CS 5644: Analysis of Algorithms

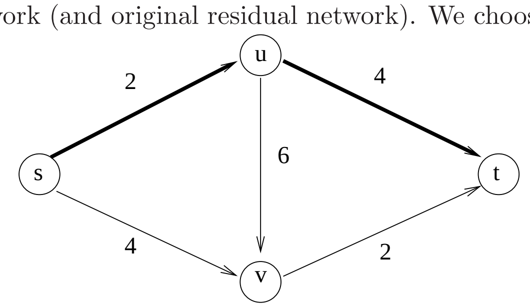
Homework 11 Solution

1. For part (a), let the first object have weight $k/2$, the second object has weight k, and the third object has weight $k/2$. The algorithm will put each of the objects on their own trucks, but an optimal solution would combine the first and third objects into a single truck.

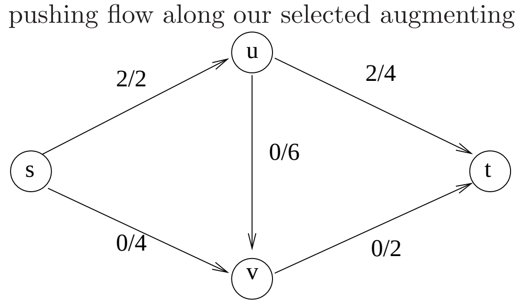
   For part (b), let W denote the sum of the weights of all of the objects. We need a good lower bound on the number of trucks used in an optimal solution OPT. We can get this lower bound by realizing that an optimal solution must use at least $\frac{W}{k}$ trucks (packing exactly k into each truck). We will show that we use at most $\frac{2W}{k}$ trucks. Consider the first two trucks we will pack. It must be that the sum of the weight put on these two trucks is greater than k (if it was at most k then we would have put it all onto one truck). Partition the objects into groups where the first two trucks are a group, the next two are the second group, etc. Each of these groups has weight at least k, and therefore there are at most W/k such groups. Our algorithm will use at most 2 trucks to pack each of these sets, and therefore the algorithm will use at most $\frac{2W}{k}$ trucks. Since OPT is at least $\frac{W}{k}$, it follows that the algorithm is a 2-approximation algorithm.

2. We compute the flow using the augmenting path method of Ford-Fulkerson, and the minimum cut is all of the nodes reachable from s in the final residual network.

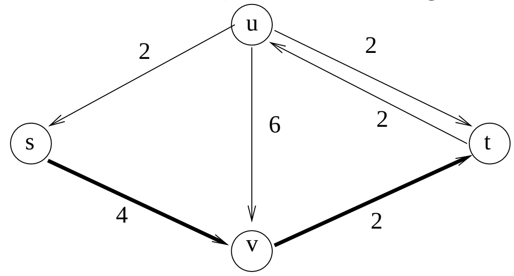
Original flow network (and original residual network). We choose the bolded path.



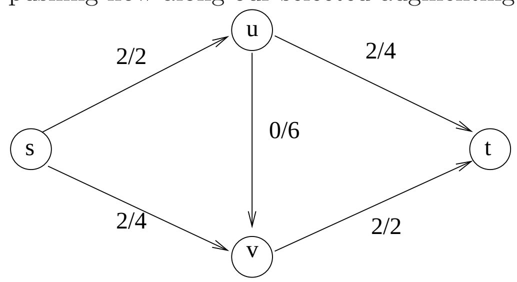
After pushing flow along our selected augmenting path:



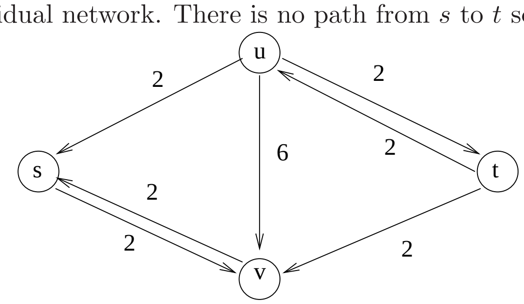
Our new residual network, and our next augmenting path:



After pushing flow along our selected augmenting path:



The final residual network. There is no path from s to t so we are done.



The min cut is $S = \{s, v\}$ and $T = \{u, t\}$.

3. We will view each unit of flow as an assignment of a client to a base station. We can view the underlying graph as a bipartite graph where the clients are one set of vertices and the base stations are the other set of vertices. We add a source vertex s and a sink vertex t. We add an edge of capacity 1 from s to each of the client vertices (this captures the constraint that each client can be assigned to at most one base station). We add an edge of capacity 1 from a client to each of the base stations whose distance is at most r. We then add edges from each base station to t with capacity k (this captures the constraint that each base station can have at most k clients assigned to it). This completes the construction.

   We will now show that each client can be connected to a base station if and only if the maximum flow of this flow network is n (the number of clients). First suppose that there is a flow of value n. Then there must be one unit of flow to each of the clients from the source, and therefore one unit of flow leaving the client and going through a particular base station. Since the capacity on the edge leaving a base station is at most k, then we know that there is at most k units of incoming flow. Therefore we can feasibly assign each client to the base station as indicated by the flow to get a valid assignment of the clients to the base stations.

   Now suppose that there is a valid way of assigning each of the clients to base stations. We will show how to construct a flow of value n. For each "client/base station" pair, set the flow on the corresponding edge in the flow network to be 1. Let b_i denote the number of clients assigned to base station i. We know that $b_i \leq k$, so we can set the flow on the edge from i to t to be b_i. We set the flow on all of the edges from s to a client to be 1. Clearly this is a feasible flow, and its value is n.