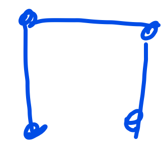
Suppose we have a set of n locations, and we wish to build a connected network on top of them. The network should be connected (there should be a path between any two locations in the network), and subject to this constraint, we wish to build the network as cheaply as possible.



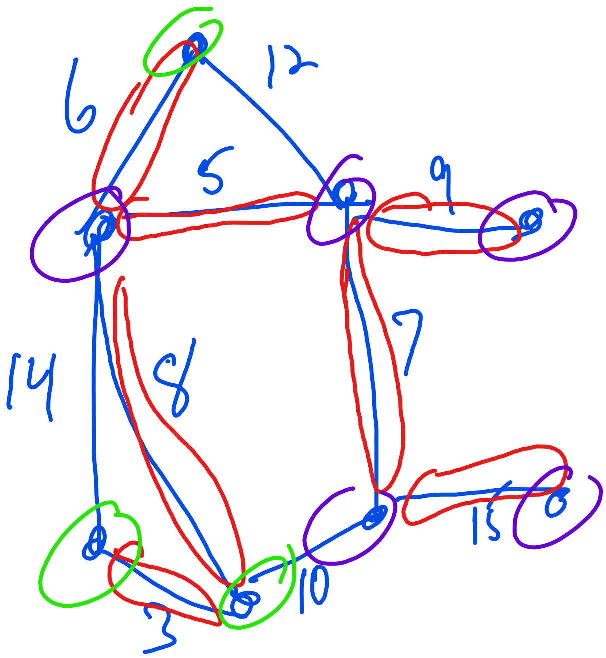
optimal

Note that a solution to this must be a tree (if the network contains a cycle, we can remove one of the connections to obtain a cheaper network and still satisfy the connectivity constraint).

In graph theory, a tree which contains every vertex of the graph is known as a **spanning tree**.

If we assign non-negative weights $w(u, v)$ to each edge $\{u, v\}$ in the graph (i.e. the cost to connect two locations in the network), then a **minimum spanning tree** (MST) is a spanning tree such that the sum of the weights of the edges in the tree is minimized.

Example of a MST:



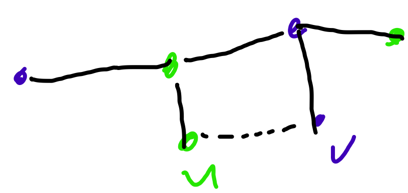
A key observation of MSTs (for simplicity, assume the weights on the edges are distinct):

Theorem

Let T be a MST of $G=(V,E)$ and let $A \subset V$. Suppose $\{u,v\} \in E$ is the minimum weight edge connecting A to $V \setminus A$. Then $\{u,v\} \in T$.

Proof

For the sake of contradiction, suppose it is not in T. Let green vertices be A and purple vertices be $V \setminus A$.

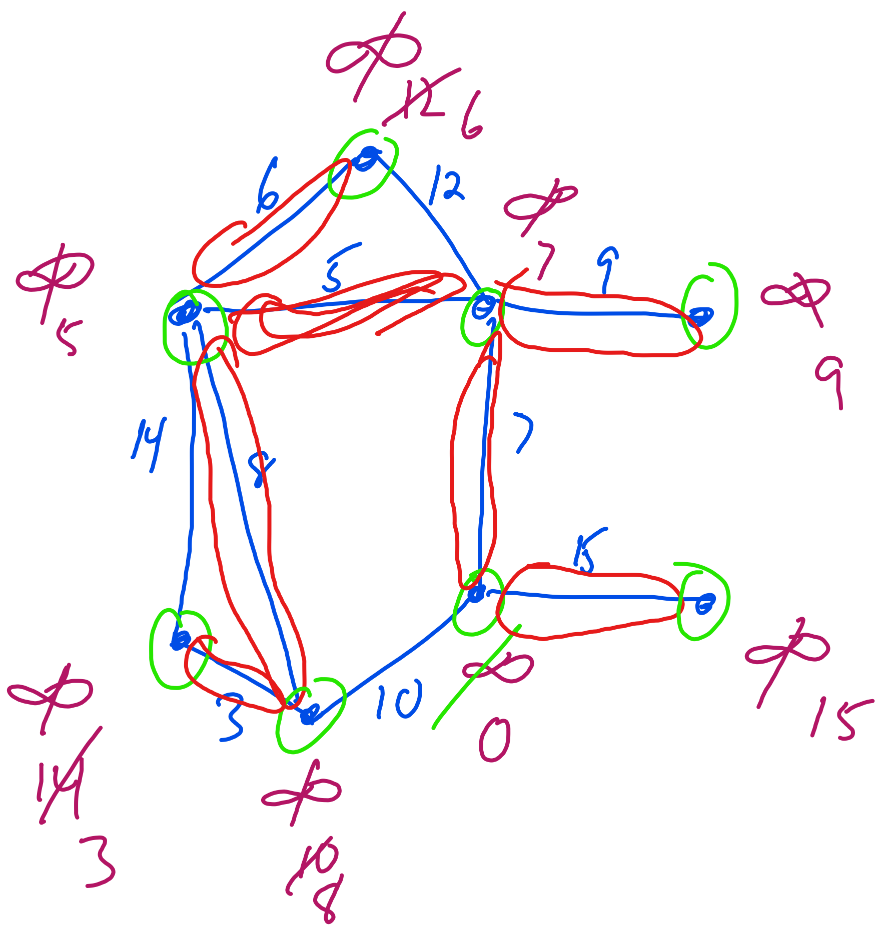


Solid $= T$
dotted $= \{u,v\}$

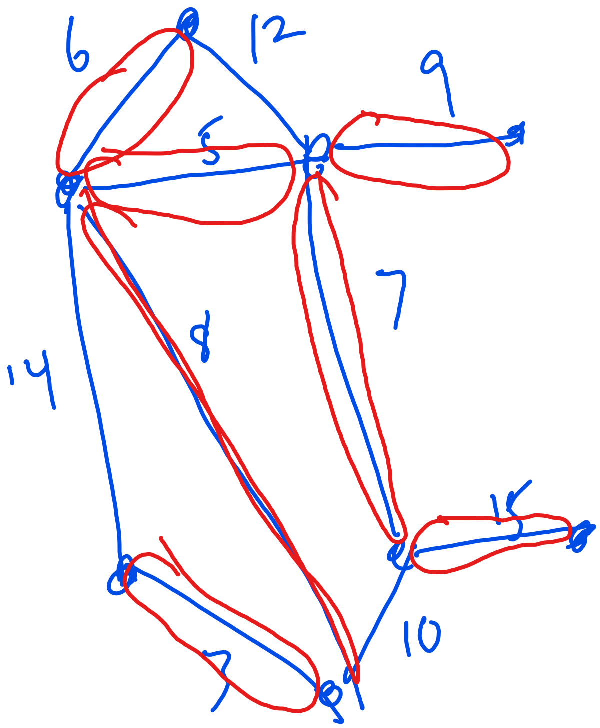
Follow path from u to v in T, and remove an edge connecting a green to a purple. Such an edge must exist because the path starts on a green and ends on a purple. We can then add $\{u,v\}$ to obtain a cheaper spanning tree, a contradiction.

Prim's algorithm:

- Maintain a key for each vertex (initially set to ∞). Arbitrarily pick a vertex and change its key to 0. Let Q initially be the set of all vertices.
- Find the vertex u in Q with the smallest weight, and remove u from Q.
  - For each neighbor v of u, check if $w(u, v) < key(v)$. If so, set $key(v) = w(u, v)$, and mark u as the "parent" of v in the tree.



Kruskal's algorithm: Repeatedly pick the edge with the smallest weight as long as it does not form a cycle.



Here is an overview of MST algorithms:

- Prim's algorithm:
  - Maintains one tree
  - Utilizing a proper data structure (binary heap), runs in time $O(m \log n)$.

- Kruskal's algorithm:
  - Maintains a forest.
  - When using a data structure which we will discuss later in the class, runs in time $O(m \log m)$.

    union-find

    $\log n^2$

    $= 2 \log n$

- There is a randomized algorithm due to Karger, Klein, and Tarjan [1993] which runs in expected time $O(n + m)$.