Mersions.Know

Uncultured Tramp

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1 Immersions

1.1 Immersions

$$\texttt{ImmersionAt} :: \prod M, N : \texttt{SManifold} \, . \, M \to ?C^1(M, N)$$

$$f : \texttt{ImmersionAt}(p) \iff T_p f : \texttt{Injective}(T_p M, T_{f(p)} N)$$

$$\texttt{Immersion} :: \prod M, N : \texttt{SManifold} \, . \, ?C^1(M, N)$$

$$f : \texttt{Immersion} \iff \forall p \in M \, . \, f : \texttt{ImmersionAt}(p)$$

$$\operatorname{rank} f = \dim M$$

$$f :: \mathbb{R}^2 \to \mathbb{R}^3$$

$$f := (u, v) \mapsto (\cos u, \sin u, v)$$

$$T_{(u,v)} f = \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{bmatrix}$$

$$-\sin u = 0 \Rightarrow -\cos u \neq 0 \Rightarrow \operatorname{rank} T_{(u,v)} f = 2$$

$$-\cos u = 0 \Rightarrow -\sin u \neq 0 \Rightarrow \operatorname{rank} T_{(u,v)} f = 2$$

$$\operatorname{rank} f = 2 \Rightarrow f : \texttt{Immersion}(\mathbb{R}^2, \mathbb{R}^3)$$

$$f :: \mathbb{R}^2 \to \mathbb{R}^3$$

$$f := (u, v) \mapsto (\cos u \sin v, \sin u \sin v, (1 - 2\cos^2 v) \cos v)$$

$$T_{(u,v)} f = f'(u, v) = \begin{bmatrix} -\sin u \sin v & \cos u \cos v \\ \sin v \cos u & \cos v \sin u \\ 0 & 6 \sin v \cos^2 v - \sin v \end{bmatrix}$$

$$T_0 f = f'(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\operatorname{rank} T_0 f = 1 \Rightarrow f \; ! \; \texttt{Immersion}$$

$f :: \mathbb{T}^2 \to \mathbb{R}^3$

$f := (e^{i\theta_1}, e^{i\theta_2}) \mapsto ((a + b\cos\theta_1)\cos\theta_2, (a + b\cos\theta_1)\sin\theta_2, b\sin\theta_1)$

$T_{(\theta_1,\theta_2)} f = f'(\theta_1, \theta_2) = \begin{bmatrix} -b\sin\theta_1\cos\theta_2 & -(a+b\cos\theta_1)\sin\theta_2 \\ -b\sin\theta_1\sin\theta_2 & (a+b\cos\theta_1)\cos\theta_2 \\ b\cos\theta_1 & 0 \end{bmatrix}$

$\Delta^2_{1,2}(T_{(\theta_1,\theta_2)} f) = -b(a + b\cos\theta_1)\sin\theta_1$

$\Delta^2_{1,3}(T_{(\theta_1,\theta_2)} f) = b(a + b\cos\theta_1)\sin\theta_2\cos\theta_1$

$\Delta^2_{2,3}(T_{(\theta_1,\theta_2)} f) = -b(a + b\cos\theta_1)\cos\theta_2\cos\theta_1$

$|a| > |b|, b \neq 0 \Rightarrow f :$ `Immersion`$(\mathbb{T}^2, \mathbb{R}^3)$

`LocalIIdentity` $:: \forall M, N :$ `SManifolds` $. \forall p \in M \forall f : C^\infty \;\&$ `ImmersionAt`$(M, N)(p)$.

 . . $\exists (U, x) :$ `ChartCentredAt`$(M, p) : \exists (V, y) :$ `ChartCentredAt`$(N, f(p)) :$

 $: f(U) \subset V : x^{-1} f y = \mathrm{id} \oplus 0$

1.2 Embedings

$\texttt{Embedding} ::?\texttt{Immersion}(M,N)$

$f : \texttt{Embedding} \iff f \in \texttt{ISO}(M)_{\mathsf{TOP}}(M, f(M))$

$\texttt{EmbededManifolds} :: \forall f : \texttt{Embeding}(M,N) \;.\; f(M) : \texttt{RegularSubmanifold}(N)$

$\texttt{EmbeddingMark} :: \forall f : \texttt{Immersion} \;\&\; \texttt{Injective}(M,N) \;.\; M : \texttt{Compact} \Rightarrow f : \texttt{Embeding}(M,N)$

$\texttt{Proof} =$

$\texttt{Assume}\ f : \texttt{Immersion} \;\&\; \texttt{Injective}(M,N),$

$\texttt{Assume}\ M : \texttt{Compact},$

$f : \texttt{Injective}(M,N) \Rightarrow f : \texttt{Bijective}(M, f(M)),$

$f : \texttt{Immersion}(M,N) \Rightarrow f : \mathcal{M}_{\mathsf{TOP}}(M, f(M)) \Rightarrow f^{-1} : \texttt{ClosedMap}(f(M), M))\texttt{as}(1),$

$M : \texttt{Compact}, f(M) : \texttt{Hausdorff} \Rightarrow f : \texttt{ClosedMap}(M, f(M))\texttt{as}2,$

$(1,2) \Rightarrow f : \texttt{Iso}_{\mathsf{TOP}}(M, f(M)) \Rightarrow f : \texttt{Embedding}(M,N);;$

$\forall f : \texttt{Immersion} \;\&\; \texttt{Injective}(M,N) \;.\; M : \texttt{Compact} \Rightarrow f : \texttt{Embeding}(M,N)\square$

$\texttt{LocallEmbedding} :: \forall f : \texttt{Immersion}(M,N) \;.\; \forall p \in M \;.\; \exists U \in \mathcal{U}(p) : f_{|U} : \texttt{Embedding}(U,N)$

$\texttt{Proof} =$

$\texttt{Assume}\ f : \texttt{Immersion}(M,N),$

$\texttt{Assume}\ p \in M,$

$\texttt{Immersion}(M,N)(f)(p) \Rightarrow f : \texttt{ImmersionAt}(M,N)(p),$

$\texttt{ImmersionAt}(M,N)(p)(f) \Rightarrow T_p f : \texttt{Injective}(T_p M, T_{f(p)} N) \Rightarrow$

$\Rightarrow \exists U \in \mathcal{U}(p) : f_{|U} : \texttt{Injective}(U,N)\ \texttt{as}\ (1),$

$\texttt{LocallyCompact}(M)(p) \Rightarrow \exists V \in \mathcal{U}(p) : \overline{V} : \texttt{Compact}\ \texttt{as}\ (2),$

$W := U \cap V,$

$\texttt{EmbeddingMark}(f_{|W})(1,2) \Rightarrow f_{|W} : \texttt{Embedding}(U,N);;$

$\forall f : \texttt{Immersion}(M,N) \;.\; \forall p \in M \;.\; \exists U \in \mathcal{U}(p) : f_{|U} : \texttt{Embedding}(U,N)\square$

$\texttt{FieldEmbedding} :: \forall f : \texttt{Immersion}(M,N) \;.\; \forall X : \mathfrak{X}_f(M,N) \;.$

$\;.\; \exists U : \texttt{Open}(f(M)) : \exists Y \in \mathfrak{X}(U) : X = Y \circ f$

$\texttt{Proof} =$

$\texttt{Assume}\ f : \texttt{Immersion}(M,N),$

$\texttt{Assume}\ X : \mathfrak{X}_f(M,N),$

$\texttt{EmbededManifold}(f) \Rightarrow f(M) : \texttt{RegularSubmanifold}(N),$

$\texttt{Assume}\ p \in M,$

$Y(f(p)) := X(p),$

$(V_{f(p)}, x_{f(p)}) := \texttt{RegularSubmanifold}(N)(f(M))(f(p));$

$$U := \bigcup_{p \in M} V_{f(p)}$$

$\texttt{Assume}\ q \in U,$

$q \in U \Rightarrow \exists p \in M : q \in V_{g(p)}\ \texttt{E},$

$\texttt{Extend}\ Y(q) := Y(x^{-1}_{f(p)}(\pi_{x_{f(p)} \circ f(M)} x_{f(p)}(q)));$

$Y : \mathfrak{X}(U),$

$X = Y \circ f;;$

$\forall f : \texttt{Immersion}(M, N)\ .\ \forall X : \mathfrak{X}_f(M, N)\ .$

$\quad .\ \exists U : \texttt{Open}(f(M)) : \exists Y \in \mathfrak{X}(U) : X = Y \circ f \square$

$\texttt{ProperIsEmbedding} :: \forall f : \texttt{Injective}\ \&\ \texttt{Immersion}\ \&\ \texttt{Proper}(M, N)\ .$

$\quad .\ f : \texttt{Embedding}(M, N)$

$\texttt{Proof} =$

$\texttt{Assume}\ f : \texttt{Injective}\ \&\ \texttt{Immersion}\ \&\ \texttt{Proper}(M, N),$

$\texttt{Assume}\ U : \texttt{Open}(M),$

$\texttt{Assume}\ p \in U,$

$\texttt{LocallyCompact}(N)(f(p)) \Rightarrow \exists O \in \mathcal{U}(f(p)) : \overline{O} : \texttt{Compact}(N),$

$K := f^{-1}(\overline{O}),$

$f : \texttt{Proper}(M, f(M)) \Rightarrow K : \texttt{Compact}(M),$

$f : C(M, f(M)) \Rightarrow f^{-1}(O) : \texttt{Open}(M),$

$W_p := U \cap f^{-1}(O),$

$f^{-1}(O) \subset U,$

$\texttt{EmbeddingMark}(f_{|K})(K) \Rightarrow f_{|K} : \texttt{Embedding}(K, f(K)),$

${f^{-1}}_{|K} : C(K, f(K)) \Rightarrow f(W) : \texttt{Open}(f(K)) \Rightarrow$

$\Rightarrow \exists V : \texttt{Open}(N) : f(W) = f(K) \cap V = \overline{O} \cap f(M) \cap V$

$f(W_p) \subset O \subset \overline{O},$

$f(W_p) = O \cap f(M) \cap V : \texttt{Open}(f(M)),$

$W_p := f(W);$

$f(U)_= \bigcup_{p \in U} W_p \Rightarrow f(U) : \texttt{Open}(f(M));$

$f : \texttt{Embedding}(M, N)\ \square$

$\texttt{SmoothlyUniversal} :: ?C^\infty(M, N)$

$f : \texttt{SmoothlyUniversal} \iff \forall S : \texttt{SManifold}\ .\ \forall g : S \to M\ .$

$\quad .\ f \circ g : C^\infty(S, N) \iff g : C^\infty(S, M)$

$\texttt{WeakEmbedding} = \texttt{SmoothlyUniversal}\ \&\ \texttt{Immersion}\ \&\ \texttt{Injective}(M, N)$

$\texttt{EveryEmbeddingIsWeak} :: \forall f : \texttt{Embedding}(M, N) \ . \ f : \texttt{WeakEmbedding}(M, N)$

$\texttt{Proof} =$

$\texttt{Assume}\ f : \texttt{Embedding}(M, N),$

$\texttt{Embedding}(M, N)(f) \to f : \texttt{Immersion}(M, N)\ \texttt{as}\ (1),$

$\texttt{Embedding}(M, N)(f) \to f : \texttt{ISO}_{\mathsf{TOP}}(M, f(M)) \to f : \texttt{ISO}_{\mathsf{SET}}(M, f(M)) \to$

$\to f : \texttt{Injective}(M, N)\ \texttt{as}\ (2),$

$\texttt{Assume}\ S : \texttt{SManifold},$

$\texttt{Assume}\ g : S \to M,$

$\texttt{Assume}\ f \circ g : C^\infty(S, N),$

$\texttt{Assume}\ p \in S,$

$A_p := (T_{g(p)} f)^\dagger T_p (f \circ g) : \mathcal{L}(T_p, T_{g(p)})$

$\texttt{Assume}\ (U, x) : \texttt{ChartCentredAt}(g(p)),$

$\texttt{Assume}\ (V, z) : \texttt{ChartCentredAt}(p),$

$\texttt{Assume}\ v \in S^{n-1},$

$$\lim_{h \to 0} \frac{x(g(z^{-1}(hv)) - x(g(p))}{h} = \lim_{h \to 0} \frac{x(f^\dagger f g(z^{-1}(hv)) - x(f^\dagger f g(p))}{h} =$$

$$= D_{g(p)} x (T_{g(p)} f)^\dagger T_p (f \circ g)(D_p z)^{-1} v = D_{g(p)} x A_p (D_p z)^{-1} v \to$$

$$\to T_p v = A - pv;$$

$T_p = A_p;$

$g : C^\infty(S, M);$

$f \circ g : C^\infty(S, N) \to g : C^\infty(S, M)\ \texttt{as}\ (3),$

$\texttt{Assume}\ g : C^\infty(S, M) \to f \circ g : C^\infty(S, N);\ \texttt{as}\ (4),$

$(3, 4) \to f : \texttt{SmoothlyUniversal}(M, N)\ \texttt{as}\ (5),$

$(1, 2, 5) \to f : \texttt{WeakEmbedding}(M, N);$

1.3 Immersed submanifolds

$\texttt{ImmersedSubmanifold} :: \prod M : \texttt{SManifold} \; . \; ?\texttt{Subset}(M)$

$S : \texttt{ImmersedSubmanifold} \iff S : \texttt{IS} \iff i_M : \texttt{Immersion}(S, M)$

$\texttt{SmoothThroughImmersion} :: \forall M, N : \texttt{SManifold} \; . \; \forall S : \texttt{IS}(N) \; .$
$\quad . \; \forall f : \mathcal{M}_{\mathsf{TOP}}(M, S) \; . \; i_S \circ f : C^\infty(M, N) \Rightarrow f : C^\infty(M, S)$

$\texttt{Proof} =$

$\texttt{Assume } M, N : \texttt{SManifold},$

$\texttt{Assume } S : \texttt{IS}(N),$

$\texttt{Assume } f : \mathcal{M}_{\mathsf{TOP}}(M, S),$

$\texttt{Assume } i_S \circ f : C^\infty(M, N),$

$\texttt{Assume } p \in S,$

$\texttt{IS}(N)(S) \to i_S : \texttt{Immersion}$

$\texttt{LocalEmbedding}(i_S, p) \to \exists U \in \mathcal{U}(p) : i_{S|U} : \texttt{Embedding Extract},$

$\mathcal{M}_{\mathsf{TOP}}(M, S)(f)(U) \to f^{-1}(U) : \texttt{Open}(M),$

$\texttt{EveryEmbeddingIsWeak}(i_{S|U}) \to i_{S|U} : \texttt{WeakEmbedding}(U, N) \to$
$\quad \to i_{S|U} : \texttt{SmoothlyUniversal}(U, N) \texttt{ as } (1),$

$V := f^{-1}(U),$

$i_S \circ f : C^\infty(M, N) \to i_S \circ f_{|V} : C^\infty(V, N) \to_{(1)} f_{|V} : C^\infty(V, S);$

$f : C^\infty(V, S) \square$

$\texttt{SmoothThroughEmbedding} :: \forall M, N : \texttt{SManifold} \; . \; \forall S : \texttt{RegularSubmanifold}(N) \; .$
$\quad . \; \forall f : M \to S \; . \; i_S \circ f : C^\infty(M, N) \Rightarrow f : C^\infty(M, S)$

$\texttt{Proof} =$

$\texttt{Assume } M, N : \texttt{SManifold},$

$\texttt{Assume } S : \texttt{RegularSubmanifold}(N),$

$\texttt{Assume } f : M \to S,$

$\texttt{Assume } i_S \circ f : C^\infty(M, N),$

$\texttt{RegularSubmanifold}(N)(S) \to i_S : \texttt{Embedding}(S, N),$

$\texttt{EveryEmbeddingIsWeak}(i_S) \to i_S : \texttt{WeakEmbedding}(S, N) \to$
$\quad \to i_S : \texttt{SmoothlyUniversal}(S, N) \texttt{ as } (1),$

$i_S \circ f \to_{(1)} f : C^\infty(M, N) \square$

$\texttt{WESubmanifold} :: ?\texttt{IS}(M)$

$S : \texttt{WESubmanifold} \iff i_S : \texttt{WeakEmbedding}(S, M)$

$\texttt{SmoothlyConnectedComponent} :: \prod M : \texttt{SManifold} \; . \; \prod S : \texttt{Subset}(M) . S \to ?S$

$\texttt{SmoothlyConnectedComponent}(x) = C_x(S) := \{ y \in S : \exists p : \texttt{SPath}(M)(x, y) : \operatorname{Im} p \subset S \}$

$\texttt{InitialSubmanifold} :: \prod M : \texttt{SManifold} \,.\, \dim M \to ??M$

$\texttt{InitialSubmanifold}(k) = W(k) :=$

$:= \left\{ S : \forall s \in S : \exists (U, x) : \texttt{ChartCentredAt}(M, s) : x(C_s(U \cap S)) = \mathbb{R}^k \times \{0\} \right\}$

$\texttt{WeakEmbeddingIsInitial} :: \forall I : \texttt{WeakEmbedding}(S, M) \,.\, I(S) \in W(\dim S)$

$\texttt{Proof} =$

$\texttt{Assume}\ I : \texttt{WealEmbedding}(S, M),$

$k := \dim S,$

$\texttt{Assume}\ s \in S,$

$n := \dim M,$

$k = \texttt{LocallIdentity}(I) \to \texttt{ChartCentredAt}(s, S) : \exists (V, y) : \texttt{ChartCentredAt}(M, I(s)) :$

$: x^{-1} f y = \mathrm{id} \oplus 0\ \texttt{as}\ (1)$

$x(U) : \texttt{Open}(\mathbb{R}^d) \to \exists r \in \mathbb{R}_{++} : \mathbb{B}^k(0, r) \subset x(U)\ \texttt{Extract}\ r',$

$y(V) : \texttt{Open}(\mathbb{R}^d) \to \exists r \in \mathbb{R}_{++} : \mathbb{B}^n(0, r) \subset y(V)\ \texttt{Extract}\ r'',$

$r := \min(r', r''),$

$A := x^{-1}\Big(\mathbb{B}^k(0, r)\Big) : \texttt{Open}(S),$

$B := y^{-1}\Big(\mathbb{B}^n(0, r)\Big) : \texttt{Open}(M),$

$(1) \to I(A) = B \cap I(S)\ \texttt{as}\ (2),$

$f := \Lambda(r, u) \in \mathbb{B}^n(0, r) \,.\, \texttt{if}\ (\rho, u) == 0\ \texttt{then}\ 0\ \texttt{else}\ \left(\tan \frac{\pi \rho}{2r}, u \right),$

$z := f \circ x : \texttt{Coordinates}(M, B),$

$z(I(A)) = \mathbb{R}^k \times \{0\},$

$\texttt{Assume}\ I(p) :\in I(A),$

$L := \Lambda t \in [0, 1] \,.\, x^{-1}\big(t x(s) + (1 - t) x(p)\big) : \texttt{SPath}(S)(s, p),$

$P := I \circ L,$

$\mathrm{Im}\, P \subset I(A) \to_{(2)} \mathrm{Im}\, P \subset B \cup I(S),$

$\texttt{WeakEmbedding}(S, M)(I) \to I : \texttt{SmoothlyUniversal}(S, M) \to$

$\to P : \texttt{SPath}(M)(I(s), I(p)),$

$I(p) \in C_s(B \cup I(S));$

$I(A) = C_s(B \cup I(S)) \to z(C_s(B \cup I(S)) = \mathbb{R}^k \times \{0\};$

$I(S) \in W(k) \square$

$\texttt{InitialToWeaklyEmbeded} :: \forall M \in \texttt{SManifold} \; . \; \forall k \in \dim M \; . \; \forall S \in W(k) \; .$
$\quad . \; \exists! \mathcal{A} : \texttt{Atlas}(S) \; . \; (S, \mathcal{A}) : \texttt{WESubmanifold}(M)$

$\texttt{Proof} =$
$\texttt{Assume}\ M : \texttt{SManifold},$
$\texttt{Assume}\ k \in \dim M,$
$\texttt{Assume}\ S \in W(k),$
$\texttt{Assume}\ s \in S,$
$W(k)(S) \to \exists (U.x) \in \texttt{ChartsCentredAt}(s) \; . \; x() = \mathbb{R}^k \times \{0\}$
$\quad \texttt{Extract}\ (U_s, x_s),$
$A_s := C_s(U_s \cap S),$
$a_s := \pi_{1,\dots,k} \circ x_{|A};$
$\texttt{Assume}\ p, q \in S,$
$A_p \cap A_q = (S \cap U_p \cap U_q),$
$x_p \circ x_q^{-1} : \texttt{Auto}_{\mathsf{MAN}}(U_p \cap U_q) \to a_p \circ a_q^{-1} : \texttt{Auto}_{\mathsf{MAN}}(A_p \cap A_q);$
$(A, a) : \texttt{Atlas}(S),$
$\texttt{Assume}\ X : \texttt{SManifold},$
$\texttt{Assume}\ f : X \to S,$
$\texttt{Assume}\ i_S \circ f : C^\infty(X, M),$
$\texttt{Assume}\ p \in X,$
$V := f^{-1}(A_{f(p)}) = (i_S \circ f)^{-1}(U_{f(p)}) \in \mathcal{U}(p),$
$A_{f(p)} : \texttt{Regular} \to i_{A_{f(p)}} : \texttt{SmoothlyUniversal}(A_{f(p)}, M) \to$
$\quad \to f_{|V} \in C^\infty(V, S);$
$f : C^\infty(X, S);;;$
$(S, (A, a)) : \texttt{WESubmanifold},$
$\texttt{Assume}\ (B, b) : \texttt{Atlas}(S) : (S, (B, b)) : \texttt{WESubmanifold},$

$? \dots$

Proof after Riemannian geometry is learned

$\texttt{WhitneyImmersion} :: \forall M \in \texttt{SManifold} : \exists A : \texttt{Finite} : \texttt{CompatibleAtlas}(M)\ .$

$\quad .\ \exists I : \texttt{Immersion}\&\texttt{Injection}\Big(M, \mathbb{R}^{2n+1}\Big)$

$\texttt{Proof} =$

$\texttt{Assume}\ M \in \texttt{SManifold} : \exists A : \texttt{Finite} : \texttt{CompatibleAtlas}(M)\ \texttt{Extract},$

$(U, x) := \texttt{toList}(A),$

$N := \texttt{length}(U, x),$

$V := \texttt{RefinedFiniteCover}(U) : C^\infty(M),$

$O := \texttt{RefinedFiniteCover}(V) : C^\infty(M, \mathbb{R}^n),$

$\texttt{Assume}\ i \in N,$

$f_i := \texttt{cutFunction}\big(\overline{O}_i, \overline{V}_i\ \big),$

$\phi_i := \Lambda p \in M\ .\ \texttt{if}\ p \in U_i\ \texttt{then}\ f_i(p)x_i(p)\ \texttt{else}\ 0;$

$$F := \bigoplus_{i \in N} f_i \oplus \bigoplus_{i \in N} \phi_i : C^\infty(M, \mathbb{R}^{n+Nn}),$$

$\texttt{Assume}\ p \in M,$

$O : \texttt{OCover}(M) \to \exists i \in N : p \in O_i\ \texttt{Extract},$

$p \in O_i \to T_p\phi_i = D_p x_i,$

$x_i : \texttt{ISO}_{\mathsf{SMAN}}(M, \mathbb{R}^n) \to D_p x_i : \texttt{Invertible} \to T_p\phi_i : \texttt{Invertivle},$

$T_p\phi_i : \texttt{Invertivle} \to \operatorname{rank} T_p\phi_i = n \to \operatorname{rank} T_p F = n \to$

$\to T_p F : \texttt{Injection} \to F : \texttt{ImmersionAt}(p);$

$F : \texttt{Immersion}(M, \mathbb{R}^{n+Nn})\ \texttt{as}\ (1),$

$\texttt{Assume}\ p, q \in M,$

$\texttt{Assume}\ F(p) = F(q),$

$O : \texttt{OCover}(M) \to \exists i \in N : p \in O_i\ \texttt{Extract},$

$p \in O_i \to f_i(p) = 1 \to f_i(q) = 1 \to q \in O_i,$

$q \in O_i \to \phi_i(q) = x_i(q)\ \texttt{as}\ (2),$

$p \in O_i \to \phi_(p) = x_i(p)\ \texttt{as}\ (3),$

$(2, 3) \to x_i(q) = x_i(p)\ \texttt{as}\ (4),$

$x_i : \texttt{Injection} \to_{(4)} p = q;;$

$F : \texttt{Injection}(A, \mathbb{R}^{n+Nn})\ \texttt{as}\ (2),$

$\texttt{Assume}\ n + Nn > 2n + 1,$

$\mathfrak{I} := \Lambda k \in (n(N-1) - 1)_+\ .$

$\quad .\ \exists L : \texttt{Subspace}(\mathbb{R}^{n+Nn}) : \pi_L \circ F : \texttt{Immersion}\&\texttt{Injective}(M, \mathbb{R}^{n+Nn-k}),$

$(1, 2) \to \mathfrak{I}(0),$

$\texttt{Assume}\ k \in (n(N-1) - 2)_+,$

$\texttt{Assume}\ \mathfrak{I}(k) \twoheadrightarrow L : \texttt{Subspace}(\mathbb{R}^{n+Nn}) : \operatorname{codim} V = k,$

$I := \pi_L \circ F,$

$d := nN + n - k,$

$h := \Lambda p \in M \ . \ \Lambda q \in M \ . \ t \in \mathbb{R} \ . \ t(I(p) - I(q)) : C^\infty(M \times M \times t, \mathbb{R}^d),$

$\texttt{Assume}\ p \in M$

$D_{(t,p,q)}h = [I(p) - I(q), tD_pI, tD_qI] \to D_{(0,p,p)} = 0 \to$

$\to (0, p, p) : \texttt{Critical}(h)\ \texttt{as}\ (3),$

$\texttt{Taylor2}(h)(3) \to \exists (Y, y) : \texttt{ChartCentredAt}(M, p) : \forall t \in \mathbb{R} \ . \ \forall q, q' \in Y \ .$

$. \ h(t', q, q') = h(0, p, p) + D_p^2 h(t, y(q) - y(p), y(q') - y(p))^2\ \texttt{Extract},$

$$D^2_{(t,p,q)}h = \begin{bmatrix} & D_pI & D_qI \\ D_pI & tD_p^2 & \\ D_qI & & tD_q^2 \end{bmatrix} \to D^2_{(0,p,p)}h = \begin{bmatrix} & D_pI & D_pI \\ D_pI & & \\ D_pI & & \end{bmatrix} \to$$

$\forall q, q \in Y \ . \ h(t, q, q') = 2t'\Big(D_pI\big(y(q) - y(p)\big) - D_pI\big(y(q') - y(p)\big)\Big);$

$\operatorname{Im} DI \subset \operatorname{Im} h\ \texttt{as}\ (*),$

$d > 2n + 1 \to)(*)\exists v \in \mathbb{R}^d : v \notin \operatorname{Im} h : v \notin \operatorname{Im} DI$

$h(\ldots, \ldots, 0) = 0 \to v \neq 0,$

$L_+ := \perp z,$

$I_+ := \pi_{L_+} \circ I,$

$\texttt{Assume}\ p, q \in N,$

$\texttt{Assume}\ I_+(p) = I_+(q) \to \exists a \in \mathbb{R} \ . \ I(p) - I(q) = av\ \texttt{Extract},$

$\texttt{Assume}\ a \neq 0 \to h(p, q, a^{-1}) = v \to \perp;$

$a = 0 \to I(p) = I(q)\ \texttt{as}\ (3),$

$I : \texttt{Injective}(M, \mathbb{R}^d) \to_{(3)} p = q;$

$I_+ : \texttt{Injective}(M, \mathbb{R}^{d+1})\ \texttt{as}\ (3),$

$\texttt{Assume}\ p \in M,$

$\texttt{Assume}\ w \in T_pM : D_pI_+(w) = 0,$

$D_pI_+(w) = D_p\pi_{L_+}I(w) = \pi_{L_+}D_pI(w)\ \texttt{as}\ (4),$

$D_pI(w) : \texttt{Injection} \to_{(4)} \exists a \in \mathbb{R} : D_pI(w) = av\ \texttt{Extract},$

$\texttt{Assume}\ a \neq 0,$

$D_pU(a^{-1}w) = b \to \perp;$

$a = 0 \to D_pI(w) \to w = 0;;$

$D_pI_+ : \texttt{Injection}\ \texttt{as}\ (4),$

$(3, 4) \to \mathfrak{I}(k + 1);;$

$\mathfrak{I}(n(N - 1) - 1) \to \texttt{WhitneyImmersion}(M)\square$

2 Submersions

$\texttt{SubmersionAt} :: \prod M : \texttt{Manifold} \ . \ M \to ?C^\infty(M,N)$

$f : \texttt{Submersion}(p) \iff T_p f : \texttt{Surjective}(T_pM, T_{f(p)}M)$

$\texttt{Submersion} :: ?C^\infty(M,N)$

$f : \texttt{Submersion} \iff \forall p \in M \ . \ f : \texttt{ImmersionAt}(p)$

$\texttt{LocalProjection} :: \forall f : \texttt{Submersion}(M,N) \ . \ \forall p \in M :$
$\exists (U,x) : \texttt{ChartCentredAt}(M,p) : \exists (V,y) : \texttt{ChartCentredAt}(N,f(p)) :$
$: \exists L : \texttt{Subspace}\big(\mathbb{R}^n\big) : y \circ f \circ x^{-1} = \pi_{L|x(U)}$

$\texttt{Fiber} :: \texttt{Submersion}(M,N) \to ??M$

$F : \texttt{Fiber}(f) \iff \exists q \in N \ . \ F = f^{-1}(q)$

$\texttt{SubmersionIsOpen} :: \forall f : \texttt{Submersion}(M,N) \ . \ f : \texttt{OpenMap}(M,N)$

$\texttt{Proof} =$

$\texttt{Assume } f : \texttt{Sumersion}(M,N),$

$\texttt{Assume } U : \texttt{Open}(M),$

$\texttt{Assume } p \in U,$

$\big((X,x),(Y,y),L\big) := \texttt{LocalProjection}(f,p),$

$V_p := U \cap X : \texttt{Open}(),$

$x : \texttt{OpenMap}(M,\mathbb{R}^m) \to x(V_) : \texttt{Open}(\mathbb{R}^m),$

$\pi_L : \texttt{OpenMap}(\mathbb{R}^m,\mathbb{R}^n) \to \pi_L x(V_p) : \texttt{Open}(\mathbb{R}^n),$

$y : \texttt{Continuous}(\mathbb{R}^n,N) \to y^{-1}\pi_L x(V_p) : \texttt{Open}(N),$

$f_{|X} = y^{-1}\pi_L x \to f(V_p) : \texttt{Open}(N);$

$f(U) = \bigcup_{p \in U} f(V_p) \to f(U) : \texttt{Open}(N);$

$f : \texttt{OpenMap}(M,N) \ \square$

$\texttt{LocalSectionOfManifold} :: \forall P : \texttt{Submersion}(M,N) . \forall p \in M .$

$. \exists V : \texttt{Open}(N) : \exists \sigma : \texttt{LocalSection}(P,V) : p \in \operatorname{Im} \sigma$

$\texttt{Proof} =$

$\texttt{Assume } P : \texttt{Submersion}(M,N),$

$\texttt{Assume } p \in M,$

$((U,x),(V,y),L) := \texttt{LocalProjection}(P,p),$

$u = \pi_{\perp L} x(p),$

$\sigma := y(\Lambda v \in y(V) . v \oplus_L u)x^{-1},$

$\texttt{Assume } q \in V,$

$q\sigma P = q\big(y(\Lambda v \in y(V) . v \oplus_L u)x^{-1} x \pi_L y^{-1}\big) = p;$

$\sigma : \texttt{LocalSection}(P,V),$

$P(p)\sigma = pP\sigma = px\pi_L(\Lambda v \in y(V) . v \oplus_L u)x^{-1} =$

$px\pi_L(\Lambda v \in y(V) . v \oplus \pi_{\perp L} x(p))x^{-1} = (\pi_L x(p) \oplus \pi_{\perp L} x(p))x^{-1} =$

$= x(p)x^{-1} = p \to p \in \operatorname{Im} \sigma;; \square$

$\texttt{UniversalSubmersion} :: \forall P : \texttt{Surjection}\&\texttt{Submersion}(M,N) . \forall T : \texttt{SManifold} .$

$. \forall f : N \to T . f : C^\infty(N,M) \iff Pf : C^\infty(M,T)$

$\texttt{Proof} =$

$\texttt{Assume } P : \texttt{Submersion}(M,N),$

$\texttt{Assume } T : \texttt{SManifold}(N,T),$

$\texttt{Assume } f : N \to T,$

$\texttt{Assume } Pf : C^\infty(M,T),$

$\texttt{Assume } q \in N,$

$p := \texttt{Surjection}(M,N)(P)(q) \to q = pP,$

$(\sigma, U_q) := \texttt{LocalSectionOfSubmersion}(P,p),$

$\texttt{LocalSection}(P,U_q)(\sigma) \to f_{|U_q} = \sigma P f \texttt{ as } (1),$

$\sigma : C^\infty(N,M), Pf : C^\infty(M,T), (1) \to f_{|U_p} : C^\infty(U_q,T);$

$| : \forall q \in N . \exists U \in \mathcal{U}(q) . f_{|U} : C^\infty(U,T) \to$

$\to f : C^\infty(N,T);;; \square$

$\texttt{FiberDecomposition} :: \forall P : \texttt{Submersion}\&\texttt{Surjective}(M, N) \ . \ \forall g : C^\infty(M, T) :$

$\quad : \forall F : \texttt{Fiber}(P) \ . \ g_{|F} : \texttt{Constant} \ . \ \exists! f : C^\infty(N, T) : g = Pf$

$\texttt{Proof} =$

$\texttt{Assume}\ P : \texttt{Submersion}\&\texttt{Surjective}(M, N),$

$\texttt{Assume}\ g : C^\infty(M, T) : \forall F : \texttt{Fiber}(P) \ . \ g_{|F} : \texttt{Constant},$

$\texttt{Assume}\ q \in N,$

$\texttt{Surjective}(M, N)(P)(q) \to P^{-1}(q) \neq \emptyset \ \texttt{as}\ (1),$

$P^{-1}(q) : \texttt{Fiber}(P) \to_{(1)} g_{|P^{-1}(q)} : \texttt{Constant},$

$c := \texttt{Constant}(g_{|P^{-1}(q)}),$

$f(q) := c;$

$f : N \to T,$

$\texttt{UniversalSubmersion}(P)(T, f)(g) \to f \in C^\infty(N, T),$

$\texttt{Assume}\ h \in C^\infty(N, T) : g = Ph : h \neq f,$

$f \neq p \to \exists q \in N \ . \ f(q) \neq h(q)\ \texttt{Extract},$

$\texttt{Surjective}(M, N)(P)(q) \to P^{-1}(q) \neq \emptyset \ \texttt{as}\ (1),$

$P^{-1}(q) : \texttt{Fiber}(P) \to_{(1)} g_{|P^{-1}(q)} : \texttt{Constant},$

$c := \texttt{Constant}(g_{|P^{-1}(q)}),$

$f(q) = c = h(q) \to \bot;$

$\exists! f : C^\infty(N, T) : g = Pf; ; \square$

$\texttt{SubmersionMark} :: \forall M : \texttt{SndCtble}\&\texttt{SManifold} \ .$

$\quad . \ \forall f \in C^\infty(M, N) : \texttt{Surjection}\&\texttt{ConstantRank}(M, N) \ . \ f : \texttt{Submersion}$