0.1 Overview

Chapter 1

This dissertation is a literature study to investigate the validity of different linear models for beams and plates. The term validity in this dissertation means how well a model compares to a more realistic model for real world applications. The first chapter of the dissertation introduces the models. The simplest model is the Timoshenko model. The other models are a two-dimensional elastic beam model, three-dimensional elastic beam and plate models and a Reissner-Mindlin plate model. The aim in this dissertation is to validate the use of the Timoshenko beam model and the Reissner-Mindlin plate model in applications. These models are simplified one-dimensional beam and two-dimensional plate models. But more realistic models exist, such as a two-dimensional and three-dimensional beam model and a three-dimensional plate model. Using modal analysis (see Chapter 2), it is shown that the solutions of the models can be represented as a linear combination of the modal solutions. Using this idea, it is only required to compare the eigenvalues and eigenfunctions of the models to be able to compare the difference between the solutions. In Chapter 1, the models of the dissertation are given (in dimensionless form). Model problems are then defined by including boundary conditions. These model problems are the models used in the rest of the dissertation. The variational forms are also derived which enables us to see the similarity between the models and the general theory.

Chapter 2

First, a variational form for a model problem of a cantilever Timoshenko beam is given. This model problem in variational form is then extended to complete function spaces, and a weak variational problem is obtained. This weak variational problem is used as an example to explain the main theory of this chapter which is from the article [VV02]. In this article, the authors present a general weak variational form. The weak variational form of all the model problems of this dissertation are special cases of this general weak variational problem. Therefore the assumptions and results can be formulated and applied to all the applications in this dissertation. The article gives four assumptions. Under these assumptions it is shown that the general vibration problem has a unique solution using semi-group theory. The theory is then applied to the example by proving that the assumptions hold.

Finally the concept of modal analysis is introduced, which is fundamental to this dissertation. First the idea of modal analysis is explained by hands of another example, again using a cantilever Timoshenko beam. Then the general case is discussed that follows the article [CVV18]. Given a general vibration problem, it is split into two problems by introducing a trial solution. It can then be verified that these two problems are the eigenvalue problem and an ordinary differential equation. An additional assumption (additional to the assumptions of [VV02]) is introduced by [CVV18]. Under this assumption, the eigenvalue problem has a complete orthonormal sequence of eigen-solutions. The solution of the general vibration problem is an infinite series of modal solutions. This formal series solution is then shown to also be valid for a initial value problem. This general case is also applicable to all the models in this dissertation.

This theory is crucial for this dissertation. It ensures that the solutions of the models will compare well if the eigenvalues and eigenfunctions compare well. Therefore for the comparisons in Chapter 4 and 6, it is only required to compare the eigenvalues and eigenfunctions. The validity of the formal series solution is also important as it ensures that the comparisons remain valid, as long as the models are disturbed in the same way.

Chapter 3

In this chapter, some theory for the Finite Element Method is discussed. This chapter contains two parts, that covers two different theoretical results. The first is on the convergence of the Galerkin approximation of a general vibration problem. This theory is presented in the article [BV13]. The second part concerns the convergence of the eigenvalues and eigenfunctions of a vibration problem when applying the Finite Element Method. This theory is presented in the textbook [SF73].

In the first part, the authors of [BV13] consider the general vibration problem that is studied in Chapter 2. The general vibration problem has a solution as shown in Chapter 2. The Galerkin Approximation is derived from the general vibration problem and is rewritten into a system of ordinary differential using the Finite Element Method. This ordinary differential equation can be proven to have a unique solution. The main results of [BV13] shows that this solution of the Galerkin Approximation converges to the solution of the general vibration problem. The approach of the authors is to calculate the error estimates.

The second part of the chapter considers work done in a textbook [SF73] on eigenvalue problems for elliptic partial differential equations. The specific work discussed, covers the convergence of the eigenvalues and eigenfunctions of a general vibration problem when applying the Finite Element Method. The authors consider a general eigenvalue problem. Then using the Rayleigh-quotient, from the Rayleigh-Ritz method, as well as an approximation theorem from [OR76], the main result is proven. The specific work done in this section is updating the notation of the textbook, as well as expanding some proofs so that the results are easier to understand.

Chapter 4

This chapter is a focus on the main theory of this dissertation, the Timoshenko beam theory. The first section is a discussion of modal analysis applied to the Timoshenko beam theory, and specifically a discussion of the article [VV06]. In this article, the authors present a method to calculate the exact eigenvalues and eigenfunctions of a Timoshenko beam. Starting with a general eigenvalue problem for a Timoshenko beam model, the authors derive a general solution for the ordinary differential equation. The authors then explain the method by hands of an example by applying the method to a cantilever beam model. The next sections of this chapter also then looks at examples of applying the method, first to a cantilever beam model, and then to a pinned-pinned beam model.

Section 4.5 is a discussion of the article [LVV09]. In this article, the authors investigate the validity of a cantilever Timoshenko beam model, by comparing it to a two-dimensional cantilever Timoshenko beam model. The authors compare the models by comparing the eigenvalues and eigenfunctions. (See Chapter 2 on modal analysis). The two-dimensional model is more complex and there are eigenvalues that are not shared between the models. The authors use the mode shapes to match up the eigenvalues. The eigenvalues matching the eigenvalues of the Timoshenko beam model are referred to by the authors as beam-type eigenvalues. The authors also consider different shapes of beams, from a short thick beam to a long slender beam. The results show that the models compare very well, even for a short and thick beam.

The next section is a discussion on the article [SP06]. In this article, the authors investigate the validity of the Timoshenko beam theory, by comparing the eigenvalues (natural frequencies) for a physical beam, to a Timoshenko beam and a three-dimensional beam using Finite Element Analysis. The authors report on an experiment with forced vibration a free-free beam where the natural frequencies are measured. These empirical results are then compared to the theoretical results. This result, together with the results in chapter 6, give a good picture to the validity of the Timoshenko beam theory.

Chapter 5

In this chapter, the Finite Element Method is applied to cantilever two-dimensional elastic body, cantilever three-dimensional elastic body and a cantilever Reissner-Mindlin plate. The aim of this section is to obtain an algorithm to calculate the eigenvalues and eigenfunctions of the models. The Finite Element Method is not applied to the Timoshenko beam theory, as chapter 4 provides an alternative method. For all the models, the Finite Element Method is applied using bi-cubic or tri-cubic basis functions to improve the rate of convergence and reduce the processing required to obtain accurate results. Each section ends with a eigenvalue problem for the models that can be easily applied to a computer program to calculate the eigenvalues and eigenvectors for the models. This is in preparation of Chapter 6 where the eigenvalues and eigenfunctions are calculated and compared.

Chapter 6

This chapter is an extension of the work of Section 4.5. In Section 4.5, the validity of the Timoshenko beam theory is investigated by comparing a Timoshenko beam model to a two-dimensional beam model.

In real-world applications, a beam is a three-dimensional model. Therefore it should be more realistic to use a three-dimensional model to investigate the validity of the Timoshenko beam theory. This is mentioned by the authors of [LVV09]. Their suggestion is to use the two-dimensional model as an intermediate step, to avoid complexities. Therefore the validity of the two-dimensional model is investigated, using a three-dimensional beam model as a reference. Again the results show that the comparison relies on the shape of the beams. The two-dimensional model compares well to the three-dimensional beam if the beam is not wide.

If the width of the beam is very large, the use of a beam model can be questioned. A plate model might be more suited. Therefore the last section of this chapter investigates the validity of the Reissner-Mindlin plate model. The Reissner-Mindlin plate model is compared to a three-dimensional plate model. The results show that the Reissner-Mindlin plate model compares well to the three-dimensional plate model.

The same method is used in this chapter as in Section 4.5. The mode shapes are sketched and matched. The corresponding eigenvalues can then be matched up and compared. The eigenvalues relating to the Reissner-Mindlin plate model are referred to as plate-type eigenvalues.

0.2 Contributions

There are four models used in this dissertation. For each of the models, the dimensionless variational form is derived. Also presented are the model problems that are used in this dissertation.

The first result investigates the existence and uniqueness of solutions for general vibration problems. The article that is discussed proves this result for a general vibration problem, using four assumptions. To explain the theory, an example is presented using one of the model problems of the dissertation. The cantilever Timoshenko beam model is chosen for the simple boundary conditions, as well as it's recurring importance in this dissertation. The weak variational form of the model is derived as well as the function spaces defined. This is presented in the same format as the general vibration problem. In fact all the models in this dissertation are special cases of this general vibration problem. The theory is then applied to this example problem, as a demonstration. To apply the theory, the four assumptions are proven to be true.

The next result looks at modal analysis. Before the general case is discussed, again the cantilever Timoshenko beam is used to illustrated the concept of modal analysis.A trial solution to the boundary value problem is suggested. This trial solution is substituted into the partial differential equation and two problems are obtained. The first problem is the eigenvalue problem and the second is an ordinary differential equation. The eigenvalue problem can be solved with theory discussed later in the dissertation and the ordinary differential equation can then be solved. Substitution of these two results into the boundary value problem confirms that the trial solution is correct. The same idea is then discussed for the general case.

We then look at the convergence of the Galerkin Approximation for our general vibration problem. The results of the article discussed are updated with improved notation using a different article. The general case of the Galerkin approximation use a lot of symbols that are not immediately obvious, so again the cantilever Timoshenko beam model problem is used and the Galerkin Approximation is derived in an attempt to explain some of the conventions. The results of the article are then discussed and presented. The results are presented in a concise and practical format to reduce the need to define any unnecessary symbols or notation. The results are also presented in four theorems, summarizing the results of the article that are important but that are not necessarily presented as a theorem in the article.

The next result is on the convergence of the eigenvalues and eigenfunctions of a general vibration problem when using the Finite Element Method. The results are from a textbook. The results are presented with updated notation, coinciding with the notation used in the dissertation. The results are also expanded and extra results are added in an attempt to better explain the theory.

We then look at an important result for the Timoshenko beam theory. This provides a method to calculate the exact eigenvalues and eigenfunctions for the Timoshenko beam theory. Two examples are then used, a cantilever beam and a free-free beam, as an example of the application of the theory. The eigenvalues are calculated and the corresponding mode shapes are plotted. To obtain these results, the equation of motion is plotted, the isolation intervals are determined and the eigenvalues are then calculated using interval division to a desired level of accuracy. The mode shapes can then also be plotted with back substitution. These examples are important preparation for the main comparisons of this dissertation. We also discuss a result comparing the Timoshenko beam theory to results from an empirical study. We add to this article by giving the model problems.

For the rest of the models in this dissertation, the two- and three-dimensional elastic bodies, and the Reissner-Mindlin plate model, a different approach is required to solve the eigenvalue problems. For these models we use the Finite Element Method. For each of the models, their reference configurations are given. All of the models are assumed to have a square cross-section and in a cantilever configuration. This reference configuration is then discretised into a grid of rectangular shaped elements. A set of admissable piecewise Hermite cubic functions are then used and each model is rewritten into a Galerkin Approximation. We the define the standard Finite Element Method Matrices for each case. These are referred to as the mass and stiffness matrices in engineering. Finally our boundary value problems are written into a system of ordinary differential equations in a matrix representation. At this point is it easy to derive the eigenvalue problem for each of the problems in this matrix form. This is in preparation for the main comparisons made in this dissertation.

For our main comparisons, we first look at an article comparing a cantilever

Timoshenko beam model to a cantilever two-dimensional model. We discuss the article and replicate the results. The eigenvalues and eigenfunctions of the Timoshenko beam model is obtained using the exact method already described. The eigenvalues and the eigenfunctions for the two-dimensional model are approximated using the Finite Element Method matrix representation of the eigenvalue problem. A MATLAB program is written to approximate the eigenvalues and plot the mode shapes. The accuracy of this approximation is also investigated by looking at the rate of convergence for different grid sizes. Following the article, the eigenvalues are matched up by first comparing the mode shapes of the two models. The eigenvalues can then be matched up and also any eigenvalues not relating to beam-type problems can be filtered out. Based on the results in modal analysis, only a few eigenvalues need to be considered. The relative error between the eigenvalues are then calculated for different shapes of beams. The results improve on the article by showing more significant digits and also including some more results.

We then extend the results of the article to investigate the validity of a cantilever two-dimensional beam model as well as a cantilever Reissner-Mindlin plate model. The same method of the article is followed. To investigate the validity of a cantilever two-dimensional model, a cantilever three-dimensional model is considered. Since both of the models are not beam models, careful consideration needs to be taken to identify the eigenvalues. The same approach is used by comparing the mode shapes. For interest, the non-beam type eigenvalues shared between the two models are also included and the rest that the three-dimensional model does not share with the two-dimensional model are omitted. A clear distinction is made to show the beam type eigenvalues and the beam type results. The shape of the models are also carefully chosen to represent a variety of realistic cases that are interesting. The results show that the shape of the models play an important role in how well the models compare. An interesting result shows that this comparison is not good when the beam gets too wide. We therefore suggest a different model, like a plate model. This lead to the introduction of the Reissner-Mindlin plate model into this dissertation. The validity of a cantilever Reissner-Mindlin plate model is investigated using the same method. The cantilever three-dimensional plate model is again used as the reference model with the restriction that the body is wide.

0.3 Further Research

Future work would include the addition of damping into the models. A lot of the articles do include results for damping, however the results of the modal analysis (the crucial theory of this dissertation) might not be so trivial to prove.

The use of the Hermite cubic basis functions results in the derivatives of the displacement functions to also be available. This brings into question if the stresses of the models can also be compared.

Further improvements to the code can be made. Although a lot of effort was put into optimizing the code, this lead to the code being difficult to understand. Ideally a refactor and simplification of the code, while maintaining its functionality is desired.