CS 5633: Analysis of Algorithms

Homework 10 Solution

1. First we have to show that the problem is in NP. Given a subset of the customers, we can easily verify in polynomial time if the set has size at least k and if any two customers have ever purchased the same product. Therefore the problem is in NP.

   Now we will show that some NP-hard problem can be reduced to our problem. Note that the problem has many similarities with the independent set (IS) problem. We will begin our reduction with IS. Given a graph $G = (V, E)$ for IS, we will construct a set of customers and products in a way such that G will contain an independent size of size at least k if and only if there is a subset of k customers such that no two customers in this set bought the same product.

   For each vertex $v \in V$, let there be a customer c_v. For each edge $\{u, v\} \in E$, let there be a product $p_{\{u,v\}}$. We set customer c_v to have purchased one of each product which corresponds to an edge incident on v in G (note that each product will have exactly 2 customers who purchased it). This completes the reduction.

   We will now show that there is an independent size of size at least k in G if and only if there is a subset of k customers such that no two customers in this set bought the same product. First suppose there is an independent set of size k in G, and let us call this set I. For each vertex $i \in I$, we let customer c_i be in our subset. Since I is an independent set, then for any pair of vertices $i_1, i_2 \in I$ we have that there is no edge connecting i_1 and i_2. By construction, it will be that c_{i_1} and c_{i_2} did not purchase any of the same products, and therefore we have a subset of customers of size k such that no two customers in this set bought the same product.

   Now suppose that there is a subset of customers of size k such that no two customers in this set bought the same product. We will show that there is an independent set of size at least k in G. Consider the subset of vertices $V' \subseteq V$ of G such that $v \in V'$ if and only if c_v is in the set of customers of size at least k. Two customers purchased the same product only when their corresponding vertices in G had an edge connecting them. This implies that if v_1, v_2 are vertices in V', then $\{v_1, v_2\} \notin E$. Therefore V' is an independent set of size k. This completes the proof that this problem is NP-complete.

2. First we have to show that the problem is in NP. Given a subset of counselors, we can easily verify in polynomial time if the set has size at most k and if each of the sports are covered by the set. Therefore the problem is in NP.

   Now we will show that some NP-hard problem can be reduced to our problem. Note that this problem is a covering problem (we want to cover the sports with a small set of counselors). The vertex cover (VC) problem is similarly a covering problem (covering edges with vertices), and so we will begin our reduction from VC. Given a graph $G = (V, E)$ for VC, we will construct a set of counselors and sports in a way such that G will contain a vertex cover of size at most k if and only if there is a subset of at most k counselors such that they collectively can teach each of the sports.

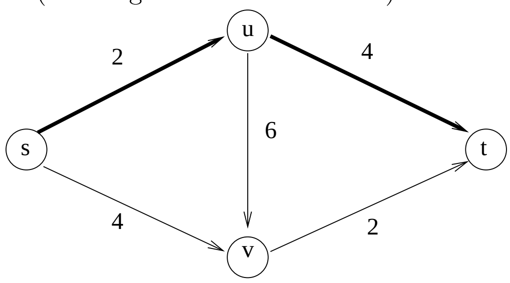
   For each vertex $v \in V$, we create a counselor c_v. For each edge $\{u, v\} \in E$, we create a sport, and we set c_u and c_v to be eligible to teach this sport (note that each sport will have exactly two counselors who can teach it). This completes the reduction.

   Now we will show that there is a vertex cover of size at most k if and only if there is a set of at most k counselors who can collectively teach each of the sports. First suppose there is a vertex cover C of size at most k in G. For each $v \in C$, let c_v be a counselor in our set. Clearly this set has size at most k, and we will argue that this set covers all of the sports. Since C was a vertex cover, we know that for each edge in E, at least one of its endpoints is in C. Following our reduction, this implies that for each sport, we have chosen at least one of the two counselors who can teach the sport. Therefore the counselors will collectively teach each of the sports.

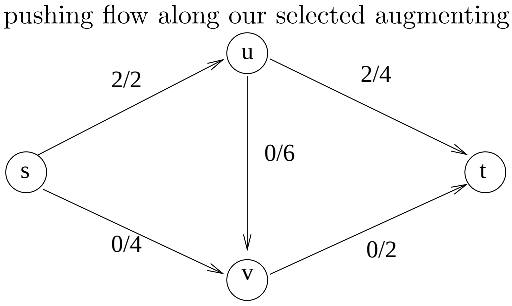
Now suppose there is a set S of at most k counselors who collectively can teach each of the sports. Let V' denote the vertices in G that correspond with the counselors in S. We know that for each sport, we have chosen a counselor who can teach it, but based off of the reduction that means that any edge must have had one of its endpoints chosen. Therefore V' is a vertex cover of G, and its size is at most k. This completes the proof that this problem is NP-complete.

3. We compute the flow using the augmenting path method of Ford-Fulkerson, and the minimum cut is all of the nodes reachable from s in the final residual network.

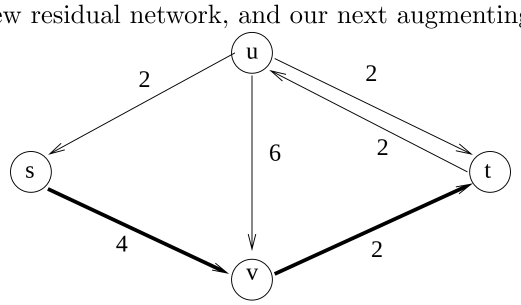
Original flow network (and original residual network). We choose the bolded path.



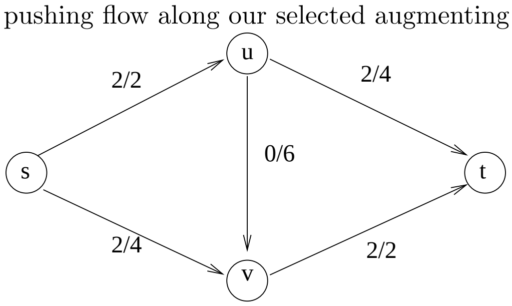
After pushing flow along our selected augmenting path:



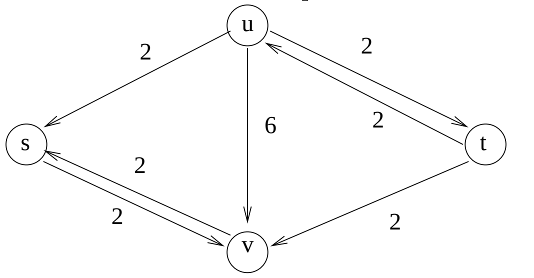
Our new residual network, and our next augmenting path:



After pushing flow along our selected augmenting path:



The final residual network. There is no path from s to t so we are done.



The min cut is $S = \{s, v\}$ and $T = \{u, t\}$.

4. We will view each unit of flow as an assignment of a client to a base station. We can view the underlying graph as a bipartite graph where the clients are one set of vertices and the base stations are the other set of vertices. We add a source vertex s and a sink vertex t. We add an edge of capacity 1 from s to each of the client vertices (this captures the constraint that each client can be assigned to at most one base station). We add an edge of capacity 1 from a client to each of the base stations whose distance is at most r. We then add edges from each base station to t with capacity k (this captures the constraint that each base station can have at most k clients assigned to it). This completes the construction.

   We will now show that each client can be connected to a base station if and only if the maximum flow of this flow network is n (the number of clients). First suppose that there is a flow of value n. Then there must be one unit of flow to each of the clients from the source, and therefore one unit of flow leaving the client and going through a particular base station. Since the capacity on the edge leaving a base station is at most k, then we know that there is at most k units of incoming flow. Therefore we can feasibly assign each client to the base station as indicated by the flow to get a valid assignment of the clients to the base stations.

   Now suppose that there is a valid way of assigning each of the clients to base stations. We will show how to construct a flow of value n. For each "client/base station" pair, set the flow on the corresponding edge in the flow network to be 1. Let b_i denote the number of clients assigned to base station i. We know that $b_i \leq k$, so we can set the flow on the edge from i to t to be b_i. We set the flow on all of the edges from s to a client to be 1. Clearly this is a feasible flow, and its value is n.