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1 Team Round Problems

Problem 1
Find the digit A such that the seven digit number $197121A$ is divisible by 7.

Solution: Write the number instead as $1971210 + A = 1960000 + 11210 + A = 1960000 + 11200 + (10 + A)$. Clearly, the first two terms here are divisible by 7 ($7 \cdot 280000$ and $7 \cdot 1600$ respectively) and so we need $(10 + A)$ to be divisible by 7, meaning $A = \boxed{4}$. It is easily checked that $\frac{1971214}{7} = 281602$.

Problem 2
How many axes of rotational symmetry does the type of quadrilateral that has equal diagonals and has two pairs of congruent, adjacent sides have?

Solution: Such a quadrilateral is necessarily a square. A square has $\boxed{4}$ axes of rotational symmetry (both diagonals as well as the lines dividing the square in half from top to bottom and from left to right).

Problem 3
Let $f(x) = (x+64)(x^2+64)(x^3+64)$. How many real values of x are there such that $f(x) = 0$?

Solution: $f(x) = 0$ can happen when one or more of the factors of $f(x)$ is 0. We consider them each separately:

- If $x + 64 = 0$, then $x = 0 - 64 = -64$. This is the only such value for which this can be the case
- If $x^2 + 64 = 0$, then $x^2 = 0 - 64 = -64$. You can check that no real number squared can equal a negative number, and so there are no real solutions here[1]
- If $x^3 + 64 = 0$, then $x^3 = 0 - 64 = -64$. Some testing of numbers (or knowing your perfect cubes) yields that $x = -4$ is a solution, and that there are no other real solutions.[1]

The answer to our question is then that there are $\boxed{2}$ such values.

[1]*: The solutions to this equation are $8i$ and $-8i$, a concept known as imaginary numbers if you wish to explore further. These are not real numbers, however, so they do not affect the answer.*

Problem 4
In the 2018-19 NBA season, LeBron James made $p\%$ of his free throws. If the chance that he misses at least one of his two free throws is 50%, find p to the nearest 10 percent.

Solution: The probability that he makes a given free throw is $p\%$ which can be written as the fraction $\frac{p}{100}$. The probability that he makes both free throws is $(\frac{p}{100})^2$. The probability that he misses at least one free throw is the probability that he does not make both free throws, which is $1 - (\frac{p}{100})^2 = 1 - \frac{p^2}{10000}$, and if this chance is 50% or $\frac{1}{2}$, then $\frac{p^2}{10000} = \frac{1}{2}$ and $p = \sqrt{5000} \approx \boxed{70}$ to the nearest ten.

Problem 5
A hose starts pumping 2 liters per second, and linearly increases its rate to 6 liters per second over 5 seconds. How many liters does the hose pump over the 5 seconds?

Solution: Since the hose linearly increases its rate of pumping over the time period, the amount the hose pumps in total can be calculated by multiplying the average amount pumped per second times the number of seconds. The average is simply $\frac{2\frac{\text{liters}}{\text{second}}+6\frac{\text{liters}}{\text{second}}}{2} = 4\frac{\text{liters}}{\text{second}}$, while the time period is 5 seconds, meaning there is a total of $4\frac{\text{liters}}{\text{second}} \cdot 5 \text{ seconds} = \boxed{20}$ liters of water pumped.

Problem 6
Isabella picks a set of one-digit positive integers. She then claims that none of the integers in her set are divisible by any other integers in the set (except for itself). What is the largest possible size of Isabella's set?

Solution: We will first show that Isabella cannot include more than 5 numbers. Consider grouping the numbers $\{1, 2, 4, 8\}, \{3, 6\}$. Isabella can only select one number from each of these groups, and since there are 3 other numbers $(5, 7, 9)$, she can select at maximum 5 numbers. To show this is achievable, consider if she picked $\{5, 6, 7, 8, 9\}$. This satisfies the conditions of the problem, and hence our desired answer is $\boxed{5}$.

Problem 7
Find the number of ways to color each face of a tetrahedron red or blue if rotations are not considered distinct.

Solution: A tetrahedron has only 4 faces. There are then only a few cases:

- If all 4 faces are red, there is only one coloring
- If 3 faces are red and 1 is blue, there is also only one coloring since rotations are symmetric
- If 2 faces are red and 2 are blue, there is also only one coloring since rotations are symmetric and all faces are adjacent to each other
- If 1 face is red and 3 are blue, there is also only one coloring since rotations are symmetric
- If all 4 faces are blue, there is only one coloring

There are then only $\boxed{5}$ distinct colorings.

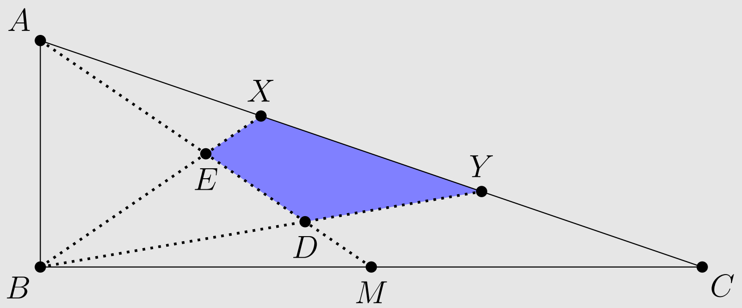
Problem 8
In order to get to the next floor of NC(SMC)^2 Headquarters, one must climb 2 sets of stairs. Starting from the first floor, Anisha must climb x sets of stairs to get to her office on the yth floor. If x and y are consecutive positive square numbers, find x.

Solution: To get to the yth floor, Anisha must go to the next floor $y - 1$ times. If she climbs two sets of stairs each time, she climbs $2(y-1) = 2y-2$ sets of stairs in total. We then have $x = 2y-2$ for consecutive positive square numbers x and y. Clearly $x > y$ since $y > 2$ (checking $y = 1$ does not work). Then, let $y = a^2$ and $x = (a + 1)^2 = a^2 + 2a + 1$ for some a. Substituting yields $a^2 + 2a + 1 = 2a^2 - 2 \implies a^2 - 2a - 3 = 0 \implies (a - 3)(a + 1) = 0 \implies a = -1, 3$ but $a = -1$ is clearly extraneous so $a = 3 \implies y = a^2 = 9 \implies x = (a+1)^2 = 16$ which can be checked to work, so $x = \boxed{16}$.

Problem 9
Jane and May are playing a game with a weighted dice. The probability that the weighted dice lands on a given side is proportional to the square of the number on that side. A player wins the game if they roll a prime number before the other. Given that Jane rolls first and the probability that Jane wins the game can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m+n$.

Solution: Let the probability that the die lands on side k be k^2x. The probabilities of sides $1-6$ respectively are then $1x, 4x, 9x, 16x, 25x, 36x$ respectively, but by fundamental principles of counting, we have $1x+4x+9x+16x+25x+36x = 1 \implies 91x = 1 \implies x = \frac{1}{91}$. The probability of rolling a prime number is then $4x+9x+25x = 38x = \frac{38}{91}$ since the prime numbers are 2, 3, and 5. The probability that Jane wins rolling first can be calculated by adding the probability that Jane wins on the first roll, the probability that neither her nor May win on their first rolls and that Jane wins on her second roll, the probability that neither her nor May win on their first two rolls and that Jane wins on her third roll, etc. These probabilities can be expressed as $\frac{38}{91} + \frac{53}{91}\cdot\frac{53}{91}\cdot\frac{38}{91} + \frac{53}{91}\cdot\frac{53}{91}\cdot\frac{53}{91}\cdot\frac{53}{91}\cdot\frac{38}{91} = (\frac{38}{91})(1+\frac{53}{91}^2+\frac{53}{91}^4\dots) = (\frac{38}{91})(\frac{1}{1-\frac{53}{91}^2}) = \frac{38}{91}\cdot\frac{91^2}{91^2-53^2} = \frac{38}{91}\cdot\frac{91^2}{(91+53)(91-53)} = \frac{38}{91}\cdot\frac{91\cdot 91}{144\cdot 38} = \frac{91}{144}$ so our desired answer is $91+144 = \boxed{235}$

Problem 10
Let ABC be a triangle with $AB = 45$, $BC = 336$, and $AC = 339$. Let M be the midpoint of BC, let X, Y be the trisection points of AC closest to and furthest from A respectively, and D, E be the intersections of AM with BX and BY respectively. Compute the area of $XYDE$.



Solution: We proceed with mass points.

Consider first if D, Y were not a part of the diagram. If the mass of M is $2x$, then the masses of B and C are both x, and the mass of A is then $2x$ from $\frac{AX}{XC} = \frac{1}{2}$, meaning the mass of E is $4x$ and the mass of X is $3x$. This then tells us that $AE = EM$ from the masses of A and M and that $BE : EX = 3 : 1$ from the masses of B and X.

Consider now if E, X were not a part of the diagram. If the mass of M is $2y$, then the masses of B and C are both y, the mass of A is $\frac{y}{2}$ from $\frac{AY}{YC} = 2$, then the mass of D is $2.5y$ and that of Y is $1.5y$. This tells us that $AD : DM = 4 : 1$ from the masses of A and M and that $BD : DY = 5 : 2$ from the masses of B and Y.

Combining our findings, we see that $AE : ED : DM = 5 : 3 : 2$. See that $[XYDE] = [BXY] - [BED]$ and that $[BXY] = \frac{1}{3}\cdot[ABC]$ since XY is $\frac{1}{3}\cdot AC$. We also have $[BED] = \frac{3}{10}\cdot[ABM]$ since ED is $\frac{3}{10}\cdot AM$. $[ABM] = \frac{1}{2}\cdot[ABC]$ so the desired $[BXY]-[BED] = (\frac{1}{3}-\frac{3}{10}\cdot\frac{1}{2})[ABC] = \frac{11}{60}[ABC]$ while $[ABC] = \frac{1}{2}\cdot 45\cdot 336 = 45\cdot 168$ so our desired quantity is $\frac{11}{60}\cdot 45\cdot 168 = \frac{45}{60}\cdot 11\cdot 168 = \frac{3}{4}\cdot 11\cdot 168 = 126\cdot 11 = \boxed{1386}$

2 Accuracy Round Problems

Problem 1
If $x = 20$ and $y = 23$, what is the value of the expression $5x - 13y$?

Solution: Substituting appropriate values for x and y, our final expression becomes $5 \times 20 - 13 \times 23 = 100 - 299 = \boxed{-199}$

Problem 2
Assume that 1 meter = 1.1 yards. If a swimmer can swim 100 meters in 110 seconds, at the same rate, how fast can she swim 100 yards?

Solution: Since 1 meter = 1.1 yards, we know that 100 meters = 110 yards. So, the swimmer can swim 110 yards in 110 seconds. At this rate, she can swim 100 yards in $\boxed{100}$ seconds.

Problem 3
A rectangle has a perimeter of 40 and an area of 96. What is the square of the length of the diagonal of this rectangle?

Solution: Let the length and width of this rectangle be l and w. We have that $2l + 2w = 40$ giving $l + w = 20$ and $lw = 96$. It is feasible to guess the solution $l = 12$ and $w = 8$ (and vice versa) but this can also be solved with substitution.

Plugging in $l = 20 - w$ into the second equation, we have

$$w(20 - w) = 96$$

$$w^2 - 20w + 96 = 0$$

$(w - 8)(w - 12) = 0$ The $w = 8$ root gives $l = 12$ and the $w = 12$ root gives $l = 8$, so the square of the diagonal is $\sqrt{(l^2 + w^2)}^2 = \sqrt{64 + 144}^2 = \boxed{208}$.

Problem 4
Sheep B runs a grass mowing (eating?) business. For each customer, he is paid a base price of \$7. On top of that, he receives \$11 for every hour he spends eating. Assume that Sheep B works a positive integer number of hours at each customer. If Sheep B made \$94 today, how many hours did he work?

Solution: Since Sheep B must work at least one hour at each customer, the base price can be considered to be \$18 with \$11 add-ons. Hence, we can write the word problem as $18x + 11y = 94$ where x is the number of customers and y is the number of hours Sheep B worked outside of the base one hour. Because x, y must be non-negative integers, we can plug in values until it works. The only possible solution is $x = 4$ and $y = 2$, giving our answer of $x + y = \boxed{6}$.

Problem 5
Ana's birthday is 1/1/2000, Bob's birthday is 1/1/2002, and Carla's birthday is 1/1/2009. During what year did Ana, Bob, and Carla's ages form a Pythagorean triple (that is, the numbers could be the side lengths of a right triangle)?

Solution: Let Carla's age be x. Then, Ana and Bob's ages are $x+9$ and $x+7$ respectively. Using the Pythagorean Theorem, we have

$$(x)^2 + (x+7)^2 = (x+9)^2$$

$$x^2 + x^2 + 14x + 49 = x^2 + 18x + 81$$

$$x^2 - 4x - 32 = 0$$

$(x-8)(x+4) = 0$ Discarding the negative root, we have $x = 8$ (so the Pythagorean triple is 8-15-17). The year this occurs is then $\boxed{2017}$.

Problem 6
How many positive integer factors does $3^8 - 1$ have?

Solution: We apply difference of squares several times to help find the prime factorization

$$3^8 - 1 = (3^4 - 1)(3^4 + 1) = (3^2 - 1)(3^2 + 1)(3^4 + 1) = (3^1 - 1)(3^1 + 1)(3^2 + 1)(3^4 + 1)$$

This means $3^8 - 1 = 2 \cdot 4 \cdot 10 \cdot 82$. Rewriting as a prime factorization, $3^8 - 1 = 2^5 \cdot 5 \cdot 41$, meaning there are $6 \cdot 2 \cdot 2 = \boxed{24}$ factors.

Problem 7
If 6 fair coins are flipped, the probability that there are more heads than tails can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. What is $m + n$?

Solution: If we compute the probability q that there are equal heads and tails, our desired answer p is equal to $\frac{1-q}{2}$ since the probability of having more heads than tails is symmetric to the probability of having more tails than heads.

To compute q, we need to find all possibilities of 3 heads and 3 tails. This occurs in $\binom{6}{3} = 20$ ways and each way happens with $\frac{1}{64}$ probability, so $q = \frac{20}{64} = \frac{5}{16}$, meaning our desired answer $p = \frac{1-\frac{5}{16}}{2} = \frac{11}{32}$, giving final answer $\boxed{43}$.

Problem 8
Albert has 2 pairs of socks in a drawer: 1 black pair and 1 white pair. Every second, he removes a sock randomly from the drawer, until he has matching socks. The average number of seconds Albert must wait until he has a matching pair of socks can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. What is $m + n$?

Solution: Note that Albert is guaranteed to draw a matching pair by the time he draws his third sock. The probability that his first two socks match is $\frac{1}{3}$, so the expected value of number of seconds is $2 \cdot \frac{1}{3} + 3 \cdot \frac{2}{3} = \frac{8}{3}$, giving final answer $\boxed{11}$.

Problem 9
Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let G be the centroid (the intersection of the three medians, lines that connect vertex to midpoint of sides) and let H be the orthocenter (the intersection of the three altitudes of the triangle). The length of GH can be expressed as $\frac{\sqrt{m}}{n}$ for some squarefree positive integer m that is relatively prime to n. Find $m + n$

Solution: Consider placing this triangle on the coordinate plane such that BC lies on the x-axis and A lies on the y-axis. By Pythagorean properties of a 13-14-15 triangle, we have that $B = (-5, 0)$ and $C = (9, 0)$ (try drawing an altitude and computing these out!). Further, $A = (0, 12)$. Since the centroid is the average of the vertices, we have $G = (\frac{4}{3}, 4)$.

The orthocenter is the intersection of the altitudes. Since the y-axis is already one of the altitudes, we just need to find the y-intercept of the altitude of AC through B. This line must have slope $\frac{3}{4}$ and must pass through the point $(-5, 0)$, so the equation of this line is

$$(y - 0) = \frac{3}{4}(x + 5)$$

$$y = \frac{3}{4}x + \frac{15}{4}$$

Hence, the y-intercept of this line, which is the orthocenter, is $(0, \frac{15}{4})$. The distance between this point and $\frac{\frac{4}{3}}{4}$ is

$$\sqrt{(\frac{4}{3} - 0)^2 + (\frac{15}{4} - 4)^2} = \sqrt{\frac{16}{9} + \frac{1}{16}} = \sqrt{\frac{256}{144} + \frac{9}{144}} = \frac{\sqrt{265}}{12}$$

This gives our final answer of $\boxed{277}$.

Problem 10

Let a, b, c, d be pairwise distinct positive integers such that $a, b, c, d \leq 100$. Find the maximum possible value of $\gcd(a, b) + \gcd(b, c) + \gcd(c, d) + \gcd(d, a)$.

Solution: The following argument can be reasonably made by inspection: The answer must be $x, 2x, 4x, 3x$ or $4x, 2x, 6x, 3x$ for some integer x. This is because it maximizes the sum of adjacent GCDs divided by it's sum. Setting maximal x for the $x, 2x, 4x, 3x$ case ($x = 25$), we have the pairwise GCDs are $25, 50, 25, 25$ giving a sum of 125. Setting maximal x for the $4x, 2x, 6x, 3x$ case ($x = 16$) we have the pairwise GCDs are $32, 32, 48, 16$ giving a sum of 128. No other combination comes close to our answer of $\boxed{128}$.

3 Speed Round Problems

Problem 1
What is the sum of the digits of the difference $2023 - 513$?

Solution: The desired difference is $2023 - 513 = 1510$. The digit sum is then $1 + 5 + 1 + 0 = \boxed{7}$

Problem 2
A regular hexagon, square, and equilateral triangle are joined at their edges to form the diagram below. If the area of the square is 25, what is the perimeter of the figure?

Solution: The area of the square being 25 means each side length in the figure is $\sqrt{25} = 5$. There are 9 outer edges, and hence the perimeter is $9 \cdot 5 = \boxed{45}$

Problem 3
Thomas makes a visit to the vending machine every day he is at school. He always buys his favorite snack – a Snickers candy bar for \$1.24. If he is at school 5 days a week, how much money in dollars does he spend at the vending machine weekly, rounded to the nearest dollar?

Solution: Notice that $5 \cdot \$1.20 = \$6$, and the extra 4 cents per candy bar will only add 20 cents to this count. Our desired answer is then $\boxed{6}$

Problem 4
Sophia finds a dress on sale for 35% off, and purchases it for \$40.95. In dollars, what is the price of the dress before the sale?

Solution: The dress being 35% off means that it is 65% of the original price. 65% can be rewritten as $\frac{65}{100} = \frac{13}{20}$. Let the original price be P, then we have $\frac{13P}{20} = \$40.95 \implies P = \frac{20}{13} \cdot \$40.95 = 20 \cdot \frac{\$40.95}{13} = 20 \cdot \$3.15 = \boxed{63}$.

Problem 5
A fruit store is selling apples for \$3 each and selling packages of three apples for \$8. What is the largest number of apples that a customer can buy from the fruit store with \$17?

Solution: The price per apple in the package of three apples is cheaper since each apple costs less (≈\$2.67 per apple rather than \$3.00 per individual apple). We then want as many of those packages as we can have. Since each package is \$8 and we have \$17 total, we can buy at most two of these packages, but this gives us 6 apples and \$1 remaining. Since we want the largest number of apples, it might be better to use all the money instead. If we instead choose to buy 1 package of 3 and 3 individual apples, the price would be $\$8 + 3 \cdot \$3 = \$17$, using all the money, but this yields the same count of $\boxed{6}$ apples.

Problem 6
Last week, Ganning biked 6 miles a day from Sunday through Friday. On Saturday, he biked 20 miles. How many miles per day did Ganning bike last week on average?

Solution: There are 6 days from Sunday through Friday, and he biked 6 miles on each of these days, for a total of $6 \cdot 6 = 36$ miles biked on these days. On Saturday, he biked 20 more miles, making his new total $36 + 20 = 56$ miles. The average number of miles per day is the total number of miles divided by the total number of days, or $\frac{56}{7} = \boxed{8}$.

Problem 7
The expression $\frac{20}{22} + \frac{20}{23}$ can be written as one simplified fraction $\frac{m}{n}$ for relatively prime integers m, n. What is $m + n$?

Solution: We want a common denominator first. If we make the common denominator $22 \cdot 23 = 506$, we get $\frac{20}{22} = \frac{20 \cdot 23}{22 \cdot 23}$ while $\frac{20}{23} = \frac{20 \cdot 22}{23 \cdot 22}$. Adding the two then yields $\frac{20 \cdot 23 + 20 \cdot 22}{22 \cdot 23} = \frac{20 \cdot 45}{22 \cdot 23} = \frac{10 \cdot 45}{11 \cdot 23} = \frac{450}{253} \implies 450 + 253 = \boxed{703}$.

Problem 8
Let $s(n)$ be the sum of the odd divisors of n. What is $s(9) + s(18)$?

Solution: The divisors of 9 are $1, 3, 9$ so the sum of the odd divisors is $1 + 3 + 9 = 13$. The divisors of 18 are $1, 2, 3, 6, 9, 18$ so the sum of the odd divisors is once again $1 + 3 + 9 = 13$. Then we have $s(9) = s(18) = 13$ and $s(9) + s(18) = \boxed{26}$.

Problem 9
What is the difference between the mean of the first 5 positive perfect squares and the median of the first 5 positive perfect squares?

Solution: The first 5 perfect squares are $1, 4, 9, 16, 25$. Their mean is then $\frac{1+4+9+16+25}{5} = \frac{55}{5} = 11$ while their median is simply the third perfect square, or 9. The difference between these two numbers is $\boxed{2}$.

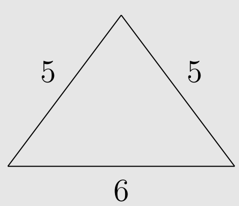
Problem 10
A cube has surface area 150. What is the volume of the cube?

Solution: Let the side of the cube be s. The surface area of the cube is then $6 \cdot s^2 = 6s^2$ since the cube has 6 faces that are each squares with side length s and area s^2. We then have $6s^2 = 150 \implies s^2 = 25 \implies s = 5$. The volume of a cube with side length s is s^3, so the desired volume is $5^3 = \boxed{125}$

Problem 11
Find the value of x that satisfies this equation: $-3(x-2)^3 = 81$.

Solution: Dividing both sides of the equation by -3 yields $(x-2)^3 = \frac{81}{-3} = -27$. Ignoring complex numbers, cube rooting both sides yields $\sqrt[3]{(x-2)^3} = \sqrt[3]{-27} \implies (x-2) = -3 \implies x = \boxed{-1}$.

Problem 12
Find the area of a triangle with side lengths 5, 5, and 6.



Solution: Suppose we drop an altitude from the top vertex shown to the bottom side. This then splits the bottom side into two equal parts of length $\frac{6}{2} = 3$ since the triangle is isosceles. The altitude forms a right angle with the bottom side. By the pythagorean theorem, the height of the triangle is then $\sqrt{5^2 - 3^2} = \sqrt{16} = 4$, and the area is then $A = \frac{bh}{2} = \frac{6 \cdot 4}{2} = \boxed{12}$.

Problem 13
At 2022 NC(SMC)^2, Samuel and Max tied for first, Calvin and Jason tied for third, and Anna, Brandon, and Avery tied for fifth. How many different possible ways could the 1st, 2nd, 3rd, 4th, and 5th place awards be given out after ties are broken if each award is given to exactly one person?

Solution: The first place award can go to either Samuel or Max and the other will receive the second place award, so there are 2 ways to give the first and second place awards. By the same logic, there are 2 ways to give the third and fourth place awards to Calvin and Jason. Finally, any of Anna, Brandon, or Avery can receive the 5th place award. By the fundamental principle of counting, there are then $2 \cdot 2 \cdot 3 = \boxed{12}$ ways to distribute the awards.

Problem 14
An arithmetic sequence has first few terms $20, 23, 26, 29, \cdots$. What is the 2023rd term?

Solution: The difference between two consecutive terms of the arithmetic sequence is $23 - 20 = 3$. To get from the 1st term to the nth term, you need to add $(n-1)$ differences to the first term (consider each difference between terms being a "jump," to get to the nth term you need to "jump" $n-1$ times). The 2023rd term then requires 2022 "jumps," adding $2022 \cdot 3 = 6066$ to the first term. The desired term is then $20 + 6066 = \boxed{6086}$

Problem 15
Two fair six-sided dice are rolled. The probability that both dice show odd prime numbers can be expressed as a common fraction $\frac{m}{n}$ for relatively prime integers m, n. What is $m + n$?

Solution: The odd prime numbers from $1 - 6$ are 3 and 5 only. The probability of rolling an odd prime number on one dice is then $\frac{2}{6} = \frac{1}{3}$, and since the rolls are independent, we multiply the probabilities on each roll. The desired probability is then $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \implies 1 + 9 = \boxed{10}$

Problem 16
Emma originally planned to travel at 75 mph on a trip, which would have taken her 4 hours to complete. However, the NEED FOR SPEED calls for a 5 mph increase. How many minutes would Emma save by driving 5 mph faster?

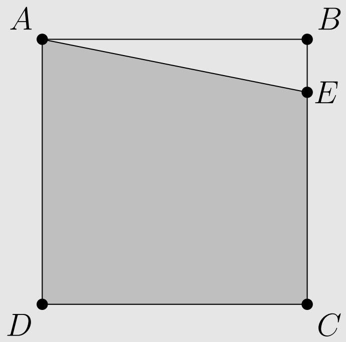
Solution: The trip is $75 \cdot 4 = 300$ miles long. If she were going 5 mph faster, she would be going $75 + 5 = 80$ mph. The trip would then take $\frac{300}{80} = 3.75$ hours, and 3.75 hours is $3.75 \cdot 60 = 225$ minutes. The original trip takes 4 hours which is $4 \cdot 60 = 240$ minutes. The difference between these times is $\boxed{15}$ minutes saved.

Problem 17
Eli and Nicole are in a classroom, with an unknown number written on the whiteboard. The teacher asks everyone to take that number, add 4, divide by 2, multiply by 3, and subtract 5. However, Eli forgets to follow the order of operations and performs his operations in the order they were said in. Nicole remembers the order of operations, and gets an answer of 35. What was Eli's incorrect answer?

Solution: Let the starting number be x. The expression the teacher gives is $x+4\div2\times3-5$. Nicole's answer is then $x + 6 - 5 = x + 1$ following the order of operations. Eli's answer is $3 \cdot \frac{(x+4)}{2} - 5$ since he follows the operations in the order they were said in. Nicole obtaining 35 as her answer means that $x + 1 = 35 \implies x = 34$, so Eli's incorrect answer is then $3 \cdot \frac{(34+4)}{2} - 5 = 3 \cdot 19 - 5 = \boxed{52}$

Problem 18

Square $ABCD$ has side length 10. Point E is on side BC such that $BE : EC = 1 : 4$. What is the area of quadrilateral $AECD$?



Solution: The area of quadrilateral $AECD$ is the area of $ABCD$ minus the area of ABE. The area of $ABCD$ is simply $10^2 = 100$. The area of ABE can be found by taking $\frac{AB \cdot BE}{2}$ since it is a right triangle. We have that $EC = 4 \cdot BE$ from the ratio and $BE + EC = BC = 10 \implies BE + 4 \cdot BE = 5 \cdot BE = 10 \implies BE = 2$ and so $\frac{AB \cdot BE}{2} = \frac{10 \cdot 2}{2} = 10$, so the area of $AECD$ is $100 - 10 = \boxed{90}$.

Problem 19

How many positive integers less than 100 are both odd and not a perfect square?

Solution: There are 50 positive integers less than 100 that are odd. Of these, the perfect squares are $1^2 = 1, 3^2 = 9, 5^2 = 25, \ldots, 9^2 = 81$. There are then 5 of these 50 positive integers that are perfect squares, so the desired number is $\boxed{45}$ positive integers less than 100 that are odd and not a perfect square.

Problem 20

A *common* football game is a game where only some combination of 7 point touchdowns and 3 point field goals are scored by both teams. Which of the following Super Bowls must not have been a common game? Do not write your answer in Roman Numerals.

- Super Bowl 46: Giants 21 - 17 Patriots
- Super Bowl 48: Seahawks 43 - 8 Broncos
- Super Bowl 50: Broncos 24 - 10 Panthers
- Super Bowl 52: Eagles 43 - 38 Patriots
- Super Bowl 54: Chiefs 31 - 20 49ers

Solution: Super Bowl 48 is not possible because while 43 can be made with 4 touchdowns and 5 field goals, 8 cannot be made. Hence, Super Bowl 48 must not be a common game, and our answer is $\boxed{48}$.

It can be shown that all other scores can be obtained by 7 point touchdowns and 3 point field goals. In particular, by the Chicken McNugget Theorem, all scores 12 and above can be attained.

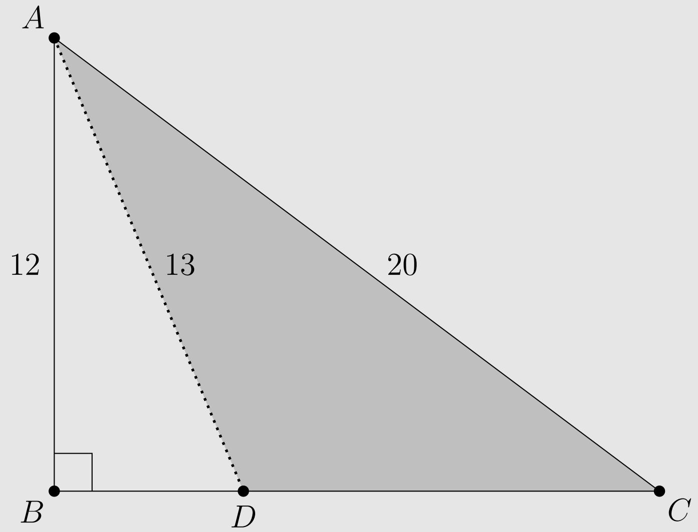
Problem 21

Find the number of positive even factors of 216.

Solution: The prime factorization of $216 = 2^3 \cdot 3^3$. Any factor of 216 is of the form $2^a \cdot 3^b$ for $0 \leq a, b \leq 3$. This means there are then $(3+1)(3+1) = 16$ total positive factors. A factor can only be odd if it does not have any powers of 2, meaning it is of the form $2^0 \cdot 3^b$. This then means that the odd factors are $2^0 \cdot 3^0, 2^0 \cdot 3^1, 2^0 \cdot 3^2, 2^0 \cdot 3^3 = 1, 3, 9, 27$. There are only 4 odd factors out of 16 total factors and so the remaining $16 - 4 = \boxed{12}$ must be even.

Problem 22

There is a triangle ABC such that $\angle ABC = 90°$, $AB = 12$, and $AC = 20$. There is a point D on segment BC such that $AD = 13$. Find the area of triangle ADC.



Solution: Two applications of the pythagorean theorem quickly yield $BC = \sqrt{AC^2 - AB^2} = \sqrt{20^2 - 12^2} = \sqrt{400 - 144} = \sqrt{256} = 16$ and $BD = \sqrt{AD^2 - AB^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$. The length $DC = BC - BD = 16 - 5 = 11$, and the area of ADC is $\frac{bh}{2} = \frac{DC \cdot AB}{2} = \frac{11 \cdot 12}{2} = \boxed{66}$.

Problem 23

Paul's shadow is 9 feet long. He then stands on 2 feet stilts, which makes his shadow 12 feet long. How tall is Paul in feet?

Solution: We are able to make similar triangles between height and shadow since we assume the sun shines at the same angle. If Paul's height is h, we have $\frac{h}{9} = \frac{h+2}{12} \implies 12h = 9(h + 2) = 9h + 18$ from cross-multiplying. We can then solve $12h = 9h + 18 \implies 3h = 18 \implies h = \boxed{6}$.

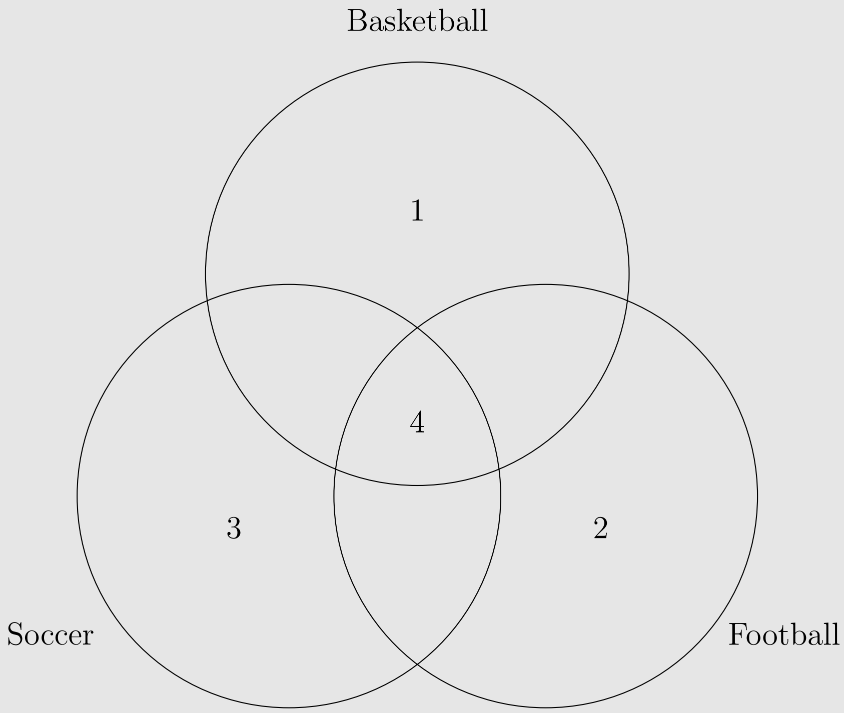
Problem 24

Alice, Alina, and Vikas are standing in line in random order. Given Alice is the second tallest in the group, the probability that Alice is taller than everyone in front of her can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. What is $m + n$?

Solution: If Alice is the second tallest in the group, one of Alina and Vikas is taller than Alice while the other is shorter than Alice. Without loss of generality, assume Alina is taller than Alice while Vikas is shorter. For Alice to be taller than everyone in front of her, she can either be first in line, which can happen 2 ways, or she can be second in line with the only person in front of her being Vikas, which can happen in 1 way. There are then $2 + 1 = 3$ working line orderings out of $3! = 6$ total orderings, meaning our desired probability is $\frac{3}{6} = \frac{1}{2} \implies 1 + 2 = \boxed{3}$.

Problem 25

A set of 40 students responded to a survey on whether they like to watch basketball, football, and/or soccer. Each student responded with one or more sports, and each sport was liked by the same number of students. Some of the data is provided in the Venn diagram below. How many people said they liked to watch basketball and football?



Solution: Let the number of students who like Basketball and Football only be x, the number of students who like Soccer and Football only be y, and the number of students who like Basketball and Soccer only be z. We desire x. Since the same number of students like Soccer and Basketball, we have $1 + z + 4 + x = 3 + z + 4 + y$ by setting the counts equal. This given $y = x - 2$. Similarly, from Basketball and Football we have $1 + z + 4 + x = 2 + x + 4 + y \implies z + 1 = 2 + y \implies z = y + 1 = (x - 2) + 1 = x - 1$. Since there are 40 students total, we have $1+2+3+4+x+y+z = 40 \implies 1+2+3+4+x+(x-2)+(x-1) = 40 \implies 3x+7 = 40 \implies x = \boxed{11}$.

Problem 26

Compute the sum of all positive integer divisors of 108.

Solution: The prime factorization of 108 is $2^2 \cdot 3^3$. Consider the expansion of the sum $(1+2+2^2)(1+3+3^2+3^3)$. It is easy to verify that expanding this sums every divisor of 108; this is because of how the distributive property works. In each set of parentheses, you choose one factor (one power of 2 in the first set and one power of 3 in the second set). The product of these will create a unique divisor of 108, and when you sum these up, you get the sum of all divisors of 108. In fact, this method generally works for finding the sum of the divisors of any number. Regardless, our desired sum is then $(1 + 2 + 2^2)(1 + 3 + 3^2 + 3^3) = 7 \cdot 40 = \boxed{280}$

Problem 27

There are positive integers x, y such that $y = \sqrt{x^2 + 13}$. Find y.

Solution: Squaring both sides of the equation yields $y^2 = x^2 + 13$ for positive integers x, y. Subtracting x^2 from both sides yields $y^2 - x^2 = 13$ and we can factor the left side using difference of squares to get $(y-x)(y+x) = 13$. Since x, y are integers, both of these factors are as well. Clearly,

$y > x$ since both numbers are positive and $y^2 > x^2$. Then, $y - x < y + x$ and so $y - x = 1$ while $y + x = 13$. You can add these two equations to get $2y = 1 + 13 = 14 \implies y = \boxed{7}$. You can easily verify that $y = 7$ and $x = 6$ satisfy the constraints.

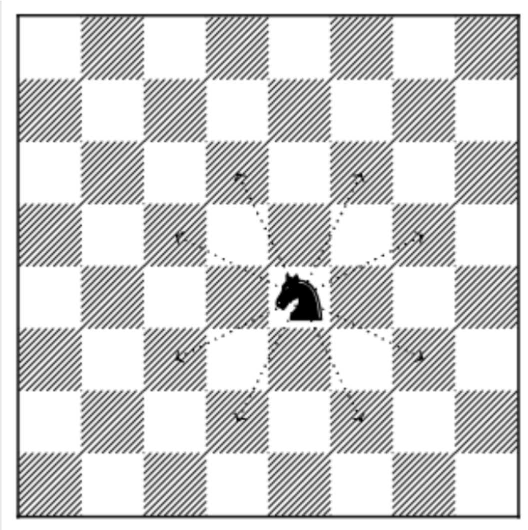
Problem 28
There is a list of 6 positive integers. The median of this list is 3 and the unique mode is 4. The smallest possible average of the 6 integers can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

Solution: The 3rd and fourth integers must sum to $3 \cdot 2 = 6$. They are then either 1 and 5, 2 and 4, or 3 and 3 respectively. Suppose they are 3 and 3. It would then not be possible to have 4 be a unique mode since there would be at most 2 4s and there are already at least 2 3s. If they are 1 and 5, then you cannot have any 4s and it once again cannot be a unique mode. The 3rd and 4th numbers must be 2 and 4 respectively. The 1st and 2nd numbers must be both either 1 or 2. Regardless of what they are, there will be at least 2 1s or at least 2 2s. This means you must have 3 4s, so the 4th, 5th, and 6th numbers must be 4s. To minimize the average, we want the numbers to have the smallest sum, so we pick the 1st and 2nd numbers to both be 1 yielding our average of $\frac{1+1+2+4+4+4}{6} = \frac{16}{6} = \frac{8}{3} \implies 8 + 3 = \boxed{11}$

Problem 29
Nikhil can climb stairs 1, 2, or 3 stairs at a time. How many different ways are there for him to climb a staircase of 7 stairs?

Solution: We will use a method called recursion where we relate the number of ways Nikhil can climb n stairs to the number of ways he can climb some smaller numbers of stairs. Let a_n be the number of ways Nikhil can climb a staircase of n stairs. If his last step is climbing 1 stair, the previous $n-1$ stairs could have been climbed in a_{n-1} ways. If his last step is climbing 2 stairs, the previous $n-2$ stairs could have been climbed in a_{n-2} ways. If his last step is climbing 3 stairs, the previous $n-3$ stairs could have been climbed in a_{n-3} ways. We then have $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. There is clearly only 1 way to climb a staircase of 1 stair, so $a_1 = 1$. There are 2 ways to climb a staircase of 2 stairs: 2 steps of 1 and a step of 2, so $a_2 = 2$. There are 4 ways to climb a staircase of 3 stairs; 3 steps of 1, 1 step of 3, 1 step of 1 and then 1 step of 2, or 1 step of 2 and then 1 step of 1, so $a_3 = 4$. Then, using our relationship, $a_4 = a_3 + a_2 + a_1 = 4 + 2 + 1 = 7$, $a_5 = a_4 + a_3 + a_2 = 7 + 4 + 2 = 13$, $a_6 = a_5 + a_4 + a_3 = 13 + 7 + 4 = 24$, and finally, $a_7 = a_6 + a_5 + a_4 = 24 + 13 + 7 = \boxed{44}$

Problem 30
Michelle is playing an altered game of chess where she only plays with knights. Find the maximum number of knights Michelle can place on the standard 8 by 8 chess board so that no two of her knights attack each other.
Note: The knight is only able to attack squares that are 2 squares in one direction and 1 square in a perpendicular direction.



Solution: The key observation here is that if a knight is on a black square then it can only attack a white square and vice versa. The problem constraints are then satisfied if the knights are all on white squares or all on black squares, each of which there are 32. To see that this is the maximum, note that if there are knights on every white square all black squares are attacked and vice versa, so another knight cannot be added to this configuration. It is relatively easy to see that a configuration placing knights on both black and white squares is less optimal than placing all on one color square, so the desired answer is $\boxed{32}$