0.1 FEM computation of eigenvalues and eigenfunctions

Let W be a Hilbert space with the inner product $c(\cdot, \cdot)$ and induced norm $||\cdot||_W$. In [SF73], the authors use a Hilbert space H, with inner product $(\cdot, \cdot)$ and induced norm $||\cdot||$. We remain with the notation that is consistent with this dissertation. Let V be a linear subspace of W, with inner product defined by the bilinear form $b(\cdot, \cdot)$. It is assumed that the bilinear form b is symmetric and that the assumptions in Section **??** holds.

The following eigenvalue problem is considered in [SF73]. The same problem was treated in Section **??**, although the notation differs slightly.

Problem E

Find a vector $u \in V$ and number $\lambda \in R$ such that $u \neq 0$ and

$$b(u, v) = \lambda c(u, v) \tag{0.1.1}$$

for each $v \in V$.

Recall that the eigenvectors can be ordered in such a way that

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

where λ_i are the corresponding eigenvalues.

Properties of eigenvalues

Since any multiple of a eigenfunction is still an eigenfunction, the eigenfunctions can be normalized so that $||u_i||_W = 1$ for all i.

Let $\{\phi_k \in V \mid k = 1, 2, ..., N_e\}$ be a set of linear independent admissible basis functions. Define $S^h := \text{span}\{\phi_k \in V \mid k = 1, 2, ..., N_e\}$ so that S^h is a finite dimensional subspace of V.

Consider the Galerkin approximation for (0.1.1):

Problem E^h

Find $u^h \in S^h$ such that $u^h \neq 0$ and

$$b(u^h, v) = \lambda^h c(u^h, v) \text{ for all } v \in S^h.$$

For examples, see Chapter 5.

Problem E^h can be written as a matrix eigenvalue problem,

$$\lambda^n M\bar{u}_n = K\bar{u}_n. \tag{0.1.2}$$

Since N_e is never small and usually large to very large, a compute algorithm is required to calculate the eigenvalues (and eigenfunctions) of (0.1.2).

The pair $(\lambda^n, \bar{u}_n)$ correspond to the pair (λ_k^h, u_k^h) which is the solution of Problem E^h. It is necessary to understand some of the theory to make the connection.

In S^h, the ordering of vectors is the same as in the original space. Denote the normalized eigenvectors in the space S^h as u_k^h with corresponding eigenvalues λ_k^h for $k = 1, 2, ..., N_e$.

In section 0.2 it is proved that the error $|\lambda_k^h - \lambda_k|$ is large when k is large. It is for example possible that $|\lambda_1^h - \lambda_1|$ is sufficiently small while λ_k^h cannot even be considered as an approximation for λ_k when $k > \dfrac{1}{2}N_e$.

Finally, any subspace of S^h will also be a subspace of V. So the minmax principle applies and a lower bound for the approximate eigenvalues hold [SF73]:

$$\lambda_i \leq \lambda_i^h. \tag{0.1.3}$$

0.2 Estimating the eigenvalues.

In this section, the work in the textbook [SF73] is discussed. The results are the same as given in the textbook, how ever the proofs are expanded for greater clarity.

0.2.1 Projection of the eigenfunctions

Some theory is required before the main results can be proven. The theory is from [SF73].

Rayleigh quotient

$$R(v) = \frac{b(v,v)}{c(v,v)} \quad \text{for } v \in V. \tag{0.2.1}$$

Projection

If $u \in V$, then Pu is its projection in the subspace S^h.

$$b(u - Pu, v^h) = 0 \text{ for all } v \in S^h.$$

Let $E_j \in V$ denote the eigenspace spanned by the exact eigenvectors $\{u_1, u_2, ..., u_j\}$ for $j = 1, 2, ..., m$. Clearly $m \leq N_e$.

Consider the subspace S_j of S^h where

$$S_j = PE_j \text{ for } j = 1, 2, ..., m.$$

The elements Pu_j are the projections of the eigenfunctions u_j into the space S^h. These projections Pu_j are not necessarily equal to $u^h \in S^h$. In fact, u_j^h can be vastly different from u_j. The situation is not simple. It is possible that $Pu_k = 0$ for large k Assume that the dimension of S^h is large enough, substantially larger than m.

Let $B_m = \{u \in E_m \mid ||u||_W = 1\}$ and define $\mu_m = \inf\{(Pu, Pu \mid u \in B_m)\}$.

The first step to obtain estimates for the eigenvalues, it to show that the elements of B_m are linearly independent. In the first part we follow the approach in [Zie00]. The author introduced the quantity μ_m above.

Proposition 1. $\mu_m > 0$ if and only if $\dim S_m = m$.

Proof. To show that the dimension of $S_m = m$, suppose that the elements of B_m are linearly dependent. Then there exists a $u \in B_m$ such that $Pu = 0$ and consequently $\mu_m = 0$. The result follows from the contra-positive. □

0.2.2 Upper bounds for approximate eigenvalues

Recall the definition of the Rayleigh quotient R in (0.2.1).

Proposition 2. $\lambda_m^h \leq \max R(Pu) \text{ for } u \in B_m$

Proof. Since $\dim S_m = m$, following from the minmax principle that

$$\lambda_m^h \leq \max R(v) \text{ for } v \in S_m. \tag{0.2.2}$$

Take an arbitrary nonzero $v \in S_m$. Then there exists a $Py \in E_m$ such that $v = Py$.

Now we take an arbitrary $v \in S_m$, $v \neq 0$. Then there exists a $u \in E_m$ such that $Pu = v$. This Pu is the projection into S_m of some $u \in E_m$ (which is also $\dfrac{1}{||u||_W} u \in B_m$).

Next we show that $R(||u||_W^{-1} u) = R(u)$:

$$\frac{b\left(\frac{1}{||u||_W} u, \frac{1}{||u||_W} u\right)}{c\left(\frac{1}{||u||_W} u, \frac{1}{||u||_W} u\right)} = \frac{b(u,u)}{c(u,u)} = R(u).$$

Finally, from (0.2.2)

$$\begin{aligned} \lambda_m^h &\leq \max R(v) \quad \text{for} \quad v \in S_m, \\ &= \max R(Pu) \quad \text{for} \quad u \in E_m, \\ &= \max R(Pu) \quad \text{for} \quad u \in B_m. \end{aligned}$$

$\square$

In the textbook, the authors show that the eigenfunctions are orthogonal. But they do so using matrix representations of the eigenvalue problem. A different method can be used to show this.

Pick any $i, j \in \mathbb{N}$ such that $i, j \leq m$. Then

$$\begin{aligned} b(u_i, \nu) &= \lambda_i c(u_i, \nu), \\ \text{and } b(u_j, \nu) &= \lambda_j c(u_j, \nu). \end{aligned}$$

for each $\nu \in V$. Clearly

$$\begin{aligned} b(u_i, u_j) &= \lambda_i c(u_i, u_j), \\ \text{and } b(u_j, u_i) &= \lambda_j c(u_j, u_i). \end{aligned}$$

Then using the symmetry of $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$,

$$0 = (\lambda_i - \lambda_j) c(u_i, u_j).$$

So if $i \neq j$ then $\lambda_i \neq \lambda_j$. Therefore u_i and u_j are orthogonal when $\lambda_i \neq \lambda_j$.

The next steps in the textbook [SF73] contains proofs with multiple results. In an attempt to better understand the results, the proofs are broken up into smaller proofs.

Lemma 1. $\lambda_m^h \leq \dfrac{\lambda_m}{\mu_m^h}$

Proof. Consider the linearity of the bilinear form b, and fact that any $u \in E_m$ can be expressed as a linear combination $u = \sum_{i=1}^m c_i u_i$. Then

$$\begin{aligned} b(u,u) &= b\left(\sum_{i=1}^m c_i u_i, \sum_{j=1}^m c_j u_j\right), \\ &= \sum_{i=1}^m c_i \sum_{j=1}^m c_j b(u_i, u_j). \end{aligned}$$

The summation parameters can be merged into a single parameter. Then

$$\begin{aligned} b(u,u) &= \sum_{i=1}^m c_i^2 \lambda_i u_i, \\ &\leq \lambda_m \sum_{i=1}^m c_i^2 u_i, \\ &= \lambda_m ||u||_W^2. \end{aligned}$$

for all $u \in B_m$.

And since $B_m \subset E_m$, $b(Pu, Pu) \leq \lambda_m$ for all $u \in B_m$

Using the Rayleigh quotient, and the definition of μ_m^h,

$$\begin{aligned} R(Pu) &= \frac{b(Pu,Pu)}{c(Pu,Pu)}, \\ &= \frac{b(Pu,Pu)}{||Pu||_W^2}, \\ &\leq \frac{\lambda_m}{\mu_m^h}. \end{aligned}$$

Together with Proposition 2 it follows that

$$\lambda_m^h \leq \frac{\lambda_m}{\mu_m^h}.$$

□

0.2.3 The error bound

Following Lemma 1, and since $\lambda_i^h \geq \lambda_i$ it follows that $0 < \mu_m^h \leq 1$. It is now possible to define the 'error bound' in [SF73]:

$$\sigma_m^h = 1 - \mu_m^h. \tag{0.2.3}$$

Proposition 3. $0 \leq \sigma_m^h < 1$ and $\lambda_m^h - \lambda_m \leq \lambda_m^h \sigma_m^h$

Proof. Starting with the result of Lemma 1, $\lambda_m^h \mu_m^h \leq \lambda_m$.

Since $-\lambda_m \leq -\lambda_m^h \mu_m^h$, it follows that

$$\lambda_m^h - \lambda_m \leq \lambda_m^h - \lambda_m^h \mu_m^h = \lambda_m^h (1 - \mu_m^h).$$

□

This result gives an error estimate for the eigenvalues. To prove the convergence of the eigenvalues, it is necessary to prove that the error estimate σ_m^h converges to zero as $h \to 0$.

Proposition 4. $\sigma_m^h = \max \{2c(u, u - Pu) - ||u - Pu||_W^2 \mid u \in B_m\}$

Proof. Let $u \in B_m$. Then

$$\begin{aligned} ||u - Pu||_W^2 &= c(u - Pu, u - Pu), \\ &= c(u, u) - 2c(u, Pu) + c(Pu, Pu), \\ &= 2c(u, u) - 2c(u, Pu) + c(Pu, Pu) - c(u, u), \\ &= 2c(u, u - Pu) + c(Pu, Pu) - c(u, u). \end{aligned}$$

Consequently,

$$c(u, u) - c(Pu, Pu) = 2c(u, u - Pu) - ||u - Pu||_W^2.$$

Since $u \in B_m$, $||u||_W^2 = 1$ and hence

$$1 - ||Pu||_W^2 = 2c(u, u - Pu) - ||u - Pu||_W^2.$$

On the right hand side, $1 - ||Pu||_W^2 \leq 1 - \mu_m^h = \sigma_m^h$ for all $u \in B_m$. Therefore

$$\sigma_m^h = \max \left\{ 2c(u, u - Pu) - ||u - Pu||_W^2 \mid u \in B_m \right\}.$$

□

Proposition 4 is a result given in [SF73] without explaining how it is derived.

Introduce some new notation for convenience. For any $u \in E_m$, let $u^* = \sum_{i=1}^m c_i \lambda_i^{-1} u_i$ where $u = \sum_{i=1}^m c_i u_i$.

Proposition 5. For any $u \in E_m$

$$c(u, u - Pu) = b(u^* - Pu^*, u - Pu)$$

Proof. For any $i = 1, 2, ..., m$,

$$\begin{aligned}
\lambda_i c(u_i, u - Pu) &= b(u_i, u - Pu), \\
&= b(u_i, u - Pu) - b(u - Pu, Pu_i) \quad \text{(Rayleigh-Ritz Projection)}, \\
&= b(u_i, u - Pu) - b(Pu_i, u - Pu), \\
&= b(u_i - Pu_i, u - Pu).
\end{aligned}$$

Multiplying by $c_i \lambda_i^{-1}$ and summation over i gives:

$$\begin{aligned}
\sum_{i=1}^m c_i \lambda_i^{-1} \lambda_i c(u_i, u - Pu) &= \sum_{i=1}^m c_i \lambda_i^{-1} b(u_i - Pu_i, u - Pu), \\
&= b(\sum_{i=1}^m c_i \lambda_i^{-1} u_i - \sum_{i=1}^m c_i \lambda_i^{-1} Pu_i, u - Pu), \\
&= b(u^* - Pu^*, u - Pu).
\end{aligned}$$

Therefore $c(u, u - Pu) = b(u^* - Pu^*, u - Pu)$. □

Lemma 2. $\sigma_m^h \leq \max\{2||u^* - Pu^*||_W ||u - Pu||_W \mid u \in B_m\}$.

Proof. From Proposition 4,

$$\begin{aligned}
\sigma_m^h &= \max\left\{2c(u, u - Pu) - ||u - Pu||_W^2 \mid u \in B_m\right\}, \\
&\leq \max\{2c(u, u - Pu) \mid u \in B_m\}.
\end{aligned}$$

From Proposition 5,

$$\sigma_m^h = \max\{2b(u^* - Pu^*, u - Pu) \mid u \in B_m\}.$$

Using the Schwartz inequality,

$$b(u^* - Pu^*, u - Pu) \leq ||u^* - Pu^*||_W ||u - Pu||_W.$$

Finally,

$$\sigma_m^h \;\; \le \;\; \max\left\{2||u^* - Pu^*||_W ||u - Pu||_W \mid u \in B_m\right\}.$$

□

0.2.4 Convergence of the eigenvalues

Assumption

For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $h < \delta$, then

$$||u - Pu||_W < \epsilon \;\text{ for each }\; u \in B_m.$$

Remark: Pu is the closest element in S^h to u. In particular, $||u - Pu||_W \le ||u - \Pi u||_W$, where Πu is the interpolant of u in S^h. The operator Π is treated in Section 0.4.

Lemma 3. For any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sigma_m^h < \epsilon \;\text{ if }\; h < \delta. \tag{0.2.4}$$

Substitute the assumption into the estimate for σ_m^h in Lemma 2.

Lemma 4. There exists a $\delta > 0$ such that for $h < \delta$

$$\lambda_m^h - \lambda_m \le 2\lambda_m \sigma_m^h. \tag{0.2.5}$$

Proof. Using Lemma 3, choose δ such that $\sigma_m^h < \frac{1}{2}$. Then (by Lemma 1) $\lambda_m^h < 2\lambda_m$ and therefore $\lambda_m^h - \lambda_m \le 2\lambda_m \sigma_m^h$. □

The convergence of the eigenvalues follows from (0.2.4) and (0.2.5). An estimate of the error depends on an estimate for $u - \Pi u$, see Section 0.4

0.3 Convergence of the eigenfunctions

The next step is to show the convergence of the eigenfunctions. The problem can be formulated using the following result.

Lemma 5.

$$b(u_m - u_m^h, u_m - u_m^h) \;\; = \;\; \lambda_m c(u_m - u_m^h, u_m - u_m^h) + \lambda_m^h - \lambda_m.$$

Proof.

$$\begin{aligned}
b(u_m - u_m^h, u_m - u_m^h) &= b(u_m, u_m) - 2b(u_m, u_m^h) + b(u_m^h, u_m^h), \\
&= \lambda_m c(u_m, u_m) - 2\lambda_m c(u_m, u_m^h) + \lambda_m^h c(u_m^h, u_m^h), \\
&= \lambda_m - 2\lambda_m c(u_m, u_m^h) + \lambda_m^h, \\
&= 2\lambda_m - 2\lambda_m c(u_m, u_m^h) + \lambda_m^h - \lambda_m, \\
&= \lambda_m c(u_m - u_m^h, u_m - u_m^h) + \lambda_m^h - \lambda_m.
\end{aligned}$$

□

It has been shown that the eigenvalues converge to the exact eigenvalues as $h \to 0$. So from this result, it only remains to show that $c(u_m - u_m^h, u_m - u_m^h) \to 0$ as $h \to 0$.

At this point, another assumption must be made. Assume that there are not eigenvalues with multiplicity more than 1. In other words, all the eigenvalues correspond only to one eigenfunction. In [SF73], the authors mention that for repeated eigenvalues, then the eigenfunctions can be chosen so that the main convergence results hold. This case is ommited in this dissertation.

Lemma 6. For all m and j

$$(\lambda_j^h - \lambda_m) c(Pu_m, u_j^h) = \lambda_m c(u_m - Pu_m, u_j^h).$$

Proof. Since the term $\lambda_m c(Pu, u_j^h)$ appears on both sides of the equation, it is only required to show that

$$\lambda_j^h c(Pu, u_j^h) = \lambda_m c(u, u_j^h).$$

Since both u and u_j^h are eigenfunctions, then

$$\begin{aligned}
\lambda_j^h c(Pu, u_j^h) &= b(Pu, u_j^h), \\
\lambda_m c(u, u_j^h) &= b(u, u_j^h).
\end{aligned}$$

Then equality follows from the definitions of the projection P. □

The set $\{u_1^h, u_2^h, ..., u_N^h\}$ forms an orthonormal basis for S^h. The projection Pu_m can be written as:

$$Pu_m = \sum_{j=1}^{N} c(Pu_m, u_j^h) u_j^h. \tag{0.3.1}$$

From Lemma 6, it follows that $c(P_m, u_j^h)$ is small if λ_m^h is not close to λ_j. Therefore (0.3.1) tells us that Pu_m is close to u_m^h. The estimate for $Pu_m - u_m^h$ will follow from this result.

Following the convergence of the eigenvalues, $\exists\rho > 0$ and $\exists\delta > 0$ such that if $h < \delta$,

$$|\lambda_m - \lambda_j^h| \;>\; \rho \quad \text{for all} \quad j = 1, 2, ..., N. \tag{0.3.2}$$

Therefore

$$\frac{\lambda_m}{|\lambda_m - \lambda_j^h|} \;\leq\; \rho \quad \text{for all} \quad j = 1, 2, ..., N. \tag{0.3.3}$$

To simplify the notation, let $\beta = c(Pu_m, Pu_m^h)$.

Lemma 7.

$$||Pu - \beta Pu_m^h||_W^2 \;\leq\; \rho^2 ||u_m - Pu_m||_W^2.$$

Lemma 8.

$$||u_m - \beta u_m^h||_W \;\leq\; (1 + \rho)\, ||u_m - Pu_m||_W.$$

The proofs for lemma's 7 and 8 are given in [SF73].

So again using the Approximation Theorem, it follows that $||u_m - \beta u_m^h||_W \leq Ch^k ||u^k||_W$.

Lemma 9.

$$||u_m - u_m^h||_W \;\leq\; 2||u_m - \beta u_m^h||_W.$$

Proof.

$$\begin{aligned} ||u_m - u_m^h||_W &= ||u_m - \beta u_m^h + \beta u_m^h - u_m^h||_W, \\ &\leq ||u_m - \beta u_m^h||_W + ||\beta u_m^h - u_m^h||_W, \\ &= 2||u_m - \beta u_m^h||_W. \end{aligned}$$

□

Therefore $||u_m - u_m^h||_W \leq Ch^k ||u^k||_W$. So for any $\epsilon > 0$, a $\delta > 0$ can be found such that if $h < \delta$, $||u_m - u_m^h||_W < \epsilon$.

0.4 The approximation theorem

Consider a interpolation operator Π. This projection is linear, i.e.

$$\begin{aligned}\Pi(f+g) &= \Pi f + \Pi g,\\ \Pi(\alpha f) &= \alpha \Pi f \quad \text{for a constant } \alpha.\end{aligned}$$

Define the interval $I_e = [a, a+h]$. A necessary condition is for the operator Π is that when Πu is restricted to the interval I_e, this must equal the projection of u restricted to the interval I_e. This can be written as

$$[\Pi u]_{I_e} = \Pi_e [u]_{I_e}.$$

The following notation is introduced.

$\mathcal{P}_j(I_e)$: Is the set of all polynomials on the interval I_e of degree at most j.

$r(\Pi_e)$: If the range of Π_e is contained in $\mathcal{P}_j(I_e)$ and $k < j$ is the largest integer such that $\Pi_e f = f$ for each $f \in \mathcal{P}_j(I_e)$, then $r(\Pi_e) = k$.

$s(\Pi_e)$: Is a integer and the largest order derivative used in the definition of Π_e.

From the textbook [OR76], following approximation theorem for finite elements is given verbatim:

Theorem (The Interpolation Theorem for Finite Elements). Let Ω be an open bounded domain in $\mathcal{R}^n$ satisfying the cone condition. Let k be a fixed integer and m an integer such that $0 \le m \le k+1$. Let $\Pi \in L(H^{k+1}(\Omega), H^m(\Omega))$ be such that

$$\Pi u = u \qquad \text{for all } u \in \mathcal{P}_k(\Omega) \tag{0.4.1}$$

Then for any $u \in H^{k+1}(\Omega)$ and for sufficiently small h, there exists positive a constant C, independent of u and h, such that

$$||u - \Pi u||_{H^m(\Omega)} \le C \frac{h^{k+1}}{p^m} |u|_H^{k+1}(\Omega) \tag{0.4.2}$$

where $|u|_H^{k+1}(\Omega)$ is the seminorm.

For the requirements of this dissertation, this can be simplified with an assumption. The assumption is that the basis of S^h consists of polynomials. With this assumptions, the semi-norm $|u|_H$ is equal to the norm $||u||_W$. This approximation theorem can be rewritten as

Theorem 1. Suppose there exists an integer k such that for each element

$$s(\Pi_e) + 1 \leq k \leq r(\Pi_e) + 1.$$

Then there exists a constant C such that for any $u \in C_+^k(I)$,

$$||(\Pi u)^{(m)} - u^{(m)}||_W \leq Ch^{k-m}||u^{(k)}||_W \quad \text{for } m = 0, 1, ..., k.$$