Entropy Makes the Collatz Sequence Go Down

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Abstract

We look at the Collatz Sequence from an information theory perspective to lay out its underlying computational mechanics. The mechanisms are similar to those used in pseudo random number generators and one way hashes. An individual run is divided into three phases. In the first phase the influence of the seed runs its course after information contained in the initial seed is lost. Values are randomized in the second phase. They follow the statistical model where the average gain is just over 0.866; causing them to decline. The third phase begins once a value goes below the seed; providing that the series is not circular. At this point we know the run will terminate at one. An equivalent restatement of the Collatz sequence steps through alternating chains of even and odd values. This variation constitutes a pseudo random number generator. The operations used to scramble values are unbiased which results in an even distibution of ones and zeros. The entropy of this mechanism is high so that in the second phase values are fairly randomized.

1 Introduction

The computational mechanics of the Collatz sequence are analyzed to determine the odds of taking an even or odd step. With the following definition of the sequence we show the the average odds of taking either step are even. In this case statistically the sequence will on average decline and eventually terminate. This variation is often referred to as the Syracuse sequence.

$$\text{N is Even:} \quad N' = \frac{N}{2}$$
$$\text{N is Odd:} \quad N' = \frac{3N+1}{2}$$

The gain of a each transition is its Output divided by the Input (N' / N). As N gets larger in the odd transition the "+ 1" term quickly becomes insignificant. To compute the gain for the odd transition in the limit we can safely drop the "+1"term.

	Output / Input	Gain
N is Even:	$\frac{N}{2N}$	0.5
N is Odd:	$\frac{3N+1}{2N}$	1.5

The total gain of a series of transitions is the product of the gains for each transition. Based on this the average gain of a sequence depends on the probability of taking either an odd or even path.

$$G_s = 1.5^{p_{\text{odd}}} \cdot 0.5^{p_{\text{even}}}$$

The choice of which path is taken is determined by the low order bit of the input value. If the sequence should produces uniformly randomized values then the chances of taking either transitions is 50:50. This implies the low bit would need to be uniformally random over the sequence. The average gain of a uniformly randomized sequence is then:

$$\text{Average transaction gain} = \langle\text{transaction gain}\rangle^{p(\langle\text{transaction}\rangle)}$$
$$\text{Average odd gain} = 1.5^{0.5} = 1.22474$$
$$\text{Average even gain} = 0.5^{0.5} = 0.70711$$
$$\text{Average sequence gain: } G_a = 1.22474 \cdot 0.70711 = 0.86603+$$

The statistical average gain in each step is less than one so on average the sequence declines. However, the gain in a single run for a given Seed can vary significantly. There could potentially still be a series where the the total gain indefinitely exceeds one and never terminates.

$$\begin{aligned}
\text{Breakeven Gain} &= 1.5^p \cdot 0.5^{1-p} = 1 \\
\ln\left(1.5^p \cdot 0.5^{1-p}\right) &= \ln(1) = 0 \\
&= \ln(1.5^p) + \ln(0.5^{1-p}) = 0 \\
&= p \cdot \ln(1.5) + (1-p) \cdot \ln(0.5) = 0 \\
p \cdot 0.40546 &= -(1-p) \cdot (-0.69315) \\
\frac{0.40546}{-0.69315} &= \frac{-(1-p)}{p} \\
-0.58497 &= -\frac{1-p}{p} \\
-0.58497 - 1 &= -\frac{1}{p} \\
p &= \frac{1}{1.58497} \approx 0.63093
\end{aligned}$$

$$\text{Gain} = 1.5^{0.63} \cdot 0.5^{0.37} \approx 0.99898 \quad \text{Under breaking even}$$

$$\text{Gain} = 1.5^{0.64} \cdot 0.5^{0.36} \approx 1.01001 \quad \text{Over breaking even}$$

For any series of values to continually increase and never terminate it would have to sustain an average gain over one. To break even odd transitions would need to occur about 64% of the time. They would need to be applied over 1.7 times more than evens; which is substantially skewed. It remains to be shown that the sequence does not intrinsically favor odd transitions.

1.1 Even and Odd Chains

Consecutive iterations of the same kind of transition in a run form a chain. Even chains start with an even seed value that in binary have one or more trailing zeroes. After applying the transitions in an even chain the result simply has the low order zeros removed.

Odd chains consume an odd input and have multiple odd intermediate values. Eventually an Odd chain transitions to an even number. The number of consecutive low order one bits determines the chain length. For example, an input of 19 is a binary 10011 so the subsequent chain has two Odd transitions: $19 \to 29 \to 44$

Let k be the number of low order one bits and j is the input value with the low one bits removed plus 1.

The input to an odd chain has the form: $j \cdot 2^k - 1$

The output of the chain simplifies to: $j \cdot 3^k - 1$

$$\begin{aligned}
N_1 &= \frac{3N+1}{2} && \text{First transition} \\
&= \frac{3(j \cdot 2^k - 1) + 1}{2} && \text{Substitute } N = j \cdot 2^k - 1 \\
&= \frac{3j \cdot 2^k - 3 + 1}{2} \\
&= \frac{3j \cdot 2^k - 2}{2} \\
&= 3^1 \cdot j \cdot 2^{k-1} - 1
\end{aligned}$$

$$\begin{aligned}
N_{i+1} &= \frac{3\left(3^i \cdot j \cdot 2^{k-i} - 1\right) + 1}{2} && \text{Subsequent transitions} \\
&= \frac{9 \cdot j \cdot 2^{k-i} - 3 + 1}{2} \\
&= \frac{9 \cdot j \cdot 2^{k-i} - 2}{2} \\
&= 3^{i+1} \cdot j \cdot 2^{k-i-1} - 1 \\
N_k &= 3^k \cdot j - 1 && \text{Odd chain output}
\end{aligned}$$

Each run of the Collatz sequence will have segments with alternating even and odd chains. For reference, here are the first few chains for the series beginning with a seed of 27.

Syracuse Sequence	**Even**	**Odd**	**j**	**k**	**Base**
$27 \to 41 \to 124 \to 62$		$27 \to 62$	7	2	110_
$\to 31$	$\to 31$		31	1	11111
$\to 47 \to 71 \to 107 \to 161 \to 242$		$\to 242$	1	5	0_1111
$\to 121$	$\to 121$		121	1	111100
$\to 364 \to 182$		$\to 182$	61	1	1011011
$\to 91$	$\to 91$		91	1	10110_
$\to 137 \to 206$		$\to 206$	23	2	1100111
$\to 103$	$\to 103$		103	1	1100_11
$\to 155 \to 233 \to 350$		$\to 350$	13	3	10101111
$\to 175$	$\to 175$		175	1	1010_11
$\to 263 \to 395 \to 593 \to 890$		$\to 890$	11	4	110111101

1.2 Combining Even And Odd Chains

Each run alternates between even and odd chains. We represent this algebraically by merging both into a single step. This gives us a series defined as a single transition. Since all intermediate values using this combined definition are even, an initial odd value needs to first transition one step using the "3n + 1" rule to reach the first even number.

Every input has the binary form: [j] [ones] [zeros]

$$N = ([j+1] \cdot 2^{k_o} - 1) \cdot 2^{k_z} \to [j+1] \cdot 3^{k_o} - 1$$

where:

- k_z - number of trailing zeros in N
- k_o - number of the next higher set of ones in N
- j - N shifted right by $k_z + k_o$ bits: $\frac{N}{2^{k_z+k_o}}$

Using the previous example, initially we transition 27 to 82. From there the next few steps are:

	j	**ko**	**kz**	**binary**
$82 \to 41 \to 124 \to 62$	20	1	2	10100_10
$\to 31 \to 484 \to 242$	0	5	1	0_111110
$\to 121 \to 364 \to 182$	60	1	1	111100_10
$\to 91 \to 274 \to 206$	22	2	1	10110_110
$\to 103 \to 310 \to 466$	12	3	1	1100_1110
$\to 233 \to 700 \to 350$	116	1	1	1110100_10
$\to 593 \to 1780 \to 890$	10	4	1	1010_11110

2 Sequence Entropy

Statistical averages only hold when the odds are fair. In this section, we show why the dice are not loaded. Shannon entropy is a measure of information denoting the level of uncertainty about the possible outcomes of a random variable [1].

$$H = -p0 \cdot \log_2(p0) - p1 \cdot \log_2(p1)$$

where:

- $p0$ is the probability a bit is zero.
- $p1$ is the probability a bit is one $(1 - p0)$.

A set of coin tosses has $p0 = p1 = 0.5$ so its entropy is 1; totally random. When looking at the entropy of bits in a number then $p0$ is the percentage of zero bits. For the binary number, 1010_1111, $p0$ is 0.25 ($H = 0.811$). Strings of all ones or zeros have no entropy ($H = 0$). For a binary number, we are measuring the bits in a number horizontally.

Bits in a series of numbers have two dimensions - horizontal bits in each individual value and vertical bits over the duration of the series. We can also measure entropy vertically over a number series. That means we can observe a select bit position in each value as the series progresses.

For Collatz, the low-order bit is of interest because it determines if a number is odd or even. In turn, that determines which transition to take. When the entropy of the low-order bit is high, then on average there are nearly as many even transitions as odds.

Each kind of chain takes a value where the low-order bits are a string of zeros or ones and either removes or replaces them. Even inputs remove low-order zeros. The expression for odd chain inputs has a 2^k term that transitions to a 3^k term.

Since strings of zeros have no entropy and the j term has positive entropy, entropy increases each time an even transition is applied. In odd transitions, entropy is also increased by removing the repeated ones and again by scrambling the remaining bits. The upper bits, j, are scrambled by multiplying j by a power of 3. As a run progresses, this increase in entropy randomizes the values. The number of odd and even transitions balance out driving the sequence downward and eventually forcing it to terminate.

2.1 Losing Information

A Seed can be contrived to produce a run of any desired length. The longer the run, the larger the seed has to be in order to contain enough information to influence the desired outcome. Initially, as a run progresses, the information contained in the Seed is lost. When there are two possible ways to reach a value in a run we lose the information about which path was taken to reach it [2].

Odd numbers always transition 3n + 1 to even numbers, so an odd value can only be reached from one even value. However, certain even numbers can be reached from either an odd or even transition. For example, an output of 16 can be reached from either 32 or 5:

$$32 \to 32/2 = 16$$
$$5 \to 3 \cdot 5 + 1 = 16$$

Whenever transitioning to an even value such that (even%3 = 1), then the previous value could have been either:

$$2 \cdot \text{even or } (\text{even} - 1)/3$$

For Collatz, a bit of information contained in the seed is lost each time one of these select even numbers is reached. After all bits in the seed are scrubbed, this initial phase is complete. Any attempt to contrive a Seed to skew results can only directly affect values during this phase.

One-way hash functions rely on this concept of lost information [3]. Secure hashes have many ways to reach each hashed value. This is how passwords are encoded and used for authentication. Using this metaphor, you can think of the Seed as a password and the Collatz sequence as a trivial one-way hash schema used to mask it.

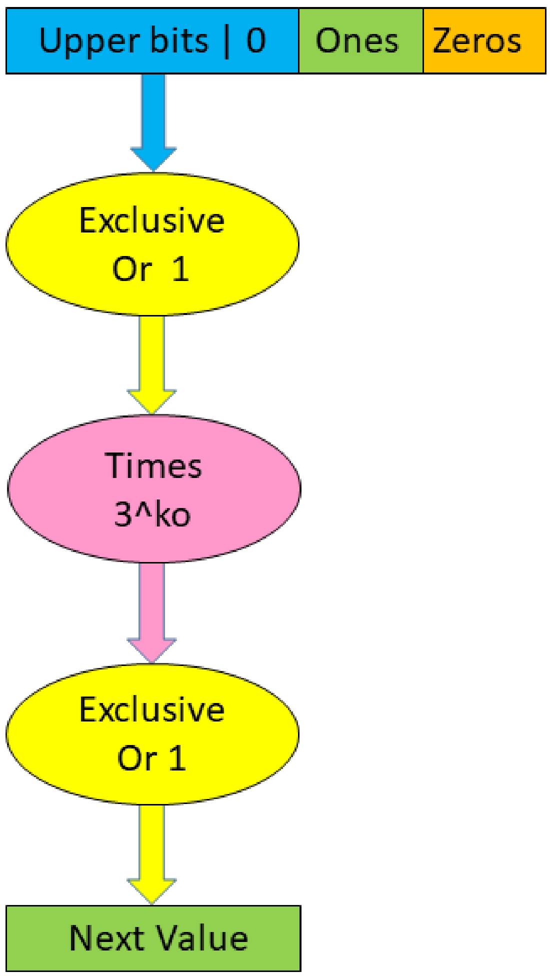
2.2 RandomizationPhase

This next phase is key as this is where the sequence runs below the Seed. The sequence is rewritten as a pseudo random number (PRNG) generator. Hastad et al. (1999) [4] show that any one-way hash can be used to create a PRNG. Uniformly randomized values eventually trend towards their average. In turn, this drives transitions towards their average gain. In the introduction, we showed that Collatz has an average gain of 0.86603, eventually driving the series below the Seed value.

We'll be using the combined series from section 1.2 for each randomization step. The value, j, is always even so the $[j + 1]$ term will simply set the low order bit to one as there is no carry. Also, the product will have an odd result so that decrementing by 1 will likewise just clear the low order bit.

$$\text{Input: } ([j + 1] \cdot 2^{k_o} - 1) \cdot 2^{k_z}$$

$$\text{Result: } [j + 1] \cdot 3^{k_o} - 1 = [j \oplus 1] \cdot 3^{k_o} \oplus 1$$



Count trailing zeros then ones.
Select the Upper bits.

Flip the low order zero bit to one.

Scale by 3 to the number of one bits.
This fairly scrambles the value.

Flip the low order one bit to zero.

Done if the result is less than the Seed.

In this next example, the top line has steps for a randomization phase that begins with 647. Calculations for each combined transition $(1942, 2186, 1640, 308, 116)$ are shown in binary:

	1942	2186	1640	116
Input	11110010_110	1000100010_10	1100110_1000	100110_100
Shift Right	11110010	1000100010	1100110	100110
Xor 1	11110011	1000100011	1100111	100111
Times 3^{ko}	100010001011	11001101001	100110101	1110101
Xor 1	1000100010_10	1100110_1000	100110_100	1110100

A pseudo-random number generator repeatedly applies a function to produce a series of values. In order to produce uniformly random numbers, operators cannot be biased towards producing either more ones or zeros. In a uniform sequence, the entropy will be one. If it is not uniform, the bias will show up in the operators.

Select

If you remove some low-order bits of a random number, the remaining part will still be random. Using the upper bits from the input still gives random values. However, the way the value is split, the selected upper value will be even. The low-order bit is zero, and only the other bits are a randomized portion.

Logical Exclusive Or 1

The first Exclusive Or sets the low bit of the selected region. This is balanced out by clearing it in the final step with another Exclusive.

Product

The product of a random variable by a constant is also random, but with a larger gap between them. Multiplying random numbers from 1 to 10 by 3 yields random numbers from 3 to 30. They simply have a gap of 3 between them instead of 1.

The product used to scramble values is equivalent to repeated sums of the input. The following table shows all combinations for the three inputs (A, B, Carry In) and the two outputs (Sum, Carry Out). It also shows changes (Exclusive Or) between the sum and inputs A and B:

A	**B**	**Carry In**	**Sum**	**Carry Out**	**A $\oplus$ Sum**	**B $\oplus$ Sum**
0	0	0	0	0	0	0
0	0	1	1	0	1	1
0	1	0	1	0	1	0
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	1
1	1	1	1	1	0	0

Input bits A and B are vertically aligned and are altered by addition. Carries are applied horizontally and propagate to higher order bits. This way, bits in both directions become scrambled.

Note that all the columns in the table are different. This shows how bits are scrambled to produce randomized results. Also note that all columns contain 4 zeros and 4 ones. This balance produces results that are unbiased towards either zero or one bits. The end result is a series of uniformly distributed pseudo-random numbers.

Outputs in any individual series depend on the values k_z and k_o. The more random they are, the more random the series. The k values measure the width of a horizontal subset of bits in each value. Pseudo-random number generators that conflate operations on horizontal and vertical sets of bits rely on the independence of these orthogonal values.

The repeated zeros and upwardly ones in the lowest bits that might have low entropy are continuously removed and replaced with scrambled bits. This creates a self-regulating system that continuously randomizes the lower bits. Since those bits control the selection operation in the next round.

When runs have uniformly random values, then revisiting the Syracuse sequence, the average number of even and odd transitions will balance out. In turn, this causes the run to decrease since the average gain is less than one. If the sequence was not uniformly random, then we would see bias amongst the arithmetic operations used in each round.

Examples where seeds produce long runs will have highly skewed values in the first phase, but that cannot be sustained. As values become more randomized and the series progresses, they will trend towards average results. With coin tossing, even if you get lucky and call the results of several coin tosses, your luck will run out in the long run.

2.3 Reduction To One

The previous randomization phase leaves us with a value of N that is below the seed. It is well understood that once this happens, we know the series will eventually terminate at one. Firstly, we know all values below some arbitrary small number M (say 10) transition to one.

Next, starting with the next higher Seed, $M + 1$, we transition until it reaches M or less. Since we already know Seeds of M or less will reach one, by induction, once a series goes below its Seed we know it will reach one. This is why even Seeds are uninteresting as they immediately decline.

When measuring the length of a run, including this phase is not useful and can distort any result. When winding down as numbers get smaller, they can become more irregular. Instead of defining the run length as the number of steps to one, use only the number of odd steps until the series goes below the Seed. It is usually more practical to count only odd steps because it corresponds to the number of terms in the algebraic expansion of a run.

2.4 Observed Entropy

The first few values will have lower entropy until enough bits are included to average out. In the Introduction, we've shown that to sustain an infinite run there needs to be 64% or more ones. This gives an entropy of:

$$H = -0.36 \cdot \log_2(0.36) - 0.64 \cdot \log_2(0.64) = 0.94268$$

Individual runs will typically have some jitter since we are performing discrete computations. There will be higher entropy at the end of very long runs, which are rare. In the first phase, long runs will be skewed towards more odd steps to make the values grow larger up front. Short runs where evens dominate won't even make it to the randomization phase.

To compute the entropy, the low-order zero bit is discarded as it is fixed. Also, since the values have a variable width, the uppermost bit where one is also discarded. This differs from practical PRNGs where the values have a fixed width.

To see the randomization in action, this trace lists entropy in the first two phases. Entropy is computed using the accumulated number of ones and zeros in the run. The counts of ones and zeros are reset at the start of the Randomization phase so that those computations are completely separate. Even in the Information Loss phase, entropy is well above the 0.94268 bound right out of the gate. The computed length of the Information Loss phase is quite conservative.

Seed $= 4\_50449\_75045\_09599 = \#10\_00d1\_0da5\_de9f$

Step	**Entropy**	**Ones**	**Zeros**	**Notes**
1	0.99750	24	27	Information Loss phase 1
2	0.99993	52	51	
3	0.99988	77	79	
4	0.99998	105	104	
5	0.99934	136	128	
6	0.99965	163	156	
7	0.99950	194	184	
8	0.99986	222	216	
9	0.99999	250	248	
10	0.99994	281	276	
11	0.99973	314	302	
12	0.99984	342	332	
13	0.99993	369	362	
14	0.99966	402	385	
15	0.99983	428	415	
16	0.99989	456	445	
17	0.99982	487	472	
18	0.99975	518	499	
19	0.99988	545	531	
20	0.99920	31	29	Randomization phase 2
30	1.00000	321	321	
40	0.99978	606	585	
50	0.99966	899	861	
60	0.99983	1192	1156	
70	0.99969	1489	1429	
80	0.99996	1770	1744	
90	0.99963	2105	2012	
100	0.99967	2410	2309	
110	0.99876	2772	2551	
120	0.99828	3105	2816	
130	0.99890	3387	3133	

If the hash function had a bias, it would show up by running it over many consecutive numbers. Here the hash was run over a million consecutive even numbers. This next chart shows the cumulative entropy of the resulting values, which is very near one as expected.

Iteration	Entropy	Ones	Zeros
50,000	0.99350	475,206	574,794
100,000	0.99721	984,678	1,115,322
150,000	0.99886	1,512,405	1,637,595
200,000	0.99966	2,054,339	2,145,661
250,000	0.99984	2,585,879	2,664,121
300,000	0.99970	3,086,254	3,213,746
350,000	0.99987	3,625,941	3,724,059
400,000	0.99970	4,113,665	4,286,335
450,000	0.99976	4,638,681	4,811,319
500,000	0.99983	5,168,585	5,331,415
550,000	0.99991	5,711,156	5,838,844
600,000	0.99996	6,255,089	6,344,911
650,000	1.00000	6,816,718	6,833,282
700,000	0.99994	7,415,041	7,284,959
750,000	1.00000	7,882,886	7,867,114
800,000	1.00000	8,392,173	8,407,827
850,000	1.00000	8,925,576	8,924,424
900,000	1.00000	9,428,185	9,471,815
950,000	0.99999	9,944,542	10,005,458
1,000,000	0.99999	10,466,775	10,533,225
1,048,576	1.00000	11,012,891	11,007,205

This next chart shows a contorted sequence where some bits are artificially forced to one. This shows how skewed the operators have to get in order for entropy to drop below 0.94268, where it would increase indefinitely. 11 out of 20 bits are needed to be forced to one to sufficiently skew the result.

Value Of Forced Bits	Final Entropy	Ones	Zeros
0	1.00000	11,012,891	11,007,205
10,000	0.99995	11,102,306	10,917,790
11,000	0.99971	11,229,280	10,790,816
11,100	0.99931	11,350,975	10,669,121
11,500	0.99916	11,385,687	10,634,409
15,500	0.99892	11,435,329	10,584,767
55,500	0.99703	11,716,482	10,303,614
155,500	0.99727	11,687,638	10,332,458
175,500	0.98541	12,573,465	9,446,631
177,500	0.96792	13,323,124	8,696,972
177,700	0.95400	13,775,529	8,244,567
177,740	0.94266	14,093,550	7,926,546

3 Conclusion

The Collatz sequence incorporates principle mechanisms commonly used to create pseudo random number generators.

- To overcome a contrived Seed, one way hashing smothes out any regularities.
- Repeated low order one and zero bits are erased at each step.
- A product using independent values randomizes values.

Any individual run is partitioned into three phases. In the initial phase the Seed value can influence the outcome to produce arbitrarily long runs. After that the series generates randomized values until it goes below the Seed value. From there it is guaranteed to reduce to one unless the series is circular. A uniformly randomized series eventually moves towards a statistically average gain. For a Collatz series to sustain an average gain above one would require over 1.7 times more odd transitions than even. This is well above parity. Instead, randomization forces the series to average out and decrease until it inevitably goes below the seed. Once it does that we know it will terminate. The random behavior of the Collatz sequence makes it impossible to prove algebraically. Conway[5] showed that a generalization of the 3N + 1 problem is undecidable. Trying to make sense of the values in the series is akin to analysing values produced by a random number generator. The irony is that this randomness is the force that leads to convergence.

References

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Appendix

Here are the entropy values for the randomization phase of some long runs. The Sample Size is the number of values produced by the algorithm for randomization. All of the entropy values are well above 0.94268; the entropy required to produce an infinitely long run.

Entropy is computed from values in each run. Since at the end of each step the values are all even, the low order bit is discarded. Values also have variable widths; unlike values in a practical PRNG. To account for this the upper one bit is also discarded. The entropy is then computed using the total number of ones and zeros in each run. To illustrate this the chart shows the entropy after five steps into the randomization phase.

Run Length	**Seed**	**Entropy**	**Sample Size**	**Run Run**	**Seed**	**Entropy**	**Sample Size**
200	371_871_359	0.99979	68	292	331_224_689_767	0.99971	117
201	247_914_239	0.99979	69	293	188_890_883_743	0.99946	92
202	165_276_159	0.99980	69	294	215_384_833_215	0.99991	104
203	2_173_615_775	0.99998	61	295	167_903_007_771	0.99952	92
204	293_824_283	0.99974	69	296	144_460_775_535	0.99995	104
205	195_882_855	0.99980	70	297	125_291_645_607	0.99938	101
206	620_752_511	0.99903	62	298	196_281_297_639	0.99975	118
207	348_236_187	0.99969	70	299	1_063_641_582_407	0.99992	102
208	413_835_007	0.99871	62	300	709_094_388_271	0.99993	102
209	1_651_171_495	0.99810	68	301	473_644_547_375	0.99953	103
210	127_456_255	0.99983	95	302	380_103_773_863	0.99979	103
211	245_235_559	0.99882	65	303	315_763_031_583	0.99956	104
212	5_425_672_039	1.00000	71	304	284_396_952_295	0.99929	105
213	217_987_163	0.99892	66	305	33_980_539_439	0.99951	103
214	290_649_551	0.99894	66	306	150_164_453_871	0.99984	105
215	193_766_367	0.99899	67	307	22_653_692_959	0.99946	105
216	145_324_775	0.99894	66	308	224_708_703_047	0.99948	109
217	96_883_183	0.99899	67	309	20_136_615_963	0.99955	106
218	1_583_507_967	0.99875	79	310	199_741_069_375	0.99937	110
219	661_398_811	0.99972	73	311	1_150_284_049_727	0.99959	104
220	1_968_165_887	0.99995	82	312	23_865_618_919	0.99942	107
221	2_079_441_767	0.99964	75	313	620_398_672_495	0.99999	108
222	326_610_023	0.99905	88	314	21_213_883_483	0.99963	108
223	984_082_943	0.99993	83	315	149_805_802_031	0.99934	109
224	1_232_261_787	0.99961	75	316	766_856_033_151	0.99950	105
225	656_055_295	0.99993	86	317	1_131_779_353_631	0.99986	112
226	2_848_461_311	0.99983	68	318	99_870_534_687	0.99937	110
227	409_344_047	0.99999	90	319	478_337_265_823	0.99977	114
228	272_896_031	0.99999	92	320	566_918_240_975	0.99973	115
229	181_930_687	1.00000	93	321	425_188_680_731	0.99971	115
230	1_304_621_055	0.99981	83	322	140_284_537_063	0.99932	112
231	1_324_921_887	0.99734	73	323	188_972_746_991	0.99974	121
232	95_592_191	0.99992	95	324	251_963_662_655	0.99971	117
233	1_104_180_463	1.00000	82	325	167_975_775_103	0.99973	123
234	5_328_487_839	0.99997	80	326	3_654_218_733_311	0.99990	113
235	13_551_207_911	0.99998	82	327	125_981_831_327	0.99971	122
236	9_034_138_607	0.99997	82	328	566_619_806_719	0.99980	122
237	63_728_127	0.99984	96	329	2_253_835_349_759	0.99974	105
238	15_218_280_607	1.00000	75	330	1_325_730_144_347	0.99997	115
239	17_108_656_891	0.99890	84	331	1_649_531_356_143	0.99990	108

Run Length	Seed	Entropy	Sample Size	Run Run	Seed	Entropy	Sample Size
240	3_246_339_311	0.99929	80	332	1_176_549_020_911	0.99970	122
241	2_164_226_207	0.99921	80	333	1_287_402_586_111	0.99945	108
242	1_442_817_471	0.99917	81	334	26_130_934_783	0.99950	114
243	20_445_954_119	0.99967	73	335	1_303_333_417_199	0.99990	109
244	13_630_636_079	0.99952	73	336	83_987_887_551	0.99973	123
245	9_087_090_719	0.99952	73	337	881_989_193_575	0.99998	117
246	6_058_060_479	0.99950	74	338	20_646_664_519	0.99957	115
247	18_019_682_047	0.99994	90	339	1_267_630_141_951	0.99994	116
248	17_825_084_863	0.99996	86	340	18_352_590_683	0.99958	116
249	217_740_015	0.99914	90	341	24_470_120_911	0.99963	116
250	1_801_487_687	0.99891	83	342	883_820_096_231	0.99998	115
251	1_200_991_791	0.99876	84	343	21_751_218_587	0.99983	118
252	16_670_963_135	0.99965	75	344	12_235_060_455	0.99973	119
253	6_250_517_663	0.99957	93	345	5_086_317_509_375	0.99979	127
254	32_060_507_419	0.99952	86	346	898_696_369_947	0.99921	121
255	14_884_335_615	0.99982	92	347	18_570_171_467_519	0.99934	122
256	87_147_171_839	0.99957	89	348	5_157_142_856_607	0.99805	118
257	6_216_083_103	0.99957	81	349	4_018_818_772_839	0.99979	129
258	8_781_412_679	0.99977	79	350	5_509_607_710_143	0.99996	117
259	24_083_989_231	0.99918	85	351	10_681_465_356_287	0.99811	119
260	23_962_604_007	1.00000	92	352	7_120_976_904_191	0.99807	120
261	21_407_990_427	0.99914	86	353	4_747_317_936_127	0.99800	125
262	5_854_275_119	0.99974	80	354	7_244_052_517_375	0.99801	118
263	3_902_850_079	0.99960	82	355	19_754_675_554_139	0.99926	120
264	349_414_071_423	1.00000	89	356	13_169_783_702_759	0.99934	120
265	25_244_554_015	1.00000	94	357	17_559_711_603_679	0.99914	120
266	81_774_557_807	0.99981	90	358	12_380_114_311_679	0.99926	122
267	60_142_063_643	0.99941	88	359	8_253_409_541_119	0.99938	122
268	4_111_644_527	0.99947	84	360	18_026_976_767_615	0.99973	129
269	5_482_192_703	0.99940	83	361	5_521_395_748_159	0.99990	124
270	2_741_096_351	0.99950	86	362	8_011_989_674_495	0.99974	130
271	23_759_827_611	0.99944	89	363	4_141_046_811_119	0.99994	125
272	139_869_168_255	0.99948	94	364	7_121_768_599_551	0.99972	132
273	1_827_397_567	0.99953	86	365	31_508_135_471_707	0.99968	130
274	28_159_795_687	0.99936	89	366	2_760_697_874_079	0.99996	126
275	59_834_174_399	0.99830	92	367	33_508_530_061_951	0.99998	119
276	4_704_765_167	0.99948	96	368	18_307_067_699_951	0.99611	119
277	6_273_020_223	0.99944	96	369	12_204_711_799_967	0.99612	120
278	39_889_449_599	0.99845	94	370	2_813_538_212_167	0.99983	137
279	53_185_932_799	0.99837	94	371	23_838_942_284_287	0.99939	129