Metrics.Know

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$\texttt{predicate} \quad \texttt{Metric} :: \prod X : \texttt{Set}(T) \;.\; ?\,(X \times X \to \mathbb{R}_+)$

$\quad d : \texttt{Metric} \iff \prod x, y, z \in X \;.\; (\texttt{iff}\ d(x,y) = 0 \;.\; x = y) \times$

$\quad \times\, d(x,y) \leq d(x,z) + d(z,y)$

$\texttt{data} \quad \texttt{MetricSpace} := \sum X : \texttt{Set}(T) \;.\; \texttt{Metric}(X)$

$\texttt{function} \quad \texttt{metric} :: \prod A : \texttt{MetricSpace}(T) \;.\; \texttt{Metric}(X)$

$\quad \texttt{metric}\,(X, f) := d_{(X,f)} := f$

$\texttt{function} \quad \texttt{implicit} :: \prod A : \texttt{MetricSpace}(T) \;.\; \texttt{Set}(T)$

$\quad \texttt{implicit}\,(X, d) := X$

$\texttt{function} \quad \texttt{ball} :: \prod X : \texttt{MetricSpace}(T) \;.\; X \to \mathbb{R}_{++} \to ?X$

$\quad \texttt{ball}\, x\, r := \mathbb{B}(x)(r) := \{b \in X : d_X(b,x) < r\}$

$\texttt{predicate} \quad \texttt{Bounded} :: \prod X : \texttt{MetricSpace}(T) \;.\; ??X$

$\quad A : \texttt{Bounded} \iff \sum B \in \mathbb{R}_+ \;.\; \prod x, y \in A \;.\; d_X(x,y) \leq B$

$$\texttt{function} \quad \mathrm{diam} :: \prod X : \texttt{MetricSpace}(T) \; . \; ?X \to \overline{\mathbb{R}}_+$$
$$\mathrm{diam}(A) = \sup\{d_X(x,y) | x, y \in A\}$$

$$\texttt{function} \quad \texttt{bound} :: \prod X : \texttt{Set}(T) \; . \; \texttt{Metric}(X) \to \texttt{Metric}(X)$$
$$\texttt{bound}\, d := \overline{d} := \Lambda\, x, y \in X \; . \; \texttt{if } d(x,y) \leq 1 \texttt{ then } d(x,y) \texttt{ else } 1$$

$$\texttt{predicate} \quad \texttt{Valuation} :: \prod R : \texttt{Ring}(T) \; . \; R \to \mathbb{R}_+$$
$$v : \texttt{Valuation} \iff \prod x, y \in R \; . \; (\texttt{iff } v(x) = 0 \; . \; x = 0) \times$$
$$\times\, (v(x+y) \leq v(x) + v(y)) \times (v(xy) = v(x)v(y))$$

$$\texttt{predicate} \quad \texttt{Norm} :: \prod M : R\text{-}\texttt{Module} \; . \; \prod v : \texttt{Valuation}(R) \; . \; M \to \mathbb{R}_+$$
$$N : \texttt{Norm} \iff \prod r \in R \; . \; \prod a, b \in M \; . \; N(ra) = v(r)N(a) \times$$
$$\times\, (N(a+b) \leq N(a) + v(b)) \times (a \neq 0 \to N(a) \neq 0)$$

$$\texttt{function} \quad \texttt{toDistance} :: \prod M : R\text{-}\texttt{Module} \; . \; \texttt{Norm}(M) \to \texttt{Metric}(M)$$
$$\texttt{toDistance}(N) = \Lambda a, b \in M \; . \; N(a-b)$$

$$\texttt{predicate} \quad \texttt{Cauchy} :: \prod X : \texttt{MetricSpace} \; . \; ?(\mathbb{N} \to X)$$
$$x : \texttt{Cauchy} \iff \prod \epsilon \in \mathbb{R}_{++} \; . \; \sum c \in x : \sum N \in \mathbb{N} : \prod n > N \; . \; x_n \in \mathbb{B}\, c\, \epsilon$$

$$\texttt{predicate} \quad \texttt{Limit} :: \prod X : \texttt{MetricSpace} \; . \; \mathbb{N} \to X \to ?X$$
$$L : \texttt{Limit}(x) \iff \prod \epsilon \in \mathbb{R}_{++} \; . \; \sum N \in \mathbb{N} : \prod n > N \; . \; d(x_n, L) \leq \epsilon$$

$$\texttt{predicate} \quad \texttt{Convergent} :: \prod X : \texttt{MetricSpace} \; . \; ?(\mathbb{N} \to X)$$
$$x : \texttt{Convergent} \iff \sum L \in X \; . \; L : \texttt{Limit}(x)$$

$\texttt{predicate}\quad \texttt{Complete} :: ?\texttt{MetricSpace}$

$$X : \texttt{Complete} \iff \prod x : \texttt{Cauchy}(X)\ .\ x : \texttt{Convergent}(X)$$

$\texttt{function}\quad (\bowtie) :: (\mathbb{N} \to X) \times (\mathbb{N} \to X) \to \mathbb{N} \to X$

$$x \bowtie y = \Lambda n\ \in \mathbb{N}\ .\ \texttt{if}\ 2|n\ \texttt{then}\ y_{n/2}\ \texttt{else}\ x_{\lfloor n/2 \rfloor}$$

$\texttt{function}\quad \texttt{complete} :: \texttt{MetricSpace}(T) \to \texttt{Complete}(T)$

$$\texttt{complete}(X) := \overline{X} := (\texttt{Cauchy}\, X) \,/\, \big\{(x, y) : (\texttt{Cauchy}\, X)^2 : x \bowtie y : \texttt{Cauchy}\big\}$$