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# Rings and modules

Notes

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# **Part I**

## **Rings**





# Chapter 1

## Rings and ideals

**Definition 1.1.** A **ring** is a set  $R$  with two binary operations, the addition  $R \times R \rightarrow R$ ,  $(x, y) \mapsto x + y$ , and the multiplication  $R \times R \rightarrow R$ ,  $(x, y) \mapsto xy$ , such that the following properties hold:

- 1)  $(R, +)$  is an abelian group.
- 2)  $(xy)z = x(yz)$  for all  $x, y, z \in R$ .
- 3)  $x(y + z) = xy + xz$  for all  $x, y, z \in R$ .
- 4)  $(x + y)z = xz + yz$  for all  $x, y, z \in R$ .
- 5) There exists  $1_R \in R$  such that  $x1_R = 1_Rx = x$  for all  $x \in R$ .

Our definition of a ring is that of a ring with identity. In general one writes the identity element  $1_R$  as 1 if there is no risk of confusion.

**Definition 1.2.** A ring  $R$  is said to be **commutative** if  $xy = yx$  for all  $x, y \in R$ .

**Example 1.3.**  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are commutative rings.

**Example 1.4.** The set

$$\mathbb{R}[X] = \left\{ \sum_{i=0}^n a_i X^i : n \in \mathbb{N}_0, a_1, \dots, a_n \in \mathbb{R} \right\}$$

of real polynomials in one variable is a commutative ring with the usual operations.

More generally, if  $R$  is a commutative ring, then  $R[X]$  is a commutative ring. This construction allows us to define the polynomial ring  $R[X, Y]$  in two commuting variables  $X$  and  $Y$  and coefficients in  $R$  as  $R[X, Y] = (R[X])[Y]$ . One can also define the ring  $R[X_1, \dots, X_n]$  of real polynomials in  $n$  commuting variables  $X_1, \dots, X_n$  with coefficients in  $R$  as  $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$ .

**Example 1.5.** If  $A$  is an abelian group, then the set  $\text{End}(A)$  of group homomorphisms  $A \rightarrow A$  is a ring with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(g(x)), \quad f, g \in \text{End}(A) \text{ and } x \in A.$$

Let  $R$  be a ring. Some facts:

- 1)  $x0 = 0x = x$  for all  $x \in R$ .
- 2)  $x(-y) = -xy$  for all  $x, y \in R$ .
- 3) If  $1 = 0$ , then  $|R| = 1$ .

**Example 1.6.** The real vector space  $H(\mathbb{R}) = \{a1 + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$  with basis  $\{1, i, j, k\}$  is a ring with the multiplication induced by the formulas

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

As an example, let us perform a calculation in  $H(\mathbb{R})$ :

$$(1 + i + j)(i + k) = i + k - 1 + ik + ji + jk = i + k - 1 - j - k + i = -1 + 2i - j,$$

as  $ij = i(ij) = -j$ . This is the ring of real **quaternions**.

**Example 1.7.** Let  $n \geq 2$ . The abelian group  $\mathbb{Z}/n = \{0, 1, \dots, n\}$  of integers modulo  $n$  is a ring with the usual multiplication modulo  $n$ .

**Example 1.8.** Let  $n \geq 1$ . The set  $M_n(\mathbb{R})$  of real  $n \times n$  matrices is a ring with the usual matrix operations. Recall that if  $a = (a_{ij})$  and  $b = (b_{ij})$ , the multiplication  $ab$  is given by

$$(ab)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

Similarly, for any ring  $R$  one defines the ring  $M_n(R)$  of  $n \times n$  matrices with coefficients in  $R$ .

**Definition 1.9.** Let  $R$  be a ring. A **subring**  $S$  of  $R$  is a subset  $S$  such that  $(S, +)$  is a subgroup of  $(R, +)$  such that  $1 \in S$  and if  $x, y \in S$ , then  $xy \in S$ .

Clearly,  $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$  is a chain of subrings.

**Example 1.10.**  $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$  is a subring of  $\mathbb{C}$ . This is known as the ring of **Gauss integers**.

**Example 1.11.**  $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$  is a subring of  $\mathbb{R}$ .

**Example 1.12.** If  $R$  is a ring, then the **center**  $Z(R) = \{x \in R : xy = yx \text{ for all } y \in R\}$  is a subring of  $R$ .

If  $S$  is a subring of a ring  $R$ , then the zero element of  $S$  is the zero element of  $R$ , i.e.  $0_R = 0_S$ . Moreover, the additive inverse of an element  $s \in S$  is the additive inverse of  $s$  as an element of  $R$ .

**Exercise 1.13.**

- 1) If  $S$  and  $T$  are subrings of  $R$ , then  $S \cap T$  is a subring of  $R$ .
- 2) If  $R_1 \subseteq R_2 \subseteq \dots$  is a sequence of subrings of  $R$ , then  $\cup_{i \geq 1} R_i$  is a subring of  $R$ .

**Definition 1.14.** Let  $R$  be a ring. An element  $x \in R$  is a **unit** if there exists  $y \in R$  such that  $xy = yx = 1$ .

The set  $\mathcal{U}(R)$  of units of a ring  $R$  form a group with the multiplication. For example,  $\mathcal{U}(\mathbb{Z}/8) = \{1, 3, 5, 7\}$ .

**Definition 1.15.** A **division ring** is a ring  $R$  such that  $\mathcal{U}(R) = R \setminus \{0\}$ .

The ring  $H(\mathbb{R})$  real quaternions is a non-commutative division ring. Find the inverse of an arbitrary element  $a1 + bi + cj + dk \in H(\mathbb{R})$ .

**Definition 1.16.** A **field** is a commutative division ring with  $1 \neq 0$ .

Clearly,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are fields. If  $p$  is a prime number, then  $\mathbb{Z}/p$  is a field.

**Exercise 1.17.** Prove that  $\mathbb{Q}[\sqrt{2}]$  is a field. Find the multiplicative inverse of a non-zero element of the form  $x + y\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ .

More challenging: Prove that

$$\mathbb{Q}[\sqrt[3]{2}] = \{x + y\sqrt[3]{2} + z\sqrt[3]{4} : x, y, z \in \mathbb{Q}\}$$

is a field. What is the inverse of a non-zero element of the form  $x + y\sqrt[3]{2} + z\sqrt[3]{4}$ ?

**Definition 1.18.** Let  $R$  be a ring. A **left ideal** of  $R$  is a subset  $I$  such that  $(I, +)$  is a subgroup of  $(R, +)$  and such that  $RI \subseteq I$ , i.e.  $ry \in I$  for all  $r \in R$  and  $y \in I$ .

Similarly one defines right ideals, one needs to replace the condition  $RI \subseteq I$  by the inclusion  $IR \subseteq I$ .

**Example 1.19.** Let  $R = M_2(\mathbb{R})$ . Then

$$I = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

is a right ideal  $R$  that is not a left ideal.

Can you find an example of a right ideal that is not a left ideal?

**Definition 1.20.** Let  $R$  be a ring. An ideal of  $R$  is a subset that is both a left and a right ideal of  $R$ .

If  $R$  is a ring, then  $\{0\}$  and  $R$  are both ideals of  $R$ .

**Exercise 1.21.** Let  $R$  be a ring.

- 1) If  $\{I_\alpha : \alpha\}$  is a collection of ideals of  $R$ , then  $\cap_\alpha I_\alpha$  is an ideal of  $R$ .
- 2) If  $I_1 \subseteq I_2 \subseteq \dots$  is a sequence of ideals of  $R$ , then  $\cup_{i \geq 1} I_i$  is an ideal of  $R$ .

**Example 1.22.** Let  $R = \mathbb{R}[X]$ . If  $f(X) \in R$ , then the set

$$(f(X)) = \{f(X)g(X) : g(X) \in R\}$$

of multiples of  $f(X)$  is an ideal of  $R$ . One can prove that this is the smallest ideal of  $R$  containing  $f(X)$ .

If  $R$  is a ring and  $X$  is a subset of  $R$ , one defines the ideal generated by  $X$  as the smallest ideal of  $R$  containing  $X$ , that is

$$(X) = \bigcap \{I : I \text{ ideal of } R \text{ such that } X \subseteq I\}.$$

One proves that

$$(X) = \left\{ \sum_{i=1}^m r_i x_i s_i : m \in \mathbb{N}_0, r_1, \dots, r_m, s_1, \dots, s_m \in R \right\},$$

where by convention the empty sum is equal to zero. If  $X = \{x_1, \dots, x_n\}$  is a finite set, then we write  $(X) = (x_1, \dots, x_n)$ .

xca:ideals\_Z

**Exercise 1.23.** Prove that every ideal of  $\mathbb{Z}$  is of the form  $n\mathbb{Z}$  for some  $n \geq 0$ .

**Exercise 1.24.** Let  $n \geq 2$ . Find the ideals of  $\mathbb{Z}/n$ .

**Exercise 1.25.** Find the ideals of  $\mathbb{R}$ .

A similar exercise is to find the ideals of any division ring.

**Definition 1.26.** Let  $R$  be a ring and  $I$  be an ideal of  $R$ . Then  $I$  is **principal** if  $I = (x)$  for some  $x \in R$ .

The division algorithm shows that every ideal of  $\mathbb{Z}$  is principal, see Exercise 1.23.

**Exercise 1.27.** Prove that every ideal of  $\mathbb{R}[X]$  is principal.

If  $K$  is a field, there is a division algorithm in the polynomial ring  $K[X]$ . Then one proves that every ideal of  $K[X]$  is principal.

**Exercise 1.28.** Let  $R$  be a ring and  $x \in R$ . Prove that  $x \in \mathcal{U}(R)$  if and only if  $(x) = R$ .

A division ring (and, in particular, a field) has only two ideals.

**Definition 1.29.** Let  $R$  and  $S$  be rings. A map  $f: R \rightarrow S$  is a **ring homomorphism** if  $f(1) = 1$ ,  $f(x+y) = f(x) + f(y)$  and  $f(xy) = f(x)f(y)$  for all  $x, y \in R$ .

Our definition of a ring is that of a ring with identity. This means that the identity element 1 of a ring  $R$  is part of the structure. For that reason, in the definition of a ring homomorphism  $f$  one needs  $f(1) = 1$ .

**Example 1.30.** The map  $f: \mathbb{Z}/6 \rightarrow \mathbb{Z}/6$ ,  $x \mapsto 3x$ , is not a ring homomorphism because  $f(1) = 3$ .

If  $R$  is a ring, then the identity map  $\text{id}: R \rightarrow R, x \mapsto x$ , is a ring homomorphism.

**Example 1.31.** The inclusions  $\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$  are ring homomorphisms.

More generally, if  $S$  is a subring of a ring  $R$ , then the inclusion map  $S \hookrightarrow R$  is a ring homomorphism.

**Example 1.32.** Let  $R$  be a ring. The map  $\mathbb{Z} \rightarrow R, k \mapsto k1$ , is a ring homomorphism.

**Example 1.33.** Let  $x_0 \in \mathbb{R}$ . The evaluation map  $\mathbb{R}[X] \rightarrow \mathbb{R}, f \mapsto f(x_0)$ , is a ring homomorphism.

The **kernel** of a ring homomorphism  $f: R \rightarrow S$  is the subset

$$\ker f = \{x \in R : f(x) = 0\}.$$

One proves that the kernel of  $f$  is an ideal of  $R$ . Moreover,  $\ker f = \{0\}$  if and only if  $f$  is injective. The image

$$f(R) = \{f(x) : x \in R\}$$

is a subring of  $S$ . In general,  $f(R)$  is not an ideal of  $S$ .

**Example 1.34.** The map  $\mathbb{C} \rightarrow M_2(\mathbb{R}), a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , is an injective ring homomorphism.

**Example 1.35.** The map  $\mathbb{Z}[i] \rightarrow \mathbb{Z}/5, a + bi \mapsto a + 2b \pmod{5}$ , is a ring homomorphism with  $\ker f = \{a + bi : a + 2b \equiv 0 \pmod{5}\}$ .

**Exercise 1.36.** There is no ring homomorphism  $\mathbb{Z}/6 \rightarrow \mathbb{Z}/15$ . Why?

**Exercise 1.37.** If  $f: \mathbb{R}[X] \rightarrow \mathbb{R}$  is a ring homomorphism such that the restriction  $f|_{\mathbb{R}}$  of  $f$  onto  $\mathbb{R}$  is the identity, then there exists  $x_0 \in \mathbb{R}$  such that  $f$  is the evaluation map at  $x_0$ .

We now define ring quotients. Let  $R$  be a ring and  $I$  be an ideal of  $R$ . Then  $R/I$  is an abelian group with

$$(x + I) + (y + I) = (x + y) + I$$

and the **canonical map**  $R \rightarrow R/I, x \mapsto x + I$ , is a surjective group homomorphism. Recall that  $R/I$  is the set of cosets  $x + I$ , where  $x + I = y + I$  if and only if  $x - y \in I$ . Note that here we only used that  $I$  is an additive subgroup of  $R$ . We need an ideal to put a ring structure on the set  $R/I$  of cosets modulo  $I$ . As in the case of integers, we use the following notation. For  $x, y \in R$  we write

$$x \equiv y \pmod{I} \iff x - y \in I.$$

How can we put a ring structure on  $R/I$ ? It makes sense to define a multiplication on  $R/I$  in such a way that the canonical map  $R \rightarrow R/I$  is a surjective ring homomorphism. For that purpose, we define

$$(x+I)(y+I) = (xy) + I.$$

Since  $I$  is an ideal of  $R$ , this multiplication is well-defined. In fact, let  $x+I = x_1+I$  and  $y+I = y_1+I$ . We want to show that  $xy+I = x_1y_1+I$ . Since  $x-x_1 \in I$ ,

$$xy - x_1y = (x-x_1)y \in I$$

because  $I$  is a right ideal. Similarly, since  $y-y_1 \in I$ , it follows that

$$x_1y - x_1y_1 = x_1(y-y_1) \in I,$$

as  $I$  is a left ideal. Thus

$$xy - x_1y_1 = xy - x_1y + x_1y - x_1y_1 = (x-x_1)y + x_1(y-y_1) \in I.$$

**Theorem 1.38.** *Let  $R$  be a ring and  $I$  be an ideal of  $R$ . Then  $R/I$  with*

$$(x+I) + (y+I) = (x+y) + I, \quad (x+I)(y+I) = (xy) + I,$$

*is a ring and the canonical map  $R \rightarrow R/I$ ,  $x \mapsto x+I$ , is a surjective ring homomorphism with kernel  $I$ .*

We have already seen that the multiplication is well-defined. The rest of the proof is left as an exercise.

**Example 1.39.** Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$$

A direct calculation shows that the map  $R \rightarrow \mathbb{Q}$ ,  $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto a$ , is a surjective ring homomorphism with  $\ker f = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Q} \right\}$ . Thus  $R/\ker f \simeq \mathbb{Q}$ .

**Example 1.40.** Let  $R = (\mathbb{Z}/3)[X]$  and  $I = (2X^2 + X + 2)$  be the ideal of  $R$  generated by the polynomial  $2X^2 + X + 2$ . If  $f(X) \in R$ , the division algorithm allows us to write

$$f(X) = (2X^2 + X + 2)q(X) + r(X),$$

for some  $q(X), r(X) \in R$ , where either  $r(X) = 0$  or  $\deg r(X) < 2$ . This means that  $r(X) = aX + b$  for some  $a, b \in R$ . Note that  $f(X) \equiv aX + b \pmod{2X^2 + X + 2}$  for some  $a, b \in \mathbb{Z}/3$ , so the quotient ring  $R/I$  has nine elements.

An **isomorphism** between the rings  $R$  and  $S$  is a bijective ring homomorphism  $R \rightarrow S$ . Notation:  $R \simeq S$  if and only if there exists an isomorphism  $R \rightarrow S$ . As it happens in the case of groups, to understand quotient rings one has the first isomorphism theorem.

**Theorem 1.41 (first isomorphism theorem).** *If  $f: R \rightarrow S$  is a ring homomorphism, then  $R/\ker f \simeq f(R)$ .*

This is somewhat similar to the result one knows from group theory. One needs to show that the map  $R/I \rightarrow f(R)$ ,  $x + I \mapsto f(x)$ , is a well-defined bijective ring homomorphism.

**Example 1.42.** The evaluation map  $\mathbb{R}[X] \rightarrow \mathbb{C}$ ,  $f(X) \mapsto f(i)$ , is a surjective ring homomorphism with kernel  $(X^2 + 1)$ . Thus

$$\mathbb{R}[X]/(X^2 + 1) \simeq \mathbb{C}$$

by the first isomorphism theorem. In practice, this is how it works. Let  $f(X) \in \mathbb{R}[X]$ . The division algorithm on  $\mathbb{R}[X]$  allows us to write

$$f(X) = (X^2 + 1)q(X) + r(X)$$

for some  $q(X), r(X) \in \mathbb{R}[X]$ , where  $r(X) = 0$  or  $\deg r(X) < 2$ . Thus  $r(X) = aX + b$  for some  $a, b \in \mathbb{R}$ . This implies that

$$f(X) \equiv aX + b \pmod{(X^2 + 1)}.$$

It is quite easy to describe the ring operation of  $\mathbb{R}[X]/(X^2 + 1)$ . Clearly

$$(aX + b) + (cX + d) \equiv (a + c)X + (b + d) \pmod{(X^2 + 1)},$$

Since  $X^2 \equiv -1 \pmod{(X^2 + 1)}$ ,

$$(aX + b)(cX + d) \equiv X(ad + bc) + (bd - ac),$$

which reminds us the usual multiplication rule of the field of complex numbers.

**Exercise 1.43.** Prove that  $\mathbb{Z}[\sqrt{-5}] \simeq \mathbb{Z}[X]/(X^2 - 5)$ .

Similarly, if  $N$  is a square-free integer, then  $\mathbb{Z}[\sqrt{N}] \simeq \mathbb{Z}[X]/(X^2 - N)$ .

**Exercise 1.44.** Prove the following isomorphisms:

- 1)  $\mathbb{Z}[X]/(7) \simeq (\mathbb{Z}/7)[X]$ .
- 2)  $\mathbb{Q}[\sqrt{2}] \simeq \mathbb{Q}[X]/(X^2 - 2)$ .
- 3)  $\mathbb{R}[X]/(X^2 - 1) \simeq \mathbb{R} \times \mathbb{R}$ .
- 4)  $\mathbb{Q}[X]/(X - 2) \simeq \mathbb{Q}$ .
- 5)  $\mathbb{R}[X, Y]/(X) \simeq \mathbb{R}[Y]$ .

**Exercise 1.45.** Are the rings  $\mathbb{Q}[\sqrt{2}]$  and  $\mathbb{Q}[\sqrt{3}]$  isomorphic?

**Exercise 1.46.** Let  $R$  be the ring of continuous maps  $[0, 2] \rightarrow \mathbb{R}$ . Prove that the set  $I = \{f \in R : f(1) = 0\}$  is an ideal of  $R$  and that  $R/I \simeq \mathbb{R}$ .

**Exercise 1.47.** Let  $n \geq 1$ . Let  $R$  be a ring and  $I$  be an ideal of  $R$ . Prove that  $M_n(I)$  is an ideal of  $M_n(R)$  and that  $M_n(R)/M_n(I) \simeq M_n(R/I)$ .

**Exercise 1.48.** Let  $R = \mathbb{Z}[\sqrt{10}]$  and  $I = (2, \sqrt{10})$ . Prove that  $R/I \simeq \mathbb{Z}/2$ .

$\text{ca: } \mathbb{Z}[\sqrt{10}]/(2, \sqrt{10})$

xca:  $\mathbb{Z}[i]/(1+3i)$

**Exercise 1.49.** Prove that  $\mathbb{Z}[i]/(1+3i) \simeq \mathbb{Z}/10$ .

**Exercise 1.50.** Prove that there is no ideal  $I$  of  $\mathbb{Z}[i]$  such that  $\mathbb{Z}[i]/I \simeq \mathbb{Z}/15$ .

**Exercise 1.51.** Let  $R = (\mathbb{Z}/2)[X]/(X^2 + X + 1)$ .

- 1) How many elements does  $R$  have?
- 2) Can you recognize the additive group of  $R$ ?
- 3) Prove that  $R$  is a field.

As it happens in group theory, one has the following important result.

**Theorem 1.52 (correspondence theorem).** *Let  $f: R \rightarrow S$  be a surjective ring homomorphism. There exists a bijective correspondence between the set of ideals of  $R$  containing  $\ker f$  and the set of ideals of  $S$ .*

*Sketch of the proof.* Let  $I$  be an ideal of  $R$  containing  $\ker f$  and let  $J$  be an ideal of  $S$ . We need to prove the following facts:

- 1)  $f(I)$  is an ideal of  $S$ .
- 2)  $f^{-1}(J)$  is an ideal of  $R$  containing  $\ker f$ .
- 3)  $f(f^{-1}(J)) = J$  and  $f^{-1}(f(I)) = I$ .
- 4) If  $f(I) = J$ , then  $R/I \simeq S/J$ .

We only prove the fourth statement, the others are left as exercises. Note that the third claim implies that  $f(I) = J$  if and only if  $I = f^{-1}(J)$ . Let  $\pi: S \rightarrow S/J$  be the canonical map. The composition  $g = \pi \circ f: R \rightarrow S/J$  is a ring homomorphism and

$$\ker g = \{x \in R : g(x) = 0\} = \{x \in R : f(x) \in J\} = \{x \in R : x \in f^{-1}(J) = I\} = I.$$

Since  $g(R) = S/J$ , the first isomorphism theorem implies that  $R/I \simeq S/J$ . □



## Chapter 2

### Chinese remainder theorem

Note that if  $R$  is a commutative ring and  $I$  and  $J$  are ideals of  $R$ , then

$$I + J = \{u + v : u \in I, v \in J\}$$

is an ideal of  $R$ .

**Definition 2.1.** Let  $R$  be a commutative ring. The ideals  $I$  and  $J$  of  $R$  are said to be **coprime** if  $R = I + J$ .

The terminology is motivated by the following example. If  $I$  and  $J$  are ideals of  $\mathbb{Z}$ , then  $I = (a)$  and  $J = (b)$  for some  $a, b \in \mathbb{Z}$ . Then

$a$  and  $b$  are coprime  $\iff 1 = ra + sb$  for some  $r, s \in \mathbb{Z} \iff I$  and  $J$  are coprime.

If  $I$  and  $J$  are ideals of  $R$ , then

$$IJ = \left\{ \sum_{i=1}^m u_i v_i : m \in \mathbb{N}_0, u_1, \dots, u_m \in I, v_1, \dots, v_m \in J \right\}$$

is an ideal of  $R$ . Note that  $IJ \subseteq I \cap J$ . The equality does not hold in general. Take for example  $R = \mathbb{Z}$  and  $I = J = (2)$ . Then  $IJ = (4) \subsetneq (2) = I \cap J$ .

**Proposition 2.2.** Let  $R$  be a commutative ring. If  $I$  and  $J$  are coprime ideals, then  $IJ = I \cap J$ .

*Proof.* Let  $x \in I \cap J$ . Since  $I$  and  $J$  are coprime,  $1 = u + v$  for some  $u \in I$  and  $v \in J$ ,  $x = x1 = x(u + v) = xu + xv \in IJ$ .  $\square$

**Theorem 2.3 (chinese remainder theorem).** Let  $R$  be a commutative ring and  $I$  and  $J$  be coprime ideals. If  $u, v \in R$ , then there exists  $x \in R$  such that

$$\begin{cases} x \equiv u \pmod{I}, \\ x \equiv v \pmod{J}. \end{cases}$$

*Proof.* Since the ideals  $I$  and  $J$  are coprime,  $1 = a + b$  for some  $a \in I$  and  $b \in J$ . Let  $x = av + bu$ . Then

$$x - u = av + (b - 1)u = av - au = a(v - u) \in I,$$

that is  $x \equiv u \pmod{I}$ . Similarly,  $x - v \in J$  and  $x \equiv v \pmod{J}$ .  $\square$

**Corollary 2.4.** *Let  $R$  be a commutative ring. If  $I$  and  $J$  are coprime ideals of  $R$ , then  $R/(I \cap J) \simeq R/I \times R/J$ .*

*Proof.* Let  $\pi_I: R \rightarrow R/I$  and  $\pi_J: R \rightarrow R/J$  be the canonical maps. A straightforward calculation shows that the map  $\varphi: R \rightarrow R/I \times R/J, x \mapsto (\pi_I(x), \pi_J(x))$ , is an injective ring homomorphism with  $\ker \varphi = I \cap J$ . The chinese remainder theorem implies that  $\varphi$  is surjective. If  $(u + I, v + J) \in R/I \times R/J$ , then there exists  $x \in R$  such that  $x - u \in I$  and  $x - v \in J$ . This translates into the surjectivity of  $\varphi$ . Now  $R/(I \cap J) \simeq R/I \times R/J$  by the first isomorphism theorem.  $\square$

Let  $R$  be a commutative ring and  $I_1, \dots, I_n$  be ideals of  $R$ . Then

$$I_1 \cdots I_n = \left\{ \sum_{i=1}^m u_{i_1} \cdots u_{i_n} : m \in \mathbb{N}_0, u_{i_1}, \dots, u_{i_n} \in I_{i_j} \right\}$$

is an ideal of  $R$ . If  $I_1$  and  $I_j$  are coprime for all  $j \in \{2, \dots, n\}$ , then  $I_1$  and  $I_2 \cdots I_n$  are coprime. If  $I_i$  and  $I_j$  are coprime whenever  $i \neq j$ , then

$$R/(I_1 \cap \cdots \cap I_n) \simeq R/I_1 \times \cdots \times R/I_n.$$

**Exercise 2.5 (Lagrange's interpolation theorem).** The chinese remainder theorem proves the following well-known result. Let  $x_1, \dots, x_k \in \mathbb{R}$  be such that  $x_i \neq x_j$  whenever  $i \neq j$  and  $y_1, \dots, y_k \in \mathbb{R}$ . Then there exists  $f(X) \in \mathbb{R}[X]$  such that

$$\begin{cases} f(X) \equiv y_1 \pmod{(X - x_1)}, \\ f(X) \equiv y_2 \pmod{(X - x_2)}, \\ \vdots \\ f(X) \equiv y_k \pmod{(X - x_k)}. \end{cases}$$

The solution  $f(X)$  is unique modulo  $(X - x_1)(X - x_2) \cdots (X - x_n)$ .

xca:gather\_people

**Exercise 2.6.** Let us gather people in the following way. When I count by three, there are two persons left. When I count by four, there is one person left over and when I count by five there is one missing. How many persons are there?

xca:no\_solution

**Exercise 2.7.** Prove that

$$\begin{cases} x \equiv 29 \pmod{52}, \\ x \equiv 19 \pmod{72}. \end{cases}$$

does not have solution.

`xca:consecutive`

**Exercise 2.8.** Find three consecutive integers such that the first one is divisible by a square, the second one is divisible by a cube and the third one is divisible by a fourth power.

`xca:perfect_square`

**Exercise 2.9.** Prove that for each  $n \in \mathbb{N}$  there are  $n$  consecutive integers such that each integer is divisible by a perfect square  $\neq 1$ .



## Chapter 3

### Noetherian rings

In this chapter we will work with commutative rings.

**Definition 3.1.** A ring  $R$  is said to be **noetherian** if every (increasing) sequence  $I_1 \subseteq I_2 \subseteq \cdots$  of ideals of  $R$  stabilizes, that is  $I_n = I_m$  for some  $m \in \mathbb{N}$  and all  $n \geq m$ .

The ring  $\mathbb{Z}$  of integers is noetherian.

**Example 3.2.** Let  $R = \{f: [0, 1] \rightarrow \mathbb{R}\}$  with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x), \quad f, g \in R, x \in [0, 1].$$

For  $n \in \mathbb{N}$  let  $I_n = \{f \in R : f|_{[0, 1/n]} = 0\}$ . Then each  $I_n$  is an ideal of  $R$  and the sequence  $I_1 \subsetneq I_2 \subsetneq \cdots$  does not stabilize. Thus  $R$  is not noetherian.

**Definition 3.3.** Let  $R$  be a ring. An ideal  $I$  of  $R$  is said to be **finitely generated** if  $I = (X)$  for some finite subset  $X$  of  $R$ .

The zero ideal is always finitely generated.

**Proposition 3.4.** Let  $R$  be a ring. Then  $R$  is noetherian if and only if every ideal of  $R$  is finitely generated.

*Proof.* Assume first that  $R$  is noetherian. Let  $I$  be an ideal of  $R$  that is not finitely generated. Thus  $I \neq \{0\}$ . Let  $x_1 \in I \setminus \{0\}$  and let  $I_1 = (x_1)$ . Since  $I$  is not finitely generated,  $I \neq I_1$  and hence  $\{0\} \subsetneq I_1 \subsetneq I$ . Once I have the ideals  $I_1, \dots, I_{k-1}$ , let  $x_k \in I \setminus I_{k-1}$  (such an element exists because  $I_{k-1}$  is finitely generated and  $I$  is not) and  $I_k = (I_{k-1}, x_k)$ . The sequence  $\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$  does not stabilize.

Assume now that every ideal of  $R$  is finitely generated and let  $I_1 \subseteq I_2 \subseteq \cdots$  be a sequence of ideals of  $R$ . Then  $I = \bigcup_{i \geq 1} I_i$  is an ideal of  $R$ , so it is finitely generated, say  $I = (x_1, \dots, x_n)$ . We may assume that  $x_j \in I_{j_j}$  for all  $j$ . Let  $N = \max\{j_1, \dots, j_n\}$  and  $n \geq N$ . Then  $I_N \subseteq I \subseteq I_N$  and therefore the sequence stabilizes.  $\square$

**Exercise 3.5.** Let  $R = \mathbb{C}[X_1, X_2, \dots]$  be the ring of polynomial in an infinite number of commuting variables. Prove that the ideal  $I = (X_1, X_2, \dots)$  of polynomials with zero constant term is not finitely generated.

The correspondence theorem and the previous proposition allow us to prove easily the following result.

**Proposition 3.6.** *Let  $I$  be an ideal of  $R$ . If  $R$  is noetherian, then  $R/I$  is noetherian.*

*Proof.* Let  $\pi: R \rightarrow R/I$  be the canonical surjection and let  $J$  be an ideal of  $R/I$ . Then  $\pi^{-1}(J)$  is an ideal of  $R$  containing  $I$ . Since  $R$  is noetherian,  $\pi^{-1}(J)$  is finitely generated, say  $\pi^{-1}(J) = (x_1, \dots, x_k)$  for  $x_1, \dots, x_k \in R$ . Thus

$$J = \pi(\pi^{-1}(J)) = (\pi(x_1), \dots, \pi(x_k))$$

and hence  $J$  is finitely generated.  $\square$

Since  $\mathbb{Z}$  is noetherian,  $\mathbb{Z}/n$  is noetherian for all  $n \geq 2$ .

**Exercise 3.7.** Prove that  $\mathbb{R}[X]$  is noetherian.

**Theorem 3.8 (Hilbert).** *Let  $R$  be a commutative ring. If  $R$  is noetherian ring, then  $R[X]$  is noetherian.*

*Proof.* We need to show that every ideal of  $R[X]$  is finitely generated. Assume that there is an ideal  $I$  of  $R[X]$  that is not finitely generated. In particular,  $I \neq \{0\}$ . Let  $f_1(X) \in I \setminus \{0\}$  be of minimal degree. For  $i > 1$  let  $f_i(X) \in I$  be of minimal degree such that  $f_i(X) \notin (f_1(X), \dots, f_{i-1}(X))$  (note that such an  $f_i(X)$  exists because  $I$  is not finitely generated). For each  $i$  let  $a_i$  be the leading coefficient of  $f_i(X)$ , that is

$$f_i(X) = a_i X^{n_i} + \dots,$$

where the dots denote lowest degree terms. Note that  $a_i \neq 0$ . Let  $J = (a_1, a_2, \dots)$ . Since  $R$  is noetherian, the sequence

$$(a_1) \subseteq (a_1, a_2) \subseteq \dots (a_1, a_2, \dots, a_k) \subseteq \dots$$

stabilizes, so  $J$  is finitely generated, say  $J = (a_1, \dots, a_m)$  for some  $m \in \mathbb{N}$ . There exist  $u_1, \dots, u_m \in R$  such that

$$a_{m+1} = \sum_{i=1}^m u_i a_i.$$

Let

$$g(X) = \sum_{i=1}^m u_i f_i(X) X^{n_{m+1}-n_i} \in (f_1(X), \dots, f_m(X)).$$

The leading coefficient of  $g(X)$  is  $\sum_{i=1}^m u_i a_i = a_{m+1}$  and, moreover, the degree of  $g(X)$  is  $n_{m+1}$ . Thus  $\deg(g(X)) < n_{m+1}$ . Since  $f_{m+1}(X) \notin (f_1(X), \dots, f_m(X))$ ,

$$g(X) - f_{m+1}(X) \notin (f_1(X), \dots, f_m(X)),$$

a contradiction to the minimality of the degree of  $f_{m+1}$ .  $\square$

Since  $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$ , by induction one proves that if  $R$  is a commutative noetherian ring, then  $R[X_1, \dots, X_n]$  is noetherian.

**Example 3.9.** Since  $\mathbb{Z}$  is noetherian, so is  $\mathbb{Z}[X]$  by Hilbert's theorem. Now  $\mathbb{Z}[\sqrt{N}]$  is noetherian, as  $\mathbb{Z}[\sqrt{N}] \simeq \mathbb{Z}[X]/(X^2 - N)$  and quotients of noetherian rings are noetherian.

**Example 3.10.** The ring  $\mathbb{Z}[X, X^{-1}]$  is noetherian, as  $\mathbb{Z}[X, X^{-1}] \simeq \mathbb{Z}[X, Y]/(XY - 1)$ .

**Exercise 3.11.** Prove that  $R[[X]]$  is noetherian if  $R$  is noetherian.

**Exercise 3.12.** Let  $f: R \rightarrow R$  be surjective ring homomorphism. Prove that  $f$  is an isomorphism if  $R$  is noetherian.





## Chapter 4

### Factorization

**Definition 4.1.** A commutative ring  $R$  is said to be an **integral domain** if  $xy = 0$  implies  $x = 0$  or  $y = 0$ .

The rings  $\mathbb{Z}$  and  $\mathbb{Z}[i]$  are both integral domains. More generally, if  $N$  is a square-free integer, then the ring  $\mathbb{Z}[\sqrt{N}]$  is an integral domain. The ring  $\mathbb{Z}/4$  of integers modulo 4 is not an integral domain.

**Definition 4.2.** Let  $R$  be an integral domain and  $x, y \in R$ . Then  $x$  **divides**  $y$  if  $y = xz$  for some  $z \in R$ . Notation:  $x \mid y$  if and only if  $x$  divides  $y$ . If  $x$  does not divide  $y$  one writes  $x \nmid y$ .

Note that  $x \mid y$  if and only if  $(y) \subseteq (x)$ .

**Definition 4.3.** Let  $R$  be an integral domain and  $x, y \in R$ . Then  $x$  and  $y$  are **associate** in  $R$  if  $x = yu$  for some  $u \in \mathcal{U}(R)$ .

Note that  $x$  and  $y$  are associate if and only if  $(x) = (y)$ .

**Example 4.4.** The integers 2 and  $-2$  are associate in  $\mathbb{Z}$ .

**Example 4.5.** Let  $R = \mathbb{Z}[i]$ .

- 1) Let  $d \in \mathbb{Z}$  and  $a + ib \in R$ . Then  $d \mid a + ib$  in  $R$  if and only if  $d \mid a$  and  $d \mid b$  in  $\mathbb{Z}$ .
- 2) 2 and  $-2i$  are associate in  $R$ .

**Example 4.6.** Let  $R = \mathbb{R}[X]$  and  $f(X) \in R$ . Then  $f(X)$  and  $\lambda f(X)$  are associate in  $R$  for all  $\lambda \in \mathbb{R}^\times$ .

**Definition 4.7.** Let  $R$  be an integral domain and  $x \in R \setminus \{0\}$ . Then  $x$  is **irreducible** if and only if  $x \notin \mathcal{U}(R)$  and  $x = ab$  with  $a, b \in R$  implies that  $a \in \mathcal{U}(R)$  or  $b \in \mathcal{U}(R)$ .

Note that  $x$  is irreducible if and only if  $(x) \neq R$  and there is no principal ideal  $(y)$  such that  $(x) \subsetneq (y) \subsetneq R$ .

**Example 4.8.** Let  $R = \mathbb{R}[X]$  and  $f(X) \in R \setminus \{0\}$ . Then  $f(X)$  is irreducible if  $\lambda \in \mathbb{R}^\times$  or  $\lambda f(X)$  for  $\lambda \in \mathbb{R}^\times$  are the only divisors of  $f(X)$ .

The irreducibles of  $\mathbb{Z}$  are the prime numbers.

**Definition 4.9.** Let  $R$  be an integral domain and  $p \in R \setminus \{0\}$ . Then  $p$  is **prime** if  $p \notin \mathcal{U}(R)$  and  $yz \in (p)$  implies that  $y \in (p)$  or  $z \in (p)$ .

In  $\mathbb{Z}$  primes and irreducible coincide. This does not happen in full generality. However, the following result holds.

**Proposition 4.10.** Let  $R$  be an integral domain and  $x \in R$ . If  $x$  is prime, then  $x$  is irreducible.

*Proof.* Let  $p$  be a prime. Then  $p \neq 0$  and  $p \notin \mathcal{U}(R)$ . Let  $x$  be such that  $x \mid p$ . Then  $p = xy$  for some  $y \in R$ . This means  $xy \in (p)$ , so  $x \in (p)$  or  $y \in (p)$  because  $p$  is prime. If  $x \in (p)$ , then  $x = pz$  for some  $z \in R$  and hence

$$p = xy = (pz)y.$$

Since  $p - pzy = p(1 - zy)$  and  $R$  is an integral domain, it follows that  $1 - zy = 0$ . Thus  $y \in \mathcal{U}(R)$ . Similarly, if  $y \in (p)$ , then  $x \in \mathcal{U}(R)$ .  $\square$

To show that there rings where some irreducibles are not prime, we need the following lemma.

**Lemma 4.11.** Let  $N \in \mathbb{Z}$  be a square-free integer and  $R = \mathbb{Z}[\sqrt{N}]$ . The map

$$N: R \rightarrow \mathbb{N}, \quad a + b\sqrt{N} \mapsto |a^2 - Nb^2|,$$

satisfies the following properties:

- 1)  $N(\alpha) = 0$  if and only if  $\alpha = 0$ .
- 2)  $N(\alpha\beta) = N(\alpha)N(\beta)$  for all  $\alpha, \beta \in R$ .
- 3)  $\alpha \in \mathcal{U}(\mathbb{Z}[\sqrt{N}])$  if and only if  $N(\alpha) = 1$ .
- 4) If  $N(\alpha)$  is prime in  $\mathbb{Z}$ , then  $\alpha$  is irreducible in  $R$ .

*Proof.* The first three items are left as an exercises. Let us prove 4). If  $\alpha = \beta\gamma$  for some  $\beta, \gamma \in R$ , then  $N(\alpha) = N(\beta)N(\gamma)$ . Since  $N(\alpha)$  is a prime number, it follows that  $N(\alpha) = 1$  or  $N(\gamma) = 1$ . Thus  $\beta \in \mathcal{U}(R)$  or  $\gamma \in \mathcal{U}(R)$ .  $\square$

**Example 4.12.** Let  $R = \mathbb{Z}[i]$ .

- 1)  $\mathcal{U}(R) = \{-1, 1, i, -i\}$ .
- 2) 3 is irreducible in  $R$ . In fact, if  $3 = \alpha\beta$ , then  $9 = N(\alpha)N(\beta)$ . This implies that  $N(\alpha) \in \{1, 3, 9\}$ . Write  $\alpha = a + bi$  for  $a, b \in \mathbb{Z}$ . If  $N(\alpha) = 1$ , then  $\alpha \in \mathcal{U}(R)$  by the lemma. If  $N(\alpha) = 9$ , then  $N(\beta) = 1$  and hence  $\beta \in \mathcal{U}(R)$  by the lemma. Finally, if  $N(\alpha) = 3$ , then  $a^2 + b^2 = 3$ , which is a contradiction since  $a, b \in \mathbb{Z}$ .
- 3) 2 is not irreducible in  $R$ . In fact,  $2 = (1 + i)(1 - i)$  and since

$$N(1 + i) = N(1 - i) = 2,$$

it follows that  $1 + i \notin \mathcal{U}(R)$  and  $1 - i \notin \mathcal{U}(R)$ .

**Exercise 4.13.** Let  $R = \mathbb{Z}[\sqrt{-5}]$ .

- 1) Prove that  $1 + \sqrt{-5}$  and  $1 - \sqrt{-5}$  are irreducible in  $R$ .
- 2) Prove that  $2, 3, 1 + \sqrt{-5}$  and  $1 - \sqrt{-5}$  are not associate in  $R$ .

**Example 4.14.** Let  $R = \mathbb{Z}[\sqrt{-3}]$  and  $x = 1 + \sqrt{-3}$ .

- 1)  $x$  is irreducible. If  $x = \alpha\beta$  for some  $\alpha, \beta \in R$ , then  $4 = N(x) = N(\alpha)N(\beta)$ . Write  $\alpha = a + b\sqrt{-3}$  for some  $a, b \in \mathbb{Z}$ . Then  $N(\alpha) = a^2 + 3b^2 \neq 2$ . If  $N(\alpha) = 2$ , then  $a^2 + 3b^2 = 2$  and then  $a$  and  $b$  both have the same parity. If both  $a$  and  $b$  are even, say  $a = 2k$  and  $b = 2l$  for some  $k, l \in \mathbb{Z}$ , then

$$2 = a^2 + 3b^2 = 4k^2 + 12l^2$$

is divisible by 4, a contradiction.

If both  $a$  and  $b$  are odd, say  $a = 2k + 1$  and  $b = 2l + 1$  for some  $k, l \in \mathbb{Z}$ , then

$$2 = a^2 + 3b^2 = 4k^2 + 4k + 12l^2 + 12l + 4$$

is divisible by 4, a contradiction.

- 2)  $x$  is not prime. Note that  $(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4$ , then  $x$  divides  $4 = 2 \cdot 2$ . But  $1 + \sqrt{-3} \nmid 2$ , as

$$(a - 3b) + (a + b)\sqrt{3} = (1 + \sqrt{-3})(a + b\sqrt{-3}) = 2$$

implies that  $a - 3b = 2$  and  $a + b = 0$ , which yields  $a = 1/2 \notin \mathbb{Z}$ , a contradiction.

**Exercise 4.15.** Let  $R = \mathbb{Z}[\sqrt{5}]$ . Prove that  $1 + \sqrt{5}$  is irreducible and not prime in  $R$ .

**Definition 4.16.** Let  $R$  be an integral domain. Then  $R$  is **principal** (or a principal domain) if every ideal of  $R$  is principal.

The rings  $\mathbb{Z}$  and  $\mathbb{R}[X]$  are both principal.

**Example 4.17.** The ring  $\mathbb{Z}[X]$  is not principal. For example, the ideal  $I = (2, X)$  is not principal.

First note that  $I \neq \mathbb{Z}[X]$ . In fact, if  $I = \mathbb{Z}[X]$ , then  $1 = 2f(X) + Xg(X)$  for some  $f(X), g(X) \in \mathbb{Z}[X]$ . Then

$$0 = \deg(1) = \deg(2f(X) + Xg(X)) = \deg(f(X)) + \deg(g(X)) + 1 > 0,$$

a contradiction.

If  $I = (h(X))$  for some  $h(X) \in \mathbb{Z}[X]$ , then  $2 = h(X)g(X)$  for some  $g(X) \in \mathbb{Z}[X]$ . This implies that  $\deg(h(X)) = 0$ , so  $h(X) = h(1) \in \mathbb{Z}$ . In particular,  $2 = h(1)g(1)$  and hence  $h(1) \in \{-1, 1, 2, 2\}$ . Since  $I \neq \mathbb{Z}[X]$ , it follows that  $h(X) = h(1) \notin \{-1, 1\}$ . Now  $X = h(X)f(X)$  for some  $f(X) \in \mathbb{Z}[X]$ . In particular,  $\deg(f(X)) = 1$ , so we may assume that  $f(X) = a_0 + a_1X$  for  $a_0, a_1 \in \mathbb{Z}$  and  $a_1 \neq 0$ . It follows that

$$X = \pm 2f(X) = \pm 2(a_0 + a_1X)$$

and therefore  $\pm 1/2 = a_1 \in \mathbb{Z}$ , a contradiction.

**Example 4.18.** Let  $R = \mathbb{Z}[\sqrt{-5}]$ . The ideal  $I = (2, 1 + \sqrt{-5})$  is not principal, so  $R$  is not principal.

We first note that  $I \neq R$ . If not, there exist  $x, y, u, v \in \mathbb{Z}$  such that

$$1 = 2(x + y\sqrt{-5}) + (1 + \sqrt{-5})(u + v\sqrt{-5}) = (2x + u - 5v) + \sqrt{-5}(2y + u + v).$$

This implies that  $1 = 2x + u - 5v$  and  $0 = 2y + u + v$ . These formulas imply that  $1 = 2(x + y + u - 2v)$ , a contradiction because  $x + y + u - 2v \in \mathbb{Z}$ .

Now assume that  $I = (\alpha)$ . Then  $\alpha \mid 2$  and  $\alpha \mid 1 + \sqrt{-5}$ . Then  $N(\alpha) \mid 4$  and  $N(\alpha) \mid 6$ , because  $N(2) = 4$  and  $N(1 + \sqrt{-5}) = 6$ . Thus  $N(\alpha) \in \{1, 2\}$ . If we write  $\alpha = a + b\sqrt{-5}$  for  $a, b \in \mathbb{Z}$ , then it follows that  $a^2 + 5b^2 = N(\alpha) = 1$  and hence  $\alpha \in \mathcal{U}(R)$ . Therefore  $I = R$ , a contradiction.

Sometimes primes and irreducibles coincide.

**Proposition 4.19.** Let  $R$  be a principal domain and  $x \in R$ . Then  $x$  is irreducible if and only if  $x$  is prime.

*Proof.* We only need to prove that if  $x$  is irreducible, then  $x$  is prime. Let us assume that  $x \mid yz$ . Let  $I = (x, y)$ . Since  $R$  is principal,  $I = (a)$  for some  $a \in R$ . In particular,  $x = ab$  for some  $b \in R$ . Since  $x$  is irreducible,  $a \in \mathcal{U}(R)$  or  $b \in \mathcal{U}(R)$ . If  $a \in \mathcal{U}(R)$ , then  $I = R$  and hence  $1 = xr + ys$  for some  $r, s \in R$ . Thus

$$z = z1 = z(xr + ys) = zxr + zys$$

and therefore  $x \mid z$ . If  $b \in \mathcal{U}(R)$ , then  $x$  and  $a$  are associate, in  $R$ . Thus  $I = (x) = (a)$  and hence  $xt = y$  for some  $t \in R$ , that is  $x \mid y$ .  $\square$

In the integers primes and irreducible coincide. This happens because  $\mathbb{Z}$  is a principal domain.

**Example 4.20.** Since  $\mathbb{Z}[\sqrt{-3}]$  have irreducible elements that are not prime, it follows that  $\mathbb{Z}[\sqrt{-3}]$  is not a principal domain.

**Definition 4.21.** Let  $R$  be an integral domain. We say that  $R$  is an **euclidean domain** if there exists a map  $\varphi: R \setminus \{0\} \rightarrow \mathbb{N}_0$  such that for every  $x, y \in R$  with  $y \neq 0$  there exist  $q, r \in R$  such that  $x = qy + r$ , where  $r = 0$  or  $\varphi(r) < \varphi(y)$ .

Note that in the definition of euclidean domains we do not ask for the uniqueness of the quotient and the remainder. In fact, we will meet important examples of euclidean domains where uniqueness in the division algorithm is not achieved.

**Example 4.22.**  $\mathbb{Z}$  is an euclidean domain with  $\varphi(x) = |x|$ .

Do we have uniqueness in the previous example?

**Example 4.23.**  $\mathbb{R}[X]$  is an euclidean domain with  $\varphi(f(X)) = \deg(f(X))$ .

The previous examples shows why in the definition of an euclidean domain we consider  $\varphi: R \setminus \{0\} \rightarrow \mathbb{N}_0$ .

**Example 4.24.** Let  $R = \mathbb{Z}[i]$ . Then  $R$  is an euclidean domain with  $\varphi(\alpha) = N(\alpha)$ . Let  $\alpha = a + ib$  and  $\beta = c + id \neq 0$ . Then

$$\frac{\alpha}{\beta} = \frac{a+ib}{c+id} = r + is$$

for some  $r, s \in \mathbb{Z}$ . Let  $m, n \in \mathbb{Z}$  be such that  $|r - m| \leq 1/2$  and  $|s - n| \leq 1/2$ . If  $\delta = m + in$  and  $\gamma = \alpha - \beta\delta$ , then  $\gamma \in R$ ,  $\delta \in R$  and  $\alpha = \beta\delta + \gamma$ . If  $\gamma \neq 0$ , then

$$\begin{aligned} \varphi(\gamma) &= \varphi\left(\beta\left(\frac{\alpha}{\beta} - \delta\right)\right) = \varphi(\beta)\varphi\left(\frac{\alpha}{\beta} - \delta\right) \\ &= \varphi(\beta)\varphi((r-m) + i(s-n)) = \varphi(\beta)((r-m)^2 + (s-n)^2) \\ &\leq \varphi(\beta)(1/4 + 1/4) \\ &< \varphi(\beta). \end{aligned}$$

In  $\mathbb{Z}[i]$  the division algorithm does not have uniqueness. In fact, if  $\alpha, \beta \in \mathbb{Z}[i]$  and  $\alpha = \beta\delta + \gamma$  for some  $\delta, \gamma \in \mathbb{Z}[i]$ , then there are up to four possibilities for the remainder  $\gamma$ .

**Example 4.25.** Let  $R = \mathbb{Z}[i]$  and  $\alpha = -1 + i$  and  $\beta = 1 + 2i$ . Let  $I = (\beta)$  be the ideal of  $R$  generated by  $\beta$ . First note that

$$I = (\beta) = (1 + 2i)R = (1 + 2i)\mathbb{Z} + (1 + 2i)\mathbb{Z}i = (1 + 2i)\mathbb{Z} + (-2 + i)\mathbb{Z}.$$

This allows us to draw the lattice of elements of  $I$ , that is the lattice formed by the multiples of  $\beta$ , see Figure ???. Since  $\alpha - \gamma \in I = (\beta)$ , there are at most four possibilities for writing the division algorithm. In our particular example, we find three possible cases:

1) If  $\alpha - \gamma = \beta_0$ , where  $0 = \beta_0 = \beta \cdot 0$ , then  $\gamma = \alpha = -1 + i$  and

$$-1 + i = (1 + 2i)0 + (-1 + i)$$

with  $N(-1 + i) = 2 < N(\beta) = 5$ .

2) If  $\alpha - \gamma = \beta_1$ , where  $-2 + i = \beta_1 = \beta i$ , then  $\gamma = -1$  and

$$-1 + i = (1 + 2i)i + (-1)$$

with  $N(-1) = 1 < N(\beta) = 5$ .

3) If  $\alpha - \gamma = \beta_2$ , where  $-1 + 3 = \beta_2 i = \beta(1 + i)$ , then  $\gamma = 2i$  and

$$-1 + i = (1 + 2i)(1 + i) + (-2i)$$

and  $N(-2i) = 4 < N(\beta) = 5$ .

We know that  $\mathbb{Z}$  and  $\mathbb{R}[X]$  are both principal. The proofs are very similar, as both use the division algorithm essentially in the same way. The following result takes advantage of this fact.



Figure 4.1

fig:lattice

**Proposition 4.26.** *Let  $R$  be an euclidean domain. Then  $R$  is principal.*

*Proof.* Let  $I$  be an ideal of  $R$ . If  $I = \{0\}$ , then  $I = (0)$  and hence it is principal. So we may assume that  $I \neq \{0\}$ . Let  $y \in I \setminus \{0\}$  be such that  $\varphi(y)$  is minimal. We claim that  $I = (y)$ . If  $z \in I$ , then  $z = yq + r$ , where  $r = 0$  or  $\varphi(r) < \varphi(y)$ . The minimality of  $\varphi(y)$  implies that  $r = 0$ . Thus  $z = yq \in (y)$  and it follows that  $I = (y)$ .  $\square$

**Example 4.27.** Since  $\mathbb{Z}[i]$  is euclidean, then it is principal.

**Example 4.28.** The rings  $\mathbb{Z}[\sqrt{-5}]$  and  $\mathbb{Z}[\sqrt{-3}]$  are not principal. Why?

The ring  $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$  is an example of a ring that is principal and not euclidean. We will not prove this fact in these notes. For a proof see...

**Definition 4.29.** Let  $R$  be an integral domain. Then  $R$  is a **unique factorization domain** if the following statements hold:

- 1) Each  $x \in R \setminus \{0\}$  that is not a unit can be written as  $x = c_1 \cdots c_n$  for irreducibles  $c_1, \dots, c_n$ .
- 2) If  $x = c_1 \cdots c_n = d_1 \cdots d_m$  for irreducibles  $c_1, \dots, c_n$  and  $d_1, \dots, d_m$ , then  $n = m$  and there exists  $\sigma \in \mathbb{S}_n$  such that  $c_i$  and  $d_{\sigma(i)}$  are associate for all  $i \in \{1, \dots, n\}$ .

It is important to remark that some rings have factorizations and this factorization is not unique. In fact, if  $N$  is a square-free integer,  $\mathbb{Z}[\sqrt{N}]$  is noetherian and hence it has factorization. This fact will be proved in the proof of the following theorem. However, not all these rings will be euclidean domains.

**Theorem 4.30.** *Let  $R$  be a principal domain. Then  $R$  is a unique factorization domain.*

*Proof.* We divide the proof in three steps.

*Claim.*  $R$  is noetherian.

Let  $I_1 \subsetneq I_2 \subsetneq \dots$  be a sequence of ideals of  $R$ . Since  $R$  is principal, each  $I_j$  is principal, say  $I_j = (a_j)$  for some  $a_j \in R$ , so the sequence is of the form  $(a_1) \subsetneq (a_2) \subsetneq \dots$ . Since  $I = \bigcup_{i \geq 1} (a_i)$  is an ideal of  $R$ , there exists  $x \in R$  such that  $I = (x)$ . Since  $x \in (a_n)$  for some  $n \in \mathbb{N}$ , it follows that  $(a_k) \subseteq I = (x) \subseteq (a_n)$  for all  $k \in \mathbb{N}$ .

*Claim.*  $R$  admits factorizations.

Let  $x \in R \setminus \{0\}$  be such that  $x \notin \mathcal{U}(R)$ . If  $x$  is irreducible, there is nothing to prove. If not,  $x = x_1 x_2$  with  $x_1 \notin \mathcal{U}(R)$  and  $x_2 \notin \mathcal{U}(R)$ . If  $x_1$  and  $x_2$  are both irreducibles, we are done. If not, say  $x_1$  can be written as  $x_1 = x_{11} x_{12}$  with  $x_{11} \notin \mathcal{U}(R)$  and  $x_{12} \notin \mathcal{U}(R)$ . If this process does not terminate, it means that there is a sequence of ideals

$$(x) \subsetneq (x_1) \subsetneq (x_{11}) \subsetneq \cdots$$

that does not stabilize, which contradicts the fact that  $R$  is noetherian.

*Claim.*  $R$  admits unique factorization.

Let  $x \in R$  be such that  $x$  factorizes into irreducibles as  $x = c_1 \cdots c_n = d_1 \cdots d_m$ . We may assume that  $n \leq m$ . We proceed by induction on  $m$ . If  $m = 1$ , then  $n = 1$  and  $c_1 = d_1$ . If  $m > 1$ , then, since  $c_1$  is prime and  $c_1 \mid d_1 \cdots d_m$ , it follows that  $c_1 \mid d_j$  for some  $j$ , say  $c_1 \mid d_1$  (here is precisely where the permutation  $\sigma$  appears). Since  $d_1$  is irreducible,  $c_1$  and  $d_1$  are associate, that is  $c_1 = u d_1$  for some  $u \in \mathcal{U}(R)$ . Then

$$c_1 c_2 \cdots c_n = (u d_1) c_2 \cdots c_n = d_1 d_2 \cdots d_m.$$

Since  $d_1 \neq 0$ ,

$$d_1 (u c_2 \cdots c_n - d_2 \cdots d_m) = 0,$$

which implies that  $(u c_2) \cdots c_n = d_2 \cdots d_m$  because  $R$  is an integral domain. Note that  $u c_2$  is irreducible and hence the claim follows by the inductive hypothesis.  $\square$

It is interesting to remark that the proof of the previous theorem is exactly the proof one does for  $\mathbb{Z}$ .

**Example 4.31.** The ring  $\mathbb{Z}[i]$  is a unique factorization domain.

**Example 4.32.** The ring  $R = \mathbb{Z}[\sqrt{-6}]$  is not a unique factorization domain. In fact,

$$10 = 2 \cdot 5 = (2 + \sqrt{-6})(2 - \sqrt{-6}).$$

Note that  $N(a + b\sqrt{-6}) = a^2 + 6b^2 \neq 2$ . This implies that 2 is irreducible, as if  $2 = \alpha\beta$ , then  $4 = N(2) = N(\alpha)N(\beta)$ . Similarly, 5 is irreducible. It is an exercise to prove that  $2 + \sqrt{-6}$  and  $2 - \sqrt{-6}$  are both irreducible.

We conclude the chapter with the following theorem.

**Theorem 4.33 (Fermat).** Let  $p \in \mathbb{Z}$  be a prime number. the following statements are equivalent:

- 1)  $p = 2$  or  $p \equiv 1 \pmod{4}$ .
- 2) There exists  $a \in \mathbb{Z}$  such that  $a^2 \equiv 1 \pmod{p}$ .
- 3)  $p$  is not irreducible in  $\mathbb{Z}[i]$ .
- 4)  $p = a^2 + b^2$  for some  $a, b \in \mathbb{Z}$ .

*Proof.* We first prove that 1)  $\implies$  2). If  $p = 2$ , take  $a = 1$ . If  $p = 4k + 1$  for some  $k \in \mathbb{Z}$ , then by Fermat's little theorem, the polynomial  $X^{p-1} - 1 \in (\mathbb{Z}/p)[X]$  has roots  $1, 2, \dots, p-1$ . Write

$$(X-1)(X-2)\cdots(X-(p-1)) = X^{p-1} - 1 = X^{4k} - 1 = (X^{2k} + 1)(X^{2k} - 1)$$

in  $(\mathbb{Z}/p)[X]$ . Since  $p$  is prime,  $\mathbb{Z}/p$  is a field and hence  $(\mathbb{Z}/p)[X]$  is a unique factorization domain (because it is euclidean). Thus there exists  $\alpha \in \mathbb{Z}/p$  such that  $\alpha^{2k} + 1 = 0$ . To finish the proof take  $a = \alpha^k$ .

We now prove that 2)  $\implies$  3). If  $a^2 \equiv -1 \pmod{p}$ , then  $a^2 + 1 = kp$  for some  $k \in \mathbb{Z}$ . Since  $(a-i)(a+i) = a^2 + 1 = kp$ , then  $p$  divides  $(a-i)(a+i)$ . We claim that  $p$  does not divide  $a-i$  in  $\mathbb{Z}[i]$ . Indeed, if  $p \mid a-i$ , then  $a-i = p(e+fi)$  for some  $e, f \in \mathbb{Z}$  and this implies that  $1 = pf$ , a contradiction. Similarly,  $p$  does not divide  $a+i$ . Thus  $p$  is not prime in  $\mathbb{Z}[i]$  and hence it is not irreducible in  $\mathbb{Z}[i]$  (because in  $\mathbb{Z}[i]$  primes and irreducible coincide).

We now prove that 3)  $\implies$  4). If  $p = (a+bi)(c+di)$  with  $a+bi \notin \mathcal{U}(\mathbb{Z}[i])$  and  $c+di \notin \mathcal{U}(\mathbb{Z}[i])$ , then

$$p^2 = N(p) = N(a+bi)N(c+di) = (a^2 + b^2)(c^2 + d^2)$$

in  $\mathbb{Z}$ . Since  $\mathbb{Z}$  has unique factorization, it follows that  $p = a^2 + b^2$ .

Finally we prove that 4)  $\implies$  1). The only possible remainders after division by four are 0, 1, 2 and 3. For all  $a$ , either  $a^2 \equiv 0 \pmod{4}$  or  $a^2 \equiv 1 \pmod{4}$ . If  $p \equiv 3 \pmod{4}$ , then  $p$  is never a sum of two squares, as  $a^2 + b^2 \equiv 0 \pmod{4}$ ,  $a^2 + b^2 \equiv 1 \pmod{4}$  or  $a^2 + b^2 \equiv 2 \pmod{4}$ .  $\square$



## Chapter 5

### Zorn's lemma

**Definition 5.1.** A non-empty set  $R$  is said to be a **partially ordered set** (or poset, for short) if there is a subset  $X \subseteq R \times R$  such that

- 1)  $(r, r) \in X$  for all  $r \in R$ ,
- 2) if  $(r, s) \in X$  and  $(s, t) \in X$ , then  $(r, t) \in X$ , and
- 3) if  $(r, s) \in X$  and  $(s, r) \in X$ , then  $r = s$ .

The set  $X$  is a partial order relation on  $R$ . We will use the following notation:  $(r, s) \in X$  if and only if  $r \leq s$ . Moreover,  $r < s$  if and only if  $r \leq s$  and  $r \neq s$ .

**Definition 5.2.** Let  $R$  be a poset and  $r, s \in R$ . Then  $r$  and  $s$  are **comparable** if either  $r < s$  or  $s < r$ .

**Example 5.3.** Let  $U = \{1, 2, 3, 4, 5\}$  and  $T$  be the set of subsets of  $U$ . Then  $T$  is a poset with the usual inclusion, that is  $C \leq D$  if and only if  $C \subseteq D$ . The subsets  $\{1, 2\}$  and  $\{3, 4\}$  of  $U$  are elements of  $T$  that are not comparable.

**Definition 5.4.** Let  $R$  be a poset and  $r \in R$ . Then  $r$  is **maximal** in  $R$  if such that  $r \leq t$  implies  $r = t$ .

**Example 5.5.**  $\mathbb{Z}$  has no maximal elements.

**Example 5.6.** Let  $R = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$  with  $(x_1, y_1) \leq (x_2, y_2)$  if and only if  $x_1 = x_2$  and  $y_1 \leq y_2$ . Then  $R$  is a poset and each  $(x, 0)$  is maximal. Thus  $R$  has infinitely many maximal elements.

**Definition 5.7.** Let  $R$  be a poset and  $S$  be a non-empty subset of  $R$ . An **upper bound** for  $S$  is an element  $u \in R$  such that  $s \leq u$  for all  $s \in S$ .

**Example 5.8.** Let  $S = \{6\mathbb{Z}, 12\mathbb{Z}, 24\mathbb{Z}\}$  be a subset of the set of subgroups of  $\mathbb{Z}$ . Then  $6\mathbb{Z} = 6\mathbb{Z} \cap 12\mathbb{Z} \cap 24\mathbb{Z}$  is an upper bound of  $S$ .

**Definition 5.9.** Let  $R$  be a poset. A **chain** is a non-empty subset  $S$  of  $R$  such that any two elements of  $S$  are comparable.

We now state Zorn's lemma:

Let  $R$  be a poset such that every chain in  $R$  admits an upper bound in  $R$ . Then  $R$  contains a maximal element.

It is not intuitive, but it is logically equivalent to a more intuitively statement in set theory, the Axiom of Choice, which says every cartesian product of non-empty sets is non-empty. It is more an axiom than a lemma. The reason for calling Zorn's lemma a lemma rather than an axiom is purely historical.

**Definition 5.10.** Let  $R$  be a ring. An ideal  $I$  of  $R$  is said to be **maximal** if  $I \neq R$  and if  $J$  is an ideal of  $R$  such that  $I \subseteq J$ , then either  $I = J$  or  $J = R$ .

If  $p$  is a prime number, then  $p\mathbb{Z}$  is a maximal ideal of  $\mathbb{Z}$ .

**Exercise 5.11.** Let  $R$  be a commutative ring. Prove that  $R$  is a field if and only if  $\{0\}$  is a maximal ideal of  $R$ .

**Exercise 5.12.** Let  $R$  be a commutative ring and  $I$  be an ideal of  $R$ . Prove that  $I$  is maximal if and only if  $R/I$  is a field.

The following application of Zorn's lemma uses the identity of a ring.

**Theorem 5.13.** Let  $R$  be a ring. Each proper ideal  $I$  of  $R$  is contained in a maximal ideal. In particular, all rings have maximal ideals.

*Proof.* Let  $X = \{J \subseteq R : J \text{ is an ideal of } R \text{ such that } I \subseteq J \subsetneq R\}$ . Since  $I \in X$ , it follows that  $X$  is non-empty. Moreover,  $X$  is a poset with respect to the inclusion. If  $C$  is a chain in  $X$ , then  $\bigcup_{J \in C} J$  is an upper bound for  $C$ , as  $\bigcup_{J \in C} J$  is an ideal and  $\bigcup_{J \in C} J \neq R$  because  $1 \notin \bigcup_{J \in C} J$ . Zorn's lemma implies that there exists  $M \in X$  maximal. We claim that  $M$  is a maximal ideal of  $R$ . The definition of  $X$  implies that  $M$  is a proper ideal of  $R$ . If  $M_1$  is an ideal of  $R$  such that  $M \subseteq M_1$ , it follows that  $M \subseteq M_1$  and hence  $M_1 \in X$ . The maximality of  $M$  implies that  $M = M_1$ .  $\square$

**Exercise 5.14.** Compute the maximal ideals of  $\mathbb{R}[X]$  and  $\mathbb{C}[X]$ .

One can also compute the maximal ideals of  $K[X]$  for any field  $K$ .

**Exercise 5.15.** Let  $R$  be a principal domain and  $p \in R$ . Then  $p$  is irreducible if and only if  $(p)$  is maximal.

**Example 5.16.** The ideal  $(X^2 + 2X + 2)$  is maximal in  $\mathbb{Q}[X]$  because

$$X^2 + 2X + 2 = (X + 1)^2 + 1$$

has degree two and no rational roots. Hence  $X^2 + 2X + 2$  is irreducible in  $\mathbb{Q}[X]$  and it generates a maximal ideal.

**Exercise 5.17.** Let  $R$  be a commutative ring and  $I$  be an ideal of  $R$ . Then  $I$  is maximal if and only if  $R/I$  is a field.

**Example 5.18.** Let  $R = (\mathbb{Z}/2)[X]$  and  $f(X) = X^2 + X + 1$ . Since  $f(X)$  is irreducible (because  $\deg f(X) = 2$  and  $f(X)$  has no roots in  $\mathbb{Z}/2$ , it follows that  $(f(X))$  is a maximal ideal. Thus  $R/I$  is a field.

**Exercise 5.19.** Compute the maximal ideals of  $\mathbb{Z}/n$ .

**Exercise 5.20.** Let  $R$  be a commutative ring and  $J(R)$  be the intersection of all maximal ideals of  $R$ . Prove that  $x \in J(R)$  if and only if  $1 - xy \in \mathcal{U}(R)$  for all  $y \in R$ . The ideal  $J(R)$  is proper and it is known as the **Jacobson radical** of  $R$ .

We conclude the chapter with a different application of Zorn's lemma.

**Exercise 5.21.** Prove that every non-zero vector space has a basis.

The previous exercise can be used to solve the following exercises:

**Exercise 5.22.** Prove that there exists a group homomorphism  $f: \mathbb{R} \rightarrow \mathbb{R}$  that is not of the form  $f(x) = \lambda x$  for some  $\lambda \in \mathbb{R}$ .

**Exercise 5.23.** Prove that the abelian groups  $\mathbb{R}^n$  and  $\mathbb{R}$  are isomorphic.

**Exercise 5.24.** Prove that if  $G$  is a group such that  $|G| > 2$ , then  $|\text{Aut}(G)| > 1$ .



## Chapter 6

### Algebras

We now discuss an important family of examples. Fix a field  $K$ . For a finite group  $G$  let  $K[G]$  be the vector space (over  $K$ ) with basis  $\{g : g \in G\}$ . Then  $K[G]$  is a ring with

$$\left(\sum_{g \in G} \lambda_g g\right) \left(\sum_{h \in G} \mu_h h\right) = \sum_{g, h \in G} \lambda_g \mu_h (gh).$$

Thus  $K[G]$  is a ring and also a vector space (over  $K$ ) and these structures are somewhat compatible. Note that

$$(\lambda a + \mu g = c = \lambda(ac) + \mu(bc), \quad a(\lambda b + \mu c) = \lambda(ab) + \mu(ac)$$

for all  $\lambda, \mu \in K$  and  $a, b, c \in K[G]$ .

**Definition 6.1.** Let  $A$  be a ring. Then  $A$  is an algebra (over the field  $K$ ) if  $A$  is a vector space and the map  $K \rightarrow Z(A)$ ,  $k \mapsto k1_A$ , is an injective ring homomorphism.

Thus  $K[G]$  is an algebra, as it is ring that contains  $K$  in its center (or more precisely, the map  $K \rightarrow Z(K[G])$ ,  $k \mapsto k1$ , is an injective ring homomorphism. Other examples of algebras are the polynomial rings  $K[X]$  and  $K[X, Y]$  and matrix rings  $M_n(K)$ .

**Example 6.2.** If  $A$  is an algebra, then  $M_n(A)$  is an algebra.

The ring  $K[G]$  is commutative if and only if  $G$  is abelian. Moreover,  $K[G]$  is a vector space of dimension  $\dim K[G] = |G|$ .

**Example 6.3.** Let  $G = \langle g : g^3 = 1 \rangle = \{1, g, g^2\} \simeq C_3$  be the cyclic group of order three. If  $\alpha = a_1 1 + a_2 g + a_3 g^2$  and  $\beta = b_1 1 + b_2 g + b_3 g^2$ , then

$$\alpha\beta = (a_1 b_1 + a_2 b_3 + a_3 b_2)1 + (a_1 b_2 + a_2 b_1 + a_3 b_3)g + a_1 b_3 + a_2 b_2 + a_3 b_1 g^2.$$

One can check that  $\mathbb{C}[G] \simeq \mathbb{C}[X]/(X^3 - 1)$ .

In general, one proves that  $\mathbb{C}[C_n] \simeq \mathbb{C}[X]/(X^n - 1)$  for  $n \geq 2$ .

**Example 6.4.** Let  $K$  be a field and  $G = \{1, g\} \simeq C_2$  be the cyclic group of order two. The product of  $K[G]$  is

$$(a1 + bg)(c1 + gd) = (ac + bd)1 + (ad + bc)g.$$

If  $K = \mathbb{C}$ , then the map  $K[G] \rightarrow K \times K$ ,  $a1 + bg \mapsto (a + b, a - b)$ , is a linear isomorphism of rings.

If  $K = \mathbb{Z}/2$ , then the map  $K[G] \rightarrow \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ ,  $a1 + bg \mapsto \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}$ , is a linear isomorphism of rings.

**Exercise 6.5.** Prove that  $\mathbb{R}[C_3] \simeq \mathbb{R} \times \mathbb{C}$ .

The group ring has the following property, which is left as an exercise. ~~zzZ~~Let  $R$  be a ring and  $G$  be a finite group. If  $f: G \rightarrow \mathcal{U}(R)$  is a group homomorphism, then there exists a unique ring homomorphism  $\varphi: K[G] \rightarrow R$  such that  $\varphi|_G = f$ .

**Example 6.6.** Let  $\mathbb{D}_3 = \langle r, s : r^3 = s^2 = 1, srs^{-1} = r^{-1} \rangle$  be the dihedral group of six elements. We claim that

$$\mathbb{C}[\mathbb{D}_3] \simeq \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}).$$

Let  $\omega$  be a primitive root of one. Let

$$R = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One easily checks that  $SRS^{-1} = R^{-1}$  and  $R^3 = S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . It follows that there exists a group homomorphism  $G \rightarrow \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$  such that  $r \mapsto (1, 1, R)$  and  $s \mapsto (1, -1, S)$ . This group homomorphism is a ring isomorphism.

## **Part II**

# **Representations**





## Chapter 7

# Group representations

**Definition 7.1.** A **representation** (over the field  $K$ ) of a group  $G$  is a group homomorphism  $\rho : G \rightarrow \mathbf{GL}(V)$ ,  $g \mapsto \rho_g$ , for some vector space  $V$  (over  $K$ ).

The **degree** of  $V$  will be the dimension of  $V$ . Note that if  $\dim V = n$ , fixing a basis of  $V$  we get a **matrix representation**  $\rho : G \rightarrow \mathbf{GL}(V) \simeq \mathbf{GL}_n(V)$  of  $G$ .

**Example 7.2.** We use group representations to show that  $G = \langle x, y : x^2 = y^2 = 1 \rangle$  is infinite. Note that

$$\rho : G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

is a group homomorphism, as  $\rho_x^2 = \rho_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . We claim that the elements of the form  $(xy)^n$  are different for all  $n$ . It is enough to show that  $(xy)^n = (xy)^m$ , then  $n = m$ . Note that

$$\rho_{xy} = \rho_x \rho_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus  $\rho_{xy}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  and  $\rho_{xy}^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ . From this the claim follows.

The previous example shows the power of group representations, even for infinite groups. However, in this course we will work with complex finite-dimensional representations of finite groups.

**Example 7.3.** If  $G$  is a group, then  $\rho : G \rightarrow \mathbb{C}^\times$ ,  $g \mapsto 1$ , is a representation. This representation is known as the **trivial representation** of  $G$ .

**Example 7.4.** The sign yields a representation of  $\mathbb{S}_n$ . It is the group homomorphism  $\mathbb{S}_n \rightarrow \mathbb{C}^\times$ ,  $\sigma \mapsto \text{sign}(\sigma)$ .

**Example 7.5.** Let  $G = \langle g : g^6 = 1 \rangle$  be the cyclic group of order six. Then

$$\rho : G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad g \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a group representation.

**Proposition 7.6.** Let  $G$  be a finite group and  $\rho : G \rightarrow \mathbf{GL}(V)$  be a representation. Then each  $\rho_g$  is diagonalizable.

*Proof.* Let  $n = \dim V$ . Fix a basis of the finite-dimensional vector space  $V$  and consider a matrix representation  $\rho : G \rightarrow \mathbf{GL}_n(\mathbb{C})$ . Since  $g$  is finite,  $g^m = 1$  for some  $m \in \mathbb{N}$ . This means that  $\rho_g$  is a root of  $X^m - 1 \in \mathbb{C}[X]$ . Since the roots of the polynomial  $X^m - 1$  are all different and  $X^m - 1$  factorizes linearly on  $\mathbb{C}[X]$ , it follows that the minimal polynomial of  $\rho_g$  also factorizes linearly in  $\mathbb{C}[X]$ . Hence  $\rho_g$  is diagonalizable.  $\square$

**Definition 7.7.** Let  $\rho : G \rightarrow \mathbf{GL}(V)$  and  $\psi : G \rightarrow \mathbf{GL}(W)$  be representations of a finite group  $G$ . A linear map  $T : V \rightarrow W$  is said to be invariant if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ T \downarrow & & \downarrow T \\ W & \xrightarrow{\psi_g} & W \end{array}$$

commutes, i.e.  $\psi_g T = T \rho_g$  for all  $g \in G$ .

**Definition 7.8.** The representations  $\rho : G \rightarrow \mathbf{GL}(V)$  and  $\psi : G \rightarrow \mathbf{GL}(W)$  are **equivalent** if there exists a bijective invariant map  $T : V \rightarrow W$ .

**Example 7.9.** Let  $G = \mathbb{Z}/n$ . The representations

$$\rho : G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad m \mapsto \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix}$$

and

$$\psi : G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad m \mapsto \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix}$$

are equivalent, as  $\rho_m T = T \psi_m$  for all  $m \in G$  if  $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ .

**Definition 7.10.** Let  $\rho : G \rightarrow \mathbf{GL}(V)$  a representation. A subspace  $W$  of  $V$  is said to be **invariant** (with respect to  $\rho$ ) if  $\rho_g(W) \subseteq W$  for all  $g \in G$ .

If  $\rho : G \rightarrow \mathbf{GL}(V)$  is a representation and  $W \subseteq V$  is invariant, then the map  $\rho|_W : G \rightarrow \mathbf{GL}(W)$ ,  $g \mapsto (\rho_g)|_W$ , is a representation. The map  $\rho|_W$  is the **restriction** of  $\rho$  to  $W$ .

**Example 7.11.** If  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  are representations and  $T: V \rightarrow W$  is an invariant map, then the **kernel**

$$\ker T = \{v \in V : T(v) = 0\}$$

is an invariant of  $V$  and the **image**

$$T(V) = \{T(v) : v \in V\}$$

is an invariant subspace of  $W$ .

**Definition 7.12.** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  a representation. A **subrepresentation** of  $\rho$  is a restricted representation of the form  $\rho|_W: G \rightarrow \mathbf{GL}(W)$  for some invariant subspace  $W$  of  $V$ .

**Example 7.13.** Let  $G = \langle g : g^3 = 1 \rangle$  be the cyclic group of order three and

$$\rho: G \rightarrow \mathbf{GL}_3(\mathbb{R}), \quad g \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The subspace

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$$

is a invariant subspace of  $\mathbb{R}^3$ .

**Definition 7.14.** A representation  $\rho: G \rightarrow \mathbf{GL}(V)$  is **irreducible** if  $\{0\}$  and  $V$  are the only invariant subspaces of  $V$ .

Degree-one representations are irreducible.

**Example 7.15.** Let  $G = \langle g : g^3 = 1 \rangle$  be the cyclic group of order three and

$$\rho: G \rightarrow \mathbf{GL}_3(\mathbb{R}), \quad g \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We claim that the invariant subspace

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\} \subseteq \mathbb{R}^3$$

is irreducible. Let  $S$  be a non-zero invariant subspace of  $W$  and let  $s = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in S$  be a non-zero element. Then

$$t = \begin{pmatrix} y_0 \\ z_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in S.$$

We claim that  $\{s, t\}$  are linearly independent. If not, there exists  $\lambda \in \mathbb{R}$  such that  $\lambda s = t$ . Thus  $\lambda x_0 = y_0$ ,  $\lambda y_0 = z_0$  and  $\lambda z_0 = x_0$ . This implies that  $\lambda^3 x_0 = x_0$ . Since  $x_0 \neq 0$  (because if  $x_0 = 0$ , then  $y_0 = z_0 = 0$ , a contradiction), it follows that  $\lambda = 1$  and hence  $x_0 = y_0 = z_0$ , a contradiction because  $x_0 + y_0 + z_0 = 0$ . Therefore  $\dim S = 2$  and hence  $S = W$ .

**Exercise 7.16.** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  be a degree-two representation. Prove that  $\rho$  is irreducible if and only if there is no common eigenvector for the  $\rho_g$ ,  $g \in G$ .

The previous exercise can be used to show that the representation  $\mathbb{S}_3 \rightarrow \mathbf{GL}_2(\mathbb{C})$  of the symmetric group  $\mathbb{S}_3$  given by

$$(12) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

is irreducible.

**Example 7.17.** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be representations. One defines the **direct sum**  $\rho \oplus \psi$  of  $\rho$  and  $\psi$  as

$$\rho \oplus \psi: G \rightarrow \mathbf{GL}(V \oplus W), \quad g \mapsto (\rho \oplus \psi)_g,$$

where  $(\rho \oplus \psi)_g: V \oplus W \rightarrow V \oplus W$  is given by  $(v, w) \mapsto (\rho_g(v), \psi_g(w))$ .

**Definition 7.18.** A representation  $\rho: G \rightarrow \mathbf{GL}(V)$  is **completely irreducible** if  $V$  can be decomposed as  $V = V_1 \oplus \cdots \oplus V_n$ , where each  $V_i$  is a invariant subspace of  $V$  and each  $\rho|_{V_i}$  is irreducible.

Since we are considering finite-dimensional vector spaces, our vector spaces are Hilbert spaces, so they have an inner product  $V \times V \rightarrow \mathbb{C}$ ,  $(v, w) \mapsto \langle v, w \rangle$ .

**Definition 7.19.** A representation  $\rho: G \rightarrow \mathbf{GL}(V)$  is **unitary** if  $\langle \rho_g v, \rho_g w \rangle = \langle v, w \rangle$  for all  $g \in G$  and  $v, w \in V$ .

**Definition 7.20.** A representation  $\rho: G \rightarrow \mathbf{GL}(V)$  is **decomposable** if  $V$  can be decomposed as  $V = S \oplus T$  where  $S$  and  $T$  are non-zero invariant subspaces of  $V$ .

A representation is **indecomposable** if it is not decomposable.

**Exercise 7.21.** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  be a unitary representation. Prove that  $\rho$  is either irreducible or decomposable.

**Example 7.22.** Let  $G$  be a finite group and  $V = \mathbb{C}[G]$ . The **left regular representation** of  $G$  is the representation

$$L: G \rightarrow \mathbf{GL}(V), \quad g \mapsto L_g,$$

where  $L_g(h) = gh$ . With the inner product

$$\left\langle \sum_{g \in G} \lambda_g g, \sum_{g \in G} \mu_g g \right\rangle = \sum_{g \in G} \lambda_g \overline{\mu_g}$$

the representation  $L$  is unitary.

**Proposition 7.23 (Weyl's trick).** *Every representation of a finite group is equivalent to a unitary representation.*

*Proof.* Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $V \times V \rightarrow \mathbb{C}, (v, w) \mapsto \langle v, w \rangle_0$  be an inner product on  $V$ . A straightforward calculation shows that

$$\langle v, w \rangle = \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle_0$$

is an inner product of  $V$ . Since

$$\begin{aligned} \langle \rho_g v, \rho_g w \rangle &= \sum_{h \in G} \langle \rho_h \rho_g v, \rho_h \rho_g w \rangle_0 \\ &= \sum_{h \in G} \langle \rho_{hg} v, \rho_{hg} w \rangle_0 = \sum_{x \in G} \langle \rho_x v, \rho_x w \rangle_0 = \langle v, w \rangle, \end{aligned}$$

the representation  $\rho$  is unitary.  $\square$

Weyl's trick has several interesting corollaries. Let  $\rho: G \rightarrow \mathbf{GL}(V)$  be a representation of a finite group  $G$ . Then 1) every non-zero representation is either irreducible or decomposable, and 2) every  $\rho_g$  is diagonalizable (as unitary operators are diagonalizable).

**Exercise 7.24.** If  $G$  is an infinite group it is not longer true that every non-zero representation is either irreducible or decomposable. Find an example.

Recall that we only consider finite-dimensional representations of finite groups.

**Theorem 7.25 (Maschke).** *Every representation of a finite group is completely reducible.*

*Proof.* Let  $G$  be a finite group and  $\rho: G \rightarrow \mathbf{GL}(V)$  be a representation of  $G$ . We proceed by induction on  $\dim V$ . If  $\dim V = 1$ , the result is trivial, as degree-one representations are irreducible. Assume that the result holds for representations of degree  $\leq n$ . Let  $\rho: G \rightarrow \mathbf{GL}(V)$  be a representation of degree  $n + 1$ . If  $\rho$  is irreducible, we are done. If not, write  $V = S \oplus T$ , where  $S$  and  $T$  are non-zero invariant subspaces. Since  $\dim S < \dim V$  and  $\dim T < \dim V$ , it follows from the inductive hypothesis that both  $S$  and  $T$  are completely irreducible. Thus  $V$  is completely irreducible.  $\square$

**Example 7.26.** Let  $G = \mathbb{S}_3$  and  $\rho: G \rightarrow \mathbf{GL}_3(\mathbb{C})$  be the representation given by

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then  $\rho_g$  is unitary for all  $g \in G$  (because  $\rho_{(12)}$  and  $\rho_{(123)}$  are both unitary). Moreover,

$$S = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad T = S^\perp = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle,$$

are irreducible invariant subspaces of  $V = \mathbb{C}^3$ . A direct calculation shows that in the orthogonal basis  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$  the matrices  $\rho_{(12)}$  and  $\rho_{(123)}$  can be written as

$$\rho_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

**Exercise 7.27.** Let  $G$  be a finite group. Prove that there is a bijection between degree-one representations of  $G$  and degree-one representations of  $G/[G, G]$ .

**Lemma 7.28 (Schur).** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be irreducible representations. If  $T: V \rightarrow W$  is a non-zero invariant map, then  $T$  is bijective.

*Proof.* Since  $T$  is non-zero and  $\ker T$  is an invariant subspace of  $V$ , it follows that  $\ker T = \{0\}$ , as  $\rho$  is irreducible. Thus  $T$  is injective. Since  $T(V)$  is a non-zero invariant subspace of  $W$ , it follows from the fact that  $\psi$  is irreducible that  $T$  is surjective. Therefore  $T$  is bijective.  $\square$

Two applications:

**Proposition 7.29.** If  $\rho: G \rightarrow \mathbf{GL}(V)$  is an irreducible representation and  $T: V \rightarrow V$  is invariant, then  $T = \lambda \text{id}$  for some  $\lambda \in \mathbb{C}$ .

*Proof.* Let  $\lambda$  be an eigenvector of  $T$ . Then  $T - \lambda \text{id}$  is invariant and it is not bijective. Thus  $T - \lambda \text{id} = 0$  by Schur's lemma.  $\square$

**Proposition 7.30.** Let  $G$  be a finite abelian group. If  $\rho: G \rightarrow \mathbf{GL}(V)$  is an irreducible representation, then  $\dim V = 1$ .

*Proof.* Let  $h \in G$ . Note that since  $G$  is abelian,  $T = \rho_h$  is invariant:

$$T\rho_g = \rho_h\rho_g = \rho_{hg} = \rho_{gh} = \rho_g\rho_h = \rho_gT.$$

By Schur's lemma, there exists  $\lambda_h \in \mathbb{C}$  such that  $\rho_h = \lambda_h \text{id}$ . If  $v \in V \setminus \{0\}$ , then  $V = \langle v \rangle$ , as  $\rho$  is irreducible.  $\square$

## Chapter 8

### Characters

**Definition 8.1.** Let  $\rho : G \rightarrow \mathbf{GL}(V)$  be a representation. The **character** of  $\rho$  is the map  $\chi_\rho : G \rightarrow \mathbb{C}, g \mapsto \text{trace } \rho_g$ .

If a representation  $\rho$  is irreducible, its character is said to be an **irreducible character**. The **degree** of a character is the degree of the affording representation.

**Proposition 8.2.** Let  $\rho : G \rightarrow \mathbf{GL}(V)$  be a representation,  $\chi$  be its character and  $g \in G$ . The following statements hold:

- 1)  $\chi(1) = \dim V$ .
- 2)  $\chi(g) = \chi(hgh^{-1})$  for all  $h \in G$ .
- 3)  $\chi(g)$  is the sum of  $\chi(1)$  roots of one of order  $|g|$ .
- 4)  $\chi(g^{-1}) = \overline{\chi(g)}$ .
- 5)  $|\chi(g)| \leq \chi(1)$ .

*Proof.* The first statement is trivial. To prove 2) note that

$$\chi(hgh^{-1}) = \text{trace}(\rho_{hgh^{-1}}) = \text{trace}(\rho_h \rho_g \rho_h^{-1}) = \text{trace } \rho_g = \chi(g).$$

Statement 3) follows from the fact that the trace of  $\rho_g$  is the sum of the eigenvalues of  $\rho_g$  and these numbers are roots of the polynomial  $X^{|g|} - 1 \in \mathbb{C}[X]$ . To prove 4) write  $\chi(g) = \lambda_1 + \dots + \lambda_k$ , where the  $\lambda_j$  are roots of one. Then

$$\overline{\chi(g)} = \sum_{j=1}^k \overline{\lambda_j} = \sum_{j=1}^k \lambda_j^{-1} = \text{trace}(\rho_g^{-1}) = \text{trace}(\rho_{g^{-1}}) = \chi(g^{-1}).$$

Finally, we prove 5). Use 3) to write  $\chi(g)$  as the sum of  $\chi(1)$  roots of one, say  $\chi(g) = \lambda_1 + \dots + \lambda_k$  for  $k = \chi(1)$ . Then

$$|\chi(g)| = |\lambda_1 + \dots + \lambda_k| \leq |\lambda_1| + \dots + |\lambda_k| = \underbrace{1 + \dots + 1}_{k\text{-times}} = k.$$

□

If two representations are equivalent, their characters are equal.

**Definition 8.3.** Let  $G$  be a group and  $f: G \rightarrow \mathbb{C}$  be a map. Then  $f$  is a **class function** if  $f(g) = f(hgh^{-1})$  for all  $g, h \in G$ .

Characters are class functions.

**Proposition 8.4.** If  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  are representations, then  $\chi_{\rho \oplus \psi} = \chi_\rho + \chi_\psi$ .

*Proof.* For  $g \in G$ , it follows that  $(\rho \oplus \psi)_g = \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}$ . Thus

$$\chi_{\rho \oplus \psi}(g) = \text{trace}((\rho \oplus \psi)_g) = \text{trace}(\rho_g) + \text{trace}(\psi_g) = \chi_\rho(g) + \chi_\psi(g). \quad \square$$

Let  $V$  be a vector space with basis  $\{v_1, \dots, v_k\}$  and  $W$  be a vector space with basis  $\{w_1, \dots, w_l\}$ . A **tensor product** of  $V$  and  $W$  is a vector space  $X$  together with a bilinear map

$$V \times W \rightarrow X, \quad (v, w) \mapsto v \otimes w,$$

such that  $\{v_i \otimes w_j : 1 \leq i \leq k, 1 \leq j \leq l\}$  is a basis of  $X$ . The tensor product of  $V$  and  $W$  is unique up to isomorphism and it is denoted by  $V \otimes W$ . Note that

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

**Definition 8.5.** Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be representations. The **tensor product** of  $\rho$  and  $\psi$  is the representation of  $G$  given by

$$\rho \otimes \psi: G \rightarrow \mathbf{GL}(V \otimes W), \quad g \mapsto (\rho \otimes \psi)_g,$$

where

$$(\rho \otimes \psi)_g(v \otimes w) = \rho_g(v) \otimes \psi_g(w)$$

for  $v \in V$  and  $w \in W$ .

A direct calculation shows that the tensor product of representations is indeed a representation.

**Proposition 8.6.** If  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  are representations, then

$$\chi_{\rho \otimes \psi} = \chi_\rho \chi_\psi.$$

*Proof.* For each  $g \in G$  the map  $\rho_g$  is diagonalizable. Let  $\{v_1, \dots, v_n\}$  be a basis of eigenvectors of  $\rho_g$  and let  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  be such that  $\rho_g(v_i) = \lambda_i v_i$  for all  $i \in \{1, \dots, n\}$ . Similarly, let  $\{w_1, \dots, w_m\}$  be a basis of eigenvectors of  $\psi_g$  and  $\mu_1, \dots, \mu_m \in \mathbb{C}$  be such that  $\psi_g(w_j) = \mu_j w_j$  for all  $j \in \{1, \dots, m\}$ . Each  $v_i \otimes w_j$  is eigenvector of  $\rho \otimes \psi$  with eigenvalue  $\lambda_i \mu_j$ , as

$$(\rho \otimes \psi)_g(v_i \otimes w_j) = \rho_g v_i \otimes \psi_g w_j = \lambda_i v_i \otimes \mu_j w_j = (\lambda_i \mu_j) v_i \otimes w_j.$$



Thus  $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$  is a basis of eigenvectors and the  $\lambda_i \mu_j$  are the eigenvalues of  $(\phi \otimes \psi)_g$ . It follows that

$$\chi_{\rho \otimes \psi}(g) = \sum_{i,j} \lambda_i \mu_j = \left( \sum_i \lambda_i \right) \left( \sum_j \mu_j \right) = \chi_\rho(g) \chi_\psi(g). \quad \square$$

For completeness we mention without proof that it is also possible to define the dual  $\rho^*: G \rightarrow \mathbf{GL}(V^*)$  of a representation  $\rho: G \rightarrow \mathbf{GL}(V)$  by the formula

$$(\rho_g^* f)(v) = f(\rho_g^{-1} v), \quad g \in G, f \in V^* \text{ and } v \in V.$$

We claim that the character of the dual representation is then  $\overline{\chi_\rho}$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $V$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  be such that  $\rho_g v_i = \lambda_i v_i$  for all  $i \in \{1, \dots, n\}$ . If  $\{f_1, \dots, f_n\}$  is the dual basis of  $\{v_1, \dots, v_n\}$ , then

$$(\rho_g^* f_i)(v_j) = f_i(\rho_g^{-1} v_j) = \overline{\lambda_j} f_i(v_j) = \overline{\lambda_j} \delta_{ij}$$

and the claim follows.



## Chapter 9

### Schur's orthogonality relations

Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be representations of a finite group  $G$ . Since  $V$  and  $W$  are vector spaces, the set

$$\mathrm{Hom}(V, W) = \{T: V \rightarrow W : T \text{ is linear}\}$$

is a vector space with

$$\begin{aligned} (\lambda T)(v) &= \lambda T(v) && \text{for all } \lambda \in \mathbb{C} \text{ and all } v \in V, \\ (T + T_1)(v) &= T(v) + T_1(v) && \text{for all } v \in V. \end{aligned}$$

We claim that the set  $\mathrm{Hom}_G(V, W)$  of invariant maps is a subspace of  $\mathrm{Hom}(V, W)$ . Indeed, the zero map is clearly invariant. If  $T, T_1 \in \mathrm{Hom}_G(V, W)$  and  $\lambda \in \mathbb{C}$ , then

$$(T + \lambda T_1)(\rho_g v) = T(\rho_g v) + \lambda T_1(\rho_g v) = \psi_g T(v) + \lambda \psi_g T_1(v) = \psi_g((T + \lambda T_1)(v))$$

for all  $v \in V$ .

**Proposition 9.1.** *Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be representations and  $T: V \rightarrow W$  be a linear map. Then*

$$T^\# = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \rho_g \in \mathrm{Hom}_G(V, W).$$

Moreover, the map  $\mathrm{Hom}(V, W) \rightarrow \mathrm{Hom}_G(V, W)$ ,  $T \mapsto T^\#$ , is linear and surjective.

*Proof.* Let  $h \in G$  and  $v \in V$ . Then

$$\begin{aligned} T^\# \phi_h(v) &= \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \phi_g \phi_h(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \phi_{gh}(v) \\ &= \frac{1}{|G|} \sum_{x \in G} \psi_{hx^{-1}} T \phi_x(v) = \frac{1}{|G|} \sum_{x \in G} \psi_h \psi_{x^{-1}} T \phi_x(v) = \psi_h T^\#(v). \end{aligned}$$

If  $T \in \mathrm{Hom}_{\mathbb{C}[G]}(V, V)$ , then

$$T^\#(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \phi_g(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \psi_g T(v) = T(v)$$

for all  $v \in V$ .  $\square$

If  $\phi: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  are non-equivalent irreducible representations and  $T: V \rightarrow W$  is a linear map, then  $T^\# = 0$ , as  $T^\# \in \text{Hom}_{\mathbb{C}[G]}(V, W) = \{0\}$  by the previous proposition and Schur's lemma.

**Theorem 9.2 (Ergodic theorem).** *Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be irreducible representations. If  $T: V \rightarrow W$  is linear, then  $T^\# = (\dim V)^{-1} \text{trace}(T) \text{id}$ .*

*Proof.* The previous proposition and Schur's lemma imply that  $T^\# = \lambda \text{id}$  for some  $\lambda \in \mathbb{C}$ . We now compute the trace of  $T^\#$ . On the one hand,

$$\text{trace}(T^\#) = \text{trace}(\lambda \text{id}) = (\dim V) \lambda.$$

On the other hand,

$$\text{trace}(T^\#) = \frac{1}{|G|} \sum_{g \in G} \text{trace}(\rho_{g^{-1}} T \rho_g) = \frac{1}{|G|} \sum_{g \in G} \text{trace}(T) = \text{trace}(T),$$

as  $\text{trace}(ABA^{-1}) = \text{trace}(B)$  for all  $A$  and  $B$ . Hence

$$\text{trace}(T^\#) = (\dim V)^{-1} \text{trace}(T) \text{id}. \quad \square$$

We now prove Schur's orthogonality relations. We need some preliminary material. First recall that the matrix  $E_{ij}$  is given by

$$(E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad \delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

If  $\rho: G \rightarrow \mathbf{GL}(V) \simeq \mathbf{GL}_n(\mathbb{C})$  is a representation, then  $\rho_g$  is the matrix  $(\rho_{ij}(g))$  and hence the character of  $\rho$  is given by

$$\chi_\rho(g) = \sum_{i=1}^n \rho_{ii}(g).$$

**Lemma 9.3.** *Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be irreducible representations. Then  $(E_{ik}^\#)_{lj} = \langle \rho_{ij}, \psi_{kl} \rangle$ .*

*Proof.* We compute

$$\begin{aligned}
(E_{ki}^\#)_{lj} &= \frac{1}{|G|} \sum_{g \in G} (\psi_{g^{-1}} E_{ki} \rho_g)_{lj} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{p,q} \psi_{lq}(g^{-1}) (E_{ki})_{qp} (\rho_{ij}(g))_{pj} \\
&= \frac{1}{|G|} \sum_{g \in G} \psi_{lk}(g^{-1}) \rho_{ij}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\psi_{kl}(g)} \rho_{ij}(g) = \langle \rho_{ij}, \psi_{kl} \rangle. \quad \square
\end{aligned}$$

**Theorem 9.4 (Schur).** *Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be irreducible representations. Then the following statements hold:*

1)  $\langle \rho_{ij}, \psi_{kl} \rangle = 0$  if  $\rho$  and  $\psi$  are not equivalent.

2)  $\langle \rho_{ij}, \rho_{kl} \rangle = \frac{1}{\dim V} \delta_{ik} \delta_{jl}$ .

*Proof.* Let us prove the first claim. Since  $\rho$  and  $\psi$  are not equivalent, it follows from Schur's lemma that  $\text{Hom}_G(V, W) = \{0\}$ . Thus  $E_{ki}^\# \in \text{Hom}_G(V, W) = \{0\}$  by the Ergodic theorem.

To prove the second claim, we use the previous lemma:

$$(E_{ki}^\#)_{lj} = \langle \rho_{ij}, \psi_{kl} \rangle = \frac{1}{\dim V} (\text{trace } E_{ki}) \delta_{lj} = \frac{1}{\dim V} \delta_{ki} \delta_{lj}. \quad \square$$

Now we can prove Schur's first orthogonality relation.

**Theorem 9.5 (Schur).** *Let  $\rho: G \rightarrow \mathbf{GL}(V)$  and  $\psi: G \rightarrow \mathbf{GL}(W)$  be irreducible representations. Then*

$$\langle \chi_\rho, \chi_\psi \rangle = \begin{cases} 1 & \text{if } \rho \simeq \psi, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $n = \dim V$  and  $m = \dim W$ . We compute

$$\begin{aligned}
\langle \chi_\rho, \chi_\psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\psi(g)} \\
&= \sum_{i=1}^n \sum_{j=1}^m \frac{1}{|G|} \rho_{ii}(g) \overline{\psi_{jj}(g)} = \sum_{i=1}^n \sum_{j=1}^m \frac{1}{|G|} \langle \rho_{ii}, \psi_{jj} \rangle = \begin{cases} 1 & \text{if } \rho \simeq \psi, \\ 0 & \text{otherwise.} \end{cases} \quad \square
\end{aligned}$$

Schur's theorem has several important corollaries.

If  $\rho: G \rightarrow \mathbf{GL}(V)$  is an irreducible representation of degree  $n$ , then

$$\{\sqrt{n} \rho_{ij} : 1 \leq i, j \leq n\}$$

is an orthonormal set.

**Corollary 9.6.** *A finite group has finitely many classes of irreducible representations.*

*Proof.* Let  $G$  be a finite group. Every isomorphism class of representations of  $G$  contains a unitary representation. Since  $\dim L(G) = |G|$ , it follows that  $G$  admits  $\leq |G|$  equivalence classes of irreducible representations. Let  $\rho_1, \dots, \rho_r$  be the representatives of the isomorphism classes of the irreducible representations of  $G$ . For each  $k$  let  $n_k = \deg \rho_k$ . Since the  $n_1^2 + \dots + n_r^2$  maps  $\sqrt{n_k}(\rho_k)_{ij}$ ,  $1 \leq k \leq r$ ,  $1 \leq i, j \leq n_k$ , form an orthonormal set of  $L(G)$ , it follows that  $r \leq n_1^2 + \dots + n_r^2 \leq |G|$ .  $\square$

Let  $G$  be a finite group. Since  $G$  has only finitely many non-equivalent irreducible representation, we will often say that

$$\rho_1, \dots, \rho_r$$

are the irreducible representations of  $G$ , where it is assumed that the  $\rho_i$  form a complete set of representatives of irreducible representations of  $G$ . For each  $i$  we write  $\chi_i = \chi_{\rho_i}$ . The set of irreducible characters will be denoted by

$$\text{Irr}(G) = \{\chi_1, \dots, \chi_r\}.$$

Recall that  $L(G) = \{f: G \rightarrow \mathbb{C}\}$  is a vector space with

$$(f+g)(x) = f(x) + g(x), \quad \lambda f(x) = \lambda f(x), \quad f, g \in L(G), \lambda \in \mathbb{C}, x \in G.$$

Let  $C(G)$  be the subspace of class functions. We claim that  $\dim C(G) = K(G)$ , the number of conjugacy classes of  $G$ . If  $C$  is a conjugacy class, then

$$\delta_C: G \rightarrow \mathbb{C}, \quad \delta_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise.} \end{cases}$$

is a class function. Let us prove that the set  $\{\delta_C : C \text{ is a conjugacy class of } G\}$  is a basis of  $C(G)$ . It is a generating set because each  $f$  can be written as

$$f = \sum_C f(C) \delta_C.$$

The  $\delta_C$  are linearly independent because they are orthogonal: If  $C$  and  $D$  are conjugacy classes of  $G$ , then

$$\langle \delta_C, \delta_D \rangle = \frac{1}{|G|} \sum_{x \in G} \delta_C(x) \overline{\delta_D(x)} = \begin{cases} |C|/|G| & \text{if } C = D, \\ 0 & \text{otherwise.} \end{cases}$$

**Corollary 9.7.** *Let  $G$  be a finite group. There are at most  $K(G)$  equivalence classes of irreducible representations of  $G$ .*

*Proof.* Non-equivalent representations have different characters. Irreducible characters form an orthonormal set, thus they are linearly independent. Since irreducible characters are class functions, it follows that there are at most  $K(G)$  irreducible different characters.  $\square$

Let  $m \in \mathbb{N}$ . If  $V$  is a vector space, we write  $mV = V \oplus \cdots \oplus V$  ( $m$ -times). Similarly, if  $\rho$  is a representation, we write  $m\rho = \rho \oplus \cdots \oplus \rho$  ( $m$ -times).

**Theorem 9.8.** *Let  $\rho_1, \dots, \rho_r$  be the irreducible representations of  $G$ . If  $\rho = \sum_{i=1}^r m_i \rho_i$  where  $m_1, \dots, m_r \in \mathbb{N}_0$ , then  $m_j = \langle \rho, \rho_j \rangle$  for all  $j \in \{1, \dots, r\}$ .*

*Proof.* Write  $\chi_\rho = \sum_{j=1}^r m_j \chi_j$ . Then

$$\langle \chi_\rho, \chi_i \rangle = \sum_{j=1}^r \langle \chi_j, \chi_i \rangle m_j = m_i$$

for all  $i \in \{1, \dots, r\}$ . □

The theorem states that the decomposition of a representation  $\rho$  into irreducibles is unique and that it is determined (up to equivalence) by its character.

**Corollary 9.9.** *A representation  $\rho$  is irreducible if and only if  $\langle \chi_\rho, \chi_\rho \rangle = 1$ .*

*Proof.* We first decompose  $\rho$  as a sum of irreducibles, say  $\rho = \sum_{j=1}^r m_j \rho_j$  with  $m_1, \dots, m_r \geq 0$ . Then  $\langle \chi_\rho, \chi_\rho \rangle = \sum_{j=1}^r m_j^2$ . Now  $\langle \chi_\rho, \chi_\rho \rangle = 1$  if and only if there is exactly one  $j$  such that  $m_j = 1$  and  $m_i = 0$  for all  $i \neq j$ . □

**Exercise 9.10.** Let  $\rho: H \rightarrow \mathbf{GL}(V)$  be an irreducible representation of  $H$  and  $f: G \rightarrow H$  be a surjective group homomorphism. Prove that the composition  $\rho \circ f: G \rightarrow \mathbf{GL}(V)$  is an irreducible representation.

**Theorem 9.11.** *Let  $G$  be a finite group and  $L$  be its regular representation. Then  $L = \sum_{j=1}^r n_j \rho_j$ , where  $n_j = \deg \rho_j$ .*

*Proof.* Let  $G = \{g_1, \dots, g_n\}$ ,  $n = |G|$ . If  $g \in G$ , since  $L_g(g_i) = gg_i$  for all  $i$ , the matrix of  $L_g$  in the basis  $\{g_1, \dots, g_n\}$  is then

$$(L_g)_{ij} = \begin{cases} 1 & \text{if } g_i = gg_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\chi_L(g) = \text{trace}(L_g) = \sum_{i=1}^r (L_g)_{ii} = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\langle \chi_L, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \overline{\chi_i(1)} = n_i$$

for all  $i \in \{1, \dots, n\}$ . □

Now several corollaries.

**Corollary 9.12.** *Let  $G$  be a finite group and  $\rho_1, \dots, \rho_r$  be the irreducible representations of  $G$ . For each  $k$  let  $n_k = \deg \rho_k$ . The following statements hold:*

- 1)  $|G| = n_1^2 + \dots + n_r^2$ .
- 2)  $\{\sqrt{n_k}(\rho_k)_{ij} : 1 \leq k \leq r, 1 \leq i, j \leq n_k\}$  is an orthonormal basis of  $L(G)$ .
- 3)  $r$  is equal to the number of conjugacy classes of  $G$ .

*Proof.* Since  $\chi_L = \sum_{j=1}^r n_j \chi_j$ , the first claim follows. The second claim follows from the orthogonality relations. Let us prove the third claim. Let  $f \in C(G)$  and write  $f$  as a linear combination of the  $(\rho_k)_{ij}$ , say

$$f = \sum_{i,j,k} \lambda_{ijk} (\rho_k)_{ij}, \quad \lambda_{ijk} \in \mathbb{C}.$$

If  $x \in G$ , then

$$\begin{aligned} f(x) &= \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} \lambda_{ijk} (\rho_k)_{ij}(g^{-1}xg) = \sum_{i,j,k} \lambda_{ijk} \frac{1}{|G|} \sum_{g \in G} (\rho_k)_{ij}(g^{-1}xg). \end{aligned}$$

Let  $T = (\rho_k)_x : V \rightarrow V$ . Then

$$T^\# = \frac{1}{|G|} \sum_{g \in G} (\rho_k)_{g^{-1}} (\rho_k)_x (\rho_k)_g = \frac{1}{|G|} \sum_{g \in G} (\rho_k)(g^{-1}xg) = \frac{1}{\dim V} \chi_k(x) \text{id}$$

by the Ergodic theorem and because  $\rho_k$  is a group homomorphism. Thus

$$f(x) = \sum_{i,j,k} \lambda_{ijk} ((\rho_k)_x)_{ij} = \sum_{i,j,k} \lambda_{ijk} \frac{1}{\dim V} \chi_k(x) \delta_{ij} = \sum_{i,k} \lambda_{ijk} \frac{1}{n_k} \chi_k(x). \quad \square$$

This implies that  $\dim C(G) \leq r$  and the claim follows.

In the following exercise, the reader is asked to provide a proof of the second Schur's orthogonality relation.

**Exercise 9.13.** Let  $G$  be a finite group and  $C$  and  $D$  be conjugacy classes of  $G$ . If  $g \in C$  and  $h \in D$ , then

$$\sum_{\chi \in \text{Irr}(G)} \chi(g) \overline{\chi(h)} = \begin{cases} |G|/|C| & \text{if } C = D, \\ 0 & \text{otherwise.} \end{cases}$$



## Chapter 10

### Some examples

Let  $G$  be a finite group and  $\chi_1, \dots, \chi_r$  be the irreducible characters of  $G$ . Without loss of generality we may assume that  $\chi_1$  is the trivial character, i.e.  $\chi_1(g) = 1$  for all  $g \in G$ . Recall that  $r$  is the number of conjugacy classes of  $G$ . Each  $\chi_j$  is constant on conjugacy classes. The **character table** of  $G$  is given by

	1	$k_2$	$\dots$	$k_r$
	1	$g_2$	$\dots$	$g_r$
$\chi_1$	1	1	$\dots$	1
$\chi_2$	$n_2$	$\chi_2(g_2)$	$\dots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\chi_r$	$n_r$	$\chi_r(g_2)$	$\dots$	$\chi_r(g_r)$

where the  $n_j$  are the degrees of the irreducible representations of  $G$  and each  $k_j$  is the size of the conjugacy class of the element  $g_j$ . By convention, the character table contains not only the values of the irreducible characters of the group.

**Example 10.1.** Sea  $G = \langle g : g^4 = 1 \rangle$  be the cyclic group of order four. The character table of  $G$  is given by

	1	1	1	1
	1	$g$	$g^2$	$g^3$
$\chi_1$	1	1	1	1
$\chi_2$	1	$\lambda$	$\lambda^2$	$\lambda^3$
$\chi_3$	1	$\lambda^2$	$\lambda^4$	$\lambda^2$
$\chi_4$	1	$\lambda^3$	$\lambda^2$	$\lambda$

**Exercise 10.2.** Let  $n \in \mathbb{N}$  be such that  $n \geq 2$ . Let  $C_n = \langle g : g^n = 1 \rangle$  be the cyclic group of order  $n$ .

- 1) Prove that the maps  $\chi_i : C_n \rightarrow \mathbb{C}^\times$ ,  $g^k \mapsto e^{2\pi i k/n}$ , where  $i \in \{0, 1, \dots, n-1\}$ , are the irreducible representations of  $C_n$ .
- 2) Let  $\lambda$  be a primitive root of 1 of order  $n$ . Prove that the character table of  $C_n$  of order  $n$  is given by

	1	1	1	...	1
	1	$g$	$g^2$	...	$g^{n-1}$
$\chi_1$	1	1	1	...	1
$\chi_2$	1	$\lambda$	$\lambda^2$	...	$\lambda^{n-1}$
$\chi_3$	1	$\lambda^2$	$\lambda^4$	...	$\lambda^{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\chi_n$	1	$\lambda^{n-1}$	$\lambda^{n-2}$	...	$\lambda$

**Exercise 10.3.** Let  $A$  and  $B$  be abelian groups. We write  $\text{Irr}(A) = \{\rho_1, \dots, \rho_r\}$  and  $\text{Irr}(B) = \{\phi_1, \dots, \phi_s\}$ . Prove that the maps

$$\varphi_{ij}: A \times B \rightarrow \mathbb{C}^\times, \quad (a, b) \mapsto \rho_i(a)\phi_j(b),$$

where  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, s\}$ , are the irreducible representations of  $A \times B$ .

Let us show a particular example of the previous exercise.

**Example 10.4.** The character table of the group  $C_2 \times C_2 = \{1, a, b, ab\}$  is

	1	1	1	1
	1	$a$	$b$	$ab$
$\chi_1$	1	1	1	1
$\chi_2$	1	1	-1	-1
$\chi_3$	1	-1	1	-1
$\chi_4$	1	-1	-1	1

Clearly, the order in which the computer returns the irreducible characters is not necessarily the same we used!

**Example 10.5.** The symmetric group  $\mathbb{S}_3$  has three conjugacy classes. The representatives are  $\text{id}$ ,  $(12)$  and  $(123)$ . There are three irreducible representations. We already found all the irreducible characters! The character table of  $\mathbb{S}_3$  is given by

	1	3	2
	1	$(12)$	$(123)$
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

Let us recall how this table was computed. Degree-one irreducibles were easy to compute. To compute the third row of the table one possible approach is to use the irreducible representation

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$\begin{aligned}\chi_3((12)) &= \text{trace} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 0, \\ \chi_3((123)) &= \chi_3((12)(23)) = \text{trace} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1.\end{aligned}$$

We should remark that the irreducible representation mentioned is not really needed to compute the third row of the character table. We can, for example, use the regular representation  $L$ . The character of  $L$  is given by

$$\chi_L(g) = \begin{cases} 6 & \text{si } g = \text{id}, \\ 0 & \text{si } g \neq \text{id}. \end{cases}$$

The equality  $0 = \chi_L((12)) = 1 - 1 + 2\chi_3((12))$  implies that  $\chi_3((12)) = 0$  and the equality  $0 = \chi_L((123)) = 1 + 1 + 2\chi_3((123))$  implies that  $\chi_3((123)) = -1$ .

Another approach uses the orthogonality relations. We need to compute  $\chi_3((12))$  and  $\chi_3((123))$ . Let  $a = \chi_3((12))$  and  $b = \chi_3((123))$ . Then we get that  $a = 0$  and  $b = -1$ . We just need to solve

$$\begin{aligned}0 &= \langle \chi_3, \chi_1 \rangle = \frac{1}{6}(2 + 3a + 2b), \\ 0 &= \langle \chi_3, \chi_2 \rangle = \frac{1}{6}(2 - 3a + 2b).\end{aligned}$$

**Exercise 10.6.** Compute the character table of  $\mathbb{S}_4$ .

**Example 10.7.** We now compute the character table of the alternating group  $\mathbb{A}_4$ . This group has 12 elements and four conjugacy classes.

representative	id	(123)	(132)	(123)
size	1	4	4	3

Since  $[\mathbb{A}_4, \mathbb{A}_4] = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ ,  $\mathbb{A}_4/[\mathbb{A}_4, \mathbb{A}_4]$  has three elements. Thus  $\mathbb{A}_4$  has three degree-one irreducibles and an irreducible character of degree three. Let  $\omega = \exp(2\pi i/3)$  be a primitive cubic root of 1. If  $\chi$  is a non-trivial degree-one character, then  $\chi((123)) = \omega^j$  for some  $j \in \{1, 2\}$  and  $\chi((132)) = \omega^{2j}$ . Since  $(132)(134) = (12)(34)$  and the permutations  $(134)$  and  $(123)$  are conjugate,

$$\chi_i((12)(34)) = \chi_i((132)(134)) = \chi_i((132))\chi_i((134)) = \omega^3 = 1$$

for all  $i \in \{1, 2\}$ .

To compute  $\chi_4$  we use the regular representation.

$$\begin{aligned}0 &= \chi_L((12)(34)) = 1 + 1 + 1 + 3\chi_4((12)(34)), \\ 0 &= \chi_L((123)) = 1 + \omega + \omega^2 + 3\chi_4((123)), \\ 0 &= \chi_L((132)) = 1 + \omega + \omega^2 + 3\chi_4((132)).\end{aligned}$$

Then we obtain that  $\chi_4((123)) = \chi_4((132)) = 0$  and  $\chi_4((12)(34)) = -1$ . Therefore, the character table of  $\mathbb{A}_4$  is given by

	id	(123)	(132)	(12)(34)
$\chi_1$	1	1	1	1
$\chi_2$	1	$\omega$	$\omega^2$	1
$\chi_3$	1	$\omega^2$	$\omega$	1
$\chi_4$	3	0	0	-1

**Example 10.8.** Let  $Q_8 = \{-1, 1, i, -i, j, -j\}$  be the quaternion group. Let us compute the character table of  $Q_8$ . The group  $Q_8$  is generated by  $\{i, j\}$  and the map  $\rho: Q_8 \rightarrow \mathbf{GL}_2(\mathbb{C})$ ,

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a representation. The conjugacy classes of  $Q_8$  are  $\{1\}$ ,  $\{-1\}$ ,  $\{-i, i\}$ ,  $\{-j, j\}$  and  $\{-k, k\}$ . So there are five irreducible representations. We can compute the character of  $\rho$ :

	1	-1	i	j	k
$\chi_\rho$	2	2	0	0	0

Then  $\rho$  is irreducible, as  $\langle \chi_\rho, \chi_\rho \rangle = 1$ .

Since  $[Q_8, Q_8] = \{-1, 1\} = Z(Q_8)$ , the quotient group  $Q_8/[Q_8, Q_8]$  has four elements and hence there are four irreducible degree-one representations. Since  $Q_8$  is non-abelian,  $Q_8/Z(Q_8)$  cannot be cyclic. This implies that  $Q_8/[Q_8, Q_8] \simeq C_2 \times C_2$ . This allows us to compute almost all the character table of  $Q_8$ .

	1	-1	i	j	k
$\chi_1$	1	1	1	1	1
$\chi_2$	1	1	-1	1	-1
$\chi_3$	1	1	1	-1	-1
$\chi_4$	1	1	-1	-1	1
$\chi_5$	2	-2	0	0	0

It remains to compute  $\chi_j(-1)$  for  $j \in \{2, 3, 4\}$ , these missing values are presented in shaded cells. To compute these values that  $\langle \chi_i, \chi_j \rangle = 0$  whenever  $i \neq j$ . The calculations are left as an exercise.

**Exercise 10.9.** Compute the character table of the dihedral group of eight elements.

## **Part III**

### **Modules**



## Chapter 11

# Modules, submodules, homomorphisms

**Definition 11.1.** Let  $R$  be a ring. A **module** (over  $R$ ) is an abelian group  $M$  with a map  $R \times M \rightarrow M$ ,  $(x, m) \mapsto x \cdot m$ , such that the following conditions hold:

- 1)  $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$  for all  $r_1, r_2 \in R$  y  $m \in M$ .
- 2)  $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$  for all  $r \in R$  y  $m_1, m_2 \in M$ .
- 3)  $r_1 \cdot (r_2 \cdot m) = (r_1 r_2) \cdot m$  for all  $r_1, r_2 \in R$  y  $m \in M$ .
- 4)  $1 \cdot m = m$  for all  $m \in M$ .

Our definition is that of left module. Similarly one defines right modules. We will always consider left modules, so they will be referred simply as modules.

**Example 11.2.** A module over a field is a vector space.

**Example 11.3.** Every abelian group is a modulo over  $\mathbb{Z}$ .

**Example 11.4.** Let  $R$  be a ring. Then  $R$  is a module (over  $R$ ) with  $x \cdot m = xm$ . This is the **(left) regular representation** of  $R$  and it usually be denoted by  ${}_R R$ .

**Example 11.5.** If  $R$  is a ring, then  $R^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in R\}$  is a module (over  $R$ ) with  $r \cdot (x_1, \dots, x_n) = (rx_1, \dots, rx_n)$ .

**Example 11.6.** If  $R$  is a ring, then  $M_{m,n}(R)$  is a module (over  $R$ ) with usual matrix operations.

Students usually ask why in the definition of a ring homomorphism one needs the condition  $1 \mapsto 1$ . The following example provides a good explanation.

**Example 11.7.** If  $f: R \rightarrow S$  is a ring homomorphism and  $M$  is a module (over  $S$ ) with  $(s, m) \mapsto sm$ , then  $M$  is also a module (over  $R$ ) with  $r \cdot m = f(r)m$  for all  $r \in R$  and  $m \in M$ . In fact,

$$\begin{aligned} 1 \cdot m &= f(1)m = 1m = m, \\ r_1 \cdot (r_2 \cdot m) &= f(r_1)(r_2 \cdot m) = f(r_1)(f(r_2)m) = (f(r_1)f(r_2))m = f(r_1 r_2)m \end{aligned}$$

for all  $r_1, r_2 \in R$  and  $m \in M$ .

**Example 11.8.** Let  $R = \mathbb{R}[X]$  and  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map. Then  $M = \mathbb{R}^n$  with

$$\left( \sum_{i=0}^n a_i X^i \right) \cdot v = \sum_{i=0}^n a_i T^i(v)$$

is a module (over  $R$ ).

**Example 11.9.** If  $\{M_i | i \in I\}$  is a family of modules, then

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i \text{ for all } i \in I\}$$

is a module with  $x \cdot (m_i)_{i \in I} = (x \cdot m_i)_{i \in I}$ , where  $(m_i)_{i \in I}$  denotes the map  $I \rightarrow M_i$ ,  $i \mapsto m_i$ . This module is the **direct product** of the family  $\{M_i : i \in I\}$ .

**Example 11.10.** If  $\{M_i | i \in I\}$  is family of modules, then

$$\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i \text{ for all } i \in I \text{ and } m_i = 0 \text{ except finitely many } i \in I\}$$

is a module with  $x \cdot (m_i)_{i \in I} = (x \cdot m_i)_{i \in I}$ . This module is the **direct sum** of the family  $\{M_i : i \in I\}$ .

If  $M$  is a module, then  $0 \cdot m = 0$  and  $-m = (-1) \cdot m$  for all  $m \in M$  and  $x \cdot 0 = 0$  for all  $x \in R$ .

**Example 11.11.** Let  $M = \mathbb{Z}/6$  as a module (over  $\mathbb{Z}$ ). Note that  $3 \cdot 2 = 0$  but  $3 \neq 0$  (in  $\mathbb{Z}$ ) and  $2 \neq 0$  (in  $\mathbb{Z}/6$ ).

**Definition 11.12.** Let  $M$  be a module. A subset  $N$  of  $M$  is a **submodule** of  $M$  if  $(N, +)$  is a subgroup of  $(M, +)$  and  $x \cdot n \in N$  for all  $x \in R$  and  $n \in N$ .

Clearly, if  $M$  is a module, then  $\{0\}$  and  $M$  are submodules of  $M$ .

**Example 11.13.** Let  $R$  be a field and  $M$  be a module over  $R$ . Then  $N$  is a submodule of  $M$  if and only if  $N$  is a subspace of  $M$ .

**Example 11.14.** Let  $R = \mathbb{Z}$  and  $M$  be a module (over  $R$ ). Then  $N$  is a submodule of  $M$  if and only if  $N$  is a subgroup of  $M$ .

**Example 11.15.** If  $M = {}_R R$ , then a subset  $N \subseteq M$  is a submodule of  $M$  if and only if  $N$  is a left ideal of  $R$ .

**Example 11.16.** If  $V$  is a vector space and  $T: V \rightarrow V$  is a linear map, then  $V$  is a module (over  $\mathbb{R}[X]$ ) with

$$\left( \sum_{i=0}^n a_i X^i \right) \cdot v = \sum_{i=0}^n a_i T^i(v).$$

A submodule is a subspace  $W$  of  $V$  such that  $T(W) \subseteq W$ .



Clearly, a subset  $N$  of  $M$  is a submodule if and only if  $r_1 n_1 + r_2 n_2 \in N$  for all  $r_1, r_2 \in R$  and  $n_1, n_2 \in N$ .

**Exercise 11.17.** If  $N$  and  $N_1$  are submodules of  $M$ , then

$$N + N_1 = \{n + n_1 : n \in N, n_1 \in N_1\}$$

is a submodule of  $M$ .

**Definition 11.18.** Let  $M$  and  $N$  be modules over  $R$ . A map  $f: M \rightarrow N$  is a **module homomorphism** if  $f(x+y) = f(x) + f(y)$  and  $f(r \cdot x) = r \cdot f(x)$  for all  $x, y \in M$  and  $r \in R$ .

We denote by  $\text{Hom}_R(M, N)$  the set of module homomorphisms  $M \rightarrow N$ .

**Exercise 11.19.** Let  $f \in \text{Hom}_R(M, N)$ .

- 1) If  $V$  is a submodule of  $M$ , then  $f(V)$  is a submodule of  $N$ .
- 2) If  $W$  is a submodule of  $N$ , then  $f^{-1}(W)$  is a submodule of  $M$ .

If  $f \in \text{Hom}_R(M, N)$ , the **kernel** of  $f$  is the submodule

$$\ker f = f^{-1}(\{0\}) = \{m \in M : f(m) = 0\}$$

of  $M$ . We say that  $f$  is a **monomorphism** (resp. **epimorphism**) if  $f$  is injective (resp. surjective). Moreover,  $f$  is an **isomorphism** if  $f$  is bijective.

**Exercise 11.20.** Let  $f \in \text{Hom}_R(M, N)$ . Prove that the following statements are equivalent:

- 1)  $f$  is a monomorphism.
- 2)  $\ker f = \{0\}$ .
- 3) For every module  $V$  and every  $g, h \in \text{Hom}_R(V, M)$ ,  $f \circ g = f \circ h \implies g = h$ .
- 4) For every module  $V$  and every  $g \in \text{Hom}(V, M)$ ,  $f \circ g = 0 \implies g = 0$ .

**Example 11.21.** Let  $R = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$ . We claim that  $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} \not\cong \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix}$  as modules over  $R$ , where the module structure is given by the usual matrix multiplication. Assume that they are isomorphic. Let  $f: \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$  be an isomorphism of modules and let  $x_0 \in \mathbb{R} \setminus \{0\}$  be such that  $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ . Thus

$$f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

a contradiction, as  $f$  is injective.



## **Chapter 12**

### **Finitely generated modules**



## **Chapter 13**

### **Free modules**



## **Chapter 14**

### **Structure theorems**





## Chapter 15

### Some hints

#### Rings and ideals

**1.48** Use the ring homomorphism  $\mathbb{Z}[\sqrt{10}] \rightarrow \mathbb{Z}/2$ ,  $a + b\sqrt{10} \mapsto a \bmod 2$ .

**1.49** Use the ring homomorphism  $\mathbb{Z} \hookrightarrow \mathbb{Z}[i] \xrightarrow{\pi} \mathbb{Z}[i]/(1+3i)$ , where  $\pi$  is the canonical map.



## **Chapter 16**

### **Some solutions**

**Rings and ideals**

**Chinese remainder theorem**

**Noetherian rings**

**Factorization**

**Zorn's lemma**



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