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Rings and modules

Notes

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Part I

Rings

Chapter 1

Rings and ideals

Definition 1.1. A **ring** is a set R with two binary operations, the addition $R \times R \rightarrow R$, $(x, y) \mapsto x + y$, and the multiplication $R \times R \rightarrow R$, $(x, y) \mapsto xy$, such that the following properties hold:

- 1) $(R, +)$ is an abelian group.
- 2) $(xy)z = x(yz)$ for all $x, y, z \in R$.
- 3) $x(y + z) = xy + xz$ for all $x, y, z \in R$.
- 4) $(x + y)z = xz + yz$ for all $x, y, z \in R$.
- 5) There exists $1_R \in R$ such that $x1_R = 1_Rx = x$ for all $x \in R$.

Our definition of a ring is that of a ring with identity. In general one writes the identity element 1_R as 1 if there is no risk of confusion.

Definition 1.2. A ring R is said to be **commutative** if $xy = yx$ for all $x, y \in R$.

Example 1.3. \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} are commutative rings.

Example 1.4. The set

$$\mathbb{R}[X] = \left\{ \sum_{i=0}^n a_i X^i : n \in \mathbb{N}_0, a_1, \dots, a_n \in \mathbb{R} \right\}$$

of real polynomials in one variable is a commutative ring with the usual operations.

More generally, if R is a commutative ring, then $R[X]$ is a commutative ring. This construction allows us to define the polynomial ring $R[X, Y]$ in two commuting variables X and Y and coefficients in R as $R[X, Y] = (R[X])[Y]$. One can also define the ring $R[X_1, \dots, X_n]$ of real polynomials in n commuting variables X_1, \dots, X_n with coefficients in R as $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$.

Example 1.5. If A is an abelian group, then $\text{End}(A)$ is a ring with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(g(x)), \quad f, g \in \text{End}(A) \text{ and } x \in A.$$

Let R be a ring. Some facts:

- 1) $x0 = 0x = x$ for all $x \in R$.
- 2) $x(-y) = -xy$ for all $x, y \in R$.
- 3) If $1 = 0$, then $|R| = 1$.

Example 1.6. The real vector space $H(\mathbb{R}) = \{a1 + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ with basis $\{1, i, j, k\}$ is a ring with the multiplication induced by the formulas

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j.$$

As an example, let us perform a calculation in $H(\mathbb{R})$:

$$(1 + i + j)(i + k) = i + k - 1 + ik + ji + jk = i + k - 1 - j - k + i = -1 + 2i - j,$$

as $ij = i(ij) = -j$. This is the ring of real **quaternions**.

Example 1.7. Let $n \geq 2$. The abelian group $\mathbb{Z}/n = \{0, 1, \dots, n\}$ of integers modulo n is a ring with the usual multiplication modulo n .

Example 1.8. Let $n \geq 1$. The set $M_n(\mathbb{R})$ of real $n \times n$ matrices is a ring with the usual matrix operations. Recall that if $a = (a_{ij})$ and $b = (b_{ij})$, the multiplication ab is given by

$$(ab)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

Similarly, for any ring R one defines the ring $M_n(R)$ of $n \times n$ matrices with coefficients in R .

Definition 1.9. Let R be a ring. A **subring** S of R is a subset S such that $(S, +)$ is a subgroup of $(R, +)$ such that $1 \in S$ and if $x, y \in S$, then $xy \in S$.

Clearly, $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ is a chain of subrings.

Example 1.10. $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is a subring of \mathbb{C} . This is known as the ring of **Gauss integers**.

Example 1.11. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subring of \mathbb{R} .

Example 1.12. If R is a ring, then the **center** $Z(R) = \{x \in R : xy = yx \text{ for all } y \in R\}$ is a subring of R .

If S is a subring of a ring R , then the zero element of S is the zero element of R , i.e. $0_R = 0_S$. Moreover, the additive inverse of an element $s \in S$ is the additive inverse of s as an element of R .

Exercise 1.13.

- 1) If S and T are subrings of R , then $S \cap T$ is a subring of R .
- 2) If $R_1 \subseteq R_2 \subseteq \dots$ is a sequence of subrings of R , then $\cup_{i \geq 1} R_i$ is a subring of R .

Definition 1.14. Let R be a ring. An element $x \in R$ is a **unit** if there exists $y \in R$ such that $xy = yx = 1$.

The set $\mathcal{U}(R)$ of units of a ring R form a group with the multiplication. For example, $\mathcal{U}(\mathbb{Z}/8) = \{1, 3, 5, 7\}$.

Definition 1.15. A **division ring** is a ring R such that $\mathcal{U}(R) = R \setminus \{0\}$.

The ring $H(\mathbb{R})$ real quaternions is a non-commutative division ring. Find the inverse of an arbitrary element $a1 + bi + cj + dk \in H(\mathbb{R})$.

Definition 1.16. A **field** is a commutative division ring with $1 \neq 0$.

Clearly, \mathbb{Q} , \mathbb{R} and \mathbb{C} are fields. If p is a prime number, then \mathbb{Z}/p is a field.

Exercise 1.17. $\mathbb{Q}[\sqrt{2}]$ is a field. Find the multiplicative inverse of $x + y\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$.

More challenging: Prove that

$$\mathbb{Q}[\sqrt[3]{2}] = \{x + y\sqrt[3]{2} + z\sqrt[3]{4} : x, y, z \in \mathbb{Q}\}$$

is a field. What is the inverse of $x + y\sqrt[3]{2} + z\sqrt[3]{4}$?

Definition 1.18. Let R be a ring. A **left ideal** of R is a subset I such that $(I, +)$ is a subgroup of $(R, +)$ and such that $RI \subseteq I$, i.e. $ry \in I$ for all $r \in R$ and $y \in I$.

Similarly one defines right ideals, one needs to replace the condition $RI \subseteq I$ by the inclusion $IR \subseteq I$.

Example 1.19. Let $R = M_2(\mathbb{R})$. Then

$$I = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

is a right ideal R that is not a left ideal.

Can you find an example of a right ideal that is not a left ideal?

Definition 1.20. Let R be a ring. An ideal of R is a subset that is both a left and a right ideal of R .

If R is a ring, then $\{0\}$ and R are both ideals of R .

Exercise 1.21. Let R be a ring.

- 1) If $\{I_\alpha : \alpha\}$ is a collection of ideals of R , then $\cap_\alpha I_\alpha$ is an ideal of R .
- 2) If $I_1 \subseteq I_2 \subseteq \dots$ is a sequence of ideals of R , then $\cup_{i \geq 1} I_i$ is an ideal of R .

Example 1.22. Let $R = \mathbb{R}[X]$. If $f(X) \in R$, then the set

$$(f(X)) = \{f(X)g(X) : g(X) \in R\}$$

of multiples of $f(X)$ is an ideal of R . One can prove that this is the smallest ideal of R containing $f(X)$.

If R is a ring and X is a subset of R , one defines the ideal generated by X as the smallest ideal of R containing X , that is

$$(X) = \bigcap \{I : I \text{ ideal of } R \text{ such that } X \subseteq I\}.$$

One proves that

$$(X) = \left\{ \sum_{i=1}^m r_i x_i s_i : m \in \mathbb{N}_0, r_1, \dots, r_m, s_1, \dots, s_m \in R \right\},$$

where by convention the empty sum is equal to zero. If $X = \{x_1, \dots, x_n\}$ is a finite set, then we write $(X) = (x_1, \dots, x_n)$.

xca:ideals_Z

Exercise 1.23. Prove that every ideal of \mathbb{Z} is of the form $n\mathbb{Z}$ for some $n \geq 0$.

Exercise 1.24. Let $n \geq 2$. Find the ideals of \mathbb{Z}/n .

Exercise 1.25. Find the ideals of \mathbb{R} .

Definition 1.26. Let R be a ring and I be an ideal of R . Then I is **principal** if $I = (x)$ for some $x \in R$.

The division algorithm shows that every ideal of \mathbb{Z} is principal, see Exercise 1.23.

Exercise 1.27. Prove that every ideal of $\mathbb{R}[X]$ is principal.

If K is a field, there is a division algorithm in the polynomial ring $K[X]$. Then one proves that every ideal of $K[X]$ is principal.

Exercise 1.28. Let R be a ring and $x \in R$. Prove that $x \in \mathcal{U}(R)$ if and only if $(x) = R$.

One proves that a field has only two ideals.

Definition 1.29. Let R and S be rings. A map $f: R \rightarrow S$ is a **ring homomorphism** if $f(1) = 1$, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$.

Our definition of a ring is that of a ring with identity. This means that the identity element 1 of a ring R is part of the structure. For that reason, in the definition of a ring homomorphism f one needs $f(1) = 1$.

Example 1.30. The map $f: \mathbb{Z}/6 \rightarrow \mathbb{Z}/6$, $x \mapsto 3x$, is not a ring homomorphism because $f(1) = 3$.

If R is a ring, then the identity map $\text{id}: R \rightarrow R$, $x \mapsto x$, is a ring homomorphism.

Example 1.31. The inclusions $\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$ are ring homomorphisms.

More generally, if S is a subring of a ring R , then the inclusion map $S \hookrightarrow R$ is a ring homomorphism.

Example 1.32. Let R be a ring. The map $\mathbb{Z} \rightarrow R, k \mapsto k1$, is a ring homomorphism.

Example 1.33. Let $x_0 \in \mathbb{R}$. The evaluation map $\mathbb{R}[X] \rightarrow \mathbb{R}, f \mapsto f(x_0)$, is a ring homomorphism.

The **kernel** of a ring homomorphism $f: R \rightarrow S$ is the subset

$$\ker f = \{x \in R : f(x) = 0\}.$$

One proves that the kernel of f is an ideal of R . Moreover, $\ker f = \{0\}$ if and only if f is injective. The image

$$f(R) = \{f(x) : x \in R\}$$

is a subring of S . In general, $f(R)$ is not an ideal of S .

Example 1.34. The map $\mathbb{C} \rightarrow M_2(\mathbb{R}), a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, is an injective ring homomorphism.

Example 1.35. The map $\mathbb{Z}[i] \rightarrow \mathbb{Z}/5, a + bi \mapsto a + 2b \bmod 5$, is a ring homomorphism with $\ker f = \{a + bi : a + 2b \equiv 0 \bmod 5\}$.

Exercise 1.36. There is no ring homomorphism $\mathbb{Z}/6 \rightarrow \mathbb{Z}/15$. Why?

Exercise 1.37. If $f: \mathbb{R}[X] \rightarrow \mathbb{R}$ is a ring homomorphism such that the restriction $f|_{\mathbb{R}}$ of f onto \mathbb{R} is the identity, then there exists $x_0 \in \mathbb{R}$ such that f is the evaluation map at x_0 .

We now define ring quotients. Let R be a ring and I be an ideal of R . Then R/I is an abelian group with

$$(x + I) + (y + I) = (x + y) + I$$

and the **canonical map** $R \rightarrow R/I, x \mapsto x + I$, is a surjective group homomorphism. Recall that R/I is the set of cosets $x + I$, where $x + I = y + I$ if and only if $x - y \in I$. Note that here we only used that I is an additive subgroup of R . We need an ideal to put a ring structure on the set R/I of cosets modulo I . As in the case of integers, we use the following notation. For $x, y \in R$ we write

$$x \equiv y \bmod I \iff x - y \in I.$$

How can we put a ring structure on R/I ? It makes sense to define a multiplication on R/I in such a way that the canonical map $R \rightarrow R/I$ is a surjective ring homomorphism. For that purpose, we define

$$(x + I)(y + I) = (xy) + I.$$

Since I is an ideal of R , this multiplication is well-defined. In fact, let $x + I = x_1 + I$ and $y + I = y_1 + I$. We want to show that $xy + I = x_1y_1 + I$. Since $x - x_1 \in I$,

$$xy - x_1y = (x - x_1)y \in I$$

because I is a right ideal. Similarly, since $y - y_1 \in I$, it follows that

$$x_1y - x_1y_1 = x_1(y - y_1) \in I,$$

as I is a left ideal. Thus

$$xy - x_1y_1 = xy - x_1y + x_1y - x_1y_1 = (x - x_1)y + x_1(y - y_1) \in I.$$

Theorem 1.38. *Let R be a ring and I be an ideal of R . Then R/I with*

$$(x + I) + (y + I) = (x + y) + I, \quad (x + I)(y + I) = (xy) + I,$$

is a ring and the canonical map $R \rightarrow R/I$, $x \mapsto x + I$, is a surjective ring homomorphism with kernel I .

We have already seen that the multiplication is well-defined. The rest of the proof is left as an exercise.

Example 1.39. Let $R = (\mathbb{Z}/3)[X]$ and $I = (2X^2 + X + 2)$ be the ideal of R generated by the polynomial $2X^2 + X + 2$. If $f(X) \in R$, the division algorithm allows us to write

$$f(X) = (2X^2 + X + 2)q(X) + r(X),$$

for some $q(X), r(X) \in R$, where either $r(X) = 0$ or $\deg r(X) < 2$. This means that $r(X) = aX + b$ for some $a, b \in R$. Note that $f(X) \equiv aX + b \pmod{2X^2 + X + 2}$ for some $a, b \in \mathbb{Z}/3$, so the quotient ring R/I has nine elements.

As it happens in the case of groups, to understand quotient rings one has the first isomorphism theorem.

Theorem 1.40 (first isomorphism theorem). *If $f: R \rightarrow S$ is a ring homomorphism, then $R/\ker f \simeq f(R)$.*

This is somewhat similar to the result one knows from group theory. One needs to show that the map $R/I \rightarrow f(R)$, $x + I \mapsto f(x)$, is a well-defined bijective ring homomorphism.

Example 1.41. The evaluation map $\mathbb{R}[X] \rightarrow \mathbb{C}$, $f(X) \mapsto f(i)$, is a surjective ring homomorphism with kernel $(X^2 + 1)$. Thus

$$\mathbb{R}[X]/(X^2 + 1) \simeq \mathbb{C}$$

by the first isomorphism theorem. In practice, this is how it works. Let $f(X) \in \mathbb{R}[X]$. The division algorithm on $\mathbb{R}[X]$ allows us to write

$$f(X) = (X^2 + 1)q(X) + r(X)$$

for some $q(X), r(X) \in \mathbb{R}[X]$, where $r(X) = 0$ or $\deg r(X) < 2$. Thus $r(X) = aX + b$ for some $a, b \in \mathbb{R}$. This implies that

$$f(X) \equiv aX + b \pmod{(X^2 + 1)}.$$

It is quite easy to describe the ring operation of $\mathbb{R}[X]/(X^2 + 1)$. Clearly

$$(aX + b) + (cX + d) \equiv (a + c)X + (b + d) \pmod{(X^2 + 1)},$$

Since $X^2 \equiv -1 \pmod{(X^2 + 1)}$,

$$(aX + b)(cX + d) \equiv X(ad + bc) + (bd - ac),$$

which reminds us the usual multiplication rule of the field of complex numbers.

Example 1.42. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$$

A direct calculation shows that the map $R \rightarrow \mathbb{Q}$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto a$, is a surjective ring homomorphism with $\ker f = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Q} \right\}$. Thus $R/\ker f \simeq \mathbb{Q}$.

Exercise 1.43. Let R be the ring of continuous maps $[0, 2] \rightarrow \mathbb{R}$. Prove that the set $I = \{f \in R : f(1) = 0\}$ is an ideal of R and that $R/I \simeq \mathbb{R}$.

Exercise 1.44. Let $n \geq 1$. Let R be a ring and I be an ideal of R . Prove that $M_n(I)$ is an ideal of $M_n(R)$ and that $M_n(R)/M_n(I) \simeq M_n(R/I)$.

Exercise 1.45. Let $R = \mathbb{Z}[\sqrt{10}]$ and $I = (2, \sqrt{10})$. Prove that $R/I \simeq \mathbb{Z}/2$.

Exercise 1.46. Prove that $\mathbb{Z}[i]/(1 + 3i) \simeq \mathbb{Z}/10$.

Exercise 1.47. Prove that there is no ideal I of $\mathbb{Z}[i]$ such that $\mathbb{Z}[i]/I \simeq \mathbb{Z}/15$.

As it happens in group theory, one has the following important result.

Theorem 1.48 (correspondence theorem). *Let $f: R \rightarrow S$ be a surjective ring homomorphism. There exists a bijective correspondence between the set of ideals of R containing $\ker f$ and the set of ideals of S .*

Sketch of proof. Let I be an ideal of R containing $\ker f$ and let J be an ideal of S . We need to prove the following facts:

- 1) $f(I)$ is an ideal of S .
- 2) $f^{-1}(J)$ is an ideal of R containing $\ker f$.
- 3) $f(f^{-1}(J)) = J$ and $f^{-1}(f(I)) = I$.
- 4) If $f(I) = J$, then $R/I \simeq S/J$.

We only prove the fourth statement, the others are left as exercises. Note that the third claim implies that $f(I) = J$ if and only if $I = f^{-1}(J)$. Let $\pi: S \rightarrow S/J$ be the canonical map. The composition $g = \pi \circ f: R \rightarrow S/J$ is a ring homomorphism and

$$\ker g = \{x \in R : g(x) = 0\} = \{x \in R : f(x) \in J\} = \{x \in R : x \in f^{-1}(J) = I\} = I.$$

Since $g(R) = S/J$, the first isomorphism theorem implies that $R/I \simeq S/J$. \square

Chapter 2

Chinese remainder theorem

Note that if R is a commutative ring and I and J are ideals of R , then

$$I + J = \{u + v : u \in I, v \in J\}$$

is an ideal of R .

Definition 2.1. Let R be a commutative ring. The ideals I and J of R are said to be **coprime** if $R = I + J$.

The terminology is motivated by the following example. If I and J are ideals of \mathbb{Z} , then $I = (a)$ and $J = (b)$ for some $a, b \in \mathbb{Z}$. Then

a and b are coprime $\iff 1 = ra + sb$ for some $r, s \in \mathbb{Z} \iff I$ and J are coprime.

If I and J are ideals of R , then

$$IJ = \left\{ \sum_{i=1}^m u_i v_i : m \in \mathbb{N}_0, u_1, \dots, u_m \in I, v_1, \dots, v_m \in J \right\}$$

is an ideal of R . Note that $IJ \subseteq I \cap J$. The equality does not hold in general. Take for example $R = \mathbb{Z}$ and $I = J = (2)$. Then $IJ = (4) \subsetneq (2) = I \cap J$.

Proposition 2.2. Let R be a commutative ring. If I and J are coprime ideals, then $IJ = I \cap J$.

Proof. Let $x \in I \cap J$. Since I and J are coprime, $1 = u + v$ for some $u \in I$ and $v \in J$, $x = x1 = x(u + v) = xu + xv \in IJ$. \square

Theorem 2.3 (chinese remainder theorem). Let R be a commutative ring and I and J be coprime ideals. If $u, v \in R$, then there exists $x \in R$ such that

$$\begin{cases} x \equiv u \pmod{I}, \\ x \equiv v \pmod{J}. \end{cases}$$

Proof. Since the ideals I and J are coprime, $1 = a + b$ for some $a \in I$ and $b \in J$. Let $x = av + bu$. Then

$$x - u = av + (b - 1)u = av - au = a(v - u) \in I,$$

that is $x \equiv u \pmod{I}$. Similarly, $x - v \in J$ and $x \equiv v \pmod{J}$. \square

Corollary 2.4. *Let R be a commutative ring. If I and J are coprime ideals of R , then $R/(I \cap J) \simeq R/I \times R/J$.*

Proof. Let $\pi_I: R \rightarrow R/I$ and $\pi_J: R \rightarrow R/J$ be the canonical maps. A straightforward calculation shows that the map $\varphi: R \rightarrow R/I \times R/J, x \mapsto (\pi_I(x), \pi_J(x))$, is an injective ring homomorphism with $\ker \varphi = I \cap J$. The chinese remainder theorem implies that φ is surjective. If $(u + I, v + J) \in R/I \times R/J$, then there exists $x \in R$ such that $x - u \in I$ and $x - v \in J$. This translates into the surjectivity of φ . Now $R/(I \cap J) \simeq R/I \times R/J$ by the first isomorphism theorem. \square

Let R be a commutative ring and I_1, \dots, I_n be ideals of R . Then

$$I_1 \cdots I_n = \left\{ \sum_{i=1}^m u_{i_1} \cdots u_{i_n} : m \in \mathbb{N}_0, u_{i_1}, \dots, u_{i_n} \in I_{i_j} \right\}$$

is an ideal of R . If I_1 and I_j are coprime for all $j \in \{2, \dots, n\}$, then I_1 and $I_2 \cdots I_n$ are coprime. If I_i and I_j are coprime whenever $i \neq j$, then

$$R/(I_1 \cap \cdots \cap I_n) \simeq R/I_1 \times \cdots \times R/I_n.$$

Exercise 2.5 (Lagrange's interpolation theorem). The chinese remainder theorem proves the following well-known result. Let $x_1, \dots, x_k \in \mathbb{R}$ be such that $x_i \neq x_j$ whenever $i \neq j$ and $y_1, \dots, y_k \in \mathbb{R}$. Then there exists $f(X) \in \mathbb{R}[X]$ such that

$$\begin{cases} f(X) \equiv y_1 \pmod{(X - x_1)}, \\ f(X) \equiv y_2 \pmod{(X - x_2)}, \\ \vdots \\ f(X) \equiv y_k \pmod{(X - x_k)}. \end{cases}$$

The solution $f(X)$ is unique modulo $(X - x_1)(X - x_2) \cdots (X - x_n)$.

xca:gather_people

Exercise 2.6. Let us gather people in the following way. When I count by three, there are two persons left. When I count by four, there is one person left over and when I count by five there is one missing. How many persons are there?

xca:no_solution

Exercise 2.7. Prove that

$$\begin{cases} x \equiv 29 \pmod{52}, \\ x \equiv 19 \pmod{72}. \end{cases}$$

does not have solution.

`xca:consecutive`

Exercise 2.8. Find three consecutive integers such that the first one is divisible by a square, the second one is divisible by a cube and the third one is divisible by a fourth power.

`xca:perfect_square`

Exercise 2.9. Prove that for each $n \in \mathbb{N}$ there are n consecutive integers such that each integer is divisible by a perfect square $\neq 1$.

Chapter 3

Noetherian rings

In this chapter we will work with commutative rings.

Definition 3.1. A ring R is said to be **noetherian** if every (increasing) sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of R stabilizes, that is $I_n = I_m$ for some $m \in \mathbb{N}$ and all $n \geq m$.

The ring \mathbb{Z} of integers is noetherian.

Example 3.2. Let $R = \{f: [0, 1] \rightarrow \mathbb{R}\}$ with

$$(f + g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x), \quad f, g \in R, x \in [0, 1].$$

For $n \in \mathbb{N}$ let $I_n = \{f \in R : f|_{[0, 1/n]} = 0\}$. Then each I_n is an ideal of R and the sequence $I_1 \subsetneq I_2 \subsetneq \cdots$ does not stabilize. Thus R is not noetherian.

Definition 3.3. Let R be a ring. An ideal I of R is said to be **finitely generated** if $I = (X)$ for some finite subset X of R .

The zero ideal is always finitely generated.

Proposition 3.4. Let R be a ring. Then R is noetherian if and only if every ideal of R is finitely generated.

Proof. Assume first that R is noetherian. Let I be an ideal of R that is not finitely generated. Thus $I \neq \{0\}$. Let $x_1 \in I \setminus \{0\}$ and let $I_1 = (x_1)$. Since I is not finitely generated, $I \neq I_1$ and hence $\{0\} \subsetneq I_1 \subsetneq I$. Once I have the ideals I_1, \dots, I_{k-1} , let $x_k \in I \setminus I_{k-1}$ (such an element exists because I_{k-1} is finitely generated and I is not) and $I_k = (I_{k-1}, x_k)$. The sequence $\{0\} \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ does not stabilize.

Assume now that every ideal of R is finitely generated and let $I_1 \subseteq I_2 \subseteq \cdots$ be a sequence of ideals of R . Then $I = \bigcup_{i \geq 1} I_i$ is an ideal of R , so it is finitely generated, say $I = (x_1, \dots, x_n)$. We may assume that $x_j \in I_{j_j}$ for all j . Let $N = \max\{j_1, \dots, j_n\}$ and $n \geq N$. Then $I_N \subseteq I \subseteq I_N$ and therefore the sequence stabilizes. \square

Exercise 3.5. Let $R = \mathbb{C}[X_1, X_2, \dots]$ be the ring of polynomial in an infinite number of commuting variables. Prove that the ideal $I = (X_1, X_2, \dots)$ of polynomials with zero constant term is not finitely generated.

The correspondence theorem and the previous proposition allow us to prove easily the following result.

Proposition 3.6. *Let I be an ideal of R . If R is noetherian, then R/I is noetherian.*

Proof. Let $\pi: R \rightarrow R/I$ be the canonical surjection and let J be an ideal of R/I . Then $\pi^{-1}(J)$ is an ideal of R containing I . Since R is noetherian, $\pi^{-1}(J)$ is finitely generated, say $\pi^{-1}(J) = (x_1, \dots, x_k)$ for $x_1, \dots, x_k \in R$. Thus

$$J = \pi(\pi^{-1}(J)) = (\pi(x_1), \dots, \pi(x_k))$$

and hence J is finitely generated. \square

Since \mathbb{Z} is noetherian, \mathbb{Z}/n is noetherian for all $n \geq 2$.

Exercise 3.7. Prove that $\mathbb{R}[X]$ is noetherian.

Theorem 3.8 (Hilbert). *Let R be a commutative ring. If R is noetherian ring, then $R[X]$ is noetherian.*

Proof. We need to show that every ideal of $R[X]$ is finitely generated. Assume that there is an ideal I of $R[X]$ that is not finitely generated. In particular, $I \neq \{0\}$. Let $f_1(X) \in I \setminus \{0\}$ be of minimal degree. For $i > 1$ let $f_i(X) \in I$ be of minimal degree such that $f_i(X) \notin (f_1(X), \dots, f_{i-1}(X))$ (note that such an $f_i(X)$ exists because I is not finitely generated). For each i let a_i be the leading coefficient of $f_i(X)$, that is

$$f_i(X) = a_i X^{n_i} + \dots,$$

where the dots denote lowest degree terms. Note that $a_i \neq 0$. Let $J = (a_1, a_2, \dots)$. Since R is noetherian, the sequence

$$(a_1) \subseteq (a_1, a_2) \subseteq \dots (a_1, a_2, \dots, a_k) \subseteq \dots$$

stabilizes, so J is finitely generated, say $J = (a_1, \dots, a_m)$ for some $m \in \mathbb{N}$. There exist $u_1, \dots, u_m \in R$ such that

$$a_{m+1} = \sum_{i=1}^m u_i a_i.$$

Let

$$g(X) = \sum_{i=1}^m u_i f_i(X) X^{n_{m+1}-n_i} \in (f_1(X), \dots, f_m(X)).$$

The leading coefficient of $g(X)$ is $\sum_{i=1}^m u_i a_i = a_{m+1}$ and, moreover, the degree of $g(X)$ is n_{m+1} . Thus $\deg(g(X)) < n_{m+1}$. Since $f_{m+1}(X) \notin (f_1(X), \dots, f_m(X))$,

$$g(X) - f_{m+1}(X) \notin (f_1(X), \dots, f_m(X)),$$

a contradiction to the minimality of the degree of f_{m+1} . \square

Since $R[X_1, \dots, X_n] = (R[X_1, \dots, X_{n-1}])[X_n]$, by induction one proves that if R is a commutative noetherian ring, then $R[X_1, \dots, X_n]$ is noetherian.

Example 3.9. Since \mathbb{Z} is noetherian, so is $\mathbb{Z}[X]$ by Hilbert's theorem. Now $\mathbb{Z}[\sqrt{N}]$ is noetherian, as $\mathbb{Z}[\sqrt{N}] \simeq \mathbb{Z}[X]/(X^2 - N)$ and quotients of noetherian rings are noetherian.

Example 3.10. The ring $\mathbb{Z}[X, X^{-1}]$ is noetherian, as $\mathbb{Z}[X, X^{-1}] \simeq \mathbb{Z}[X, Y]/(XY - 1)$.

Exercise 3.11. Prove that $R[[X]]$ is noetherian if R is noetherian.

Exercise 3.12. Let $f: R \rightarrow R$ be surjective ring homomorphism. Prove that f is an isomorphism if R is noetherian.

Chapter 4

Factorization

Definition 4.1. A commutative ring R is said to be an **integral domain** if $xy = 0$ implies $x = 0$ or $y = 0$.

The rings \mathbb{Z} and $\mathbb{Z}[i]$ are both integral domains. More generally, if N is a square-free integer, then the ring $\mathbb{Z}[\sqrt{N}]$ is an integral domain. The ring $\mathbb{Z}/4$ of integers modulo 4 is not an integral domain.

Definition 4.2. Let R be an integral domain and $x, y \in R$. Then x **divides** y if $y = xz$ for some $z \in R$. Notation: $x \mid y$ if and only if x divides y . If x does not divide y one writes $x \nmid y$.

Note that $x \mid y$ if and only if $(y) \subseteq (x)$.

Definition 4.3. Let R be an integral domain and $x, y \in R$. Then x and y are **associate** in R if $x = yu$ for some $u \in \mathcal{U}(R)$.

Note that x and y are associate if and only if $(x) = (y)$.

Example 4.4. The integers 2 and -2 are associate in \mathbb{Z} .

Example 4.5. Let $R = \mathbb{Z}[i]$.

- 1) Let $d \in \mathbb{Z}$ and $a + ib \in R$. Then $d \mid a + ib$ in R if and only if $d \mid a$ and $d \mid b$ in \mathbb{Z} .
- 2) 2 and $-2i$ are associate in R .

Example 4.6. Let $R = \mathbb{R}[X]$ and $f(X) \in R$. Then $f(X)$ and $\lambda f(X)$ are associate in R for all $\lambda \in \mathbb{R}^\times$.

Definition 4.7. Let R be an integral domain and $x \in R \setminus \{0\}$. Then x is **irreducible** if and only if $x \notin \mathcal{U}(R)$ and $x = ab$ with $a, b \in R$ implies that $a \in \mathcal{U}(R)$ or $b \in \mathcal{U}(R)$.

Note that x is irreducible if and only if $(x) \neq R$ and there is no principal ideal (y) such that $(x) \subsetneq (y) \subsetneq R$.

Example 4.8. Let $R = \mathbb{R}[X]$ and $f(X) \in R \setminus \{0\}$. Then $f(X)$ is irreducible if $\lambda \in \mathbb{R}^\times$ or $\lambda f(X)$ for $\lambda \in \mathbb{R}^\times$ are the only divisors of $f(X)$.

The irreducibles of \mathbb{Z} are the prime numbers.

Definition 4.9. Let R be an integral domain and $p \in R \setminus \{0\}$. Then p is **prime** if $p \notin \mathcal{U}(R)$ and $yz \in (p)$ implies that $y \in (p)$ or $z \in (p)$.

In \mathbb{Z} primes and irreducible coincide. This does not happen in full generality. However, the following result holds.

Proposition 4.10. Let R be an integral domain and $x \in R$. If x is prime, then x is irreducible.

Proof. Let p be a prime. Then $p \neq 0$ and $p \notin \mathcal{U}(R)$. Let x be such that $x \mid p$. Then $p = xy$ for some $y \in R$. This means $xy \in (p)$, so $x \in (p)$ or $y \in (p)$ because p is prime. If $x \in (p)$, then $x = pz$ for some $z \in R$ and hence

$$p = xy = (pz)y.$$

Since $p - pzy = p(1 - zy)$ and R is an integral domain, it follows that $1 - zy = 0$. Thus $y \in \mathcal{U}(R)$. Similarly, if $y \in (p)$, then $x \in \mathcal{U}(R)$. \square

To show that there rings where some irreducibles are not prime, we need the following lemma.

Lemma 4.11. Let $N \in \mathbb{Z}$ be a square-free integer and $R = \mathbb{Z}[\sqrt{N}]$. The map

$$N: R \rightarrow \mathbb{N}, \quad a + b\sqrt{N} \mapsto |a^2 - Nb^2|,$$

satisfies the following properties:

- 1) $N(\alpha) = 0$ if and only if $\alpha = 0$.
- 2) $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in R$.
- 3) $\alpha \in \mathcal{U}(\mathbb{Z}[\sqrt{N}])$ if and only if $N(\alpha) = 1$.
- 4) If $N(\alpha)$ is prime in \mathbb{Z} , then α is irreducible in R .

Proof. The first three items are left as an exercises. Let us prove 4). If $\alpha = \beta\gamma$ for some $\beta, \gamma \in R$, then $N(\alpha) = N(\beta)N(\gamma)$. Since $N(\alpha)$ is a prime number, it follows that $N(\alpha) = 1$ or $N(\beta) = 1$. Thus $\beta \in \mathcal{U}(R)$ or $\gamma \in \mathcal{U}(R)$. \square

Example 4.12. Let $R = \mathbb{Z}[i]$.

- 1) $\mathcal{U}(R) = \{-1, 1, i, -i\}$.
- 2) 3 is irreducible in R . In fact, if $3 = \alpha\beta$, then $9 = N(\alpha)N(\beta)$. This implies that $N(\alpha) \in \{1, 3, 9\}$. Write $\alpha = a + bi$ for $a, b \in \mathbb{Z}$. If $N(\alpha) = 1$, then $\alpha \in \mathcal{U}(R)$ by the lemma. If $N(\alpha) = 9$, then $N(\beta) = 1$ and hence $\beta \in \mathcal{U}(R)$ by the lemma. Finally, if $N(\alpha) = 3$, then $a^2 + b^2 = 3$, which is a contradiction since $a, b \in \mathbb{Z}$.
- 3) 2 is not irreducible in R . In fact, $2 = (1 + i)(1 - i)$ and since

$$N(1 + i) = N(1 - i) = 2,$$

it follows that $1 + i \notin \mathcal{U}(R)$ and $1 - i \notin \mathcal{U}(R)$.

Exercise 4.13. Let $R = \mathbb{Z}[\sqrt{-5}]$.

- 1) Prove that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible in R .
- 2) Prove that $2, 3, 1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are not associate in R .

Example 4.14. Let $R = \mathbb{Z}[\sqrt{-3}]$ and $x = 1 + \sqrt{-3}$.

- 1) x is irreducible. If $x = \alpha\beta$ for some $\alpha, \beta \in R$, then $4 = N(x) = N(\alpha)N(\beta)$. Write $\alpha = a + b\sqrt{-3}$ for some $a, b \in \mathbb{Z}$. Then $N(\alpha) = a^2 + 3b^2 \neq 2$. If $N(\alpha) = 2$, then $a^2 + 3b^2 = 2$ and then a and b both have the same parity. If both a and b are even, say $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$, then

$$2 = a^2 + 3b^2 = 4k^2 + 12l^2$$

is divisible by 4, a contradiction.

If both a and b are odd, say $a = 2k + 1$ and $b = 2l + 1$ for some $k, l \in \mathbb{Z}$, then

$$2 = a^2 + 3b^2 = 4k^2 + 4k + 12l^2 + 12l + 4$$

is divisible by 4, a contradiction.

- 2) x is not prime. Note that $(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4$, then x divides $4 = 2 \cdot 2$. But $1 + \sqrt{-3} \nmid 2$, as

$$(a - 3b) + (a + b)\sqrt{3} = (1 + \sqrt{-3})(a + b\sqrt{-3}) = 2$$

implies that $a - 3b = 2$ and $a + b = 0$, which yields $a = 1/2 \notin \mathbb{Z}$, a contradiction.

Exercise 4.15. Let $R = \mathbb{Z}[\sqrt{5}]$. Prove that $1 + \sqrt{5}$ is irreducible and not prime in R .

Definition 4.16. Let R be an integral domain. Then R is **principal** (or a principal domain) if every ideal of R is principal.

The rings \mathbb{Z} and $\mathbb{R}[X]$ are both principal.

Example 4.17. The ring $\mathbb{Z}[X]$ is not principal. For example, the ideal $I = (2, X)$ is not principal.

First note that $I \neq \mathbb{Z}[X]$. In fact, if $I = \mathbb{Z}[X]$, then $1 = 2f(X) + Xg(X)$ for some $f(X), g(X) \in \mathbb{Z}[X]$. Then

$$0 = \deg(1) = \deg(2f(X) + Xg(X)) = \deg(f(X)) + \deg(g(X)) + 1 > 0,$$

a contradiction.

If $I = (h(X))$ for some $h(X) \in \mathbb{Z}[X]$, then $2 = h(X)g(X)$ for some $g(X) \in \mathbb{Z}[X]$. This implies that $\deg(h(X)) = 0$, so $h(X) = h(1) \in \mathbb{Z}$. In particular, $2 = h(1)g(1)$ and hence $h(1) \in \{-1, 1, 2, 2\}$. Since $I \neq \mathbb{Z}[X]$, it follows that $h(X) = h(1) \notin \{-1, 1\}$. Now $X = h(X)f(X)$ for some $f(X) \in \mathbb{Z}[X]$. In particular, $\deg(f(X)) = 1$, so we may assume that $f(X) = a_0 + a_1X$ for $a_0, a_1 \in \mathbb{Z}$ and $a_1 \neq 0$. It follows that

$$X = \pm 2f(X) = \pm 2(a_0 + a_1X)$$

and therefore $\pm 1/2 = a_1 \in \mathbb{Z}$, a contradiction.

Example 4.18. Let $R = \mathbb{Z}[\sqrt{-5}]$. The ideal $I = (2, 1 + \sqrt{-5})$ is not principal, so R is not principal.

We first note that $I \neq R$. If not, there exist $x, y, u, v \in \mathbb{Z}$ such that

$$1 = 2(x + y\sqrt{-5}) + (1 + \sqrt{-5})(u + v\sqrt{-5}) = (2x + u - 5v) + \sqrt{-5}(2y + u + v).$$

This implies that $1 = 2x + u - 5v$ and $0 = 2y + u + v$. These formulas imply that $1 = 2(x + y + u - 2v)$, a contradiction because $x + y + u - 2v \in \mathbb{Z}$.

Now assume that $I = (\alpha)$. Then $\alpha \mid 2$ and $\alpha \mid 1 + \sqrt{-5}$. Then $N(\alpha) \mid 4$ and $N(\alpha) \mid 6$, because $N(2) = 4$ and $N(1 + \sqrt{-5}) = 6$. Thus $N(\alpha) \in \{1, 2\}$. If we write $\alpha = a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$, then it follows that $a^2 + 5b^2 = N(\alpha) = 1$ and hence $\alpha \in \mathcal{U}(R)$. Therefore $I = R$, a contradiction.

Sometimes primes and irreducibles coincide.

Proposition 4.19. Let R be a principal domain and $x \in R$. Then x is irreducible if and only if x is prime.

Proof. We only need to prove that if x is irreducible, then x is prime. Let us assume that $x \mid yz$. Let $I = (x, y)$. Since R is principal, $I = (a)$ for some $a \in R$. In particular, $x = ab$ for some $b \in R$. Since x is irreducible, $a \in \mathcal{U}(R)$ or $b \in \mathcal{U}(R)$. If $a \in \mathcal{U}(R)$, then $I = R$ and hence $1 = xr + ys$ for some $r, s \in R$. Thus

$$z = z1 = z(xr + ys) = zxr + zys$$

and therefore $x \mid z$. If $b \in \mathcal{U}(R)$, then x and a are associate, in R . Thus $I = (x) = (a)$ and hence $xt = y$ for some $t \in R$, that is $x \mid y$. \square

In the integers primes and irreducible coincide. This happens because \mathbb{Z} is a principal domain.

Example 4.20. Since $\mathbb{Z}[\sqrt{-3}]$ have irreducible elements that are not prime, it follows that $\mathbb{Z}[\sqrt{-3}]$ is not a principal domain.

Definition 4.21. Let R be an integral domain. We say that R is an **euclidean domain** if there exists a map $\varphi: R \setminus \{0\} \rightarrow \mathbb{N}_0$ such that for every $x, y \in R$ with $y \neq 0$ there exist $q, r \in R$ such that $x = qy + r$, where $r = 0$ or $\varphi(r) < \varphi(y)$.

Note that in the definition of euclidean domains we do not ask for uniqueness. In fact, we will meet important examples of euclidean domains where uniqueness in the division algorithm is not achieved.

Chapter 5

Zorn 's lemma

Chapter 6

Some solutions

Rings and ideals

Chinese remainder theorem

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