What I think about when I think about calculus

Keith Goodman

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Introduction

I haven't done calculus in decades. Well, that's no longer true because I recently took a single-variable calculus course. The 13-week online class was taught by Robert Ghrist. He's the calculus teacher you wish you had.

This document contains some of the ideas that we covered in class and some of the ideas that occured to me while taking the class.

Finding partial sums using descrete calculus

After helping my daughter with arithmetic and geometric sequences and series in Algebra 2 and Precalculus I didn't think I could get excited about finding partial sums. Yet that's exactly what happened when we covered the subject in class.

Our motivation in this section is to simplify the partial sum

$$S(k) = \sum_{n=0}^{k} n^3 - 4n^2.$$

Notice that the series is neither arithmetic nor geometric—finally something new.

The partial sum is analogous to the definite integral $I(k)$ in continuous calculus:

$$I(k) = \int_{n=0}^{k} f(n)dn = \int_{n=0}^{k} \frac{dF(n)}{dn}dn = F(k) - F(0)$$

where dn corresponds to $\Delta n = 1$ in the discrete case and $F(n)$ is an antiderivative of $f(n)$.

We will proceed in an analogous fashion for the discrete case:

$$\begin{aligned} S(k) &= \sum_{n=0}^{k} a_n = \sum_{n=0}^{k} (\Delta b)_n \\ &= (b_1 - b_0) + (b_2 - b_1) + \ldots + (b_k - b_{k-1}) + (b_{k+1} - b_k) \\ &= b_{k+1} - b_0 \end{aligned}$$

where $(\Delta b)_n = b_{n+1} - b_n$, the forward difference, is the discrete derivative and b is a discrete antiderivative of a.

The problem of simplifying $S(k)$ has now been reduced to finding a discrete antiderivative of $n^3 - 4n^2$. In other words we need to find a sequence b_n such that $(\Delta b)_n = n^3 - 4n^2$. It is easy to find a recursion relation: $b_{n+1} = n^3 - 3n^2 + b_n$. However, to fully simplify $S(k)$, we need an analytic solution for b, which we will obtain by using falling powers.

Falling powers are defined as

$$n^{\underline{k}} = n(n-1)(n-2)\ldots(n-k+1)$$

where k is an integer and, by a second definition, $n^{\underline{0}} = 1$. For example

$$n^{\underline{3}} = n(n-1)(n-2) = n^3 - 3n^2 + 2n.$$

Solving for n^3 gives $n^3 = n^{\underline{3}} + 3n^2 - 2n$. Similarly $n^2 = n^{\underline{2}} + n$. After a little bit of algebra we get

$$(\Delta b)_n = n^3 - 4n^2 = n^{\underline{3}} - n^{\underline{2}} - 3n^{\underline{1}}.$$

The whole point of using falling powers is to (gently) coerce the problem into a form solvable by the familiar mechanics of continuous calculus. After three lines of algebra, the discrete derivative of $n^{\underline{k}}$ is given by the discrete power rule: $\Delta n^{\underline{k}} = kn^{\underline{k-1}}$:

$$\begin{aligned}\Delta n^{\underline{k}} &= (n+1)^{\underline{k}} - n^{\underline{k}} \\ &= (n+1)n^{\underline{k-1}} - n^{\underline{k-1}}(n-k+1) \\ &= kn^{\underline{k-1}}\end{aligned}$$

Therefore b is quite simply given by

$$b_n = \frac{1}{4}n^{\underline{4}} - \frac{1}{3}n^{\underline{3}} - \frac{3}{2}n^{\underline{2}} + C$$

where C is a constant. (To see that b_n is correct we would differentiate it to obtain $n^{\underline{3}} - n^{\underline{2}} - 3n^{\underline{1}}$.)

Putting it all together:

$$\begin{aligned}S(k) &= \sum_{n=0}^{k} n^3 - 4n^2 \\ &= \sum_{n=0}^{k} \Delta(\frac{1}{4}n^{\underline{4}} - \frac{1}{3}n^{\underline{3}} - \frac{3}{2}n^{\underline{2}} + C) \\ &= \frac{1}{4}(k+1)^{\underline{4}} - \frac{1}{3}(k+1)^{\underline{3}} - \frac{3}{2}(k+1)^{\underline{2}} \\ &= \frac{1}{12}k(k+1)(3k^2 - 13k - 8)\end{aligned}$$

where, if we wish to use the last line, $k > 2$ lest we run into troubles such as $3^{\underline{4}} = 0$. For $2 \geq k \geq 0$, the partial sum $S(k)$ is easily calculated by hand.

Let's try a simpler example:

$$\sum_{n=0}^{k} n^2 = \sum_{n=0}^{k} n^{\underline{2}} + n^{\underline{1}} = \sum_{n=0}^{k} \Delta\left(\frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2}\right) = \frac{(k+1)^{\underline{3}}}{3} + \frac{(k+1)^{\underline{2}}}{2} = \frac{k(k+1)(2k+1)}{6}$$

where, if we wish to use the last part, $k > 1$.

Without much effort we adapted widely known results from continuous calculus to find partial sums of discrete series. The techniques in this section allow us to find partial sums of sequences a_n that can be written as polynomials in n and some that can be written as rational functions of n.

Taylor series of composite functions

Let's say the bottleneck of a computer program is the calculation of $\cos(\sin x)$ and $\sin(\cos x)$. And that we are only interested in values of x near zero. Our job is to speed things up. We decide to replace both composite functions with the first few terms of their Taylor series. There are two ways we could proceed: (1) a messy brute-force calculation of derivatives; or (2) making a composition from Taylor series we already know, followed by messy algebra. We opt for the second choice.

The first few terms of the Taylor series of $\cos x$ and $\sin x$ centered at $x = 0$ are given by

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \\ \sin x &= x - \frac{x^3}{6} + O(x^5)\end{aligned}$$

Plugging the Taylor series of $\sin x$ into that of $\cos x$ and keeping terms up to order x^4 gives:

$$\begin{aligned}\cos(\sin x) &= 1 - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{24} + O(x^6) \\ &= 1 - \frac{1}{2}(x^2 - \frac{x^4}{3} + O(x^6)) + \frac{x^4}{24} + O(x^6) \\ &= 1 - \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6)\end{aligned}$$

Similarly, plugging the Taylor series of $\cos x$ into that of $\sin x$ and keeping terms up to order x^4 gives:

$$\begin{aligned}
\sin(\cos x) &= \cos x - \frac{\cos^3 x}{6} + O(x^5) \\
&= 1 - \frac{x^2}{4} + \frac{x^4}{24} + O(x^6) - \frac{1}{6}(1 - \frac{3x^2}{2} + \frac{7x^4}{8} + O(x^6)) + O(x^5) \\
&= \frac{5}{6} - \frac{x^2}{4} - \frac{5x^4}{48} + O(x^5)
\end{aligned}$$

Plugging these series into the program makes it run 100 times faster! The only problem is that the program now returns values that are way off even though the input values of x are near zero. Further checking reveals that the expansion of $\cos(\sin x)$ is accurate but the expansion of $\sin(\cos x)$ is way off.

What went wrong?

We expanded both $\cos x$ and $\sin x$ about $x = 0$. In the case of $\cos(\sin x)$, $\sin x$ is near zero when x is near zero. Therefore it was appropriate to expand $\cos x$ about zero as well. However in the case of $\sin(\cos x)$, $\cos x$ is near 1 when x is near zero. So for this case we should have expanded $\sin x$ about $x = 1$.

Being aware of this issue before setting out to solve the problem would have been nice (and is the whole point of this section), but it shouldn't be necessary. When we expanded $\sin(\cos x)$ we got terms that contain $(1 - x^2 + O(x^4))^{2n+1}$, where n is a non-negative integer. No matter how large a value of n we choose, the leading order term (for small x) will be $-(2n+1)x^2$. Therefore to find the coefficent of the x^2 term we would have needed to evaluate an infinite series.

The problem goes away if we exapand $\sin x$ about $x = 1$:

$$\sin(\cos x) = \sin(1) - \frac{\cos(1)}{2}x^2 + \frac{\cos(1) - 3\sin(1)}{24}x^4 + O(x^6)$$

Rising powers

Although we studied falling powers in class it never occured to me that there were rising powers. After seeing rising powers mentioned on the class forum, a web search lead to me to the following unanswered question on Mathematics Stack Exchange:

(1) Show that

$$n^{\underline{p}} = (-1)^p(-n)^{\overline{p}}$$

(2) Evaluate the sum

$$\sum_{a \le n < b} n^{\overline{p}}$$

What better way to learn about rising powers than to answer the question? Let's begin by considering the case where p is even:

$$\begin{aligned}
n^{\underline{p}} &= n(n-1)(n-2)\cdots(n-p+1) && \text{Definition of falling powers} \\
&= (-n)(-n+1)(-n+2)\cdots(-n+p-1) && \text{Multiply each term by } -1 \\
&= (-n)^{\overline{p}} && \text{Definition of rising powers}
\end{aligned}$$

In the case where p is odd, multiplying each term by -1 changes the sign of the product, therefore $n^{\underline{p}} = -(-n)^{\overline{p}}$. In general

$$n^{\underline{p}} = (-1)^p(-n)^{\overline{p}}$$

Equivalently, substituting $m = -n$ and rearranging gives

$$m^{\overline{p}} = (-1)^p(-m)^{\underline{p}}$$

Let's now find the partial sum:

$$
\begin{aligned}
\sum_{n=a}^{b-1} n^{\overline{p}} &= \sum_{n=a}^{b-1} (-1)^p (-n)^{\underline{p}} \\
&= \sum_{n=a}^{b-1} \Delta \left[-\frac{(-1)^p}{p+1} (-n)^{\underline{p+1}} \right] \\
&= \frac{(-1)^{p+1}}{p+1} (-n)^{\underline{p+1}} \Bigg|_{n=a}^{b} \\
&= \frac{(-1)^{p+1}}{p+1} \left[(-b)^{\underline{p+1}} - (-a)^{\underline{p+1}} \right] \\
&= \frac{(-1)^{p+1}}{p+1} \left[(-1)^{p+1} b^{\overline{p+1}} - (-1)^{p+1} a^{\overline{p+1}} \right] \\
&= \frac{b^{\overline{p+1}} - a^{\overline{p+1}}}{p+1}
\end{aligned}
$$

where I used the discrete version of the Fundamental Theorem of Integral Calculus, the discrete derivative, and $(\Delta a)_n = a_{n+1} - a_n$ is the forward difference. These results were derived in the section on Partial Sums.

An alternative method to calculate the sum could have started with

$$
\begin{aligned}
n^{\overline{p}} &= n(n+1)(n+2)\cdots(n+p-1) && \text{Definition of rising powers} \\
&= (n+p-1)^{\underline{p}} && \text{Definition of falling powers}
\end{aligned}
$$

Using Taylor series to find derivatives

We often find the Taylor series of a function $f(x)$ about $x = a$ by a brute-force calculation of derivatives $f^{(n)}(a)$. If, however, we already know the Taylor series of $f(x)$ about $x = a$ then we can use it to easily find the derivative $f^{(n)}(a)$. For example what is the second derivative of $\sin x^2$ evaluated at $x = 0$?

In class we memorized the Taylor series of $\sin x$ about $x = 0$:

$$\sin x = x - \frac{x^2}{2} + \frac{x^3}{6} + O(x^5).$$

Therefore $\sin x^2 = x^2 + O(x^4)$. The term containing x^2 in any Taylor series about $x = 0$ is $f''(0)x^2/2$. Setting this equal to the x^2 term in the Taylor series of $\sin x^2$ gives

$$
\begin{aligned}
\frac{f''(0)x^2}{2} &= x^2 \\
f''(0) &= 2
\end{aligned}
$$

Similarly we would easily find that the sixth derivative of e^{-x^2} evaluated at $x = 0$ was -120.