0.1 Introduction

In this chapter, two important convergence results of the Finite Element Method (FEM) are discussed. The first result looks at the convergence of the Galerkin approximation for second order hyperbolic type problems (or general vibation problems). The papers considered for the first result are [BV13] and [BSV17]. The second result is from the textbook [SF73] and examines the convergence of eigenvalues and eigenfunctions for a vibration problem, using the Finite Element Method.

0.2 Galerkin approximation for second order hyperbolic type problems

In the article [BV13], the authors investigate the convergence of the Galerkin approximation for second order hyperbolic type problems. The article [BSV17] extends the work of [BV13] by including general damping and damping at the endpoints. For the models in Chapter 1, [BV13] is sufficient while [BSV17] provides more insight and improved notation.

In Section 2.2 of this dissertation, Problem GVar is presented. It is identical to the problem in [BV13]. For convenience, the problem is repeated here.

0.2.1 Formulation of the Galerkin approximation

Recall the spaces V, W and X from Section **??** where $V \subset W \subset X$.

Problem GVar

Given a function $f : J \to X$, find a function $u \in C(J,\ X)$ such that u' is continuous at 0 with respect to $\|\cdot\|_W$ and for each $t \in J,\ u(t) \in V,\ u'(t) \in V,\ u''(t) \in W$ and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X \quad \text{for each } v \in V, \quad (0.2.1)$$

while $u(0) = u_0$, $u'(0) = u_1$.

Assume that the assumptions **A1-A4** from Section 2.2 are satisfied ensuring that Problem GVar has a unique solution.

Before the theory of [BV13] can be discussed, some preliminary work is necessary. The structure of this section is as follows. First the Galerkin approximation for Problem GVar is derived. Then an equivalent system of ordinary differential equations is derived using the Finite Element Method. Finally, the convergence of the Galerkin approximation is discussed using the work of the article [BV13].

Consider the example of the cantilever Timoshenko beam model from Section **??**. In this section the variational problem Problem T-2V is given in terms of bilinear forms.

Problem T-2V

Find a function $u \in T[0,1]$ such that for all $t \in [0,1]$,

$$c(u''(t), v) + b(u(t), v) \quad = \quad (Q(t), v),$$

for each $v \in T[0,1]$.

The interval $[0,1]$ is divided into n equal subintervals $[x_i, x_{i+1}]$, each of length $h = \dfrac{1}{n}$, such that $x_i = ih$ for $i = 0, 1, ..., n$.

Consider a set of $n+1$ linear independent, piecewise linear basis functions δ_i. The subset of these functions that satisfies the boundary conditions are called admissible basis functions. For Problem 2-T, the admissible basis functions are δ_i for $i = 1, 2, ..., n$. Define the space S^h as the space spanned by the admissible basis functions, i.e.

$$S^h = \text{span}\{\delta_1, \delta_2, ..., \delta_n\}.$$

This space $S^h \times S^h$ is a finite dimensional subspace of $T[0,1]$. Define the following functions $w^h \in S^h$ and $\phi^h \in S^h$ as

$$\begin{aligned} w^h(t) &= \sum_{i=1}^{n} w(x_i^*, t)\delta_i(t), \\ \phi^h(t) &= \sum_{i=1}^{n} w(x_i^*, t)\delta_i(t), \end{aligned}$$

where $x_i^* \in [x_i, x_{i+1}]$. Then let $u_h = (w^h, \phi^h)$.

Using these functions, the Galerkin approximation for Problem T-2, referred to as Problem T-2V^h, can be derived.

Problem T-2V^h

Find a function $u_h \in S^h \times S^h$ such that for all $t \in [0,1]$,

$$c(u_h''(t), v) + b(u_h(t), v) \quad = \quad (Q^I(t), v),$$

for each $v \in S^h \times S^h$. For each t, $Q^I(t)$ is the interpolant of $Q(t)$ in S^h.

This example serves as an illustration of the derivation of the Galerkin approximation and the convention of symbols before the general case is presented. Piecewise linear basis functions are used for this example, but other basis functions can be used. In Chapter 5, the basis functions used are piecewise cubic Hermite polynomials.

For the general case presented below, S^h is a finite dimensional subspace of V.

Problem GVar^h

Given a function $f : J \to X$, find a function $u_h \in C^2(J, S^h)$ such that for each $t \in J$

$$c(u_h''(t), v) + a(u_h'(t), v) + b(u_h(t), v) = (f(t), v)_X \quad \text{for each } v \in S^h, \tag{0.2.2}$$

with the initial values $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$. The initial conditions u_0^h and u_1^h are projections of u_0 and u_1 in the finite dimensional space S^h.

0.2.2 System of ordinary differential equations

Problem GVar^h is equivalent to a system of second order differential equations. Consider the standard FEM matrices defined by

$$\begin{aligned} K_{ij} &= b(\phi_j, \phi_i), \\ C_{ij} &= a(\phi_j, \phi_i), \\ M_{ij} &= c(\phi_j, \phi_i), \\ F_i(t) &= c(f(t), \phi_i), \end{aligned}$$

where ϕ_i and ϕ_j are admissible basis functions.

Using these matrices, Problem GVar^h is rewritten as a system of ordinary differential equations denoted by Problem ODE.

Recall that $u^h(t) = \sum_k u_k(t)\phi_k$ where $\bar{u}_k = (u_1(t), u_2(t), ..., u_n(t))$ where each ϕ_k corresponds to a node number k. More complex cases are treated in Chapter **??**.

Problem ODE

Find a function $\bar{u} \in S^h$ such that

$$M\bar{u}'' + C\bar{u}' + K\bar{u} = F(t) \quad \text{with } \bar{u}(0) = \bar{u}_0 \text{ and } \bar{u}(1) = \bar{u}_1 \tag{0.2.3}$$

The following propositions related to Problem ODE are given in [BV13].

Proposition 1. If $F \in C(J)$, then Problem ODE has a unique solution for each pair of vectors $\bar{u}_0$ and $\bar{u}_1$

Proposition 2. Suppose $M, K, C, F, \bar{u}_0$ and $\bar{u}_1$ are defined as above. Then, the function u_h is a solution of Problem GVar^h if and only if the function $\bar{u}$ is a solution of Problem ODE.

Proposition 2 provides a link between the solution of Problem ODE and the solution of Problem GVar^h. Theorem 1 below follows.

Theorem 1. If $f \in C(J, X)$, then there exists a unique solution $u_h \in C^2(J, S^h)$ for Problem GVar^h for each u_0^h and u_1^h in S^h. If $f = 0$ then $u_h \in C^2((-\infty, \infty))$.

It is now required to find an approximation for the solution of $\bar{u}$ of Problem ODE.

Consider the time interval $J = [0, T]$. Divide J into N steps with length $\tau = \frac{T}{N}$. Each interval can be expressed as $[t_{k-1},\ t_k]$ for $k = 1, ..., N$. Denote the approximation of u_h on the interval $[t_{k-1},\ t_k]$ by u_k^h, i.e. $u_h(t_k)$ corresponds to u_k^h for each k. (Recall that $u_h(t) = \sum u_i(t)\phi_i$.) A finite difference method is used to compute u_k^h for each k.

0.2.3 Error estimates

In this subsection we consider estimates for the error $u(t_k) - u_k^h$. To simplify the process, the error is divided into errors for the semi-discrete problem and the fully discrete problem.

Under the assumptions **A1**-**A4** of Section 2 and continuity of f, there exists a unique solution for Problem GVar^h. The next step is to show that the solution of Problem GVar^h converges to the solution of Problem GVar.

Let u be the solution of Problem GVar and u^h be the solution of Problem GVar^h. The authors of [BV13] define the following error,

$$e^h(t) = u(t) - u^h(t). \tag{0.2.4}$$

In [BV13] it is assumed that there exists a subspace H of V and a positive integer α such that

$$\inf_{v\in S^h} \|w-v\|_V \leq Ch^\alpha |||w|||_H,$$

for each $w \in H$ where $|||w|||_H$ is a norm or semi-norm for H.

Theorem 2. If $u(t) \in H$ and $u'(t) \in H$ for each t, then

$$||e^h(t)||_W \quad \leq \quad Ch^\alpha \left(|||u(t)||_H + |||u'(t)||_H\right),$$

for each t.

From Theorem 2 for the semi-discrete problem an error estimate for $e(t) = [u(t) - u_h(t)]$ with respect to the norm of W was obtained. The authors of [BV13] then proceed to obtain an error estimate for $e_k = [u_h(t_k) - u_k^h]$.

The error can then be expressed as

$$e(t_k) = u(t_k) - u_k^h \quad = \quad [u(t_k) - u_h(t_k)] + [u_h(t_k) - u_k^h]. \tag{0.2.5}$$

In (0.2.5), the the error for the semi-discrete problem is the term $u(t_k) - u_h(t_k)$ and the term $u_h(t_k) - u_k^h$ is the error for the fully disrete appreximation of the semi-discrete approximation.

Since the dimension of S^h is not fixed, the equivalence of norms cannot be used, and therefore this error estimate for $e_k = [u_h(t_k) - u_k^h]$ should also be with respect to the norm of W. The local error e_1 can be estimated using Taylor polynomials, but then e_k 'grows' as k increases.

A stability result is derived in [BV13]. Recall that $\tau = \dfrac{T}{N}$.

Lemma.

$$\max \|e_n\|_W^2 \leq KT\tau$$

where K depends on u, u_h and their derivatives.

Using this lemma, [BV13] prove the error estimate. The error estimate for the term $[u_h(t_k) - u_k^h]$ with respect to the norm of W as proven by the authors of [BV13] is presented here as Theorem 3.

Theorem 3. If $f \in C^2([0,T], X)$, then

$$\|u_h(t_k) - u_k^h\|_W \quad \leq \quad K\tau^2$$

for each $t \in (0,\ T)$.

0.2.4 Main result

Finally, Theorem 2 of the semi-discrete problem and Theorem 3 of the fully discrete problem gives an error estimate for the error $e(t)$. Consequently, the error estimate $e^h(t) = u(t) - u^h(t)$ is obtained.

The main result proving the convergence of the solution of the Galerkin Approximation is given in [BV13] as follows.

Theorem 4. Main Result
If $f \in C^2([0, T], X)$, then

$$\|u(t_k) - u_k^h\|_W \quad \leq \quad K\tau^2$$

for each $t \in (0,\ T)$.

The constant K depend on u, u_h and their derivatives.