Convex Analysis

Uncultured Tramp

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Contents

1 Convex Functions

1.1 Subject

$\texttt{epigraph} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \left(D \to \overset{\infty}{\mathbb{R}}\right) \to ?(V \oplus \mathbb{R})$

$\texttt{epigraph}(f) = \operatorname{epi} f := \{(x, \phi) | x \in D, \phi \in \mathbb{R}, \phi \geq f(x)\}$

$\texttt{Convex} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . ?\left(D \to \overset{\infty}{\mathbb{R}}\right)$

$f : \texttt{Convex} \iff \texttt{Convex}(V \oplus \mathbb{R}, \operatorname{epi} f)$

$\texttt{effectiveDomain} :: \prod V : \mathbb{R}\text{-VS} . \prod D \subset V . \texttt{Convex}(V, D) \to ?D$

$\texttt{effectiveDomain}(f) = \operatorname{dom} f := \pi_1 \operatorname{epi} f$

$\texttt{DomainIsConvex} :: \forall V \in \mathbb{R}\text{-VS} . \forall D \subset V . \forall f : \texttt{Convex}(V, D) . \texttt{Convex}(V, \operatorname{dom} f)$

Proof $=$

As a linear image of convex set.

□

$\texttt{ProperConvexFunction} :: \prod V : \mathbb{R}\text{-VS} . ?\texttt{Convex}(V, V) .$

$f : \texttt{ProperConvexFunction} \iff \forall x \in V . f(x) > -\infty \ \& \ \exists x \in V . f(x) < +\infty$

$\texttt{InterpolationProperty} ::$

$:: \forall V : \mathbb{R}\text{-VS} . \forall C : \texttt{Convex}(V) . \forall f : C \to (-\infty, +\infty] .$

$. \texttt{Convex}(V, C, f) \iff \forall x, y \in C . \forall \lambda \in [0, 1] .$

$. f\Big(\lambda x + (1 - \lambda) y\Big) \leq \lambda f(x) + (1 - \lambda) f(y)$

Proof $=$

$(\Rightarrow)$: assume that f is convex.

Then f has convex epigraph.

Take arbitrary $x, y \in C$ and $\lambda \in [0, 1]$.

If f takes value $+\infty$ either in x or y, then the inequality follows, so assume the contrary.

Then $\Big(x, f(x)\Big), \Big(y, f(y)\Big)$ trivially belong to the epigraph,

so by convexity $\Big(\lambda x + (1 - \lambda) y, \lambda f(x) + (1 - \lambda) f(y)\Big)$ is also in epigraph.

The definition of epigraph produces the inequality $f\Big(\lambda x + (1 - \lambda) y\Big) \leq \lambda f(x) + (1 - \lambda) f(y)$.

$(\Leftarrow)$: now assume that inequality always hold.

Assume $(x, \phi), (y, \psi)$ belong to the epigraph and $\lambda \in [0, 1]$.

Then $\lambda \phi + (1 - \lambda) \psi \geq \lambda f(x) + (1 - \lambda) f(y) \geq f\Big(\lambda x + (1 - \lambda) y\Big)$.

So $\lambda (x, \phi) + (1 - \lambda)(y, \psi)$ belong to the epigraph.

Thus, epigraph is convex and f is convex.

`JensensIneq` ::

$:: \forall V : \mathbb{R}\text{-VS} \; . \; \forall C : \texttt{Convex}(V) \; . \; \forall f : C \to (-\infty, +\infty] \; .$

$$. \; \forall n \in \mathbb{N} \; . \; \forall \lambda \in \mathbb{R}^n_+ \; . \; \forall \aleph : \sum_{k=1}^{n} \lambda_k = 1 \; . \; \forall v \in V^n \; . \; f\left(\sum_{k=1}^{n} \lambda_k v_k\right) \leq \sum_{k=1}^{n} \lambda_k f(v_k)$$

`Proof` =

Iterate the interpolation property.

□

`SecondDerivativeConvexityTest` :: $\forall I : \texttt{OpenInterval}(\mathbb{R}) \; . \; \forall f \in C^2(I) \; .$

$. \; \texttt{Convex}(\mathbb{R}, I, f) \iff f'' \geq 0$

`Proof` =

$(\Rightarrow)$: assume there is a $t \in I$ such that $f''(t) < 0$.

As f'' must be continous there is whole open interval (a, b) such that $f''(j) < 0$ for all $j \in (a, b)$.

Take some $x, y \in (a, b)$ with $x < y$ and define$z = \lambda x + (1 - \lambda)y$ for siome $\lambda \in (0, 1)$.

Then $f(z) - f(x) = \int_x^z f'(t) \, dt > f'(z)(z - x)$ and $f(y) - f(z) = \int_z^y f'(t) \, dt < f'(z)(y - z)$.

Then from definiton of z we get $f(z) > f(x) - (1 - \lambda)f'(z)(y - x)$ and $f(z) > f(y) + \lambda f'(z)(y - x)$.

By adding two inequalities with multipliers λ and $(1 - \lambda)$ one gets $f\Big(\lambda x + (1 - \lambda)y\Big) > \lambda f(x) + (1 - \lambda)f(y)$.

But this contradicts a convexity. .

$(\Rightarrow)$: use same inequalities but with different sign to prove the convexity.

□

`ExponentIsConvexity` :: $\forall \alpha \in \mathbb{R} \; . \; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}, \Lambda t \in \mathbb{R} \; . \; e^{\alpha t}\Big)$

`Proof` =

write $f(t) = e^{\alpha t}$.

Then $f''(t) = \alpha^2 e^{\alpha t} \geq 0$.

□

`MonomialConvexity1` :: $\forall p \in [1, +\infty) \; . \; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \; . \; t^p\Big)$

`Proof` =

Write $f(t) = t^p$.

Then $f''(t) = p(p - 1)t^{p-2} \geq 0$ for $t > 0$.

□

`MonomialConvexity2` :: $\forall p \in [0, 1) \; . \; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \; . \; -t^p\Big)$

`Proof` =

Write $f(t) = t^p$.

Then $f''(t) = p(1 - p)t^{p-2} \geq 0$ for $t > 0$.

□

$\texttt{MonomialConvexity3} :: \forall p \in (-\infty, 0] \;.\; \texttt{Convex}\Big(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t^p\Big)$

$\texttt{Proof} \;=$

write $f(t) = t^p$.

Then $f''(t) = p(p-1)t^{p-2} \geq 0$ for $t > 0$.

$\square$

$\texttt{GeneralizedArcsinDerivativeIsConvex} :: \forall \alpha \in \mathbb{R}_{++} \;.\; \texttt{Convex}\left(\mathbb{R}, (-\alpha, \alpha), \Lambda t \in \mathbb{R} \;.\; \frac{1}{\sqrt{\alpha^2 - t^2}}\right)$

$\texttt{Proof} \;=$

Write $f(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}}$.

Then $f'(t) = \dfrac{t}{\sqrt{\alpha^2 - t^2}^3}$.

And $f''(t) = \dfrac{1}{\sqrt{\alpha^2 - t^2}^3} + \dfrac{3t^2}{\sqrt{\alpha^2 - t^2}^5} > 0$ for $t \in (-\alpha, \alpha)$.

$\square$

$\texttt{NegativeLogIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; -\ln(t)\right)$

$\texttt{Proof} \;=$

Write $f(t) = -\ln(t)$.

Then $f''(t) = \dfrac{1}{t^2} > 0$ for $t > 0$.

$\square$

$\texttt{NegativeEntropyIsConvex} :: \texttt{Convex}\left(\mathbb{R}, \mathbb{R}_{++}, \Lambda t \in \mathbb{R} \;.\; t\ln(t)\right)$

$\texttt{Proof} \;=$

Write $f(t) = t\ln(t)$.

Then $f'(t) = \ln(t) + 1$.

And $f''(t) = \dfrac{1}{t} > 0$ for $t > 0$.

$\square$

$\texttt{Concave} :: \prod V : \mathbb{R}\text{-}\mathsf{VS} \;.\; \prod D \subset V \;.\; ?\Big(D \to \overset{\infty}{\mathbb{R}}\Big)$

$f : \texttt{Concave} \iff \texttt{Convex}(V, D, -f)$

$\texttt{SecondDerivativeConvexityTest2} :: \forall V : \texttt{EucledeanSpace} \;.\; \forall U : \texttt{Open} \;\&\; \texttt{Convex}(V) \;.\; \forall f \in C^2(U) \;.$
$\;.\; \texttt{Convex}(\mathbb{R}, U, f) \iff \mathbf{D}^2 f \geq 0$

$\texttt{Proof} \;=$

For $x \in U$ and $v \in V \setminus \{0\}$ define $\phi_{x,v}(t) = f(x + tv)$ with a domain $I_{x,v} = \{t \in \mathbb{R} | x + tv \in C\}$.

Then f is convex iff every $\phi_{x,v}$ does.

But $\phi''_{x,v}(t) = \langle v, \mathbf{D}^2 f|_y v\rangle$, where $y = x + tv$.

So f is convex iff $\mathbf{D}^2 f$ is positive-semidefinite.

$\square$

GeometricMeanIsConcave ::

$$:: \forall V : \texttt{EucledeanSpace} \; . \; \texttt{Concave}\left(V, V_{++}, \Lambda x \in V \; . \; \prod_{k=1}^{n} \sqrt[n]{x_k}\right) \quad \texttt{where} \quad n = \dim V$$

Proof $=$

write $f(x) = \prod_{k=1}^{n} \sqrt[n]{x_k}$.

Then $\nabla f|_x = \left(\frac{1}{n \sqrt[n]{x_i}^{n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}\right)_{i=1}^{n}$.

And $\mathbf{D}^2_{i,j} f|_x = \frac{1}{n^2 \sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k}$ when $i \neq j$, and $\mathbf{D}^2_{i,i} f|_x = -\frac{n-1}{n^2 \sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j}$.

So, $\mathbf{D}^2 f|_x(v,v) = -\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{\sqrt[n]{x_i}^{2n-1}} \prod_{j \neq i}^{n} \sqrt[n]{x_j} + \frac{1}{n^2} \sum_{i \neq j}^{n} \frac{v_i v_j}{\sqrt[n]{x_i x_j}^{n-1}} \prod_{k \neq i,j}^{n} \sqrt[n]{x_k} =$

$$= f(x)\left(-\frac{n-1}{n^2} \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} + \frac{1}{n^2} \sum_{i \neq j}^{n} \frac{v_i v_j}{x_i x_j}\right) = -\frac{f(x)}{n^2}\left(n \sum_{i=1}^{n} \frac{v_i^2}{x_i^2} - \left(\sum_{i=1}^{n} \frac{v_i}{x_i}\right)^2\right) \leq 0 \; .$$

This follows from obvious matching schema.

□

NormsAreConvex $:: \forall V : \mathbb{R}\text{-VS} \; . \; \forall \eta : \texttt{Norm}(V)\texttt{Convex}(V, V, \eta)$

Proof $=$

Write $\eta(v) = \|v\|$.

Just use triangle inequality $\Big\|\lambda x + (1-\lambda)y\Big\| \leq \|\lambda x\| + \Big\|(1-\lambda)y\Big\| = \lambda\|x\| + (1-\lambda)\|y\|$.

□

convexIndicator $:: \forall V : \mathbb{R}\text{-VS} \; . \; \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

convexIndicator $(C) = \Lambda x \in V \; . \; \chi(x|C) := \Lambda x \in V \; . \; \infty\big[x \in C^{\complement}\big]$

supportFunction $:: \forall V : \mathbb{R}\text{-HIL} \; . \; \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

supportFunction $(C) = \Lambda x \in V \; . \; \chi^*(x|C) := \sup_{y \in C} \langle x, y \rangle$

gauge $:: \forall V : \mathbb{R}\text{-VS} \; . \; \texttt{Convex}(V) \to \texttt{Convex}(V, V)$

gauge $(C) = \Lambda x \in V \; . \; \gamma(x|C) := \Lambda x \in V \; . \; \inf\left\{\lambda \in \mathbb{R}_{++} \middle| x \in \lambda C\right\}$

ConvexFunctionHasConvexLevelSets ::

$$:: \forall V \in \mathbb{R}\text{-VS} \; . \; \forall f : \texttt{Convex}(V, V) \; . \; \forall \alpha \in \overset{\infty}{\mathbb{R}} \; . \; \texttt{Convex}\Big(V, \{v \in V : f(v) \geq \alpha\}\Big)$$

Proof $=$

...

□

ConvexFunctionHasConvexStrictLevelSets ::

$$:: \forall V \in \mathbb{R}\text{-VS} \; . \; \forall f : \texttt{Convex}(V, V) \; . \; \forall \alpha \in \overset{\infty}{\mathbb{R}} \; . \; \texttt{Convex}\Big(V, \{v \in V : f(v) > \alpha\}\Big)$$

Proof $=$

...

ConvexlyBoundedRegionIsConvex ::

$$:: \forall V \in \mathbb{R}\text{-VS} . \forall I \in \mathsf{SET} . \forall \alpha : I \to \overset{\infty}{\mathbb{R}} . \forall f : I \to \mathtt{Convex}(V, V) . \mathtt{Convex}\Big(V, \{v \in V : \forall i \in I . f_i(v) > \alpha_i\}\Big)$$

Proof =

GeneralizedAMGMIneq :: $\forall n \in \mathbb{N} . \forall \lambda : \mathbb{R}^n_+ . \forall x : \mathbb{R}^n_{++} . \forall \aleph : \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$

Proof =

By Jensen inequality for natural logarithm $\ln\left(\sum_{i=1}^n \lambda_i x_i\right) \geq \sum_{i=1}^n \lambda_i \ln(x_i)$.

Then by exponentiating both parts $\sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}$.

□

PositivelyHomogeneous :: $\prod V : \mathbb{R}\text{-VS} . ?\Big(V \to (-\infty, +\infty]\Big)$

$f : \mathtt{PositivelyHomogeneous} \iff \forall v \in V . \forall \alpha \in \mathbb{R}_{++} . f(\alpha v) = \alpha f(v)$

PositiveHomogeneousZeroPositivity :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V) . f(0) \geq 0$

Proof =

Note that $f(0) = f(t0) = tf(0)$ for all $t \in \mathbb{R}_{++}$.

This means that $f(0)$ is either 0 or $+\infty$.

□

PositiveHomogeneousConvexity :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{PositivelyHomogeneous}(V)$.

$. \mathtt{Convex}(V, V, f) \iff \forall x, y \in V . f(x + y) \leq f(x) + f(y)$

Proof =

$(\Rightarrow)$: assume f is convex.

Then $f(x + y) = f\left(\frac{2}{2}x + \frac{2}{2}y\right) \leq \frac{1}{2}f(2x) + \frac{1}{2}f(2y) = f(x) + f(y)$ for any $x, y \in V$.

$(\Leftarrow)$: assume the inequality holds .

Then $f\Big(\lambda x + (1 + \lambda)y\Big) \leq f(\lambda x) + f\Big((1 - \lambda)y\Big) = \lambda f(x) + (1 - \lambda)f(y)$ when $\lambda \in (0, 1)$ and $x, y \in V$.

Otherwise, when $\lambda = 0, 1$, convexity condition holds trivially.

□

Conic $:= \lambda V \in \mathbb{R}\text{-VS} . \mathtt{Convex}(V, V) \times \mathtt{PositivelyHomogeneous}(V) : \mathbb{R}\text{-VS} \to \mathtt{Type}$;

ConicIneq :: $\forall V : \mathbb{R}\text{-VS} . \forall f : \mathtt{Convex}(V, V) \,\&\, \mathtt{PositivelyHomogeneous}(V) . \forall n \in \mathbb{N} . \forall x \in V^n$.

$$. \forall \lambda \in \mathbb{R}^n_{++} . f\left(\sum_{i=1}^n \lambda_i x\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

Proof =

Iterate previous theorem.

□

ConicEpigraph :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : V \to (-\infty, +\infty) . \texttt{Conic}(V, f) \iff \texttt{ConvexCone}(V, \operatorname{epi} f)$

Proof $=$

$\ldots$

$\square$

ConicIsSupersymmetric :: $\forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . \forall v \in V . f(v) \geq -f(-v)$

Proof $=$

Write $f(x) + f(-x) \geq f(x - x) = f(x) \geq 0$.

So $f(x) \geq -f(-x)$.

$\square$

ConicIsLinearIffSymmetric :: $\forall V \in \mathbb{R}\text{-VS} . \forall f \in \texttt{Conic}(f) . f \in V^* \iff \forall v \in V . f(-v) = -f(v)$

Proof $=$

$(\Rightarrow)$: this is trival.

$(\Leftarrow)$: assume that the property holds .

Let $x, y \in V$.

Then $f(x) + f(y) \geq f(x + y) \geq -f(-x - y) \geq -f(-x) - f(-y) = f(x) + f(y)$.

This mean $f(x) + f(y) = f(x + y)$.

But as x and y were arbitrary f must be additive and hence linear.

$\square$

1.2 Convexity Preserving Operations

$\texttt{ConvexComposition} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall D \subset V \forall f : \texttt{Convex}(V, D) \,.\, \forall \phi : \texttt{Convex} \,\&\, \texttt{Increasing}(\mathbb{R}, \mathbb{R}) \,.\, \texttt{Convex}(V,$

$\texttt{Proof} =$

Assume $x, y \in \operatorname{dom} f, \lambda \in [0, 1]$.

Then $\phi\Big(f\big(\lambda x + (1-\lambda)y\big)\Big) \le \phi\Big(\lambda f(x) + (1-\lambda)f(y)\Big) \le \lambda\phi \circ f(x) + (1-\lambda)\phi \circ f(y)$.

$\square$

$\texttt{ConvexFunctionFromSet} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall C : \texttt{Convex}(V \oplus \mathbb{R}) \,.\, \texttt{Convex}\Big(V, V, \Lambda v \in V \,.\, \inf\big\{t | (v, t) \in C\big\}\Big)$

$\texttt{Proof} =$

This is function has cinvex epigraph.

$\square$

$\texttt{InfimalConvolutionIsConvex} :: \forall V \in \mathbb{R}\text{-VS} \,.\, \forall n \in \mathbb{N} \,.\, \forall f : \{1, \dots, n\} \to \texttt{ProperConvexFunction}(V) \,.$

$$. \texttt{Convex}\left(V, V, \Lambda x \in V \,.\, \inf\left\{\sum_{k=1}^{n} f_k(v_k) \middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}\right)$$

$\texttt{Proof} =$

Let $g = \inf\left\{\sum_{k=1}^{n} f_k(v_k) \middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$.

$C = \sum_{k=1}^{n} \operatorname{epi} f_k$ is convex.

A tuple $(x, \phi) \in C$ if there is a sequence $(v, \psi) \in (V \oplus \mathbb{R})^n$ such that $x = \sum_{k=1}^{n} v_k, \phi = \sum_{k=1}^{n} \psi_k$

and $f(v_k) \le \psi_k$ for every $k \in \{1, \dots, n\}$.

Thus $\phi = \sum_{k=1}^{n} \psi_k \ge \sum_{k=1}^{n} f(v_k) \ge g(x)$, so $(x, \phi) \in \operatorname{epi} g$.

Then g can be constructed from set C.

$\square$

$$\texttt{infimalConvolution} :: \prod_{V \in \mathbb{R}\text{-VS}} \prod_{n=1}^{\infty} \Big(\{1, \dots, n\} \to \texttt{ProperConvexFunction}(V)\Big) \to \texttt{Convex}(V, V)$$

$$\texttt{infimalConvolution}(f) = \mathop{\square}_{k=1}^{n} f_i := \inf\left\{\sum_{k=1}^{n} f_k(v_k) \middle| v \in V^n, \sum_{k=1}^{n} v_k = x\right\}$$

$$\texttt{ConvexDelta} :: \prod_{V \in \mathbb{R}\text{-VS}} V \to \texttt{ProperConvexFunction}(V)$$

$\texttt{convexDelta}(a) = \delta_a := \Lambda x \in V \,.\, \texttt{if } x = a \texttt{ then } 0 \texttt{ else } +\infty$

GraphTranslationByInfimalConvolution :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{ProperConvexFunction}(V) . \forall a, v \in V . (\delta_a \Box f$

Proof $=$

Clearly $(\delta_a \Box f)(v) = \min \left\{ f(v-a), +\infty \right\}$.

$\square$

InfimalConvolutionDomain :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{ProperConvexFunction}(V) . \forall a, v \in V \ . \ \mathrm{dom}(f \Box g) = \mathrm{dom}\, f$

Proof $=$

Obvious .

$\square$

DistanceExpression :: $\forall V : \texttt{NormedSpace}(\mathbb{R}) . \forall C : \texttt{Convex}(V) . d_V(C, \bullet) = \| \bullet \| \Box \chi(\bullet | C)$

Proof $=$

$\square$

InfimalConvolutionDefinesCommutativeMonoid ::

$:: \forall V \in \mathbb{R}\text{-VS} .$

$. \texttt{CommutativeMonoid}\Big(\texttt{Convex}(V, V), \Lambda f, g \in \texttt{Convex}(V, V) . \Lambda x \in V \ . \ \inf \left\{ \phi \middle| (v, \phi) \in (\mathrm{epi}\, f + \mathrm{epi}\, g) \right\}\Big)$

Proof $=$

δ_0 is a neutral element, comutativity and associativity is almost obvious.

$\square$

rightScalarMultiplication :: $\prod_{V \in \mathbb{R}\text{-VS}} . \texttt{Convex}(V, V) \to \mathbb{R}_+ \to \texttt{Convex}(V, V)$

rightScalarMultiplication $(f, \lambda) = f\lambda :=$ ConvexFunctionFromSet$(V, \lambda\, \mathrm{epi}\, f)$

RightScalarMultiplicationExpression ::

$:: \forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) . \forall \lambda \in \mathbb{R}_{++} . \forall x \in X \ . \ f\lambda(x) = \lambda f(\lambda^{-1} x)$

Proof $=$

Obvious.

$\square$

RightScalarMultiplicationByZero :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) \ . \ f0(x) = \delta(0)$

Proof $=$

Obvious.

$\square$

ConicityByRightMultiplication :: $\forall V \in \mathbb{R}\text{-VS} . \forall f : \texttt{Convex}(V, V) . \texttt{Conic}(V, f) \iff \forall \lambda \in \mathbb{R}_{++} . f\lambda = f$

Proof $=$

Follows from the expression for right multiplication.

$\square$

$\texttt{conicClosure} :: \prod_{V \in \mathbb{R}\text{-}\mathsf{VS}} \texttt{ConvexFunction}(V) \to \texttt{Conic}(V)$

$\texttt{conicClosure}\,(\operatorname{cone} f) := \texttt{ConvexFunctionFromSet}\Big(V. \operatorname{cone} \operatorname{epi} f\Big)$

$\texttt{GaugeExpression} :: \forall V \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \forall C : \texttt{Convex} \,\&\, \texttt{NonEmpy}(V) \,.\, \gamma(\bullet|C) = \operatorname{cone}\Big(\chi(\bullet|C) + 1\Big)$

$\texttt{Proof} =$

$(x, \phi) \in \operatorname{epi} \gamma(\bullet|C)$ iff $x \in \lambda C$ and $0 < \lambda \leq \phi$.

This means that $(x, \lambda) \in \operatorname{cone} C \times \{1\} \subset \operatorname{cone} \operatorname{epi}\Big(\chi(\bullet|C) + 1\Big)$.

So $(x, \phi) \in \operatorname{cone} C \times \{\phi/\lambda\} \subset \operatorname{cone} \operatorname{epi}\Big(\chi(\bullet|C) + 1\Big) = \operatorname{epi} \operatorname{cone}\Big(\chi(\bullet|C) + 1\Big)$.

On the other hand id $(x, \psi) \in \operatorname{epi} \operatorname{cone}\Big(\chi(\bullet|C) + 1$ then there exists $\lambda \in \mathbb{R}_{++}$ such that $\lambda x \in C$ and $\lambda\psi \geq 1$.

But this means that $\psi \geq \lambda^{-1} \geq \gamma(x|C)$.

Thus $(x, \psi) \in \gamma(\bullet|C)$.

And both functions are equal by equality of epigraphs.

$\square$

$\texttt{SupremumIsConvex} :: \forall V \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to \texttt{ConvexFunction}(V) \,.\, \texttt{ConvexFunction}(V, \sup_{i \in I} f_i)$

$\texttt{Proof} =$

$\operatorname{epi} \sup_{i \in I} f_i = \bigcap_{i \in I} \operatorname{epi} f_i$ is convex.

$\texttt{convexHull} :: \prod_{V \in \mathbb{R}\text{-}\mathsf{VS}} \prod_{I \in \mathsf{SET}} \Big(I \to V \to \overset{\infty}{\mathbb{R}}\Big) \to \texttt{ConvexFunction}(V)$

$\texttt{convexHull}\,(f) = \operatorname{conv}_{i \in I} f_i := \texttt{ConvexFunctionFromSet}\left(V, \operatorname{conv} \bigcup_{i \in \mathcal{I}} \operatorname{epi} f_i\right)$

$\texttt{ConvexHullExpression} :: \forall V \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \forall I \in \mathsf{SET} \,.\, \forall f : I \to V \to (-\infty, +\infty] \,.\, \forall v \in V \,.$

$$. \operatorname{conv}_{i \in I} f_i(x) = \inf\left\{ \sum_{i \in I} \lambda_i f_i(v_i) \middle| \lambda \in \mathbb{R}_+^{\oplus I}, v : I \to V, \sum_{i \in I} \lambda_i = 1, \sum_{i \in I} \lambda_i v_i = x \right\}$$

$\texttt{Proof} =$

This follows from the thorough examination of the definition.

$\texttt{convexPullback} :: \prod V, W \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \mathbb{R}\text{-}\mathsf{VS}(V, W) \to \texttt{ConvexFunction}(W) \to \texttt{ConvexFunction}(V)$

$\texttt{convexPullback}\,(f, T) = fT := f \circ T$

$\texttt{convexPushforward} :: \prod V, W \in \mathbb{R}\text{-}\mathsf{VS} \,.\, \mathbb{R}\text{-}\mathsf{VS}(V, W) \to \texttt{ConvexFunction}(V) \to \texttt{ConvexFunction}(W)$

$\texttt{convexPullback}\,(f, T) = T_* f := \Lambda w \in W \,.\, \inf\{f(v) | w = Tv\}$

1.3 Closures

1.4 Continuity

2 Duality

3 (Sub)differential Calculus

4 From Optimization to Convex Algebra

Sources

1. Convex Analysis — R. T. Rockaffeler 1972