Prob. 1	Prob. 2	Prob. 3

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Problem 1.

A string in A cannot begin with a zero; it needs to switch to 1s eventually in order to have an even number of 1s, but ”01” is banned. Also, since ”01” is banned, all 1s must precede all 0s. Therefore, A can be redefined as the concatenation of

$$A_1 \subseteq \{1\}^* := \{\omega \mid \omega \text{ contains an odd number of 1s}\}$$
$$A_2 \subseteq \{0\}^* := \{\omega \mid \omega \text{ contains an even number of 0s}\}.$$

Both A_1 and A_2 are regular, because we can construct the following DFAs for them:

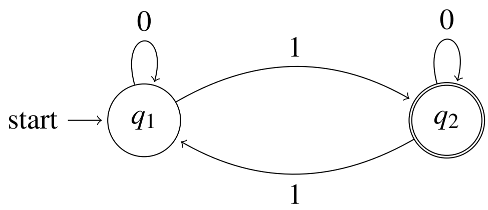

Figure 1: DFA A_1

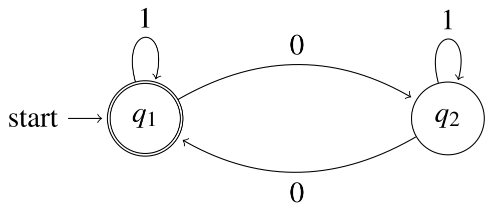

Figure 2: DFA A_2

Since A is the concatenation of two regular languages ($A_1 \circ A_2$), it is also regular as we saw in the class.

Problem 2.

Since L is regular, there is a DFA M that can match L. Let us create an NFA M' that can match L^R, which will prove that L^R is regular. M' is composed of all of M states, plus one new state. This new state is the start state of the NFA, and it contains only ε transitions to all states that were accepting states of M. The accepting state of M' is the start state of M. The transition function of M' is the opposite of M if an input x caused M to go from q_0 to q_1, now it causes M' to go from q_1 to q_0.

To express it more formal we have:

$$
\begin{aligned}
M &= (Q, \Sigma, \delta, s, F) \\
M' &= (Q', \Sigma, \delta;, s', F') \\
s' &= \text{new state, connected to the end of all the states } F \\
Q' &= Q + \{s'\} \\
F' &= \{s\} \\
\delta' &= \delta^R
\end{aligned}
$$

$$
\delta'(q,x) = \begin{cases} F & \text{if } q = s' \\ \{q_2 | \delta(q_2, x)\} = q & \text{otherwise} \end{cases}
$$

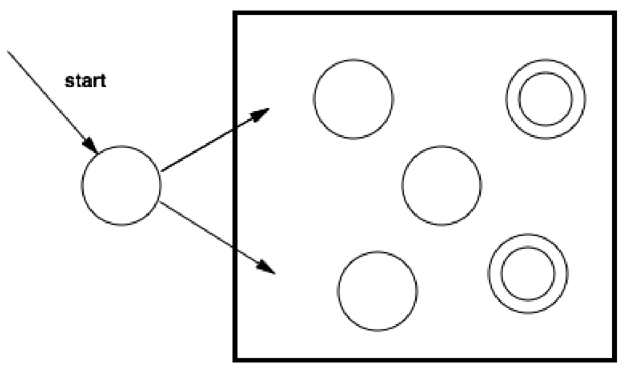

Figure 3: Original DFA M

Set 2, Problem 2

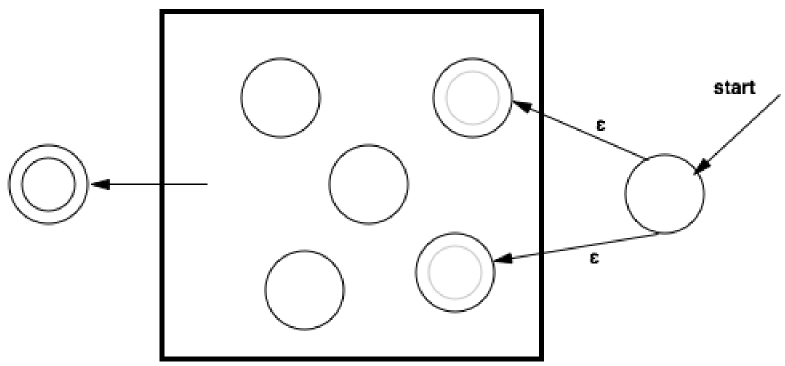

Figure 4: New NFA M'

This new machine $'M$' will be able to read every word the original machine $'M'$ was able to read in the inverse direction. As for every word the M was able to read we ended up in one of the accepting states to which we can transition to at no cost (ε) and therefore we will end up after the reverse read of the word in the initial state of M meaning in the accepting state of M'

Problem 3.

We haven proven in problem 2 that any regular Language can be inversed, and it will stay regular. Therefore if we can show that the language

$$\frac{L^R}{2} := \{x' \in \Sigma * \,|\, (yx)^R = x^R y^R \in L \text{for some } y' \text{ with } |x'| = |y'|\}$$

is regular, its inverse $L/2$ will also be regular. So if we can prove that a DFA (or NFA) exists then with the result obtained from the second problem we have the required proof. Instead of proving:

$$\frac{L}{2} := \{x \in \Sigma * \,|\, \text{ for some y such that } |x| = |y|, xy \in L\}$$

Let L be a regular language. Let $\frac{L}{2} := \{x | \exists y : xy \in L, |x| = |y|\}$. We want to prove that $\frac{L}{2}$ is regular.

Let $M < Q, \Sigma, \delta, q_0, F >$ be a DFA that accepts L, which must exist because L is regular. Supposing that $\hat{\delta}(q_0, x) = q_i$, i.e. the input x leads M to the state q_i, we must check if $\exists y : |y| = |x|$ that satisfies $\hat{\delta}(q_i, y) \in F$.

Let $S_n \subset Q$ be the set of states that lead to an accepting state for some input (not necessarily any input) of length n. It is clear that for x of length n and $\hat{\delta}(q_0, x) = q_i$, $q_i \in S_n \rightarrow x \in \frac{L}{2}$. $S_0 = F$ by definition of S_n there are no more steps needed. S_{n+1} can be easily computed from S_n and δ it is the set of states that can transition to a state in S_n. We need a DFA that will keep track of S_n as we go through L. Let M' a new DFA whose states are in $(Q, Q*)$ i.e. they're pairs of one state in M and a set of states in M. The transition function δ' of M' takes an input x of length n and yields $(\hat{\delta}(q_0, x), S_n)$. The start state of M' is (q_0, F), and the accepting states are $(q, S) \in (2, Q^*)$ where $q \in S$.

In common English, "the states reached after n inputs that can also reach an accepting state after n inputs". $(*)$ above allows us to show that M' indeed accepts $\frac{L}{2}$, and therefore $\frac{L}{2}$ is regular.

source: `http://www-bcf.usc.edu/~breichar/teaching/2011cs360/half%28L%29example.pdf`