What I think about when I think about calculus

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August 5, 2015

Introduction

I haven't done calculus in decades. Well, that's no longer true because I recently took a single-variable calculus course. The 13-week online class was taught by Robert Ghrist. He's the calculus teacher you wish you had.

This document contains some of the ideas that we covered in class and some of the ideas that occured to me while taking the class.

Partial sums

After helping my daughter with arithmetic and geometric sequences and series in Algebra 2 and Precalculus I didn't think I could get excited about finding partial sums. Yet that's exactly what happened when we covered the subject in class.

Our motivation in this section is to simplify the partial sum

$$S(k) = \sum_{n=0}^{k} n^3 - 4n^2.$$

Notice that the series is neither arithmetic nor geometric—finally something new.

The partial sum is analogous to the definite integral $I(k)$ in continuous calculus:

$$I(k) = \int_{n=0}^{k} f(n) dn = \int_{n=0}^{k} \frac{dF(n)}{dn} dn = F(k) - F(0)$$

where dn corresponds to $\Delta n = 1$ in the discrete case and $F(n)$ is an antiderivative of $f(n)$.

We will proceed in an analogous fashion for the discrete case:

$$\begin{aligned} S(k) &= \sum_{n=0}^{k} a_n = \sum_{n=0}^{k} (\Delta b)_n \\ &= (b_1 - b_0) + (b_2 - b_1) + \ldots + (b_k - b_{k-1}) + (b_{k+1} - b_k) \\ &= b_{k+1} - b_0 \end{aligned}$$

where $(\Delta b)_n = b_{n+1} - b_n$, the forward difference, is the discrete derivative and b is a discrete antiderivative of a.

The problem of simplifying $S(k)$ has now been reduced to finding a discrete antiderivative of $n^3 - 4n^2$. In other words we need to find a sequence b_n such that $(\Delta b)_n = n^3 - 4n^2$. It is easy to find a recursion relation: $b_{n+1} = n^3 - 3n^2 + b_n$. However, to fully simplify $S(k)$, we need an analytic solution for b, which we will obtain by using falling powers.

Falling powers are defined as

$$n^{\underline{k}} = n(n-1)(n-2)\ldots(n-k+2)(n-k+1)$$

where k is an integer and, by a second definition, $n^{\underline{0}} = 1$. For example

$$n^{\underline{3}} = n(n-1)(n-2) = n^3 - 3n^2 + 2n.$$

Solving for n^3 gives $n^3 = n^{\underline{3}} + 3n^2 - 2n$. Similarly $n^2 = n^{\underline{2}} + n$. After a little bit of algebra we get

$$(\Delta b)_n = n^3 - 4n^2 = n^{\underline{3}} - n^{\underline{2}} - 3n^{\underline{1}}.$$

The whole point of using falling powers is to (gently) coerce the problem into a form solvable by the familiar mechanics of continuous calculus. After three lines of algebra, which we will skip, the discrete derivative of $n^{\underline{k}}$ is given by the discrete power rule: $\Delta n^{\underline{k}} = kn^{\underline{k-1}}$. Therefore b is quite simply given by

$$b_n = \frac{1}{4}n^{\underline{4}} - \frac{1}{3}n^{\underline{3}} - \frac{3}{2}n^{\underline{2}} + C$$

where C is a constant. (To see that b_n is correct we would differentiate it to obtain $n^{\underline{3}} - n^{\underline{2}} - 3n^{\underline{1}}$.)

Putting it all together:

$$\begin{aligned} S(k) &= \sum_{n=0}^{k} n^3 - 4n^2 \\ &= \sum_{n=0}^{k} \Delta(\frac{1}{4}n^{\underline{4}} - \frac{1}{3}n^{\underline{3}} - \frac{3}{2}n^{\underline{2}} + C) \\ &= \frac{1}{4}(k+1)^{\underline{4}} - \frac{1}{3}(k+1)^{\underline{3}} - \frac{3}{2}(k+1)^{\underline{2}} \\ &= \frac{1}{12}k(k+1)(3k^2 - 13k - 8) \end{aligned}$$

where, if we wish to use the last line, $k > 2$ lest we run into troubles such as $3^{\underline{4}} = 0$. For $2 \geq k \geq 0$, the partial sum $S(k)$ is easily calculated by hand.

Let's try a simpler example:

$$\sum_{n=0}^{k} n^2 = \sum_{n=0}^{k} n^{\underline{2}} + n^{\underline{1}} = \sum_{n=0}^{k} \Delta(\frac{1}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}}) = \frac{1}{3}(k+1)^{\underline{3}} + \frac{1}{2}(k+1)^{\underline{2}} = \frac{1}{6}k(k+1)(2k+1)$$

where, if we wish to use the last part, $k > 1$.

Without much effort we were able to use the results we already know from continuous calculus to find partial sums of discrete series. The techniques we used in this section allow us to find partial sums of sequences a_n that can be written as polynomials in n and some of the sequences that can be written as rational functions of n.