Abstract Measure Theory

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Contents

Intro

1 Measure Algebras

1.1 Subject

1.1.1 Definition and Basic Property

MeasureAlgebra $:: ? \sum A : \sigma$-DedekindComplete . $A \to \overset{\infty}{\mathbb{R}}_+$

(A, μ) : MeasureAlgebra $\iff \forall a \in A \,.\, \mu(a) = 0 \iff a = 0$ &

& $\forall a$: PairwiseDisjointElements$(\mathbb{N}, A)$. $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$

MeasureMonotonicity :: $\forall (A, \mu)$: MeasureAlgebra . $\forall a, b \in A \,.\, a \leq b \Rightarrow \mu(a) \leq \mu(b)$

Proof =

Write $\mu(b) = \mu(a) + \mu(b \setminus a) \geq \mu(a)$.

□

MeasureStrictMonotonicity :: $\forall (A, \mu)$: MeasureAlgebra . $\forall a, b \in A \,.\, a > b \Rightarrow \mu(a) > \mu(b)$

Proof =

Definition of measure algebra implies that $\mu(b \setminus a) > 0$.

Write $\mu(b) = \mu(a) + \mu(b \setminus a) > \mu(a)$.

□

MinkovskyIneq :: $\forall (A, \mu)$: MeasureAlgebra . $\forall a, b \in A \,.\, \mu(a \vee b) \leq \mu(a) + \mu(b)$

Proof =

Write $\mu(a) + \mu(b) = \mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab) + \mu(ab) \geq mu(a \setminus ab) + \mu(ab) + \mu(b \setminus ab = \mu(a \vee b)$.

□

MonotonicSupremumAsLimit :: $\forall (A, \mu)$: MeasureAlgebra . $\forall a : \mathbb{N} \uparrow A \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n \to \infty} \mu(a_n)$

Proof =

Construct disjoint sequence $b_n = a_n \setminus \bigvee_{k=1}^{n-1} a_k$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty} \mu(b_n) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(b_n) = \lim_{n \to \infty} \mu\left(\bigvee_{k=1}^{n} b_k\right) = \lim_{n \to \infty} \mu(a_n)$.

□

$\texttt{SupremumIneq} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \to A \,.\, \mu\left(\bigvee_{n=1}^{\infty} a_n\right) \le \sum_{n=1}^{\infty} \mu(a_n)$

$\texttt{Proof} =$

Construct increasing sequence $b_n = \bigvee_{k=1}^{n} a_n$.

Then by construction $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) = \lim_{n\to\infty} \mu\left(\bigvee_{k=1}^{n} a_k\right) \le \lim_{n\to\infty} \sum_{k=1}^{n} \mu(a_k) = \sum_{n=1}^{\infty} \mu(a_n)$.

$\square$

$\texttt{MonotonicInfimumAsLimit} ::$

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall a : \mathbb{N} \downarrow A \,.\, \forall \aleph : \inf_{n\in\mathbb{N}} \mu(a_n) < \infty \,.\, \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$$

$\texttt{Proof} =$

Without loss of generality assume that $\mu(a_1) < \infty$.

Then construc the increasing sequence $b_n = a_1 \setminus a_n$.

Then $\mu(a_1) - \mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(a_1 \setminus \bigwedge_{n=1}^{\infty} a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} a_1 \setminus a_n\right) = \mu\left(\bigvee_{n=1}^{\infty} b_n\right) = \lim_{n\to\infty} \mu(b_n) =$

$= \lim_{n\to\infty} \mu\left(a_1 \setminus a_n\right) = \lim_{n\to\infty} \mu(a_1) - \mu(a_n) = \mu(a_1) - \lim_{n\to\infty} \mu(a_n)$.

So basic algebraic manipulations $\mu\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n\to\infty} \mu(a_n)$.

$\square$

$\texttt{SupremumExistance} ::$

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall C : \texttt{UpwardsDirected}(A) \,.\, \forall \aleph : \sup_{c\in C} \mu(c) < \infty \,.\, \exists a \in A : a = \sup C$$

$\texttt{Proof} =$

1 Assume $\gamma = \sup_{c\in C} \mu(c)$.

2 Then there exists a sequrnce of $a : \mathbb{N} \to C$ such that $\mu(a_n) \ge \gamma - 2^{-n}$.

3 As C is upwards closed, it is possible to find $c : \mathbb{N} \to C$ such that $c_{n+1} \ge a_n \vee c_n$.

4 Then c is monotonic-nondecreasing and so it has $\mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \lim_{n\to\infty} \mu(c_n) = \gamma$.

4.1 Note that $\gamma \ge \mu(c_n) \ge \gamma - 2^{-n}$.

5 let $d = \bigvee_{n=1}^{\infty} c_n$.

6 $d \ge f$ for everty $f \in C$.

6.1 Assume this is false.

6.2 Then $f \setminus d \ne 0$ and so $\alpha = \mu(f \setminus d) > 0$.

6.3 Then there exists n such that $\gamma - \mu(c_n) < \alpha$.

6.4 As C is upwards derected there is $g \in C$ such that $g \ge f \vee c_n$.

6.5 But $\mu(g) \ge \mu(f \vee c_n) = \mu(c_n) + \mu(f \setminus c_n) \ge \mu(c_n) + \mu(f \setminus d) > \gamma$ which is impossible.

7 If there is another f with the property (6), then $d = \bigvee_{n=1}^{\infty} c_n \le f$ as $c_n \le f$ for each $n \in \mathbb{N}$.

$\square$

`UpperContinuity` ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall C : \texttt{UpwardsDirected}(A) \;.\; \forall \aleph : \exists a \in A : a = \sup C \;.\; \sup_{c \in C} \mu(c) = \mu(\sup C)$

`Proof` $=$

Case $\sup_{c \in C} \mu(c) = \infty$ is trivial.

Finite case follows from the cconstruction in the previous theorem.

$\square$

`DisjointUpperContinuity` ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall C : \texttt{PairwiseDisjointElements}(A) \;.\; \forall \aleph : \exists a \in A : a = \sup C \;.$

$. \; \mu(\sup C) = \sum_{c \in C} \mu(c)$

`Proof` $=$

Construct a new set $D = \left\{ \bigvee_{n=1}^{\infty} c_k \middle| c : \mathbb{N} \to C \right\}$.

Then D is upwards directed and $\sup C = \sup D$.

But this is evedent that $\mu(\sup D) = \sup_{d \in D} \mu(d) = \sup_{c : \mathbb{N} \to C} \mu\left(\bigvee_{n=1} c_n \right) = \sup_{n \in \mathbb{N}, c : \{1, \ldots, n\} \to C} \sum_{k=1}^{n} \mu(c_k) = \sum_{c \in C} \mu(c)$.

$\square$

`InfimumExistance` ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall C : \texttt{DownwaedDirected}(A) \;.\; \forall \aleph : \inf_{c \in C} \mu(c) < \infty \;.\; \exists a \in A : a = \inf C$

`Proof` $=$

1 There exists some $a \in C$ such that $\mu(a) < \infty$.

2 Construct another set $D = a \setminus C$.

3 Then D is upwards directed and $\sup_{d \in D} \mu(d) \leq \mu(a) < \infty$.

4 So there is $d = \sup d$.

5 Define $f = a \setminus d$.

6 $f \leq c$ for any $c \in C$ as $a \setminus f \geq a \setminus c$.

7 if some g has property (6) then $a \setminus g \geq d$ and so $g \leq f$.

$\square$

`LowerContinuity` ::

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall C : \texttt{DownwardsDirected}(A) \;.\; \forall \aleph : \exists a \in A : a = \inf C \;.$

$. \; \forall \beth : \inf_{c \in C} \mu(c) < \infty \;.\; \inf_{c \in C} \mu(c) = \mu(\inf C)$

`Proof` $=$

Use the construction in the previous theorem.

$\square$

1.1.2 Measure Algebras Generated by Measure Spaces

measureAlgebra :: MEAS $\to$ MeasureAlgebra

measureAlgebra $(X,\Sigma,\mu) = (A_\mu, \bar{\mu}) := \left(\dfrac{\Sigma}{\Sigma \cap \mathcal{N}_\mu}, [E] \mapsto \mu(E)\right)$

This is obviously well defined as $[E]=[F]$ iff $\mu(E \triangle F) = 0$.

canononicalProjection :: $\forall (X,\Sigma,\mu) \in$ MEAS . σ-BOOL(Σ, A_μ)

canonicalProjection $(E) = \pi_\mu(E) := [E]$

1 The algebraic properites are obvious as $\Sigma \cap \mathcal{N}_\mu$ is an ideal.

2 In order to prove sigma-continuity assume $E : \mathbb{N} \to \Sigma$.

2.1 Let $Z : \mathbb{N} \to \Sigma \cap \mathcal{N}_\mu$.

2.2 Then $F_Z = \bigvee_{n=1}^{\infty} (E_n \triangle Z_n) = \left(\bigvee_{n=1}^{\infty} E_n\right) \triangle \left(\bigvee_{n=1}^{\infty} Z_n\right)$.

2.3 Note that $\mu\left(\bigvee_{n=1}^{\infty} Z_n\right) \le \sum_{n=1}^{\infty} \mu(Z_n) = 0$.

2.4 So $\bigvee_{n=1}^{\infty} Z_n \in \Sigma \cap \mathcal{N}_\mu$ as $\mu \ge 0$.

2.5 Thus $[F_Z] = \left[\bigcap_{n=1}^{\infty} E_n\right]$ for any selection of Z.

2.6 This means that $\pi_\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \bigvee_{n=1}^{\infty} \pi_\mu(E_n)$.

□

MeasureAlgebraMonotonicity :: $\forall (X,\Sigma,\mu) \in$ MEAS . $\forall T \subset_\sigma \Sigma$. $\pi_\mu(T) \subset_\sigma A_\mu$

Proof =

1 Clearly $B = \pi_\mu(T) \subset A_\mu$.

2 Also as T is σ-algebra and $\pi - \mu$ is a σ-continuous homomorphism B is again.

□

MeasureAlgebraInverseMonotonicity :: $\forall (X,\Sigma,\mu) \in$ MEAS . $\forall B \subset_\sigma A_\mu$. $\pi_\mu^{-1}(B) \subset_\sigma \Sigma$

Proof =

1 Clearly $T = \pi_\mu^{-1}(B) \subset \Sigma$.

2 Assume F is a set constructed by applying σ-algebra operations to setes $E_1, E_2, \ldots \in T$.

3 Then $\pi_\mu(F)$ can be constructed by applying same operations to $\pi(E_1), \pi(E_2), \ldots$.

4 This implies that $\pi_\mu(F) \in B$ and reciprorary $F \in T$.

5 Thus T is a σ-algebra.

□

1.1.3 Stone Representation Theorem

$\texttt{StoneRepresentationTheorem} :: \forall(A,\mu) : \texttt{MeasureAlgebra} \,.\, \exists(X,\Sigma,\nu) \in \mathsf{MEAS} \,.\, (A,\mu) = (A_\nu, \bar{\nu})$

$\texttt{Proof} \ =$

1 By Loomis-Sikorski representation there exists a set X with a sigma-algebra Σ and sigma-ideal I such that $\dfrac{\Sigma}{I} \cong_{\mathsf{BOOL}} A$.

2 Then there is a canonical projetion $\pi_I \in \mathsf{BOOL}(\Sigma, A)$.

3 Define $\nu = \pi_I \mu$.

4 ν is measure on Σ.

4.1 $\nu(\emptyset) = \mu(0) = 0$.

4.2 Assume E is a disjoint sequence in Σ.

4.3 Then $\pi_I(E_n)\pi_I(E_m) = \pi_i(E_n \cap E_m) = \pi_i(\emptyset) = 0$, so $\pi_I(E)$ is disjoint in A.

4.4 Thus, $\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \pi_I \mu \left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigvee_{n=1}^{\infty} \pi_I(E_n)\right) = \sum_{n=1}^{\infty} \pi_I \mu(E_n) = \sum_{n=1}^{\infty} \nu(E_n)$.

5 Also by consytuction $\mathcal{N}_\nu \cap \Sigma = I$, so $(A,\mu) = (A_\nu, \bar{\nu})$.

□

$\texttt{spaceOfStone} :: \texttt{MeasureAlgebra} \to \mathsf{MEAS}$

$\texttt{SpaceOfStone}\,(A,\mu) = (Z_A, \dot{\Sigma}_\mu, \dot{\mu}) := \texttt{StoneRepresentationTheorem}(A,\mu)$

1.1.4 Ideals

$\texttt{PrincipleIdealRestriction} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \; . \; \forall a \in A \; . \; \texttt{MeasureAlgebra}\Big((a), \mu_{|(a)}\Big)$

$\texttt{Proof} \; =$

This is obvious.

$\square$

$\texttt{measureQuotient} \; ::$

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \; . \; \forall I : \texttt{Ideal}(A) \; . \; \forall [a] \in \frac{A}{I} \; . \; \exists \gamma \in \overset{\infty}{\mathbb{R}}_{++} \; . \; \gamma = \min\{\mu(b) | b \in A, \pi_I(b) = [a]\}$$

$\texttt{Proof} \; =$

1 $\gamma = \inf\{\mu(b) | b \in A, \pi_I(b) = [a]\}$ exists as a set is bounded by below by 0.

2 If $\gamma = \infty$ then the result is obvious.

3 Otherwise there is a decreasing sequence $b : \mathbb{N} \to A$ such that $\pi_I(b_n) = [a]$ for any n and $\lim_{n\to\infty} \mu(b_n) = \gamma$.

4 Then $c = \bigwedge_{n=1}^{\infty} b_n$ is such that $\mu(c) = \gamma$ and $\pi_I(c) = a$.

4.1 Clearly $\pi_I\left(\bigwedge_{n=1}^{\infty} b_n\right) = \bigwedge_{n=1}^{\infty} \pi_I(b_n) = \bigwedge_{n=1}^{\infty} [a] = [a]$.

5 So the infimum is atteined.

$\square$

$\texttt{measureQuotient} :: \prod (A,\mu) : \texttt{MeasureAlgebra} \; . \; \prod I : \texttt{Ideal}(A) \; . \; \frac{A}{I} \to \mathbb{R}_{++}$

$\texttt{measureQuotient}\,(a) = \mu_I(a) := \min\{\mu(b) | b \in A, \pi_I(b) = a\}$

$\texttt{finiteElementsIdeal} :: \prod (A,\mu) : \texttt{MeasureAlgebra} \; . \; \texttt{Ideal}(A)$

$\texttt{finiteElementsIdeal}\,() = A^f := \{a \in A | \mu(a) < \infty\}$

MeasureIdealQuotient :: $\forall (A, \mu)$: MeasureAlgebra . $\forall I$: Ideal(A) . MeasureAlgebra $\left(\frac{A}{I}, \mu_I\right)$

Proof =

1 Clearly $\mu_I(0) = 0$.

2 Assume that $[a] \neq 0$.

2.1 Then there exists $b \in A$ such that $\pi_I(a) = [a]$ and $\mu(b) = \mu_I[a]$.

2.2 As $[a] \neq 0$, then $b \neq 0$, and henceforth $\mu(b) \neq 0$.

2.3 Thus, $\mu_I[a] \neq 0$.

3 Assume $[a] : \mathbb{N} \to \frac{A}{I}$ is disjoint.

3.1 It is possible to select representatives b_n for each $[a_n]$ such that $\mu(b_n) = \mu_I[a_n]$.

3.2 Then $b_n b_m \in I$ if $n \neq m$.

3.3 Construct a new sequence $c_n = b_n + \sum_{k=1}^{n-1} b_n b_k$ is a disjoint represintative sequance for $[a_n]$.

3.3.1 In fact $c = b$.

3.4 $\bigvee_{n=1}^{\infty} c_n$ is the minimal representative of $\bigvee_{n=1}^{\infty} [a_n]$.

3.4.1 Assume d is a representative for $\bigvee_{n=1}^{\infty} a_n$.

3.4.2 If $\mu(d) < \mu\left(\bigvee_{n=1}^{\infty} c_n\right)$ then we may construct $c_n \wedge d$ which is smaller then c.

3.4.3 But this is a contradiction.

3.5 So $\mu_I\left(\bigvee_{n=1}^{\infty} [a_n]\right) = \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \mu_I[a_n]$.

$\square$

1.1.5 Measure Properties

$\texttt{ProbabilityAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \pi) : \texttt{ProbabilityAlgebra} \iff \pi(e) = 1$

$\texttt{FiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \mu) : \texttt{FiniteMeasureAlgebra} \iff \mu(e) < \infty$

$\sigma\texttt{-FiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$

$$(A, \mu) : \sigma\texttt{-FiniteMeasureAlgebra} \iff \exists a : \mathbb{N} \to A \,.\, \forall n \in \mathbb{N} \,.\, \mu(a_n) < \infty \;\&\; \bigvee_{n=1}^{\infty} a_n = e$$

$\texttt{SemifiniteMeasureAlgebra} :: ?\texttt{MeasureAlgebra}$
$(A, \mu) : \texttt{SemifiniteMeasureAlgebra} \iff \forall a \in A \,.\, \mu(a) = \infty \Rightarrow \exists b \in A \,.\, b < a \;\&\; 0 < \mu(b) < \infty$

$\texttt{LocalizableMeasureAlgebra} := \texttt{OrderDedekindComplete} \;\&\; \texttt{SemifiniteMeasureAlgebra} : \texttt{Type};$

$\texttt{ProbabilityConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Probability}(X, \Sigma, \mu) \iff \texttt{ProbabilityAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
$\square$

$\texttt{FiniteConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Finite}(X, \Sigma, \mu) \iff \texttt{FiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
$\square$

$\texttt{SigmaFiniteConstruction} :: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \sigma\texttt{-Finite}(X, \Sigma, \mu) \iff \sigma\texttt{-FiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
$\square$

$\texttt{SemifiniteConstruction} ::$
$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Semifinite}(X, \Sigma, \mu) \iff \texttt{SemifiniteMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
$\square$

$\texttt{LocalizableConstruction} ::$
$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Localizable}(X, \Sigma, \mu) \iff \texttt{LocalizableMeasureAlgebra}(A_\mu, \bar{\mu})$
$\texttt{Proof} =$
This is obvious.
$\square$

`AtomInConstruction` ::

:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, \forall E \in \Sigma \, . \, E \in \mathrm{Atom}(X, \Sigma, \mu) \iff [E] \in \mathrm{Atom}(A_\mu, \bar{\mu})$

`Proof` =

This is obvious.

$\square$

`AtomlessConstruction` ::

:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, \forall E \in \Sigma \, . \, E \in \mathtt{Atomless}(X, \Sigma, \mu) \iff [E] \in \mathtt{Atomless}(A_\mu, \bar{\mu})$

`Proof` =

This is obvious.

$\square$

`PurelyAtomicConstruction` ::

:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, \forall E \in \Sigma \, . \, E \in \mathtt{PurelyAtomic}(X, \Sigma, \mu) \iff [E] \in \mathtt{PurelyAtomic}(A_\mu, \bar{\mu})$

`Proof` =

This is obvious.

$\square$

`FinitenessPropertiesIerarchy` ::

:: $\forall (A, \mu) : \mathtt{MeasureAlgebra} \, . \, \mathtt{PobabilityAlgebra}(A, \mu) \Rightarrow \mathtt{FiniteMeasureAlgebra}(A, \mu) \Rightarrow$

$\Rightarrow \sigma\text{-}\mathtt{FiniteMeasureAlgebra}(A, \mu) \Rightarrow \mathtt{LocalizableMeasureAlgebra}(A, \mu) \Rightarrow \mathtt{Semifinite}(A, \mu)$

`Proof` =

1 Most implications here are obvious expect the one deriving Localizability from σ-finiteness.

2 So assume that (A, μ) is σ-finite .

2.1 Then the corresponding Stone space $(\mathsf{Z}A, \Sigma_\mu, \bar{\mu})$ is σ-finite.

2.2 But then $(\mathsf{Z}A, \Sigma_\mu, \bar{\mu})$ is localizable .

2.3 So (A, μ) is also localizable.

$\square$

`MeasureAlgebraOfCompletion` :: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, A_\mu \cong_{\mathsf{BOOL}} A_{\hat{\mu}}$

`Proof` =

This is basically follows from definitions.

$\square$

`MeasureAlgebraOfLocallyDeterminedCompletion` ::

:: $\forall (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, \exists A_\mu \xrightarrow{\phi} A_{\tilde{\mu}} : \mathsf{BOOL} \, . \, \forall a \in A_{\tilde{\mu}} \, . \, \hat{\tilde{\mu}}(a) < \infty \Rightarrow \exists b \in A_\mu \, . \, \phi(b) = a \,\&$

$\& \, \forall b \in A_\mu \, . \, \hat{\mu}(b) < \infty \Rightarrow \hat{\tilde{\mu}}(\phi(b)) = \hat{\mu}(b)$

`Proof` =

...

$\square$

`localDeterminationMorphism` :: $\prod (X, \Sigma, \mu) \in \mathsf{MEAS} \, . \, \mathsf{BOOL}(A_\mu, A_{\tilde{\mu}})$

`localDeterminationMorphism` $() = \phi_\mu :=$ `MeasureAlgebraOfLocallyDeterminedCompletion`

$\texttt{localDeterminationMorhismInjectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Semifinite}(X, \Sigma, \mu) \iff \texttt{Injective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

Proof $=$

...

$\square$

$\texttt{localDeterminationMorhismBijectivity} ::$

$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} \,.\, \texttt{Localizable}(X, \Sigma, \mu) \iff \texttt{Bijective}(A_\mu, A_{\bar{\mu}}, \phi_\mu)$

Proof $=$

...

$\square$

$\texttt{SemifinitenessCriterion} :: \forall (A, \mu) : \texttt{MeasureAlgebra} \,.$

$. \texttt{SemifiniteMeasureAlgebra}(A, \mu) \iff \exists P : \texttt{PartitionOfUnity}(A) \,.\, \forall p \in P \,.\, \mu(p) < \infty$

Proof $=$

1 $(\Rightarrow)$ assume first that (A, μ) is semifinite.

1.1 Then A^f is order dense in A.

1.2 By order density theorem there is a desired partition of unity.

2 $(\Leftarrow)$ Let P be the partition of unity.

2.1 Assume $a \in A$ is such that $\mu(a) = \infty$.

2.2 Then there exists $p \in P$ such that $pa \neq 0$.

2.3 Note that this means that $\mu(pa) > 0$.

2.4 Also it is clear that $\mu(pa) \le \mu(p) < \infty$.

$\square$

$\texttt{SemifiniteneSupElementExpression} ::$

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \,.\, \forall a \in A \,.\, a = \bigvee \{b \in A : b \le a, \mu(b) < \infty\}$

Proof $=$

This follows from the previous theorem.

$\square$

$\texttt{SemifiniteneSupMeasureComputation} ::$

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra}(A, \mu) \,.\, \forall a \in A \,.\, \mu(a) = \bigvee \{\mu(b) \in A : b \le a, \mu(b) < \infty\}$

Proof $=$

This follows from the previous theorem.

$\square$

1.1.6 Connections with other Boolean Properties

SemifiniteIsWeaklyDistributive ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra(A, μ) . (σ, ∞)-WeaklyDistributive(A, μ)

Proof =

1 Assume $X : \mathbb{N} \to 2^A$ is a sequence of downwards selected sets with $\inf X_n = 0$ for every $n \in \mathbb{N}$.

2 Let $C = \{a \in A : \forall n \in \mathbb{N} \,.\, \exists x \in X_n \,.\, a \geq x\}$.

3 Assume $d \in A$ is such that $d \neq 0$.

4 Then there is an element $d' \leq d$ such that $0 < \mu(d') < 0$.

5 $\inf\limits_{x \in X} d'x = 0$ for each $n \in N$.

6 Select a sequence $x : \prod\limits_{n=1}^{\infty} X_n$ suc that $\mu(d'x_n) \leq 2^{-n-2}\mu(d')$.

7 Define $c = \sup\limits_{n=1} a_n \in C$.

8 Then $\mu(d'c) \leq \sum\limits_{n=0}^{\infty} \mu(cx_n) < \mu(d')$.

9 This means that $d \not\leq c$.

10 And as d was arbitrary $\inf C = 0$.

$\square$

SemifiniteIffCCC :: $\forall (A, \mu)$: SemifiniteMeasureAlgebra(A, μ) .

. σ-FiniteMeasureAlgebra(A, μ) $\iff$ WithCountableChainCondition(A)

Proof =

1 $(\Leftarrow)$ assume that A has ccc.

1.1 Then there is a partition of unitity P in A consisting of finite elements as A is semifinite.

1.2 But as A has ccc P must be atmost countable.

1.3 This proves that A is σ-finite.

2 $(\Rightarrow)$ assume that (A, μ) is σ-finite .

2.1 Then there exists a countable partition of unity P of A wirh finite elements.

2.2 If A is not ccc, then there exists an uncountable refinement Q of A with finite elements.

2.3 Then by pigionhole principle there exists $p \in P$
such that set $Q' = \{q \in Q : q \subset p\}$ such that Q' is uncountable.

2.4 as for $\mu(q) > 0$ for any $q \in Q'$ by pigionhole principle there exists some $n \in \mathbb{Z}$
such that there are an infinite number of $q \in Q'$ with $\mu(q) \in [2^{-n-1}, 2^{-n}]$.

2.5 So $\mu(p) \geq \sum\limits_{q \in Q'} \mu(q) = \infty$, but this is a contradiction.

$\square$

`SemifiniteIffProbabilityRenormalizationExists` ::

:: $\forall (A, \mu) :$ `SemifiniteMeasureAlgebra`(A, μ) . $A \neq \{0\} \Rightarrow$

$\Rightarrow \exists \pi : A \to \overset{\infty}{\mathbb{R}}_+$. `ProbabilityAlgebra`(A, π)

`Proof` $=$

1 Corresponding Stone space is σ-finite.

2 So there exists a proper renormalization of $\bar{\mu}$ to a probability π with the same sets of measure zero.

3 Then the measure algebra of $(\mathsf{Z}A, \pi)$ is a probability algebra and $A_\pi \cong_{\mathsf{BOOL}} A$.

$\square$

1.1.7 Subspace Measures and Indefinite Integrals

MeasurableEnvelopePrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall Y \subset X . \forall E : \texttt{MeasurableEnvelope}(X, \Sigma, \mu, Y) . (A_{\mu|Y}, \widehat{\mu|Y}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

This result is technically convoluted but actually is pretty intuituve.

□

PrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall E \in \Sigma . (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} \Big(([E]), \hat{\mu}_{|([E])} \Big)$$

Proof =

A straightforward application of a previous theorem.

□

ThickEquivalence ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall Y : \texttt{Thick}(X, \Sigma, \mu) . (A_{\mu|E}, \widehat{\mu|E}) \cong_{\mathsf{MA}} (X, \hat{\mu})$$

Proof =

A straightforward application of a previous theorem.

□

IndefiniteIntegralPrincipleIdealIsomorphism ::

$$:: \forall (X, \Sigma, \mu) \in \mathsf{MEAS} . \forall f \in \mathsf{I}_+(X, \Sigma, \mu) . \exists E \in \Sigma . A_{f\,d\mu} \cong_{\mathsf{BOOL}} ([E])$$

Proof =

We may assume that $\operatorname{supp} f$ has a measurable envelope E.

Then the result is obvious as $\mathcal{N}_\mu \subset \mathcal{N}_{f\,d\mu}$.

□

1.1.8 Simple Products

$$\texttt{simpleProduct} :: \prod_{I \in \mathsf{SET}} (I \to \texttt{MeasureAlgebra}) \to \texttt{MeasureAlgebra}$$

$$\texttt{simpleProduct}\,(A,\mu) = \prod_{i \in I} (A_i, \mu_i) := \left(\prod_{i \in I} A_i, \sum_{i \in I} \mu_i \right)$$

Obviously $\sum_{i \in I} \mu_i(0) = \sum_{i \in I} 0 = 0$.

Also assume $a : \mathbb{N} \to \prod_{i \in I} A_i$ is disjoint.

Then $\sum_{i \in I} \mu_i \left(\bigvee_{n=1}^{\infty} a_n \right) = \sum_{i \in I} \sum_{n=1}^{\infty} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty} \sum_{i \in I} \mu_i(a_{n,i}) = \sum_{n=1}^{\infty} \sum_{i \in I} \mu_i(a_n)$.

□

`PrincipleIdealsInMeasureAlgebras` ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,\mu) : I \to \texttt{MeasureAlgebra} \,.\, (A_i, \mu_i) \cong_{\mathsf{MA}} \left((e_i), \left(\sum_{i \in I} \mu_i \right)_{|(e_i)} \right)$$

Proof =

This is pretty ovious.

□

`SimpleProductCoproductCorrespondance` ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall (X,\Sigma,\mu) : I \to \mathsf{MEAS} \,.\, \prod_{i \in I} (A_{\mu_i}, \hat{\mu}_i) \cong \texttt{measureAlgebra} \coprod_{i \in I} (X_i, \Sigma_i, \mu_i)$$

Proof =

Obvious by Stone Theory.

□

`SimpleProductOfSemifinite` ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,\mu) : I \to \texttt{SemifiniteMeasureAlgebra} \,.\, \texttt{SemifiniteMeasureAlgebra} \left(\prod_{i \in I} (A,\mu) \right)$$

Proof =

Assume a has infinite measure in (A,μ).

Then there exists $i \in I$ such that $a_i \neq 0$.

As (A_i, μ_i) is semifinite there is $b \leq a_i$ such that $0 < \mu_i(b) < \infty$.

Then $be_i \leq a$ and $0 < \sum_{j \in I} \mu_j(be_i) = \mu_i(b) < \infty$.

□

SimpleProductOfLocalizable ::

$$:: \forall I \in \mathsf{SET} \,.\, \forall (A,\mu) : I \to \texttt{LocalizableMeasureAlgebra} \,.\, \texttt{LocalizableMeasureAlgebra}\left(\prod_{i\in I}(A,\mu)\right)$$

Proof =

Let J be a set and $a : J \to \prod_{i\in I}(A_i,\mu_i)$.

Then $\sup_{j\in J} a_j = (\sup_{j\in J} a_{j,i})_{i\in I}$.

$\square$

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall (e_n)_{n=1}^{\infty} : \texttt{PartitionOfUnity}(A) \,.\, (A,\mu) \cong_{\mathsf{MA}} \prod_{n=1}^{\infty}\Big((e_n), \mu_{|(e_m)}\Big)$$

Proof =

This is pretty obvious.

$\square$

PoUProductRepresentation ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} \,.\, \exists I \in \mathsf{SET} \,.\, \exists (B,\nu) : I \to \texttt{FiniteMeasureAlgebra} \,.$$
$$.\, (A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$$

Proof =

It is possible to select a partition of unity P of A consisting of finite elements.

Then by previous theorem $(A,\mu) \cong \prod_{p\in P}\Big((p), \mu_{|(p)}\Big)$.

And each $\Big((p), \mu_{|(p)}\Big)$ are obviously finite.

$\square$

LocalizableMeasureAlgebrasHasLocallyDeterminedRepresentations ::

$$:: \forall (A,\mu) : \texttt{LocalizableMeasureAlgebra} \,.\, \exists (X,\Sigma,\nu) : \texttt{LocallyDetermined} \,.\, (A,\mu) \cong_{\mathsf{MA}} (A_\nu, \hat{\nu})$$

Proof =

Represent $(A,\mu) \cong_{\mathsf{MA}} \prod_{i\in I}(B_i,\nu_i)$.

Then Stone's spaces $\mathsf{Z}\ B_i$ correspond to finite measure spaces.

And Stone's space of product correspond to a disjoint union of $\mathsf{Z}\ B_i$.

But such spaces are trivially locally determined.

$\square$

1.1.9 Strictly Localizable Spaces

$\texttt{StrictlyLocalizableSpacePoU} ::$

$:: \forall (X, \Sigma, \mu) : \texttt{StrictlyLocalizable} \; . \; \forall P : \texttt{PartitionOfUnity}(A_\mu) \; .$

$. \; \exists E : P \to \Sigma \; . \; \forall p \in P \; . \; [E_p] = p \;\&\; \texttt{Decomposition}(X, \Sigma, \mu, \operatorname{Im} E)$

$\texttt{Proof} \; =$

$\ldots$

$\square$

1.1.10 Subalgebras

$\texttt{SubalgebaMeasureAlgebra} :: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall B \subset_\sigma A \,.\, \texttt{MeasureAlgebra}(B, \mu_{|B})$

Proof $=$

This is obvious.

□

$\texttt{SubalgebaFinifteMeasureAlgebra} ::$

$:: \forall (A,\mu) : \texttt{FiniteMeasureAlgebra} \,.\, \forall B \subset_\sigma A \,.\, \texttt{FiniteMeasureAlgebra}(B, \mu_{|B})$

Proof $=$

This is obvious.

□

$\texttt{SigmaFiniteSubalgebraMeasureAlgebra} ::$

$:: \forall (A,\mu) : \sigma\texttt{-FiniteMeasureAlgebra} \,.\, \forall B \subset_\sigma A \,.$

$. \texttt{SemifiniteMeasureAlgebra}(B, \mu_{|B}) \Rightarrow \sigma\texttt{-FiniteMeasureAlgebra}(B, \mu_{|B})$

Proof $=$

1 The set B^f is order-dense in B.

2 But then B^f is also order-dense in A.

3 Select a finite-measured countable partition of unity P in A.

4 If B is not σ-finite, then there is a subordinate uncountale partition of unity Q.

5 Then there would exist a uncountable refinement of P subordinate to Q.

6 Then P must contain an infinite element, but this is imposible!.

7 So Q must be countable, and so $(B, \mu_{|B})$ must be countable.

□

$\texttt{FinifteMeasureAlgebraBySubalgebra} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall B \subset_\sigma A \,.\, \texttt{FiniteMeasureAlgebra}(B, \mu_{|B}) \Rightarrow \texttt{FiniteMeasureAlgebra}(A, \mu)$

Proof $=$

This is obvious.

□

$\texttt{ProbabilityAlgebraBySubalgebra} ::$

$:: \forall (A,\mu) : \texttt{MeasureAlgebra} \,.\, \forall B \subset_\sigma A \,.$

$. \texttt{ProbabilityAlgebra}(B, \mu_{|B}) \Rightarrow \texttt{ProbabilityAlgebra}(A, \mu)$

Proof $=$

This is obvious.

□

$\texttt{SigmaFiniteAlgebraBySubalgebra} ::$

$:: \forall (A, \mu) : \texttt{MeasureAlgebra} \;.\; \forall B \subset_\sigma A \;.$

$.\; \sigma\text{-}\texttt{Finite}(B, \mu_{|B}) \Rightarrow \sigma\text{-}\texttt{Finite}(A, \mu)$

$\texttt{Proof} \;=$

This is obvious.

$\square$

1.1.11 Localization

MeasureAlgebraCompletion ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\exists! \hat{\mu} : \tau(A) \to \overset{\infty}{\mathbb{R}}_{++}$.

. $\hat{\mu}_{|A} = \mu$ & LocalizableMeasureAlgebra$(\tau(A), \hat{\mu})$

Proof =

1 Define $\hat{\mu}(t) = \sup\{\mu(a) | a \in A, a \leq t\}$.

2 As A is order dense in $\tau(A)$, it holds that $\hat{\mu}(a) = 0 \iff a = 0$ for any $a \in \tau(A)$.

3 If $t : \mathbb{N} \to \tau(A)$ is disjoint then $\hat{\mu}\left(\bigvee_{n=1}^{\infty} t_n\right) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

3.1 Write $S = \{a \in A : \exists c : \mathbb{N} \to A \ . \ a = \lim_{n\to\infty} c_n \ \& \ c \leq t\}$.

3.2 Then there is $s = \sup S \in \tau(A)$.

3.3 We write $\hat{\mu}(s) = \sup_{c \leq t} \mu\left(\bigvee_{n=1}^{\infty} c_n\right) = \sup_{c \leq t} \sum_{n=1}^{\infty} \mu(c_n) = \sum_{n=1}^{\infty} \sup_{c \leq t_n} \mu(c) = \sum_{n=1}^{\infty} \hat{\mu}(t_n)$.

4 Obviously $(\tau(A), \hat{\mu})$ is semifinite and order-complete, and hence Localizable.

□

localization :: SemifiniteMeasureAlgebra → LocalizableMeasureAlgebra

localization $(A, \mu) = \Big(\tau(A), \tau(\mu)\Big)$:= MeasureAlgebraCompletion

LocalizationFiniteEmbedding ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\iota_\tau(A^f) = \tau^f(A)$

Proof =

1 Assume $t \in \tau(A)$ such that $\hat{\mu}(t) < \infty$.

2 Note, $\hat{\mu}(t) = \sup_{a \leq t} \mu(a)$.

3 So we may select an increasing $a : \mathbb{N} \to A$ such that $\lim_{n\to\infty} \mu(a_n) = \hat{\mu}(t)$.

4 Then $b = \bigvee_{n=1}^{\infty} a_n \in A$ and $\hat{\mu}(b) = \mu(b) = \hat{\mu}(t)$.

5 So $\mu(t \setminus b) = 0$, and so $t = b \in A$ as clearly $b < t$.

□

1.1.12 Stone Spaces

LocallalizableMeasureAlgebraHasStrictlyLocalizableStoneSpace ::

:: $\forall (A, \mu)$: LocalizableMeasureAlgebra . StrictlyLocalizable($\mathsf{Z}\ A, \Sigma_\mu, \bar{\mu}$)

Proof =

1 We already proved that $\bar{\mu}$ is locally determined.

2 As (A, μ) is semifinite there is a partition of unity P consisting of finite elements.

3 Use Stone representation $S_A(P)$ to construct a corresponding set in $\mathsf{Z}\ A$.

4 Assume $E \in \Sigma_\mu$ such that $\bar{\mu}(E) > 0$.

5 By definition of Stone's Space there is a clopen set $F \in \mathsf{Z}\ A$ such that $E \triangle F$ is meager.

6 And there is a Stone representation $a \in A$ such that $F = S_A(a)$.

7 Then $\mu(a) = \nu(S_A(a)) = \nu(E) > 0$.

8 So, there exists $p \in P$ such that $ap \neq 0$.

9 Ths means that $\nu(E \cap S_A(p)) > 0$.

10 As E was arbitrary this means that $S_A(P)$ provides a strict localization for $\bar{\mu}$.

□

MeagerSetsAreNowhereDense ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall M \in \mathbf{MGR}(\mathsf{Z}\ A)$. NowhereDense($\mathsf{Z}\ A, M$)

Proof =

1 As it was shown A is (σ, ∞)-WeaklyDistributive boolean algebra.

2 And this is a property of (σ, ∞)-WeaklyDistributive boolean algebra.

□

StoneSpaceMeasurableExpression ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\exists U$: Clopen($\mathsf{Z}\ A$) . $\exists F$: NowhereDense($\mathsf{Z}\ A$) . $E = U \cap F$

Proof =

1 This is clear from the previous theorem.

□

StoneSpaceMeasureComputation ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . $\forall E \in \Sigma_\mu$.

. $\bar{\mu}(E) = \sup \left\{ \mu(U) \middle| U : \text{Clopen}(\mathsf{Z}\ A), U \subset E \right\}$

Proof =

1 This is clear from the previous theorem.

□

StoneSpaceCLDIsStrictlyLocalizable ::

:: $\forall (A, \mu)$: SemifiniteMeasureAlgebra . StrictlyLocalizable($\mathsf{Z}\ A, \bar{\Sigma}_\mu, \bar{\bar{\mu}}$)

Proof =

...

□

StoneSpaceCLDZeroSets ::

$:: \forall (A, \mu) : \texttt{SemifiniteMeasureAlgebra} \, . \, \mathcal{N}_{\bar{\bar{\mu}}} = \mathcal{N}_{\bar{\mu}}$

Proof =

...

$\square$

FiniteStoneSpaceMeasureComputation ::

$:: \forall (A, \mu) : \texttt{FiniteMeasureAlgebra} \, . \, \forall E \in \Sigma_\mu \, .$

$. \, \bar{\mu}(E) = \inf \left\{ \mu(U) \middle| U : \texttt{Clopen}(\mathsf{Z}\ A), E \subset U \right\}$

Proof =

1 This is clear from the previous theorem.

$\square$

1.1.13 Purely Infinite Elements

$$\texttt{purelyInfiniteElements} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \sigma\texttt{-Ideal}(A)$$

$$\texttt{purelyInfiniteElements}\,() = I_\infty(\mu := \{a \in A : \forall b \in A \,.\, b \leq a \;\&\; \mu(b) < \infty \Rightarrow b = 0\}$$

$$\texttt{semifiniteMeasure} :: \prod(A,\mu) : \texttt{MeasureAlgebra} \,.\, \frac{A}{I_\infty(\mu)} \to \overset{\infty}{\mathbb{R}}_+$$

$$\texttt{semifiniteMeasure}\,([a]) = \mu_{\mathrm{sf}} := \sup\{\mu(b) | b \in A : b \leq a \;\&\; \mu(b) < \infty\}$$

If $[a] = [b]$, then $a \triangle b \in I_\infty(\mu)$.

So μ_{sf} is well-defined.

$\texttt{SemifiniteMeasureIsMeasure}$ ::

$$:: \forall(A,\mu) : \texttt{MeasureAlgebra} \,.\, \texttt{SemifiniteMeasureAlgebra}\left(\frac{A}{I}, \mu_{\mathrm{sf}}\right)$$

Proof =

1 If $\mu_{\mathrm{sf}}[a] = 0$, then clearly $a \in I_\infty$.

2 Assume $[a] : \mathbb{N} \to A$ is disjoint.

2.1 Then $a_n a_m \in I_\infty$ if $n \neq m$.

2.2 Select increasing $b : \mathbb{N} \to A^f$ such that $b_n \leq \bigvee_{k=1}^{\infty} a_k$ and $\lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}}\left[\bigvee_{k=1}^{\infty} a_k\right] = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k]$.

2.3 By (2.1) we mat assert that ab_n is disjoint and then $\bigvee_{k=1}^{\infty} a_k b_n = b_n$ for any $n \in \mathbb{N}$.

2.4 So $\mu(b) = \sum_{k=1}^{\infty} \mu(a_k b_n)$.

2.5 By taking limits and using monotonic convergence theorem

$$\sum_{k=1}^{\infty} \mu_{\mathrm{sf}}[a_k] = \sum_{k=1}^{\infty} \lim_{n\to\infty} \mu(a_k b_n) = \lim_{n\to\infty} \mu(b_n) = \mu_{\mathrm{sf}} \bigvee_{k=1}^{\infty} [a_k].$$

3 Clearly $\mu_{\mathrm{sf}}[a] < \mu(a)$.

3.1 If $\mu_{\mathrm{sf}}[a] = \infty$, then $a \notin I_\infty$.

3.2 So it is possible to select $b \in A$ such that $b \leq a$ and $0 < \mu(b) \leq a$.

3.3 $0 < \mu_{\mathrm{sf}}[b] \leq \mu(b) < \infty$.

3.4 This proves that $\left(\frac{A}{I}, \mu_{\mathrm{sf}}\right)$ is semifinite.

□

1.2 Topology

1.2.1 Subject

$$\texttt{measureAlgebraAsTopologicalSpace} :: \texttt{MeasureAlgebra} \to \mathsf{TOP}$$
$$\texttt{measureAlgebraAsTopologicalSpace}\,((A,\mu)) = (A,\mu) :=$$
$$:= \Big(A, \mathcal{W}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f\,.\,\Lambda b \in A^f\,.\,\Lambda c \in A\,.\,\mu(ac+ab)\big)\Big)$$

$$\texttt{measureAlgebraAsUniformlSpace} :: \texttt{MeasureAlgebra} \to \mathsf{UNI}$$
$$\texttt{measureAlgebraAsUniformSpace}\,((A,\mu)) = (A,\mu) :=$$
$$:= \Big(A, \mathcal{I}\big(A^f \times A^f, \mathbb{R}, \Lambda a \in A^f\,.\,\Lambda b \in A^f\,.\,\Lambda c \in A\,.\,\mu(ac \triangle ab)\big)\Big)$$

$$\texttt{metricOfFrechetNikodym} :: \prod (A,\mu) : \texttt{MeasureAlgebra}\,.\,\texttt{Metric}(A^f)$$
$$\texttt{metricOfFrechetNikodym}\,() = \rho_\mu := \Lambda a, b \in A^f\,.\,\mu(a \triangle b)$$

$\texttt{BooleanOperationsAreUniformlyContinuous}$ ::

$:: \forall (A,\mu) : \texttt{MeasureAlgebra}\,.\,(*), (\setminus), (\vee), (\wedge) \in \mathsf{UNI}(A \times A, A)$

Proof $=$

1 Let $\circ$ stay for any binary operation above.

2 Select $c, d \in A$.

3 Then $\mu\Big(a(c \circ d) + b\Big) \le \mu\Big(a(c \vee d) + b\Big) \le \mu(ac+d) + \mu(ad+b)$.

4 So μ is bounded by the sum of uniform functions and is uniformly continuous.

$\square$

$\texttt{FiniteElementsAreDense}$ ::

$:: \forall (A,\mu) : \texttt{MeasureAlgebra}\,.\,\texttt{Dense}(A, A^f)$

Proof $=$

1 Select $c \in A$.

2 Then c has a base of neighborhoods of form $U = \{u \in A : \mu(au+ac) \le r\}$ with $a \in A^f, r \in \mathbb{R}_{++}$.

3 But then $ac \in U$ and $ac \in A^f$.

$\square$

$\texttt{FiniteMeasureAlgebraHasUniformlyContinuousMeasure}$ ::

$\forall (A,\mu) : \texttt{FiniteMeasureAlgebra}\,.\,\mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

Proof $=$

This is pretty obvious as $\mu = \rho_\mu(0,a)$.

$\square$

$\texttt{FiniteMeasureAlgebraHasUniformlyContinuousMeasure}$ ::

$\forall (A,\mu) : \texttt{FiniteMeasureAlgebra}\,.\,\mu \in \mathsf{UNI}(A, \mathbb{R}_{++})$

Proof $=$

This is pretty obvious as $\mu = \rho_\mu(0,a)$.

$\square$

`SemifinitMeasureAlgebraHasLoweSemicontinuousMeasure` ::

$\forall (A, \mu) :$ `SemifiniteMeasureAlgebra` $. \ \mu \in$ `LowerSemicontinuous`$(A, \overset{\infty}{\mathbb{R}}_{++})$

`Proof` $=$

This is pretty obvious as $\mu = \rho_\mu(0, a)$.

$\square$

1.3 Category

1.4 Radon-Nikodym Parallels

2 Maharam's Theory

3 Abstract Ergodic Theory

4 Measurable Algebras

Sources:

1. D. H. Fremlin — Measure Theory (32,33,34) 2016