Matrix Chain

CS 5633 Analysis of Algorithms

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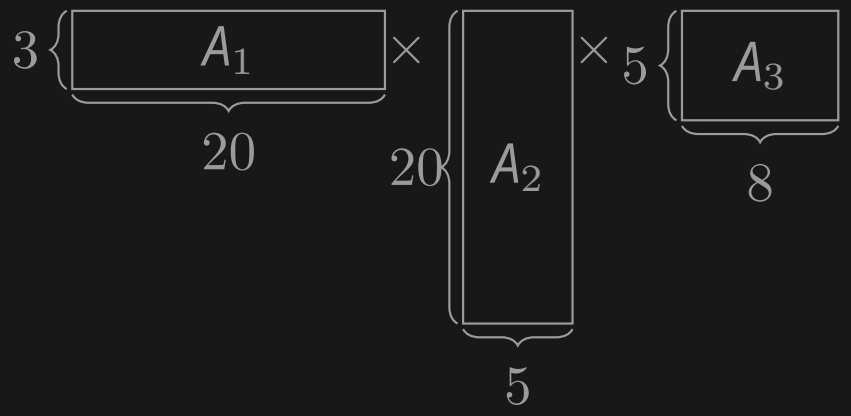
The Matrix Multiplication Chain Problem

Matrix Multiplication Chain (MMC)

- Suppose we are given a sequence/chain of n matrices $A_1, A_2, \ldots, A_n$, and we are interested in computing the product $A_1 \cdot A_2 \cdots A_n$.
- Note that if this product can be computed, then there are $n+1$ values $p_0, p_1, \ldots, p_n$ such that matrix A_i is a $p_{i-1} \times p_i$ matrix.
  - Example: $n = 3, p_0 = 3, p_1 = 20, p_2 = 5, p_3 = 8$.



- Computing $A_1 \cdot A_2$ takes $3 \cdot 20 \cdot 5$ multiplications and results in a 3×5 matrix.

Time Complexity of Matrix Multiplication Chain

- In general, computing $A_i \cdot A_{i+1}$ takes $p_{i-1} \cdot p_i \cdot p_{i+1}$ multiplications and results in a $p_{i-1} \times p_{i+1}$ matrix.
- Matrix multiplication is associative, and therefore the order in which we multiply the matrices does not affect the resulting matrix.
  - Computing $(A_1 \cdot A_2) \cdot A_3$ results in the same matrix as computing $A_1 \cdot (A_2 \cdot A_3)$.
- That being said, the order in which we do the multiplications can greatly affect the total number of scalar multiplications performed by the algorithm.
  - Computing $(A_1 \cdot A_2) \cdot A_3$ takes $3 \cdot 20 \cdot 5 + 3 \cdot 5 \cdot 8 = 300 + 120 = 420$ multiplications.
  - Computing $A_1 \cdot (A_2 \cdot A_3)$ takes $20 \cdot 5 \cdot 8 + 3 \cdot 20 \cdot 8 = 800 + 480 = 1280$ multiplications.

The Optimal Solution

- Given that the order in which we multiply the matrices can have a huge impact on the running time of the algorithm, it often is worth the time in practice to compute an optimal "parenthesization" of the matrices that minimizes the number of scalar multiplications needed to compute the product.
- What is the running time of the brute force algorithm which checks the cost of each possible parenthesization and returns the best one?
  - The algorithm is correct, but we can show that the number of parenthesizations is $\Omega(2^n)$. Too slow.

Finding the Optimal Solution

- What about a “greedy” strategy which iteratively tries to find a “good” pair of matrices to multiply?
  - We can construct counterexamples which show that a greedy strategies can fail.
  - For example, considers matrices $A_1 = 1 \times 20$, $A_2 = 20 \times 21$, and $A_3 = 21 \times 5$
    - If we try to eliminate the largest number: $A_1 \cdot (A_2 \cdot A_3) = 20 \cdot 21 \cdot 5 + 1 \cdot 20 \cdot 5 == 2200$.
    - The other solution: $(A_1 \cdot A_2) \cdot A_3 = 1 \cdot 20 \cdot 21 + 1 \cdot 21 \cdot 5 == 525$.

The Dynamic Programming Solution to Matrix Multiplication Chain

Constructing Sub-problems

- How about a dynamic programming algorithm? Can we express an optimal solution as a combination of optimal solutions for some set of
- Let $A_{i,j} = A_i \cdot A_{i+1} \cdots A_j$ for $i \leq j$ (note we are interested in computing $A_{1,n}$).
- Suppose an optimal parenthesization for $A_{i,j}$ splits this matrix at k, so

$$A_{i,j} = (A_i \cdots A_k) \cdot (A_{k+1} \cdots A_j)$$

- Notice that this parenthesization will compute $A_{i,k}$ and $A_{k+1,j}$ and then multiply these matrices together using $p_{i-1} \cdot p_k \cdot p_j$ scalar multiplications.

Constructing the Optimal Solution from Sub-problems

- When computing $A_{i,k} = A_i \cdots A_k$, an optimal solution for $A_{i,j}$ must use an optimal parenthesization of $A_i \cdots A_k$ (otherwise we contradict the assumption that the solution was optimal).
- Let $m[i, j]$ denote the minimum number of scalar multiplications to compute $A_{i,j}$. Therefore the cost of splitting $A_{i,j}$ at k is

$$m[i, k] + m[k + 1, j] + p_{i-1} \cdot p_k \cdot p_j$$

Constructing the Optimal Solution from Sub-problems cont.

- We have the following recurrence relation:
  - $m[i,i] = 0$.
  - $m[i,j] = \min_{i \leqslant k < j}(m[i,k] + m[k+1,j] + p_{i-1} \cdot p_k \cdot p_j)$
- We can see that the running time of the straight forward recursive algorithm is $\Omega(2^n)$, but there are only $\Theta(n^2)$ subproblems (recursive algorithm can call the same sub-problem multiple times).
- The dynamic programming solution essentially computes the values for the two-dimensional array m.

An Example

- Consider the following matrix multiplication problem:
  - $A_1 = 2 \times 100$, $A_2 = 100 \times 4$, $A_3 = 4 \times 10$, $A_4 = 10 \times 3$.
  - $n = 2$, $p_0 = 2$, $p_1 = 100$, $p_2 = 4$, $p_3 = 10$, $p_4 = 3$,.
- The m matrix:

	1	2	3	4
1	0	800	880	940
2	x	0	4000	1320
3	x	x	0	120
4	x	x	x	0

- Note that only the top-right of the matrix needs to be populated.
- $m[1,1]$, $m[2,2]$, $m[3,3]$ and $m[4,4]$ are base cases.
- $m[1,2]$, $m[2,3]$, and $m[3,4]$ are trivial cases.

An Example cont.

- $m[1,3]$ is the minimum of
  - $k = 1$,
    $m[1,1] + m[2,3] + 2 \cdot 100 \cdot 10 = 0 + 4000 + 2000 = 6000.$
  - $k = 2$, $m[1,2] + m[3,3] + 2 \cdot 4 \cdot 10 = 800 + 0 + 80 = 880.$
- $m[2,4]$ is the minimum of
  - $k = 2$, $m[2,2] + m[3,4] + 100 \cdot 4 \cdot 3 = 0 + 120 + 1200 = 1320.$
  - $k = 3$,
    $m[2,3] + m[4,4] + 100 \cdot 10 \cdot 3 = 4000 + 0 + 3000 = 7000.$
- $m[1,4]$ is the minimum of
  - $k = 1$, $m[1,1] + m[2,4] + 2 \cdot 100 \cdot 3 = 0 + 1320 + 600 = 1920.$
  - $k = 2$, $m[1,2] + m[3,4] + 2 \cdot 4 \cdot 3 = 800 + 120 + 24 = 944.$
  - $k = 3$, $m[1,3] + m[4,4] + 2 \cdot 10 \cdot 3 = 880 + 0 + 60 = 940.$

Time Complexity

- Half of the m matrix have to be filled. Therefore, there are $\Theta(n^2)$ values to compute.
- To compute each value, $i - j$ decompositions have to be evaluated to find the minimum. Therefore, each value takes at most $O(n)$ time.
- Then the total run time is $O(n^3)$.