Red-Black Tree

1. Every node is red or black
2. Root is black
3. Leaves (nil) are black
4. If a node is red, both children are black
5. All simple path, from any node x, excluding x, to a descendant leaf has the same # of black nodes. [black-height (x)]

Time complexity = $O(\log n)$

Theorem

→ A Red back-tree with n keys has height $h \leq 2 \cdot \log(n+1)$

→ For each path of length h, there are at most $h/2$ red nodes. (Property 4) so, $bh \geq h/2$

→ The # of nil leaves is $\leq n+1$

→ $n+1 \geq 2^{bh} \Rightarrow \log_2(n+1) \geq bh$

$\log_2(h+1) \geq bh \geq h/2$

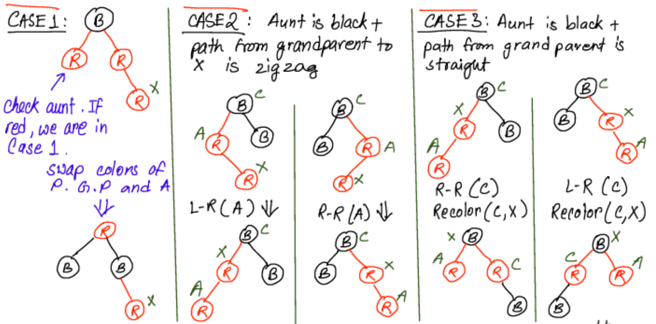
$\Rightarrow h \leq 2\log_2(n+1)$

Insertion: Insert node as a red node. Only property 4 can be violated.

CASE 1: Check aunt. If red, we are in Case 1. swap colors of P. G.P and A

CASE 2: Aunt is black + path from grandparent to X is zigzag

CASE 3: Aunt is black + path from grand parent is straight



Number of vertices in a red black tree with all black nodes: $2^{bh}-1$

Number of vertices in a red black tree with alternating black-red nodes: $2^{2bh}-1$

Numer of red nodes in alternating R-B tree: $2+8+32\cdots+n$ (internal)

$\Rightarrow 2(1+4+16)$

h = black height $\Rightarrow 2\sum_{i=0}^{h} 4^i = 2\left(\frac{4^h-1}{3}\right)$

Number of internal black nodes: $1+4+16+64 \Rightarrow \sum_{i=0}^{h} 4^i = \frac{4^h-1}{3}$

DP: ① Hotel problem:

1) There are 2^n different subsets of hostel and each subset takes $O(n)$ time to check its feasibility and costs. Therefore $O(n \cdot 2^n)$

2) $a[i] = \begin{cases} 0 & \text{if hostel } i \text{ is} \leq 20 \text{ miles from start} \\ \min\limits_{k \in H(i)} (C_k + a[k]) & \text{otherwise} \end{cases}$

$H(i)$ are the hostels within 20 miles before hostel i.

3) Assuming a dummy hostel h_{n+1} at the very end of the trail.

```
for (i=1 to n+1)
{ if (h_i is ≤ 20 miles from the start)
    { a[i] = 0 } end if
  else
    { mincost = ∞
      j = i-1
      while (h_i - h_j ≤ 20)
      { if (a[j] + C_j < mincost)
          { mincost = a[j] + C_j } end if
        j--
      } end while
      a[i] = min cost
    } end else
} end for
return a[n+1]
```

Trip problem:

1) There are 2^n different ways of picking station and it takes $O(n)$ time to determine if a choice is feasible and compute the cost. Therefore, $O(n \cdot 2^n)$

2) $C[1]=20$, $C[2]=50$, $C[3]=20+70=90$, $C[4]=50+30=80$

3) $C[i] = \begin{cases} C_i & \text{if } M_i \leq 300 \\ C_i + \min\limits_{k \in S(i)} \{C[k]\} & \text{otherwise} \end{cases}$

set of all gas station within 300 mile before St. i

4)
```
for (i=1 to n)
{ if (M_i ≤ 300)
    { C[i] = C_i } end if
  else { j = i-1
         min = i-1
         while (m_i - m_j ≤ 300)
         { if (C[j] < C[min])
             { min = j } end if
           j--;
         } end while
         C[i] = C[min] + C_i
       } end else
} end for

j = n-1
min = n-1
while (M - m_j ≤ 300)
{ if (C[min] > C[j])
    { min = j } end if
  j--;
} end while
return C[min]
```

Subsequence palindrome

1) There are 2^n different ways we can create a subset win n characters. Then to determine whether it's a palindrome or not, we need $O(n)$ time. Therefor $O(n \cdot 2^n)$

2) $C[1,1]=1$, $C[2,2]=1$, $C[3,3]=1$, $C[1,2]=1$, $C[2,3]=1$, $C[1,3]=3$

3) $C[i][j] = \begin{cases} 1 & \text{if } i==j \\ \max(C[i][j-1], C[i+1][j]) & \text{if } S_1[i] \neq S_2[j] \\ \max(C[i][j-1], C[i+1][j]) + 2 & \text{if } S_1[i] == S_2[j] \end{cases}$

LCS Theorem:

$C[i][j] = \begin{cases} C[i-1][j-1]+1 & \text{if } X[i] = Y[j] \\ \max(C[i-1][j], C[i, j-1]) & \text{otherwise} \end{cases}$

Matrix-mul Theorem

$M[i,j] = \begin{cases} 0 & \text{if } i==j \\ \min\limits_{i \leq k < j} (M[i,k] + m[k+1,j] + P_{i-1} \times P_k \times P_j) \end{cases}$

$P[n]$ = array sequence $\{3, 20, 5, 8\}$

```
Mul(i, j)
{ if (i == j)
    { M[i, j] = 0 } end if
  else
  { min = ∞
    for k = 1 to k < j
    { x = Mul(i, k) + Mul(k+1, j) + P[i-1]×P[k]×P[k+1]
      if (x < min)
        { min = x } end if
    } end for
    M[i, j] = min
  } end else
  return M[i, j]
}
```

coin change:

$DP[i][j] = (i/k[j]) + dp[i][j-1]$

Since i represents n and $k[j]$ represents the current coin

Sort k = range of each element

→ Counting sort: if n is bounded and small then use counting sort. Running time: $O(n+k)$. It is a stable sort. $k = n$

→ Radix Sort: Sort from LSD to MSD. if n is not bounded and range is given for each digit the use Radix Sort. $O(n)$

→ Merge Sort: $O(n \cdot \log n)$ → Insertion sort: $O(n^2)$

if we convert decimal to binary or hexadecimal.

* decimal → binary ① # of digits for radix sort increases.
  ② Range of values for counting sort decreases.
* decimal → hex ① # of digits for radix sort decreases
  ② Range of values for counting sort increases.

Q1) An array of n binary numbers: Binary numbers can have an unbound range, so we can use merge sort for $\Theta(n \log n)$

Q2) An array of n credit card numbers: 16 digits between 0 to 9. So we use radix sort. Runtime will be $O(n+k) = O(n)$

Q3) An ranking of n candidates for a job: Here $k=n$ (ranking are 1, 2, 3, 4 … n) which implies radix or counting sort. Run time = $O(n)$.

Q4) A list of n UTSA students by their banner IDs: 8 digits between 0 to 9. So we use radix sort. Runtime will be $O(n+k) \simeq O(n)$

Q5) An array of n natural numbers: Range is unknown. Therefore we use merge sort to get $O(n \log n)$.

Q6) An array of n students by their grades on an exam: n students with range 0 to 100; $k=101$. Therefore counting sort will be $O(n)$ time.

Q7) A list of n sports team according to their rank: Here $k=n$ (ranking are 1, 2, 3, 4, …, n) which implies radix or counting sort. Runtime = $O(n)$.

Q8) An array of n rational numbers: Range is not given, which is unbound. So we use merge sort, runtime = $O(n \cdot \log n)$

Q9) A phone book consisting of n telephone numer: 10 digit number, each ranging between 0 to 9. So we use radix sort, runtime = $O(n+k) \simeq O(n)$

Q8) A array of n numbers in the range $[0 \ldots n^2]$: Radix Sort, Convert numbers to base $\log n$. Runtime $O(n)$

B-tree: Values for each node = $2t-1$, Root must store at least 1.

① Every node except the root, stores $t-1 \leq$ # of element $\leq 2t-1$

② each node has at most $2t$ child nodes, minimum t child nodes.

③ children of a node = # of elements + 1 (except leaf)

Theorem: A B-tree with minimum degree $t > 2$ which stores n values has height $h \leq \log_t \frac{n+1}{2}$.

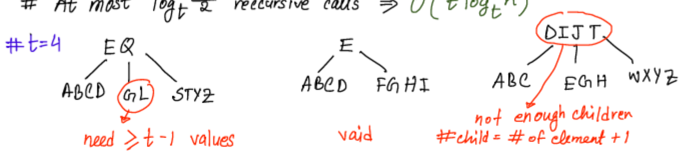
Proof: # of nodes $\geq 1+2+2t+2t^2+2t^3 \ldots + 2t^{i-1} + \ldots + 2t^{h-1}$

$= 1 + \sum_{i=0}^{h} 2t^{i-1} = 1 + \sum_{i=1}^{h-1} 2t^i = 1 + 2\left[\frac{t^h-1}{t-1}\right]$

# of values $= n \geq 1 \cdot 1 + 1 \cdot 2\left[\frac{t^h-1}{t-1}\right] \cdot t-1 = 2t^h - 1$

$\Rightarrow n \geq 2t^h - 1 \Rightarrow \frac{n+1}{2} \geq t^h \Rightarrow \log_t\left(\frac{n+1}{2}\right) \geq h$

At most $\log_t \frac{n+1}{2}$ recursive calls $\Rightarrow O(t \log_t n)$



coin-change

① we can make multiple copies of each coin. Certainly, we would not need more than n/d_i copies of coin d_i. Considering every way of choosing these coins, Therefore, n^m where at most n copies of each coin over m different coins

② $a[1]=1$, $a[2]=2$, $a[3]=3$, $a[4]=1$, $a[5]=1$, $a[6]=2$, $a[7]=3$

③ $a[i] = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = d_j \text{ for some } j \\ \min\limits_{j: d_j \leq i} \{a[i-d_j]+1\} & \text{otherwise} \end{cases}$

④
```
a[0]=0
for j=1 to m
  { a[dj] = 1 }
for (i=2 to n)
{ if (a[i] == NULL)
  { min = n;
    for (j=1 to m)
    { if (dj ≤ i AND a[i-dj] < min)
        { min = a[i-dj]; } end if
    } end for
    a[i] = min+1
  } end if
} end for
return a[n]
```

Runtime = $O(n^2)$