Minimum Spanning Tree

CS 5633 Analysis of Algorithms

Computer Science
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Minimum Spanning Tree

Minimum Spanning Tree (MST)

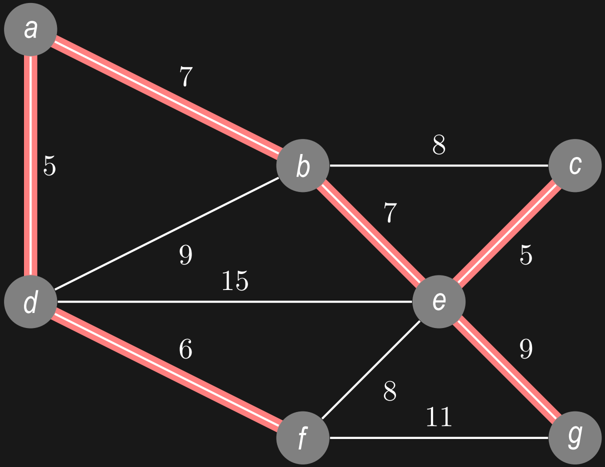
- Suppose we have a set of n locations, and we wish to build a connected network on top of them. The network should be connected (there should be a path between any two locations in the network), and subject to this constraint, we wish to build the network as cheaply as possible.
- Note that a solution to this must be a tree (if the network contains a cycle, we can remove one of the connections to obtain a cheaper network and still satisfy the connectivity constraint).

Minimum Spanning Tree (MST) cont.

- In graph theory, a tree which contains every vertex of the graph is known as a **spanning tree**.
- If we assign weights to the edges of a graph (i.e. the cost to connect two locations in the network), then a **minimum spanning tree** (MST) is a spanning tree such that the sum of the weights of the edges in the tree is minimized.

Example of MST

- The MST consists of red edges.



MST Algorithms

Growing the MST

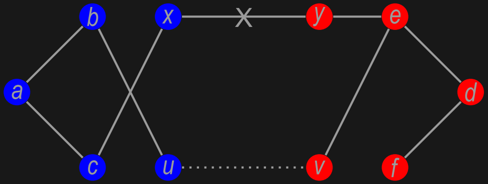
- Considering that MST is an optimization problem, a greedy algorithm strategy may work.
- Intuitively, a greedy strategy would always choose the edge with the lowest weight.
- We can then grow the MST by gradually adding low weight edges to it.

Proof of Greedy Strategy

- We need to prove that always adding the least-weight edges can give us the minimum tree.
- Theorem: Let T be a MST of $G = \{V, E\}$, and let $A \in V$. Suppose $(u, v) \in E$ is the least-weight edge connects $\{A\}$ and $\{V - A\}$. Then $(u, v) \in T$.
- Proof: We will prove by contradiction.
  1. Assume (u, v) is no in T. Let blue vertices be in $\{A\}$, red vertices be in $\{V - A\}$.
  2. One of u and v is blue and one is red because (u, v) connect A and $\{V - A\}$.
  3. Follow the path in T from u to v, and remove an edge $\{x, y\}$ that connects a red vertex to a blue vertex. Such an edge much exist because this path begin from a blue vertex and ends at a red vertex.

Proof of Greedy Strategy cont.

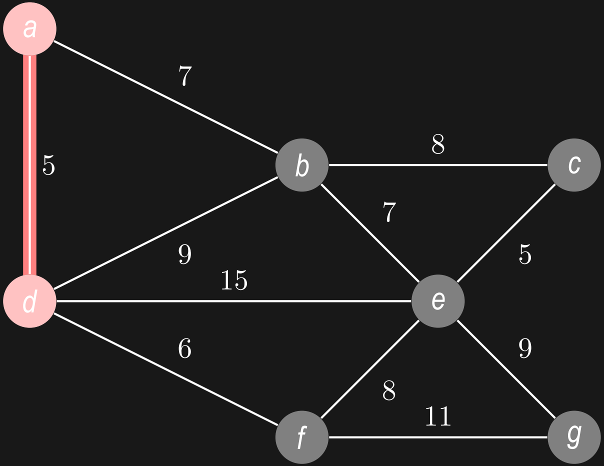
- Theorem: Let T be a MST of $G = \{V, E\}$, and let $A \in V$. Suppose $(u, v) \in E$ is the least-weight edge connects $\{A\}$ and $\{V - A\}$. Then $(u, v) \in T$.
- Proof: continuing from previous page,
  4. We can then add $\{u, v\}$ to T to construct T'. T' is a spanning tree as it connect the red vertices and blue vertices through $\{u, v\}$, while red (blue) vertices are connected to each other using the original edges in T.
  5. Because $\{u, v\}$ has lower weight than $\{x, y\}$, T' is smaller than T, which contradicts the premise that T is the minimum spanning tree. Therefore, the assumption that $\{u, v\}$ is not in T is wrong.



Kruskal's Algorithm

1. Invented by Kruskal in 1956.
2. Maintain a tree T.
3. Repeat the following steps until all edges are examined.
   3.1 Pick the least weight edge (u, v) from the remaining edges $\{E - T\}$ of the graph.
   3.2 If (u, v) causes a cycle in T discard it. Otherwise, add (u, v) to T.

An Example of Kruskal's Algorithm



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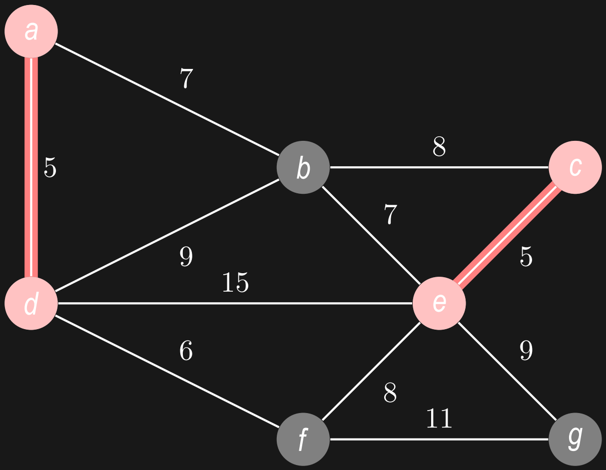

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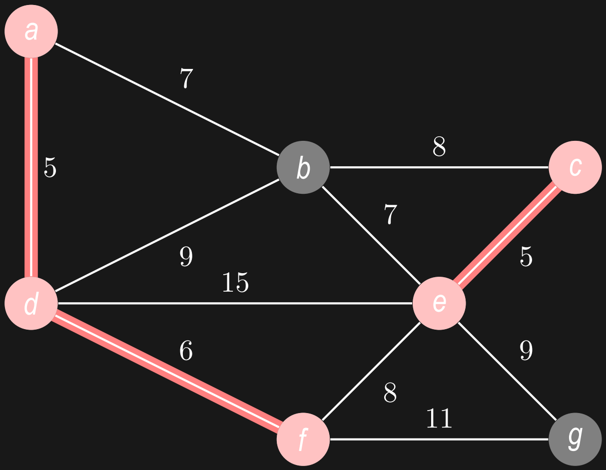

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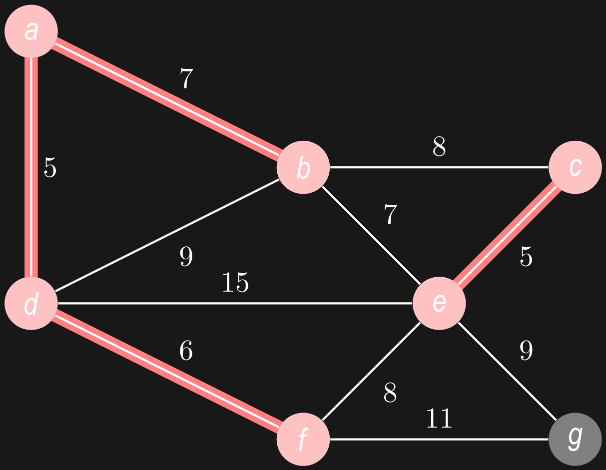

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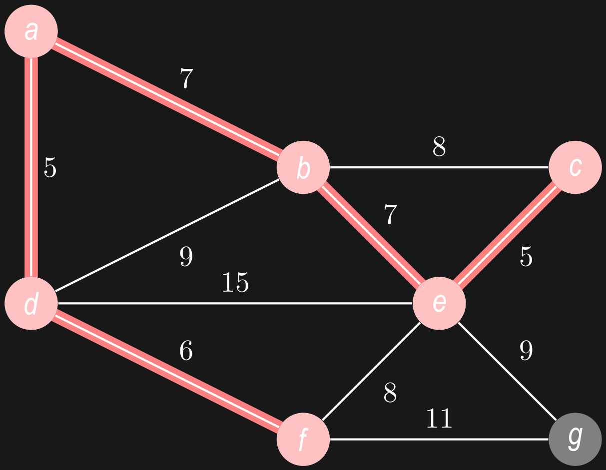

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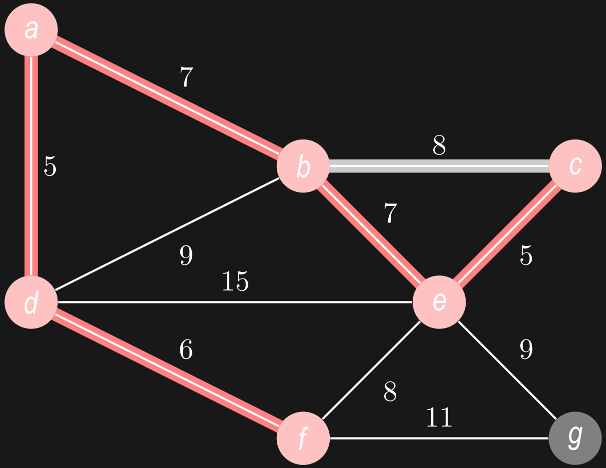

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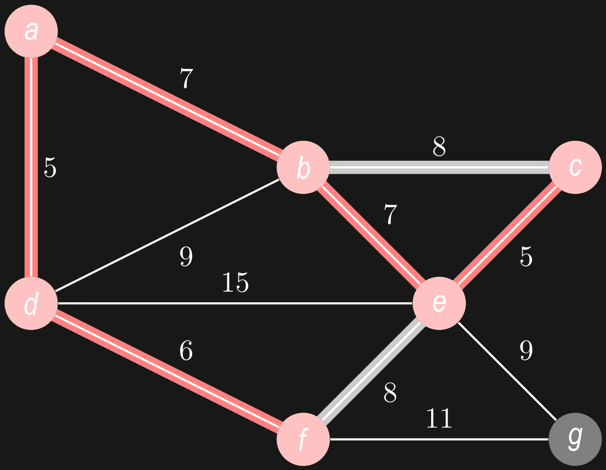

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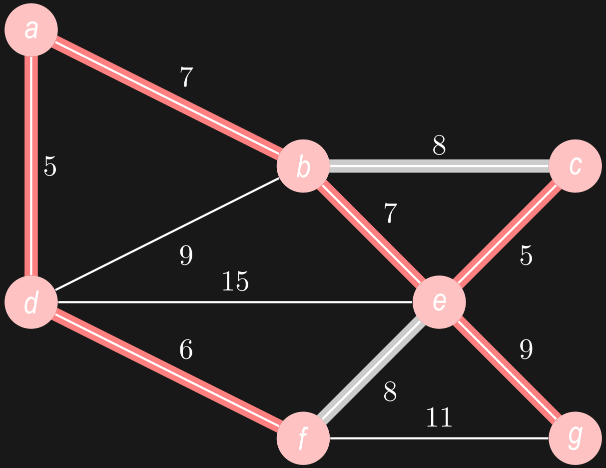

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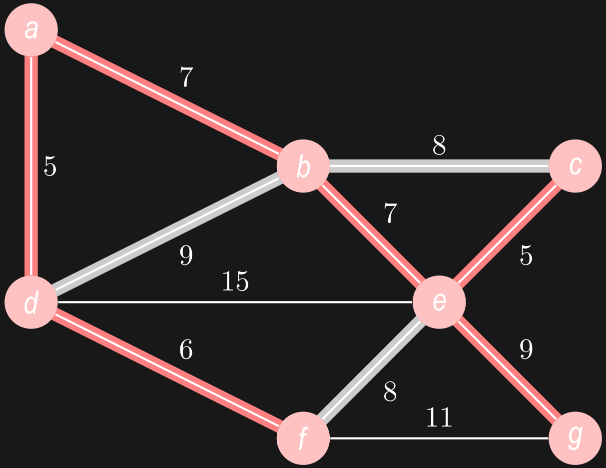

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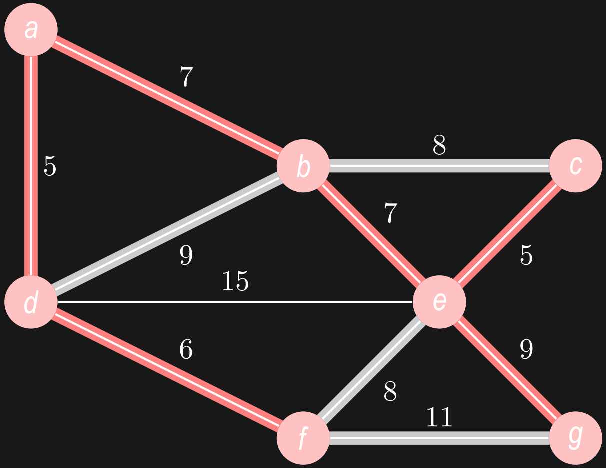

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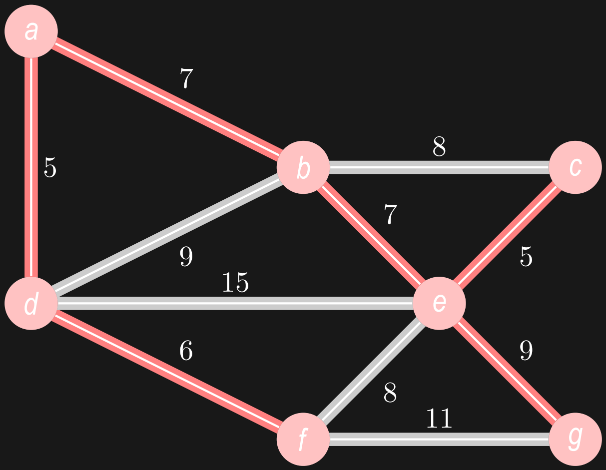


Implementing Kruskal's Algorithm

- The difficult part of Kruskal's algorithm is to detect whether adding an edge creates a cycle.
- Usually, cycles are detected with DFS search with a cost of $O(|E| + |V|)$.
- Given Kruskal's algorithm requires examine very edge, the total cost with DFS is $O(|E|^2 + |E||V|)$.
- To reduce this cost, Kruskal's algorithm is usually implemented with discrete sets.
  - Because Kruskal's algorithm may construct several trees along the way, each discrete set is used to represent one such tree.
  - If an edge connects two vertices within the same discrete set, this edge mush create a cycle in that set and should be discarded.

Pseudo-code of Kruskal's Algorithm

Algorithm 1: Kruskal's algorithm with discrete sets.

```
Function MST_Kruskal(graph G(V, E))
    T = empty set;
    for each u ∈ V do
        // Initially each vertex is a tree by itself represented by its own
          discrete set;
        MAKE_SET(u);
    Sort(E); // In-place sort the edges based on weight from low to high;
    for each edge (u, v) ∈ E do
        // Examine the least-weight edge;
        if FIND_SET(u) ≠ FIND_SET(v) then
            T.add((u, v));
            Merge u's set and v's set into one set;
```

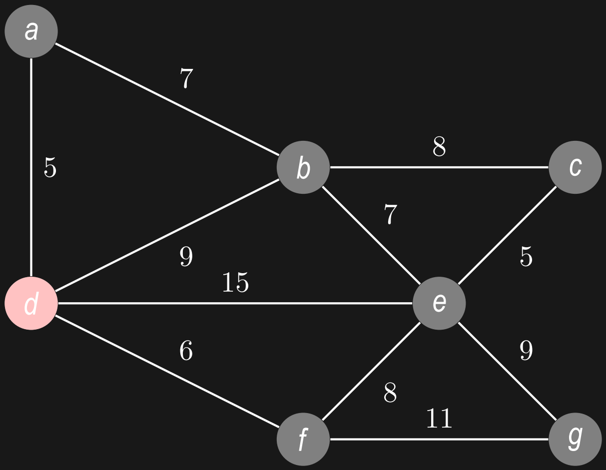
Run-time of Kruskal's Algorithm

- Creating the initial sets takes $O(|V|)$ time.
- Sorting the edges takes $O(|E| \lg |E|)$ time.
- Examining the edges takes $O(|E| \lg(|V|))$ time. Consider using tree sets. Each discrete set find requires $O(\lg(|V|))$ time, and each merge requires $O(1)$ time.
- Therefore, the total cost of Kruskal's algorithm is $O(|E| \lg |E|)$.
- Since $|E| < |V|^2$, we have $\lg |E| < 2 \lg |V|$. The total run-time of Kruskal's algorithm is then $O(|E| \lg |E|) = O(|E| \lg |V|)$.

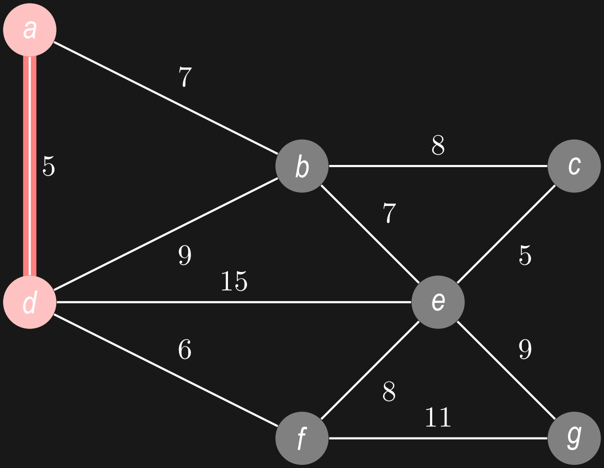
Prim's Algorithm

1. First discovered in 1930 by Jarnik, rediscovered in 1957 by Prim and in 1959 by Dijkstra.
2. Maintain a tree, T, and a set of T's vertices, C. Initially, add a random vertex to C.
3. Repeat the following operation until all vertices are in C.
   3.1 Always pick the least weight edge that connects a vertex in C and a vertex in $\{V - C\}$.

An Example of Prim's Algorithm



An Example of Prim's Algorithm



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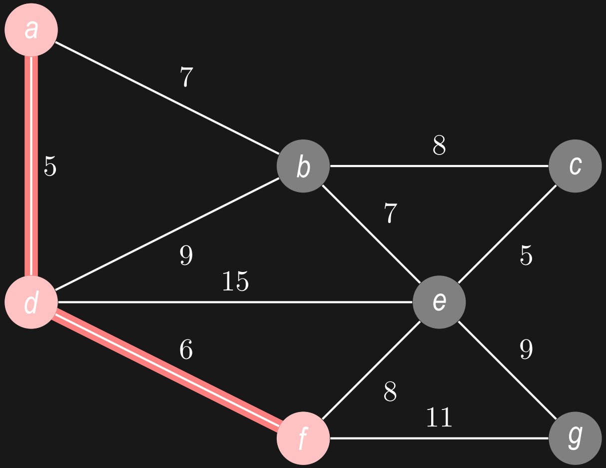

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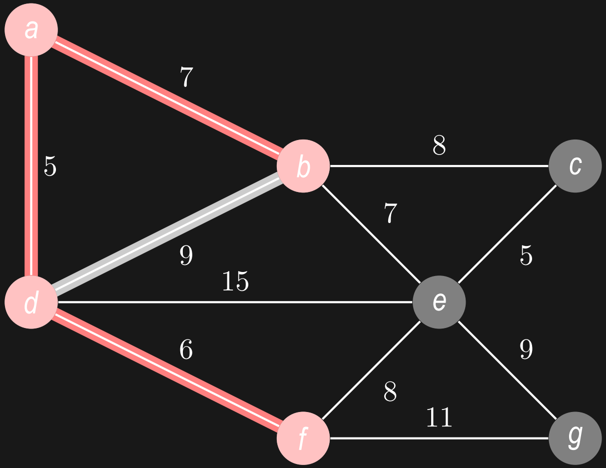

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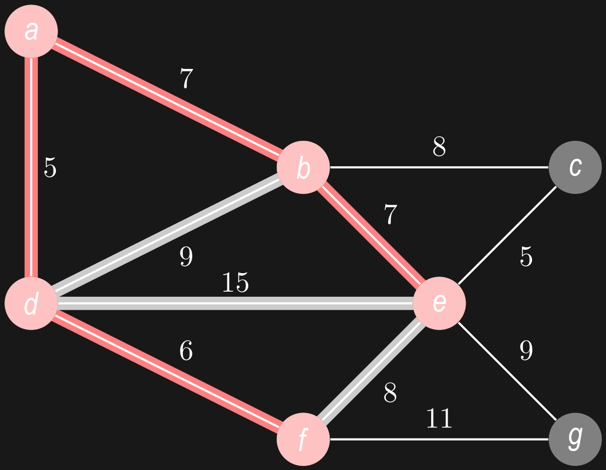

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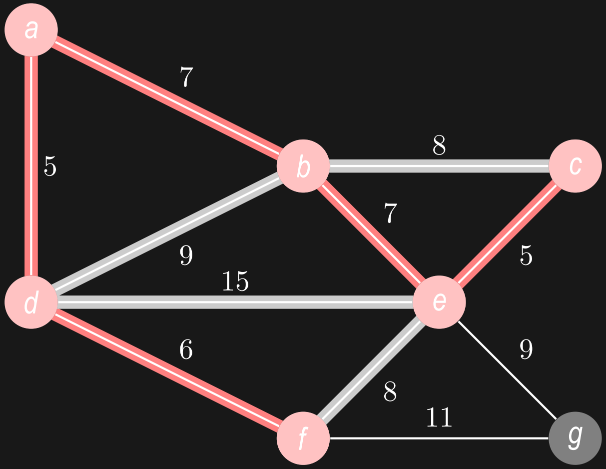

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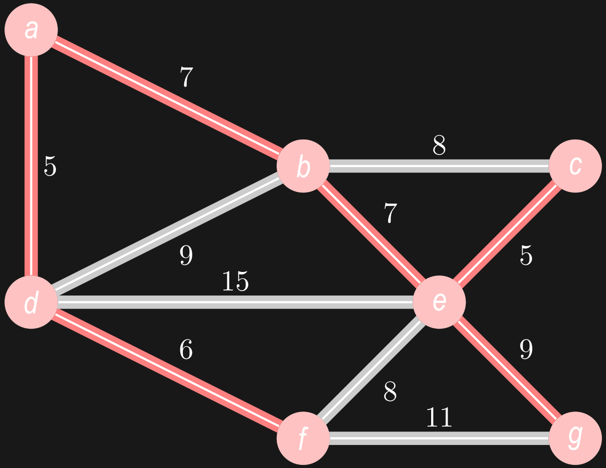

Implementing Prim's Algorithm

- The difficult part of Prim's algorithm is to find the pair of vertices in C and $\{V - C\}$ with least weight edge.
- With a brute-force implementation, we may need to examine $|C| \times (|V| - |C|) = O(V^2)$ edges for each added new edge.
- To reduce this cost, Prim's algorithm is usually implemented with a priority queue to manage the vertices in $\{V - C\}$.
  - The vertices in $\{V - C\}$ are sorted based on the weights of the edges that connect them to the vertices in C.
  - The priority queue Q is used to maintains this sorted order of vertices of $\{V - C\}$. The priority queue allows fast extracting-min and update (reorder) operations.

Pseudo-code of Prim's Algorithm

Algorithm 2: Prim's algorithm with a priority queue.

```
Function MST_Prim(graph G(V, E))
    C = empty set; T = empty set;
    Q = V; // initially all vertices are in {V − C}, which is managed by Q;
    for each u ∈ Q do
        u.key = ∞; // key is weight of the least-weight edge that
          connects u to a vertices in C;
        u.π = NIL; // π is the least-weight edge that connects u to a
          vertices in C;
    while Q is not empty do
        // Add the vertex (of {V − C}) with the least-weight edge to C to
          MST;
        u = Q.Extract_Min();
        C.add(u);T.add(u.π);
        // Update the vertices (of {V − C}) with respect to the new
          vertex (u) of C;
        for each v connected to u do
            if v ∈ Q and wegith(u, v) < v.key then
                v.π=(u, v);
                v.key=weight(u, v);
```

Run-time of Prim's Algorithm

- Assuming a search tree is used as the priority queue.
- Building the search tree takes $O(|V|)$ time.
- There are $|V|$ calls to Extract_Min, while each call costs $O(\lg |V|)$ in a search tree. Total costs of all Extract_Min is $O(|V| \lg |V|)$.
- The inner loop at line 11 examines the edges and updates the tree. Given the are $|E|$ edges and each update costs $O(\lg |V|)$, the total cost of the inner loop is than $O(|E| \lg |V|)$.
- The total cost of Prim's Algorithm is then $O(|V| \lg |V| + |E| \lg |V|) = O(|E| \log |V|)$ (a fully-connected graph has more edges than vertices, $|E| \geqslant |V|$).