Randomized Algorithms

CS 5633 Analysis of Algorithms

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Probability Basics

Random Experiments

- **Random Experiment** (RE): an experiment whose outcome cannot be predicted with certainty before the experiment is run.
  - Flip of a coin
  - Roll of a die
  - Winner of the lottery
- We call the set of all possible outcomes of a RE the **sample space** S.
  - Flip of a coin: $S = \{H, T\}$
  - Roll of a die: $S = \{1, 2, 3, 4, 5, 6\}$
  - Winner of the lottery: $S =$ set of all lotto players.

Random Events

- We call $E \subseteq S$ an **event**.
- We often are interested in the **probability** that an event E occurs (denoted $P(E)$). Intuitively, given that $s \in S$ is the outcome of a RE, how often will it be that $s \in E$.
  - Suppose we are rolling a single die in our experiment, and suppose $E = \{2, 6\}$. Then $P(E)$ is the probability that a 2 or a 6 is rolled.
- If S is finite and each element of S is equally likely to occur, then $P(E) = \frac{|E|}{|S|}$.
  - In our previous example, $P(E) = \frac{|E|}{|S|} = \frac{2}{6} = 1/3$.
- Note that there are random experiments which do not satisfy these properties.
  - Consider the RE in which we flip a coin repeatedly until we get a heads.
  - $S = \{H, TH, TTH, TTTH, \ldots\}$

Basic Proprieties of Probabilities

- $0 \leqslant P(s) \leqslant 1$, for all $s \in S$.
- $P(S) = 1$.
- $P(E) = \sum_{s \in E} P(s)$.
- Let $\overline{E}$ be the complement of random event E. That is $\overline{E} = S \backslash E$. E.g., $E = \{2, 6\}$, then $\overline{E} = \{1, 3, 4, 5\}$.
  - $P(\overline{E}) = 1 - P(E)$.

Random variables

- **Random variable** is a variable whose possible values are numerical outcomes of a random phenomenon.
  - Less formally, a random variable X is a function that projects the random events from a sample space to real numbers, i.e., $X : S \to \mathbb{R}$.
- Example: flip a coin 3 times, and let X denote the number of heads in the outcome.
  - Sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - $X(HHH) = 3$, $X(HTT) = 1$, $X(TTT) = 0$.

Expected Value of a Random Variable

- The **expected value** of a random variable is the long-run average value of repetitions of the experiment it represents.
  - The expected value of random variable X is denoted as $E[X]$.
  - By definition, $E[X] = \sum_{x \in X(S)} P(X = x) \cdot X$. Note the $x \in X(S)$, represents all possible values of X.
- In the previous example, the expected value of X is the average random variable value of all elements in S, which is $E[x] = (3 + 2 + 2 + 1 + 2 + 1 + 1 + 0)/8 = 1.5$.
  - That is, on average, we have 1.5 heads when flip a coin 3 times.

Another Example of Expected Value

- Let's play a game with the flip-coin-3-times.
  - Win \$4 when TTT is the outcome
  - Lose \$1 otherwise.
  - Let Y be the random variable representing the earnings. That is, $Y(TTT) = 4$; for all other outcomes, $Y(outcome) = -1$.
- What is the expected earnings then?
  - By definition, $E[Y] = P(Y = 4) \cdot 4 + P(Y = -1) \cdot -1$.
  - Out of the 8 possible outcomes, only one gives $Y = 4$, i.e., $P(Y = 4) = \frac{1}{8}$.
  - Out of the 8 possible outcomes, seven give $Y = -1$, i.e., $P(Y = 4) = \frac{7}{8}$.
  - Put in the probabilities, we have $E[Y] = -\frac{3}{8}$.

Linearity of Expectation

- Let X and Y be two random variables (regardless of their dependencies), we have,
  - E[X+Y] = E[X] + E[Y]

Indicator Random Variable

- We define a special random variable, called **Indicator Random Variable**, which is always associated with an event A.
- The indicator random variable I_A associated with event A is defined as

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } A \text{ does not occur} \end{cases}$$

- The expected value of I_A is,

$$\begin{aligned} E(I_A) &= P(I_A = 1) \cdot 1 + P(I_A = 0) \cdot 0 = P(A) \\ &= P(A) \cdot 1 + P(\bar{A}) \cdot 0 = P(A) \end{aligned} \tag{1}$$

Random Algorithm Analysis

A Case Study: The “Hire Assistant Problem”

- We want to hire an assistant.
- We will interview a sequence of n people.
- If a person if better than everyone we have seen so far, hire that person.
- Suppose cost for an interview is only 1.
- Suppose the cost of hire a person is C_h which is also expensive.

The Algorithm for “Hire Assistant Problem”

Algorithm 1: Hire Assistant Problem

```
best = 0;
for i ← 1 to n do
    interview applicant i;
    if applicant i is better than best then
        best = i;
        hire applicant i;
    end
end
```

Analysis of "Hire Assistant Problem" Algorithm

- The worst case, the each applicant is better than the previous ones. We have make n hires. Therefore, the cost is (including interviewing and hiring) $n + n * C_h$.
- The best case, the first applicant is the best. We have to make just one hires. Therefore, the cost is $n + C_h$.
- What is the average cost?
  - In other words, what is the expected cost?
  - Cost include interviewing cost and hiring cost.
  - Interviewing cost is always n.
  - The hiring cost has an linear relationship with the number of hires.
  - Therefore, the real question is, what is the average or expected number of hires?

Analysis of “Hire Assistant Problem” Algorithm cont.

- Assuming the applicants from a uniform distribution. That is, each ordering of candidates is equally likely.
- Let I_i be the indicator random variable of whether the i’th applicant is hired. That is,

$$I_i = \begin{cases} 1, & \text{if applicant } i \text{ is hired} \\ 0, & \text{if applicant } i \text{ is not hired} \end{cases}$$

Analysis of "Hire Assistant Problem" Algorithm cont.

- Let X denotes the total number of hires.
  - Clearly, $X = I_1 + I_2 + \ldots + I_n = \sum_{i=1}^{n} I_i$
  - The expected number of hires is $E(X) = E(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i)$ (by linearity of expectation).

Analysis of "Hire Assistant Problem" Algorithm cont.

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  - From slide 10, we know that $E(I_i) = P(I_i)$.
    - Applicant 1 will definitely be hired, therefore, $P(I_1) = 1$.
    - The probability of applicant 2 being hired is the probability that applicant 2 is the best of the first two. Therefore, $P(I_2) = \frac{1}{2}$.
    - The probability of applicant 3 being hired is the probability that applicant 3 is the best of the first three. Therefore, $P(I_3) = \frac{1}{3}$.
    - Similarly, the probability of applicant i being hired is the probability that applicant i is the best of the first i applicants. Therefore, $P(I_i) = \frac{1}{i}$.
  - Based on the above, we have $E(X) = \sum_{i=1}^{n} E(I_i) = \sum_{i=1}^{n} P(I_i) = \sum_{i=1}^{n} \frac{1}{i} = O(\ln n) + O(1)$ (See Appendix eq A.7).

Analysis of "Hire Assistant Problem" Algorithm cont.

- As the expected hire count is $O(\ln n)$, then the expect hiring cost is $O(C_h \cdot \ln n)$.
- The total expected cost is $O(C_h \cdot \ln n) + O(n)$.
- To ensure a random applicant order, we can permute the list of applicants first.

Algorithm 2: Hire Assistant Problem

```
randomly permute the list of applicants;
best = 0;
for i ← 1 to n do
    interview applicant i;
    if applicant i is better than best then
        best = i;
        hire applicant i;
    end
end
```

The Importance of Random Algorithms

- The behavior of some algorithms depends heavily on the input set, while the properties of the inputs sets are random.
- In other words, many algorithms have non-deterministic behavior.
- However, the distribution of the input sets are usually hard to determine, which makes analyzing the behavior of non-deterministic algorithms very difficult.
- In the next lecture, we will see how to handle these non-deterministic algorithms.