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Associative algebras

Notes

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Chapter 1
Semisimple algebras

Definition 1.1. An **algebra** (over the field K) is a vector space (over K) with an associative multiplication $A \times A \to A$ such that $a(\lambda b + \mu c) = \lambda(ab) + \mu(ac)$ and $(\lambda a + \mu b)c = \lambda(ac) + \mu(bc)$ for all $a, b, c \in A$, and that contains an element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.

Note that an algebra over K is a ring A that is a vector space (over K) such that the map $K \to A$, $\lambda \mapsto \lambda 1_A$, is injective.

Definition 1.2. An algebra A is **commutative** if $ab = ba$ for all $a, b \in A$.

Example 1.3. The field $\mathbb{R}$ is a real algebra and similarly $\mathbb{C}$ is a complex algebra. Moreover, $\mathbb{C}$ is a real algebra.

Any field K is an algebra over K.

Example 1.4. Let K be a field. Then $K[X]$, $K[X,Y]$ and $K[[X]]$ are algebras over K.

Example 1.5. If A is an algebra, then $M_n(A)$ is an algebra.

The dimension of an algebra is by definition the dimension of the underlying vector space.

Definition 1.6. Let A and B be algebras. A map $f\colon A \to B$ is an **algebra homomorphism** if f is linear and it is a ring homomorphism.

The map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$, is a ring homomorphism that is not $\mathbb{C}$-linear, so it is not an $\mathbb{C}$-algebra homomorphism.

Example 1.7. Let G be a finite group. The vector space $\mathbb{C}[G]$ with basis $\{g : g \in G\}$ is an algebra with multiplication

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

Note that $\dim \mathbb{C}[G] = |G|$ and $\mathbb{C}[G]$ is commutative if and only G is abelian. This is the **complex group algebra** of G.

Two basic exercises about group algebras.

Exercise 1.8. Let G be a non-trivial finite group. Then $\mathbb{C}[G]$ has zero divisors.

Exercise 1.9. Let A be an algebra and G be a finite group. If $f\colon G \to \mathscr{U}(R)$ is a group homomorphism, then there exists an algebra homomorphism $\varphi\colon K[G] \to A$ such that $\varphi|_G = f$.

Definition 1.10. Let A be an algebra. An (left) **ideal** of A is an (left) ideal of the ring A that is also a subspace.

Let A be an algebra over K. If I is a left ideal of the ring A, then I is a subspace (over K), as $\lambda a = \lambda(1_A a) = (\lambda 1_A)a$ for all $\lambda \in K$ and $a \in A$.

Definition 1.11. Let A be an algebra. A **module** over A is a module M of the ring A.

Note that if M is a module over A, then M is a vector space with $\lambda m = (\lambda 1_A)m$ for all $\lambda \in K$ and $m \in M$.

Exercise 1.12. Let A be an algebra and M be a module over A. Then M is finitely generated if and only if M is finite-dimensional.

An important example of a module is given by the left representation. The algebra A is a module over A with the left multiplication.

Definition 1.13. Let A be an algebra and M be a module over A. Then M is **simple** if $M \neq \{0\}$ and $\{0\}$ and M are the only submodules of M.

Definition 1.14. Let A be a finite-dimensional algebra and M be a finite-dimensional module over A. Then M is **semisimple** if M is a direct sum of finitely many simple submodules.

Clearly, a finite direct sum of semisimples is semisimple.

Lemma 1.15 (Schur). *Let A be an algebra. If S and T are simple modules and $f\colon S \to T$ is a non-zero module homomorphism, then f is an isomorphism.*

Proof. $\square$

References