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Associative algebras

Notes

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Chapter 1
Semisimple algebras

Definition 1.1. An **algebra** (over the field K) is a vector space (over K) with an associative multiplication $A \times A \to A$ such that $a(\lambda b + \mu c) = \lambda(ab) + \mu(ac)$ and $(\lambda a + \mu b)c = \lambda(ac) + \mu(bc)$ for all $a, b, c \in A$, and that contains an element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.

Note that an algebra over K is a ring A that is a vector space (over K) such that the map $K \to A$, $\lambda \mapsto \lambda 1_A$, is injective.

Definition 1.2. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$.

Example 1.3. The field $\mathbb{R}$ is a real algebra and similarly $\mathbb{C}$ is a complex algebra. Moreover, $\mathbb{C}$ is a real algebra.

Any field K is an algebra over K.

Example 1.4. Let K be a field. Then $K[X]$, $K[X,Y]$ and $K[[X]]$ are algebras over K.

Example 1.5. If A is an algebra, then $M_n(A)$ is an algebra.

The dimension of an algebra is by definition the dimension of the underlying vector space.

Definition 1.6. Let A and B be algebras. A map $f \colon A \to B$ is an **algebra homomorphism** if f is linear and it is a ring homomorphism.

The map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is a ring homomorphism that is not $\mathbb{C}$-linear, so it is not an $\mathbb{C}$-algebra homomorphism.

Example 1.7. Let G be a finite group. The vector space $\mathbb{C}[G]$ with basis $\{g : g \in G\}$ is an algebra with multiplication

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

Note that $\dim \mathbb{C}[G] = |G|$ and $\mathbb{C}[G]$ is commutative if and only G is abelian. This is the **complex group algebra** of G.

Two basic exercises about group algebras.

Exercise 1.8. Let G be a non-trivial finite group. Then $\mathbb{C}[G]$ has zero divisors.

Exercise 1.9. Let A be an algebra and G be a finite group. If $f\colon G \to \mathscr{U}(R)$ is a group homomorphism, then there exists an algebra homomorphism $\varphi\colon K[G] \to A$ such that $\varphi|_G = f$.

Definition 1.10. Let A be an algebra. An (left) **ideal** of A is an (left) ideal of the ring A that is also a subspace.

Let A be an algebra over K. If I is a left ideal of the ring A, then I is a subspace (over K), as $\lambda a = \lambda(1_A a) = (\lambda 1_A)a$ for all $\lambda \in K$ and $a \in A$.

Definition 1.11. Let A be an algebra. A **module** over A is a module M of the ring A.

Note that if M is a module over A, then M is a vector space with $\lambda m = (\lambda 1_A)m$ for all $\lambda \in K$ and $m \in M$.

Exercise 1.12. Let A be an algebra and M be a module over A. Then M is finitely generated if and only if M is finite-dimensional.

An important example of a module is given by the left representation. The algebra A is a module over A with the left multiplication.

Definition 1.13. Let A be an algebra and M be a module over A. Then M is **simple** if $M \neq \{0\}$ and $\{0\}$ and M are the only submodules of M.

Definition 1.14. Let A be a finite-dimensional algebra and M be a finite-dimensional module over A. Then M is **semisimple** if M is a direct sum of finitely many simple submodules.

Clearly, a finite direct sum of semisimples is semisimple.

Lemma 1.15 (Schur). *Let A be an algebra. If S and T are simple modules and $f\colon S \to T$ is a non-zero module homomorphism, then f is an isomorphism.*

Proof. □

Chapter 2
The Jacobson radical

radical

We will consider rings possibly without identity. Thus a **ring** is an abelian group R with an associative multiplication $(x,y) \mapsto xy$ such that $(x+y)z = xz+yz$ and $x(y+z) = xy+xz$ for all $x,y,z \in R$. If there is an element $1 \in R$ such that $x1 = 1x = x$ for all $x \in R$, we say that R is a ring (or a unitary ring). A **subring** S of R is an additive subgroup of R closed under multiplication.

Example 2.1. $2\mathbb{Z} = \{2m : m \in \mathbb{Z}\}$ is a ring.

A **left ideal** (resp. **right ideal**) is a subring I of R such that $rI \subseteq I$ (resp. $Ir \subseteq I$) for all $r \in R$. An **ideal** (also two-sided ideal) of R is a subring I of R that is both a left and a right ideal of R.

Example 2.2. If I and J are both ideals of R, then the sum $I+J = \{x+y : x \in I, y \in J\}$ and the intersection $I \cap J$ are both ideals of R. The product IJ, defined as the additive subgroup of R generated by $\{xy : x \in I, y \in J\}$, is also an ideal of R.

Example 2.3. If R is a ring, the set $Ra = \{xa : x \in R\}$ is a left ideal of R. Similarly, the set $aR = \{ax : x \in R\}$ is a right ideal of R. The set RaR, which is defined as the additive subgroup of R generated by $\{xay : x,y \in R\}$, is a ideal of R.

Example 2.4. Ir R is a unitary ring, then Ra is the left ideal generated by a, aR is the right ideal generated by a and RaR is the ideal generated by a. If R is not unitary, the left ideal generated by a is $Ra + \mathbb{Z}a$, the right ideal generated by a is $aR + \mathbb{Z}a$ and the ideal generated by a is $RaR + Ra + aR + \mathbb{Z}a$.

A ring R is said to be **simple** if $R^2 \neq \{0\}$ and the only ideals of R are 0 and R. The condition $R^2 \neq \{0\}$ is trivially satisfied in the case of rings with identity, as $1 \in R^2$.

Example 2.5. Division rings are simple.

Let S be a unitary ring. Recall that $M_n(S)$ is the ring of $n \times n$ square matrices with entries in S. If $A = (a_{ij}) \in M_n(S)$ y E_{ij} is the matrix such that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, then

$$E_{ij} A E_{kl} = a_{jk} E_{il} \tag{2.1}$$ eq:trick

for all $i,j,k,l \in \{1,\dots,n\}$.

Exercise 2.6. If D is a division ring, then $M_n(D)$ is simple.

Let R be a ring. A left R-module (or module, for short) is an abelian group M together with a map $R\times M\to M$, $(r,m)\mapsto rm$, such that

$$(r+s)m=rm+sm,\qquad r(m+n)=rm+rs,\qquad r(sm)=(rs)m$$

for all $r,s\in R$, $m,n\in M$. If R has an identity 1 and $1m=m$ holds for all $m\in M$, the module M is said to be **unitary**. If M is a unitary module, then $M=RM\neq\{0\}$.

The module M is said to be **simple** if $RM\neq\{0\}$ and the only submodules of M are 0 and M. If M is a simple module, then $M\neq\{0\}$.

lemma:simple

Lemma 2.7. *Let M be a non-zero module. Then M is simple if and only if $M=Rm$ for all $0\neq m\in M$.*

Proof. Assume that M is simple. Let $m\neq 0$. Since Rm is a submodule of the simple module M, either $Rm=\{0\}$ or $Rm=M$. Let $N=\{n\in M:Rn=\{0\}\}$. Since N is a submodule of M and $RM\neq\{0\}$, $N=\{0\}$. Therefore $Rm=M$, as $m\neq 0$. Now assume that $M=Rm$ for all $m\neq 0$. Let L be a non-zero submodule of M and let $0\neq x\in L$. Then $M=L$, as $M=Rx\subseteq L$. □

Example 2.8. Let D be a division ring and let V be a non-zero vector space (over D). If $R=\mathrm{End}_D(V)$, then V is a simple R-módulo with $fv=f(v)$, $f\in R$. $v\in V$.

exa:I_k

Example 2.9. Let $n\geq 2$. If D is a division ring and $R=M_n(D)$, then each

$$I_k=\{(a_{ij})\in R:a_{ij}=0\text{ for }j\neq k\}$$

is an R-module isomorphic to D^n. Thus $M_n(D)$ is a simple ring that is not a simple $M_n(D)$-module.

A left ideal L of a ring R is said to be **minimal** if $L\neq\{0\}$ and L does not strictly contain other left ideals of R. Similarly one defines right minimal ideals and minimal ideals.

Example 2.10. Let D be a division ring and let $R=M_n(D)$. Then $L=RE_{11}$ is a minimal left ideal.

Example 2.11. Let L be a non-zero left ideal. If $RL\neq\{0\}$, then L is minimal if and only if L is a simple R-module.

A left (resp. right) ideal L of R is said to be **regular** if there exists $e\in R$ such that $r-re\in L$ (resp. $r-er\in L$) for all $r\in R$. If R is a ring with identity, every left (or right) ideal is regular. A left (resp. right) ideal I of R is said to be **maximal** if $I\neq M$ and I is not properly contained in any other left (resp. right) ideal of R. A standard application of Zorn's lemma proves that every unitary ring contains a maximal left (or right) ideal. Similarly one defines maximal ideals.

proposition:R/I

Proposition 2.12. *Let R be a ring and M be a module. Then M is simple if and only if $M\simeq R/I$ for some maximal regular left ideal I.*

Proof. Assume that M is simple. Then $M = Rm$ for some $m \neq 0$ by Lemma 2.7. The map $\phi\colon R \to M$, $r \mapsto rm$, is an epimorphism of R-modules, so the first isomorphism theorem implies that $M \simeq R/\ker\phi$.

We claim that $I = \ker\phi$ is a maximal ideal. The correspondence theorem and the simpllicity of M imply that I is a maximal ideal (because each left ideal J such that $I \subseteq J$ yields a submodule of R/I).

We claim that I is regular. Since $M = Rm$, there exists $e \in R$ such that $m = em$. If $r \in R$, then $r - re \in I$ since $\phi(r - re) = \phi(r) - \phi(re) = rm - r(em) = 0$.

Supongamos ahora que L es maximal y regular. Por el teorema de la correspondencia, R/L no tiene submódulos propios no nulos. Veamos entonces que $R(R/L) \neq 0$. Si $R(R/L) = 0$ y $r \in R$, entonces, como L es regular, $r - re \in L$ y luego $r \in L$ pues

$$0 = r(e + I) = re + I = r + I,$$

una contradicción a la maximalidad de L. $\square$

We will now discuss primitive rings.

Let R be a ring and M be a left R-module. For a subset $N \subseteq M$ we define the **annihilator** of N as the subset

$$\mathrm{Ann}_R(N) = \{r \in R : rn = 0\ \forall n \in N\}.$$

Example 2.13. $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n) = n\mathbb{Z}$.

The following exercise is standard.

Exercise 2.14. Let R be a ring and M be a module. If $N \subseteq M$ is a subset, then $\mathrm{Ann}_R(N)$ is a left ideal of R. If $N \subseteq M$ is a submodule of R, then $\mathrm{Ann}_R(N)$ is an ideal of R.

A module M is said to be **faithful** if $\mathrm{Ann}_R(M) = \{0\}$.

Example 2.15. If K is a field, then K^n is a faithful unitary $M_n(K)$-module.

Example 2.16. If V is vector space over a field K, then V is faithful unitary $\mathrm{End}_K(V)$-module.

A ring R is said to be **primitive** if there exists a faithful simple R-módulo. Since we are considering left modules, our definition of primitive rings is that of left primitive rings. By convention, a primitive ring will always mean a left primitive ring. The use of right modules yields to the notion of right primitive rings.

proposition:simple=>prim **Proposition 2.17.** *If R is a simple unitary ring, then R is primitive.*

Proof. Since R is unitary, there exists a maximal left ideal I and, moreover, R is regular. By Proposition 2.12, R/I is a simple R-module. Since $\mathrm{Ann}_R(R/I)$ is an ideal of R and R is simple, either $\mathrm{Ann}_R(R/I) \in \{0\}$ or $\mathrm{Ann}_R(R/I) = R$. Moreover, since $1 \notin \mathrm{Ann}(R/I)$, it follows that $\mathrm{Ann}_R(R/I) = \{0\}$. $\square$

osition:prim+conm=cuerpo

Proposition 2.18. *If R is a commutative ring, then R is primitive if and only if R is a field.*

Proof. If R is a field, then R is primitive because it is a unitary simple ring, see Proposition 2.17. If R is a primitive commutative ring, Proposition 2.12 implies that there exists a maximal regular ideal I such that R/I is a faithful simple R-module. Since $I \subseteq \mathrm{Ann}_R(R/I) = \{0\}$ and I is regular, there exists $e \in R$ such that $r = re = er$. Therefore R is a unitary commutative ring. Since $I = \{0\}$ is a maximal ideal, R is a field. □

Example 2.19. The ring $\mathbb{Z}$ is not primitive.

An ideal P of a ring R is said to be **primitive** if $P = \mathrm{Ann}_R(M)$ for some simple R-module M.

lemma:primitivo

Lemma 2.20. *Let R be a ring and P be an ideal of R. Then P is primitive if and only if R/P is a primitive ring.*

Proof. Assume that $P = \mathrm{Ann}_R(M)$ for some R-module M. Then M is a simple R/P-module with $(r+P)m = rm$, $r \in R$, $m \in M$. This is well-defined, as $P = \mathrm{Ann}_R(M)$. Since M is a simple R-module, it follows that M is a simple R/P-module. Moreover, $\mathrm{Ann}_{R/P} M = \{0\}$. Indeed, if $(r+P)M = 0$, then $r \in \mathrm{Ann}_R M = P$ and hence $r+P = P$.

Assume now that R/P is primitive. Let M be a faithful simple R/P-module. Then $rm = (r+P)m$, $r \in R$, $m \in M$, turns M into an R-module. It follows that M is simple and that $P = \mathrm{Ann}_R(M)$. □

Example 2.21. Let $R_1, \dots, R_n$ be primitive ring and $R = R_1 \times \cdots \times R_n$. Then each $P_i = R_1 \times \cdots \times R_{i-1} \times \{0\} \times R_{i+1} \times \cdots \times R_n$ is a primitive ideal of R since $R/P_i \simeq R_i$.

lemma:maxprim

Lemma 2.22. *Let R be a ring. Si P es un ideal primitivo, existe un ideal a izquierda L maximal tal que $P = \{x \in R : xR \subseteq L\}$. Recíprocamente, si L es un ideal a izquierda maximal y regular, entonces $\{x \in R : xR \subseteq L\}$ es un ideal primitivo.*

Proof. Assume that $P = \mathrm{Ann}_R(M)$ for some simple R-module M. By Proposition 2.12, there exists a regular maximal left ideal L such that $M \simeq R/L$. Then $P = \mathrm{Ann}_R(R/L) = \{x \in R : xR \subseteq L\}$.

Conversely, let L a regular maximal left ideal.By Proposition 2.12, R/L is a simple R-module simple. Then

$$\mathrm{Ann}_R(R/L) = \{x \in R : xR \subseteq L\}$$

if a primitive ideal. □

Proposition 2.23. *Maximal ideals of unitary rings are primitive.*

Proof. Let R be a ring with identity and M be a maximal ideal of R. Then R/M is a simple unitary ring by Proposition 2.12. Then R/M is primitive by Proposition 2.17. By lema 2.20, M is primitive. □

Exercise 2.24. Prove that every primitive ideal of a commutative ring is maximal.

Exercise 2.25. Prove that $M_n(R)$ is primitive if and only if R is primitive.

Let us discuss the Jacobson radical and radical rings.

Let R be a ring. The **Jacobson radical** $J(R)$ is the intersection of all the annihilators of simple left R-modules. If R does not have simple left R-modules, then $J(R) = R$. From the definition it follows that $J(R)$ is an ideal. Moreover,

$$J(R) = \bigcap \{P : P \text{ left primitive ideal}\}.$$

If I is an ideal of R and $n \in \mathbb{N}$, I^n is the additive subgroup of R generated by the set $\{y_1 \dots y_n : y_j \in I\}$. An ideal I of R is **nilpotent** if $I^n = \{0\}$ for some $n \in \mathbb{N}$. Similarly one defines right or left nil ideals. Note that an ideal I is nilpotent if and only if there exists $n \in \mathbb{N}$ such that $x_1 x_2 \cdots x_n = 0$ for all $x_1, \dots, x_n \in I$.

An element x of a ring is said to be **nil** (or nilpotent) if $x^n = 0$ for some $n \in \mathbb{N}$. An ideal I of a ring is said to be nil if every element of I is nil. Every nilpotent ideal is nil, as $I^n = 0$ implies $x^n = 0$ for all $x \in I$.

Example 2.26. Let $R = \mathbb{C}[x_1, x_2, \dots]/(x_1, x_2^2, x_3^3, \dots)$. The ideal $I = (x_1, x_2, x_3, \dots)$ is nil in R, as it is generated by nilpotent element. However, it is not nilpotente. Indeed, if I is nilpotent, then there exists $k \in \mathbb{N}$ such that $I^k = 0$ and hence $x_i^k = 0$ for all i, a contradiction since $x_{k+1}^k \neq 0$.

pro:nilJ **Proposition 2.27.** *Let R be a ring. Then every nil left ideal (resp. right ideal) is contained in $J(R)$.*

Proof. Assume that there is a nil left ideal (resp. right ideal) I such that $I \not\subseteq J(R)$. There exists a simple R-module M such that $n = xm \neq 0$ for some $x \in I$ and some $m \in M$. Since M is simple, $Rn = M$ and hence there exists $r \in R$ such that

$$(rx)m = r(xm) = rn = m \quad (\text{resp. } (xr)n = x(rn) = xm = n).$$

Thus $(rx)^k m = m$ (resp. $(xr)^k n = n$) for all $k \geq 1$, a contradiction since $rx \in I$ (resp. $xr \in I$) is a nilpotent element. □

Let R be a ring. An element $a \in R$ is said to be **left quasi-regular** if there exists $r \in R$ such that $r + a + ra = 0$. Similarly, a is said to be **right quasi-regular** if there exists $r \in R$ such that $a + r + ar = 0$.

exercise:circ **Exercise 2.28.** Let R be a ring. Prove that $R \times R \to R$, $(r,s) \mapsto r \circ s = r + s + rs$, is an associative operation with neutral element 0.

Exercise 2.29. Let $R = \mathbb{Z}/3 = \{0, 1, 2\}$. Compute the multiplication table with respect to the circle operation given by the previous exercise.

If R is unitary, an element $x \in R$ is left quasi-regular (resp. right quasi-regular) if and only if $1 + x$ is left invertible (resp. right invertible). In fact, if $r \in R$ is such that

$r+x+rx=0$, then $(1+r)(1+x)=1+r+x+rx=1$. Conversely, if there exists $y\in R$ such that $y(1+x)=1$, then

$$(y-1)\circ x=y-1+x+(y-1)x=0.$$

Example 2.30. If $x\in R$ is a nilpotent element, then $y=\sum_{n\geq 1}x^n\in R$ is quasi-regular. En efecto, si existe N tal que $x^N=0$, la suma que define al elemento y es finita y cumple que $y+(-x)+y(-x)=0$.

A left ideal I of R is said to be **left quasi-regular** (resp. right quasi-regular) if every element of I is left quasi-regular (resp. right quasi-regular). A left ideal is said to be **quasi-regular** if it is left and right quasi-regular. Similarly one defines right quasi-regular ideals and quasi-regular ideals.

lemma:casiregular

Lemma 2.31. *Let I be a left ideal of R. If I is left quasi-regular, then I is quasi-regular.*

Proof. Let $x\in I$. Let us prove that x is right quasi-regular. Since I is left quasi-regular, there exists $r\in R$ such that $r\circ x=r+x+rx=0$. Since $r=-x-rx\in I$, there exists $s\in R$ tal que $s\circ r=s+r+sr=0$. Then s is right quasi-regular and

$$x=0\circ x=(s\circ r)\circ x=s\circ(r\circ x)=s\circ 0=s. \qquad \square$$

Let $(A,\leq)$ be a partially order set, this means that A is a set together with a reflexive, transitive and anti-symmetric binary relation R en $A\times A$, where $a\leq b$ if and only if $(a,b)\in R$. Recall that the relation is reflexive if $a\leq a$ for all $a\in A$, the relation is transitive if $a\leq b$ and $b\leq c$ imply that $a\leq c$ and the relation is anti-symmetric if $a\leq b$ and $b\leq a$ imply $a=b$.

The elements $a,b\in A$ are said to be **comparable** if $a\leq b$ or $b\leq a$. An element $a\in A$ is said to be **maximal** if $c\leq a$ for all $c\in A$ that is comparable with a. An **upper bound** for a non-empty subset $B\subseteq A$ is an element $d\in A$ such that $b\leq d$ for all $b\in B$. A **chain** in A is a subset B such that every pair of elements of B are comparable. **Zorn's lemma** states the following property:

> If A is a non-empty partially ordered set such that every chain in A contains an upper bound in A, then A contains a maximal element.

Our application of Zorn's lemma:

lemma:maxreg

Lemma 2.32. *Let R be a ring and $x\in R$ be an element that is not left quasi-regular Then there exists a maximal left ideal M such that $x\notin M$. Moreover, R/M is a simple R-module and $x\notin \operatorname{Ann}_R(R/M)$.*

Proof. Let $T=\{r+rx:r\in R\}$. A straightforward calculation shows that T is a left ideal of R such that $x\notin T$ (if $x\in T$, then $r+rx=-x$ for some $r\in R$, a contradiction since x is not left quasi-regular).

The only left ideal of R contanining $T\cup\{x\}$ is R. Indeed, if there exists a left ideal U containing T, then $x\notin U$, since otherwise every $r\in R$ could be written as $r=(r+rx)+r(-x)\in U$.

Let $\mathscr{S}$ be the set of proper left ideals of R containing T partially ordered by inclusion. If $\{K_i : i \in I\}$ is a chain in $\mathscr{S}$, then $K = \cup_{i\in I} K_i$ is an upper bound for the chain (K is a proper, as $x \notin K$). Zorn's lemma implies that $\mathscr{S}$ admits a maximal element M. Thus M is a maximal left ideal such that $x \notin M$. Moreover, M is regular since $r + r(-x) \in T \subseteq M$ for all $r \in R$. Therefore R/M is a simple R-module by Proposition 2.12. Since $x(x+M) \neq 0$ (if $x^2 \in M$, then $x \in M$, as $x + x^2 \in T \subseteq M$), it follows that $x \notin \mathrm{Ann}_R(R/M)$. □

If $x \in R$ is not left quasi-regular, Lemma 2.32 implies that there exists a simple R-module M such $x \notin \mathrm{Ann}_R(M)$. Thus $x \notin J(R)$.

thm:casireg_eq **Theorem 2.33.** *Let R be a ring and $x \in R$. The following statements are equivalent:*

1) *The left ideal generated by x is quasi-regular.*
2) *Rx is quasi-regular.*
3) *$x \in J(R)$.*

Proof. The implication $(1) \implies (2)$ is trivial, as Rx is included in the left ideal generated by x.

We now prove $(2) \implies (3)$. If $x \notin J(R)$, then Lemma 2.32 implies that there exists a simple R-module M such that $xm \neq 0$ for some $m \in M$. The simplicity of M implies that $R(xm) = M$. Thus there exists $r \in R$ such that $rxm = -m$. There is an element $s \in R$ such that $s + rx + s(rx) = 0$ and hence

$$-m = rxm = (-s - srx)m = -sm + sm = 0,$$

a contradiction.

Finally, to prove $(3) \implies (1)$ it is enough to note that x is left quasi-regular. Thus the left ideal generated by x is quasi-regular by Lemma 2.31. □

The theorem immediately implies the following corollary.

Corollary 2.34. *If R is a ring, then $J(R)$ if a quasi-regular ideal that contains every left quasi-regular ideal.*

The following result is somewhat what we all had in mind.

thm:J(R) **Theorem 2.35.** *Let R be a ring such that $J(R) \neq R$. Then*

$$J(R) = \bigcap \{I : I \text{ regular maximal left ideal of } R\}.$$

Proof. We only prove the non-trivial inclusion. Let

$$K = \bigcap \{I : I \text{ regular maximal left ideal of } R\}.$$

By Proposition 2.12,

$$J(R) = \bigcap \{\mathrm{Ann}_R(R/I) : I \text{ regular maximal left ideal of } R\}.$$

Let I be a regular maximal left ideal. If $r \in J(R) \subseteq \mathrm{Ann}_R(R/I)$, then, since I is regular, there exists $e \in R$ such that $r - re \in I$. Since

$$re + I = r(e+I) = 0,$$

$re \in I$ and hence $r \in I$. Thus $J(R) \subseteq K$. □

Example 2.36. Each maximal ideals of $\mathbb{Z}$ is of the form $p\mathbb{Z} = \{pm : m \in \mathbb{Z}\}$ for some prime number p. Thus $J(\mathbb{Z}) = \cap_p p\mathbb{Z} = \{0\}$.

We now review some basic results useful to compute radicals.

Proposition 2.37. *Let $\{R_i : i \in I\}$ be a family of rings. Then*

$$J\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} J(R_i).$$

Proof. Let $R = \prod_{i\in I} R_i$ and $x = (x_i)_{i\in I} \in R$. The left ideal Rx is quasi-regular if and only if each left ideal $R_i x_i$ is quasi-regular in R_i, as x is quasi-regular in R if and only if each x_i is quasi-regular in R_i. Thus $x \in J(R)$ if and only if $x_i \in J(R_i)$ for all $i \in I$. □

For the next result we shall need a lemma.

lemma:trickJ1

Lemma 2.38. *Let R be a ring and $x \in R$. If $-x^2$ is a left quasi-regular element, then x también.*

Proof. Sea $r \in R$ tal que $r + (-x^2) + r(-x^2) = 0$ y sea $s = r - x - rx$. Entonces x es casi-regular a izquierda pues

$$\begin{aligned} s + x + sx &= (r - x - rx) + x + (r - x - rx)x \\ &= r - x - rx + x + rx - x^2 - rx^2 = r - x^2 - rx^2 = 0. \end{aligned}$$ □

proposition:J(I)

Proposition 2.39. *If I is an ideal of R, then $J(I) = I \cap J(R)$.*

Proof. Since $I \cap J(R)$ if an ideal of I, if $x \in I \cap J(R)$, then x is left quasi-regular in R. Let $r \in R$ be such that $r + x + rx = 0$. Since $r = -x - rx \in I$, x is left quasi-regular in I. Thus $I \cap J(R) \subseteq J(I)$.

Let $x \in J(I)$ and $r \in R$. Since $-(rx)^2 = (-rxr)x \in I(J(I)) \subseteq J(I)$, the element $-(rx)^2$ is left quasi-regular a izquierda en I. Thus rx is left quasi-regular by Lemma 2.38. □

A ring R is said to be **radical** if $J(R) = R$.

Example 2.40. If R is a ring, then $J(R)$ is a radical ring, by Proposition 2.39.

Example 2.41. The Jacobson radical of $\mathbb{Z}/8$ is $\{0, 2, 4, 6\}$.

There are several characterizations of radical rings.

theorem:anillo_radical

Theorem 2.42. *Let R be ring. The following statements are equivalent:*

1) *R is radical.*
2) *R admits no simple R-modules.*
3) *R no tiene ideales a izquierda maximales y regulares.*
4) *R no tiene ideales a izquierda primitivos.*
5) *Every element of R is quasi-regular.*
6) *$(R,\circ)$ is a group.*

Proof. The equivalence $(1) \iff (5)$ follows from Theorem 2.33.

The equivalence $(5) \iff (6)$ is left as an exercise.

Let us prove that $(1) \implies (2)$. Assume that there exists a simple R-module N. Since $R = J(R) \subseteq \mathrm{Ann}_R(N)$, $R = \mathrm{Ann}_S(N)$. Hence $RN = \{0\}$, a contradiction to the simplicity of N.

To prove $(2) \implies (3)$ we note that for each regular and maximal left ideal I, the quotient R/I is a simple R-module by Proposición 2.12.

To prove $(3) \implies (4)$ assume that there is a primitive left ideal $I = \mathrm{Ann}_R(M)$, where M is some simple R-module. Since $R = J(R) \subseteq I$, it follows that $I = R$, a contradiction to the simplicity of M.

Finally we prove $(4) \implies (2)$. If M is a simple R-module, then $\mathrm{Ann}_R(M)$ is a primitive left ideal. □

Example 2.43. Let
$$A = \left\{ \frac{2x}{2y+1} : x, y \in \mathbb{Z} \right\}.$$
Then A is a radical ring, as the inverse of the element $\frac{2x}{2y+1}$ with respect to the circle operation $\circ$ is
$$\left(\frac{2x}{2y+1} \right)' = \frac{-2x}{2(x+y)+1}.$$

A ring R is said to be **nil** if for every $x \in R$ there exists $n = n(x)$ such that $x^n = 0$.

Exercise 2.44. Prove that a nil ring is a radical ring.

Exercise 2.45. Let $\mathbb{R}[X]$ be the ring of power series with real coefficients. Prove that the ideal $X\mathbb{R}[X]$ consisting of power series with zero constant term is a radical ring that is not nil.

The following problem is maybe the most important open problem in non-commutative ring theory.

The conjecture is known to be true in several cases. Exercises?

thm:Jnilpotente

Theorem 2.46. *If R is a left artinian ring, then $J(R)$ is nilpotent.*

Proof. Let $J = J(R)$. Since R is a left artinian ring, the sequence $(J^m)_{m\in\mathbb{N}}$ of left ideals stabilizes. There exists $k \in \mathbb{N}$ such that $J^k = J^l$ for all $l \geq k$. We claim that $J^k = \{0\}$. If $J^k \neq \{0\}$ let $\mathscr{S}$ the set of left ideals I such that $J^k I \neq \{0\}$. Since

$$J^k J^k = J^{2k} = J^k \neq \{0\},$$

the set $\mathscr{S}$ is non-empty. Since R is left artinian, $\mathscr{S}$ has a minimal element I_0. Since $J^k I_0 \neq \{0\}$, let $x \in I_0 \setminus \{0\}$ be such that $J^k x \neq \{0\}$. Moreover, $J^k x$ is a left ideal of R contained in I_0 and such that $J^k x \in \mathscr{S}$, as $J^k(J^k x) = J^{2k}x = J^k x \neq \{0\}$. The minimality of I_0 implies that, $J^k x = I_0$. In particular, there exists $r \in J^k \subseteq J(R)$ such that $rx = x$. Since $-r \in J(R)$ is left quasi-regular, there exists $s \in R$ such that $s - r - sr = 0$. Thus

$$x = rx = (s - sr)x = sx - s(rx) = sx - sx = 0,$$

a contradiction. □

Corollary 2.47. *Let R be a left artinian ring. Each nil left ideal is nilpotent and $J(R)$ is the unique maximal nilpotent ideal of R.*

Proof. Let L be a nil left ideal of R. By Proposition 2.27, L is contained in $J(R)$. Thus L is nilpotent, as $J(R)$ is nilpotent by Theorem 2.46. □

Theorem 2.48. *Let R be a ring and $n \in \mathbb{N}$. Then $J(M_n(R)) = M_n(J(R))$.*

Proof. We first prove that $J(M_n(R)) \subseteq M_n(J(R))$. If $J(R) = R$, the theorem is clear. Let us assume that $J(R) \neq R$ and let $J = J(R)$. If M is a simple R-module, then M^n is a simple $M_n(R)$-module with the usual multiplication. Let $x = (x_{ij}) \in J(M_n(R))$ and $m_1, \dots, m_n \in M$. Then

$$x \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

In particular, $x_{ij} \in \operatorname{Ann}_R(M)$ for all $i, j \in \{1, \dots, n\}$. Hence $x \in M_n(J)$.

We now prove that $M_n(J) \subseteq J(M_n(R))$. Let

$$J_1 = \begin{pmatrix} J & 0 & \cdots & 0 \\ J & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix} \in J_1.$$

Since x_1 es quasi-regular, there exists $y_1 \in R$ such that $x_1 + y_1 + x_1 y_1 = 0$. If

$$y = \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $u = x + y + xy$ is lower triangular, as

$$u = \begin{pmatrix} 0 & 0 \cdots & 0 \\ x_2y_1 & 0 \cdots & 0 \\ x_3y_1 & 0 \cdots & 0 \\ \vdots & \vdots \ddots & \vdots \\ x_ny_1 & 0 \cdots & 0 \end{pmatrix}.$$

Since $u^n = 0$, the element

$$v = -u + u^2 - u^3 + \cdots + (-1)^{n-1}u^{n-1}$$

is such that $u + v + uv = 0$. Thus x is right quasi-regular, as

$$x + (y + v + yv) + x(y + v + yv) = 0,$$

and therefore J_1 is right quasi-regular. Similarly one proves that each J_i is right quasi-regular and hence $J_i \subseteq J(M_n(R))$ for all $i \in \{1, \ldots, n\}$. In conclusion,

$$J_1 + \cdots + J_n \subseteq J(M_n(R))$$

and therefore $M_n(J) \subseteq J(M_n(R))$. □

For completeness we recall basic results on the Jacobson radical in the case of unitary rings.

Exercise 2.49. Let R be a unitary ring. Then

$$J(R) = \bigcap \{M : M \text{ is a left maximal ideal}\}.$$

Exercise 2.50. Let R be a unitary ring. The following statements are equivalent:

1) $x \in J(R)$.
2) $xM = 0$ for all simple R-module M.
3) $x \in P$ for all primitive left ideal P.
4) $1 + rx$ is invertible for all $r \in R$.
5) $1 + \sum_{i=1}^{n} r_ixs_i$ is invertible for all $n \in \mathbb{N}$ and all $r_i, s_i \in R$.
6) x belongs to every left maximal ideal maximal.

prob:Koethe **Open problem 2.1 (Köthe).** Let R be a ring. Is the sum of two arbitrary nil left ideals of R is nil?

Notes

The material on non-commutative ring theory is standard, see for example [**?**]. Radical rings were introduced by Jacobson in [**?**]. Nil rings were used by Zelmanov in his solution to Burnside's problem, see for example [**?**].

Open problem 2.1 is the well-known Köthe's conjecture. The conjecture was first formulated in 1930, see [**?**]. It is known to be true in several cases. In full generality, the problem is still open. In [**?**] Krempa proved that the following statements are equivalent:

1) Köthe's conjecture is true.
2) If R is a nil ring, then $R[X]$ is a radical ring.
3) If R is a nil ring, then $M_2(R)$ is a nil ring.
4) Let $n \geq 2$. If R is a nil ring, then $M_n(R)$ is a nil ring.

In 1956 Amitsur formulated the following conjecture, see for example [**?**]: If R is a nil ring, then $R[X]$ is a nil ring. In [**?**] Smoktunowicz found a counterexample to Amitsur's conjecture. This counterexample suggests that Köthe's conjecture might be false. A simplification of Smoktunowicz's example appears in [**?**]. See [**?**, **?**] for more information on Köthe's conjecture and related topics.

References

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