Fractional Variable Order Derivative Simulink Toolkit ver. 2.00 User Guide

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May 14, 2015

Abstract

This paper presents the User Guide for the Fractional Variable Order Derivative Toolkit. The installation procedure, blocks' description and examples are presented.

Keywords: fractional calculus, variable order differentiation

1 Fractional variable order Grunwald-Letnikov type derivatives

As a base of generalization onto variable order derivative the following definition of fractional constant order derivative is taken into consideration:

$$ {}_0\mathrm{D}_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh), \tag{1} $$

where $n = t/h$, t is a time, and h is a sample time. For ideal case, when $t \to \infty$, number n of samples taken into consideration also increases to infinity. In practical implementation n has to be restricted to some predefined number—in our implementation it will be a constant **Nbuf**.

According to this definition one obtains: fractional derivatives for $\alpha > 0$, fractional integrals for $\alpha < 0$, and the original function $f(t)$ for $\alpha = 0$.

For the case of order changing with time (variable order case), three types of definition can be found in literature Lorenzo and Hartley (2002); Valerio and da Costa (2011).

*This work was supported by the Polish National Science Centre with the decision number DEC-2011/03/D/ST7/00260

1.1 The $\mathcal{A}$-type of fractional variable order derivative

The $\mathcal{A}$-type derivative is obtained by replacement in (1) the constant order α by variable order $\alpha(t)$. In this approach all coefficients for past samples are obtained for present value of the order which yields:

Definition 1 *The $\mathcal{A}$-type of fractional variable order derivative is defined as follows:*

$$ {}^{\mathcal{A}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha(t)}} \sum_{r=0}^{n} (-1)^r \binom{\alpha(t)}{r} f(t-rh). $$

Let us consider the following switching scheme presented in Fig. 1 based on the chain of derivatives blocks related by the following switching rule. The switches S_i, $i=1,\dots,N$, take the following positions

$$ S_i = \begin{cases} b & \text{for } t_{i-1} \le t < t_i, \\ a & \text{otherwise,} \end{cases} \qquad i = 1,\dots,N, $$

and

$$ \alpha_i = \alpha_{i+1} + \hat{\alpha}_i, \qquad i = 1,\dots,N-1. $$

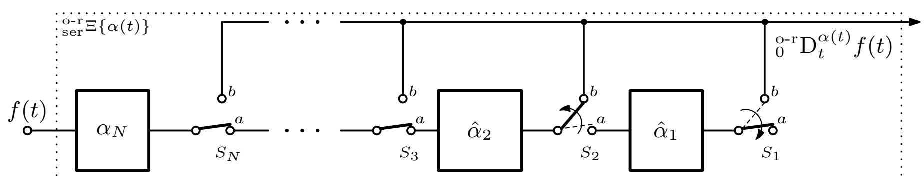

Figure 1: Structure of output-reductive switching order scheme in serial form ${}^{\text{o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2).

Theorem 1 (Sierociuk et al. (2015)) *Output-reductive switching order scheme presented in Fig. 1 is equivalent to the $\mathcal{A}$-type of variable order derivative (given by Def. 1), i.e.,*

$$ {}^{\mathcal{A}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) \equiv {}^{\text{o-r}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t). $$

The detailed proof of this theorem, together with application of them to analog modelling, is presented in Sierociuk et al. (2015).

1.2 The $\mathcal{B}$-type of fractional variable order derivative

The second type of definition assumes that coefficients for past samples are obtained for order that was present for these samples. In this case, the definition has the following form:

Definition 2 *The $\mathcal{B}$-type of fractional variable order derivative is defined as follows:*

$$ {}^{\mathcal{B}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) = \lim_{h\to 0} \sum_{r=0}^{n} \frac{(-1)^r}{h^{\alpha(t-rh)}} \binom{\alpha(t-rh)}{r} f(t-rh). $$

Let us consider the following so-called input-additive switching order scheme in the serial form, denoted ${}^{\text{i-a}}_{\text{ser}}\Xi\{\alpha(t)\}$ and presented in Fig. 2. The switches S_i, $i = 1, \ldots, N$, take the following positions

$$S_i = \begin{cases} b & \text{for } t_{i-1} \leq t < t_i, \\ a & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, N.$$

and

$$\alpha_i = \alpha_{i-1} + \bar{\alpha}_i, \qquad i = 2, \ldots, N.$$

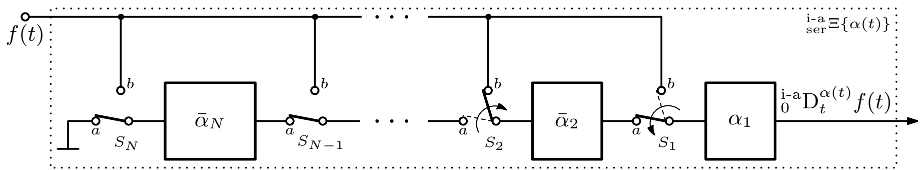

Figure 2: Structure of input-additive switching order scheme in serial form ${}^{\text{i-a}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2).

Theorem 2 (Sierociuk et al. (2014)) *Input-additive switching order scheme presented in Fig. 2 is equivalent to the $\mathcal{B}$-type of variable order derivative (given by Def. 2), i.e.,*

$${}^{\mathcal{B}}_0\mathrm{D}^{\alpha(t)}_t f(t) \equiv {}^{\text{i-a}}_0\mathrm{D}^{\alpha(t)}_t f(t).$$

1.3 The $\mathcal{C}$-type of fractional variable order derivative

The third definition is less intuitive and will not be considered in this toolkit. This definition assumes that coefficients for the newest samples are obtained respective for the oldest orders. For such a case, the following definition applies:

Definition 3 *The $\mathcal{C}$-type of fractional variable order derivative is defined as follows:*

$${}^{\mathcal{C}}_0\mathrm{D}^{\alpha(t)}_t f(t) = \lim_{h\to 0} \sum_{r=0}^{n} \frac{(-1)^r}{h^{\alpha(rh)}} \binom{\alpha(rh)}{r} f(t - rh).$$

For this type of definition equivalent switching strategy is not known. What is also important, practical implementation is questionable, because usually needs reduction of samples that are taken into consideration. In this case limiting the number of samples will effect that the newest order will not be taken into consideration, and makes the definition useless. That is why this definition will not be implemented in this toolkit.

1.4 The $\mathcal{D}$-type of fractional variable order derivative

Definition 4 (Sierociuk et al. (2013)) *The $\mathcal{D}$-type of fractional variable order derivative is defined as follows:*

$${}^{\mathcal{D}}_0\mathrm{D}^{\alpha(t)}_t f(t) = \lim_{h\to 0} \left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{n} (-1)^j \binom{-\alpha(t)}{j} {}^{\mathcal{D}}_0\mathrm{D}_{t-jh}{}^{\alpha(t-jh)} f(t - jh) \right).$$

Let us consider the following so-called input-reductive switching order scheme in the serial form, denoted ${}^{\text{i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ and presented in Fig. 3. The switches S_i, $i = 1, \ldots, N$, take the following positions

$$S_i = \begin{cases} b & \text{for } t_{i-1} \le t < t_i, \\ a & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, N.$$

and

$$\alpha_i = \alpha_{i+1} + \hat{\alpha}_i, \qquad i = 1, \ldots, N-1.$$

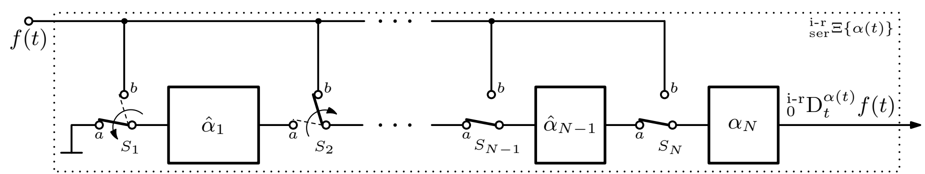

Figure 3: Structure of input-reductive switching order scheme in serial form ${}^{\text{i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2).

Theorem 3 (Sierociuk et al. (2015)) *Input-reductive switching order scheme presented in Fig. 3 is equivalent to the $\mathcal{D}$-type of variable order derivative (given by Def. 4), i.e.,*

$${}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) \equiv {}^{\text{i-r}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t).$$

1.5 The $\mathcal{E}$-type of fractional variable order derivative

Definition 5 (Macias and Sierociuk (2014)) *The $\mathcal{E}$-type fractional variable order derivative is defined as follows:*

$${}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) = \lim_{h \to 0} \left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{n} (-1)^j \binom{-\alpha(t-jh)}{j} \frac{h^{\alpha(t-jh)}}{h^{\alpha(t)}} {}^{\mathcal{E}}_{0}\mathrm{D}^{\alpha(t)}_{t-jh} f(t) \right).$$

Let us consider the following so-called output-additive switching order scheme in the serial form (serial o-a scheme), denoted ${}^{\text{o-a}}_{\text{ser}}\Xi\{\alpha(t)\}$ and presented in Fig. 4. The switches S_i, $i = 1, \ldots, N$, take the following positions

$$S_i = \begin{cases} b & \text{for } t_{i-1} \le t < t_i, \\ a & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, N,$$

and

$$\alpha_{i+1} = \alpha_i + \bar{\alpha}_{i+1}, \qquad i = 1, \ldots, N-1.$$

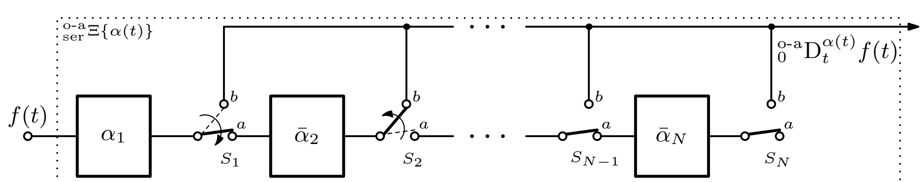

Figure 4: Structure of output-additive switching order scheme in serial form ${}^{\text{o-a}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2).

Theorem 4 (Macias and Sierociuk (2014)) *Output-additive switching order scheme presented in Fig. 4 (for one switch, i.e., $N = 2$) is equivalent to the $\mathcal{E}$-type fractional variable-order derivative (given by Def. 5), i.e.,*

$$ {}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)} f(t) \equiv {}_0^{\text{o-a}}\mathrm{D}_t^{\alpha(t)} f(t). $$

1.6 The $\mathcal{F}$-type of fractional variable order derivative

Definition 6 *The $\mathcal{F}$-type fractional variable order derivative is defined as follows:*

$$ {}_0^{\mathcal{F}}\mathrm{D}_b^{\alpha(t)} f(t) = \lim_{h\to 0}\left(\frac{f(b)}{h^{\alpha(b)}} - \sum_{j=1}^{n} (-1)^j \binom{-\alpha(jh)}{j} \frac{h^{\alpha(jh)}}{h^{\alpha(b)}} {}_0^{\mathcal{F}}\mathrm{D}_{b-jh}^{\alpha(t)} f(t) \right). $$

This definition will not be implemented in this toolkit from the same reasons as $\mathcal{C}$-type of definition.

2 Duality of variable order derivatives

In general, the order composition does not hold in variable order derivatives case, e.g., ${}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{A}}\mathrm{D}_t^{-\alpha(t)} f(t) \neq f(t)$. But the following properties, called duality between variable order definitions, occur:

$$ \begin{aligned} {}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t), \\ {}_0^{\mathcal{D}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{A}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t), \\ {}_0^{\mathcal{B}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{E}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t), \\ {}_0^{\mathcal{E}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{B}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t), \\ {}_0^{\mathcal{C}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{F}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t), \\ {}_0^{\mathcal{F}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{C}}\mathrm{D}_t^{-\alpha(t)} f(t) &= f(t). \end{aligned} $$

3 Fractional Variable Order Derivative Toolkit

In this section, the blocks' description, its parameters and installation procedures have been described.

3.1 Blocks' description

The Fractional Variable Order Derivative Toolkit contains the following blocks:

- **Fractional Constant Order Derivative**—implementation of the fractional constant order derivative.
- **A-type Fractional Variable Order Derivative**—implementation of the $\mathcal{A}$-type of fractional variable order derivative.
- **B-type Fractional Variable Order Derivative**—implementation of the $\mathcal{B}$-type of fractional variable order derivative.

- **D-type Fractional Variable Order Derivative**—implementation of the $\mathcal{D}$-type of fractional variable order derivative.
- **E-type Fractional Variable Order Derivative**—implementation of the $\mathcal{E}$-type of fractional variable order derivative.

Blocks parameters have the following meaning:

- Fractional orders vector—the parameter represents the vector of derivative orders.
- Number of inputs/orders—the parameter represents the number of inputs/orders
- Time sampling **Ts**—the parameter represents the value of block's time sampling step of simulation.
- Parameter **Nbuf**—the parameter is a width of a circular buffer and can be interpreted as a number of samples taking into consideration. In order to achieve best results it should have a number of all time samples used in simulation. It can be computed as "simulation time" divided by "time sampling".

3.2 Installation

The toolkit contains the following files:

- `foderiv.c` (C-MEX S-function)—implementation of the fractional constant order derivative;
- `fvoderiv_a.c` (C-MEX S-function)—implementation of $\mathcal{A}$-type fractional variable order derivative;
- `fvoderiv_b.c` (C-MEX S-function)—implementation of $\mathcal{B}$-type fractional variable order derivative;
- `fvoderiv_d.c` (C-MEX S-function)—implementation of $\mathcal{D}$-type fractional variable order derivative;
- `fvoderiv_e.c` (C-MEX S-function)—implementation of $\mathcal{E}$-type fractional variable order derivative;
- `voderiv.mdl`—Simulink library of the toolkit;
- `slblocks.m`—data for Simulink Library Browser;
- `example1.mdl`—an example of systems simulation;
- `switching_strategies_equivalences.mdl`—an example of switching strategies equivalences simulation;
- `duality.mdl`—an example of duality property simulation;
- `make_mex.m`—a matlab routine for compiling C-MEX S-functions.

In order to install the toolkit, unpack the file `fvoderiv.zip`. Next, add directory with the toolkit to the Matlab path (menu **File** > **Set Path**). In the Matlab environment, change the path to the path of the toolkit and run file `make_mex.m` in order to compile required C-MEX S-functions. It is also possible to compile C-MEX S-functions separately using `mex name_of_file.c` command.

After successful installation, the Fractional Variable Order Derivative Toolkit should appear in the Simulink Library Browser.

The S-functions were written in C-language (as non-inlined C-MEX S-functions Level-2) and can be successfully used in Real Time Workshop (e.g, with dSPACE DS1104 cards).

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