Random Experiment (RE): an experiment whose outcome cannot be predicted with certainty before the experiment is run.

- Flip of a coin
- Roll of a die
- Winner of the lottery

We call the set of all possible outcomes of a RE the **sample space** S.

- Flip of a coin: $S = \{H, T\}$
- Roll of a die: $S = \{1, 2, 3, 4, 5, 6\}$
- Winner of the lottery: $S =$ set of all lotto players.

We call $E \subseteq S$ an **event**.

We often are interested in the **probability** that an event E occurs (denoted $P(E)$). Intuitively, given that $s \in S$ is the outcome of a RE, how often will it be that $s \in E$.

- Suppose we are rolling a single die in our experiment, and suppose $E = \{2, 6\}$. Then $P(E)$ is the probability that a 2 or a 6 is rolled.

If S is finite and each element of S is equally likely to occur, then $P(E) = \frac{|E|}{|S|}$.

- In our previous example, $P(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3$.

Note that there are random experiments which do not satisfy these properties.

- Consider the RE in which we flip a coin repeatedly until we get a heads.
- $S = \{H, TH, TTH, TTTH, \ldots\}$

Some properties of probability:

$$0 < P(\{s\}) \le 1 \quad \text{for all } s \in S$$

$$P(S) = 1$$

$$P(E) = \sum_{s \in E} P(\{s\})$$

Let $\bar{E}$ denote the complement of E. That is

$$\bar{E} = S \setminus E$$

(setminus)

$$E = \{2, 6\} \Rightarrow \bar{E} = \{1, 3, 4, 5\}$$

$$P(\bar{E}) = 1 - P(E)$$

Random Variables:

A function $X: S \to \mathbb{R}$

Ex: Flip a coin 3 times.

Let X denote the # of times I flipped heads.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$X(HHT) = 2$

$X(TTT) = 0$

Ex: Play a game.

Win \$4 if TTT

Lose \$1 otherwise.

Let Y be a RV that denotes our "gain".

$$Y(TTT) = 4, \quad Y(HTH) = -1$$

On average, will we win or lose and by how much?

We want to find the <u>Expected Value</u> of Y, denoted $E[Y]$

$$\boxed{E[X] = \sum_{x \in X(S)} P(X = x) \cdot x}$$

all possible outcomes of X

$$Y(S) = \{4, -1\}$$

$$E[Y] = P(Y=4) \cdot 4 + P(Y=-1) \cdot -1$$

$$= \frac{1}{8} \cdot 4 + \frac{7}{8} \cdot -1$$

$$= \frac{4}{8} - \frac{7}{8} = -\frac{3}{8}$$

Linearity of Expectation

Let X and Y be 2 random variables,

$$E[X+Y] = E[X] + E[Y]$$

Randomized Analysis

Hire Assistant Problem:

- We want to hire an assistant
- Interview a sequence of n people.
- If a person is better than everyone we have seen so far hire them.

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1. best = 0
2. for i = 1 to n
3.        Interview Candidate i
4.        if ( i is better than best)
5.              best = i
6.              hire candidate i
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Best Case: First person is best, 1 hire

Worst Case: Candidates are in reverse order. n hires

Average Case? Is it $\frac{n}{2}$?

Label candidates 1 to n such that n is my favorite and 1 is my least favorite.

Suppose candidate orderings come from a uniform distribution, every ordering is equally likely.

$S = \{\text{All permutations of } n \text{ candidates}\}$ $|S| = n!$

Let E_{Best} denote the best case scenario. $|E_{Best}| = (n-1)!$

$$P(E_{Best}) = \frac{(n-1)!}{n!} = \frac{\cancel{(n-1)!}}{n \cdot [\cancel{(n-1)!}]} = \frac{1}{n}$$

Let E_{worst} denote the worst case scenario. $|E_{worst}| = 1$

$$P(E_{worst}) = \frac{1}{n!}$$

Let X denote the # of hires we made.

We want to compute $E[X]$.

Let X_i be an indicator RV that denotes whether we hired candidate i;

$$X_i = \begin{cases} 0 & \text{if did not hire} \\ 1 & \text{otherwise.} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i \qquad E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

linearity of expectation

$$E[X_i] = P(X_i = 1)\cdot 1 + \cancel{P(X_i = 0)\cdot 0}$$

$$= \boxed{P(X_i = 1)}$$

$P(X_n = 1) = 1$,

$P(X_{n-1} = 1) = \frac{1}{2}$

$P(X_{n-2} = 1) = \frac{1}{3}$

$P(X_{n-3} = 1) = \frac{1}{4}$

⋮

$P(X_1 = 1) = \frac{1}{n}$

$$\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$$