NC(SMC)^2 2025 Team Round Answers

Jett Mu

May 10th, 2025

1. If $4\spadesuit + 3\heartsuit = 13$ and $3\spadesuit + \heartsuit = 6$, find $\spadesuit + \heartsuit$.

   Answer:

$$4\spadesuit + 3\heartsuit = 13$$
$$3\spadesuit + \heartsuit = 6$$

$$2(4\spadesuit + 3\heartsuit) - (3\spadesuit + \heartsuit) = 2 \cdot 13 - 6$$
$$5\spadesuit + 5\heartsuit = 20$$
$$\spadesuit + \heartsuit = \boxed{4}$$

2. Trevor writes a 7-digit number, but his mischievous little brother erases one of the digits, so it appears as 138_943. Given that Trevor's original number was divisible by 9, what was the digit his brother erased?

   Answer:
   If the sum of the digits is a multiple of 9, the number is also a multiple of 9. Let the unknown digit be a.

$$1 + 3 + 8 + a + 9 + 4 + 3 = 28 + a$$

   The next multiple of 9 after 28 is 36.

$$a = 36 - 28 = \boxed{8}$$

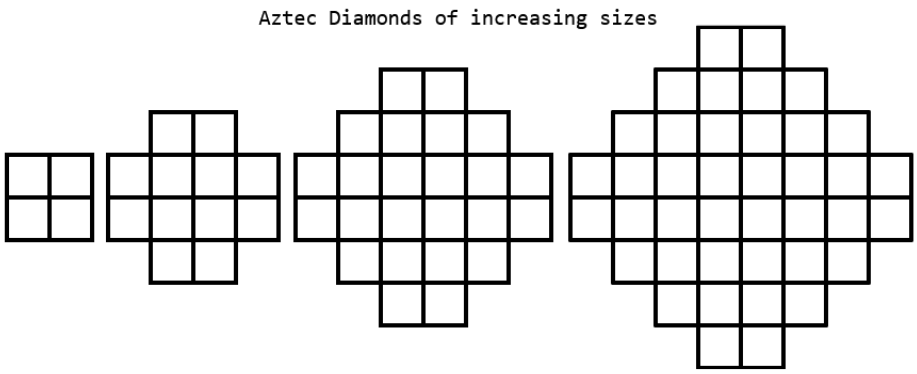
3. For a potluck, Kush bought 225 boxes of chicken nuggets, packed in either 8 nuggets or 11 nuggets, totaling 2025 nuggets. How many boxes of 8 nuggets did Kush buy?

Answer:
Let E be the number of eight nugget boxes, and L be the number of 'leven nugget boxes.

$$E + L = 225$$
$$8E + 11L = 2025$$

$$8E + 11L - 8(E + L) = 2025 - 225 \cdot 8$$
$$3L = 225$$
$$L = 75$$
$$E = 225 - L = \boxed{150}$$

4. A builder initially tiled a floor in the shape of an Aztec diamond, as shown below. Realizing that the tiling was too small, he added an additional layer of tiles around the entire shape. Now, there are 144 total tiles. How many tiles were there before the extra layer was added?



Answer:
Let's count from index 1, or the 2x2 square.

$$\text{Index} \Rightarrow \# \text{ of tiles}$$
$$1 \Rightarrow 4$$
$$2 \Rightarrow 4 + 8 = 12$$
$$3 \Rightarrow 12 + 12 = 24$$
$$4 \Rightarrow 24 + 16 = 40$$

We see that the change in number of tiles increases by 4 each time. You can proceed on with this pattern until you get 144.

$$5 \Rightarrow 40 + 20 = 60$$
$$6 \Rightarrow 60 + 24 = 84$$
$$7 \Rightarrow 84 + 28 = 112$$
$$8 \Rightarrow \boxed{112} + 32 = 144$$

5. Trevor plays a game by tossing a coin; if heads, she earns 10 dollars - if tails, she loses 12 dollars. After 15 rounds of the game, what's the expected change in dollars?

   Answer:
   Expected value of one game: $\frac{1}{2} \cdot \$10 + \frac{1}{2} \cdot (-\$12) = \$5 - \$6 = -1$
   After 15 games: $-\$1 \cdot 15 = \$\boxed{-15}$

6. How many integers satisfy the following?

$$\frac{1}{3} < \frac{x}{55} < \frac{6}{7}$$

   Answer:

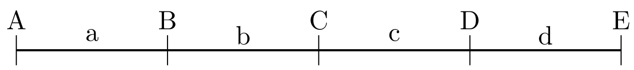
$$\begin{aligned} \frac{1}{3} &< \frac{x}{55} \\ x &> \frac{55}{3} > \frac{54}{3} = 18 \end{aligned}$$

$$\begin{aligned} \frac{x}{55} &< \frac{6}{7} \\ x &< \frac{330}{7} < \frac{336}{7} = 48 \end{aligned}$$

$18 < x < 48$. There are $\boxed{29}$ integers

7. Line l contains points A, B, C, D, E in this exact order from left to right. The sum of the distances from A to the four other points is 92, while the sum of the distances from B to the four other points is 50. Find the length of AB.

   Answer:



$$\begin{aligned} AB + AC + AD + AE &= a + (a+b) + (a+b+c) + (a+b+c+d) = 4a + 3b + 2c + d = 92 \\ BA + BC + BD + BE &= a + b + (b+c) + (b+c+d) = a + 3b + 2c + d = 50 \end{aligned}$$

Subtract! $3a = 42$, so $a = \boxed{14}$

8. A rectangular prism box has faces of area 54, 88, and 132. What is the volume of this box?

Answer:
Let a, b, c be the side lengths of the box. Without loss of generality:

$$\begin{aligned} ab &= 54 = 2 \cdot 3^3 \\ ac &= 88 = 2^3 \cdot 11 \\ bc &= 132 = 2^2 \cdot 3 \cdot 11 \end{aligned}$$

$$\begin{aligned} ab \cdot ac \cdot bc &= 2^6 \cdot 3^4 \cdot 11^2 \\ (abc)^2 &= (2^3 \cdot 3^2 \cdot 11)^2 \\ abc &= 8 \cdot 9 \cdot 11 = \boxed{792} \end{aligned}$$

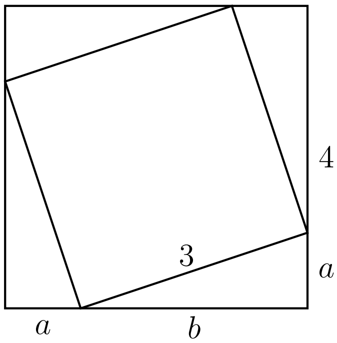
9. At the NCSSM dorms, each student's room number is printed on their key fob in a square grid format, ranging from 01 to 99. Because of this, some pairs of room numbers appear identical when rotated 180 deg (for example, 01 and 10). One day, Grace picks up a lost key fob. With how many room numbers will she be unable to determine the correct room number?

Answer:
Define an *ambiguous* digit to be a digit which resembles itself or another digit when rotated 180 deg. For the room number to be unidentifiable, both digits must be *ambiguous*, with the exception of a few cases where both digits are the same.

From careful deliberations, these seven digits $1, 2, 5, 6, 8, 9, 0$ are *ambiguous*, with a total of $7 \cdot 7$ room numbers. However, if the two digits are rotationally symmetrical, either rotation orientation displays the correct fob. So we must subtract out $11, 22, 55, 69, 88, 96, 00$ from the total, giving us $49 - 7 = \boxed{42}$ *ambiguous* room numbers.

10. A square X with area 9 is drawn in a square Y with area 16 such that each vertex of X lies on a side of Y. Each side of Y is divided by a vertex of X into two segments, one of length a and the other of length b. What is the value of $\frac{7}{a} + \frac{7}{b}$?

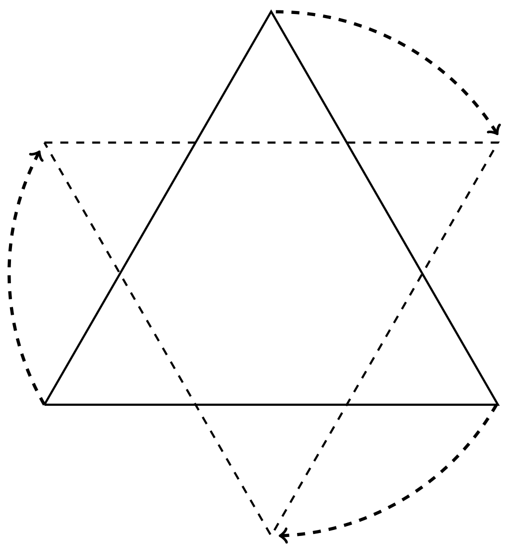
Answer:



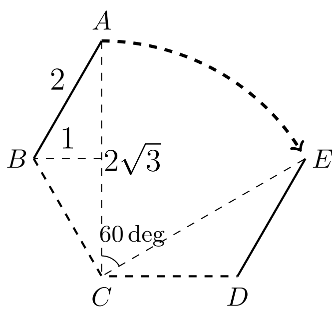
$a + b = 4$. By the Pythagorean theorem, $a^2 + b^2 = 9$.

$$(a+b)^2 = a^2 + 2ab + b^2 = 9 + 2ab = 16$$
$$2ab = 7 \Rightarrow ab = \frac{7}{2}$$
$$7\left(\frac{1}{a} + \frac{1}{b}\right) = 7 \cdot \frac{a+b}{ab} = 7 \cdot 4 \cdot \frac{2}{7} = \boxed{8}$$

11. An equilateral triangle of side length 6 is placed in sand, twisted 60 degrees about its center, and removed to reveal a gear pattern. If the area of this pattern can be written as $a\pi + \sqrt{b}$ for integers a and b, what is $a + b$?



Answer:
Let's split this shape into equal thirds by splitting its center into three angles of 120 deg each, isolating each protruding part. We'll look at only one of the thirds.



The rotated triangle divides each side into thirds, so $AB = DE = \frac{6}{3} = 2$. Through angle chasing, we have $\angle ACE = 60\,\text{deg}$.
$[ACE] = \frac{60\,\text{deg}}{360\,\text{deg}} \cdot (2\sqrt{3})^2\pi = \frac{1}{6} \cdot 12\pi = 2\pi$
$[ABC] = 2\sqrt{3} \cdot 1 \cdot \frac{1}{2} = \sqrt{3}$
$[CDE] = [ABC] = \sqrt{3}$
Total area: $[ACE] + [ABC] + [CDE] = 2\pi + 2\sqrt{3} = 2\pi + 2\sqrt{3}$

The combined area of all 3 equal parts:

$$3(2\pi + 2\sqrt{3}) = 6\pi + 6\sqrt{3} = 6\pi + \sqrt{108}$$

$$a + b = 6 + 108 = \boxed{114}$$

12. Prismarine is made with a $1:5$ blue-to-green ratio, and Teal with a $4:5$ blue-to-green ratio. Dark Cerulean is a $2:3$ mix of Prismarine and Teal. If Megan has 14 gallons of blue paint and wants to create Dark Cerulean using only blue and green paint, how many gallons of green should she use?

    Answer: 1 gallon of Prismarine is made with $\frac{1}{6}$ blue and $\frac{5}{6}$ green gallons. 1 gallons of Teal is made with $\frac{4}{9}$ blue and $\frac{5}{9}$ green gallons.
    1 gallons of Dark Cerulean is made with $\frac{2}{5}$ Prismarine and $\frac{3}{5}$ Teal gallons, or $\frac{1}{6}\cdot\frac{2}{5}+\frac{4}{9}\cdot\frac{3}{5}=\frac{1}{3}$ blue and $\frac{5}{6}\cdot\frac{2}{5}+\frac{5}{9}\cdot\frac{3}{5}=\frac{2}{3}$ green gallons. To make Dark Cerulean, the ratio of paint required is a $\frac{1}{3}:\frac{2}{3}=1:2$ blue-to-green ratio.

    $$14 \text{ gallons of blue} \cdot \frac{2 \text{ gallons of green}}{1 \text{ gallons of blue}} = \boxed{28} \text{ gallons of green}$$

13. Two unfair coins are each weighted differently such that:

    - The chance of both showing heads equals the chance of both showing tails
    - The chance of both showing heads is also three times as likely as getting one head and one tail.

    After flipping both coins 84 times, how many times can you expect both coins to show heads?

    Answer:
    Let p_1 and p_2 be the probabilities of the two coins landing heads, respectively.
    The chance of both heads = chance of both tails:

    $$\begin{aligned} p_1p_2 &= (1-p_1)(1-p_2) \\ p_1p_2 &= 1-p_1-p_2+p_1p_2 \\ p_1+p_2 &= 1 \end{aligned}$$

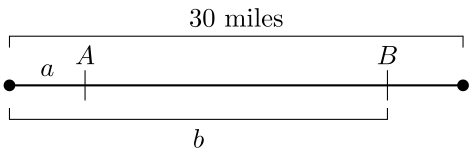
    The chance of both showing heads = 3·chance of getting one head one tail:

    $$\begin{aligned} p_1p_2 &= 3\big(p_1(1-p_2)+(1-p_1)p_2\big) \\ p_1p_2 &= 3p_1-3p_1p_2+3p_2-3p_1p_3 \\ 7p_1p_2 &= 3(p_1+p_2)=3 \Rightarrow p_1p_2=\frac{3}{7} \end{aligned}$$

    Therefore, our of 84 trials of flipping both coins, the expected number of times both show heads is $84\cdot p_1p_2 = 84\cdot\frac{3}{7}=\boxed{36}$

14. Jay can't decide between running 30 miles to school or staying home playing Brawl Stars. He runs $\frac{4}{5}$ of the way to school, turns back and runs $\frac{4}{5}$ of the way home, then repeats this cycle forever. Eventually, he runs back and forth between two points, A and B. Find the distance between A and B.

Answer:



Since the ratio of distances are the same, the distances are symmetrical: Home to $A = B$ to school $= a$. So, $b + a = 30$.
When Jay starts at point B, he runs $\frac{4}{5}b$ to point A. So, $b - \frac{4}{5}b = \frac{1}{5}b = a$.
Combine the first equation with the second:

$$\frac{1}{5}b + b = \frac{6}{5}b = 30 \text{ miles} \Rightarrow b = 25 \text{ miles}$$

$$b - a = b - \frac{1}{5}b = 25 \text{ miles} - 5 \text{ miles} = \boxed{20} \text{ miles}$$

15. Grace and Tatiana live 3 time zones apart. They both have 12 hour digital clocks that read $HH : MM$, which can be put together as $HHMM$ to form one 3 or 4 digit integer. How many minutes in a 24 hour period will both of their times form a perfect square? ($8:00$ PM is read the same as $8:00$ AM)

Answer:
3 time zones means one person is 3 hours ahead. Let N_a be the clock time head and N_b be the clock time behind. There are two cases:
Case 1: $N_a - N_b = 300$
Since both are square numbers, let N_a and N_b be $(x+n)^2$ and x^2 respectively, for positive integers x and n.

$$\begin{aligned}(x+n)^2 - x^2 &= x^2 + 2nx + n^2 - x^2 \\ &= 2nx + n^2 = 300\end{aligned}$$

n must be a factor of 300 and $n^2 < 300$. Since both $2nx$ and 300 are even, n^2 and thus n must be even as well. So, $n \in \{2, 4, 6, 10, 12\}$. We will test these cases.

$$\begin{aligned}
&\text{For } n = 2:\\
&2(2)x + 2^2 = 4x + 4 = 300 \Rightarrow x = 74\\
&\text{For } n = 4:\\
&2(4)x + 4^2 = 8x + 16 = 300 \Rightarrow x = 35.5\\
&\text{For } n = 6:\\
&2(6)x + 6^2 = 12x + 36 = 300 \Rightarrow x = 22\\
&\text{For } n = 10:\\
&2(10)x + 10^2 = 20x + 100 = 300 \Rightarrow x = 10\\
&\text{For } n = 12:\\
&2(12)x + 12^2 = 24x + 144 = 300 \Rightarrow x = 6.5
\end{aligned}$$

Discard any non-integer x's. Discard $x = 74$ as well, because $74^2 = 5476$ is not a valid time. So, this case has 2 solutions: $N_b = 22^2 = 0484$ and $N_b = 10^2 = 0100$.
Case 2: $N_b - N_a = 900$
In this case $1259 > N_b > 1000$, meaning that N_a wraps around the clock cycle, now less than N_b by 900. $N_a = x^2$, $N_b = (x + n)^2$

$$(x + n)^2 - x^2 = 2nx + n^2 = 900$$

n must be a factor of 900, $n^2 < 900$, and n must be even. So, $n \in \{2, 4, 6, 10, 12, 18, 20\}$. We will test these cases.

$$\begin{aligned}
&\text{For } n = 2: x \text{ is definitely too big}\\
&\text{For } n = 4: x \text{ is definitely too big}\\
&\text{For } n = 6:\\
&2(6)x + 6^2 = 12x + 36 = 900 \Rightarrow x = 72\\
&\text{For } n = 10:\\
&2(10)x + 10^2 = 20x + 100 = 900 \Rightarrow x = 40\\
&\text{For } n = 12:\\
&2(12)x + 12^2 = 24x + 144 = 900 \Rightarrow x = 31.5\\
&\text{For } n = 18:\\
&2(12)x + 12^2 = 36x + 324 = 900 \Rightarrow x = 16\\
&\text{For } n = 20:\\
&2(12)x + 12^2 = 40x + 400 = 900 \Rightarrow x = 12.5
\end{aligned}$$

$x^2 < 1259$, so $x < 36$. Only one of these numbers works: $N_a = 16^2 = 256$.

So we have 3 solutions for each AM and PM. In total, $\boxed{6}$ solutions:

$$\begin{gathered} N_b||N_a \\ 1:00AM||4:00AM \\ 4:84AM||7:84AM \\ 11:56AM||2:56PM \\ 1:00PM||4:00PM \\ 4:84PM||7:84PM \\ 11:56PM||2:56AM \end{gathered}$$

16. Let A, B, C be distinct non-zero digits $(1, 2, \dots 9)$, where AB refers to the 2-digit number with tens digit A and units digit B. Henry writes the equation $AB \cdot CA + BCA$. Yrneh writes the same equation backwards: $ACB + AC \cdot BA$. If both of these equations evaluate to the same number, what is the largest possible value of ABC?

Answer:
These digits can be represented as a sum of their parts:

$$\begin{aligned} AB \cdot CA + BCA &= (10A + B)(10C + A) + 100B + 10C + A \\ ABC + AC \cdot BA &= 100A + 10B + C + (10A + C)(10B + A) \end{aligned}$$

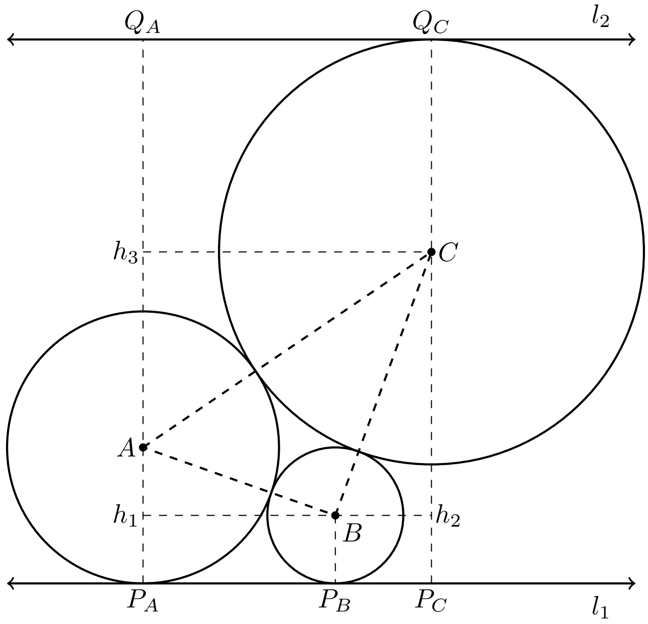
$$\begin{aligned} AB \cdot CA + BCA &= ABC + AC \cdot BA \\ 100AC + 10A^2 + 10BC + AB + 100B + 10C + A &= 100A + 10B + C + 100AB + 10A^2 + 10BC + AC \\ 99AB - 99AC + 99A - 99B &= 0 \\ AB - AC + A &= B \\ A &= \frac{B}{B - C + 1} \end{aligned}$$

We need to maximize A, so we need to minimize $B - C + 1$.
$B - C + 1$ cannot be 1, because that would mean $A = B$, which is not allowed.
Let $B - C + 1 = 2$, so $A = \frac{B}{2}$. The largest value B can be is 8, so $A = 4$.

$B - C + 1 = 8 - C + 1 = 2 \Rightarrow C = 7$
With $(A, B, C) = (4, 8, 7)$, $ABC = \boxed{487}$

17. Circle A of radius 2 is tangent to line l_1. Circle B of radius 1 lies on the same side of l_1 as circle A and is externally tangent to both circle A and l_1. Circle C is externally tangent to both circles A and B. Line l_2, parallel to l_1, is tangent to circle C such that the distance between l_1 and l_2 is 8. If the radius of circle C can be expressed as $\frac{a}{b}$, where a and b are relatively prime integers, what is $10a+b$?

Answer:



Let r be the radius of circle C. $AC = r+2, BC = r+1, AB = 3$. Note that $h_1h_2Ch_3$ forms a rectangle, so the goal is to find $h_1h_2 = h_3C$ in two different ways.

Finding h_1h_2 :

$h_1B = \sqrt{AB^2 - (AP_A - BP_B)^2} = \sqrt{3^2 - 1^2} = \sqrt{8}$

$Q_CP_C = P_Ch_2 + h_2C + CQ_C = 1 + h_2C + r = 8 \Rightarrow h_2C = 7 - r$

$Bh_2 = \sqrt{BC^2 - Ch_2^2} = \sqrt{(1+r)^2 - (7-r)^2} = \sqrt{16r - 48}$

So, $h_1h_2 = h_1B + Bh_2 = \sqrt{8} + \sqrt{16r-48}$

Finding h_3C :

$Q_AP_A = P_AA + Ah_3 + h_3Q_A = 2 + Ah_3 + r = 8 \Rightarrow Ah_3 = 6 - r$

$h_3C = \sqrt{AC^2 - Ah_3^2} = \sqrt{(2+r)^2 - (6-r)^2} = \sqrt{16r - 32}$

$$\begin{aligned}
h_1h_2 &= h_3C \\
\sqrt{8} + \sqrt{16r-48} &= \sqrt{16r-32} \\
8 + 2\sqrt{8(16r-48)} + 16r - 48 &= 16r - 32 \\
\sqrt{128r - 384} &= 4 \Rightarrow r = \frac{400}{128} = \frac{25}{8} \\
10a + b &= 250 + 8 = \boxed{258}
\end{aligned}$$

18. A fair 6-sided die is repeatedly rolled until an odd number appears. The probability that every even number appears exactly once before the first odd number appears can be written as $\frac{1}{p}$. What is p?

Answer:
Consider this to be a three probability problems combined into one: the probability of rolling the first even number before the first odd number (P_1), the probability of rolling the second even number before the first odd number given that you had already rolled one even number (P_2), and the probability of rolling the third even number before the first odd number given that you had already rolled two even numbers (P_3). So, the total probability starting from no even numbers is $P_1 \cdot P_2 \cdot P_3$.

For P_1: If you didn't roll an odd number, you definitely rolled an even number. So, $P_1 = \frac{1}{2}$, or the probability you didn't roll an odd number on the first roll.

For P_2: The chance that you roll a new even number at any moment is $\frac{2}{6} = \frac{1}{3}$. Note that if you roll the same even number that you had already rolled prior (with a probability of $\frac{1}{6}$, the probability of rolling a new even number in this new moment is still P_2. Thus:

$$P_2 = \frac{1}{3} + \frac{1}{6}P_2 \Rightarrow P_2 = \frac{2}{5}$$

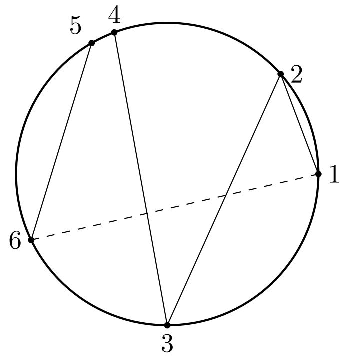
For P_3: From the revelation in P_2, this case is very similar. The chance that you roll a new even number at any moment is $\frac{1}{6}$. If you roll one of the two same even numbers that you had already rolled prior (with a probability of $\frac{2}{6} = \frac{1}{3}$), the probability of rolling a new even number in this new moment is still P_3. Thus:

$$P_3 = \frac{1}{6} + \frac{1}{3}P_2 \Rightarrow P_2 = \frac{1}{4}$$

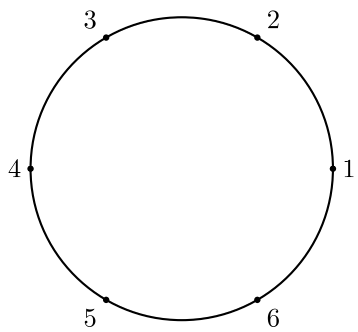
So, $P_{total} = P_1P_2P_3 = \frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{\boxed{20}}$

19. Ian picks a random point on the circumference of a circle and, without lifting his pencil, creates a path of straight lines to 5 more random points on the circumference. Given that no lines had overlapped so far, the probability that drawing a line from the last point to the first crosses no previous lines can be represented as $1/a$. What is a?

Example of invalid drawing

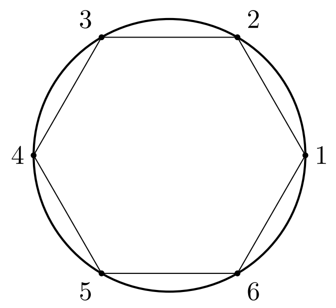


Answer:
Any arrangement of 6 points is equally likely. After creating a path, moving any point to another spot between its neighboring points doesn't change which lines are crossed and which lines are uncrossed. So, without loss of generality, Ian chooses 6 points equally spaced.

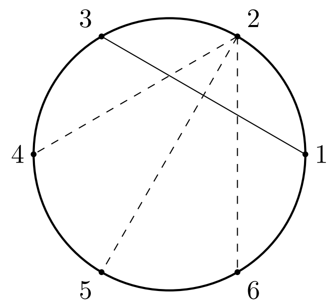


Among these 6 points, any order of path is equally likely. So, we have transformed this problem to finding the ratio of 6 uncrossed lines out of 6 to 5 uncrossed lines out of 5. Without loss of generality, the path always starts at point 1.

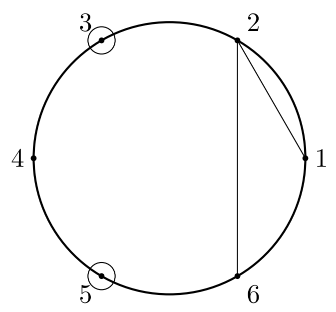
There are 2 possible paths such that all 6 lines don't cross: regular hexagon going clockwise and regular hexagon going counterclockwise. Let's consider the number of 5 uncrossed lines.



Consider if the path "skips" an unused point, such as drawing $1 \to 3$ and skipping the unused point 2. This divides the group of unused points in two separate groups. Drawing a line from two point between these two groups would necessarily cross a line. Therefore, we must avoid skipping points.



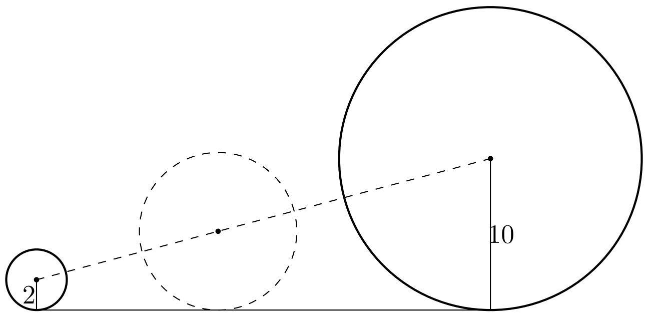
To avoid line crossings, we choose 1 of the 2 adjacent points to the mass of already-connected points. For example, if points $1, 2, 6$ are used, the two available points are 3 and 5.



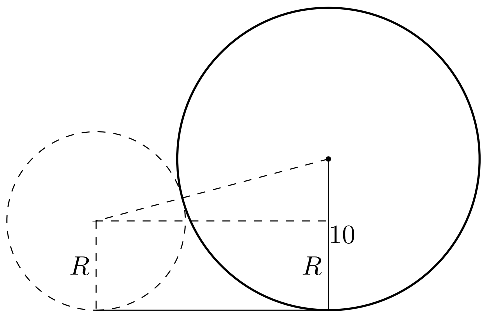
There are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 16$ ways to choose a path of 5 uncrossed lines this way.
Therefore, the ratio we are looking for is $\frac{2}{16} = \frac{1}{\boxed{8}}$

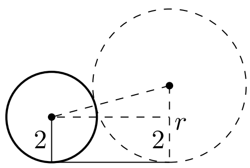
20. Spencer draws two dots on his floor, spaced 27 units apart, and places two spheres of radii 2 and 10 on top of the dots. He then places a third sphere on the floor between the two spheres such that the three centers are collinear. What's the product of the largest and smallest possible radius of the third sphere?

Answer:
Below is a 2-dimensional cross section containing the centers of the spheres.



Shown below are diagrams for the largest possible radius R and the smallest possible radius r.





Since all three centers are collinear, the slope from center to center remains constant. Thus, right triangles formed in these diagrams are similar. The ratio of the hypotenuse to the shortest leg for both cases are equal:

$$\frac{R+10}{10-R} = \frac{r+2}{r-2}$$

$$Rr - 2R + 10r - 20 = -Rr - 2R + 10r + 20$$

$$2Rr = 40 \Rightarrow Rr = \boxed{20}$$

The distance between the spheres didn't even matter.

NC(SMC)^2 2025 Speed Round Answers

Jett Mu

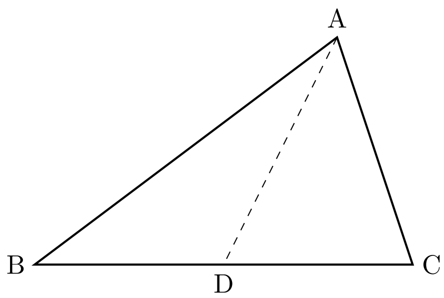
May 10th, 2025

1. A bar of chocolate costs 14 cents. Ignoring taxes, how many bars of chocolate can you buy with a two-dollar bill?

   Answer:
   2 dollars = 200 cents.
   200 cents $\cdot \frac{\text{1 chocolate bar}}{\text{14 cents}} = 14.2\ldots$ chocolate bars
   We can buy $\boxed{14}$ chocolate bars.

2. On triangle ABC, point D is on BC such that $DB = DC$. If the area of ABD is 24, find the area of ABC.

   Answer:



   Let h be the length of the altitude from A to BC.

$$\begin{aligned}[ABD] &= \frac{1}{2} \cdot h \cdot \overline{BD} \\ [ABC] &= \frac{1}{2} \cdot h \cdot \overline{BC}. \\ &= \frac{1}{2} \cdot h \cdot (\overline{BD} \cdot 2) \\ &= [ABD] \cdot 2 \\ &= \boxed{48}\end{aligned}$$

3. Five consecutive integers sum up to 85. What's the greatest of these five numbers?

Answer:
Let n be the largest of the five. The numbers are: $n-4, n-3, n-2, n-1, n$.
The sum is: $5n - 10 = 85$, $n = \boxed{19}$.

4. If $f(x) = x^3 + 3^x$, what is the value of $f(3)$?

   Answer:
   $f(3) = 3^3 + 3^3 = 3 \cdot 3 \cdot 3 + 3 \cdot 3 \cdot 3 = 27 + 27 = \boxed{54}$

5. A leaky faucet drips 7 drops every 10 seconds. At this rate, how many drops will the faucet drip in 3 minutes?

   Answer:
   $\frac{7 \text{ drops}}{10 \text{ sec}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} \cdot 3 \text{ min} = \boxed{126}$ drops

6. Let $x@y = xy - x - y + 1$. Compute $((3@2) - 1)@(4@2)$.

   Answer:
   $3@2 = 3 \cdot 2 - 3 - 2 + 1 = 2$
   $4@2 = 4 \cdot 2 - 4 - 2 + 1 = 3$
   $((3@2) - 1)@(4@2) = (2 - 1)@3 = 1 \cdot 3 - 1 - 3 + 1 = \boxed{0}$

7. My square has the same perimeter as a right triangle with a leg of length 6 and a hypotenuse of length 10. What's the area of my square?

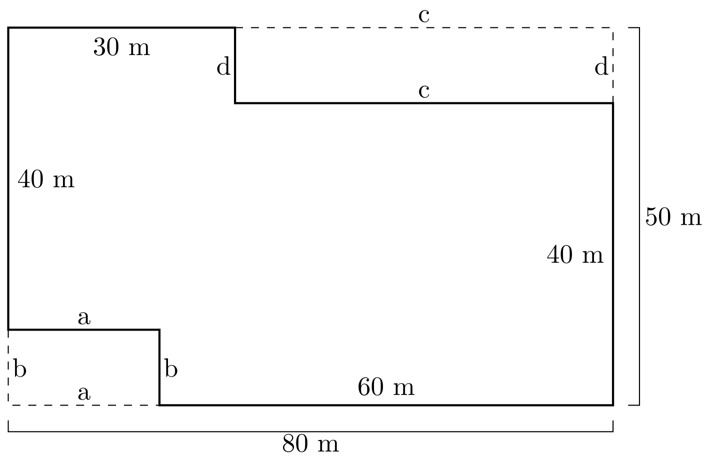
   Answer:
   Perimeter of triangle: $10 + 6 + \sqrt{10^2 - 6^2} = 16 + 8 = 24$
   Perimeter of square: $4s = 24$, where s is the side length
   Area of square: $s^2 = \left(\frac{24}{4}\right)^2 = \boxed{36}$

8. Every meal, Anushka eats two slices of pie and splits another slice equally with her cat. After many meals, they finish an 18-slice pie. How many slices did Anushka have in total?

   Answer:
   Every meal, Anushka and her cat finish 3 slices. So, there are $\frac{18}{3} = 6$ total meals.
   Each meal, Anushka finishes $2 + \frac{1}{2} = 2.5$ slices.
   Total slices $= 2.5 \cdot 6 = \boxed{15}$

9. This is the floorplan of my summer house in Fairvale, Alaska. I go there to fish and birdwatch. What is the area of the floor, in m^2?

Answer:



$a + 60$ m$= 80$ m$\Rightarrow a = 20$ m
$b + 40$ m$= 50$ m$\Rightarrow b = 10$ m
$c + 30$ m$= 80$ m$\Rightarrow c = 50$ m
$d + 40$ m$= 50$ m$\Rightarrow d = 10$ m
Area of floor:

$$80 \text{ m} \cdot 50 \text{ m} - a \cdot b - c \cdot d = 4000 \text{ m}^2 - 200 \text{ m}^2 - 500 \text{ m}^2 = \boxed{3300} \text{m}^2$$

10. The North Carolina School of Math and Science had 1820 applicants this year, which was a 30% increase from last year. How many applicants were there last year?

Answer:

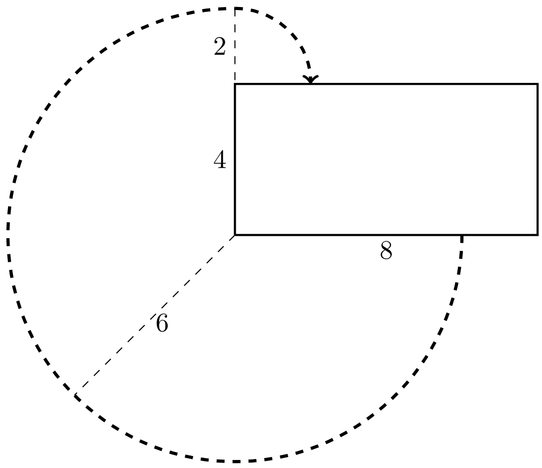
$$\begin{aligned} \text{Amount Last Year } \cdot (1 + 30\%) &= 1820 \\ \text{Amount Last Year } &= 1820 \cdot \frac{10}{13} \\ &= \boxed{1400} \end{aligned}$$

11. The sum of 2 positive integers a and b is 9. If the sum of their squares is 41, what is their absolute difference between a and b?

Answer:
Let $k = a - b$.
$(a + b)^2 + (a - b)^2 = 9^2 + k^2 = 81 + k^2$
$(a + b)^2 + (a - b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2a^2 + 2b^2 = 2(41)$
$81 + k^2 = 82$
$k^2 = 82 - 81 = 1$, $|k| = \boxed{1}$

12. My dog is outside and tied to a 6 m long leash, which is attached to a corner of a 4 m by 8 m shed. If the area of the space the dog can travel is $X\pi$ m^2, what is X?

Answer:
Stretching the leash as tight as it can, the dog can trace a three-quarter arc of radius 6, and



wraps around the shorter side to trace a quarter arc of radius 2.
Since the dog can travel anywhere within this traced region, the total area is:

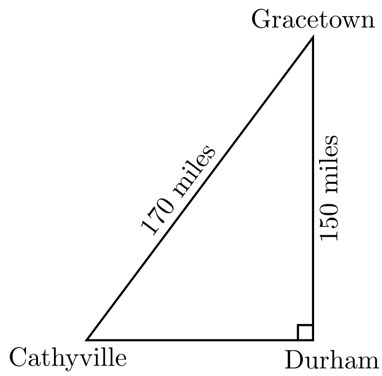
$$\frac{3}{4} \cdot 6^2 \cdot \pi + \frac{1}{4} \cdot 2^2 \cdot \pi = 27\pi + \pi = \boxed{28}\pi$$

13. In 2010, 40 apples costed 10 dollars. Today, a dozen apples cost 36 dollars. By how many cents has the price of one apple increased since 2010?

Answer:
One apple in 2010: $\frac{10 \text{ dollars}}{40 \text{ apples}} = 0.25$ dollars $= 25$ cents. One apple today: $\frac{36 \text{ dollars}}{12 \text{ apples}} =$ 3 dollars $=$ 300 cents. $300 - 25 = \boxed{275}$ cents.

14. In the mystical country of NCSSMland, Cathyville is 170 miles from Gracetown, which is 150 miles from Durham. The three towns form a right angle at Durham. If Brandon drives in a straight line from Cathyville to Durham at 25 miles per hour, how long is his journey in minutes?

Answer:



Cathyville to Durham $= \sqrt{170^2 - 150^2} = 10\sqrt{17^2 - 15^2} = 80$ miles
80 miles $\cdot \frac{1 \text{ hour}}{25 \text{ miles}} \cdot \frac{60 \text{ min}}{1 \text{ hour}} = \boxed{192}$ minutes

15. There are 100 people attending a water-bottle conference who each enjoy Dasani, Aquafina, or both. If 74 people enjoy Dasani, and 64 people enjoy Aquafina, how many people enjoy both?

Answer:
Let x be how many enjoy both.
Amount who only like Dasani: $74 - x$, Amount who only like Aquafina: $64 - x$.
Total: $(74 - x) + (64 - x) + x = 100$
$138 - x = 100$, $x = \boxed{38}$

16. If $81x = 9^7$ and $3^y = x$, compute y.

Answer:

$$\begin{aligned} 81x &= 9^7 \\ 3^4 x &= (3^2)^7 \\ x &= \frac{3^{14}}{3^4} = 3^{10} \\ 3^y &= x = 3^{10}, \\ y &= \boxed{10} \end{aligned}$$

17. For how many positive integers $x < 200$ is $2025x$ a perfect square?

Answer:
$2025 = 45^2$. So, for $2025x$ to be square, x must also be square.
$14^2 = 196 < 200, 15^2 = 225 > 200$.
So, $x = 1^2, 2^2, 3^2 \dots 14^2$, $\boxed{14}$ numbers

18. What is the least possible value of $x^2 + 8x + 35$ for real values of x??

Answer:
$x^2 + 8x + 35 = x^2 + 8x + 16 + 19 = (x + 4)^2 + 19$
The minimum value of $(x + 4)^2$ is 0.
So, the minimum value of $(x + 4)^2 + 19$ is $0 + 19 = \boxed{19}$

19. Both Ian and Jett have a favorite whole number that has 8 distinct positive factors. Given that Jett's number is larger than Ian's, what's the smallest possible value of Jett's favorite number?

Answer:
This is asking for the second smallest number with 8 positive factors.
For a number to have 8 factors, it must either be 3 prime numbers multiplied together, 2 prime numbers with one multiplied 3 times and the other one, or 1 prime number multiplied 7 times.
Let's consider the smallests for each case.
3 primes: $2 \cdot 3 \cdot 5 = 30$, next $2 \cdot 3 \cdot 7 = 42$
2 primes: $2^3 \cdot 3 = 24$, next $2^3 \cdot 5 = 40$
1 prime: $2^7 = 128$
The smallest is 24, followed by $\boxed{30}$

20. I counted one pet bacterium at noon on Monday. If this bacteria colony doubles in number every 15 hours, on what day of the week will I count exactly 2048 bacteria?
(Monday = 1, Tuesday = 2, ..., Sunday = 7)

Answer:
$2048 = 2^{11}$, so it has doubled 11 times.
$11 \cdot 15$ hours $= 165$ hours $\cdot \frac{1 \text{ day}}{24 \text{ hours}} = 6$ days $+ 21$ hours have passed.
6 days after Monday, we are on Sunday, at noon.
21 hours later, the next day rolls over and we are on Monday$= \boxed{1}$.

21. There are 768 girls accepted into NCSSM, divided into 192 non-overlapping friend groups of 4 girls each. Each girl is randomly and independently assigned to one of four dorms: Bryan, Beall, Reynolds, or Royall. What is the expected number of friend groups in which all four girls are assigned to the same dorm?

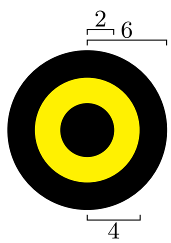
Answer:
Consider just 1 friend group of 4 girls.
$P(\text{All in Bryan}) = (\frac{1}{4})^4$
$P(\text{All in Beall}) = (\frac{1}{4})^4$
$P(\text{All in Reynolds}) = (\frac{1}{4})^4$
$P(\text{All in Royall}) = (\frac{1}{4})^4$
$P(\text{All in same dorm}) = (\frac{1}{4})^4 \cdot 4 = (\frac{1}{4})^3 = \frac{1}{64}$
Expected number of friend groups in the same dorm $= 768 \cdot P(\text{All in same dorm}) = \frac{192}{64} = \boxed{3}$

22. Aaron's dying wish is to split his last 20 one-dollar bills among his 4 children. If each child receives at least 4 dollar bills, how many ways can Aaron split the cash?

Answer:
Aaron gives 4 dollars to each child. Now, the question is equivalent to if Aaron had only 4 one-dollar bills to split among his 4 children.
Imagine a list of 4 dollars and 3 bars. The 4 sections created by the 3 bars represents the amount of dollars Aaron gives to the corresponding child.
For example, $DD|D||D$ means the first kid gets 2 dollars, the second kid gets 1 dollar, the third kid gets 0 dollars, and the fourth kid gets 1 dollar.
Thus, there are $\frac{7!}{4!\cdot 3!} = \boxed{35}$ ways to split the cash.
If you didn't know stars and bars, straight up counting works fine too.

23. Avery's bee drawing consists of three circles of diameters 12, 8, and 4 centered at the same point. The inner circle and outermost ring are painted black while the middle ring is painted yellow. What is the ratio of the total area painted black to the total area painted yellow?

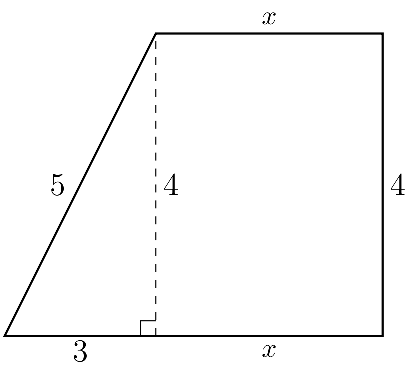
Answer:



Area of black regions: $6^2\pi - 4^2\pi + 2^2\pi = 24\pi$
Area of yellow region: $4^2\pi - 2^2\pi = 12\pi$
Ratio: $\frac{24\pi}{12\pi} = \boxed{2}$

24. A right trapezoid has nonparallel sides of lengths 4 and 5 and area 26. What is its perimeter?

Answer:



Let the length of the shorter base be x. With the Pythagorean theorem, the longer base is $x + 3$.
The area is thus: $\frac{(3+x)+x}{2} \cdot 4 = (3 + 2x) \cdot 2 = 26$. So, $x = 5$.
The perimeter is: $5 + x + 4 + x + 3 = \boxed{22}$

25. How many integer solutions (x, y) does $8x^2 = 9y^4$ have?

Answer:
$(0, 0)$ definitely works. Let's find non-zero cases.
Let $x = 2^n \cdot 3 \cdot N$, where n is a non-negative integer and N is an odd positive integer.

$$\begin{aligned}\frac{8}{9}x^2 &= \frac{2^3}{3^2}(2^n \cdot 3 \cdot N)^2 \\ &= 2^{3+2n}N^2 \\ &= y^4\end{aligned}$$

Since y is an integer and N^2 is not even, 2^{3+2n} must be a fourth power. So, $3+2n \equiv 0 \mod 4$. However, this isn't possible since $3 + 2n$ is always odd.
Thus, the only answer is $(0, 0)$: $\boxed{1}$

26. A square is drawn on top of a regular hexagon. What is the maximum possible number of intersections?

Answer:
Tracing along the hexagon, it must cross each side of the square exactly twice, once to enter and once to exit the square's interior. A square has 4 sides, so the maximum number of intersections is $4 \cdot 2 = \boxed{8}$

27. A pot contains a certain number of dumplings. When the dumplings are evenly divided among 7 people, 6 dumplings remain. When divided among 6 people, 5 dumplings remain. When divided among 5 people, 4 dumplings remain. What is the smallest possible number of dumplings in the pot?

Answer:
Let N be the number of dumplings in the pot.

$$\begin{aligned} N &\equiv -1 \mod 7 \\ N &\equiv -1 \mod 6 \\ N &\equiv -1 \mod 5 \end{aligned}$$

$N = -1$ clearly fits the criteria. However, N must be positive.
These remainders repeat every increase in $7 \cdot 6 \cdot 5 = 210$ dumplings. So, the smallest number of dumplings is $210 - 1 = \boxed{209}$

28. A *gnarly* number is a 5-digit integer that uses each digit 0-4 exactly once such that 0 cannot be the first digit. How many *gnarly* numbers are divisible by 55?

Answer:
Our 5-digit integer is $\underline{ABCDE}$. For this to be divisible by 55, it must be divisible by both 11 and 5.
Divisible by 5: $E = 0$
Divisible by 11: $A + C + E - B - D \mod 11 = 0\ldots$
Since the maximum of $|A+C+E-B-D| = |4+3+2-1-0| = 8$, $A+C+E-B-D = 0$.
$E = 0$ and $A + B + C + D + E = 10$, so $A + C = B + D = 5$.
We have $\boxed{8}$ cases:
$(A, B, C, D) = (4, 1, 3, 2), (1, 4, 3, 2), (4, 1, 2, 3), (1, 4, 2, 3)$ and flip each.

29. A frog stands at point $(0, 0)$. Every jump, it can only travel either one unit right or one unit up. However, the frog wants to avoid the toad at point $(4, 3)$ at all costs! How many different paths can the frog take to reach the point $(5, 5)$?

Answer:
We proceed with complementary counting. The frog does 5 right jumps and 5 up jumps to get to $(5, 5)$. So, the number of ways to get to $(5, 5)$ is the number of ways to arrange 5 rights and 5 ups: $\frac{10!}{5!5!} = 252$.
The number of paths that pass through $(4, 3)$ is the number of paths that get to $(4, 3)$ multiplied by the number of paths that start from $(4, 3)$ and arrive at $(5, 5)$. $\frac{7!}{4!3!} \cdot \frac{3!}{1!2!} = 35 \cdot 3 = 105$
$252 - 105 = \boxed{147}$

30. Let N be a positive integer such that, for all integer values of x, $N + \frac{45x+2025}{x^2+45x}$ is either negative, prime, or not an integer. How many values of N satisfy this condition?

Answer:
Let $S = \frac{45x+2025}{x^2+45x} = \frac{45(x+45)}{x(x+45)} = \frac{45}{x}$ for $x \neq -45$. When $x = 45$, S is undefined and, therefore, not an integer $\ddot\smile$. Otherwise, S is an integer if and only if x is a factor of 45.

The possible integer values of S are the factors of 45, without -1. For convenience, let's consider the positive values of S:

$$1, 3, 5, 9, 15, 45 \equiv 1, 0, 2, 0, 0, 0 \mod 3$$

Since each $1, 2, 0 \mod 3$ is represented, at least one integer $N + S$ will not be prime, given that $N + 1 \equiv 0 \mod 3$ and is not prime. This is only true when $N + 1 = 3$ or $N = 2$. So, we test all positive integer values of $2 + S$.

$$2 + 1, 2 + 3, 2 + 5, 2 + 9, 2 + 15, 2 + 45 = 3, 5, 7, 11, 17, 47$$

Remember, we don't consider $S = -1$ because $x = -45$ makes S not an integer. Miraculously, they are all prime. So, our $\boxed{1}$ answer is 2.