Representation theory of algebras

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CONTENTS

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Introduction

The notes correspond to the master course **Representation theory of algebras** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve two-hour lectures.

Most of the material is based on standard results of the representation theory of finite groups. Basic texts on representation theory are [2] and [19].

The notes include Magma code, which we use to verify examples and offer alternative solutions to certain exercises. Magma [5] is a powerful software tool designed for working with algebraic structures. There is a free online version of Magma available.

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1. Lecture: Week 1

§ 1.1. The Artin–Wedderburn theorem. We first review the basic definitions concerning finite-dimensional semisimple algebras. Proofs can be found in the notes to the course **Associative Algebras** (see Lectures 1, 2 and 3).

Our base field will be the field $\mathbb{C}$ of complex numbers.

A (complex) **algebra** A is a (complex) vector space with an associative multiplication $A \times A \to A$ such that

$$a(\lambda b + \mu c) = \lambda(ab) + \mu(ac), \quad (\lambda a + \mu b)c = \lambda(ac) + \mu(bc)$$

for all $a, b, c \in A$. If A contains an element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$, then A is a unitary algebra. Our algebras will be unitary.

Our algebras will also be finite-dimensional. Clearly, $\mathbb{C}$ is an algebra. Other examples of algebras are $\mathbb{C}[X]$ and $M_n(\mathbb{C})$.

A (left) **module** M (over a unitary algebra A) is an abelian group M together with a map $A \times M \to M$, $(a, m) \mapsto am$, such that $1_A m = m$ for all $m \in M$ and $a(bm) = (ab)m$ and $a(m + m_1) = am + am_1$ for all $a, b \in A$ and $m, m_1 \in M$. A **submodule** N of M is a subgroup N such that $an \in N$ for all $a \in A$ and $n \in N$.

> 1.1. EXERCISE. Let A be a finite-dimensional algebra. If M is an A-module, then M is a vector space with $\lambda m = (\lambda 1_A)m$ for $\lambda \in \mathbb{C}$ and $m \in M$. Moreover, M is finitely generated (as an A-module) if and only if M is finite-dimensional.

A module M is said to be **simple** if $M \neq \{0\}$ and $\{0\}$ and M are the only submodules of M. A finite-dimensional module M is said to be **semisimple** if M is a direct sum of finitely many simple submodules. Clearly, simple modules are semisimple. Moreover, any finite direct sum of semisimples is semisimple.

A finite-dimensional algebra A is said to be **semisimple** if every finitely-generated A-module is semisimple.

1.2. THEOREM (Artin–Wedderburn). *Let A be a complex finite-dimensional semisimple algebra, say with k isomorphism classes of simple modules. Then*

$$A \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$$

for some $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$.

The unique simple module of the algebra $M_{n_j}(\mathbb{C})$ is the column space $\mathbb{C}^{n_j}$. This means that the simple component of dimension n_j^2 has a simple module of dimension n_j.

We also give some basic facts on the Jacobson radical of finite-dimensional algebras. If A is a finite-dimensional algebra, the **Jacobson radical** is defined as

$$J(A) = \bigcap \{M : M \text{ is a maximal left ideal of } A\}.$$

It turns out that $J(A)$ is an ideal of A. If A is unitary, then Zorn's lemma implies that there a maximal left ideal of A and hence $J(A) \neq A$.

An ideal I of A is said to be **nilpotent** if $I^m = \{0\}$ for some m, that is $x_1 \cdots x_m = 0$ for all $x_1, \ldots, x_m \in I$. One proves that the Jacobson radical of A contains every nilpotent ideal

of A. An important fact is that

$$A \text{ is semisimple} \iff J(A) = \{0\}$$
$$\iff A \text{ has no non-zero nilpotent ideals.}$$

§ 1.2. Group algebras. Let G be a finite group. The (complex) **group algebra** $\mathbb{C}[G]$ is the $\mathbb{C}$-vector space with basis $\{g : g \in G\}$ and multiplication

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

Clearly, $\dim \mathbb{C}[G] = |G|$. Moreover, $\mathbb{C}[G]$ is commutative if and only if G is abelian.

If G is non-trivial, then $\mathbb{C}[G]$ contains proper non-trivial ideals. For example, the **augmentation ideal**

$$I(G) = \left\{ \sum_{g \in G} \lambda_g g \in \mathbb{C}[G] : \sum_{g \in G} \lambda_g = 0 \right\}$$

is a non-zero proper ideal of $\mathbb{C}[G]$.

1.3. EXERCISE. Let G be a finite non-trivial group. Prove that $\mathbb{C}[G]$ has zero divisors.

For $n \in \mathbb{Z}_{\geq 2}$, we write C_n to denote the (multiplicative) cyclic group of order n.

1.4. EXERCISE. Prove that $\mathbb{C}[C_n] \simeq \mathbb{C}[X]/(X^n - 1)$.

1.5. EXERCISE. Let G be a finite group. The set
$$\mathrm{Fun}(G, \mathbb{C}) = \{\alpha \colon G \to \mathbb{C}\}$$
is a complex vector space with the operations
$$(\alpha + \beta)(x) = \alpha(x) + \beta(x), \quad (\lambda \alpha)(x) = \lambda \alpha(x),$$
for all $\alpha, \beta \in \mathrm{Fun}(G, \mathbb{C})$, $x \in G$ and $\lambda \in \mathbb{C}$. It is an algebra with the **convolution product**
$$(\alpha * \beta)(x) = \sum_{y \in G} \alpha(xy^{-1})\beta(y).$$
Let
$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$
Prove the following statements:
1) The set $\{\delta_x : x \in G\}$ is a basis of $\mathrm{Fun}(G, \mathbb{C})$.
2) The map $\mathbb{C}[G] \to \mathrm{Fun}(G, \mathbb{C})$, $g \mapsto \delta_g$, extends linearly to an algebra isomorphism.

Recall that a finite-dimensional module M is semisimple if and only if for every submodule S of M there is a submodule T of M such that $M = S \oplus T$.

1.6. THEOREM (Maschke). *Let G be a finite group and M be a finite-dimensional $\mathbb{C}[G]$-module. Then M is semisimple.*

PROOF. We must show that every submodule S of M admits a complement. Since S is a subspace of M, there exists a subspace T_0 of M such that $M = S \oplus T_0$ (as vector spaces). We use T_0 to construct a submodule T of M that complements S. Since $M = S \oplus T_0$, every $m \in M$ can be written uniquely as $m = s + t_0$ for some $s \in S$ and $t_0 \in T$. Let

$$p_0 \colon M \to S, \quad p_0(m) = s,$$

where $m = s + t_0$ with $s \in S$ and $t_0 \in T$. If $s \in S$, then $p_0(s) = s$. In particular, $p_0^2 = p_0$, as $p_0(m) \in S$.

Generally, p_0 is not a $\mathbb{C}[G]$-modules homomorphism. Let

$$p \colon M \to S, \quad p(m) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot p_0(g \cdot m).$$

We claim that p is a homomorphism of $\mathbb{C}[G]$-modules. For that purpose, we need to show that $p(g \cdot m) = g \cdot p(m)$ for all $g \in G$ and $m \in M$. In fact,

$$p(g \cdot m) = \frac{1}{|G|} \sum_{h \in G} h^{-1} \cdot p_0(h \cdot (g \cdot m)) = \frac{1}{|G|} \sum_{h \in G} (gh^{-1}) \cdot p_0(h \cdot m) = g \cdot p(m).$$

We now claim that $p(M) = S$. The inclusion $\subseteq$ is trivial to prove, as S is a submodule of M and $p_0(M) \subseteq S$. Conversely, if $s \in S$, then $g \cdot s \in S$, as S is a submodule. Thus $s = g^{-1} \cdot (g \cdot s) = g^{-1} \cdot p_0(g \cdot s)$ and hence

$$s = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot (g \cdot s) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot (p_0(g \cdot s)) = p(s).$$

Since $p(m) \in S$ for all $m \in M$, it follows that $p^2(m) = p(m)$, so p is a projector onto S. Hence S admits a complement in M, that is $M = S \oplus \ker(p)$. $\qquad\square$

1.7. EXERCISE. Let $G = \langle g \rangle$ be the cyclic group of order four and $\rho_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $M = \mathbb{C}^{2 \times 1}$ as an $\mathbb{C}[G]$-module with

$$g \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

Prove that M is a semisimple non-simple $\mathbb{C}[G]$-module.

1.8. EXERCISE. Let $G = \langle g \rangle$ be the cyclic group of order four and $\rho_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $M = \mathbb{R}^{2 \times 1}$ as an $\mathbb{R}[G]$-module with

$$g \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

Prove that M is a simple $\mathbb{R}[G]$-module.

If G is a finite group, then $\mathbb{C}[G]$ is semisimple. By Artin–Wedderburn theorem,

$$\mathbb{C}[G] \simeq \prod_{i=1}^{r} M_{n_i}(\mathbb{C}),$$

where r is the number of isomorphism classes of simple modules of $\mathbb{C}[G]$. Moreover,

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{r} n_i^2,$$

and the integers $n_1, n_2, \ldots, n_r$ are the dimensions of the non-isomorphic simple modules of the complex group algebra $\mathbb{C}[G]$.

1.9. THEOREM. *Let G be a finite group. The number of simple modules of $\mathbb{C}[G]$ coincides with the number of conjugacy classes of G.*

PROOF. By Artin–Wedderburn theorem, $\mathbb{C}[G] \simeq \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$. Thus

$$Z(\mathbb{C}[G]) \simeq \prod_{i=1}^{r} Z(M_{n_i}(\mathbb{C})) \simeq \mathbb{C}^r.$$

In particular, $\dim Z(\mathbb{C}[G]) = r$. If $\alpha = \sum_{g \in G} \lambda_g g \in Z(\mathbb{C}[G])$, then $h^{-1}\alpha h = \alpha$ for all $h \in G$. Thus

$$\sum_{g \in G} \lambda_{hgh^{-1}} g = \sum_{g \in G} \lambda_g h^{-1} g h = \sum_{g \in G} \lambda_g g$$

and hence $\lambda_g = \lambda_{hgh^{-1}}$ for all $g, h \in G$. A basis for $Z(\mathbb{C}[G])$ is given by elements of the form

$$\sum_{g \in K} g,$$

where K is a conjugacy class of G. Therefore $\dim Z(\mathbb{C}[G])$ equals the number of conjugacy classes of G. □

If G is a finite group, then

$$\mathbb{C}[G] \simeq \prod_{i=1}^{k} M_{n_i}(\mathbb{C}),$$

where k is the number of conjugacy classes of G. In particular,

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{k} n_i^2.$$

1.10. EXERCISE. Prove that $\mathbb{C}[C_4] \simeq \mathbb{C}^4$.

For $n \geq 1$, let $\mathbb{S}_n$ denote the symmetric group in n letters.

1.11. EXAMPLE. The group $\mathbb{S}_3$ has three conjugacy classes: $\{\mathrm{id}\}$, $\{(12), (13), (23)\}$ and $\{(123), (132)\}$. Since $6 = a^2 + b^2 + c^2$, it follows that $\mathbb{C}[G] \simeq \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

There is a multiplicative version of Maschke's theorem. A group G **acts by automorphisms** on A if there is a group homomorphism $\lambda \colon G \to \mathrm{Aut}(A)$. In this case, a subgroup B of A is said to be G-invariant if $\lambda_g(B) \subseteq B$ for all $g \in G$.

1.12. BONUS EXERCISE. Let K be a finite group of order m. Assume that K acts by automorphisms on $V = U \times W$, where U and W are subgroups of V and U is abelian and K-invariant. Prove that if the map $U \to U$, $u \mapsto u^m$, is bijective, there exists a normal K-invariant subgroup N of V such that $V = U \times N$.

1.13. BONUS EXERCISE. Let p be a prime number and K be a finite group with order not divisible by p. Let V be a p-elementary abelian group. Assume that K acts by automorphism on V. Prove that if U be a K-invariant subgroup of V, there exists a K-invariant subgroup N of V such that $V = U \times N$.

§ 1.3. Representations. Unless we state differently, we will always work with finite groups. All our vector spaces will be complex vector spaces.

1.14. DEFINITION. Let G be a finite group. A **representation** of G is a group homomorphism $\rho\colon G \to \mathbf{GL}(V)$, where V is a finite-dimensional vector space. The **degree** (or dimension) of the representation is the integer $\deg \rho = \dim V$.

Let $G \to \mathbf{GL}(V)$ be a representation. If we fix a basis of V, then we obtain a **matrix representation** of G, that is a group homomorphism

$$\rho\colon G \to \mathbf{GL}(V) \simeq \mathbf{GL}_n(\mathbb{C}), \quad g \mapsto \rho_g,$$

where $n = \dim V$.

1.15. EXAMPLE. Since $\mathbb{S}_3 = \langle (12), (123) \rangle$, the map $\rho\colon \mathbb{S}_3 \to \mathbf{GL}_3(\mathbb{C})$,

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is a representation of $\mathbb{S}_3$.

1.16. EXAMPLE. Let $G = \langle g \rangle$ be cyclic of order six. The map $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$,

$$g \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a representation of G.

1.17. EXAMPLE. Let $G = \langle g \rangle$ be cyclic of order four. The map $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$,

$$g \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a representation of G.

1.18. EXAMPLE. Let $G = \langle a, b : a^2 = b^3 = (ab)^3 = 1 \rangle$. The map

$$a \mapsto \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

defines a representation $G \to \mathbf{GL}_3(\mathbb{C})$.

1.19. EXAMPLE. Let $Q_8 = \{-1, 1, i, -i, j, -j, k, -k\}$ be the quaternion group. Recall that

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

The group Q_8 is generated by $\{i, j\}$ and the map $\rho\colon Q_8 \to \mathbf{GL}_2(\mathbb{C})$,

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a representation.

1.20. EXAMPLE. Let G be a finite group that acts on a finite set X. Let $V = \mathbb{C}X$ the complex vector space with basis $\{x : x \in X\}$. The map

$$\rho\colon G \to \mathbf{GL}(V), \quad \rho_g\left(\sum_{x \in X} \lambda_x x\right) = \sum_{x \in X} \lambda_x \rho_g(x) = \sum_{x \in X} \lambda_{g^{-1} \cdot x} x,$$

is a representation of degree $|X|$.

1.21. EXAMPLE. The map $\rho\colon G \to \mathbb{C}^\times$, $g \mapsto 1$, is a representation, that is $\mathbb{C}$ is a $\mathbb{C}[G]$-module with $g \cdot \lambda = \lambda$ for all $g \in G$ and $\lambda \in \mathbb{C}^\times$. This representation is known as the **trivial representation**.

1.22. EXAMPLE. The map $\mathrm{sign}\colon \mathbb{S}_n \to \mathbf{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is a representation of $\mathbb{S}_n$.

An important fact is that there exists a bijective correspondence between representations of a finite group G and finite-dimensional modules over $\mathbb{C}[G]$. The correspondence is given as follows. If $\rho\colon G \to \mathbf{GL}(V)$ is a representation, then V is a $\mathbb{C}[G]$-module with

$$\left(\sum_{g \in G} \lambda_g g\right) \cdot v = \sum_{g \in G} \lambda_g \rho_g(v).$$

Conversely, if V is a $\mathbb{C}[G]$-module, then $\rho\colon G \to \mathbf{GL}(V)$, $\rho_g\colon V \to V$, $v \mapsto g \cdot v$, is a representation.

This bijection between representations of groups and modules over group algebras allows us to construct a dictionary between concepts in the language of representations and that of modules. Both languages are useful, so depending on our convenience, we will use one or the other.

> 1.23. EXERCISE. Let G be a finite group and $\rho\colon G \to \mathbf{GL}(V)$ be a representation. Prove that each ρ_g is diagonalizable.

The previous exercise uses properties of the minimal polynomial. We will see a different proof later.

1.24. DEFINITION. Let G be a group and $\phi\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ be representations of G. We say that ϕ and ψ are **equivalent** if there exists a linear isomorphism $T\colon V \to W$ such that

$$\psi_g T = T \phi_g$$

for all $g \in G$. In this case, we write $\phi \simeq \psi$.

Note that $\phi \simeq \psi$ if and only if V and W are isomorphic as $\mathbb{C}[G]$-modules.

1.25. EXAMPLE. The representation

$$\phi\colon \mathbb{Z}/n \to \mathbf{GL}_2(\mathbb{C}), \quad \phi(m) = \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix},$$

is equivalent to the representation

$$\psi\colon \mathbb{Z}/n \to \mathbf{GL}_2(\mathbb{C}), \quad \psi(m) = \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix}.$$

The equivalence is obtained with the matrix $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, as a direct calculation shows that $\phi_m T = T\psi_m$ for all m.

> **1.26. EXERCISE.** Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation. Fix a basis of V and consider the corresponding matrix representation ϕ of ρ. Prove that ρ and ϕ are equivalent.

1.27. DEFINITION. Let $\phi\colon G \to \mathbf{GL}(V)$ be a representation. A subspace $W \subseteq V$ is said to be *G-**invariant*** if $\phi_g(W) \subseteq W$ for all $g \in G$.

Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation. If W is a G-invariant subspace of V, then the restriction $\rho|_W \colon G \to \mathbf{GL}(W)$ is a representation. In particular, W is a submodule (over $\mathbb{C}[G]$) of V.

1.28. DEFINITION. A non-zero representation $\rho\colon G \to \mathbf{GL}(V)$ is said to be **irreducible** if $\{0\}$ and V are the only G-invariant subspaces of V.

Note that a representation $\rho\colon G \to \mathbf{GL}(V)$ is irreducible if and only if V is simple.

1.29. EXAMPLE. Degree-one representations are irreducible.

> **1.30. EXERCISE.** Let G be a finite group. Prove that there exists a bijective correspondence between degree-one representations of G and degree-one representations of $G/[G,G]$.

> **1.31. EXERCISE.** Let G be a finite group of order n with k conjugacy classes. Let $m = (G : [G,G])$. Prove that $n + 3m \geq 4k$.

In the following example, we work over the real numbers.

1.32. EXAMPLE. Let $G = \langle g \rangle$ be the cyclic group of three elements and
$$\rho\colon G \to \mathbf{GL}(\mathbb{R}^3), \quad \rho_g(x,y,z) = (y,z,x).$$
The set
$$N = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\}$$
is a G-invariant subspace of $\mathbb{R}^3$.

We claim that N is irreducible. If N contains a non-zero G-invariant subspace S, let $(x_0, y_0, z_0) \in S \setminus \{(0,0,0)\}$. Since S is G-invariant,
$$(y_0, z_0, x_0) = g \cdot (x_0, y_0, z_0) \in S.$$
We claim that $\{(x_0, y_0, z_0), (y_0, z_0, x_0)\}$ is linearly independent. If there exists $\lambda \in \mathbb{R}$ such that $\lambda(x_0, y_0, z_0) = (y_0, z_0, x_0)$, then $x_0 = \lambda^3 x_0$. Since $x_0 = 0$ implies $y_0 = z_0 = 0$, it follows that $\lambda = 1$. In particular, $x_0 = y_0 = z_0$, a contradiction, as $x_0 + y_0 + z_0 = 0$. Hence $\dim S = 2$ and therefore $S = N$.

What happens in the previous example if we consider complex numbers?

> **1.33. EXERCISE.** Let $\phi\colon G \to \mathbf{GL}(V)$, $g \mapsto \phi_g$, be a degree-two representation. Prove that ϕ is irreducible if and only if there is no common eigenvector for all the ϕ_g.

1.34. EXAMPLE. Recall that $\mathbb{S}_3$ is generated by (12) and (23). The map
$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$
defines a representation ϕ of $\mathbb{S}_3$. Exercise 1.33 shows that ϕ is irreducible.

2. Lecture: Week 2

We now describe some crucial examples of representations.

2.1. EXAMPLE. Let $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ be representations. The **direct sum** $\rho \oplus \psi\colon G \to \mathbf{GL}(V \oplus W)$, $g \mapsto (\rho_g, \psi_g)$, is a representation. This is equivalent to say that the vector space $V \oplus W$ is a $\mathbb{C}[G]$-module with

$$g \cdot (v, w) = (g \cdot v, g \cdot w), \quad g \in G, \ v \in v, \ w \in W.$$

Let V be a vector space with basis $\{v_1, \ldots, v_k\}$ and W be a vector space with basis $\{w_1, \ldots, w_l\}$. A **tensor product** of V and W is a vector space X with together with a bilinear map

$$V \times W \to X, \quad (v, w) \mapsto v \otimes w,$$

such that $\{v_i \otimes w_j : 1 \leq i \leq k, \ 1 \leq j \leq l\}$ is a basis of X. The tensor product of V and W is unique up to isomorphism and it is denoted by $V \otimes W$. Note that

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

2.2. EXAMPLE. Let V and W be $\mathbb{C}[G]$-modules. The **tensor product** $V \otimes W$ is a $\mathbb{C}[G]$-module with

$$g \cdot v \otimes w = g \cdot v \otimes g \cdot w, \quad g \in G, \ v \in V, \ w \in W.$$

Let $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ be representations. The **tensor product** of ρ and ψ is the representation of G given by

$$\rho \otimes \psi\colon G \to \mathbf{GL}(V \otimes W), \quad g \mapsto (\rho \otimes \psi)_g,$$

where

$$(\rho \otimes \psi)_g(v \otimes w) = \rho_g(v) \otimes \psi_g(w)$$

for $v \in V$ and $w \in W$.

2.3. EXERCISE. Let G be a finite group and V be a $\mathbb{C}[G]$-module. Prove that the dual V^* is a $\mathbb{C}[G]$-module with

$$(g \cdot f)(v) = f(g^{-1}v), \quad f \in V^*, \ v \in V, \ g \in G.$$

2.4. EXERCISE. Let G be a finite group and V and W be $\mathbb{C}[G]$-modules. Prove that the set $\mathrm{Hom}(V, W)$ of complex linear maps $V \to W$ is a $\mathbb{C}[G]$-module with

$$(g \cdot f)(v) = gf(g^{-1}v), \quad f \in \mathrm{Hom}(V, W), \ v \in V, \ g \in G.$$

If, moreover, V and W are finite-dimensional, then

$$V^* \otimes W \simeq \mathrm{Hom}(V, W)$$

as $\mathbb{C}[G]$-modules.

2.5. DEFINITION. A representation $\rho\colon G \to \mathbf{GL}(V)$ is said to be **completely reducible** if ρ can be decomposed as $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ for some irreducible representations $\rho_1, \ldots, \rho_n$ of G.

Note that if $\rho\colon G \to \mathbf{GL}(V)$ is completely reducible and $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ for some irreducible representations $\rho_i\colon G \to \mathbf{GL}(V_i)$, $i \in \{1, \ldots, n\}$, then each V_i is an invariant subspace of V and $V = V_1 \oplus \cdots \oplus V_n$. Moreover, in some basis of V, the matrix ρ_g can be written as

$$\rho_g = \begin{pmatrix} (\rho_1)_g & & & \\ & (\rho_2)_g & & \\ & & \ddots & \\ & & & (\rho_n)_g \end{pmatrix}.$$

2.6. DEFINITION. A representation $\rho\colon G \to \mathbf{GL}(V)$ is **decomposable** if V can be decomposed as $V = S \oplus T$ where S and T are non-zero invariant subspaces of V.

A representation is **indecomposable** if it is not decomposable.

2.7. EXERCISE. Let $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ be equivalent representations. Prove the following facts:
  1) If ρ is irreducible, then ψ is irreducible.
  2) If ρ is decomposable, then ψ is decomposable.
  3) If ρ is completely reducible, then ψ is completely reducible.

§ 2.1. Characters. Fix a finite group G and consider (matrix) representations of G. We use linear algebra to study these representations.

2.8. DEFINITION. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation. The **character** of ρ is the map $\chi_\rho\colon G \to \mathbb{C}$, $g \mapsto \operatorname{trace} \rho_g$.

If a representation ρ is irreducible, its character is said to be an **irreducible character**. The **degree** of a character is the degree of the affording representation.

2.9. EXAMPLE. We can compute the character of the representation

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (23) \mapsto \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix},$$

of Example 1.34. Since

$$\rho_{(132)} = \rho_{(23)(12)} = \rho_{(23)}\rho_{(12)} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

we conclude that $\rho_{(132)} = -1$. Similar calculations show that

$$\chi_{\mathrm{id}} = 2, \quad \chi_{(12)} = \chi_{(13)} = \chi_{(23)} = 0, \quad \chi_{(123)} = \chi_{(132)} = -1.$$

2.10. PROPOSITION. *Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation, χ be its character and $g \in G$. The following statements hold:*
  1) $\chi(1) = \dim V$.
  2) $\chi(g) = \chi(hgh^{-1})$ *for all $h \in G$.*
  3) $\chi(g)$ *is the sum of $\chi(1)$ roots of one of order $|g|$.*
  4) $\chi(g^{-1}) = \overline{\chi(g)}$.
  5) $|\chi(g)| \le \chi(1)$.

PROOF. The first statement is trivial.

To prove 2) note that

$$\chi(hgh^{-1}) = \text{trace}(\rho_{hgh^{-1}}) = \text{trace}(\rho_h\rho_g\rho_h^{-1}) = \text{trace}\,\rho_g = \chi(g).$$

Statement 3) follows from the fact that the trace of ρ_g is the sum of the eigenvalues of ρ_g and these numbers are roots of the polynomial $X^{|g|} - 1 \in \mathbb{C}[X]$. To prove 4) write $\chi(g) = \lambda_1 + \cdots + \lambda_k$, where the λ_j are roots of one. Then

$$\overline{\chi(g)} = \sum_{j=1}^{k} \overline{\lambda_j} = \sum_{j=1}^{k} \lambda_j^{-1} = \text{trace}(\rho_g^{-1}) = \text{trace}(\rho_{g^{-1}}) = \chi(g^{-1}).$$

Finally, we prove 5). We use 3) to write $\chi(g)$ as the sum of $\chi(1)$ roots of one, say $\chi(g) = \lambda_1 + \cdots + \lambda_k$ for $k = \chi(1)$. Then

$$|\chi(g)| = |\lambda_1 + \cdots + \lambda_k| \leq |\lambda_1| + \cdots + |\lambda_k| = \underbrace{1 + \cdots + 1}_{k\text{-times}} = k. \qquad \square$$

If two representations are equivalent, their characters are equal.

2.11. DEFINITION. Let G be a group and $f\colon G \to \mathbb{C}$ be a map. Then f is a **class function** if $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

Characters are class functions. If G is a finite group, we write

$$\text{ClassFun}(G) = \{f\colon G \to \mathbb{C} : f \text{ is a class function}\}.$$

One proves that $\text{ClassFun}(G)$ is a complex vector space.

2.12. EXERCISE. Let G be a finite group. For a conjugacy class K of G let

$$\delta_K\colon G \to \mathbb{C}, \quad \delta_K(g) = \begin{cases} 1 & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\{\delta_K : K$ is a conjugacy class of $G\}$ is a basis of $\text{ClassFun}(G)$. In particular, $\dim \text{ClassFun}(G)$ is the number of conjugacy classes of G.

2.13. PROPOSITION. *If $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ are representations, then $\chi_{\rho\oplus\psi} = \chi_\rho + \chi_\psi$.*

PROOF. For $g \in G$, it follows that $(\rho \oplus \psi)_g = \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}$. Thus

$$\chi_{\rho\oplus\psi}(g) = \text{trace}((\rho \oplus \phi)_g) = \text{trace}(\rho_g) + \text{trace}(\psi_g) = \chi_\rho(g) + \chi_\psi(g). \qquad \square$$

2.14. PROPOSITION. *If $\rho\colon G \to \mathbf{GL}(V)$ and $\psi\colon G \to \mathbf{GL}(W)$ are representations, then*

$$\chi_{\rho\otimes\psi} = \chi_\rho\chi_\psi.$$

PROOF. For each $g \in G$, the map ρ_g is diagonalizable. Let $\{v_1, \ldots, v_n\}$ be a basis of eigenvectors of ρ_g and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be such that $\rho_g(v_i) = \lambda_i v_i$ for all $i \in \{1, \ldots, n\}$. Similarly, let $\{w_1, \ldots, w_m\}$ be a basis of eigenvectors of ψ_g and $\mu_1, \ldots, \mu_m \in \mathbb{C}$ be such that $\psi_g(w_j) = \mu_j w_j$ for all $j \in \{1, \ldots, m\}$. Each $v_i \otimes w_j$ is eigenvector of $(\rho\otimes\psi)_g$ with eigenvalue $\lambda_i\mu_j$, as

$$(\rho \otimes \psi)_g(v_i \otimes w_j) = \rho_g v_i \otimes \psi_g w_j = \lambda_i v_i \otimes \mu_j v_j = (\lambda_i\mu_j)v_i \otimes w_j.$$

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Thus $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$ is a basis of eigenvectors and the $\lambda_i \mu_j$ are the eigenvalues of $(\rho \otimes \psi)_g$. It follows that

$$\chi_{\rho \otimes \psi}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum_i \lambda_i\right)\left(\sum_j \mu_j\right) = \chi_\rho(g)\chi_\psi(g). \qquad \square$$

We know that it is also possible to define the dual $\rho^* \colon G \to \mathbf{GL}(V^*)$ of a representation $\rho \colon G \to \mathbf{GL}(V)$ by the formula

$$(\rho_g^* f)(v) = f(\rho_g^{-1} v), \quad g \in G,\ f \in V^* \text{ and } v \in V.$$

We claim that the character of the dual representation is then $\overline{\chi_\rho}$. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be such that $\rho_g v_i = \lambda_i v_i$ for all $i \in \{1, \ldots, n\}$. If $\{f_1, \ldots, f_n\}$ is the dual basis of $\{v_1, \ldots, v_n\}$, then

$$(\rho_g^* f_i)(v_j) = f_i(\rho_g^{-1} v_j) = \overline{\lambda_j} f_i(v_j) = \overline{\lambda_j} \delta_{ij}$$

and the claim follows.

Let G be a finite group. If $\chi, \psi \colon G \to \mathbb{C}$ are characters of G and $\lambda \in \mathbb{C}$, we define

$$(\chi + \psi)(g) = \chi(g) + \psi(g), \quad (\chi\psi)(g) = \chi(g)\psi(g), \quad (\lambda\chi)(g) = \lambda\chi(g).$$

We can then form linear combinations of characters. These functions, of course, are not necesarilly characters.

2.15. THEOREM. *Let G be a finite group. Then irreducible characters of G are linearly independent.*

PROOF. Let $S_1, \ldots, S_k$ be a complete set of representatives of classes of simple $\mathbb{C}[G]$-modules. Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. By Artin–Wedderburn theorem, there is an algebra isomorphism $f \colon \mathbb{C}[G] \to M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$, where $\dim S_j = n_j$ for all j. Moreover,

$$M_{n_j}(\mathbb{C}) \simeq \underbrace{S_j \oplus \cdots \oplus S_j}_{n_j-\text{times}}$$

for all j. For each j let $e_j = f^{-1}(I_j)$, where I_j is the identity matrix of $M_{n_j}(\mathbb{C})$. We claim that

$$\chi_i(e_j) = \begin{cases} \dim S_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $\chi_i(g)$ is the trace of the action of g on S_j. Since $e_i e_j = 0$ if $i \ne j$, it follows that $\chi_i(e_j) = 0$ if $i \ne j$. Moreover, e_j acts as the identity on S_j, thus $\chi_j(e_j) = \dim S_j$.

Now if $\sum \lambda_i \chi_i = 0$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, then

$$(\dim S_j)\lambda_j = \sum \lambda_i \chi_i(e_j) = 0$$

and hence $\lambda_j = 0$, as $\dim S_j \ne 0$. $\qquad \square$

2.16. THEOREM. *Let G be a finite group and $S_1, \ldots, S_k$ be the simple $\mathbb{C}[G]$-modules (up to isomorphism). If $V = \oplus_{i=1}^k a_j S_j$, then $\chi_V = \sum a_i \chi_i$, where $\chi_i = \chi_{S_i}$ for all i. Moreover, if U and V are $\mathbb{C}[G]$-modules,*

$$U \simeq V \iff \chi_U = \chi_V.$$

PROOF. The first part is left as an exercise.

It is also an exercise to prove that $U \simeq V$ implies $\chi_U = \chi_V$. Let us prove the converse. Assume that $\chi_U = \chi_V$. Since $\mathbb{C}[G]$ is semisimple, $U \simeq \oplus_{i=1}^k a_i S_i$ and $V \simeq \oplus_{i=1}^k b_i S_i$ for some integers $a_1, \ldots, a_k \geq 0$ and $b_1, \ldots, b_k \geq 0$. Since

$$0 = \chi_U - \chi_V = \sum_{i=1}^k (a_i - b_i)\chi_i$$

and the χ_i are linearly independent, it follows that $a_i = b_i$ for all i. Hence $U \simeq V$. $\quad\square$

2.17. EXERCISE. Let G be a finite group and U be a $\mathbb{C}[G]$-module. Prove $\chi_{U^*} = \overline{\chi_U}$.

We will use the following exercise later:

2.18. EXERCISE. Prove that if G is a finite group and U and V are $\mathbb{C}[G]$-modules, then

$$\chi_{\mathrm{Hom}(U,V)} = \overline{\chi_U}\chi_V.$$

For a finite group G we write $\mathrm{Irr}(G)$ to denote the complete set of isomorphism classes of characters of irreducible representations of G.

2.19. EXERCISE. Let G be a finite group. Prove that the set $\mathrm{Irr}(G)$ is a basis of $\mathrm{ClassFun}(G)$.

Let G be a finite group and U be a $\mathbb{C}[G]$-module. Let

$$U^G = \{u \in U : g \cdot u = u \text{ for all } g \in G\}.$$

Then U^G is a subspace of U. The following lemma is important:

2.20. LEMMA. $\dim U^G = \frac{1}{|G|} \sum_{x \in G} \chi_U(x)$

PROOF. Let ρ be the representation associated with U and let

$$\alpha = \frac{1}{|G|} \sum_{x \in G} \rho_x \colon U \to U.$$

We claim that $\alpha^2 = \alpha$. Let $g \in G$. Then

$$\rho_g(\alpha) = \frac{1}{|G|} \sum_{x \in G} \rho_g \rho_x = \frac{1}{|G|} \sum_{x \in G} \rho_{gx} = \alpha.$$

Thus

$$\alpha(\alpha(u)) = \frac{1}{|G|} \sum_{x \in G} \rho_x(\alpha(u)) = \alpha(u)$$

for all $u \in U$. This implies that α has eigenvalues 0 and 1. In fact, if $u \in U$ is an eigenvector of α of eigenvalue $\lambda \in \mathbb{C}$, then

$$\lambda u = \alpha(u) = \alpha(\alpha(u)) = \alpha(\lambda u) = \lambda \alpha(u) = \lambda^2 u.$$

Thus $\lambda(\lambda - 1) = 0$.

Let V be the eigenspace of eigenvalue 1. We now claim that $V = U^G$. Let us first prove that $V \subseteq U^G$. For that purpose, let $v \in V$ and $g \in G$. Then

$$g \cdot v = \rho_g(v) = \rho_g(\alpha(v))$$

$$= \frac{1}{|G|} \sum_{x \in G} \rho_g \rho_x(v) = \frac{1}{|G|} \sum_{y \in G} \rho_y(v) = \alpha(v) = v.$$

Now we prove that $V \supseteq U^G$. Let $u \in U^G$, so $\rho_g(u) = u$ for all $g \in G$. Then

$$\alpha(u) = \frac{1}{|G|} \sum_{x \in G} \rho_x(u) = \frac{1}{|G|} \sum_{x \in G} u = u.$$

Thus

$$\dim U^G = \dim V = \operatorname{trace} \alpha = \frac{1}{|G|} \sum_{x \in G} \operatorname{trace} \rho_x = \frac{1}{|G|} \sum_{x \in G} \chi_U(x). \qquad \square$$

One proves that the operation

$$\langle \chi_U, \chi_V \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \overline{\chi_V(g)}$$

defines an inner product.

2.21. THEOREM. *Let G be a finite group and U and V be $\mathbb{C}[G]$-modules. Then*

$$\langle \chi_U, \chi_V \rangle = \dim \operatorname{Hom}_G(U, V).$$

PROOF. We claim that

$$\operatorname{Hom}_G(U, V) = \operatorname{Hom}(U, V)^G.$$

Let us first prove that $\operatorname{Hom}_G(U, V) \subseteq \operatorname{Hom}(U, V)^G$. Let $f \in \operatorname{Hom}_G(U, V)$ and $g \in G$. Then

$$(g \cdot f)(u) = g \cdot f(g^{-1} \cdot u) = g \cdot (g^{-1} \cdot f(u)) = f(u)$$

for all $u \in U$. Now we prove that $\operatorname{Hom}_G(U, V) \supseteq \operatorname{Hom}(U, V)^G$. Let $f \in \operatorname{Hom}(U, V)^G$. Then $f \colon U \to V$ is a linear such that $g \cdot f = f$ for all $g \in G$. Then we compute

$$(g \cdot f)(u) = f(u) \implies g \cdot f(g^{-1} \cdot u) = f(u)$$
$$\implies f(g^{-1} \cdot u) = g^{-1} \cdot f(u) \quad \text{for all } g \in G \text{ and } u \in U$$

This means that one has

$$f(g \cdot u) = g \cdot f(u)$$

for all $g \in G$ and $u \in U$.

Using Exercise 2.18,

$$\dim \operatorname{Hom}_G(U, V) = \dim \operatorname{Hom}(U, V)^G$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}(U,V)}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_U(g)} \chi_V(g)$$

$$= \langle \chi_V, \chi_U \rangle.$$

Since $\dim \operatorname{Hom}_G(U, V) \in \mathbb{R}$, one has

$$\langle \chi_U, \chi_V \rangle = \overline{\langle \chi_V, \chi_U \rangle} = \langle \chi_V, \chi_U \rangle$$

and the claim follows.   □

Let G be a finite group and $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. Note that k is the number of conjugacy classes of G. Let $g_1, \ldots, g_k$ be representatives of conjugacy classes of G. The **matrix of characters** of G is $X = (X_{ij})$, where

$$X_{ij} = \chi_i(g_j)$$

for $i, j \in \{1, \ldots, k\}$.

2.22. EXAMPLE. Let $G = \mathbb{S}_3$. The group G has three conjugacy classes, so $|\operatorname{Irr}(G)| = 3$. Let $g_1 = \operatorname{id}$, $g_2 = (12)$ and $g_3 = (123)$. We know that $6 = n_1^2 + n_2^2 + n_3^2$. We know two degree-one (irreducible) representations of G, the trivial one and the sign. This implies that $n_1 = n_2 = 1$ and $n_3 = 2$. The matrix of characters is then

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	?	?

Two entries of the character table of Example 2.22 are still unknown. As we further develop the theory of characters, we will discover several tricks that can be used to find these missing entries.

3. Lecture: Week 3

§ 3.1. Schur's orthogonality relations. We start with a crucial exercise. It is known as Schur's lemma.

3.1. EXERCISE. If G is a group and U and V are simple $\mathbb{C}[G]$-modules, then a non-zero module homomorphism $U \to V$ is an isomorphism.

We now discuss a handy application of Schur's lemma. Let G be a finite group and S be a simple $\mathbb{C}[G]$-module. We claim that $\operatorname{Hom}_G(S, S) \simeq \mathbb{C}$. Let $f \in \operatorname{Hom}_G(S, S)$ and $\lambda \in \mathbb{C}$ be an eigenvalue of f. Then $f - \lambda \operatorname{id}\colon S \to S$ is not invertible. By Schur's lemma, $f - \lambda \operatorname{id} = 0$ and hence $f = \lambda \operatorname{id}$.

3.2. THEOREM (Schur). *Let G be a finite group and $\chi, \psi \in \operatorname{Irr}(G)$. Then*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $S_1, \ldots, S_k$ be the simples of $\mathbb{C}[G]$. For each j, let χ_j be the irreducible character of S_j. Then

$$\langle \chi_i, \chi_j \rangle = \dim \operatorname{Hom}_G(S_i, S_j) = \begin{cases} 1 & \text{if } S_i \simeq S_j, \\ 0 & \text{otherwise.} \end{cases}$$

But we know that $S_i \simeq S_j$ if and only if $\chi_i = \chi_j$. □

With the theorem, one can construct the character table of $\mathbb{S}_3$. For example, this can be done using that $\langle \chi_3, \chi_3 \rangle = 1$ and that $\langle \chi_1, \chi_3 \rangle = 0$. As an exercise, check that the character table of $\mathbb{S}_3$ is given by

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	−1	1
χ_3	2	0	−1

3.3. EXERCISE. Let G be a finite group. Prove that $\operatorname{Irr}(G)$ is an orthonormal basis of $\operatorname{ClassFun}(G)$.

The previous exercise has some consequences. Let G be a finite group and assume that $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. If $\alpha = \sum a_i \chi_i$, then $\alpha = \sum \langle \alpha, \chi_i \rangle \chi_i$.

3.4. THEOREM. *Let G be a finite group and $S_1, \ldots, S_k$ be the simples of G. Then the left regular $\mathbb{C}[G]$-module decomposes as*

$$\mathbb{C}[G] \simeq \bigoplus_{i=1}^{k} (\dim S_i) S_i.$$

PROOF. Let $n = |G|$. Assume that $G = \{g_1, \ldots, g_n\}$. Decompose the $\mathbb{C}[G]$-module corresponding to the left regular representation as

$$\mathbb{C}[G] \simeq a_1 S_1 \oplus \cdots \oplus a_k S_k$$

for some integers $a_1, \ldots, a_k \geq 0$. Let $L \colon G \to \mathbb{S}_G$, $g \mapsto L_g$, where $L_g(g_j) = gg_j$ for all j. Since the matrix of L_g in the basis $\{g_1, \ldots, g_n\}$ is

$$(L_g)_{ij} = \begin{cases} 1 & \text{if } g_i = gg_j, \\ 0 & \text{otherwise,} \end{cases}$$

one obtains that

$$\chi_L(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\chi_L = \sum_{i=1}^{k} a_i \chi_i = \sum_{i=1}^{k} \langle \chi_L, \chi_i \rangle \chi_i$$

and

$$a_i = \langle \chi_L, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \overline{\chi_i(1)} = \dim S_i.$$

Thus $\mathbb{C}[G] \simeq \bigoplus_{i=1}^{k} (\dim S_i) S_i$.   □

If G is a finite group, let $\mathrm{Char}(G)$ be the set of characters of G.

3.5. EXERCISE. Let $n \in \{1, 2, 3\}$. Let G be a finite group and $\alpha \in \mathrm{Char}(G)$. Prove that α is the sum of n irreducible characters if and only if $\langle \alpha, \alpha \rangle = n$.

We now prove Schur's second orthogonality relation.

3.6. THEOREM (Schur). *Let G be a finite group and $g, h \in G$. Then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g) \overline{\chi(h)} = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $g_1, \ldots, g_r$ be the representatives of the conjugacy classes of G. Assume that $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. For each $k \in \{1, \ldots, r\}$, let $c_k = (G : C_G(g_k))$ denote the size of the conjugacy class of g_k. Then

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \frac{1}{|G|} \sum_{k=1}^{r} c_k \chi_i(g_k) \overline{\chi_j(g_k)}.$$

We write this as $I = \frac{1}{|G|} X D X^*$, where I denotes the identity matrix, $X_{ij} = \chi_i(g_j)$, $X^* = \overline{X}^T$ and

$$D = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_r \end{pmatrix}.$$

Since, in matrices, $AB = I$ implies $BA = I$, it follows that $I = \frac{1}{|G|}X^*XD$. Thus, using that $|G| = c_k|C_G(g_k)|$ holds for all k,

$$(|G|D^{-1})_{ij} = (X^*X)_{ij} = \sum_{k=1}^{r} \overline{\chi_k(g_i)}\chi_k(g_j) = \begin{cases} |C_G(g_j)| & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

3.7. THEOREM (Solomon). *Let G be a finite group and $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$. If $g_1, \ldots, g_r$ are the representatives of the conjugacy classes of G and $i \in \{1, \ldots, r\}$, then*

$$\sum_{j=1}^{r} \chi_i(g_j) \in \mathbb{Z}_{\geq 0}.$$

PROOF. Let $n = |G|$. Assume that $G = \{g_1, g_2, \ldots, g_r, g_{r+1}, \ldots, g_n\}$. Let V be the complex vector space with basis $\{g_1, \ldots, g_n\}$. The action of G on G by conjugation induces a group homomorphism $\rho\colon G \to \mathbf{GL}(V)$, $g \mapsto \rho_g$, where $\rho_g(h) = ghg^{-1}$. The matrix of ρ_g in the basis $\{g_1, \ldots, g_n\}$ is

$$(\rho_g)_{ij} = \begin{cases} 1 & \text{if } g_jg = gg_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\chi_\rho(g) = \operatorname{trace}\rho_g = \sum_{k=1}^{|G|}(\rho_g)_{kk} = |\{k : g_kg = gg_k\}| = |C_G(g)|.$$

Write $\chi_\rho = \sum_{i=1}^{r} m_i\chi_i$ for $m_1, \ldots, m_r \geq 0$. For each j let $c_j = (G : C_G(g_j))$. Then

$$m_i = \langle \chi_\rho, \chi_i \rangle = \frac{1}{|G|}\sum_{g \in G} \chi_\rho(g)\overline{\chi_i(g)}$$

$$= \frac{1}{|G|}\sum_{j=1}^{r} c_j|C_G(g_j)|\overline{\chi_i(g_j)} = \sum_{j=1}^{r}\overline{\chi_i(g_j)}. \qquad \square$$

§ 3.2. Algebraic integers and characters.

3.8. DEFINITION. Let $\alpha \in \mathbb{C}$. We say that α is **algebraic integer** if $f(\alpha) = 0$ for some monic polynomial $f \in \mathbb{Z}[X]$.

Let $\mathbb{A}$ be the set of algebraic integers. Note that $\mathbb{Z} \subseteq \mathbb{A}$.

3.9. EXAMPLE. Every root of one is an algebraic integer.

3.10. PROPOSITION. $\mathbb{Q} \cap \mathbb{A} = \mathbb{Z}$.

PROOF. Let $m/n \in \mathbb{Q}$ with $\gcd(m, n) = 1$ and $n > 0$. If $f(m/n) = 0$ for some

$$f = X^k + a_{k-1}X^{k-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X]$$

of degree $k \geq 1$, then

$$0 = n^kf(m/n) = m^k + a_{k-1}m^{k-1}n + \cdots + a_1mn^{k-1} + a_0n^k.$$

This implies that

$$m^k = -n\left(a_{k-1}m^{k-1} + \cdots + a_1mn^{k-2} + a_0n^{k-1}\right)$$

and hence n divides m^k. Thus $n \in \{-1, 1\}$ and therefore $m/n \in \mathbb{Z}$. $\qquad \square$

3.11. PROPOSITION. *Let $x \in \mathbb{C}$. Then $x \in \mathbb{A}$ if and only if x is an eigenvalue of an integer matrix.*

PROOF. Let us prove the non-trivial implication. Let

$$f = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$$

be such that $f(x) = 0$. Then x is an eigenvalue of the companion matrix of f, that is the matrix

$$C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in \mathbb{Z}^{n \times n}. \qquad \square$$

3.12. THEOREM. $\mathbb{A}$ *is a subring of* $\mathbb{C}$.

PROOF. Let $\alpha, \beta \in \mathbb{A}$. By the previous proposition, α is an eigenvalue of an integer matrix $A \in \mathbb{Z}^{n \times n}$, say $Av = \alpha v$ for some $v \neq 0$, β is an eigenvalue of an integer matrix $B \in \mathbb{Z}^{m \times m}$, say $Bw = \beta w$ for some $w \neq 0$. Then

$$(A \otimes I_{m \times m} + I_{n \times n} \otimes B)(v \otimes w) = (\alpha + \beta)(v \otimes w),$$

where $I_{k \times k}$ denotes the $(k \times k)$ identity matrix, and

$$(A \otimes B)(v \otimes w) = (\alpha\beta)v \otimes w.$$

This implies that $\alpha + \beta \in \mathbb{A}$ and $\alpha\beta \in \mathbb{A}$, again by the previous proposition. $\qquad \square$

3.13. THEOREM. *Let G be a finite group. If $\chi \in \operatorname{Char}(G)$ and $g \in G$, then $\chi(g) \in \mathbb{A}$.*

PROOF. Let φ be a representation of G such that $\chi_\varphi = \chi$. Since φ_g is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{A}$ (because G is finite and the λ_j are roots of one),

$$\chi(g) = \operatorname{trace} \varphi_g = \sum_{i=1}^{k} \lambda_i \in \mathbb{A}. \qquad \square$$

4. Lecture: Week 4

We will use the following notation: if χ is a character of a group G and C is a conjugacy class of G, then $\chi(g) = \chi(xgx^{-1})$ for all $x \in G$. We write $\chi(C)$ to denote the value $\chi(g)$ for any $g \in C$.

4.1. THEOREM. *Let G be a finite group, $\chi \in \mathrm{Irr}(G)$ and K be a conjugacy class of G. Then*

$$\frac{\chi(K)}{\chi(1)}|K| \in \mathbb{A}.$$

We need a lemma.

4.2. LEMMA. *Let $x \in \mathbb{C}$. Then $x \in \mathbb{A}$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{C}$ not all zero such that $xz_i = \sum_{j=1}^{k} a_{ij}z_j$ for some $a_{ij} \in \mathbb{Z}$ and all $i \in \{1, \ldots, k\}$.*

PROOF. Let us first prove $\implies$. Let $f = X^k + a_{k-1}X^{k-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$ be such that $f(x) = 0$. For $i \in \{1, \ldots, k\}$ let $z_i = x^{i-1}$. Then $xz_i = x^i = z_{i+1}$ for all $i \in \{1, \ldots, k-1\}$. Moreover, $xz_k = x^k = -a_0 - a_1 x - \cdots - a_{k-1}x^{k-1}$.

We now prove $\impliedby$. Let $A = (a_{ij}) \in \mathbb{Z}^{k \times k}$ and Z be the column vector $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$.

Note that Z is non-zero. Moreover, $AZ = xZ$, as

$$(AZ)_i = \sum_{j=1}^{k} a_{ij}z_j = xz_i = (xZ)_i$$

for all i. Thus x is an eigenvalue of $A \in \mathbb{Z}^{k \times k}$ and hence $x \in \mathbb{A}$. $\qquad\square$

The previous lemma could be used to give an alternative proof of the fact that the algebraic integers form a ring.

PROOF OF THEOREM 4.1. Let φ be a representation of G and χ be its character. Let $C_1, \ldots, C_r$ be the conjugacy classes of G and for every $i \in \{1, \ldots, r\}$ let

$$T_i = \sum_{x \in C_i} \varphi_x.$$

CLAIM. $T_i = \left(\frac{|C_i|}{\chi(1)} \chi(C_i) \right) \mathrm{id}.$

We proceed in several steps. First, we prove that $T_i = \lambda \,\mathrm{id}$ for some $\lambda \in \mathbb{C}$. We prove that T_i is a morphism of representations:

$$\varphi_g T_i \varphi_g^{-1} = \sum_{x \in C_i} \varphi_g \varphi_x \varphi_g^{-1} = \sum_{x \in C_i} \varphi_{gxg^{-1}} = \sum_{y \in C_i} \varphi_y = T_i.$$

Now Schur's lemma implies that $T_i = \lambda \,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.

We now prove that

$$\lambda = \frac{|C_i|\chi(C_i)}{\chi(1)}.$$

To prove this we compute λ:

$$\lambda \chi(1) = \mathrm{trace}(\lambda\,\mathrm{id}) = \mathrm{trace}\,T_i = \sum_{x \in C_i} \mathrm{trace}\,\varphi_x = \sum_{x \in C_i} \chi(x) = |C_i|\chi(C_i).$$

Then the claim follows.

Now we claim that

$$T_i T_j = \sum_{k=1}^{r} a_{ijk} T_k$$

for some $a_{ijk} \in \mathbb{Z}_{\geq 0}$. In fact,

$$T_i T_j = \sum_{x \in C_i} \sum_{y \in C_j} \varphi_x \varphi_y = \sum_{x \in C_i} \sum_{y \in C_j} \varphi_{xy} = \sum_{g \in G} a_{ijg} \varphi_g,$$

where a_{ijg} is the number of elements $(x, y) \in C_i \times C_j$ such that $g = xy$.

CLAIM. The elements a_{ijg} depend only on the conjugacy class of g.

Let $X_g = \{(x, y) \in C_i \times C_j : g = xy\}$. If $h = kgk^{-1}$, the map

$$X_g \to X_h, \quad (x, y) \mapsto (kxk^{-1}, kyk^{-1}),$$

is well-defined. It is bijective with inverse

$$X_h \to X_g, \quad (a, b) \mapsto (k^{-1}ak, k^{-1}bk).$$

Hence $|X_g| = |X_h|$.

Let a_{ijk} be the number of elements $(x, y) \in C_i \times C_j$ such that $xy = g$ for some $g \in C_k$. Then

$$T_i T_j = \sum_{g \in G} a_{ijg} \varphi_g = \sum_{k=1}^{r} \sum_{g \in C_k} a_{ijg} \varphi_g = \sum_{k=1}^{r} a_{ijk} \sum_{g \in C_k} \varphi_g = \sum_{k=1}^{r} a_{ijk} T_k.$$

Therefore

(4.1) $$\left(\frac{|C_i|}{\chi(1)} \chi(C_i) \right) \left(\frac{|C_j|}{\chi(1)} \chi(C_j) \right) = \sum_{k=1}^{r} a_{ijk} \left(\frac{|C_k|}{\chi(1)} \chi(C_k) \right).$$

By the previous lemma, $x = \frac{|C_j|}{\chi(1)} \chi(C_j) \in \mathbb{A}$. $\qquad\square$

§ 4.1. Frobenius' theorem.

4.3. THEOREM (Frobenius). *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)$ divides $|G|$.*

PROOF. Let φ be an irreducible representation with character χ. Since $\langle \chi, \chi \rangle = 1$,

$$\frac{|G|}{\chi(1)} = \frac{|G|}{\chi(1)} \langle \chi, \chi \rangle = \sum_{g \in G} \frac{\chi(g)}{\chi(1)} \overline{\chi(g)}.$$

Let $C_1, \ldots, C_r$ be the conjugacy classes of G. Then

$$\frac{|G|}{\chi(1)} = \sum_{i=1}^{r} \sum_{g \in C_i} \frac{\chi(g)}{\chi(1)} \overline{\chi(g)} = \sum_{i=1}^{r} \left(\frac{|C_i|}{\chi(1)} \chi(C_i) \right) \overline{\chi(C_i)} \in \mathbb{A} \cap \mathbb{Q} = \mathbb{Z},$$

as $\overline{\chi(C_i)} \in \mathbb{A}$. This implies that $\chi(1)$ divides $|G|$. $\qquad\square$

The character table gives information on the structure of the group. For example, with the previous result, one can easily prove that groups of order p^2 (where p is a prime number) are abelian.

4.4. EXERCISE. Let p and q be prime numbers such that $p < q$. If $q \not\equiv 1 \bmod p$, then a group of order pq is abelian.

Another application:

4.5. THEOREM. *Let G be a finite simple group. Then $\chi(1) \neq 2$ for all $\chi \in \mathrm{Irr}(G)$.*

PROOF. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(1) = 2$. Let $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$ be an irreducible representation of G with character χ. Since G is simple, $\ker \rho = \{1\}$. Since $\chi(1) = 2$, G is non-abelian and hence $[G, G] = G$. Since G has $(G : [G, G]) = 1$ degree-one characters, it follows that G has only one degree-one character, the trivial one. The composition

$$G \overset{\rho}{\hookrightarrow} \mathbf{GL}_2(\mathbb{C}) \overset{\det}{\longrightarrow} \mathbb{C}^\times$$

is a degree-one representation, which means that $\det \rho_g = 1$ for all $g \in G$. By Frobenius' theorem, $|G|$ is even (because $2 = \chi(1)$ divides $|G|$). Let $x \in G$ be such that $|x| = 2$ (Cauchy's theorem). Then $|\rho_x| = 2$, as ρ is injective. Since ρ_x is diagonalizable, there exists $C \in \mathbf{GL}_2(\mathbb{C})$ such that

$$C\rho_x C^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

for some $\lambda, \mu \in \{-1, 1\}$. Since $1 = \det \rho_x = \lambda\mu$ and ρ is non-trivial, $\lambda = \mu = -1$. In particular, $C\rho_x C^{-1}$ is central and hence ρ_x is central. Since ρ is injective, x is central and thus $Z(G) \neq \{1\}$, a contradiction. $\square$

4.6. THEOREM (Schur). *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)$ divides $(G : Z(G))$.*

Let G and G_1 be groups. If V is a $\mathbb{C}[G]$-module and V_1 is a $\mathbb{C}[G_1]$-module, then $V \otimes V_1$ is a $\mathbb{C}[G \times G_1]$-module with

$$(g, g_1) \cdot v \otimes v_1 = (g \cdot v) \otimes (g_1 \cdot v_1)$$

for $(g, g_1) \in G \times G_1$, $v \in V$ and $v_1 \in V_1$.

4.7. LEMMA. *Let G and G_1 be finite groups. If ρ is an irreducible representation of G and ρ_1 is an irreducible representation of G_1, then $\rho \otimes \rho_1$ is an irreducible representation of $G \times G_1$.*

PROOF. Write $\chi = \chi_\rho$ and $\chi_1 = \chi_{\rho_1}$. Since χ is irreducible, $\langle \chi, \chi \rangle = 1$. Similarly, $\langle \chi_1, \chi_1 \rangle = 1$. Now $\rho \otimes \rho_1$ is irreducible, as

$$
\begin{aligned}
\langle \chi\chi_1, \chi\chi_1 \rangle &= \frac{1}{|G \times G_1|} \sum_{(g,g_1) \in G \times G_1} (\chi\chi_1)(g, g_1) \overline{(\chi\chi_1)(g, g_1)} \\
&= \frac{1}{|G||G_1|} \sum_{g \in G} \sum_{g_1 \in G} \chi(g)\chi_1(g_1) \overline{\chi(g)\chi_1(g_1)} \\
&= \frac{1}{|G||G_1|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \sum_{g_1 \in G} \chi_1(g_1)\overline{\chi_1(g_1)} \\
&= \langle \chi, \chi \rangle \langle \chi_1, \chi_1 \rangle = 1. \qquad \square
\end{aligned}
$$

4.8. EXERCISE. Let G and G_1 be finite groups. Prove that irreducible characters of $G \times G_1$ are of the form $\chi \otimes \chi_1$ for $\chi \in \mathrm{Irr}(G)$ and $\chi_1 \in \mathrm{Irr}(G_1)$.

We now prove Schur's theorem. The proof goes back to Tate; it uses the **tensor power trick**. See Tao's blog `https://terrytao.wordpress.com` for other applications of this powerful trick.

PROOF OF THEOREM 4.6. Let $\rho \colon G \to \mathbf{GL}(V)$ be an irreducible representation with character χ. Let $z \in Z(G)$. Then ρ_z commutes with ρ_g for all $g \in G$. By Schur's lemma, $\rho_z(v) = \lambda(z)v$ for all $v \in V$. Note that $\lambda \colon Z(G) \to \mathbb{C}^\times$, $z \mapsto \lambda(z)$, is a well-defined group homomorphism, as

$$
\lambda(z_1 z_2)v = \rho_{z_1 z_2}(v) = \rho_{z_1}\rho_{z_2}(v) = \lambda(z_2)\rho_{z_1}(v) = \lambda(z_1)\lambda(z_2)v
$$

for all $v \in V$ and $z_1, z_2 \in Z(G)$.

Let $n \in \mathbb{Z}_{\geq 1}$. Write $G^n = G \times \cdots \times G$ (n-times). Let

$$
\sigma \colon G^n \to \mathbf{GL}(V^{\otimes n}), \quad (g_1, \ldots, g_n) \mapsto \rho_{g_1} \otimes \cdots \otimes \rho_{g_n}.
$$

Then σ is a representation. The character of σ is χ^n. By the previous lemma, σ is irreducible. For $z_1, \ldots, z_n \in Z(G)$, we compute

$$
\begin{aligned}
\sigma(z_1, \ldots, z_n)(v_1 \otimes \cdots \otimes v_n) &= z_1 v_1 \otimes \cdots \otimes z_n v_n \\
&= \lambda(z_1) \cdots \lambda(z_n) v_1 \otimes \cdots \otimes v_n \\
&= \lambda(z_1 \cdots z_n) v_1 \otimes \cdots \otimes v_n.
\end{aligned}
$$

Let

$$
H = \{(z_1, \ldots, z_n) \in Z(G)^n : z_1 \cdots z_n = 1\} \subseteq G^n.
$$

The central subgroup H acts trivially on $V^{\otimes n}$, so there exists a representation

$$
\tau \colon G^n/H \to \mathbf{GL}(V^{\otimes n}),
$$

of degree $\chi(1)^n$. Since σ is irreducible, so is τ. By Frobenius' theorem, $\chi(1)$ divides $|G|$ and $\chi(1)^n$ divides $|G^n/H| = \frac{|G|^n}{|Z(G)|^{n-1}}$. Write

$$
|G| = \chi(1)s \quad \text{and} \quad |G|(G : Z(G))^{n-1} = \chi(1)^n r
$$

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for some $r, s \in \mathbb{Z}$. Let a and b be such that $\gcd(a, b) = 1$ and $\frac{a}{b} = \frac{(G:Z(G))}{\chi(1)}$. Then

$$s\left(\frac{a}{b}\right)^{n-1} = s\frac{(G:Z(G))^{n-1}}{\chi(1)^{n-1}} = \frac{|G|}{\chi(1)}\frac{(G:Z(G))^{n-1}}{\chi(1)^{n-1}} = r \in \mathbb{Z}.$$

Thus b^{n-1} divides s and hence $b = 1$ (because n is arbitrary). $\square$

§ 4.2. Examples of character tables. Let G be a finite group and $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Without loss of generality we may assume that χ_1 is the trivial character, i.e. $\chi_1(g) = 1$ for all $g \in G$. Recall that r is the number of conjugacy classes of G. Each χ_j is constant on conjugacy classes. The **character table** of G is given by

	1	k_2	$\cdots$	k_r
	1	g_2	$\cdots$	g_r
χ_1	1	1	$\cdots$	1
χ_2	n_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	n_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

where the n_j are the degrees of the irreducible representations of G and each k_j is the size of the conjugacy class of the element g_j. By convention, the character table contains only the values of the irreducible characters of the group.

4.9. EXAMPLE. Let $G = \langle g : g^4 = 1 \rangle$ be the cyclic group of order four. The character table of G is given by

	1	1	1	1
	1	g	g^2	g^3
χ_1	1	1	1	1
χ_2	1	λ	λ^2	λ^3
χ_3	1	λ^2	λ^4	λ^2
χ_4	1	λ^3	λ^2	λ

Let us see how to see this calculation on the computer:

```
> C4 := CyclicGroup(4);
> T := CharacterTable(C4);
> T;


Character Table of Group C4
---------------------------


--------------------
Class |   1  2  3  4
Size  |   1  1  1  1
Order |   1  2  4  4
--------------------
p  =  2   1  1  2  2
--------------------
X.1   +   1  1  1  1
X.2   +   1  1 -1 -1
X.3   0   1 -1  I -I
X.4   0   1 -1 -I  I
```

```
Explanation of Character Value Symbols
--------------------------------------
```

```
I = RootOfUnity(4)
```

Some remarks:

     1) The symbol `I` denotes a primitive fourth root of 1.

     2) The function `CharacterTable` computes more than just the character table of the group; it also provides additional information.

```
> T[1];
( 1, 1, 1, 1 )
> Degree(T[1]);
1
> Degree(T[2]);
1
> Degree(T[3]);
1
> Degree(T[4]);
1
```

    4.10. EXAMPLE. The character table of the group $C_2 \times C_2 = \{1, a, b, ab\}$ is

	1	1	1	1
	1	a	b	ab
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Let us do this by computer:

```
> C2xC2 := AbelianGroup([2,2]);
> T := CharacterTable(C2xC2);
> T;


Character Table of Group C2xC2
------------------------------


--------------------
Class |   1  2  3  4
Size  |   1  1  1  1
Order |   1  2  2  2
--------------------
p  =  2   1  1  1  1
--------------------
X.1    +   1  1  1  1
X.2    +   1 -1  1 -1
X.3    +   1  1 -1 -1
X.4    +   1 -1 -1  1
```

4.11. EXERCISE. Let A and B be abelian groups. We write $\text{Irr}(A) = \{\rho_1, \ldots, \rho_r\}$ and $\text{Irr}(B) = \{\phi_1, \ldots, \phi_s\}$. Prove that the maps

$$\varphi_{ij} \colon A \times B \to \mathbb{C}^\times, \quad (a, b) \mapsto \rho_i(a)\phi_j(b),$$

where $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, are the irreducible representations of $A \times B$.

4.12. EXAMPLE. The character table of $\mathbb{S}_3$ is given by

	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let us recall one possible way to compute this table. Degree-one irreducibles were easy to compute. To compute the third row of the table, one possible approach is to use the irreducible representation

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$\chi_3\left((12)\right) = \text{trace}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 0,$$

$$\chi_3\left((123)\right) = \chi_3\left((12)(23)\right) = \text{trace}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1.$$

We should remark that the irreducible representation mentioned is not needed to compute the third row of the character table.

```
> S3 := Sym(3);
> T := CharacterTable(S3);
> T;


Character Table of Group S3
---------------------------


-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1    +   1  1  1
X.2    +   1 -1  1
X.3    +   2  0 -1
```

4.13. EXERCISE. Compute the character table of $\mathbb{S}_4$.

4.14. EXERCISE. Compute the character table of $\mathbb{A}_4$.

4.15. EXERCISE. Compute the character table of the quaternion group Q_8.

4.16. EXERCISE. Compute the character table of the dihedral group of eight elements.

5. Lecture: Week 5

§ 5.1. McKay's conjecture. Let G be a finite group and let p be a prime number dividing $|G|$. Write $\mathrm{Syl}_p(G)$ to denote the (non-empty) set of Sylow p-subgroups of G. Recall that the **normalizer** of P is the subgroup

$$N_G(P) = \{g \in G : gPg^{-1} = P\}.$$

McKay made the following conjecture for the prime $p = 2$ and simple groups and later generalized by Alperin in [**1**] and independently by Isaacs in [**18**].

5.1. CONJECTURE (McKay). Let p be a prime. If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then

$$|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}| = |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1)\}|.$$

Isaacs proved the conjecture for solvable groups; see [**18, 21**]. Malle and Späth prove the conjecture for $p = 2$.

5.2. THEOREM (Malle–Späth). *If G is finite and $P \in \mathrm{Syl}_2(G)$, then*

$$|\{\chi \in \mathrm{Irr}(G) : 2 \nmid \chi(1)\}| = |\{\psi \in \mathrm{Irr}(N_G(P)) : 2 \nmid \psi(1)\}|.$$

The proof appears in [**28**] and uses the classification of finite simple groups. It uses a deep result of Isaacs, Malle and Navarro [**22**].

In full generality, McKay's conjecture was proved in 2024.

5.3. THEOREM (Cabanes–Späth). *If G is finite, p a prime number and $P \in \mathrm{Syl}_p(G)$, then*

$$|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}| = |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1)\}|.$$

We cannot prove Theorems 5.2 and 5.3 here. However, we can use the computer to prove some particular cases with the following function:

```
> McKay := function(G, p)
function> local N, n, m;
function> N := Normalizer(G, SylowSubgroup(G, p));
function> degG := CharacterDegrees(G);
function> degN := CharacterDegrees(N);
function> n := &+[ d[2] : d in degG | d[1] mod p ne 0 ];
function> m := &+[ d[2] : d in degN | d[1] mod p ne 0 ];
function> return n eq m;
function> end function;
```

As a concrete example, let us verify McKay's conjecture for the Mathieu simple group M_{11} of order 7920.

```
> M11 := sub<Sym(11) | (1,10)(2,8)(3,11)(5,7), (1,4,7,6)(2,11,10,9)>;
> McKay(M11,2);
true
> McKay(M11,3);
true
> McKay(M11,5);
true
> McKay(M11,11);
true
```

5.4. BONUS EXERCISE. Verify the McKay's conjecture for all sporadic simple groups.

The following conjecture refines McKay's conjecture. It was formulated by Isaacs and Navarro:

5.5. CONJECTURE (Isaacs–Navarro). Let p be a prime and $k \in \mathbb{Z}$. If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then

$$|\{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1) \text{ and } \chi(1) \equiv \pm k \bmod p\}|$$
$$= |\{\psi \in \mathrm{Irr}(N_G(P)) : p \nmid \psi(1) \text{ and } \psi(1) \equiv \pm k \bmod p\}|.$$

The Isaacs–Navarro conjecture is still open. However, it is known to be true for solvable groups, sporadic simple groups and symmetric groups, see [23].

5.6. BONUS EXERCISE. Verify the Isaacs–Navarro conjecture in some small groups such the Mathieu simple group M_{11}.

§ 5.2. Commutators. Let G be a finite group with conjugacy classes $C_1, \ldots, C_s$. For $i \in \{1, \ldots, s\}$ and $\chi \in \mathrm{Irr}(G)$ let

$$\omega_\chi(C_i) = \frac{|C_i|\chi(C_i)}{\chi(1)} \in \mathbb{A}.$$

In the proof of Theorem 4.1, Equality (4.1), we obtained that

(5.1)
$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{k=1}^{s} a_{ijk}\omega_\chi(C_k),$$

where a_{ijk} is the number of solutions of $xy = z$ with $x \in C_i$, $y \in C_j$ and $z \in C_k$.

5.7. THEOREM (Burnside). *Let G be a finite group with conjugacy classes $C_1, \ldots, C_s$. Then*

$$a_{ijk} = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(C_i)\chi(C_j)\overline{\chi(C_k)}}{\chi(1)}.$$

PROOF. By (5.1),

$$\frac{|C_i||C_j|}{\chi(1)}\chi(C_i)\chi(C_j) = \sum_{k=1}^{s} a_{ijk}|C_k|\chi(C_k).$$

Multiply by $\overline{\chi(C_l)}$ and sum over all $\chi \in \mathrm{Irr}(G)$ to obtain

$$|C_i||C_j| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\overline{\chi(C_l)}}{\chi(1)}\chi(C_i)\chi(C_j) = \sum_{\chi \in \mathrm{Irr}(G)} \sum_{k=1}^{s} a_{ijk}|C_k|\chi(C_k)\overline{\chi(C_l)}$$

$$= \sum_{k=1}^{s} a_{ijk}|C_k| \sum_{\chi \in \mathrm{Irr}(G)} \chi(C_k)\overline{\chi(C_l)}$$

$$= a_{ijl}|G|,$$

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because

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(C_k)\overline{\chi(C_l)} = \begin{cases} \frac{|G|}{|C_l|} & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

5.8. THEOREM (Burnside). *Let G be a finite group and $g, x \in G$. Then g and $[x, y]$ are conjugate for some $y \in G$ if and only if*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(x)|^2 \chi(g)}{\chi(1)} > 0.$$

PROOF. Let $C_1, \ldots, C_s$ be the conjugacy classes of G. Assume that $x \in C_i$ and $g \in C_k$ for some i and k. Then $C_i^{-1} = \{z^{-1} : z \in C_i\} = C_j$ for some j. By Burnside's theorem,

$$a_{ijk} = \frac{|C_i|^2}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(C_i)|^2 \overline{\chi(C_k)}}{\chi(1)}.$$

We first prove $\impliedby$. Since $a_{ijk} > 0$, there exist $u \in C_i$ and $v \in C_j$ such that $g = uv$ (since $zgz^{-1} = u_1 v_1$ for some $u_1 \in C_i$ and $v_1 \in C_j$, it follows that $g = (z^{-1}u_1 z)(z^{-1}v_1 z)$, so take $u = z^{-1}u_1 z \in C_i$ and $v = z^{-1}v_1 z \in C_j$). If x and u are conjugate, say $u = zxz^{-1}$ for some z, then x^{-1} and v are conjugate, as

$$zxz^{-1} = u \implies zx^{-1}z^{-1} = u^{-1} \in C_i^{-1} = C_j.$$

Let $z_2 \in G$ be such that $z_2 x^{-1} z_2^{-1} = v$. If $y = z^{-1}z_2$, then g and $[x, y]$ are conjugate, as

$$g = uv = (zxz^{-1})(z_2 x^{-1} z_2^{-1}) = (zxyx^{-1}y^{-1})yz_2^{-1} = z[x, y]z^{-1}.$$

We now prove $\implies$. Let $y \in G$ be such that g and $[x, y]$ are conjugate, say $g = z[x, y]z^{-1}$ for some $z \in G$. Let $v = yxy^{-1}$. Then g and $xv^{-1} = xyx^{-1}y^{-1} = [x, y]$ are conjugate. In particular, since $g \in C_i C_j$, $a_{ijk} > 0$. $\qquad \square$

5.9. EXERCISE. Let G be a finite group, $g \in G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$\sum_{h \in G} \chi([g, h]) = \frac{|G|}{\chi(1)}|\chi(g)|^2.$$

Prove also that

$$\chi(g)\chi(h) = \frac{\chi(1)}{|G|} \sum_{z \in G} \chi(zgz^{-1}h)$$

holds for all $h \in G$.

We now prove a theorem of Frobenius that uses character tables to recognize commutators. For that purpose, let

$$\tau(g) = |\{(x, y) \in G \times G : [x, y] = g\}|.$$

5.10. THEOREM (Frobenius). *Let G be a finite group. Then*

$$\tau(g) = |G| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}.$$

PROOF. Let $\chi \in \mathrm{Irr}(G)$. Since χ is irreducible,

$$1 = \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{z \in G} \chi(z)\overline{\chi(z)} = \frac{1}{|G|} \sum_C |C|\chi(C)\overline{\chi(C)},$$

where the last sum is taken over all conjugacy classes of G. Let $g \in G$ and C be the conjugacy class of g. The equation $xu^{-1} = g$ with $x \in C$ and $u \in C^{-1}$ has

$$\frac{|C||C^{-1}|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)}$$

solutions. If (x, u) is a solution of $xu^{-1} = g$, then there are $|C_G(x)|$ elements y such that $yxy^{-1} = u$. ($yxy^{-1} = u = y_1xy_1^{-1}$ implies that $y_1^{-1}y \in C_G(x)$ which implies $yC_G(x) = y_1C_G(x)$.) Now $[x, y] = x(yx^{-1}y^{-1}) = g$ has

$$|C| \sum_{\chi} \frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)}$$

solutions, where the sum is taken over all irreducible characters of G. Now we sum over all conjugacy classes of G:

$$\sum_C \sum_{\chi} |C| \frac{\chi(C)\chi(C^{-1})\chi(g^{-1})}{\chi(1)} = \sum_{\chi} \frac{\chi(g^{-1})}{\chi(1)} \left(\sum_C |C|\chi(C)\chi(C^{-1}) \right)$$

$$= |G| \sum_{\chi} \frac{\chi(g^{-1})}{\chi(1)}.$$

From this, the formula follows.   □

Application:

5.11. COROLLARY. *Let G be a finite group and $g \in G$. Then g is a commutator if and only if*

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0.$$

§ 5.3. Ore's conjecture. In 1951 Ore and independently Itô proved that every element of any alternating simple group is a commutator. Ore also mentioned that "it is possible that a similar theorem holds for any simple group of finite order, but it seems that at present we do not have the necessary methods to investigate the question".

5.12. CONJECTURE (Ore). Let G be a finite simple non-abelian group. Then every element of G is a commutator.

Ore's conjecture was proved in 2010:

5.13. THEOREM (Liebeck–O'Brien–Shalev–Tiep). *Every element of a non-abelian finite simple group is a commutator.*

The proof appears in [**26**]. It needs about 70 pages and uses the classification of finite simple groups (CFSG) and character theory. See [**27**] for more information on Ore's conjecture and its proof.

Although the proof of Ore's conjecture is too complicated for this course, we can use the computer to prove the conjecture in some particular cases:

5.14. PROPOSITION. *Ore's conjecture is true for sporadic simple groups.*

PROOF. Let G be a finite simple group. We know that $g \in G$ is a commutator if and only if $\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \neq 0$. Let us write a computer script to check whether every element in a group is a commutator. Our function needs the character table of a group and returns `true` if every element of the group is a commutator and `false` otherwise.

```
gap> Ore := function(char)
> local s,f,k;
> for k in [1..NrConjugacyClasses(char)] do
> s := 0;
> for f in Irr(char) do
> s := s+f[k]/Degree(f);
> od;
> if s<=0 then
> return false;
> fi;
> od;
> return true;
> end;
function( char ) ... end
```

Now we check Ore's conjecture for Mathieu simple groups and for the Monster group:

```
gap> Ore(CharacterTable("M11"));
true
gap> Ore(CharacterTable("M12"));
true
gap> Ore(CharacterTable("M22"));
true
gap> Ore(CharacterTable("M23"));
true
gap> Ore(CharacterTable("M24"));
true
gap> Ore(CharacterTable("M"));
true
```

It is an exercise to check the conjecture for the other finite sporadic simple groups McL, Ru, Ly, Suz, He, HN, Th, Fi_{22}, Fi_{23}, Fi'_{24}, B, M □

See [**25**] for other applications of character theory.

§ 5.4. The Cauchy–Frobenius–Burnside theorem.

5.15. THEOREM (Cauchy–Frobenius–Burnside). *Let G be a finite group that acts on a finite set X. If m is the number of orbits, then*

$$m = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|,$$

where $\operatorname{Fix}(g) = \{x \in X : g \cdot x = x\}$.

PROOF. Let $X = \{x_1, \ldots, x_n\}$ and V be the complex vector space with basis $\{x_1, \ldots, x_n\}$. Let $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$, $g \mapsto \rho_g$, be the representation

$$(\rho_g)_{ij} = \begin{cases} 1 & \text{if } g \cdot x_j = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $(\rho_g)_{ii} = 1$ if $x_i \in \text{Fix}(g)$ and $(\rho_g)_{ii} = 0$ if $x_i \notin \text{Fix}(g)$. Thus

$$\chi_\rho(g) = \text{trace}\, \rho_g = \sum_{i=1}^{n} (\rho_g)_{ii} = |\text{Fix}(g)|.$$

Recall that

$$V^G = \{v \in V : g \cdot v = v \text{ for all } g \in G\}$$

and that

$$\dim V^G = \frac{1}{|G|} \sum_{z \in G} \chi_\rho(z) = \langle \chi_\rho, \chi_1 \rangle$$

where χ_1 is the trivial character of G.

We can assume that, after a possible re-enumeration, $x_1, \ldots, x_m$ are the representatives of the orbits of G on X. For $i \in \{1, \ldots, m\}$, let $v_i = \sum_{x \in G \cdot x_i} x$.

CLAIM. $\{v_1, \ldots, v_m\}$ is a basis of V^G.

If $g \in G$, then $g \cdot v_i = \sum_{x \in G \cdot x_i} g \cdot x = \sum_{y \in G \cdot x_i} y = v_i$. Hence $\{v_1, \ldots, v_m\} \subseteq V^G$. Moreover, $\{v_1, \ldots, v_m\}$ is linearly independent because the v_j are orthogonal and non-zero:

$$\langle v_i, v_j \rangle = \begin{cases} |G \cdot x_i| & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We now prove that $V^G = \langle v_1, \ldots, v_m \rangle$. Let $v \in V^G$. Then $v = \sum_{x \in X} \lambda_x x$ for some coefficients $\lambda_x \in \mathbb{C}$. If $g \in G$, then $g \cdot v = v$. Since

$$\sum_{x \in X} \lambda_x x = v = g \cdot v = \sum_{x \in X} \lambda_x (g \cdot x) = \sum_{x \in X} \lambda_{g^{-1} \cdot x} x,$$

it follows that $\lambda_x = \lambda_{g^{-1} \cdot x}$ for all $x \in X$ and $g \in G$. This means that if $y, z \in X$ and $g \in G$ is such that $g \cdot y = z$, then $\lambda_y = \lambda_z$. Thus

$$v = \sum_{x \in X} \lambda_x x = \sum_{i=1}^{m} \lambda_{x_i} \sum_{y \in G \cdot x_i} y = \sum_{i=1}^{m} \lambda_{x_i} v_i.$$

Hence

$$m = \dim V^G = \langle \chi_\rho, \chi_1 \rangle = \frac{1}{|G|} \sum_{z \in G} \chi_\rho(z) = \frac{1}{|G|} \sum_{z \in G} |\text{Fix}(z)|. \qquad \square$$

It is possible to give an alternative short proof of the theorem. For example, for transitive actions (i.e., $m = 1$), we proceed as follows:

$$\sum_{g \in G} |\text{Fix}(g)| = \sum_{g \in G} \sum_{\substack{x \in X \\ g \cdot x = x}} 1 = \sum_{x \in X} \sum_{\substack{g \in G \\ g \cdot x = x}} 1 = \sum_{x \in X} |G_x| = |G_x||X| = |G|.$$

5.16. EXERCISE. Use the previous idea to prove Theorem 5.15.

Let G act on a finite set X. Then G acts on $X \times X$ by

(5.2) $$g \cdot (x, y) = (g \cdot x, g \cdot y).$$

The orbits of this action are called the **orbitals** of G on X. The **rank** of G on X is the number of orbitals.

5.17. PROPOSITION. *Let G be a group that acts on a finite set X. The rank of G on X is*

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|^2.$$

PROOF. The action (5.2) has $\operatorname{Fix}(g) \times \operatorname{Fix}(g)$ as fixed points, as

$$g \cdot (x, y) = (x, y) \iff (g \cdot x, g \cdot y) = (x, y)$$
$$\iff g \cdot x = x \text{ and } g \cdot y = y \iff (x, y) \in \operatorname{Fix}(g) \times \operatorname{Fix}(g).$$

Now the claim follows from Cauchy–Frobenius–Burnside theorem. $\qquad\square$

5.18. DEFINITION. Let G act on a finite set X. We say that G is **2-transitive** on X if given $x, y \in X$ with $x \ne y$ and $x_1, y_1 \in X$ with $x_1 \ne y_1$ there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = y_1$.

The symmetric group $\mathbb{S}_n$ acts 2-transitively on $\{1, \ldots, n\}$.

5.19. PROPOSITION. *If G is 2-transitive on X, then the rank of G on X is two.*

PROOF. The set $\Delta = \{(x, x) : x \in X\}$ is an orbital. The complement $X \times X \setminus \Delta$ is another orbital: if $x, x_1, y, y_1 \in X$ are such that $x \ne y$ and $x_1 \ne y_1$, then there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = y_1$, so $g \cdot (x, y) = (x_1, y_1)$. $\qquad\square$

6. Lecture: Week 6

Cauchy–Frobenius–Burnside theorem is helpful to find characters.

6.1. Proposition. *Let G be 2-transitive on X with character $\chi(g) = |\operatorname{Fix}(g)|$. Then $\chi - \chi_1$ is an irreducible character.*

Proof. In particular, G is transitive on X. Since the trivial character χ_1 is irreducible, $\langle \chi_1, \chi_1 \rangle = 1$. By Cauchy–Frobenius–Burnside, the rank of G on X is
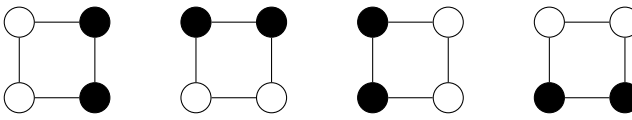
$$2 = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)|^2 = \langle \chi, \chi \rangle.$$

Thus $\langle \chi - \chi_1, \chi - \chi_1 \rangle = \langle \chi, \chi \rangle - 1 - 1 + 1 = 1$. Since χ is a character, $\chi - \chi_1$ is an integer linear combination of the irreducible characters of G. Hence there exists an irreducible character χ_i of G such that $\chi - \chi_1 = \pm \chi_i$. But $(\chi - \chi_1)(1) = |X| - 1 \geq 0$ and so $\chi - \chi_1 = \chi_i$.   □

6.2. Example. The symmetric group $\mathbb{S}_n$ is 2-transitive on $\{1, \ldots, n\}$. The alternating group $\mathbb{A}_n$ is 2-transitive on $\{1, \ldots, n\}$ if $n \geq 4$. These groups then have an irreducible character χ given by $\chi(g) = |\operatorname{Fix}(g)| - 1$.

6.3. Example. Let p be a prime number and let $q = p^m$. Let V be the vector space of dimension 2 over the finite field of q elements. The group $G = \mathbf{GL}_2(q)$ acts 2-transitively on the set X of one-dimensional subspaces of V. In fact, if $\langle v \rangle \neq \langle v_1 \rangle$ and $\langle w \rangle \neq \langle w_1 \rangle$, then $\{v, v_1\}$ and $\{w, w_1\}$ are bases of V. The matrix g that corresponds to the linear map $v \mapsto w$, $v_1 \mapsto w_1$, is invertible. Thus $g \in \mathbf{GL}_2(q)$. The previous proposition produces the irreducible character $\chi(g) = |\operatorname{Fix}(g)| - 1$.

6.4. Example. In how many ways can we color (in black and white) the vertices of a square? We will count colorings up to symmetric. This means that, for example, the colorings

(6.1)

will be considered equivalent. Let $G = \langle g \rangle$ the cyclic group of order four. Let X be the set of colorings of the square. Then $|X| = 16$.

Let g act on X by anti-clockwise rotations of 90°. All the colorings of (6.1) belong to the same orbit. Another orbit of X is
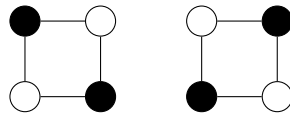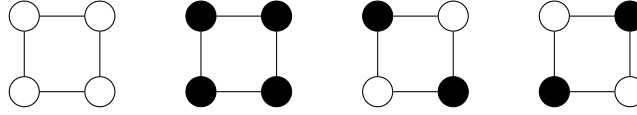

Cauchy–Frobenius–Burnside theorem states that there are

$$\frac{1}{|G|} \sum_{x \in G} |\operatorname{Fix}(x)|$$

orbits.

For each $x \in G = \{1, g, g^2, g^3\}$ we compute $\operatorname{Fix}(x)$. The identity fixes the 16 elements of X, both g and g^3 fix only two elements of X and g^2 fixes four elements of X. For example,

the elements of X fixed by g^2 are

Thus X is the union of

$$\frac{1}{|G|} \sum_{x \in G} |\operatorname{Fix}(x)| = \frac{1}{4}(16 + 2 + 4 + 2) = 6$$

orbits.

> 6.5. BONUS EXERCISE. In how many ways (up to symmetry) can you arrange eight non-attacking rooks on a chessboard? Symmetries are given by the dihedral group $\mathbb{D}_4$ of eight elements.

There are 5282 ways (up to symmetry) to arrange eight non-attacking rooks on a chessboard.

§ 6.1. Commuting probability. For a finite group G, let $\operatorname{cp}(G)$ be the probability that two random elements of G commute. This number is also known as the **commutativity** of G. As an application of Cauchy–Frobenius–Burnside theorem, we prove that $\operatorname{cp}(G) = k/|G|$, where k is the number of conjugacy classes of G. Let

$$C = \{(x, y) \in G \times G : xy = yx\}.$$

We claim that

$$\operatorname{cp}(G) = \frac{|C|}{|G|^2} = \frac{k}{|G|}.$$

Let G act on G by conjugation. By Cauchy–Frobenius–Burnside theorem,

$$k = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)| = \frac{1}{|G|} \sum_{g \in G} |C_G(g)| = \frac{|C|}{|G|},$$

as $\operatorname{Fix}(g) = \{x \in G : gxg^{-1} = x\} = C_G(g)$ and $\sum_{g \in G} |C_G(g)| = |C|$. Alternatively, using Theorem 5.10 with $g = 1$,

$$\operatorname{cp}(G) = \frac{\tau(1)}{|G|^2} = \frac{1}{|G|} \sum_{\chi \in \operatorname{Irr}(G)} 1 = \frac{k}{|G|},$$

as $k = |\operatorname{Irr}(G)|$.

6.6. THEOREM. *If G is a non-abelian finite group, then $\operatorname{cp}(G) \leq 5/8$.*

PROOF. Let $y_1, \dots, y_m$ the representatives of conjugacy classes of G of size ≥ 2. By the class equation,

$$|G| = |Z(G)| + \sum_{i=1}^{m} (G : C_G(y_i)) \geq |Z(G)| + 2m.$$

Thus $m \leq (1/2)(|G| - |Z(G)|)$ and hence

$$k = |Z(G)| + m \leq |Z(G)| + \frac{1}{2}(|G| - |Z(G)|) = \frac{1}{2}(|Z(G)| + |G|).$$

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Since G is non-abelian, $G/Z(G)$ is not cyclic. In particular, $(G : Z(G)) \geq 4$. Therefore

$$k \leq \frac{1}{2}(|Z(G)| + |G|) \leq \frac{1}{2}\left(\frac{1}{4} + 1\right)|G|,$$

that is $k/|G| \leq 5/8$. $\qquad\square$

6.7. EXERCISE.
  1) Prove that $\mathrm{cp}(Q_8) = 5/8$.
  2) Prove that $\mathrm{cp}(\mathbb{A}_5) = 1/12$.

6.8. EXERCISE. Let G be a finite non-abelian group and p be the smallest prime number dividing $|G|$. Prove that $\mathrm{cp}(G) \leq (p^2 + p - 1)/p^3$. Moreover, the equality holds if and only if $(G : Z(G)) = p^2$.

6.9. EXERCISE. Let G be a finite group and H be a subgroup of G.
  1) $\mathrm{cp}(G) \leq \mathrm{cp}(H)$.
  2) If H is normal in G, then $\mathrm{cp}(G) \leq \mathrm{cp}(G/H)\,\mathrm{cp}(H)$.

Degrees of irreducible characters give a lower bound:

6.10. PROPOSITION. *If G is a finite group, then*

$$\mathrm{cp}(G) \geq \left(\frac{\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)}{|G|}\right)^2.$$

PROOF. Let k be the number of conjugacy classes of G. By Cauchy–Schwarz inequality,

$$\left(\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\right)^2 \leq \left(\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2\right)\left(\sum_{\chi \in \mathrm{Irr}(G)} 1\right) = \left(\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2\right)k = |G|k.$$

From this, the claim follows. $\qquad\square$

Using basic facts about irreducible characters, we obtain a generalization of Theorem 6.6.

6.11. THEOREM. *Let G be a finite group. Then*

$$|[G, G]| \leq 3/(4\,\mathrm{cp}(G) - 1).$$

PROOF. For $n \in \mathbb{Z}_{>0}$, let ρ_n be the number of irreducible characters of degree n. Then the number of conjugacy classes of G is $k = \sum_{i \geq 1} \rho_i$ and $|G| = \sum_{i \geq 1} i^2 \rho_i$. It follows that

$$|G| - \rho_1 = \sum_{i \geq 2} i^2 \rho_i \geq 4 \sum_{i \geq 2} \rho_i = 4(k - \rho_1) = 4(|G|\,\mathrm{cp}(G) - \rho_1).$$

Since $\rho_1 = (G : [G, G])$,

$$\mathrm{cp}(G) \leq \frac{1}{4} + \frac{3}{4}\frac{\rho_1}{|G|} = \frac{1}{4} + \frac{3}{4|[G, G]|}.$$

From this, the claim follows. $\qquad\square$

6.12. EXERCISE. Use Theorem 6.11 to prove Theorem 6.6.

Theorem 6.11 can also be used to prove similar statements.

6.13. EXERCISE. Let G be a finite group. Prove the following statements:
  1) If $\mathrm{cp}(G) > 1/2$, then G is nilpotent.
  2) If $\mathrm{cp}(G) > 21/80$, then G is solvable.

In the following exercise, we will discuss the notion of isoclinic groups. We first need a preliminary result:

6.14. EXERCISE. Let G be a group. Prove that the commutator map
$$c_G \colon G/Z(G) \times G/Z(G) \to [G,G], \quad c_G(xZ(G), yZ(G)) = [x,y],$$
is well-defined.

The idea is that two groups are said to be isoclinic if their commutator functions are somewhat equal.

6.15. EXERCISE. Let G and H be groups. A pair (σ, τ) of maps is an **isoclinism** between G and H if $\sigma \colon G/Z(G) \to H/Z(H)$ and $\tau \colon [G,G] \to [H,H]$ are group isomorphisms and the diagram

(6.2)
$$
\begin{array}{ccc}
G/Z(G) \times G/Z(G) & \xrightarrow{\ \sigma \times \sigma\ } & H/Z(H) \times H/Z(H) \\
\downarrow{\scriptstyle c_G} & & \downarrow{\scriptstyle c_H} \\
[G,G] & \xrightarrow[\ \ \ \ \ \ \ \tau\ \ \ \ \ \ \ ]{} & [H,H]
\end{array}
$$

commutes. We write $G \sim H$ when there exists an isoclinism between G and H.
  Prove the following statements:
  1) If $G \simeq H$, then $G \sim H$.
  2) If $G \sim H$, then $\mathrm{cp}(G) = \mathrm{cp}(H)$.

6.16. EXERCISE. Let S be a non-abelian simple group and G be a group such that $G \sim S$. Prove that $G \simeq S \times A$ for some abelian group A.

6.17. EXERCISE. Let H be a subgroup of G. If $G = HZ(G)$, then $G \sim H$. Conversely, if $G \sim H$ and H is finite, then $G = HZ(G)$.

The following theorem appeared in 1970 as a problem in volume 13 of the *Canadian Math. Bulletin*. The solution appeared in 1973. Iván Sadosfchi Costa found the proof we present here.

6.18. THEOREM (Dixon). *The commuting probability of every finite non-abelian simple group is at most* $1/12$.

PROOF. Let G be a finite non-abelian simple group. We claim that $\mathrm{cp}(G) \le 1/12$. We assume that $\mathrm{cp}(G) > 1/12$. Since G is a non-abelian simple group, the identity of G is the only central element of G.

Let us assume first that there is a conjugacy class of G of size m, where m is such that $1 < m \leq 12$. Then G is a transitive subgroup of $\mathbb{S}_m$. For these groups, the problem is easy: we show that there are no non-abelian simple groups that act transitively on sets of size $m \in \{2, \ldots, 12\}$ with commuting probability $> 1/12$. To do this, we list these transitive groups and their commuting probabilities and verify that all commuting probabilities are $\leq 1/12$:

```
gap> l := AllTransitiveGroups(NrMovedPoints, [2..12], \\
> IsAbelian, false, IsSimple, true);;
[ A5, L(6) = PSL(2,5) = A_5(6), A6,
  L(7) = L(3,2), A7, L(8)=PSL(2,7), A8,
  L(9)=PSL(2,8), A9, A_5(10), L(10)=PSL(2,9),
  A10, L(11)=PSL(2,11)(11), M(11), A11, A_5(12),
  L(2,11), M_11(12), M(12), A12 ]
gap> List(l, CommutingProbability);
[ 1/12, 1/12, 7/360, 1/28, 1/280, 1/28, 1/1440,
  1/56, 1/10080, 1/12, 7/360, 1/75600, 2/165,
  1/792, 31/19958400, 1/12, 2/165, 1/792, 1/6336,
  43/239500800 ]
gap> ForAny(l, x->CommutingProbability(x)>1/12);
false
```

Now assume that all non-trivial conjugacy classes of G have at least 13 elements. Let k be the number of conjugacy classes of G. Then the class equation implies that

$$|G| \geq 1 + (k-1)13 = 13k - 12.$$

Since $\mathrm{cp}(G) = k/|G| > 1/12$, $k > |G|/12$. Thus

$$|G| > \frac{13}{12}|G| - 12$$

and therefore $|G| < 144$. Thus one needs to check what happens with groups of order < 144. But we know that the only non-abelian simple group of size < 144 is the alternating simple group $\mathbb{A}_5$.

```
gap> AllGroups(Size, [2..143], \\
> IsAbelian, false, \\
> IsSimple, true);
[ Alt( [ 1 .. 5 ] ) ]
```

This completes the proof.   □

The alternating group $\mathbb{A}_5$ is important in this setting:

6.19. THEOREM (Guralnick–Robinson). *If G is a finite non-solvable group such that* $\mathrm{cp}(G) > 3/40$*, then $G \simeq \mathbb{A}_5 \times T$ for some abelian group T and $\mathrm{cp}(G) = 1/12$.*

The proof appears in [14].

Results on probability of commuting elements generalize in other directions. In [34, 35, 36, 37], Thompson proved the following result:

6.20. THEOREM (Thompson). *If G is a finite group such that every pair of elements of G generate a solvable group, then G is solvable.*

The proof uses the classification of finite simple groups (CFSG). A simpler proof independent of the CFSG appears in [9].

There is a probabilistic version of Thompson's theorem:

6.21. THEOREM (Guralnick–Wilson). *Let G be a finite group.*

1) *If the probability that two random elements of G generate a solvable group is $> 11/30$, then G is solvable.*
2) *If the probability that two random elements of G generate a nilpotent group is $> 1/2$, then G is nilpotent.*
3) *If the probability that two random elements of G generate a group of odd order is $> 11/30$, then G has odd order.*

The proof uses the CFSG and appears in [**15**].

§ 6.2. Jordan's theorem and applications. We now follow [**30**] to present other applications.

6.22. THEOREM (Jordan). *Let G be a non-trivial finite group. If G acts transitively on a finite set X and $|X| > 1$, then there exists $g \in G$ with no fixed points.*

PROOF. Cauchy–Frobenius–Burnside theorem implies that

$$1 = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}(g)| = \frac{1}{|G|} \left(|X| + \sum_{g \neq 1} |\operatorname{Fix}(g)| \right).$$

If every $g \in G \setminus \{1\}$ contains at least one fixed-point, then

$$1 = \frac{1}{|G|} \left(|X| + \sum_{g \neq 1} |\operatorname{Fix}(g)| \right) \geq \frac{1}{|G|}(|X| + |G| - 1) = 1 + \frac{|X| - 1}{|G|}$$

and thus $|X| \leq 1$, a contradiction. $\qquad\square$

6.23. COROLLARY. *Let G be a finite group and H be a proper subgroup of G. Then $G \neq \cup_{g \in G} gHg^{-1}$.*

PROOF. The group G acts transitively by left multiplication on $X = G/H$. The stabilizer of xH is

$$G_{xH} = \{g \in G : gxH = xH\} = xHx^{-1}.$$

Since $H \neq G$, it follows that $|X| = |G/H| > 1$. Jordan's theorem now implies that there exists $g \in G$ with no fixed-points, that is there is an element $g \in G$ such that $g \notin \cup_{x \in G} xHx^{-1}$. $\qquad\square$

Let G be a finite group. We say that the conjugacy classes C and D **commute** if there exist $c \in C$ and $d \in D$ such that $[c, d] = 1$. Note that C and D commute if and only if for all $c \in C$ there exists $d \in D$ such that $[c, d] = 1$.

6.24. COROLLARY (Wildon). *Let G be a finite group and C be a conjugacy class of G. Then $|C| = 1$ if and only if C commutes with every conjugacy class of G.*

PROOF. We prove $\Longleftarrow$. Assume that C commutes with every conjugacy class of G. Let $c \in C$ and $H = C_G(c)$. Then $H \cap D \neq \emptyset$ for every conjugacy class D. We claim that $G = \cup_{g \in G} gHg^{-1}$. In fact, let $x \in G$. Then $x \in D$ for some conjugacy class D. Let $h \in H \cap D$. There exists $y \in G$ such that $h = yxy^{-1}$, that is $x = y^{-1}hy \in \cup_{g \in G} gHg^{-1}$. By Jordan's theorem, $H = G$. Thus c is central and hence $C = \{c\}$.

We now prove $\Longrightarrow$. If $C = \{c\}$, then $c \in Z(G)$ and C commute with every conjugacy class of G. $\qquad\square$

With the CFSG one proves a result similar to that of Jordan.

6.25. THEOREM (Fein–Kantor–Schacher). *Let G be a non-trivial finite group. If G acts transitively on a finite set X and $|X| > 1$, then there exist a prime number p and an element $g \in G$ with no fixed-points with order a power of p.*

The proof appears in [**7**].

§ 6.3. Derangements: Cameron–Cohen theorem. Let G be a finite group that acts faithfully and transitively on a finite set X, say $G \leq \mathbb{S}_n$, where $X = \{1, 2, \ldots, n\}$. Let G_0 be the set of elements $g \in G$ with no fixed-points, that is $g(x) \neq x$ for all $x \in X$. Such permutations are known as **derangements**. Let $c_0 = |G_0|/|G|$.

6.26. THEOREM (Cameron–Cohen). *If G is a subgroup of $\mathbb{S}_n$ that acts transitively on $\{1, \ldots, n\}$, then $c_0 \geq \frac{1}{n}$.*

PROOF. Let $X = \{1, \ldots, n\}$. By definition, the rank of G is the number of orbitals of G on X. It follows that the rank is ≥ 2, as $X \times X$ decomposes as

$$X \times X = \Delta \cup ((X \times X) \setminus \Delta)$$

Let $\chi(g) = |\operatorname{Fix}(g)|$ and $G_0 = \{g \in G : \chi(g) = 0\}$. If $g \notin G_0$, then $1 \leq \chi(g) \leq n$. Since $(\chi(g) - 1)(\chi(g) - n) \leq 0$,

$$\frac{1}{|G|} \sum_{g \in G \setminus G_0} (\chi(g) - 1)(\chi(g) - n) \leq 0.$$

On the one hand,

$$\frac{1}{|G|} \sum_{g \in G} (\chi(g) - 1)(\chi(g) - n)$$

$$= \frac{1}{|G|} \left\{ \sum_{g \in G_0} + \sum_{g \in G \setminus G_0} \right\} (\chi(g) - 1)(\chi(g) - n)$$

$$\leq n \frac{|G_0|}{|G|} = nc_0.$$

On the other hand, since the rank of G is ≥ 2,

$$(6.3) \qquad 2 - \frac{n+1}{|G|} \sum_{g \in G} \chi(g) + n \leq \frac{1}{|G|} \sum_{g \in G} (\chi(g) - 1)(\chi(g) - n) \leq nc_0.$$

Since G is transitive on X, Cauchy–Frobenius–Burnside theorem implies that $\sum_{g \in G} \chi(g) = |G|$. Thus $2 - (n + 1) + n \leq nc_0$ and hence $1/n \leq c_0$. □

Cameron–Cohen theorem contains another claim: If n is not the power of a prime number, then $c_0 > 1/n$. The proof uses Frobenius' theorem.

With the CFSG the bound in Cameron–Cohen theorem can be improved:

6.27. THEOREM (Guralnick–Wan). *Let G be a finite transitive group of degree $n \geq 2$. If n is not a power of a prime number and $G \neq \mathbb{S}_n$ for $n \in \{2, 4, 5\}$, then $c_0 \geq 2/n$.*

The proof appears in [**12**] and uses the classification of finite 2-transitive groups, which depends on the CFSG.

7. Lecture: Week 7

§ 7.1. The Brauer–Fowler theorem. Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation with character χ. The $\mathbb{C}[G]$-module $V \otimes V$ has character χ^2. Let $\{v_1, \ldots, v_n\}$ be a basis of V and

$$T\colon V \otimes V \to V \otimes V, \quad v_i \otimes v_j \mapsto v_j \otimes v_i.$$

It is an exercise to check that

$$T(v \otimes w) = w \otimes v$$

for all $v, w \in V$. It follows that T does not depend on the chosen basis. Note that T is a homomorphism of $\mathbb{C}[G]$-modules, as

$$T(g \cdot (v \otimes w)) = T((g \cdot v) \otimes (g \cdot w)) = (g \cdot w) \otimes (g \cdot v) = g \cdot T(v \otimes w)$$

for all $g \in G$ y $v, w \in V$. In particular, the **symmetric part**

$$S(V \otimes V) = \{x \in V \otimes V : T(x) = x\}$$

and the **antisymmetric** part

$$A(V \otimes V) = \{x \in V \otimes V : T(x) = -x\}$$

of $V \otimes V$ are both $\mathbb{C}[G]$-submodules of $V \otimes V$. The terminology is motivated by the following fact:

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V).$$

In fact, $S(V \otimes V) \cap A(V \otimes V) = \{0\}$, as $x \in S(V \otimes V) \cap A(V \otimes V)$ implies $x = T(x)$ and $x = -T(x)$. Hence $x = 0$. Moreover, $V \otimes V = S(V \otimes V) + A(V \otimes V)$, as every $x \in V \otimes V$ can be written as

$$x = \frac{1}{2}(x + T(x)) + \frac{1}{2}(x - T(x))$$

with $\frac{1}{2}(x + T(x)) \in S(V \otimes V)$ and $\frac{1}{2}(x - T(x)) \in A(V \otimes V)$.

We claim that $\{v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$ is a basis of $S(V \otimes V)$ and that

$$\{v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}$$

is a basis of $A(V \otimes V)$. Since both sets are linearly independent,

$$\dim S(V \otimes V) \geq n(n+1)/2 \text{ and } \dim A(V \otimes V) \geq n(n-1)/2.$$

Moreover,

$$n^2 = \dim(V \otimes V) = \dim S(V \otimes V) + \dim A(V \otimes V),$$

so it follows that $\dim S(V \otimes V) = n(n+1)/2$ and $\dim A(V \otimes V) = n(n-1)/2$.

7.1. PROPOSITION. *Let G be a finite group and V be a finite-dimensional $\mathbb{C}[G]$-module with character χ. If $S(V \otimes V)$ has character χ_S and $A(V \otimes V)$ has character χ_A, then*

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)) \qquad and \qquad \chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2)).$$

PROOF. Let $g \in G$ and $\rho\colon G \to \mathbf{GL}(V)$ be the representation associated with V, that is $\rho(g)(v) = \rho_g(v) = g \cdot v$. Since ρ_g is diagonalizable, let $\{e_1, \ldots, e_n\}$ be a basis of eigenvectors of ρ_g, say $g \cdot e_i = \lambda_i e_i$ with $\lambda_i \in \mathbb{C}$ for all $i \in \{1, \ldots, n\}$. In particular, $\chi(g) = \sum_{i=1}^{n} \lambda_i$.

Since $\{e_i \otimes e_j - e_j \otimes e_i : 1 \leq i < j \leq n\}$ is a basis of $A(V \otimes V)$ and

$$g \cdot (e_i \otimes e_j - e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i),$$

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it follows that
$$\chi_A(g) = \sum_{1 \le i < j \le n} \lambda_i \lambda_j.$$

On the other hand, $g^2 \cdot e_i = \lambda_i^2 e_i$ for all i, $\chi(g^2) = \sum_{i=1}^{n} \lambda_i^2$. Thus

$$\chi^2(g) = \chi(g)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j = 2 \sum_{1 \le i < j \le n} \lambda_i \lambda_j + \sum_{i=1}^{n} \lambda_i^2 = 2\chi_A(g) + \chi(g^2).$$

Since $V \otimes V = S(V \otimes V) \oplus A(V \otimes V)$, it follows that $\chi^2(g) = \chi_S(g) + \chi_A(g)$, that is $\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$. $\qquad \square$

An **involution** of a group is an element $x \ne 1$ such that $x^2 = 1$. It is possible to use the character table to count the number of involutions.

7.2. PROPOSITION. *If G is a finite group with t involutions, then*

$$1 + t = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi(1),$$

where χ_1 is the trivial character of G.

PROOF. Assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$. For $x \in G$ let
$$\theta(x) = |\{y \in G : y^2 = x\}|.$$

Since θ is a class function, θ is a linear combination of the χ_j's, say
$$\theta = \sum_{\chi \in \mathrm{Irr}(G)} \langle \theta, \chi \rangle \chi.$$

For every $\chi \in \mathrm{Irr}(G)$ we compute:

$$\langle \chi_S - \chi_A, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

$$= \frac{1}{|G|} \sum_{x \in G} \sum_{\substack{g \in G \\ g^2 = x}} \chi(g^2) = \frac{1}{|G|} \sum_{x \in G} \theta(x)\chi(x) = \langle \theta, \chi \rangle.$$

Thus $\theta = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi$. Now the claim follows after evaluating this expression in $x = 1$. $\qquad \square$

Before proving the Brauer–Fowler theorem, we need a lemma. We will use the Cauchy–Schwarz inequality:

$$x_1, \dots, x_n \in \mathbb{R} \implies \sum x_i^2 \ge \frac{1}{n}\left(\sum x_i\right)^2.$$

7.3. LEMMA. *Let G be a finite group with k conjugacy classes. If t is the number of involutions of G, then $t^2 \le (k-1)(|G|-1)$.*

PROOF. Assume that $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$, where χ_1 is the trivial character of G. If $\chi \in \mathrm{Irr}(G)$, then

$$\langle \chi^2, \chi_1 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g) = \langle \chi, \overline{\chi} \rangle = \begin{cases} 1 & \text{if } \chi = \overline{\chi}, \\ 0 & \text{otherwise.} \end{cases}$$

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Since $\chi^2 = \chi_S + \chi_A$, if $\langle \chi^2, \chi_1 \rangle = 1$, then the trivial character either is part of χ_S or χ_A, but not both. Thus
$$\langle \chi_S - \chi_A, \chi_1 \rangle \in \{-1, 1, 0\}.$$
We claim that $t \leq \sum_{i=2}^{k} \chi_i(1)$. In fact, since $|\langle \chi_S - \chi_A, \chi_1 \rangle| \leq 1$,

$$1 + t = \theta(1) = \left| \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_S - \chi_A, \chi_1 \rangle \chi(1) \right|$$

$$\leq \sum_{\chi \in \mathrm{Irr}(G)} |\langle \chi_S - \chi_A, \chi_1 \rangle| \chi(1) \leq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1).$$

It follows that $t \leq \sum_{i=2}^{k} \chi_i(1)$. By the Cauchy–Schwarz inequality,

$$t^2 \leq \left(\sum_{i=2}^{k} \chi_i(1) \right)^2 \leq (k-1) \sum_{i=2}^{k} \chi(1)^2 = (k-1)(|G| - 1). \qquad \square$$

Now we prove the Brauer–Fowler theorem.

7.4. THEOREM (Brauer–Fowler). *Let G be a finite simple group and x be an involution of G. If $|C_G(x)| = n$, then $|G| \leq (n^2)!$*

PROOF. If G is abelian, the claim is trivial. Let G be a finite non-abelian simple group. We first assume the existence of a proper subgroup H of G such that

$$(G : H) \leq n^2.$$

The group G acts on G/H by left multiplication, so there is a group homomorphism $\rho \colon G \to \mathbb{S}_{n^2}$. Since G is simple, either $\ker \rho = \{1\}$ or $\ker \rho = G$. If $\ker \rho = G$, then $\rho(g)(yH) = yH$ for all $g \in G$ and $y \in G$. Hence $H = G$, a contradiction. Therefore ρ is injective and hence G is isomorphic to a subgroup of $\mathbb{S}_{n^2}$. In particular, $|G|$ divides $(n^2)!$.

Let $m = (|G| - 1)/t$, where t is the number of involutions of G. Since $|C_G(x)| = n$, the group G has at least $|G|/n$ involutions (because the conjugacy class of x has size $|G|/n$ and all its elements are involutions), that is $t \geq |G|/n$. Hence

$$m = (|G| - 1)/t < n.$$

It is enough to show that G contains a subgroup of index $\leq m^2$.

Let $C_1, \ldots, C_k$ be the conjugacy classes of G, where $C_1 = \{1\}$. Since G is simple and non-abelian, $|C_i| > 1$ for all $i \in \{2, \ldots, k\}$. By the previous lemma,

$$t^2 \leq (k-1)(|G| - 1) \implies |G| - 1 = \frac{mt^2}{t} \leq \frac{(k-1)(|G| - 1)^2}{t^2} = (k-1)m^2.$$

If $|C_i| > m^2$ for all $i \in \{2, \ldots, k\}$, then

$$|G| - 1 = \sum_{i=2}^{k} |C_i| > (k-1)m^2,$$

a contradiction. Thus there exists a non-trivial conjugacy class C of G such that $|C| \leq m^2$. If $g \in C$, then $C_G(g)$ is a proper subgroup of G of index $|C| \leq m^2$. $\qquad \square$

The bound of the Brauer–Fowler theorem is not essential. What matters is the following consequence:

7.5. COROLLARY. *Let $n \geq 1$ be an integer. There are at most finitely many finite simple groups with an involution with a centralizer of order n.*

As an exercise, a simple application:

7.6. EXERCISE. If G is a finite simple group and x is an involution with centralizer of order two, then $G \simeq \mathbb{Z}/2$.

§ 7.2. A comment: An elementary proof of Brauer–Fowler theorem. We need to find a subgroup of index $\leq 2n^2$. Let X be the conjugacy class of x. For $g \in G$ let

$$J(g) = \{z \in X : zgz^{-1} = g^{-1}\}.$$

We claim that $|J(g)| \leq |C_G(g)|$. The map $J(g) \to C_G(g)$, $z \mapsto gz$, is well-defined, as

$$(gz)g(gz)^{-1} = g(zgz^{-1})g^{-1} = g^{-1} \in C_G(g).$$

It is injective, as $gz = gz_1$ implies $z = z_1$.

Let $J = \{(g,z) \in G \times X : zgz^{-1} = g^{-1}\}$. Since $X \times X \to J$, $(y,z) \mapsto (yz, z)$, is well-defined (since $z(yz)z^{-1} = zy = (yz)^{-1}$) and it is trivially injective,

$$|X|^2 \leq |J| = \sum_{(g,z) \in J} 1 \leq \sum_{g \in G} |J(g)| \leq \sum_{g \in G} |C_G(g)| = k|G|,$$

where k is the number of conjugacy classes of G, as $(g,z) \in J$ if and only if $z \in J(g)$. Thus $|G| \leq kn^2$, as

$$\left(\frac{|G|}{|C_G(x)|} \right)^2 = |X|^2 = \frac{|G|^2}{n^2} \leq k|G|.$$

CLAIM. There exists a non-trivial conjugacy class with $\leq 2n^2$ elements.

Assume that the claim is not true. Let $C_1, \ldots, C_k$ be the conjugacy classes of G, where $C_1 = \{1\}$ and $|C_i| > 2n^2$ for all $i \in \{2, \ldots, k\}$. Then

$$|G| = 1 + \sum_{i=2}^{k} |C_i| > 1 + \sum_{i=2}^{k} 2n^2 = 1 + (k-1)2n^2 \geq |G|,$$

a contradiction.

CLAIM. There exists a subgroup H of G such that $(G : H) \leq 2n^2$.

Let C be a conjugacy class of G such that $|C| \leq 2n^2$. Let $g \in C$. Then $H = C_G(g)$ is a subgroup of G such that $(G : H) \leq 2n^2$. This finishes the proof of the Brauer–Fowler theorem.

§ 7.3. Frobenius's reciprocity. We now present a very quick version of Frobenius' reciprocity theorem. We first define the restriction of class functions.

7.7. DEFINITION. Let G be a finite group and $f : G \to \mathbb{C}$ be a map. For a subgroup H of G, the **restriction** of f to H is the map $\mathrm{Res}_H^G = f|_H : H \to \mathbb{C}$, $h \mapsto f(h)$.

7.8. EXERCISE. Let G be a finite group. Prove that the map $\mathrm{Res}_H^G : \mathrm{ClassFun}(G) \to \mathrm{ClassFun}(H)$, $f \mapsto \mathrm{Res}_H^G(f)$, is a well-defined linear map.

We now define induction. Let G be a finite group and H be a subgroup of G. If $f\colon H \to \mathbb{C}$ is a map, then

$$\dot{f}(x) = \begin{cases} f(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

It is an exercise to prove that the map $f \mapsto \dot{f}$ is linear.

7.9. DEFINITION. Let G be a finite group and H be a subgroup of G. Let $f\colon H \to \mathbb{C}$ be a map. The **induction** of f to G is the map

$$g \mapsto \operatorname{Ind}_H^G f(g) = \frac{1}{|H|} \sum_{x \in G} \dot{f}(x^{-1}gx).$$

7.10. EXERCISE. Let G be a finite group. Prove that the map $\operatorname{Ind}_H^G\colon \operatorname{ClassFun}(H) \to \operatorname{ClassFun}(G)$, $f \mapsto \operatorname{Ind}_H^G(f)$, is a well-defined linear map.

7.11. THEOREM (Frobenius' reciprocity). *Let G be a finite group and H be a subgroup of G. If $a \in \operatorname{ClassFun}(H)$ and $b \in \operatorname{ClassFun}(G)$, then*

$$\langle \operatorname{Ind}_H^G a, b \rangle = \langle a, \operatorname{Res}_H^G b \rangle.$$

PROOF. It follows from a direct calculation:

$$(7.1) \qquad \langle \operatorname{Ind}_H^G a, b \rangle = \frac{1}{|G|} \sum_{x \in G} \operatorname{Ind}_H^G a(x) \overline{b(x)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{x,y \in G} \dot{a}(y^{-1}xy) \overline{b(x)}.$$

Since

$$\dot{a}(y^{-1}xy) \neq 0 \implies y^{-1}xy \in H \iff x \in yHy^{-1},$$

setting $h = y^{-1}xy$ we can write (7.1) as

$$\begin{aligned}
\langle \operatorname{Ind}_H^G a, b \rangle &= \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{h \in H} a(h) \overline{b(xhx^{-1})} \\
&= \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{h \in H} a(h) \overline{b(h)} \\
&= \frac{1}{|G|} \sum_{x \in G} \langle a, \operatorname{Res}_H^G b \rangle.
\end{aligned}$$

From this the claim follows. $\qquad\square$

8. Lecture: Week 8

§ 8.1. The correspondence theorem. Let N be a normal subgroup of G and $\pi\colon G \to G/N$, $g \mapsto gN$, be the canonical map. If $\widetilde{\rho}\colon G/N \to \mathbf{GL}(V)$ is a representation of G/N with character $\widetilde{\chi}$, the composition $\rho = \widetilde{\rho}\pi\colon G \to \mathbf{GL}(V)$, $\rho(g) = \widetilde{\rho}(gN)$, is a representation of G. Thus
$$\chi(g) = \operatorname{trace}\rho_g = \operatorname{trace}(\widetilde{\rho}_{gN}) = \widetilde{\chi}(gN).$$
In particular, $\chi(1) = \widetilde{\chi}(1)$. The character χ is the **lifting** to G of the character $\widetilde{\chi}$ of G/N.

8.1. PROPOSITION. *If $\chi \in \operatorname{Char}(G)$, then*
$$\ker\chi = \{g \in G : \chi(g) = \chi(1)\}$$
is a normal subgroup of G.

PROOF. Let $\rho\colon G \to \mathbf{GL}_n(\mathbb{C})$ be a representation with character χ. Then $\ker\rho \subseteq \ker\chi$, as $\rho_g = \operatorname{id}$ implies $\chi(g) = \operatorname{trace}(\rho_g) = n = \chi(1)$. We claim that $\ker\chi \subseteq \ker\rho$. If $g \in G$ is such that $\chi(g) = \chi(1)$, since ρ_g is diagonalizable, there exist eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that
$$n = \chi(1) = \chi(g) = \sum_{i=1}^{n} \lambda_i.$$
Since each λ_i is a root of one, $\lambda_1 = \cdots = \lambda_n = 1$. Hence $\rho_g = \operatorname{id}$. $\qquad\square$

If χ is a character, the subgroup $\ker\chi$ is the **kernel** of χ.

8.2. THEOREM (Correspondence theorem). *Let N be a normal subgroup of a finite group G. There exists a bijective correspondence*
$$\operatorname{Char}(G/N) \longleftrightarrow \{\chi \in \operatorname{Char}(G) : N \subseteq \ker\chi\}$$
that maps irreducible characters to irreducible characters.

PROOF. If $\widetilde{\chi} \in \operatorname{Char}(G/N)$, let χ be the lifting of $\widetilde{\chi}$ to G. If $n \in N$, then
$$\chi(n) = \widetilde{\chi}(nN) = \widetilde{\chi}(N) = \chi(1)$$
and thus $N \subseteq \ker\chi$.

If $\chi \in \operatorname{Char}(G)$ is such that $N \subseteq \ker\chi$, let $\rho\colon G \to \mathbf{GL}(V)$ be a representation with character χ. Let $\widetilde{\rho}\colon G/N \to \mathbf{GL}(V)$, $gN \mapsto \rho(g)$. We claim that $\widetilde{\rho}$ is well-defined:
$$gN = hN \iff h^{-1}g \in N \implies \rho(h^{-1}g) = \operatorname{id} \iff \rho(h) = \rho(g).$$
Moreover, $\widetilde{\rho}$ is a representation, as
$$\widetilde{\rho}((gN)(hN)) = \widetilde{\rho}(ghN) = \rho(gh) = \rho(g)\rho(h) = \widetilde{\rho}(gN)\widetilde{\rho}(hN).$$
If $\widetilde{\chi}$ is the character of $\widetilde{\rho}$, then $\widetilde{\chi}(gN) = \chi(g)$.

We now prove that χ is irreducible if and only if $\widetilde{\chi}$ is irreducible. If U is a subspace of V, then
$$U \text{ is } G\text{-invariant} \iff \rho(g)(U) \subseteq U \text{ for all } g \in G$$
$$\iff \widetilde{\rho}(gN)(U) \subseteq U \text{ for all } g \in G.$$
Thus
$$\chi \text{ is irreducible} \iff \rho \text{ is irreducible}$$
$$\iff \widetilde{\rho} \text{ is irreducible} \iff \widetilde{\chi} \text{ is irreducible .} \qquad\square$$

8.3. EXAMPLE. Let $G = \mathbb{S}_4$ and $N = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. We know that N is normal in G and that $G/N = \langle a, b \rangle \simeq \mathbb{S}_3$, where $a = (123)N$ and $b = (12)N$. The character table of G/N is

	N	$(12)N$	$(123)N$
$\widetilde{\chi}_1$	1	1	1
$\widetilde{\chi}_2$	1	-1	1
$\widetilde{\chi}_3$	2	0	-1

For each $i \in \{1, 2, 3\}$ we compute the lifting χ_i to G of the character $\widetilde{\chi}_i$ of G/N. Since $(12)(34) \in N$ and $(13)(1234) = (12)(34) \in N$,

$$\chi((12)(34)) = \widetilde{\chi}(N), \quad \chi((1234)) = \widetilde{\chi}((13)N) = \widetilde{\chi}((12)N).$$

Since the characters $\widetilde{\chi}_i$ are irreducibles, the liftings χ_i are also irreducibles. With this process we obtain the following irreducible characters of G:

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0

The character table of a group can be used to find the lattice of normal subgroups. In particular, the character table detects simple groups.

8.4. LEMMA. *Let G be a finite group and let $g, h \in G$. Then g and h are conjugate if and only if $\chi(g) = \chi(h)$ for all $\chi \in \mathrm{Char}(G)$.*

PROOF. If g and h are conjugate, then $\chi(g) = \chi(h)$, as characters are class functions of G. Conversely, if $\chi(g) = \chi(h)$ for all $\chi \in \mathrm{Char}(G)$, then $f(g) = f(h)$ for all class function f of G, as characters G generate the space of class functions of G. In particular, $\delta(g) = \delta(h)$, where

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ and } g \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

This implies that g and h are conjugate.   □

As a consequence, we get that

(8.1) $$\bigcap_{\chi \in \mathrm{Irr}(G)} \ker \chi = \{1\}.$$

Indeed, if $g \in \ker \chi$ for all $\chi \in \mathrm{Irr}(G)$, then $g = 1$ since the lemma implies that g and 1 are conjugate because $\chi(g) = \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

8.5. PROPOSITION. *Let G be a finite group. If N is a normal subgroup of G, then there exist characters $\chi_1, \ldots, \chi_k \in \mathrm{Irr}(G)$ such that*

$$N = \bigcap_{i=1}^{k} \ker \chi_i.$$

PROOF. Apply the previous remark to the group G/N to obtain that

$$\bigcap_{\widetilde{\chi} \in \mathrm{Irr}(G/N)} \ker \widetilde{\chi} = \{N\}.$$

Assume that $\mathrm{Irr}(G/N) = \{\widetilde{\chi}_1, \ldots, \widetilde{\chi}_k\}$. We lift the irreducible characters of G/N to G to obtain (some) irreducible characters $\chi_1, \ldots, \chi_k$ of G such that

$$N \subseteq \ker\chi_1 \cap \cdots \cap \ker\chi_k.$$

If $g \in \ker\chi_i$ for all $i \in \{1, \ldots, k\}$, then

$$\widetilde{\chi}_i(N) = \chi_i(1) = \chi_i(g) = \widetilde{\chi}_i(gN)$$

for all $i \in \{1, \ldots, k\}$. This implies that

$$gN \in \bigcap_{i=1}^{k} \ker\widetilde{\chi}_i = \{N\},$$

that is $g \in N$. □

Recall that a non-trivial group is **simple** if it contains no non-trivial normal proper subgroups. Examples of simple groups are cyclic groups of prime order and the alternating groups $\mathbb{A}_n$ for $n \geq 5$. As a corollary of Proposition 8.5, we can use the character table to detect simple groups.

8.6. PROPOSITION. *Let G be a finite group. Then G is not simple if and only if there exists a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some $g \in G \setminus \{1\}$.*

PROOF. If G is not simple, there exists a normal subgroup N of G such that $N \neq G$ and $N \neq \{1\}$. By Proposition 8.5, there exist characters $\chi_1, \ldots, \chi_k \in \mathrm{Irr}(G)$ such that $N = \ker\chi_1 \cap \cdots \cap \ker\chi_k$. In particular, there exists a non-trivial character χ_i such that $\ker\chi_i \neq \{1\}$. Thus there exists $g \in G \setminus \{1\}$ such that $\chi_i(g) = \chi_i(1)$.

Assume now that there exists a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some $g \in G \setminus \{1\}$. In particular, $g \in \ker\chi$ and hence $\ker\chi \neq \{1\}$. Since χ is non-trivial, $\ker\chi \neq G$. Thus $\ker\chi$ is a proper non-trivial normal subgroup of G. □

8.7. EXAMPLE. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	1	1	-1	-1	1
χ_5	2	-2	2	0	0	0
χ_6	8	0	-1	0	0	0

then G cannot be simple. Note that such a group G would have order $\sum_{i=1}^{6} \chi_i(1)^2 = 72$. Mathieu's group M_9 has precisely this character table!

8.8. EXAMPLE. Let $\alpha = \frac{1}{2}(-1 + \sqrt{7}i)$. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	8	0	0	-1	1	1
χ_4	3	-1	1	0	α	$\overline{\alpha}$
χ_5	3	-1	1	0	$\overline{\alpha}$	α
χ_6	6	2	0	0	0	0

then G is simple. Note that such a group G would have order $\sum_{i=1}^{6} \chi_i(1)^2 = 168$. The group
$$\mathbf{PSL}_2(7) = \mathbf{SL}_2(7)/Z(\mathbf{SL}_2(7))$$
is a simple group that has precisely this character table!

§ 8.2. Frobenius' groups. If p is a prime number, then the units $(\mathbb{Z}/p)^\times$ of $\mathbb{Z}/p$ form a multiplicative group. Moreover, $(\mathbb{Z}/p)^\times$ is cyclic of order $p - 1$.

Let
$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x \in (\mathbb{Z}/p)^\times, \, y \in \mathbb{Z}/p \right\}.$$

Then G is a group with the usual matrix multiplication and $|G| = p(p-1)$. Let p and q be prime numbers such that q divides $p - 1$, $z \in \mathbb{Z}$ be an element of multiplicative order q modulo p and
$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} z & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \langle a, b \rangle.$$

A direct calculation shows that

(8.2)
$$a^p = b^q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad bab^{-1} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = a^z.$$

Every element of H is of the form $a^i b^j$ for $i \in \{0, \ldots, p-1\}$ and $j \in \{0, \ldots, q-1\}$. Thus $|H| = pq$. Using (8.2) we can compute the multiplication table of G.

> 8.9. EXERCISE. Let p and q be prime numbers such that q divides $p - 1$. Let $u, v \in \mathbb{Z}$ be elements of order q modulo p. Prove that
> $$\langle a, b : a^p = b^q = 1, bab^{-1} = a^u \rangle \simeq \langle a, b : a^p = b^q = 1, bab^{-1} = a^v \rangle.$$

The group
$$F_{p,q} = \langle a, b : a^p = b^q = 1, bab^{-1} = a^u \rangle,$$
where $u \in \mathbb{Z}$ has order q modulo p, is a particular case of a **Frobenius group**.

8.10. PROPOSITION. *Let p and q be prime numbers such that $p > q$. Let G be a group of order pq. Then either G is abelian or q divides $p - 1$ and $G \simeq F_{p,q}$.*

PROOF. Assume that G is not abelian. By Sylow's theorems, q divides $p - 1$ and there exists a unique Sylow p-subgroup P of G. Let $a, b \in G$ be such that $P = \langle a \rangle \simeq \mathbb{Z}/p$ and $G/P = \langle bP \rangle \simeq \mathbb{Z}/q$. By Lagrange's theorem, $G = \langle a, b \rangle$. We compute the order of b^q. Since G is not cyclic (because it is not abelian) and $b^q \in P$, we conclude that $|b^q| = 1$. Since P is normal in G, $bab^{-1} \in P$ and hence $bab^{-1} = a^z$ for some $z \in \mathbb{Z}$. Therefore $b^q a b^{-q} = a^{z^q}$. This implies that $z^q \equiv 1 \bmod p$. The order of z in $(\mathbb{Z}/p)^\times$ divides q and hence it is equal to q (otherwise, $z = 1$ and thus $bab^{-1} = a$, which implies that G is abelian). In conclusion, $G \simeq F_{p,q}$. $\qquad\square$

With the proposition, we prove, for example, that every group of order 15 is abelian. We can also prove that up to isomorphism $\mathbb{Z}/20$ and $F_{5,4}$ are the only groups of order 20.

8.11. DEFINITION. We say that a finite group G is a **Frobenius group** if G has a non-trivial proper subgroup H such that $H \cap xHx^{-1} = \{1\}$ for all $x \in G \setminus H$. In this case, the subgroup H is called a **Frobenius complement**.

8.12. THEOREM (Frobenius). *Let G be a Frobenius group with complement H. Then*

$$N = \left(G \setminus \bigcup_{x \in G} xHx^{-1} \right) \cup \{1\}$$

is a normal subgroup of G.

PROOF. For each $\chi \in \mathrm{Irr}(H)$, $\chi \neq 1_H$, let $\alpha = \chi - \chi(1)1_H \in \mathrm{ClassFun}(H)$, where 1_H denotes the trivial character of H.

We claim that $\mathrm{Res}_H^G \mathrm{Ind}_H^G \alpha = \alpha$. First, $\mathrm{Ind}_H^G \alpha(1) = \alpha(1) = 0$. If $h \in H \setminus \{1\}$, then

$$\mathrm{Ind}_H^G \alpha(h) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}hx \in H}} \alpha(x^{-1}hx) = \frac{1}{|H|} \sum_{x \in H} \alpha(h) = \alpha(h),$$

since, if $x \notin H$, then $x^{-1}hx \in H$ implies that $h \in H \cap xHx^{-1} = \{1\}$.

By Frobenius' reciprocity,

(8.3) $\langle \mathrm{Ind}_H^G \alpha, \mathrm{Ind}_H^G \alpha \rangle = \langle \alpha, \mathrm{Res}_H^G \mathrm{Ind}_H^G \alpha \rangle = \langle \alpha, \alpha \rangle = 1 + \chi(1)^2.$

Again, by Frobenius' reciprocity,

$$\langle \mathrm{Ind}_H^G \alpha, 1_G \rangle = \langle \alpha, \mathrm{Res}_H^G 1_G \rangle = \langle \alpha, 1_H \rangle = \langle \chi - \chi(1)1_H, 1_H \rangle = -\chi(1),$$

where 1_G is the trivial character of G. If we write

$$\mathrm{Ind}_H^G \alpha = \sum_{\eta \in \mathrm{Irr}(G)} \langle \mathrm{Ind}_H^G \alpha, \eta \rangle \eta = \langle \mathrm{Ind}_H^G \alpha, 1_G \rangle 1_G + \underbrace{\sum_{\substack{1_G \neq \eta \\ \eta \in \mathrm{Irr}(G)}} \langle \mathrm{Ind}_H^G \alpha, \eta \rangle \eta}_{\phi},$$

then $\mathrm{Ind}_H^G \alpha = -\chi(1)1_G + \phi$, where ϕ is a linear combination of non-trivial irreducible characters of G. We compute

$$1 + \chi(1)^2 = \langle \mathrm{Ind}_H^G \alpha, \mathrm{Ind}_H^G \alpha \rangle = \langle \phi - \chi(1)1_G, \phi - \chi(1)1_G \rangle = \langle \phi, \phi \rangle + \chi(1)^2$$

and hence $\langle \phi, \phi \rangle = 1$.

CLAIM. *If $\eta \in \mathrm{Irr}(G)$ is such that $\eta \neq 1_G$, then $\langle \mathrm{Ind}_H^G \alpha, \eta \rangle \in \mathbb{Z}$.*

By Frobenius' reciprocity, $\langle \mathrm{Ind}_H^G \alpha, \eta \rangle = \langle \alpha, \mathrm{Res}_H^G \eta \rangle$. If we decompose $\mathrm{Res}_H^G \eta$ into irreducibles of H, say

$$\mathrm{Res}_H^G \eta = m_1 1_H + m_2 \chi + m_3 \theta_3 + \cdots + m_t \theta_t$$

for some $m_1, m_2, \ldots, m_t \geq 0$, then, since

$$\langle \alpha, 1_H \rangle = \langle \chi - \chi(1)1_H, 1_H \rangle = -\chi(1), \qquad \langle \alpha, \chi \rangle = \langle \chi - \chi(1)1_H, \chi \rangle = 1,$$

and

$$\langle \alpha, \theta_j \rangle = \langle \chi - \chi(1)1_H, \theta_j \rangle = 0$$

for all $j \in \{3, \ldots, t\}$, we conclude that

$$\langle \mathrm{Ind}_H^G \alpha, \eta \rangle = -m_1 \chi(1) + m_2 \in \mathbb{Z}.$$

CLAIM. $\phi \in \mathrm{Irr}(G)$.

Since $\langle \operatorname{Ind}_H^G \alpha, \eta \rangle \in \mathbb{Z}$ for all $\eta \in \operatorname{Irr}(G)$ such that $\eta \neq 1_G$ and

$$1 = \langle \phi, \phi \rangle = \sum_{\substack{\eta, \theta \in \operatorname{Irr}(G) \\ \eta, \theta \neq 1_G}} \langle \operatorname{Ind}_H^G \alpha, \eta \rangle \langle \operatorname{Ind}_H^G \alpha, \theta \rangle \langle \eta, \theta \rangle = \sum_{\substack{\eta \neq 1_G \\ \eta \in \operatorname{Irr}(G)}} \langle \operatorname{Ind}_H^G \alpha, \eta \rangle^2,$$

there is a unique $\eta \in \operatorname{Irr}(G)$ such that $\langle \operatorname{Ind}_H^G \alpha, \eta \rangle^2 = 1$ and all the other products are zero, that is $\phi = \pm \eta$ for some $\eta \in \operatorname{Irr}(G)$. Since

$$\chi - \chi(1) 1_H = \alpha = \operatorname{Res}_H^G \operatorname{Ind}_H^G \alpha = \operatorname{Res}_H^G (\phi - \chi(1) 1_G) = \operatorname{Res}_H^G \phi - \chi(1) 1_H,$$

it follows that $\phi(1) = \operatorname{Res}_H^G \phi(1) = \chi(1) \in \mathbb{Z}_{\geq 1}$. Thus $\phi \in \operatorname{Irr}(G)$.

We have proved that if $\chi \in \operatorname{Irr}(H)$ is such that $\chi \neq 1_H$, then there exists $\phi_\chi \in \operatorname{Irr}(G)$ such that $\operatorname{Res}_H^G(\phi_\chi) = \chi$.

We prove that N is equal to

$$M = \bigcap_{\substack{\chi \in \operatorname{Irr}(H) \\ \chi \neq 1_H}} \ker \phi_\chi.$$

We first prove that $N \subseteq M$. Let $n \in N \setminus \{1\}$ and $\chi \in \operatorname{Irr}(H) \setminus \{1_H\}$. Since n does not belong to a conjugate of H,

$$\operatorname{Ind}_H^G \alpha(n) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}nx \in H}} \alpha(x^{-1}nx) = 0,$$

as $n \in N$ implies that the set $\{x \in G : x^{-1}nx \in H\}$ is empty. Since

$$0 = \operatorname{Ind}_H^G \alpha(n) = \phi_\chi(n) - \chi(1) = \phi_\chi(n) - \phi_\chi(1),$$

we conclude that $n \in \ker \phi_\chi$.

We now prove that $M \subseteq N$. Let $h \in M \cap H$ and $\chi \in \operatorname{Irr}(H) \setminus \{1_H\}$. Then

$$\phi_\chi(h) - \chi(1) = \operatorname{Ind}_H^G \alpha(h) = \alpha(h) = \chi(h) - \chi(1),$$

and $h \in \ker \chi$, as

$$\chi(h) = \phi_\chi(h) = \phi_\chi(1) = \chi(1).$$

Therefore

$$h \in \bigcap_{\chi \in \operatorname{Irr}(H)} \ker \chi = \{1\}.$$

By (8.1), the kernels of irreducible characters have trivial intersection. We now prove that $M \cap xHx^{-1} = \{1\}$ for all $x \in G$. Let $x \in G$ and $m \in M \cap xHx^{-1}$. Since $m = xhx^{-1}$ for some $h \in H$, $x^{-1}mx \in H \cap M = \{1\}$. This implies that $m = 1$. $\qquad \square$

At the moment, there is no proof of Frobenius' theorem essentially independent of character theory.

8.13. DEFINITION. Let G be a Frobenius group. The normal subgroup N of Frobenius' theorem is called the **Frobenius kernel**.

8.14. COROLLARY. *Let G be a Frobenius group with complement H. Then there exists a normal subgroup N of G such that $G = HN$ and $H \cap N = \{1\}$.*

PROOF. Frobenius' theorem yields the subgroup N. Let us prove that $H \supseteq N_G(H)$. If $h \in H \setminus \{1\}$ and $g \in G$ are such that $ghg^{-1} \in H$, then $h \in g^{-1}Hg \cap H$ and hence $g \in H$. Since $H = N_G(H)$, the subgroup H has $(G : H)$ conjugates. Thus $|G| = |H||N|$, where $N = \left(G \setminus \bigcup_{x \in G} xHx^{-1}\right) \cup \{1\}$, as

$$|N| = |G| - (G : H)(|H| - 1) = (G : H).$$

Since $N \cap H = \{1\}$,

$$|HN| = |N||H|/|H \cap N| = |N||H| = |G|$$

and therefore $G = NH$. $\square$

Some comments.

8.15. COROLLARY (Combinatorial Frobenius' theorem). *Let G be a group acting transitively on a finite set X. Assume that each $g \in G \setminus \{1\}$ fixes at most one element of X. The set N formed by the identity and the permutations that move every element of X is a normal subgroup of G.*

PROOF. Let $x \in X$ and $H = G_x$. We claim that if $g \in G \setminus H$, then $H \cap gHg^{-1} = 1$. If $h \in H \cap gHg^{-1}$, then $h \cdot x = x$ and $g^{-1}hg \cdot x = x$. Since $g \cdot x \neq x$, h fixes two elements of X. Thus $h = 1$, as every non-trivial element fixes at most one element of X.

By Theorem 8.12,

$$N = \left(G \setminus \bigcup_{g \in G} gHg^{-1}\right) \cup \{1\}$$

is a subgroup of G. Let us compute the elements of N. If $h \in \cup_{g \in G} gHg^{-1}$, then there exists $g \in G$ such that $g^{-1}hg \in H$, that is $(g^{-1}hg) \cdot x = x$; equivalently, $h \in G_{g \cdot x}$. Therefore, the non-identity elements of N are the elements of G moving every element of X. $\square$

8.16. EXAMPLE. Let F be a finite field and G be the group of maps $f \colon F \to F$ of the form $f(x) = ax + b$, $a, b \in F$ with $a \neq 0$. The group G acts on F and every $f \neq \mathrm{id}$ fixes at most one element of F, as

$$x = f(x) = ax + b \implies a \neq 1 \text{ and } x = b/(1 - a).$$

In this case, $N = \{f : f(x) = x + b, b \in F\}$ is a subgroup of G.

8.17. EXERCISE. Prove that Theorem 8.12 can be obtained from Corollary 8.15.

In his doctoral thesis Thompson proved the following result, conjectured by Frobenius.

8.18. THEOREM (Thompson). *Let G be a Frobenius group. If N is the Frobenius kernel, then N is nilpotent.*

See [20, Theorem 6.24] for the proof.

9. Lecture: Week 9

10. Lecture: Week 10

§ 10.1. Some theorems of Burnside. For $n \geq 1$ let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$. The **natural representation** of $\mathbb{S}_n$ is $\rho \colon \mathbb{S}_n \to \mathbf{GL}_n(C)$, $\sigma \mapsto \rho_\sigma$, where $\rho_\sigma(e_j) = e_{\sigma(j)}$ for all $j \in \{1, \ldots, n\}$. The matrix of ρ_σ in the standard basis is

$$(10.1) \qquad\qquad (\rho_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

10.1. LEMMA. *For $n \geq 1$ let $\rho \colon \mathbb{S}_n \to \mathbf{GL}_n(C)$ be the natural representation of the symmetric group. If $A \in \mathbb{C}^{n \times n}$ and $\sigma \in \mathbb{S}_n$, then*

$$A_{ij} = (\rho_\sigma A)_{\sigma(i)j} = (A\rho_\sigma)_{i\sigma^{-1}(j)}$$

for all $i, j \in \{1, \ldots, n\}$.

PROOF. With (10.1) we compute:

$$(A\rho_\sigma)_{ij} = \sum_{k=1}^{n} A_{ik}(\rho_\sigma)_{kj} = A_{i\sigma(j)}, \quad (\rho_\sigma A)_{ij} = \sum_{k=1}^{n} (\rho_\sigma)_{ik} A_{kj} = A_{\sigma^{-1}(i)j}. \qquad \square$$

10.2. DEFINITION. Let G be a finite group. A character χ of G is said to be **real** if $\chi = \overline{\chi}$, that is $\chi(g) \in \mathbb{R}$ for all $g \in G$.

> 10.3. EXERCISE. Let G be a finite group. If $\chi \in \mathrm{Irr}(G)$, then $\overline{\chi}$ is irreducible.

10.4. DEFINITION. Let G be a group. A conjugacy class C of G is said to be **real** if for every $g \in C$ one has $g^{-1} \in C$.

We use the following notation: if G is a group and $C = \{xgx^{-1} : x \in G\}$ is a conjugacy class of G, then $C^{-1} = \{xg^{-1}x^{-1} : x \in G\}$.

10.5. THEOREM (Burnside). *Let G be a finite group. The number of real conjugacy classes equals the number of real irreducible characters.*

PROOF. Let $C_1, \ldots, C_r$ be the conjugacy classes of G and let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Let $\alpha, \beta \in \mathbb{S}_r$ be such that $\overline{\chi_i} = \chi_{\alpha(i)}$ and $C_i^{-1} = C_{\beta(i)}$ for all $i \in \{1, \ldots, r\}$. Note that χ_i is real if and only if $\alpha(i) = i$ and that C_i is real if and only if $\beta(i) = i$. The number n of fixed points of α is equal to the number of real irreducible characters of G, and the number m of fixed points of β is equal to the number of real classes. Let $\rho \colon \mathbb{S}_r \to \mathbf{GL}_r(\mathbb{C})$ be the natural representation of $\mathbb{S}_r$, with character χ_ρ. Then $\chi_\rho(\alpha) = n$ and $\chi_\rho(\beta) = m$. We claim that $\mathrm{trace}\, \rho_\alpha = \mathrm{trace}\, \rho_\beta$. Let $X \in \mathbf{GL}(r, \mathbb{C})$ be the character matrix of G. By Lemma 10.1 and the fact that $\overline{\chi(g)} = \chi(g^{-1})$ for all $g \in G$,

$$\rho_\alpha X = \overline{X} = X\rho_\beta.$$

Since X is invertible, $\rho_\alpha = X\rho_\beta X^{-1}$. Thus

$$n = \chi_\rho(\alpha) = \mathrm{trace}\, \rho_\alpha = \mathrm{trace}\, \rho_\beta = \chi_\rho(\beta) = m. \qquad \square$$

10.6. COROLLARY. *Let G be a finite group. Then $|G|$ is odd if and only if the only real $\chi \in \mathrm{Irr}(G)$ is the trivial character.*

PROOF. We first prove $\Longleftarrow$. If $|G|$ is even, there exists $g \in G$ of order two (Cauchy's theorem). The conjugacy class of g is real.

We now prove $\Longrightarrow$. Assume that G has a non-trivial real conjugacy class C. Let $g \in C$. We claim that G has an element of even order. Let $h \in G$ be such that $hgh^{-1} = g^{-1}$. Then $h^2 \in C_G(g)$, as $h^2gh^{-2} = g$. If $h \in \langle h^2 \rangle \subseteq C_G(g)$, then g has even order, as $g^{-1} = g$. If $h \notin \langle h^2 \rangle$, then h^2 does not generate $\langle h \rangle$. Hence h has even order, as $|h| \neq |h^2| = |h|/\gcd(|h|,2)$, so $\gcd(|h|,2) \neq 1$. $\qquad\square$

10.7. THEOREM (Burnside). *Let G be a finite group of odd order with r conjugacy classes. Then $r \equiv |G|$ mod 16.*

PROOF. Since $|G|$ is odd, every non-trivial $\chi \in \mathrm{Irr}(G)$ is not real by the previous corollary. The irreducible characters of G are

$$\chi_1, \chi_2, \overline{\chi_2}, \ldots, \chi_k, \overline{\chi_k}, \quad r = 1 + 2(k-1),$$

where χ_1 denotes the trivial character. For every $j \in \{2, \ldots, k\}$ let $d_j = \chi_j(1)$. Since each d_j divides $|G|$ by Frobenius' theorem and $|G|$ is odd, every d_j is an odd number, say $d_j = 1 + 2m_j$. Thus

$$|G| = 1 + \sum_{j=2}^{k} 2d_j^2 = 1 + \sum_{j=2}^{k} 2(2m_j + 1)^2$$

$$= 1 + \sum_{j=2}^{k} 2(4m_j^2 + 4m_j + 1) = 1 + 2(k-1) + 8\sum_{j=2}^{k} m_j(m_j + 1).$$

Hence $|G| \equiv r$ mod 16, as $r = 1 + 2k$ and every $m_j(m_j + 1)$ is even. $\qquad\square$

10.8. EXERCISE. Prove that every group of order 15 is abelian.

11. Lecture: Week 11

§ 11.1. Solvable groups and Burnside's theorem. For a group G let $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ for $i \geq 0$. The **derived series** of G is the sequence

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$$

Each $G^{(i)}$ is a characteristic subgroup of G. We say that G is **solvable** if $G^{(n)} = \{1\}$ for some n.

11.1. EXAMPLE. Abelian groups are solvable.

11.2. EXAMPLE. The group $\mathbf{SL}_2(3)$ is solvable, as the derived series is

$$\mathbf{SL}_2(3) \supseteq Q_8 \supseteq C_4 \supseteq C_2 \supseteq \{1\}.$$

Here is what the computer says:

```
gap> IsSolvable(SL(2,3));
true
gap> List(DerivedSeries(SL(2,3)),StructureDescription);
[ "SL(2,3)", "Q8", "C2", "1" ]
```

11.3. EXAMPLE. Non-abelian simple groups cannot be solvable.

11.4. EXERCISE. Let G be a group. Prove the following statements:
   1) A subgroup H of G is solvable, when G is solvable.
   2) Let K be a normal subgroup of G. Then G is solvable if and only if K and G/K are solvable.

11.5. EXAMPLE. For $n \geq 5$ the group $\mathbb{A}_n$ is not solvable. It follows that $\mathbb{S}_n$ is not solvable for $n \geq 5$.

11.6. EXERCISE. Let p be a prime number. Prove that finite p-groups are solvable.

11.7. THEOREM (Burnside). *Let G be a finite group. If $\phi\colon G \to \mathbf{GL}_n(\mathbb{C})$ is a representation with character χ and C is a conjugacy class of G such that $\gcd(|C|, n) = 1$, then for every $g \in C$ either $\chi(g) = 0$ or ϕ_g is a scalar matrix.*

We need a lemma.

11.8. LEMMA. *Let $\epsilon_1, \ldots, \epsilon_n$ be roots of one such that $(\epsilon_1 + \cdots + \epsilon_n)/n \in \mathbb{A}$. Then either $\epsilon_1 = \cdots = \epsilon_n$ or $\epsilon_1 + \cdots + \epsilon_n = 0$.*

PROOF. Let $\alpha = (\epsilon_1 + \cdots + \epsilon_n)/n$. If the ϵ_js are not all equal, then $\|\alpha\| < 1$. Moreover, $\|\beta\| < 1$ for every algebraic conjugate β of α. Since the product of the algebraic conjugates of α is an integer of absolute value < 1, it follows that it is zero. $\qquad\square$

Now we prove the theorem.

PROOF OF THEOREM 11.7. Let $\epsilon_1, \ldots, \epsilon_n$ be the eigenvalues of ϕ_g. By assumption, $\gcd(|C|, n) = 1$, there exist $a, b \in \mathbb{Z}$ such that $a|C| + bn = 1$. Since $|C|\chi(g)/n \in \mathbb{A}$, after multiplying by $\chi(g)/n$ we obtain that

$$a|C|\frac{\chi(g)}{n} + b\chi(g) = \frac{\chi(g)}{n} = \frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n) \in \mathbb{A}.$$

The previous lemma implies that there are two cases to consider: either $\epsilon_1 = \cdots = \epsilon_n$ or $\epsilon_1 + \cdots + \epsilon_n = 0$. In the first case, since ϕ_g is diagonalizable, ϕ_g is a scalar matrix. In the second case, $\chi(g) = 0$. $\qquad\square$

11.9. THEOREM (Burnside). *Let p be a prime number. If G is a finite group and C is a conjugacy class of G with $p^k > 1$ elements, then G is not simple.*

PROOF. Let $g \in C \setminus \{1\}$. Column orthogonality implies that

(11.1)
$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi(g)$$
$$= \sum_{p \mid \chi(1)} \chi(1)\chi(g) + \sum_{p \nmid \chi(1) : \chi \neq \chi_1} \chi(1)\chi(g) + 1,$$

where the one corresponds to the trivial representation of G.

Look at this equation modulo p. If $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G)$ such that $\chi \neq \chi_1$ and $p \nmid \chi(1)$, then

$$-\frac{1}{p} = \sum \frac{\chi(1)}{p}\chi(g) \in \mathbb{A} \cap \mathbb{Q} = \mathbb{Z},$$

where the sum is taken over all non-trivial irreducibles of G of degree divisible by p, a contradiction. Hence there exists an irreducible non-trivial representation ϕ with character χ such that p does not divide $\chi(1)$ and $\chi(g) \neq 0$. By the previous theorem, ϕ_g is a scalar matrix. If ϕ is faithful, then g is a non-trivial central element, a contradiction since $|C| > 1$. If ϕ is not faithful, then G is not simple (because $\ker \phi$ is a non-trivial proper normal subgroup of G). $\qquad\square$

11.10. THEOREM (Burnside). *Let p and q be prime numbers. If G has order $p^a q^b$, then G is solvable.*

PROOF. If G is abelian, then it is solvable. Suppose now G is non-abelian. Let us assume that the theorem is not true. Let G be a group of minimal order $p^a q^b$ that is not solvable. Since $|G|$ is minimal, G is a non-abelian simple group. By the previous theorem, G has no conjugacy classes of size p^k nor conjugacy classes of size q^l with $k, l \geq 1$. The size of every conjugacy class of G is one or divisible by pq. Note that, since G is a non-abelian simple group, the center of G is trivial. Thus there is only one conjugacy class of size one. By the class equation,

$$|G| = 1 + \sum_{C : |C| > 1} |C| \equiv 1 \bmod pq,$$

where the sum is taken over all conjugacy classes with more than one element, a contradiction. $\qquad\square$

Some generalizations of Burnside's theorem.

11.11. THEOREM (Kegel–Wielandt). *If G is a finite group and there are nilpotent subgroups A and B of G such that $G = AB$, then G is solvable.*

See [**3**, Theorem 2.4.3] for the proof.

Another generalization of Burnside's theorem is based on **word maps**. A word map of a group G is a map

$$G^k \to G, \quad (x_1, \ldots, x_k) \mapsto w(x_1, \ldots, x_k)$$

for some word $w(x_1, \ldots, x_k)$ of the free group F_k of rank k. Some word maps are surjective in certain families of groups. For example, Ore's conjecture is precisely the surjectivity of the word map $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$ in every finite non-abelian simple group.

11.12. THEOREM (Guralnick–Liebeck–O'Brien–Shalev–Tiep). *Let $a, b \geq 0$, p and q be prime numbers and $N = p^a q^b$. The map $(x, y) \mapsto x^N y^N$ is surjective in every finite simple group.*

The proof appears in [**13**].

The theorem implies Burnside's theorem. Let G be a group of order $N = p^a q^b$. Assume that G is not solvable. Fix a composition series of G. There is a non-abelian factor S of order that divides N. Since S is simple non-abelian and $s^N = 1$, it follows that the word map $(x, y) \mapsto x^N y^N$ has trivial image in S, a contradiction to the theorem.

§ 11.2. Feit–Thompson theorem.

11.13. THEOREM (Feit–Thompson). *Groups of odd order are solvable.*

The proof of Feit–Thompson theorem is extremely hard. It occupies a full volume of the **Pacific Journal of Mathematics** [**8**]. A formal verification of the proof (based on the computer software Coq) was announced in [**11**].

Back in the day it was believed that if a certain divisibility conjecture is true, the proof of Feit–Thompson theorem could be simplified.

11.14. CONJECTURE (Feit–Thompson). There are no prime numbers p and q such that $\frac{p^q - 1}{p - 1}$ divides $\frac{q^p - 1}{q - 1}$.

The conjecture remains open. However, now we know that proving the conjecture will not simplify further the proof of Feit–Thompson theorem.

In 2012 Le proved that the conjecture is true for $q = 3$, see [**24**].

In [**32**] Stephens proved that a certain stronger version of the conjecture does not hold, as the integers $\frac{p^q - 1}{p - 1}$ and $\frac{q^p - 1}{q - 1}$ could have common factors. In fact, if $p = 17$ and $q = 3313$, then

$$\gcd\left(\frac{p^q - 1}{p - 1}, \frac{q^p - 1}{q - 1}\right) = 112643.$$

Nowadays we can check this easily in almost every desktop computer:

```
> p := 17;
> q := 3313;
> bool, a := IsCoercible(Integers(), (p^q-1)/(p-1));
> bool, b := IsCoercible(Integers(), (q^p-1)/(q-1));
> Gcd(a,b);
112643
```

No other counterexamples have been found of Stephen's stronger version of the conjecture.

12. Lecture: Week 12

Some topics for final projects

We collect here some topics for final presentations. Some topics can also be used as bachelor or master theses.

Kolchin's theorem. If V is a finite-dimensional complex vector space and G is a subgroup of $\mathbf{GL}(V)$ such that every element g of G is unipotent (i.e., $g-1$ is a nilpotent linear transformation), then there exists a basis of V in which all the element of G are represented by upper triangular matrices with ones on the diagonal. See my notes for Associative Algebra) or [**2**, Chapter 2].

Staircase groups. This topic describes a situation similar to that of Kolchin's theorem (see the course Associative Algebra), but more general. See [**2**, Chapter 5].

Solvable and nilpotent groups. The character table of a finite group detects solvability and nilpotency of groups, see [**2**, Chapter 6].

Ito's theorem. Ito's theorem generalize Frobenius' theorem (Theorem 4.3) and Schur's theorem (Theorem 4.6). The theorem states that if χ is an irreducible character of a finite group G, then $\chi(1)$ divides $(G : A)$ for every normal abelian subgroup A of G. See [**29**, §8.1].

Characters of $\mathbf{GL}_2(q)$ and $\mathbf{SL}_2(q)$. One possible topic is the character table of $\mathbf{GL}_2(q)$, see [**31**, §5.2]. Alternatively, one can present the character table of the group $\mathbf{SL}_2(p)$ following Humphreys's paper [**17**]. The character theory of $\mathbf{SL}_2(q)$ appears in [**31**, §5.2], see [**4**, Chapter 20] for details.

Representations of the symmetric group. See for example [**31**, §10] and [**10**].

Random walks on finite groups. The goal is to construct the character table or the irreducible representations of the symmetric group. The topic has connections with combinatorics and applications to voting and card shuffling. See [**10**, 4] and [**31**, §11].

Fourier analysis on finite groups. See [**31**, §5] for a very elementary approach and some basic applications. Other applications appear in [**33**].

Mackey's irreducibility criterion. It is not at all clear that induction of an irreducible character will produce an irreducible character. In fact, inducing the trivial character of the trivial subgroup to the whole group produces the regular representation, which in general is not irreducible. Mackey found a criterion that describes when an induced character is irreducible. See [**31**, §8.3].

McKay's conjecture. Prove McKay's conjecture 5.1 for all sporadic simple groups. This was first proved by Wilson in [**38**]. Note that for some "small" sporadic simple groups this can be done with the script presented in §5.1. However, for several sporadic simple groups a different approach is needed. One needs to know the structure of normalizers.

Hirsh's theorem. In [**16**] Hirsch found a generalization of Burnside's Theorem 10.7. If G is a finite group and d is the greatest common divisor of all the numbers $p^2 - 1$, where the p's are prime divisors of $|G|$ and r the number of conjugate sets in G. Then

$$|G| \equiv \begin{cases} r \bmod 2d & \text{if } |G| \text{ odd,} \\ r \bmod 3 & \text{if } |G| \text{ even and } \gcd(|G|, 3) = 1. \end{cases}$$

The proof is elementary and does not use character theory. Is it possible to prove Hirsch's theorem using characters?

Clifford's theorem. Clifford's theorem provides a description of the restriction of irreducible representations to normal subgroups. It is a tool that tries to construct representations of the group from representations of a normal subgroup. See for example [**6**, Chapter 7].

Irreducible characters of semidirect products. What can be said about irreducible characters of semidirect products? The case of semidirect products by abelian groups is treated in [**29**, Section 8.2]. Clifford's theorem is also useful here.

Hurwitz's theorem. It states that if there is an identity of the form

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_n^2,$$

where the x_j's and the y_j's are real (or complex) numbers and each z_k is a bilinear function in the x_j's and the y_j's, then $n \in \{1, 2, 4, 8\}$. There are several proofs of this result, and one of them uses representation theory!

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