTHM}(A, D, e) : \sum P : \texttt{PairwiseDisjointElements}(A) \,.\, P \subset D \;\&\; e = \sup P,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [2] : a \neq 0,$

$\texttt{Assume}\ [3] : \forall p \in P \,.\, pa = 0,$

$b := a^{\complement} \in A,$

$\texttt{Assume}\ p \in P,$

$[4] := \texttt{E}(b)\texttt{E}\mathsf{RNG}(A)[3](p) : pb = p(e + a) = p + ap = p,$

$[p.*] := \texttt{IBooleanOrder}[4] : p \leq b;$

$\rightsquigarrow [4] := \texttt{ISetLe} : P \leq b,$

$[5] := \texttt{E}b[2] : b \neq e,$

$[6] := \texttt{UnityIsMax}[5] : b < e,$

$[a.*] := \texttt{E}\sup[1.2][4][6] : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot\texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}(A, P);$

$\square$

1.3.5 Regular embeddings

$$\texttt{RegularEmbedding} := \prod_{A,B} \in \mathsf{BOOL} \,.\, \mathsf{BOOL} \,\&\, \texttt{Injective} \,\&\, \sigma\texttt{-Continuous}(A,B) : \mathsf{BOOL}^2 \to \mathsf{SET};$$

$$\texttt{RegularEmbeded} :: \prod_{A \in \mathsf{BOOL}} ?\texttt{Subring}(A)$$
$$B : \texttt{RegularEmbeded} \iff \texttt{RegularEmbedding}(A, B, \iota_B)$$

$$\texttt{RegularEmbedable} :: \mathsf{BOOL} \to ?\mathsf{BOOL}$$
$$A : \texttt{RegularEmbedable} \iff \Lambda B \in \mathsf{BOOL} \,.\, \exists \texttt{RegularEmbedding}(A,B)$$

$\texttt{OrderDenseIsEmbeddable} :: \forall B \in \mathsf{BOOL} \,.\, \forall D : \texttt{OrderDense} \,\&\, \texttt{Subring}(B) \,.\, \texttt{RegularEmbeded}(B,D)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \mathbb{N} \downarrow D,$

$\texttt{Assume}\ [1] : \inf_{n=1} x_n =_B 0,$

$\texttt{Assume}\ [2] : \inf_{n=1} x_n \neq_A 0,$

$\Big(a, [3]\Big) := \mathtt{E}\inf[2] : \sum a \in A \,.\, 0 < a < x,$

$\Big(d, [4]\Big) := \mathtt{E}\texttt{OrderDense}(A,D)(a) : \sum d \in D \,.\, 0 \neq d \leq a,$

$[5] := [3][4] : d < x,$

$[6] := \mathtt{I}\inf[5] : \inf_{n=1} x_n \neq_B 0,$

$[x.*] := [6][1] : \perp;$

$\rightsquigarrow [1] := \mathtt{E}\perp\mathtt{I} \Rightarrow \mathtt{I}\forall : \forall x : \mathbb{N} \downarrow D \,.\, \inf_{n=1} x_n =_B 0 \Rightarrow \inf_{n=1} x_n =_A 0,$

$[*] := \texttt{OCByOCAtZero}[1]\mathtt{I}\texttt{RegularEmbeded} : \texttt{RegularEmbeded}(B,D);$

$\square$

$\texttt{OrderCKernelIsSigmaIdeal} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\texttt{-Continuous} \,\&\, \mathsf{BOOL}(A,B) \,.\, \sigma\texttt{-Ideal}\Big(A, \ker f\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ x : \mathbb{N} \uparrow \ker f,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [1] : \sup_{n=1} x_n = a,$

$[3] := \mathtt{E}\ker f(x) : f(x) = 0,$

$[4] := \mathtt{E}\sigma\texttt{-Continuous}(A,B,f)[3] : 0 = \inf_{n=1} 0 = \inf_{n=1} f(x_n) = f\Big(\sup_{n=1} x_n\Big) = f(a),$

$[x.*] := \mathtt{E}\ker f[4] : a \in \ker f;$

$\rightsquigarrow [*] := \texttt{SigmaIdealBySup} : \sigma\texttt{-Ideal}\Big(A, \ker f\Big);$

$\square$

$\texttt{OrderCBySigmaIdeal} :: \forall A, B \in \mathsf{BOOL} \; . \; \forall f \in \mathsf{BOOL}(A, B) \; .$

$\; . \; \sigma\texttt{-Ideal}\Big(A, \ker f\Big) \;\&\; \texttt{RegularEmbeded}\Big(B, \operatorname{Im} f\Big) \Rightarrow \sigma\texttt{-Continuous}(A, B, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; x : \mathbb{N} \downarrow A,$

$\texttt{Assume}\; a \in A,$

$\texttt{Assume}\; [1] : \inf_{n=1} x_n = 0,$

$\texttt{Assume}\; b \in f(A),$

$\texttt{Assume}\; [2] : f(x) \geq b > 0,$

$\Big(s, [3]\Big) := \texttt{EImage}(b) : \sum s \in A \; . \; b = f(s),$

$[4] := \texttt{E}\mathsf{BOOL}(A, B)[3][2] : f(s \setminus x) = f(s) \setminus f(x) = b \setminus f(x) = 0,$

$[5] := \texttt{E}\ker f[4] : s \setminus x \in (\ker f)^{\mathbb{N}},$

$[6] := \texttt{ComplementSup}[1]\texttt{E}\setminus : \sup_{n=1} s \setminus x_n = s \setminus \inf_{n=1} x_n = s \setminus 0 = s,$

$[7] := \texttt{E}\sigma\texttt{-Ideal}(A, \ker f)[6] : s \in \ker f,$

$[8] := \texttt{E}\ker f[7][3] : b = 0,$

$[b.*] := [2][8] : \perp;$

$\rightsquigarrow [2] := \texttt{I}\inf : \inf_{n=1} f(x_n) =_{f(A)} 0,$

$[x.*] := \texttt{ERegularEmbeded}(B, \operatorname{Im} f) : \inf_{n=1} f(x_n) =_B 0;$

$\rightsquigarrow [*] := \texttt{OCByOCAtZero} : \sigma\texttt{-Continuous}(A, B, f);$

$\square$

$\texttt{SigmaIdealTHM} :: \forall A \in \mathsf{BOOL} \; . \; \forall I : \texttt{Ideal}(A) \; . \; \sigma\texttt{-Ideal}(A, I) \iff \sigma\texttt{-Continuous}\left(A, \frac{A}{I}, \pi_I\right)$

$\texttt{Proof} \;=$

$\dots$

$\square$

1.3.6 Order-Continuity and Stone Spaces

$$\texttt{OrderContiniousOpenImage} :: \forall A,B \in \mathsf{BOOL} \ . \ \forall f : \texttt{OrderContinuous}(A,B) \ .$$
$$. \ \forall U \in \mathcal{T}\ \mathsf{Z}\ B \ . \ U \neq \emptyset \Rightarrow \mathrm{int}(\mathsf{Z}\ f)(U) \neq \emptyset$$

$\texttt{Proof} \ =$

$u := \texttt{ENonEmpty} : U,$

$\Big(b, [2]\Big) := \texttt{StoneBase}(B,U,u) : u \in S_B(b) \subset U,$

$[3] := \texttt{InterioIsMonotonic}[2] : \mathrm{int}(\mathsf{Z}\ f)\Big(S_B(b)\Big) \subset \mathrm{int}(\mathsf{Z}\ f)(U),$

$[4] := \texttt{StoneRepresentationIsCompact}(B,b)\ \texttt{CompactImage}\Big(\mathsf{Z}\ B, \mathsf{Z}\ A, \mathsf{Z}\ f\Big) :$

$: \texttt{CompactSubset}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)\Big(S_B(b)\Big)\Big),$

$[5] := \texttt{HausdorffCompactIsDense}[4] : \texttt{Closed}\Big(\mathsf{Z}; A, (\mathsf{Z}\ f)\Big(S_B(b)\Big)\Big),$

$\texttt{Assume}\ [6] : \mathrm{int}(\mathsf{Z}\ f)(U) = \emptyset,$

$[7] := \texttt{INowhereDense}[5][6] : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)\Big(S_B(b)\Big),$

$D := \Big\{a \in A : S_A(a) \cap (\mathsf{Z}\ f)\Big(S_B(b)\Big) = \emptyset\Big\} :?A,$

$[8] := \texttt{ENowhereDense}[7]\texttt{E}D\texttt{IOrderDense} : \texttt{OrderDense}(A,D),$

$[9] := \texttt{OrderDenseSup}[8] : \sup D = e,$

$[10] := \texttt{EOrderContinuous}(A,B,f)[9]\texttt{E}\mathsf{BOOL}(A,B,f) : \sup f(D) = e,$

$\Big(d, [11]\Big) := \mathrm{E}_2 \sup[10](u) : \sum d \in D \ . \ f(d)b \neq 0,$

$\Big(v, [12]\Big) := \texttt{StoneRepresentationTHM}[11] : \sum v \in \S_B(b) \ . \ v\Big(f(d)b\Big) = 1,$

$[13] := \texttt{I}\mathsf{Z}[12] : (\mathsf{Z}\ f)(v) \in S_A(d) \cap (\mathsf{Z}\ f)\Big(S_B(b)\Big),$

$[6.*] := \texttt{E}D(d)[13] : \bot;$

$\leadsto [*] := \texttt{E}\bot : \mathrm{int}\ (\mathsf{Z}\ f)(U) \neq \emptyset;$

$\square$

$$\texttt{OrderContinuousNDPreimage} :: \forall A,B \in \mathsf{BOOL} \ . \ \forall f : \texttt{OrderContinuous}(A,B) \ .$$
$$. \ \forall N : \texttt{NowhereDense}(\mathsf{Z}\ A) \ . \ \texttt{NowhereDense}\Big(\mathsf{Z}\ B, (\mathsf{Z}\ f)^{-1}(N)\Big)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ N : \texttt{NowhereDense}(\mathsf{Z}\ A),$

$[1] := \texttt{ENowhereDense}\Big(\mathsf{Z}\ A, N\Big) : \mathrm{int}\ \overline{N} = \emptyset,$

$C := (\mathsf{Z}\ f)^{-1}(\overline{N}) : \texttt{Cloed}\Big(\mathsf{Z}\ B\Big),$

$U := \mathrm{int}\ C : \texttt{Open}\Big(\mathsf{Z}\ B\Big),$

$[2] := \texttt{E}\,\mathrm{int}\ \texttt{E}U[1] : \mathrm{int}(\mathsf{Z}\ f)(U) = \emptyset,$

$[3] := \texttt{OrderContinuousOpenImage}[2] : U = \emptyset,$

$[*] := \texttt{EclosureE}U\texttt{E}C[3]\texttt{INowhereDense} : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)^{-1}(N)\Big);$

$\square$

$\texttt{OrderContinuityByNowhereDenseOreimage} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f \in \mathsf{BOOL}(A, B) \,.$

$\quad . \Big(\forall N : \texttt{NowhereDense}\Big(\mathsf{Z}\ A\Big) \,.\, \texttt{NowhereDense}\Big(\mathsf{Z}\ B, (\mathsf{Z}\ f)^{-1}(N)\Big)\Big) \Rightarrow \texttt{OrderContinuous}\Big(A, B, f\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ X :?A,$

$\texttt{Assume}\ [1] : \inf X = 0,$

$N := \bigcap_{x \in X} S_A(x) :? \mathsf{Z}\ A,$

$[2] := [1]\texttt{E}N\texttt{ZeroInfinumCriterion} : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, N\Big),$

$[3] := [0][2] : \texttt{NowhereDense}\Big(\mathsf{Z}\ B, (\mathsf{Z}\ f)^{-1}(N)\Big),$

$[4] := \texttt{E}N\texttt{EIntersectPreimageE}(\mathsf{Z}\ f) : (\mathsf{Z}\ f)^{-1}(N) = \bigcap_{x \in X} (\mathsf{Z}\ f)^{-1}\Big(S_A(x)\Big) == \bigcap_{x \in X S_A(f(x))},$

$[X.*] := \texttt{ZeroInfimumCriterion}[3][4] : \inf f(X) = 0;$

$\rightsquigarrow [*] := \texttt{OCByOCAtZero} : \texttt{OrderContinuous}(A, B, f);$

$\square$

1.3.7 Upper Envelopes

$$\texttt{upperEnvelope} :: \prod_{A \in \mathsf{BOOL}} \prod_{B \subset_{\mathsf{BOOL}} A} A \to ?B$$

$$b : \texttt{upperEnvelope} \iff \Lambda a \in A \,.\, b = \mathrm{upr}_B(a) \iff b = \inf\{x \in B : x \geq a\}$$

$$\texttt{UprAndSupCommute} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \forall X \subset A \,.\, \forall y \in A \,.\, \forall b \in B \,.$$
$$.\ \left(\Big(\forall x \in X \,.\, \exists\, \mathrm{upr}_B(x) \Big) \ \&\ y = \sup X \ \&\ b = \sup_{x \in X} \mathrm{upr}_B(x) \right) \Rightarrow b = \mathrm{upr}_B(y)$$

Proof $=$

. . .

□

$$\texttt{UprAndIntersectCommute} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, \forall b \in B \,.$$
$$.\ \exists\, \mathrm{upr}_B(a) \Rightarrow \mathrm{upr}_B(a \cap b) = \mathrm{upr}_B(a) \cap b$$

Proof $=$

. . .

□

1.4 Order-Completenes

1.4.1 Sigma-Completenes

σ-DedekindComplete $:: ?\mathsf{LATT}$

$L : \sigma$-DedekindComplete $\iff \forall x : \mathbb{N} \to L \,.\, \exists \inf x \;\&\; \exists \sup x$

σ-DedekindCompleteSubset $:: \prod_{L \in \mathsf{L}} ?L$

$A : \sigma$-DedekindCompleteSubset $\iff \forall x : \mathbb{N} \to A \,.\, \exists \inf x \;\&\; \exists \sup x$

SigmaCompleteQuotient $:: \forall B : \sigma$-DedekindComplete $. \forall I : \sigma$-Ideal(B) . σ-DedekindComplete$\left(\frac{B}{I}\right)$

Proof $=$

Assume $[x] : \mathbb{N} \to \frac{B}{I}$,

$\Big(b, [1]\Big) :=$ Eσ-DedekindComplete$(B) : \sum b \in B \,.\, b = \sup_{n=1} x_n$,

$[x.*] :=$ SigmaIdealTHM $: \sup_{n=1} [x_n] = [b]$;

$\rightsquigarrow [*] :=$ Iσ-DedekindComplete : σ-DedekindComplete$\left(\frac{B}{I}\right)$;

$\square$

SigmaAlgebraFactorization $:: \forall X \in \mathsf{SET} \,.\, \forall A : \sigma$-Algebra$(X)$ $. \forall I : \sigma$-Ideal(X) .

. σ-DedekindComplete$\left(\frac{A}{I \cap A}\right)$

Proof $=$

...

$\square$

SigmaCompleteSubalgebras $:: \forall A \in \mathsf{BOOL} \;\&\; \sigma$-DedekindComplete $. \forall B \subset^{\sigma}_{\mathsf{BOOL}} A$.

. σ-DedekindCompleteSubset(A, B)

Proof $=$

...

$\square$

SigmaCompleteIdeal $:: \forall A \in \mathsf{BOOL} \;\&\; \sigma$-DedekindComplete $. \forall I : \sigma$-Ideal(A) .

. σ-DedekindCompleteSubset(A, I)

Proof $=$

...

$\square$

$\mathtt{SigmaCompleteImage} :: \forall A, B \in \mathsf{BOOL} \,.\, \forall f : \sigma\text{-}\mathtt{Continuous}(A, B) \,.\, \sigma\text{-}\mathtt{DedekindComplete}(A) \Rightarrow$
$\quad \Rightarrow \mathtt{SequentiallyOrderClosed}(B, f(A))$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ y : \mathbb{N} \uparrow f(A),$

$\mathtt{Assume}\ b \in B,$

$\mathtt{Assume}\ [1] : \sup_{n=1} y_n = b,$

$\Big(x, [2]\Big) := \mathtt{E} f(A)(y) : \sum x : \mathbb{N} \uparrow A \,.\, y = f(x),$

$\Big(a, [3]\Big) := \mathtt{E}\sigma\text{-}\mathtt{DedekindComplete}(A)(x) : \sum_{a \in A} a = \sup_{n=1} x_n,$

$[4] := [1][2]\mathtt{E}\sigma\text{-}\mathtt{Continuous}(A, B, f)[3] : b = \sup_{n=1} y_n = \sup_{n=1} f(x_n) = f\Big(\sup_{n=1} x_n\Big) = f(a),$

$[y.*] := \mathtt{E} f(A)[4] : b \in f(A);$

$\rightsquigarrow [*] := \mathtt{ISequentiallyOrderClosed} : \mathtt{SequentiallyOrderClosed}\Big(B, f(A)\Big);$

$\square$

$\mathtt{SigmaContinuousByCompleteImage} :: \forall A \in \mathsf{BOOL} \,.\, \forall B : \sigma\text{-}\mathtt{DedekindComplete} \,.\, \forall f : \mathtt{Injective}\ \&\ \mathsf{BOOL}(A, I$
$\quad \mathtt{SequentiallyOrderClosed}\Big(B, f(A)\Big) \Rightarrow \sigma\text{-}\mathtt{Continuous}(A, B, f)$

$\mathtt{Proof} \ =$

$\mathtt{Assume}\ x : \mathbb{N} \downarrow A,$

$\mathtt{Assume}\ [1] : \inf_{n=1} x_n = 0,$

$\Big(b, [2]\Big) := \mathtt{E}\sigma\text{-}\mathtt{DedekindComplete}(B)\Big(f(x)\Big) : \sum_{b \in B} \inf_{n=1} f(x_n) = b,$

$[3] := \mathtt{ESequentiallyOrderClosed}\Big(B, f(A)\Big)[2] : b \in f(A),$

$\Big(a, [4]\Big) := \mathtt{E} f(A)[3] : \sum a \in A \,.\, b = f(a),$

$\mathtt{Assume}\ [5] : a \neq 0,$

$[6] := \mathtt{EInjective}\ \&\ \mathsf{BOOL}(A, B, f)[5] : b \neq 0,$

$\Big(n, [7]\Big) := \mathtt{E}\inf[1][5](a) : \sum_{n=1}^{\infty} x_n < a,$

$[8] := \mathtt{E}\mathsf{BOOL}\ \&\ \mathtt{Injective}(A, B, f)(a)[4] : f(x_n) < f(a) = b,$

$[*.5] := \mathtt{E}\inf[2][8] : \perp;$

$\rightsquigarrow [5] := \mathtt{E}\perp : a = 0,$

$[x.*] := [2][4]\mathtt{E}\mathsf{BOOL}(A, B, f)[5] : \inf_{n=1} f(x_n) = 0;$

$\rightsquigarrow [*] := \mathtt{OCByOCAtZero} : \sigma\text{-}\mathtt{Continuous}(A, B, f),$

$\square$

$\texttt{SigmaCompleteSubalgebraCriterion} :: \forall A : \sigma\text{-}\texttt{DedekindComplete} \ \& \ \mathsf{BOOL} \ . \ \forall B \subset_{\mathsf{BOOL}} A \ .$

$\texttt{SequentiallyOrderClosed}(A, B) \iff \sigma\text{-}\texttt{DedekindComplete}(B) \ \& \ \sigma\text{-}\texttt{Continuous}\Big(B, A, \iota_B\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{SigmaCompleteSigmaGenetrationCommutation} ::$

$:: \forall A : \sigma\text{-}\texttt{DedekindComplete} \ \& \ \mathsf{BOOL} \ . \ \forall B \in \mathsf{BOOL} \ . \ \forall X \subset A \ . \ \forall f : \sigma\text{-}\texttt{Continuous}(A, B, f) \ .$

$. \ f \ \sigma \ X = \sigma \ f \ X$

$\texttt{Proof} =$

$[1] := \texttt{E}\sigma\texttt{Eimage}(f, X) : f \ X \subset f \ \sigma \ X,$

$[2] := \texttt{SigmaCompleteImage}\Big(\sigma X, B, f\Big) : \texttt{SequentiallyOrderClosed}\Big(B, f\sigma X\Big),$

$[3] := \texttt{E}\sigma f X[1][2] : \sigma \ f \ X \subset f\sigma X,$

$[4] := \texttt{OrderContinuousOrderSubalgebraImage}(A, B, f, X) : f \ \sigma \ X \subset f \ \sigma \ X,$

$[*] := \texttt{ISetEq}[3][4] : f \ \sigma \ X = \sigma \ f \ X;$

$\square$

$\texttt{IsomorphismByOrderDenseInjecttion} :: \forall A : \tau\text{-}\texttt{Algebra} \ . \ \forall B : \mathsf{BOOL} \ .$

$. \ \forall f : \texttt{Injective} \ \& \ \mathsf{BOOL}(A, B) \ . \ \texttt{OrderDense}\Big(f(A), B\Big) \Rightarrow \texttt{Isomorphism}\Big(\mathsf{BOOL}, A, B, f\Big)$

$\texttt{Proof} =$

$\dots$

$\square$

1.4.2 Morphism Extension

$\texttt{SigmaCompleteExtensionLemma} :: \forall A : \sigma\texttt{-DedekindComplete} \,\&\, \mathsf{BOOL} \,.\, \forall B \subset^{\sigma}_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, B_a \subset^{\sigma}_{\mathsf{BOOL}} A$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x : \mathbb{N} \to B_a,$

$\Big(y, z, [1]\Big) := \texttt{SubalgebraGeneratedByOneElement}(A, B, a, x) : \sum y, z : \mathbb{N} \to B \,.\, x = (y \setminus a) \cup (z \cap a),$

$\Big(y', [2]\Big) := \mathtt{E}\sigma\texttt{-DedekindComplete}(A, y)\mathtt{E}\texttt{SequentiallyOrderClosed}(A, B) : \sum_{y' \in B} y' = \sup_{n=1} y_n,$

$\Big(z', [3]\Big) := \mathtt{E}\sigma\texttt{-DedekindComplete}(A, z)\mathtt{E}\texttt{SequentiallyOrderClosed}(A, B) : \sum_{z' \in B} z' = \sup_{n=1} z_n,$

$x' := (y' \setminus a) \cup (z' \cap a) \in B_a,$

$[4] := \mathtt{E}\sup[2]\mathtt{E}\sup[3]\mathtt{E}x' : x \le x',$

$\texttt{Assume}\ x'' \in A,$

$\texttt{Assume}\ [5] : x \le x'',$

$[6] := [5]\texttt{IntersectionSup}[3] : x'' \ge \sup_{n=1} z_n \cap a = z' \cap a,$

$[7] := [5]\texttt{ComplementSup}[2] : x'' \ge \sup_{n=1} y_n \setminus a = y' \setminus a,$

$[x''.*] := \mathtt{I}x'[6][7] : x'' \ge x';$

$\rightsquigarrow [5] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall x'' \in A \,.\, x'' \ge x \Rightarrow x'' \ge x,$

$[x.*] := \mathtt{I}\sup[4][5] : x' = \sup_{n=1} x_n;$

$\rightsquigarrow [*] := \mathtt{I}\texttt{SequentiallyOrderClosed} : B_a \subset^{\sigma}_{\mathsf{BOOL}} A;$

$\square$

$\texttt{HomomorphismExtensionTHM} :: \forall A : \mathsf{BOOL} \,.\, \forall B : \mathsf{BOOL} \,\&\, \texttt{OrderDedekindComplete} \,.$

$\quad \forall C \subset_{\mathsf{BOOL}} A \,.\, \forall C \xrightarrow{f} B : \mathsf{BOOL} \,.\, \exists A \xrightarrow{\hat{f}} B : \mathsf{BOOL} : \hat{f}_{|C} = f$

$\texttt{Proof} \ =$

$P := \left\{ D \xrightarrow{f} B : \mathsf{BOOL} \middle| C \subset_{\mathsf{BOOL}} D \subset_{\mathsf{BOOL}} A \right\} : \mathsf{SET}\Big(\uparrow \big(\mathsf{BOOL} \big) \Big),$

$[1] := \texttt{E}P(f) : f \in P,$

$\hat{f} := \texttt{ZornLemma}(P) : \max P,$

$D := \operatorname{dom} \hat{f} : \texttt{Subring}(A),$

$\texttt{Assume}\ [2] : D \neq A,$

$a := \texttt{I}(\setminus)[2]\texttt{E}(A \setminus D) \in A \setminus D,$

$X := \{ d \in D : d \le a \} :?D,$

$y := \sup \hat{f}(X) \in B,$

$\texttt{Assume}\ d, d' \in D,$

$\texttt{Assume}\ [3] : d \le a \le d',$

$[4] := \texttt{E}X(d)[3] : d \in X,$

$\Big[(d, d'). * .2 \Big] := \texttt{E}y\texttt{E}_1 \sup[4] : \hat{f}(d) \le y,$

$[6] := \texttt{E}X[3]\texttt{I}d' : X \le d',$

$[7] := \texttt{E}\mathsf{BOOL}(D, B, \hat{f})[6] : f(X) \le f(d'),$

$\Big[(d, d'). * .2 \Big] := \texttt{E}_2 \sup[7] : y \le \hat{f}(d');$

$\rightsquigarrow [3] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall d, d' \in D \,.\, \forall (d \le a \le d') \Rightarrow f(d) \le y \le f(d'),$

$\Big(g, [4] \Big) := \texttt{HomonorphismExtension}(A, B, D, \hat{f}, a, y)[3] : \sum D_a \xrightarrow{g} B : \mathsf{BOOL} \,.\, g_{|D} = \hat{f} \,\&\, g(a) = y,$

$[5] := \texttt{E}P[4.1] : g \in P,$

$[6] := \texttt{E}a\texttt{E}g : \hat{f} < g,$

$[2.*] := [6]\texttt{E} \max P(\hat{f}) : \perp;$

$\rightsquigarrow [*] := \texttt{E}\perp : \operatorname{dom} \hat{f} = A;$

$\square$

1.4.3 Loomis-Sikorski Representation

$$\texttt{LoomisSikorskiAlgebra} :: \prod A \in \mathsf{BOOL} \,\&\, \sigma\text{-}\texttt{DedekindComplete}\sigma\text{-}\texttt{Subalgebra}(?\mathsf{Z}\ A)$$

$$\texttt{LoomisSikorskiAlgebra}\,() = \mathcal{LS}(A) := \left\{U \triangle M \middle| U \in \mathcal{TK}\ \mathsf{Z}\ A \,\&\, M \in \mathbf{MGR}\ \mathsf{Z}\ A\right\}$$

$[1] := \texttt{E}\mathcal{LS}(A) : \emptyset, A \in \mathcal{LS}(A),$

$\texttt{Assume}\ U \triangle M, U' \triangle M' \in \mathcal{LS}(A),$

$[\ldots * .1] := \texttt{CheckingTruthTableESubring}\Big(?\mathsf{Z}\ A, \mathcal{TK}\ \mathsf{Z}\ A\Big)\texttt{E}\sigma\texttt{-Ideal}\Big(?\mathsf{Z}\ A, \mathsf{MGR}(\mathsf{Z}\ A)\Big)\texttt{E}\mathcal{LS}(A) :$

$\quad : (U \triangle M) \triangle (U' \triangle M') = (U \triangle U') \triangle (M \triangle M') \in \mathcal{LS}(A),$

$[\ldots * .2] := \texttt{CheckingTruthTableESubring}\Big(?\mathsf{Z}\ A, \mathcal{TK}\ \mathsf{Z}\ A\Big)\texttt{E}\sigma\texttt{-Ideal}\Big(?\mathsf{Z}\ A, \mathsf{MGR}(\mathsf{Z}\ A)\Big)\texttt{E}\mathcal{LS}(A) :$

$\quad : (U \triangle M) \cap (U' \triangle M') = (U \cap U') \triangle (M \cap M') \triangle (U \cap M') \triangle (U' \cap M) \in \mathcal{LS}(A);$

$\rightsquigarrow [2] := \texttt{ISubring} : \mathcal{LS}(A) \subset_{\mathsf{BOOL}} ?\mathsf{Z}\ A,$

$\texttt{Assume}\ U \triangle M : \mathbb{N} \to \mathcal{LS}(A),$

$\Big(a, [3]\Big) := \texttt{ClopenAreStoneRepresentations}(A, U) : \sum a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, S_A(a_n) = U_n,$

$a' := \sup_{n=1} a_n \in A,$

$[4] := \texttt{SupremumStonrRepresentationE}a' : S_A(a') = \overline{\bigcup_{n=1} U_n},$

$[5] := \texttt{NowhereDenseClosure}[4] : \texttt{NowhereDense}\left(\mathsf{Z}\ A, S_A(a') \setminus \bigcup_{n=1} U_n\right),$

$E := S_A(a') \setminus \bigcup_{n=1} U_n \triangle M_n : \mathbb{N} \to ?\mathsf{Z}\ A,$

$[6] := \texttt{E}E\texttt{E} \triangle\ : E \subset \left(S_A(a') \setminus \bigcup_{n=1}\right) \cup \bigcup_{n=1} M_n,$

$[7] := \texttt{MeagerSubset}[5][6] : E \in \mathsf{MGR}(A),$

$\Big[(U \triangle M) . *\Big] := \texttt{UnionAsSupI}E\texttt{E} \triangle\ : S_A(a') \sup_{n=1} U_n \triangle M_n = \bigcup_{n=1}^{\infty} U_n \triangle M_n = S_A(a') \triangle E;$

$\rightsquigarrow [3] := \texttt{I}\exists\texttt{I}\forall : \forall U \triangle M : \mathbb{N} \to \mathcal{LS}(A) \,.\, \exists \sup_{n=1} U_n \triangle M_n,$

$[*] := \texttt{I}(\subset^{\sigma}_{\mathsf{BOOL}}) : \mathcal{LS}(A) \subset^{\sigma}_{\mathsf{BOOL}} ?\mathsf{Z}\ A;$

$\square$

$\texttt{LoomisSikorskiRepresentation} :: \forall A : \mathsf{BOOL}\ \&\ \sigma\text{-}\texttt{DedekindComplete} \,.\, \dfrac{\mathcal{LS}(A)}{\mathbf{MGR}\ \mathsf{Z}\ A} \cong_{\mathsf{BOOL}} A$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [U \bigtriangleup M] \in \dfrac{\mathcal{LS}(A)}{\mathbf{MGR}\ \mathsf{Z}\ A},$

$\Big(\varphi[U \bigtriangleup M], [1]\Big) := \texttt{OpenCompactsAreStoneRepresentations}(A, U) : \sum \varphi[U \bigtriangleup M] \in A \,.$

$\quad . \, U = S_A\Big(\varphi[U \bigtriangleup M]\Big),$

$\texttt{Assume}\ [U' \bigtriangleup M'] \in \dfrac{\mathcal{LS}(A)}{\mathbf{MGR}\ \mathsf{Z}\ A},$

$\texttt{Assume}\ [2] : [U \bigtriangleup M] = [U' \bigtriangleup M'],$

$\Big(M'', [3]\Big) := \texttt{EquotientRing}[2] : \sum M'' \in \mathbf{MGR}\ \mathsf{Z}\ A \,.\, U = U' \bigtriangleup M \bigtriangleup M' \bigtriangleup M'',$

$[4] := \texttt{EBair}[3]\texttt{BairCategoryTHM}(\mathsf{Z}\ A) : M \bigtriangleup M' \bigtriangleup M = \emptyset,$

$\Big[[U \bigtriangleup M].*\Big] := [4][3][1] : \varphi[U \bigtriangleup M] = \varphi[U' \bigtriangleup M'];$

$\leadsto \varphi := \texttt{I}(\to) : \texttt{Isomorphism}\left(\mathsf{BOOL}, \dfrac{\mathcal{LS}\ A}{\mathbf{MGR}\ \mathsf{Z}\ A}, A\right);$

$\square$

$\texttt{LoomisSikorskiTHM} :: \forall A \in \mathsf{BOOL} \,.\, \sigma\text{-}\texttt{DedekindComplete}(A) \iff$

$\quad \iff \exists X \in \mathsf{SET} : \exists \mathcal{A} : \sigma\text{-}\texttt{Algebra}(X) : \exists \mathcal{I} \in \mathcal{I}_\sigma(\mathcal{A}) \,.\, A \cong_{\mathsf{BOOL}} \dfrac{\mathcal{A}}{\mathcal{I}}$

$\texttt{Proof} \ =$

$\ldots$

$\square$

1.4.4 Algebra of Open Domains

$$\texttt{nowhereDenseIdeal} :: \prod_{X \in \mathsf{TOP}} \texttt{Ideal}(?X)$$

$\texttt{nowhereDenseIdeal}\,(X) = \mathbf{ND}(X) := \texttt{NowhereDense}(X)$

$\texttt{Assume}\ A, B \in \mathbf{ND}(X),$

$[1] := \texttt{NowhereDenseUnion}(X, A, B) : A \cup B \in \mathbf{ND}(X),$

$[2] := \texttt{SymmetricDifferenceSubset}(X, A, B) : A \triangle B \subset A \cup b,$

$\Big[(A, B).*\Big] := \texttt{NowhereDenseSubset}[1][2] : A \triangle B \in \mathbf{ND}(X);$

$\rightsquigarrow [1] := \texttt{I}\forall : \forall A, B \in \mathbf{ND}(X) \,.\, A \triangle B \in \mathbf{ND}(X),$

$\texttt{Assume}\ A \in \mathbf{ND}(X),$

$\texttt{Assume}\ B :?X,$

$[2] := \texttt{IntersectionIsSubset}(X, A, B) : A \cap B \subset A,$

$[A.*] := \texttt{NowhereDenseSubset}[2] : A \cap B \subset \mathbf{ND}(X);$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall A \in \mathbf{ND}(X) \,.\, \forall B \subset X \,.\, AB \in \mathbf{ND}(X),$

$[*] := \texttt{IIdeal}[1][2] : \texttt{Ideal}\Big(?X, \mathbf{ND}(X)\Big);$

$\square$

$\texttt{weaklyBoundedAlgebra} :: \mathsf{TOP} \to \mathsf{BOOL}$

$\texttt{weaklyBoundedAlgebra}\,(X) = \Sigma(X) := \Big\{A \subset X : \partial A \in \mathbf{ND}(X)\Big\}$

$[1] := \texttt{E}\,\partial\,\emptyset\texttt{TypeIdeal}\Big(?X, \mathbf{ND}(X)\Big) : \partial\,\emptyset = \emptyset \in \mathbf{ND}(X),$

$[2] := \texttt{E}\Sigma(X)[1] : \emptyset \in \sigma(X),$

$\texttt{Assume}\ A, B \in \Sigma(X),$

$[3] := \texttt{ClosureUnion}(X, A, B) : \overline{A \cup B} = \overline{A} \cup \overline{B},$

$[4] := \texttt{SubsetOfUnionInteriorMonotonic}(X) : \mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B),$

$[5] := \texttt{E}\,\partial(A \cup B)[3][4]\texttt{CheckingTruthTablesI}\,\partial\,A\texttt{I}\,\partial\,B :$

$$: \partial(A \cup B) = \overline{A \cup B} \setminus \mathrm{int}(A \cup B) \subset \Big(\overline{A} \cup \overline{B}\Big) \setminus \Big(\mathrm{int}\,A \cup \mathrm{int}\,B\Big) \subset$$

$$\subset \Big(\overline{A} \setminus \mathrm{int}(A)\Big) \cup \Big(\overline{B} \setminus \mathrm{int}\,B\Big) = \partial\,A \cup \partial\,B,$$

$[6] := \texttt{E}\Sigma(A)\texttt{E}\Sigma(B)\texttt{E}\mathbf{ND}(X)\texttt{NowhereDenseUnion}(X)\texttt{NowhwereDenseSubset}(X)[5] : \partial(A \cup B) \in \mathbf{ND}(X),$

$\Big[(A, B).*\Big] := \texttt{E}\Sigma(X)[6] : A \cup B \in \Sigma(X);$

$\rightsquigarrow [3] := \texttt{I}\forall : \forall A, B \in \Sigma(X) \,.\, A \cup B \in \Sigma(X),$

$\texttt{Assume}\ A \in \Sigma(X),$

$[4] := \texttt{E}\Sigma(X)(A) : \partial\,A \in \mathbf{ND}(X),$

$[5] := \texttt{BoundaryComplement}[4] : \partial\,A^{\complement} = \partial\,A \in \mathbf{ND}(X),$

$[A.*] := \texttt{E}\Sigma(X)[5] : A^{\complement} \in \Sigma(X);$

$\rightsquigarrow [4] := \texttt{I}\forall : \forall A \in \Sigma(X) \,.\, A^{\complement} \in \Sigma(X),$

$[*] := \texttt{SubalgebraCritertion}[2][3][4] : \Sigma(X) \in \mathsf{BOOL};$

$\square$

$\texttt{NowhereDenseAreWeaklyBounded} :: \forall X \in \mathsf{TOP} \,.\, \mathbf{ND}(X) \subset \Sigma(X)$

$\texttt{Proof} \; =$

$\texttt{Assume}\ A \in \mathbf{ND}(X),$

$[1] := \texttt{NowhereDenseClosureIsNowhereDense}(X, A) : \overline{A} \in \mathbf{ND}(X),$

$[2] := \texttt{BoundaryInClosure}(X, A) : \partial\, A \subset \overline{A},$

$[3] := \texttt{NowhereDenseSubset}[1][2] : \partial\, A \in \mathbf{ND}(X),$

$[A.*] := \texttt{E}\Sigma(X)[3] : A \in \Sigma(X);$

$\leadsto [*] := \texttt{I} \subset : \mathbf{ND}(X) \subset \Sigma(X);$

$\square$

$\texttt{openDomainAlgebra} :: \mathsf{TOP} \to \mathsf{BOOL}$

$$\texttt{openDomainAlgebra}\,(X) = \mathbf{OD}(X) := \frac{\Sigma(X)}{\mathbf{ND}(X)}$$

$\texttt{OpenDomainRepresentation} :: \forall X \in \mathsf{TOP} \,.\, \forall A \in \Sigma(X) \,.\, \exists! U : \texttt{OpenDomain} \,.\, U \triangle A \in \mathbf{ND}(X)$

$\texttt{Proof} \; =$

$U := \operatorname{int} \overline{A} \in \mathcal{T}(X),$

$[1] := \texttt{ClosedSetInteriorIsOpenDomain}(X)\texttt{E}U : \texttt{OpenDomain}(X, U),$

$[2] := \texttt{E}U\texttt{E} \operatorname{int} \texttt{I} \operatorname{int} \texttt{I}\, \partial : U \setminus A = \operatorname{int} \overline{A} \setminus A \subset \overline{A} \setminus \operatorname{int} A = \partial\, A,$

$[3] := \texttt{E}\Sigma(X)(A)\texttt{NowhereDenseSubset}[2] : U \setminus A \in \mathbf{ND}(X),$

$[4] := \texttt{E}U\texttt{E} \operatorname{int} \texttt{I} \operatorname{int} \texttt{I}\, \partial : A \setminus U = A \setminus \operatorname{int} \overline{A} \subset \overline{A} \setminus \operatorname{int} A = \partial\, A,$

$[5] := \texttt{E}\Sigma(X)(A)\texttt{NowhereDenseSubset}[4] : A \setminus U \in \mathbf{ND}(X),$

$[6] := \texttt{SymmetricDifferenceExpression}(X, A, U)\texttt{NowhereDenseUnion}(X)[3][5] :$
$\quad : A \triangle U = (A \setminus U) \cup (U \setminus A) \in \mathbf{ND}(X),$

$\texttt{Assume}\ V : \texttt{OpenDomain}(X),$

$\texttt{Assume}\ [7] : A \triangle V \in \mathbf{ND}(X),$

$[8] := \texttt{EIdeal}\Big(?X, \mathbf{ND}(X)\Big)[6][7] : U \triangle V \in \mathbf{ND}(X),$

$[9] := \texttt{E}\overline{U}\texttt{AntitoneSetDifference}(X)\texttt{DifferenceSubsets}(X) : V \setminus \overline{U} \subset V \setminus U \subset U \triangle V,$

$[10] := \texttt{OpenAndNowhereDense}[9][8] : V \setminus \overline{U} = \emptyset,$

$[11] := \texttt{OpenInterior}(X, V)\texttt{E}(\setminus)[10]\texttt{MonotonicInterior}(X)\texttt{EOpenDomain}(X, U) : V = \operatorname{int} V \subset \operatorname{int} \overline{U} = U,$

$[12] := \texttt{E}\overline{U}\texttt{AntitoneSetDifference}(X)\texttt{DifferenceSubsets}(X) : U \setminus \overline{V} \subset U \setminus V \subset U \triangle V,$

$[13] := \texttt{OpenAndNowhereDense}[12][8] : V \setminus \overline{U} = \emptyset,$

$[14] := \texttt{OpenInterior}(X, U)\texttt{E}(\setminus)[13]\texttt{MonotonicInterior}(X)\texttt{EOpenDomain}(X, V) : U = \operatorname{int} U \subset \operatorname{int} \overline{V} = V,$

$[*] := \texttt{ISetEq}[11][14] : U = V;$

$\square$

$\texttt{SumOfOpenDomains} :: \forall X \in \mathsf{TOP} \,.\, \forall A, B \in \mathbf{OD}\ X \,.\, A + B = \operatorname{int} \Big(A \triangle B\Big)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{ProductOfOpenDomains} :: \forall X \in \mathsf{TOP} \,.\, \forall A, B \in \mathbf{OD}\ X \,.\, AB = A \cap B$

$\texttt{Proof} \; =$

$\ldots$

$\square$

$\texttt{OpenDomainsInfinum} :: \forall X \in \mathsf{TOP} . \forall A \subset \mathbf{OD}\ X .\ \inf A = \operatorname{int} \bigcap A$

$\texttt{Proof}\ =$

$F := \operatorname{int} \bigcap A : \texttt{Open}(X),$

$[1] := \mathtt{E}F\texttt{OpenDomainIntersectionInterior}(X)\texttt{InteriorOfClosedSetIsOpenDomain} :$

$: F = \operatorname{int} \bigcap A = \operatorname{int} \overline{\bigcap A} \in \mathbf{OD}\ X,$

$[2] := \mathtt{E}F\texttt{IntersectioIsSubsetInteriorIsSubset}\mathtt{I}F : \forall U \in A . F \subset U,$

$[3] := \mathtt{I}\mathbf{OD}(X)[1][2] : \forall U \in A . F \le_{\mathbf{OD}(X)} U,$

$\texttt{Assume}\ G \in \mathbf{OD}(X),$

$\texttt{Assume}\ [4] : \forall U \in A . G \le_{\mathbf{OD}(X)} U,$

$[5] := \texttt{SubsetOfIntersection}(X)\mathtt{E}\mathbf{OD}(X)[4] : G \subset \bigcap A,$

$[6] := \texttt{OpenInteriorSubset}(A, G)[5]\mathtt{I}F : G \subset F,$

$[G.*] := \mathtt{I}\mathbf{OD}(X)[6] : G \le_{\mathbf{OD}(X)} F;$

$\rightsquigarrow [*] := \mathtt{I}\inf \mathtt{E}F[3] : \inf A = \operatorname{int} \bigcap A;$

$\square$

$\texttt{OpenDomainsSupremum} :: \forall X \in \mathsf{TOP} . \forall A \subset \mathbf{OD}\ X .\ \sup A = \operatorname{int} \overline{\bigcup A}$

$\texttt{Proof}\ =$

$F := \operatorname{int} \overline{\bigcup A} \in \mathbf{OD}(X),$

$\texttt{Assume}\ U \in A,$

$[1] := \texttt{SubsetOfUnion}(X, A, U) : U \subset \bigcup A,$

$[2] := [1]\texttt{ClosureIsUperset}\left(\bigcup\right) : U \subset \overline{\bigcup A},$

$[3] := [2]\texttt{OpenInteriorSubset}\mathtt{I}F : U \subset F,$

$[*] := [3]\mathtt{I}\mathbf{OD}(X) : U \le_{\mathbf{OD}(X)} F;$

$\rightsquigarrow [1] := \mathtt{I}\forall : \forall U \in A . U \le_{\mathbf{OD}(X)} F,$

$\texttt{Assume}\ G \in \mathbf{OD}(X),$

$\texttt{Assume}\ [2] : \forall U \in A . U \le_{\mathbf{OD}(X)} G,$

$[3] := \texttt{SubsetUnion}[2] : \bigcup A \subset G,$

$[G.*] := \texttt{InteriorIsMonotonic}(X)\texttt{ClosureIsMonotonic}(X)\mathtt{E}\mathbf{OD}(X)(G)\mathtt{I}F\mathtt{I}\mathbf{OD}(X) :$

$: F \le_{\mathbf{OD}(X)} G;$

$\rightsquigarrow [*] := \mathtt{I}\sup : \sup A = \operatorname{int} \overline{\bigcup A};$

$\square$

$\texttt{OpenDomainAlgebraIsDedekindComplete} :: \forall X \in \mathsf{TOP} . \texttt{OrderDedekindComplete}(\mathbf{OD}\ X)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$$\mathtt{PseudoOpen} :: \prod_{X,Y\in\mathsf{TOP}} ?\mathsf{TOP}(X,Y)$$

$$f : \mathtt{PseudoOpen} \iff \forall N : \mathtt{NowhereDense}(Y) \,.\, \mathtt{NowhereDense}\Big(X, f^{-1}(N)\Big);$$

$$\mathtt{HomoOD} :: \prod_{X,Y\in\mathsf{TOP}} \mathtt{PseudoOpen}(X,Y) \to \mathsf{BOOL} \,\&\, \mathtt{OrderContinuous}\Big(\mathbf{OD}(Y), \mathbf{OD}(X)\Big)$$

$\mathtt{HomoOD}\,(f) = \tilde{f} := \Lambda U \in \mathbf{OD}(Y) \,.\, \operatorname{int}_X \overline{f^{-1}(U)}$

$\mathtt{Assume}\ U, V : \mathbf{OD}(Y),$

$[1] := \mathtt{IntersectionIsSubet}(U,V) : U \cap V \subset U \,\&\, U \cap V \subset V,$

$[2] := \mathtt{EMonotonic}\Big(\tilde{f}\Big)[1] : \tilde{f}(U\cap V) \subset \tilde{f}(U) \,\&\, \tilde{f}(U\cap V) \subset \tilde{f}(V),$

$[3] := \mathtt{SubsetIntersection}[2] : \tilde{f}(U\cap V) \subset \tilde{f}(U) \cap \tilde{f}(V),$

$\mathtt{Assume}\ [4] : \tilde{f}(U\cap V) \neq \tilde{f}(U) \cap \tilde{f}(V),$

$G := \tilde{f}(U\cap V) \setminus \overline{\tilde{f}(U) \cap \tilde{f}(V)} :?X,$

$[5] := [3][4]\mathtt{E}\mathbf{OD}(X)\Big(\tilde{f}(U\cap V)\Big)\mathtt{I}G : G \neq \emptyset,$

$M := \overline{f(G)} : \mathtt{Closed}(Y),$

$[6] := \mathtt{E}M : G \subset f^{-1}(M),$

$[7] := \mathtt{RegularOpenDifferenceIsNotMeage}[6] : \neg\mathtt{NowhereDense}\Big(X, f^{-1}(M)\Big),$

$[8] := \mathtt{EPseudoOpen}(X,Y,f)[7] : \neg\mathtt{NowhereDense}\Big(Y, M\Big),$

$H := \operatorname{int} M \in \mathcal{T}(Y),$

$[9] := \mathtt{E}G\mathtt{E}(\tilde{f})\mathtt{InetriorSubset} : G \subset \tilde{f}(U) \subset \overline{f^{-1}(U)},$

$[10] := \overline{f([9])}\mathtt{I}M : M = \overline{f(G)} \subset \overline{f\overline{f^{-1}(U)}} \subset \overline{U},$

$[11] := \mathtt{I}H[10]\mathtt{E}(Y,U) : H \subset \operatorname{int}\overline{U} = U,$

$[12] := \mathtt{E}G\mathtt{E}(\tilde{f})\mathtt{InetriorSubset} : G \subset \tilde{f}(V) \subset \overline{f^{-1}(V)},$

$[13] := \overline{f([9])}\mathtt{I}M : M = \overline{f(G)} \subset \overline{f\overline{f^{-1}(V)}} \subset \overline{V},$

$[14] := \mathtt{I}H[10]\mathtt{E}(Y,V) : H \subset \operatorname{int}\overline{V} = V,$

$[15] := f^{-1}\Big([14][11]\Big)\mathtt{I}(\tilde{f}) : f^{-1}(H) \subset f^{-1}(V\cap U) \subset \tilde{f}(V\cap U),$

$[16] := \mathtt{E}H\mathtt{I}G[15] : \emptyset \neq G \cap f^{-1}(H) \subset G \cap \tilde{f}(V\cap W),$

$[17] := \mathtt{E}G[16] : \perp,$

$\rightsquigarrow [1] := \mathtt{E}\perp : \tilde{f}(UV) = \Big(\tilde{f}(U)\Big)\Big(\tilde{f}(V)\Big);$

$\mathtt{Assume}\ U \in \mathbf{OD}(Y),$

$V := \overline{U}^{\complement} : \mathbf{OD}(Y),$

$[2] := \mathtt{E}V\mathtt{E}\tilde{f}\mathtt{Epreimage} : \tilde{f}(V) \cap \tilde{f}(U) = \emptyset,$

$[3] := \mathtt{E}V\mathtt{IDense} : \mathtt{Open} \,\&\, \mathtt{Dense}\Big(Y, U \cup V\Big),$

$[4] := \mathtt{EpreimageI}\tilde{f} : f^{-1}(U\cup V) = f^{-1}(U) \cup f^{-1}(V) \subset \tilde{f}(U) \cup \tilde{f}(V),$

$[U.*] := [3][4]\mathtt{EPseudoOpen}(X,Y,f)\mathtt{DifferenceOfOpenDomiansIsNotMeager} : \tilde{f}(V) = \overline{\tilde{f}(U)}^{\complement};$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall U \in \mathbf{OD}(Y) \,.\, \tilde{f}(U^{\complement}) = (\tilde{f})^{\complement}(U),$

$[3] := \mathtt{I}\mathsf{BOOL}[1][2] : \tilde{f} \in \mathsf{BOOL}\Big(\mathbf{OD}(Y), \mathbf{OD}(X)\Big),$

$\texttt{Assume}\ A :?\mathbf{OD}(Y),$

$\texttt{Assume}\ [4] : Y = \sup A,$

$[5] := \texttt{OpenDominSupremum}[4]\texttt{IDense} : \texttt{Open}\ \&\ \texttt{Dense}\left(Y, \bigcup A\right),$

$[6] := \texttt{PreimageUnion}(X, Y, f, A)\texttt{I}\tilde{f} : f^{-1}\bigcup A = \bigcup f^{-1}(A) \leq \bigcup \tilde{f}(A),$

$[7] := \texttt{EPseudoOpen}(X, Y, f)[5][6] : \texttt{Open}\ \&\ \texttt{Dense}\left(X, \bigcup \tilde{f}(A)\right),$

$[A.*] := \texttt{OpenDomainSuperemum}[7] : \sup \tilde{f}(A) = X;$

$\rightsquigarrow [*] := \texttt{IOrderContinuous} : \texttt{OrderContinuous}\Big(\mathbf{OD}(Y), \mathbf{OD}(X), \tilde{f}\Big);$

$\square$

$\texttt{OrderCompleteBooleanAlgebraIsExtremlyDisconnected} ::$

$:: \forall B \in \mathsf{BOOL} \,.\, \texttt{OrderDedekindComplete}(B) \Rightarrow \texttt{ExtremelyDisconnected}(\mathsf{Z}\ B)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ U \in \mathcal{T}\ \mathsf{Z}\ B,$

$A := \{a \in A : S_B(a) \subset U\} :?B,$

$a := \sup A \in B,$

$[1] := \texttt{E}A\texttt{StoneTHM}(B)\texttt{OpenAsUnionCover}(U) : U = \bigcup_{a \in A} S_B(a),$

$[2] := \texttt{E}a\texttt{E}\sup \texttt{Iclosure} : \overline{U} = S_B(a),$

$[U.*] := \texttt{EClopen}[2] : \overline{U} \in \mathcal{T}\ \mathsf{Z}\ B;$

$\rightsquigarrow [*] := \texttt{IExtremelyDisconnected} : \texttt{ExtremelyDisconnected}(\mathsf{Z}\ B);$

$\square$

$\texttt{CompactOpenAlgebraOfAlgebraWithExtremelyDisconnectedStoneSpace} ::$

$:: \forall B \in \mathsf{BOOL} \,.\, \texttt{ExtremelyDisconnected}(\mathsf{Z}\ B) \Rightarrow \mathbf{OD}(\mathsf{Z}\ B) = \mathcal{TK}(\mathsf{Z}\ B)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ U \in \mathcal{TK}(\mathsf{Z}\ B),$

$[1] := \texttt{ClosedClosure}(\mathsf{Z}\ B)\texttt{OpenInterior}(\mathsf{Z}\ B) : \operatorname{int}\overline{U} = \operatorname{int} U = U,$

$[2] := \texttt{E}\mathbf{OD}(\mathsf{Z}\ B)[1] : U \in \mathbf{OD}(Z\ B);$

$\rightsquigarrow [1] := \texttt{I} \subset : \mathcal{TK}(\mathsf{Z}\ B) \subset \mathbf{OD}(\mathsf{Z}\ B),$

$\texttt{Assume}\ U \in \mathbf{OD}(\mathsf{Z}\ B),$

$[2] := \texttt{E}\mathbf{OD}(\mathsf{Z}\ B, U) : \operatorname{int}\overline{U} = U,$

$[3] := \texttt{EExtremelyDisconnected}(\mathsf{Z}\ B)\texttt{OpenInterion} : \overline{U} = U,$

$[4] := \texttt{E}\mathcal{TK}\ \mathsf{Z}\ B[3] : U \in \mathcal{TK}\ \mathsf{Z}\ B;$

$\rightsquigarrow [2] := \texttt{I} \subset : \mathbf{OD}(\mathsf{Z}\ B) \subset \mathcal{TK}\ \mathsf{Z}\ B,$

$[*] := \texttt{ISetEq}[1][2] : \mathbf{OD}(\mathsf{Z}\ B) = \mathcal{TK}\ \mathsf{Z}\ B;$

$\square$

$\texttt{AlgebraIsDedekindCompleteIfStoneSpaceOpenDomainAlgebraIsCompectOpenAlgebra} ::$

$$:: \forall B \in \mathsf{BOOL} \,.\, \mathbf{OD}(\mathsf{Z}\ B) = \mathcal{TK}\ \mathsf{Z}\ B \Rightarrow \texttt{OrderDedekindComplete}(B)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A :?B,$

$\mathcal{A} := S_B(A) :?\mathcal{TK}\ \mathsf{Z}\ B,$

$[1] := \texttt{E}\mathcal{A}[0]\texttt{I}\mathcal{A} : \mathcal{A} \subset \texttt{OrderDense}(\mathsf{Z}\ B,),$

$U := \sup \mathcal{A} \in \texttt{OrderDense}(\mathsf{Z}\ B),$

$[2] := \texttt{E}U[0]\texttt{I}U : U \in \mathcal{TK}\ \mathsf{Z}\ B,$

$\Big(a, [3]\Big) := \texttt{StoneRepresentationTHM}(B, U) : \sum a \in B \,.\, U = S_B(a),$

$[A.*] := \texttt{E}a\texttt{I} \sup \texttt{I}a : \sup A = a;$

$\rightsquigarrow [*] := \texttt{IOrderDedekindComplete} : \texttt{OrderDedekindComplete}(B),$

$\square$

$\texttt{OpenDomainContravariantFunctor} ::$

$$:: \forall X, Y : \texttt{ExtremelyDisconnected} \,.\, \forall X \xrightarrow{f} Y \,.\, f_* \in \mathsf{BOOL}\Big(\mathbf{OD}(Y), \mathbf{OD}(X)\Big)$$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{OpenDomainOrderContinuityImplyClopen} ::$

$$:: \forall X, Y : \texttt{ExtremelyDisconnected}\ \&\ \mathsf{HC} \,.\, \forall X \xrightarrow{f} Y : \mathsf{TOP} \,.$$

$$\texttt{Surjective}(X, Y, f)\ \&\ \texttt{OrderContinuous}\Big(\mathbf{OD}(Y), \mathbf{OD}(X), f_*\Big) \Rightarrow \forall U \in \mathcal{TK}(X) \,.\, f(U) \in \mathcal{TK}(Y)$$

$\texttt{Proof} \ =$

$[1] := \texttt{ClosedMappingTHM}(X, Y, f) : \texttt{ClosedMapping}(X, Y, f),$

$\texttt{Assume}\ U \in \mathcal{TK}(X),$

$[2] := \texttt{EClosedMappingTHM}(X, Y, f) : \texttt{Closed}\Big(Y, f(U)\Big),$

$[3] := \texttt{EExtremelyDisconnected}(Y)[2] : \operatorname{int} f(U) \in \mathcal{TK}(Y),$

$\mathcal{V} := \{V \in \mathcal{TK}(Y) : V \subset f(U)\} :?\mathcal{TK}(Y),$

$[4] := \texttt{E}\mathbf{OD}(Y)\Big(\operatorname{int} f(U)\Big)\texttt{InteriorAsUnionI}\mathcal{V}\texttt{OpenDomainSup} : \operatorname{int} f(U) = \operatorname{int} \overline{\operatorname{int} f(U)} = \operatorname{int} \overline{\bigcup \mathcal{V}} = \sup \mathcal{V},$

$[5] := \texttt{E}\mathcal{V} : f_*(\mathcal{V}) \leq U,$

$[] := \ldots :,$

$[] := \texttt{I}f_*[4]\texttt{EOrderContinuous}\Big(\mathbf{OD}(Y), \mathbf{OD}(X), f_*\Big) :$

$$: f^{-1}\Big(\operatorname{int} f(U)\Big) = f_*\Big(\operatorname{int} f(U)\Big) = f_*\Big(\sup \mathcal{V}\Big) = \sup f_*(\mathcal{V}) = \operatorname{int} \overline{\bigcup_{V \in \mathcal{V}} f^{-1}(V)} = \overline{\bigcup_{V \in \mathcal{V}} f^{-1}(V)} = U,$$

$\ldots$

$\square$

$\texttt{ClopenImplyOpen} :: \forall X, Y : \texttt{ExtremelyDisconnected} \ \& \ \mathsf{HC} \ . \ \forall X \xrightarrow{f} Y : \mathsf{TOP} \ .$

$\quad \texttt{Surjective}(X, Y, f) \ \& \ U \in \mathcal{TK}(X) \ . \ f(U) \in \mathcal{TK}(Y) \Rightarrow \texttt{OpenMap}\Big(X, Y, f\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{OpenImplyOpenDomainOrderContinuity} :: \forall X, Y : \texttt{ExtremelyDisconnected} \ \& \ \mathsf{HC} \ . \ \forall X \xrightarrow{f} Y : \mathsf{TOP} \ .$

$\quad \texttt{Surjective}(X, Y, f) \ \& \ \texttt{OpenMap}\Big(X, Y, f\Big) \Rightarrow \texttt{OrderContinuous}\Big(\mathbf{OD}(Y), \mathbf{OD}(X), f_*\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

$\texttt{DenseSubetOpenDomainTransition} :: \forall X \in \mathsf{TOP} \ . \ \forall Y : \texttt{Dense}(X) \ .$

$\quad . \ \texttt{Isomorphism}\Big(\mathsf{BOOL}, \mathbf{OD}(X), \mathbf{OD}(Y), \Lambda U \in \mathbf{OD}(X) \ . \ U \cap Y\Big)$

$\texttt{Proof} \ =$

$\dots$

$\square$

1.4.5 Dedekind Completion

StoneRepresentationIsInjectiveOC ::

$:: \forall B \in \mathsf{BOOL} \,.\, \mathsf{BOOL}\ \&\ \texttt{OrderContinuous}\ \&\ \texttt{Injective}\Big(B, \mathbf{OD}\ \mathsf{Z}\ B, S_B\Big)$

Proof $=$

Assume $A :?B$,

Assume $[1] : \inf A = 0$,

$[2] := \texttt{ZeroInfinumCriterion}[1] : \texttt{NowhereDense}\,(\mathsf{Z}\ B)\,,$

$$[A.*] :=: \inf S_B(A) = \operatorname{int} \bigcap_{a \in A} S_B(a) = \emptyset;$$

$\rightsquigarrow [*] :=: \texttt{OrderContinuous}\Big(B, \mathbf{OD}\ \mathsf{Z}\ B, S_B\Big),$

$\square$

BooleanAlgebraCompletionUniversalProperty ::

$:: \forall B \in \mathsf{BOOL} \,.\, \forall C \in \mathsf{BOOL}\ \&\ \texttt{OrderDedekindComplete} \,.\, \forall f \in \mathsf{BOOL}\ \&\ \texttt{OrderContinuous}(B, C) \,.$

$.\ \exists! \hat{f} \in \mathsf{BOOL}\ \&\ \texttt{OrderContinuous}\Big(\mathbf{OD}(\mathsf{Z}\ B), C\Big) : S_B \hat{f} = f$

Proof $=$

Assume $U \in \mathbf{OD}(\mathsf{Z}\ B)$,

$A := \{a \in B : S_B(a) \subset U\} :?B,$

$\hat{f}(U) := \sup f(A) : C;$

$\rightsquigarrow \hat{f} := \mathtt{I}(\rightarrow) : \mathsf{BOOL}\ \&\ \texttt{OrderContinuous}\Big(\mathbf{OD}(\mathsf{Z}\ B), C\Big),$

$[*] := \mathtt{E}\hat{f} : S_B \hat{f} = f;$

$\square$

1.4.6 Principle Ideals

$\texttt{PrincipleIdealsAreOrderComplete} :: \forall B \in \mathsf{BOOL} \,.\, \forall I : \texttt{PrincipleIdeal}(B) \,.\, \texttt{OrderClosed}(B, I)$

$\texttt{Proof} \;=$

$\Big(b, [1]\Big) := \texttt{EPrincipleIdeal}(B.I) : \sum b \in B \,.\, I = \langle b \rangle,$

$\texttt{Assume}\ A :?I,$

$\texttt{Assume}\ s \in A,$

$\texttt{Assume}\ [2] : s = \sup A,$

$[3] := \texttt{PrincipleIdealStructure}(B, I, A)[1] : A \le b,$

$[4] := \mathtt{E}\sup A[2][3] : s \le b,$

$[s.*] := \texttt{PrincipleIdealStructure}(B, I, s)[1][4] : s \in I;$

$\leadsto [*] := \texttt{IOrderClosed} : \texttt{OrderClosed}(B, I);$

$\square$

$\texttt{DedekindCompleteByPrincipleIdeals} :: \forall B \in \mathsf{BOOL} \,.\, \texttt{OrderDedekindComplete}(B) \iff$

$\quad \iff \forall I : \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B) \,.\, \texttt{PrincipleIdeal}(I)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{OrderDedekindComplete}(B),$

$\texttt{Assume}\ I : \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B),$

$s := \sup I \in B,$

$[2] := \mathtt{E}s\texttt{EOrderClosed}(B, I) : s \in I,$

$[3] := \mathtt{E}s\mathtt{E}\sup I[2]\texttt{EIdeal}(I) : I = \{b \in B : b \le s\},$

$[1.*] := \texttt{PrincipleIdealStructure}[3] : \texttt{PrincipleIdeal}(B, I);$

$\leadsto I := \mathtt{I}\forall\mathtt{I} \Rightarrow: \forall B \in \mathsf{BOOL} \,.\, \texttt{OrderDedekindComplete}(B) \Rightarrow$

$\quad \Rightarrow \forall I \in \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B) \,.\, \texttt{PrincipleIdeal}(I),$

$\texttt{Assume}\ [2] : \forall I : \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B) \,.\, \texttt{PrincipleIdeal}(I),$

$\texttt{Assume}\ A :?B,$

$I := \langle A \rangle_\tau : \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B),$

$\Big(s, [2]\Big) := [2](A) : \sum s \in B \,.\, I = \langle s \rangle,$

$[3] := \texttt{PrincipleIdealStructure}[2]\mathtt{E}A : \forall a \in A \,.\, a \le s,$

$\texttt{Assume}\ z \in B,$

$\texttt{Assume}\ [4] : \forall a \in A.a \le z,$

$[5] := [2][2]\mathtt{E}I[4]\texttt{PrincipleIdealsAreOrderComplete} : \langle s \rangle = I \subset \langle z \rangle,$

$[6] := [5]\texttt{EPrincipleIdeal}\Big(B, \langle s \rangle\Big) : s \in \langle z \rangle,$

$[z.*] := \texttt{PrincipleIdealStructure}[6] : s \le z;$

$\leadsto [A.*] := \mathtt{I}\sup[3] : s = \sup A;$

$\leadsto [*] := \mathtt{I} \Rightarrow \mathtt{I} \iff [1] : \texttt{OrderDedekindComplete}(B) \iff$

$\quad \iff \forall I : \texttt{Ideal} \,\&\, \texttt{OrderClosed}(B) \,.\, \texttt{PrincipleIdeal}(I);$

$\square$

$\texttt{OrderClosedSubalgerbraOfPrincipleIdeal} ::$

$:: \forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall B \subset^{\tau}_{\mathsf{BOOL}} A \;.\; \forall a \in A \;.\; \{ab|b \in B\} \subset^{\tau}_{\mathsf{BOOL}} \langle a\rangle$

$\texttt{Proof} \;=$

$C := \{ab|b \in B\} : \texttt{Subalgebra}(A),$

$[1] := \texttt{E}C\texttt{PrincipleIdealStructure} : \texttt{Subalgebra}\Big(\langle a\rangle, C\Big),$

$\texttt{Assume}\; X :?C,$

$Y := \{b \in B : ab \in X\} :?B,$

$b := \sup Y \in B,$

$[2] := \texttt{BooleanRingIsALattice}\texttt{E}C\texttt{E}b : \forall x \in X \;.\; x \le ab,$

$\texttt{Assume}\; z \in \langle a\rangle,$

$\texttt{Assume}\; [3] : \forall x \in X \;.\; x \le z,$

$Z := \{u \in A : ua = z\} :?A,$

$z' := \sup Z \in A,$

$[4] := \texttt{E}z' : z' = a^{\complement} \vee z,$

$\texttt{Assume}\; y \in Y,$

$\Big(x, u, [5]\Big) := \texttt{E}Y(y) : \sum x \in X \;.\; \sum u \in \langle a^{\complement}\rangle \;.\; y = x \vee u,$

$[y.*] := \texttt{BooleanRingIsALattice}[4][5] : y \le z';$

$\rightsquigarrow [5] := \texttt{I}\forall : \forall y \in Y \;.\; y \le z',$

$[6] := \texttt{E}\sup \texttt{E}z'[5] : b \le z',$

$[z.*] := \texttt{BooleanRingIsALattice}\texttt{E}z' : ab \le az' = z;$

$\rightsquigarrow [X.*] := \texttt{I}\sup[2] : \sup X = ab;$

$\rightsquigarrow [*] := \texttt{IOrderClosedSubalgebra} : C \subset^{\tau}_{\mathsf{BOOL}} \langle a\rangle;$

$\square$

$\texttt{KernelIsAPrincipleIdeal} :: \forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall B \in \mathsf{BOOL} \;.$

$. \forall f : \texttt{OrderContinuous} \;\&\; \mathsf{BOOL}(A, B) \;.\; \texttt{PrincipleIdeal}(A, \ker f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; X :? \ker f,$

$a := \sup X \in A,$

$[1] := \texttt{E}a\texttt{BooleanOrderContinuousSup}(A, B, f, X)\texttt{E}\ker f\texttt{E}\sup : f(a) = f(\sup X) = \sup f(X) = \sup\{0\} = 0,$

$[X.*] := \texttt{E}\ker f[1] : a \in \ker f;$

$\rightsquigarrow [1] := \texttt{IOrderClosed} : \texttt{OrderClosed}(A, \ker f),$

$[*] := \texttt{DedekindCompleteByPrincipleIdeals}(B)(\ker f) : \texttt{PrincipleIdeal}(A, \ker f);$

$\square$

$\texttt{kernelElement} :: \prod A : \tau\text{-}\texttt{Algebra} \;.$

$$. \prod_{B \in \mathsf{BOOL}} \Big(\texttt{OrderContinuous} \;\&\; \mathsf{BOOL}(A, B)\Big) \to A$$

$\texttt{kernelElement}\,(f) = k_f := \texttt{KernelIsAPrincipleIdeal}\texttt{E}\texttt{PrincipleIdeal}(A)$

$\texttt{AlgebraDeterminedByKernelElement} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall B \in \mathsf{BOOL} \,.$

$\quad .\, \forall f : \texttt{Surjective} \ \& \ \texttt{OrderContinuous} \ \& \ \mathsf{BOOL}(A, B) \,.\, B \cong_{\mathsf{BOOL}} \langle k_f^{\complement} \rangle$

$\texttt{Proof} \ =$

$[1] := \texttt{IsomorphismTHM} : B \cong_{\mathsf{BOOL}} \dfrac{A}{\ker f},$

$\texttt{Assume}\ [a] \in \dfrac{A}{\ker f},$

$\Big(u, v, [2]\Big) := \texttt{E}\complement(k_f)(a) : \sum u \in \langle k_f \rangle \,.\, \sum v \in \langle k_f^{\complement} \rangle \,.\, a = u + v,$

$[a.*] := \texttt{E}k_f[2] : [a] = [v];$

$\rightsquigarrow [*] := \texttt{I} \cong_{\mathsf{BOOL}} [1] : B \cong \langle k_f^{\complement} \rangle;$

$\square$

1.4.7 Upper Envelopes in Complete Algebras

$\texttt{DisjointUpperEnvelopesTHM} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall B \subset^{\tau}_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\; \mathrm{upr}_B(a)\,\mathrm{upr}_B(a^{\complement}) = 0 \Rightarrow a \in B$

$\texttt{Proof} \;=$

$X := \{b \in B : a \leq b\} :?B,$

$[1] := \mathtt{E}X : a \leq X,$

$[2] := \texttt{EOrderClosed}(A, B)\mathtt{E}\,\mathrm{upr}_B(a)\mathtt{E}\sup[1] : a \leq \mathrm{upr}_B(a),$

$X := \{b \in B : a^{\complement} \leq b\} :?B,$

$[3] := \mathtt{E}Y : a^{\complement} \leq Y,$

$[4] := \texttt{EOrderClosed}(A, B)\mathtt{E}\,\mathrm{upr}_B(a^{\complement})\mathtt{E}\sup[3] : a^{\complement} \leq \mathrm{upr}_B(a^{\complement}),$

$[5] := \texttt{ComplementProduct}[2][4] : a = \mathrm{upr}_B(a),$

$[*] := \mathtt{E}\,\mathrm{upr}_B(a)[5] : a \in B;$

$\square$

$\texttt{UpperEnvelopesIdentityExtension} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall B \subset^{\tau}_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, \forall b \in B \,.$

$\mathrm{upr}^{\complement}_B(a^{\complement}) \leq b \leq \mathrm{upr}_B(a) \iff \exists A \xrightarrow{f} B : \mathsf{BOOL} : f_{|B} = \mathrm{id}_B \;\&\; f(a) = b$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UpperEnvelopeAndSubalgebraExtension} ::$

$:: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall B \subset^{\tau}_{\mathsf{BOOL}} A \,.\, \forall a \in A \,.\, \forall c \in B_a \,.\, ac = a\,\mathrm{upr}_B(ac)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

1.4.8 Basically Disconnected Spaces

$$\texttt{Cozero} :: \prod_{X \in \mathsf{TOP}} ??X$$

$$A : \texttt{Cozero} \iff \exists X \xrightarrow{f} \mathbb{R} : \mathsf{TOP} : A = \{x \in X : f(x) \neq 0\}$$

$$\texttt{BasicallyDisconnected} :: ?\mathsf{TOP}$$

$$X : \texttt{BasicallyDisconnected} \iff \forall A \subset X \,.\, \overline{A} \in \mathcal{T}(X) \iff \texttt{Cozero}(X, A)$$

$$\texttt{SigmaCompleteByBasicallyDisconnectedStoneSpace} ::$$

$$:: \forall B \in \mathsf{BOOL} \,.\, \sigma\texttt{-Algebra}(B) \iff \texttt{BasicallyDisconnected}(\mathsf{Z}\ B)$$

Proof $=$

. . .

□

1.4.9 Algebra of Ideals

$$\texttt{idealComplement} :: \prod_{B \in \mathsf{BOOL}} \mathcal{I}_\tau(B) \to \mathcal{I}_\tau(B)$$

$$\texttt{idealComplement}\,(I) = \overline{I} := \{b \in B : \forall i \in I \,.\, ib = 0\}$$

$$\texttt{idealJoin} :: \prod_{B \in \mathsf{BOOL}} \mathcal{I}_\tau^2(B) \to \mathcal{I}_\tau(B)$$

$$\texttt{idealJoin}\,(I, J) = I \vee J := \overline{\overline{I} \cap \overline{J}}$$

$\texttt{TauIdealsAreBooleanLattice} :: \forall B \in \mathsf{BOOL} \,.\, \texttt{BooleanLattice}\Big(\mathcal{I}_\tau(B), \cap, \vee\Big)$

$\texttt{Proof} =$

...

□

$\texttt{TauIdealsAreTauAlgebra} :: \forall B \in \mathsf{BOOL} \,.\, \tau\texttt{-Algebra}\Big(\mathcal{I}_\tau(B)\Big)$

$\texttt{Proof} =$

...

□

$\texttt{PrincipalIdealGenerationIsInjection} ::$

$:: \forall B \in \mathsf{BOOL} \,.\, \texttt{Injective} \,\&\, \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}\Big(B, \mathcal{I}_\tau(B), \Lambda b \in B \,.\, \langle b \rangle\Big)$

$\texttt{Proof} =$

...

□

$\texttt{PrincipleIdealsAreOrderDense} :: \forall B \in \mathsf{BOOL} \,.\, \texttt{OrderDense}\Big(\mathcal{I}_\tau(B), \texttt{PrincipleIdeal}(B)\Big)$

$\texttt{Proof} =$

...

□

$\texttt{PrinicapalIdealAreOrderCompletion} :: \forall B \in \mathsf{BOOL} \,.\, \mathcal{I}_\tau(B) \cong_{\mathsf{BOOL}} \mathbf{OD}(\mathsf{Z}\ B)$

$\texttt{Proof} =$

...

□

1.5 Category Limits

1.5.1 Products

$\texttt{ProductOfBooleanAlgebrasIsBooleanAlgebra} :: \forall I \in \mathsf{SET} \;.\; \forall B : I \to \mathsf{BOOL} \;.\; \prod_{i \in I} B_i \in \mathsf{BOOL}$

$\texttt{Proof} \;=$

$\texttt{Assume}\; b \in \prod_{i \in I} B_i,$

$[b.*] := \texttt{E} \prod_{i \in I} B_i, b \Lambda i \in \mathcal{I} \;.\; \texttt{E}\mathsf{BOOL}(B_i)\texttt{I} \prod_{i \in I} B_i, b : b^2 = (b_i^2)_{i=1} = (b_i)_{i=1} = b;$

$\rightsquigarrow [*] := \texttt{I}\mathsf{BOOL} : \prod_{i \in I} B_i \in \mathsf{BOOL};$

$\square$

$\texttt{BooleanProductOrderIsAProductOrder} :: \forall I \in \mathsf{SET} \;.\; \forall B : I \to \mathsf{BOOL} \;.$

$\forall a, b \in \prod_{i \in I} B \;.\; a \leq b \iff \forall i \in I \;.\; a_i \leq b_i$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{algebraProductEmbedding} :: \prod_{I \in \mathsf{SET}} \prod_{B : I \to \mathsf{BOOL}} \prod_{i \in I} \mathsf{BOOL}\left(B_i, \prod_{i \in I} B_i\right)$

$\texttt{algebraProductEmbedding}\,(i) = \theta_i := \Lambda b \in B_i \;.\; \Lambda j \in I \;.\; \texttt{if}\; j == i \;\texttt{then}\; b \;\texttt{else}\; 0$

$\texttt{AlgebraProductPartitionOfUnity} ::$

$:: \forall I \in \mathsf{SET} \;.\; \forall B : I \to \mathsf{BOOL} \;.\; \texttt{PartitionOfUnity}\left(\prod_{i \in I} B_i, \{\theta_i(e_{B_i}) | i \in I\}\right)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{ProductStructureByFinitePartitionOfUnity} :: \forall B \in \mathsf{BOOL} \,.\, \forall P : \texttt{PartitionOfUnity}(B) \,\&\, \texttt{Finite} \,.$

$$. \texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p\in P}\langle p\rangle, \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp\right)$$

$\texttt{Proof} =$

$$\varphi := \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp : \mathsf{BOOL}\left(B, \prod_{p\in P}\langle p\rangle\right),$$

$[1] := \texttt{E}PoU(P, A)\texttt{E}\varphi\texttt{I}\ker : \ker\varphi = \{0\},$

$$[2] := \texttt{ZeroKernelTHM}[1] : \texttt{Injective}\left(B, \prod_{p\in P}\langle P\rangle, \varphi\right),$$

$$\texttt{Assume}\ t \in \prod_{p\in P}\langle p\rangle,$$

$$b := \sum_{p\in P} t_p \in B,$$

$\texttt{Assume}\ p \in P,$

$[p.*] := \texttt{E}\varphi\texttt{E}b\texttt{E}\mathsf{RNG}(B)\texttt{PrincipleIdealStructure}(B, p)\texttt{EDisjoint}(B, P)\texttt{EBooleanOrder} :$

$$: \varphi_p(b) = p\sum_{q\in P} t_q = \sum_{q\in P} pt_q = pt_p = t_p;$$

$\rightsquigarrow [t.*] := \texttt{I}(=, \rightarrow) : \varphi(b) = t;$

$$\rightsquigarrow [3] := \texttt{ISurjective} : \texttt{Surjective}\left(B, \prod_{p\in P}\langle p\rangle, \varphi\right),$$

$$[*] := \texttt{IIsomorphism}[1][2] : \texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p\in P}\langle p\rangle, \varphi\right);$$

$\square$

$\texttt{ProductStructureByCountablePartitionOfUnity} ::$

$:: \forall B : \sigma\texttt{-Algebra} \,.\, \forall P : \texttt{PartitionOfUnity}(B) \;\&\; \texttt{Countable} \,.$

$$. \,\texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p \in P} \langle p \rangle, \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp\right)$$

$\texttt{Proof} \;=$

$$\varphi := \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp : \mathsf{BOOL}\left(B, \prod_{p \in P} \langle p \rangle\right),$$

$[1] := \texttt{E}PoU(P, A)\texttt{E}\varphi\texttt{I}\ker : \ker \varphi = \{0\},$

$$[2] := \texttt{ZeroKernelTHM}[1] : \texttt{Injective}\left(B, \prod_{p \in P} \langle P \rangle, \varphi\right),$$

$\texttt{Assume}\ t \in \prod_{p \in P} \langle p \rangle,$

$b := \sup_{p \in P} t_p \in B,$

$\texttt{Assume}\ p \in P,$

$[p.*] := \texttt{E}\varphi\texttt{E}b\texttt{E}\sigma\texttt{-Algebra}(B)\texttt{PrincipleIdealStructure}(B, p)\texttt{EDisjoint}(B, P)\texttt{EBooleanOrderE}\sup :$

$: \varphi_p(b) = p \sup_{q \in P} t_q = \sup_{q \in P} p t_q = \sup\{0, t_p\} = t_p;$

$\rightsquigarrow [t.*] := \texttt{I}(=, \rightarrow) : \varphi(b) = t;$

$$\rightsquigarrow [3] := \texttt{ISurjective} : \texttt{Surjective}\left(B, \prod_{p \in P} \langle p \rangle, \varphi\right),$$

$$[*] := \texttt{IIsomorphism}[1][2] : \texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p \in P} \langle p \rangle, \varphi\right);$$

$\square$

$\texttt{ProductStructureByPartitionOfUnity} :: \forall B : \tau\texttt{-Algebra} \,.\, \forall P : \texttt{PartitionOfUnity}(B) \,.$

$$. \texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p \in P} \langle p \rangle, \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp\right)$$

$\texttt{Proof} \ =$

$$\varphi := \Lambda b \in B \,.\, \Lambda p \in P \,.\, bp : \mathsf{BOOL}\left(B, \prod_{p \in P} \langle p \rangle\right),$$

$[1] := \texttt{E}PoU(P, A)\texttt{E}\varphi\texttt{I}\ker : \ker \varphi = \{0\},$

$$[2] := \texttt{ZeroKernelTHM}[1] : \texttt{Injective}\left(B, \prod_{p \in P} \langle P \rangle, \varphi\right),$$

$\texttt{Assume}\ t \in \prod_{p \in P} \langle p \rangle,$

$b := \sup_{p \in P} t_p \in B,$

$\texttt{Assume}\ p \in P,$

$[p.*] := \texttt{E}\varphi\texttt{E}b\texttt{E}\sigma\texttt{-Algebra}(B)\texttt{PrincipleIdealStructure}(B, p)\texttt{EDisjoint}(B, P)\texttt{EBooleanOrderE}\sup :$

$: \varphi_p(b) = p \sup_{q \in P} t_q = \sup_{q \in P} pt_q = \sup\{0, t_p\} = t_p;$

$\rightsquigarrow [t.*] := \texttt{I}(=, \rightarrow) : \varphi(b) = t;$

$$\rightsquigarrow [3] := \texttt{ISurjective} : \texttt{Surjective}\left(B, \prod_{p \in P} \langle p \rangle, \varphi\right),$$

$$[*] := \texttt{IIsomorphism}[1][2] : \texttt{Isomorphism}\left(\mathsf{BOOL}, B, \prod_{p \in P} \langle p \rangle, \varphi\right);$$

$\square$

1.5.2 Products of Subset Algebras

$$\texttt{SetAlgebraProductRepresentation} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{SET} \,.\, \forall A : \prod_{i \in I} \texttt{Algebra}(X_i) \,.$$

$$. \prod_{i \in I} A_i \cong_{\mathsf{BOOL}} \left\{ S \subset \bigsqcup_{i \in I} X_i : \forall i \in I \,.\, \left\{ x | (i, x) \in S \right\} \in A_i \right\}$$

$\texttt{Proof} =$

$$B := \left\{ S \subset \bigsqcup_{i \in I} X_i : \forall i \in I \,.\, \left\{ x | (i, x) \in S \right\} \in A_i \right\} \in \mathsf{BOOL},$$

$$\varphi := \Lambda S \in \prod_{i \in I} A_i \,.\, \bigsqcup_{i \in I} S_i : \texttt{Isomorphism}\left(\mathsf{BOOL}, \prod_{i \in I} A_i, B\varphi \right),$$

□

$$\texttt{SetAlgebraProductFactrorizationRepresentation} ::$$

$$:: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{SET} \,.\, \forall A : \prod_{i \in I} \texttt{Algebra}(X_i) \,.\, \forall J : \prod_{i \in I} \texttt{Ideal}(A_i) \,.$$

$$. \prod_{i \in I} \frac{A_i}{J_i} \cong_{\mathsf{BOOL}} \frac{\left\{ S \subset \bigsqcup_{i \in I} X_i : \forall i \in I \,.\, \left\{ x | (i, x) \in S \right\} \in A_i \right\}}{\left\{ S \subset \bigsqcup_{i \in I} X_i : \forall i \in I \,.\, \left\{ x | (i, x) \in S \right\} \in J_i \right\}}$$

$\texttt{Proof} =$

$\ldots$

□

1.5.3 Products of Open Domain Algebras

$\texttt{OpenDomainAlgebraAsProduct} ::$

$:: \forall X \in \mathsf{TOP} \,.\, \forall \mathcal{U} : \texttt{Disjoint}\ \mathcal{T}\ X \,.\, \texttt{Dense}\left(X, \bigcup \mathcal{U}\right) \Rightarrow \mathbf{OD}(X) \cong \prod_{U \in \mathcal{U}} \mathbf{OD}(U)$

$\texttt{Proof} \ =$

$[1] := \Lambda U \in \mathcal{U} \,.\, \texttt{HomoOD}\Big(U, X, \iota_U\Big) : \forall U \in \mathcal{U} \,.\, \mathsf{BOOL}\Big(\mathbf{OD}(X), \mathbf{OD}(U), V \mapsto V \cap U\Big),$

$\Big(f, [2]\Big) := \texttt{ProductUniversalProperty}[1] :$

$: \sum ! f \in \mathsf{BOOL}\left(\mathbf{OD}(X), \prod_{U \in \mathcal{U}} \mathbf{OD}(U)\right) \,.\, \forall U \in \mathcal{U} \,.\, f\pi_U = \Lambda V \in \mathbf{OD}(X) \,.\, V \cap U,$

$[4] := \texttt{EDense}[0] : \forall V \in \mathbf{OD}(X) \,.\, V \cap \bigcup \mathcal{U} = \emptyset \iff V = \emptyset,$

$[5] := \texttt{E}\bigcup[3] : \forall V \in \mathbf{OD}(X) \,.\, \Big(\forall U \in \mathcal{U} \,.\, U \cap V = \emptyset\Big) \iff V = \emptyset,$

$[6] := [3][5]\texttt{I}\ker : \ker f = \{\emptyset\},$

$[7] := \texttt{ZeroKernelTM}[6] : \texttt{Injective}\left(\mathbf{OD}(X), \prod_{U \in \mathcal{U}} \mathbf{OD}(U), f\right),$

$\texttt{Assume}\ W \in \prod_{U \in \mathcal{U}} \mathbf{OD}(U),$

$[8] := \texttt{EDisjoint}(\mathcal{U})\texttt{E}W : \texttt{Disjoint}(\mathcal{T}(X), W),$

$W' := \bigcup_{U \in \mathcal{U}} W_U \in \mathcal{T}(X),$

$H := \operatorname{int} \overline{U} \in \mathbf{OD}(X),$

$\texttt{Assume}\ U \in \mathcal{U},$

$[U.*] := \texttt{E}H\texttt{InteriorSubset}(X, U)\texttt{ClosureSubset}(X, U)[8]\texttt{E}\mathbf{OD}(U) :$

$: U \cap H = U \cap \operatorname{int} \overline{W'} = \operatorname{int}_U \overline{U \cap W'} = \operatorname{int}_U \overline{W_U} = W_U;$

$\rightsquigarrow [9] := \texttt{I}\forall : \forall U \in \mathcal{U} \,.\, U \cap H = W_U,$

$[W.*] := \texttt{E}H[2][9] : f(H) = W;$

$\rightsquigarrow [8] := \texttt{ISurjective} : \texttt{Surjective}\left(\mathbf{OD}(X), \prod_{U \in \mathcal{U}} \mathbf{OD}(U), f\right),$

$[*] := \texttt{IIsomorphism}[7][8] : \texttt{Isomorphism}\left(\mathsf{BOOL}, \mathbf{OD}(X), \prod_{U \in \mathcal{U}} \mathbf{OD}(U), f\right);$

$\square$

1.5.4 Coproducts

$$\texttt{booleanCorpoduct} :: \prod_{I \in \mathsf{SET}} (I \to \mathsf{BOOL}) \to \mathsf{BOOL}$$

$$\texttt{booleanCoproduct}\,(B) = \bigotimes_{i \in I} B_i := \mathcal{TK} \prod_{i \in I} \mathsf{Z}\; B_i$$

$$\texttt{booleanCanonicalEmbedding} :: \prod_{I \in \mathsf{SET}} \prod_{B : I \to \mathsf{BOOL}} \prod_{i \in I} \mathsf{BOOL}\left(B_i, \bigotimes_{j \in I} B_j\right)$$

$$\texttt{booleanCanonicalEmbedding}\,(b) = \iota_i(b) := \pi_i^{-1}\Big(S_{B_i}(b)\Big)$$

$$\texttt{CoproductStoneSpace} :: \forall I \in \mathsf{SET}\ .\ \forall B : I \to \mathsf{BOOL}\ .\ \mathsf{Z} \bigotimes_{i \in I} B_i \cong_{\mathsf{TOP}} \prod_{i \in \mathcal{I}} \mathsf{Z}\; B_i$$

$\texttt{Proof}\ =$

$$[1] := \texttt{TychonoffTHM}(I, \mathsf{Z}\; B) : \texttt{Compact}\left(\prod_{i \in I} \mathsf{Z}\; B_i\right),$$

$\texttt{Assume}\ p \in \prod_{i \in I} \mathsf{Z}\; B_i,$

$\texttt{Assume}\ U : \mathcal{U}(p),$

$$\Big(J, V, [2]\Big) := \texttt{ProductTopologyBase}(I, \mathsf{Z}\; B, p, U) :$$

$$: \sum J : \texttt{Finite}(I)\ .\ \sum V : \prod_{j \in J} \mathcal{T}\; \mathsf{Z}\; B_j\ .\ p \in \prod_{j \in J} V_j \times \prod_{j \in J^{\mathcal{C}}} \mathsf{Z}\; B_j \subset U,$$

$$\Big(W, [3]\Big) := \Lambda j \in J\ .\ \texttt{EZeroDimensional}(\mathsf{Z}\; B_j, V_j) :$$

$$: \sum W \in \prod_{j \in J} \texttt{Clopen}(\mathsf{Z}\; B_j, p_j)\ .\ \forall j \in J\ .\ p_j \in W_j \subset V_j,$$

$$H := \bigcup_{j \in J} \prod_{i \in I} \texttt{if}\ j == i\ \texttt{then}\ W_j\ \texttt{else}\ \mathsf{Z}\; B_i : \texttt{Clopen}\left(\prod_{i \in I} \mathsf{Z}\; B_i\right),$$

$$[*] := \mathtt{E}H : p \in H \subset U;$$

$$\rightsquigarrow [2] := \mathtt{I}\,\dim_{\mathsf{TOP}} : \dim_{\mathsf{TOP}} \prod_{i \in i} \mathsf{Z}\; B_i = 0,$$

$$[3] := \texttt{T2Product}(I, \mathsf{Z}\; B) : \texttt{T2}\left(\prod_{i \in I} \mathsf{Z}\; B_i\right),$$

$$[4] := \texttt{IStoneSpace}[1][2][3] : \texttt{StoneSpace}\left(\prod_{i \in I} \mathsf{Z}\; B_i\right),$$

$$[*] := \texttt{StoneHomomorphism}[4]\texttt{IbooleanCoproduct} : \prod_{i \in I} \mathsf{Z}\; B_i \cong_{\mathsf{TOP}} \mathsf{Z} \bigotimes_{i \in I} B_i;$$

$\square$

$\texttt{BooleanCoproduct} :: \texttt{Coproduct}(\mathsf{BOOL}, \texttt{booleanCoproduct})$

$\texttt{Proof} =$

$\texttt{Assume}\ I \in \mathsf{SET},$

$\texttt{Assume}\ B : I \to \mathsf{BOOL},$

$\texttt{Assume}\ A \in \mathsf{BOOL},$

$\texttt{Assume}\ f : \prod_{i \in I} \mathsf{BOOL}(B_i, A),$

$\Big(H, [1]\Big) := \texttt{ProductUniversalProperty}(\mathsf{TOP}, \mathsf{Z}\ B, \mathsf{Z}\ A, \mathsf{Z}\ f) :$

$: \sum \mathsf{Z}\ A \xrightarrow{H} \prod_{i=1} \mathsf{Z}\ B_i : \mathsf{TOP}\ .\ \forall i \in I\ .\ \mathsf{Z}\ f_i = H\pi_i,$

$$\varphi := \texttt{CoproductStoneSpace}(I, B) : \texttt{Isomorphism}\left(\mathsf{TOP}, \prod_{i=1} \mathsf{Z}\ B_i, \mathsf{Z} \bigotimes_{i=1} B_i\right),$$

$$h := S_{\bigotimes_{i \in I} B_i} \varphi^{-1} H^{-1} S_A^{-1} : \mathsf{BOOL}\left(\bigotimes_{i \in I} B_i, A\right),$$

$\texttt{Assume}\ i \in I,$

$[i.*] := \texttt{E}\iota_i\texttt{E}h\texttt{E}S[1]\texttt{E}\mathsf{Z}\ f :$

$: \iota_i h = S_{B_i} \pi_i^{-1} S_{\bigotimes_{i \in I} B_i} \varphi^{-1} H^{-1} S_A^{-1} = S_{B_i} \pi_i^{-1} H^{-1} S_A^{-1} = S_{B_i} (\mathsf{Z}\ f_i)^{-1} S_A^{-1} = f_i;$

$\rightsquigarrow [2] := \texttt{I}\forall : \forall i \in I\ .\ \iota_i h = f_i,$

$$\texttt{Assume}\ g : \mathsf{BOOL}\left(\bigotimes_{i \in I} B_i, A\right),$$

$\texttt{Assume}\ [3] : \forall i \in I\ .\ \iota_i g = f_i,$

$[4] := \mathsf{Z}[3] : \forall i \in I\ .\ (\mathsf{Z}\ g)\pi_i = \mathsf{Z}\ f_i,$

$[5] := \texttt{E}\exists![1][4] : \mathsf{Z}\ g = H,$

$$[g.*] := \texttt{StoneHomoAndCCorespondance}\left(\bigotimes_{i \in I} B_i, A\right) \texttt{I}h : g = h;$$

$$\rightsquigarrow [I.*] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall \bigotimes_{i \in I} B_i \xrightarrow{g} A : \mathsf{BOOL}\ .\ (\forall i \in I\ .\ \iota_i g = f_i) \Rightarrow g = h;$$

$\rightsquigarrow [*] := \texttt{ICoproduct} : \texttt{Coproduct}(\texttt{booleanCoproduct});$

$\square$

$$\texttt{CopoductGeneration} :: \forall I \in \mathsf{SET}\ .\ \forall B : I \to \mathsf{BOOL}\ .\ \bigotimes_{i \in I} B_i = \left\langle \bigcup_{i=1} \iota_i(B_i) \right\rangle_{\mathsf{RING}}$$

$\texttt{Proof} =$

$$[1] := \texttt{E}\iota\texttt{EgenerateSubring} : \left\langle \bigcup_{i=1} \iota_i(B_i) \right\rangle_{\mathsf{RING}} \subset \bigotimes_{i \in I} B_i,$$

$$\Big(h, [2]\Big) := \texttt{CoproductUniversalProperty} : \sum! \bigotimes_{i \in I} B_i \xrightarrow{h} \left\langle \bigcup_{i=1} \iota_i(B_i) \right\rangle_{\mathsf{RING}} : \mathsf{BOOL}\ .\ \forall i \in \mathcal{I}\ .\ \iota_i h = \iota_i,$$

$[3] := \texttt{E}\exists![2] : h = \mathrm{id},$

$$[*] := \texttt{E}h[1][3] : \left\langle \bigcup_{i=1} \iota_i(B_i) \right\rangle_{\mathsf{RING}} = \bigotimes_{i \in I} B_i;$$

$\square$

$$\texttt{coproductBase} :: \prod_{I \in \mathsf{SET}} \prod_{B : I \to \mathsf{BOOL}} ? \bigotimes_{i \in I} B_i$$

$$\texttt{coproductBase}\,() = C(I, B) := \left\{ b \in \bigotimes_{i \in I} B_i : \exists J \subset I : a : \prod_{j \in J} B_j : b = \inf_{j \in J} \iota_j(a_j) \right\}$$

$$\texttt{CoproductBaseExpression} :: \forall I \in \mathsf{SET} \,.\, \forall B : I \to \mathsf{BOOL} \,.\, \forall b \in \bigotimes_{i \in I} B_i \,.\, \exists S \subset C(I, B) : b = \sup S$$

Proof $=$

$$\mathcal{D} := \left\{ P : \texttt{PartitionOfUnity}\left(\bigotimes_{i \in I} B_i \right) \,\&\, \texttt{Finite} : P \subset C(I, B) \right\} :$$

$$:? \left(\texttt{PartitionOfUnity}\left(\bigotimes_{i \in I} B_i \right) \,\&\, \texttt{Finite} \right),$$

$$A := \left\{ b \in \bigotimes_{i \in I} B_i : \exists D \in \mathcal{D} : \exists D' \subset \mathcal{D} : b = \sup D \right\} :? \bigotimes_{i \in I} B_i,$$

$[1] := \mathtt{E}A : e, 0 \in A,$

$[2] := \mathtt{E}C(I, B)\mathtt{I}(\cdot_{\bigotimes_{i \in I} B_i}) : \forall x, y \in C(I, B) \,.\, xy \in C(I, B),$

Assume $x, y \in A,$

$$\Big(D, D', [3]\Big) := \mathtt{E}A(x) : \sum_{D \in \mathcal{D}} \sum_{D' \subset D} x = \sup D',$$

$$\Big(E, E', [4]\Big) := \mathtt{E}A(y) : \sum_{E \in \mathcal{D}} \sum_{E' \subset E} y = \sup E',$$

$$F := DE :? \bigotimes_{i \in i} B_i,$$

$[5] := \mathtt{E}F[2] : DE \in \mathcal{D},$

$$\Big[(x, y). * .1\Big] := \mathtt{EPartitionOfUnity}\left(\bigotimes_{i \in I} B_i, D \right) [1]\mathtt{I}x^{\complement}\mathtt{I}A : x^{\complement} = \sup D \setminus D' \in A,$$

$[6] := \mathtt{I} \subset [3][4]\mathtt{I}F : D'E' \subset F,$

$$\Big[(x, y). * .2\Big] := [5][6]\texttt{BooleanRingIsALattice}\left(\bigotimes_{i \in I} B_i \right) \mathtt{I}xy\mathtt{I}A : xy = \sup D'E' \in A;$$

$$\rightsquigarrow [3] := \texttt{BooleanSubalgebraCriterion2} : A \subset_{\mathsf{BOOL}} \bigotimes_{i \in I} B_i,$$

$$[4] := \mathtt{E}\mathcal{D}\mathtt{E}\complement : \forall i \in I \,.\, \forall b \in B_i \,.\, \left\{ \iota_i(b), \iota_i^{\complement}(b) \right\} \in \mathcal{D},$$

$[5] := [4]\mathtt{E}A : \forall i \in I \,.\, \forall b \in B_i \,.\, \iota_i(b) \in A,$

$$[6] := \texttt{CoptoductGenerates}(I, B)[3][5] : A = \bigotimes_{i \in I} B_i,$$

$$[*] := [6]\mathtt{E}A\mathtt{E}\mathcal{D} : \forall b \in \bigotimes_{i \in I} B_i \,.\, \exists S \subset C(I, B) \,.\, b = \sup S;$$

□

$$\texttt{CoproductBaseIsOrderDense} :: \forall I \in \mathsf{SET} \,.\, \forall B : I \to \mathsf{BOOL} \,.\, \texttt{OrderDense}\left(\bigotimes_{i \in I} B_i, C(I, B) \right)$$

Proof $=$

...

□

$\texttt{CanonicalEmbeddingIsOrderC} :: \forall I \in \mathsf{SET} \, . \, \forall B : I \to \mathsf{BOOL} \, . \, \forall i \in I \, . \, \texttt{OrderContinuous}\left(B_i, \bigotimes_{i \in I} B_i\right)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ A :?B_i,$

$\texttt{Assume}\ [1] : \inf A = 0,$

$\texttt{Assume}\ p \in \bigotimes_{i \in I} B_i,$

$\texttt{Assume}\ [2] : p \neq 0,$

$\Big(c, [3]\Big) := \texttt{EOrderDense}\left(\bigotimes_{i \in I} B_i, C(I, B)\right)(c) : \sum c \in C(I, B) \, . \, 0 < c \leq p,$

$\Big(J, b, [4]\Big) := \texttt{E}C(I, B, c) : \sum J \subset I \, . \, \sum b : \prod_{j \in J} B_j \, . \, c = \inf_{j \in j} \iota_j(b_j),$

$J' := J \cup \{i\} :?I,$

$b' := \Lambda j \in J' \, . \, \texttt{if}\ j \in J\ \texttt{then}\ b_j\ \texttt{else}\ \iota_i(e) : \prod_{j \in J'} B_j,$

$[5] := \texttt{E}b'[4] : \forall j \in J' \, . \, b'_j \neq 0,$

$\Big(a, [6]\Big) := \texttt{E}b[4][1] : \sum a \in B_i \, . \, a \not\geq b'_i,$

$[7] := \texttt{EBooleanOrder}[6]\texttt{I}(\setminus) : b'_i \setminus a \neq \emptyset,$

$t := \texttt{ENonEmpty}[7] \in b'_i \setminus a,$

$\Big(z, [8]\Big) := \texttt{ENonEmpty}(c)\texttt{E}t \in \sum z \in c \, . \, z_i = t,$

$[9] := [8]\texttt{I}\iota_i : z \in c \setminus \iota_i(a),$

$[p.*] := \texttt{IBooleanOrder}[3][9] : p \not\leq \iota_i(a);$

$\rightsquigarrow [A.*] := \texttt{I}\inf : \inf \iota_i(A) = 0;$

$\rightsquigarrow [*] := \texttt{IOrderContinuous} : \texttt{OrderContinuous}\left(B_i, \bigotimes_{i \in I} B_i, \iota_i\right);$

$\square$

$\texttt{ZeroCorproduct} :: \forall I \in \mathsf{SET} \, . \, \forall B : I \to \mathsf{BOOL} \, . \, (\exists i \in I : B_i = \star) \Rightarrow \bigotimes_{i \in I} B_i = \star$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CanonicalEmbeddingIsInjection} :: \forall I \in \mathsf{SET} \, . \, \forall B : I \to \mathsf{BOOL} \, . \, \bigotimes_{i \in I} B_i \neq \star \Rightarrow$

$\Rightarrow \forall i \in I \, . \, \texttt{Injective}\left(B_i, \bigotimes_{i \in I} B_i, \iota_i\right)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CoproductNonzeroInf} :: \forall I \in \mathsf{SET} \,.\, \forall B : I \to \mathsf{BOOL} \,.\, \bigotimes_{i \in I} B_i \neq \star \Rightarrow$

$$\Rightarrow \left(\forall J : \texttt{Finite}(I) \,.\, \forall a : \prod_{j \in J} B_j \,.\, \forall j \in J \,.\, a_j \neq 0 \Rightarrow \inf_{j \in J} \iota_j(a_j) \neq 0 \right)$$

Proof $=$

...

□

$\texttt{CanonicalEmbeddingEquality} :: \forall I \in \mathsf{SET} \,.\, \forall B : I \to \mathsf{BOOL} \,.\, \bigotimes_{i \in I} B_i \neq \star \Rightarrow$

$$\Rightarrow \left(\forall i, j \in I \,.\, i \neq j \Rightarrow \Big(\forall a \in B_i \,.\, \forall b \in B_j \,.\, \iota_i(a) = \iota_j(b) \Rightarrow \big(a = e \,\&\, b = e \big| a = 0 \,\&\, b = 0 \big) \Big) \right)$$

Proof $=$

...

□

$\texttt{CoproductPartition} :: \forall I \in \mathsf{SET} \,.\, \forall B : I \to \mathsf{BOOL} \,.\, \forall \mathcal{I} : \texttt{Partition}(I) \,.\, \bigotimes_{i \in I} B_i \cong_{\mathsf{BOOL}} \bigotimes_{J \in \mathcal{I}} \bigotimes_{j \in J} B_j$

Proof $=$

...

□

1.5.5 Coproducts of Subset Algebras

$$\texttt{SetAlgebraCoproductRepresentation} :: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{SET} \,.\, \forall A : \prod_{i \in I} \texttt{Algebra}(X_i) \,.$$

$$\bigotimes_{i \in I} A_i \cong_{\mathsf{BOOL}} \left\langle \left\{ \left\{ x \in \prod_{j \in I} X_j : x_i \in S \right\} \middle| i \in I, S \in A_i \right\} \right\rangle_{\mathsf{RING}}$$

$\texttt{Proof} \;=$

$$B := \left\langle \left\{ \left\{ x \in \prod_{j \in I} X_j : x_i \in S \right\} \middle| i \in I, S \in A_i \right\} \right\rangle_{\mathsf{RING}} \in \mathsf{BOOL},$$

$$f := \Lambda i \in I \,.\, \Lambda S \in A_i \,.\, \left\{ x \in \prod_{j \in I} X_j : x_i \in S \right\} : \prod_{i \in I} \mathsf{BOOL}(A_i, B),$$

$$\Big(h, [1]\Big) := \texttt{CoproductUniversalProperty}(I, A, B, f) : \sum h \in \mathsf{BOOL}\left(\bigotimes_{i \in I} A_i, B \right),$$

$$[2] := \mathtt{E} f [1] \mathtt{E} B : \texttt{Surjective}\left(\bigotimes_{i \in I} A_i, B, h \right),$$

$$\texttt{Assume } p \in \bigotimes_{i \in I} A_i,$$

$$\texttt{Assume } [3] : p \neq 0,$$

$$\Big(c, [4]\Big) := \texttt{EOrderDense}\left(\bigotimes_{i \in I} A_i, C(I, A) \right)[3] : \sum c \in C(I, A) \,.\, 0 < c \leq p,$$

$$\Big(J, S, [5]\Big) := \mathtt{E}C(I, A, c) : \sum J : \texttt{Finite}(I) \,.\, \sum S : \prod_{j \in J} A_j \,.\, c = \inf_{j \in J} \iota_j(S_j),$$

$$t := \prod_{j \in J} \iota_j(S_j) \in \bigotimes_{i \in I} B_i,$$

$$[6] := \mathtt{E}\inf[4][5] : \forall j \in J \,.\, S_j \neq \emptyset,$$

$$[7] := \texttt{BooleanRingIsALatticeE}t[5] : t \leq c,$$

$$[8] := \mathtt{E}t\mathtt{E}\mathsf{BOOL}\left(\bigotimes_{i \in I} A_i, B \right)[1]\mathtt{E}f[6] :$$

$$: h(t) = h\left(\prod_{j \in J} \iota_j(S_j) \right) = \prod_{j \in J} \iota_j h(S_j) = \prod_{j \in J} f_j(S_j) = \left\{ x \in \prod_{i \in I} X_i : \forall j \in J \,.\, x_j \in S_j \right\} \neq \emptyset,$$

$$[p.*] := \texttt{BooleanMorphismIsMonotonic}[4][7][8] : f(p) \neq 0;$$

$$\rightsquigarrow [3] := \mathtt{I}\ker h : \ker h = \{0\},$$

$$[4] := \texttt{ZeroKernelTHM}[3] : \texttt{Injective}\left(\bigotimes_{i \in I} A_i, B, h \right),$$

$$[*] := \texttt{IIsomorphism}[2][4] : \texttt{Isomorphism}\left(\mathsf{BOOL}, \bigotimes_{i \in I} A_i, B, h \right);$$

$\square$

SetAlgebraCoproductFactorizationRepresentation ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall X : I \to \mathsf{SET} \,.\, \forall A : \prod_{i \in I} \mathtt{Algebra}(X_i) \,.\, \forall J : \prod_{i \in I} \mathtt{Ideal}(A_i)$$

$$\bigotimes_{i \in I} \frac{A_i}{J_i} \cong_{\mathsf{BOOL}} \frac{\left\langle \left\{ \left\{ x \in \prod_{j \in I} X_j : x_i \in S \right\} \middle| i \in I, S \in A_i \right\} \right\rangle_{\mathsf{RING}}}{\left\langle \left\{ \left\{ x \in \prod_{j \in I} X_j : x_i \in S \right\} \middle| i \in I, S \in J_i \right\} \right\rangle_{\mathcal{I}}}$$

Proof $=$

...

□

1.5.6 Tensors

$$\texttt{tensor} :: \prod_{I\in\mathsf{SET}} \prod_{B:I\to\mathsf{BOOL}} \prod J : \texttt{Finite}(I) \;.\; \prod_{j\in J} B_j \to \bigotimes_{i\in I} B_i$$

$$\texttt{tensor}\,(b) = \bigotimes_{j\in J} b_j := \prod_{j\in J} \iota_j(b_j)$$

$$\texttt{TensorDistributivity} :: \forall I \in \mathsf{SET} \;.\; \forall B : I \to \mathsf{BOOL} \;.\; \forall J : \texttt{Finite}(B) \forall n \in \mathbb{N} \;.\; \forall b : \prod_{j\in J} B_j^n \;.$$

$$. \bigotimes_{j\in J} \prod_{k=1}^{n} b_{j,k} = \prod_{k=1}^{n} \bigotimes_{j\in J} b_{j,k}$$

$\texttt{Proof} \;=$

...

$\square$

$$\texttt{ZeroTensor} :: \forall I \in \mathsf{SET} \;.\; \forall B : I \to \mathsf{BOOL} \;.\; \forall J : \texttt{Finite}(B) \;.\; \forall b : \prod_{j\in J} B_j \;.$$

$$. \bigotimes_{j\in J} b_j = 0 \iff \exists j \in J : b_j = 0$$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{TensorApproximation} :: \forall A, B \in \mathsf{BOOL} \, . \, \forall p \in A \otimes B \, .$
$\quad . \, \exists \mathcal{A} : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) : \exists b : \mathcal{A} \to B : p = \sup_{a \in \mathcal{A}} a \otimes b_a$

$\texttt{Proof} \;=$

$C := \{p \in A \otimes B : \exists \mathcal{A} : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) : \exists b : \mathcal{A} \to B : p = \sup_{a \in \mathcal{A}} a \otimes b_a\} :?(A \otimes B),$

$[1] := \mathtt{E}\sup \mathtt{E}C : e = \sup_{a \in \{e_A\}} a \otimes e_B \in C,$

$\texttt{Assume } c, c' \in C,$

$\Big(\mathcal{A}, b, [2]\Big) := \mathtt{E}C(c) : \sum \mathcal{A} : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) \, . \, \sum b : \mathcal{A} \to B \, . \, c = \sup_{a \in \mathcal{A}} a \otimes b_a,$

$\Big(\mathcal{A}', b', [3]\Big) := \mathtt{E}C(c') : \sum \mathcal{A}' : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) \, . \, \sum b' : \mathcal{A}' \to B \, . \, c' = \sup_{a \in \mathcal{A}'} a \otimes b'_a,$

$[4] := \mathtt{E}\texttt{PartitionOfUnity}(A, \mathcal{A})\mathtt{E}\texttt{PartitionOfUnity}(A, \mathcal{A})\mathtt{I}\texttt{PartitionOfUnity} :$
$\quad : \texttt{PartitionOfUnity}(A, \mathcal{A}\mathcal{A}'),$

$b'' := \Lambda a a' \in \mathcal{A}\mathcal{A}' \, . \, b_a b'_{a'} : \mathcal{A}\mathcal{A}' \to B,$

$\Big[(c, c') . *\Big] := [2][3]\texttt{BooleanRingIsALattice}(A \otimes B)\texttt{TensorDistribuibity}(A, B)\mathtt{I}b''\mathtt{E}C :$
$\quad : cc' = (\sup_{a \in \mathcal{A}} a \otimes b_a)(\sup_{a \in \mathcal{A}'} a \otimes b'_a) = \sup_{a \in \mathcal{A}} \sup_{a' \in \mathcal{A}'} (a \otimes b_a)(a' \otimes b'_{a'}) = \sup_{a \in \mathcal{A}} \sup_{a' \in \mathcal{A}'} aa' \otimes b_a b'_{a'} = \sup_{a \in \mathcal{A}\mathcal{A}'} a \otimes b''_a \in C;$

$\rightsquigarrow [2] := \mathtt{I}\forall : \forall c, c' \in C \, . \, cc' \in C,$

$\texttt{Assume } c \in C,$

$\Big(\mathcal{A}, b, [3]\Big) := \mathtt{E}C(c) : \sum \mathcal{A} : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) \, . \, \sum b : \mathcal{A} \to B \, . \, c = \sup_{a \in \mathcal{A}} a \otimes b_a,$

$[4] := [3]\texttt{BooleanRingIsALattice}(A \otimes B)\texttt{TensorDistribuibity}(A, B)\mathtt{E}\texttt{PartitionOfUnity}(A, \mathcal{A})$
$\quad \mathtt{E}\complement\mathtt{E}\mathsf{BOOL}(A)\mathtt{E} \otimes \mathtt{E}\sup : c(\sup_{a \in \mathcal{A}} a \otimes b^\complement_a) = (\sup_{a \in \mathcal{A}} a \otimes b^\complement_a)(\sup_{a \in \mathcal{A}} a \otimes b_a) \le \sup_{a \in \mathcal{A}} \sup_{a' \in \mathcal{A}} (a \otimes b^\complement_a)(a' \otimes b^\complement_{a'}) =$
$\quad = \sup_{a \in \mathcal{A}} \sup_{a' \in \mathcal{A}} aa' \otimes b^\complement_a b_{a'} = \sup_{a \in \mathcal{A}} a^2 \otimes b^\complement_a b_a = \sup_{a \in \mathcal{A}} a \otimes 0 = \sup_{a \in \mathcal{A}} 0 = 0,$

$[5] := \texttt{LatticeMainimalElement}(A \otimes B)[4] : c(\sup_{a \in \mathcal{A}} a \otimes b^\complement_a),$

$[6] := [3]\texttt{BooleanRingIsALattice}(A \otimes B)\mathtt{E} \otimes \Lambda a \in \mathcal{A} \, . \, \texttt{ComplementSum}(B, b_a)\mathtt{E}\texttt{PartitionOfUnity}(\mathcal{A}) :$
$\quad : c + (\sup_{a \in \mathcal{A}} a \otimes b^\complement_a) = (\sup_{a \in \mathcal{A}} a \otimes b_a) + (\sup_{a \in \mathcal{A}} a \otimes b^\complement_a) \ge \sup_{a \in \mathcal{A}} a \otimes b_a + a \otimes b^\complement_a = \sup_{a \in \mathcal{A}} a \otimes (b_a + b^\complement_a) = \sup_{a \in \mathcal{A}} a \otimes e = e,$

$[7] := \texttt{LatticeMaximalElement}(A \otimes B)[6] : c + (\sup_{a \in \mathcal{A}} a \otimes b^\complement_a) = e,$

$[c.*] := \texttt{LawOfExcludedMiddle}\mathtt{E}C : c^\complement = \sup_{a \in \mathcal{A}} a \otimes b^\complement_a \in C;$

$\rightsquigarrow [3] := \mathtt{I}\forall : \forall c \in C \, . \, c^\complement \in C,$

$[4] := \texttt{BooleanSubalgebraCriterion2}[1][2][3] : C \subset_{\mathsf{BOOL}} A \otimes B,$

$[5] := \Lambda a \in A \, . \, \mathtt{E}\sup \; \mathtt{E} \otimes \texttt{LatticeMinimalElement}(A \otimes B)\mathtt{E}C : \forall a \in A \, . \, a \otimes e = \sup\{a \otimes e, a^\complement \otimes 0\} \in C,$

$[6] := \Lambda b \in B \, . \, \mathtt{E}\sup \mathtt{E}C : e \otimes b = \sup_{a \in \{e\}} a \otimes b \in C,$

$[7] := \texttt{CoproductGeneration}(A, B)[4][5][6] : B = C,$

$[*] := \mathtt{E}C[7] : \forall p \in A \otimes B \, . \, \exists \mathcal{A} : \texttt{Finite} \,\&\, \texttt{PartitionOfUnity}(A) : \exists b : \mathcal{A} \to B : p = \sup_{a \in \mathcal{A}} a \otimes b_a;$

$\square$

$\texttt{TensorBound} :: \forall A, B \in \mathsf{BOOL} \, . \, \forall p \in A \otimes B \, . \, p \ne 0 \Rightarrow (\exists a \in A : \exists b \in B : a \ne 0 \,\&\, b \ne 0 \,\&\, a \otimes b \le p)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CoproductOfPowerSetsIsIncomplete} :: \neg\sigma\text{-}\texttt{Algebra}(?\mathbb{N}\otimes ?\mathbb{N})$

$\texttt{Proof} \ =$

$A := \left\{\{n\}\otimes\{n\}\middle| n\in\mathbb{N}\right\} :?(?\mathbb{N}\otimes ?\mathbb{N}),$

$\texttt{Assume}\ c \in ?\mathbb{N}\otimes ?\mathbb{N},$

$\texttt{Assume}\ [1] : c = \inf A,$

$\Big(k, a, b, [2]\Big) := \texttt{CoproductBasrExpression}(?\mathbb{N}, ?\mathbb{N}, c) : \sum_{k=1}^{\infty} \sum_{a,b:[1,\ldots,k]\to ?\mathbb{N}} c = \sup_{1\le i\le k} a_i \otimes b_i,$

$\Big(m, n, [3]\Big) := \texttt{PigionholePrinciple}[1][2] : \sum_{m,n=1}^{\infty} m \neq n \ \&\ \Big\{i\in[1,\ldots,k] : m\in a_i\Big\} = \Big\{i\in[1,\ldots,k] : n\in a_i\Big\},$

$[4] := \texttt{E}A[1]\texttt{E}\inf : \{n\}\otimes\{n\} \le c,$

$\Big(j, [5]\Big) := \texttt{TensorDistribuivity}?\mathbb{N}, ?\mathbb{N}[4] :$

$: \sum j \in [1,\ldots,k] \ . \ \Big(a_j \cap \{n\}\Big) \otimes \Big(b_j \cap \{n\}\Big) = \Big(\{n\}\cap\{n\}\Big) \otimes (a_j\otimes b_j) \neq 0,$

$[6] := \texttt{ZeroTensor}[4] : n \in a_j \ \&\ n \in b_j,$

$[7] := [6][3] : m \in a_j,$

$[8] := \texttt{TensorDistirbutivity}(?\mathbb{N}, ?\mathbb{N})[6][7]\texttt{ZeroTensor}(?\mathbb{N}, ?\mathbb{N}) :$

$: (a_j\otimes b_j)\Big(\{m\}\otimes\{n\}\Big) = \Big(a_j\cap\{m\}\Big)\otimes\Big(b_j\cap\{n\}\Big) \neq 0,$

$[9] := \texttt{E}\sup[2][8] : c\Big(\{m\}\otimes\{n\}\Big) \neq 0,$

$[10] := \texttt{E}\inf[1]\texttt{E}A : c\Big(\{m\}\otimes\{n\}\Big) = 0,$

$[c.*] := \texttt{I}\bot[9][10] : \bot;$

$\rightsquigarrow [*] := \texttt{I}\sigma\text{-}\texttt{Algebra} : \neg\sigma\text{-}\texttt{Algebra}(?\mathbb{N}\otimes ?\mathbb{N});$

$\square$

1.5.7 General Limits

$\texttt{BooleanAlgebrasIsBicomplete} :: \texttt{Bicomplete}(\mathsf{BOOL})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ I \in \mathsf{SET},$

$\texttt{Assume}\ B : I \to \mathsf{BOOL},$

$\texttt{Assume}\ R :?I^2,$

$\texttt{Assume}\ f : \prod_{(i,j)\in R} \mathsf{BOOL}(B_i, B_j),$

$L := \left\{ x \in \prod_{i\in I} B_i : \forall (i,j) \in R \ .\ x_j = f_{i,j}(x_i) \right\} \in \mathsf{BOOL},$

$[0] := \mathtt{E}L : \forall (i,j) \in R \ .\ \pi_j = \pi_i f_{i,j},$

$\texttt{Assume}\ A \in \mathsf{BOOL},$

$\texttt{Assume}\ g : \prod_{i\in I} \mathsf{BOOL}(A, B_i),$

$\texttt{Assume}\ [1] : \forall (i,j) \in R \ .\ g_i f_{i,j} = g_j,$

$\Big(h, [A.*.1]\Big) := \texttt{ProductUniversalProperty}(\mathsf{BOOL}, I, B, A, g) : \sum ! A \xrightarrow{h} \prod_{i\in I} B_i : \mathsf{BOOL} \ .\ \forall i \in I \ .\ h\pi_i = g_i,$

$[2] := [A.*.1][1][A.*.1] : \forall (i,j) \in R \ .\ h\pi_j = g_j = g_i f_{i,j} = h\pi_i f_{i,j},$

$[A.*.2] := \mathtt{I}L[2] : \operatorname{Im} h \subset L;$

$\rightsquigarrow [I.*.2] := \mathtt{I}\lim[0] : (L, \pi) = \lim(\mathsf{BOOL}, B, f);$

$\rightsquigarrow [1] := \texttt{IComplete} : \texttt{Complete}(\mathsf{BOOL}),$

$\texttt{Assume}\ I \in \mathsf{SET},$

$\texttt{Assume}\ B : I \to \mathsf{BOOL},$

$\texttt{Assume}\ R :?I^2,$

$\texttt{Assume}\ f : \prod_{(i,j)\in R} \mathsf{BOOL}(B_i, B_j),$

$J := \left\langle \left\{ \iota_i(b) + f_{i,j}\iota_j(b) \middle| (i,j) \in R, b \in B_i \right\} \right\rangle_{\mathcal{I}} : \texttt{Ideal}\left(\bigotimes_{i\in I} B_i \right),$

$C := \dfrac{\bigotimes_{i\in I} B_i}{J} \in \mathsf{BOOL},$

$\iota' := \Lambda i \in I \ .\ \iota_i \pi_J : \prod_{i\in i} \mathsf{BOOL}(B_i, C),$

$[2] := \mathtt{E}C\mathtt{E}\iota' : \forall (i,j) \in R \ .\ f_{i,j}\iota'_j = \iota'_i,$

$\texttt{Assume}\ A \in \mathsf{BOOL},$

$\texttt{Assume}\ g : \prod_{i\in I} \mathsf{BOOL}(B_i, A),$

$\texttt{Assume}\ [3] : \forall (i,j) \in R \ .\ f_{i,j} g_j = g_i,$

$\Big(h, [4]\Big) := \texttt{CoproductUniversalProperty}(\mathsf{BOOL}, I, B, A, g) :$

$\quad : \sum ! \bigotimes_{i\in I} B_i \xrightarrow{h} A : \mathsf{BOOL} \ .\ \forall i \in I \ .\ \iota_i h = g_i,$

$\Big(\hat{h}, [5]\Big) := \texttt{QuotientMorphis}\left(\bigotimes_{i\in i} B_i, A, J, h \right) [3] : \sum ! C \xrightarrow{\hat{h}} A : \mathsf{BOOL} \ .\ \pi_J \hat{h} = h,$

$[A.*.1] := \Lambda i \in I \ .\ \mathtt{E}\iota'_i[5][4.i] : \forall i \in I \ .\ \iota'_i h' = \iota_i \pi_J h' = \iota_i h = g_i;$

$\rightsquigarrow [I.*] := \mathtt{I}\,\mathrm{colim}[2] : (C, \iota') = \mathrm{colim}(\mathsf{BOOL}, B, f);$

$\rightsquigarrow [2] := \mathtt{IComomplete} : \mathtt{Cocomple}(\mathsf{BOOL}),$

$[*] := \mathtt{IBicomplete}[1][2] : \mathtt{Bicomplete}(\mathsf{BOOL});$

$\square$

1.6 Further Properties

1.6.1 Countable Chain Condition

$\texttt{WithCountableChainCondition} :: ?\mathsf{BOOL}$
$A : \texttt{WithCountableChainCondition} \iff \forall D : \texttt{PairwiseDisjointElements}(A) \ . \ |D| \le \aleph_0$

$\texttt{SouslinProperty} :: ?\mathsf{TOP}$
$X : \texttt{SouslinProperty} \iff \forall D : \texttt{Disjoint}\ \mathcal{T}\ X \ . \ |D| \le \aleph_0$

$\texttt{StoneCCCTHM} :: \forall A \in \mathsf{BOOL} \ . \ \texttt{WithCountableChainCondition}(A) \iff \texttt{SouslinProperty}(\mathsf{Z}\ A)$
$\texttt{Proof} \ =$
$\dots$
$\square$

$$\texttt{CountablySaturatedIdeal} :: \prod_{A:\mathsf{BOOL}} ?\texttt{Ideal}(A)$$
$I : \texttt{CountablySaturatedIdeal} \iff \omega_1\texttt{-SaturatedIdeal} \iff$
$\forall D : \texttt{PairwiseDisjointElements}(A) \ . \ D \subset A \setminus I \Rightarrow |D| \le \aleph_0$

$\texttt{CCCQuotient} :: \forall A : \sigma\texttt{-Algebra} \ . \ \forall I : \sigma\texttt{-Ideal}(A) \ .$
$$. \ \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right) \iff \omega_1\texttt{-SaturatedIdeal}(A, I)$$
$\texttt{Proof} \ =$
$\texttt{Assume}\ [1] : \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right),$
$\texttt{Assume}\ D : \texttt{PairwiseDisjointElements}(A),$
$\texttt{Assume}\ [2] : D \subset A \setminus I,$
$\texttt{Assume}\ a, b \in D,$
$\texttt{Assume}\ [3] : a \ne b,$
$\texttt{Assume}\ [4] : \pi_I(a) = \pi_I(b),$
$\Big(i, [5]\Big) := \mathtt{E}\pi_I[4] :\sim i \in I \ . \ b = a + i,$
$[6] := \texttt{IdealContainsZeroE} \setminus [2](a \,\&\, b) : a \ne 0 \,\&\, b \ne 0,$
$[7] := [5]\mathtt{E}\mathsf{BOOL}(A, a)\texttt{EIdeal}(I, A)[6][2] : ab = a(a + i) = a^2 + ia = a + ia \ne 0,$
$[8] := \texttt{EPairwiseDisjointElements}(A, D, a, b)[3] : ab = 0,$
$\Big[(a, b).*\Big) := \mathtt{I}\bot[7][8] : \bot;$
$\leadsto [3] := \mathtt{E}\bot\mathtt{I} \Rightarrow \mathtt{I}\forall : \forall a, b \in D \ . \ a \ne b \Rightarrow \pi_I(a) \ne \pi_I(b),$
$[4] := \texttt{EWithCountableChainCondition}[1]\Big(\pi_I(D)\Big) : \Big|\pi_I(D)\Big| \le \aleph_0,$
$[5] := \texttt{IInjective}[3] : \texttt{Injective}\left(D, \frac{A}{I}, \pi_{I|D}\right),$
$[D.*] := \texttt{InjectivePreservesCard}[4][5] : |D| \le \aleph_0;$
$\leadsto [1.*] := \mathtt{I}\omega_1\texttt{-SaturatedIdeal} : \omega_1\texttt{-SaturatedIdeal}(A, I);$
$[1] := \mathtt{I} \Rightarrow : \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right) \Rightarrow \omega_1\texttt{-SaturatedIdeal}(A, I),$

$\texttt{Assume}\ [2] : \omega_1\text{-}\texttt{SaturatedIdeal}(A, I),$

$\texttt{Assume}\ [3] : \neg\texttt{WithCountableChainCondition}\left(\frac{A}{I}\right),$

$\Big(D, [4]\Big) := \texttt{EWithCountableChainCondition}[3] : \sum D : \texttt{PairwiseDisjointElements}\left(\frac{A}{I}\right) . |D| > \aleph_0,$

$(\kappa, d) := \texttt{WellOrderingEnumeration}(D) : \sum \kappa \in \mathsf{ORD} . \texttt{Bijection}(\kappa, D),$

$[5] := \texttt{BijectionPreservesCardinality}(\kappa, D, d)[4]\texttt{OrdinalityCardianlityBound} : \omega_1 \leq \kappa,$

$\texttt{Assume}\ \xi : \omega_1,$

$\Big(a, [5]\Big) := \Lambda\xi \in \omega_1 . \texttt{ESurjective}\left(A, \frac{A}{I}, \pi_I, d_\xi\right) : \sum_{a:\omega_1 \to A} \forall \xi \in \omega_1 . d_\xi = \pi_I(a_\xi),$

$[6] := \texttt{E}a\texttt{EPairwiseDisjointElements}(D) : \forall \xi \in \omega_1 . a_\xi \notin I,$

$\texttt{Assume}\ \xi \in \omega_1,$

$\texttt{Assume}\ \eta \in \xi,$

$[7] := \texttt{E}\mathsf{ORD}(\xi, \eta) : \eta \neq \xi,$

$[8] := \texttt{EPairwiseDisjointElements}(D, d_\xi, d_\eta) : d_\xi d_\eta = 0,$

$[\eta.*] := [5][8] : a_\xi a_\eta \in I;$

$\rightsquigarrow [7] := \texttt{I}\forall : \forall \eta \in \xi . a_\xi a_\eta \in I,$

$v_\xi := \sup_{\eta \in \xi} a_\xi a_e ta \in I,$

$c_\xi := a_\xi \setminus v_\xi : A \setminus I;$

$\rightsquigarrow (v, c) := \texttt{I} \to : \omega_1 \to I \times (A \setminus I),$

$\texttt{Assume}\ \xi \in \omega_1,$

$\texttt{Assume}\ \eta \in \xi,$

$[\xi/*] := \texttt{E}c_\eta \texttt{I}v_\xi : c_\xi c_\eta \leq c_\xi a_\eta \leq c_\xi v_\xi = 0;$

$\rightsquigarrow [8] := \texttt{IPairwiseDisjointElements} : \texttt{PairwiseDisjointElements}(A, \operatorname{Im} c),$

$[3.*] := \texttt{E}\omega_1\text{-}\texttt{SaturatedIdeal}(A, I)[8] : \perp;$

$\rightsquigarrow [2.*] := \texttt{E}\perp : \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right);$

$\rightsquigarrow [*] := \texttt{I} \Rightarrow \texttt{I} \iff : \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right) \iff \omega_1\text{-}\texttt{SaturatedIdeal}(A, I);$

$\square$

$\texttt{CCCSupInfAnalogy} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall X \subset A \,.\, \exists Y : \texttt{Countable}(X) \,.$
$\quad .\ \mathrm{ub}\, Y = \mathrm{ub}\, X \;\&\; \mathrm{lb}\, Y = \mathrm{lb}\, X$

$\texttt{Proof} =$

$C := \bigcup_{x \in X} \{a \in A : a \leq x\} :?A,$

$\mathcal{D} := \{D : \texttt{PairwiseDisjointElements}(X) : D \subset C\} :?\texttt{PairwiseDisjointElements}(X),$

$\big(D, [1]\big) := \texttt{ZornLemma}(\mathcal{D}, \subset) : \sum D \in \mathcal{D} \,.\, D = \max \mathcal{D},$

$[2] := \texttt{EWithCountableChainCondition}(A, D) : |D| \leq \aleph_0,$

$\big(x, [3]\big) := \Lambda d \in D \,.\, \texttt{E}C(d) : \sum_{x : D \to X} \forall d \in D \,.\, d \leq x_d,$

$Y_0 := \mathrm{Im}\, x :?X,$

$[4] := \texttt{E}Y_0\texttt{SurjectiveCardinalityBound}[2] : |Y_0| \leq \aleph_0,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [5] : Y_0 \leq a,$

$\texttt{Assume}\ [6] : X \not\leq a,$

$\big(x', [7]\big) := \texttt{E}(\not\leq)[6] : \sum x' \in X \,.\, x' \not\leq a,$

$a' := x' \setminus a : A,$

$[8] := \texttt{E}a'\texttt{EBooleanOrder}(A)[7]\texttt{E}(\setminus) : a' \neq 0,$

$[9] := \texttt{E}a'\texttt{SetminusOrder}(A) : a' \leq x',$

$[10] := \texttt{E}C[9] : a' \in C,$

$\texttt{Assume}\ d : D,$

$[11] := [5][10]\texttt{E}(\setminus) : 0 = x_d a',$

$[d.*] := \texttt{E}x[3][11] : da' = 0;$

$\rightsquigarrow [12] := \texttt{I}\forall : \forall d \in D \,.\, da' = 0,$

$D' := D \cup \{a'\} :?C,$

$[13] := \texttt{E}D'[2]\texttt{IPairwiseDisjointElements}(A)\texttt{I}D' : \texttt{PairwiseDisjointElements}(A, D'),$

$[a.*] := \texttt{E}D'[1][13]\texttt{I}\bot : \bot;$

$\rightsquigarrow [5] := \texttt{E}\bot\texttt{I} \Rightarrow \texttt{I}\forall : \forall a \in A \,.\, Y_0 \leq a \Rightarrow X \leq a,$

$[6] := \texttt{IupperBoundsISetEq}[5] : \mathrm{ub}\, X = \mathrm{ub}\, Y_0,$

$\big(Y_1, [7]\big) := \texttt{ByAnalogyAndDuality}(Y_0) : \sum Y_1 \subset X \,.\, \mathrm{lb}\, X = \mathrm{lb}\, Y_0,$

$Y := Y_1 \cup Y_0 :?X,$

$[*] := \texttt{E}Y_1[6][7] : \mathrm{ub}\, X = \mathrm{ub}\, Y \;\&\; \mathrm{lb}\, X = \mathrm{lb}\, Y;$

$\square$

$\texttt{CCCAlgebraUpgrade} :: \forall A : \sigma\texttt{-Algebra} \;\&\; \texttt{WithCountableChainCondition} \,.\, \tau\texttt{-Algebra}(A)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CCCSOCIsOC} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall X : \texttt{SequentiallyOrderClosed}(A) \,.$
$\quad .\ \texttt{OrderClosed}(A, X)$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CCCMonotonicContinuityUpgrade} ::$

$. \forall A : \texttt{WithCountableChainCondition} \;.\; \forall P \in \mathsf{POSET} \;.$

$. \forall A \xrightarrow{f} P : \mathsf{POSET} \;.\; \sigma\text{-}\texttt{Continuous}(A, P, f) \Rightarrow \texttt{OrderContinuous}(A, P, f)$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{CCCIffOrdinalCondition} :: \forall A : \sigma\text{-}\texttt{Algebra} \;.\; \texttt{WithCountableChainCondition}(A) \iff$

$\iff \{a : \omega_1 \to A : \forall \eta, \xi \in \omega_1 \;.\; \eta < \xi \Rightarrow a_\eta < a_\xi\} = \emptyset$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{WithCountableChainCondition}(A),$

$\texttt{Assume}\ a : \omega_1 \to A,$

$\texttt{Assume}\ [2] : \forall \eta, \xi \in \omega_1 \;.\; \eta < \xi \Rightarrow a_\eta < a_\xi,$

$b := \Lambda \eta \in \omega_1 \;.\; a_{\sigma(\eta)} \setminus a_\eta : b : \omega_1 \to A,$

$B := \operatorname{Im} b :?A,$

$[3] := \mathtt{E}B[2]\mathtt{I}\texttt{PairwiseDisjointElements}\mathtt{I}B : \texttt{PairwiseDisjointElements}(A, B),$

$[4] := \mathtt{E}B[2]\texttt{ordinalCardinality}(\omega_1)\mathtt{I}B : |B| > \aleph_0,$

$[a.*] := \mathtt{E}\texttt{WithCountableChainCondition}(A)[3][4] : \bot;$

$\leadsto [1.*] := \mathtt{E}\bot\mathtt{I}\emptyset : \{a : \omega_1 \to A : \forall \eta, \xi \in \omega_1 \;.\; \eta < \xi \Rightarrow a_\eta < a_\xi\} = \emptyset;$

$\leadsto [1] := \mathtt{I} \Rightarrow : \texttt{WithCountableChainCondition}(A) \Rightarrow \{a : \omega_1 \to A : \forall \eta, \xi \in \omega_1 \;.\; \eta < \xi \Rightarrow a_\eta < a_\xi\} = \emptyset,$

$\texttt{Assume}\ [2] : \{a : \omega_1 \to A : \forall \eta, \xi \in \omega_1 \;.\; \eta < \xi \Rightarrow a_\eta < a_\xi\} = \emptyset,$

$\texttt{Assume}\ D : \texttt{PairwiseDisjointElements}(A),$

$\texttt{Assume}\ [3] : |D| > \aleph_0,$

$\Big(D', [4]\Big) := \texttt{FirstUncountableOrdinal}[3] : \sum D' \subset D \;.\; |D'| = |\omega_1| \;\&\; \forall d \in D' \;.\; d \neq 0,$

$d := \mathtt{E}\texttt{EqCardinality}[4] : \texttt{Bijective}(\omega_1, D'', d),$

$\Big(P, [01]\Big) := \texttt{DijointElementsHavePartitionOfUnity}(D') : \sum P : \texttt{PartitionOfUnity} \;.\; D' \subset P,$

$a_0 := d_0 \in A,$

$[i.0] := \texttt{EmptyTruth} : \forall \eta \in 0 \;.\; a_\eta < a_0,$

$[j.0] := \mathtt{E}p\mathtt{E}\texttt{PairwiseDisjointElements}(A, P) : \forall \eta \in \kappa \;.\; \eta > 0 \Rightarrow a_0 p_\eta = 0,$

$\Big(\kappa, p, [02]\Big) := \texttt{ExtendEnumeration}(D', d, P)[01] : \sum \kappa \in \mathsf{ORD} \;.\; \sum P : \texttt{Bijective}(\kappa, P)\omega_1 \leq \kappa \;.\; p_{|\omega_1} = d,$

$\texttt{Assume}\ \xi \in \omega_1,$

$\texttt{Assume}\ [5] : \neg\texttt{Limit}(\xi),$

$\Big(\eta, [6]\Big) := \mathtt{E}\texttt{Limit}[5] : \sum \eta \in \omega_1 \;.\; \xi = \sigma(\eta),$

$a_\xi := d_\xi \vee a_\eta \in A,$

$\texttt{Assume}\ \zeta \in \xi,$

$[7] := \mathtt{E}\mathsf{ORD}(\xi, \zeta)[6] : \zeta \leq \eta < \xi,$

$[\zeta.*] := \mathtt{E}a_\xi\mathtt{E}\texttt{PairwiseDisjointElements}(A, D)\mathtt{E}a_\eta[j.\eta][i.\eta](\zeta) : a_\xi = d_\xi \vee a_\eta > a_\eta \geq a_\zeta;$

$\leadsto [i.\xi] := \mathtt{I}\forall : \forall \zeta \in \xi \;.\; a_\zeta < a_\xi,$

$\texttt{Assume}\ \zeta \in \kappa,$

$\texttt{Assume}\ [7] : \xi < \zeta,$

$[\zeta.*] := \mathtt{E}a_\xi\mathtt{E}\texttt{PartitionOfUnity}(A, P)[7][j.\eta] : p_\zeta a_\xi = p_\zeta(d_\xi \vee a_\eta) = p_\zeta d_\xi \vee p_\zeta a_\eta = 0 \vee 0 = 0;$

$\leadsto [j.\xi] := \mathtt{I}\forall : \forall \zeta \in \kappa \;.\; \xi < \zeta \Rightarrow p_\zeta a_\xi = 0;$

$\leadsto [5] := \mathtt{I}\exists\mathtt{I} \Rightarrow : \neg\texttt{Limit}(\xi) \Rightarrow \exists a_\xi \in A : \Big(\forall \zeta \in \xi \;.\; a_\zeta < a_\xi\Big) \;\&\; \Big(\forall \zeta \in \kappa \;.\; \xi < \zeta \Rightarrow a_\xi p_\zeta = 0\Big),$

$\texttt{Assume } [6] : \texttt{Limit}(\xi),$

$a_\xi := \sup_{\eta \in \xi} d_\eta \in A,$

$\texttt{Assume } \eta \in \xi,$

$\texttt{Assume } [7] : a_\eta \not\leq a_\xi,$

$b := a_\eta \setminus a_\xi \in A,$

$[8] := \texttt{E}b[7]\texttt{I}b : b \neq 0,$

$\Big(\zeta, [9]\Big) :=: \sum \zeta \in \kappa \,.\, \zeta > \xi \Rightarrow d_\zeta a_\eta \neq 0,$

$[10] := [9.1]\texttt{E}(\eta) : \zeta > \eta,$

$[11] := [j.\eta](\zeta)[10] : d_\zeta a_\eta = 0,$

$[7.*] := [10][11] : \bot;$

$\leadsto [7] := \texttt{E}\bot : a_\eta \leq a_\xi,$

$\Big(\zeta, [8]\Big) := \texttt{ELimit}(\xi, \eta) : \sum \zeta \in \xi \,.\, \eta < \zeta,$

$[9] := [j.\eta](\zeta)[8] : d_\zeta a_\eta = 0,$

$[*.\xi] := \texttt{E}a_\xi\texttt{E}\sup[7][9] : a_\eta < a_\xi;$

$\leadsto [i.\xi] := \texttt{E}\bot\texttt{I}\forall : \forall \eta \in \xi \,.\, a_\eta < a_\xi;$

$\texttt{Assume } \eta \in \kappa,$

$\texttt{Assume } [7] \in \eta > \xi,$

$[8] := \texttt{E}D'\texttt{E}P : \forall \zeta \in \xi \,.\, d_\zeta p_\eta = 0,$

$\eta.*] := \texttt{E}a_\xi\texttt{BooleanAlgebraIsALattice} : a_\xi p_\eta = 0;$

$\leadsto [j.\xi] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall \eta \in \kappa \,.\, \eta > \xi \Rightarrow a_\xi p_\eta = 0;$

$\leadsto [6] := \texttt{I}\exists\texttt{I} \Rightarrow: \neg\texttt{Limit}(\xi) \Rightarrow \exists a_\xi \in A : \Big(\forall \zeta \in \xi \,.\, a_\zeta < a_\xi\Big) \,\&\, \Big(\forall \zeta \in \kappa \,.\, \xi < \zeta \Rightarrow a_\xi p_\zeta = 0\Big);$

$(\xi, i, j) := \texttt{E}(|)\texttt{LEM}\Big(\texttt{Limit}(\xi)\Big)[5][6] : \sum a_\xi \in A : \Big(\forall \zeta \in \xi \,.\, a_\zeta < a_\xi\Big) \,\&\, \Big(\forall \zeta \in \kappa \,.\, \xi < \zeta \Rightarrow a_\xi p_\zeta = 0\Big);$

$\leadsto \Big(a, [5]\Big) := \texttt{TransfiniteInduction}[i.1][j.1] :$

$: \prod_{\xi \in \omega_1} \sum a_\xi \in A : \Big(\forall \zeta \in \xi \,.\, a_\zeta < a_\xi\Big) \,\&\, \Big(\forall \zeta \in \kappa \,.\, \xi < \zeta \Rightarrow a_\xi p_\zeta = 0\Big),$

$[D.*] := [5.1][2] : \bot;$

$\leadsto [2.*] := \texttt{IWithCountableChainCondition} : \texttt{WithCountableChainCondition}(A);$

$\leadsto [*] := [1]\texttt{I} \iff : \texttt{WithCountableChainCondition}(A) \iff$

$: \iff \{a : \omega_1 \to A : \forall \eta, \xi \in \omega_1 \,.\, \eta < \xi \Rightarrow a_\eta < a_\xi\} = \emptyset;$

□

$\texttt{CCCIdealUpgrade} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall I : \sigma\texttt{-Ideal}(A) \,.\, \tau\texttt{-Ideal}(A, I)$

$\texttt{Proof } =$

...

□

$\texttt{CCCQuuotineTau} :: \forall A \in \mathsf{BOOL}\forall I : \tau\texttt{-Ideal} \,\&\, \omega_1\texttt{-SaturatedIdeal}(a) \,.\, \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right)$

$\texttt{Proof } =$

...

□

$\texttt{CCCQuotient2} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall I : \sigma\text{-}\texttt{Ideal}(A) \,.$

$\quad .\, \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right)$

$\texttt{Proof} \;=$

$[1] := \texttt{EWithCountableChainCondition}(A)\texttt{I}\omega_1\text{-}\texttt{SaturatedIdeal} : \omega_1\text{-}\texttt{SaturatedIdeal}(A, I),$

$[2] := \texttt{CCCIdealUpgrade}(A, I) : \tau\text{-}\texttt{Ideal},$

$[*] := \texttt{CCCQuotinentTau}(A, I)[1][2] : \texttt{WithCountableChainCondition}\left(\frac{A}{I}\right);$

$\square$

$\texttt{CCCSubalgebra} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \texttt{WithCountableChainCondition}(B)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{CCCByIdealUpgrade} :: \forall A : \mathsf{BOOL} \,.\, \Big(\forall I : \sigma\text{-}\texttt{Ideal}(A) \,.\, \tau\text{-}\texttt{Ideal}(A)\Big) \Rightarrow \texttt{WithCountableChainCondition}(A)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ D : \texttt{PairwiseDisjointElements}(A),$

$\texttt{Assume}\ [1] : |D| > \aleph_0,$

$\Big(P, [2]\Big) := \texttt{DisjointElementsHavePartitionOfUnity} : \sum P : \texttt{PartitionOfUnity}(A) \,.\, D \subset P,$

$[3] := \texttt{SupersetStrictCardinality}[1][2] : |P| > \aleph_0,$

$I := \langle P \rangle_{\mathcal{I},\sigma} : \sigma\text{-}\texttt{Ideal}(A),$

$[4] := \texttt{E}\mathcal{I}\texttt{EPartitionOfUnity}(A, P)[3] : e \notin I,$

$[5] := [0](I) : \tau\text{-}\texttt{Ideal}(A, I),$

$[6] := \texttt{E}\mathcal{I}\texttt{EPartitionOfUnity}(A, P)\texttt{E}\tau\text{-}\texttt{Ideal}(A, I)[5] : e \in I,$

$[D.*] := [4][6] : \bot;$

$\leadsto * := \texttt{E}\bot\texttt{IWithCountableChainCondition} : \texttt{WithCountableChainCondition}(A);$

$\square$

$\texttt{CCCBySubalgebraUpgrade} :: \forall A : \mathsf{BOOL} \,.\, \left(\forall B \subset^{\sigma}_{\mathsf{BOOL}} A \,.\, B \subset^{\tau}_{\mathsf{BOOL}} A\right)$

$\Rightarrow \texttt{WithCountableChainCondition}(A)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ D : \texttt{PairwiseDisjointElements}(A),$

$\texttt{Assume}\ [1] : |D| > \aleph_0,$

$\left(P, [2]\right) := \texttt{DisjointElementsHavePartitionOfUnity} : \sum P : \texttt{PartitionOfUnity}(A) \,.\, D \subset P,$

$[3] := \texttt{SupersetStrictCardinality}[1][2] : |P| > \aleph_0,$

$\left(p, [4]\right) := \texttt{ENumerous}(P, 0)[3] : \sum p \in P \,.\, p \neq 0,$

$P' := P \setminus \{p\} : \texttt{PairwiseDisjointElements}(A),$

$p' := p^{\complement} \in A,$

$[5] := \texttt{InfiniteSubsetFiniteDifference}\big(P, \{p\}\big)[4] : |P'| > \aleph_0,$

$\texttt{Assume}\ a \in \langle p' \rangle_{\mathcal{I}},$

$[6] := \texttt{E}p'\texttt{E}\complement\texttt{PrincipleIdealExpression}(A, p', a)\texttt{EBooleanOrder}(A) : ap = 0,$

$\left(q, [7]\right) := \texttt{EPartitionOfUnity}(A, P)(a) : \sum q \in P \,.\, qp \neq 0,$

$[8] := \texttt{E}(\#, \rightarrow)[6][7] : p \neq q,$

$[a.*] := \texttt{E}P'[8] : q \in P';$

$\rightsquigarrow [6] := \texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}\left(\langle p' \rangle, P'\right),$

$[7] := \texttt{PartitionOfUnitySupremum}[6] : \sup P' = p',$

$B := \sigma(P') : \sigma\texttt{-Subalgebra}(A),$

$[8] := \texttt{E}B\texttt{EPartitionOfUnity}(A, P)[3] : p' \notin B,$

$[9] := [0](B) : \tau\texttt{-Subalgebra}(A, B),$

$[10] := \texttt{E}B\texttt{E}[7]\texttt{E}\tau\texttt{-Subalgebra}(A, B)[9] : p' \in B,$

$[D.*] := [8][10] : \bot;$

$\rightsquigarrow * := \texttt{E}\bot\texttt{IWithCountableChainCondition} : \texttt{WithCountableChainCondition}(A);$

$\square$

$\texttt{CCCByMorphismUpgrade} :: \forall A : \mathsf{BOOL} \,.$

$.\left(\forall B \in \mathsf{BOOL} \forall f : \sigma\texttt{-Continuous}(A, B) \,.\, \texttt{OrderContinuous}(A, B, f)\right) \Rightarrow$

$\Rightarrow \texttt{WithCountableChainCondition}(A)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{CCCByBeingSurjectiveImage} :: \forall A : \texttt{WithCountableChainCondition} \,.\, \forall B \in \mathsf{BOOL} \,.$

$.\, \forall f : \mathsf{BOOL} \,\&\, \texttt{Surjective} \,\&\, \texttt{OrderContinuous}(A, B) \,.\, \texttt{WithCountableChainCondition}(B)$

$\texttt{Proof} \ =$

$[1] := \texttt{IsomorphismTHM}(A, B, f) : B \cong_{\mathsf{BOOL}} \dfrac{A}{\ker f},$

$[2] := \texttt{TauIdealTHM}(A, B, f) : \tau\texttt{-Ideal}(A, B, f),$

$[3] := \texttt{EWithCountableChainCondition}(A)\texttt{I}\omega_1\texttt{-SaturatedIdeal} : \omega_1\texttt{-SaturatedIdeal}(A, \ker f),$

$[4] := \texttt{CCCQuotientTau}[2][3] : \texttt{WithCountableChainCondition}\left(\dfrac{A}{\ker f}\right),$

$[*] := [1][4] : \texttt{WithCountableChainCondition}(B);$

$\square$

$\texttt{CCCByDenseSubalgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.$

$. \texttt{WithCountableChainCondition}(B) \;\&\; \texttt{OrderDense}(A, B) \Rightarrow \texttt{WithCountableChainCondition}(A)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; D : \texttt{PairwiseDisjointElements}(A),$

$D' := D \setminus \{0\} : \texttt{PairwiseDisjointElements}(A),$

$\Big(b, [1]\Big) := \texttt{EOrderDense}(A, B)(D') : \sum b : D' \to B \,.\, \forall d \in D' \,.\, 0 < b_d \le d,$

$[2] := \texttt{EPairwiseDisjointElements}(A)[1] : \texttt{Injective}(D', B, b) \;\&\; \texttt{PairwiseDisjointElements}(B, \operatorname{Im} b),$

$[3] := \texttt{EWithCountableChainCondition}(B)[2.2] : |\operatorname{Im} b| \le \aleph_0,$

$[4] := \texttt{InjectionReflectsCardinalityBounds} : |D'| \le \aleph_0,$

$[D.*] := \texttt{E}D'\texttt{E}\aleph_0[4] : |D| \le \aleph_0;$

$\rightsquigarrow [*] := \texttt{IWithCountableChainCondition} : \texttt{WithCountableChainCondition}(A);$

$\square$

$\texttt{CCCByCoproductStruct} :: \forall A, B \in \mathsf{BOOL} \,.\, A \not\cong_{\mathsf{BOOL}} \star \not\cong_{\mathsf{BOOL}} B \;\&\; \texttt{WithCountableChainCondition}(A \otimes B) \Rightarrow$

$\texttt{Proof} \;=$

1.6.2 Weakly Distributive Algebras

$(\sigma,\infty)\text{-}\mathtt{WeaklyDistributive} :: ?\mathsf{BOOL}$

$A : (\sigma,\infty)\text{-}\mathtt{WeaklyDistributive} \iff \forall X : \mathbb{N} \to \mathtt{DownwardsDirected}(A)\ .$

$.\ \Big(\forall n \in \mathbb{N}\ .\ \inf X_n = 0\Big) \Rightarrow \inf\Big\{b \in A : \forall n \in \mathbb{N}\ .\ \exists a \in X_n : a \le b\Big\} = 0$

$\mathtt{WDPoUProperty} :: \forall A : (\sigma,\infty)\text{-}\mathtt{WeaklyDistributive}\ .\ \forall P : \mathbb{N} \to \mathtt{PartitionOfUnity}(X)\ .$

$.\ \exists Q : \mathtt{PartitionOfUnity}(X) : \forall n \in \mathbb{N}\ .\ \forall q \in Q\ .\ \Big|\{p \in P_n : pq \ne 0\}\Big| < \infty$

$\mathtt{Proof}\ =$

$C := \Lambda n \in \mathbb{N}\ .\ \Big\{(\sup F)^{\complement}\Big| F : \mathtt{Finite}(P_n)\Big\} : \mathbb{N} \to ?A,$

$[1] := \mathtt{E}C\Lambda n \in \mathbb{N}\ .\ \mathtt{EPartitionOfUnity}(A, P_n) : \forall n \in \mathbb{N}\ .\ \inf C_n = 0,$

$B := \Big\{b \in A : \forall n \in \mathbb{N}\ .\ \exists a \in C_n : a \le b\Big\} :?A,$

$[2] := \mathtt{E}(\sigma,\infty)\text{-}\mathtt{WeaklyDistributive}(A)\mathtt{E}B[1] : \inf B = 0,$

$Q' := \Big\{a \in A : \exists b \in B : ab = 0\Big\} :?A,$

$[3] := \mathtt{E}Q'[2]\mathtt{IOrderDense} : \mathtt{OrderDense}\Big(A, Q'\Big),$

$\Big(Q, [4]\Big) := \mathtt{OrderDenseContainsPoU}(Q) : \sum Q : \mathtt{PartitionOfUnity}(A)\ .\ Q \subset Q',$

$\mathtt{Assume}\ n \in \mathbb{N},$

$\mathtt{Assume}\ q \in Q,$

$\Big(b, [5]\Big) := \mathtt{E}Q(q)[4] : \sum b \in B\ .\ bq = 0,$

$\Big(c, [6]\Big) := \mathtt{E}B(n, b) : \sum c \in C_n\ .\ c \le b,$

$\Big(F, [7]\Big) := \mathtt{E}C_n(c) : \sum F : \mathtt{Finite}(P_n)\ .\ c = (\sup F)^{\mathcal{C}},$

$[8] := [5]\mathtt{I}\complement[6][7] : q \le b^{\complement} \le c^{\complement} \le \sup F,$

$[n.*] := \mathtt{E}\sup[8]\mathtt{EPartitionOfUnity}(A, P_n) : \Big|\{p \in P_n : pq \ne 0\}\Big| < \infty;$

$\rightsquigarrow [*] := \mathtt{I}\forall\mathtt{I}\forall : \forall n \in \mathbb{N}\ .\ \forall q \in Q\ .\ \Big|\{p \in P_n : pq \ne 0\}\Big| < \infty;$

$\square$

$\texttt{WDSupProperty1} :: \forall A : (\sigma,\infty)\texttt{-WeaklyDistributive} \,.\, \forall X : \mathbb{N} \to ?\texttt{UpwardDirected}(A) \,.$

$\quad .\, \forall x : \mathbb{N} \to A \,.\, (\forall n \in \mathbb{N} \,.\, x_n = \sup X_n) \Rightarrow \inf \left\{ x_n \setminus b \middle| n \in \mathbb{N}, b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \right\} = 0$

$\texttt{Proof} \;=$

$B := \{b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a\} :?A,$

$D := \Lambda n \in \mathbb{N} \,.\, \langle x_n^{\complement} \rangle \cup \bigcup_{a \in X_n} \langle a \rangle : \mathbb{N} \to ?A,$

$[1] := \texttt{E}D[0]\texttt{I}D\texttt{IOrderDense} : \forall n \in \mathbb{N} \,.\, \texttt{OrderDense}(A, P_n),$

$\Big(P, [2]\Big) := \Lambda n \in \mathbb{N} \,.\, \texttt{OrderDenseConatinsPoU}(A, D_n) :$

$\quad : \sum P : \mathbb{N} \to \texttt{PartitionOfUnity}(A) \,.\, \forall n \in \mathbb{N} \,.\, P_n \subset D_n,$

$\Big(Q, [3]\Big) := \texttt{WDPoUProperty}(A, P) :$

$\quad : \sum Q : \texttt{PartitionOfUnity}(A) \,.\, \forall n \in \mathbb{N} \,.\, \forall q \in Q \,.\, \Big|\{p \in P_n : pq \neq 0\}\Big| < \infty,$

$C := \{x_n \setminus b | n \in \mathbb{N}, b \in B\} :?A,$

$\texttt{Assume}\ c \in A,$

$\texttt{Assume}\ [4] : c \leq C,$

$\texttt{Assume}\ [5] : c \neq 0,$

$\Big(q, [6]\Big) := \texttt{EPartitionOfUnity}(A, Q)[5] : \sum q \in Q \,.\, qc \neq 0,$

$P' := \Lambda n \in \mathbb{N} \,.\, \{p \in P_n : pqc \neq 0\} : \mathbb{N} \to ?A,$

$[7] := \texttt{E}P'[3] : \forall n \in \mathbb{N} \,.\, |P'| < \infty,$

$p := \Lambda n \in \mathbb{N} \,.\, \sup P'_n : \mathbb{N} \to A,$

$[8] := \texttt{E}p\texttt{EPartitionOfUnity}(P) : \forall n \in \mathbb{N} \,.\, qc \leq p_n,$

$[9] := [0]\texttt{E}P' : \forall n \in \mathbb{N} \,.\, \forall t \in P' \,.\, \exists a \in X_n : t \leq a,$

$[10] := \texttt{EUpwardsDirected}(X)[9]\texttt{E}p : \texttt{UpwardsDirectred}(p),$

$[11] := \texttt{E}B[8] : qc \in B,$

$[12] := \texttt{E}B\texttt{E}C(c) : \forall b \in B \,.\, bc = 0,$

$[c.*] := [11][12][6] : \perp;$

$\leadsto [*] := \texttt{I}\inf : \inf C = 0,$

$\square$

$\texttt{WDSupProperty2} :: \forall A : (\sigma,\infty)\texttt{-WeaklyDistributive} \,.\, \forall X : \mathbb{N} \to ?\texttt{UpwardDirected}(A) \,.$

$\quad .\, \forall x : \mathbb{N} \to A \,.\, \forall y \in A \,.\, (\forall n \in \mathbb{N} \,.\, x_n = \sup X_n \;\&\; y = \inf_{n=1} x_n) \Rightarrow$

$\quad \Rightarrow \sup \left\{ b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \right\} = y$

$\texttt{Proof} \;=$

$[1] := \texttt{WDSupProperty1}(A, X) : \inf \left\{ x_n \setminus b \middle| n \in \mathbb{N}, b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \right\} = 0,$

$[2] := \texttt{E}\inf[1][0]\texttt{I}\inf : \inf \left\{ y \setminus b \middle| n \in \mathbb{N}, b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \right\} = 0,$

$[*] := \texttt{InfComplementation}[1][2] : \sup \left\{ b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \right\} = y;$

$\square$

$\texttt{WDBySupProperty} :: \forall A \in \mathsf{BOOL} \,.\, \Big(\forall X : \mathbb{N} \to ?\texttt{UpwardDirected}(A) \,.\, \forall x : \mathbb{N} \to A \,.\, \forall y \in A \,.$

$.\, (\forall n \in \mathbb{N} \,.\, x_n = \sup X_n \;\&\; y = \inf_{n=1} x_n) \Rightarrow \sup \Big\{ b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in X_m : b \leq a \Big\} = y \Big) \Rightarrow$

$\Rightarrow (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ X : \mathbb{N} \to \texttt{DownwardsDirected}(A),$

$\texttt{Assume}\ [1] : \forall n \in \mathbb{N} \,.\, \inf X_n = 0,$

$B := \{ b \in A : \forall n \in \mathbb{N} \,.\, \exists a \in X_n : a \leq b \} :?A,$

$Y := \{ x^{\complement} : x \in X \} : \mathbb{N} \to \texttt{UpwardDirected}(A),$

$[2] := \texttt{E}Y\texttt{ComplementInf}[1]\texttt{I}Y : \forall n \in \mathbb{N} \,.\, \sup Y_n = e,$

$[3] := \texttt{ConstantInf}(A, e) : \inf e = e,$

$[4] := [0](Y, e, e)[2][3]\texttt{I}B : e = \sup \Big\{ b \in A : \forall m \in \mathbb{N} \,.\, \exists a \in Y_m : b \leq a \Big\} = \sup\{ b^{\complement} | b \in B \},$

$[X.*] := \texttt{ComplementSup}[4] : \inf B = 0;$

$\rightsquigarrow [*] := \texttt{I}(\sigma, \infty)\text{-}\texttt{WeaklyDistributive} : (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A);$

$\square$

$\texttt{NowhereDenseStoneSpaceCondition} :: \forall A \in \mathsf{BOOL} \,.\, \forall X \subset \mathsf{Z}\, A \,.$

$.\, \Big(\exists P : \texttt{PartitionOfUnity}(A) \,.\, \forall p \in P \,.\, S_A(p) \cap X = \emptyset \Big) \iff \texttt{NowhereDense}(\mathsf{Z}\, A, X)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{NowhereDense}(\mathsf{Z}\, A, X),$

$D := \Big\{ a \in A : S_A(a) \cap X = \emptyset \Big\} :?A,$

$[2] := \texttt{ENowhereDense}(\mathsf{Z}\, A, X)\texttt{I}D\texttt{IOrderDense} : \texttt{OrderDense}(A, D),$

$\Big(P, [3]\Big) := \texttt{OrderDenseConatinsPoU}(A, D_n) : \sum P : \texttt{PartitionOfUnity}(A) \,.\, P \subset D,$

$[1.*] := \texttt{E}D[3] : \forall p \in P \,.\, S_A(p) \cap X = \emptyset;$

$\rightsquigarrow [1] := \texttt{I}\exists\texttt{I} \Rightarrow: \texttt{NowhereDense}(\mathsf{Z}\, A, X) \Rightarrow \Big(\exists P : \texttt{PartitionOfUnity}(A) \,.\, \forall p \in P \,.\, S_A(p) \cap X = \emptyset \Big),$

$\texttt{Assume}\ P : \texttt{PartitionOfUnity}(A),$

$\texttt{Assume}\ [2] : \forall p \in P \,.\, S_A(p) \cap X = \emptyset,$

$[3] := \texttt{EPartitionOfUnity}(A, P)\texttt{I}\sup : \sup P = 1,$

$Y := \bigcap_{p \in P} S_A(p) \in \mathcal{T}(\mathsf{Z}\, A),$

$[4] := \texttt{E}Y\texttt{IDense}[3]\texttt{I}Y : \texttt{Dense}(\mathsf{Z}\, A, Y),$

$[5] := \texttt{E}Y[2] : X \subset Y^{\complement},$

$[2.*] := \texttt{INowhereDense}[4][5] : \texttt{NowhereDense}(\mathsf{Z}\, A, X);$

$\rightsquigarrow [*] := \texttt{I} \Rightarrow \texttt{I} \iff [1] : \Big(\exists P : \texttt{PartitionOfUnity}(A) \,.\, \forall p \in P \,.\, S_A(p) \cap X = \emptyset \Big) \iff$

$\iff \texttt{NowhereDense}(\mathsf{Z}\, A, X);$

$\square$

$\texttt{WDStoneSpaceCondition} :: \forall A \in \mathsf{BOOL}\ .$

$\quad .\ (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A) \iff \forall X \in \texttt{Meager}(\mathsf{Z}\ A)\ .\ \texttt{NowhereDense}(\mathsf{Z}\ A, X)$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A),$

$\texttt{Assume}\ X : \texttt{Meager}(\mathsf{Z}\ A),$

$\Big(N,[2]\Big) := \texttt{EMeager}(\mathsf{Z}\ A, X) : \sum N : \mathbb{N} \to \texttt{NowhereDense}(\mathsf{Z}\ A)\ .\ X = \bigcup_{n=1} N_n,$

$\Big(P,[3]\Big) := \Lambda n \in \mathbb{N}\ .\ \texttt{NowhereDenseStoneSpaceCondition}(A, N_n) :$

$\quad : \sum P : \mathbb{N} \to \texttt{PartitionOfUnity}(n)\ .\ \forall n \in \mathbb{N}\ .\ \forall p \in P_n\ .\ N_n \cap S_A(p) = \emptyset,$

$\Big(Q,[4]\Big) := \texttt{WDPoUProperty}(A, P) : \sum Q : \texttt{PartitionOfUnity}(A)\ .$

$\quad .\ \forall n \in \mathbb{N}\ .\ \forall q \in Q\ .\ \Big|\{p \in P_n : pq \neq 0\}\Big| < \infty,$

$\texttt{Assume}\ q \in Q,$

$\texttt{Assume}\ [5] : S_A(q) \cap X \neq \emptyset,$

$f := \texttt{ENonEmpty}[0] \in S_A(q) \cap X,$

$\Big(n,[6]\Big) := \texttt{E}[2](f)[5] : \sum n \in \mathbb{N}\ .\ f \in N_n,$

$[7] := [3](n) : \forall p \in P_n\ .\ f \notin S_A(p),$

$C := \{p \in P_n : pq \neq 0\} :?P_n,$

$[8] := \texttt{E}C[4] : |C| < \infty,$

$F := \bigcup_{a \in P_n} S_A(p) : \texttt{Clopen}(\mathsf{Z}\ A),$

$U := S_A(q) \setminus F : \texttt{Clopen}(\mathsf{Z}\ A),$

$\Big(u,[9]\Big) := \texttt{ClopenSetHasStoneRepresentation}(U) : \sum u \in A : U = S_A(u),$

$[10] := \texttt{EPartitionOfUnity}(P_n)[9]\texttt{E}S_A\texttt{E}F : u = 0,$

$[11] := \texttt{E}U[10][9] : F = S_A(q),$

$[12] := [7][11] : f \notin S_A(q),$

$[q.*] := \texttt{I}\bot[12] : \bot;$

$\rightsquigarrow [5] := \texttt{E}\bot\texttt{I}\forall : \forall q \in Q\ .\ S_A(q) \cap X = \emptyset,$

$[1.*] := \texttt{NowhereDenseStoneSpaceCondition}[5] : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, X\Big);$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A) \Rightarrow \forall X : \texttt{Meager}(X)\ .\ \texttt{NowhereDense}\Big(\mathsf{Z}\ A, X\Big),$

$\texttt{Assume}\ [2] : \forall X : \texttt{Meager}(X)\ .\ \texttt{NowhereDense}\Big(\mathsf{Z}\ A, X\Big),$

$\texttt{Assume}\ P : \mathbb{N} \to \texttt{PartitionOfUnity}(A),$

$X := \bigcup_{n=1}^{\infty} \left(\bigcup_{p \in P_n} S_A(p)\right)^{\complement} : \texttt{NowhereDense}(\mathsf{Z}\ A),$

$\Big(Q,[3]\Big) := \texttt{NowhereDenseStoneSpaceCondition}(A, X) :$

$\quad : \sum Q : \texttt{PartitionOfUnity}(A)\ .\ \forall q \in Q\ .\ S_A(q) \cap X = \emptyset,$

$\texttt{Assume}\ n \in \mathbb{N}$,

$\texttt{Assume}\ q \in Q$,

$$[4] := [3](q) : S_A(q) \subset X^{\complement} \subset \bigcap_{m=1}^{\infty} \bigcup_{p \in P_m} S_A(p) \subset \bigcup_{p \in P_n} S_A(p),$$

$$\Big(m, p, [5]\Big) := \texttt{ECompactSubset}\Big(\mathsf{Z}\ A, S_A(q)\Big)[4] : \sum_{m=1}^{\infty} \sum p : [1, \dots, m] \to P_n \ .\ S_A(q) \subset \bigcup_{i=1}^{m} S_A(p_i),$$

$$[P.*] := \texttt{EPartitionOfUnity}(A, P_n)\texttt{E}S_A[5] : \Big|\{p \in P_n ; pq \neq 0\}\Big| \leq m \leq \infty;$$

$\rightsquigarrow [2.*] := \texttt{WDbySupProperty} : (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A)$;

$\rightsquigarrow [*] := \texttt{I} \Rightarrow \texttt{I} \iff : (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A) \iff \forall X \in \texttt{Meager}(\mathsf{Z}\ A)\ .\ \texttt{NowhereDense}(\mathsf{Z}\ A, X)$;

$\square$

$\texttt{RealOpenDomainsAreNotWD} :: \neg(\sigma, \infty)\text{-}\texttt{WeaklyDistributive}\Big(\mathbf{OD}(\mathbb{R})\Big)$

$\texttt{Proof} =$

$q := \texttt{enumerate}(\mathbb{Q}) : \texttt{Bijection}(\mathbb{N}, \mathbb{Q})$,

$$X := \Lambda n \in \mathbb{N}\ .\ \Big\{U \in \mathbf{OD}(\mathbb{R}) : \forall i \in [1, \dots, n]\ .\ q_i \in U\Big\} : \mathbb{N} \downarrow (?A),$$

$$[1] := \Lambda n \in \mathbb{N}\ .\ \texttt{OpenDomainsIndinum}(\mathbb{R}, X_n)\texttt{E}\bigcap\texttt{E}\,\mathrm{int} :$$

$$: \forall n \in \mathbb{N}\ .\ \inf X_n = \mathrm{int} \bigcap X_n = \mathrm{int}\Big\{q_i | i \in [1, \dots, n]\Big\} = \emptyset,$$

$$Y := \Big\{U \in \mathbf{OD}(\mathbb{R}) : \forall n \in \mathbb{N}\ .\ \exists V \in X_n : V \leq U\Big\} :?\mathbf{OD}(\mathbb{R}),$$

$[2] := \texttt{E}Y\texttt{E}X : \forall U \in Y\ .\ \mathbb{Q} \subset U$,

$[3] := \texttt{RationalsAreDense}[2] : Y = \Big\{[\mathbb{R}]\Big\}$,

$[4] := \inf[3]\texttt{RealsExist} : \inf Y = [\mathbb{R}] \neq [\emptyset]$,

$$[*] := \texttt{E}(\sigma, \infty)\text{-}\texttt{WeaklyDistributive}[1][4] : \neg(\sigma, \infty)\text{-}\texttt{WeaklyDistributive}\Big(\mathbf{OD}(\mathbb{R})\Big);$$

$\square$

$\texttt{WeaklyDistributiveByRegularEmbedding} :: \forall A : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive} \,.\, \forall B : \texttt{RegularEmbeded}(A) \,.$
$\quad . \,(\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(B)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ X : \mathbb{N} \to \texttt{DownwardsDirected}(B),$

$$Y' := \left\{ a \in B : \forall n \in \mathbb{N} \,.\, \exists b \in X_n : b \le a \right\} :?A,$$

$$Y := \left\{ a \in A : \forall n \in \mathbb{N} \,.\, \exists b \in X_n : b \le a \right\} :?B,$$

$\texttt{Assume}\ [1] : \forall n \in \mathbb{N} \,.\, \inf_B X_n = 0,$

$[2] := \texttt{E}\texttt{RegularEmbeded}(A,B)[1] : \forall n \in \mathbb{N} \,.\, \inf_A X_n = 0,$

$[3] := \texttt{E}(\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A)[2] : \inf_A Y = 0,$

$\texttt{Assume}\ b \in B,$

$\texttt{Assume}\ [4] : 0 < b \le Y,$

$$\Big(a, \beta, [6]\Big) := [4][3] : \sum a \in A \,.\, \sum \beta : \prod_{n=1}^{\infty} X_n \,.\, \Big(\forall n \in \mathbb{N} \,.\, \beta_n \le a\Big) \ \&\ a \le b,$$

$$[7] := [6.1][6.2] : \Big(\forall n \in \mathbb{N} \,.\, \beta_n < b\Big),$$

$[8] := \texttt{E}Y[7] : b \in Y,$

$[b.*] := \texttt{I}\inf[4] : \inf Y = b;$

$\rightsquigarrow [4] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall b \in B : 0 < b \le Y \Rightarrow b = \inf Y \ \&\ b \in Y,$

$\texttt{Assume}\ [5] : \inf_B Y \ne 0,$

$$\Big(b, [6]\Big) := \texttt{E}\inf Y : \sum b \in B \,.\, 0 < b \le Y,$$

$$[7] := [4]\Big(b, [6]\Big) : b = \inf Y \ \&\ b \in Y,$$

$$\Big(a, \beta, [8]\Big) := [6][3] : \sum a \in A \,.\, \sum \beta : \prod_{n=1}^{\infty} X_n \,.\, \Big(\forall n \in \mathbb{N} \,.\, \beta_n \le a\Big) \ \&\ a < b,$$

$[9] := [4][6]\texttt{I}\,\mathrm{Atom} : b \in \mathrm{Atom}(B),$

$$[10] := [9.1][9.2] : \Big(\forall n \in \mathbb{N} \,.\, \beta_n < b\Big),$$

$[11] := \texttt{E}\,\mathrm{Atom}(B)[9][10] : \beta = 0,$

$[12] := \texttt{E}Y[11]\texttt{I}\inf \texttt{I}Y : \inf Y = 0,$

$[5.*] := [5][12] : \bot;$

$\rightsquigarrow [X.*] := \texttt{E}\bot : \inf Y = 0;$

$\rightsquigarrow [*] := \texttt{I}(\sigma,\infty)\text{-}\texttt{WeaklyDistributive} : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(B);$

$\square$

$\texttt{WDByDenseSubalgebra} :: \forall A \in \mathsf{BOOL} . \forall B \subset_{\mathsf{BOOL}} A .$

$. (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(B) \ \& \ \texttt{OrderDense}(A, B) \Rightarrow (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A)$

$\texttt{Proof} \ =$

$\texttt{Assume} \ P : \mathbb{N} \to \texttt{PartitionOfUnity}(A),$

$\Big(P', [1]\Big) := \texttt{EPartitionOfUnity}(A, P)\texttt{EOrderDense}(A, B) :$

$: \sum P' : \mathbb{N} \to \texttt{PartitionOfUnity}(B) . \forall n \in \mathbb{N} . \forall p' \in P' . \exists p \in P_n : p' \leq p,$

$\Big(Q, [2]\Big) := \texttt{WDPoUProperty}(B, P') :$

$: \sum Q : \texttt{PartitionOfUnity}(B) . \forall q \in Q . \forall n \in \mathbb{N} . \Big|\{p \in P'_n : pq \neq 0\}\Big| < \infty,$

$\texttt{Assume} \ a \in A,$

$\texttt{Assume} \ [3] : a \neq 0,$

$\Big(b, [4]\Big) := \texttt{EOrderDense}(A, B)(a) : \sum b \in B . 0 < b \leq a,$

$\Big(q, [5]\Big) := \texttt{EPartitionOfUnity}(B, Q)(b) : \sum q \in Q . qb \neq 0,$

$[a.*] := [4][5] : qa \neq 0;$

$\rightsquigarrow [3] := \texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}(A, Q),$

$[P.*] := \texttt{EPartitionOfUnity}(P, A)[2][1] : \forall q \in Q . \forall n \in \mathbb{N} . \Big|\{p \in P_n : pq \neq 0\}\Big| < \infty;$

$\rightsquigarrow [*] := \texttt{I}(\sigma, \infty)\text{-}\texttt{WeaklyDistributive} : (\sigma, \infty)\text{-}\texttt{WeaklyDistributive}(A);$

$\square$

$\texttt{WDByBeingSurjectiveImage} :: \forall A : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive} \,.\, \forall B \in \mathsf{BOOL} \,.$
$\quad .\, \forall f : \texttt{Surjective} \,\&\, \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}(A,B) \,.\, (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(B)$

$\texttt{Proof} \;=$

$[1] := \texttt{ClosedMapLemma}\Big(\mathsf{Z}\ B, \mathsf{Z}\ A.\mathsf{Z}\ f\Big) : \texttt{ClosedMap}\Big(\mathsf{Z}\ B, \mathsf{Z}\ A.\mathsf{Z}_{A,B}\ f\Big),$

$\texttt{Assume}\ N : \texttt{NowhereDense}(\mathsf{Z}\ B),$

$[2] := \texttt{EClosedMap}\Big(\mathsf{Z}\ B, \mathsf{Z}\ A.\mathsf{Z}_{A,B}\ f\Big)(\bar{(}N)) : \texttt{Closed}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)(\overline{N})\Big),$

$[3] := \texttt{Eclosure} : \overline{(\mathsf{Z}\ f)(N)} \subset (\mathsf{Z}\ f)(\overline{N}),$

$\texttt{Assume}\ [4] : \operatorname{int} \overline{(\mathsf{Z}\ f)(N)} \neq \emptyset,$

$[5] := \texttt{InteriorIsSubset}[3] : \operatorname{int} \overline{(\mathsf{Z}\ f)(N)} \subset (\mathsf{Z}\ f)(\overline{N}),$

$[6] := \texttt{InjectivePreimage}(\mathsf{Z}\ f)[5] : (\mathsf{Z}\ f)^{-1}\Big(\operatorname{int} \overline{(\mathsf{Z}\ f)(N)}\Big) \subset \overline{N},$

$[7] := \mathtt{E}\mathsf{TOP}(\mathsf{Z}\ B, \mathsf{Z}\ A, \mathsf{Z}\ f)\mathtt{I}\operatorname{int}[5][4] : \operatorname{int} \overline{N} \neq \emptyset,$

$[8] := \texttt{ENowhereDense}(\mathsf{Z}\ B, N)[7] : \bot;$

$\rightsquigarrow [4] := \mathtt{E}\bot : \operatorname{int} \overline{(\mathsf{Z}\ f)(N)} = \emptyset,$

$[N.*] := \texttt{INowhereDense}[4] : \texttt{NowhereDense}\big(\mathsf{Z}\ A, (\mathsf{Z}\ f)(N)\big);$

$\rightsquigarrow [1] := \mathtt{I}\forall : \forall N \in \texttt{NowhereDense}(\mathsf{Z}\ A) \,.\, \texttt{NowhereDense}\big(\mathsf{Z}\ A, (\mathsf{Z}\ f)(N)\big),$

$\texttt{Assume}\ M : \texttt{Meager}(\mathsf{Z}\ B),$

$\Big(N, [2]\Big) := \texttt{EMeager}(\mathsf{Z}\ B, M) : \sum N : \mathbb{N} \to \texttt{NowhereDense}(\mathsf{Z}\ B) \,.\, M = \bigcup_{n=1}^{\infty} N_n,$

$[3] := [2]\texttt{UnionMap}\mathsf{Z}\ B, \mathsf{Z}\ A, N, (\mathsf{Z}\ f) : (\mathsf{Z}\ f)(M)(\mathsf{Z}\ f)\bigcup_{n=1}^{\infty} N_n = \bigcup_{n=1}^{\infty}(\mathsf{Z}\ f)(N_n),$

$[4] := \texttt{IMeager}[1][3] : \texttt{Meager}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)(M)\Big),$

$[5] := \texttt{WDStoneSpacrCondition}(A)[4] : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, (\mathsf{Z}\ f)(M)\Big),$

$[M.*] := \texttt{OrderContinuousNDPreimage}[5] : \texttt{NowhereDense}(\mathsf{Z}\ B, M);$

$\rightsquigarrow [*] := \texttt{WDStoneSpaceCondition} : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(B);$

$\square$

1.6.3 Atoms

$$\texttt{Atom} :: \prod_{A \in \mathsf{BOOL}} ?A$$

$$a : \texttt{Atom} \iff a \in \mathrm{Atom}(A) \iff \left|\langle a \rangle_{\mathcal{I}}\right| = 2$$

$$\texttt{Atomless} :: ?\mathsf{BOOL}$$

$$A : \texttt{Atomless} \iff \mathrm{Atom}(A) = \emptyset$$

$$\texttt{PurelyAtomic} :: ?\mathsf{BOOL}$$

$$A : \texttt{PurelyAtomic} \iff \texttt{OrderDense}\Big(A, \mathrm{Atom}(A)\Big)$$

$$\texttt{booleanDelta} :: \prod_{A} \mathrm{Atom}(A) \to \mathsf{Z}\ A$$

$$\texttt{booleanDelta}\,(a) = \delta_a := \Lambda b \in A\ .\ [a \le b]$$

$$\texttt{DeltaIsIsolatedPoint} :: \forall A \in \mathsf{BOOL}\ .\ \forall a \in A\ .\ \texttt{IsolatedPoint}\Big(\mathsf{Z}\ A, \delta_a\Big)$$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [2] : \Big|S_A(a)\Big| > 1,$

$\Big(f, [3]\Big) := \mathtt{E} S_A(a)[2] : \sum f \in S_A(a)\ .\ f \ne \delta_a,$

$\Big(b, [4]\Big) := \mathtt{E}\delta_a[3] : \sum b \in A\ .\ f(b) = 1\ \&\ a \not\le b,$

$[5] := \mathtt{E}\mathsf{BOOL}(A, \mathbb{B}, f)\mathtt{E}f[4.1]\mathtt{E}\mathbb{B} : f(ab) = f(a)f(b) = 1 \wedge 1 = 1,$

$[6] := \texttt{ZeroInKer}(A, \mathbb{B}, f)[5] : ab \ne 0,$

$[7] := \texttt{BooleanOrderProduct}[4.2] : ab < a,$

$[8] := [6][7] : 0 < ab < a,$

$[2.*] := \mathtt{E}\,\mathrm{Atom}(A, a)[8]\mathtt{I}\bot : \bot;$

$\leadsto [2] := \mathtt{E}\bot : \Big|S_A(a)\Big| \le 1,$

$[3] := \mathtt{E}\,\mathrm{Atom}(A, a)[2]\texttt{ZeroStoneRepresentation} : \Big|S_A(a)\Big| = 1,$

$[*] := \texttt{IsolatedPoinProperty}[3]\mathtt{I}\delta_a\texttt{StoneTopologyBasis}(A) : \texttt{IsolatedPoint}\Big(\mathsf{Z}\ A, \delta_a\Big);$

$\square$

$\texttt{AtomsByStoneIsolatedPoint} :: \forall A \in \mathsf{BOOL} \,.\, \forall f : \texttt{IsolatedPoint}(\mathsf{Z}\ A) \,.\, \exists a \in \mathrm{Atom}(A) \,.\, f = \delta_a$

$\texttt{Proof} \ =$

$\Big(U, [1]\Big) := \texttt{IsolatedPointProperty}(\mathsf{Z}\ A, \mathsf{Z}\ A, f) : \sum U \in \mathcal{U}(f) \,.\, U = \{f\},$

$[2] := \texttt{ET2}(\mathsf{Z}\ A)[1] : \texttt{Clopen}(\mathsf{Z}\ A, U),$

$[3] := \texttt{ClosedIsCompact}[2]\texttt{I}\mathcal{TK}(A) : U \in \mathcal{TK}(A),$

$\Big(a, [4]\Big) := \texttt{CompactOpenAreStoneRepresentations}(A, U) : \sum_{a \in A} S_A(a) = U,$

$\texttt{Assume } b \in A,$

$\texttt{Assume } [5] : b < a,$

$[6] := \texttt{StoneRepresentationBooleanOrder}[5] : S_A(b) \subsetneq S_A(a),$

$[7] := [1][4][6] : S_A(b) = \emptyset,$

$[b.*] := \texttt{StoneRepresentationTHM}[7] : b = 0;$

$\leadsto [5] := \texttt{I}\,\mathrm{Atom} : a \in \mathrm{Atom}(A),$

$[6] := \texttt{E}S_A[1][4] : f(a) = 1,$

$[7] := \Lambda b \in A \,.\, \Lambda T : a \le b \,.\, \texttt{E}\mathsf{POSET}(A, \mathbb{B}, f)[6]T : \forall b \in A \,.\, a \le b \Rightarrow f(b) = 1,$

$\texttt{Assume } b \in A,$

$\texttt{Assume } [8] : a \not\le b,$

$[9] := \texttt{ProductBooleanOrder}(A, a, a, b) : ab \le a,$

$[10] := \texttt{E}\,\mathrm{Atom}(A, a)[8][9] : ab = 0,$

$[b.*] := \texttt{UnityMult}\Big(A, f(b)\Big)[6]\texttt{E}\mathsf{BOOL}(A, \mathbb{B}, f)[10]\texttt{ZeroRingImage}(A, \mathbb{B}, f) :$

$\quad : f(b) = 1 \cdot f(b) = f(a)f(b) = f(ab) = f(0) = 0;$

$\leadsto [8] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall b \in A \,.\, a \not\le b \Rightarrow f(b) = 0,$

$[*] := \texttt{I}\delta_a[7][8] : f = \delta_a;$

$\square$

$\texttt{AtomsIsolatedPointsCorrespondance} :: \forall A \in \mathsf{BOOL} \,.\, \texttt{Bijection}\Big(\mathrm{Atom}(A), \texttt{IsolatedPoint}(\mathsf{Z}\ A), \delta\Big)$

$\texttt{Proof} \ =$

$[1] := \texttt{DeltaIsIsolatedPoin}(A)\texttt{AtomsByStoneIsolatedPoints} :$

$\quad : \texttt{Sujection}\Big(\mathrm{Atom}(A), \texttt{IsolatedPoint}(\mathsf{Z}\ A), \delta\Big),$

$\texttt{Assume } a, b \in \mathrm{Atom}(A),$

$\texttt{Assume } [2] : \delta_a = \delta_b,$

$[3] := \texttt{E}\delta_a(a)\texttt{E}(=)[2] : 1 = \delta_a(a) = \delta_b(a),$

$[4] := \texttt{E}\delta_b[3] : b \le a,$

$[5] := \texttt{E}\delta_b(b)\texttt{E}(=)[2] : 1 = \delta_b(b) = \delta_a(b),$

$[6] := \texttt{E}\delta_b[5] : a \le b,$

$\Big[(a, b).*\Big] := \texttt{ESymmetric}(A, \le)[4][6] : a = b;$

$\leadsto [2] := \texttt{IInjective} : \texttt{Injective}\Big(\mathrm{Atom}(A), \texttt{IsolatedPoint}(\mathsf{Z}\ A), \delta\Big),$

$[*] := \texttt{IBijective} : \texttt{Bijective}\Big(\mathrm{Atom}(A), \texttt{IsolatedPoint}(\mathsf{Z}\ A), \delta\Big);$

$\square$

$\texttt{AtomlessStoneExpression} :: \forall A \in \mathsf{BOOL} \,.\, \texttt{Atomless}(A) \iff \texttt{IsolatedPoint}(\mathsf{Z}\ A) = \emptyset$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{PurelyAtomicStoneExpression} :: \forall A \in \mathsf{BOOL} \,.\, \texttt{PurelyAtomic}(A) \iff \texttt{Dense}\Big(\mathsf{Z}\ A, \texttt{IsolatedPoint}(\mathsf{Z}\ A)\Big)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{CantorAlgebraTHM} :: \forall A \in \mathsf{BOOL} \,.\, A \neq \star \,\&\, \texttt{Atomless}(A) \,\&\, |A| \leq \aleph_0 \iff A \cong_{\mathsf{BOOL}} \mathcal{TK}(\mathcal{C})$

$\texttt{Proof}\ =$

$\texttt{Assume}\ [1] : A \neq \star \,\&\, \texttt{Atomless}(A) \,\&\, |A| \leq \aleph_0,$

$[2] := \texttt{AtomlessStoneExpression}(A)[1.2]\texttt{IPerfect} : \texttt{Perfect}(\mathsf{Z}\ A),$

$[3] := \texttt{UrysohnMetrizationTheorem}[1.2] : \texttt{Metrizable}(\mathsf{Z}\ A),$

$[4] := \mathtt{E}\mathsf{Z}[1.1] : \mathsf{Z}\ A \neq \emptyset,$

$[5] := \texttt{BrouwersTopologicalCharOfCantorSet}[2][3][4] : \mathcal{C} \cong_{\mathsf{TOP}} \mathsf{Z}\ A,$

$[6] := \mathcal{TK}(5]) : \mathcal{TK}\ \mathcal{C} \cong_{\mathsf{BOOL}} \mathcal{TK}\ \mathsf{Z}\ A,$

$[*] := \mathtt{E}\mathcal{TK}[4] : \mathcal{TK}\ \mathcal{C} \cong_{\mathsf{BOOL}} A;$

$\square$

$\texttt{AtomsInSubalgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall B : \texttt{RegularEmbeded}(A) \,.\, \forall a \in \mathrm{Atom}(A) \,.\, \exists b \in \mathrm{Atom}(B) : a \leq b$

$\texttt{Proof}\ =$

$X := \{b \in B : b \geq a\} :?B,$

$\texttt{Assume}\ [1] : \inf X = 0,$

$[2] := \mathtt{E}X\mathtt{I}(\leq) : a \leq X,$

$[3] := \mathtt{E}\texttt{RegularEmbeded}(A, B)[1][2] : a = 0,$

$[1.*] := \mathtt{E}\,\mathrm{Atom}(A, a)[3] : \bot;$

$\rightsquigarrow \Big(b, [1]\Big) := \mathtt{E}\bot : \sum b \in B \,.\, 0 < b \leq X,$

$\texttt{Assume}\ b' \in B,$

$\texttt{Assume}\ [2] : 0 < b' < b,$

$[3] := \mathtt{I}(\complement)[2] : b \not\leq (b')^\complement,$

$[4] := [1][3]\mathtt{I}X : (b')^\complement \notin X,$

$[5] := \mathtt{E}X[4] : a \not\leq (b')^\complement,$

$[6] := \mathtt{E}\complement[5] : ab' \neq 0,$

$[7] := \mathtt{E}\,\mathrm{Atom}(A, a)[6] : b' \geq a,$

$[8] := \mathtt{I}X[7] : b' \in X,$

$[9] := [8][1] : b \leq b',$

$[b'].* := \texttt{TrichtomyPrinciple}[9][2] : \bot;$

$\rightsquigarrow [*] := \mathtt{E}\bot\mathtt{I}\forall\mathtt{I}\,\mathrm{Atom} : b \in \mathrm{Atom}(B);$

$\square$

$\texttt{AtomlessBySubalgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall B : \texttt{RegularEmbeded}(A) \,.\, \texttt{Atomless}(B) \Rightarrow \texttt{Atomless}(A)$

$\texttt{Proof}\ =$

$\ldots$

$\square$

$\texttt{PurelyAtomicIsWeaklyDistributive} :: \forall A : \texttt{PurelyAtomic} \,.\, (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A)$

$\texttt{Proof} \;=$

$U := \bigcup_{a \in \mathrm{Atom}(A)} \{\delta_a\} \in \mathcal{T}\, \mathsf{Z}\, A,$

$[1] := \mathtt{E}U\mathtt{E}\texttt{PurelyAtomic}(A)\texttt{PurelyAtomicStoneExpression}\mathtt{I}U : \texttt{Dense}(\mathsf{Z}\, A),$

$N := U^{\complement} \in \texttt{Closed} \;\&\; \texttt{NowhereDense}(\mathsf{Z}\, A),$

$\texttt{Assume}\ M : \texttt{Meager}(\mathsf{Z}\, A),$

$[2] := \texttt{IntersectionIsSubset}(\mathsf{Z}\, A, N, M)\texttt{NowhereDenseSubset}\mathsf{Z}\, A, N, N \cap M : \texttt{NowhereDense}(\mathsf{Z}\, A, N \cap M),$

$[3] := \texttt{IntersectionIsSubset}(\mathsf{Z}\, A, M, U)\texttt{MeagerSubset}\mathsf{Z}\, A, M, M \cap U : \texttt{Meager}\Big(\mathsf{Z}\, A, M \cap U\Big),$

$\Big(S, [4]\Big) := \mathtt{E}\texttt{Meager}[4] : \sum S\mathbb{N} \to \texttt{NowhereDense}(\mathsf{Z}\, A) \,.\, M \cap U = \bigcup_{n=1}^{\infty} S_n,$

$[5] := \texttt{IntersectionIsSubset}(\mathsf{Z}\, A, U, M) : M \cap U \subset U,$

$[6] := \texttt{UnionSubset}[4][5] : \forall n \in \mathbb{N} \,.\, S_n \subset U,$

$[7] := \mathtt{E}U\texttt{AtomsIsolatedPointsCorrespondance}(A)\mathtt{I}\texttt{Discrete}\mathtt{I}U : \texttt{Discrete}(U),$

$[8] := \Lambda n \in \mathbb{N} \,.\, [6]\mathtt{E}\texttt{NowhereDense}(S_n)\mathtt{E}\texttt{Discrete}(U)[7] : \forall n \in \mathbb{N} \,.\, S_n = \emptyset,$

$[9] := \texttt{EmptysetUnion}(\mathsf{Z}\, A)[8][4] : M \cap U = \emptyset,$

$[10] := \texttt{ComplementDisjointRepresentation}(\mathsf{Z}\, A, M, U)\mathtt{E}N : M = M \cap N,$

$[M.*] := \mathtt{E}(=)[10][2] : \texttt{NowhereDense}(\mathsf{Z}\, A, M);$

$\rightsquigarrow [*] := \mathtt{I}\forall\texttt{NowhereDenseStoneSpaceCondition}(A) : (\sigma,\infty)\text{-}\texttt{WeaklyDistributive}(A);$

$\square$

$\texttt{PurelyAtomicByRegularEmbedding} :: \forall A : \texttt{PurelyAtomic} \,.\, \forall B : \texttt{RegularEmbeded}(A) \,.\, \texttt{PurelyAtomic}(B)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \neg\texttt{PurelyAtomic}(B),$

$\Big(b, [2]\Big) := \mathtt{E}\texttt{PurelyAtomic}(B) : \sum_{b \in B} \inf_B \{c \in B \,.\, 0 < c \le b\} = 0,$

$[3] := \mathtt{E}\texttt{RegularEmbeded}(A)[2] : \inf_A \{c \in B \,.\, 0 < c \le b\} = 0,$

$\Big(a, [4]\Big) := \mathtt{E}\texttt{PurelyAtomic}(A, b) : \sum a \in \mathrm{Atom}(A) \,.\, a < b,$

$[5] := \mathtt{E}\,\mathrm{Atom}(A, a)[4]\mathtt{I}\inf_A : \inf_A \{c \in B \,.\, 0 < c \le b\} \ge a > 0,$

$[1.*] := \texttt{TrichotomyPrinciple}[3][5] : \bot;$

$\rightsquigarrow [*] := \mathtt{E}\bot : \texttt{PurelyAtomic}(B);$

$\square$

$\texttt{AtomsOfDenseSubalgebra} :: \forall A \in \mathsf{BOOL} \,.\, \forall B \subset_{\mathsf{BOOL}} A \,.\, \texttt{OrderDense}(A, B) \Rightarrow \mathrm{Atom}(A) = \mathrm{Atom}(B)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ a \in \mathrm{Atom}(A),$

$[1] := \texttt{E}\,\mathrm{Atom}(A, a) : a \neq 0,$

$\Big(b, [2]\Big) := \texttt{EOrderDense}(A, B)\Big(a, [1]\Big) : \sum b \in B \,.\, 0 < b \leq a,$

$[3] := \texttt{E}\,\mathrm{Atom}(A, a)\Big(b, [2]\Big) : a = b,$

$[a.*] := \texttt{ESubring}(A, B)[3]\texttt{I}\,\mathrm{Atom} : a \in \mathrm{Atom}(B);$

$\rightsquigarrow [1] := \texttt{I} \subset : \mathrm{Atom}(A) \subset \mathrm{Atom}(B),$

$\texttt{Assume}\ b \in \mathrm{Atom}(B),$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [2] : a < b,$

$\texttt{Assume}\ [3] : a \neq 0,$

$\Big(b', [4]\Big) := \texttt{EOrderDense}(A, B)\Big(a, [3]\Big) : \sum b' \in B \,.\, 0 < b \leq a,$

$[5] := [2][4] : b' < b,$

$[a.*] := \texttt{E}\,\mathrm{Atom}(B, b)\Big(b', [5]\Big) : \perp;$

$\rightsquigarrow [b.*] := \texttt{E}\perp\texttt{I} \Rightarrow \texttt{I}\forall\texttt{I}\,\mathrm{Atom} : b \in \mathrm{Atom}(A);$

$\rightsquigarrow [*] := \texttt{I} \subset \texttt{ISubsetEq} : \mathrm{Atom}(A) = \mathrm{Atom}(B);$

$\square$

$\texttt{AtomlessByBeingSurjectiveImage} :: \forall A : \texttt{Atomless} \,.\, \forall B \in \mathsf{BOOL} \,.$

$\quad .\, \forall f : \texttt{Surjective} \,\&\, \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}(A, B) \,.\, \texttt{Atomless}(B)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ \delta : \texttt{IsolatedPoint}(\mathsf{Z}\ B),$

$[1] := \texttt{T2PointsAreClosed}\Big(\mathsf{Z}\ A, f(\delta)\Big)\texttt{AtomlessStoneExpression}(A)\texttt{E}\,\mathrm{int} :$

$\quad : \mathrm{int}\,\overline{\Big\{(\mathsf{Z}\ f)(\delta)\Big\}} = \mathrm{int}\,\Big\{(\mathsf{Z}\ f)(\delta)\Big\} = \emptyset,$

$[2] := \texttt{INowhereDense}[1] : \texttt{NowhereDense}\Big(\mathsf{Z}\ A, \{(\mathsf{Z}\ f)(\delta)\}\Big),$

$[3] := \texttt{OrderContinuousNDPreimage}(A, B, f)\texttt{InjectivePreimage}(A, B, f) : \texttt{NowhereDense}\Big(\mathsf{Z}\ B, \{\delta\}\Big),$

$[4] := \texttt{T2PointsAreClosed}(\mathsf{Z}\ B, \delta)\texttt{EIsolatedPoint}(\mathsf{Z}\ B, \delta)\texttt{E}\,\mathrm{int} :$

$\quad : \mathrm{int}\,\overline{\Big\{\delta\Big\}} = \mathrm{int}\{\delta\} = \{\delta\},$

$[5] := \texttt{ENowhereDense}\Big(\mathsf{Z}\ B, \{\delta\}\Big) : \mathrm{int}\,\overline{\{\delta\}} = \emptyset,$

$[*.\delta] := \texttt{ESingleton}(\mathsf{Z}\ B, \delta)[4][5] : \perp;$

$\rightsquigarrow [1] := \texttt{E}\perp\texttt{IPerfect} : \texttt{Perfect}(\mathsf{Z}\ B),$

$[*] := \texttt{AtomlessStoneExpression}[1] : \texttt{Atomless}(B);$

$\square$

$\texttt{PurelyAtomicByBeingSurjectiveImage} :: \forall A : \texttt{PurelyAtomic} . \forall B \in \mathsf{BOOL} .$
$. \forall f : \texttt{Surjective} \,\&\, \texttt{OrderContinuous} \,\&\, \mathsf{BOOL}(A, B) . \texttt{PurelyAtomic}(B)$

$\texttt{Proof} =$

$\texttt{Assume}\ b \in B,$

$\texttt{Assume}\ [1] : b \neq 0,$

$\texttt{Assume}\ [2] : \forall y \in \mathrm{Atom}(B) . y \not\leq b,$

$\Big(a, [3]\Big) := \texttt{ESurjective}(A, B, f)(a) : \sum_{a \in A} f(a) = b,$

$[4] := \texttt{HomoZeroImage}[1][3] : a \neq 0,$

$\mathcal{A} := \Big\{x \in \mathrm{Atom}(A) : x \leq a\Big\} :? \mathrm{Atom}(A),$

$[5] := \texttt{E}\mathcal{A}\texttt{EPurelyAtomic}(A) : \sup \mathcal{A} = a,$

$\texttt{Assume}\ x \in \mathcal{A},$

$[6] := \texttt{E}\mathcal{A}(x) : x \leq a,$

$[7] := \texttt{BooleanHomoIsMonotonic}[6][1] : f(x) \leq b,$

$\texttt{Assume}\ [8] : f(x) \neq 0,$

$\Big(y, [9]\Big) := [2][8] : \sum_{y \in B} 0 < y < f(x),$

$[10] := \texttt{ESurjective}(A, B, f)(y) : \sum_{x' \in A} f(x') = y,$

$[11] := \texttt{HomoZeroImage}[10][9] : x' \neq 0,$

$[12] := [10]\texttt{EBooleanOrder}(B)[9][10][4]\texttt{E}\mathsf{BOOL}(A, B, f) : f(x') = y = yf(x) = f(x')f(x) = f(x'x),$

$[13] := \texttt{BooleanHomoIsMonotonic}[12] : x'x \not< x',$

$[14] := \texttt{BooleanProductOrder}(A, x, x') : x'x \leq x',$

$[15] := \texttt{TrichtomyPrinciple}[13][14] : x'x = x',$

$[16] := \texttt{EBooleanOrder}(A)[15] : x' \leq x,$

$[17] := \texttt{BooleanHomoIsMonotonic}[12][16] : x' < x,$

$[x.*] := \texttt{E}\,\mathrm{Atom}(A, x)[17][11] : \perp;$

$\leadsto [8] := \texttt{E}\bot\texttt{I}\forall : \forall x \in \mathcal{A} . f(x) = 0,$

$[9] := [8]\texttt{E}\sup : \sup f(\mathcal{A}) = \sup\{0\} = 0,$

$[10] := \texttt{EOrderContinuous}(A, B, f)[5][3] : \sup f(\mathcal{A}) = f(\sup\ \mathcal{A}) = f(a) = b,$

$[b.*] := [1][5][9] : \perp;$

$\leadsto [*] := \texttt{E}\bot\texttt{I} \Rightarrow \texttt{I}\forall\texttt{IPurelyAtomic} : \texttt{PurelyAtomic}(B);$

$\square$

1.6.4 Homogeneous Algebras

$\texttt{Homogeneous} :: ?\mathsf{BOOL}$

$A : \texttt{Homogeneous} \iff \forall a \in A \,.\, a \neq 0 \Rightarrow A \cong_{\mathsf{BOOL}} \langle a \rangle_{\mathcal{I}}$

$\texttt{HomogeneousAlternatives} :: \forall A : \texttt{Homogeneous} \,.\, A \cong_{\mathsf{BOOL}} \mathbb{B} | \texttt{Atomless}(A)$

$\texttt{Proof} =$

$\texttt{Assume}\ a \in \mathrm{Atom}(A),$

$[1] := \texttt{E}\,\mathrm{Atom}(A, a) : a \neq 0,$

$\Big(b, [2]\Big) := \texttt{EHomogeneous}(A)[1] : A \cong \langle a \rangle_{\mathcal{I}},$

$[3] := \texttt{E}\,\mathrm{Atom}(A, a)[2] : A \cong \mathbb{B};$

$\rightsquigarrow [*] := \texttt{IAtomless} : A \cong_{\mathsf{BOOL}} \mathbb{B} | \texttt{Atomless}(A);$

$\square$

$\texttt{HomegeneousByDenseSubset} :: \forall A : \tau\texttt{-Algebra} .$

$$. \texttt{OrderDense}\left(A, \left\{d \in A : \langle d\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} A\right\}\right) \Rightarrow \texttt{Homogeneous}(A)$$

$\texttt{Proof} \ =$

$D := \left\{d \in A : \langle d\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} A\right\} :?A,$

$\texttt{Assume}\ [00] : A \ncong_{\mathsf{BOOL}} \star,$

$\texttt{Assume}\ [000] : \texttt{Atomless}(A),$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [1] : a \neq 0,$

$\left(x, [2]\right) := \texttt{EAtomless}[000][00] : \sum x : \mathbb{N} \downdownarrows A \,.\, x_1 = a,$

$D' := \{d \in D \,.\, \forall n \in \mathbb{N} \,.\, \forall b \in A \,.\, b \leq d \Rightarrow a_{n+1} \not\leq b \not\leq a_n\} :?D,$

$[3] := \texttt{E}D'\texttt{E}\tau\texttt{-Algebra}(A)\texttt{EOrderDense}(A, D)[0]\texttt{I}D' : \texttt{OrderDense}(A, D'),$

$\left(P, [4]\right) := \texttt{OrderDenseContainsPartitionOfUnity}(A.D') : \sum P : \texttt{PartitionOfUnity}(A) \,.\, P \subset D',$

$[5] := \texttt{E}P\texttt{EStrictlyDecreasing}(\mathbb{N}, A, x)\texttt{IInfinite} : |P| = \infty,$

$P' := \{p \in P : p \leq a\} :?P,$

$[6] := \texttt{E}P'\texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}\left(\langle a\rangle_{\mathcal{I}}, P'\right),$

$[a.*] := \texttt{ProductStructureByPartitionOfUnity}\left(\langle a\rangle_{\mathcal{I}}, P'\right)\texttt{E}P'\texttt{E}D$

$\quad \texttt{ProductStructureByPartitionOfUnity}\left(A, P\right)\texttt{E}P\texttt{E}D\texttt{ProductDecomposition}(\mathsf{BOOL})$

$\quad \texttt{InfiniteProductCardinality}(P, P')[5]\texttt{E}P\texttt{E}D\texttt{ProductStructureByPartitionOfUnity}\left(A, P\right) :$

$$\langle a\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \prod_{p \in P'} \langle p\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} A^{P'} \cong_{\mathsf{BOOL}} \left(\prod_{p \in P} \langle p\rangle_{\mathcal{I}}\right)^{P'} \cong_{\mathsf{BOOL}} \left(A^P\right)^{P'} \cong_{\mathsf{BOOL}} A^{P \times P'} \cong_{\mathsf{BOOL}} A^P \cong_{\mathsf{BOOL}}$$

$$\cong_{\mathsf{BOOL}} \prod_{p \in P} \langle p\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} A;$$

$\leadsto [*] := \texttt{IHomogeneous} : \texttt{Homogeneous}(A);$

$\square$

$\texttt{OrderClosureIsHomog} :: \forall A : \texttt{Homogeneous} \,.\, \texttt{Homogeneous}\left(\mathbf{OD}(\mathsf{Z}\ A)\right)$

$\texttt{Proof} \ =$

$\ldots$

$\square$

$\texttt{HomogeneousCoproduct} :: \forall I \in \mathsf{SET} . \forall A : I \to \texttt{Homogeneous} . \texttt{Homogeneous}\left(\bigotimes_{i\in I} A_i\right)$

$\texttt{Proof} =$

$\texttt{Assume } [0] : \forall i \in I . A_i \neq \star,$

$\texttt{Assume } i \in I,$

$\texttt{Assume } [00] : \texttt{Atomless}(A_i),$

$\texttt{Assume } a : \prod_{j\in I} A_j,$

$J := \{j \in I : a_j \neq e\} :?I,$

$\texttt{Assume } [1] : \Big|J\Big| < \infty,$

$t := \bigotimes_{j\in I} a_j \in \bigotimes_{j\in I} A_i,$

$f := \Lambda j \in I . \Lambda b \in \langle a_j\rangle_{\mathcal{I}} . t\iota_j(b) : \prod_{j\in I} \mathsf{BOOL}\Big(\langle a_j\rangle_{\mathcal{I}}, \langle t\rangle_{\mathcal{I}}\Big),$

$\Big(g, [2]\Big) := \texttt{CoproductUniversalProperty}\Big(\mathsf{BOOL}, I, \langle a\rangle_{\mathcal{I}}, \langle t\rangle_{\mathcal{I}}, f\Big) : \sum \bigotimes_{j\in I} \langle a_j\rangle_{\mathcal{I}} \xrightarrow{g} \langle t\rangle_{\mathcal{I}} . \forall j \in I . \iota_j g = f_j,$

$\texttt{Assume } K : \texttt{Finite}(I),$

$\texttt{Assume } [3] : J \subset K,$

$\texttt{Assume } b : \prod_{k\in K} \langle a_k\rangle,$

$s := \bigotimes_{j\in J} b_j \in \bigotimes_{j\in I} A_i,$

$[K.*] := \texttt{E}s\texttt{E} \otimes [2]\texttt{E}f\texttt{E}\mathsf{BOOL}\left(\bigotimes_{j\in K} A_j\right) \texttt{I} \otimes \texttt{I}s\texttt{TensorDistributivity}(I, A, a, b)$

$: \Lambda j \in K . \texttt{PrincipleIdealStructure}(A_j, a_j)\texttt{EBooleanOrder} : g(s) = g\left(\bigotimes_{j\in K} b_j\right) = g\left(\prod_{j\in K} \iota_j(b_j)\right) =$

$= \prod_{j\in K} \iota_j g(b_j) = \prod_{j\in K} f_j(b_j) = \prod_{j\in K} t\iota_j(b_j) = t \prod_{j\in K} \iota_j(b_j) = t \bigotimes_{j\in K} b_j = ts = s;$

$\rightsquigarrow [3] := \texttt{I}\forall : \forall K : \texttt{Finite}(I) . J \subset K \Rightarrow \forall b : \prod_{k\in K} \langle a_k\rangle_{\mathcal{I}} . g\left(\otimes_{k\in k} b_k\right) = \otimes_{k\in k} b_k,$

$\texttt{Assume } b \in \bigotimes_{j\in I} \langle a_j\rangle_I,$

$\texttt{Assume } [5] \in b \neq 0,$

$\Big(K, c, [6]\Big) := \texttt{TensorApproximation}(I, A, b)[5] :$

$: \sum K : \texttt{Finite}(A) . \sum c : \prod_{k\in K} \langle a_k\rangle_{\mathcal{I}} . (\forall k \in K . c_k \neq 0) \ \& \ \bigotimes_{k\in K} c_k \leq b,$

$K' := J \cap K :?K,$

$c' := \Lambda j \in I . \texttt{ if } j \in K' \texttt{ then } c_j \texttt{ else } a_j : \prod_{j\in I} \langle a_j\rangle_{\mathcal{I}},$

$$[7] := \mathtt{E}\mathsf{POSET}\left(\bigotimes_{j\in I}\langle a_j\rangle_{\mathcal{I}}, \langle t\rangle_{\mathcal{I}}, g\right)[6.2]\mathtt{E}c'[6.1]:$$

$$: g(b) \geq g\left(\bigotimes_{k\in K} c_k\right) = g\left(\bigotimes_{j\in I} c'_j\right) = \bigotimes_{j\in I} c'_j > 0,$$

$$[b.*] := \mathtt{E}(>)[7] : g(b) \neq 0;$$

$$\rightsquigarrow [4] := \mathtt{I} \Rightarrow \mathtt{I}\forall\mathtt{I}\ker : \ker g = \{0\},$$

$$[5] := \mathtt{ZeroKernelTHM}[4] : \mathtt{Injective}\left(\bigotimes_{j\in I}\langle a_j\rangle_{\mathcal{I}}, \langle t\rangle_{\mathcal{I}}, g\right),$$

$$\mathtt{Assume}\ K : \mathtt{Finite}(I),$$

$$\mathtt{Assume}\ c : \prod_{k\in K} A_k,$$

$$K' := K \cap J :?I,$$

$$c' := \Lambda j \in I\ .\ \mathtt{if}\ j \in K'\ \mathtt{then}\ c_j\ \mathtt{else}\ e : \prod_{j\in I} A_j,$$

$$d := \bigotimes_{j\in I} c'_j \in \bigotimes_{j\in I} A_j,$$

$$\mathtt{Assume}\ [6] : d \leq t,$$

$$[K.*] := \mathtt{EBooleanOrder}[6]\mathtt{E}d\mathtt{E}t\mathtt{TrnsorDestributivity}(I, A)[3](K', c'a) :$$

$$: d = dt = \left(\bigotimes_{j\in I} c'_j\right)\cdot\left(\bigotimes_{j\in I} a_j\right) = \bigotimes_{j\in I} c'_j a_j = g\left(\bigotimes_{j\in I} c'_j a_j\right);$$

$$\rightsquigarrow [6] := \mathtt{I} \Rightarrow \mathtt{I}^2\forall\mathtt{TensorApproximation}(I, A)\mathtt{ISurjective} : \mathtt{Surjective}\left(\bigotimes_{j\in I}\langle a_j\rangle_{\mathcal{I}}, \langle t\rangle_{\mathcal{I}}, g\right),$$

$$[7] := \mathtt{IIsomorphic}[5][6] : \langle t\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \bigotimes_{j\in I}\langle a_j\rangle_{\mathcal{I}},$$

$$[a.*] := \left(\Lambda i \in I\ .\ \mathtt{EHomogeneous}(A_i)\right)[7] : \langle t\rangle_{\mathcal{I}} \cong \bigotimes_{j\in I} A_j;$$

$$\rightsquigarrow [1] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall a : \prod_{j\in I} A_j\ .\ \Big|\{j \in I : a_j \neq e\}\Big| < \infty \Rightarrow \left\langle\bigotimes_{j\in I} a_j\right\rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \bigotimes_{j\in I} A_j,$$

$$T := \left\{t \in \bigotimes_{j\in I} A_j : \exists a : \prod_{j\in I} A_j : \Big|\{j \in I : a_j \neq e\}\Big| < \infty\ \&\ t = \bigotimes_{j\in I} a_j\right\} :? \bigotimes_{j\in I} A_j,$$

$\texttt{Assume}\ n \in \mathbb{N}$,

$\Big(D, [2]\Big) := \texttt{IPartitionOfUnity}[00]\texttt{EAtomless} : \sum D : \texttt{PartitionOfUnity}(A_i) \ . \ |D| = n$,

$[3] := \texttt{CoproductPartitionOfUnity}(I, A, i, D) : \texttt{PartitionOfUnity}\left(\bigotimes_{j \in I} A_j, \iota_i(D)\right)$,

$[n.*] := \texttt{ProductStructureByPartitionOfUnity}\left(\bigotimes_{j \in I} A_j, \iota_i(D)\right)[0][00][1](\Lambda d \in D \ . \ \star \mapsto d)$

$\texttt{InjectiveCannonicalEmbedding}(I, A) : \bigotimes_{j \in I} A_j \cong_{\mathsf{BOOL}} \prod_{d \in D} \langle d \rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \prod_{d \in D} \bigotimes_{j \in I} A_j \cong_{\mathsf{BOOL}} \left(\bigotimes_{i \in I} A_i\right)^n$;

$\rightsquigarrow [2] := \texttt{I}\forall : \forall n \in \mathbb{N} \ . \ \left(\bigotimes_{i \in I} A_i\right)^n \cong_{\mathsf{BOOL}} \bigotimes_{i \in I} A_i$,

$\texttt{Assume}\ a \in \bigotimes_{j \in I} A_j$,

$\texttt{Assume}\ [3] : a \neq 0$,

$\Big(n, t, [4]\Big) := \texttt{TensorApproximation}(I, A, a)[2][0][00]\texttt{I}T :$

$: \sum n \in \mathbb{N} \ . \ \sum t : [1, \dots, n] \to T \ . \ a = \bigvee_{k=1}^{n} t_k \ \& \ \forall k, l \in [1, \dots, n] \ . \ t_i t_j = 0$,

$[5] := \texttt{IPartitionOfUnity}[5] : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} t\Big)$,

$[a.*] := \texttt{ProductStructureByPartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} t\Big)\texttt{E}T(t)[2](n) : \langle a \rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \prod_{k=1}^{n} \langle t_k \rangle_{\mathcal{I}} \cong_{\mathsf{BOOL}} \left(\bigotimes_{i \in I} A_i\right)$

$\rightsquigarrow [*] := \texttt{I}\forall\texttt{IHomogeneous} : \texttt{Homogeneous}\left(\bigotimes_{i \in I} A_i\right)$;

$\square$

1.7 Automorphisms Group of a Boolean Algebra[!!]

1.7.1 Gluing Lemmas

$\texttt{FiniteGluingLemma} :: \forall A : \mathsf{BOOL} \,.\, \forall I : \texttt{Finite} \,.\, \forall a : I \to A \,.\, \forall b : I \to A \,.$

$\quad .\, \forall [0] : \texttt{PartitionOfUnity}(A, \operatorname{Im} a \;\&\; \operatorname{Im} b) \,.\, \forall f : \prod_{i \in I} \texttt{Isomorphism}\Big(\mathsf{BOOL}, \langle a_i \rangle, \langle b_i \rangle\Big) \,.$

$\quad .\, \exists g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall i \in I \,.\, g_{|\langle a_i \rangle} = f_i$

$\texttt{Proof} \;=$

$[1] := \texttt{ProductStructureByFinitePartitionOfUnity}(A, \operatorname{Im} a) : A \cong_{\mathsf{BOOL}} \prod_{i \in I} \langle a_i \rangle,$

$[2] := \texttt{ProductStructureByFinitePartitionOfUnity}(A, \operatorname{Im} b) : A \cong_{\mathsf{BOOL}} \prod_{i \in I} \langle b_i \rangle,$

$h := \Lambda t \in \prod_{i \in I} A_i \,.\, \Lambda_{i \in I} f_i(t_i) : \mathsf{BOOL}\left(\prod_{i \in I} \langle a_i \rangle, \prod_{i \in I} \langle b_i \rangle \right),$

$g := [1] h [2]^{-1} \in \mathrm{End}_{\mathsf{BOOL}}(A),$

$[3] := \texttt{EgEIsomorphism}\Big(\mathsf{BOOL}, \langle a \rangle, \langle b \rangle\Big) : g \in \mathrm{Aut}_{\mathsf{BOOL}}(A),$

$[*] := \texttt{EPartitionOfUnity}(A, \operatorname{Im} a \;\&\; \operatorname{Im} b)\texttt{E}g : \forall i \in I \,.\, g_{|\langle a_i \rangle} = f_i;$

$\square$

$\texttt{CountableGluingLemma} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall I : \texttt{Countable} \,.\, \forall a : I \to A \,.\, \forall b : I \to A \,.$

$\quad .\, \forall [0] : \texttt{PartitionOfUnity}(A, \operatorname{Im} a \;\&\; \operatorname{Im} b) \,.\, \forall f : \prod_{i \in I} \texttt{Isomorphism}\Big(\mathsf{BOOL}, \langle a_i \rangle, \langle b_i \rangle\Big) \,.$

$\quad .\, \exists g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall i \in I \,.\, g_{|\langle a_i \rangle} = f_i$

$\texttt{Proof} \;=$

$\dots$

$\square$

$\texttt{GluingLemma} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall I \in \mathsf{SET} \,.\, \forall a : I \to A \,.\, \forall b : I \to A \,.$

$\quad .\, \forall [0] : \texttt{PartitionOfUnity}(A, \operatorname{Im} a \;\&\; \operatorname{Im} b) \,.\, \forall f : \prod_{i \in I} \texttt{Isomorphism}\Big(\mathsf{BOOL}, \langle a_i \rangle, \langle b_i \rangle\Big) \,.$

$\quad .\, \exists g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall i \in I \,.\, g_{|\langle a_i \rangle} = f_i$

$\texttt{Proof} \;=$

$\dots$

$\square$

FinitePoUPermutationExtension :: $\forall A$: Homogeneous . $\forall P, Q$: PartitionOfUnity & Finite(A) .

. $\forall \theta$: Bijective(P, Q) . $\exists f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$. $f_{|P} = \theta$

Proof =

. . .

□

CountablePoUPermutationExtension :: $\forall A$: Homogeneous & σ-Algebra .

. $\forall P, Q$: PartitionOfUnity & Countable(A) . $\forall \theta$: Bijective(P, Q) . $\exists f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$. $f_{|P} = \theta$

Proof =

. . .

□

FinitePoUPermutationExtension :: $\forall A$: Homogeneous & τ-Algebra . $\forall P, Q$: PartitionOfUnity(A) .

. $\forall \theta$: Bijective(P, Q) . $\exists f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$. $f_{|P} = \theta$

Proof =

. . .

□

1.7.2 Support of Endomorphisms

$$\texttt{Supports} :: \prod_{A \in \mathsf{BOOL}} \mathrm{End}_{\mathsf{BOOL}}(A) \to ?A$$

$$a : \texttt{Supports} \iff a \in \mathrm{Supp}(A, f) \iff \Lambda f \in \mathrm{End}_{\mathsf{BOOL}}(A) \;.\; \forall b \in \langle a^{\complement} \rangle \;.\; f(b) = b$$

$$\texttt{WithSupport} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{End}_{\mathsf{BOOL}}(A)$$

$$f : \texttt{WithSupport} \iff \exists a \in A : \;.\; a = \min \texttt{Supports}(A, f)$$

$$\texttt{support} :: \prod_{A \in \mathsf{BOOL}} \texttt{WithSupport}(A) \to A$$

$$\texttt{support}\,(f) = \mathrm{supp}\, f := \min \mathrm{Supp}(A, f)$$

$\texttt{SupportIsPreserved} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \;.\; \forall a : \texttt{Supports}(A, f) \Rightarrow f(a) = a$

$\texttt{Proof} \;=$

$[1] := \texttt{E}\mathrm{End}_{\mathsf{BOOL}}(A, f)\texttt{E}\complement\texttt{ESupports}(A, f, a)(a^{\complement}) : f^{\complement}(a) = f(a^{\complement}) = a^{\complement},$

$[*] := [1]^{\complement} : f(a) = a;$

$\square$

$\texttt{UnderSupportIsPreserved} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \;.\; \forall a : \texttt{Supports}(A, f) \;.\; \forall b \in \langle a \rangle \;.\; f(b) \in \langle a \rangle$

$\texttt{Proof} \;=$

$[1] := \texttt{E}\mathrm{End}_{\mathsf{BOOL}}(A, f)\texttt{E}\complement\texttt{ESupports}(A, f, a)(a^{\complement}) : f^{\complement}(b) = f(b^{\complement}) \geq f(a^{\complement}) = a^{\complement},$

$[*] := [1]^{\complement} : f(b) \leq a;$

$\square$

$\texttt{SupportComposition} :: \forall A \in \mathsf{BOOL} \;.\; \forall f, g \in \mathrm{End}_{\mathsf{BOOL}}(A) \;.\; \forall a : \texttt{Supports}(A, f \,\&\, g, a) \;.$

$\quad . \;\texttt{Supports}(A, fg)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; b \in \langle a^{\complement} \rangle_{\mathcal{I}},$

$[1] := \texttt{ESupports}(A, f \,\&\, g, a)(b) : f(b) = b \,\&\, g(b) = b,$

$[b.*] := [1.1][1.2] : fg(b) = g(b) = b;$

$\rightsquigarrow [*] := \texttt{I}\forall\texttt{ISupports} : \texttt{Supports}(A, fg, a);$

$\square$

$\texttt{SupportIsNonEmpty} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \;.\; \mathrm{Supp}(A, f) \neq \emptyset$

$\texttt{Proof} \;=$

$\texttt{Assume}\; b : \langle e^{\complement} \rangle_{\mathcal{I}},$

$[1] := \texttt{E}\langle e^{\complement} \rangle_{\mathcal{I}}(b) : b = 0,$

$[b.*] := [1]\texttt{E}\mathrm{End}_{\mathsf{BOOL}}(A, f)\texttt{ZeroImage}[1] : f(b) = f(0) = 0 = b;$

$\rightsquigarrow [1] := \texttt{I}\,\mathrm{Supp} : e \in \mathrm{Supp}(A, f),$

$[*] := \texttt{INonEmpty}[1] : \mathrm{Supp}(A, f) \neq \emptyset;$

$\square$

$\texttt{SupportIsClosedUnderIntersection} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a, b \in \mathrm{Supp}(A, f) \,.$
$\quad . \; ab \in \mathrm{Supp}(A, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ c \in \langle (ab)^{\complement} \rangle,$

$[1] := \texttt{E}\complement\texttt{E}\langle (ab)^{\complement} \rangle : cab = 0,$

$[1.*] := \texttt{DisjointPairUnionDecomposition}(A, a, b, c)[1]\texttt{E}\mathsf{BOOL}(A, a, b)$
$\quad \texttt{E}\,\mathrm{Supp}(A, f, a)(c \setminus a)\texttt{E}\,\mathrm{Supp}(A, f, b)(c \setminus b)\texttt{DisjointPairUnionDecomposition}(A, a, b, c)[1] :$
$\quad : f(c) = f\Big((c \setminus a) \vee (c \setminus b)\Big) = f(c \setminus a) \vee f(c \setminus b) = c \setminus a \vee c \setminus b = c;$

$\rightsquigarrow [*] := \texttt{I}\,\mathrm{Supp} : ab \in \mathrm{Supp};$

$\square$

$\texttt{SupportContainsGreater} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in \mathrm{Supp}(A, f) \,.$
$\quad . \; \forall b \in A \,.\, a \leq b \Rightarrow b \in \mathrm{Supp}(A, f)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SupportIsInfClosed} :: \forall A \in \mathsf{BOOL} \,.\, \forall f : \texttt{OrderContinuous}(A, A) \,.\, \forall B \subset \mathrm{Supp}(A, f) \,.\, \forall a \in A \,.$
$\quad . \; a = \inf B \,. \;\; \Rightarrow a \in \mathrm{Supp}(A, f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ c \in \langle a^{\complement} \rangle,$

$[1] := \texttt{PrincipleIdealRepresentation}(A, a, c)\texttt{E}\complement : a \leq c^{\complement},$

$C := B \vee c^{\complement} :?A,$

$[2] := \texttt{E}C\,\texttt{JoinOrder}(A) : B \leq C,$

$[3] := \texttt{SupportContainsGreater}[2] : C \subset \mathrm{Supp}(A, f),$

$[4] := \texttt{E}C\texttt{LatticeSup}(A)[0]\texttt{GreaterJoin}[1] : \inf C = \inf(B \vee c^{\complement}) = (\inf B) \vee c^{\complement} = a \vee c^{\complement} = c^{\complement},$

$[5] := [4]\texttt{EOrderContinuous}(A, A, f)\texttt{SupportIsPreserved}(A, f, C)[2][4] :$
$\quad : f\Big(c^{\complement}\Big) = f(\inf C) = \inf f(C) = \inf C = c^{\complement},$

$[c.*] := \Big(\texttt{E}\mathrm{End}_{\mathsf{BOOL}}(A, f)[5]\Big)^{\complement} : f(c) = c;$

$\rightsquigarrow [*] := \texttt{I}\,\mathrm{Supp} : a \in \mathrm{Supp}(A, f);$

$\square$

$\texttt{InjectiveSupportSwitch} :: \forall A \in \mathsf{BOOL} . \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) . \forall \iota \in \mathrm{End}_{\mathsf{BOOL}}(A) \;\&\; \texttt{Injective}(A, A) .$
$\quad . \forall a \in \mathrm{Supp}(A, f\iota) . f(a) \in \mathrm{Supp}(A, \iota f)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; b \in \langle f^{\complement}(a)\rangle,$

$[1] := \texttt{PrincipleIdealRepresentation}\Big(A, f(a), b\Big) : bf(a) = 0,$

$[2] := \texttt{SupportIsPreserve}(A, f\iota, a)\texttt{E}\mathrm{End}_{\mathsf{BOOL}}(A, \iota)[1]\texttt{ZeroHomo}(A, A, \iota) :$
$\quad : \iota(b)a = \iota(b)f\iota(a) = \iota\Big(bf(a)\Big) = \iota(0) = 0,$

$[3] := \texttt{PrincipleIdealRepresentation}[2] : \iota(b) \in \langle a^{\complement}\rangle,$

$[4] := \texttt{E}\,\mathrm{Supp}(A, f, a)[3] : \iota f\iota(b) = \iota(b),$

$[b.*] := \texttt{EInjective}(A, A, \iota)[4] : \iota f(b) = b;$

$\leadsto [*] := \texttt{I}\,\mathrm{Supp}(A, \iota f) : f(a) \in \mathrm{Supp}(A, \iota f);$

$\square$

$\texttt{InjectiveSupportReductionByCommutation} :: \forall A \in \mathsf{BOOL} . \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) .$
$\quad . \forall \iota \in \mathrm{End}_{\mathsf{BOOL}}(A) \;\&\; \texttt{Injective}(A, A) . \iota f = f\iota \Rightarrow \Big(\forall a \in A . \iota(a) \in \mathrm{Supp}(A, f) \Rightarrow a \in \mathrm{Supp}(A, f)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\; a \in A,$

$\texttt{Assume}\; [1] : \iota(a) \in \mathrm{Supp}(A, f),$

$\texttt{Assume}\; b : \langle a^{\complement}\rangle_{\mathcal{I}},$

$[2] := \texttt{PrincipleIdealRepresentation}(A, a^{\complement}, b) : ab = 0,$

$[3] := \texttt{ZeroHomo}(A, A, f)[2]\texttt{E}\mathrm{End}_{\mathsf{BOOL}} A, \iota : 0 = \iota(ab) = \iota(a)\iota(b),$

$[4] := \texttt{PrincipleIdealRepresentation}(A, \iota(a), \iota(b)[3] : \iota(b) \in \langle \iota^{\complement}(a)\rangle_{\mathcal{I}},$

$[5] := [0]\texttt{E}\,\mathrm{Supp}\Big(A, f, \iota(a)\Big)[4] : f\iota(b) = \iota f(b) = \iota(b),$

$[b.*] := \texttt{EInjective}(A, A, \iota)[5] : f(b) = b;$

$\leadsto [*] := \texttt{I}\,\mathrm{Supp}(A, f) : a \in \mathrm{Supp}(A, f);$

$\square$

$\texttt{CommutationByDisjointSupport} :: \forall A \in \mathsf{BOOL} . \forall f, g \in \mathrm{End}_{\mathsf{BOOL}}(A) .$
$\quad . \forall a \in \mathrm{Supp}(A, f) . \forall b \in \mathrm{Supp}(A, g) . ab = 0 \Rightarrow fg = gf$

$\texttt{Proof} \;=$

$c := (a \vee b)^{\complement} \in A,$

$\texttt{Assume}\; t \in A,$

$[1] := [0]\texttt{E}c : t = at + bt + ct,$

$[2] := \texttt{UnderSupportIsPreserved}(A, a, at) : f(at) \leq a,$

$[3] := \texttt{UnderSupportIsPreserved}(A, b, bt) : f(bt) \leq b,$

$[*.1] := [1]\texttt{E}\mathrm{End}_A(fg)\texttt{E}\,\mathrm{Supp}(A, f)\texttt{E}\,\mathrm{Supp}(A, g)[2][3]\texttt{E}\mathrm{End}_A(gf)[1] :$
$\quad : fg(t) = fg(at + bt + ct) = fg(at) + fg(bt) + fg(ct) = f(at) + g(bt) + ct = gf(at) + gf(bt) + gf(ct) =$
$\quad = gf((at + bt + ct) = gf(t);$

$\leadsto [*] := \texttt{I}(=, \rightarrow) : fg = gf;$

$\square$

$\texttt{IterativeInjectiveSupport} :: \forall A \in \mathsf{BOOL} \,.\, \forall \iota \in \mathrm{End}_{\mathsf{BOOL}}(A) \,\&\, \texttt{Injective}(A, A) \,.\, \forall n \in \mathbb{N} \,.$
$\quad .\, \forall a \in A \,.\, a \in \mathrm{Supp}(A, \iota^n) \iff \iota(a) \in \mathrm{Supp}(A, \iota^n)$

$\texttt{Proof} \ =$

$[1] := \texttt{InjectiveSupportReductionByCommutation}(A, \iota, \iota^n, a) : \iota(a) \in \mathrm{Supp}(A, \iota^n) \Rightarrow a \in \mathrm{Supp}(A, \iota^n),$

$[2] := \texttt{InjectiveSupportSwitch}(A, \iota, \iota^{n-1}, a) : a \in \mathrm{Supp}(A, \iota^n) \Rightarrow \iota(a) \in \mathrm{Supp}(A, \iota^n),$

$[*] := \texttt{I} \iff : a \in \mathrm{Supp}(A, \iota^n) \iff \iota(a) \in \mathrm{Supp}(A, \iota^n);$

$\square$

$\texttt{InverseSupport} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in \mathrm{Supp}(A, f) \,.\, a \in \mathrm{Supp}(A, f^{-1})$

$\texttt{Proof} \ =$

$\texttt{Assume}\ b \in \langle a \rangle_{\mathcal{I}},$

$[b.*] := \texttt{EInverse}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), f\Big)\texttt{E}\,\mathrm{Supp}(A, f, a, b) : b = f f^{-1}(b) = f^{-1}(b);$

$\rightsquigarrow [*] := \texttt{I}\,\mathrm{Supp}(A, f) : a \in \mathrm{Supp}(A, f^{-1}),$

$\square$

$\texttt{EndomorphismSupportImpliesSumBound} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in \mathrm{Supp}(A, f) \,.$
$\quad .\, \forall x \in A \,.\, f(x) + x \le a$

$\texttt{Proof} \ =$

$b := a^{\complement} \in A,$

$[1] := x\texttt{LawOfExcludedMiddle}(a)\texttt{E}b : x = x(a + b) = xa + xb,$

$[2] := \texttt{UnderSupportIsPreserved}(A, f, a, xa) : f(xa) \le a,$

$[*] := [1]\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\texttt{E}\,\mathrm{Supp}(A, f, a)\Big(xa^{\complement}\Big)\texttt{BooleanRingHasChar2}(A)\texttt{BooleanRingIsALattice}(A)$
$\quad \texttt{BooleanSumBound}(A)[2] : f(x) + x = f(xa + xa^{\complement}) + xa + xa^{\complement} = f(xa) + f(xa^{\complement}) + xa^{\complement} + xa =$
$\quad = f(xa) + xa^{\complement} + xa^{\complement} + xa = f(xa) + xa \le a,$

$\square$

$\texttt{DecompositionBoundPropertyImplication} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.$
$\quad .\, \Big(\forall x \in A \,.\, f(x) + x \le a\Big) \Rightarrow \Big(\forall x \in A \,.\, f(x)x = 0 \Rightarrow x \le a\Big)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in A,$

$\texttt{Assume}\ [1] : f(x)x = 0,$

$[x.*] := \texttt{DisjointSumBound}[1]\Big(A, x, f(x)\Big)[0](x) : x \le f(x) + x \le a;$

$\rightsquigarrow [*] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall x \in A \,.\, f(x)x = 0 \Rightarrow x \le a,$

$\square$

$\texttt{YetAnotherImplication} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.$

$. \left(\forall x \in A \,.\, f(x)x = 0 \Rightarrow x \leq a\right) \Rightarrow \left(\forall x \in A \,.\, x \neq 0 \,\&\, x \leq a^{\complement} \Rightarrow f(x)x \neq 0\right)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ x \in A,$

$\texttt{Assume}\ [1] : x \neq 0,$

$\texttt{Assume}\ [2] : x \leq a^{\complement},$

$\texttt{Assume}\ [3] : f(x)x = 0,$

$[4] := [0](x)[3] : x \leq a,$

$[5] := \texttt{BooleanRingIsIsALattice}[2][4]\texttt{LawOfExcludedMiddle}(A, a) : x \leq a^{\complement}a = 0,$

$[6] := \texttt{BooleanRingMinmalElement}[5] : x = 0,$

$[x.*] := [6][1] : \perp;$

$\rightsquigarrow [*] := \texttt{E}\perp\texttt{I} \Rightarrow \texttt{I}\forall : \forall x \in A \,.\, x \neq 0 \,\&\, x \leq a^{\complement} \Rightarrow f(x)x \neq 0;$

$\square$

$\texttt{AutomotphismSupportCondition} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.$

$. \left(\forall x \in A \,.\, x \neq 0 \,\&\, x \leq a^{\complement} \Rightarrow f(x)x \neq 0\right) \Rightarrow a \in \mathrm{Supp}(A, f)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] \in a \notin \mathrm{Supp}(A, f),$

$\left(b, [2]\right) := \texttt{E}\,\mathrm{Supp}(A, f) : \sum b \in \left\langle a^{\complement}\right\rangle_{\mathcal{I}} \,.\, f(b) \neq b,$

$[3] := \texttt{ZeroHomo}[2] : b \neq 0,$

$[4] := \texttt{I}[2][3] : b \setminus f(b) \neq 0 \big| f(b) \setminus b \neq 0,$

$\texttt{Assume}\ [5] : b \setminus f(b) \neq 0,$

$c := b \setminus f(b) \in A,$

$[6] := \texttt{E}c\texttt{E} \setminus \texttt{E}b : c = b \setminus f(b) \leq b \leq a^{\complement},$

$[7] := [0](c)[5][6]\texttt{E}c\texttt{E}\setminus : 0 \neq cf(c) = \left(b \setminus f(b)\right)\left(f(b) \setminus f^2(b)\right) = 0,$

$[5.*] := \texttt{I}\perp[7] : \perp;$

$\rightsquigarrow [5] := \texttt{E}\perp : b \setminus f(b) = 0,$

$[6] := \texttt{E}|[5] : f(b) \setminus b \neq 0,$

$[7] := \texttt{E}\mathrm{Aut}_{\mathbb{B}}(A, f^{-1})[6] : b \setminus f^{-1}(b) \neq 0,$

$[8] := \texttt{E} \setminus \texttt{E}b : b \setminus f^{-1}(b) \leq b \leq a^{\complement},$

$[9] := [0][7][8] : 0 \neq \left(b \setminus f^{-1}(b)\right)\left(f(b) \setminus b\right) = 0,$

$[1.*] := \texttt{I}\perp[9] : \perp;$

$\rightsquigarrow [*] := \texttt{E}\perp : a \in \mathrm{Supp}(A, f);$

$\square$

$\texttt{SuppConjugation} :: \forall A \in \mathsf{BOOL} \,.\, \forall g \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$

$\quad . \forall a \in \mathrm{Supp}(A, g) \,.\, f(a) \in \mathrm{Supp}\left(A, f^{-1}gf\right)$

$\texttt{Proof} \; =$

$\texttt{Assume}\; b \in \left\langle f^{\complement}(a) \right\rangle_{\mathcal{I}},$

$[1] := \texttt{PrincipleIdealStrucure}\Big(A, f(a), b\Big)\mathtt{E}\mathrm{End}_{\mathsf{BOOL}}(A, f) : b \le f^{\complement}(a) = f\Big(a^{\complement}\Big),$

$[2] := \texttt{MonotonicBooleanMorphism}\Big(A, f^{-1}\Big)[1] : f^{-1}(b) \le a^{\complement},$

$[3] := \mathtt{E}\,\mathrm{Supp}(A, g, a)\Big(f^{-1}(b)\Big) : f^{-1}g(b) = f^{-1}(b),$

$[b.*] := \mathrm{Aut}_{\mathsf{BOOL}}(A, f)[4] : f^{-1}gf(b) = b;$

$\rightsquigarrow [*] := \mathtt{I}\,\mathrm{Supp} : f(a) \in \mathrm{Supp}\Big(A, f^{-1}gf\Big);$

$\square$

$\texttt{SuppConjugationEq} :: \forall A \in \mathsf{BOOL} \,.\, \forall g, f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall h \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.$

$\quad . \forall a \in \mathrm{Supp}(A, g) \,.\, g_{|\langle a\rangle_{\mathcal{I}}} = f_{|\langle a\rangle_{\mathcal{I}}} \Rightarrow f^{-1}hf = g^{-1}hg$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{ContinuousHaveSupport} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall f : \mathrm{End}_{\mathsf{BOOL}}(A) \;\&\; \texttt{OrderContinuous}(A, A) \,.$

$\quad . \texttt{WithSupport}(A, f)$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{SupportIsPreserved2} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall f : \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, f(\mathrm{supp}\, f) = \mathrm{supp}\, f$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{SupportBySums} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall f : \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \mathrm{supp}\, f = \sup \left\{ a + f(a) \middle| a \in A \right\}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{SupportByZeroImages} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall f : \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \mathrm{supp}\, f = \sup \left\{ a \in A : a + f(a) = 0 \right\}$

$\texttt{Proof} \; =$

$\dots$

$\square$

$\texttt{SupportOfInverse} :: \forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall f : \mathrm{Aut}_{\mathsf{BOOL}}(A) \;.\; \operatorname{supp} f^{-1} = \operatorname{supp} f$

$\texttt{Proof} \;=$

...

$\square$

$\texttt{SupportOfTheConjugate} :: \forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall f, g : \mathrm{Aut}_{\mathsf{BOOL}}(A) \;.\; \operatorname{supp} f^{-1} g f = f(\operatorname{supp} g)$

$\texttt{Proof} \;=$

...

$\square$

1.7.3 Periodic and Aperiodic Parts Theorem

$$\texttt{Periodic} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$$

$$f : \texttt{Periodic} \iff A \neq \emptyset \;\&\; (\exists n \in \mathbb{N} : f^n = \mathrm{id} \;\&\; \forall i \in [1, \dots, n-1] \;.\; \mathrm{supp}\, f^i = e)$$

$$\texttt{period} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$$

$$\texttt{period}\,(f) = \pi(f) := \min(\exists n \in \mathbb{N} : f^n = \mathrm{id})$$

$$\texttt{Aperiodic} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$$

$$f : \texttt{Aperiodic} \iff \forall n \in \mathbb{N} \;.\; \mathrm{supp}\;\, f^n = e$$

$$\texttt{WithAllSupports} :: \prod_{A \in \mathsf{BOOL}} ?\Big(\mathrm{End}_{\mathsf{BOOL}}(A) \;\&\; \texttt{Injective}(A, A)\Big)$$

$$f : \texttt{WithAllSupports} \iff \forall n \in \mathbb{N} \;.\; \texttt{WithSupport}(A, f^n)$$

$\mathtt{PeriodicAperiodicPartsTHM} :: \forall A : \sigma\text{-}\mathtt{Algebra} \,.\, \forall f : \mathtt{WithAllSupports} \,.\, \exists p : \mathtt{Injective}\Big(\omega_0 + 1, A\Big) :$

$: \mathtt{PartitionOfUnity}(A, \operatorname{Im} p) \;\&\; \Big(\forall k \in \omega_0 + 1 \,.\, f(p_k) \le p_k\Big) \;\&$

$\& \left(\forall n \in \mathbb{Z}_+ \,.\, p_n \ne 0 \Rightarrow \mathtt{Periodic}\Big(\langle p_n\rangle_{\mathcal{I}}, f_{|\langle p_n\rangle_{\mathcal{I}}}\Big) \;\&\; \pi(f_{|\langle p_n\rangle_{\mathcal{I}}}) = n\right) \;\&\; \mathtt{Aperiodic}\Big(\langle p_{\omega_0}\rangle_{\mathcal{I}}, f_{|\langle p_{\omega_0}\rangle_{\mathcal{I}}}\Big)$

$\mathtt{Proof}\ =$

$p := \mathtt{BoundedTransfiniteInduction}\Bigg(\omega_0 + 1, \neg \operatorname{supp} f,$

$, \Lambda n \in \omega_0 \,.\, \Lambda a : [1, \ldots, n-1] \to A \,.\, \bigwedge_{i=1}^{n} (\neg a_i) \setminus \operatorname{supp} f^n, \Lambda k : \mathtt{Limit} \,.\, \Lambda k \to A \,.\, \inf_{n \in k} \operatorname{supp} f^n\Bigg) : (\omega_0 + 1) \to A,$

$P := \operatorname{Im} p :?A,$

$[*.1] := \mathtt{E}P\mathtt{E}p\mathtt{IPartitionOfUnity} : \mathtt{PartitionOfUnity}(A, P),$

$[2] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}p\mathtt{IterativeInjectiveSupport}(A, f, p_{n-1})\mathtt{SupportIsPreserved}(A, f^n) :$

$\forall n \in \mathbb{N} \,.\, f(p_n) = p_n,$

$[3] := \mathtt{E}p_{\omega_1}\mathtt{E}\inf : \forall n \in \mathbb{N} \,.\, p_{\omega_0} \le \operatorname{supp} f^n,$

$[4] := \mathtt{UnderSupportIsPreserved}[3] : \forall n \in \mathbb{N} \,.\, f(p_{\omega_0}) \le \operatorname{supp} f^n,$

$[5] := \mathtt{E}\inf \mathtt{E}p_{\omega_0}[4] : f(p_{\omega_0}) \le p_{\omega_0},$

$[*.2] := [2][5] : \forall n \in \omega_0 + 1 \,.\, f(p_n) \le p_n,$

$\mathtt{Assume}\ n \in \mathbb{N},$

$\mathtt{Assume}\ [00] : p_n \ne 0,$

$\mathtt{Assume}\ a \in \langle p_n\rangle_{\mathcal{I}},$

$\mathtt{Assume}\ [000] : a \ne 0,$

$[a.*.2] := \mathtt{E}p_n\mathtt{E}a : \forall k \in [1, \ldots, n-1] \,.\, f^k(a) \ne a,$

$[7] := \mathtt{E}p_n\mathtt{E}a\mathtt{EI}\operatorname{supp} f^n : a \operatorname{supp} f^n = 0,$

$[a.*.1] := \mathtt{E}\operatorname{Supp}[7] : f^n(a) = a;$

$\leadsto [6] := \mathtt{I}\forall : \forall a \in \langle p_n\rangle_{\mathcal{I}} \,.\, a \ne 0 \Rightarrow \forall k \in [1, \ldots, n-1] \,.\, f^k(a) \ne a \;\&\; f^n(a) = a,$

$[n.*.1] := \mathtt{IPeriodic}[6][00] : \mathtt{Periodic}\Big(\langle p_n\rangle_{\mathcal{I}}, f^n_{|\langle p_n\rangle_{\mathcal{I}}}\Big),$

$[n.*.2] := [n.*.1][6]\mathtt{I}\pi : \pi(f_{|\langle p_n\rangle_{\mathcal{I}}}) = n;$

$\leadsto [*.3] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall n \in \mathbb{N} \,.\, p_n \ne 0 \Rightarrow \mathtt{Periodic}\Big(\langle p_n\rangle_{\mathcal{I}}, f_{|\langle p_n\rangle_{\mathcal{I}}}\Big) \;\&\; \pi(f_{|\langle p_n\rangle_{\mathcal{I}}}) = n,$

$[*.4] := \mathtt{E}\operatorname{supp} \mathtt{E}p_{\omega_0}\mathtt{IAperiodic} : \mathtt{Aperiodic}\Big(\langle p_{\omega_0}\rangle_{\mathcal{I}}, f_{|\langle p_{\omega_0}\rangle_{\mathcal{I}}}\Big);$

$\square$

1.7.4 Full Subgroups

$$\texttt{FullSubgroup} :: \prod_{A \in \mathsf{BOOL}} \texttt{Subgroup}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big)$$

$G : \texttt{FullSubgroup} \iff \forall I \in \mathsf{SET} \,.\, \forall a : \texttt{Injective}(I, A) \,.\, \forall f : I \to G \,.$
$\quad . \forall [0] : \texttt{PartitionOfUnity}(A, \mathrm{Im}\, a) \,.\, \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$
$\quad . \forall [00] : \forall b \in A \,.\, \Big((\exists i \in I : b \leq a_i) \Rightarrow g(b) = f_i(b) \Big) \Rightarrow g \in G$

$$\texttt{CountablyFullSubgroup} :: \prod_{A \in \mathsf{BOOL}} \texttt{Subgroup}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big)$$

$G : \texttt{CountablyFullSubgroup} \iff \forall I : \texttt{Countable} \,.\, \forall a : \texttt{Injective}(I, A) \,.\, \forall f : I \to G \,.$
$\quad . \forall [0] : \texttt{PartitionOfUnity}(A, \mathrm{Im}\, a) \,.\, \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$
$\quad . \forall [00] : \forall b \in A \,.\, \Big((\exists i \in I : b \leq a_i) \Rightarrow g(b) = f_i(b) \Big) \Rightarrow g \in G$

$$\texttt{generateFullSubgroup} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A) \to \texttt{FullSubgroup}(A)$$

$$\texttt{generateFullSubgroup}\,(X) = \langle X \rangle_{\mathrm{F}} := \bigcap \Big\{ G : \texttt{FullSubgroup}(A) : X \subset G \Big\}$$

$$\texttt{generateCoubtablyFullSubgroup} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A) \to \texttt{CountablyFullSubgroup}(A)$$

$$\texttt{generateCountablyFullSubgroup}\,(X) = \langle X \rangle_{\mathrm{CF}} := \bigcap \Big\{ G : \texttt{CountablyFullSubgroup}(A) : X \subset G \Big\}$$

$\texttt{FullSubgroupGeneratedByGroupExpression} :: \forall A \in \mathsf{BOOL} \,.\, \forall G \subset_{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$
$\quad . \; \langle G \rangle_{\mathrm{F}} = \Big\{ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall a \in A \setminus \{0\}. \exists b \in \langle a \rangle_{\mathcal{I}} \setminus \{0\} : \exists g \in G : \forall c \in \langle b \rangle_{\mathcal{I}} \,.\, f(c) = g(c) \Big\}$

$\texttt{Proof} \; =$

$H := \Big\{ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall a \in A \setminus \{0\}. \exists b \in \langle a \rangle_{\mathcal{I}} \setminus \{0\} : \exists g \in G : \forall c \in \langle b \rangle_{\mathcal{I}} \,.\, f(c) = g(c) \Big\} :?\mathrm{Aut}_{\mathsf{BOOL}}(A),$

$\texttt{Assume}\; h, h' \in H,$

$\texttt{Assume}\; a \in A,$

$\texttt{Assume}\; [1] : a \neq 0,$

$\Big(b, g, [2]\Big) := \mathtt{E}H(h, a)[1] : \sum b \in \langle a \rangle_{\mathcal{I}} \,.\, \sum g \in G \,.\, b \neq 0 \;\&\; \forall c \in \langle b \rangle_{\mathcal{I}} \,.\, h(c) = g(c),$

$[3] := \mathtt{E}\mathrm{End}_{\mathsf{BOOL}}(A, g)[2.1] : g(b) \neq 0,$

$\Big(b', g', [4]\Big) := \mathtt{E}H\Big(h', g(b)\Big)[3] : \sum b' \in \Big\langle g(b) \Big\rangle_{\mathcal{I}} \,.\, \sum g' \in G \,.\, b' \neq 0 \;\&\; \forall c \in \langle b' \rangle_{\mathcal{I}} \,.\, h'(c) = g'(c),$

$\texttt{Assume}\; c \in \langle b \rangle_{\mathcal{I}},$

$[5] := \texttt{BooleanMorphismIsMonotonic}(A, A, g)\texttt{PrincipleIsealExpression}(\mathtt{A}, \mathtt{b}, \mathtt{c}) : g(c) \in \Big\langle g(b) \Big\rangle_{\mathcal{I}},$

$\Big[h, h'\Big].* := [4.2][5][2.2](c) : gg'(c) = gh'(c) = hh'(c);$

$\rightsquigarrow [1] := \mathtt{I}\forall\mathtt{I}^2\exists\mathtt{I}\forall\mathtt{I}H\mathtt{I}\forall : \forall h, h' \in H \,.\, hh' \in H,$

Assume $h \in H$,

Assume $a \in A$,

Assume $[2] : a \neq 0$,

$[3] := \mathtt{E}\mathrm{End}_{\mathsf{BOOL}}(A, h^{-1})[2] : h^{-1}(a) \neq 0$,

$\Big(b, g, [4]\Big) := \mathtt{E}H\Big(h, h^{-1}(a)\Big)[2] : \sum b \in \Big\langle h^{-1}(a)\Big\rangle_{\mathcal{I}} \, . \, \sum g \in G \, . \, b \neq 0 \,\&\, \forall c \in \langle b\rangle_{\mathcal{I}} \, . \, h(c) = g(c)$,

$[h.*.1] := \mathtt{BoolenMorphismIsMonotonic}(A, h) : h(b) \leq a$,

$[h.*.2] := \mathtt{E}\mathrm{End}_{\mathsf{BOOL}}(A, h)[4.1] : h(b) \neq 0$,

Assume $c \in \langle h(b)\rangle_{\mathcal{I}}$,

$[5] := \mathtt{PrincipleIdealStructur}\Big(A, h(b), c\Big)\mathtt{BooleanMorphismIsMonotonic}\Big(A, A, h^{-1}\Big) : h^{-1}(c) \leq b$,

$[6] := [4.2][5]\mathtt{EInverse} : h^{-1}g(c) = h^{-1}h(c) = c$,

$[h.*] := g^{-1}[6] : h^{-1}(c) = g^{-1}(c)$;

$\leadsto [2] := \mathtt{I}\forall\mathtt{I}^2\exists\mathtt{I}\forall\mathtt{I}H\mathtt{I}\forall : \forall h \in H \, . \, h^{-1} \in H$,

$[3] := \mathtt{I}\mathsf{GRP}[1][2] : H \in \mathsf{GRP}$,

Assume $I \in \mathsf{SET}$,

Assume $a : \mathtt{Injective}(I, A)$,

Assume $h : I \to H$,

Assume $[4] : \mathtt{PartitionOfUnity}(A, \operatorname{Im} a)$,

Assume $f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$,

Assume $[5] : \forall b \in A \, . \, (\exists i \in I : b < a_i) \Rightarrow f(b) = h_i(b)$,

Assume $b \in A$,

Assume $[6] : b \neq 0$,

$\Big(i, [7]\Big) := \mathtt{EPartitionOfUnity}[4](b)[6] : \sum_{i \in I} a_i b \neq 0$,

$[8] := \mathtt{LatticeMeetsIneq}(A, a_i, b) : a_i b \leq a_i$,

$\Big(b', g, [9]\Big) := \mathtt{E}H\Big(h_i, a_i b\Big)[7] : \sum b' \in \langle a_i b\rangle_{\mathcal{I}} \, . \, \sum g \in G \, . \, b' \neq 0 \,\&\, \forall c \in \langle b'\rangle_{\mathcal{I}} \, . \, h_i(c) = g(c)$,

$[b.*.1] := \mathtt{PrincipleIdealStructure}(A, a_i b, b')\mathtt{LatticeJoinIneq}(A, b', a_i) : b' \leq a_i b \leq b$,

Assume $c \in \langle b'\rangle_{\mathcal{I}}$,

$[10] := \mathtt{PrincipleIdealStructure}(A, b', c)\mathtt{PrincipleIdealStructure}(A, a'b, b')\mathtt{LatticeJoinIneq}(A, a_i, b) :$
$\quad : c \leq b' \leq a'b \leq a'$,

$[b.*.1] := [5](c)[10][8](c) : f(c) = h_i(c) = g(c)$;

$\leadsto [I.*] := \mathtt{I}H : f \in H$;

$\leadsto [4] := \mathtt{IFullSubgroup} : \mathtt{FullSubgroup}(A, H)$,

$[5] := \mathtt{E}\,\langle G\rangle_{\mathrm{F}}\,[4] : \langle G\rangle_{\mathrm{F}} \subset H$,

Assume $F : \mathtt{FullSubgroup}(A)$,

Assume $[6] : G \subset F$,

Assume $h \in H$,

$D := \Big\{b \in A : \exists g \in G : \forall c \in \langle b\rangle_{\mathcal{I}} \, . \, f(c) = g(c)\Big\} :?A$,

$[7] := \mathtt{E}H(h)\mathtt{E}D\mathtt{IOrderDense} : \mathtt{OrderDense}(A, D)$,

$\Big(P, [8]\Big) := \mathtt{OrderDenseContainsPartitionOfUnity}[7] : \sum P : \mathtt{PartitionOfUnity}(A) \, . \, P \subset A$,

$\Big(g, [9]\Big) := \mathtt{E}P\mathtt{E}H : \sum g : P \to G \, . \, \forall p \in P \, . \, \forall c \in \langle p\rangle_{\mathcal{I}} \, . \, g_p(c) = h(c)$,

$[10] := \mathtt{E}g[6] : \forall p \in P \, . \, g_p \in F$,

$[h.*] := \mathtt{EFullSubgroup}(A, F)(P, P, g, h)[10][9] : h \in F$;

$\rightsquigarrow [F.*] := \mathtt{I} \subset : H \subset F;$

$\rightsquigarrow [6] := \mathtt{I} \Rightarrow \mathtt{I}\forall : \forall F : \mathtt{FullSubgroup}(A) \,.\, G \subset F \Rightarrow H \subset F,$

$[7] := \mathtt{E}\,\langle G\rangle_{\mathrm{F}} : H \subset \langle G\rangle_{\mathrm{F}}\,,$

$[*] := \mathtt{ISetEq}[5][7] : \langle G\rangle_{\mathrm{F}} = H;$

$\square$

$\mathtt{CountablyFullSubgroupGeneratedByGroupElement} :: \forall A : \sigma\text{-}\mathtt{Algebra} \,.\, \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$

$$. \ \langle g\rangle_{\mathrm{CF}} = \Big\{ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \exists p : \mathbb{Z} \to A : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p) \ \&$$

$$\& \ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p_n\rangle_{\mathcal{I}} \,.\, f(b) = g^n(b) \Big\}$$

$\mathtt{Proof} \ =$

$$H := \Big\{ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \exists p : \mathbb{Z} \to A : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p) \ \&$$

$$\& \ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p_n\rangle_{\mathcal{I}} \,.\, f(b) = g^n(b) \Big\} :?\mathrm{Aut}_{\mathsf{BOOL}}(A),$$

$\mathtt{Assume}\ h, h' : H,$

$\Big(p, [1]\Big) := \mathtt{E}H(h) : \sum p : \mathbb{Z} \to A \,.\, \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p) \ \&\ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p_n\rangle_{\mathcal{I}} \,.\, h(b) = g^n(b),$

$\Big(p', [2]\Big) := \mathtt{E}H(h') : \sum p' : \mathbb{Z} \to A \,.\, \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p') \ \&\ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p'_n\rangle_{\mathcal{I}} \,.\, h'(b) = g^n(b),$

$$p'' := \Lambda n \in \mathbb{Z} \,.\, \bigvee_{n=l+k} p_l \wedge h^{-1}(p'_k) : \mathbb{Z} \to A,$$

$\mathtt{Assume}\ n, m : \mathbb{Z},$

$\mathtt{Assume}\ [3] : n \neq m,$

$\mathtt{Assume}\ k, l, t, s : \mathbb{Z},$

$\mathtt{Assume}\ [4] : n = k + l,$

$\mathtt{Assume}\ [5] : m = t + s,$

$[6] := [3][4][5] : k \neq t \Big| l \neq s,$

$\Big[(k, l, t, s).*\Big] := \mathtt{EPairwiseDisjointElements}(A, \mathrm{Im}\, p \ \&\ \mathrm{Im}\, p')[6] : p_l h^{-1}(p'_k) p_s h^{-1}(p'_t) = 0;$

$\rightsquigarrow [4] := \mathtt{I}\forall\mathtt{I}^2 \Rightarrow : \forall k, l, t, s \in \mathbb{Z} \,.\, (n = k + l \ \&\ m = t + s) \Rightarrow p_l h^{-1}(p'_k) p_s h^{-1}(p'_t) = 0,$

$\Big[(n, m).*\Big] := \mathtt{E}p''\mathtt{EDistributiveLattice}(A)[4]\mathtt{E}\sup :$

$$: p''_n p''_m = \bigvee_{n=l+k} p_l h^{-1}(p'_k) \bigvee_{m=t+s} p_t h^{-1}(p'_s) = \bigvee_{n=l+k, m=t+s} p_l h^{-1}(p'_k) p_s h^{-1}(p'_t) = \bigvee_{n=l+k, m=t+s} 0 = 0;$$

$\rightsquigarrow [3] := \mathtt{IPairwiseDisjointElements} : \mathtt{PairwiseDisjointElements}(A, \mathrm{Im}\, p''),$

$\mathtt{Assume}\ a \in A,$

$\mathtt{Assume}\ [4] : a \neq 0,$

$\Big(n, [5]\Big) := \mathtt{EOrderDense}(A, \mathrm{Im}\, p)\Big(a, [4]\Big) : \sum n \in \mathbb{Z} \,.\, p_n a \neq 0,$

$\Big(m, [6]\Big) := \mathtt{EOrderDense}(A, h^{-1}\, \mathrm{Im}\, p')\Big(p_n a, [5]\Big) : \sum m \in \mathbb{Z} \,.\, h^{-1}(p'_m) p_n a \neq 0,$

$[7.*] := \mathtt{E}p''\mathtt{OrderContinuousMult}(A)\mathtt{E}\sup[6] :$

$$p''_{m+n} a = \left(\bigvee_{n+m=l+k} p_l h^{-1}(p'_k) \right) a = \bigvee_{n+m=l+k} p_l h^{-1}(p'_k) a \geq p_n g^{-1}(p'_m) a > 0;$$

$\rightsquigarrow [4] := \mathtt{IOrderDense} : \mathtt{OrderDense}(A, \mathrm{Im}\, p''),$

$[5] := \mathtt{PoUIffODAndDisjoint}[3][4] : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p''),$

$\mathtt{Assume}\ k, l \in \mathbb{Z}$,

$\mathtt{Assume}\ c \in \langle h^{-1}(p_k)p_l \rangle_{\mathcal{I}}$,

$[7] := \mathtt{PrincipleIdealExpression}(h^{-1}(p_k)p_l)\mathtt{LatticeJointIneq}(A, p_l, cp'_k) : c \leq h^{-1}(p'_k)p_l \leq p_l$,

$[8] := \mathtt{PrincipleIdealExpression}(h^{-1}(p_k)p_l)\mathtt{BooooleanMorphismIsMonotonic}(A, A, h)$
$\mathtt{LatticeJointIneq}(A, p_l, cp'_k) : h(c) \leq p'_k h(p_l) \leq p'_k$,

$\Big[(l, k). *\Big] := [2](k)[8][1](l)[9]\mathtt{ExpMult}\Big(\mathrm{End}_{\mathsf{BOOL}}(A)\Big) : hh'(c) = hg^k(c) = g^l g^l(c) = g^{l+k}(c)$;

$\leadsto [6] := \mathtt{I}\forall\mathtt{I}\forall : \forall k, l \in \mathbb{Z} \,.\, \forall c \in \langle h^{-1}(p_k)p_l \rangle_{\mathcal{I}} \,.\, hh'(c) = g^{l+k}(c)$,

$\mathtt{Assume}\ n : \mathbb{N}$,

$\mathtt{Assume}\ c : \langle p''_n \rangle_{\mathcal{I}}$,

$[7] := \mathtt{PrincipleIdealExpression}(A, p''_n, c) : c \leq p''_n$,

$[n.*] := \mathtt{EBooleanOrder}(A)[7]\mathtt{E}p''_n\mathtt{EOrderContinuous}(A, A, hh')\mathtt{MultiplicationIsOrderContinuous}(A)$
$[6]\mathtt{LatticeJoinIneqEOrderContinuous}(A, A, g^{k+l})\mathtt{MultiplicationIsOrderContinuous}(A)\mathtt{I}p''_n$
$\mathtt{EBooleanOrder}(A)[7]$:

$$: hh'(c) = hh'(cp''_n) = hh'\left(c\left(\bigvee_{k+l=n} h^{-1}(p'_k)p_l\right)\right) = \bigvee_{k+l=n} hh'\Big(ch^{-1}(p'_k)p_l\Big) = \bigvee_{k+l=n} g^{k+l}\Big(ch^{-1}(p'_k)p_l\Big) =$$

$$= g^{k+l}\left(c\bigvee_{k+l=n} h^{-1}(p'_k)p_l\right) = g^{k+l}(cp''_n) = g^{k+1}(c);$$

$\leadsto \Big[(h, h'). *\Big] := \mathtt{I}H : hh' \in H$;

$\leadsto [1] := \mathtt{I}\forall : \forall h, h' \in H \,.\, hh' \in H$,

$\mathtt{Assume}\ h \in H$,

$\Big(p, [2]\Big) := \mathtt{E}H(h) : \sum p : \mathbb{Z} \to A \,.\, \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p)\ \&\ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p_n \rangle_{\mathcal{I}} \,.\, h(b) = g^n(b)$,

$\mathtt{Assume}\ n : \mathbb{N}$,

$\mathtt{Assume}\ c : \Big\langle h(p_n) \Big\rangle$,

$[3] := \mathtt{PrincipleIdealStructru}(A, p_n, c)\mathtt{BooleanMorphismIsMonotonic}(A, A, h^{-1}) : h^{-1}(c) \leq p_n$,

$[4] := \mathtt{EInverse}[2](n)[3] : c = hh^{-1}(c) = g^n h^{-1}(c)$,

$[n.*] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A)(g^n)[4] : h^{-1}(c) = g^{-n}(c)$;

$\leadsto [h.*] := \mathtt{I}h : h^{-1} \in H$;

$\leadsto [2] := \mathtt{I}\forall : \forall h \in H \,.\, h^{-1} \in H$,

$[3] := \mathtt{I}\mathsf{GRP}[2][3] : H \in \mathsf{GRP}$,

$\mathtt{Assume}\ I : \mathtt{Countable}$,

$\mathtt{Assume}\ a : \mathtt{Injective}(I, A)$,

$\mathtt{Assume}\ h : I \to H$,

$\mathtt{Assume}\ [4] : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, a)$,

$\mathtt{Assume}\ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$,

$\mathtt{Assume}\ [5] : \forall b \in A \,.\, (\exists i \in I : b < a_i) \Rightarrow f(b) = h_i(b)$,

$\mathtt{Assume}\ i \in I$,

$\Big(p^i, [6]\Big) := \mathtt{E}H(h_i) : \sum p^i : \mathbb{Z} \to A \,.\, \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p^i)\ \&\ \forall n \in \mathbb{N} \,.\, \forall b \in \langle p^i_n \rangle_{\mathcal{I}} \,.\, h_i(b) = g^n(b)$,

$q_i := \Lambda n \in \mathbb{Z} \,.\, a_i p^i_n : \mathbb{Z} \to A$;

$\leadsto q := \mathtt{I}(\to) : I \to \mathbb{Z} \to A$,

$p := \Lambda n \in \mathbb{Z} \,.\, \bigvee_{i \in I} q_{i,n} : \mathbb{Z} \to A$,

$[6] := \mathtt{E}p\mathtt{EPartitionOfUnity}(A, \mathrm{Im}\, a) : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p)$,

$\texttt{Assume}\ n \in \mathbb{Z}$,

$\texttt{Assume}\ I \in I$,

$\texttt{Assume}\ c \in \langle p_n^i a_i \rangle_{\mathcal{I}}$,

$[7] := \texttt{PrincipleIdealStructue}(A, p_n^i a_i, c)\texttt{JoinIneq}(A, a_i, p_n^i) : c \leq p_n^i a_i \leq a_i$,

$[8] := \texttt{PrincipleIdealStructue}(A, p_n^i a_i, c)\texttt{JoinIneq}(A, p_n^i, a_i) : c \leq p_n^i a_i \leq p_n^i$,

$[n.*] := [5](c)[7] \ldots : f(c) = h_i(c) = g^n(c)$;

$\rightsquigarrow [7] := \texttt{I}^3\forall : \forall n \in \mathbb{Z} \,.\, \forall i \in I \,.\, \forall c \in \langle p_n^i a_i \rangle_{\mathcal{I}} \,.\, g(c) = g^n(c)$,

$\texttt{Assume}\ n \in \mathbb{Z}$,

$\texttt{Assume}\ c \in \langle p_n \rangle_{\mathcal{I}}$,

$[n.*] := \texttt{EBooleanOrder}(A)\texttt{E}c\texttt{E}p_n\texttt{EOrderContinuous}(A, A, f')\texttt{MultiplicationIsOrderContinuous}(A)$
$[7]\texttt{LatticeJoinIneqEOrderContinuous}(A, A, g^n)\texttt{MultiplicationIsOrderContinuous}(A)\texttt{I}p_n$

$$: \texttt{EBooleanOrder}(A)\texttt{E}c : f(c) = f(cp_n) = f\left(c \bigvee_{i \in I} p_n^i a_i\right) = \bigvee_{i \in I} f\left(cp_n^i a_i\right) = \bigvee_{i \in I} f\left(cp_n^i a_i\right) = \bigvee_{i \in I} g^n\left(cp_n^i a_i\right) =$$

$$= g^n\left(c \bigvee_{i \in I} p_n^i a_i\right) = g^n(cp_n) = g^n(c);$$

$\rightsquigarrow [I.*] := \texttt{I}H : f \in H$;

$\rightsquigarrow [4] := \texttt{ICountablyFullSubgroup} : \texttt{CountablyFullSubgroup}(A, H)$,

$[5] := \texttt{I}\ \langle g \rangle_{\mathrm{CF}}\ [4] : \langle g \rangle_{\mathrm{CF}} \subset H$,

$\texttt{Assume}\ G : \texttt{CountablyFullSubgroup}(A)$,

$\texttt{Assume}\ [6] : g \in G$,

$\texttt{Assume}\ h \in H$,

$\left(p^i, [7]\right) := \texttt{E}H(h) : \sum p : \mathbb{Z} \to A \,.\, \texttt{PartitionOfUnity}(A, \operatorname{Im} p) \,\&\, \forall n \in \mathbb{N} \,.\, \forall b \in \langle p_n \rangle_{\mathcal{I}} \,.\, h(b) = g^n(b)$,

$[G.*] := \texttt{ECountablyFullSubgroup}(A, G)[6][7] : h \in G$;

$\rightsquigarrow [6] := \texttt{I}\ \langle g \rangle_{\mathrm{CF}} : H \subset \langle g \rangle_{\mathrm{CF}}$,

$[*] := \texttt{ISetEq}[5][6] : H = \langle g \rangle_{\mathrm{CF}}$;

$\square$

$\texttt{FullSubgroupGeneratedByGroupElement} :: \forall A \in \mathsf{BOOL} \,.\, \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.$

$$.\ \langle g \rangle_{\mathrm{F}} = \left\{ f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \forall a \in A \setminus \{0\} \,.\, \exists b : \langle a \rangle_{\mathcal{I}} \setminus \{0\} : \exists n \in \mathbb{Z} : \forall c \in \langle a \rangle_{\mathcal{I}} \,.\, f(c) = g^n(c) \right\}$$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{CompleteAlgebraFullGroupIndifference} :: \forall A : \tau\text{-}\texttt{Algebra} . \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) . \langle g\rangle_{\mathrm{F}} = \langle g\rangle_{\mathrm{CF}}$

$\texttt{Proof} =$

$[1] := \mathtt{E}\langle g\rangle_{\mathrm{F}}\,\mathtt{E}\langle g\rangle_{\mathrm{CF}}\,\mathtt{I}\subset : \langle g\rangle_{\mathrm{CF}} \subset \langle g\rangle_{\mathrm{F}},$

$\texttt{Assume } f \in \langle g\rangle_{\mathrm{F}},$

$D := \Big\{a \in A : \exists n \in \mathbb{Z} : \forall b \in \langle a\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} :?A,$

$[2] := \texttt{FullSubgroupGeneratedByGroupElement}(A, g, f)\mathtt{E}D\mathtt{I}\texttt{OrderDense} : \texttt{OrderDense}(A, D),$

$\Big(P, [3]\Big) := \texttt{OrderDenseContainsPartitionOfUnity} : \sum P : \texttt{PartitionOfUnity}(A) . P \subset D,$

$p := \Lambda n \in \mathbb{Z} . \bigvee \Big\{q \in P : \forall b \in \langle q\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} : \mathbb{Z} \to A,$

$[4] := \mathtt{E}\texttt{PartitionOfUnity}(A, P)\mathtt{E}p\texttt{MultIsOrderC}(A)\mathtt{I}\texttt{PartitionOfUnity} : \texttt{PartitionOfUnity}(A, \mathrm{Im}\, p),$

$[5] := \Lambda n \in \mathbb{Z} . \mathtt{E}p\mathtt{E}\texttt{OrderContinuous}(A, A, f \ \&\ g^n) : \forall n \in \mathbb{N} . \forall b \in \langle p_n\rangle_{\mathcal{I}} . f(b) = g^n(b),$

$[f.*] := \texttt{CountablyFullSubgroupGeneratedByGroupElement}(A)[4][5] : f \in \langle g\rangle_{\mathrm{CF}};$

$\rightsquigarrow [2] := \mathtt{I}\subset : \langle g\rangle_{\mathrm{F}} \subset \langle g\rangle_{\mathrm{CF}},$

$[*] := \mathtt{ISetEq}[1][2] : \langle g\rangle_{\mathrm{F}} = \langle g\rangle_{\mathrm{CF}};$

$\square$

$\texttt{CompleteAlgebraElementsFullGroupSuppExpression} ::$

$$:: \forall A : \tau\text{-}\texttt{Algebra} . \forall g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) . \langle g\rangle_{\mathrm{F}} = \left\{f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \bigwedge_{n\in\mathbb{Z}} \mathrm{supp}\, f g^n = 0\right\}$$

$\texttt{Proof} =$

$$H := \left\{f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : \bigwedge_{n\in\mathbb{Z}} \mathrm{supp}\, f g^n = 0\right\} :?\mathrm{Aut}_{\mathsf{BOOL}}(A),$$

$\texttt{Assume } f \in \langle g\rangle_{\mathrm{F}},$

$[1] := \mathtt{E}\,\mathrm{supp}\,\mathtt{I}\texttt{inverse}(\mathbb{Z})\mathtt{E}\mathbb{Z}\texttt{SupInverse}(A)\mathtt{E}\complement\texttt{FullSubgroupGeneratedByElement}(A, g, f)$

$\texttt{DensitySupTHM}(A, e)\mathtt{E}\complement :$

$$: \bigwedge_{n\in\mathbb{Z}} \mathrm{supp}\, f g^n = \bigwedge\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a^{\complement}\rangle_{\mathcal{I}} . f g^n(b) = b\} =$$

$$\bigwedge\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a^{\complement}\rangle_{\mathcal{I}} . f(b) = g^{-n}(b)\} = \bigwedge\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a^{\complement}\rangle_{\mathcal{I}} . f(b) = g^n(b)\} =$$

$$\neg\bigvee\Big\{a^{\complement} | a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a^{\complement}\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} = \neg\bigvee\Big\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} =$$

$$= e^{\complement} = 0,$$

$[2] := \mathtt{E}H[1] : f \subset H;$

$\rightsquigarrow [1] := \mathtt{I}\subset : \langle g\rangle_{\mathrm{F}} \subset H,$

$\texttt{Assume } f : H,$

$X := \Big\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} :?A,$

$[2] := \mathtt{E}H(f)\mathtt{I}X : e = \bigvee\Big\{a \in A : \exists n \in \mathbb{N} . \forall b \in \langle a\rangle_{\mathcal{I}} . f(b) = g^n(b)\Big\} = \sup X,$

$\texttt{Assume } a \in A,$

$\texttt{Assume } [3] : a \neq 0,$

$\Big(x, n, [4]\Big) := \mathtt{E}\sup[2] : \sum x \in X . \sum n \in \mathbb{Z} . ax \neq 0 \ \&\ \forall b \in \langle x\rangle . f(b) = g^n(b),$

$[a.*] := \texttt{MeetIneq}(A)[4]\mathtt{E}x : \forall b \in \langle ax\rangle_{\mathcal{I}} . f(b) = g^n(b);$

$\rightsquigarrow [f.*] := \texttt{FullSubgroupGeneratedByElement}(A, g, f) : f \in \langle g\rangle_{\mathrm{F}};$

$\rightsquigarrow [*] := \mathtt{I}\subset \mathtt{ISetEq}[1] : H = \langle g\rangle_{\mathrm{F}};$

$\square$

ElementsFullSubgroupFixedPoint $:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall \phi \in \langle f \rangle_{\mathrm{F}} \,.\, \forall a \in A \,.$
$\quad . \, f(a) = a \Rightarrow \phi(a) = a$

Proof $=$

$G := \Big\{ g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) : g(a) = a \Big\} : \mathtt{Subgroup}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big),$

Assume $I \in \mathtt{Set}$,

Assume $p : \mathtt{Injective}(I, A)$,

Assume $[1] : \mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p)$,

Assume $g : \mathtt{Injective}(I, G)$,

Assume $f \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$,

Assume $[2] : \forall b \in A \,.\, \forall i \in I \,.\, b \leq a_i \Rightarrow f(b) = g_i(b)$,

$[3] := \mathtt{E}\mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p)\mathtt{E}\mathtt{OrderContinuous}(A, A, f)\mathtt{OrderContinuousMult}(A, I, p, a)[2]$
$\quad \Lambda i \in I \,.\, \mathtt{E}\mathsf{BOOL}(A, A, g_i)\mathtt{E}G(g_i)\mathtt{OrderContinuousMult}(A, I, p, a)[2]\mathtt{E}\mathtt{OrderContinuous}(A, A, f)$
$\quad \mathtt{E}\mathtt{PartitionOfUnity}(A, \mathrm{Im}\, p)\mathtt{E}\mathsf{BOOL}(A, A, f)\mathtt{E}\mathsf{RING}(A) :$

$$: f(a) = f\left(a \bigvee_{i \in I} p_i\right) = \bigvee_{i \in I} f(ap_i) = \bigvee_{i \in I} g_i(ap_i) = a \bigvee_{i \in I} g_i(p_i) = a \bigvee_{i \in I} f(p_i) = af\left(\bigvee_{i \in I} p_i\right) = af(e) = ae = a,$$

$[I.*] := \mathtt{E}G[3] : f \in G;$

$\rightsquigarrow [1] := \mathtt{I}\mathtt{FullSubgroup} : \mathtt{FullSubgroup} A, G,$

$[2] := \mathtt{E}\, \langle g \rangle_{\mathrm{F}}\, [0][1] : \langle g \rangle_{\mathrm{F}} \subset G,$

$[*] := \mathtt{E}G[2] : \forall \phi \in \langle G \rangle_{\mathrm{F}} \,.\, \phi(a) = a;$

$\square$

ElementsFullSubgroupSupport $:: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall g \in \langle f \rangle_{\mathrm{F}} \,.\, \mathrm{Supp}(f) \subset \mathrm{Supp}(g)$

Proof $=$

Assume $a \in \mathrm{Supp}(f)$,

Assume $b \in \langle a^{\complement} \rangle_{\mathcal{I}}$,

$[1] := \mathtt{E}\,\mathrm{Supp}(f, a, b) : f(b) = b,$

$[b.*] := \mathtt{ElementsFullSubgroupFixedPoint}[1] : g(b) = b;$

$\rightsquigarrow [a.*] := \mathtt{I}\,\mathrm{Supp} : a \in \mathrm{Supp}(g);$

$\rightsquigarrow [*] := \mathtt{I} \subset : \mathrm{Supp}(f) \subset \mathrm{Supp}(g);$

$\square$

1.7.5 Recurrence

$\texttt{limsup} :: \prod A : \tau\text{-}\texttt{Algebra}.(\mathbb{N} \to A) \to A$

$\texttt{limsup}\,(a) = \limsup_{n=1} a_n := \inf_{n=1} \sup_{k\geq n} a_k$

$\texttt{LimsupIsAFixedPoint} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall f : \texttt{OrderContinuous}(A, A) \,\&\, \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.$

$\quad . \; f\left(\limsup_{n=1} f^n(a)\right) = \limsup_{n=1} f^n(a)$

$\texttt{Proof} \;=$

$[*] := \texttt{E}\limsup \texttt{EOrderContinuous}(A, A, f)\texttt{I2}\texttt{SupIsMonotonic}(A)\texttt{E}\inf :$

$\quad : f\left(\limsup_{n=1} f^n(a)\right) = f\left(\inf_{n=1}\sup_{k\geq n} f^k(a)\right) = \inf_{n=1}\sup_{k\geq n} f^{k+1}(a) = \inf_{n=2}\sup_{k\geq n} f^k(a) = \inf_{n=1}\sup_{k\geq n} f^k(a) = \limsup_{n=1} f^n(a);$

$\square$

$\texttt{IteratedSupLemma} :: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall f : \texttt{OrderContinuous}(A, A) \,\&\, \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.$

$\quad a \leq \sup_{n=1} f^n(a) \Rightarrow \forall k \in \mathbb{N} \,.\, \sup_{n=k} f^n(a) = \sup_{n=1} f^n(a)$

$\texttt{Proof} \;=$

$\nearrow := \Lambda m \in \mathbb{N} \,.\, \left(\forall k \in [1, \dots, m] \,.\, \sup_{n=k} f^n(a) = \sup_{n=1} f^n(a)\right) : \mathbb{N} \to \texttt{Type},$

$[1] := \texttt{I}(=, A, \sup_{n=1} f^n(a)) : \sup_{n=1} f^n(a) = \sup_{n=1} f^n(a),$

$[2] := \texttt{I}\nearrow[1] : \nearrow(1),$

$[00] := f[0]\texttt{EOrderContinuous}(A, A, f) : f(a) \leq \sup_{n=2} f^n(a),$

$\texttt{Assume}\; m \in \mathbb{N},$

$\texttt{Assume}\; [2] : \nearrow(m),$

$[m.*] := \Lambda n \in [m+1, \dots, \infty) \,.\, \texttt{I}n + 1\texttt{EOrderContinuous}(A, A, f)\texttt{E}\nearrow[2]\texttt{EOrderContinuous}(A, A, f)[00] :$

$\quad : \sup_{n=m+1} f^n(a) = \sup_{n=m} f^{n+1}(a) = f\left(\sup_{n=m} f^n(a)\right) = f\left(\sup_{n=1} f^n(a)\right) = \sup_{n=2} f^n(a) = \sup_{n=1} f^n(a);$

$\rightsquigarrow [2] := \texttt{I} \Rightarrow \texttt{I}\forall : \forall m \in \mathbb{N} \,.\, \nearrow(m) \Rightarrow \nearrow(m+1),$

$[*] := \texttt{E}\mathbb{N}[1][2] : \forall m \in \mathbb{N} \,.\, \nearrow(m);$

$\square$

$\texttt{RecurrentOn} :: \prod_{A \in \mathsf{BOOL}} A \to ?\mathrm{End}_{\mathsf{BOOL}}(A)$

$f : \texttt{RecurrentOn} \iff \Lambda a \in A \,.\, \forall b \in \langle a \rangle \,.\, b \neq 0 \Rightarrow \exists k \in \mathbb{N} \,.\, a f^k(b) \neq 0$

$\texttt{DoublyRecurrentOn} :: \prod_{A \in \mathsf{BOOL}} A \to ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$

$f : \texttt{DoublyRecurrentOn} \iff \Lambda a \in A \,.\, \texttt{RecurrentOn}(A, a, f \,\&\, f^{-1})$

$\texttt{RecurrenceOnImpliesBound} :: \forall A : \tau\texttt{-Algebra} . \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) . \forall a \in A .$
$\quad . \texttt{RecurrentOn}(A, a, f) \Rightarrow \forall m \in \mathbb{N} . a \leq \sup_{n=m} f^{-n}(a)$

$\texttt{Proof} =$

$b := a \setminus \sup_{n=m} f^{-n}(a) :? A,$

$\texttt{Assume } [1] \in b \neq 0,$

$[2] := \texttt{E}b\texttt{SetminusIneq} : b \leq a,$

$\left(k, [3]\right) := \texttt{ERecurrentOn}(A, a, f)[1][2] : \sum k \in \mathbb{N} . f^k(b)a \neq 0,$

$[4] := \texttt{E}b\texttt{EInverseE}\setminus : af^k(b) = af^k(a) \setminus_{n=1} f^{n-k}(a) = 0,$

$[1.*] := [3][4] : \bot;$

$\rightsquigarrow [1] := \texttt{E}\bot : b = 0,$

$[2] := \texttt{E} \setminus \texttt{E}b[1] : a \leq \sup_{n=1} f^{-n}(a),$

$[*] := [2]\texttt{IteratedSupLemma}(A, a, f^{-1}) : \forall m \in \mathbb{N} . a \leq \sup_{m=1} f^{-n}(a);$

$\square$

$\texttt{RecurrenceOnImpliesBound} :: \forall A : \tau\texttt{-Algebra} . \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) . \forall a \in A .$
$\quad . a \leq \sup_{n=1} f^{-n}(a) \Rightarrow \texttt{RecurrentOn}(A, a, f)$

$\texttt{Proof} =$

$\texttt{Assume } b \in A,$

$\texttt{Assume } [1] : b \leq a,$

$\texttt{Assume } [2] : b \neq 0,$

$[3] := [1][0] : b \leq \sup_{n=1} f^{-n}(a),$

$[4] := \texttt{EBooleanOrder}(A)\texttt{oCMult}(A) : b = b \sup_{n=1} f^{-n}(a) = \sup_{n=1} f^{-n}(a),$

$\left(n, [5]\right) := \texttt{E}\sup[4] : \sum_{n=1}^{\infty} f^{-n}(a)b \neq 0,$

$[b.*] := f^n[5] : af^n(b) \neq 0;$

$\rightsquigarrow [*] := \texttt{IRecurrentOn} : \texttt{RecurrentOn}(A, a, f);$

$\square$

$\texttt{DoubleRecurrenceSymmetry} :: \forall A \in \mathsf{BOOL} . \forall a \in A . \forall f : \texttt{DoublyRecurrentOn}(A, a) .$
$\quad . \texttt{DoublyRecurrentOn}\left(A, a, f^{-1}\right)$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{inducedAutomorphism} :: \prod A : \tau\texttt{-Algebra} . \forall a \in A . \texttt{DounblyRecurrentOn}(A, a) \to \mathrm{Aut}_{\mathsf{BOOL}}\langle a \rangle_{\mathcal{I}}$

$\texttt{inducedAutomorphism}\,(f) = f_a := \Lambda b \in \langle a \rangle_{\mathcal{I}} . f^n(b) \texttt{ where } b \le af^{-n}(a) \setminus \sup_{1 \le k < n} f^{-k}(a)$

$p := \Lambda n \in \mathbb{N} . af^{-n}(a) \setminus \sup_{1 \le k < n} f^{-k}(a) : \mathbb{N} \to A,$

$q := \Lambda n \in \mathbb{N} . af^{n}(a) \setminus \sup_{1 \le k < n} f^{k}(a) : \mathbb{N} \to A,$

$[1] := \texttt{E}p\texttt{IPairwiseDisjointElements} : \texttt{PairwiseDisjointElements}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big),$

$\texttt{Assume } b \in \langle a \rangle_{\mathcal{I}},$

$N := \{n \in \mathbb{N} . af^n(b) \ne 0\} :?\mathbb{N},$

$[2] := \texttt{EDoublyRecurrentOn}(A, a, f)\texttt{E}N : N \ne \emptyset,$

$n := \min N \in \mathbb{N},$

$[3] := \texttt{E}n : af^n(b) \ne 0,$

$[4] := \Lambda k \in [1, \dots, n-1] . \texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f^{-k})\texttt{E}n\texttt{E} \min \texttt{E}N\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f^{-k}) :$

$: \forall k \in [1, \dots, n-1] . bf^{-k}(a) = f^{-k}(f^k(b)a) = f^{-k}(0) = 0,$

$[b.*] := \texttt{E}p_n\texttt{EBooleanOrder}(A, a, b)\texttt{MeetDifference}(A)[4]\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f^{-k})[3]\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f^{-k}) :$

$: bp_n = baf^{-n}(a) \setminus \sup_{1 \le k < n} f^{-k}(a) = bf^{-n}(a) \setminus \sup_{1 \le k < n} bf^{-k}(a) = bf^{-n}(a) = f^{-n}\Big(bf^n(a)\Big) \ne 0;$

$\rightsquigarrow [2] := \texttt{IPartitionOfUnity}[1] : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big),$

$[3] := \texttt{E}q\texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, q\Big),$

$\texttt{Assume } n \in \mathbb{N},$

$[n.*] :=: f^n(p_n) = f^n\left(af^{-n}(a) \setminus \sup_{1 \le k < n} f^{-k}(a)\right) = f^n(a)a \setminus_{1 \le k < n} f^k(a) = q_n;$

$\rightsquigarrow [4] := \texttt{I}\forall : \forall n \in \mathbb{N} . f^n(p_n) = q_n,$

$f_a := \Lambda \bigvee_{n=1}^{\infty} b_i p_i \in \langle a \rangle_{\mathcal{I}} . \bigvee_{n=1}^{\infty} f^n(b_i) q_i : \langle a \rangle_{\mathcal{I}} \to \langle a \rangle_{\mathcal{I}},$

$[5] := \texttt{EPartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big)\texttt{EPartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, q\Big) : f_a \in \mathrm{Aut}_{\mathsf{BOOL}}\langle a \rangle_{\mathcal{I}},$

$\square$

$\texttt{InducedHomomorphismInverse} :: \forall A : \tau\texttt{-Algebra} . \forall a \in A . \forall f : \texttt{DoublyRecurrentOn}(A, a) . (f_a)^{-1} = (f^{-1})_a$

$\texttt{Proof } =$

$\dots$

$\square$

$\texttt{InducedHomomorphismPoU} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall n \in \mathbb{Z}_+ \,.$

$. \exists p : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big) \,.\, \forall k \in \mathbb{Z}_+ \,.\, \forall c \in \langle p_k \rangle_{\mathcal{I}} \,.\, f_a^n(c) = f^{n+k}(c)$

$\texttt{Proof} \;=$

$\Big(p, q, [1]\Big) := \mathtt{E} f_a : \sum p, q : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} \,.\, \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big) \;\&\; \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, q\Big) \;\&$

$\& \Big(\forall n \in \mathbb{N} \,.\, f^n(p_n) = q_n\Big) \;\&\; \forall \bigvee_{n=1}^{\infty} b_n p_n \in \langle a \rangle_{\mathcal{I}} \,.\, f_a\left(\bigvee_{n=1}^{\infty} b_n p_n\right) = \bigvee_{n=1}^{\infty} f^n(b_n) q_n,$

$w_0 := \Lambda k \in \mathbb{Z}_0 \,.\, \texttt{if } k == 0 \texttt{ then } a \texttt{ else } 0 : \mathbb{Z}_+ \to \langle a \rangle_{\mathcal{I}},$

$[2] := \mathtt{E} w_0 : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, w_0\Big),$

$[3] := \Lambda k \in \mathbb{Z}_+ \,.\, \Lambda c \in \langle w_{0,k} \rangle_{\mathcal{I}} \texttt{ZeroExponentTHM}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), f\Big) \mathtt{E}\, \mathrm{id}\, \mathtt{E} w_0(x) :$

$: \forall k \in \mathbb{Z}_0 \,.\, \forall c \in \langle w_{0,k} \rangle_{\mathcal{I}} \,.\, f_a^0(c) = \mathrm{id}(c) = c = f^k(c),$

$\texttt{Assume } n : \mathbb{Z}_+,$

$\texttt{Assume } w_n : \mathbb{Z}_+ \to \langle a \rangle_{\mathcal{I}},$

$\texttt{Assume } [4] : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, w_n\Big),$

$\texttt{Assume } [5] : \forall k \in \mathbb{Z}_+ \forall c \in \langle w_{n,k} \rangle_{\mathcal{I}} \,.\, f_a^n(c) = f^{n+k}(c),$

$b := \Lambda k, t \in \mathbb{Z}_+ \,.\, f_a^{-1}(w_{n,k}) p_t : \mathbb{Z}_+^2 \to \langle a \rangle_{\mathcal{I}},$

$w_{n+1} := \Lambda k \in \mathbb{Z}_+ \,.\, \bigvee_{k+1=t+s} b_{t,s} : \mathbb{Z}_+ \to \langle a \rangle_{\mathcal{I}},$

$[n.*.1] := \mathtt{E}\texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, w_n\Big) \mathtt{E}\texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big)$

$: \mathtt{E} w_{n+1} \mathtt{I}\texttt{PartitionOfUnity}\mathtt{I} w_{n+1} : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, w_{n+1}\Big),$

$\texttt{Assume } k \in \mathbb{Z}_+,$

$\texttt{Assume } c \in \langle w_{n+1,k} \rangle_{\mathcal{I}},$

$[n.*.2] := \mathtt{E}\texttt{BooleanOrder}(w_{n+1,k}, c) \mathtt{E} w_{n+1,k} \mathtt{E} b \mathtt{E}\texttt{OrderContinuous}\Big(\langle a \rangle_{\mathcal{I}}, f_a\Big) \mathtt{E}\texttt{Inverse}\Big(\mathrm{Aut}_{\mathsf{BOOL}} \langle a \rangle_{\mathcal{I}}, f_a\Big)$

$[1][3] \mathtt{E}\texttt{OrderContinuous}\Big(A, f\Big) [1]^3 \mathtt{I} b \mathtt{I} w_{n+1,k} \mathtt{E}\texttt{BooleanOrder}(w_{n+1,k}, c) :$

$: f_a^{n+1}(c) = f_a^{n+1}(c w_{n+1,k}) = f_a^{n+1}\left(c \bigvee_{k+1=t+s} b_{t,s}\right) = f_a f_a^n \left(c \bigvee_{k+1=t+s} f_a^{-1}(w_{n,s}) p_t\right) =$

$= \bigvee_{k+1=t+s} f_a f_a^n \Big(c f_a^{-1}(w_{n,s}) p_t\Big) = \bigvee_{k+1=t+s} f_a^n(w_{n,s} f_a(p_t c)) = \bigvee_{k+1=t+s} f^{n+s}\Big(w_{n,s} f^t(c p_t)\Big) =$

$= f^{n+k+1}\left(c \bigvee_{k+1=t+s} f^{-t}(w_{n,s}) p_t\right) = f^{n+k+1}\left(c \bigvee_{k+1=t+s} f^{-t}(w_{n,s} q_t)\right) = f^{n+k+1}\left(c \bigvee_{k+1=t+s} f_a^{-1}(w_{n,s} q_t)\right) =$

$= f^{n+k+1}\left(c \bigvee_{k+1=t+s} f_a^{-1}(w_{n,s}) p_t)\right) = f^{n+k+1}\left(c \bigvee_{k+1=t+s} b_{t,s}\right) = f^{n+k+1}(c w_{n+1,k}) = f^{n+k+1}(c);$

$\rightsquigarrow [*] := \mathtt{E}\mathbb{N} : \forall n \in \mathbb{Z}_+ \,.\, \exists p : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} : \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big) \,.\, \forall k \in \mathbb{Z}_+ \,.\, \forall c \in \langle p_k \rangle_{\mathcal{I}} \,.$

$. f_a^n(c) = f^{n+k}(c);$

$\square$

$\texttt{InducedHomomorphismRecurrence} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall n \in \mathbb{N} \,.$

$\quad . \,\forall b \in \left\langle af^{-n}(a) \right\rangle_{\mathcal{I}} \,.\, \exists b' \in \langle b \rangle_{\mathcal{I}} : \exists k \in [0, \ldots, n] : \forall c \in \langle b' \rangle_{\mathcal{I}} \,.\, f^n(c) = f_a^k(c)$

$\texttt{Proof} \;=$

$\nearrow := \Lambda n \in \mathbb{N} \,.\, \forall m \in [1, \ldots, n] \,.\, \forall b \in \left\langle af^{-n}(a) \right\rangle_{\mathcal{I}} \,.\, \exists b' \in \langle b \rangle_{\mathcal{I}} : \exists k \in [0, \ldots, n] : \forall c \in \langle b' \rangle_{\mathcal{I}} \,.\, f^n(c) = f_a^k(c) :$

$\quad : \mathbb{N} \to \texttt{Type},$

$\Big(p, q, [1]\Big) := \texttt{E} f_a : \sum p, q : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} \,.\, \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} p\Big) \;\&\; \texttt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} q\Big) \;\&$

$\quad \& \left(\forall n \in \mathbb{N} \,.\, f^n(p_n) = q_n\right) \;\&\; \forall \bigvee_{n=1}^{\infty} b_n p_n \in \langle a \rangle_{\mathcal{I}} \,.\, f_a\left(\bigvee_{n=1}^{\infty} b_n p_n\right) = \bigvee_{n=1}^{\infty} f^n(b_n) q_n,$

$[2] := \Lambda c \in \left\langle af^{-1}(a) \right\rangle_{\mathcal{I}} \,.\, [1](1, c) : \forall c \in \langle b \rangle_{\mathcal{I}} \,.\, f(c) = f_a(c),$

$[3] := \texttt{I}\nearrow[2] : \nearrow(1),$

$\texttt{Assume}\; n : \mathbb{N},$

$\texttt{Assume}\; [4] : \nearrow(n),$

$\texttt{Assume}\; b \in \left\langle af^{-n-1}(a) \right\rangle_{\mathcal{I}},$

$\Big(v, [5]\Big) := \texttt{ByConstruction}\Big(p, [1]\Big) : \sum v : \{1, \ldots, n+1\} \to \left\langle af^{-n}(a) \right\rangle_{\mathcal{I}} \,.\, b = \bigvee_{i=1}^{n} v_i p_i,$

$\texttt{Assume}\; 6 : b \neq 0,$

$[7] := [5][6] : \Big\{k \in \{1, \ldots, n+1\} : v_k \neq 0\Big\} \neq \emptyset,$

$k := \min \Big\{k \in \{1, \ldots, n+1\} : v_k \neq 0\Big\} \in \{1, \ldots, n+1\},$

$b' := v_k p_k : \langle b p_k \rangle_{\mathcal{I}},$

$[8] := \texttt{E} b' \texttt{E} f_a : \forall c \in \langle b' \rangle_{\mathcal{I}} \,.\, f_a(c) = f^k(c),$

$[9] := \texttt{ExponentMult}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), f, k, n-k\Big) \texttt{E} b' \texttt{MonotonicBooleanMorphism}(A, A, f) \texttt{E} b :$

$\quad : f^k f^{n-k+1}(b') = f^{n+1}(b') \leq f^n(b) \leq a,$

$\Big(c, l, [10]\Big) := \texttt{E}\nearrow[4][9](n+1-k) :$

$\quad : \sum c \in \left\langle f^k(b') \right\rangle_{\mathcal{I}} \,.\, \sum l \in \{1, \ldots, n+k-k\} \,.\, \forall d \in \langle c \rangle_{\mathcal{I}} \,.\, f^{n-1-k}(d) = f_a^k(d),$

$b'' := f^k(c) \in \langle b \rangle_{\mathcal{I}},$

$[n.*] := \Lambda d \in \langle b'' \rangle_{\mathcal{I}} \texttt{ExponentMult}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), f, k, n-k\Big)[10][8] : f^{n+1}(d) = f^k f^{n+1-k}(d) f_a f_a^l = f_a^{l+1}(d);$

$\rightsquigarrow [*] := \texttt{E}\mathbb{N}[3]\texttt{E}\nearrow : \forall n \in \mathbb{N} \,.\, \forall b \in \left\langle af^{-n}(a) \right\rangle_{\mathcal{I}} \,.\, \exists b' \in \langle b \rangle_{\mathcal{I}} : \exists k \in [0, \ldots, n] : \forall c \in \langle b' \rangle_{\mathcal{I}} \,.\, f^n(c) = f_a^k(c);$

$\square$

$\texttt{LateRecurrenceDivision} :: \forall A : \tau\texttt{-Algebra} . \forall a \in A . \forall f : \texttt{DoublyRecurrentOn}(A, a) . \forall m, n \in \mathbb{Z}_+ .$

$$. \left(\forall k \in \{1, \dots, m-1\} . af^k(a) = 0\right) \Rightarrow \exists d : \left\{1, \dots, \left\lfloor \frac{n}{m} \right\rfloor\right\} \to A :$$

$$: \texttt{PairwiseDisjointElements}(A, \operatorname{Im} d) \ \& \ \sup \operatorname{Im} d = af^{-n}(a) \ \&$$

$$\& \ \forall k \in \left\{1, \dots, \left\lfloor \frac{n}{m} \right\rfloor\right\} . \forall c \in \langle d_k \rangle_{\mathcal{I}} . f^n(c) = f_a^k(c)$$

$\texttt{Proof} =$

$$\nearrow := \Lambda n \in \mathbb{N} . \forall t \in [1, \dots, t] . \exists d : \left\{1, \dots, \left\lfloor \frac{t}{m} \right\rfloor\right\} \to A : \texttt{PairwiseDisjointElements}(A, \operatorname{Im} d) \ \&$$

$$\& \ \sup \operatorname{Im} d = af^{-t}(a) \ \& \ \forall k \in \left\{1, \dots, \left\lfloor \frac{t}{m} \right\rfloor\right\} . \forall c \in \langle d_k \rangle_{\mathcal{I}} . f^t(c) = f_a^k(c) : \mathbb{N} \to \texttt{Type},$$

$[1] := [0]\texttt{E}\nearrow : \forall t \in [1, \dots, m-1] . \nearrow(t),$

$\texttt{Assume } [2] : m = n,$

$d := af^{-n}a \in A,$

$\texttt{Assume } c \in \langle d \rangle_{\mathcal{I}},$

$[3] := \Lambda k \in \{1, \dots, n-1\} . f\big([0]\big)[2] : \forall k \in \{1, \dots, n-1\} . af^{-k}(a) = 0,$

$[2.*] := \texttt{E}f_a[3]\texttt{E}d : f^n(c) = f_a(c);$

$\leadsto [2] := \texttt{I}\nearrow : \nearrow(m),$

$$\big(p, q, [3]\big) := \texttt{E}f_a : \sum p, q : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} . \texttt{PartitionOfUnity}\big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} p\big) \ \& \ \texttt{PartitionOfUnity}\big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} q\big) \ \&$$

$$\& \ \big(\forall n \in \mathbb{N} . f^n(p_n) = q_n\big) \ \& \ \forall \bigvee_{n=1}^{\infty} b_n p_n \in \langle a \rangle_{\mathcal{I}} . f_a\left(\bigvee_{n=1}^{\infty} b_n p_n\right) = \bigvee_{n=1}^{\infty} f^n(b_n) q_n,$$

$[4] := \texttt{I}\nearrow[2] : \nearrow(1),$

$\texttt{Assume } n, s \in \mathbb{N},$

$\texttt{Assume } [5] : \nearrow(n),$

$\texttt{Assume } [6] : n + 1 = ms,$

$$t := \Lambda i \in \{m, \dots, n\} . \left\lfloor \frac{i}{m} \right\rfloor : \{m, \dots, n\} \to \{1, \dots, s-1\},$$

$$\big(d, [7]\big) := \texttt{E}\nearrow[4] : \sum d : \{m, \dots, n\} \times \{1, \dots, s-1\} \to A . \forall i \in \{m, \dots, n\} .$$

$$. \texttt{PartitionOfUnity}\Big(\big\langle af^{-1}(a) \big\rangle_{\mathcal{I}}, \operatorname{Im} d_i\Big) \ \& \ \forall j \in \{1, \dots, t_i\} . \forall c \in \langle d_{i,j} \rangle_{\mathcal{I}} . f^i(c) = f_a^j(c),$$

$$[8] := \texttt{I}p_n\texttt{E}\mathsf{BOOL}\big(A, f^{-i}(a)\big)\Lambda i \in \{m, \dots, n\}\texttt{EPartitionOfUnity}\Big(\big\langle af^{-i}(a) \big\rangle_{\mathcal{I}}, \operatorname{Im} d_i\Big)$$

$\texttt{EDistiributiveLattice}(A) :$

$$: af^{-n-1}(a) = p_{n+1} \vee \bigvee_{i=m}^{n} p_i f^{-n-1}(a) = p_{n+1} \vee \bigvee_{i=m}^{n} p_i f^{-i}\big(af^{i-n-1}(a)\big) = p_{n+1} \vee \bigvee_{i=m}^{n} p_i f^{-i}\left(\bigvee_{j=1}^{t_i} d_{n+1-i,j}\right) =$$

$$= p_{n+1} \vee \bigvee_{i=m}^{n} \bigvee_{j=1}^{t_i} p_i f^{-i}(d_{n+1-i,j}),$$

$$d_{n+1} := \Lambda i \in \{1, \dots, s\} . \texttt{if } i == 1 \texttt{ then } p_{n+1} \texttt{ else } \bigvee_{i=m}^{n+1-m} p_i f^{-i}(d_{n+1-i,j-1}) : \{1, \dots, s\} \to A,$$

$$[9] := \texttt{E}d_{n+1}[8][7] : \texttt{PartitionOfUnity}\Big(\big\langle af^{-n-1}(a) \big\rangle_{\mathcal{I}}, \operatorname{Im} d_{n+1}\Big),$$

$\texttt{Assume}\ k \in \{1,\dots,s\}$,
$\texttt{Assume}\ i \in \{m,\dots,n\}$,
$\texttt{Assume}\ b : \left\langle p_i f^{-i}(d_{n+1-i,k-1})\right\rangle_{\mathcal{I}}$,
$[10] := \mathtt{E} f_a \texttt{PrincipleIdealStructrue}\Big(A, p_i f^{-i}(d_{n-i,k-1}), b\Big)\texttt{BooleanMorphismIsMonotonic}(A, A, f^i)$
$\quad \texttt{BooleanMeet}(A) : f_a(b) = f^i(b) \le f^i(p_i) d_{n+1-i,k-l} \le d_{n-i,k-i}$,
$[k.*] := \texttt{ExponenentMult}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), f, i, n-i\Big)\mathtt{I} f_a[10]\texttt{ExponenentMult}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big) :$
$\quad : f^{n+1}(b) = f^i f^{n+1-i}(b) = f_a f^{n+1-i}(b) = f_a^k(b)$;
$\leadsto [n.*] := \mathtt{I}\forall : \forall k \in \{1,\dots,s\} \,.\, f^{n+1}(d_{n+1,k}) = f_a^k(d_{n+1,k})$;
$\leadsto [*] := \mathtt{E}\mathbb{N} : \forall m, n \in \mathbb{Z}_+ \,.$
$\quad .\, \Big(\forall k \in \{1,\dots,m-1\} \,.\, af^k(a) = 0\Big) \Rightarrow \exists d : \left\{1,\dots,\left\lfloor \frac{n}{m}\right\rfloor\right\} \to A :$
$\quad : \texttt{PairwiseDisjointElements}(A, \mathrm{Im}\, d)\ \&\ \sup \mathrm{Im}\, d = af^{-n}(a)\ \&$
$\quad \&\ \forall k \in \left\{1,\dots,\left\lfloor \frac{n}{m}\right\rfloor\right\} \,.\, \forall c \in \langle d_k\rangle_{\mathcal{I}} \,.\, f^n(c) = f_a^k(c)$;
$\square$

$\texttt{InnerRecurrenceCriterion} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall b \in \langle a\rangle_{\mathcal{I}} \,.$
$\quad .\, \texttt{DoublyReccurent}(A, b, f) \iff \texttt{DoublyReccurent}\Big(\langle a\rangle_{\mathcal{I}}, b, f_a\Big)$

$\texttt{Proof} \ =$
$\texttt{Assume}\ [1] : \texttt{DoublyReccurent}(A, b, f)$,
$\texttt{Assume}\ c \in \langle b\rangle_{\mathcal{I}} \setminus \{0\}$,
$\Big(p, q, [2]\Big) := \mathtt{E} f_b : \sum p, q : \mathbb{N} \to \langle b\rangle_{\mathcal{I}} \,.\, \texttt{PartitionOfUnity}\Big(\langle b\rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big)\ \&\ \texttt{PartitionOfUnity}\Big(\langle b\rangle_{\mathcal{I}}, \mathrm{Im}\, q\Big)\ \&$
$\quad \&\ \Big(\forall n \in \mathbb{N} \,.\, f^n(p_n) = q_n\Big)\ \&\ \forall \bigvee_{n=1}^{\infty} d_n p_n \in \langle a\rangle_{\mathcal{I}} \,.\, f_a\left(\bigvee_{n=1}^{\infty} d_n p_n\right) = \bigvee_{n=1}^{\infty} f^n(d_n) q_n$,
$n := \min\{n \in \mathbb{N} : f^{-n}(c) b \ne 0\} \in \mathbb{N}$,
$\Big([3]\Big) := \mathtt{E}q\mathtt{E}n : c q_n \ne 0$,
$\Big(d, [4]\Big) := \texttt{LateRecurrenceDivision}(A, b, f, n, 1)\mathtt{E}n : \sum d : \{1,\dots,n\} \to \left\langle af^{-n}(a)\right\rangle_{\mathcal{I}} \,.$
$\quad .\ \sup \mathrm{Im}\, d = af^{-n}(a)\ \&\ \forall k \in \{1,\dots,n\} \,.\, \forall x \in \langle d_k\rangle_{\mathcal{I}} \,.\, f^n(x) = f_a^k(c)$,
$\Big(k, [5]\Big) := [4][3] : \sum k \in \{1,\dots,n\} \,.\, d_k c f^{-n}(c) \ne 0$,
$x := d_k c : \langle c\rangle_{\mathcal{I}}$,
$[6] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\mathtt{E}x\mathtt{E}c\mathtt{E}^2\texttt{BooleanOrder}(A)[5] : f^{-n}(bf^n(x)) = f^{-n}(b)x = f^{-n}(b)bx = x \ne 0$,
$[7] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)[1][4](x) : 0 \ne bf^n(x) = bf_a^k(x)$,
$[1.*.1] := \mathtt{E}\texttt{BooleanOrder}(A)(x, c)[7] : bf_a^k(c) \ne 0$,
$m := \min\{m \in \mathbb{N} : f^m(c) b \ne 0\} \in \mathbb{N}$,
$\Big([8]\Big) := \mathtt{E}p\mathtt{E}m : c p_n \ne 0$,
$\Big(d, [9]\Big) := \texttt{LateRecurrenceDivision}(A, b, f^{-1}, m, 1)\mathtt{E}n : \sum d' : \{1,\dots,m\} \to \langle af^m(a)\rangle_{\mathcal{I}} \,.$
$\quad .\ \sup \mathrm{Im}\, d' = af^m(a)\ \&\ \forall k \in \{1,\dots,n\} \,.\, \forall x \in \langle d'_k\rangle_{\mathcal{I}} \,.\, f^n(x) = f_a^k(c)$,
$\Big(l, [10]\Big) := [8][9] : \sum k \in \{1,\dots,n\} \,.\, d'_l c f^m(b) \ne 0$,
$y := d'_l c : \langle c\rangle_{\mathcal{I}}$,
$[10] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\mathtt{E}y\mathtt{E}c\mathtt{E}^2\texttt{BooleanOrder}(A)[10] : f^m(bf^{-m}(y)) = f^m(b)y = f^m(b)by \ne 0$,
$[11] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)[10][9](y) : 0 \ne bf^{-m}(y) = bf_a^{-l}(y)$,
$[1.*.2] := \mathtt{E}\texttt{BooleanOrder}(A)(y, c)[7] : bf_a^{-l}(c) \ne 0$;

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow: \mathtt{DoublyReccurent}(A, b, f) \Rightarrow \mathtt{DoublyReccurent}\Big(\langle a \rangle_{\mathcal{I}}, b, f_a\Big),$

$\mathtt{Assume}\ [2] : \mathtt{DoublyReccurent}\Big(\langle a \rangle_{\mathcal{I}}, b, f_a\Big),$

$\mathtt{Assume}\ c \in \langle b \rangle_{\mathcal{I}} \setminus \{0\},$

$\Big(n, [3]\Big) := \mathtt{EReccurent}\Big(\langle a \rangle_{\mathcal{I}}, b, f_a, c\Big) : \sum n \in \mathbb{N} \ .\ f_a^n(c)b \neq 0,$

$\Big(p, [4]\Big) := \mathtt{InducedHomomorphismPoU}(A, a, f, n) : \sum p : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} \ .\ \mathtt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} p\Big) \ .\ \forall k \in \mathbb{N}$

$[5] := f_a^{-n}[3] : cf_a^{-n}(b) \neq 0,$

$\Big(k, [6]\Big) := \mathtt{EPartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} p, cf_a^{-n}(b), [5]\Big) : \sum k \in \mathbb{N} \ .\ p_k c f_a^{-n}(b) \neq 0,$

$d := p_k c \in \langle c \rangle_{\mathcal{I}},$

$[2.*.1] := [4][6] : f^{n+k}(d)b = f_a^n(d)(b) \neq 0,$

$\Big(m, [7]\Big) := \mathtt{EReccurent}\Big(\langle a \rangle_{\mathcal{I}}, b, f_a^{-1}, c\Big) : \sum m \in \mathbb{N} \ .\ f_a^{-m}(c)b \neq 0,$

$\Big(q, [8]\Big) := \mathtt{InducedHomomorphismPoU}(A, a, f^{-1}, m) : \sum q : \mathbb{N} \to \langle a \rangle_{\mathcal{I}} \ .\ \mathtt{PartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} q\Big) \ .\ \forall l \in$

$[9] := f_a^m[3] : cf_a^m(b) \neq 0,$

$\Big(l, [10]\Big) := \mathtt{EPartitionOfUnity}\Big(\langle a \rangle_{\mathcal{I}}, \operatorname{Im} q, cf_a^m(b), [9]\Big) : \sum l \in \mathbb{N} \ .\ q_l c f_a^m(b) \neq 0,$

$d' := q_l c \in \langle c \rangle_{\mathcal{I}},$

$[2.*.2] := [4][6] : f^{-m-l}(d')b = f_a^{-m}(d')(b) \neq 0;$

$\rightsquigarrow [*] := \mathtt{I} \iff [1] : \mathtt{DoublyReccurent}(A, b, f) \iff \mathtt{DoublyReccurent}\Big(\langle a \rangle_{\mathcal{I}}, b, f_a\Big);$

$\square$

$\mathtt{InnerRecurrenceExpression} :: \forall A : \tau\text{-}\mathtt{Algebra} \ .\ \forall a \in A \ .\ \forall f : \mathtt{DoublyRecurrentOn}(A, a) \ .\ \forall b \in \langle a \rangle_{\mathcal{I}} \ .$

$\quad .\ \mathtt{DoublyReccurent}(A, b, f) \Rightarrow f_b = (f_a)_b$

$\mathtt{Proof}\ =$

$\dots$

$\square$

$\mathtt{FixedPointRecurrence} :: \forall A : \tau\text{-}\mathtt{Algebra} \ .\ \forall a \in A \ .\ \forall f : \mathtt{DoublyRecurrentOn}(A, a) \ .\ \forall b \in \operatorname{Fix}(f) \ .$

$\quad .\ \mathtt{DoublyReccurent}(A, ab, f)$

$\mathtt{Proof}\ =$

$\mathtt{Assume}\ c \in \langle ab \rangle_{\mathcal{I}} \setminus \{0\},$

$[2] := \mathtt{MeetIneq}(A)\mathtt{E}(a, b, c) : c \leq a,$

$\Big(n, [3]\Big) := \mathtt{ERecurrentOn}\Big(A, a, f, c\Big) : \sum n \in \mathbb{N} \ .\ f^n(c)a \neq 0,$

$[c.*.1] := \mathtt{EFix}(f, b)\mathtt{EBOOL}(A, A, f^n)\mathtt{EBooleanOrder}(A)[3] : f^n(c)ab = f^n(c)af^n(b) = f^n(cb)a = f^n(b)a \neq 0,$

$\Big(m, [4]\Big) := \mathtt{ERecurrentOn}\Big(A, a, f^{-1}, c\Big) : \sum m \in \mathbb{N} \ .\ f^{-m}(c)a \neq 0,$

$[c.*.2] := \mathtt{EFix}(f, b)\mathtt{EBOOL}(A, A, f^n)\mathtt{EBooleanOrder}(A)[4] :$

$\quad : f^{-m}(c)ab = f^{-m}(c)af^{-m}(b) = f^{-m}(cb)a = f^{-m}(b)a \neq 0;$

$\rightsquigarrow [*] := \mathtt{IDoublyRecurrentOn} : \mathtt{DoublyReccurent}(A, f, ab);$

$\square$

$\texttt{FixedPointInducedMorphism} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall b \in \mathrm{Fix}(f) \,.$

$\quad .\, f_{ab} = f_{a|\langle ab\rangle_{\mathcal{I}}}$

$\texttt{Proof} \;=$

$[1] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}\mathrm{Fix}(f^{-n}, b)\mathtt{E}\mathsf{BOOL}(A, a) : \forall n \in \mathbb{N} \,.\, f^{-n}(a)a^2b = f^{-n}(ab)ab,$

$[*] := \mathtt{E}f_{ab}[1] : f_{ab} = f_{a|\langle ab\rangle_{\mathcal{I}}};$

$\square$

$\texttt{InucedMorphismFixedPoint} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall b \in \mathrm{Fix}(f) \,.$

$\quad .\, ab \in \mathrm{Fix}(f_a)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{AperiodicInducedMorphism} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall b \in \mathrm{Fix}(f) \,.$

$\quad .\, \texttt{Aperiodic}(A, f) \Rightarrow \texttt{Aperiodic}\Big(\langle a\rangle_{\mathcal{I}}, f_a\Big)$

$\texttt{Proof} \;=$

$[1] := \mathtt{E}\texttt{Aperiodic}(A, f)\mathtt{E}\,\mathrm{supp} : \forall b \in A \setminus 0 \,.\, \forall n \in \mathbb{N} \,.\, \exists c \in \langle b\rangle_{\mathcal{I}} \,.\, f^n(c) \neq c,$

$\texttt{Assume}\ b : \langle a\rangle_{\mathcal{I}} \setminus \{0\},$

$\texttt{Assume}\ n \in \mathbb{N},$

$\Big(p, [2]\Big) := \texttt{InducedHomomorphismPoU}(A, a, f, n) : \sum p : \mathbb{N} \to \langle a\rangle_{\mathcal{I}} \,.\, \texttt{PartitionOfUnity}\Big(\langle a\rangle_{\mathcal{I}}, \mathrm{Im}\, p\Big) \,.$

$\quad .\, \forall k \in \mathbb{N} \,.\, \forall d \in \langle p_k\rangle_{\mathcal{I}} \,.\, f_a^n(d) = f^{n+k}(d),$

$\Big(k, [3]\Big) := \mathtt{E}\texttt{PartitionOfUnity}\Big(\langle a\rangle_{\mathcal{I}}, \mathrm{Im}\, p, b\Big) : \sum k \in \mathbb{N} \,.\, bp_m \neq 0,$

$\Big(d, [4]\Big) := [1](bp_m, n+k) : \sum d \in \langle bp_m\rangle_{\mathcal{I}} \,.\, f^{n+k}(d) \neq dc,$

$[b.*] := [4][2] : f_a^n(d) \neq d;$

$\rightsquigarrow [*] := \mathtt{I}\texttt{Apperiodic} : \texttt{Aperiodic}\Big(\langle a\rangle_{\mathcal{I}}, f_a\Big);$

$\square$

$\texttt{TrinitaryLemma} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{DoublyRecurrentOn}(A, a) \,.\, \forall b \in \langle a\rangle_{\mathcal{I}} \,.$

$\quad f(a)a = 0 \,\&\, bf_a(b) = 0 \Rightarrow \texttt{PairwiseDisjointElements}\Big(A, \{b, f(b), f^2(b)\}\Big)$

$\texttt{Proof} \;=$

$[1] := f[0.1] : f^2(b)f(b) = 0,$

$\texttt{Assume}\ c : \big\langle f^2(b)b\big\rangle_{\mathcal{I}},$

$[2] := [0.1]\mathtt{E}f_a\mathtt{E}c : f^{-2}(c) = f_a^{-1}(c),$

$[3] := \texttt{BooleanMorphismIsMonotonic}(A, A, f^{-2})\mathtt{E}c[2] : f^{-2}(c) \leq f^{-2}(b)b = f_a^{-1}(b)b,$

$[4] := \texttt{BooleanMorphismIsMonotonic}(\langle a\rangle_{\mathcal{I}}, \langle a\rangle_{\mathcal{I}}, f_a)[3][0.2] : f^{-2}f_a(c) \leq bf_a(b) = 0,$

$[5] := \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f^{-2}f_a)[4] : c = 0;$

$\rightsquigarrow [2] := \mathtt{E}\big\langle f^2(b)b\big\rangle_{\mathcal{I}} : f^2(b)b = 0,$

$[*] := \mathtt{I}\texttt{PairwiseDisjointElements}[1][2] : \texttt{PairwiseDisjointElements}\Big(A, \{b, f(b), f^2(b)\}\Big);$

$\square$

$\texttt{extendedInducedIsomorphism} :: \sum A : \tau\text{-}\texttt{Algebra} . \sum a \in A . \texttt{DoublyRecurrentOn}(A, a) . \to \mathrm{Aut}_{\mathsf{BOOL}}(A)$

$\texttt{extendedInducedIsomorphism}\,(f) = \tilde{f}_a := \Lambda ba + ca^{\complement} \in A . f_a(ba) + ca^{\complement}$

$\texttt{ExtendedInducedIsomorphismInFullSubgroup} ::$

$:: \forall A : \tau\text{-}\texttt{Algebra} . \forall a \in A . \forall f : \texttt{DoublyRecurrentOn}(A, a) . \tilde{f}_a \in \langle f \rangle_{\mathrm{F}}$

$\texttt{Proof} =$

...

□

$\texttt{Recurrent} :: \prod_{A:\mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$

$f : \texttt{Recurrent} \iff \forall a \in A . \texttt{RecurrentOn}(A, a, f)$

$\texttt{DoublyRecurrent} :: \prod_{A:\mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$

$f : \texttt{DoublyRecurrent} \iff \forall a \in A . \texttt{DoublyRecurrentOn}(A, a, f)$

$\texttt{RecurrentCondition} :: \forall A \in \mathsf{BOOL} . \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) .$

$. \texttt{Recurrent}(A, f) \iff \forall a \in A . a = \sup_n a f^n(a)$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{Recurrent}(A, f),$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ b \in \langle a \rangle_{\mathcal{I}},$

$[2] := \texttt{ERecurrent}(A, f, b) : \texttt{RecurrentOn}(A, b, f),$

$\texttt{Assume}\ [3] : \forall n \in \mathbb{N} . f^n(a) b = 0,$

$\texttt{Assume}\ [4] : b \neq 0,$

$\Big(n, [5]\Big) := \texttt{ERecurrentOn}(A, a, f)[2][4] : \sum n \in \mathbb{N} . b f^n(b) \neq 0,$

$[6] := \texttt{ZeroIsMinimal}(A)[5]\texttt{MeetIneq}(A) : 0 < b f^n(b) \leq a f^n(b),$

$[7] := \texttt{TrichtomyPrinciple}[6] : a f^n(b) \neq 0,$

$[1.*] := [7][3](n) : \bot;$

$\rightsquigarrow [1] := \texttt{I} \Rightarrow : \texttt{Recurrent}(A, f) \Rightarrow \forall a \in A . a = \sup_n a f^n(a),$

$\texttt{Assume}\ [2] : \forall a \in A . a = \sup_n a f^n(a),$

...

□

$\texttt{RelativeAtom} :: \prod A : \mathsf{BOOL} . \prod B : \texttt{Subalgebra}(A) . ?A$

$a : \texttt{RelativeAtom} \iff a \in \mathrm{Atom}_A(B) \iff \forall c \in \langle a \rangle_{\mathcal{I}} . \exists b \in B : c = ab$

$\texttt{RelativelyAtomless} :: \prod A : \mathsf{BOOL} . ?\texttt{Subalgebra}(A) . ?A$

$B : \texttt{RelativelyAtomless} \iff \mathrm{Atom}_A(B) = \emptyset$

$\texttt{AperidocConditionForRecurrent} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \texttt{Recurrent}(A) \;.\; \texttt{Aperiodic}(A, f) \iff$

$\iff \texttt{RelativelyAtomless}\Big(A, \mathrm{Fix}(f)\Big)$

$\texttt{Proof} \;=$

$\texttt{Assume}\ [1] : \texttt{Aperiodic}(A, f),$

$\texttt{Assume}\ a : \mathrm{Atom}_A\Big(\mathrm{Fix}(f)\Big),$

$[2] := \texttt{ERecurrent}(A, f, a) : \texttt{RecurrentOn}(A, a, f),$

$n := \min\{n \in \mathbb{N} : f^n(a)a \neq 0\} \in \mathbb{N},$

$\texttt{Assume}\ b \in \langle af^n(a)\rangle_{\mathcal{I}}\,,$

$\Big(b, [4]\Big) := \mathrm{E}\,\mathrm{Atom}_A\Big(\mathrm{Fix}(f)\Big) : \sum c \in \mathrm{Fix}(f) \;.\; b = ca,$

$[5] := [4]\mathrm{E}\mathsf{BOOL}(A, A, f^n)\mathrm{E}\mathrm{Fix}(f, c)\texttt{MeetIneq}(A)\mathrm{E}b[4] : f^n(b) = f^n(ca) = cf^n(a) \geq caf^n(a) \geq b,$

$$d := \bigvee_{k=0}^{n-1} f^k(b) \in A,$$

$[6] := \mathrm{E}\sup \mathrm{E}d[5] : f(d) \geq d,$

$[7] := \texttt{SupremumLemma}[6] : f(d) = d,$

$[8] := \Lambda k \in [1, \dots, n-1] \;.\; \mathrm{E}\mathsf{BOOL}(A, A, f^k)\mathrm{E}b\mathrm{E}n :$

$\quad \forall k \in [1, \dots, n-1] \;.\; f^n(b)f^k(b) = f^k\Big(f^{n-k}(b)b\Big) \leq f^k\Big(f^{n-k}(a)a\Big) = 0,$

$[9] := \mathrm{E}d[7] : f^n(b) \leq \sup_{0<k<n} f^k(b),$

$[b.*] := [5][8][9] : f^n(b) = b;$

$\rightsquigarrow [4] := \mathrm{I}\forall : \forall b \in \langle af^n(a)\rangle_{\mathcal{I}} \;.\; f^n(b) = b,$

$[a.*] := \texttt{EAperiodic} : \bot;$

$\rightsquigarrow [1.*] := \mathrm{E}\bot : \mathrm{Atom}_A\Big(\mathrm{Fix}(f)\Big) = \emptyset;$

$\rightsquigarrow [1] := \mathrm{I} \Rightarrow: \texttt{Aperiod}(A, f) \Rightarrow \mathrm{Atom}_A\Big(\mathrm{Fix}(f)\Big) = \emptyset,$

$\texttt{Assume}\ [2] : \mathrm{Atom}_A\Big(\mathrm{Fix}(f)\Big) = \emptyset,$

$\texttt{Assume}\ [3] : \texttt{Aperiodic}(A, f),$

$n := \min\{n \in \mathbb{N} : \mathrm{supp}\, f^n \neq e\} \in \mathbb{N},$

$\Big(a, [4]\Big) := \mathrm{E}n\mathrm{E}\,\mathrm{supp}\, f^n : \sum a \in A \;.\; a \neq 0 \;\&\; \forall b \in \langle a\rangle_{\mathcal{I}} \;.\; f^n(b) = b,$

$\texttt{Assume}\ b : \langle a\rangle_{\mathcal{I}}\,,$

$\texttt{Assume}\ [5] : b \neq 0,$

$\texttt{Assume}\ k \in [1, \dots, n-1],$

$\Big(c, [6]\Big) := \mathrm{E}n\mathrm{E}\,\mathrm{supp}(k) : \sum c \in \langle b\rangle_{\mathcal{I}} \;.\; f^k(c) \neq c,$

$d := \texttt{if}\ c \setminus f^k(c) \neq 0\ \texttt{then}\ c \setminus f^k(c)\ \texttt{else}\ c \setminus f^{n-k}(c) : \langle c\rangle_{\mathcal{I}}\,,$

$\dots$

$\square$

$\texttt{Ergodic} :: \prod_{A \in \mathsf{BOOL}} ?\mathrm{Aut}_{\mathsf{BOOL}}(A)$

$f : \texttt{Ergodic} \iff \forall a, b \in A \setminus \{0\} \;.\; \exists n \in \mathbb{N} : f^n(a)b \neq 0$

$\texttt{AperidocConditionForErgodic} :: \forall A \in \mathsf{BOOL} \;.\; \forall f \in \texttt{Ergodic}(A) \;.\; \texttt{Aperiodic}(A, f) \iff \texttt{Atomless}(A)$

$\texttt{Proof} =$

...

$\square$

1.7.6 Interaction with Stone Spaces

$\texttt{StoneAutomorphismAgreement} :: \forall A \in \mathsf{BOOL} \,.\, \forall a, b \in A \,.\, \forall \langle a \rangle_{\mathcal{I}} \xrightarrow{f} \langle b \rangle_{\mathcal{I}} : \mathsf{BOOL} \,.$
$\quad . \forall \varphi \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \varphi_{|\langle a \rangle_{\mathcal{I}}} = f \iff \mathsf{Z}(\varphi)_{|\mathsf{Z}\ \langle b \rangle_{\mathcal{I}}} = \mathsf{Z}(f)$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \varphi_{|\langle a \rangle_{\mathcal{I}}} = f,$

$\texttt{Assume}\ u \in \mathsf{Z}\ \langle b \rangle_{\mathcal{I}},$

$[u.*] := \mathtt{E}\mathsf{Z}(f)[0]\mathtt{I}\mathsf{Z}(\varphi) : \mathsf{Z}(f)(b) = u \circ f = u \circ \varphi = \mathsf{Z}(\varphi)(b);$

$\rightsquigarrow [1.*] := \mathtt{I}(=, \to) : \mathsf{Z}(\varphi)_{|\mathsf{Z}\ \langle b \rangle_{\mathcal{I}}} = \mathsf{Z}(f);$

$\rightsquigarrow [2] := \mathtt{I} \Rightarrow : \varphi_{|\langle a \rangle_{\mathcal{I}}} = f \Rightarrow \mathsf{Z}(\varphi)_{|\mathsf{Z}\ \langle b \rangle_{\mathcal{I}}} = \mathsf{Z}(f),$

$\texttt{Assume}\ [2] : \mathsf{Z}(\varphi)_{|\mathsf{Z}\ \langle b \rangle_{\mathcal{I}}} = \mathsf{Z}(f),$

$[2.*] := \mathcal{TK}[2] : f = \varphi_{|\langle a \rangle_{\mathcal{I}}};$

$\rightsquigarrow [*] := \mathtt{I} \iff [1] : \varphi_{|\langle a \rangle_{\mathcal{I}}} = f \iff \mathsf{Z}(\varphi)_{|\mathsf{Z}\ \langle b \rangle_{\mathcal{I}}} = \mathsf{Z}(f);$

$\square$

$\texttt{StoneSupports} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, a \in \mathrm{Supp}\, f \iff$
$\quad \iff S_A(a) \subset \overline{\left\{ v \in \mathsf{Z}(A) : \mathsf{Z}(f)(v) \neq v \right\}}$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : a \in \mathrm{Supp}(f),$

$[2] := \mathtt{E}\,\mathrm{Supp}(f, a) : \forall b \in \left\langle a^{\complement} \right\rangle_{\mathcal{I}} \,.\, f(b) = b,$

$[3] := \mathtt{I}S_A(a)\mathtt{I}\mathsf{Z}(f)[2] : \forall v \in S_A^{\complement}(a) \,.\, \mathsf{Z}(f)(v) = v,$

$[4] := [3]^{\complement} : \left\{ v \in \mathsf{Z}(A) : \mathsf{Z}(f)(v) \neq v \right\} \subset S_A(a),$

$[1.*] := \mathtt{I}\mathrm{cl}_{\mathsf{Z}(A)}[4] : \overline{\left\{ v \in \mathsf{Z}(A) : \mathsf{Z}(f)(v) \neq v \right\}} \subset S_A(a);$

$\rightsquigarrow [*] :=: a \in \mathrm{Supp}\, g \iff S_A(a) \subset \overline{\left\{ v \in \mathsf{Z}(A) : \mathsf{Z}(f)(v) \neq v \right\}};$

$\square$

$\texttt{StoneSupports} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, a = \mathrm{supp}\, f \iff$
$\quad \iff S_A(a) = \overline{\left\{ v \in \mathsf{Z}(A) : \mathsf{Z}(f)(v) \neq v \right\}}$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{FullSubgroupDenseProperty} :: \forall A : \tau\text{-}\texttt{Algebra} . \forall f, g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) .$

$$. \, g \in \langle f \rangle_{\mathrm{F}} \iff \texttt{Dense}\left(\mathsf{Z}(A), \bigcup_{n=-\infty}^{\infty} \mathrm{int}\left\{v \in \mathsf{Z}(A) : \mathsf{Z}(g)(v) = \mathsf{Z}(f^n)(v)\right\}\right)$$

$\texttt{Proof} =$

$\Big(p, [1]\Big) := \texttt{CountablyFullSubgroupGeneratedByGrouopElement}(A, f, g) :$

$: \sum p : \mathbb{Z} \to A . \texttt{PartitionOfUnity}(A, \mathrm{Im}\, p) \ \& \ \forall n \in \mathbb{Z} . \forall b \in \langle p_n \rangle_{\mathcal{I}} . g(b) = f^n(b),$

$[2] := \mathsf{Z}[1.2] : \forall n \in \mathbb{Z} . S_A(a) = \left\{v \in \mathsf{Z}(A) : (\mathsf{Z}\ g)(v) = (\mathsf{Z}\ f^n)(v)\right\},$

$[3] := \texttt{SupremumStoneExpression}(A)\texttt{EPartitionOfUnity}(A, \mathrm{Im}\, p)[2] :$

$$: \mathsf{Z}(A) = \mathrm{cl} \bigcup_{n=-\infty}^{\infty} \mathrm{int}\, S_A(p_n) = \bigcup_{n=-\infty}^{\infty} \mathrm{int}\left\{v \in \mathsf{Z}(A) : \mathsf{Z}(g)(v) = \mathsf{Z}(f^n)(v)\right\},$$

$$[*] := \texttt{IDense}[3] : \texttt{Dense}\left(\mathsf{Z}(A), \bigcup_{n=-\infty}^{\infty} \mathrm{int}\left\{v \in \mathsf{Z}(A) : \mathsf{Z}(g)(v) = \mathsf{Z}(f^n)(v)\right\}\right);$$

$\square$

$\texttt{FullSugroupComeager} :: \forall A : \tau\text{-}\texttt{Algebra} . \forall f, g \in \mathrm{Aut}_{\mathsf{BOOL}}(A) .$

$$. \, g \in \langle f \rangle_{\mathrm{F}} \iff \texttt{Comeager}\left(\mathsf{Z}(A), \left\{v \in \mathsf{Z}(A) : \mathsf{Z}(g)(v) \in \{\mathsf{Z}(f^n)(v) | n \in \mathbb{N}\}\right\}\right)$$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{Comeager}\left(\mathsf{Z}(A), \left\{v \in \mathsf{Z}(A) : \mathsf{Z}(g)(v) \in \{\mathsf{Z}(f^n)(v) | n \in \mathbb{N}\}\right\}\right),$

$F := \Lambda n \in \mathbb{Z} . \left\{v \in \mathsf{Z}(A) . \mathsf{Z}(g)(v) = \mathsf{Z}(f^n)(v)\right\} : \mathbb{Z} \to ?\mathsf{Z}(A),$

$[2] := \texttt{NowhereDenseConstruction}(\mathsf{Z}\ A, F) : \forall n \in \mathbb{Z} . \texttt{NowhereDense}\Big(\mathsf{Z}\ A, F_n \setminus \mathrm{int}\, F_n\Big),$

$$[3] := \texttt{EComeager}[2][1]\texttt{IComeager} : \texttt{Comeager}\left(\mathsf{Z}\ A, \bigcup_{n=-\infty}^{\infty} \mathrm{int}\, F_n\right),$$

$$[4] := \texttt{BairTHM}[3] : \texttt{Dense}\left(\mathsf{Z}\ A, \bigcup_{n=-\infty}^{\infty} \mathrm{int}\, F_n\right),$$

$\texttt{Assume}\ a \in A,$

$\Big(n, [6]\Big) := \texttt{EDense}[4](a) : \sum a \in A . S_A\Big(g(a)\Big) \cap \mathrm{int}\, F_n \neq \emptyset,$

$\Big(b, [7]\Big) := \texttt{StoneTHM}[6] : \sum b \in A . S_A(b) \subset S_A\Big(g(a)\Big) \cap \mathrm{int}\, F_n,$

$[8] := \texttt{E}\mathsf{Z}(g)[7] : g^{-1}(b) \leq a,$

$[a.*] := \texttt{E}F_n[8] : \forall c \in \langle b \rangle_{\mathcal{I}} . g(c) = f^n(c);$

$\rightsquigarrow [*] := \texttt{FullSubgroupGeneratedByGroupElement} : g \in \langle f \rangle_{\mathrm{F}};$

$\square$

$\texttt{RecurrentStoneCriterion} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{End}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, \texttt{RecurrentOn}(A, f, a) \iff$

$$\iff S_A(a) \subset \overline{\bigcup_{n=1}^{\infty} \mathsf{Z}(f^n)\Big(S_A(a)\Big)}$$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{RecurrentOn}(A, f, a),$

$[2] := \texttt{ERecurrentOn}(A, f, a) : \forall b \in \langle a \rangle_{\mathcal{I}} \setminus \{0\} \,.\, \exists k \in \mathbb{N} : af^k(b) \neq 0,$

$[3] := \texttt{I}S_A\texttt{I}\mathsf{Z}[2] : \forall b \in \langle a \rangle_{\mathcal{I}} \setminus \{0\} \,.\, \exists k \in \mathbb{N} : S_A(a) \cap \Big(\mathsf{Z}(f^k)\Big)^{-1} S_A(b) \neq \emptyset,$

$[4] := \texttt{Iimage}[3] : \forall b \in \langle a \rangle_{\mathcal{I}} \setminus \{0\} \,.\, \exists k \in \mathbb{N} : \Big(\mathsf{Z}(f^k)\Big)\Big(S_A(a)\Big) \cap S_A(b) \neq \emptyset,$

$[5] := \texttt{IDense}[4] : \texttt{Dense}\left(S_A(a), S_A(a) \cap \bigcup_{k=1}^{\infty} \mathsf{Z}(f^k)\Big(S_A(a)\Big)\right),$

$[*] := \texttt{I}\,\mathrm{cl}[5] : S_A(a) \subset \overline{\bigcup_{n=1}^{\infty} \mathsf{Z}(f^k)\Big(S_A(a)\Big)};$

$\square$

$\texttt{InducedHomeomorphismSetting} :: \forall A : \sigma\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{RecurrentOn}(A, a) \,.$

$$.\ \bigcup_{n=1}^{\infty} G_n = S_A(a) \cap \bigcup_{n=1}^{\infty} \mathsf{Z}(f^{-n})\Big(S_A(a)\Big)$$

$\texttt{where}$

$$G = \Lambda k \in \mathbb{N} \,.\, \Big\{v \in \langle a \rangle_{\mathcal{I}} : f^k(v) \in S_A(a)\ \&\ \forall i \in \{1, \ldots, k-1\} f^i(v) \notin S_A(a)\Big\}$$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{InducedHomeomorphismSetting} :: \forall A : \sigma\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{RecurrentOn}(A, a) \,.$

$$.\ \bigcup_{n=1}^{\infty} G_n = S_A(a) \cap \bigcup_{n=1}^{\infty} \mathsf{Z}(f^{-n})\Big(S_A(a)\Big)$$

$\texttt{where}$

$$G = \Lambda k \in \mathbb{N} \,.\, \Big\{v \in \langle a \rangle_{\mathcal{I}} : f^k(v) \in S_A(a)\ \&\ \forall i \in \{1, \ldots, k-1\} f^i(v) \notin S_A(a)\Big\}$$

$\texttt{Proof} =$

$\ldots$

$\square$

$\texttt{InducedHomeomorphismProperty} :: \forall A : \sigma\texttt{-Algebra} \,.\, \forall a \in A \,.\, \forall f : \texttt{RecurrentOn}(A, a) \,.$

$.\ \forall k \in \mathbb{N} \,.\, \forall v \in G_k \,.\, \mathsf{Z}(f_a)(v) = \mathsf{Z}^k(f)(v)\texttt{where}$

$$G = \Lambda k \in \mathbb{N} \,.\, \Big\{v \in \langle a \rangle_{\mathcal{I}} : f^k(v) \in S_A(a)\ \&\ \forall i \in \{1, \ldots, k-1\} f^i(v) \notin S_A(a)\Big\}$$

$\texttt{Proof} =$

$\ldots$

$\square$

1.7.7 Exchanging Automorphisms

$\texttt{AutomorphismChain} :: \mathsf{BOOL} \to ?\texttt{MorphismChain}\left(\mathsf{BOOL}^*\right)$

$(n, B_\bullet, a_\bullet, f_\bullet) : \texttt{AutomorphismChain} \iff a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{AutomorphismChain}(A) \iff$
$: \Lambda A \in \mathsf{BOOL} \,.\, n < \infty \;\&\; \forall i \in \sigma(n) \,.\, A_i = B \;\&\; \forall i \in n \,.\, f_i \in \mathrm{Aut}_{\mathsf{BOOL}}(A)$

$\texttt{ExchangeChain} :: \prod_{A \in \mathsf{BOOL}} ?\texttt{AutomorphismChain}\left(\mathsf{BOOL}^*\right)$

$(n, a_\bullet, f_\bullet) : \texttt{ExchangeChain} \iff a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{ExchangeChain}(A) \iff$
$\iff \texttt{PairwiseDisjointElements}(A, \mathrm{Im}\, a)$

$\texttt{exchangingAutomorphism} :: \prod_{A \in \mathsf{BOOL}} \texttt{ExchangeChain} \to \mathrm{Aut}_{\mathsf{BOOL}}(A)$

$\texttt{exchangingAutomorphism}\,(n, a_\bullet, f_\bullet) = \overleftarrow{a_{1 f_1} a_{2 f_2} \dots_{f_n} a_{n+1}} := \Lambda b \in A \,.\, \texttt{if } \exists i \in n : b \le a_i \texttt{ then } f_i(b) \texttt{ else}$
$\texttt{if } b \subset a_{n+1} \texttt{ then } \prod_{i=1}^{n} f^{-1}_{n+1-i}(b) \texttt{ else } b$

$\texttt{ExchangingAutomorphismOrder} :: \forall A \in \mathsf{BOOL} \,.\, \forall a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{ExchangeChain}(A) \,.$
$.\ \mathrm{ord}\, \overleftarrow{a_{1 f_1} a_{2 f_2} \dots_{f_n} a_{n+1}} = n + 1$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{PreservationUnderCycling} :: \forall A \in \mathsf{BOOL} \,.\, \forall a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{ExchangeChain}(A) \,.$
$.\ \forall \gamma : \texttt{Cycle}(n) \,.\, \overleftarrow{a_{1 f_1} a_{2 f_2} \dots_{f_n} a_{n+1}} = \overleftarrow{a_{\gamma(1) f_{\gamma(1)}} a_{\gamma(2) f_{\gamma(2)}} \dots_{f_{\gamma(n)}} a_{\gamma(n)+1}}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ExchangingAutomorphismConjugation} :: \forall A \in \mathsf{BOOL} \,.\, \forall a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{ExchangeChain}(A) \,.$
$.\ \forall \phi \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \phi^{-1} \overleftarrow{a_{1 f_1} a_{2 f_2} \dots_{f_n} a_{n+1}} \phi = \overleftarrow{\phi(a_1)_{\phi^{-1} f_1 \phi} \phi(a_2)_{\phi^{-1} f_2 \phi} \dots_{\phi^{-1} f_n \phi} \phi(a_{n+1})}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ExchangingAutomorphismsComposition} :: \forall A \in \mathsf{BOOL} \,.$
$.\ \forall a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1}, \forall b_1 \xrightarrow{g_1} b_2 \xrightarrow{b_2} \dots \xrightarrow{g_n} b_{n+1} : \texttt{ExchangeChain}(A) \,.$
$.\ \texttt{PairwiseDisjointElements}\left(A, \mathrm{Im}\, a \cup \mathrm{Im}\, b\right) \Rightarrow$
$\Rightarrow \overleftarrow{a_{1 f_1} a_{2 f_2} \dots_{f_n} a_{n+1}} \overleftarrow{b_{1 g_1} b_{2 f_2} \dots_{g_n} b_{n+1}} = \overleftarrow{(a_1 + b_2)_{f_1 g_1} (a_2 + b_2)_{f_2 g_2} \dots_{f_n g_n} (a_{n+1} + b_{n+1})}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ExchangingAutomotphismInAFullSubgroup} :: \forall A \in \mathsf{BOOL} \;.\; \forall a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} a_{n+1} : \texttt{ExchangeChain}(A) \;.$
$\overleftarrow{a_{1 f_1} a_{2 f_2} \dots {}_{f_n} a_{n+1}} \in \langle f_1, \dots, f_n \rangle_{\mathrm{CF}}$

$\texttt{Proof} =$

$\dots$

$\square$

$\texttt{ExchangingInvolutionAsjoining} :: \forall A \in \mathsf{BOOL} \;.\; \forall a \xrightarrow{f} b, b \xrightarrow{g} c : \texttt{ExchangeChain}(A) \;.$
$\texttt{PairwiseDisjointElements}\Big(A, \{a, b, c\}\Big) \overleftarrow{a_f b} \overleftarrow{b_g c} = \overleftarrow{a_f b_g c}$

$\texttt{Proof} =$

$\dots$

$\square$

$$\texttt{ExchangingInvolution} :: \prod_{A \in \mathsf{BOOL}} \mathrm{Aut}_{\mathsf{BOOL}}(A)$$

$f : \texttt{ExchangingInvolution} \iff \exists a \xrightarrow{g} b : \texttt{ExchangeChain}(A) \;.\; f = \overleftarrow{a_g b}$

1.8 Factorization Theorems in an Automorphisms Group[!!]

1.8.1 Separators and Transversals

$$\texttt{Separator} :: \prod_{A\in\mathsf{BOOL}} \mathrm{Aut}_{\mathsf{BOOL}}(A) \to ?A$$

$$a : \texttt{Separator} \iff \Lambda f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, a \in \mathrm{Sep}(f) \,.\, \iff$$

$$\iff \Lambda f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, af(a) = 0 \;\&\; \forall b \in A \,.\, \forall n \in \mathbb{Z}_+ \,.\, bf^n(a) = 0 \Rightarrow f(b) = b$$

$$\texttt{Transversal} :: \prod_{A\in\mathsf{BOOL}} \mathrm{Aut}_{\mathsf{BOOL}}(A) \to ?A$$

$$a : \texttt{Transversal} \iff \Lambda f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, a \in \mathrm{Tr}(f) \iff$$

$$\iff \sup_{n\in\mathbb{Z}} f^n(a) = e \;\&\; \forall n \in \mathbb{Z} \,.\, \forall b \in \langle af^n(a)\rangle_{\mathcal{I}} \,.\, f^n(b) = b$$

$$\texttt{TransversalConstructionLemma} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall n \in \mathbb{N} \,.\, \forall a : \prod_{k=1}^{n} \mathrm{Sep}(f^k) \,.$$

$$.\, f^{n+1} = \mathrm{id} \Rightarrow \exists\, \mathrm{Tr}(f)$$

$\texttt{Proof} =$

$X := \left\{ f^i(a_j) \middle| i, j \in [1, \dots, n] \right\} : \texttt{Finite}(A),$

$B := \langle X\rangle_{\mathsf{RING}} : \texttt{Subring}(A),$

$[1] := \texttt{EFinite}(A, X)\mathtt{E}B : |B| < \infty,$

$[2] := [1]\texttt{IPurelyAtomic} : \texttt{PurelyAtomic}(B),$

$G := \langle f\rangle_{\mathsf{GRP}} \in \mathsf{GRP},$

$\alpha := \Lambda g \in G \,.\, \Lambda b \in B \,.\, g(b) : G \curvearrowright \mathrm{Atom}(B),$

$\mathcal{O} := \left\{ O_\alpha(b) | b \in \mathrm{Atom}(B) \right\} : \texttt{Partition}\Big(\mathrm{Atom}(B)\Big),$

$\texttt{Assume}\ C \in \mathcal{O},$

$m := |C| \in \mathbb{N},$

$[3] := [0]\mathtt{E}m : m \le n + 1,$

$\texttt{Assume}\ c \in C,$

$\texttt{Assume}\ d \in \langle c\rangle_{\mathcal{I},A},$

$[4] := \mathtt{E}m\texttt{Eorbit} : \forall k \in \mathbb{Z} \,.\, f^{m+k}(c) = c,$

$[5] := \mathtt{E}\,\mathrm{Sep}(f^m, a_m)[4] : \forall k \in \mathbb{Z} \,.\, a_m f^k(c) = 0,$

$[6] := \mathtt{E}d[5] : \forall k \in \mathbb{Z} \,.\, a_m f^k(d) = 0,$

$[C.*] := \mathtt{E}\,\mathrm{Sep}(f^m, a_m)[6] : f^m(d) = d;$

$\rightsquigarrow [3] := \mathtt{I}\forall : \forall C \in \mathcal{O} \,.\, \forall c \in C \,.\, \forall d \le_A c \,.\, f^{|C|}(d) = d,$

$b := \texttt{FiniteChoice}(\mathcal{O}) \in \prod_{C\in\mathcal{O}} C,$

$t := \bigvee_{C\in\mathcal{O}} b_C \in A,$

$[4] := \mathtt{E}t\mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\mathtt{E}\mathcal{O}\texttt{EPartition}\Big(\mathcal{O}, \mathrm{Atom}(B)\Big)\texttt{EPurelyAtomic}(B)\texttt{ESubring}(A, B) \in$

$$: \bigvee_{k=1}^{n} f^k(t) = \bigvee_{k=1}^{n} \bigvee_{C\in\mathcal{O}} f^k(b_C) = \bigvee_{C\in\mathcal{O}} \bigvee C = \bigvee B = e_A,$$

$\texttt{Assume}\ m \in \mathbb{Z},$

$\texttt{Assume}\ c : \langle t f^m(t)\rangle_{\mathcal{I}},$

$k := m \mod n+1 : [0,\dots,n],$

$[5] := \mathtt{E}c[0] : c \le t f^k(t),$

$\Big(\mathcal{O}', d, [6]\Big) := \mathtt{E}c\mathtt{E}t\mathtt{E}\,\mathrm{Atom}(B,b) : \sum \mathcal{O}' \subset \mathcal{O}\ .\ \sum d : \prod_{C\in\mathcal{O}'} \langle b_C\rangle_{\mathcal{I}A} c = \bigvee_{C\in\mathcal{O}'} d_C,$

$[7] := \Lambda C \in \mathcal{O}'\ .\ \mathtt{E}d_C\mathtt{E}b_C\mathtt{E}\,\mathrm{Atom}(B,b_C)\mathtt{EPartition}\Big(\mathcal{O}, \mathrm{Atom}\, b\Big)[5] : \forall C \in \mathcal{O}'\ .\ f^k(d_C) \le f^k(b_C) = b_C,$

$[8] := \mathtt{Eorbit}[7] : \forall C \in \mathcal{O}'\ .\ k \vdots |C|,$

$[9] := [3][8] : \forall C \in \mathcal{O}\ .\ f^k(d_C) = d_C,$

$[m.*] := \mathtt{E}m[0][9]\mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A,f)[6] : f^m(c) = f^k(c) = c;$

$\rightsquigarrow [*] := [4]\mathtt{I}\,\mathrm{Tr}(f) : t \in \mathrm{Tr}(f);$

$\square$

$\texttt{ExchangingInvolutionBySeparator} :: \forall A \in \mathsf{BOOL}\ .\ \forall f : \texttt{Involution}(A)\ .$

$\quad .\ \texttt{ExchangingInvolution}(A,f) \iff \exists\,\mathrm{Sep}(f)$

$\texttt{Proof} =$

$\texttt{Assume}\ [1] : \texttt{ExchangingInvolution}(A,f),$

$\Big(a, b, g, [2]\Big) := \mathtt{E}[1] : \sum a \xrightarrow{g} b : \texttt{ExchangeChain}\ .\ f = \overleftarrow{a_g b},$

$[3] := \mathtt{EExchangeChain}(a,b,g) : ab = 0,$

$[4] := \mathtt{E}[2]\big(f(a)\big) : f(a) = b,$

$[5] := [4][3] : af(a) = 0,$

$[6] := [2]\mathtt{EexchangingAutomorphism} : \forall c \in A\ .\ \Big(\forall n \in \mathbb{Z}\ .\ cf^n(a) = 0\Big) \Rightarrow f(c) = c,$

$[1.*] := \mathtt{I}\,\mathrm{Sep}(f) : a \in \mathrm{Sep}(f);$

$\rightsquigarrow [1] := \mathtt{I} \Rightarrow : \texttt{ExchangingInvolution}(A,f) \Rightarrow \exists\,\mathrm{Sep}(f),$

$\texttt{Assume}\ a \in \mathrm{Sep}(f),$

$[2] := \mathtt{E}_1\,\mathrm{Sep}(f,a) : af(a) = 0,$

$[3] := \mathtt{EInvolution}(A,f)(a) : f^2(a) = a,$

$[5] := \Lambda c \in \Big\langle (a+f(a))^{\complement}\Big\rangle_{\mathcal{I}}\ .\ \mathtt{DeMorganaLaw}(A)[2]\mathtt{E}\complement : \forall c \in \Big\langle (a+f(a))^{\complement}\Big\rangle_{\mathcal{I}}\ .\ ac \,\&\, f(a)c = 0,$

$[6] := [5][3] : \forall c \in \Big\langle (a+f(a))^{\complement}\Big\rangle_{\mathcal{I}}\ .\ \forall n \in \mathbb{Z}\ .\ f^n(a)c = 0,$

$[7] := \mathtt{E}_2\,\mathrm{Sep}(f,a)[6] : \forall c \in \Big\langle (a+f(a))^{\complement}\Big\rangle_{\mathcal{I}}\ .\ f(c) = c,$

$[a.*] := \mathtt{EexchangingAutomorphism}[2][3][7] : f = \overleftarrow{a_f f(a)};$

$[*] := \mathtt{I} \iff [1] : \forall A \in \mathsf{BOOL}\ .\ \forall f : \texttt{Involution}(A)\ .\ \texttt{ExchangingInvolution}(A,f) \iff \exists\,\mathrm{Sep}(f);$

$\square$

$\texttt{ExchangingInvolutionByTransversal} :: \forall A \in \mathsf{BOOL} \,.\, \forall f : \texttt{Involution}(A) \,.$

$\quad . \texttt{ExchangingInvolution}(A, f) \iff \exists\, \mathrm{Tr}(f)$

$\texttt{Proof} \ =$

$\texttt{Assume}\ [1] : \texttt{ExchangingInvolution}(A, f),$

$a := \texttt{ExchangingInvolutionBySeparator}(A, f)[1] \in \mathrm{Sep}(f),$

$[2] := \texttt{EInvolution}(A, f) : f^2 = \mathrm{id},$

$t := \texttt{TransversalConstructionLemma}(A, f, a)[1] : \mathrm{Tr}(f);$

$\leadsto [1] := \texttt{I}\exists\texttt{I} \Rightarrow : \texttt{ExchangingInvolution}(A, f) \Rightarrow \exists\, \mathrm{Tr}(f),$

$\texttt{Assume}\ t \in \mathrm{Tr}(f),$

$[2] := \texttt{EInvolution}(A, f)(t) : f^2(t) = t,$

$[3] := \texttt{E}\,\mathrm{Tr}(f, t)[2] : f(t) \vee t = e,$

$a := t^{\complement} \in A,$

$[4] := \texttt{E}a\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)[3]\texttt{E}\complement : f(a)a = f(t^{\complement})t^{\complement} = \Big(f(t)t\Big)^{\complement} = e^{\complement} = 0,$

$[5] := \texttt{DeMorganaLaw}(A)[4]\texttt{E}t\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A)\texttt{DoubleNegationLaw}(A) : \forall b \leq (a + f(a))^{\complement} \,.\, b \leq tf(t),$

$[6] := \texttt{E}\,\mathrm{Tr}(f, t)[5] : \forall b \leq (a + f(a))^{\complement} \,.\, f(b) = b,$

$[7] := \texttt{EInvolution}(A, f)(a) : f^2(a) = a,$

$[a.*] := \texttt{EexchangingAutomorphism}[4][6][7] : f = \overleftarrow{a_f f(a)};$

$\leadsto [*] := \texttt{I} \iff [1] : \texttt{ExchangingInvolution}(A, f) \iff \exists\, \mathrm{Tr}(f);$

$\square$

1.8.2 Frolik's Theorem

$\texttt{FroliksLemma1} :: \forall A : \sigma\texttt{-Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall s \in \mathrm{Sep}(f) \,.$
$\quad .\, \exists y \in A : y \vee f(y) \vee f^2(y) \in \mathrm{Supp}(f)\ \&\ yf(y) = 0$

$\texttt{Proof} \ =$

$$a := \bigvee_{n=1}^{\infty} f^n(s) \in A,$$

$$b := \bigvee_{n=1}^{\infty} f^{-n}(s) \in A,$$

$[1] := \mathtt{E}_1\, \mathrm{Sep}(f,s) : f(s)s = 0,$

$[2] := \mathtt{E}_2\, \mathrm{Sep}(f,s)\mathtt{I}\,\mathrm{Supp} : s \cup a \cup b \in \mathrm{Supp}\, f,$

$$x := \Lambda n \in \mathbb{Z}_+ \,.\, f^n(s) \setminus \bigvee_{k=1}^{n-1} f^k(s) : \mathbb{N} \to A,$$

$$[3] := \mathtt{E}x\mathtt{E}a : a \vee s = \bigvee_{n=1}^{\infty} x_n,$$

$[4] := \mathtt{E}x\mathtt{IPairwiseDisjointElements} : \texttt{PairwiseDisjointElements}\Big(A, \mathrm{Im}\, x\Big),$

$$y_1 := \bigvee_{n=1}^{\infty} x_{2n} \setminus f^{-1}(s) \in A,$$

$[5] := f^{-1}[1] : f^{-1}(s)s = 0,$

$[6] := \mathtt{E}y_1[5] : s \le y_1 \le s \vee a,$

$[7] := \Lambda n \in \mathbb{N} \,.\, \mathtt{E}x\mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A,f)\mathtt{I}x :$

$$: \forall n \in \mathbb{N} \,.\, f\Big(x_{2n} \setminus f^{-1}(s)\Big) = f\left(f^{2n}(s) \setminus \bigvee_{k=-1}^{2n-1} f^k(s)\right) = f^{2n+1}(s) \setminus \bigvee_{k=0}^{2n} f^k(s) = x_{2n+1},$$

$[8] := \mathtt{E}y_1[7]\mathtt{E}\texttt{PairwiseDisjointElements}(A, \mathrm{Im}\, x)[4] : f(y_1)y_1 = 0,$

$[9] := \mathtt{E}y_1\mathtt{E}a : f(y_1) \le a,$

$[10] := \mathtt{E}y_1\mathtt{E}a : a \setminus f^{-1}(s) \le y_1 \vee f(y_1),$

$c := s \setminus a \in A,$

$[11] := \Lambda i, j \in \mathbb{Z} \,.\, \Lambda T : i < j \,.\, \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A,f)\mathtt{E}c\texttt{DifferenceProductBound}(A)\mathtt{E}a\texttt{ZeroImage}(A,A,f) :$

$$: \forall i, j \in \mathbb{Z} \,.\, i < j \Rightarrow f^i(c)f^j(c) = f^j\Big(cf^{i-j}(c)\Big) = f^i\Big((s \setminus a)\big(f^{j-i}(s) \setminus f^{j-i}(a)\big)\Big) \le f^i\Big(s \setminus f^{j-i}(a)\Big) = 0,$$

$[12] := \mathtt{E}c\texttt{CommonDiferenceUnion}(A)\texttt{TelescopingUnion}(A) \setminus \mathtt{I}b :$

$$: \bigvee_{n=1}^{\infty} f^{-n}(c) = \bigvee_{n=1}^{\infty} \left(f^{-n}(s) \setminus \bigvee_{k=-n+1}^{\infty} f^k(s)\right) = \bigvee_{n=1}^{\infty} \left(f^{-n}(c) \setminus \bigvee_{k=-n+1}^{-1} f^k(c)\right) \setminus (s \vee a) =$$

$$= \bigvee_{n=1}^{\infty} f^{-n}(a) \setminus (s \vee a) = b \setminus (s \vee a),$$

$[13] := \Lambda k \in \mathbb{N} \,.\, \Lambda i \in \mathbb{Z}_+ \,.\, \mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A,f) \,.\, \mathtt{E}c\texttt{ZeroImage}(A,A,f) :$

$$: \forall k \in \mathbb{N} \,.\, \forall i \in \mathbb{Z}_+ \,.\, f^{-k}(c)f^i(s) = f^{-k}\Big(cf^{i+k}(s)\Big) = f^{-k}(0) = 0,$$

$[14] := [13]\mathtt{I}y_1 : \forall k \in \mathbb{N} \,.\, f^{-k}(cy_1) = 0,$

$$y := y_1 \vee \bigvee_{n=1}^{\infty} f^{-2n}(c) \in A,$$

$[15] := \mathtt{E}y\mathtt{E}\texttt{AssociativeLattice}(A)[14][11] : yf(y) = 0,$

$[16] := \mathtt{E}y[12] : y \vee f(y) \vee f^{-1}(y) \geq y_1 \vee f(y_1) \vee f^{-1}(s) \vee \bigvee_{n=11}^{\infty} f^{-1}(c) \geq s \vee a \vee \Big(b \setminus (s \vee a)\Big) = s \vee a \vee b,$

$[17] := \mathtt{SupportContainsGreater}[2][16] : y \vee f(y) \vee f^{-1}(y) \in \mathrm{Supp}(f),$

$[*] := \mathtt{I}f^{-1}[17] : f^{-1}(y) \vee f\Big(f^{-1}(y)\Big) \vee f^2\Big(f^{-1}(y)\Big) \in \mathrm{Supp}(f);$

□

$\mathtt{TripleSupportImpliesSequenceSupport}$ ::

:: $\forall A : \sigma\text{-}\mathtt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall y \in A \,.\, y \vee f(y) \vee f^2(y) \in \mathrm{Supp}(f) \;\&\; yf(y) = 0 \Rightarrow$

$\Rightarrow \exists a : \mathbb{N} \to A : \bigvee_{n=1}^{\infty} f(a_n) \setminus a_n \in \mathrm{Supp}(f)$

$\mathtt{Proof}$ =

$a := \Lambda n \in \mathbb{N} \,.\, f^{n-2}(y) : \mathbb{N} \to A,$

$[1] := \mathtt{E}a[0.2]\mathtt{LatticeJoinIsGreater} : \bigvee_{n=1}^{\infty} f(a_n) \setminus a_n = \bigvee_{n=0}^{\infty} f^n(y) \geq y \vee f(y) \vee f^2(y),$

$[*] := \mathtt{SupportContainsGreater}[1][0.1] : \bigvee_{n=1}^{\infty} f(a_n) \setminus a_n \in \mathrm{Supp}(f);$

□

$\texttt{FroliksLemma2} :: \forall A : \sigma\texttt{-Algebra} . \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) . \forall a : \mathbb{N} \to A . \bigvee_{n=1}^{\infty} f(a_n) \setminus a_n \in \mathrm{Supp}(f) \Rightarrow$

$\Rightarrow \exists\, \mathrm{Sep}(f)$

$\texttt{Proof} =$

$$b := \Lambda n \in \mathbb{N} . \bigvee_{k=-\infty}^{\infty} f^k\Big(f(a_n) \setminus a_n\Big) \in \mathbb{N} \to A,$$

$[1] := \mathtt{E} b : \forall n \in \mathbb{N} . f(b_n) = b_n,$

$$c := \Lambda n \in \mathbb{N} . b_n \setminus \bigvee_{k=1}^{n} b_n : \mathbb{N} \to A,$$

$[2] := \mathtt{E} c[1] : \forall n \in \mathbb{N} . f(c_n) = c_n,$

$[3] := \mathtt{E} c \mathtt{E} \setminus : \texttt{PairwiseDisjointElements}(A, \mathrm{Im}\, c),$

$$s := \bigvee_{n=1}^{\infty} c_n\Big(a_n \setminus f^{-1}(a_n)\Big) \in A,$$

$[4] := \mathtt{E} s \mathtt{E} \sigma\texttt{-Continuous}(A, A, f)[2]\texttt{MultiplicationIsOrderContinuous}(A)$

$$\mathtt{E}\texttt{PairwiseDisjointElements}(A, \mathrm{Im}\, c)[3]\mathtt{E}\setminus : s f(s) = \left(\bigvee_{n=1}^{\infty} c_n\Big(a_n \setminus f^{-1}(a_n)\Big)\right)\left(\bigvee_{n=1}^{\infty} f(c_n)\Big(f(a_n) \setminus a_n\Big)\right) =$$

$$= \left(\bigvee_{n=1}^{\infty} c_n\Big(a_n \setminus f^{-1}(a_n)\Big)\right)\left(\bigvee_{n=1}^{\infty} c_n\Big(f(a_n) \setminus a_n\Big)\right) = \bigvee_{n,m=1}^{\infty} c_n c_m\Big(a_n \setminus f^{-1}(a_n)\Big)\Big(f(a_m) \setminus a_m\Big) =$$

$$= \bigvee_{n=1}^{\infty} c_n\Big(a_n \setminus f^{-1}(a_n)\Big)\Big(f(a_n) \setminus a_n\Big) = 0,$$

$\texttt{Assume}\ x \in A,$

$\texttt{Assume}\ [5] : \forall n \in \mathbb{Z} . f^n(s)x = 0,$

$[6] := \texttt{OrderContinuousMult}[5]\mathtt{E} s \mathtt{E} \sigma\texttt{-Continuous}(A, A, f)\mathtt{I} b \mathtt{E}\texttt{BooleanOrder}(A, b, c)\mathtt{E} c \mathtt{E} b :$

$$: 0 = \left(\bigvee_{n=-\infty}^{\infty} f^n(s)\right) x = \left(\bigvee_{n=-\infty}^{\infty} \bigvee_{m=1}^{\infty} f^n(c_m)\Big(f^n(a_m) \setminus f^{n-1}(a_m)\Big)\right) x =$$

$$= \left(\bigvee_{m=1}^{\infty} c_m \bigvee_{n=-\infty}^{\infty} \Big(f^n(a_m) \setminus f^{n-1}(a_m)\Big)\right) x = \left(\bigvee_{m=1}^{\infty} c_m b_m\right) x = \left(\bigvee_{m=1}^{\infty} c_m\right) x = \left(\bigvee_{m=1}^{\infty} b_m\right) x \geq$$

$$\geq \left(\bigvee_{m=1}^{\infty} f(a_m) \setminus a_m\right) x,$$

$[x.*] := \mathtt{E}\,\mathrm{Supp}[1] : f(x) = x;$

$\rightsquigarrow [*] := \mathtt{I}\,\mathrm{Sep}(f)[3] : s \in \mathrm{Sep}(f);$

$\square$

$\texttt{SixfoldLemma} :: \forall A : \sigma\text{-}\texttt{Algebra} \;.\; \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \;.\; \exists s : \mathrm{Sep}(f) \iff$

$\iff \exists a, a', b, b', c, d \in A : f(a) = b \;\&\; f(a') = b' \;\&\; f(b') = c \;\&\; f(b \vee c) = a \vee a' \;\&\; \forall u \leq d \;.\; f(u) = u$

$\texttt{Proof} \;=$

$\texttt{Assume}\; s \in \mathrm{Sep}(f),$

$\Big(u, [1]\Big) := \texttt{FroliksLemma1}(A, f, s) : \sum u \in A \;.\; uf(u) = 0 \;\&\; u \vee f(u) \vee f^2(u) \in \mathrm{Supp}(f),$

$c := f^2(u) \setminus (f(u) \vee u) \in A,$

$b' := f^{-1}(c) \in A,$

$a' := f^{-1}(b') \in A,$

$b := f(u) \setminus b' \in A,$

$a := u \setminus a' \in A,$

$d := \Big(u \vee f(u) \vee f^2(u)\Big)^{\complement} \in A,$

$[4] := \texttt{E}d\texttt{E}c : \texttt{PartitionOfUnity}\Big(A, \{c, a, a', b, b', d\}\Big),$

$[s.*] := \ldots : f(b \vee c) = f\Big((u \vee b' \vee d)^{\complement}\Big) = \Big(f(u) \vee f(b') \vee f(d)\Big)^{\complement} = \Big(f(u) \vee c \vee d\Big)^{\complement} = u = a \vee a';$

$\ldots$

$\square$

$\texttt{SupportBySeparator} :: \forall A \in \sigma\text{-}\texttt{Algebra} \;.\; \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(f) \;.\; \forall s \in \mathrm{Sep}(f) \;.\; \exists s' \in A : s' = \mathrm{supp}\, f$

$\texttt{Proof} \;=$

$\Big(u, [1]\Big) := \texttt{FroliksLemma1}(A, f, s) : \sum u \in A \;.\; uf(u) = 0 \;\&\; u \vee f(u) \vee f^2(u) \in \mathrm{Supp}(f),$

$s' := u \vee f(u) \vee f^2(u) \in \mathrm{Supp}(f),$

$\texttt{Assume}\; a \in \mathrm{Supp}(f),$

$\texttt{Assume}\; [2] : a < s',$

$[3] := f[1.1] : f(u)f^2(u) = 0,$

$[4] := f[2] : f^2(u)f^3(u) = 0,$

$[5] := \texttt{E}\,\mathrm{Supp}(f, a)\texttt{E}s'[1.1][3][4] : s' \setminus a = 0,$

$[a.*] := \texttt{TrichtomyPrinciple}(A)\texttt{ReminderRule}(A)[5] : \bot;$

$\rightsquigarrow [*] := \texttt{E}\bot\texttt{E} < \texttt{SupportIsClosedUnderIntersections}(A, f)\texttt{I}\,\mathrm{supp} : s' = \min \mathrm{Supp}\, f = \mathrm{supp}\, f;$

$\square$

$\texttt{FroliksTHM} :: \forall A : \tau\texttt{-Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \exists\, \mathrm{Sep}(f)$

$\texttt{Proof} \;=$

$P := \Big\{ a \in A : f(a)a = 0 \Big\} :?A,$

$[1] := \mathtt{E}P\texttt{ZeroImage}(A, f) : 0 \in P,$

$[2] := \mathtt{I}\exists[1] : \exists P,$

$\texttt{Assume}\ C : \texttt{Chain}(P),$

$c := \bigvee C \in A,$

$\texttt{Assume}\ a, b \in C,$

$\Big(d, [3]\Big) := \mathtt{E}\texttt{Chain}(C, a, b) : \sum d \in C \,.\, a \le d \;\&\; b \le d,$

$\Big[(a, b). *\Big] := \texttt{BooleanMorphismIsMonotonic}(A, A, f)[3]\texttt{MonotonicMeet}(A)[3]\mathtt{E}P(d) : af(b) \le df(d) = 0;$

$\rightsquigarrow [3] := \mathtt{I}\forall : \forall a, b \in C \,.\, af(b) = 0,$

$[4] := \mathtt{E}C\mathtt{E}\texttt{OrderContinuous}(A, A, f)[3]\mathtt{E}\vee : f(c)c = f\left(\bigvee C\right) \bigvee C = \bigvee_{a \in C} \bigvee_{b \in C} af(b) = \bigvee_{a \in C} \bigvee_{b \in C} 0 = 0,$

$[C.*] := \mathtt{E}P[4] : c \in P;$

$\rightsquigarrow \Big(s, [3]\Big) := \texttt{ZornsLemma}[2] : \sum s \in P \,.\, s = \max P,$

$\texttt{Assume}\ a \in A,$

$\texttt{Assume}\ [4] : \forall n \in \mathbb{Z} \,.\, f^n(s)a = 0,$

$\texttt{Assume}\ [5] : a \ne f(a),$

$[6] := \mathtt{I} \setminus [5] : a \setminus f(a) \ne 0 \Big| f(a) \setminus a \ne 0,$

$\texttt{Assume}\ [7] : a \setminus f(a) \ne 0,$

$b := a \setminus f(a) \in A,$

$c := s \vee b \in A,$

$[8] := \mathtt{E}c[4] : c > s,$

$[9] := \mathtt{E}c\mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\mathtt{E}\texttt{AssociativeLattice}(A)\mathtt{EE}P(s)\mathtt{E}b[4] :$

$\quad : f(c)c = f(s \vee b)s \vee b = f(s)s \vee f(b)s \vee f(s)b \vee f(b)b = 0,$

$[10] := \mathtt{E}P[9] : c \in P,$

$[7.*] := [10][8][3] : \perp;$

$\rightsquigarrow [7] := \mathtt{I} \Rightarrow : a \setminus f(a) \ne 0 \Rightarrow \perp,$

$\texttt{Assume}\ [8] : f(a) \setminus a \ne 0,$

$b := f(a) \setminus a \in A,$

$c := s \vee b \in A,$

$[9] := \mathtt{E}c[4] : c > s,$

$[10] := \mathtt{E}c\mathtt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\mathtt{E}\texttt{AssociativeLattice}(A)\mathtt{EE}P(s)\mathtt{E}b[4] :$

$\quad : f(c)c = f(s \vee b)s \vee b = f(s)s \vee f(b)s \vee f(s)b \vee f(b)b = 0,$

$[11] := \mathtt{E}P[10] : c \in P,$

$[8.*] := [11][9][3] : \perp;$

$\rightsquigarrow [8] := \mathtt{I} \Rightarrow : f(a) \setminus a \ne 0 \Rightarrow \perp,$

$[a.*] := \mathtt{E}|[6][7][8] : \perp;$

$\rightsquigarrow [*] := \mathtt{E}\perp\mathtt{I}\forall\mathtt{I}\,\mathrm{Sep}\,\mathtt{E}P(s) : s \in \mathrm{Sep}(f);$

$\square$

1.8.3 Towards Factorization by Exchanging Involutions

$\texttt{ExchangingInvolutionsInTheCompleteAlgebra} ::$

$:: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall f : \texttt{Involution}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big) \,.\, \texttt{ExchangingInvolution}(A, f)$

$\texttt{Proof} \;=$

$[1] := \texttt{FrolicsTHM}(A, f) : \exists\,\mathrm{Sep}(f),$

$[*] := \texttt{ExchangingInvolutionBySeparator}(A, f)[1] : \texttt{ExchangingInvolution}(A, f);$

$\square$

$\texttt{ExchangingAutomorphisimInTheCompleteAlgebra} ::$

$:: \forall A : \tau\text{-}\texttt{Algebra} \,.\, \forall f : \texttt{Periodic}(A) \,.\, \pi(f) \geq 2 \Rightarrow \exists a \in A :$

$: \texttt{PartitionOfUnity}\left(A, \left\{f^k(a) \middle| k \in \big[0, \ldots, \pi(f) - 1\big]\right\}\right) \;\&\; f = \overleftarrow{a_{1f}\, f(a)_f \ldots_f f^{\pi(f)-1}(a)}$

$\texttt{Proof} \;=$

$[1] := \Lambda k \in \big[0, \ldots, \pi(f) - 1\big]\texttt{FrolicsTHM}(A, f^k) : \forall k \in \big[0, \ldots, \pi(f) - 1\big] \,.\, \exists\,\mathrm{Sep}(f^k),$

$a := \texttt{TransversalConstructionLemma}(A, f)[1]\texttt{EPeriodic}(A, f) \in \mathrm{Tr}(f),$

$[2] := \Lambda k \in \big[0, \ldots, \pi(f) - 1\big]\texttt{EPeriodic}(A, f, k) : \forall k \in \big[0, \ldots, \pi(f) - 1\big] \,.\, \operatorname{supp} f^k = e,$

$[3] := \texttt{E}\operatorname{supp}[2]\mathrm{E}_2\,\mathrm{Tr}(f, a) : \forall k, l \in \big[0, \ldots, \pi(f) - 1\big] \,.\, k \neq l \Rightarrow f^k(a) f^l(a) = 0,$

$[*.1] := \mathrm{E}_1\,\mathrm{Tr}(f, a)[3]\texttt{IPartitionOfUnity} : \texttt{PartitionOfUnity}\left(A, \left\{f^k(a) \middle| k \in \big[0, \ldots, \pi(f) - 1\big]\right\}\right),$

$[*] := \texttt{IexchangingInvolution}[*.1] : f = \overleftarrow{a_{1f}\, f(a)_f \ldots_f f^{\pi(f)-1}(a)};$

$\square$

$\texttt{TransversalAggregation} :: \forall A : \sigma\texttt{-Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a : \mathbb{N} \to A \,.$

$$. \left(\forall n \in \mathbb{N} \,.\, f(a_n) = a_n \;\&\; \exists\, \mathrm{Tr}(f_{|\langle a_n\rangle_{\mathcal{I}}})\right) \Rightarrow \exists\, \mathrm{Tr}(f_{|\langle b\rangle_{\mathcal{I}}}) \quad \texttt{where} \quad b = \bigvee_{n=1}^{\infty} a_n$$

$\texttt{Proof} \;=$

$$t := \mathtt{E}\exists[0] : \prod_{n=1}^{\infty} \mathrm{Tr}(\langle a_n\rangle_{\mathcal{I}}),$$

$$[1] := \mathtt{E}b\mathtt{E}\sigma\texttt{-Continuous}(A, A, f)[0.1]\mathtt{I}b : f(b) = f\left(\bigvee_{n=1}^{\infty} a_n\right) = \bigvee_{n=1}^{\infty} f(a_n) = \bigvee_{n=1}^{\infty} a_n = b,$$

$$u := \bigvee_{n=1}^{\infty}\left(t_n \setminus \bigvee_{k=1}^{n-1} a_k\right) \in A,$$

$[2] := \mathtt{E}u\texttt{MonotonicSup}(A)\mathtt{I}u : u \le b,$

$[3] := \mathtt{E}u\Lambda n \in \mathbb{Z} \,.\, \mathtt{E}\texttt{OrderContinuous}(A, A, f^n)[0.1]\Lambda m \in \mathbb{N} \,.\, \mathtt{E}_1\, \mathrm{Tr}(f_{|\langle a_m\rangle_{\mathcal{I}}}, t_m)\mathtt{I}b :$

$$: \bigvee_{n=-\infty}^{\infty} f^n(u) = \bigvee_{n=-\infty}^{\infty} f^n\left(\bigvee_{m=1}^{\infty}\left(t_n \setminus \bigvee_{k=1}^{m-1} a_k\right)\right) = \bigvee_{n=-\infty}^{\infty}\bigvee_{m=0}^{\infty}\left(f^n(t_m) \setminus \bigvee_{k=1}^{m-1} a_k\right) = \bigvee_{m=0}^{\infty} a_m = b,$$

$[4] := \Lambda n \in \mathbb{Z} \,.\, \Lambda c \le u f^n(u) \,.\, \mathtt{E}\texttt{BooleanOrder}\Big(A, c, f^n(u)u\Big)\mathtt{E}u\mathtt{E}\texttt{OrderContinuous}(A, A, f^n)$

$\texttt{OrderContinuousMult}(A)\mathtt{E} \setminus \mathtt{E}\texttt{OrderContinuous}(A, A, f^n)\mathtt{E}\,\mathrm{Tr}(f)[0.2]\mathtt{E}\setminus$

$\mathtt{E}\texttt{OrderContinuous}(A, A, f^n)\texttt{OrderContinuousMult}(A)\mathtt{I}u\mathtt{E}\texttt{BooleanOrder}\Big(A, c, f^n(u)u\Big) :$

$$: \forall n \in \mathbb{Z} \,.\, \forall c \le u f^n(u) \,.\, f^n(c) = f^n\Big(c f^n(u) u\Big) = f^n\left(c f^n\left(\bigvee_{m=1}^{\infty}\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\right)\bigvee_{k=1}^{\infty}\left(t_k \setminus \bigvee_{h=1}^{k-1} a_h\right)\right) =$$

$$= f^n \bigvee_{m=1}^{\infty}\bigvee_{k=1}^{\infty} c f^n\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\left(t_k \setminus \bigvee_{h=1}^{k-1} a_h\right) = f^n \bigvee_{m=1}^{\infty} c f^n\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right) =$$

$$= \bigvee_{m=1}^{\infty} f^n\left(c f^n\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\right) = \bigvee_{m=1}^{\infty} c f^n\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right) =$$

$$= \bigvee_{m=1}^{\infty}\bigvee_{k=1}^{\infty} c f^n\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\left(t_k \setminus \bigvee_{h=1}^{k-1} a_h\right) = c f^n\left(\bigvee_{m=1}^{\infty}\left(t_m \setminus \bigvee_{l=1}^{m-1} a_l\right)\right)\bigvee_{k=1}^{\infty}\left(t_k \setminus \bigvee_{h=1}^{k-1} a_h\right) = c f^n(u) u = c,$$

$[*] := \mathtt{I}\,\mathrm{Tr}(f)[3][4] : u \in \mathrm{Tr}(f_{\langle b\rangle_{\mathcal{I}}});$

$\square$

$\texttt{InverseTransversality} :: \forall A \in \mathsf{BOOL} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall t \in \mathrm{Tr}(f) \,.\, t \in \mathrm{Tr}(f^{-1})$

$\texttt{Proof} \;=$

$\dots$

$\square$

$$\texttt{downstreamElement} :: \prod A : \sigma\texttt{-Algebra} \,.\, \Big(A \times \mathrm{Aut}_{\mathsf{BOOL}}(A)\Big) \to A$$

$$\texttt{downstreamElement}\,(a, f) = a^*_f := \bigvee_{n=-\infty}^{\infty}\left(f^n(a) \setminus \bigvee_{k=n+1}^{\infty} f^k(a)\right)$$

$$\texttt{upstreamElement} :: \prod A : \sigma\texttt{-Algebra} \,.\, \Big(A \times \mathrm{Aut}_{\mathsf{BOOL}}(A)\Big) \to A$$

$$\texttt{upstreamElement}\,(a, f) = a^f_* := \bigvee_{n=-\infty}^{\infty}\left(f^n(a) \setminus \bigvee_{k=-\infty}^{n-1} f^k(a)\right)$$

$\texttt{DownstreamElementIsFixed} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, a^*_f \in \mathrm{Fix}(f)$

$\texttt{Proof} \;=$

$[*] := \texttt{E}a^*_f\texttt{EOrderContinuous}(A, A, f)\texttt{I}a^*_f :$

$$: f(a^*_f) = f \bigvee_{n=-\infty}^{\infty} \left(f^n(a) \setminus \bigvee_{k=n+1}^{\infty} f^k(a) \right) = \bigvee_{n=-\infty}^{\infty} \left(f^n(a) \setminus \bigvee_{k=n+1}^{\infty} f^k(a) \right) = a^*_f;$$

$\square$

$\texttt{DownstreamElementAdmitsTransversals} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, \exists\, \mathrm{Tr}(f_{|\langle a^*_f \rangle_{\mathcal{I}}})$

$\texttt{Proof} \;=$

$$t := a \setminus \bigvee_{n=1}^{\infty} f^n(a) \in A,$$

$[1] := \texttt{E}t\texttt{E}a^*_f\texttt{E}\vee : t \le a^*_f,$

$[2] := \texttt{E}a^*_f\texttt{EOrderContinuous}(A, A, f)\texttt{E}\mathrm{Aut}_{\mathsf{BOOL}}(A, f)\texttt{I}t :$

$$: a^*_f = \bigvee_{n=-\infty}^{\infty} \left(f^n(a) \setminus \bigvee_{k=n+1}^{\infty} f^k(a) \right) = \bigvee_{n=-\infty}^{\infty} f^n \left(a \setminus \bigvee_{n=1}^{\infty} f^n(a) \right) = \bigvee_{n=-\infty}^{\infty} f^n(t),$$

$[3] := \texttt{E}t\texttt{E}\setminus : \forall n \in \mathbb{Z} \,.\, n \ne 0 \Rightarrow t f^n(t) = 0,$

$[*] := \texttt{I}\; \mathrm{Tr}(f_{|\langle a^*_f \rangle_{\mathcal{I}}})[2][3] : t \in f_{|\langle a^*_f \rangle_{\mathcal{I}}};$

$\square$

$\texttt{UpstreamElementIsFixed} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, a^f_* \in \mathrm{Fix}(f)$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{UpstreamElementAdmitsTransversals} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall a \in A \,.\, \exists\, \mathrm{Tr}(f_{|\langle a^f_* \rangle_{\mathcal{I}}})$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{TransversalFactorizationTHM} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall f \in \mathrm{Aut}_{\mathsf{BOOL}}(A) \,.\, \forall t \in \mathrm{Tr}(A) \,.$

$\quad . \; \exists \alpha, \beta : \texttt{ExchangingInvolution}(A) \,.\, f = \alpha\beta \;\&\; \alpha, \beta \in \langle f \rangle_{\mathrm{CF}}$

$\texttt{Proof} \;=$

$\ldots$

$\square$

$\texttt{SubgroupWithSeparators} :: \prod A \in \mathsf{BOOL} \,.\, ??_{\mathsf{GRP}} \mathrm{Aut}_{\mathsf{BOOL}}(A)$

$G : \texttt{SubgroupWithSeparators} \iff \forall g \in G \,.\, \exists\, \mathrm{Sep}(g)$

$\texttt{SwSHasSupports} :: \forall A : \sigma\text{-}\texttt{Algebra} \,.\, \forall G : \texttt{SubgroupWithSeparators}(A) \,.\, \forall g \in G \,.\, \exists a \in A : a = \operatorname{supp} g$

$\texttt{Proof} \;=$

$\ldots$

$\square$

ExistanceOfTransversalInSwSCondition ::

$$:: \forall A : \sigma\text{-}\mathtt{Algebra} \,.\, \forall G : \mathtt{SubgroupWithSeparators} \,.\, \forall g \in G \,.\, \forall n \in \mathbb{N} \,.\, \left(g^n = \mathrm{id} \Rightarrow \exists\, \mathrm{Tr}(g)\right)$$

Proof =

$[1] := \Lambda k \in [1, \ldots, n-1] \,.\, \mathtt{ESubgroupWithSeparators}(G, g^k) : \forall k \in [1, \ldots, n-1] \,.\, \exists\, \mathrm{Sep}(g^k),$

$[*] := \mathtt{TransversalConstructionLemma}[0][1] : \exists\, \mathrm{Tr}(g);$

$\square$

SWSLocalization $:: \forall A : \sigma\text{-}\mathtt{Algebra} \,.\, \forall G : \mathtt{SubgroupWithSeparators} \,.\, \forall g \in G \,.\, \exists\, \mathrm{Tr}\left(g_{|\langle a \rangle_{\mathcal{I}}}\right)$

$$\text{where} \quad a = \left(\bigwedge_{n=1}^{\infty} \operatorname{supp} g^n\right)^{\complement}$$

Proof =

...

$\square$

CountablyFullSwSLemma ::

$$:: \forall : \sigma\text{-}\mathtt{Algebra} \,.\, \forall G : \mathtt{SubgroupWithSeparators} \;\&\; \mathtt{CountablyFullSubgroup}(A) \,.$$

$$.\, \forall a \in \mathrm{Fix}(G) \,.\, \mathtt{SubgroupWithSeparators} \;\&\; \mathtt{CountablyFullSubgroup}\left(\langle a \rangle_{\mathcal{I}}, G_{|\langle a \rangle_{\mathcal{I}}}\right)$$

Proof =

...

$\square$

1.8.4 The Great Exchange

$$\texttt{TriplingSequence} :: \prod_{A \in \mathsf{BOOL}} \mathrm{Aut}_{\mathsf{BOOL}}(A) \to ?(\mathbb{Z}_+ \downarrow A)$$

$$a : \texttt{TriplingSequence} \iff a_0 = e \;\&\; \Bigg(\forall n \in \mathbb{Z} \;.\; \texttt{DoublyRecurrentOn}(A, a_n, g) \;\&$$

$$\& \bigvee_{m=1}^{\infty} g^m(a_n) = \bigvee_{m=1}^{\infty} g^{-m}(a_n) = e \;\&\; \texttt{PairwiseDisjointElements}\Big(A, \big\{a_{n+1}, g_{a_n}(a_{n+1}), g^2_{a_n}(a_{n+1})\big\}\Big)\Bigg)$$

$\texttt{TriplingSequenceConstruction}$ ::

$: \forall A : \sigma\texttt{-Algebra} \;.\; \forall G : \texttt{SubgroupWithSeparators} \;\&\; \texttt{CountablyFullSubgroup}(A) \;.\; \forall g \in G \;.$

$. \;\texttt{Aperiodic}(A, g) \Rightarrow \exists \texttt{TriplingSequence}(A, g)$

Proof =

...

□

$\texttt{TheGreatExchangeLemma} :: \forall A : \sigma\texttt{-Algebra} \;.\; \forall f : \texttt{Aperiodic}(A) \;.\; \forall a : \texttt{TriplingSequence}(A, f) \;.$

$$. \;\exists \phi \in \langle f \rangle_{\mathrm{CF}} : \texttt{ExchangingInvolution}(A, \phi) \;\&\; \bigwedge_{n=1}^{\infty} \mathrm{supp}(\phi f)^n = 0$$

Proof =

...

□

$\texttt{TransversalCompletionLemma}$ ::

$\forall A : \sigma\texttt{-Algebra} \;.\; \forall G : \texttt{SubgroupWithSeparators} \;\&\; \texttt{CountablyFullSubgroup}(A) \;.\; \forall g \in G \;.$

$. \;\exists \phi \in G : \texttt{ExchangingInvolution}(A, \phi) \;\&\; \exists\, \mathrm{Tr}(\phi g)$

Proof =

...

□

$\texttt{MainFactorizationTHM}$ ::

$\forall A : \sigma\texttt{-Algebra} \;.\; \forall G : \texttt{SubgroupWithSeparators} \;\&\; \texttt{CountablyFullSubgroup}(A) \;.\; \forall g \in G \;.$

$. \;\exists \alpha, \beta, \gamma \in G : \texttt{ExchangingInvolution}(A, \alpha \;\&\; \beta \;\&\; \gamma) \;\&\; g = \alpha\beta\gamma$

Proof =

...

□

$\texttt{CompleteFactorizationTHM}$ ::

$\forall A : \tau\texttt{-Algebra} \;.\; \forall G : \texttt{FullSubgroup}(A) \;.\; \forall g \in G \;.$

$. \;\exists \alpha, \beta, \gamma \in G : \texttt{ExchangingInvolution}(A, \alpha \;\&\; \beta \;\&\; \gamma) \;\&\; g = \alpha\beta\gamma \;\&\; \mathrm{supp}\, g \in \mathrm{Supp}\,\alpha \cap \mathrm{Supp}\,\beta \cap \mathrm{Supp}\,\gamma$

Proof =

...

□

1.8.5 Subgroups with many involutions and simplicity of it all

$$\texttt{SubgroupWithManyInvolutions} :: \prod_{A \in \mathsf{BOOL}} ??\mathsf{GRP}\mathrm{Aut}_{\mathsf{BOOL}}(A)$$

$$G : \texttt{SubgroupWithManyInvolutions} \iff \forall a \in A \;.\; a \neq 0 \Rightarrow \exists g \in G : \texttt{Involution}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A), g\Big) \;\&$$
$$\& \; a \in \mathrm{Supp}(g)$$

$\texttt{AtomlessHomogeneousHasManyInvolutions}$ ::

$$:: \forall A : \texttt{Atomless} \;\&\; \texttt{Homogeneous} \;.\; \texttt{SubgroupWithManyInvolutions}\Big(A, \mathrm{Aut}_{\mathsf{BOOL}}(A)\Big)$$

$\texttt{Proof}$ =

...

□

$\texttt{SubgroupWithManyExchangingInvolutions}$ :: ::

$$\forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall G : \texttt{FullSubgroup} \;\&\; \texttt{SubgroupWithManyInvolutions}(A) \;.$$
$$.\; \forall a \in A \;.\; a \neq 0 \Rightarrow \exists g \in G : \texttt{ExchangingInvolution}(A, g) \;\&$$
$$\& \; a \in \mathrm{Supp}(g)$$

$\texttt{Proof}$ =

...

□

$\texttt{NormalSubgroupsAreInvariantIdeals}$ ::

$$:: \forall A : \tau\text{-}\texttt{Algebra} \;.\; \forall G : \texttt{FullSubgroup} \;\&\; \texttt{SubgroupWithManyInvolutions}(A) \;.\; \forall H \subset G \;.$$
$$.\; H \lhd G \iff \exists I : \texttt{Ideal}(A) \;\&\; \texttt{Invariant}(G, A) : H = \{g \in G : \operatorname{supp} g \in I\}$$

$\texttt{Proof}$ =

...

□

$$\texttt{SimplicityTHM} :: \forall A : \tau\text{-}\texttt{Algebra} \;\&\; \texttt{Homogeneous} \;.\; \texttt{Simple}\Big(\mathrm{Aut}_{\mathsf{BOOL}}(A)\Big)$$

$\texttt{Proof}$ =

...

□

1.9 Simple Functions[!]

This chapter represents knowledge from Fremlin's measure theory 361,362; prereq: OTVS

2 Applications towards Analysis[!]

Possible sources for this chapter is the book by Vladimirov; prereq: Spectral Analysis

3 Applications towards Logic and Set Theory [!]

Possible sources for this chapter is handbook of Boolean Algebras by Monk et al. and lecture notes by Podzorov; prereq: Forcing and M-Logic

Sources:

1. MEASURE THEORY by D.H.Fremlin, chapters 31 and 38