HW 1 Solution

1(a) At the beginning of each iteration of the while loop, $k = 2^c$.

(b) <u>Base Case</u>: $k=1$, $c=0 \Rightarrow 1 = 2^0$ ✓

<u>Inductive Step</u>: Let k_{old} + c_{old} be the values of k and c at the beginning of the while loop iteration, and let k_{new} and c_{new} be these values after the iteration.

Assume by inductive hypothesis that $k_{old} = 2^{c_{old}}$. Note we have $k_{new} = 2 * k_{old}$ and $c_{new} = c_{old} + 1$.

$$k_{new} = 2 \cdot k_{old} = 2(2^{c_{old}}) = 2^{c_{old}+1} = 2^{c_{new}}.$$ ✓

At the end of the while loop, we have $c = n$, and therefore $k = 2^n$. Therefore the algorithm is correct.

(c) The while loop runs from $c=0$ to $c=n$, incrementing c by 1 each iteration. The algorithm runs in $\Theta(n)$ time.

(a) Outer loop iterates $\frac{3n}{4}$ times

Inner loop iterates $\log_3 20n$ times

$\therefore$ Running time $= \frac{3n}{4} \cdot \log_3 20n = \Theta(n \log n)$

(b) Running time $= \sum_{i=1}^{3n^2} \log_2 i$

$= \log_2 1 + \log_2 2 + \log_2 3 + \ldots + \log_2($

$\leq \log_2(3n^2) + \log_2(3n^2) + \ldots + \log_2($

$= 3n^2 \log_2(3n^2)$

Again Running time

$$= \sum_{i=1}^{3n^2} \log_2 i$$

$$\geq \sum_{i=3n^2/2}^{3n^2} \log_2 i$$

$$= \log_2\left(\frac{3n^2}{2}\right) + \log_2\left(\frac{3n^2}{2}+1\right) + \cdots + \log(3n^2)$$

$$\geq \log\left(\frac{3n^2}{2}\right) + \log_2\left(\frac{3n^2}{2}\right) + \cdots + \log\left(\frac{3n^2}{2}\right)$$

$$= \frac{3n^2}{2} \log \frac{3n^2}{2}$$

$$= \Omega(n^2 \log n)$$

$\therefore$ Running time $= \theta(n^2 \log n)$

3. (a) $4n^5 - 50n^2 + 10n \in \theta(n^5)$

Now, $4n^5 - 50n^2 + 10n \leq 4n^5 + 10n$

$\leq 4n^5 + 10n^5 \quad [n \geq 1$

$= 14n^5$

$= O(n^5)$

Again, $4n^5 - 50n^2 + 10n \geq 4n^5 - 50n^2$

$= n^5 + (3n^5 - 50n^2)$

$\geq n^5$ if $3n^5 - 50n^2 \geq$

$\Rightarrow n \geq \sqrt[3]{\frac{50}{3}}$

$\Rightarrow n \geq 3$

$= \Omega(n^5)$

(b) $5n^{2/3} + 8\log n \in o(n)$

Now, $\lim\limits_{n \to \infty} \dfrac{5n^{2/3} + 8\log n}{n}$

$= \lim\limits_{n \to \infty} 5n^{-1/3} + \lim\limits_{n \to \infty} \dfrac{8\log n}{n}$

$= 0 + \lim\limits_{n \to \infty} \dfrac{8 \cdot \frac{1}{n}}{1}$

$= 0 + 8 \cdot \lim\limits_{n \to \infty} \frac{1}{n}$

$= 0$

$\therefore 5n^{2/3} + 8\log n \in o(n)$ (proved)

(c) $n^5 + 4n^2 + 15 \geq n^5 + 4n^2$

$\geq n^5$ for $n \geq 1 (= n_0)$

$\geq n^3$ [if $n \geq 1$]

$\therefore n^5 + 4n^2 + 15 \in \Omega(n^3)$

(proved)

(4) Sorted order :

$\log_2 n \leq 2^{\sqrt{\log_2 n}} \leq n^{1/3} \leq n^5 \leq 10^n \leq n^n$.