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Competitive Programming

From Problem 2 Solution in O(1)

Elementary Math

Introduction

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Mathematics and CS

- Directly
  - Many CS components need math
  - Discrete Mathematics is critical area (E.g. Graph Theory)
  - Machine Learning needs Algebra, Calculus, Probability..
  - 3-D motion and graphics...Robot Trajectory...etc
- Indirectly
  - Builds logical/critical thinking skills
  - Thinking abstractly and concretely
  - Problem solving skills
- Commercial apps?
  - Most of them don’t need math.

Mathematics and CP

- In competitive programming (CP), Math is an important topic.
- Many problems needs **basic high school** skills
- Others may focus on **Discrete Mathematics**
  - Graph theory is the most important field
  - Number theory & Combinatorics are nice frequent topics
  - Little Geometry, Probability and Game Theory
- Problem Setters may avoid geometry and probability problems, due to output **precision** problems

#include<cmath> in C++

- C++ offers for us some ready-made functions
- Trigonometric, Hyperbolic, Exponential , Logarithmic, Power, Rounding, Remainder
- Please, play with Majority of these functions
  - **At least**: floor, ceil, round, fabs, sqrt, exp, log, log2, log10, pow, cos, cosh, acosh, isnan
  - We will explore some of them
- Other languages should also have similar functions

Machine Arithmetic

- + - * / are the usual arithmetic operations
- 32 bit numbers are suitable most of time
  - -2147483648 to 2147483647
  - Short fact: You have up to 2 billions
- 64 bit numbers can cover much wider range
  - 9,223,372,036,854,775,808 (9 * 10^18)
  - This is tooo big and fit **most** of time for your purpose
  - But slower (8 bytes vs 4), so use it only if needed
- Still we can face over/underflow
  - In intermediate computations or final results
- doubles range: 1.7E +/- 308 (15 digits)

Real Numbers

- **Rational** Numbers: can be represented as **fraction**: ⅙, 7/2, 9/3, 5/1. **Irrational**: Pi = 3.1415926, sqrt(2)
- **Decimal expansion** of fraction, to write it
  - 1/16 = 0.0625, ½ = 0.5
  - 1/12 = 0.08333333333 .. 3 repeats for ever
  - 5/7 = 0.714285714285714285…. 714285 repeat forever
  - ⅙ = 0.1(6), 1/12 = 0.08(3). 5/7 = 0.(71428), ½ = 0.5(0)
- How to know # of digits before cycle of n/d?
  - Programming? mark reminders of long division
  - Mathematically? See

Double Comparison

- Operations can result in double value
  - Internal value can be shifted with +/- EPS
  - EPS is a very small amount (e.g. 1e-10)
  - e.g. x = 4.7 may be internally 4.7000001 or 4.69999999
  - so if(x == 4.7) fails! Although printing shows 4.7
- Printing zero is tricky (-0.00 problem)
  - Compare x first with zero. If zero, then x = 0

```
// return 0 for a==b, 1 for a > b, -1 for a < b
int comp_double(double a, double b)
{   // if very small difference, then equal
    if (fabs(a-b) <= 1e-10)
        return 0;
    return a < b ? -1 : 1;
}
```

Big Integers

- Sometimes computations are too big
  - 1775! is 4999 digits
- Java/C# implement BigInteger library to handle such big computations
- In C++, you may implement it by yourself.
  - Exercise: Try to represent/add/multiply big numbers
  - I wrote this code when was young. Use freely (not in TC)
  - Main idea: Think in number as **reversed array**
- In competitions, nowadays problem setters avoid such problems most of time

Big Integers: Factorial

- Create a big array. Initialize it to 1
- Think in it as **reversed** array. Arr[0] first digit
  - e.g. 120 represented as 021 (arr[0] = 0)
- From i = 2 to N
  - Multiply i in every cell
  - For every cell, if its value > 9 handle its carry
    - v => (v%10, v/10)
  - For last cell, check if it has carry (typically will have), and put it in next cell, **AS LONG AS** there is a carry
- See code example in previous page link

Big Integers: Factorial

1	0	0	0	0	0	0	0	0	0	Initalize 1 = 1!
2	0	0	0	0	0	0	0	0	0	Multiply 2 = 2!
6	0	0	0	0	0	0	0	0	0	Multiply 3 = 3!
24	0	0	0	0	0	0	0	0	0	Multiply 4

Remember...Keep only a digit in cell...move carry to next cell

4	2	0	0	0	0	0	0	0	0	4! But REVERSED
20	10	0	0	0	0	0	0	0	0	Multiply 5
0	12	0	0	0	0	0	0	0	0	Move Carry 1st cell
0	2	1	0	0	0	0	0	0	0	Move Carry 2nd cell = 5!

Be careful from last cell...we may keep shift carry to right many times when N is large

Rounding Values

- Rounding is replacing value with another approximate value...many types and styles
- In C++, we have 4 rounding functions
  - **round**: nearest integer to x [halfway cases away from 0]
  - **floor**: round down
  - **ceil**: round up
  - **trunc**: rounds toward zero (remove fraction)
- In integers, x/y is floor of results
  - ceil(x, y) = (x+y-1)/y
  - round(x, y) = (x+y/2)/y [if x > 0] and (x-y/2)/y [x < 0]

Rounding Values: Examples

- Be careful from -ve and 0.5

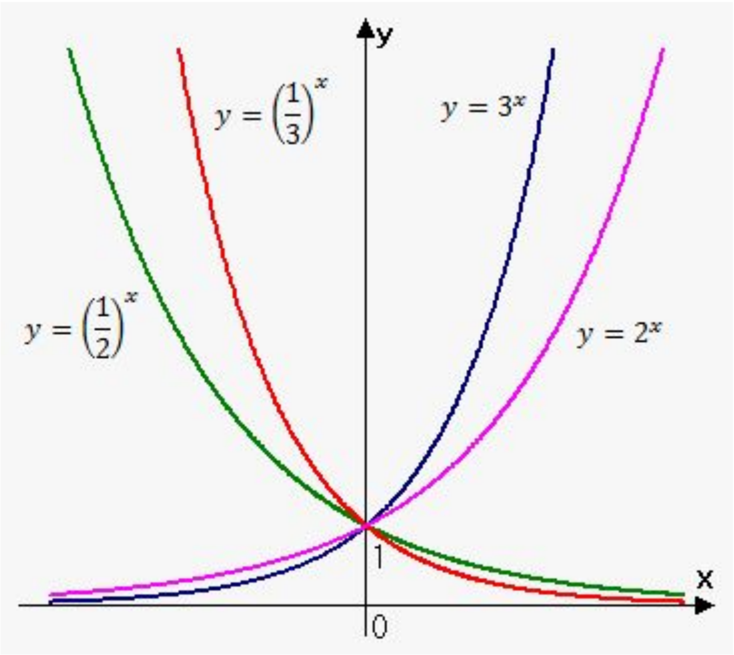
value	round	floor	ceil	trunc
2.3	2.0	2.0	3.0	2.0
3.8	4.0	3.0	4.0	3.0
5.5	6.0	5.0	6.0	5.0
-2.3	-2.0	-3.0	-2.0	-2.0
-3.8	-4.0	-4.0	-3.0	-3.0
-5.5	-6.0	-6.0	-5.0	-5.0

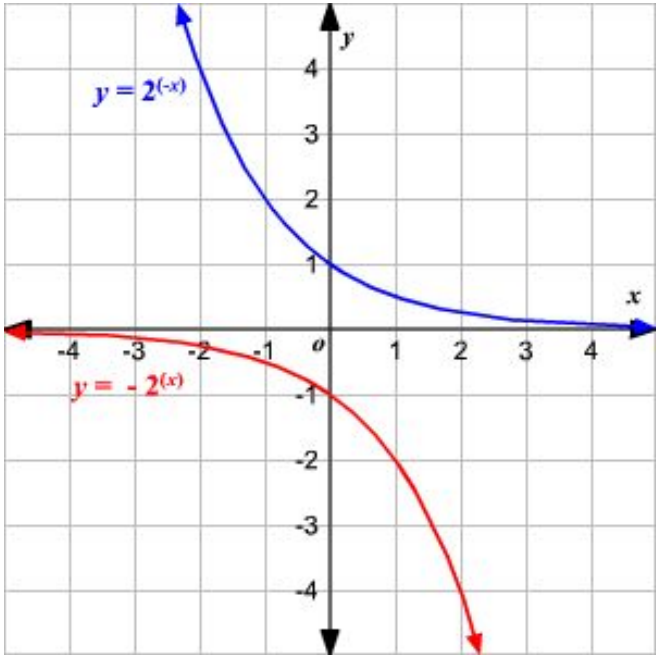
- round(x) == x < 0 ? ceil(x-0.5) : floor(x+0.5);
- To round to multiple of a specified amount
  - round(x, m) = round(x / m) * m
  - round(48.2 seconds, 15) = 45 seconds
  - round(2.1784 dollars, 0.01 (1 cent)) = 2.18 dollars

Exponential function

- If we have 2 jackets, 2 jeans & 2 shoes, I can have $2x2x2 = 2^3 = 8$ different clothing styles
- Exponential function: $y = b^x$
- b (base), x (exponent): real,integer,+ve,-ve
- Popular values: 2, 10, e [e is **Euler's** number ~= 2.7]
  - a 64 bit integer stores $\mathbf{2^{64}}$ numbers … a very **big** number
- $2^{-x} = (½)^{\mathbf{x}} = 0.5^x$ and $2^{\mathbf{x}} = (½)^{-x} = 0.5^{-x}$
- In C++: pow(2, 3) $= 2^3 = 8$
- Exponential indicates growing **fast**

Exponential function: Graphs

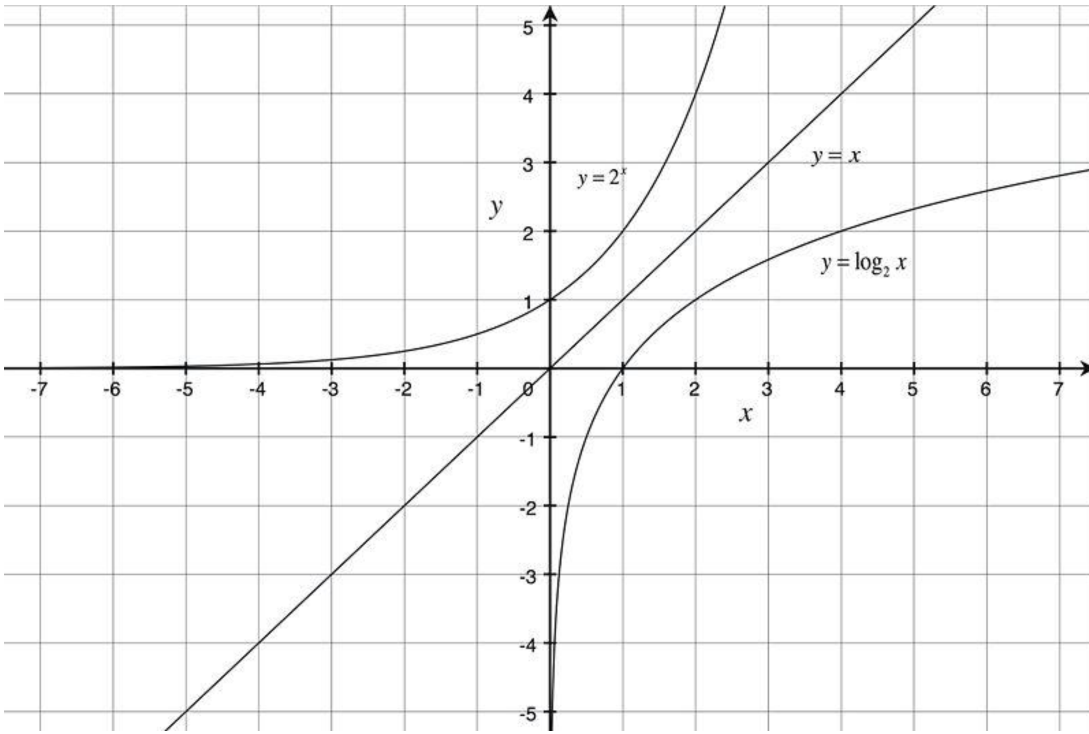


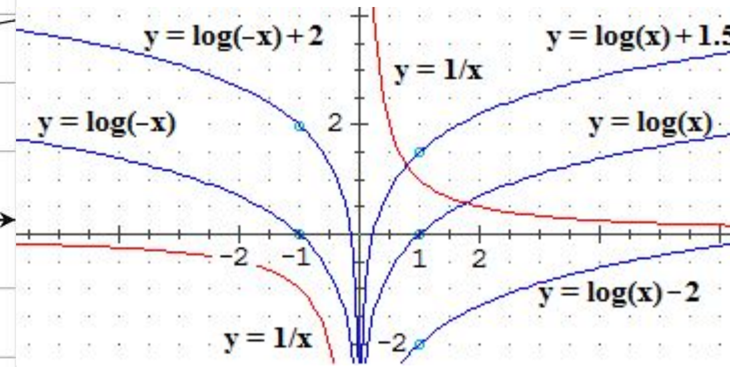


Logarithm

- It is the **inverse** operation to **exponentiation**
- $y = b^x$ ==> $\log_b y = x$
  - $\log_{10} 1000$ = how many 10 multiplications = 1000? 3
  - $\log_2 16$ = how many 2 multiplications = 16? 4
  - $\log_{10} 0.001$ = how many 10 divisions = 1/1000? -3
- b=[10, e, 2] => (common, natural, binary) log
  - Math notations: lg(x), ln(x), lb(x)
  - In c++: log10(x), log(x), log2(x)
- log is **strictly** increasing for $b > 1$
- log is strictly decreasing for $0 < b < 1$

Logarithm: Graphs





log is **strictly** increasing function. Strictly [nothing equal], increasing, go up.
1 2 5 5 7 9 [Increasing]
1 2 5 6 7 9 [strictly Increasing]
9 7 5 5 2 1[decreasing]
9 7 6 5 2 1 [strictly decreasing]

Logarithm Operations

	Formula	Example
product	$\log_b(xy) = \log_b(x) + \log_b(y)$	$\log_3(243) = \log_3(9 \cdot 27) = \log_3(9) + \log_3(27) = 2 + 3 = 5$
quotient	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$\log_2(16) = \log_2\left(\frac{64}{4}\right) = \log_2(64) - \log_2(4) = 6 - 2 = 4$
power	$\log_b(x^p) = p\log_b(x)$	$\log_2(64) = \log_2(2^6) = 6\log_2(2) = 6$
root	$\log_b \sqrt[p]{x} = \frac{\log_b(x)}{p}$	$\log_{10}\sqrt{1000} = \frac{1}{2}\log_{10}1000 = \frac{3}{2} = 1.5$

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}. \qquad \log_b(x) = \frac{\log_{10}(x)}{\log_{10}(b)} = \frac{\log_e(x)}{\log_e(b)}. \qquad b = x^{\frac{1}{\log_b(x)}}.$$

- $\log_b b = 1$, $\log_b 1 = 0$, $\log_b 0 = -oo$, $\log_1 x =$ undefined
- $b^{logb(x)} = x$ => take $\log_b$ for the equation to get x
- xb^y => $b^{logb(x) + y}$

Logarithm and # of digits

- $\log_{10}(10x) = \log_{10}(10) + \log_{10}(x) = 1 + \log_{10}(x).$
- base 10 can be used to know # of digits
  - # digits = 1 + floor($\log_{10}$(x))
  - log10(1000) = 3 => 4 digits
  - log10(1430) = 3.15 => 4 digits
  - log10(9999) = 3.99 => 4 digits
  - log10(10000) = 4 => 4 digits
  - So from 1000 to 10000-1, we have log10(x) = 3.xyz..
- Generally, # of digits in base b.
  - log2(16) = 4 => 5 bits. [16 in base 10 = 10000 in base 2]
- Homework: # of digits of factorial n?

Math Materials

- Knowledge of interest here
- Mathematics part in
  - Programming Challenge
  - Competitive Programming
  - Algorithms Books (e.g. CLR)
- Other Math Books
  - Concrete Mathematics
  - Discrete Mathematics and Its Applications
  - Mathematics for Computer Science (2013)
- Solving...Solving...Solving
  - if can't solve..see editorial/solution...take notes

Time To Solve

- There are some ad-hoc problems in the video
- There also some problems on Big Integers
  - This type of problems usually don't appear nowadays
  - Why?: problem setters avoid languages advantages
- Warmup by solving some of them :)
- Also read this cheat sheet

تم بحمد الله

علمكم الله ما ينفعكم

ونفعكم بما تعلمتم

وزادكم علماً