Abstract Algebra

Uncultured Tramp

December 9, 2017

Contents

1 General Concepts

1.1 Equation Algebra

$X, Y, Z : \texttt{Set}(T)$

$\texttt{EquationRightMultiplication} :: \forall \odot : X \times Y \to Z \,.\, \forall a, b \in X \,.\, \forall c \in Y \,.\, a = b \Rightarrow a \odot c = b \odot c$

$\texttt{Proof} \; =$

$(1) := I(=, \times)(a, b, c) : (a, c) = (b, c),$

$(2) := I(=)(\cdot) : \cdot = \cdot,$

$(*) := E(=, \to)(\cdot, \cdot)(2)((a, c), (b, x))(1) : a \cdot c = b \cdot c;$

$\square$

$\texttt{rightEquationMult} :: \prod \odot : X \times Y \to Z \,.\, \prod a, b \in X \,.\, \prod c \in Y \,.\, a = b \to a \odot c = b \odot c$

$\texttt{rightEquationMult}\,((1)) = (1) \odot c := \texttt{EquationRightMultiplication}$

$\texttt{EquationLeftMultiplication} :: \forall \odot : X \times Y \to Z \,.\, \forall a, b \in Y \,.\, \forall c \in X \,.\, a = b \Rightarrow c \odot a = c \odot b$

$\texttt{Proof} \; =$

$(1) := I(=, \times)(a, b, c) : (c, a) = (c, b),$

$(2) := I(=)(\cdot) : \cdot = \cdot,$

$(*) := E(=, \to)(\cdot, \cdot)(2)((c, a), (c, b)(1) : c \cdot a = c \cdot b;$

$\square$

$\texttt{leftEquationMult} :: \prod \odot : X \times Y \to Z \,.\, \prod a, b \in Y \,.\, \prod c \in X \,.\, a = b \to a \odot c = b \odot c$

$\texttt{leftEquationMult}\,((1)) = (1) \odot c := \texttt{EquationRightMultiplication}$

1.2 Binary Operations

$$\texttt{Commutative} :: ?(X \times X \to X)$$
$$\odot : \texttt{Commutative} \iff \forall x, y \in X \;.\; x \odot y = y \odot x$$

$$\texttt{Associative} :: ?(X \times X \to X)$$
$$\odot : \texttt{Associative} \iff \forall x, y, z \in X \;.\; x \odot (y \odot z) = (x \odot y) \odot z$$

$$\texttt{LeftDistributive} :: ?\Big((X \times X \to X) \times (X \times X \to X)\Big)$$
$$\odot, \oplus : \texttt{LeftDistributive} \iff \forall x, y, z \in X \;.\; x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

$$\texttt{RightDistributive} :: ?\Big((X \times X \to X) \times (X \times X \to X)\Big)$$
$$\odot, \oplus : \texttt{RighDistributive} \iff \forall x, y, z \in X \;.\; (x \oplus y) \odot z == (x \odot z) \oplus (y \odot z)$$

$$\texttt{Idempotent} :: ?(X \times X \to X)$$
$$\odot : \texttt{Idempotent} \iff \forall x \in X \;.\; x \odot x = x$$

1.3 Identity And Inverse

$$\texttt{Identity} :: \Big((X \times X) \to X\Big) \to ?X$$

$$e : \texttt{Identity} \iff \Lambda \odot : (X \times X) \to X \;.\; \forall x \in X \;.\; x \odot e = x = e \odot x$$

$\texttt{IdentityIsUnique} :: \forall \odot : (X \times X) \to X \;.\; \forall e, f : \texttt{Identity}(X)(\odot) \;.\; e = f$

$\texttt{Proof} \;=$

$(1) := \eth\texttt{Identity}(X)(\odot)(e) : e \odot f = f,$

$(2) := \eth\texttt{Identity}(X)(\odot)(f) : e \odot f = e,$

$(*) := (1)(2) : f = e;$

$\square$

$$\texttt{Inverse} :: \Big((X \times X) \to X\Big) \to X \to ?X$$

$$a : \texttt{Inverse} \iff \Lambda \odot : \Big((X \times X) \to X\Big) \;.\; \Lambda x \in X \;.\; x \odot a : \texttt{Identity}(X) \;\&\; a \odot x : \texttt{Identity}(X)$$

$\texttt{IverseIsUnique} :: \forall \odot : (X \times X) \to X \;.\; \forall x \in X \;.\; \forall a, b : \texttt{Inverse}(X)(\odot)(x) \;.\; a = b$

$\texttt{Proof} \;=$

'$(1) := \eth\texttt{Inverse}(X)(\odot)(x)(a)\texttt{UniqueIdentity}\eth\texttt{Inverse}(X)(\odot)(x)(b) : a \odot x = b \odot x,$

$(2) := \eth_1\texttt{Inverse}(X)(\odot)(x)(a) : xa : \texttt{Identity},$

$(*) := (1) \odot a\eth\texttt{Identity}(X)(\odot)(xa) : a = b,$

$\square$

1.4 Magmas, Semigroups and Monoids

$$\texttt{Magma} = \sum X : \texttt{Set}(T) \, . \, X \times X \to X$$

$$\texttt{synecdoche} :: \texttt{Magma} \to \texttt{Set}(T)$$
$$\texttt{synecdoche}\,(M, \cdot) := M$$

$$\texttt{operation} :: \prod (X, \odot) : \texttt{Magma} \, . \, (X \times X) \to X$$
$$\texttt{operation}\,() = \cdot_{(X,\odot)} := \odot$$

$$\texttt{Semigroup} :: ?\texttt{Magma}(T)$$
$$X : \texttt{Semigroup} \iff \cdot_X : \texttt{Associative}(X)$$

$$\texttt{Monoid} :: ?\texttt{Semigroup}(T)$$
$$X : \texttt{Monoid} \iff \exists \texttt{Unity}(X)(\cdot_X)$$

$$\texttt{unity} :: \prod X : \texttt{Monoid} \, . \, \texttt{Unity}(X)(\cdot_X)$$
$$\texttt{unity}\,() = e_X := \eth\texttt{Monoid}$$

$$\texttt{iteratedProduct} :: \prod X : \texttt{Monoid} \, . \, \prod n \in \mathbb{Z}_+ \, . \, (n \to X) \to X$$
$$\texttt{iteratedProduct}\,(\emptyset) = \prod_{i=1}^{0} := e_X$$
$$\texttt{iteratedProduct}\,(x) = \prod_{i=1}^{n} x_i := x_n \prod_{i=1}^{n-1} x_{|n-1,i}$$