№4.

$\xi \sim p(x) = R[\theta; 2\theta]$

$\vec{X}_n$ – выборка

(а) $\alpha_1(\theta) = M\xi = \frac{3\theta}{2}$

(б) $\tilde{\alpha}_1 = \frac{1}{n}\sum_i X_i = \overline{X}$

$\frac{3\theta_1}{2} = \overline{X}$

$\tilde{\theta}_1 = \frac{2}{3}\overline{X}$ ОММ

$M[\tilde{\theta}_1] = \frac{2}{3} M[\overline{X}] = \frac{2}{3} M\xi = \theta \Rightarrow$ несмещен.

$D[\tilde{\theta}_1] = \frac{4}{9} D[\overline{X}] = \frac{4}{9} D[\frac{1}{n}\sum X_i] =$
$= \frac{4}{9n^2}\sum D X_i = \frac{4}{9n} D\xi = \frac{\theta^2}{27n} \xrightarrow[n\to+\infty]{} 0 \Rightarrow$

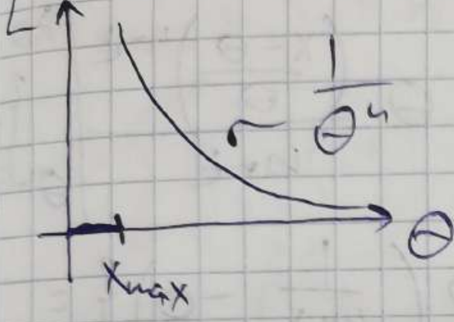
$M\xi^2 = \int_\theta^{2\theta} \frac{1}{\theta} x^2 dx = \frac{1}{3\theta}(8\theta^3 - \theta^3) = \frac{7\theta^2}{3}$

$D\xi = \frac{7\theta^2}{3} - \frac{9\theta^2}{4} = \frac{\theta^2}{12}$

$\Rightarrow$ сост. по дост. усл.

$$L(\theta)=\prod_{i=1}^{n} p(x_i,\theta)=\frac{1}{\theta^n}(\theta<x_i<2\theta)=$$

$$=\frac{1}{\theta^n}(\max x_i<2\theta)\longrightarrow \sup$$




$$\tilde{\theta}_2=\frac{x_{max}}{2} \quad \text{ОММП}$$

$$M[\tilde{\theta}_2]=\frac{1}{2}M[x_{max}]=\frac{1}{2}\int_{\theta}^{2\theta}\frac{1}{\theta}\,x\,n\left(\frac{x-\theta}{\theta}\right)^{n-1}dx=$$

$$=\frac{1}{2}\int_{\theta}^{2\theta}x\,d\left(\left(\frac{x-\theta}{\theta}\right)^{n}\right)=\frac{1}{2}\left(x\left(\frac{x-\theta}{\theta}\right)^{n}\Bigg|_{\theta}^{2\theta}-\int_{\theta}^{2\theta}\left(\frac{x-\theta}{\theta}\right)^{n}dx\right)=$$

$$=\frac{1}{2}\left(2\theta-\theta\,\frac{\left(\frac{x-\theta}{\theta}\right)^{n+1}}{n+1}\Bigg|_{\theta}^{2\theta}\right)=$$

$$=\frac{1}{2}\left(2\theta-\theta\left(\frac{1}{n+1}-0\right)\right)=\frac{1}{2}\left(2\theta-\frac{\theta}{n+1}\right)=$$

$$=\theta-\frac{\theta}{2(n+1)}=\frac{\theta(2n+1)}{2(n+1)}\Rightarrow \text{ассимпт. несмещ.}$$

но смещена $\Rightarrow \tilde{\theta}_2'=\frac{2(n+1)}{2n+1}\tilde{\theta}_2$

$$M[\tilde{\theta}_2^2]=\frac{1}{4}M[x_{max}^2]=\frac{1}{4}\int_{\theta}^{2\theta}\frac{1}{\theta}\,x^2\,n\left(\frac{x-\theta}{\theta}\right)^{n-1}dx=$$

$$=\frac{1}{4}\int_{\theta}^{2\theta}x^2\,d\left(\left(\frac{x-\theta}{\theta}\right)^{n}\right)=\frac{1}{4}\left\{x^2\left(\frac{x-\theta}{\theta}\right)^{n}\Bigg|_{\theta}^{2\theta}-\right.$$

$$\left.-2\int_{\theta}^{2\theta}x\left(\frac{x-\theta}{\theta}\right)^{n}dx\right\}=\frac{1}{4}\left\{4\theta^2-2\frac{\theta}{n+1}\int_{\theta}^{2\theta}x\,d\left(\left(\frac{x-\theta}{\theta}\right)^{n+1}\right)\right\}=$$

$$= \frac{1}{4}\left\{4\Theta^2 - 2\frac{\Theta}{n+1}\left\{x\left(\frac{x-\Theta}{\Theta}\right)^{n+1}\Big|_{\Theta}^{2\Theta} - \right.\right.$$

$$\left.\left. - \int_{\Theta}^{2\Theta}\left(\frac{x-\Theta}{\Theta}\right)^{n+1} dx\right\}\right\} =$$

$$= \Theta^2 - \frac{1}{2}\frac{\Theta}{n+1}\left\{2\Theta - \Theta\frac{\left(\frac{x-\Theta}{\Theta}\right)^{n+2}}{n+2}\Big|_{\Theta}^{2\Theta}\right\} =$$

$$= \Theta^2 - \frac{1}{2}\frac{\Theta}{n+1}\left\{2\Theta - \Theta\left(\frac{1}{n+2} - 0\right)\right\} =$$

$$= \Theta^2 - \frac{\Theta^2}{n+1} + \frac{\Theta^2}{2(n+1)(n+2)} =$$

$$= \Theta^2\left(1 - \frac{1}{n+1} + \frac{1}{2(n+1)(n+2)}\right) =$$

$$= \frac{1}{2}\Theta^2\frac{(2n^2+4n+1)}{(n+1)(n+2)}.$$

$$\mathcal{D}[\tilde{\Theta}_2] = \frac{1}{2}\Theta^2\frac{(2n^2+4n+1)}{(n+1)(n+2)} - \frac{\Theta^2}{4}\cdot\frac{(2n+1)^2}{(n+1)^2} =$$

$$= \frac{\Theta^2}{4(n+1)}\left(\frac{4n^2+8n+2}{n+2} - \frac{(2n+1)^2}{n+1}\right) =$$

$$= \frac{n\Theta^2}{4(n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$ – сост. по дост. усл.

$$\mathcal{D}[\tilde{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$ сост. по дост. усл.

$$\widetilde{\theta}_3 = \frac{1}{5}\left(X_{min} + 2X_{max}\right)$$

$$M\widetilde{\theta}_3 = \frac{1}{5} M X_{min} + \frac{2}{5} M X_{max} = \frac{1}{5}\frac{\theta(n+2)}{n+1} + \frac{2}{5}\frac{\theta(2n+1)}{(n+1)} = \frac{\theta(5n+4)}{5(n+1)}$$

$$M X_{min} = \int_{\theta}^{2\theta} x\, n\left(1 - \frac{x-\theta}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx =$$

$$= -\int_{\theta}^{2\theta} x \cdot d\left(\left(1 - \frac{x-\theta}{\theta}\right)^{n}\right) = -x\left(1 - \frac{x-\theta}{\theta}\right)^{n}\Bigg|_{\theta}^{2\theta} +$$

$$+ \int_{\theta}^{2\theta}\left(1 - \frac{x-\theta}{\theta}\right)^{n} dx = +\theta + \theta(-1)\frac{\left(1-\frac{x-\theta}{\theta}\right)^{n+1}}{n+1}\Bigg|_{\theta}^{2\theta}$$

$$= \theta - \theta\left(-\frac{1}{n+1}\right) = \theta + \frac{\theta}{n+1} = \frac{\theta(n+2)}{n+1}$$

$$M\widetilde{\theta}_3 = \frac{\theta(5n+4)}{5(n+1)}$$ — асимпт. несмещ.

смещ $\Rightarrow \widetilde{\theta}_3' = \frac{5(n+1)}{5n+4}\widetilde{\theta}_3$

$$M[X_{max}^2] = 2\theta^2 \frac{(2n^2+4n+1)}{(n+1)(n+2)}$$

$$M[X_{min}^2] = \int_{\theta}^{2\theta} \frac{1}{\theta} x^2 n\left(1 - \frac{x-\theta}{\theta}\right)^{n-1} dx =$$

$$= \int_{\theta}^{2\theta} \frac{1}{\theta} x^2 n\left(\frac{2\theta - x}{\theta}\right)^{n-1} dx = \frac{n}{\theta^n}\int_{\theta}^{2\theta} x^2 (2\theta - x)^{n-1} dx$$

$$= \frac{-1}{\theta^n}\int_{\theta}^{2\theta} x^2\, d(2\theta - x)^n = -\frac{1}{\theta^n}\left((2\theta - x)^n x^2\Big|_{\theta}^{2\theta} - \right.$$

$$-\int_{\theta}^{2\theta}(2\theta-x)^{n}\, 2x\, dx\Big) =$$

$$= -\frac{1}{\theta^n}\Big(-\theta^{n+2} + \frac{1}{n+1}\int_{\theta}^{2\theta} 2x\, d(2\theta-x)^{n+1}\Big) =$$

$$= \theta^2 - \frac{2}{\theta^n (n+1)}\Big( x(2\theta-x)^{n+1}\Big|_{\theta}^{2\theta} -$$

$$-\int_{\theta}^{2\theta}(2\theta-x)^{n+1}\, dx\Big) =$$

$$= \theta^2 - \frac{2}{\theta^n(n+1)}\Big(-\theta^{n+2} + \frac{1}{n+2}\int_{\theta}^{2\theta} d(2\theta-x)^{n+2}\Big) =$$

$$= \theta^2 + \frac{2\theta^2}{n+1} - \frac{2}{\theta^n(n+1)(n+2)}(2\theta-x)^{n+2}\Big|_{\theta}^{2\theta} =$$

$$= \theta^2 + \frac{2\theta^2}{n+1} + \frac{2}{\theta^n(n+1)(n+2)}\cdot\theta^{n+2} =$$

$$= \theta^2 + \frac{2\theta^2}{n+1} + \frac{2\theta^2}{(n+1)(n+2)} =$$

$$= \theta^2 \cdot \frac{n^2+5n+8}{(n+1)(n+2)}$$

$$M[X_{min}\cdot X_{max}] = \iint_{\theta}^{2\theta} xy\, n(n-1)\frac{1}{\theta^n}(x-y)^{n-2}\, dx\, dy$$

$$= \int_{\theta}^{2\theta} x\,dx \int_{\theta}^{x} n(n-1)\frac{1}{\theta^n}\, y\,(x-y)^{n-2}\,dy =$$

$$= \frac{n(n-1)}{\theta^n} \int_{\theta}^{2\theta} x\,dx \int_{\theta}^{x} y\,(x-y)^{n-2}\,dy \;=$$

$$\left\{ \frac{-1}{n-1}\int_{\theta}^{x} y\, d(x-y)^{n-1} = \frac{-1}{n-1}\left( y(x-y)^{n-1}\Big|_{\theta}^{x} - \right.\right.$$

$$\left. - \int_{\theta}^{x} (x-y)^{n-1}\,dy \right) = \frac{-1}{n-1}\,\theta\,(x-\theta)^{n-1} +$$

$$+ \frac{-1}{(n-1)\cdot n}\int_{\theta}^{x} d\left((x-y)^n\right) = \frac{\theta(x-\theta)^{n-1}}{n-1} +$$

$$\left. + \frac{1}{(n-1)n}\,(x-\theta)^n \right\}$$

$$= \frac{n}{\theta^{n-1}}\int_{\theta}^{2\theta} x\,(x-\theta)^{n-1}\,dx + \frac{1}{\theta^n}\int_{\theta}^{2\theta} x\,(x-\theta)^n\,dx =$$

$$= \frac{n}{\theta^{n-1}\cdot n}\int_{\theta}^{2\theta} x\,d(x-\theta)^n + \frac{1}{\theta^n(n+1)}\int_{\theta}^{2\theta} x\,d(x-\theta)^{n+1} =$$

$$= \frac{1}{\theta^{n-1}} \left( x (x-\theta)^n \Big|_{\theta}^{2\theta} - \right.$$

$$\left. - \int_{\theta}^{2\theta} (x-\theta)^n \, dx \right) + \frac{1}{\theta^n (n+1)} \left( x (x-\theta)^{n+1} \Big|_{\theta}^{2\theta} - \right.$$

$$\left. - \int_{\theta}^{2\theta} (x-\theta)^{n+1} \, dx \right) =$$

$$= \frac{2\theta^{n+1}}{\theta^{n-1}} - \frac{1}{\theta^{n-1}(n+1)} \int_{\theta}^{2\theta} d(x-\theta)^{n+1} +$$

$$+ \frac{2\theta^{n+2}}{\theta^n (n+1)} - \frac{1}{\theta^n (n+1)(n+2)} \int_{\theta}^{2\theta} d(x-\theta)^{n+2} =$$

$$= 2\theta^2 + \frac{2\theta^2}{n+1} - \theta^2 \frac{1}{n+1} - \frac{\theta^2}{(n+1)(n+2)} =$$

$$= 2\theta^2 + \frac{\theta^2}{n+1} - \frac{\theta^2}{(n+1)(n+2)} =$$

$$= \theta^2 \frac{2n^2 + 7n + 5}{(n+1)(n+2)} = \theta^2 \frac{2 \cdot n + 5}{n+2}$$

$$M[\tilde{\Theta}_3^2] = M\left[\frac{1}{25}(X_{min} + 2X_{max})^2\right] =$$

$$= \frac{1}{25}\left(M[X_{min}^2] + 4M[X_{max}^2] + 4M[X_{min} \cdot X_{max}]\right)$$

$$= \frac{1}{25}\left(\Theta^2 \frac{n^2+5n+8}{(n+1)(n+2)} + 4 \cdot 2\Theta^2 \frac{2n^2+4n+1}{(n+1)(n+2)} + \right.$$

$$\left. + 4\Theta^2 \frac{5+7n+2n^2}{(n+2)(n+1)}\right) = \frac{1}{25}\Theta^2\left(\frac{25n^2+65n+36}{(n+1)(n+2)}\right)$$

$$D[\tilde{\Theta}_3] = \frac{1}{25}\Theta^2\left(\frac{25n^2+65n+36}{(n+1)(n+2)}\right) -$$

$$- \frac{1}{25}\Theta^2 \frac{25n^2+40n+16}{(n+1)^2} = \frac{\Theta^2}{25} \cdot \frac{5n+4}{(n+1)^2(n+2)}$$

$$D[\tilde{\Theta}_3'] = \frac{25(n+1)^2}{(5n+4)^2} D[\tilde{\Theta}_3] = \frac{\Theta^2}{(5n+4)(n+2)} \xrightarrow[n\to\infty]{} 0$$

сост. по дост. усл.

c) $D[\tilde{\Theta}_1] = \frac{\Theta^2}{27n}$ $D[\tilde{\Theta}_2'] = \frac{n\Theta^2}{(n+2)(2n+1)^2}$

$$D[\tilde{\Theta}_3'] = \frac{\Theta^2}{(5n+4)(n+2)}$$

$\tilde{\Theta}_3'$ – самая эффективн.

(d) $\xi \sim R[\Theta; 2\Theta]$ ; $\beta = 0{,}95$

$\vec{X}_n$ – выборка

Пусть $y_i = \dfrac{x_i - \Theta}{\Theta}$ $i = 1 \dots n$ – элементы выборки объема $n$ $R[0;1]$..

$$y_{max} = \frac{X_{max} - \Theta}{\Theta}$$

Найдем. $f_1$ и $f_2$ : $P(f_1 < y_{max} < f_2) \geq \beta$

$y_i \sim F(z) = z$ $(0;1)$

$y_{max} \sim f(z) = n z^{n-1}$ ф-ция расп. $z^n$.

плотность расп-е

$$\int_0^{f_1} n z^{n-1} \, dz = \frac{1-\beta}{2}$$

$$z^n \Big|_0^{f_1} = \frac{1-\beta}{2}$$

$$f_1^n = \frac{1-\beta}{2} \Rightarrow f_1 = \sqrt[n]{\frac{1-\beta}{2}}$$

$$\int_{f_2}^{1} n z^{n-1} \, dz = \frac{1-\beta}{2}$$

$$z^n \Big|_{f_2}^{1} = \frac{1-\beta}{2}$$

$$1 - f_2^n = \frac{1-\beta}{2} \quad f_2^n = \frac{1+\beta}{2} \quad f_2 = \sqrt[n]{\frac{1+\beta}{2}}$$

$$P\left(\sqrt[n]{\frac{1-\beta}{2}} < \frac{X_{max}-\Theta}{\Theta} < \sqrt[n]{\frac{1+\beta}{2}}\right) \geqslant \beta$$

$$P\left(\sqrt[n]{\frac{1-\beta}{2}}+1 < \frac{X_{max}}{\Theta} < \sqrt[n]{\frac{1+\beta}{2}}+1\right) \geqslant \beta$$

$$P\left(\frac{X_{max}}{\sqrt[n]{\frac{1+\beta}{2}}+1} < \Theta < \frac{X_{max}}{\sqrt[n]{\frac{1-\beta}{2}}+1}\right) \geqslant \beta$$

доверительный интервал:

$$\left(\frac{X_{max}}{\sqrt[n]{\frac{1+\beta}{2}}+1} ; \frac{X_{max}}{\sqrt[n]{\frac{1-\beta}{2}}+1}\right).$$

e) ОММ $\tilde{\Theta} = \frac{2}{3}\overline{X}$ $\qquad \tilde{\alpha}_1 = \overline{X} = \frac{1}{n}\sum_i X_i$

$f(\alpha) = \frac{2}{3}\alpha_1 \qquad \nabla f = \frac{2}{3} \qquad \tilde{\alpha}_2 = \frac{1}{n}\sum_i X_i^2$

$$\underbrace{\frac{\frac{2}{3}\overline{X}-\Theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}}\sqrt{n}}_{A} \sim N(0;1)$$

$$u_{0{,}025} < A < u_{0{,}975}$$

доверит. интервал

$$-1{,}96 < A < 1{,}96$$

$$\frac{-1{,}96\sqrt{\frac{4}{9}(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}}{\sqrt{n}} + \frac{2}{3}\overline{X} < \Theta < \frac{1{,}96\sqrt{\frac{4}{9}(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}}{\sqrt{n}} + \frac{2}{3}\overline{X}$$

ОММП

$$\tilde{\theta} = \frac{X_{max}}{2}$$

возьмем несмещ. $\tilde{\theta} = \frac{n+1}{2n+1} X_{max}$

$$I(\theta) = \int_{\theta}^{2\theta} \left( \frac{\partial(-\ln\theta)}{\partial\theta} \right)^2 \frac{1}{\theta}\, dx = $$

$f = \theta \quad f' = 1$

$$= \frac{1}{\theta^3}\,\theta = \frac{1}{\theta^2}$$

$$\sigma = \theta$$

$$\underbrace{\frac{\frac{(n+1)X_{max}}{2n+1} - \theta}{\frac{n+1}{2n+1}X_{max}}\sqrt{n}}_{A} \rightsquigarrow N(0;1)$$

$$-1{,}96 < A < 1{,}96$$

$$\frac{-1{,}96}{\sqrt{n}} \cdot \frac{n+1}{2n+1} X_{max} - \frac{(n+1)}{2n+1} X_{max} < -\theta < \frac{1{,}96}{\sqrt{n}}\tilde{\theta} - \tilde{\theta}$$

$$\frac{n+1}{2n+1} X_{max}\left(1 - \frac{1{,}96}{\sqrt{n}}\right) < \theta < \frac{n+1}{2n+1} X_{max}\left(1 + \frac{1{,}96}{\sqrt{n}}\right).$$