

Ring and module theory

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ABSTRACT. The notes correspond to the bachelor course **Ring and Module Theory** (2024) of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences.

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Introduction

The notes correspond to the bachelor course **Ring and Module Theory** of the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences. The course is divided into twelve or thirteen two-hours lectures.

The material is somewhat standard. Basic texts on abstract algebra are for example [1], [3] and [6]. Lang's book [7] is also a standard reference, but maybe a little bit more advanced. We based the lectures on the representation theory of finite groups on [9] and [10].

The notes include many exercises, some with full detailed solutions. Mandatory exercises have a green background, while optional ones (bonus exercises) have a yellow background.

We also mention a set of great expository papers written by Keith Conrad. The notes are extremely well-written and are useful at every stage of a mathematical career.

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1. Lecture: 25/09/2024

The objective of the first five lectures is to extract certain properties inherent to integers and adapt them for broader applications. Initially, we will identify similarities between the ring of integers and the ring of real polynomials, followed by a more in-depth exploration of these specific attributes in more general contexts. A pivotal similarity between integers and real polynomials lies in the existence of the division algorithm. Nevertheless, it is essential to emphasize that \mathbb{Z} and $\mathbb{R}[X]$ are fundamentally distinct entities, like apples and oranges. For instance, in $\mathbb{R}[X]$, the presence of the variable X allows us to utilize formal derivatives (with respect to X), a feature absent in the ring of integers.

§ 1.1. Rings.

1.1. DEFINITION. A **ring** is a non-empty set R with two binary operations, the addition $R \times R \rightarrow R$, $(x, y) \mapsto x + y$, and the multiplication $R \times R \rightarrow R$, $(x, y) \mapsto xy$, such that the following properties hold:

- 1) $(R, +)$ is an abelian group.
- 2) $(xy)z = x(yz)$ for all $x, y, z \in R$.
- 3) $x(y + z) = xy + xz$ for all $x, y, z \in R$.
- 4) $(x + y)z = xz + yz$ for all $x, y, z \in R$.
- 5) There exists $1_R \in R$ such that $x1_R = 1_Rx = x$ for all $x \in R$.

Our definition of a ring is that of a ring with identity. In general one writes the identity element 1_R as 1 if there is no risk of confusion.

1.2. DEFINITION. A ring R is said to be **commutative** if $xy = yx$ for all $x, y \in R$.

1.3. EXAMPLE. \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} are commutative rings.

Most of the students will have some familiarity with real polynomials from their school days.

1.4. EXAMPLE. The set

$$\mathbb{R}[X] = \left\{ \sum_{i=0}^n a_i X^i : n \in \mathbb{Z}_{\geq 0}, a_0, \dots, a_n \in \mathbb{R} \right\}$$

of real polynomials in one variable is a commutative ring with the usual operations. For example, if $f(X) = 1 + 5X^3$ and $g(X) = 3X - 2X^3$, then

$$\begin{aligned} f(X) + g(X) &= 1 + 3X + 3X^3, \\ f(X)g(X) &= 3X - 2X^3 + 15X^4 - 10X^6. \end{aligned}$$

Recall that, if

$$f(X) = a_0 + a_1X + a_2X^2 + \dots + a_nX^n$$

and $a_n \neq 0$, then a_n is the **leading coefficient** of $f(X)$ and n is the **degree** of $f(X)$. In this case, we use the notation $n = \deg f(X)$.

More generally, if R is a commutative ring, then $R[X]$ is a commutative ring. This construction allows us to define the polynomial ring $R[X, Y]$ in two commuting variables X and Y and coefficients in R as $R[X, Y] = (R[X])[Y]$. One can also define the ring $R[X_1, \dots, X_n]$ of polynomials in n commuting variables X_1, \dots, X_n with coefficients in R as

$$R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n].$$

1.5. EXAMPLE. If A is an abelian group, then the set $\text{End}(A)$ of group homomorphisms $A \rightarrow A$ is a ring with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(g(x)), \quad f, g \in \text{End}(A) \text{ and } x \in A.$$

If X is a set, we write $|X|$ to denote the size of X .

1.6. EXERCISE. Let R be a ring. Prove the following facts:

- 1) $x0 = 0x = 0$ for all $x \in R$.
- 2) $x(-y) = -xy$ for all $x, y \in R$.
- 3) If $1 = 0$, then $|R| = 1$.

1.7. EXAMPLE. The real vector space $H(\mathbb{R}) = \{a1 + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ with basis $\{1, i, j, k\}$ is a ring with the multiplication induced by the formulas

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

As an example, let us perform a calculation in $H(\mathbb{R})$:

$$(1 + i + j)(i + k) = i + k - 1 + ik + ji + jk = i + k - 1 - j - k + i = -1 + 2i - j,$$

as $ik = i(ij) = -j$ and $ji = -k$. This is the ring of real **quaternions**.

1.8. EXAMPLE. Let $n \geq 2$. The abelian group $\mathbb{Z}/n = \{0, 1, \dots, n-1\}$ of integers modulo n is a ring with the usual multiplication modulo n .

1.9. EXAMPLE. Let $n \geq 1$. The set $M_n(\mathbb{R})$ of real $n \times n$ matrices is a ring with the usual matrix operations. Recall that if $a = (a_{ij})$ and $b = (b_{ij})$, the multiplication ab is given by

$$(ab)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

Note that $M_n(\mathbb{R})$ is commutative if and only if $n = 1$.

Similarly, for any ring R one defines the ring $M_n(R)$ of $n \times n$ matrices with coefficients in R .

1.10. EXAMPLE. Let R and S be rings. Then $R \times S = \{(r, s) : r \in R, s \in S\}$ is a ring with the operations

$$(r, s) + (r_1, s_1) = (r + r_1, s + s_1), \quad (r, s)(r_1, s_1) = (rr_1, ss_1).$$

The zero of $R \times S$ is $(0_R, 0_S)$ and the unit element of $R \times S$ is $(1_R, 1_S)$. The ring $R \times S$ is known as the **direct product** of R and S .

1.11. DEFINITION. Let R be a ring. A **subring** S of R is a subset S such that $(S, +)$ is a subgroup of $(R, +)$ such that $1 \in S$ and if $x, y \in S$, then $xy \in S$.

For example, $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a chain of subrings.

1.12. EXAMPLE. $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is a subring of \mathbb{C} . This is known as the ring of **Gauss integers**.

A **square-free** integer is an integer that is divisible by no perfect square other than 1. Some examples: 2, 3, 5, 6, 7, and 10. The numbers 4, 8, 9, and 12 are not square-free.

1.13. EXAMPLE. Let N be a square-free integer. Then $\mathbb{Z}[\sqrt{N}]$ is a subring of \mathbb{C} .

If N is a square-free integer, then $a + b\sqrt{N} = c + d\sqrt{N}$ in $\mathbb{Z}[\sqrt{N}]$ if and only if $a = c$ and $b = d$. Why?

1.14. EXAMPLE. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subring of \mathbb{R} .

Why in the ring $\mathbb{Q}[\sqrt{2}]$ equality $a + b\sqrt{2} = c + d\sqrt{2}$ implies $a = c$ and $b = d$?

1.15. EXAMPLE. If R is a ring, then the **center**

$$Z(R) = \{x \in R : xy = yx \text{ for all } y \in R\}$$

is a subring of R .

If S is a subring of a ring R , then the zero-element of S is the zero-element of R , i.e., $0_R = 0_S$. Moreover, the additive inverse of an element $s \in S$ is the additive inverse of s as an element of R .

1.16. EXERCISE.

- 1) If S and T are subrings of R , then $S \cap T$ is a subring of R .
- 2) If $R_1 \subseteq R_2 \subseteq \dots$ is a sequence of subrings of R , then $\cup_{i \geq 1} R_i$ is a subring of R .

1.17. DEFINITION. Let R be a ring. An element $x \in R$ is a **unit** if there exists $y \in R$ such that $xy = yx = 1$.

The set $\mathcal{U}(R)$ of units of a ring R form a group with the multiplication. For example, $\mathcal{U}(\mathbb{Z}) = \{-1, 1\}$ and $\mathcal{U}(\mathbb{Z}/8) = \{1, 3, 5, 7\}$.

1.18. EXERCISE. Compute $\mathcal{U}(\mathbb{R}[X])$.

1.19. DEFINITION. A **division ring** is a ring $R \neq \{0\}$ such that $\mathcal{U}(R) = R \setminus \{0\}$.

The ring $H(\mathbb{R})$ of **real quaternions** is a non-commutative division ring. Find the inverse of an arbitrary element $a1 + bi + cj + dk \in H(\mathbb{R})$.

1.20. DEFINITION. A **field** is a commutative division ring (with $1 \neq 0$).

For example, \mathbb{Q} , \mathbb{R} , and \mathbb{C} are fields. If p is a prime number, then \mathbb{Z}/p is a field.

1.21. EXERCISE. Prove that $\mathbb{Q}[\sqrt{2}]$ is a field. Find the multiplicative inverse of a non-zero element of the form $x + y\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$.

More challenging: Prove that

$$\mathbb{Q}[\sqrt[3]{2}] = \{x + y\sqrt[3]{2} + z\sqrt[3]{4} : x, y, z \in \mathbb{Q}\}$$

is a field. What is the inverse of a non-zero element of the form $x + y\sqrt[3]{2} + z\sqrt[3]{4}$?

§ 1.2. Ideals.

1.22. DEFINITION. Let R be a ring. A **left ideal** of R is a subset $I \subseteq R$ such that $(I, +)$ is a subgroup of $(R, +)$ and $RI \subseteq I$, i.e. $ry \in I$ for all $r \in R$ and $y \in I$.

Similarly, one defines **right ideals**, one needs to replace the condition $RI \subseteq I$ by the inclusion $IR \subseteq I$.

1.23. EXAMPLE. Let $R = M_2(\mathbb{R})$. Then

$$I = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

is a right ideal R that is not a left ideal.

Can you find an example of a right ideal that is not a left ideal?

1.24. DEFINITION. Let R be a ring. An **ideal** of R is a subset that is both a left and a right ideal of R .

If R is a ring, then $\{0\}$ and R are both ideals of R .

1.25. EXERCISE. Let R be a ring.

- 1) If $\{I_\alpha : \alpha \in \Lambda\}$ is a collection of ideals of R , then $\cap_\alpha I_\alpha$ is an ideal of R .
- 2) If $I_1 \subseteq I_2 \subseteq \cdots$ is a sequence of ideals of R , then $\cup_{i \geq 1} I_i$ is an ideal of R .

1.26. EXAMPLE. Let $R = \mathbb{R}[X]$. If $f(X) \in R$, then the set

$$(f(X)) = \{f(X)g(X) : g(X) \in R\}$$

of multiples of $f(X)$ is an ideal of R . One can prove that this is the smallest ideal of R containing $f(X)$.

If R is a ring and X is a subset of R , one defines the ideal generated by X as the smallest ideal of R containing X , that is

$$(X) = \bigcap \{I : I \text{ ideal of } R \text{ such that } X \subseteq I\}.$$

One proves that

$$(X) = \left\{ \sum_{i=1}^m r_i x_i s_i : m \in \mathbb{Z}_{\geq 0}, x_1, \dots, x_m \in X, r_1, \dots, r_m, s_1, \dots, s_m \in R \right\},$$

where by convention, the empty sum is equal to zero. If $X = \{x_1, \dots, x_n\}$ is a finite set, then we write $(X) = (x_1, \dots, x_n)$.

1.27. EXERCISE. Prove that every ideal of \mathbb{Z} is of the form $n\mathbb{Z}$ for some $n \geq 0$.

1.28. EXERCISE. Let $n \geq 2$. Find the ideals of \mathbb{Z}/n .

1.29. EXERCISE. Find the ideals of \mathbb{R} .

A similar exercise is to find the ideals of any division ring.

1.30. EXERCISE. Let R be a commutative ring. Prove that every ideal of $M_n(R)$ is of the form $M_n(I)$ for some ideal I of R .

1.31. DEFINITION. Let R be a ring and I be an ideal of R . Then I is **principal** if $I = (x)$ for some $x \in R$.

The division algorithm shows that every ideal of \mathbb{Z} is principal, see Exercise 1.27.

The ring $\mathbb{R}[X]$ of real polynomials also has a division algorithm. Given the polynomials $f(X) \in \mathbb{R}[X]$ and $g(X) \in \mathbb{R}[X]$ with $g(X) \neq 0$, there are polynomials $q(X) \in \mathbb{R}[X]$ and $r(X) \in \mathbb{R}[X]$ such that

$$f(X) = q(X)g(X) + r(X),$$

where $r(X) = 0$ or $\deg r(X) < \deg g(X)$.

For example, if $f(X) = 2X^5 - X$ and $g(X) = X^2 + 1$, then

$$f(X) = q(X)g(X) + r(X),$$

where $q(X) = 2X^3 - 2X$ and $r(X) = X$.

1.32. EXERCISE. Prove that every ideal of $\mathbb{R}[X]$ is principal.

If K is a field, there is a division algorithm in the polynomial ring $K[X]$. Then one proves that every ideal of $K[X]$ is principal.

1.33. EXERCISE. Let R be a commutative ring and $x \in R$. Prove that $x \in \mathcal{U}(R)$ if and only if $(x) = R$. What happens if R is non-commutative?

A division ring (and, in particular, a field) has only two ideals.

1.34. DEFINITION. Let R and S be rings. A map $f: R \rightarrow S$ is a **ring homomorphism** if $f(1) = 1$, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

Our definition of a ring is that of a ring with identity. This means that the identity element 1 of a ring R is part of the structure. For that reason, in the definition of a ring homomorphism f one needs $f(1) = 1$.

1.35. EXAMPLE. The map $f: \mathbb{Z}/6 \rightarrow \mathbb{Z}/6$, $x \mapsto 3x$, satisfies

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{Z}/6$. However, f is not a ring homomorphism because $f(1) = 3$.

If R is a ring, then the identity map $\text{id}: R \rightarrow R$, $x \mapsto x$, is a ring homomorphism.

1.36. EXAMPLE. The inclusions $\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$ are ring homomorphisms.

More generally, if S is a subring of a ring R , then the inclusion map $S \hookrightarrow R$ is a ring homomorphism.

1.37. EXAMPLE. Let R be a ring. The map $\mathbb{Z} \rightarrow R$, $k \mapsto k1$, is a ring homomorphism.

1.38. EXAMPLE. Let $x_0 \in \mathbb{R}$. The evaluation map $\mathbb{R}[X] \rightarrow \mathbb{R}$, $f \mapsto f(x_0)$, is a ring homomorphism.

The **kernel** of a ring homomorphism $f: R \rightarrow S$ is the subset

$$\ker f = \{x \in R : f(x) = 0\}.$$

One proves that the kernel of f is an ideal of R . Moreover, recall from group theory that $\ker f = \{0\}$ if and only if f is injective. The image

$$f(R) = \{f(x) : x \in R\}$$

is a subring of S . In general, $f(R)$ is not an ideal of S .

1.39. EXAMPLE. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[X]$ is a ring homomorphism. Then $2\mathbb{Z}$ is an ideal of \mathbb{Z} but not an ideal of $\mathbb{Z}[X]$.

1.40. EXAMPLE. The map $\mathbb{C} \rightarrow M_2(\mathbb{R})$, $a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, is an injective ring homomorphism.

1.41. EXAMPLE. The map $\mathbb{Z}[i] \rightarrow \mathbb{Z}/5$, $a + bi \mapsto a + 2b \bmod 5$, is a ring homomorphism with $\ker f = \{a + bi : a + 2b \equiv 0 \bmod 5\}$.

1.42. EXERCISE. There is no ring homomorphism $\mathbb{Z}/6 \rightarrow \mathbb{Z}/15$. Why?

1.43. EXERCISE. If $f: \mathbb{R}[X] \rightarrow \mathbb{R}$ is a ring homomorphism such that the restriction $f|_{\mathbb{R}}$ of f onto \mathbb{R} is the identity, then there exists $x_0 \in \mathbb{R}$ such that f is the evaluation map at x_0 .

2. Lecture: 02/10/2024

§ 2.1. Quotients. Let R be a ring and I be an ideal of R . Then R/I is an abelian group with

$$(x + I) + (y + I) = (x + y) + I$$

and the **canonical map** $R \rightarrow R/I$, $x \mapsto x + I$, is a surjective group homomorphism with kernel I . Recall that R/I is the set of cosets $x + I$, where

$$x + I = y + I \iff x - y \in I.$$

Note that here we only used that I is an additive subgroup of R . We need an ideal to put a ring structure on the set R/I of cosets modulo I . As in the case of the integers, we use the following notation. For $x, y \in R$ we write

$$x \equiv y \pmod{I} \iff x - y \in I.$$

How can we put a ring structure on R/I ? It makes sense to define a multiplication on R/I so that the canonical map $R \rightarrow R/I$ is a surjective ring homomorphism. For that purpose, we define

$$(x + I)(y + I) = (xy) + I.$$

Since I is an ideal of R , this multiplication is well-defined. In fact, let $x + I = x_1 + I$ and $y + I = y_1 + I$. We want to show that $xy + I = x_1y_1 + I$. Since $x - x_1 \in I$,

$$xy - x_1y = (x - x_1)y \in I$$

because I is a right ideal. Similarly, since $y - y_1 \in I$, it follows that

$$x_1y - x_1y_1 = x_1(y - y_1) \in I,$$

as I is a left ideal. Thus

$$xy - x_1y_1 = xy - x_1y + x_1y - x_1y_1 = (x - x_1)y + x_1(y - y_1) \in I.$$

2.1. THEOREM. *Let R be a ring and I be an ideal of R . Then R/I with*

$$(x + I) + (y + I) = (x + y) + I, \quad (x + I)(y + I) = (xy) + I,$$

is a ring and the canonical map $R \rightarrow R/I$, $x \mapsto x + I$, is a surjective ring homomorphism with kernel I .

We have already seen that multiplication is well-defined. The rest of the proof is left as an exercise. As an example, we show that the left distributive property holds in R/I because it holds in R , that is

$$\begin{aligned} (x + I)((y + I) + (z + I)) &= (x + I)(y + z + I) \\ &= x(y + z) + I \\ &= xy + xz + I \\ &= (xy + I)(xz + I) \\ &= (x + I)(y + I) + (x + I)(z + I). \end{aligned}$$

2.2. EXERCISE. Prove Theorem 2.1.

2.3. EXAMPLE. Let $R = (\mathbb{Z}/3)[X]$ and $I = (2X^2 + X + 2)$ be the ideal of R generated by the polynomial $2X^2 + X + 2$. If $f(X) \in R$, the division algorithm allows us to write

$$f(X) = (2X^2 + X + 2)q(X) + r(X),$$

for some $q(X), r(X) \in R$, where either $r(X) = 0$ or $\deg r(X) < 2$. This means that $r(X) = aX + b$ for some $a, b \in \mathbb{Z}/3$. Note that

$$f(X) \equiv aX + b \pmod{2X^2 + X + 2}$$

for some $a, b \in \mathbb{Z}/3$, so the quotient ring R/I has nine elements. Can you find an expression for the product $(aX + b)(cX + d)$ in R/I ?

An **isomorphism** between the rings R and S is a bijective ring homomorphism $R \rightarrow S$. If such a homomorphism exists, then R and S are said to be isomorphic, and the notation is $R \simeq S$.

2.4. EXERCISE. Prove that ring isomorphism is an equivalence relation.

As it happens in the case of groups, to understand quotient rings, one has the first isomorphism theorem.

2.5. THEOREM (First isomorphism theorem). *If $f: R \rightarrow S$ is a ring homomorphism, then $R/\ker f \simeq f(R)$.*

This is somewhat similar to the result one knows from group theory. One needs to show that, if $I = \ker f$, then the map $R/I \rightarrow f(R)$, $x + I \mapsto f(x)$, is a well-defined bijective ring homomorphism.

2.6. EXERCISE. Prove Theorem 2.5.

2.7. EXAMPLE. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$$

with the usual matrix operations. A direct calculation shows that the map $R \rightarrow \mathbb{Q}$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto a$, is a surjective ring homomorphism with

$$\ker f = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Q} \right\}.$$

Thus $R/\ker f \simeq \mathbb{Q}$.

2.8. EXAMPLE. We will prove that

$$\mathbb{R}[X]/(X^2 + 1) \simeq \mathbb{C}.$$

But before going into the technical proof, let us show how this isomorphism works. Let $f(X) \in \mathbb{R}[X]$. The division algorithm on $\mathbb{R}[X]$ allows us to write

$$f(X) = (X^2 + 1)q(X) + r(X)$$

for some $q(X), r(X) \in \mathbb{R}[X]$, where $r(X) = 0$ or $\deg r(X) < 2$. Thus $r(X) = aX + b$ for some $a, b \in \mathbb{R}$. This implies that

$$f(X) \equiv aX + b \pmod{X^2 + 1}.$$

It is quite easy to describe the ring operation of $\mathbb{R}[X]/(X^2 + 1)$. Clearly

$$(aX + b) + (cX + d) \equiv (a + c)X + (b + d) \pmod{(X^2 + 1)},$$

Since $X^2 \equiv -1 \pmod{(X^2 + 1)}$,

$$(aX + b)(cX + d) \equiv X(ad + bc) + (bd - ac),$$

which reminds us of the usual multiplication rule of the field of complex numbers. This means that, in practice, this quotient ring looks pretty much like the ring of complex numbers.

Now is time to prove that

$$\mathbb{R}[X]/(X^2 + 1) \simeq \mathbb{C}.$$

Let $\varphi: \mathbb{R}[X] \rightarrow \mathbb{C}$, $f(X) \mapsto f(i)$, be the evaluation map. Then φ is a ring homomorphism. Moreover, φ is surjective, as $\varphi(a + bX) = a + bi$ for all $a, b \in \mathbb{R}$.

We claim that $\ker \varphi = (X^2 + 1)$, the ideal of $\mathbb{R}[X]$ generated by the polynomial $X^2 + 1$. Clearly, every multiple of $X^2 + 1$, that is every polynomial of the form $(X^2 + 1)g(X)$ for some $g(X) \in \mathbb{R}[X]$, belongs to the kernel of φ . Thus $(X^2 + 1) \subseteq \ker \varphi$. Conversely, if $f(X) \in \ker \varphi$, then $f(i) = 0$. Thus

$$f(X) = (X - i)g(X)$$

for some $g(X) \in \mathbb{R}[X]$. Since $f(X) \in \mathbb{R}[X]$, $f(-i) = 0$ and hence $g(-i) = 0$. Thus $X + i$ divides $g(X)$. Therefore

$$f(X) = (X - i)(X + i)h(X) = (X^2 + 1)h(X)$$

for some $h(X) \in \mathbb{R}[X]$. Therefore $f(X) \in (X^2 + 1)$.

By the first isomorphism theorem,

$$\mathbb{R}[X]/\ker \varphi = \mathbb{R}[X]/(X^2 + 1) \simeq \mathbb{C}.$$

2.9. EXERCISE. Prove that $\mathbb{Z}[\sqrt{-5}] \simeq \mathbb{Z}[X]/(X^2 + 5)$.

Similarly, if N is a square-free integer, then

$$\mathbb{Z}[\sqrt{N}] \simeq \mathbb{Z}[X]/(X^2 - N).$$

2.10. EXERCISE. Prove the following isomorphisms:

- 1) $\mathbb{Z}[X]/(7) \simeq (\mathbb{Z}/7)[X]$.
- 2) $\mathbb{Q}[\sqrt{2}] \simeq \mathbb{Q}[X]/(X^2 - 2)$.
- 3) $\mathbb{R}[X]/(X^2 - 1) \simeq \mathbb{R} \times \mathbb{R}$.
- 4) $\mathbb{Q}[X]/(X - 2) \simeq \mathbb{Q}$.
- 5) $\mathbb{R}[X, Y]/(X) \simeq \mathbb{R}[Y]$.

2.11. EXERCISE. Are the rings $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ isomorphic?

2.12. EXERCISE. Let R be the ring of continuous maps $[0, 2] \rightarrow \mathbb{R}$, where the operations are given by

$$\begin{aligned} (f + g)(x) &= f(x) + g(x), \\ (fg)(x) &= f(x)g(x). \end{aligned}$$

Prove that the set $I = \{f \in R : f(1) = 0\}$ is an ideal of R and that $R/I \simeq \mathbb{R}$.

2.13. EXERCISE. Let $n \geq 1$. Let R be a ring and I be an ideal of R . Prove that $M_n(I)$ is an ideal of $M_n(R)$ and that

$$M_n(R)/M_n(I) \simeq M_n(R/I).$$

2.14. EXERCISE. Let $R = \mathbb{Z}[\sqrt{10}]$ and $I = (2, \sqrt{10})$. Prove that $R/I \simeq \mathbb{Z}/2$.

Hint: Use the ring homomorphism $\mathbb{Z}[\sqrt{10}] \rightarrow \mathbb{Z}/2$, $a + b\sqrt{10} \mapsto a \bmod 2$.

2.15. EXERCISE. Prove that $\mathbb{Z}[i]/(1 + 3i) \simeq \mathbb{Z}/10$.

Hint: Use the ring homomorphism

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[i] \xrightarrow{\pi} \mathbb{Z}[i]/(1 + 3i),$$

where π is the canonical map.

2.16. EXERCISE. Prove that there is no ideal I of $\mathbb{Z}[i]$ such that $\mathbb{Z}[i]/I \simeq \mathbb{Z}/15$.

2.17. EXERCISE. Let $R = (\mathbb{Z}/2)[X]/(X^2 + X + 1)$.

- 1) How many elements does R have?
- 2) Can you recognize the additive group of R ?
- 3) Prove that R is a field.

Recall that if $f: X \rightarrow Y$ is a map and $A \subseteq X$ and $B \subseteq Y$ are subsets, then

$$f(A) = \{f(a) : a \in A\}, \quad f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

The following statements are easily proved:

- 1) $A \subseteq f^{-1}(f(A))$.
- 2) $A = f^{-1}(f(A))$ if f is injective.
- 3) $f(f^{-1}(B)) \subseteq B$.
- 4) $f(f^{-1}(B)) = B$ if f is surjective.

2.18. THEOREM (Correspondence theorem). *Let $f: R \rightarrow S$ be a surjective ring homomorphism. There exists a bijective correspondence between the set of ideals of R containing $\ker f$ and the set of ideals of S . Moreover, if $f(I) = J$, then $R/I \simeq S/J$.*

SKETCH OF THE PROOF. Let I be an ideal of R containing $\ker f$ and let J be an ideal of S . We need to prove the following facts:

- 1) $f(I)$ is an ideal of S .
- 2) $f^{-1}(J)$ is an ideal of R containing $\ker f$.
- 3) $f(f^{-1}(J)) = J$ and $f^{-1}(f(I)) = I$.
- 4) If $f(I) = J$, then $R/I \simeq S/J$.

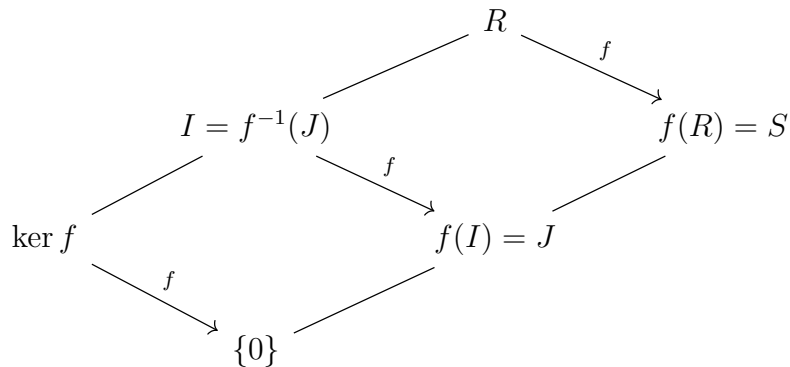
We only prove the fourth statement, the others are left as exercises. Note that the third claim implies that $f(I) = J$ if and only if $I = f^{-1}(J)$. Let $\pi: S \rightarrow S/J$ be the canonical map. The composition $g = \pi \circ f: R \rightarrow S/J$ is a surjective ring homomorphism and

$$\ker g = \{x \in R : g(x) = 0\} = \{x \in R : f(x) \in J\} = \{x \in R : x \in f^{-1}(J) = I\} = I.$$

Then the first isomorphism theorem implies that $R/I \simeq S/J$. □

2.19. EXERCISE. Prove Theorem 2.18.

As we did for groups, for the correspondence theorem, it helps to have in mind the following diagram:



§ 2.2. The Chinese remainder theorem. We now work with commutative rings. If R is a commutative ring and I and J are ideals of R , then

$$I + J = \{u + v : u \in I, v \in J\}$$

is an ideal of R . The sum of ideals makes sense also in non-commutative rings.

2.20. DEFINITION. Let R be a commutative ring. The ideals I and J of R are said to be **coprime** if $R = I + J$.

The terminology is motivated by the following example. If I and J are ideals of \mathbb{Z} , then $I = (a)$ and $J = (b)$ for some $a, b \in \mathbb{Z}$. Then Bezout's theorem states that

$$\begin{aligned}
 a \text{ and } b \text{ are coprime} &\iff 1 = ra + sb \text{ for some } r, s \in \mathbb{Z} \\
 &\iff I \text{ and } J \text{ are coprime.}
 \end{aligned}$$

In some books, coprime ideals are called **comaximal** ideals.

If I and J are ideals of R , then

$$IJ = \left\{ \sum_{i=1}^m u_i v_i : m \in \mathbb{Z}_{\geq 0}, u_1, \dots, u_m \in I, v_1, \dots, v_m \in J \right\}$$

is an ideal of R . Note that IJ is the set of all finite sum of elements of the form uv for $u \in I$ and $v \in J$. For example, if $I = (u)$ and $J = (v)$, then $IJ = (uv)$. Why we need to consider finite sums of elements of the form uv for $u \in I$ and $v \in J$?

2.21. EXERCISE. Let $R = \mathbb{R}[X, Y]$ and $I = J = (X, Y)$. Prove that the set

$$\{uv : u \in I, v \in J\}$$

is not an ideal of R .

Note that $IJ \subseteq I \cap J$. Equality does not hold in general. Take for example $R = \mathbb{Z}$ and $I = J = (2)$. Then $IJ = (4) \subsetneq (2) = I \cap J$.

2.22. PROPOSITION. Let R be a commutative ring. If I and J are coprime ideals, then $IJ = I \cap J$.

PROOF. Let $x \in I \cap J$. Since I and J are coprime, $1 = u + v$ for some $u \in I$ and $v \in J$,

$$x = x1 = x(u + v) = xu + xv \in IJ \quad \square.$$

2.23. BONUS EXERCISE. Let R be a commutative ring. Prove that $I \cap J = IJ$ for all ideals I and J of R if and only if R is **strongly regular**, that is for each $a \in R$ there exists $x \in R$ such that $a = xa^2$.

One can also prove that if R is a ring, $I \cap J = IJ$ holds for all left ideals I and J of R if and only if R is strongly regular.

2.24. THEOREM (Chinese remainder theorem). *Let R be a commutative ring and I and J be coprime ideals of R . If $u, v \in R$, then there exists $x \in R$ such that*

$$\begin{cases} x \equiv u \pmod{I}, \\ x \equiv v \pmod{J}. \end{cases}$$

PROOF. Since the ideals I and J are coprime, $1 = a + b$ for some $a \in I$ and $b \in J$. Let $x = av + bu$. Then

$$x - u = av + (b - 1)u = av - au = a(v - u) \in I,$$

that is $x \equiv u \pmod{I}$. Similarly, $x - v \in J$ and $x \equiv v \pmod{J}$. \square

For the following result, we need to use the direct product of rings. See Example 1.10.

2.25. COROLLARY. *Let R be a commutative ring. If I and J are coprime ideals of R , then $R/(I \cap J) \simeq R/I \times R/J$.*

PROOF. Let $\pi_I: R \rightarrow R/I$ and $\pi_J: R \rightarrow R/J$ be the canonical maps. A direct calculation shows that the map $\varphi: R \rightarrow R/I \times R/J$, $x \mapsto (\pi_I(x), \pi_J(x))$, is a ring homomorphism with $\ker \varphi = I \cap J$. For example, to prove that

$$\varphi(xy) = \varphi(x)\varphi(y)$$

we proceed as follows: if $x, y \in R$, then

$$\begin{aligned} \varphi(xy) &= (\pi_I(xy), \pi_J(xy)) \\ &= (\pi_I(x)\pi_I(y), \pi_J(x)\pi_J(y)) \\ &= (\pi_I(x), \pi_J(x))(\pi_I(y), \pi_J(y)) \\ &= \varphi(x)\varphi(y). \end{aligned}$$

We now claim that the map φ is surjective. To prove this, we will use the Chinese remainder theorem. If $(u + I, v + J) \in R/I \times R/J$, then there exists $x \in R$ such that $x - u \in I$ and $x - v \in J$. This translates into the surjectivity of φ , as

$$\varphi(x) = (\pi_I(x), \pi_J(x)) = (x + I, x + J) = (u + I, v + J).$$

Now $R/(I \cap J) \simeq R/I \times R/J$ by the first isomorphism theorem. \square

2.26. EXAMPLE. If n and m are coprime integers, then

$$\mathbb{Z}/(nm) \simeq \mathbb{Z}/n \times \mathbb{Z}/m$$

as rings. This follows from Corollary 2.25 with $R = \mathbb{Z}$, $I = (n)$ and $J = (m)$.

Let R be a commutative ring and I_1, \dots, I_n be ideals of R . Then $I_1 \cdots I_n$, defined as the set of all finite sums of the form $x_1 \cdots x_n$ such that $x_j \in I_j$ for all $j \in \{1, \dots, n\}$, is an ideal of R . For example, if $I_j = (x_j)$ for all $j \in \{1, \dots, n\}$, then $I_1 \cdots I_n = (x_1 \cdots x_n)$.

If I_1 and I_j are coprime for all $j \in \{2, \dots, n\}$, then I_1 and $I_2 \cdots I_n$ are coprime. If I_i and I_j are coprime whenever $i \neq j$, then $I_1 \cap \cdots \cap I_n = I_1 \cdots I_n$ and

$$R/(I_1 \cap \cdots \cap I_n) \simeq R/I_1 \times \cdots \times R/I_n.$$

Applying the previous observations in the case of the ring of integers, we get the following result: If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the factorization of n into powers of distinct primes, then

$$\mathbb{Z}/n \simeq \mathbb{Z}/(p_1^{\alpha_1}) \times \cdots \times \mathbb{Z}/(p_k^{\alpha_k})$$

as rings.

2.27. BONUS EXERCISE. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of n into powers of distinct primes. Prove that $\varphi(n) = \varphi(p_1^{\alpha_1}) \cdots \varphi(p_k^{\alpha_k})$, where φ is the Euler φ -function.

The story says that, instead of counting troops, the Chinese generals would order them to assemble in rows of different depths (hence obtaining the remainder of the number of soldiers modulo different integers) and then they would use the theorem to compute the size of the army.

2.28. BONUS EXERCISE. Let us gather people in the following way. When I count by three, there are two people left. When I count by four, there is one person left over and when I count by five there is one missing. How many people are there?

2.29. BONUS EXERCISE. Prove that

$$\begin{cases} x \equiv 29 \pmod{52}, \\ x \equiv 19 \pmod{72}, \end{cases}$$

does not have a solution.

2.30. BONUS EXERCISE. Find three consecutive integers such that the first one is divisible by a square, the second one is divisible by a cube and the third one is divisible by a fourth power.

2.31. BONUS EXERCISE. Prove that for each $n > 0$ there are n consecutive integers such that each integer is divisible by a perfect square $\neq 1$.

A special application of the Chinese remainder theorem leads to the construction of the Lagrange interpolation polynomial.

2.32. BONUS EXERCISE (Lagrange's interpolation theorem). The Chinese remainder theorem proves the following well-known result. Let $x_1, \dots, x_k \in \mathbb{R}$ be such that $x_i \neq x_j$ whenever $i \neq j$ and $y_1, \dots, y_k \in \mathbb{R}$. Then there exists $f(X) \in \mathbb{R}[X]$ such that

$$\begin{cases} f(X) \equiv y_1 \pmod{(X - x_1)}, \\ f(X) \equiv y_2 \pmod{(X - x_2)}, \\ \vdots \\ f(X) \equiv y_k \pmod{(X - x_k)}. \end{cases}$$

The solution $f(X)$ is unique modulo $(X - x_1)(X - x_2) \cdots (X - x_n)$.

With small changes, the Chinese remainder theorem can be proved in arbitrary non-commutative rings, see for example [6, Chapter III, Theorem 2.25].

3. Lecture: 09/10/2024

§ 3.1. Noetherian rings.

3.1. DEFINITION. A ring R is said to be **noetherian** if every (increasing) sequence

$$I_1 \subseteq I_2 \subseteq \cdots$$

of ideals of R stabilizes, that is $I_n = I_m$ for some $m \in \mathbb{Z}_{>0}$ and all $n \geq m$.

Finite rings are noetherian. The ring \mathbb{Z} of integers is noetherian. Why?

3.2. EXERCISE. Let $R = \{f: [0, 1] \rightarrow \mathbb{R}\}$ with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x), \quad f, g \in R, x \in [0, 1].$$

For $n \in \mathbb{Z}_{>0}$ let $I_n = \{f \in R : f|_{[0, 1/n]} = 0\}$. Then each I_n is an ideal of R and the sequence $I_1 \subsetneq I_2 \subsetneq \cdots$ does not stabilize. Thus R is not noetherian.

3.3. DEFINITION. Let R be a ring. An ideal I of R is said to be **finitely generated** if $I = (X)$ for some finite subset X of R .

If R is a ring, $\{0\}$ and R are finitely generated. Why?

If $X = \{x_1, \dots, x_n\}$ is a subset of a ring R , we write (x_1, \dots, x_n) to denote the ideal of R generated by the set $\{x_1, \dots, x_n\}$.

3.4. PROPOSITION. *Let R be a ring. Then R is noetherian if and only if every ideal of R is finitely generated.*

PROOF. Assume first that R is noetherian. Let I be an ideal of R that is not finitely generated. Thus $I \neq \{0\}$. Let $x_1 \in I \setminus \{0\}$ and let $I_1 = (x_1)$. Since I is not finitely generated, $I \neq I_1$ and hence $\{0\} \subsetneq I_1 \subsetneq I$. Let $x_2 \in I \setminus I_1$. Then $I_2 = (x_1, x_2)$ is a finitely generated ideal such that $I_1 \subsetneq I_2 \subsetneq I$. We continue with this procedure. Once I have the ideals I_1, \dots, I_{k-1} , let $x_k \in I \setminus I_{k-1}$ (such an element exists because I_{k-1} is finitely generated and I is not) and $I_k = (I_{k-1}, x_k)$. The sequence $\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ does not stabilize.

Assume now that every ideal of R is finitely generated and let $I_1 \subseteq I_2 \subseteq \cdots$ be a sequence of ideals of R . Then $I = \cup_{i \geq 1} I_i$ is an ideal of R , so it is finitely generated, say $I = (x_1, \dots, x_n)$. We may assume that $x_j \in I_{j_j}$ for all j . Let $N = \max\{j_1, \dots, j_n\}$. Then $x_j \in I_N$ for all $j \in \{1, \dots, n\}$ and hence $I \subseteq I_N$. This implies that $I_m = I_N$ for all $m \geq N$. \square

3.5. EXERCISE. Let $R = \mathbb{C}[X_1, X_2, \dots]$ be the ring of polynomial in an infinite number of commuting variables. Every element of R is a finite sum of the form

$$\sum a_{i_1 i_2 \dots i_k} X_{i_1}^{n_1} X_{i_2}^{n_2} \dots X_{i_k}^{n_k}$$

for some non-zero complex numbers $a_{i_1 i_2 \dots i_k}$. Prove that the ideal $I = (X_1, X_2, \dots)$ of polynomials with zero constant term is not finitely generated.

The correspondence theorem and the previous proposition allow us to prove easily the following result.

3.6. PROPOSITION. *Let I be an ideal of R . If R is noetherian, then R/I is noetherian.*

PROOF. Let $\pi: R \rightarrow R/I$ be the canonical surjection and let J be an ideal of R/I . Then $\pi^{-1}(J)$ is an ideal of R containing I . Since R is noetherian, $\pi^{-1}(J)$ is finitely generated, say $\pi^{-1}(J) = (x_1, \dots, x_k)$ for $x_1, \dots, x_k \in R$. Thus

$$J = \pi(\pi^{-1}(J)) = (\pi(x_1), \dots, \pi(x_k)),$$

because π is surjective and hence J is finitely generated. \square

Since \mathbb{Z} is noetherian, \mathbb{Z}/n is noetherian for all $n \geq 2$.

3.7. EXERCISE. Prove that $\mathbb{R}[X]$ is noetherian.

Hint: Use the fact that $\mathbb{R}[X]$ is a principal domain (Exercise 1.32).

3.8. THEOREM (Hilbert). *Let R be a commutative ring. If R is a noetherian ring, then $R[X]$ is noetherian.*

PROOF. We must show that every ideal of $R[X]$ is finitely generated. Assume that there is an ideal I of $R[X]$ that is not finitely generated. In particular, $I \neq \{0\}$. Let $f_1(X) \in I \setminus \{0\}$ be of minimal degree n_1 . Since I is not finitely generated, it follows that $I \neq (f_1(X))$. Let $f_2(X) \in I \setminus (f_1(X))$ be of minimal degree n_2 . In particular, the minimality of the degree of $f_1(X)$ implies that $n_2 \geq n_1$. We continue with this procedure. For $i > 1$ let $f_i(X) \in I$ be a polynomial of minimal degree n_i such that $f_i(X) \notin (f_1(X), \dots, f_{i-1}(X))$ (note that such an $f_i(X)$ exists because I is not finitely generated). Moreover, $n_i \geq n_{i-1}$. This happens because if $n_i < n_{i-1}$, then $f_i(X) \notin (f_1(X), \dots, f_{i-1}(X))$, which contradicts the minimality of $n_{i-1} = \deg f_{i-1}(X)$. For each $i \geq 1$ let a_i be the leading coefficient of $f_i(X)$, that is

$$f_i(X) = a_i X^{n_i} + \dots,$$

where the dots denote a polynomial of degree $< n_i$. Note that $a_i \neq 0$ for all $i \geq 1$.

Let $J = (a_1, a_2, \dots)$. Since R is noetherian, the sequence

$$(a_1) \subseteq (a_1, a_2) \subseteq \dots (a_1, a_2, \dots, a_k) \subseteq \dots$$

stabilizes, so we may assume that $J = (a_1, \dots, a_m)$ for some $m \in \mathbb{Z}_{>0}$. In particular, there exist $u_1, \dots, u_m \in R$ such that

$$a_{m+1} = \sum_{i=1}^m u_i a_i.$$

Let

$$g(X) = \sum_{i=1}^m u_i f_i(X) X^{n_{m+1}-n_i} \in (f_1(X), \dots, f_m(X)) \subseteq I.$$

The leading coefficient of $g(X)$ is $\sum_{i=1}^m u_i a_i = a_{m+1}$ and, moreover, the degree of $g(X)$ is n_{m+1} . Thus $\deg(g(X) - f_{m+1}(X)) < n_{m+1}$.

Since $f_{m+1}(X) \notin (f_1(X), \dots, f_m(X))$,

$$g(X) - f_{m+1}(X) \notin (f_1(X), \dots, f_m(X)),$$

a contradiction to the minimality of the degree of f_{m+1} . \square

Since $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$, by induction one proves that if R is a commutative noetherian ring, then $R[X_1, \dots, X_n]$ is noetherian.

3.9. EXAMPLE. Let $N > 0$ be a square-free integer. Since \mathbb{Z} is noetherian, so is $\mathbb{Z}[X]$ by Hilbert's theorem. Now $\mathbb{Z}[\sqrt{N}]$ is noetherian, as

$$\mathbb{Z}[\sqrt{N}] \simeq \mathbb{Z}[X]/(X^2 - N)$$

and quotients of noetherian rings are noetherian.

3.10. EXAMPLE. The ring $\mathbb{Z}[X, X^{-1}]$ is noetherian, as

$$\mathbb{Z}[X, X^{-1}] \simeq \mathbb{Z}[X, Y]/(XY - 1).$$

3.11. EXERCISE. Let R be a commutative ring and $R[[X]]$ be the ring of formal power series with the usual operations. Prove that $R[[X]]$ is noetherian if R is noetherian.

3.12. EXERCISE. Let $f: R \rightarrow R$ be a surjective ring homomorphism. Prove that f is an isomorphism if R is noetherian.

§ 3.2. Factorization.

3.13. DEFINITION. A **domain** R is a ring such that $xy = 0$ implies $x = 0$ or $y = 0$. An **integral domain** is a commutative ring that is also a domain.

The rings \mathbb{Z} and $\mathbb{Z}[i]$ are both integral domains. More generally, if N is a square-free integer, then the ring $\mathbb{Z}[\sqrt{N}]$ is an integral domain. The ring $\mathbb{Z}/4$ of integers modulo 4 is not an integral domain.

3.14. EXAMPLE. The ring $\mathbb{Z}/12$ is not a domain. Weird things can happen when we work over these rings. For example, the degree-two $f(X) = X^2 - 4 \in (\mathbb{Z}/12)[X]$ has more than two roots, as $f(2) = f(8) = f(10) = 0$. In fact,

$$f(X) = (X - 2)(X - 10) = (X - 8)(X - 10).$$

The previous example shows why it might be better to work over integral domains.

3.15. DEFINITION. Let R be an integral domain and $x, y \in R$. Then x **divides** y if $y = xz$ for some $z \in R$. Notation: $x \mid y$ if and only if x divides y . If x does not divide y one writes $x \nmid y$.

Note that $x \mid y$ if and only if $(y) \subseteq (x)$.

3.16. DEFINITION. Let R be an integral domain and $x, y \in R$. Then x and y are **associate** in R if $x = yu$ for some $u \in \mathcal{U}(R)$.

Note that x and y are associate if and only if $(x) = (y)$.

3.17. EXAMPLE. The integers 2 and -2 are associate in \mathbb{Z} .

3.18. EXAMPLE. Let $R = \mathbb{Z}[i]$.

- 1) Let $d \in \mathbb{Z}$ and $a + ib \in R$. Then $d \mid a + ib$ in R if and only if $d \mid a$ and $d \mid b$ in \mathbb{Z} .
- 2) 2 and $-2i$ are associate in R .

3.19. EXAMPLE. Let $R = \mathbb{R}[X]$ and $f(X) \in R$. Then $f(X)$ and $\lambda f(X)$ are associate in R for all $\lambda \in \mathbb{R}^\times$.

3.20. DEFINITION. Let R be an integral domain and $x \in R \setminus \{0\}$. Then x is **irreducible** if and only if $x \notin \mathcal{U}(R)$ and $x = ab$ with $a, b \in R$ implies that $a \in \mathcal{U}(R)$ or $b \in \mathcal{U}(R)$.

Note that $x \in R$ is irreducible if and only if $(x) \neq R$ and there is no principal ideal (y) such that $(x) \subsetneq (y) \subsetneq R$.

3.21. EXAMPLE. Let $R = \mathbb{R}[X]$ and $f(X) \in R \setminus \{0\}$. Then the polynomial $f(X)$ is irreducible if $\lambda \in \mathbb{R}^\times$ or $\lambda f(X)$ for $\lambda \in \mathbb{R}^\times$ are the only divisors of $f(X)$.

The irreducibles of \mathbb{Z} are the prime numbers. Note that $p \in \mathbb{Z}$ is prime if and only if $p \mid xy$ then $p \mid x$ or $p \mid y$.

3.22. EXAMPLE. The polynomial $X^4 + 1 \in \mathbb{Z}[X]$ is irreducible. If not, since $X^4 + 1$ has no integer roots, there are $a, b, c, d \in \mathbb{Z}$ such that

$$X^4 + 1 = (X^2 + aX + b)(X^2 + cX + d).$$

This implies that

$$(3.1) \quad \begin{cases} bd = 1, \\ cb + ad = 0, \\ b + d + ac = 0, \\ a + c = 0. \end{cases}$$

This is a contradiction. as (3.1) has no integer solutions.

If p is a prime number, one can prove that $X^4 + 1 \in (\mathbb{Z}/p)[X]$ is never irreducible. For example $X^4 + 1 = (X^2 + 1)^2$ in $(\mathbb{Z}/2)[X]$.

3.23. DEFINITION. Let R be an integral domain and $p \in R \setminus \{0\}$. Then p is **prime** if $p \notin \mathcal{U}(R)$ and $p \mid xy$ implies that $p \mid x$ or $p \mid y$.

Note that a non-zero element $p \in R$ is prime if and only if $(p) \neq R$ and $xy \in (p)$ implies that $x \in (p)$ or $y \in (p)$.

3.24. PROPOSITION. *Let R be an integral domain and $p \in R$. If p is prime, then p is irreducible.*

PROOF. Let p be a prime. Then $p \neq 0$ and $p \notin \mathcal{U}(R)$. Let x be such that $x \mid p$. Then $p = xy$ for some $y \in R$. In particular, $p \mid xy$. Since p is prime, $p \mid x$ or $p \mid y$. If $p \mid x$, then p and x are associate, that is $p = xu$ for some $u \in \mathcal{U}(R)$. This implies that $xu = p = xy$, so $x(u - y) = 0$. Since $x \neq 0$, $y = u \in \mathcal{U}(R)$. If $p \mid y$, then $y = pz$ for some $z \in R$. Then $y = pz = (xy)z$ and hence $y(1 - xz) = 0$. Since $y \neq 0$, $xz = 1$ and hence $x \in \mathcal{U}(R)$. \square

In \mathbb{Z} primes and irreducible coincide. This does not happen in full generality. To show that there are rings where some irreducibles are not prime, we need the following lemma.

3.25. LEMMA. *Let $N \in \mathbb{Z}$ be a non-zero square-free integer and $R = \mathbb{Z}[\sqrt{N}]$. Then the map*

$$\text{norm}: R \rightarrow \mathbb{Z}_{\geq 0}, \quad a + b\sqrt{N} \mapsto |a^2 - Nb^2|,$$

satisfies the following properties:

- 1) $\text{norm}(\alpha) = 0$ if and only if $\alpha = 0$.
- 2) $\text{norm}(\alpha\beta) = \text{norm}(\alpha)\text{norm}(\beta)$ for all $\alpha, \beta \in R$.
- 3) $\alpha \in \mathcal{U}(\mathbb{Z}[\sqrt{N}])$ if and only if $\text{norm}(\alpha) = 1$.
- 4) If $\text{norm}(\alpha)$ is prime in \mathbb{Z} , then α is irreducible in R .

PROOF. The first three items are left as exercises. Let us prove 4). If $\alpha = \beta\gamma$ for some $\beta, \gamma \in R$, then $\text{norm}(\alpha) = \text{norm}(\beta)\text{norm}(\gamma)$. Since $\text{norm}(\alpha) \in \mathbb{Z}$ is a prime number, it follows that $\text{norm}(\beta) = 1$ or $\text{norm}(\gamma) = 1$. Thus $\beta \in \mathcal{U}(R)$ or $\gamma \in \mathcal{U}(R)$. \square

3.26. EXERCISE. Prove Lemma 3.25.

3.27. BONUS EXERCISE. Prove that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.

Hint: You can prove, for example, that $(1 + \sqrt{2})^n \in \mathcal{U}(\mathbb{Z}[\sqrt{2}])$ for all $n \geq 1$.

3.28. EXAMPLE. Let $R = \mathbb{Z}[i]$.

1) $\mathcal{U}(R) = \{-1, 1, i, -i\}$.

2) 3 is irreducible in R . In fact, if $3 = \alpha\beta$, then

$$9 = \text{norm}(\alpha)\text{norm}(\beta).$$

Thus $\text{norm}(\alpha) \in \{1, 3, 9\}$. Write $\alpha = a + bi$ for $a, b \in \mathbb{Z}$. If $\text{norm}(\alpha) = 1$, then $\alpha \in \mathcal{U}(R)$ by Lemma 3.25. If $\text{norm}(\alpha) = 9$, then $\text{norm}(\beta) = 1$ and hence $\beta \in \mathcal{U}(R)$ by the lemma. Finally, if $\text{norm}(\alpha) = 3$, then $a^2 + b^2 = 3$, which is a contradiction since $a, b \in \mathbb{Z}$.

3) 2 is not irreducible in R . In fact, $2 = (1 + i)(1 - i)$ and since

$$\text{norm}(1 + i) = \text{norm}(1 - i) = 2,$$

it follows that $1 + i \notin \mathcal{U}(R)$ and $1 - i \notin \mathcal{U}(R)$.

4. Lecture: 16/10/2024

4.1. EXAMPLE. Let $R = \mathbb{Z}[\sqrt{-3}]$ and $x = 1 + \sqrt{-3}$.

1) x is irreducible. If $x = \alpha\beta$ for some $\alpha, \beta \in R$, then

$$4 = \text{norm}(x) = \text{norm}(\alpha) \text{norm}(\beta).$$

Thus $\text{norm}(\alpha) \in \{1, 2, 4\}$. Write $\alpha = a + b\sqrt{-3}$ for some $a, b \in \mathbb{Z}$. If $\text{norm}(\alpha) = a^2 + 3b^2 = 2$, then $b = 0$ and hence $a^2 = 2$, a contradiction. If $\text{norm}(\alpha) = 1$, then $\alpha \in \mathcal{U}(R)$. If $\text{norm}(\alpha) = 4$, then $\text{norm}(\beta) = 1$ and $\beta \in \mathcal{U}(R)$.

2) x is not prime. Note that

$$(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4,$$

then x divides $4 = 2 \cdot 2$. But $1 + \sqrt{-3} \nmid 2$, as

$$(a - 3b) + (a + b)\sqrt{-3} = (1 + \sqrt{-3})(a + b\sqrt{-3}) = 2$$

implies that $a - 3b = 2$ and $a + b = 0$, which yields $a = 1/2 \notin \mathbb{Z}$, a contradiction.

4.2. EXERCISE. Let $R = \mathbb{Z}[\sqrt{-5}]$.

1) Prove that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible in R .

2) Prove that $2, 3, 1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are not associate in R .

4.3. EXERCISE. Let $R = \mathbb{Z}[\sqrt{5}]$. Prove that $1 + \sqrt{5}$ is irreducible and not prime in R .

4.4. DEFINITION. Let R be an integral domain. Then R is **principal** (or a principal ideal domain) if every ideal I of R is of the form $I = (x)$ for some $x \in R$.

An ideal I of the form $I = (x)$ for some x is called a **principal ideal**.

The rings \mathbb{Z} and $\mathbb{R}[X]$ are both principal.

4.5. EXAMPLE. The ring $\mathbb{Z}[X]$ is not principal. For example, the ideal $I = (2, X)$ is not principal.

First note that $I \neq \mathbb{Z}[X]$. In fact, if $I = \mathbb{Z}[X]$, then $1 = 2f(X) + Xg(X)$ for some polynomials $f(X), g(X) \in \mathbb{Z}[X]$. Then $1 = 2f(0)$, which implies that $-1/2 = f(0) \in \mathbb{Z}$, a contradiction.

If $I = (h(X))$ for some $h(X) \in \mathbb{Z}[X]$, then $2 = h(X)g(X)$ for some $g(X) \in \mathbb{Z}[X]$. This implies that $\deg(h(X)) = 0$, so $h(X) = h(1) \in \mathbb{Z}$. In particular, $2 = h(1)g(1)$ and hence $h(1) \in \{-1, 1, 2, -2\}$. Since $I \neq \mathbb{Z}[X]$, it follows that $h(X) = h(1) \notin \{-1, 1\}$. Now $X = h(X)f(X)$ for some $f(X) \in \mathbb{Z}[X]$. In particular, $\deg(f(X)) = 1$, so we may assume that $f(X) = a_0 + a_1X$ for $a_0, a_1 \in \mathbb{Z}$ and $a_1 \neq 0$. It follows that

$$X = \pm 2f(X) = \pm 2(a_0 + a_1X)$$

and therefore $\pm 1/2 = a_1 \in \mathbb{Z}$, a contradiction.

4.6. EXERCISE. Let R be a principal domain. Prove that R is noetherian.

4.7. EXAMPLE. Let $R = \mathbb{Z}[\sqrt{-5}]$. The ideal $I = (2, 1 + \sqrt{-5})$ is not principal, so R is not principal.

We first note that $I \neq R$. If not, there exist $x, y, u, v \in \mathbb{Z}$ such that

$$1 = 2(x + y\sqrt{-5}) + (1 + \sqrt{-5})(u + v\sqrt{-5}) = (2x + u - 5v) + \sqrt{-5}(2y + u + v).$$

This implies that $1 = 2x + u - 5v$ and $0 = 2y + u + v$. These formulas imply that

$$1 = 2(x + y + u - 2v),$$

a contradiction because $x + y + u - 2v \in \mathbb{Z}$.

Now assume that $I = (\alpha)$ for some α . Then $\alpha \mid 2$ and $\alpha \mid 1 + \sqrt{-5}$. Then $\text{norm}(\alpha) \mid 4$ and $\text{norm}(\alpha) \mid 6$, because $\text{norm}(2) = 4$ and $\text{norm}(1 + \sqrt{-5}) = 6$. Thus $\text{norm}(\alpha) \in \{1, 2\}$. If we write $\alpha = a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$, then it follows that $a^2 + 5b^2 = \text{norm}(\alpha)$. Clearly, $\text{norm}(\alpha) \neq 2$, as $a^2 = 2$ has no solution in \mathbb{Z} . Now $\text{norm}(\alpha) = 1$ implies that $\alpha \in \mathcal{U}(R)$ and hence $I = R$, a contradiction.

Sometimes primes and irreducibles coincide.

4.8. PROPOSITION. *Let R be a principal domain and $x \in R$. Then x is irreducible if and only if x is prime.*

PROOF. We only need to prove that if x is irreducible, then x is prime. Let us assume that $x \mid yz$. Let $I = (x, y)$. Since R is principal, $I = (a)$ for some $a \in R$. In particular, $x = ab$ for some $b \in R$. Since x is irreducible, $a \in \mathcal{U}(R)$ or $b \in \mathcal{U}(R)$. If $a \in \mathcal{U}(R)$, then $I = R$ and hence $1 = xr + ys$ for some $r, s \in R$. Thus

$$z = z1 = z(xr + ys) = zxr + zys$$

and therefore $x \mid z$. If $b \in \mathcal{U}(R)$, then x and a are associate in R . Thus $I = (x) = (a)$ and hence $xt = y$ for some $t \in R$, that is $x \mid y$. \square

In the integers primes and irreducible coincide. This happens because \mathbb{Z} is a principal domain.

4.9. EXAMPLE. Since $\mathbb{Z}[\sqrt{-3}]$ have irreducible elements that are not prime, $\mathbb{Z}[\sqrt{-3}]$ is not a principal domain.

4.10. DEFINITION. Let R be an integral domain. We say that R is an **euclidean domain** if there exists a map $\varphi: R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$ such that for every $x, y \in R$ with $y \neq 0$ there exist $q, r \in R$ such that $x = qy + r$, where $r = 0$ or $\varphi(r) < \varphi(y)$.

Note that in the definition of euclidean domains, we do not ask for the uniqueness of the quotient and the remainder. In fact, we will meet important examples of euclidean domains where uniqueness in the division algorithm is not achieved.

4.11. EXAMPLE. \mathbb{Z} is a euclidean domain with $\varphi(x) = |x|$.

Do we have uniqueness in the previous example?

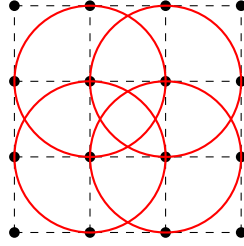
4.12. EXAMPLE. $\mathbb{R}[X]$ is a euclidean domain with $\varphi(f(X)) = \deg(f(X))$.

The previous examples suggest why for the definition of a euclidean domain one needs to consider a map $\varphi: R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$.

4.13. EXAMPLE. Let $R = \mathbb{Z}[i]$. Then R is a euclidean domain with $\varphi(\alpha) = \text{norm}(\alpha)$. Let $\alpha, \beta \in \mathbb{Z}[i]$ and assume that $\beta \neq 0$. We want to find Gauss integers $\gamma, \delta \in \mathbb{Z}[i]$ such that

$$\alpha = \beta\delta + \gamma$$

and $\varphi(\gamma) < \varphi(\beta)$. Note that every complex number $z \in \mathbb{C}$ can be written as $z = \xi + \eta$, where $\xi \in \mathbb{Z}[i]$ and $\varphi(\eta) < 1$. Indeed, this follows from the fact that \mathbb{C} is covered by open unit disks of radius one and centers in Gauss integers, see the following figure:



Now let $\alpha = a + ib$ and $\beta = c + id$, where $a, b, c, d \in \mathbb{Z}$. Write

$$\frac{\alpha}{\beta} = \frac{a + ib}{c + id} = r + is$$

for some $r, s \in \mathbb{Q}$. We don't need to compute r and s explicitly! Let $m, n \in \mathbb{Z}$ be such that $|r - m| \leq 1/2$ and $|s - n| \leq 1/2$. If $\delta = m + in$ and $\gamma = \alpha - \beta\delta$, then $\gamma \in R$, $\delta \in R$ and $\alpha = \beta\delta + \gamma$. If $\gamma \neq 0$, then

$$\begin{aligned} \varphi(\gamma) &= \varphi\left(\beta\left(\frac{\alpha}{\beta} - \delta\right)\right) = \varphi(\beta)\varphi\left(\frac{\alpha}{\beta} - \delta\right) \\ &= \varphi(\beta)\varphi((r - m) + i(s - n)) = \varphi(\beta)((r - m)^2 + (s - n)^2) \\ &\leq \varphi(\beta)(1/4 + 1/4) \\ &< \varphi(\beta). \end{aligned}$$

In $\mathbb{Z}[i]$ the division algorithm does not have uniqueness. In fact, if $\alpha, \beta \in \mathbb{Z}[i]$ and $\alpha = \beta\delta + \gamma$ for some $\delta, \gamma \in \mathbb{Z}[i]$, then there are up to four possibilities for the remainder γ .

4.14. EXAMPLE. Let $R = \mathbb{Z}[i]$ and $\alpha = -1 + i$ and $\beta = 1 + 2i$. Let $I = (\beta)$ be the ideal of R generated by β . First, note that

$$I = (\beta) = (1 + 2i)R = (1 + 2i)\mathbb{Z} + (1 + 2i)\mathbb{Z}i = (1 + 2i)\mathbb{Z} + (-2 + i)\mathbb{Z}.$$

This allows us to draw the lattice of elements of I , that is the lattice formed by the multiples of β , see Figure 4.1. Since $\alpha - \gamma \in I = (\beta)$, there are at most four possibilities for writing the division algorithm. In our particular example, we find three possible cases:

- 1)** If $\alpha - \gamma = \beta_0$, where $0 = \beta_0 = \beta 0$, then $\gamma = \alpha = -1 + i$ and

$$-1 + i = (1 + 2i)0 + (-1 + i)$$

with $\text{norm}(-1 + i) = 2 < \text{norm}(\beta) = 5$.

- 2)** If $\alpha - \gamma = \beta_1$, where $-2 + i = \beta_1 = \beta i$, then $\gamma = 1$ and

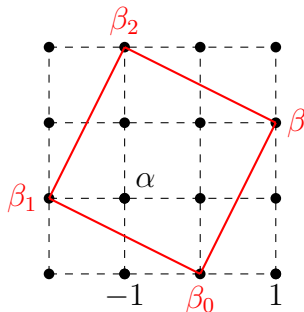
$$-1 + i = (1 + 2i)i + (-1)$$

with $\text{norm}(-1) = 1 < \text{norm}(\beta) = 5$.

- 3)** If $\alpha - \gamma = \beta_2$, where $-1 + 3i = \beta_2 = \beta(1 + i)$, then $\gamma = 2i$ and

$$-1 + i = (1 + 2i)(1 + i) + (-2i)$$

and $\text{norm}(-2i) = 4 < \text{norm}(\beta) = 5$.


 FIGURE 4.1. Division algorithm in $\mathbb{Z}[i]$.

We know that \mathbb{Z} and $\mathbb{R}[X]$ are both principal. The proofs are very similar, as both use the division algorithm essentially in the same way. The following result takes advantage of this fact.

4.15. PROPOSITION. *Let R be a euclidean domain. Then R is principal.*

PROOF. Let I be an ideal of R . If $I = \{0\}$, then $I = (0)$ and hence it is principal. So we may assume that $I \neq \{0\}$. Let $y \in I \setminus \{0\}$ be such that $\varphi(y)$ is minimal. We claim that $I = (y)$. If $z \in I$, then $z = yq + r$, where $r = 0$ or $\varphi(r) < \varphi(y)$. The minimality of $\varphi(y)$ implies that $r = 0$. Thus $z = yq \in (y)$ and it follows that $I = (y)$. \square

4.16. EXAMPLE. Since $\mathbb{Z}[i]$ is euclidean, it is principal.

4.17. EXAMPLE. The rings $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\sqrt{-3}]$ are not euclidean. Why?

The ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is an example of a ring that is principal and not euclidean. We will not prove this fact in these notes. For a proof see [2], [14] or [13].

4.18. DEFINITION. Let R be an integral domain. Then R is a **unique factorization domain** if the following statements hold:

- 1) Each $x \in R \setminus \{0\}$ that is not a unit can be written as $x = c_1 \cdots c_n$ for irreducibles c_1, \dots, c_n .
- 2) If $x = c_1 \cdots c_n = d_1 \cdots d_m$ for irreducibles c_1, \dots, c_n and d_1, \dots, d_m , then $n = m$ and there exists $\sigma \in \mathbb{S}_n$ such that c_i and $d_{\sigma(i)}$ are associate for all $i \in \{1, \dots, n\}$.

It is important to remark that some rings have factorizations and this factorization is not unique. In fact, if N is a square-free integer, $\mathbb{Z}[\sqrt{N}]$ is noetherian and hence it has factorization. However, not all these rings admit are unique factorization domains.

4.19. EXAMPLE. The ring $R = \mathbb{Z}[\sqrt{-6}]$ is not a unique factorization domain. In fact,

$$10 = 2 \cdot 5 = (2 + \sqrt{-6})(2 - \sqrt{-6}).$$

Note that $\text{norm}(a + b\sqrt{-6}) = a^2 + 6b^2 \neq 2$. This implies that 2 is irreducible, as if $2 = \alpha\beta$, then $4 = \text{norm}(2) = \text{norm}(\alpha)\text{norm}(\beta)$. Similarly, 5 is irreducible. It is an exercise to prove that $2 + \sqrt{-6}$ and $2 - \sqrt{-6}$ are both irreducible.

4.20. THEOREM. *Let R be a principal domain. Then R is a unique factorization domain.*

PROOF. We divide the proof into three steps.

CLAIM. R is noetherian.

This is trivial, as every ideal is, in particular, finitely generated by assumption.

CLAIM. R admits factorizations.

Let $x \in R \setminus \{0\}$ be such that $x \notin \mathcal{U}(R)$. If x is irreducible, there is nothing to prove. If not, $x = x_1x_2$ with $x_1 \notin \mathcal{U}(R)$ and $x_2 \notin \mathcal{U}(R)$. If x_1 and x_2 are both irreducibles, we are done. If not, say x_1 can be written as $x_1 = x_{11}x_{12}$ with $x_{11} \notin \mathcal{U}(R)$ and $x_{12} \notin \mathcal{U}(R)$. If this process does not terminate, it means that there is a sequence of ideals

$$(x) \subsetneq (x_1) \subsetneq (x_{11}) \subsetneq \cdots$$

that does not stabilize, which contradicts the fact that R is noetherian.

CLAIM. R admits unique factorization.

Let $x \in R$ be such that x factorizes into irreducibles as $x = c_1 \cdots c_n = d_1 \cdots d_m$. Note that since R is a principal domain, irreducibles and primes coincide by Proposition 4.8. We may assume that $n \leq m$. We proceed by induction on m . If $m = 1$, then $n = 1$ and $c_1 = d_1$. If $m > 1$, then, since c_1 is prime and $c_1 \mid d_1 \cdots d_m$, it follows that $c_1 \mid d_j$ for some j , say $c_1 \mid d_1$ (here is precisely where the permutation σ appears). Since d_1 is irreducible, c_1 and d_1 are associate, that is $c_1 = ud_1$ for some $u \in \mathcal{U}(R)$. Then

$$c_1c_2 \cdots c_n = (ud_1)c_2 \cdots c_n = d_1d_2 \cdots d_m.$$

Since $d_1 \neq 0$,

$$d_1(uc_2 \cdots c_n - d_2 \cdots d_m) = 0,$$

which implies that $(uc_2) \cdots c_n = d_2 \cdots d_m$ because R is an integral domain. Note that uc_2 is irreducible and hence the claim follows from the inductive hypothesis. \square

It is interesting to remark that the proof of the previous theorem is exactly the proof one does for \mathbb{Z} .

4.21. EXAMPLE. The ring $\mathbb{Z}[i]$ is a unique factorization domain.

Let us show that the converse of Theorem 4.20 does not hold.

4.22. EXERCISE. A polynomial $f \in \mathbb{Z}[X]$ is **primitive** if the greatest common divisor of its coefficients is equal to one. Let $f \in \mathbb{Z}[X]$ be a non-constant polynomial. If $f = af_1$ for some $a \in \mathbb{Z}$ and some primitive polynomial f_1 , then a is the greatest common divisor of the coefficients of f .

4.23. EXERCISE. Prove the following statements:

- 1) Let $f, g \in \mathbb{Z}[X]$ be primitive polynomials. Prove that fg is primitive.
- 2) Let $f \in \mathbb{Z}[X]$ be non-constant. Then f is irreducible in $\mathbb{Z}[X]$ if and only if f is primitive and f is irreducible in $\mathbb{Q}[X]$.

The first item of Exercise 4.23 is known as Gauss' lemma. The second one is Gauss' theorem. These results should be used to prove the following result:

4.24. EXERCISE. Prove that $\mathbb{Z}[X]$ is a unique factorization domain.

Finally, the example.

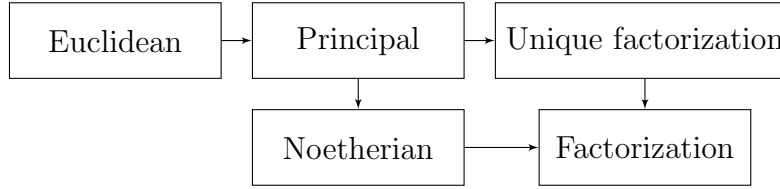
4.25. EXAMPLE. The ring $\mathbb{Z}[X]$ is a unique factorization domain that is not a principal domain.

The same technique could be used to prove that if R is a unique factorization domain, then $R[X]$ is a unique factorization domain, see for example, [6, Chapter III, Theorem 6.14].

4.26. EXERCISE. Prove that $\mathbb{R}[X, Y]$ is a unique factorization domain and the ideal $I = (X, Y)$ is not principal.

5. Lecture: 23/10/2024

The following picture shows the relationships between different classes of commutative domains:



We also know that the reverse implications do not hold.

It is time to give a very nice number-theoretical application.

5.1. THEOREM (Fermat). *Let $p \in \mathbb{Z}_{>0}$ be a prime number. The following statements are equivalent:*

- 1) $p = 2$ or $p \equiv 1 \pmod{4}$.
- 2) There exists $a \in \mathbb{Z}$ such that $a^2 \equiv -1 \pmod{p}$.
- 3) p is not irreducible in $\mathbb{Z}[i]$.
- 4) $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

PROOF. We first prove that 1) \implies 2). If $p = 2$, take $a = 1$. If $p = 4k + 1$ for some $k \in \mathbb{Z}$, then by Fermat's little theorem, the polynomial $X^{p-1} - 1 \in (\mathbb{Z}/p)[X]$ has roots $1, 2, \dots, p-1$. Write

$$(X-1)(X-2)\cdots(X-(p-1)) = X^{p-1} - 1 = X^{4k} - 1 = (X^{2k} + 1)(X^{2k} - 1)$$

in $(\mathbb{Z}/p)[X]$. Since p is prime, \mathbb{Z}/p is a field, and hence $(\mathbb{Z}/p)[X]$ is a unique factorization domain (because it is euclidean). Thus there exists $\alpha \in \mathbb{Z}/p$ such that $\alpha^{2k} + 1 = 0$. To finish the proof take $a = \alpha^k$.

We now prove that 2) \implies 3). If $a^2 \equiv -1 \pmod{p}$, then $a^2 + 1 = kp$ for some $k \in \mathbb{Z}$. Since $(a-i)(a+i) = a^2 + 1 = kp$, then p divides $(a-i)(a+i)$. Let us prove that p is not prime in $\mathbb{Z}[i]$. We claim that p does not divide $a-i$ in $\mathbb{Z}[i]$. Indeed, if $p \mid a-i$, then $a-i = p(e+fi)$ for some $e, f \in \mathbb{Z}$ and this implies that $1 = pf$, a contradiction. Similarly, p does not divide $a+i$. Thus p is not prime in $\mathbb{Z}[i]$ and hence it is not irreducible in $\mathbb{Z}[i]$ (because in $\mathbb{Z}[i]$ primes and irreducible coincide).

We now prove that 3) \implies 4). If $p = (a+bi)(c+di)$ with $a+bi \notin \mathcal{U}(\mathbb{Z}[i])$ and $c+di \notin \mathcal{U}(\mathbb{Z}[i])$, then

$$p^2 = \text{norm}(p) = \text{norm}(a+bi) \text{norm}(c+di) = (a^2 + b^2)(c^2 + d^2)$$

in \mathbb{Z} . Since \mathbb{Z} has unique factorization, it follows that $p = a^2 + b^2$.

Finally, we prove that 4) \implies 1). The only possible remainders after division by four are 0, 1, 2 and 3. For all a , either $a^2 \equiv 0 \pmod{4}$ or $a^2 \equiv 1 \pmod{4}$. If $p \equiv 3 \pmod{4}$, then p is never a sum of two squares, as $a^2 + b^2 \equiv 0 \pmod{4}$, $a^2 + b^2 \equiv 1 \pmod{4}$ or $a^2 + b^2 \equiv 2 \pmod{4}$. \square

An alternative proof of the implication 1) \implies 2) goes as follows: if $p = 4k + 1$ is prime, then $x = (2k)!$ is such that $x^2 \equiv -1 \pmod{p}$. By Wilson's theorem,

$$(p-1)! = (4k)! \equiv \text{mod } p.$$

Moreover,

$$\begin{aligned}(4k)! &\equiv (4k)(4k-1)\cdots(2k+2)(2k+1)(2k)(2k-1)\cdots 1 \pmod{p} \\ &\equiv -1(-2)\cdots(-2k+1)(-2k)(2k)(2k-1)\cdots 1 \pmod{p} \\ &\equiv (-1)^{2k}(2k)!(2k)! \pmod{p}.\end{aligned}$$

Hence $x = (2k)!$ is such that $x^2 \equiv -1 \pmod{p}$.

5.2. EXERCISE. Decompose the prime 41 as a sum of two squares.

The previous exercise can be solved by using computers and a beautiful formula of Gauss that uses binomial coefficients. Let $p = 4k + 1$ be a prime number. If

$$a = \left\langle \frac{1}{2} \binom{2k}{k} \right\rangle, \quad b = \langle (2k)!a \rangle,$$

where $\langle x \rangle$ denotes the residue of x modulo p of absolute value smaller than $p/2$, then it follows that $p = a^2 + b^2$. Simple and beautiful as it is, Gauss's formula does not seem to be of any real computational value.

In [12], the author employs the Euclidean algorithm to determine efficiently the decomposition of a prime number $p = 4k + 1$ as the sum of two squares.

5.3. EXERCISE. Let $\alpha \in \mathbb{Z}[i]$ be such that $\text{norm}(\alpha) = p^2$ for some prime number $p \in \mathbb{Z}$ with $p \equiv 3 \pmod{4}$. Prove that α is irreducible.

5.4. EXERCISE. Let $p \in \mathbb{Z}$ be a prime number such that $p \equiv 1 \pmod{4}$. If $\alpha \in \mathbb{Z}[i]$, then $\alpha = \gamma\bar{\gamma}$ for some $\gamma \in \mathbb{Z}[i]$ irreducible.

5.5. EXERCISE. Find the irreducible elements of $\mathbb{Z}[i]$.

As a consequence of Theorem 5.1 one can prove the following result: An integer ≥ 2 can be written as a sum of two squares if and only if its prime decomposition contains no factor p^k , where $p \equiv 3 \pmod{4}$ and k is odd.

The numbers that can be represented as the sums of two squares form the integer sequence A001481:

$$0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, \dots$$

They form the set of all norms of Gaussian integers.

5.6. BONUS EXERCISE (Eisenstein integers). Let $\omega = e^{2\pi i/3} \in \mathbb{C}$ and

$$R = \mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}.$$

Prove the following statements:

1) The map

$$N: R \rightarrow \mathbb{Z}_{\geq 0}, \quad \text{norm}(a + b\omega) = a^2 - ab + b^2,$$

is multiplicative and satisfies $\text{norm}(\alpha) = \alpha\bar{\alpha}$ for all $\alpha \in R$.

2) $\mathcal{U}(R) = \{1, -1, \omega, -\omega, \omega^2, -\omega^2\}$.

3) R is a euclidean domain.

4) If $\text{norm}(\alpha)$ is a prime number, then α is irreducible.

5) If $\text{norm}(\alpha) = p^2$ for some prime number p with $p \equiv 2 \pmod{3}$, then α is irreducible.

6) If $p \equiv 1 \pmod{3}$, then $p = \gamma\bar{\gamma}$ for some irreducible $\gamma \in R$.

7) Up to multiplication by units, the irreducible elements of R are $1 - 2\omega$, $a + b\omega$ with $a^2 - ab + b^2 = p$ and $p \equiv 1 \pmod{3}$, and prime numbers $p \in \mathbb{Z}$ with $p \equiv 2 \pmod{3}$.

§ 5.1. Zorn's lemma.

5.7. DEFINITION. A non-empty set R is said to be a **partially ordered set** (or poset, for short) if there is a subset $X \subseteq R \times R$ such that

- 1) $(r, r) \in X$ for all $r \in R$,
- 2) if $(r, s) \in X$ and $(s, t) \in X$, then $(r, t) \in X$, and
- 3) if $(r, s) \in X$ and $(s, r) \in X$, then $r = s$.

The set X is a partial order relation on R . We will use the following notation: $(r, s) \in X$ if and only if $r \leq s$. Moreover, $r < s$ if and only if $r \leq s$ and $r \neq s$.

5.8. DEFINITION. Let R be a poset and $r, s \in R$. Then r and s are **comparable** if either $r \leq s$ or $s \leq r$.

5.9. EXAMPLE. Let $U = \{1, 2, 3, 4, 5\}$ and T be the set of subsets of U . Then T is a poset with the usual inclusion, that is $C \leq D$ if and only if $C \subseteq D$. The subsets $\{1, 2\}$ and $\{3, 4\}$ of U are elements of T that are not comparable.

5.10. DEFINITION. Let R be a poset and $r \in R$. Then r is **maximal** in R if $r \leq t$ implies $r = t$.

5.11. EXAMPLE. \mathbb{Z} has no maximal elements.

5.12. EXAMPLE. Let $R = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ with $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 \leq y_2$. Then R is a poset and each $(x, 0)$ is maximal. Thus R has infinitely many maximal elements.

5.13. DEFINITION. Let R be a poset and S be a non-empty subset of R . An **upper bound** for S is an element $u \in R$ such that $s \leq u$ for all $s \in S$.

An upper bound for S might not be an element of S .

5.14. EXAMPLE. Let $S = \{6\mathbb{Z}, 12\mathbb{Z}, 24\mathbb{Z}\}$ be a subset of the set X of subgroups of \mathbb{Z} partially ordered by the inclusion. Then $6\mathbb{Z} = 6\mathbb{Z} \cup 12\mathbb{Z} \cup 24\mathbb{Z}$ is an upper bound of the subset S that is not maximal in X .

5.15. EXAMPLE. Let $R = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$ partially ordered with the inclusion, where R is considered as a subset of the power set of $X = \{1, 2, 3\}$. Then $\{3\}$ and $\{1, 2\}$ are maximal elements of R . Moreover, X is an upper bound for R .

5.16. DEFINITION. Let R be a poset. A **chain** is a non-empty subset S of R such that any two elements of S are comparable.

We now state **Zorn's lemma**:

*Let R be a poset such that every chain in R admits an upper bound in R .
Then R contains a maximal element.*

It is not intuitive¹ but it is logically equivalent to a more intuitive statement in set theory, the Axiom of Choice, which says every Cartesian product of non-empty sets is non-empty. It is more an axiom than a lemma. The reason for calling Zorn's lemma a lemma rather than an axiom is purely historical.

5.17. DEFINITION. Let R be a ring. An ideal I of R is said to be **maximal** if $I \neq R$ and if J is an ideal of R such that $I \subseteq J$, then either $I = J$ or $J = R$.

If p is a prime number, then $p\mathbb{Z}$ is a maximal ideal of \mathbb{Z} .

5.18. EXERCISE. Let R be a commutative ring. Prove that R is a field if and only if $\{0\}$ is a maximal ideal of R .

5.19. EXERCISE. Let R be a commutative ring and I be an ideal of R . Prove that I is maximal if and only if R/I is a field.

An ideal I of a ring R is said to be **proper** if $I \neq R$.

The following application of Zorn's lemma uses the identity of a ring.

5.20. THEOREM (Krull). *Let R be a ring. Each proper ideal I of R is contained in a maximal ideal. In particular, all rings have maximal ideals.*

PROOF. Let $X = \{J : J \text{ is an ideal of } R \text{ such that } I \subseteq J \subsetneq R\}$. Since $I \in X$, it follows that X is non-empty. Moreover, X is a poset with respect to inclusion. If C is a chain in X (say for example an increasing sequence

$$I_1 \subseteq I_2 \subseteq \cdots$$

of proper ideals of R containing I), then $\cup_{J \in C} J$ is an upper bound for C , as $\cup_{J \in C} J$ is an ideal and $\cup_{J \in C} J \neq R$ because $1 \notin \cup_{J \in C} J$. Zorn's lemma implies that there exists a maximal element $M \in X$. We claim that M is a maximal ideal of R . The definition of X implies that M is a proper ideal of R that contains I . If M_1 is a proper ideal of R such that $M \subseteq M_1$, it follows that $I \subseteq M_1$ and hence $M_1 \in X$. The maximality of M implies that $M = M_1$. \square

In the proof of the previous theorem, it is crucial to consider rings with identity.

5.21. EXERCISE. Compute the maximal ideals of $\mathbb{R}[X]$ and $\mathbb{C}[X]$.

One can also compute the maximal ideals of $K[X]$ for any field K .

¹The mathematician Jerry L. Bona has a nice joke about Zorn's lemma: *The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?*

5.22. EXERCISE. Let R be a principal domain and $p \in R \setminus \{0\}$. Then p is irreducible if and only if (p) is maximal.

5.23. DEFINITION. Let R be a commutative ring. A proper ideal I of R is said to be **prime** if $xy \in I$ implies that $x \in I$ or $y \in I$.

If p is a prime number, then $p\mathbb{Z}$ is a prime ideal of \mathbb{Z} .

5.24. EXERCISE. Let R be a commutative ring. Prove that an ideal I of R is prime if and only if R/I is a domain.

5.25. EXERCISE. Let R be a commutative ring.

- 1) Prove that maximal ideals are prime.
- 2) Let $R = \mathbb{Z}[X]$. Prove that (X) is a prime ideal that is not maximal.
- 3) Let R be a principal domain. Prove that non-zero prime ideals are maximal.

5.26. EXAMPLE. The ideal $(X^2 + 2X + 2)$ is maximal in $\mathbb{Q}[X]$ because

$$X^2 + 2X + 2 = (X + 1)^2 + 1$$

has degree two and no rational roots. Hence $X^2 + 2X + 2$ is irreducible in $\mathbb{Q}[X]$ and it generates a maximal ideal.

5.27. EXAMPLE. Let $R = (\mathbb{Z}/2)[X]$ and $f(X) = X^2 + X + 1$. Since $f(X)$ is irreducible (because $\deg f(X) = 2$ and $f(X)$ has no roots in $\mathbb{Z}/2$, it follows that $(f(X))$ is a maximal ideal. Thus R/I is a field.

5.28. EXERCISE. Compute the maximal ideals of \mathbb{Z}/n .

5.29. EXERCISE. Let R be a commutative ring and $J(R)$ be the intersection of all maximal ideals of R . Prove that $x \in J(R)$ if and only if $1 - xy \in \mathcal{U}(R)$ for all $y \in R$. The ideal $J(R)$ is known as the **Jacobson's radical** of R . Note that $J(R) \neq R$.

We conclude the lecture with a different application of Zorn's lemma.

5.30. BONUS EXERCISE. Prove that every non-zero vector space has a basis.

The previous exercise can be used to solve the following exercises:

5.31. BONUS EXERCISE. Every linearly independent subset of a non-zero vector space V can be extended to a basis of V .

Hint: If X is a linearly independent set, a basis of V that contains X will be found as a maximal linearly independent set containing X .

5.32. BONUS EXERCISE. Let V be a vector space. Prove that every subspace U of V is a direct summand of V , that is $V = U \oplus W$ for some subspace W of V .

5.33. BONUS EXERCISE. Prove that every spanning set of a non-zero vector space contains a basis.

5.34. BONUS EXERCISE. Prove that there exists a group homomorphism $f: \mathbb{R} \rightarrow \mathbb{R}$ that is not of the form $f(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.

5.35. BONUS EXERCISE. Prove that the abelian groups \mathbb{R}^n and \mathbb{R} are isomorphic.

5.36. BONUS EXERCISE. Prove that if G is a group such that $|G| > 2$, then $|\text{Aut}(G)| > 1$.

§ 5.2. The characteristic of a ring.

5.37. DEFINITION. Let R be a ring. If there is a least positive integer n such that $nx = 0$ for all $x \in R$, then R has **characteristic** n , i.e. $\text{char } R = n$. If no such n exists, then R is of characteristic zero.

Easy examples: $\text{char } \mathbb{Z} = 0$ and $\text{char } \mathbb{Z}/n = n$.

5.38. PROPOSITION. Let R be a ring such that $\text{char } R = n > 0$,

- 1) The map $f: \mathbb{Z} \rightarrow R$, $m \mapsto m1$, is a ring homomorphism and $\ker f = n\mathbb{Z}$.
- 2) $n = \min\{k \in \mathbb{Z}_{>0} : k1 = 0\}$.
- 3) If R is a domain, then n is a prime number.

PROOF. We leave 1) as an exercise.

Let us prove 2). Let $n_1 = \min\{k \in \mathbb{Z}_{>0} : k1 = 0\}$. Clearly $n \geq n_1$. For $x \in R$,

$$n_1 x = n_1(1x) = (n_1 1)x = 0x = 0$$

and hence $n_1 \geq n$.

Finally, we prove 3). If n is not prime, say $n = rs$ with $1 < r, s < n$. Then

$$0 = n1 = (rs)1 = (r1)(s1)$$

and hence $r1 = 0$ or $s1 = 0$, a contradiction. □

5.39. EXERCISE. Let p be a prime number and $R = \mathbb{Z}/p \times \mathbb{Z}/p$ be a ring with the usual point-wise operations. Prove that $\text{char } R = p$ and that R has zero divisors.

5.40. EXERCISE (binomial theorem). Let R be a commutative ring and $n \geq 1$. Prove that if $a, b \in R$, then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

5.41. EXERCISE. Let R be a commutative ring of prime characteristic p .

- 1) If $x, y \in R$, then $(x + y)^{p^n} = x^{p^n} + y^{p^n}$ for all $n \geq 0$.
- 2) The map $R \rightarrow R$, $x \mapsto x^{p^n}$, is a ring homomorphism for all $n \geq 0$.

Hint: Proceed by induction on n and use the fact that the prime number p divides $\binom{p}{k}$ for all $k \in \{1, 2, \dots, p-1\}$. Alternatively, one could use that the prime number p divides $\binom{p^n}{k}$ for all $k \in \{1, 2, \dots, p^n-1\}$.

5.42. BONUS EXERCISE. Let R be a ring and $n \geq 1$. For $a, b \in R$, let $A(x) = (a+b)x - xb$. Prove that if $a, b \in R$, then

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} A^k(1) b^{n-k}.$$

§ 5.3. Group algebras. We now discuss an important family of examples. Fix a field K . For a finite group G , let $K[G]$ be the vector space (over K) with basis $\{g : g \in G\}$. Thus $K[G]$ is a vector space of dimension $\dim K[G] = |G|$. Moreover, $K[G]$ is a ring with

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g, h \in G} \lambda_g \mu_h (gh).$$

Note that $K[G]$ is commutative if and only if G is abelian.

5.43. EXAMPLE. Let

$$\mathbb{S}_3 = \{\text{id}, (12), (13), (23), (123), (132)\}$$

the symmetric group in three letters. Every element of $\mathbb{C}[\mathbb{S}_3]$ is of the form

$$a \text{id} + b(12) + c(13) + d(23) + e(123) + f(132)$$

for some $a, b, c, d, e, f \in \mathbb{C}$. For example,

$$\alpha = 5 \text{id} + 3(123) \quad \text{and} \quad \beta = -4 \text{id} + (132)$$

are elements of $\mathbb{C}[\mathbb{S}_3]$. We compute

$$\alpha + \beta = 1 \text{id} + 3(123) + (132)$$

and

$$\begin{aligned} \alpha\beta &= (5 \text{id} + 3(123))(-4 \text{id} + (132)) \\ &= -20 \text{id} + 5(132) - 12(123) + 3(123)(132) \\ &= -20 \text{id} + 5(132) - 12(123) + 3 \text{id} \\ &= -17 \text{id} + 5(132) - 12(123). \end{aligned}$$

Another example:

$$\begin{aligned} (\text{id} + 5(13))(2 \text{id} - (12)) &= 2 \text{id} - (12) + 10(13) - 5(13)(12) \\ &= 2 \text{id} - (12) + 10(13) - 5(123), \end{aligned}$$

as $(13)(12) = (123)$.

Note that $K[G]$ is a ring and also a vector space (over K) and these structures are somewhat compatible, as the formulas

$$(\lambda a + \mu b)c = \lambda(ac) + \mu(bc), \quad a(\lambda b + \mu c) = \lambda(ab) + \mu(ac)$$

hold for all $\lambda, \mu \in K$ and $a, b, c \in K[G]$.

5.44. DEFINITION. Let A be a ring and K be a field. Then A is a **K -algebra** (or an **algebra** over the field K) if A is a vector space (over K) such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in K$ and $a, b \in A$.

The fields \mathbb{R} and \mathbb{C} are examples of \mathbb{R} -algebras. Other examples of \mathbb{R} -algebras are the polynomial rings $\mathbb{R}[X]$ and $\mathbb{R}[X, Y]$ and the matrix rings $M_n(\mathbb{R})$.

5.45. EXAMPLE. If A is an algebra, then $M_n(A)$ is an algebra.

5.46. EXAMPLE. Let G be a finite group and K be a field. Then $K[G]$ is a K -algebra.

5.47. EXAMPLE. Let $G = \langle g : g^3 = 1 \rangle = \{1, g, g^2\} \simeq C_3$ be the cyclic group of order three. If $\alpha = a_1 1 + a_2 g + a_3 g^2$ and $\beta = b_1 1 + b_2 g + b_3 g^2$, then

$$\alpha\beta = (a_1 b_1 + a_2 b_3 + a_3 b_2)1 + (a_1 b_2 + a_2 b_1 + a_3 b_3)g + a_1 b_3 + a_2 b_2 + a_3 b_1 g^2.$$

One can check that $\mathbb{C}[G] \simeq \mathbb{C}[X]/(X^3 - 1)$.

In general, one proves that the group algebra of C_n , the cyclic group of order $n \geq 2$, is isomorphic to $\mathbb{C}[X]/(X^n - 1)$.

5.48. EXERCISE. Prove that $\mathbb{R}[C_3] \simeq \mathbb{R} \times \mathbb{C}$.

5.49. EXERCISE. Let $G = \{1, g\} \simeq C_2$ be the cyclic group of order two. The product of $\mathbb{C}[G]$ is

$$(a1 + bg)(c1 + gd) = (ac + bd)1 + (ad + bc)g.$$

Prove that the map

$$\mathbb{C}[G] \rightarrow \mathbb{C} \times \mathbb{C}, \quad a1 + bg \mapsto (a + b, a - b),$$

is a linear isomorphism of rings.

5.50. EXERCISE. Let $K = \mathbb{Z}/2$ and $G = \{1, g\} \simeq C_2$ be the cyclic group of order two. Prove that the map

$$K[G] \rightarrow \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}, \quad a1 + bg \mapsto \begin{pmatrix} a + b & b \\ 0 & a + b \end{pmatrix},$$

is a linear isomorphism of rings.

The group ring has the following property, which is left as an exercise. Let A be an algebra and G be a finite group. If $f: G \rightarrow \mathcal{U}(A)$ is a group homomorphism, then there exists a unique algebra homomorphism $\varphi: K[G] \rightarrow A$ such that $\varphi|_G = f$.

5.51. EXAMPLE. Let $\mathbb{D}_3 = \langle r, s : r^3 = s^2 = 1, srs^{-1} = r^{-1} \rangle$ be the dihedral group of six elements. We claim that

$$\mathbb{C}[\mathbb{D}_3] \simeq \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}).$$

Let ω be a primitive cubic root of one. Let

$$R = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One easily checks that $SRS^{-1} = R^{-1}$ and $R^3 = S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows that there exists a group homomorphism $G \rightarrow \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ such that $r \mapsto (1, 1, R)$ and $s \mapsto (1, -1, S)$. This group homomorphism is a ring isomorphism.

6. Lecture: 30/10/2024

We will dedicate three lectures to the development of the character theory for complex representations of finite groups. Our approach will remain entirely elementary, relying solely on fundamental linear algebra principles over complex numbers.

§ 6.1. Group representations.

6.1. DEFINITION. A **representation** (over the field K) of a group G is a group homomorphism $\rho: G \rightarrow \mathbf{GL}(V)$, $g \mapsto \rho_g$, for some non-zero finite-dimensional vector space V (over K).

The **degree** of the representation $\rho: G \rightarrow \mathbf{GL}(V)$ will be the dimension of V . Note that since V is finite-dimensional, say $\dim V = n$, after fixing a basis of V we get a **matrix representation** $\rho: G \rightarrow \mathbf{GL}(V) \simeq \mathbf{GL}_n(K)$ of G . Depending on the context, we will use either representations on vector spaces or matrix representations.

6.2. EXAMPLE. We use group representations to show that $G = \langle x, y : x^2 = y^2 = 1 \rangle$ is infinite. Note that

$$\rho: G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

is a group homomorphism, as $\rho_x^2 = \rho_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We claim that the elements of the form $(xy)^n$ are different for all n . It is enough to show that $(xy)^n = (xy)^m$, then $n = m$. Note that

$$\rho_{xy} = \rho_x \rho_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $\rho_{xy}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\rho_{xy}^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Then

$$(xy)^n = (xy)^m \implies \rho_{xy}^n = \rho_{xy}^m \implies n = m.$$

The previous example shows the power of group representations, even for infinite groups. However, in this course, we will work with complex finite-dimensional representations of finite groups.

6.3. EXAMPLE. If G is a group, then $\rho: G \rightarrow \mathbf{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, $g \mapsto 1$, is a representation. This representation is known as the **trivial (complex) representation** of G .

6.4. EXAMPLE. The sign yields a representation of \mathbb{S}_n . It is the group homomorphism $\mathbb{S}_n \rightarrow \mathbb{C}^\times$, $\sigma \mapsto \text{sign}(\sigma)$.

6.5. EXAMPLE. Let $G = \langle g : g^6 = 1 \rangle$ be the cyclic group of order six. Then

$$\rho: G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad g \mapsto \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is a group representation of degree two.

6.6. DEFINITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be representations. A linear map $T: V \rightarrow W$ is said to be **invariant** if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ T \downarrow & & \downarrow T \\ W & \xrightarrow{\psi_g} & W \end{array}$$

commutes for all $g \in G$, this means that $\psi_g T = T \rho_g$ for all $g \in G$.

It is convenient to introduce an alternative notation for group representations. Let

$$\rho: G \rightarrow \mathbf{GL}(V), \quad g \mapsto \rho_g,$$

be a representation. For $g \in G$ and $v \in V$ write $g \cdot v = \rho_g(v)$. Then the following properties hold:

- 1) $1 \cdot v = v$ for all $v \in V$,
- 2) $g \cdot (h \cdot v) = (gh) \cdot v$ for all $g, h \in G$ and $v \in V$,
- 3) $g \cdot (v + w) = g \cdot v + g \cdot w$ for all $g \in G$ and $v, w \in V$, and
- 4) $g \cdot (\lambda v) = \lambda(g \cdot v)$ for all $g \in G$, $\lambda \in \mathbb{C}$ and $v \in V$.

Using this notation for the representations $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$, a linear map $T: V \rightarrow W$ is invariant if and only if

$$T(g \cdot v) = g \cdot T(v)$$

for all $v \in V$ and $g \in G$. Although we use the same notation for different representations, there is no risk of confusion.

6.7. DEFINITION. The representations $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are **equivalent** if there exists a bijective map $T: V \rightarrow W$ invariant with respect to ρ and ψ .

If the representations ρ and ψ are equivalent, we write $\rho \simeq \psi$.

Now we can understand why we can alternatively use representations or matrix representations.

6.8. EXAMPLE. Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation, say, of degree $n = \dim V$. Choose a basis $B = \{v_1, \dots, v_n\}$ of V and let

$$T: V \rightarrow \mathbb{C}^n, \quad \sum_{i=1}^n \lambda_i v_i \mapsto (\lambda_1, \dots, \lambda_n)$$

the isomorphism that takes coordinates for the basis B . For each $g \in G$, the composition of linear maps

$$\mathbb{C}^n \xrightarrow{T^{-1}} V \xrightarrow{\rho_g} V \xrightarrow{T} \mathbb{C}^n$$

produces a representation $\psi: G \rightarrow \mathbf{GL}(\mathbb{C}^n)$ is a representation of G equivalent to ρ .

6.9. EXAMPLE. Let $G = \mathbb{Z}/n$. The representations

$$\rho: G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad m \mapsto \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix},$$

and

$$\psi: G \rightarrow \mathbf{GL}_2(\mathbb{C}), \quad m \mapsto \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix},$$

are equivalent, as $\rho_m T = T \psi_m$ for all $m \in G$ if $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$.

6.10. DEFINITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ a representation. A subspace W of V is said to be **invariant** (with respect to ρ) if $\rho_g(W) \subseteq W$ for all $g \in G$.

Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation. Then $\{0\}$ and V are always invariant subspaces of V . Moreover, if $W \subseteq V$ is invariant, then the map $\rho|_W: G \rightarrow \mathbf{GL}(W)$, $g \mapsto (\rho_g)|_W$, is a representation. The map $\rho|_W$ is the **restriction** of ρ to W .

6.11. EXAMPLE. If $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are representations of a given group G and $T: V \rightarrow W$ is an invariant map, then the **kernel**

$$\ker T = \{v \in V : T(v) = 0\}$$

is an invariant of V and the **image**

$$T(V) = \{T(v) : v \in V\}$$

is an invariant subspace of W .

6.12. DEFINITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ a representation. A **subrepresentation** of ρ is a restricted representation of the form $\rho|_W: G \rightarrow \mathbf{GL}(W)$ for some invariant subspace W of V .

6.13. DEFINITION. A non-zero representation $\rho: G \rightarrow \mathbf{GL}(V)$ is **irreducible** if $\{0\}$ and V are the only invariant subspaces of V .

Degree-one representations are irreducible.

The irreducibility of a representation, of course, depends on the base field. In the following example, it is crucial using real representations.

6.14. EXAMPLE. Let $G = \langle g : g^3 = 1 \rangle$ be the cyclic group of order three and

$$\rho: G \rightarrow \mathbf{GL}_3(\mathbb{R}), \quad g \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that in this example we work with a real representation! It is a routine calculation to prove that

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$$

is an invariant subspace of \mathbb{R}^3 . We claim that W is irreducible. Let S be a non-zero invariant subspace of W and let $s = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in S$ be a non-zero element. Then

$$t = \begin{pmatrix} y_0 \\ z_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in S.$$

We claim that $\{s, t\}$ are linearly independent. If not, there exists $\lambda \in \mathbb{R}$ such that $\lambda s = t$. Thus $\lambda x_0 = y_0$, $\lambda y_0 = z_0$ and $\lambda z_0 = x_0$. This implies that $\lambda^3 x_0 = x_0$. Since $x_0 \neq 0$ (because if $x_0 = 0$, then $y_0 = z_0 = 0$, a contradiction), it follows that $\lambda = 1$ and hence $x_0 = y_0 = z_0$, a contradiction because $x_0 + y_0 + z_0 = 0$. Therefore $\dim S = 2$ and hence $S = W$.

What happens in the previous example if we work over complex numbers?

6.15. EXERCISE. Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a degree-two representation. Prove that ρ is irreducible if and only if there is no common eigenvector for the ρ_g , $g \in G$.

The previous exercise can be used to show that the representation $\mathbb{S}_3 \rightarrow \mathbf{GL}_2(\mathbb{C})$ of the symmetric group \mathbb{S}_3 given by

$$(12) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

is irreducible.

6.16. EXAMPLE. Let $\rho: G \rightarrow \mathbf{GL}_n(\mathbb{C})$ and $\psi: G \rightarrow \mathbf{GL}_m(\mathbb{C})$ be representations. One defines the **direct sum** $\rho \oplus \psi$ of ρ and ψ as

$$\rho \oplus \psi: G \rightarrow \mathbf{GL}_{n+m}(\mathbb{C}), \quad g \mapsto \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}.$$

Let us describe the previous example without using matrix representations. If V and W are complex vector spaces, the (external) direct sum of V and W is defined as the set $V \times W$ with the complex vector space structure given by

$$(v, w) + (v_1, w_1) = (v + v_1, w + w_1), \quad \lambda(v, w) = (\lambda v, \lambda w)$$

for all $v, v_1 \in V$, $w, w_1 \in W$ and $\lambda \in \mathbb{C}$. If $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are representations, the **direct sum** $\rho \oplus \psi$ of ρ and ψ is then

$$\rho \oplus \psi: G \rightarrow \mathbf{GL}(V \oplus W), \quad g \mapsto (\rho \oplus \psi)_g,$$

where $(\rho \oplus \psi)_g: V \oplus W \rightarrow V \oplus W$, $(v, w) \mapsto (\rho_g(v), \psi_g(w))$. Note that both $V \simeq V \oplus \{0\}$ and $W \simeq \{0\} \oplus W$ are invariant subspaces of $V \oplus W$.

6.17. DEFINITION. A representation $\rho: G \rightarrow \mathbf{GL}(V)$ is said to be **completely reducible** if ρ can be decomposed as $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ for some irreducible representations ρ_1, \dots, ρ_n of G .

Note that if $\rho: G \rightarrow \mathbf{GL}(V)$ is completely reducible and $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ for some irreducible representations $\rho_i: G \rightarrow \mathbf{GL}(V_i)$, $i \in \{1, \dots, n\}$, then each V_i is an invariant subspace of V and $V = V_1 \oplus \cdots \oplus V_n$. Moreover, on some basis of V , the matrix ρ_g can be written as

$$\rho_g = \begin{pmatrix} (\rho_1)_g & & & \\ & (\rho_2)_g & & \\ & & \ddots & \\ & & & (\rho_n)_g \end{pmatrix}.$$

6.18. DEFINITION. A representation $\rho: G \rightarrow \mathbf{GL}(V)$ is **decomposable** if V can be decomposed as $V = S \oplus T$ where S and T are non-zero invariant subspaces of V .

A representation is **indecomposable** if it is not decomposable.

6.19. EXERCISE. Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be equivalent representations. Prove the following facts:

- 1) If ρ is irreducible, then ψ is irreducible.
- 2) If ρ is decomposable, then ψ is decomposable.
- 3) If ρ is completely reducible, then ψ is completely reducible.

Since we are considering finite-dimensional vector spaces, our vector spaces are Hilbert spaces, so they have an inner product $V \times V \rightarrow \mathbb{C}$, $(v, w) \mapsto \langle v, w \rangle$.

6.20. DEFINITION. A representation $\rho: G \rightarrow \mathbf{GL}(V)$ is **unitary** if $\langle \rho_g v, \rho_g w \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V$.

6.21. EXAMPLE. Let G be a finite group and $V = \mathbb{C}[G]$. The **left regular representation** of G is the representation

$$L: G \rightarrow \mathbf{GL}(V), \quad g \mapsto L_g,$$

where $L_g(h) = gh$. With the inner product

$$\left\langle \sum_{g \in G} \lambda_g g, \sum_{g \in G} \mu_g g \right\rangle = \sum_{g \in G} \lambda_g \overline{\mu_g}$$

the representation L is unitary. Prove it!

If V is a vector space and W is a subspace of V ,

$$W^\perp = \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

is called the **orthogonal complement** of W . The following result is extremely important:

6.22. PROPOSITION. *Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a unitary representation. Then ρ is either irreducible or decomposable.*

PROOF. If ρ is not irreducible, there exists an invariant subspace W of V such that $\{0\} \subsetneq W \subsetneq V$. Can we find an invariant complement? Yes, and this is because our representation is unitary. Then $V = W \oplus W^\perp$ as complex vector spaces, where $W^\perp \neq \{0\}$. So ρ will be decomposable if we can prove that W^\perp is an invariant subspace of V . Let $v \in W^\perp$, $w \in W$ and $g \in G$. Then

$$\langle \rho_g(v), w \rangle = \langle v, \rho_g^{-1}(w) \rangle = 0$$

since $\rho_g^{-1}(w) \in W$, as W is an invariant subspace of V . This means that $\rho_g(v) \in W^\perp$ and hence $V = W \oplus W^\perp$ is decomposable. \square

6.23. THEOREM (Weyl's trick). *Every representation $\rho: G \rightarrow \mathbf{GL}_n(\mathbb{C})$ of a finite group is unitary.*

PROOF. Let $V = \mathbb{C}^n$ and $V \times V \rightarrow \mathbb{C}$, $(v, w) \mapsto (v \mid w)$, be an² inner product on V .

A straightforward calculation shows that

$$\langle v, w \rangle = \sum_{g \in G} (\rho_g v \mid \rho_g w)$$

²It does not matter which inner product we have chosen. We use this particular notation for our inner product of V because with this inner product, we will produce a new inner product (denoted by $(v, w) \mapsto \langle v, w \rangle$) that will turn ρ into a unitary representation.

is an inner product of V . For example,

$$\begin{aligned}
 \langle v, w + w_1 \rangle &= \sum_{g \in G} (v \mid w + w_1) \\
 &= \sum_{g \in G} ((v \mid w) + (v \mid w_1)) \\
 &= \sum_{g \in G} (v \mid w) + \sum_{g \in G} (v \mid w_1) \\
 &= \langle v, w \rangle + \langle v, w_1 \rangle
 \end{aligned}$$

holds for all $v, w, w_1 \in V$.

Since

$$\begin{aligned}
 \langle \rho_g v, \rho_g w \rangle &= \sum_{h \in G} (\rho_h \rho_g v \mid \rho_h \rho_g w) \\
 &= \sum_{h \in G} (\rho_{hg} v \mid \rho_{hg} w) = \sum_{x \in G} (\rho_x v \mid \rho_x w) = \langle v, w \rangle,
 \end{aligned}$$

the representation ρ is unitary. □

Weyl's trick has several interesting consequences:

6.24. COROLLARY. *Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation of a finite group G . The following properties hold:*

- 1) ρ is equivalent to a unitary representation.
- 2) ρ is either irreducible or decomposable.
- 3) Each ρ_g is diagonalizable.

PROOF. The first claim follows from Example 6.8 and Weyl's trick. The second claim, from 1) and Proposition 6.22. Finally, 3) follows from Weyl's trick immediately, as unitary matrices are diagonalizable. □

6.25. EXERCISE. If G is an infinite group, it is no longer true that every non-zero representation is either irreducible or decomposable. Find an example.

For the previous exercise consider the representation $\mathbb{Z} \rightarrow \mathbf{GL}_2(\mathbb{C})$, $1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

7. Lecture: 06/11/2024

Recall that, by convention, we only consider complex finite-dimensional representations of finite groups.

7.1. THEOREM (Maschke). *Every representation of a finite group is completely reducible.*

PROOF. Let G be a finite group and $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation of G . We proceed by induction on $\dim V$. If $\dim V = 1$, the result is trivial, as degree-one representations are irreducible. Assume that the result holds for representations of degree $\leq n$. Suppose that ρ has degree $n + 1$. If ρ is irreducible, we are done. If not, use Proposition 6.22 to write $V = S \oplus T$, where S and T are non-zero invariant subspaces of V . Since $\dim S < \dim V$ and $\dim T < \dim V$, it follows from the inductive hypothesis that both S and T are spaces of completely reducible representations. Thus ρ is completely reducible. \square

7.2. EXAMPLE. Let $G = \mathbb{S}_3$ and $\rho: G \rightarrow \mathbf{GL}_3(\mathbb{C})$ be the representation given by

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then ρ_g is unitary for all $g \in G$ (because $\rho_{(12)}$ and $\rho_{(123)}$ are both unitary). Moreover,

$$S = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad T = S^\perp = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle,$$

are irreducible invariant subspaces of $V = \mathbb{C}^3$. A direct calculation shows that in the orthogonal basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ the matrices $\rho_{(12)}$ and $\rho_{(123)}$ can be written as

$$\rho_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The commutator subgroup of a group G is defined as the subgroup $[G, G]$ generated by the commutators $[x, y] = xyx^{-1}y^{-1}$, that is

$$[G, G] = \langle [x, y] : x, y \in G \rangle.$$

Routine calculations show that $[G, G]$ is always a normal subgroup of G .

For example, the commutator subgroup of \mathbb{Z} is the trivial subgroup $\{0\}$, as

$$[x, y] = x + y - x - y = 0$$

for all $x, y \in \mathbb{Z}$. As an exercise, one can also prove that $[\mathbb{S}_3, \mathbb{S}_3] = \{\text{id}, (123), (132)\}$.

7.3. EXERCISE. Let H be a normal subgroup of G . Prove that G/H is abelian if and only if $[G, G] \subseteq H$.

7.4. EXERCISE. Let G be a finite group. Prove that there is a bijection between degree-one representations of G and degree-one representations of $G/[G, G]$.

The following result is simple and crucial.

7.5. LEMMA (Schur). *Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be irreducible representations. If $T: V \rightarrow W$ is a non-zero invariant map, then T is bijective.*

PROOF. Since T is non-zero and $\ker T$ is an invariant subspace of V , it follows that $\ker T = \{0\}$, as ρ is irreducible. Thus T is injective. Since $T(V)$ is a non-zero invariant subspace of W , it follows from the fact that ψ is irreducible that T is surjective. Therefore T is bijective. \square

Two applications:

7.6. PROPOSITION. *If G is finite, $\rho: G \rightarrow \mathbf{GL}(V)$ is an irreducible representation, and $T: V \rightarrow V$ is invariant, then $T = \lambda \text{id}$ for some $\lambda \in \mathbb{C}$.*

PROOF. Let λ be an eigenvalue of T . Then $T - \lambda \text{id}$ is invariant, as

$$(T - \lambda \text{id})\rho_g = T\rho_g - \lambda\rho_g = \rho_g(T - \lambda \text{id})$$

for all $g \in G$ since T is invariant. By definition, $T - \lambda \text{id}$ is not bijective. Thus $T - \lambda \text{id} = 0$ by Schur's lemma. \square

7.7. PROPOSITION. *Let G be a finite abelian group. If $\rho: G \rightarrow \mathbf{GL}(V)$ is an irreducible representation, then $\dim V = 1$.*

PROOF. Let $h \in G$. Note that since G is abelian, $T = \rho_h$ is invariant:

$$T\rho_g = \rho_h\rho_g = \rho_{hg} = \rho_{gh} = \rho_g\rho_h = \rho_g T.$$

By the previous proposition, there exists $\lambda_h \in \mathbb{C}$ such that $\rho_h = \lambda_h \text{id}$. If $v \in V \setminus \{0\}$, then $V = \langle v \rangle$. In fact, since $\langle v \rangle$ is a non-zero invariant subspace of V and ρ is irreducible, it follows that $V = \langle v \rangle$. \square

§ 7.1. Characters. This lecture is devoted to studying character theory. Fix a group and consider (matrix) representations of groups. How can we study those matrices? Since the equivalence of representations translates into the equivalence of matrices, it makes sense to use linear algebra. We can use the characteristic polynomial of the matrix,

$$A \in \mathbb{C}^{n \times n} \rightsquigarrow \chi_A(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{C}[X],$$

or any of the numbers $a_0, \dots, a_n \in \mathbb{C}$, as all of them are indeed invariants of the matrix A . The determinant and the trace of A are examples of such numbers. In the context of group representations, the trace is particularly interesting.

7.8. DEFINITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation of a finite group G . The **character** of ρ is the map $\chi_\rho: G \rightarrow \mathbb{C}$, $g \mapsto \text{trace } \rho_g$.

If a representation ρ is irreducible, its character is said to be an **irreducible character**. The **degree** of a character is the degree of the affording representation.

7.9. PROPOSITION. *Let $\rho: G \rightarrow \mathbf{GL}(V)$ be a representation of a finite group G , χ be its character and $g \in G$. The following statements hold:*

- 1) $\chi(1) = \dim V$.
- 2) $\chi(g) = \chi(hgh^{-1})$ for all $h \in G$.
- 3) $\chi(g)$ is the sum of $\chi(1)$ roots of one of order $|g|$.
- 4) $\chi(g^{-1}) = \overline{\chi(g)}$.
- 5) $|\chi(g)| \leq \chi(1)$.

PROOF. The first statement is trivial. To prove 2) note that

$$\chi(hgh^{-1}) = \text{trace}(\rho_{hgh^{-1}}) = \text{trace}(\rho_h \rho_g \rho_h^{-1}) = \text{trace} \rho_g = \chi(g).$$

Statement 3) follows from the fact that the trace of ρ_g is the sum of the eigenvalues of ρ_g and these numbers are roots of the polynomial $X^{|g|} - 1 \in \mathbb{C}[X]$. To prove 4) write $\chi(g) = \lambda_1 + \cdots + \lambda_k$, where the λ_j are roots of one. Then

$$\overline{\chi(g)} = \sum_{j=1}^k \overline{\lambda_j} = \sum_{j=1}^k \lambda_j^{-1} = \text{trace}(\rho_g^{-1}) = \text{trace}(\rho_{g^{-1}}) = \chi(g^{-1}).$$

Finally, we prove 5). Use 3) to write $\chi(g)$ as the sum of $\chi(1)$ roots of one, say

$$\chi(g) = \lambda_1 + \cdots + \lambda_k$$

for $k = \chi(1)$. Then

$$|\chi(g)| = |\lambda_1 + \cdots + \lambda_k| \leq |\lambda_1| + \cdots + |\lambda_k| = \underbrace{1 + \cdots + 1}_{k\text{-times}} = k. \quad \square$$

If two representations are equivalent, their characters are equal.

7.10. PROPOSITION. If $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are representations of a finite group G , then $\chi_{\rho \oplus \psi} = \chi_\rho + \chi_\psi$.

PROOF. For $g \in G$, it follows that $(\rho \oplus \psi)_g = \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}$. Thus

$$\chi_{\rho \oplus \psi}(g) = \text{trace}((\rho \oplus \psi)_g) = \text{trace}(\rho_g) + \text{trace}(\psi_g) = \chi_\rho(g) + \chi_\psi(g). \quad \square$$

Let V be a vector space with basis $\{v_1, \dots, v_k\}$ and W be a vector space with basis $\{w_1, \dots, w_l\}$. A **tensor product** of V and W is a vector space X with together with a bilinear map

$$V \times W \rightarrow X, \quad (v, w) \mapsto v \otimes w,$$

such that $\{v_i \otimes w_j : 1 \leq i \leq k, 1 \leq j \leq l\}$ is a basis of X . The tensor product of V and W is unique up to isomorphism and is denoted by $V \otimes W$. Note that

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

Note that every element of $V \otimes W$ is a finite sum of the form

$$\sum_{i,j} \lambda_{ij} v_i \otimes w_j$$

for some scalars $\lambda_{ij} \in \mathbb{C}$, and not of the form $v \otimes w$ for $v \in V$ and $w \in W$.

The bilinearity of tensor products is crucial. For example,

$$\begin{aligned} (v_1 + v_3) \otimes (3w_1 + w_2) &= v_1 \otimes (3w_1 + w_2) + v_3 \otimes (3w_1 + w_2) \\ &= v_1 \otimes (3w_1) + v_1 \otimes w_2 + v_3 \otimes (3w_1) + v_3 \otimes w_2 \\ &= 3v_1 \otimes w_1 + v_1 \otimes w_2 + 3v_3 \otimes w_1 + v_3 \otimes w_2. \end{aligned}$$

7.11. DEFINITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be representations. The **tensor product** of ρ and ψ is the representation of G given by

$$\rho \otimes \psi: G \rightarrow \mathbf{GL}(V \otimes W), \quad g \mapsto (\rho \otimes \psi)_g,$$

where

$$(\rho \otimes \psi)_g(v \otimes w) = \rho_g(v) \otimes \psi_g(w)$$

for $v \in V$ and $w \in W$.

A direct calculation shows that the tensor product of representations is indeed a representation.

7.12. PROPOSITION. *If $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are representations of a finite group G , then*

$$\chi_{\rho \otimes \psi} = \chi_\rho \chi_\psi.$$

PROOF. For each $g \in G$, the map ρ_g is diagonalizable. Let $\{v_1, \dots, v_n\}$ be a basis of eigenvectors of ρ_g and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that $\rho_g(v_i) = \lambda_i v_i$ for all $i \in \{1, \dots, n\}$. Similarly, let $\{w_1, \dots, w_m\}$ be a basis of eigenvectors of ψ_g and $\mu_1, \dots, \mu_m \in \mathbb{C}$ be such that $\psi_g(w_j) = \mu_j w_j$ for all $j \in \{1, \dots, m\}$. Each $v_i \otimes w_j$ is eigenvector of $\rho \otimes \psi$ with eigenvalue $\lambda_i \mu_j$, as

$$(\rho \otimes \psi)_g(v_i \otimes w_j) = \rho_g v_i \otimes \psi_g w_j = \lambda_i v_i \otimes \mu_j w_j = (\lambda_i \mu_j) v_i \otimes w_j.$$

Thus $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of eigenvectors and the $\lambda_i \mu_j$ are the eigenvalues of $(\rho \otimes \psi)_g$. It follows that

$$\chi_{\rho \otimes \psi}(g) = \sum_{i,j} \lambda_i \mu_j = \left(\sum_i \lambda_i \right) \left(\sum_j \mu_j \right) = \chi_\rho(g) \chi_\psi(g). \quad \square$$

We mention without proof that it is also possible to define the dual $\rho^*: G \rightarrow \mathbf{GL}(V^*)$ of a representation $\rho: G \rightarrow \mathbf{GL}(V)$ by the formula

$$(\rho_g^* f)(v) = f(\rho_g^{-1} v), \quad g \in G, f \in V^* \text{ and } v \in V.$$

We claim that the character of the dual representation is then $\overline{\chi_\rho}$. Let $\{v_1, \dots, v_n\}$ be a basis of V and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be such that $\rho_g v_i = \lambda_i v_i$ for all $i \in \{1, \dots, n\}$. If $\{f_1, \dots, f_n\}$ is the dual basis of $\{v_1, \dots, v_n\}$, then

$$(\rho_g^* f_i)(v_j) = f_i(\rho_g^{-1} v_j) = \overline{\lambda_j} f_i(v_j) = \overline{\lambda_j} \delta_{ij}$$

and the claim follows.

§ 7.2. Schur's orthogonality relations. Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be representations of a finite group G . Since V and W are vector spaces, the set

$$\text{Hom}(V, W) = \{T: V \rightarrow W : T \text{ is linear}\}$$

is a vector space with

$$\begin{aligned} (\lambda T)(v) &= \lambda T(v) & \text{for all } \lambda \in \mathbb{C} \text{ and all } v \in V, \\ (T + T_1)(v) &= T(v) + T_1(v) & \text{for all } v \in V. \end{aligned}$$

We claim that the set $\text{Hom}_G(V, W)$ of invariant maps is a subspace of $\text{Hom}(V, W)$. Indeed, the zero map is clearly invariant. If $T, T_1 \in \text{Hom}_G(V, W)$ and $\lambda \in \mathbb{C}$, then

$$(T + \lambda T_1)(\rho_g v) = T(\rho_g v) + \lambda T_1(\rho_g v) = \psi_g T(v) + \lambda \psi_g T_1(v) = \psi_g((T + \lambda T_1)(v))$$

for all $v \in V$.

7.13. PROPOSITION. Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be representations and $T: V \rightarrow W$ be a linear map. Then

$$T^\# = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \rho_g \in \text{Hom}_G(V, W).$$

Moreover, the map $\text{Hom}(V, W) \rightarrow \text{Hom}_G(V, W)$, $T \mapsto T^\#$, is linear and surjective.

PROOF. Let $h \in G$ and $v \in V$. Then

$$\begin{aligned} T^\# \rho_h(v) &= \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \rho_g \rho_h(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \rho_{gh}(v) \\ &= \frac{1}{|G|} \sum_{x \in G} \psi_{hx^{-1}} T \rho_x(v) = \frac{1}{|G|} \sum_{x \in G} \psi_h \psi_{x^{-1}} T \rho_x(v) = \psi_h T^\#(v). \end{aligned}$$

It is a routine calculation to show that $T \mapsto T^\#$ is a linear map. The map $T \mapsto T^\#$ is surjective since if $T \in \text{Hom}_G(V, W)$, then

$$T^\#(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} T \rho_g(v) = \frac{1}{|G|} \sum_{g \in G} \psi_{g^{-1}} \psi_g T(v) = T(v)$$

holds for all $v \in V$, so $T^\# = T$. \square

If $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ are non-equivalent irreducible representations and $T: V \rightarrow W$ is a linear map, then $T^\# = 0$, as $T^\# \in \text{Hom}_G(V, W) = \{0\}$ by the previous proposition and Schur's lemma. We will use this fact later in Schur's Theorem 8.2.

7.14. THEOREM (Ergodic theorem). Let $\rho: G \rightarrow \mathbf{GL}(V)$ be an irreducible representation. If $T: V \rightarrow V$ is linear, then $T^\# = (\dim V)^{-1} \text{trace}(T) \text{id}$.

PROOF. The previous proposition and Schur's lemma imply that $T^\# = \lambda \text{id}$ for some $\lambda \in \mathbb{C}$. We now compute the trace of $T^\#$. On the one hand,

$$\text{trace}(T^\#) = \text{trace}(\lambda \text{id}) = (\dim V) \lambda.$$

On the other hand,

$$\text{trace}(T^\#) = \frac{1}{|G|} \sum_{g \in G} \text{trace}(\rho_{g^{-1}} T \rho_g) = \frac{1}{|G|} \sum_{g \in G} \text{trace}(T) = \text{trace}(T),$$

as $\text{trace}(ABA^{-1}) = \text{trace}(B)$ for all A and B . Hence

$$\text{trace}(T^\#) = (\dim V)^{-1} \text{trace}(T) \text{id}. \quad \square$$

8. Lecture: 13/11/2024

We now prove Schur's orthogonality relations. We need some preliminary material. First, recall that the matrix E_{ij} is given by

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad \delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a finite group. Recall that $L(G) = \{f: G \rightarrow \mathbb{C}\}$ is a vector space with

$$(f + g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x), \quad f, g \in L(G), \lambda \in \mathbb{C}, x \in G.$$

The dimension of $L(G)$ is $|G|$. For $x \in G$, let

$$\delta_x: G \rightarrow \mathbb{C}, \quad \delta_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then the set $\{\delta_x : x \in G\}$ is a basis of $L(G)$.

If $\alpha: G \rightarrow \mathbb{C}$ and $\beta: G \rightarrow \mathbb{C}$ are maps, we consider the inner product

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}.$$

If $\rho: G \rightarrow \mathbf{GL}(V) \simeq \mathbf{GL}_n(\mathbb{C})$ is a representation of G , then ρ_g is the matrix $(\rho_{ij}(g))$ and hence the character of ρ is given by

$$\chi_\rho(g) = \sum_{i=1}^n \rho_{ii}(g).$$

8.1. LEMMA. *Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be irreducible representations. Then $(E_{ki}^\#)_{lj} = \langle \rho_{ij}, \psi_{kl} \rangle$.*

PROOF. Assume that ρ is unitary. Then $\rho_g^{-1} = \rho_{g^{-1}} = \rho_g^* = \bar{\rho}_g^T$ for all $g \in G$. Thus

$$\begin{aligned} (E_{ki}^\#)_{lj} &= \frac{1}{|G|} \sum_{g \in G} (\psi_{g^{-1}} E_{ki} \rho_g)_{lj} \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{p,q} \psi_{lq}(g^{-1}) (E_{ki})_{qp} \rho_{pj}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \psi_{lk}(g^{-1}) \rho_{ij}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\psi_{kl}(g)} \rho_{ij}(g) = \langle \rho_{ij}, \psi_{kl} \rangle. \end{aligned}$$

□

8.2. THEOREM (Schur). *Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be irreducible representations of a finite group G . Then the following statements hold:*

- 1) $\langle \rho_{ij}, \psi_{kl} \rangle = 0$ if ρ and ψ are not equivalent.
- 2) $\langle \rho_{ij}, \rho_{kl} \rangle = \frac{1}{\dim V} \delta_{ik} \delta_{lj}.$

PROOF. Let us prove the first claim. Since ρ and ψ are not equivalent, it follows from Schur's lemma that $\text{Hom}_G(V, W) = \{0\}$. Thus $E_{ki}^\# \in \text{Hom}_G(V, W) = \{0\}$ (see the comment before the ergodic theorem 7.14).

To prove the second claim, we use the previous lemma:

$$(E_{ki}^\#)_{lj} = \langle \rho_{ij}, \rho_{kl} \rangle = \frac{1}{\dim V} (\text{trace } E_{ki}) \delta_{lj} = \frac{1}{\dim V} \delta_{ki} \delta_{lj}. \quad \square$$

Now we can prove Schur's first orthogonality relation.

8.3. THEOREM (Schur). *Let $\rho: G \rightarrow \mathbf{GL}(V)$ and $\psi: G \rightarrow \mathbf{GL}(W)$ be irreducible representations of a finite group G . Then*

$$\langle \chi_\rho, \chi_\psi \rangle = \begin{cases} 1 & \text{if } \rho \simeq \psi, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $n = \dim V$ and $m = \dim W$. We compute

$$\begin{aligned} \langle \chi_\rho, \chi_\psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\psi(g)} \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum_{g \in G} \frac{1}{|G|} \rho_{ii}(g) \overline{\psi_{jj}(g)} \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \rho_{ii}, \psi_{jj} \rangle = \begin{cases} 1 & \text{if } \rho \simeq \psi, \\ 0 & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

Schur's theorem has several corollaries.

8.4. EXERCISE. Let G be a finite group. If $\rho: G \rightarrow \mathbf{GL}(V)$ is a unitary irreducible representation of degree n , then

$$\{\sqrt{n}\rho_{ij} : 1 \leq i, j \leq n\}$$

is an orthonormal set of size n^2 .

8.5. COROLLARY. *A finite group has finitely many classes of irreducible representations.*

PROOF. Let G be a finite group. Every isomorphism class of representations of G contains a unitary representation (Corollary 6.24). Since $\dim L(G) = |G|$, every linearly independent set of vectors from $L(G)$ has at most $|G|$ elements. By Schur's theorem 8.3, the entries of inequivalent unitary representations of G form an orthogonal set of non-zero vectors of $L(G)$. Thus there are at most $|G|$ equivalence classes of irreducible representations. \square

Let G be a finite group. Since G has only finitely many non-equivalent irreducible representation, we will often say that

$$\rho_1, \dots, \rho_r$$

are **the** irreducible representations of G , where it is assumed that the ρ_i form a complete set of representatives of irreducible representations of G . For each i we write $\chi_i = \chi_{\rho_i}$. The set of irreducible characters will be denoted by

$$\text{Irr}(G) = \{\chi_1, \dots, \chi_r\}.$$

8.6. DEFINITION. Let G be a group and $f: G \rightarrow \mathbb{C}$ be a map. Then f is a **class function** if $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

Characters are class functions.

If $g \in G$, the **conjugacy class** of g in G is the set $\{xgx^{-1} : x \in G\}$. Note that two conjugacy classes are either equal or disjoint. Moreover, every group is a disjoint union of its conjugacy classes. For example, the conjugacy classes of the group \mathbb{S}_3 are $\{\text{id}\}$, $\{(12), (13), (23)\}$ and $\{(123), (132)\}$. Thus \mathbb{S}_3 has three conjugacy classes.

Let $C(G)$ be the subspace of $L(G)$ of class functions.

8.7. PROPOSITION. Let G be a finite group. Then $\dim C(G) = K(G)$, the number of conjugacy classes of G .

PROOF. If C is a conjugacy class, then

$$\delta_C: G \rightarrow \mathbb{C}, \quad \delta_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{otherwise,} \end{cases}$$

is a class function. It is enough to prove that the set $\{\delta_C : C \text{ is a conjugacy class of } G\}$ is a basis of $C(G)$. It is a generating set because each f can be written as

$$f = \sum_C f(C) \delta_C.$$

The δ_C are linearly independent because they are orthogonal: If C and D are conjugacy classes of G , then

$$\langle \delta_C, \delta_D \rangle = \frac{1}{|G|} \sum_{x \in G} \delta_C(x) \overline{\delta_D(x)} = \begin{cases} |C|/|G| & \text{if } C = D, \\ 0 & \text{otherwise.} \end{cases}$$

From this the claim follows. □

8.8. COROLLARY. Let G be a finite group. There are at most $K(G)$ equivalence classes of irreducible representations of G .

PROOF. Non-equivalent representations have different characters. Irreducible characters form an orthonormal set, thus they are linearly independent. Since irreducible characters are class functions (that is, they are constant on conjugacy classes), it follows that there are at most $K(G)$ irreducible different characters. □

Let $m \in \mathbb{Z}_{>0}$. If V is a vector space, we write

$$mV = V \oplus \cdots \oplus V \quad (m\text{-times}).$$

Similarly, if ρ is a representation, we write

$$m\rho = \rho \oplus \cdots \oplus \rho \quad (m\text{-times}).$$

8.9. THEOREM. Let ρ_1, \dots, ρ_r be the irreducible representations of a finite group G . If

$$\rho = \sum_{j=1}^r m_j \rho_j,$$

where $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$, then $m_i = \langle \chi_\rho, \chi_i \rangle$ for all $i \in \{1, \dots, r\}$.

PROOF. Write $\chi_\rho = \sum_{j=1}^r m_j \chi_j$. Then

$$\langle \chi_\rho, \chi_i \rangle = \sum_{j=1}^r m_j \langle \chi_j, \chi_i \rangle = m_i$$

for all $i \in \{1, \dots, r\}$. □

The theorem states that the decomposition of a representation ρ into irreducibles is unique and is determined (up to equivalence) by its character.

8.10. COROLLARY. *A representation ρ of a finite group with character χ_ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

PROOF. We first decompose ρ as a sum of irreducibles, say

$$\rho = \sum_{j=1}^r m_j \rho_j$$

with $m_1, \dots, m_r \geq 0$. Then $\langle \chi_\rho, \chi_\rho \rangle = \sum_{j=1}^r m_j^2$. Now $\langle \chi_\rho, \chi_\rho \rangle = 1$ if and only if there is exactly one j such that $m_j = 1$ and $m_i = 0$ for all $i \neq j$. □

8.11. EXERCISE. Let G and H be finite groups. If $\rho: H \rightarrow \mathbf{GL}(V)$ is an irreducible representation of H and $f: G \rightarrow H$ is a surjective group homomorphism, then the composition $\rho f: G \rightarrow \mathbf{GL}(V)$ is an irreducible representation.

Recall that the left regular representation of a finite group G is the group homomorphism $L: G \rightarrow \mathbf{GL}(V)$, where V is the complex vector space with basis $\{g : g \in G\}$ and $L_g(x) = gx$ for all $g, x \in G$.

8.12. THEOREM. *Let G be a finite group and L be its regular representation. Then $L = \sum_{j=1}^r n_j \rho_j$, where $n_j = \deg \rho_j$.*

PROOF. Let $G = \{g_1, \dots, g_n\}$, $n = |G|$. If $g \in G$, since $L_g(g_i) = gg_i$ for all i , the matrix of L_g in the basis $\{g_1, \dots, g_n\}$ is then

$$(L_g)_{ij} = \begin{cases} 1 & \text{if } g_i = gg_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\chi_L(g) = \text{trace}(L_g) = \sum_{i=1}^n (L_g)_{ii} = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\langle \chi_L, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_L(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \overline{\chi_i(1)} = n_i$$

for all $i \in \{1, \dots, n\}$. □

Now several corollaries.

8.13. COROLLARY. *Let G be a finite group and ρ_1, \dots, ρ_r be the irreducible representations of G . For each k let $n_k = \deg \rho_k$. The following statements hold:*

1) $|G| = n_1^2 + \dots + n_r^2$.

- 2) $\{\sqrt{n_k}(\rho_k)_{ij} : 1 \leq k \leq r, 1 \leq i, j \leq n_k\}$ is an orthonormal basis of $L(G)$.
 3) r is equal to the number of conjugacy classes of G .

PROOF. Since $\chi_L = \sum_{j=1}^r n_j \chi_j$, the first claim follows. The second claim follows from the orthogonality relations. Let us prove the third claim. Let $f \in C(G)$ and write f as a linear combination of the $(\rho_k)_{ij}$, say

$$f = \sum_{i,j,k} \lambda_{ijk} (\rho_k)_{ij}, \quad \lambda_{ijk} \in \mathbb{C}.$$

If $x \in G$, then

$$\begin{aligned} f(x) &= \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} \lambda_{ijk} (\rho_k)_{ij}(g^{-1}xg) = \sum_{i,j,k} \lambda_{ijk} \frac{1}{|G|} \sum_{g \in G} (\rho_k)_{ij}(g^{-1}xg). \end{aligned}$$

Let $T = (\rho_k)_x = \rho_k(x) : V \rightarrow V$. Then

$$T^\# = \frac{1}{|G|} \sum_{g \in G} (\rho_k)_{g^{-1}} (\rho_k)_x (\rho_k)_g = \frac{1}{|G|} \sum_{g \in G} (\rho_k)(g^{-1}xg) = \frac{1}{n_k} \chi_k(x) \text{id}$$

by the Ergodic theorem and because ρ_k is a group homomorphism. Thus

$$f(x) = \sum_{i,j,k} \lambda_{ijk} ((\rho_k)_x)_{ij} = \sum_{i,j,k} \lambda_{ijk} \frac{1}{n_k} \chi_k(x) \delta_{ij} = \sum_{i,k} \lambda_{iik} \frac{1}{n_k} \chi_k(x). \quad \square$$

This implies that $\dim C(G) \leq r$ and the claim follows.

In the following exercise, the reader is asked to prove the second Schur's orthogonality relation.

8.14. EXERCISE. Let G be a finite group and C and D be conjugacy classes of G . If $g \in C$ and $h \in D$, then

$$\sum_{\chi \in \text{Irr}(G)} \chi(g) \overline{\chi(h)} = \begin{cases} |G|/|C| & \text{if } C = D, \\ 0 & \text{otherwise.} \end{cases}$$

§ 8.1. Examples. Let G be a finite group and χ_1, \dots, χ_r be the irreducible characters of G . Without loss of generality we may assume that χ_1 is the trivial character, i.e. $\chi_1(g) = 1$ for all $g \in G$. Recall that r is the number of conjugacy classes of G . Each χ_j is constant on conjugacy classes. The **character table** of G is given by

	1	k_2	\cdots	k_r
	1	g_2	\cdots	g_r
χ_1	1	1	\cdots	1
χ_2	n_2	$\chi_2(g_2)$	\cdots	$\chi_2(g_r)$
\vdots	\vdots	\vdots	\ddots	\vdots
χ_r	n_r	$\chi_r(g_2)$	\cdots	$\chi_r(g_r)$

where the n_j are the degrees of the irreducible representations of G and each k_j is the size of the conjugacy class of the element g_j . By convention, the character table contains not only the values of the irreducible characters of the group.

8.15. EXAMPLE. Let $G = \langle g : g^4 = 1 \rangle$ be the cyclic group of order four. The character table of G is given by

	1	1	1	1
	1	g	g^2	g^3
χ_1	1	1	1	1
χ_2	1	λ	λ^2	λ^3
χ_3	1	λ^2	λ^4	λ^2
χ_4	1	λ^3	λ^2	λ

8.16. EXERCISE. Let $n \in \mathbb{Z}_{>0}$ be such that $n \geq 2$. Let $C_n = \langle g : g^n = 1 \rangle$ be the cyclic group of order n .

- 1) Prove that the maps $\chi_i : C_n \rightarrow \mathbb{C}^\times$, $g^k \mapsto e^{2\pi i k/n}$, where $i \in \{0, 1, \dots, n-1\}$, are the irreducible representations of C_n .
- 2) Let λ be a primitive root of 1 of order n . Prove that the character table of C_n of order n is given by

	1	1	1	\dots	1
	1	g	g^2	\dots	g^{n-1}
χ_1	1	1	1	\dots	1
χ_2	1	λ	λ^2	\dots	λ^{n-1}
χ_3	1	λ^2	λ^4	\dots	λ^{n-2}
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
χ_n	1	λ^{n-1}	λ^{n-2}	\dots	λ

8.17. EXERCISE. Let A and B be finite abelian groups. We write $\text{Irr}(A) = \{\rho_1, \dots, \rho_r\}$ and $\text{Irr}(B) = \{\phi_1, \dots, \phi_s\}$. Prove that the maps

$$\varphi_{ij} : A \times B \rightarrow \mathbb{C}^\times, \quad (a, b) \mapsto \rho_i(a)\phi_j(b),$$

where $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, are the irreducible representations of $A \times B$.

Let us show a particular example of the previous exercise.

8.18. EXAMPLE. The character table of the group $C_2 \times C_2 = \{1, a, b, ab\}$ is

	1	1	1	1
	1	a	b	ab
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

8.19. EXAMPLE. The symmetric group \mathbb{S}_3 has three conjugacy classes. The representatives are id , (12) and (123) . There are three irreducible representations. We already found all the irreducible characters! The character table of \mathbb{S}_3 is given by

	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let us recall how this table was computed. Degree-one irreducibles were easy to compute. To compute the third row of the table one possible approach is to use the irreducible representation

$$(12) \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$\chi_3((12)) = \text{trace} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = 0,$$

$$\chi_3((123)) = \chi_3((12)(23)) = \text{trace} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = -1.$$

We should remark that the irreducible representation mentioned is not really needed to compute the third row of the character table. We can, for example, use the regular representation L . The character of L is given by

$$\chi_L(g) = \begin{cases} 6 & \text{si } g = \text{id}, \\ 0 & \text{si } g \neq \text{id}. \end{cases}$$

The equality $0 = \chi_L((12)) = 1 - 1 + 2\chi_3((12))$ implies that $\chi_3((12)) = 0$ and the equality

$$0 = \chi_L((123)) = 1 + 1 + 2\chi_3((123))$$

implies that $\chi_3((123)) = -1$.

Another approach uses the orthogonality relations. We need to compute $\chi_3((12))$ and $\chi_3((123))$. Let $a = \chi_3((12))$ and $b = \chi_3((123))$. Then we get that $a = 0$ and $b = -1$. We just need to solve

$$0 = \langle \chi_3, \chi_1 \rangle = \frac{1}{6}(2 + 3a + 2b),$$

$$0 = \langle \chi_3, \chi_2 \rangle = \frac{1}{6}(2 - 3a + 2b).$$

8.20. EXERCISE. Compute the character table of \mathbb{S}_4 .

8.21. EXAMPLE. We now compute the character table of the alternating group \mathbb{A}_4 . This group has 12 elements and four conjugacy classes.

representative	id	(123)	(132)	(123)
size	1	4	4	3

Since

$$[\mathbb{A}_4, \mathbb{A}_4] = \{\text{id}, (12)(34), (13)(24), (14)(23)\},$$

the quotient $\mathbb{A}_4/[\mathbb{A}_4, \mathbb{A}_4]$ has three elements. Thus \mathbb{A}_4 has three degree-one irreducibles and an irreducible character of degree three. Let $\omega = \exp(2\pi i/3)$ be a primitive cubic root of 1. If χ is a non-trivial degree-one character, then $\chi((123)) = \omega^j$ for some $j \in \{1, 2\}$ and

$\chi((132)) = \omega^{2j}$. Since $(132)(134) = (12)(34)$ and the permutations (134) and (123) are conjugate,

$$\chi_i((12)(34)) = \chi_i((132)(134)) = \chi_i((132))\chi_i((134)) = \omega^3 = 1$$

for all $i \in \{1, 2\}$.

To compute χ_4 we use the regular representation.

$$0 = \chi_L((12)(34)) = 1 + 1 + 1 + 3\chi_4((12)(34)),$$

$$0 = \chi_L((123)) = 1 + \omega + \omega^2 + 3\chi_4((123)),$$

$$0 = \chi_L((132)) = 1 + \omega + \omega^2 + 3\chi_4((132)).$$

Then we obtain that $\chi_4((123)) = \chi_4((132)) = 0$ and $\chi_4((12)(34)) = -1$. Therefore, the character table of \mathbb{A}_4 is given by

	id	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3	0	0	-1

8.22. EXAMPLE. Let $Q_8 = \{-1, 1, i, -i, j, -j\}$ be the quaternion group. The group Q_8 is generated by $\{i, j\}$ and the map $\rho: Q_8 \rightarrow \mathbf{GL}_2(\mathbb{C})$,

$$i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a representation. The conjugacy classes of Q_8 are $\{1\}$, $\{-1\}$, $\{-i, i\}$, $\{-j, j\}$ and $\{-k, k\}$. So there are five irreducible representations. We can compute the character of ρ :

	1	-1	i	j	k
χ_ρ	2	2	0	0	0

Then ρ is irreducible, as $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Since $[Q_8, Q_8] = \{-1, 1\} = Z(Q_8)$, the quotient group $Q_8/[Q_8, Q_8]$ has four elements and hence there are four irreducible degree-one representations. Since Q_8 is non-abelian, $Q_8/Z(Q_8)$ cannot be cyclic. This implies that $Q_8/[Q_8, Q_8] \simeq C_2 \times C_2$. This allows us to compute almost all the character table of Q_8 .

	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

It remains to compute $\chi_j(-1)$ for $j \in \{2, 3, 4\}$, these missing values are presented in shaded cells. To compute these values that $\langle \chi_i, \chi_j \rangle = 0$ whenever $i \neq j$. The calculations are left as an exercise.

8.23. EXERCISE. Compute the character table of the dihedral group of eight elements.

§ 8.2. Finite simple groups (optional). Recall that a non-trivial group is **simple** if it contains no non-trivial normal proper subgroups. Examples of simple groups are cyclic groups of prime order and the alternating groups A_n for $n \geq 5$. We can use the character table to detect simple groups:

8.24. PROPOSITION. *Let G be a finite group. Then G is not simple if and only if there exists a non-trivial irreducible character χ such that $\chi(g) = \chi(1)$ for some $g \in G \setminus \{1\}$.*

The proof is out of the scope of this course.

8.25. EXAMPLE. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	1	1	-1	-1	1
χ_5	2	-2	2	0	0	0
χ_6	8	0	-1	0	0	0

then G cannot be simple. Note that such a group G would have order $\sum_{i=1}^6 \chi_i(1)^2 = 72$. Mathieu's group M_9 has this character table!

8.26. EXAMPLE. Let $\alpha = \frac{1}{2}(-1 + \sqrt{7}i)$. If there exists a group G with a character table of the form

χ_1	1	1	1	1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	8	0	0	-1	1	1
χ_4	3	-1	1	0	α	$\bar{\alpha}$
χ_5	3	-1	1	0	$\bar{\alpha}$	α
χ_6	6	2	0	0	0	0

then G is simple. Note that such a group G would have order $\sum_{i=1}^6 \chi_i(1)^2 = 168$. The group

$$\mathbf{PSL}_2(7) = \mathbf{SL}_2(7)/Z(\mathbf{SL}_2(7))$$

is a simple group that has this character table!

Character theory is one of the tools in the classification of finite simple groups. One of the first non-trivial applications one finds of character theory is the following result:

8.27. THEOREM (Burnside). *Let G be a finite simple group of order $p^a q^b$ for some prime numbers p and q . Then G is cyclic.*

The proof uses character theory. It is not hard, but it is out of the scope of this course. Burnside's theorem is essential and generalizes in many different directions.

8.28. THEOREM (Feit–Thompson). *Let G be a finite simple group of odd order. Then G is cyclic.*

The proof is hard and occupies a full volume [4] of the **Pacific Journal of Mathematics** (see Figure 8.1). The theorem was formally verified by the computer proof assistant **Coq**, see [5] for the announcement.

The proof of the Feit–Thompson theorem suggests the following conjecture:

8.29. OPEN PROBLEM (Feit–Thompson). Are there distinct prime numbers p and q such that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$?

The conjecture states that there are no such prime numbers. The problem is still open; see [11] for some partial results.



FIGURE 8.1. The proof of the Feit–Thompson theorem occupies a full volume of the Pacific Journal of Mathematics.

The Feit–Thompson theorem is one of the starting points of the classification of finite simple groups (CFSG). This classification is one of the deepest theorems of the 20th century. The proof consists of tens of thousands of pages in several hundred journal articles written by about 100 authors, published mostly between 1955 and 2004.

8.30. THEOREM (CFSG). *Let G be a finite simple group. Then G lies in one (or more) of the following families:*

- 1) *Cyclic groups of prime order.*
- 2) \mathbb{A}_n for $n \geq 5$.
- 3) *Finite groups of Lie type.*
- 4) *26 sporadic simple groups.*

Groups of Lie type are finite analogs of simple Lie groups, such as $\mathbf{SL}_n(\mathbb{C})$. A typical example of a finite simple group of Lie type is $\mathbf{PSL}_n(p)$ for some prime number p , which is defined as $\mathbf{SL}_n(p)$ over its center.

The **sporadic simple groups** are 26 groups that do not follow a systematic pattern. The first five of the sporadic groups were discovered by Mathieu in the 1860s, and the other 21 were found between 1965 and 1975. Several of these groups were predicted to exist before they were constructed, sometimes just knowing their character tables! The full list of sporadic simple groups is as follows:

- Mathieu groups M_{11} , M_{12} , M_{22} , M_{23} and M_{24} .
- Janko groups J_1 , J_2 , J_3 and J_4 .
- Conway groups Co_1 , Co_2 and Co_3 .
- Fischer groups Fi_{22} , Fi_{23} and Fi_{24} .
- Higman–Sims group HS .

- McLaughlin group McL .
- Held group He .
- Rudvalis group Ru .
- Suzuki group Suz .
- O’Nan group ON .
- Harada–Norton group HN .
- Lyons group Ly .
- Thompson group Th .
- Baby Monster group B .
- Monster group M .

Mathieu groups can be realized as automorphism groups of certain very complicated combinatorial structures known as **Steiner systems**. A concrete example:

$$M_{11} = \langle (123456789\ 10\ 11), (37\ 11\ 8)(4\ 10\ 56) \rangle,$$

which is a subgroup of S_{11} of order 7920. The group M_{11} contains ten irreducible characters, so the character table of M_{11} is essentially a 10×10 matrix. Let us use the computer software **GAP** to see the character table:

```
gap> Display(CharacterTable("M11"));
M11
```

	2	4	4	1	3	.	1	3	3	.	.
	3	2	1	2	.	.	1
	5	1	.	.	.	1
	11	1	1	1

	1a	2a	3a	4a	5a	6a	8a	8b	11a	11b
2P	1a	1a	3a	2a	5a	3a	4a	4a	11b	11a
3P	1a	2a	1a	4a	5a	2a	8a	8b	11a	11b
5P	1a	2a	3a	4a	1a	6a	8b	8a	11a	11b
11P	1a	2a	3a	4a	5a	6a	8a	8b	1a	1a

X.1	1	1	1	1	1	1	1	1	1	1
X.2	10	2	1	2	.	-1	.	.	-1	-1
X.3	10	-2	1	.	.	1	A	-A	-1	-1
X.4	10	-2	1	.	.	1	-A	A	-1	-1
X.5	11	3	2	-1	1	.	-1	-1	.	.
X.6	16	.	-2	.	1	.	.	.	B	/B
X.7	16	.	-2	.	1	.	.	.	/B	B
X.8	44	4	-1	.	-1	1
X.9	45	-3	.	1	.	.	-1	-1	1	1
X.10	55	-1	1	-1	.	-1	1	1	.	.


```
A = E(8)+E(8)^3
    = Sqrt(-2) = i2
B = E(11)+E(11)^3+E(11)^4+E(11)^5+E(11)^9
    = (-1+Sqrt(-11))/2 = b11
```

Thus the character table of the Mathieu group M_{11} is given by

χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	2	0	-1	0	0	-1	-1
χ_3	10	-2	1	0	0	1	α	$-\alpha$	-1	-1
χ_4	10	-2	1	0	0	1	$-\alpha$	α	-1	-1
χ_5	11	3	2	-1	1	0	-1	-1	0	0
χ_6	16	0	-2	0	1	0	0	0	β	$1/\beta$
χ_7	16	0	-2	0	1	0	0	0	$1/\beta$	β
χ_8	44	4	-1	0	-1	1	0	0	0	0
χ_9	45	-3	0	1	0	0	-1	-1	1	1
χ_{10}	55	-1	1	-1	0	-1	1	1	0	0

where $\alpha = \sqrt{-2}$ and $\beta = \frac{-1-\sqrt{-11}}{2}$.

The largest sporadic simple group is the **Monster group** M and has order

$$808017424794512875886459904961710757005754368000000000,$$

which is roughly 8×10^{53} . The monster group was predicted by Fischer and Griess. Griess proved the existence of the Monster group, realizing it as the automorphism group of a certain space of dimension 196884. The group M can be represented as a subgroup of $\mathbf{GL}_{196883}(\mathbb{C})$. It has 194 conjugacy classes, so the character table is a 194×194 array.

8.31. OPEN PROBLEM. Formalize the proof of Theorem 8.30 in a proof assistant.

It is precisely due to the vastness of the theorem's proof, that it is very suitable for formalization. Currently, only mathematicians with an extremely deep understanding of the subject can understand parts of the proof. By formalizing the proof in a proof assistant such as **Coq** or **Lean**, the complete proof could be verified at once.

9. Lecture: 27/11/2024

§ 9.1. Modules. The rest of the course will be devoted to studying modules over rings. We first start with the main definitions and basic examples.

9.1. DEFINITION. Let R be a ring. A **module** (over R) is an abelian group (written additively) M with a map $R \times M \rightarrow M$, $(r, m) \mapsto r \cdot m$, such that the following conditions hold:

- 1) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$ for all $r_1, r_2 \in R$ and $m \in M$.
- 2) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$ for all $r \in R$ and $m_1, m_2 \in M$.
- 3) $r_1 \cdot (r_2 \cdot m) = (r_1 r_2) \cdot m$ for all $r_1, r_2 \in R$ and $m \in M$.
- 4) $1 \cdot m = m$ for all $m \in M$.

Our definition is that of left module. Similarly, one defines right modules. We will always consider left modules (over R) so that they will be referred to simply as R -modules (or just modules).

9.2. EXAMPLE. A module over a field is a vector space.

9.3. EXAMPLE. Every abelian group is a module over \mathbb{Z} .

9.4. EXAMPLE. Let R be a ring. Then R is a module (over R) with $x \cdot m = xm$. This is the **(left) regular module** over R , and it usually is denoted by ${}_R R$.

The module ${}_R R$ is also known as the (left) regular representation of R .

9.5. EXAMPLE. Let G be a finite group. If $\rho: G \rightarrow \mathbf{GL}(V)$ is a representation, then V is a $\mathbb{C}[G]$ -module with

$$g \cdot v = \rho_g(v)$$

for $g \in G$ and $v \in V$.

Conversely, if V is a $\mathbb{C}[G]$ -module and V is a complex finite-dimensional vector space, then the map

$$\rho: G \rightarrow \mathbf{GL}(V), \quad g \mapsto \rho_g,$$

where $\rho_g(v) = g \cdot v$ for $v \in V$, is a representation of G .

9.6. EXAMPLE. If R is a ring, then $R^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in R\}$ is a module (over R) with $r \cdot (x_1, \dots, x_n) = (rx_1, \dots, rx_n)$.

9.7. EXAMPLE. If R is a ring, then $M_{m,n}(R)$ is a module (over R) with usual matrix operations.

Students usually ask why in the definition of a ring homomorphism, one needs the condition $1 \mapsto 1$. The following example provides a good explanation.

9.8. EXAMPLE. If $f: R \rightarrow S$ is a ring homomorphism and M is a module (over S) with $(s, m) \mapsto sm$, then M is also a module (over R) with $r \cdot m = f(r)m$ for all $r \in R$ and $m \in M$. In fact,

$$1 \cdot m = f(1)m = 1m = m,$$

$$r_1 \cdot (r_2 \cdot m) = f(r_1)(r_2 \cdot m) = f(r_1)(f(r_2)m) = (f(r_1)f(r_2))m = f(r_1 r_2)m$$

for all $r_1, r_2 \in R$ and $m \in M$.

9.9. EXAMPLE. Let V be a finite-dimensional real vector space and $T: V \rightarrow V$ be a linear map. Let $R = \mathbb{R}[X]$. Then $M = V$ with

$$\left(\sum_{i=0}^n a_i X^i \right) \cdot v = \sum_{i=0}^n a_i T^i(v)$$

is a module (over R), where we use the following notation: for $v \in V$, we write

$$T^0(v) = v, \quad T^1(v) = T(v)v, \quad T^2(v) = T(T(v)) \dots$$

9.10. EXAMPLE. If $\{M_i : i \in I\}$ is a family of R -modules, then

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i \text{ for all } i \in I\}$$

is an R -module with $x \cdot (m_i)_{i \in I} = (x \cdot m_i)_{i \in I}$, where $(m_i)_{i \in I}$ denotes the map $I \rightarrow M_i, i \mapsto m_i$. This module is the **direct product** of the family $\{M_i : i \in I\}$.

9.11. EXAMPLE. If $\{M_i : i \in I\}$ is family of R -modules, then

$$\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i \text{ for all } i \in I \text{ and } m_i = 0 \text{ except finitely many } i \in I\}$$

is an R -module with $x \cdot (m_i)_{i \in I} = (x \cdot m_i)_{i \in I}$. This module is the **direct sum** of the family $\{M_i : i \in I\}$.

If M is a module, then $0 \cdot m = 0$ and $-m = (-1) \cdot m$ for all $m \in M$ and $x \cdot 0 = 0$ for all $x \in R$.

9.12. EXAMPLE. Let $M = \mathbb{Z}/6$ as a module (over \mathbb{Z}). Note that $3 \cdot 2 = 0$ but $3 \neq 0$ (in \mathbb{Z}) and $2 \neq 0$ (in $\mathbb{Z}/6$).

9.13. EXERCISE. Prove that $\mathbb{Z}/4$ is not a module over $\mathbb{Z}/2$.

9.14. DEFINITION. Let M be a module. A subset N of M is a **submodule** of M if N is a subgroup of M and $R \cdot N \subseteq N$, that is $x \cdot n \in N$ for all $x \in R$ and $n \in N$.

Clearly, if M is a module, then $\{0\}$ and M are submodules of M .

9.15. EXAMPLE. Let R be a field and M be a module over R . Then N is a submodule of M if and only if N is a subspace of M .

9.16. EXAMPLE. Let $R = \mathbb{Z}$ and M be a module (over R). Then N is a submodule of M if and only if N is a subgroup of M .

9.17. EXAMPLE. If $M = {}_R R$, then a subset $N \subseteq M$ is a submodule of M if and only if N is a left ideal of R .

9.18. EXAMPLE. If V is a finite-dimensional real vector space and $T: V \rightarrow V$ is a linear map, then V is a module (over $\mathbb{R}[X]$) with

$$\left(\sum_{i=0}^n a_i X^i \right) \cdot v = \sum_{i=0}^n a_i T^i(v).$$

A submodule is a subspace W of V such that $T(W) \subseteq W$.

Clearly, a subset N of M is a submodule if and only if $r_1n_1 + r_2n_2 \in N$ for all $r_1, r_2 \in R$ and $n_1, n_2 \in N$.

9.19. EXERCISE. If N and N_1 are submodules of M , then

$$N + N_1 = \{n + n_1 : n \in N, n_1 \in N_1\}$$

is a submodule of M .

9.20. DEFINITION. Let M and N be modules over R . A map $f: M \rightarrow N$ is a **module homomorphism** if $f(x+y) = f(x) + f(y)$ and $f(r \cdot x) = r \cdot f(x)$ for all $x, y \in M$ and $r \in R$.

We denote by $\text{Hom}_R(M, N)$ the set of module homomorphisms $M \rightarrow N$.

9.21. EXERCISE. Let $f \in \text{Hom}_R(M, N)$.

- 1) If V is a submodule of M , then $f(V)$ is a submodule of N .
- 2) If W is a submodule of N , then $f^{-1}(W)$ is a submodule of M .

If $f \in \text{Hom}_R(M, N)$, the **kernel** of f is the submodule

$$\ker f = f^{-1}(\{0\}) = \{m \in M : f(m) = 0\}$$

of M . We say that f is a **monomorphism** (resp. **epimorphism**) if f is injective (resp. surjective). Moreover, f is an **isomorphism** if f is bijective.

9.22. EXERCISE. Let $f \in \text{Hom}_R(M, N)$. Prove that the following statements are equivalent:

- 1) f is a monomorphism.
- 2) $\ker f = \{0\}$.
- 3) For every module V and every $g, h \in \text{Hom}_R(V, M)$, $fg = fh \implies g = h$.
- 4) For every module V and every $g \in \text{Hom}(V, M)$, $fg = 0 \implies g = 0$.

Later we will see a similar exercise for surjective module homomorphisms.

9.23. EXAMPLE. Let $R = \begin{pmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{pmatrix}$. We claim that $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} \not\cong \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix}$ as modules over R , where the module structure is given by the usual matrix multiplication. Assume that they are isomorphic. Let $f: \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$ be an isomorphism of modules and let $x_0 \in \mathbb{R} \setminus \{0\}$ be such that $f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$. Thus

$$f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

a contradiction, as f is injective.

If N and N_1 are submodules of M , we say that M is the **direct sum** of N and N_1 if $M = N + N_1$ and $N \cap N_1 = \{0\}$. In this case, we write $M = N \oplus N_1$. Note that if $M = N \oplus N_1$, then each $m \in M$ can be written uniquely as $m = n + n_1$ for some $n \in N$ and $n_1 \in N_1$. Such a decomposition exists because $M = N + N_1$. If $m \in M$ can be written as $m = n + n_1 = n' + n'_1$ for some $n, n' \in N$ and $n_1, n'_1 \in N_1$, then $-n' + n = n'_1 - n_1 \in N \cap N_1 = \{0\}$ and hence

$n = n'$ and $n_1 = n'_1$. If $M = N \oplus N_1$, the submodule N (resp. N_1) is a **direct summand** of M and the submodule N_1 (resp. N) is a **complement** of N in M .

9.24. EXAMPLE. If $M = \mathbb{R}^2$ is a vector space, then every subspace of M is a direct summand of M .

Clearly, the submodules $\{0\}$ and M are direct summands of M .

9.25. EXAMPLE. If $M = \mathbb{Z}$ as a module over \mathbb{Z} , then $m\mathbb{Z}$ is a direct summand of M if and only if $m \in \{0, 1\}$, as $n\mathbb{Z} \cap m\mathbb{Z} = \{0\}$ if and only if $nm = 0$.

9.26. EXERCISE. Let M be a module. A module N is isomorphic to a direct summand of M if and only if there are module homomorphisms $i: N \rightarrow M$ and $p: M \rightarrow N$ such that $pi = \text{id}_N$. In this case, $M = \ker p \oplus i(N)$.

The **direct sum** of submodules can be defined for finitely many summands. If V_1, \dots, V_n are submodules of M , we say that $M = V_1 \oplus \dots \oplus V_n$ if every $m \in M$ can be written uniquely as $m = v_1 + \dots + v_n$ for some $v_1 \in V_1, \dots, v_n \in V_n$.

9.27. EXERCISE. Prove that $M = V_1 \oplus \dots \oplus V_n$ if and only if $M = V_1 + \dots + V_n$ and

$$V_i \cap \left(\sum_{j \neq i} V_j \right) = \{0\}$$

for all $i \in \{1, \dots, n\}$.

If $\{N_i : i \in I\}$ is a family of submodules of a module M , then the intersection $\cap_{i \in I} N_i$ is also a submodule of M .

9.28. EXERCISE. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map and $M = \mathbb{R}^2$ with the module structure over $\mathbb{R}[X]$ given by

$$\left(\sum_{i=0}^n a_i X^i \right) \cdot (x, y) = \sum_{i=0}^n a_i T^i(x, y).$$

Find the submodules of M in the following cases:

- 1) $T(x, y) = (0, y)$.
- 2) $T(x, y) = (y, x)$.

9.29. EXERCISE. Let V be a finite-dimensional real vector space and $T: V \rightarrow V$ be a linear map. Then V is a module over $\mathbb{R}[X]$ with

$$\left(\sum_{i=0}^n a_i X^i \right) \cdot v = \sum_{i=0}^n a_i T^i(v).$$

Prove that any module homomorphism $g: V \rightarrow V$ commutes with T .

9.30. EXERCISE. Prove that $\text{Hom}_R(M, N)$ is a module over $Z(R)$ with the following action: If $r \in R$ and $f \in \text{Hom}_R(M, N)$, then $r \cdot f: M \rightarrow N$, $m \mapsto f(r \cdot m)$.

Let M be a module, and N be a submodule of M . In particular, M/N is an abelian group, and the map $\pi: M \rightarrow M/N$, $m \mapsto m + N$, is a surjective group homomorphism with kernel equal to N . We claim that the **quotient** M/N is a module with

$$r \cdot (m + N) = (r \cdot m) + N, \quad r \in R, m \in M.$$

Let us check that this operation on M/N is well-defined. If $x + N = y + N$, then $x - y \in N$ implies that

$$r \cdot x - r \cdot y = r \cdot (x - y) \in N,$$

that is $r \cdot (x + N) = r \cdot (y + N)$. It is an exercise to show that the map $\pi: M \rightarrow M/N$, $x \mapsto x + N$, is a surjective module homomorphism.

9.31. EXAMPLE. If $R = M = \mathbb{Z}$ and $N = 2\mathbb{Z}$, then $M/N \simeq \mathbb{Z}/2$.

9.32. EXAMPLE. Let R be a commutative ring and M be an R -module. We claim that

$$M \simeq \text{Hom}_R({}_R R, M).$$

Since R is commutative, it follows that $\text{Hom}_R({}_R R, M)$ is a module, see Exercise 9.30.

Let $\varphi: M \rightarrow \text{Hom}_R({}_R R, M)$, $m \mapsto f_m$, where $f_m: R \rightarrow M$, $r \mapsto r \cdot m$. To show that φ is well-defined it is enough to see that $\varphi(m) \in \text{Hom}_R({}_R R, M)$, that is

$$f_m(r + s) = (r + s) \cdot m = r \cdot m + s \cdot m, \quad f_m(rs) = (rs) \cdot m = r \cdot (s \cdot m) = r \cdot f_m(s).$$

Let us show that φ is a module homomorphism. We first note that

$$\varphi(m + n) = \varphi(m) + \varphi(n)$$

for all $m, n \in M$, as

$$\begin{aligned} \varphi(m + n)(r) &= f_{m+n}(r) = r \cdot (m + n) \\ &= r \cdot m + r \cdot n = f_m(r) + f_n(r) = \varphi(m)(r) + \varphi(n)(r). \end{aligned}$$

Moreover,

$$\varphi(r \cdot m) = r \cdot \varphi(m)$$

for all $r \in R$ and $m \in M$, as

$$\begin{aligned} \varphi(r \cdot m)(s) &= f_{r \cdot m}(s) = s \cdot (r \cdot m) = (sr) \cdot m \\ &= (rs) \cdot m = f_m(rs) = \varphi(m)(rs) = (r \cdot \varphi(m))(s). \end{aligned}$$

It remains to show that φ is bijective. We first prove that φ is injective. If $\varphi(m) = 0$, then $r \cdot m = \varphi(m)(r) = 0$ for all $r \in R$. In particular, $m = 1 \cdot m = 0$. We now prove that φ is surjective. If $f \in \text{Hom}_R({}_R R, M)$, let $m = f(1)$. Then $\varphi(m) = f$, as

$$\varphi(m)(r) = r \cdot m = r \cdot f(1) = f(r).$$

As one does for groups, it is possible to show that if M is a module and N is a submodule of M , the pair $(M/N, \pi: M \rightarrow M/N)$ has the following properties:

- 1) $N \subseteq \ker \pi$.
- 2) If $f: M \rightarrow T$ is a homomorphism such that $N \subseteq \ker f$, then there exists a unique module homomorphism $\varphi: M/N \rightarrow T$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & T \\ \pi \downarrow & \nearrow \varphi & \\ M/N & & \end{array}$$

is commutative, that is $\varphi\pi = f$.

Recall that if S and T are submodules of a module M , then both $S \cap T$ and

$$S + T = \{s + t : s \in S, t \in T\}$$

are submodules of M . The **isomorphism theorems** hold:

- 1) If $f \in \text{Hom}_R(M, N)$, then $M/\ker f \simeq f(M)$.
- 2) If $T \subseteq N \subseteq M$ are submodules, then

$$\frac{M/T}{N/T} \simeq M/N$$

- 3) If S and T are submodules of M , then

$$(S + T)/S \simeq T/(S \cap T).$$

9.33. EXAMPLE. If K is a field and V is a module over K , then V is, by definition, a vector space over K . If S and T are subspaces of V , then they are submodules of V . By the isomorphism theorem, $(S + T)/S \simeq T/(S \cap T)$ as vector spaces. By applying dimension,

$$\dim(S + T) - \dim S = \dim T - \dim(S \cap T).$$

9.34. EXAMPLE. If N is a direct summand of M and M and X is a complement for N , then $X \simeq M/N$, as

$$M/N = (N \oplus X)/N \simeq X/(N \cap X) = X/\{0\} \simeq X$$

by the second isomorphism theorem. So all complements of N in M are isomorphic.

It is also possible to prove that there exists a bijective correspondence between submodules of M/N and submodules of M containing N . The correspondence is given by $\pi^{-1}(Y) \leftrightarrow Y$ and $X \mapsto \pi(X)$.

9.35. EXERCISE. Let $f \in \text{Hom}_R(M, N)$. Prove that the following statements are equivalent:

- 1) f is an epimorphism.
- 2) $N/f(M) \simeq \{0\}$.
- 3) For every module T and every $g, h \in \text{Hom}_R(N, T)$, $gf = hf \implies g = h$.
- 4) For every module T and every $g \in \text{Hom}_R(N, T)$, $gf = 0 \implies g = 0$.

9.36. EXERCISE. Let R be a ring and M_1 and M_2 be maximal ideals of R . Prove that $R/M_1 \simeq R/M_2$ as modules over R if and only if there exists $r \in R \setminus M_2$ such that $rM_1 \subseteq M_2$.

10. Lecture: 04/12/2024

§ 10.1. **Noetherian modules.** We first start with a very easy exercise.

10.1. EXERCISE. Let M be a module. Prove that the intersection of submodules of M is a submodule of M .

10.2. DEFINITION. Let M be a module and X be a subset of M . The submodule of M generated by X is defined as

$$(X) = \bigcap \{N : N \text{ is a submodule of } M \text{ that contains } X\},$$

the smallest submodule of M containing X .

One can prove that

$$(X) = \left\{ \sum_{i=1}^m r_i \cdot x_i : m \in \mathbb{Z}_{\geq 0}, r_1, \dots, r_m \in R, x_1, \dots, x_m \in X \right\}.$$

10.3. DEFINITION. A module M is **finitely generated** if $M = (X)$ for some finite subset X of M .

If $X = \{x_1, \dots, x_m\}$ one writes $(X) = (x_1, \dots, x_m)$. For example, $\mathbb{Z} = (1) = (2, 3)$ and $\mathbb{Z} \neq (2)$.

10.4. EXERCISE. Let R be the ring of continuous maps $[0, 1] \rightarrow \mathbb{R}$ with point-wise operations and $M = {}_R R$. Prove that

$$N = \{f \in R : f(x) \neq 0 \text{ for finitely many } x\}$$

is not finitely generated.

10.5. EXERCISE. Let $G = \{g_1, \dots, g_n\}$ be a finite group. Prove that if $M = \mathbb{C}[G]$ is finitely generated, then M is a finite-dimensional complex vector space.

10.6. EXAMPLE. Let R be a ring. The R -module R^3 is finitely generated by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

If $f \in \text{Hom}_R(M, N)$ and M is finitely generated, then $f(M)$ is finitely generated. To prove this and other similar results, we introduce exact sequences (of modules and module homomorphisms). A sequence

$$(10.1) \quad 0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

of modules and homomorphism is said to be **exact** if f is injective, g is surjective and $f(M) = \ker g$. For example, the sequence

$$0 \longrightarrow M \xrightarrow{f} M \oplus N \xrightarrow{g} N \longrightarrow 0,$$

where $f(m) = (m, 0)$ and $g(m, n) = n$, is exact.

10.7. EXERCISE. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

be an exact sequence. Prove the following statements:

- 1) If N is finitely generated, then T is finitely generated.
- 2) If M and T are finitely generated, then N is finitely generated.

10.8. EXERCISE. Find a finitely generated module that contains a submodule that is not finitely generated.

10.9. PROPOSITION. *Let R be a ring and M be a module over R . Then M is finitely generated if and only if M is isomorphic to a quotient of R^k for some k .*

PROOF. Assume first that $M = (m_1, \dots, m_k)$ is finitely generated. A routine calculation shows that the map

$$\varphi: R^k \rightarrow M, \quad (r_1, \dots, r_k) \mapsto \sum_{j=1}^k r_j \cdot m_j,$$

is a surjective module homomorphism. The first isomorphism theorem implies that

$$R^k / \ker \varphi \simeq \varphi(R^k) = M.$$

Conversely, assume that there exists a surjective module homomorphism $\varphi: R^k \rightarrow M$. Since $R^k = (e_1, \dots, e_k)$, where

$$(e_i)_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

it follows that $\{\varphi(e_1), \dots, \varphi(e_k)\}$ generates $\varphi(R^k) = M$. To prove this, note that if $m \in M$, by the surjectivity of φ , we write $m = \varphi(r_1, \dots, r_k)$ for some $(r_1, \dots, r_k) \in R^k$. Hence

$$m = \varphi(r_1, \dots, r_k) = \varphi\left(\sum_{i=1}^k r_i \cdot e_i\right) = \sum_{i=1}^k r_i \cdot \varphi(e_i). \quad \square$$

10.10. DEFINITION. A module M is **noetherian** if every sequence $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M stabilizes, that is there exists n such that $M_k = M_{n+k}$ for all k .

Let M be a module and X be a non-empty family of submodules of M . We say that $S \in X$ is a **maximal element** (with respect to the inclusion) if $S \subseteq T$ for some $T \in X$ implies $S = T$.

10.11. PROPOSITION. *Let M be a module. The following statements are equivalent:*

- 1) M is noetherian.
- 2) All submodules of M are finitely generated.
- 3) Every non-empty family of submodules of M has a maximal element.

PROOF. We first prove 2) \implies 1). If $S_1 \subseteq S_2 \subseteq \dots$ is an increasing sequence of submodules of M , it follows that $S = \cup_{i \geq 1} S_i$ is a submodule³ of M . Since S is finitely

³In general, the union of submodules is not a submodule. Here $\cup_j S_j$ is a submodule because we have an increasing sequence $S_1 \subseteq S_2 \subseteq \dots$ of submodules.

generated, $S = (x_1, \dots, x_n)$ for some $x_1, \dots, x_n \in M$. It follows that $x_1, \dots, x_n \in S_N$ for some positive integer N . Thus $S \subseteq S_N$ and hence $S_N = S_{N+k}$ for all k .

We now prove 1) \implies 3). Let F be a non-empty family of submodules of M with no maximal elements. Let $S_1 \in F$. Since S_1 is not maximal in F , there exists $S_2 \in F$ such that $S_1 \subsetneq S_2$. Having constructed with this method the submodules $S_1 \subsetneq \dots \subsetneq S_k$, since S_k is not maximal in F , there exists $S_{k+1} \in F$ such that $S_k \subsetneq S_{k+1}$. This means that the sequence $S_1 \subsetneq S_2 \subsetneq \dots$ does not stabilize.

We finally prove 3) \implies 2). Let S be a submodule of M and let

$$F = \{T \subseteq S : T \text{ finitely generated submodule of } M\}.$$

Note that $F \neq \emptyset$, as $\{0\} \in F$. Then F has a maximal element N . Thus N is a finitely generated submodule of M such that $N \subseteq S$. We may assume that $N = (n_1, \dots, n_k)$. If $N = S$, then, in particular, S is finitely generated. Suppose that $N \neq S$ and let $x \in S \setminus N$. It follows that $N \subseteq (n_1, \dots, n_k, x) \subseteq S$. Since $(n_1, \dots, n_k, x) \in F$ and N is maximal, it follows that $N = (n_1, \dots, n_k, x)$, a contradiction to $x \notin N$. \square

The previous theorem shows that every noetherian module is finitely generated.

10.12. EXERCISE. Find a non-noetherian module such that every proper submodule is finitely generated.

10.13. EXERCISE. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

be an exact sequence. Prove the following statements:

- 1) If N is noetherian, then M and T are noetherian.
- 2) If M and T are noetherian, then N is noetherian.

10.14. EXERCISE. A commutative ring R is noetherian if and only if ${}_R R$ is noetherian.

10.15. EXERCISE. If M_1, \dots, M_n are noetherian, then $M_1 \oplus \dots \oplus M_n$ is noetherian.

The previous exercise cannot be extended to infinitely many modules. Why?

10.16. PROPOSITION. *If R is noetherian and M is a finitely generated module, then M is noetherian.*

PROOF. Assume that $M = (m_1, \dots, m_k)$. There exists a surjective homomorphism

$$R^k \rightarrow M, \quad (r_1, \dots, r_k) \mapsto \sum_{i=1}^k r_i \cdot m_i,$$

where $R^k = \bigoplus_{i=1}^k R$. Since R is noetherian, R^k is noetherian. Thus M is noetherian. \square

§ 10.2. Quotient fields. Let R be an integral domain and $S = R \setminus \{0\}$. On $R \times S$ we define the following relation:

$$(r, s) \sim (r_1, s_1) \iff rs_1 - r_1s = 0.$$

10.17. EXERCISE. Prove that \sim is an equivalence relation.

The equivalence class of (r, s) will be denoted by r/s or $\frac{r}{s}$.

It is possible to prove that the set $K(R) = (R \times S)/\sim$ of equivalence classes is a field with the operations

$$(10.2) \quad \frac{r}{s} + \frac{r_1}{s_1} = \frac{rs_1 + r_1s}{ss_1}, \quad \frac{r}{s} \frac{r_1}{s_1} = \frac{rr_1}{ss_1}.$$

10.18. DEFINITION. $K(R)$ is known as the **quotient field** of R .

Simple example: $K(\mathbb{Z}) = \mathbb{Q}$.

10.19. EXERCISE. Let R be an integral domain. Prove that $K(R)$ is a field.

To prove that $K(R)$ is a field, one first needs to prove that the operations (10.2) are well-defined. For example, let us check the addition is well-defined. If $r/s \sim r'/s'$ and $r_1/s_1 \sim r'_1/s'_1$, then $r/s + r_1/s_1 \sim r'/s' + r'_1/s'_1$. In fact, since $r/s \sim r'/s'$, it follows that $rs' - r's = 0$. Similarly, $r_1s'_1 - r'_1s_1 = 0$, as $r_1/s_1 \sim r'_1/s'_1$. Thus

$$\frac{rs_1 + r_1s}{ss_1} = \frac{r's'_1 + r'_1s'}{s's'_1}$$

as

$$(rs_1 + r_1s)s's'_1 = rs_1s's'_1 + r_1ss's'_1 = r's's_1s'_1 + r'_1s_1ss' = (r's'_1 + r'_1s)ss_1.$$

§ 10.3. Free modules.

10.20. DEFINITION. Let R be a ring, M be a module over R and X be a subset of M . We say that X is **linearly independent** if for each $k \in \mathbb{Z}_{>0}$, $r_1, \dots, r_k \in R$ and $m_1, \dots, m_k \in X$ such that $\sum_{i=1}^k r_i \cdot m_i = 0$, then $r_1 = \dots = r_k = 0$.

In any ring, the set $\{1\}$ is linearly independent.

A set is said to be **linearly dependent** if it is not linearly independent.

10.21. EXAMPLE.

- 1) $\{2, 3\}$ is a linearly dependent subset of \mathbb{Z} .
- 2) $\{2\}$ is a linearly dependent subset of $\mathbb{Z}/4$.
- 3) Let $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $x \in M \setminus \{0\}$. Then $\{x\}$ is a linearly independent subset of M . Is $y \in M \setminus \{x\}$, then $\{x, y\}$ is linearly dependent.

A generating set X for a module M is **minimal** if $X \setminus \{x\}$ is no longer a generating set of M for all $x \in X$. In vector spaces, a subset of the vector space is a basis if and only if it is a minimal generating set. This is no longer true for arbitrary modules.

10.22. EXAMPLE. Let $R = M_2(\mathbb{R})$ and $M = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. Then $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a minimal generating set and it is not linearly independent.

However, one can prove that each module's basis is a minimal generating set.

10.23. EXERCISE. Let $f \in \text{Hom}_R(M, N)$ and X be a subset of M .

- 1) If X is linearly dependent, then $f(X)$ is linearly dependent.
- 2) If X is linearly independent and f is injective, then $f(X)$ is linearly independent.
- 3) If $M = (X)$ and f is surjective, then $N = (f(X))$.

10.24. DEFINITION. Let M be a module and B be a subset of M . Then B is a **basis** of M if B is linearly independent and $M = (B)$. A module M is said to be **free** if it admits a basis.

As a consequence of Zorn's lemma, vector spaces are free.

10.25. EXAMPLE.

- 1) If R is a ring, then $\{1\}$ is a basis of ${}_R R$, so ${}_R R$ is free.
- 2) If R is a ring, then R^n is free as a module over R .

10.26. EXERCISE. Prove that $\mathbb{Z}/4$ is free as a $\mathbb{Z}/4$ module and that the submodule $\{0, 2\} \subseteq \mathbb{Z}/4$ is not free as a $\mathbb{Z}/4$ -module.

10.27. EXERCISE. Prove that \mathbb{Q} is not free as a module over \mathbb{Z} .

10.28. EXERCISE. Let R be a division ring and M be a non-zero and finitely generated module over R . Prove the following facts:

- 1) Every finite set of generators contains a basis.
- 2) Every linearly independent set can be extended into a basis.
- 3) Any two bases contain the same number of elements.

The previous exercise states that modules over division rings are like vector spaces over fields. In particular, such modules have dimensions.

10.29. EXAMPLE. $\mathbb{R}[X]$ is a free module (over \mathbb{R}) with basis $\{1, X, X^2, \dots\}$.

10.30. EXERCISE. Prove that $\{(a, b), (c, d)\}$ is a basis of $\mathbb{Z} \times \mathbb{Z}$ (as a module over \mathbb{Z}) if and only if $ad - bc \in \{-1, 1\}$.

10.31. EXAMPLE. If $u \in \mathcal{U}(R)$, then $\{u\}$ is a basis of ${}_R R$. Conversely, if R is an integral domain and $\{z\}$ is a basis of ${}_R R$, then $z \in \mathcal{U}(R)$. Since $1 = yz$ for some $y \in R$, it follows that $zy = 1$, as

$$(zy - 1)z = z(yz) - z = z1 - z = z - z = 0.$$

If R is a ring and I is a set,

$$R^{(I)} = \{f: I \rightarrow R : f(x) = 0 \text{ for all but finitely many } x \in I\}$$

is a module (over R) with $(f + g)(x) = f(x) + g(x)$ and $(rf)(x) = rf(x)$ for all $f, g \in R^{(I)}$, $x \in I$ and $r \in R$. Note that every element of $R^{(I)}$ can be written uniquely as a finite sum of the form

$$\sum_{i \in I} r_i \delta_i,$$

where the r_i are elements of R and only finitely many of them are non-zero, and

$$\delta_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

One proves that

$$R^{(I)} \simeq \bigoplus_{i \in I} R.$$

10.32. EXAMPLE. If I is a non-empty set, the module $R^{(I)}$ is free with basis $\{\delta_i : i \in I\}$, where

$$\delta_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

10.33. EXAMPLE. If $R = M_2(\mathbb{Z})$, then $M = {}_R R$ is free with basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. The submodule $N = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 0 \end{pmatrix}$ does not admit a basis as a module over R .

In modules, the size of a basis is not an invariant of the module.

10.34. EXAMPLE. Let V be the complex vector space with infinite basis e_0, e_1, e_2, \dots and let $R = \text{End}(V)$ with the ring structure given by

$$(f + g)(v) = f(v) + g(v), \quad (fg)(v) = f(g(v))$$

for $f, g \in R$ and $v \in V$.

Let $M = {}_R R$. The set $\{\text{id}\}$ is a basis for R . We claim that M admits a basis with two elements. If $r, s \in R$ are such that

$$\begin{aligned} r(e_{2n}) &= e_n, & r(e_{2n+1}) &= 0, \\ s(e_{2n}) &= 0, & s(e_{2n+1}) &= e_{2n}, \end{aligned}$$

then $\{r, s\}$ is a basis of M . If $f \in R$, then $f = \alpha r + \beta s$, where $\alpha: V \rightarrow V$, $e_n \mapsto f(e_{2n})$ for all n , and $\beta \in V \rightarrow V$, $e_n \mapsto f(e_{2n+1})$ for all n . In fact,

$$\begin{aligned} (\alpha r + \beta s)(e_{2n}) &= \alpha(r(e_{2n})) + \beta(s(e_{2n})) = f(e_{2n}), \\ (\alpha r + \beta s)(e_{2n+1}) &= \alpha(r(e_{2n+1})) + \beta(s(e_{2n+1})) = f(e_{2n+1}). \end{aligned}$$

Moreover, $\{r, s\}$ is linearly independent. Indeed, if $\alpha r + \beta s = 0$ for some $\alpha, \beta \in R$, Evaluation on e_{2n} yields $\alpha = 0$ and evaluation on e_{2n+1} yields $\beta = 0$.

10.35. EXAMPLE. If M is a free module with basis X and N is a free module with basis Y , then $M \oplus N$ is a free module with basis

$$\{(x, 0) : x \in X\} \cup \{(0, y) : y \in Y\}.$$

10.36. EXERCISE. Let R be a commutative ring. If M and N are free and finitely generated, then $\text{Hom}_R(M, N)$ is free and finitely generated.

Some properties of free modules:

10.37. PROPOSITION. *If M is free, then there exists a subset $\{m_i : i \in I\}$ of M such that for each $m \in M$ there exist unique $r_i \in R$, $i \in I$, where $r_i = 0$ except for finitely many $i \in I$ such that $m = \sum r_i \cdot m_i$.*

PROOF. Since M is free, there exists a basis $\{m_i : i \in I\}$ of M . If $m \in M$, $m = \sum r_i \cdot m_i$ (finite sum) for some $r_i \in R$. We claim that the r_i 's are unique. If $m = \sum s_i \cdot m_i$, then $\sum (r_i - s_i) \cdot m_i = 0$. Since $\{m_i : i \in I\}$ is linearly independent, $r_i = s_i$ for all $i \in I$. \square

10.38. PROPOSITION. *Let M be a free module over R with basis $\{m_i : i \in I\}$ and let N be a module over R . If $\{n_i : i \in I\} \subseteq N$, then there exists a unique $f \in \text{Hom}_R(M, N)$ such that $f(m_i) = n_i$ for all $i \in I$.*

SKETCH OF THE PROOF. Note that the unique homomorphism $f : M \rightarrow N$ is defined as

$$f(\sum r_i \cdot m_i) = \sum r_i \cdot n_i. \quad \square$$

Another application:

10.39. EXAMPLE. There is no surjective homomorphism $\mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ (of modules over \mathbb{Z}). If $f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ is a surjective homomorphism, let $\{u, v\}$ a basis of $\mathbb{Z} \times \mathbb{Z}$. Then $f(k) = u$ and $f(l) = v$ for some $k, l \in \mathbb{Z}$. Proposition 10.38 implies that there exists a homomorphism $g : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g(u) = k$ and $g(v) = l$. In particular, $fg = \text{id}_{\mathbb{Z} \times \mathbb{Z}}$ and hence g is injective. Since

$$g(lu - kv) = lg(u) - kg(v) = lk - kl = 0,$$

it follows that $lu - kv = 0$ and thus $k = l = 0$, as $\{u, v\}$ is linearly independent, a contradiction

Another important property:

10.40. PROPOSITION. *If M is a free R -module, then $M \simeq R^{(I)}$ for some set I .*

PROOF. Suppose that M has basis $\{m_i : i \in I\}$. Then there exists a unique homomorphism $f \in \text{Hom}_R(M, R^{(I)})$ such that $f(m_i) = \delta_i$ for all $i \in I$, where

$$\delta_i(j) = \begin{cases} 1 & \text{si } i = j, \\ 0 & \text{si } i \neq j. \end{cases}$$

We claim that f is an isomorphism. We first prove that the map f is surjective: if $(r_i)_{i \in I} \in R^{(I)}$, then $f(\sum r_i \cdot m_i) = (r_i)_{i \in I}$. Let us prove that f is injective:

$$0 = f(\sum r_i \cdot m_i) = \sum r_i \cdot f(m_i) = \sum r_i \cdot \delta_i \implies r_i = 0 \text{ for all } i \in I. \quad \square$$

10.41. COROLLARY. *Every module is (isomorphic to) a quotient of a free module.*

PROOF. Let M be a module. We claim that there exists a free module L and a surjective homomorphism $f \in \text{Hom}_R(L, M)$. Then $L/\ker f \simeq M$. Let $\{m_i : i \in I\}$ be a generating set of M (note that such a generating set always exists, as one could take, for example, the set $\{m : m \in M\}$) and let $L = R^{(I)}$. Then L is free and $f : R^{(I)} \rightarrow M$, $\delta_i \mapsto m_i$, is a surjective homomorphism. \square

Another important property of free modules: free modules are **projective**.

10.42. PROPOSITION. *Let R be a ring and let M , N and T be R -modules. Assume that M is free. If $f \in \text{Hom}_R(N, T)$ be surjective and $h \in \text{Hom}_R(M, T)$, then there exists $\varphi \in \text{Hom}_R(M, N)$ such that $f\varphi = h$.*

PROOF. We will prove that there exists a homomorphism φ that makes the diagram

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \varphi & \downarrow h & & \\ N & \xrightarrow{f} & T & \longrightarrow & 0 \end{array}$$

commutative. Let $\{m_i : i \in I\}$ be a basis of M . Since f is surjective, for each $i \in I$ there exists $n_i \in N$ such that $f(n_i) = h(m_i)$. Since M is free, there exists a unique homomorphism $\varphi: M \rightarrow N$ such that $\varphi(m_i) = n_i$ for all $i \in I$. Thus homomorphism is such that $f\varphi = h$. \square

A consequence of the previous proposition that we will use later:

10.43. PROPOSITION. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence of R -modules. If C is free, then $A \oplus C \simeq B$.

PROOF. By considering the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow h & \parallel & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

the previous proposition yields a homomorphism $h: C \rightarrow B$ such that $gh = \text{id}_C$. In fact, let $\{c_i : i \in I\}$ be a basis of C . For each $i \in I$ let $b_i \in B$ be such that $g(b_i) = c_i$. The map $h: C \rightarrow B$, $h(\sum r_i \cdot c_i) = \sum r_i \cdot b_i$, is a module homomorphism such that $gh = \text{id}_C$.

Let $\varphi: A \oplus C \rightarrow B$, $(a, c) \mapsto f(a) + h(c)$. A simple calculation shows that φ is a module homomorphism.

We prove that φ is injective. If $\varphi(a, c) = 0$, then $f(a) + h(c) = 0$. By applying g we get that $0 = g(f(a)) + g(h(c)) = c$ and hence, since f is injective and $f(a) = 0$, it follows that $(a, c) = (0, 0)$.

Now we prove that φ is surjective. Let $b \in B$. We need to find $(a, c) \in A \oplus C$ such that $\varphi(a, c) = b$. Let $c = g(b) \in C$. Since

$$g(b - h(c)) = g(b) - gh(c) = g(b) - c,$$

it follows that $b - h(c) \in \ker g = f(A)$. This means that $b - h(c) = f(a)$ for some $a \in A$. Now $b \in \varphi(A \oplus C)$, as

$$\varphi(a, c) = f(a) + h(c) = b - h(c) + h(c) = b. \quad \square$$

What can we say about the number of elements of a basis? If M has a finite basis, then every basis of M will be finite. To prove this, let M be a module and $E = \{e_i : i \in I\}$ be a basis of M . If M is finitely generated, say $M = (m_1, \dots, m_k)$, write each m_j as a (finite) linear combination of elements of E . Then there exists a finite set $\{e_1, \dots, e_m\}$ that generates M . Since E is a basis, it follows that $\{e_1, \dots, e_m\} = E$ and hence E is finite.

10.44. EXERCISE. Let M be a free module with an infinite basis E . Prove that every basis of M has cardinality $|E|$.

Under additional assumptions the previous exercise also holds for finite bases.

10.45. THEOREM. *Let R be an integral domain. If M is free with a finite basis, then any two bases of M have the same number of elements.*

PROOF. Let $K = K(R)$ be the field of fractions of R . Note that $V = \text{Hom}_R(M, K)$ is an abelian group and hence it is a vector space (over K) with

$$(\lambda f)(m) = \lambda f(m),$$

where $\lambda \in K$, $f \in V$ and $m \in M$.

The vector space V has a well-defined dimension. Let us compute $\dim V$. Let $\{e_1, \dots, e_n\}$ be a basis of M . For each $i \in \{1, \dots, n\}$ let

$$f_i: M \rightarrow K, \quad e_j \mapsto \begin{cases} 1 & \text{si } i = j, \\ 0 & \text{si } i \neq j. \end{cases}$$

We claim that $\{f_1, \dots, f_n\}$ is basis of V . It generates V , as if $f \in V$, then

$$f = \sum_{i=1}^n f(e_i) f_i,$$

as these homomorphisms coincide in the elements of a basis of M , that is

$$f(e_j) = \left(\sum_{i=1}^n f(e_i) f_i \right) (e_j)$$

for all $j \in \{1, \dots, n\}$. Moreover, $\{f_1, \dots, f_n\}$ is linearly independent, as if $0 = \sum_{i=1}^n \lambda_i f_i$, then, evaluating on each e_j , it follows that

$$0 = \left(\sum_{i=1}^n \lambda_i f_i \right) (e_j) = \lambda_j$$

for all $j \in \{1, \dots, n\}$. Thus $n = \dim V$. □

The previous result can be proved for commutative rings.

10.46. DEFINITION. Let R be an integral domain. If M is a free finitely generated module, we define the **rank** of M as the size of a basis of M . If M is not finitely generated, we say that the rank of M is infinite.

The rank of a module M will be denoted by $\text{rank}(M)$.

11. Lecture: 11/12/2024

§ 11.1. Modules over principal domains. Recall that a principal domain is a commutative ring where every ideal is principal, i.e. generated by one element and such that $xy = 0$ implies $x = 0$ or $y = 0$.

11.1. THEOREM. *Let R be a principal domain. If F is a finitely generated free R -module and N is a submodule of F , then N is free and $\text{rank}(N) \leq \text{rank}(F)$. In particular, N is finitely generated.*

PROOF. We proceed by induction on $n = \text{rank}(F)$. Si $n = 1$, then $F \simeq_R R$ and since R is commutative the submodules of F are exactly the ideals of R . In particular, $N = (r)$ for some $r \in R$. If $r = 0$, then $N = \{0\}$ and the result holds. If $r \neq 0$, then $\{r\}$ is basis of N (since R is a domain) and the result holds.

Assume now that the result holds for all free modules of rank $< n$. Let F be a free module of rank n and $\{f_1, \dots, f_n\}$ be basis of F . Let $F_n = (f_1, \dots, f_{n-1})$. By the inductive hypothesis, $U = N \cap F_n$ is free of rank $\leq n - 1$. Let $\{n_1, \dots, n_k\}$ be a basis of U (by convention, if $U = \{0\}$, then $k = 0$). If $f \in F$, there exist unique $r_1, \dots, r_n \in R$ such that

$$f = \sum_{i=1}^n r_i \cdot f_i.$$

There exists a well-defined surjective homomorphism

$$\varphi: F \rightarrow R, \quad \sum_{i=1}^n r_i \cdot f_i \mapsto r_n.$$

If $\varphi(N) = \{0\}$, then $N \subseteq (f_1, \dots, f_{n-1})$ and thus $N = U$. If $\varphi(N) \neq \{0\}$, then $\varphi(N)$ is an ideal of R , say $\varphi(N) = (x)$ for some $x \in R \setminus \{0\}$. Let $n_{k+1} \in N$ be such that $\varphi(n_{k+1}) = x$. We claim that $\{n_1, \dots, n_k, n_{k+1}\}$ is basis of N . We first proves that this is a generating set. If $n \in N$, then $\varphi(n) = rx$ for some $r \in R$. Thus $n - r \cdot n_{k+1} \in N \cap \ker \varphi = N \cap F_n = U$, as $\varphi(n - r \cdot n_{k+1}) = 0$. In particular,

$$n - r \cdot n_{k+1} \in (n_1, \dots, n_k) \implies n \in (n_1, \dots, n_k, n_{k+1}).$$

We claim that $\{n_1, \dots, n_k, n_{k+1}\}$ is linearly independent. If

$$0 = \sum_{i=1}^{k+1} r_i \cdot n_i,$$

for some $r_1, \dots, r_{k+1} \in R$, then, since $\varphi(n_i) = 0$ for all $i \in \{1, \dots, k\}$,

$$0 = \varphi(r_{k+1} \cdot n_{k+1}) = r_{k+1}x.$$

This implies that $r_{k+1} = 0$. Thus $\sum_{i=1}^k r_i \cdot n_i = 0$. Since $\{n_1, \dots, n_k\}$ is basis of U , we conclude that $r_1 = \dots = r_k = 0$. \square

The previous theorem also holds for infinite bases. However, the proof requires the use of Zorn's lemma.

11.2. COROLLARY. *Let R be a principal domain. If M is a finitely generated R -module and N is a submodule of M , then N is finitely generated.*

PROOF. There exists a free module F of finite rank and a surjective homomorphism $\varphi: F \rightarrow M$. Since $N_1 = \varphi^{-1}(N)$ is a submodule of F , the previous theorem implies that $\text{rank}(N_1) \leq \text{rank } F < \infty$. If $\{x_1, \dots, x_k\}$ is basis of N_1 , then $\{\varphi(x_1), \dots, \varphi(x_k)\}$ is a generating set of $\varphi(N_1) = \varphi(\varphi^{-1}(N)) = N$, as φ is surjective. N is generated by $\leq k = \text{rank}(N_1) \leq \text{rank}(F) < \infty$ elements. \square

Some exercises:

11.3. EXERCISE. Let R be a principal domain and let M be a free R -module. If S is a submodule of M such that M/S is free, then $M \simeq S \oplus (M/S)$. Moreover, S is free and $\text{rank}(M) = \text{rank}(S) + \text{rank}(M/S)$.

11.4. EXERCISE. Let R be a principal domain. If M is a free R -module of rank n , then every linearly independent subset of M contains at most n elements.

11.5. EXERCISE. Let R be a principal domain. and M and N be free R -modules. Prove that $M \simeq N$ if and only if $\text{rank}(M) = \text{rank}(N)$.

11.6. EXERCISE. Let R be a principal domain. If M is a free R -module of finite rank n and $\{s_1, \dots, s_n\}$ is a generating set, then $\{s_1, \dots, s_n\}$ is a basis of M .

If M is an R -module, the **annihilator** of M is defined as

$$\text{Ann}(M) = \{r \in R : r \cdot m = 0 \text{ for all } m \in M\}.$$

For $m \in M$ let

$$\text{Ann}(m) = \{r \in R : r \cdot m = 0\}.$$

Note that $\text{Ann}(M) = \bigcap_{m \in M} \text{Ann}(m)$. It is an exercise to show that both $\text{Ann}(M)$ and $\text{Ann}(m)$ are ideals of R . If $r \in R$, the annihilator of r in M is defined as

$$\text{Ann}_M(r) = \{m \in M : r \cdot m = 0\}.$$

It is an exercise to show that $\text{Ann}_M(r)$ is a submodule of M .

11.7. EXERCISE. Let M be an R -module and $m \in M$ be such that $\text{Ann}(m) = (r)$ for some $r \in R$. Let $p \in R$ be an irreducible element. Prove the following statements:

- 1) If p divides r , then $(m)/(p \cdot (m)) \simeq R/(p)$.
- 2) If p does not divide r , then $p \cdot (m) = (m)$.

The **torsion** of an R -module M is defined as the subset

$$T(M) = \{m \in M : r \cdot m = 0 \text{ for some non-zero } r \in R\}.$$

It is an exercise to show that $T(M)$ is a submodule of M . A module M is **torsion-free** if $T(M) = \{0\}$ and it is a **torsion** module if $T(M) = M$. We also say that M **has torsion** if $T(M) \neq \{0\}$.

11.8. EXERCISE. If $M \simeq N$, then $T(M) \simeq T(N)$.

11.9. EXERCISE. Prove that $T(\oplus_{i \in I} M_i) \simeq \oplus_{i \in I} T(M_i)$.

11.10. EXERCISE. Let R be a principal domain and M be an R -module. Prove that if M is finitely generated and $S \subseteq M$ is a free submodule such that M/S is torsion-free, then M is free.

The torsion generalizes the concept of elements of finite order in abelian groups. For example, $T(\mathbb{Z}/n) = \mathbb{Z}/n$, $T(\mathbb{Q}) = \{0\}$ and

$$T(\mathbb{Z} \times \mathbb{Z}/3) \simeq T(\mathbb{Z}) \times T(\mathbb{Z}/3) \simeq \{0\} \times \mathbb{Z}/3 \simeq \mathbb{Z}/3.$$

11.11. EXAMPLE. Let R be a ring, viewed as an R -module with left multiplication. Then $T(R) = \{r \in R : rs = 0 \text{ for some non-zero } s \in R\}$.

11.12. EXAMPLE. Let M be the module (over \mathbb{Z}) of integer sequences, that is $M = \mathbb{Z}^I$, where $I = \{1, 2, 3, \dots\}$. Then $T(M) = \{0\}$.

11.13. EXAMPLE. If V is a real finite-dimensional vector space and $T: V \rightarrow V$ is a linear transformation, V is a module (over $\mathbb{R}[X]$) with

$$\left(\sum_{i=0}^m a_i X^i \right) \cdot v = \sum_{i=0}^m a_i T^i(v).$$

We claim that V is a torsion module, that is $V = T(V)$. Let $n = \dim V$. If $v \in V$, then $\{v, T(v), \dots, T^n(v)\}$ is linearly dependent, as it has $n+1$ elements. In particular, there exist $a_0, \dots, a_n \in \mathbb{R}$ not all zero such that

$$0 = \sum_{i=0}^n a_i T^i(v) = \left(\sum_{i=0}^n a_i X^i \right) \cdot v.$$

Thus $v \in T(V)$.

11.14. THEOREM. *Let R be a principal domain and M be a finitely generated R -module. If $T(M) = \{0\}$, then M is free.*

PROOF. Without loss of generality, we may assume that M is non-zero. By assumption, $M = (X)$, where X is a finite set. If $x \in X$, then $r \cdot x = 0 \iff r = 0$, as $T(M) = \{0\}$. Let $S = \{x_1, \dots, x_k\} \subseteq X$ be maximal with respect to the following property:

$$r_1 \cdot x_1 + \dots + r_k \cdot x_k = 0 \text{ for } r_1, \dots, r_k \in R \implies r_1 = \dots = r_k = 0.$$

Let $F = (S)$ be the free module with basis S . If $X = S$, we are done. If $y \in X \setminus S$, then there exist $r_y, r_1, \dots, r_k \in R$ not all zero such that

$$r_y \cdot y + \sum_{i=1}^k r_i \cdot x_i = 0.$$

Since $r_y \cdot y = -\sum_{i=1}^k r_i \cdot x_i \in F$, it follows that $r_y \neq 0$, as $r_y = 0$ implies $r_1 = \dots = r_k = 0$. Since X is finite,

$$r = \prod_{y \in X \setminus S} r_y$$

is well-defined, as R is commutative and $r \cdot X \subseteq F$. If $f: M \rightarrow M$, $x \mapsto r \cdot x$, then f is a homomorphism such that $f(M) = r \cdot M$.

Since $T(M) = \{0\}$, it follows that $\ker f = \{0\}$. Thus

$$r \cdot M = f(M) \simeq M.$$

To finish the proof, we show that $r \cdot M \subseteq F$. Let $m \in M$. Since $M = (X)$, there exist $s_{i_1}, \dots, s_{i_m} \in R$ such that $m = \sum s_j \cdot x_{i_j}$. Then

$$r \cdot m = \sum (rs_j) \cdot x_{i_j} = \sum s_j \cdot (r \cdot x_{i_j}) \in F,$$

as each $r \cdot x_{i_j} \in r \cdot X \subseteq F$. In particular, since R is commutative, $r \cdot M$ is a submodule of F and hence $r \cdot M$ is a free module. \square

11.15. THEOREM. *Let R be a principal domain. If M is a finitely generated R -module, then $M = T(M) \oplus F$, where $F \simeq M/T(M)$ is finitely generated and free. The torsion submodule is unique and F is unique up to isomorphism.*

PROOF. We first prove that $T(M/T(M)) \simeq \{0\}$. If $x + T(M) \in T(M/T(M))$, then there exists $r \in R \setminus \{0\}$ such that $r \cdot (x + T(M)) = T(M)$. Then $r \cdot x \in T(M)$, that is, there exists $s \in R \setminus \{0\}$ such that $s \cdot (r \cdot x) = (sr) \cdot x = 0$. Since $sr \neq 0$, it follows that $x \in T(M)$.

Since M is finitely generated, $M/T(M)$ is finitely generated. Moreover, $M/T(M)$ is torsion-free. It follows that $M/T(M)$ is free. Consider the exact sequence

$$0 \longrightarrow T(M) \xrightarrow{\iota} M \xrightarrow{\pi} M/T(M) \longrightarrow 0$$

where ι is the inclusion map and π is the canonical map. Since $M/T(M)$ is free, there exists an R -module homomorphism $h: M/T(M) \rightarrow M$ such that $\pi h = \text{id}$, see Proposition 10.42. This homomorphism is needed to prove that $M \simeq T(M) \oplus M/T(M)$, see Proposition 10.43.

Let us prove uniqueness. Suppose that $M = T \oplus L$, where T is a torsion module and L is free. We first prove that $T = T(M)$. On the one hand, $T \subseteq T(M)$. On the other hand, if $m \in T(M)$, then $m = t + l$ for some $t \in T$ and $l \in L$. In particular, $r \cdot m = 0$ and $s \cdot t = 0$ for some non-zero $r, s \in R$. Since R is commutative,

$$0 = (rs) \cdot m = (rs) \cdot (t + l) = (rs) \cdot t + (rs) \cdot l = (rs) \cdot l.$$

and thus $l \in T(L)$. Since L is free, $T(L) = \{0\}$ (because $T(L)$ is free and hence every basis element x of $T(L)$ is such that $\{x\}$ is linearly independent, a contradiction). Thus $l = 0$ and hence $m = t \in T$. The free part is unique up to isomorphism because it is isomorphic to $M/T(M)$. \square

12. Lecture: 18/12/2024

§ 12.1. Smith's normal form. We finish the course with an algorithm that allows us to understand the structure of certain finitely generated modules. We will discuss the case of modules over euclidean domains, as in this case the algorithm is constructive.

Let M be a finitely generated module and $\{m_1, \dots, m_k\}$ be a set of generators. There exists a surjective module homomorphism

$$\varphi: R^k \rightarrow M, \quad (r_1, \dots, r_k) \mapsto \sum_{i=1}^k r_i \cdot m_i,$$

and hence $M \simeq R^k / \ker \varphi$. The submodule $\ker \varphi$ of R^k is the **relations module** of M . Since R is an euclidean domain, R is a principal domain. Since M is finitely generated, then so is the submodule $\ker \varphi$ of R^k . Let $\{e_1, \dots, e_l\}$ be a generating set of $\ker \varphi$, say

$$\begin{aligned} e_1 &= (a_{11}, a_{12}, \dots, a_{1k}), \\ e_2 &= (a_{21}, a_{22}, \dots, a_{2k}), \\ &\vdots \\ e_l &= (a_{l1}, a_{l2}, \dots, a_{lk}). \end{aligned}$$

The matrix $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq k}$ is the **relations matrix** of M with respect to $\{m_1, \dots, m_k\}$ and $\{e_1, \dots, e_l\}$.

CLAIM. If $P \in R^{l \times l}$ is invertible, then the rows $\{f_1, \dots, f_l\}$ of PA generate $\ker \varphi$. Moreover, PA is the relations matrix with respect to $\{m_1, \dots, m_k\}$ and $\{f_1, \dots, f_l\}$.

Let us prove the claim. Assume that $P = (p_{ij})$. The rows of PA are

$$\begin{aligned} f_1 &= p_{11}e_1 + \dots + p_{1l}e_l, \\ f_2 &= p_{21}e_1 + \dots + p_{2l}e_l, \\ &\vdots \\ f_l &= p_{l1}e_1 + \dots + p_{ll}e_l. \end{aligned}$$

Moreover, $f_j \in \ker \varphi$ for all $j \in \{1, \dots, l\}$. Since P is invertible, the set $\{f_1, \dots, f_l\}$ generates $\ker \varphi$. Indeed, each e_j is a linear combination of the f_i 's,

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_l \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_l \end{pmatrix}.$$

In particular, PA is the relations matrix with respect to $\{m_1, \dots, m_k\}$ and $\{f_1, \dots, f_l\}$.

CLAIM. If $Q \in R^{k \times k}$ is invertible and $Q^{-1} = (q_{ij})$ and for each $j \in \{1, \dots, k\}$ we define $n_j = \sum_{i=1}^k q_{ji} \cdot m_i$, the set $\{n_1, \dots, n_k\}$ generates M and the rows of AQ generate $\ker \varphi$. Moreover, AQ is the relations matrix with respect to $\{n_1, \dots, n_k\}$.

Now we prove the claim. Since

$$\begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix} = Q^{-1} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix},$$

it follows that

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = A \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} = (AQ) \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix}.$$

This implies that the rows of AQ are relations with respect to the generating set $\{n_1, \dots, n_k\}$. Let $\psi: R^k \rightarrow M$, $(r_1, \dots, r_k) \mapsto \sum_{i=1}^k r_i \cdot n_i$.

The rows of AQ generate $\ker \psi$ with respect to $\{n_1, \dots, n_k\}$. If $(r_1, \dots, r_k) \in \ker \psi$, then $\sum_{i=1}^k r_i \cdot n_i = 0$. Write

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (r_1 \cdots r_k) Q^{-1} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} = (r_1 \cdots r_k) \begin{pmatrix} n_1 \\ \vdots \\ n_k \end{pmatrix}.$$

We remark that each e_j belongs to R^k . Thus

$$(r_1 \cdots r_k) Q^{-1} = \left(\sum_{i=1}^k r_i q_{i1}, \sum_{i=1}^k r_i q_{i2}, \dots, \sum_{i=1}^k r_i q_{ik} \right) \in \ker \varphi.$$

Since $\ker \varphi$ is generated by $\{e_1, \dots, e_l\}$, there exist $s_1, \dots, s_l \in R$ such that

$$(r_1 \cdots r_k) Q^{-1} = \sum_{i=1}^l s_i \cdot e_i,$$

that is

$$(r_1 \cdots r_k) Q^{-1} = (s_1 \cdots s_l) \begin{pmatrix} e_1 \\ \vdots \\ e_l \end{pmatrix} = (s_1 \cdots s_l) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{l1} & a_{l2} & \cdots & a_{lk} \end{pmatrix}.$$

Rewriting this expression as

$$(r_1 \cdots r_k) = (s_1 \cdots s_l) AQ,$$

we conclude that (r_1, \dots, r_k) is a linear combination of the rows of AQ , as

$$(r_1, \dots, r_k) = \left(\sum_{i=1}^l s_i \cdot x_{i1}, \dots, \sum_{i=1}^l s_i \cdot x_{ik} \right) = \sum_{i=1}^l s_i \cdot (x_{i1}, \dots, x_{ik}).$$

Therefore $\{n_1, \dots, n_k\}$ generates M and the rows of AQ generate the corresponding relations submodule and AQ is the relations matrix with respect to $\{n_1, \dots, n_k\}$ and $\{e_1, \dots, e_l\}$.

12.1. PROPOSITION. Let A be the relations matrix of a finitely generated module M with k generators. If there exist invertible matrices $P \in R^{l \times l}$ and $Q \in R^{k \times k}$ such that

$$PAQ = \begin{pmatrix} a_1 & 0 & \cdots & \cdot & \cdot & \cdots & 0 \\ 0 & a_2 & \cdots & \cdot & \cdot & \cdots & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdot & \cdots & a_r & \cdot & \cdots & 0 \\ 0 & \cdot & \cdots & \cdot & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdot & \cdots & \cdot & \cdot & \cdots & 0 \end{pmatrix}$$

where $a_i \neq 0$ for all $i \in \{1, \dots, r\}$ and $a_i \mid a_{i+1}$ for all $i \in \{1, \dots, r-1\}$, then

$$M \simeq R/(a_1) \oplus \cdots \oplus R/(a_r) \oplus R^{k-r}.$$

PROOF. The matrix PAQ is the relations matrix with respect to the generating set $\{m_1, \dots, m_k\}$ of M and respect to the relations submodule given by the rows of PAQ . If

$$\varphi: R^k \rightarrow M, \quad (r_1, \dots, r_k) \mapsto \sum_{i=1}^k r_i \cdot m_i,$$

then $R^k / \ker \varphi \simeq M$, as φ is a surjective homomorphism. For each $j \in \{r+1, \dots, k\}$ let $a_j = 0$. Let

$$\psi: R^k \rightarrow R/(a_1) \oplus \cdots \oplus R/(a_k), \quad (s_1, \dots, s_k) \mapsto (s_1 + (a_1), \dots, s_k + (a_k)).$$

A straightforward calculation shows that

$$\ker \psi = (a_1) \oplus \cdots \oplus (a_k).$$

Thus $R^k / \ker \psi \simeq \bigoplus_{i=1}^k R/(a_i)$.

It is an exercise to show that $\ker \varphi = \ker \psi$.

Therefore $M \simeq R/(a_1) \oplus \cdots \oplus R/(a_k)$. To finish the proof we need to note that $R/(a_i) \simeq R$ for all $i \in \{r+1, \dots, k\}$. \square

The decomposition given in the previous proposition is known as the **Smith normal form** of the matrix M . How can we find the matrices P and Q ? Consider the following matrix operations:

- 1) Switch the i -th row and j -th row, that is $R_i \leftrightarrow R_j$.
- 2) Replace row R_i by $R_i + \lambda R_j$ for some $\lambda \in R$ and $j \neq i$.
- 3) Switch the i -th column and the j -th column, that is $C_i \leftrightarrow C_j$.
- 4) Replace column C_i by $C_i + \lambda C_j$ for some $\lambda \in R$ and $j \neq i$.
- 5) Replace row R_i (resp. column C_i) by λR_i (resp. λC_i) for some $\lambda \in \mathcal{U}(R)$.

These operations are invertible. For example, the first operation corresponds to multiply A on the left by a permutation matrix. The second operation corresponds to multiply A by $I + \lambda E_{ij}$ on the left, where

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Concrete examples:

$$E_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

Similarly, column operations correspond to multiply on the right the matrix A either by a permutation matrix or a matrix of the form $I + \lambda E_{ij}$.

12.2. THEOREM (Smith's normal form). *Let R be an euclidean domain. If $A \in R^{l \times k}$, there exists invertible matrices $P \in R^{l \times l}$ and $Q \in R^{k \times k}$ such that*

$$PAQ = \begin{pmatrix} a_1 & 0 & \cdots & \cdot & \cdot & \cdots & 0 \\ 0 & a_2 & \cdots & \cdot & \cdot & \cdots & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdot & \cdots & a_r & \cdot & \cdots & 0 \\ 0 & \cdot & \cdots & \cdot & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdot & \cdots & \cdot & \cdot & \cdots & 0 \end{pmatrix}$$

where $a_i \neq 0$ for all $i \in \{1, \dots, r\}$ and $a_i \mid a_{i+1}$ for all $i \in \{1, \dots, r-1\}$. Moreover, the elements a_1, \dots, a_r are unique up to multiplication by units.

SKETCH OF THE PROOF. We only prove the existence. Assume that (R, φ) be an euclidean domain. We need to show that A can be turned into a matrix of the form

$$B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n2} & \cdots & b_{nm} \end{pmatrix}$$

where each b_{ij} is divisible by b_{11} . Then we apply the same procedure to the submatrix

$$\begin{pmatrix} \frac{b_{22}}{b_{11}} & \cdots & \frac{b_{2m}}{b_{11}} \\ \vdots & & \vdots \\ \frac{b_{n2}}{b_{11}} & \cdots & \frac{b_{nm}}{b_{11}} \end{pmatrix}$$

and repeat the method until we cannot continue.

Let us show how to get the matrix B . By applying row and column operations we may assume that the coefficient of A with minimal positive Euclidean norm appears in position $(1, 1)$. If some a_{i1} is not divisible by a_{11} , we use the division algorithm to write $a_{i1} = a_{11}u + r$ for some $u \in R$ and $r \in R$ with $\varphi(r) < \varphi(a_{11})$. The transformation $R_i \leftarrow R_i - uR_1$ turns our matrix into a matrix that has r in position $(i, 1)$. Similarly, if some a_{1j} is not divisible by a_{11} , then $a_{1j} = va_{11} + s$ with $\varphi(s) < \varphi(a_{11})$. By applying $C_j \leftarrow C_j - vC_1$ our matrix turns into a matrix that has s in position $(1, j)$. If every a_{i1} is divisible by a_{11} , say $a_{i1} = a_{11}\lambda_i$, then apply $R_i \leftarrow \lambda_i R_1 - R_i$. Similarly, if every a_{1j} is divisible by a_{11} , say $a_{1j} = a_{11}\mu_j$, then apply $C_j \leftarrow \mu_j C_1 - C_j$. In this way we replace our matrix by a matrix of the form

$$\begin{pmatrix} a_{11} & 0 \\ 0 & A_1 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

If some entry of the matrix A_1 is not divisible by a_{11} , apply either $R_1 \leftarrow R_1 + R_i$ or $C_1 \leftarrow C_1 + C_j$ and repeat the procedure described before. \square

We refer to Artin's book [1, §14] for a detailed exposition of the Smith normal when the base ring R is \mathbb{Z} or $K[X]$ for any field K . For an application of the Smith's normal to explicit calculations related to homology groups, see [8, §11]. Here we will explain the algorithm with examples.

12.3. EXAMPLE. Let

$$A = \begin{pmatrix} 2 & 5 & 3 \\ 8 & 6 & 4 \\ 3 & 1 & 0 \end{pmatrix} \in \mathbb{Z}^{3 \times 3}.$$

Let us compute the Smith's normal form of A . Since the element with smallest positive norm appears in position $(3, 2)$, we apply the operations $R_1 \leftrightarrow R_3$ and $C_1 \leftrightarrow C_2$ to transform A into

$$\begin{pmatrix} 1 & 3 & 0 \\ 6 & 8 & 4 \\ 5 & 2 & 3 \end{pmatrix}.$$

To obtain zeros in positions $(1, 2)$, $(1, 3)$, $(2, 1)$ and $(2, 3)$ we apply the transformations $R_2 \leftarrow 6R_1 - R_2$, $R_3 \leftarrow 5R_1 - R_3$ and $C_2 \leftarrow 3C_1 - C_2$. Then our matrix turns into

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -10 & -4 \\ 0 & -13 & -3 \end{pmatrix}.$$

Multiply the second and the third row by -1 :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 4 \\ 0 & 13 & 3 \end{pmatrix}.$$

We perform the same procedure to the submatrix $\begin{pmatrix} 10 & 4 \\ 13 & 3 \end{pmatrix}$. We want the smallest element of the submatrix in position $(2, 2)$. For that purpose, apply $R_1 \leftrightarrow R_2$ and $C_2 \leftrightarrow C_1$:

$$\begin{pmatrix} 3 & 13 \\ 4 & 10 \end{pmatrix}.$$

Write $13 = 3 \cdot 4 + 1$ and apply $C_2 \leftarrow C_2 - 4C_1$ to obtain $\begin{pmatrix} 3 & 1 \\ 4 & -6 \end{pmatrix}$. Interchange the first two columns to obtain $\begin{pmatrix} 1 & 3 \\ -6 & 4 \end{pmatrix}$. Apply now $R_2 \leftarrow 6R_1 + R_2$. To the resulting matrix we apply $C_2 \leftarrow 3C_1 - C_2$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 22 \end{pmatrix}.$$

Hence we find the Smith's normal form of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -22 \end{pmatrix}.$$

How to interpret the Smith normal form if the matrix is not a square matrix? To understand the quotient

$$M = \mathbb{Z}^n / \langle e_1, \dots, e_m \rangle$$

with $m < n$, each of the $n - m$ missing columns of the Smith's normal form of the matrix gives a factor isomorphic to \mathbb{Z} . Thus

$$M \simeq \mathbb{Z}^{n-m} \times M_1$$

where M_1 is the module obtained from the Smith's normal form. If $n < m$, then one only needs to ignore the $m - m$ last columns, which will all be zero columns.

12.4. EXAMPLE. Let M be the abelian group with generators m_1, m_2, m_3 and relations

$$8m_1 + 4m_2 + 8m_3 = 0, \quad 4m_1 + 8m_2 + 4m_3 = 0.$$

The matrix of relations is then

$$A = \begin{pmatrix} 8 & 4 & 8 \\ 4 & 8 & 4 \end{pmatrix}.$$

We claim that $M \simeq \mathbb{Z}/4 \times \mathbb{Z}/12$. Apply $R_1 \leftarrow 2R_2 - R_1$ to obtain

$$\begin{pmatrix} 0 & 12 & 0 \\ 4 & 8 & 4 \end{pmatrix}.$$

The row operation used corresponds to multiplying on the left by the invertible matrix $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. Thus $\begin{pmatrix} 0 & 12 & 0 \\ 4 & 8 & 4 \end{pmatrix}$ corresponds to the set of generators $\{m_1, m_2, m_3\}$ and relations $12m_2 = 0$ and $4m_1 + 8m_2 + 4m_3 = 0$. Apply $C_2 \leftarrow C_2 - 2C_1$ and $C_3 \leftarrow C_3 - C_1$ to obtain the matrix

$$\begin{pmatrix} 0 & 12 & 0 \\ 4 & 0 & 0 \end{pmatrix},$$

which corresponds to generators $\{m_1 + 2m_2 + m_3, m_2\}$ and relations

$$12m_2 = 0, \quad 4(m_1 + 2m_2 + m_3) = 0.$$

The first column operation corresponds to multiplying on the right by the matrix

$$I - 2E_{12} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the second one to right multiplication by the matrix

$$I - E_{13} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, interchange the first two rows to obtain the Smith's normal form of A :

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \end{pmatrix}.$$

This matrix corresponds to the set of generators $\{m_2, m_1 + 2m_2 + m_3\}$ and relations

$$4(m_1 + 2m_2 + m_3) = 0, \quad 12m_2 = 0.$$

The column operation used corresponds to left multiplication by the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $M \simeq \mathbb{Z}/4 \times \mathbb{Z}/12$. We also obtained that

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

and that

$$Q = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

12.5. EXAMPLE. Let M be the abelian group generated by $\{m_1, \dots, m_4\}$ and let K be the subgroup of M generated by $\{e_1, e_2, e_3\}$, where

$$e_1 = 22m_3, \quad e_2 = -2m_1 + 2m_2 - 6m_3 - 4m_4, \quad e_3 = 2m_1 + 2m_2 + 6m_3 + 8m_4.$$

We want to determine the structure of M/K . The matrix of relations is

$$A = \begin{pmatrix} 0 & 0 & 22 & 0 \\ -2 & 2 & -6 & -4 \\ 2 & 2 & 6 & 8 \end{pmatrix}.$$

Apply $R_1 \leftrightarrow R_3$ and then $R_2 \leftarrow R_1 + R_2$ to obtain

$$\begin{pmatrix} 2 & 2 & 6 & 8 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 22 & 0 \end{pmatrix}.$$

Apply $C_2 \leftarrow C_2 - C_1$, $C_3 \leftarrow C_3 - 3C_1$ and $C_4 \leftarrow C_4 - 4C_1$ to obtain

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 22 & 0 \end{pmatrix}.$$

Apply $C_4 \leftarrow C_4 - C_2$ to get

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 22 & 0 \end{pmatrix}.$$

Note that $4 \nmid 22$. Apply $R_2 \leftarrow R_2 + R_3$, $C_3 \leftarrow C_3 - 5C_2$, $C_3 \leftarrow C_2$, $R_3 \leftarrow R_3 - 11R_2$ and $C_3 \leftarrow C_3 - C_2$ to get

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -44 & 0 \end{pmatrix}.$$

The group M/K has basis $\{n_1, n_2, n_3, n_4\}$ and relations $2n_1 = 0$, $2n_2 = 0$ and $44n_3 = 0$. Thus $M/K \simeq \mathbb{Z} \times (\mathbb{Z}/2)^2 \times (\mathbb{Z}/44)$.

12.6. EXAMPLE. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. Let us solve the linear system $AX = b$ in the integers. We need to compute Smith's normal form of A . For example,

computer calculations show that

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad S = PAQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that the matrix S is unique, but not the matrices P and Q . To solve $AX = b$ we proceed as follows. Let $Y = Q^{-1}X$. Then $AX = b$ implies that

$$SY = (PAQ)(Q^{-1}X) = Pb = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Write $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since $Q^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, it follows that $Y = Q^{-1}X = \begin{pmatrix} x + 2z \\ y + z \\ y \end{pmatrix}$ and hence $SY = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ turns into

$$\begin{pmatrix} x + 2z \\ y + z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x + 2z \\ y + z \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

It follows that the solution of $AX = b$ is then $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 - 2z \\ 1 - z \\ z \end{pmatrix}$ for $z \in \mathbb{Z}$.

12.7. EXERCISE. Solve $AX = b$ in \mathbb{Z} for $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$.

12.8. EXERCISE. Prove that $\mathbb{Z}^3 / \langle (6, 6, 4), (6, 12, 8) \rangle \simeq \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/6$.

12.9. EXERCISE. Prove that the abelian group with generators a, b and c with relations

$$3a + 2b + c = 0, \quad 8a + 4b + 2c = 0, \quad 7a + 6b + 2c = 0, \quad 9a + 6b + c = 0,$$

is cyclic of order four.

12.10. EXERCISE. Compute the Smith's normal form of $\begin{pmatrix} 1+i & 2-i \\ 3 & 5i \end{pmatrix} \in M_2(\mathbb{Z}[i])$.

12.11. EXERCISE. Compute the Smith's normal form of

$$\begin{pmatrix} 7 & X & 0 & -X \\ 0 & X-3 & 0 & 3 \\ 0 & 0 & X-4 & 0 \\ X-6 & -1 & 0 & X+1 \end{pmatrix} \in \mathbb{Q}[X].$$

12.12. EXERCISE. Let G be an abelian group of rank n with basis $\{x_1, \dots, x_n\}$. For $i \in \{1, \dots, n\}$ let

$$y_i = \sum_{j=1}^n a_{ij} x_j,$$

where $A = (a_{ij}) \in M_n(\mathbb{Z})$. Then $\{y_1, \dots, y_n\}$ is a basis of G if and only if A is unimodular, that is $\det A \in \{-1, 1\}$.

Some solutions

1.6.

- 1) Since $x0 = x(0 + 0) = x0 + x0$, it follows that $x0 = 0$. Similarly, $0x = 0$.
- 2) $0 = x0 = x(y + (-y)) = xy + x(-y)$. Thus $-xy = x(-y)$.
- 3) Assume that $1 = 0$ and let $x \in R$. Using the first item, $x = x1 = x0 = 0$.

1.33. If x is a unit, then $yx = xy = 1$ for some $y \in R$ and hence $ryx = r$ for all $r \in R$. Conversely, if $R = (x)$, then, in particular, $1 \in R = (x)$ and hence $1 = xy$ and $1 = zx$ for some $y, z \in R$. Now $z = z1 = z(xy) = (zx)y = 1y = y$ and hence x is a unit.

1.42. Let $f: \mathbb{Z}/6 \rightarrow \mathbb{Z}/15$ be a ring homomorphism. Then

$$0 = f(0) = f(6) = f(\underbrace{1 + \cdots + 1}_{6\text{-times}}) = 6f(1) = 6,$$

a contradiction, as $6 \neq 0$ in $\mathbb{Z}/15$.

2.2. From group theory, we know that R/I is abelian group with addition

$$(x + I) + (y + I) = (x + y) + I$$

and neutral element $0 + I = I$.

We have already seen that multiplication is well-defined. As an example, we also showed that the left distributive property holds, that is

$$(x + I)((y + I) + (z + I)) = (x + I)(y + I) + (x + I)(z + I)$$

for all $x, y, z \in R$. The right distributivity is similar:

$$((x + I) + (y + I))(z + I) = (x + I)(z + I) + (y + I)(z + I)$$

for all $x, y, z \in R$. Let us prove the associativity of the multiplication:

$$\begin{aligned} (x + I)((y + I)(z + I)) &= (x + I)(yz + I) \\ &= x(yz) + I \\ &= (xy)z + I \\ &= (xy + I)(z + I) \\ &= ((x + I)(y + I))(z + I). \end{aligned}$$

Finally, let us see why $1 + I$ is the neutral element of the multiplication:

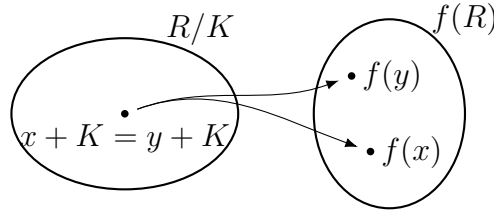
$$(x + I)(1 + I) = x1 + I = x + I = 1x + I = (1 + I)(x + I)$$

for all $x \in R$.

2.6. We first note that $K = \ker f$ is an ideal of R . Thus R/K is a ring. Let

$$\varphi: R/K \rightarrow f(R), \quad x + K \mapsto f(x).$$

We first need to prove that φ is a well-defined map, that is if $x + K = y + K$, then $f(x) = f(y)$. Thus we want to avoid situations as in the following picture:



To prove this fact, we proceed as follows:

$$\begin{aligned}
 x + K = y + K &\implies x - y \in K = \ker f \\
 &\implies f(x) - f(y) = f(x - y) = 0 \\
 &\implies f(x) = f(y).
 \end{aligned}$$

Now we need to prove that φ is a ring homomorphism. First we prove that $\varphi(1 + K) = 1$:

$$\varphi(1 + K) = f(1) = 1.$$

To finish the proof of the fact that φ is a ring homomorphism, let $x, y \in R$. Then

$$\begin{aligned}
 \varphi((x + K) + (y + K)) &= \varphi(x + y + K) \\
 &= f(x + y) \\
 &= f(x) + f(y) \\
 &= \varphi(x + K) + \varphi(y + K).
 \end{aligned}$$

and

$$\begin{aligned}
 \varphi((x + K)(y + K)) &= \varphi(xy + K) \\
 &= f(xy) \\
 &= f(x)f(y) \\
 &= \varphi(x + K)\varphi(y + K)
 \end{aligned}$$

because f is a ring homomorphism.

The map φ is surjective: if $y \in f(R)$, then $y = f(x)$ for some $x \in R$. Then

$$\varphi(x + K) = f(x) = y.$$

Finally, we show that φ is injective:

$$\begin{aligned}
 \varphi(x + K) = \varphi(y + K) &\implies f(x) = f(y) \\
 &\implies x - y \in K = \ker f \\
 &\implies x + K = y + K.
 \end{aligned}$$

2.11. Suppose there exists an isomorphism $f: \mathbb{Q}[\sqrt{2}] \rightarrow \mathbb{Q}[\sqrt{3}]$. Since f is additive and $f(1) = 1$, $f(m) = m$ for all $m \in \mathbb{Z}$. Furthermore, $f(m/n) = m/n$ for all $m/n \in \mathbb{Q}$. This implies that $f(a + b\sqrt{2}) = a + bf(\sqrt{2})$. Let $z = f(\sqrt{2})$. Then

$$z^2 = f(\sqrt{2})^2 = f(2) = 2$$

and hence $z \in \{-\sqrt{2}, \sqrt{2}\}$. But there are no numbers $\mathbb{Q}[\sqrt{3}]$ whose square is 2: if

$$(a + b\sqrt{3})^2 = 2$$

for some $a, b \in \mathbb{Q}$, then

$$2 = (a + b\sqrt{3})^2 = a^2 + 3b^2 + 2ab\sqrt{3}.$$

Since $\sqrt{3} \notin \mathbb{Q}$, either $a = 0$ or $b = 0$. Both cases yield a contradiction.

2.15. Let $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}[i] \xrightarrow{\pi} \mathbb{Z}[i]/(1+3i)$, where π denotes the canonical map. We claim that φ is a surjective ring homomorphism with kernel $\ker \varphi = 10\mathbb{Z}$. Then

$$\mathbb{Z}/10\mathbb{Z} \simeq \varphi(\mathbb{Z}) = \mathbb{Z}[i]/(1+3i)$$

by the first isomorphism theorem.

Note first that φ is a ring homomorphism, as it is the composition of ring homomorphisms.

We now show that $\ker \varphi = 10\mathbb{Z}$:

$$\begin{aligned} m \in \ker \varphi &\iff \varphi(m) + m \in (1+3i) \\ &\iff m = (1+3i)(x+iy) \text{ for some } x, y \in \mathbb{Z} \\ &\iff m = 10x \text{ for some } x \in \mathbb{Z} \\ &\iff m \in 10\mathbb{Z}. \end{aligned}$$

Finally, to see that φ is surjective it is enough to see that $i \in \varphi(\mathbb{Z})$, as then, for $a, b \in \mathbb{Z}$, one has $\varphi(a+3b) \equiv a+ib \pmod{1+3i}$. To show that $i \in \varphi(\mathbb{Z})$ note that

$$\varphi(3) = 3 \equiv i \pmod{1+3i},$$

as $3-i = -i(1+3i)$.

2.16. Suppose there is an isomorphism $\varphi: \mathbb{Z}[i]/I \rightarrow \mathbb{Z}/15$ for some ideal I . In particular,

$$\varphi(-1+I) + \varphi(1+I) = \varphi(I) = 0$$

and hence, since $\varphi(1+I) = 1$, it follows that $\varphi(-1+I) = -1$. Since $i^2 \equiv -1 \pmod{I}$, one also has $\varphi(i^2+I) = -1$. But there are no elements $x \in \mathbb{Z}/15$ such that $x^2 = -1$, a contradiction.

2.21. Let $K = \{uv : u \in I, v \in J\}$. Then $X^2 \in K$ and $Y^2 \in K$ but $X^2 + Y^2 \notin K$, so K is not an ideal.

2.23. Assume first that $I \cap J = IJ$ holds for all ideals I and J . Let $a \in R$. Since $Ra = RaR$ and Ra is an ideal,

$$Ra = (Ra)(Ra) = (RaR)a = (Ra)a = Ra^2.$$

In particular, $a \in Ra^2$ and hence $a = xa^2$ for some $x \in R$.

Conversely, assume that R is strongly regular. Since I and J are ideals, $I \cap J \supseteq IJ$. Let $a \in I \cap J$. There exists $x \in R$ such that $a = xa^2$. In particular, $a = (xa)a \in IJ$.

4.6. Let $I_1 \subsetneq I_2 \subsetneq \dots$ be a sequence of ideals of R . Since R is principal, each I_j is principal, say $I_j = (a_j)$ for some $a_j \in R$, so the sequence is of the form

$$(a_1) \subsetneq (a_2) \subsetneq \dots$$

Since $I = \bigcup_{i \geq 1} (a_i)$ is an ideal of R , there exists $x \in R$ such that $I = (x)$. Since $x \in (a_n)$ for some $n \in \mathbb{Z}_{>0}$, it follows that $(a_k) \subseteq I = (x) \subseteq (a_n)$ for all $k \in \mathbb{Z}_{>0}$.

4.22. Let d be the greatest common divisor of the coefficients of f . Since a divides all the coefficients of f , it follows that a divides d . If p is a prime that divides d , then either p divides a or p divides all the coefficients of the primitive polynomial f_1 . It follows that d divides a .

- 4.23. 1) Let $f = \sum_{i=0}^n a_i X^i$, $g = \sum_{i=0}^m b_i X^i$. Suppose that fg is not primitive and let p be a prime number dividing all the coefficients of fg . Since both f and g are primitive, there exist $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, m\}$ minimal such that $p \nmid a_i$ and $p \nmid b_j$. If c_{i+j} is the coefficient of X^{i+j} in fg , then

$$c_{i+j} = \sum_{k>i} a_k b_{i+j-k} + \sum_{k<i} a_k b_{i+j-k} + a_i b_j.$$

Thus p divides $\sum_{k>i} a_k b_{i+j-k} + \sum_{k<i} a_k b_{i+j-k}$ and does not divide the integer $a_i b_j$, that is p does not divide c_{i+j} , a contradiction.

- 2) Suppose that f is irreducible in $\mathbb{Z}[X]$. If f is not primitive, then $f = af_1$ for some $a \in \mathbb{Z} \setminus \{-1, 1\}$ and some primitive polynomial f_1 , a contradiction to the irreducibility of f . Let us prove that f is irreducible in $\mathbb{Q}[X]$. If not, say $f = gh$ for some $g, h \in \mathbb{Q}[X]$ of positive degree. After multiplying by a suitable rational number, we may assume that

$$f = \frac{a}{b} g_1 h_1,$$

where $\gcd(a, b) = 1$ and $g_1, h_1 \in \mathbb{Z}[X]$ are primitive polynomials, that is

$$bf = ag_1 h_1.$$

The greatest common divisor of the coefficients of bf is b . Since $g_1 h_1$ is primitive by the previous item, the greatest common divisor of the coefficients of $ag_1 h_1$ is a . Thus $a = b$ or $a = -b$, that is $f = g_1 h_1$ or $f = -g_1 h_1$ in $\mathbb{Z}[X]$.

4.24. Since \mathbb{Z} is noetherian, it follows that $\mathbb{Z}[X]$ is noetherian by Hilbert's theorem. Then $\mathbb{Z}[X]$ admits factorizations. We need to show that the factorization into irreducibles is unique. Let $f \in \mathbb{Z}[X]$ be non-zero and assume that

$$f = f_1 \cdots f_k = g_1 \cdots g_l$$

be factorizations of f into non-constant irreducibles integer polynomials. Since f_1, \dots, f_k and g_1, \dots, g_l are irreducible in $\mathbb{Q}[X]$ and $\mathbb{Q}[X]$ is a unique factorization domain, it follows that $k = l$ and there exists $\sigma \in \mathbb{S}_k$ such that g_i and $h_{\sigma(i)}$ are associate for all $i \in \{1, \dots, k\}$. After reordering we may assume that for each $i \in \{1, \dots, k\}$ there exists $a_i/b_i \in \mathbb{Q}$ such that $b_i g_i = a_i h_i$. Since both g_i and h_i are irreducible integer polynomials, it follows from Exercise 4.23 that both g_i and h_i are primitive. By Exercise 4.22, a_i (resp. b_i) is the greatest common divisor of the coefficients of $a_i h_i$ (resp. $b_i g_i$). This implies that a_i and b_i are associate, so a_i/b_i is a unit. Hence g_i and h_i are associate in $\mathbb{Z}[X]$.

5.4. Since $p \equiv 1 \pmod{4}$, p is not prime in $\mathbb{Z}[i]$ (see Theorem 5.1). Thus $p = \alpha\beta$ for some $\alpha, \beta \notin \mathcal{U}(\mathbb{Z}[i])$. Hence $p^2 = N(p) = N(\alpha)N(\beta)$. Since $N(\alpha) \neq 1$ and $N(\beta) \neq 1$,

$$N(\alpha) = N(\beta) = p.$$

Therefore $\alpha\bar{\alpha} = N(\alpha) = p$.

5.5. We claim that (up to units) the irreducible elements of $\mathbb{Z}[i]$ are $1+i$, $a+bi$, where $a, b \in \mathbb{Z}$ and $a^2 + b^2 = p$ with p a prime number such that $p \equiv 1 \pmod{4}$, and a prime number p with $p \equiv 3 \pmod{4}$.

Let $\alpha \in \mathbb{Z}[i]$ be irreducible. Then α is prime in $\mathbb{Z}[i]$. We claim that $p \in (\alpha)$ for some prime number $p \in \mathbb{Z}$. In fact, $(\alpha) \cap \mathbb{Z}$ is an ideal of \mathbb{Z} , so $(\alpha) \cap \mathbb{Z} = (p)$ for some positive integer $p \in \mathbb{Z}$. To see that p is a prime number, let us assume that $p \mid xy$ for some $x, y \in \mathbb{Z}$. Then $xy = pm \in (\alpha) \cap \mathbb{Z}$ for some $m \in \mathbb{Z}$. In particular, $xy = \alpha\beta$ for some $\beta \in \mathbb{Z}[i]$. Thus $\alpha \mid xy$ in $\mathbb{Z}[i]$ and therefore $\alpha \mid x$ or $\alpha \mid y$ in $\mathbb{Z}[i]$. Thus $x \in (\alpha) \cap \mathbb{Z} = (p)$ or $y \in (\alpha) \cap \mathbb{Z} = (p)$. Hence $p \mid x$ or $p \mid y$ in \mathbb{Z} , so p is a prime number.

Let $\beta \in \mathbb{Z}[i]$ be such that $p = \alpha\beta$. Then

$$p^2 = N(p) = N(\alpha\beta) = N(\alpha)N(\beta).$$

Thus $N(\alpha) \in \{p, p^2\}$, as $N(\alpha) \neq 1$ because $\alpha \notin \mathcal{U}(\mathbb{Z}[i])$.

Assume first that $N(\alpha) = p$. Write $\alpha = a + bi$ for $a, b \in \mathbb{Z}$. If $p = 2$, then

$$\alpha \in \{1+i, 1-i, -1+i, -1-i\}.$$

If $p \equiv 1 \pmod{4}$, then $p = N(\alpha) = a^2 + b^2$. If $p \equiv 3 \pmod{4}$, then $p = N(\alpha) = a^2 + b^2$, a contradiction to Theorem 5.1.

Assume now that $N(\alpha) = p^2$. Then $\beta \in \mathcal{U}(\mathbb{Z}[i])$. If $p = 2$, then α is not irreducible, as $\alpha \in \{2, -2, 2i, -2i\}$. If $p \equiv 3 \pmod{4}$, then $\alpha \in \{p, -p, pi, -pi\}$. If $p \equiv 1 \pmod{4}$, then $p = \gamma\bar{\gamma}$ for some irreducible γ such that $N(\gamma) = p$ (see Exercise 5.4). Thus α is not irreducible, a contradiction.

5.19. Assume that I is a maximal ideal of R . We need to show that every non-zero element of R/I is invertible. Recall that $1_{R/I} = 1 + I$. Let $a \notin I$. Then $(I, a) = R$, as I is maximal. Then $1 + x + ab$ for some $x \in I$ and $b \in R$. In particular,

$$(ab) + I = (a + I)(b + I) = 1 + I.$$

Conversely, let J be an ideal of R such that $I \subsetneq J$. Let $a \in J \setminus I$. Since R/I is a field, there exists $b \in R$ such that

$$(ab) + I = (a + I)(b + I) = 1 + I.$$

Thus $ab = 1 + x$ for some $x \in I \subseteq J$. Since $ab \in J$, it follows that $1 \in J$ and therefore $J = R$.

5.24. Recall that R/I is a ring with $0_{R/I} = I$. Thus $x + I = 0_{R/I}$ if and only if $x \in I$. Assume first that I is a prime ideal. Let $a, b \in R$ be such that

$$(ab) + I = (a + I)(b + I) = I.$$

Since $ab \in I$ and I is prime, $a \in I$ or $b \in I$. This means that $a + I = I$ or $b + I = I$.

Conversely, if there are $a, b \in R$ are such that $a \notin I$ and $b \notin I$ and

$$(ab) + I = (a + I)(b + I) = I,$$

then $ab \in I$ and I is not prime.

5.25.

- 1) Let $a, b \in R$ be such that $ab \in I$. If $a \notin I$, then $(I, a) = R$, as I is a maximal ideal. Thus $1 = x + ra$ for some $x \in I$ and $r \in R$. Multiplying by b and using that $x \in I$ and $ab \in I$, $b = xb + rab \in I$.

- 2) The ideal (X) is prime as $R/(X) \simeq \mathbb{Z}$ is a domain. Since \mathbb{Z} is not a field, the ideal (X) cannot be maximal (see Exercise 5.19).
- 3) Let J be an ideal such that $I \subseteq J$. There exist $a, b \in R \setminus \{0\}$ such that $I = (a)$ and $J = (b)$. In particular, since $a \in I \subseteq J$, there exists $r \in R$ such that $a = rb \in I$. Since I is prime, $r \in I$ or $b \in I$. If $b \in I$, then $I = J$. If $r \in I = (a)$, then $r = sa$ for some $s \in R$. Thus $a = rb = s(ab)$, so $a(1 - sb) = 0$. Since $a \neq 0$, $b \in \mathcal{U}(R)$ and therefore $J = R$.

5.34. Consider \mathbb{R} as a \mathbb{Q} -vector space. Since $\sqrt{2} \notin \mathbb{Q}$, the subset $\{1, \sqrt{2}\}$ of \mathbb{R} is linearly independent over \mathbb{Q} . Extend $\{1, \sqrt{2}\}$ to a basis B of \mathbb{R} as a \mathbb{Q} -vector space. The linear map $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(1) = \sqrt{2}$, $f(\sqrt{2}) = 1$ and $f(b) = b$ for all $b \in B \setminus \{1, \sqrt{2}\}$ is not of the form $f(x) = \lambda x$. Indeed, if $f(x) = \lambda x$ for some $\lambda \in \mathbb{R}$, then $\sqrt{2} = f(1) = \lambda$ and $1 = f(\sqrt{2}) = \lambda\sqrt{2}$, a contradiction.

5.35. Let $\{x_i : i \in I\}$ be a basis of \mathbb{R} as a \mathbb{Q} -vector space and let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n as a vector space over \mathbb{R} . Routine calculations prove that

$$\{x_i e_j : i \in I, 1 \leq j \leq n\}$$

is a basis of \mathbb{R}^n as a \mathbb{Q} -vector space. Since I and $I \times \{e_1, \dots, e_n\}$ have the same cardinality, there exists a bijective map $f: I \rightarrow I \times \{e_1, \dots, e_n\}$. This bijective map extends to a linear isomorphism between the rational vector spaces \mathbb{R}^n and \mathbb{R} . In particular, \mathbb{R}^n and \mathbb{R} are isomorphic as abelian groups.

5.36. Let G be a group such that $\text{Aut}(G) = \{\text{id}\}$. Then G is abelian, as the map $G \rightarrow G$, $x \mapsto gxg^{-1}$, is an automorphism of G . Since G is abelian, the map $G \rightarrow G$, $x \mapsto x^{-1}$, is an automorphism of G . Since the automorphism group of G is trivial, $x = x^2$ for all $x \in G$. At this point it is convenient to write G additively. We claim that G is a vector space over the field $\mathbb{Z}/2$ of two elements, the action is $\mathbb{Z}/2 \times G \rightarrow G$, $(\lambda, g) \mapsto \lambda g$. Note that we need $g + g = 0$ to prove that G is indeed a vector space over $\mathbb{Z}/2$, as for example

$$0 = 0g = (1 + 1)g = 1g + 1g = g + g$$

for all $g \in G$. Let B be a basis of G over $\mathbb{Z}/2$. If $|B| \geq 2$, the automorphism of G that exchanges to basis elements of B and fixes all other elements of B would be non-trivial. Hence $|B| = 1$ and G is either trivial or cyclic of order two. Both groups have trivial automorphism group.

5.41. One approach to solve this exercise requires the following fact: p is a prime number and $1 \leq k \leq p^n - 1$, then $\binom{p^n}{k}$ is divisible by p . Let us prove this claim. Write

$$\binom{p^n}{k} = \frac{(p^n)!}{k!(p^n - k)!} = \frac{p^n}{k} \binom{p^n - 1}{k - 1}.$$

Then

$$k \binom{p^n}{k} = p^n \binom{p^n - 1}{k - 1},$$

that is p^n divides $k \binom{p^n}{k}$.

If $\gcd(p, k) = 1$, then it follows that p divides $\binom{p^n}{k}$ by unique decomposition of every integer as a product of primes. We may assume then that $\gcd(p, k) \neq 1$, say $k = p^\alpha m$ for some integer m not divisible by p . Then

$$p^n \binom{p^n - 1}{k - 1} = k \binom{p^n}{k} = p^\alpha m \binom{p^n}{k}$$

and hence

$$p^{n-\alpha} \binom{p^n - 1}{k - 1} = m \binom{p^n}{k}.$$

Since $k < p^n$, it follows that $m - \alpha \geq 1$. Thus p divides $m \binom{p^n}{k}$ and hence p divides $\binom{p^n}{k}$ because p and m are coprime.

5.48. Let $G = \langle g : g^3 = 1 \rangle$ be the cyclic group of order three and ω be a primitive cubic root of one. The map $G \rightarrow \mathbb{R} \times \mathbb{C}$, $g \mapsto (1, \omega)$, extends to an algebra homomorphism $\varphi: \mathbb{R}[G] \rightarrow \mathbb{R} \times \mathbb{C}$. Since $\dim_{\mathbb{R}} \mathbb{R} \times \mathbb{C} = \dim \mathbb{R}[G] = 3$, it follows that φ is a bijective.

6.19. Let $T: V \rightarrow W$ be a bijective linear map such that $T\rho_g = \psi_g T$ for all $g \in G$.

- 1) Assume that ψ is not irreducible. Let Y be an invariant subspace of W . Then $X = T^{-1}(Y)$ is an invariant subspace of V , as

$$\rho_g(X) = \rho_g T^{-1}(Y) = T^{-1} \psi_g(Y) \subseteq T^{-1}(Y) = X$$

for all $g \in G$.

- 2) By assumption V can be decomposed as $V = X \oplus Y$ for some invariant subspaces X and Y . Let $X_1 = T(X)$ and $Y_1 = T(Y)$. Note that X_1 and Y_1 are invariant. In fact, if $g \in G$, then $\psi_g(X_1) = \psi_g T(X) = T\rho_g(X) \subseteq T(X) = X_1$. Similarly, $\psi_g(Y_1) \subseteq Y_1$ for all $g \in G$. Now we prove that $W = X_1 \oplus Y_1$. If $w \in W = T(V)$, then $w = T(v)$ for some $v \in V$ since T is surjective. Write $v = x + y$ for $x \in X$ and $y \in Y$. Then $w = T(v) = T(x) + T(y) \in X_1 + Y_1$. Now if $w \in X_1 \cap Y_1$, then $w = T(x) = T(y)$ for some $x \in X$ and $y \in Y$. Since T is injective, it follows that $x = y \in X \cap Y = \{0\}$. Hence $w = 0$.

6.25. Let $\rho: \mathbb{Z} \rightarrow \mathbf{GL}_2(\mathbb{C})$ be the homomorphism given by $m \mapsto \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$. Then ρ is not irreducible, as since $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ an eigenvector of ρ_m for all $m \in \mathbb{Z}$, the subspace $\mathbb{C}v_1$ is an invariant subspace of \mathbb{C}^2 . To prove that ρ is indecomposable, let us assume that it is not, that is ρ is equivalent to a direct sum of degree-one (and hence diagonal) representations. However, $\rho_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable, a contradiction.

9.13. Note that if M is a module over $\mathbb{Z}/2$, then $2m = 0$ for all $m \in M$. If $\mathbb{Z}/4$ is a module over $\mathbb{Z}/2$, then $0 = 2 \cdot 1 = 1 + 1 = 2$, a contradiction.

- 9.28.** 1) We prove that $\{0\}$, M , $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$ are the only submodules of M . If N is a non-zero submodule of M , then let $(x_0, y_0) \in N \setminus \{(0, 0)\}$. If $(x, y) \in M$ is such that $xy \neq 0$, then

$$\left(\frac{x}{x_0} + \left(\frac{y}{y_0} - \frac{x}{x_0} \right) X \right) \cdot (x_0, y_0) = (x, y)$$

and thus $N = M$. If $y_0 = 0$, then $N = \mathbb{R} \times \{0\}$, as $\frac{x}{x_0} \cdot (x_0, 0) = (x, 0)$. If $x_0 = 0$, then $N = \{0\} \times \mathbb{R}$, as $\frac{y}{y_0} \cdot (0, y_0) = (0, y)$.

- 2) If $N \subseteq M$ is a submodule, then N is real vector space. Assume that $N \neq \{(0, 0)\}$ and that $N \neq \mathbb{R}^2$. Since $\dim N = 1$, let $\{(a_0, b_0)\}$ be a basis of N . Since N is a submodule of M , $(b_0, a_0) = X \cdot (a_0, b_0) \in N$. In particular, there exists $\lambda \in \mathbb{R}$ such that $(b_0, a_0) = \lambda(a_0, b_0)$. Since $(a_0, b_0) \neq (0, 0)$, without loss of generality we may assume that $a_0 \neq 0$. Thus $\lambda^2 a_0 = \lambda(\lambda a_0) = \lambda b_0 = a_0$ and hence $\lambda^2 = 1$. If $\lambda = 1$, then $a_0 = b_0$. If $\lambda = -1$, then $a_0 = -b_0$. In conclusion, N is generated either by $(1, 1)$ or $(1, -1)$.

10.12. Take for example the \mathbb{Z} -module $\cup_{k \geq 1} \mathbb{Z}/(p^k)$.

12.7. The Smith's normal form yields

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -2 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad S = PAQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solution of the integer linear system is then $X = \begin{pmatrix} 1+t \\ t \\ -5+2t \end{pmatrix}$ for $t \in \mathbb{Z}$.

12.8. We do not provide the full solutions. It should be enough to show that the Smith's normal form of the matrix $\begin{pmatrix} 6 & 6 & 4 \\ 6 & 12 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

12.9. Let $A = \begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 1 \end{pmatrix}$. The Smith's normal form of A is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$. Thus the abelian group is isomorphic to $\mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \simeq \mathbb{Z}/4$.

12.10. The Smith's normal form is $\begin{pmatrix} 1 & 0 \\ 0 & -11+8i \end{pmatrix}$.

12.11. The Smith normal form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f(X) \end{pmatrix},$$

where $f(X) = X^4 - 3X^3 - 11X^2 + 7X + 84$.

12.12. Since $\det A \neq 0$, the set $\{y_1, \dots, y_n\}$ is linearly independent over \mathbb{Q} and hence they are linearly independent over \mathbb{Z} . Let $B = (b_{ij}) \in M_n(\mathbb{Q})$ be the inverse of A . Then

$$B = A^{-1} = (\det A)^{-1} \operatorname{adj}(A) = \pm \operatorname{adj}(A) \in M_n(\mathbb{Z}).$$

Then $x_i = \sum_{j=1}^n b_{ij} y_j$ for all $i \in \{1, \dots, n\}$ and hence $\{y_1, \dots, y_n\}$ generates G and therefore $\{y_1, \dots, y_n\}$ is a basis of G .

Assume now that $\{y_1, \dots, y_n\}$ is a basis of G . There exist matrices $A = (a_{ij}) \in M_n(\mathbb{Z})$ and $B = (b_{ij}) \in M_n(\mathbb{Z})$ such that

$$y_i = \sum_{j=1}^n a_{ij} x_j, \quad x_i = \sum_{j=1}^n b_{ij} y_j$$

for all $i \in \{1, \dots, n\}$. In particular, $AB = I$. Applying determinant, $(\det A)(\det B) = 1$ and hence $\det A = \det B \in \{-1, 1\}$.

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