CS 5633: Analysis of Algorithms

Homework 10 Solution

1. For part (a), we need to show that there is a polynomial time algorithm for Π'. Given an input π' for Π', we can convert π' into an input π for Π in polynomial time. Since $\Pi \in P$, we can compute the solution for π in polynomial time, and then use the output for π to determine the output for π'.

   For part (b), we need to show that a certificate c' for Π' can be verified in polynomial time. Since $\Pi' \leq \Pi$, we can convert c' into a certificate c for Π in polynomial time. Since $\Pi \in NP$, we can verify c in polynomial time and this will give us an answer for c' in polynomial time.

   For part (c), we can take an input π for Π and convert it in polynomial time to an input π' for Π'. We can again convert π' into an input π'' for Π''. The output for π' can be obtained from an output for π'', and likewise the output for π can be obtained from an output for π'. This implies that the output for π can be obtained from the output for π'', and therefore $\Pi \leq \Pi''$.

2. First we have to show that the problem is in NP. Given a subset of the customers, we can easily verify in polynomial time if the set has size at least k and if any two customers have ever purchased the same product. Therefore the problem is in NP.

   Now we will show that some NP-hard problem can be reduced to our problem. Note that the problem has many similarities with the independent set (IS) problem. We will begin our reduction with IS. Given a graph $G = (V, E)$ for IS, we will construct a set of customers and products in a way such that G will contain an independent size of size at least k if and only if there is a subset of k customers such that no two customers in this set bought the same product.

   For each vertex $v \in V$, let there be a customer c_v. For each edge $\{u, v\} \in E$, let there be a product $p_{\{u,v\}}$. We set customer c_v to have purchased one of each product which corresponds to an edge incident on v in G (note that each product will have exactly 2 customers who purchased it). This completes the reduction.

   We will now show that there is an independent size of size at least k in G if and only if there is a subset of k customers such that no two customers in this set bought the same product. First suppose there is an independent set of size k in G, and let us call this set I. For each vertex $i \in I$, we let customer c_i be in our subset. Since I is an independent set, then for any pair of vertices $i_1, i_2 \in I$ we have that there is no edge connecting i_1 and i_2. By construction, it will be that c_{i_1} and c_{i_2} did not purchase any of the same products, and therefore we have a subset of customers of size k such that no two customers in this set bought the same product.

   Now suppose that there is a subset of customers of size k such that no two customers in this set bought the same product. We will show that there is an independent set of size at least k in G. Consider the subset of vertices $V' \subseteq V$ of G such that $v \in V'$ if and only if c_v is in the set of customers of size at least k. Two customers purchased the same product only when their corresponding vertices in G had an edge connecting them. This implies that if v_1, v_2 are vertices in V', then $\{v_1, v_2\} \notin E$. Therefore V' is an independent set of size k. This completes the proof that this problem is NP-complete.

3. First we have to show that the problem is in NP. Given a subset of counselors, we can easily verify in polynomial time if the set has size at most k and if each of the sports are covered by the set. Therefore the problem is in NP.

   Now we will show that some NP-hard problem can be reduced to our problem. Note that this problem is a covering problem (we want to cover the sports with a small set of counselors). The vertex cover (VC) problem is similarly a covering problem (covering edges with vertices), and so we will begin our reduction from VC. Given a graph $G = (V, E)$ for VC, we will construct a set of counselors and sports in a way such that G will contain a vertex cover of size at most k if and only if there is a subset of at most k counselors such that they collectively can teach each of the sports.

For each vertex $v \in V$, we create a counselor c_v. For each edge $\{u, v\} \in E$, we create a sport, and we set c_u and c_v to be eligible to teach this sport (note that each sport will have exactly two counselors who can teach it). This completes the reduction.

Now we will show that there is a vertex cover of size at most k if and only if there is a set of at most k counselors who can collectively teach each of the sports. First suppose there is a vertex cover C of size at most k in G. For each $v \in C$, let c_v be a counselor in our set. Clearly this set has size at most k, and we will argue that this set covers all of the sports. Since C was a vertex cover, we know that for each edge in E, at least one of its endpoints is in C. Following our reduction, this implies that for each sport, we have chosen at least one of the two counselors who can teach the sport. Therefore the counselors will collectively teach each of the sports.

Now suppose there is a set S of at most k counselors who collectively can teach each of the sports. Let V' denote the vertices in G that correspond with the counselors in S. We know that for each sport, we have chosen a counselor who can teach it, but based off of the reduction that means that any edge must have had one of its endpoints chosen. Therefore V' is a vertex cover of G, and its size is at most k. This completes the proof that this problem is NP-complete.