Divide-and-Conquer Algorithms:

1. **Divide:** Break the problem (instance) into subproblems of sizes that are fractions of the original problem size.
2. **Conquer:** Recursively solve each of the subproblems. If the subproblems are "small enough" (base case), then solve the subproblem in a straightforward manner.
3. **Combine:** Put the solutions for each of the subproblems together to obtain a solution for the original problem.

Example: Use binary search to find an element k in a sorted array.

1. **Divide:** Compare the middle element with k.
2. **Conquer:** If the middle element is not k, we can recursively solve the subproblem where we check half of the current array.
3. **Combine:** There is only 1 subproblem in binary search, and so the combine step is trivial.

Merge Sort: Given an array of n numbers, sort the numbers in the array in non-decreasing order.

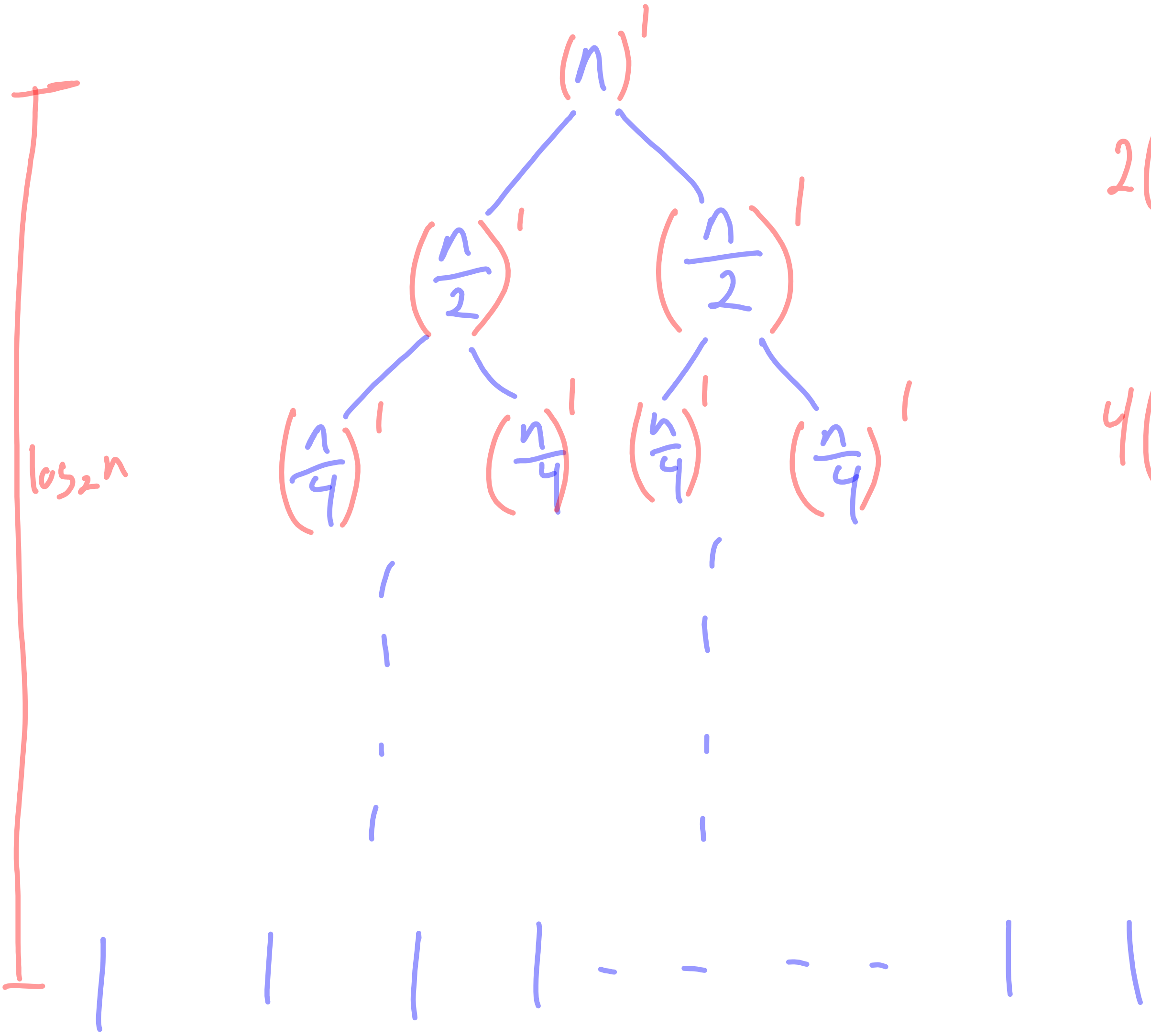
1. **Divide:** Break the problem into 2 subproblems of size $n/2$.
2. **Conquer:** Recursively sort each of the 2 subarrays.
3. **Combine:** Combine the two sorted subarrays with $n/2$ elements into one sorted array of n elements in $O(n)$ time.

~~2~~	~~8~~	~~9~~	~~10~~

~~1~~	~~7~~	~~11~~	~~12~~

1	2	7	8	9	10	11	12

Merge Sort Recursion Tree:



n

$2\left(\frac{n}{2}\right) = n$

$4\left(\frac{n}{4}\right) = n$

n

$$\sum_{i=0}^{\log_2 n} n = n + n + n + \cdots = \Theta(n \log n)$$

$$2^{\log n} = n$$

$n = 8 \quad \log n = 3 \quad 2^3 = 8$

Merge Sort:

- Merge sort runs in $\Theta(n \log n)$ time.

- $\Theta(n \log n)$ grows more slowly than the $\Theta(n^2)$ running time of insertion sort as n approaches infinity.

- Therefore, merge sort asymptotically beats insertion sort in the worse case.

- In practice, merge sort performs better than insertion sort on arrays with 30 or more numbers.

Recursion Trees:

- A recursion tree models the running time of a recursive algorithm, but in some cases can be unreliable and/or difficult to analyze.

- It is good for generating *guesses* of what the running time could be.

- In such cases, we may need to verify our guess is correct via a proof by induction.

Guess: Merge Sort runs in time $O(n \log n)$.

$T(n) \le c \cdot n \log n$ for some constant $c > 0$ and $n \ge n_0$ for some $n_0 \ge 1$.

$$T(n) = \underbrace{T\left(\tfrac{n}{2}\right)}_{\text{time to sort first half}} + \underbrace{T\left(\tfrac{n}{2}\right)}_{\text{time to sort second half}} + \underbrace{d \cdot n}_{\text{time to combine}} = 2T\left(\tfrac{n}{2}\right) + d \cdot n$$

<u>Base Case:</u> $T(1) = d'$, $c \cdot n \log n = c \cdot 1 \cdot \log 1 = 0$ X

$T(2) = 2T(1) + d \cdot 2 = 2d' + 2d$. $c n \log n = c\, 2 \log 2 = 2c$.

We need to choose c large enough so $2c \ge 2d' + 2d$.

<u>Inductive Case:</u> Assume $T(k) \le c \cdot k \log k$ for all $k < n$.

$$T(n) = 2T\left(\tfrac{n}{2}\right) + d \cdot n \le 2 \cdot \left[c \tfrac{n}{2} \cdot \underbrace{\log \tfrac{n}{2}}_{\log n - \log 2}\right] + d \cdot n$$

$$= \cancel{2} \cdot c \cdot \frac{n}{\cancel{2}} \cdot (\log n - \underbrace{\log 2}_{1}) + d \cdot n$$

$$= cn \log n - cn + dn$$

$$\le cn \log n \text{ when } cn \ge dn \Rightarrow c \ge d.$$

Example: computing a^n for some non-negative integer n.

Naive algorithm: Compute $a \cdot a \cdot a \cdots a$. Takes $\Theta(n)$ time.

Divide-and-conquer algorithm:

if n is even: $a^n = a^{\frac{n}{2}} \cdot a^{\frac{n}{2}}$

if n is odd: $a^n = a \cdot a^{\lfloor \frac{n}{2} \rfloor} \cdot a^{\lfloor n/2 \rfloor}$

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$$

$\log_2 n$

$(n)^0 \quad 1$

$(n/2)^0 \quad 1$

$(n/4)^0 \quad 1$

$\vdots \quad \vdots$

$(1)^0 \quad 1$

$\Theta(\log n)$

Example: Matrix Multiplication: Given two $n \times n$ matrices A and B, compute the $n \times n$ matrix $C = A \cdot B$.

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

Possible algorithm:

for $i = 1$; $i \leq n$; $i = i + 1$ **do**
  for $j = 1$; $j \leq n$; $j = j + 1$ **do**
    $c_{ij} = 0$
    for $k = 1$; $k \leq n$; $k = k + 1$ **do**
      $c_{ij} = c_{ij} + a_{ik} b_{kj}$
    end for
  end for
end for

$O(n^3)$

Divide-and-conquer algorithm:

$$C = A \cdot B$$

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\left.\begin{aligned} r &= a \cdot e + b \cdot g \\ s &= a \cdot f + b \cdot h \\ t &= c \cdot e + d \cdot g \\ u &= c \cdot f + d \cdot h \end{aligned}\right\}$$

8 multiplications

4 additions in time $\Theta(n^2)$

$$T(n) = 8\,T\left(\frac{n}{2}\right) + \Theta(n^2)$$

works out to $\Theta(n^3)$.

Strassen's idea:

- Recursively compute the following matrices (note only 7 multiplications):
  - $P_1 = a \cdot (f - h)$
  - $P_2 = (a + b) \cdot h$
  - $P_3 = (c + d) \cdot e$
  - $P_4 = d \cdot (g - e)$
  - $P_5 = (a + d) \cdot (e + h)$
  - $P_6 = (b - d) \cdot (g + h)$
  - $P_7 = (a - c) \cdot (e + f)$

- We can then compute C as so:
  - $r = P_5 + P_4 - P_2 + P_6$
  - $s = P_1 + P_2$
  - $t = P_3 + P_4$
  - $u = P_5 + P_1 - P_3 - P_7$

- Running time of first divide-and-conquer algorithm:
  - $T(n) = 8T(n/2) + \Theta(n^2) = \Theta(n^{\log 8}) = \Theta(n^3)$

- Running time of Strassen's algorithm:
  - $T(n) = 7T(n/2) + \Theta(n^2) = \Theta(n^{\log 7}) = o(n^3)$

- Strassen's algorithm beats ordinary iterative algorithm in practice for $n \geq 30$ or so.