1. (a) Loop invariant HW-1

At the start of every iteration value of

$$K = \frac{n}{2^c}$$

(b) Correctness of algorithm

proof by induction:

Base case:

$$K = n, \quad c = 0$$

$$\Rightarrow \frac{n}{2^0} = n \quad [\text{for } c = 0]$$

$$\Rightarrow n = n$$

so $K = n/2^c$ holds for base case

Induction step:

"While loop" iterates c times before the value of K becomes less than 1 and loop terminates. After that program returns c.

Assume loop invariant true for $c-1$ iteratio and $K = \frac{n}{2^{c-1}}$, at this point.

After next iteration value of K will be

$$K = \frac{n}{2^{c-1}} \times \frac{1}{2} = n/2^c$$

So loop invariant holds for last iteration. (c^{th} iteration)

Loop terminates when $K \ngtr 1$. That is after last iteration $K = 1$

from loop invariant $K = \frac{n}{2^c}$

$$\therefore \frac{n}{2^c} = 1 \Rightarrow c = \log_2 n$$

[proved]

(c) Running time:

The while loop iterates c times and at the end of the loop $c = \log_2 n$

$\therefore$ Running time $= O(\log_2 n)$

(a) Outer loop iterates $\frac{3n}{4}$ times

Inner loop iterates $\log_3 20n$ times

$\therefore$ Running time $= \frac{3n}{4} \cdot \log_3 20n = \Theta(n \log n)$

(b) Running time $= \sum_{i=1}^{3n^2} \log_2 i$

$= \log_2 1 + \log_2 2 + \log_2 3 + \dots + \log_2($

$\leq \log_2(3n^2) + \log_2(3n^2) + \dots + \log_2($

$= 3n^2 \log_2(3n^2)$

Again Running time

$$= \sum_{i=1}^{3n^2} \log_2 i$$

$$\geq \sum_{i=3n^2/2}^{3n^2} \log_2 i$$

$$= \log_2\left(\frac{3n^2}{2}\right) + \log_2\left(\frac{3n^2}{2}+1\right) + \cdots + \log(3n^2)$$

$$\geq \log\left(\frac{3n^2}{2}\right) + \log_2\left(\frac{3n^2}{2}\right) + \cdots + \log\left(\frac{3n^2}{2}\right)$$

$$= \frac{3n^2}{2} \log \frac{3n^2}{2}$$

$$= \Omega(n^2 \log n)$$

$\therefore$ Running time $= \theta(n^2 \log n)$

3. (a) $4n^5 - 50n^2 + 10n \in \theta(n^5)$

Now, $4n^5 - 50n^2 + 10n \leq 4n^5 + 10n$

$\leq 4n^5 + 10n^5 \quad [n \geq 1$

$= 14n^5$

$= O(n^5)$

Again, $4n^5 - 50n^2 + 10n \geq 4n^5 - 50n^2$

$= n^5 + (3n^5 - 50n^2)$

$\geq n^5$ if $3n^5 - 50n^2 \geq$

$\Rightarrow n \geq \sqrt[3]{\frac{50}{3}}$

$\Rightarrow n \geq 3$

$= \Omega(n^5)$

$\therefore 4n^5 - 50n^2 + 10n = \theta(n^5)$

(b) $5n^{2/3} + 8\log n \in o(n)$

Now, $\lim_{n \to \infty} \frac{5n^{2/3} + 8\log n}{n}$

$= \lim_{n \to \infty} 5n^{-1/3} + \lim_{n \to \infty} \frac{8\log n}{n}$

$= 0 + \lim_{n \to \infty} \frac{8 \cdot \frac{1}{n}}{1}$

$= 0 + 8 . \lim_{n \to \infty} \frac{1}{n}$

$= 0$

$\therefore 5n^{2/3} + 8\log n \in o(n)$ (proved)

(c) $n^5 + 4n^2 + 15 \geqslant n^5 + 4n^2$

$\geqslant n^5$ for $n \geqslant 1 (= n_0)$

$\geqslant n^3$ [if $n \geqslant 1$]

$\therefore n^5 + 4n^2 + 15 \in \Omega(n^3)$

(proved)

(4) Sorted order :

$\log_2 n \leqslant 2^{\sqrt{\log_2 n}} \leqslant n^{1/3} \leqslant n^5 \leqslant 10^n \leqslant n^n$.