№ 1.12

Опр-ть $h = \text{const}$ – оптимальный

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$\max_{[x-h,\, x+h]} |f'''(x)| \leq 100$$

абс. погрешность при задании $f(x)$ и $f(x \pm h)$ $\leq \Delta = 0{,}1$

$$\varepsilon_1 = \left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right| =$$

$$= \left| f'(x) - \frac{\cancel{f(x)} + f'(x)h + \cancel{\frac{f''(x)h^2}{2}} + \frac{f'''(x)}{6}h^3 - \left(\cancel{f(x)} - f'(x)h + \cancel{\frac{f''(x)h^2}{2}} - \frac{f'''(x)h^3}{6}\right)}{2h} \right| =$$

$$= \left| f'(x) - \frac{2f'(x)h + \frac{f'''(x)h^3}{3}}{2h} \right| = \left| \frac{f'''(x)h^2}{6} \right| \leq$$

$$\leq \frac{50}{3}h^2$$

$$\varepsilon_2 = \Delta \left| \frac{f(x+h) - f(x-h)}{2h} \right| \leq \frac{0{,}1}{h}$$

$$\varepsilon_\Sigma = \frac{50}{3}h^2 + \frac{0{,}1}{h} \qquad \frac{d\varepsilon_\Sigma}{dh} = \frac{100}{3}h - \frac{0{,}1}{h^2} = 0$$

$$\Rightarrow h^3 = \frac{0{,}3}{100}$$

$$h = \sqrt[3]{3 \cdot 10^{-3}} \approx \underline{0{,}14}$$

№ 1.17.

$\{x_n\}$, $n = 0, 1, 2, \ldots$

$x_{n+1} - 5x_n = 4$

x_0 известно с относ. погрешностью 10^{-6}

при каких x_0 относ. погрешность при вычислении x_n будет быстро возрастать с ростом n.

Если x_0 : $x_n = x_{n+1} = x_0$, то отн. погрешность будет слабо возрастать ↳

$x_0 - 5x_0 = 4 \Rightarrow x_0 = -1$

№ 2.1.1

$\max\limits_k (d_k |x_k|)$ $d_k > 0$ $k = \overline{1,n}$ - норма

вектора

Проверим:

1) $\forall k$ $d_k |x_k| \geq 0 \Rightarrow \max\limits_k (d_k |x_k|) \geq 0$

$\|x\| = \max\limits_k (d_k |x_k|) = 0 \Leftrightarrow \forall k \ |x_k| = 0 \Leftrightarrow$ ⓐ

$\Leftrightarrow \vec{x} = \vec{0}$

2) $\max\limits_k (d_k |a x_k|) = |a| \max\limits_k (d_k |x_k|)$

3) $\max\limits_k (d_k |x_k|) = \|x\|$ $\max\limits_k (d_k |y_k|) = \|y\|$

$\|x + y\| = \max\limits_k (d_k |x_k + y_k|) \leq$

$$\le \max_k \left(d_k \left(|x_k| + |y_k| \right) \right) = \max_k \left(d_k |x_k| \right) + \max_k \left(d_k |y_k| \right) = \|x\| + \|y\|$$

$\Rightarrow \max_k \left(d_k |x_k| \right)$ норма.

№ 2.1.19.

$$\|x\| = \sum_{k=1}^{n} d_k |x_k| \qquad d_k > 0 \quad k = \overline{1, n}$$

$$\|A\| = \sup_{\forall \vec{x}} \frac{\|A x\|}{\|x\|} = \sup \frac{d_i |A_{ij}| |x_j|}{d_j |x_j|} =$$

$$= \max \frac{d_i |A_{ij}|}{d_j}$$

№2.3.13.

$A\vec{x} = \vec{b}$

(a) $A = \begin{pmatrix} \alpha & \beta & \alpha \\ \beta & \alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} - \begin{pmatrix} 0 & -\beta & -\alpha \\ 0 & 0 & -\beta \\ 0 & 0 & 0 \end{pmatrix} -$

$- \begin{pmatrix} 0 & 0 & 0 \\ -\beta & 0 & 0 \\ 0 & -\beta & 0 \end{pmatrix}$

$$\begin{vmatrix} \lambda\alpha & \beta & 0 \\ \lambda\beta & \lambda\alpha & \beta \\ 0 & \lambda\beta & \lambda\alpha \end{vmatrix} = 0$$

$\lambda^3\alpha^3 - \lambda^2\alpha\beta^2 - \lambda^2\beta^2\alpha = 0$

$\lambda^3\alpha^2 - 2\lambda^2\beta^2 = 0$

$\lambda_{1,2} = 0$

$\lambda\alpha^2 = 2\beta^2$

$\lambda_3 = \frac{2\beta^2}{\alpha^2} \Rightarrow |\lambda_3| < 1$

$\left|\frac{2\beta^2}{\alpha^2}\right| < 1$

$2\beta^2 < \alpha^2$

$2|\beta| < |\alpha|$

8 $A = \begin{pmatrix} 0 & \beta & \alpha \\ \beta & \alpha & \beta \\ \alpha & \beta & \alpha \end{pmatrix}$

$$\begin{vmatrix} 0 & \beta & \alpha \\ \lambda\beta & \lambda\alpha & \beta \\ \lambda\alpha & \lambda\beta & \lambda\alpha \end{vmatrix} = 0$$

$\lambda\alpha\beta^2 + \lambda^2\alpha\beta^2 - \lambda^2\alpha^3 - \lambda^2\beta^2\alpha = 0$

$\lambda\alpha\beta^2 = \lambda^2\alpha^3$

$\lambda_1 = 0$

$\beta^2 = \lambda\alpha^2$

$\lambda_2 = \dfrac{\beta^2}{\alpha^2}$

$|\lambda_2| < 1$

$\left|\dfrac{\beta}{\alpha}\right| < 1.$

N2.3.19.

$$\vec{x}^{(m+1)} = (E - \tau A)\vec{x}^{(m)} + \tau \vec{b}$$

$\forall \vec{x}^{(0)}$ $\quad \tau = \frac{1}{2}$.

ⓐ $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ $\quad \lambda_1 = 3,41421356$

$\lambda_2 = 2$

$\lambda_3 = 0,58578644$

$$|1 - \tau \lambda_i| < 1$$

$$0 < \lambda_i \tau < 2$$

$$0 < \tau < \left\{ \min\left(\frac{2}{\lambda_i} \right) \right\}$$

$$0 < \tau < 0,58578643762690 53$$

...

N3.1.1.

$$f(x) = 20x^3 - 4x^2 - 5x + 1 = 0.$$

из осн. т. алгебры: все действ. корни лежат на $[\alpha; \beta]$

$$\alpha = \frac{|a_3|}{|a_3| + B} = \frac{1}{1+20} = \frac{1}{21} \quad \beta = 1 + \frac{|A|}{|a_0|} = 1 + \frac{5}{20} = \frac{5}{4}$$

$$\left[\frac{1}{21}; \frac{5}{4}\right].$$

ⓐ по т. Декарта: $\{20; -4; -5; 1\}$

число перемен знаков $= 2 \Rightarrow f(x)$ имеет 2 полож. корня на $[\frac{1}{21}; \frac{5}{4}]$

$f(-x) = -20x^3 - 4x^2 + 5x + 1 = 0$

по т. Декарта $\{-20; -4; 5; 1\}$

число перемен. знаков $= 1 \Rightarrow$

$f(x)$ имеет 1 отриц. корень

на $[-\frac{5}{4}; -\frac{1}{21}]$

$\Rightarrow \exists$ 3 корня на $\mathbb{R}$.

б) т. Бюдана-Фурье.

$f_0(x) = 20x^3 - 4x^2 - 5x + 1$ $f_0(\alpha) > 0$ $f_0(\beta) > 0$

$f_1(x) = 60x^2 - 8x - 5$ $f_1(\alpha) < 0$ $f_1(\beta) > 0$

$f_2(x) = 120x - 8$ $f_2(\alpha) < 0$ $f_2(\beta) > 0$

$f_3(x) = 120$ $f_3(\alpha) > 0$ $f_3(\beta) > 0$

$\Delta_1 = 2 - 0 = 2$

$\Delta_2 = 1 - 0 = 1$

$\Rightarrow$ 3 корень на $\mathbb{R}$.

в) по т. Штурма.

$f_0(x) = 20x^3 - 4x^2 - 5x + 1$

$f_1(x) = 60x^2 - 8x - 5$

$f_2 = -\frac{f_0}{f_1}$ $\Delta_1 = 2 - 2 = 0$

$f_3 = -\frac{f_1}{f_2}$ $\Delta_2 = 2 - 2 = 0$

Локализация

$f\left(\frac{1}{21}\right) f\left(\frac{1}{4}\right) < 0 \Rightarrow x_1 \in \left[\frac{1}{21}; \frac{1}{4}\right]$

$f\left(\frac{1}{3}\right) f\left(\frac{4}{5}\right) < 0 \Rightarrow x_2 \in \left[\frac{1}{3}; \frac{4}{5}\right]$

$f\left(-\frac{1}{3}\right) f\left(-\frac{3}{5}\right) < 0 \Rightarrow x_3 \in \left[-\frac{3}{5}; -\frac{1}{3}\right].$

N 3.2.12.

$f(x) = e^x - x^2 \qquad x_* \approx -0,7$

1) $x_{n+1} = 2\ln(-x_n)$ $\qquad g_1(x) = 2\ln(-x)$

2) $x_{n+1} = -\sqrt{e^{x_n}}$ $\qquad g_2(x) = -\sqrt{e^x}$

3) $x_{n+1} = e^{x_n}\frac{1}{x_n}$ $\qquad g_3(x) = e^x \frac{1}{x}$

4) $x_{n+1} = \frac{1}{6}x_n - \frac{5}{6}e^{\frac{x_n}{2}}$ $\qquad g_4(x) = \frac{1}{6}x - \frac{5}{6}e^{\frac{x}{2}}$

$|g_1'(x_*)| = 2,85 > 1$ – метод расх-ся

$|g_2'(x_*)| = 0,35 < 1$ – сх-ся

$|g_3'(x_*)| = 1,72 > 1$ – метод расх-ся

$|g_4'(x_*)| = 0,12 < 1$ – сх-ся

4й имеет наибольшую скорость сх-ти

N 3.2.44

$f(x) = x^2 - \cos x.$

$f'(x) = 2x + \sin x$

N 3.2.35 $\quad x_* = a^{1/5}$

$$x_{n+1} = A x_n + \frac{Ba}{x_n^4} + \frac{Ca^2}{x_n^9}$$

$$\varphi(x) = Ax + \frac{aB}{x^4} + \frac{Ca^2}{x^9}$$

$$\varphi(a^{1/5}) = Aa^{1/5} + B\frac{a}{a^{4/5}} + C\frac{a^2}{a^{9/5}} =$$

$$= Aa^{1/5} + Ba^{1/5} + Ca^{1/5} = (A+B+C)a^{1/5}$$

$$\Rightarrow A + B + C = 1 \quad (1)$$

$$\varphi'(x) = A - 4\frac{aB}{x^5} - 9\frac{Ca^2}{x^{10}}$$

$$\varphi'(a^{1/5}) = A - 4B - 9C = 0 \quad (2)$$

$$\varphi''(x) = 20\frac{aB}{x^6} + 90\frac{Ca^2}{x^{11}}$$

$$\varphi''(a^{1/5}) = 20Ba^{-\frac{1}{5}} + 90Ca^{-1/5} = a^{-\frac{1}{5}}(20B + 90C) = 0 \quad (3)$$

$$\begin{cases} (1) \\ (2) \\ (3) \end{cases}$$

№ 3.2.22.

$$f(x) = x^2 - \cos x.$$

$$f'(x) = 2x + \sin x$$

№ 3.2.35 $\quad x_* = a^{1/5}$

$$x_{n+1} = A x_n + \frac{Ba}{x_n^4} + \frac{Ca^2}{x_n^9}$$

$$\varphi(x) = Ax + \frac{aB}{x^4} + \frac{Ca^2}{x^9}$$

$$\varphi(a^{1/5}) = Aa^{1/5} + B\frac{a}{a^{4/5}} + C\frac{a^2}{a^{9/5}} =$$

$$= Aa^{1/5} + Ba^{1/5} + Ca^{1/5} = (A+B+C)a^{1/5}$$

$$\Rightarrow A + B + C = 1 \quad (1)$$

$$\varphi'(x) = A - 4\frac{aB}{x^5} - 9\frac{Ca^2}{x^{10}}$$

$$\varphi'(a^{1/5}) = A - 4B - 9C = 0 \quad (2)$$

$$\varphi''(x) = 20\frac{aB}{x^6} + 90\frac{Ca^2}{x^{11}}$$

$$\varphi''(a^{1/5}) = 20Ba^{-\frac{1}{5}} + 90Ca^{-1/5} = a^{-\frac{1}{5}}(20B + 90C) = 0 \quad (3)$$

$$\begin{cases} (1) \\ (2) \\ (3) \end{cases}$$

$A = 0{,}72 \quad B = 0{,}36 \quad C = -0{,}08$

$\Rightarrow$ 3й порядок сх-ти

$$\varphi'''(a^{1/5})\Big|_{\substack{A=0{,}72\\ B=0{,}36\\ C=-0{,}08}} \neq 0.$$

N 3.2.36.

$x_* = 1 + \sqrt{2}$

$f(x) = x^2 - 2x - 1 = 0$

$$x_{n+1} = \frac{3x_n}{8} + \frac{5}{8} + \frac{3}{2(x_n-1)} - \frac{1}{2(x_n-1)^3}$$

$$\varphi(x) = \frac{3x}{8} + \frac{5}{8} + \frac{3}{2(x-1)} - \frac{1}{2(x-1)^3}$$

$$\varphi'(x) = \frac{3}{8} - \frac{3}{2(x-1)^2} + \frac{3}{2(x-1)^4}\Big|_{x=x^*} = 0$$

$$\varphi''(x) = 3\frac{1}{(x-1)^3} - 6\frac{1}{(x-1)^5}\Big|_{x=x^*} = 0$$

$$\varphi'''(x) = -9\frac{1}{(x-1)^4} + 30\frac{1}{(x-1)^6}\Big|_{x=x^*} \neq 0$$

$\Rightarrow$ 3й порядок сх-ти.

№ 3.2.44

$$x_* = \sqrt[3]{5} - 1$$

$$f(x) = x^3 + 3x^2 + 3x - 4 = 0$$

3-й порядок сх-ти

$$x_{n+1} = x_n - \frac{f_n}{(f'_x)_n} - \frac{(f''_{xx})_n}{2(f'_x)^3_n} f_n^2$$

$$f'_x = 3x^2 + 6x + 3$$

$$f''_{xx} = 6x + 6$$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n^2 + 3x_n - 4}{3x_n^2 + 6x_n + 3} -$$

$$- \frac{6x_n + 6}{2(3x_n^2 + 6x_n^2 + 3)^2} (x_n^3 + 3x_n^2 + 3x_n - 4)^2.$$

N 3.3.2.

$x^* \in [\pi; \frac{3\pi}{2}]$

$y^* \in [\frac{\pi}{4}; \frac{\pi}{2}]$

$$\begin{cases} x = \operatorname{arctg}(y) = \varphi_1(x, y) \\ y = \operatorname{arctg}(x) = \varphi_2(x, y) \end{cases}$$

МПИ: $\begin{cases} x_{n+1} = \operatorname{arctg}(y_n) + \pi \\ y_{n+1} = \operatorname{arctg}(x_n) \end{cases}$

дост. усл. сх-ти:

$$\max\left|\frac{\partial\varphi_1}{\partial x}\right| = \max\left|\frac{\partial\varphi_2}{\partial y}\right| = 0 < 1$$

$$\max\left|\frac{\partial\varphi_2}{\partial x}\right| = \max\left|\frac{1}{1+x^2}\right| = \frac{1}{1+\pi^2} < 1$$

$$\max\left|\frac{\partial\varphi_1}{\partial y}\right| = \max\left|\frac{1}{1+y^2}\right| = \frac{1}{1+\frac{\pi^2}{4}} < 1$$

$\Rightarrow$ МПИ сх-ся.

N 3.3.8

$$\begin{cases} \sin(x+y) - 1{,}5x = 0 \\ x^2 + y^2 = 1. \end{cases}$$

$$\begin{cases} f_1(x,y) = \sin(x+y) - 1{,}5x \\ f_2(x,y) = x^2 + y^2 - 1 \end{cases}$$

$$\frac{\partial f_1}{\partial x} = \cos(x+y) - 1{,}5 \qquad \frac{\partial f_1}{\partial y} = \cos(x+y)$$

$$\frac{\partial f_2}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial y} = 2y$$

$$\frac{\partial \vec{F}}{\partial \vec{y}} = \begin{pmatrix} \cos(x+y) - 1{,}5 & \cos(x+y) \\ 2x & 2y \end{pmatrix}$$

$$\left(\frac{\partial \vec{F}}{\partial \vec{y}}\right)^{-1} = \frac{1}{2y(\cos(x+y)-1{,}5) - 2x\cos(x+y)} \cdot$$

$$\cdot \begin{pmatrix} 2y & -\cos(x+y) \\ -2x & \cos(x+y) - 1{,}5 \end{pmatrix}$$

$$x_{n+1} = x_n - \frac{2y_n(\sin(x_n+y_n) - 1{,}5x_n) - \cos(x_n+y_n)(x_n^2+y_n^2-1)}{2y(\cos(x_n+y_n)-1{,}5) - 2x_n\cos(x_n+y_n)}$$

$$y_{n+1} = y_n - \frac{-2x(\sin(x_n+y_n) - 1{,}5x_n) + (\cos(x_n+y_n)-1{,}5)(x_n^2+y_n^2-1)}{2y_n(\cos(x_n+y_n) - 1{,}5) - 2x_n\cos(x_n+y_n)}$$