We have been considering NP-hard problems recently such as vertex cover (VC), the traveling salesman problem (TSP), and independent set (IS). Most researchers believe that it is not possible to design polynomial time algorithms for these problems.

The best known algorithms are generally too slow to be of use in practice unless the input size is very small. What can we do with these problems?

One approach is to develop a *heuristic* which runs very fast, but has no guarantee on the quality of the solution (will typically return good results, but could be really bad).

Another approach is to develop polynomial time algorithms which guarantee that the solution computed by the algorithm will not be too much worse than an optimal solution. Such an algorithm is called an **approximation algorithm**.

Consider the optimization variants of VC, TSP, and IS:

- Compute a smallest vertex cover.
- Compute a shortest tour.
- Compute a largest independent set.

Let C^* denote the value of an optimal solution, and let C denote the value of a solution returned by an approximation algorithm.

We say that the algorithm has *approximation ratio* $\rho > 1$ if we can guarantee

- $C \leq \rho \cdot C^*$ for minimization problems, or
- $C \geq C^*/\rho$ for maximization problems.

In either case, the guarantee is that the returned solution will be within a factor of ρ. We would like to develop polynomial time approximation algorithms with an approximation ratio as small as possible.

We will now focus on the optimization version of the VC problem (we want to find a vertex cover of minimum cardinality).

Again, let C^* denote the value of an optimal solution, and let C denote the value of a solution returned by our algorithm.

We will give an approximation algorithm with approximation ratio 2. That is, we will give an algorithm such that $C \leq 2 \cdot C^*$.

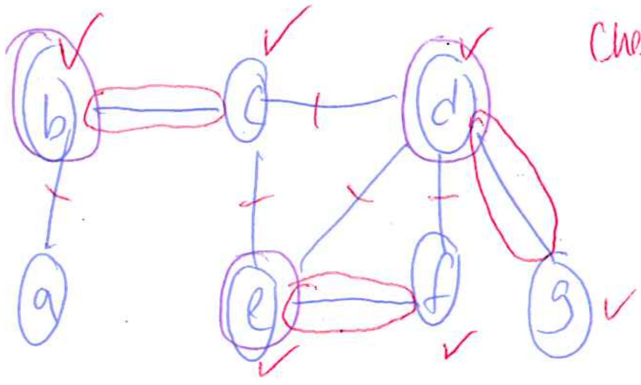
For short, we call such an algorithm a 2-approximation (an approximation algorithm with approximation ratio ρ is called an ρ-approximation in general).

Consider the following algorithm:

```
VC-Approx(G = (V,E))
    C = ∅                 // A = ∅ ← A is set of edges we choose from E'.
    E' = E
    while E' ≠ ∅ {
        let (u,v) be an arbitrary edge of E'.  // A = A ∪ (u,v) ← add the edge.
        C = C ∪ {u,v}  // Add both endpoints, not the actual edge
        Remove from E' every edge incident on either u or v.
    }
    Return C
```

Circled = OPT

Checked = Our Solution.



Theorem: VC-approx is a polynomial-time 2-approximation algorithm.

Running time is $O(n+m)$.

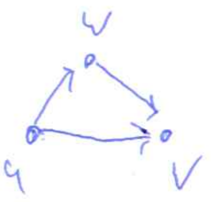
Output is a VC, because E' is empty when we terminate, and we only remove an edge from E' when we chose one of its two end points.

It must be that $|A| \leq C^*$, because each edge in A must be covered by one of its two endpoints, but a vertex is an endpoint for at most one edge in A. Therefore, we need at least $|A|$ vertices to cover only the edges in A.

Our solution has cost $2|A|$. $\Rightarrow \leq 2C^*$.

Now consider TSP with the triangle inequality.

- Triangle inequality: $c(u, v) \leq c(u, w) + c(w, v)$



The triangle inequality holds for many applications, including if the cost of moving from u to v is the Euclidean distance between the points in the plane.

We will give a 2-approximation algorithm for this problem.

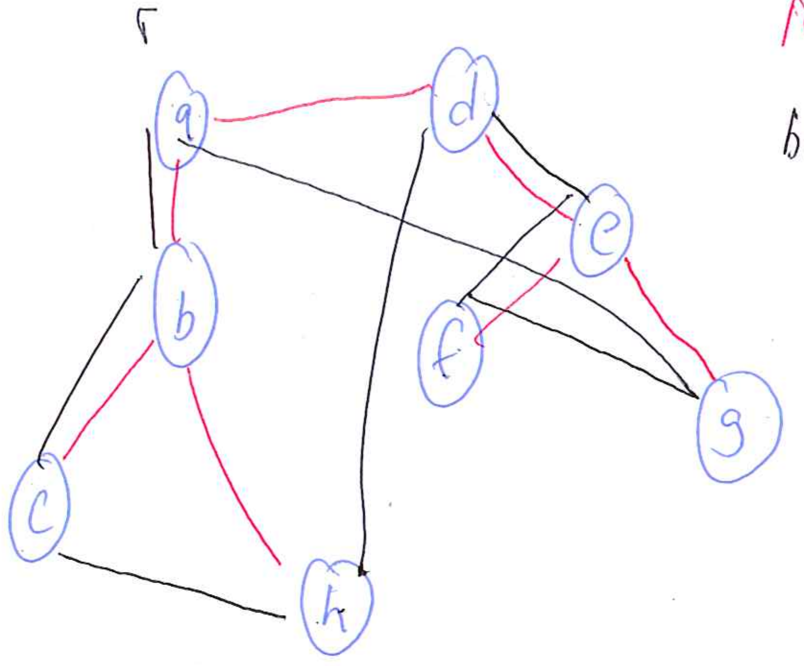
The idea is similar to last time. We will compute a lower bound on the cost of an optimal tour, and we will then compute a solution whose cost is at worst 2 times as bad as the lower bound.

If we delete any edge from an optimal tour, we are left with a spanning tree which costs less than the tour. Therefore the minimum spanning tree must cost less that the minimum TSP tour.

Consider the following algorithm:

```
TSP-Approx ($G=(V,E)$)

- Select a vertex $r \in V$ to be a "root" vertex.
- Compute a MST $T$ for $G$ from root $r$ using
      Prim's Algorithm.
- Let $H$ be a list of vertices ordered according to
  when they are first visited in a "preorder" tree walk of $T$.
- Return the tour $H$.
```



Red edges are MST

black edges are our tour.

List vertices of the "full walk" (list a vertex each time we visit it).

a b c b h b a d e f e g e d a

This walk uses each edge of MST exactly twice, so its length is $2|T|$.

Algorithm returns cycle ~~where we visit~~ containing vertices in the order we first visit them.

Using triangle inequality, we get that the length of H is at most 2 times the length of T. But the length of T is ~~less~~ less than C^*. So the length of H is at most $2 \cdot C^*$.