Q1: Suppose we are managing a company with n employees, and each employee is in at least one of 'm' mailing lists. We want to "broadcast" an email to each of the employees through one or more of the mailing lists. That is, we want to choose a subset of the mailing lists to send the email to so that each employee receives the email at least once (it is okay for an employee to receive the email more than once). We would like to know if it is possible to reach all employees using at most 'k' mailing lists. Show that this problem is NP-Complete. That is, show that it is in NP, and show that it is NP-hard.

Answer: Given a set of mailing lists, we first check to see if there are k or fewer of them. If not return No. If so, check to see if each of the employees is on some mailing list. If so, check to see if each of the employees is on some mailing list. If not, return NO, else return YES. This is a poly-time algorithm, so the problem is in NP.

This is a VC problem, so we reduce from it. Our reduction will start with an input (G,k) for VC and we will construct an input for this problem. S.t. (G,k) is yes iff our instances yes.

In VC, vertices cover edges. In mailing lists (ML), mailing list cover employees.

Reduction: For each vertex in G, we create a mailing list. For each edge in G, we create an employee. Each employee is on the two mailing lists associated with its edge's endpoint. We set the k' for ML to be the same as k for VC.

Suppose there is a VC of size k. Then pick the mailing lists associated with these vertices. This is a cover of the employees of size k. Suppose there is a ML cover of size k. Then pick the vertices in G associated with these mailing lists. This is a VC of size k=k'. This completes the proof that this problem is NP-complete.

Q2: Consider a ser A ={$a_1..a_n$} of n elements and a collection $B1, B_2, B_m$ of m subsets of A. A subset "$H \subseteq A$" is a hitting set of B_1, B_2, B_m if for each B_i we have that there is at least one element of B_i in H. Example: A= {1,2,3,4,5}, B_1= {1,2}, B_2= {2,3}, B_3= {4,5}. Then H_1 = {2,5} is a hitting set because each B_i contains either 2 or 5. However H_2= {1,3} is not a hitting set because B_3 does not contain either 1 or 3.

We are interested in computing small hitting set. The hitting set problem wants to determine if there is a hitting set of size at most k. Show that this problem is NP-complete.

Answer: First, we must prove that the problem is NP. We easily verify in polynomial time that there is a hitting set of size aftmost k for a given set of A and B's. So, problem is in NP.

Now to show that the problem is NP-complete, we need to show some NP-hard problem is reducible to our given problem. Let's us consider VC problem. We can convert G(V,E) of VC problem to our problem in polynomial time and prove that a VC of at least k exists if and only if a H, hitting set of at most k' exists. Now, let's take the graph G and all vertices V are element A. Subset B forms when there is an edge between two vertices. A vertex may have multiple elements.

Now, to cover all the edges we need at least k vertices, thus at the same time we need hitting set of size at most k'=k.

On the other hand, we have k' elements of hitting set H, which covers all the B's. That means the graph formed of input (A,B,..,B_m) is covered by k' vertices, which give our vertex cover of at least k'. Hence it is NP complete.

Q3: The half clique problem is the problem of determining whether there is a clique in the graph that contains at least half o f the vertices in the graph. In other words, given a graph with 'n' vertices, is there a clique of size at least 'n/2' in the graph. Show that the half clique problem is NP-complete.

Answer: First, we will show if it is in NP. Given a subset of vertices, we check if it is a clique of size 'n/2' in polynomial time, therefore it is in NP.

Now we will show that clique can be reduced to half clique, implying that it is NP-hard. We are given a graph G and integer 'k', and we will transform it to a graph G' such that G contains a clique size 'k' iff G'= (V', E') contains a clique of size |V'|/2. Let 'n' denote the number of vertices in G. We obtain V' by adding 'n' new vertices to have 2n total vertices. We arbitrarily choose n-k of these vertices and add all the edges between them. So that they form a clique, and we also add edges connecting each of these 'n-k' vertices to all n vertices in G. Note, now that G' has '2n' vertices, and it has clique of size 'n' iff G has a clique of size 'k' (note the clique of G' is the clique of size 'k' in G combine with the clique of size n-k that we added). Therefore, it is NP-complete.

Other way: Every instance of the clique problem consisting of the graph G (V,E) and an integer k can be converted to the required graph G' (V',E') and k' of the half clique problem. The deduction that can be made is that the graph G' will have a clique of size n/2 , if the graph G has a clique of size k. Let m be the number of nodes in the graph G. We will now prove that the problem of computing the clique indeed boils down to the computation of the independent set.

If k > = m/2, then for a constant number t, we add t nodes each of degree 0, for a graph G'. The graph G' has a total number of nodes equivalent to n = m + t, that is, the summation of all the nodes of graph G along with the extra nodes, such that it is equivalent to 2k, for any arbitrary value of k. Now k = n/2. This can be done by taking t = 2k -m. Then, the graph G has a clique of size k if and only if the graph G' has a clique of size k.

If k < m/2, then we add t additional nodes for the creation of graph G'. Edges can also be added from each new node to every other node in the graph. Therefore, any k-clique in G, for any arbitrary value of k combines with the t new nodes to make a (k+t)-clique in G', since edges have been added between each pair of vertices. A k+t-sized clique in G' must include at least k old nodes, which form a clique in the graph G. Therefore, the value of t is picked such that k+t = (m+t)/2, or t = m-2k, which makes the clique size in G' equivalent to n/2 exactly.

Cliché is a complement of independent set problem without the edges.

Q4: Suppose a store has 'n' products and has had 'm' customers buy at least one of the 'n' products. They maintain a 'm × n' array A where entry A [i, j] denotes how many times customer 'i' purchased product 'j'. For the purposes of conducting market research, the store would like to select a large subset of customers such that no two of the customers have ever bought the same product. Show that the problem of determining whether such a subset of size at least k exists is NP-complete.

Answer: First, we must show that the problem is in NP. Given a subset of the customers, we can easily verify in polynomial time if the set has size at least k and if any two customers have ever purchased the same product. Therefore, the problem is in NP.

Now we will show that some NP-hard problem can be reduced to our problem. Note that the problem has many similarities with the independent set (IS) problem. We will begin our reduction with IS. Given a graph G = (V, E) for IS, we will construct a set of customers and products in a way such that G will contain an independent size of size at least k if and only if there is a subset of k customers such that no two customers in this set bought the same product.

For each vertex v ∈ V, let there be a customer c_V. For each edge {u, v} ∈ E, let there be a product p{u,v}. We set customer c_V to have purchased one of each product which corresponds to an edge incident on v in G (note that each product will have exactly 2 customers who purchased it). This completes the reduction.

We will now show that there is an independent set of size at least k in G if and only if there is a subset of k customers such that no two customers in this set bought the same product. First suppose there is an independent set of size 'k' in G and let us call this set 'I'. For each vertex i ∈ I, we let customer c_i be in our subset. Since 'I' is an independent set, then for any pair of vertices i_1, i_2 ∈ 'I', we have that there is no edge connecting i_1 and i_2. By construction, it will be that c_{i1} and c_{i2} did not purchase any of the same products, and therefore we have a subset of customers of size 'k' such that no two customers in this set bought the same product.

Now suppose that there is a subset of customers of size k such that no two customers in this set bought the same product. We will show that there is an independent set of size at least k in G. Consider the subset of vertices V ⊆ V' of G such that v ∈ V' if and only if c_V is in the set of customers of size at least 'k'. Two customers purchased the same product only when their corresponding vertices in G had an edge connecting them. This implies that if V_1, V_2 are vertices in V', then {V_1, V_2} /∈ E. Therefore V' is an independent set of size k. This completes the proof that this problem is NP-complete.

Q5: Suppose that a sports camp will offer training for n sports. They want to hire a set of counselors who collectively can offer training for the sports. They have received applications from m potential counselors, and each candidate has indicated which of the n sports they are qualified to teach. Show that the problem of determining if there is a set of candidates of size at most k which collectively can teach each of the n sports is NP-complete.

Answer: First, we must show that the problem is in NP. Given a subset of counselors, we can easily verify in polynomial time if the set has size at most k and if each of the sports are covered by the set. Therefore, the problem is in NP.

Now we will show that some NP-hard problem can be reduced to our problem. Note that this problem is a covering problem (we want to cover the sports with a small set of counselors). The vertex cover (VC) problem is similarly a covering problem (covering edges with vertices), and so we will begin our reduction from VC. Given a graph G = (V, E) for VC, we will construct a set of counselors and sports in a way such that G will contain a vertex cover of size at most k if and only if there is a subset of at most k counselors such that they collectively can teach each of the sports. For each vertex v ∈ V, we create a counselor c_V. For each edge {u, v} ∈ E, we create a sport, and we set c_U and c_V to be eligible to teach this sport (note that each sport will have exactly two counselors who can teach it). This completes the reduction.

Now we will show that there is a vertex cover of size at most k if and only if there is a set of at most k counselors who can collectively teach each of the sports. First suppose there is a vertex cover C of size at most k in G. For each v ∈ C, let c_V be a counselor in our set. Clearly this set has size at most k, and we will argue that this set covers all of the sports. Since C was a vertex cover, we know that for each edge in E, at least one of its endpoints is in C. Following our reduction, this implies that for each sport, we have chosen at least one of the two counselors who can teach the sport. Therefore, the counselors will collectively teach each of the sports.

Now suppose there is a set S of at most k counselors who collectively can teach each of the sports. Let V' denote the vertices in G that correspond with the counselors in S. We know that for each sport, we have chosen a counselor who can teach it, but based off of the reduction that means that any edge must have had one of its endpoints chosen. Therefore V' is a vertex cover of G, and its size is at most k. This completes the proof that this problem is NP-complete.

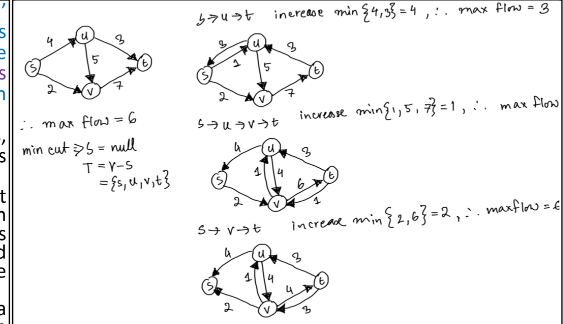
Q6: Imagine a social network where individuals are represented as nodes, and friendships between individuals are represented as edges. You want to identify groups of friends who are closely connected, forming cliques within the social network.

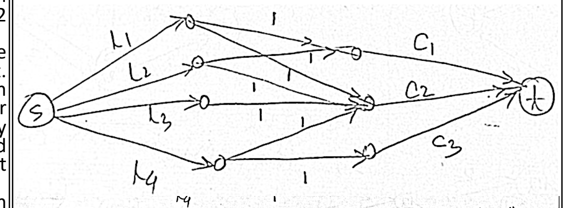
First, we must show that the problem is NP. Given a subset of friends, we can easily verify in polynomial if the set has size at least k and who are closely connected.

This is similar to clique, so we reduce from it. Our reduction will start with an input (V,E) for clique and we will construct an input for this problem. S.t. (V,E) is yes iff our instances yes. In clique, vertices cover edges. In social network, individuals cover friendship. 'k' is the maximum value of set C with every pair of nodes connected with an edge.

Reduction: For each vertex in V, we create individuals. For each edge in V, we create a friendship. Each friendship is on the two individuals associated with its edge's endpoint. We set the k' for individuals to be the same as k for clique. And take an empty set C to be the largest subset C of V such that every pair of vertices in C has an edge between them.

Suppose there is a C of size k. Then pick the individuals associated with these vertices. This is a friendship of the individuals of size k. Suppose there is a social network individual with friendship of size k. Then pick the vertices in G associated with these individuals. This is a clique of size k=k'. This completes the proof that this problem is NP-complete.





we connect the edges from a store to a location with weight 1 if their distance is at most r. From a source s, we connect edge with a weight Li to the stores. From the locations we connect edge to sink t with weight Cj. Now running Ford-Fulkerson algorithm will give us the maximum number of advertisements.

(b) we can add another vertex t' from t with weight 20 where t' is our new sink. Running Forr Fulkerson algorithm with give us maximum number of advertisement ≤ 20.