Counting

Chapter 6

Chapter Summary

- The Basics of Counting（计数）
- The Pigeonhole Principle（鸽巢原理）
- Permutations and Combinations（排列与组合）
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations
- Generating Permutations and Combinations

The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

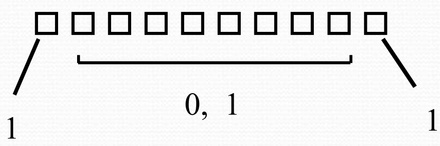
Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

〖Example 〗How many bit strings of length 10 begin and end with a 1?



Solution:

The product rule shows that there are a total of 2^8 different bit strings of length 10 begin and end with a 1.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ different possible license plates.

$$\underbrace{\_\ \_\ \_}_{\substack{\text{26 choices}\\ \text{for each}\\ \text{letter}}}\ \underbrace{\_\ \_\ \_}_{\substack{\text{10 choices}\\ \text{for each}\\ \text{digit}}}$$

Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots\ n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements? $m \leq n$

Solution: Suppose the elements in the domain are $a_1, a_2,\ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)\ (n-2)\cdots(n-m+1)$ such functions.

Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was *NYX-NNX-XXXX*.
- In the new plan, the format is *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX*.
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX*.
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX*.
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX*.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. (*In Section* 5.1, *mathematical induction was used to prove this same result*.)

Solution: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the ith element is in the subset, the bit string has a 1 in the ith position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , …, and an element in A_m.
- By the product rule, it follows that:
  $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$.

DNA and Genomes

- A *gene* is a segment of a DNA molecule that encodes a particular protein and the entirety of genetic information of an organism is called its *genome*.
- DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).
- The DNA of bacteria has between 10^5 and 10^7 links (one of the four bases). Mammals have between 10^8 and 10^{10} links. So, by the product rule there are at least 4^{105} different sequences of bases in the DNA of bacteria and 4^{108} different sequences of bases in the DNA of mammals.
- The human genome includes approximately 23,000 genes, each with 1,000 or more links.
- Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.

〖Example〗 Counting the number of elements in A,
A = {length 10 bit strings with 0-streak of length exactly 5}.

Solution:

Since the set A can be break up into the following case.

A_1 = {000001****} (* is either 0 or 1)

A_2 = {1000001***}

A_3 = {*1000001**}

A_4 = {**1000001*}

A_5 = {***1000001}

A_6 = {****100000}

Apply the sum rule:

$$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6|$$

14

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.
- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \emptyset \text{ for all } i, j.$$

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the product rule.

$$26 + 26 \cdot 10 = 286$$

Counting Passwords

- **Combining the sum and product rule allows us to solve more complex problems.**

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- **By the sum rule $P = P_6 + P_7 + P_8$.**
- **To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:**

$P_6 = 36^6 - 26^6$ =2,176,782,336 − 308,915,776 =1,867,866,560.

$P_7 = 36^7 - 26^7$ =

78,364,164,096 − 8,031,810,176 = 70,332,353,920.

$P_8 = 36^8 - 26^8$ =

2,821,109,907,456 − 208,827,064,576 =2,612,282,842,880.

Consequently, $P = P_6 + P_7 + P_8$ = 2,684,483,063,360.

〖Example 〗Choose three different numbers from the integers between 1 to 300 such that the sum of the three integers can be divisible by 3. How many the ways are there?

Solution:

$A = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 1\}$

$B = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 2\}$

$C = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 0\}$

$|A| = |B| = |C| = 100$

(1) All of the three numbers are chosen form the set A C_{100}^{3}

(2) All of the three numbers are chosen form the set B C_{100}^{3}

(3) All of the three numbers are chosen form the set C C_{100}^{3}

(4) Chose one number form the set A, B, C $C_{100}^{1} \times C_{100}^{1} \times C_{100}^{1}$

18

Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- **Class A Addresses**: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses**: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses**: used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, x_A: $2^7 - 1 = 127$ netids. $2^{24} - 2 = 16{,}777{,}214$ hostids.
  $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$.
- To find, x_B: $2^{14} = 16{,}384$ netids. $2^{16} - 2 = 16{,}534$ hostids.
  $x_B = 16{,}384 \cdot 16{,}\ 534 = 1{,}073{,}709{,}056$.
- To find, x_C: $2^{21} = 2{,}097{,}152$ netids. $2^8 - 2 = 254$ hostids.
  $x_C = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608$.
- Hence, the total number of available IPv4 addresses is

$$x = x_A + x_B + x_C$$
$$= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608$$
$$= 3{,}\ 737{,}091{,}842.$$

Not Enough Today !!
The newer IPv6 protocol solves the problem of too few addresses. uses 128 bits

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

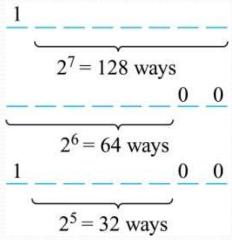
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that start with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Principles: Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

- Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

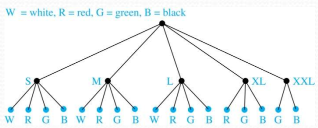
Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Tree Diagrams

- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of stores that the campus book store needs to stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.



- The store must stock 17 T-shirts.

Homework

第7版 Sec. 6.1 41, 56 ,68

第8版 Sec. 6.1 41, 58 ,70

The Pigeonhole Principle

Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

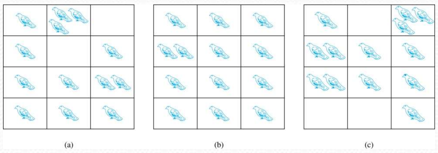
- Suppose a flock of pigeons fly into a set of pigeonholes to roost
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it

- Main numbers
- (1) the number of pigeon
- (2)the number of pigeonhole

It is also called
Dirichlet Drawer Principle

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. ◄

The Pigeonhole Principle

Corollary 1: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f.
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. ◄

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example (*optional*): Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer. Consider the $n + 1$ integers 1, 11, 111, …., 11…1 (where the last has $n + 1$ 1s). There are n possible remainders when an integer is divided by n. By the pigeonhole principle, when each of the $n + 1$ integers is divided by n, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

The direct applications of the pigeonhole principle

(1) Among any group of 11 integers, there are two integers a and b such that $10 \mid a-b$.

Pigeons : 11 integers

Pigeonholes: the possible remainders when an integer is divided by 10

(2) In a party of 2 or more people, there are 2 people with the same number of friends in the party. (Assuming you can't be your own friend and that friendship is mutual.)

Pigeons : the n people (with $n > 1$).

Pigeonholes: the possible number of friends, i.e. the set $\{0, 1, 2, 3, \ldots, n-1\}$

(注意：0和n-1的情况不会同时出现）

32

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < N/k + 1$ has been used. This is a contradiction because there are a total of n objects. ◄

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

Example: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

Solution: a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)

Example

- What is the least number of area codes needed to guarantee that 25 million phones in a state have distinct ten-digit telephone number?
- Solution: number form:NXX-NXX-XXXX, there are 8×10^6 different phone numbers of the form NXX-XXXX, hence, by generalized pigeonhole principle, at least $\lceil 25\times10^6 / 8\times10^6 \rceil=4$ area codes

(Let X denote a digit from 0 through 9.
Let N denote a digit from 2 through 9.)

Elegant application

- 〖Example〗 during 30 days a baseball team plays at least one game a day, but no more than 45 games; show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
- Solution: a_j: the number of games played on j-th day
- $b_j = \Sigma^j{}_1 a_i$th $b_1, b_2, \ldots b_{30}$ on are strictly increasing number.
- $c_j = b_j+14$, then $15\leq c_j \leq 45+14=59$, the 60 integer numbers $b_1, b_2, \ldots b_{30}, c_1, c_2, \ldots c_{30}$ are all less than or equal to 59, then two of these integers are equal, hence
- $b_i = c_j = b_j+14$,it implies $a_{j+1} + a_{j+2} + \ldots + a_i = 14$

〖Example〗 Suppose that there are n arbitrary integers $x_1, x_2, \ldots, x_n$. Show that there exist some consecutive integers such that the sum of these integers is the multiple of n.

Solution:

$$a_i = \sum_{k=1}^{i} x_k \ (i = 1,2,\ldots,n)$$

(1) $\exists i \ (n \mid a_i)$

(2) $\neg\exists i \ (n \mid a_i)$

37

〖Example〗 Every sequence of n^2+1 distinct integers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Proof:

For example, n=2

1,2,3,4,5 ; 4,8,3,6,1; 1,4,5,3,2

Let the sequence be $a_1, a_2, \ldots, a_{n^2+1}$

Associate (x_k, y_k) to the term a_k , where x_k is the length of the longest increasing subsequence starting at a_k , and y_k is the length of the longest decreasing subsequence starting at a_k

Suppose that there is no increasing or decreasing subsequence of length n+1.Then

$1 \le x_k \le n \qquad 1 \le y_k \le n$

38

Hence there are $n \times n = n^2$ pairs (x_k, y_k),

Since $a_i \neq a_j$

It follows that

(1) $a_i < a_j$

(2) $a_i > a_j$

In either case there is a contradiction.

39

Hence there are $n \times n = n^2$ pairs (x_k, y_k),

Since there are n^2+1 a_k , By the pigeonhole principle, it follows that there exist terms a_i, a_j $(1 \le i < j \le n^2+1)$ such that $(x_i, y_i) = (x_j, y_j)$

Since $a_i \neq a_j$

It follows that

(1) $a_i < a_j$

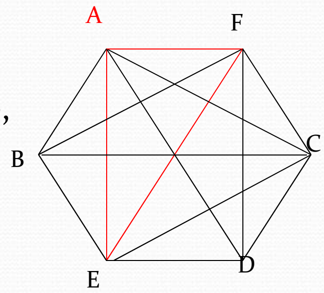
(2) $a_i > a_j$

In either case there is a contradiction.

40

Ramsey Theorem

- $K_6 \to K_3, K_3$, of 6 (or more) people, either there are 3, each pair of whom are acquainted, or there are three, each pair of whom are unacquainted.



Ramsey number R(m, n)

- The *Ramsey number R(m, n)* is the smallest number of people at a party that there either m mutual friends or n mutual enemies. *e.g.*, $R(3,3) = 6$
- **R(2, n) = R(n, 2) = n**

- 只知道9个拉姆齐数R(m,n)（3 ≤ m ≤ n）的精确值

Homework

第7版 Sec. 6.2 10, 38, 40, 42

第8版 Sec. 6.2 12, 40, 42, 44

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permuation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permuatations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

P(100,3) = 100 · 99 · 98 = 970,200

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}\ .$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}.$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:
  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775\ .$$

〖Example 〗 A soccer club has 8 female and 7 male members. For today's match, how many possible configurations are there?

(1) The coach wants to have 6 female and 5 male players on the grass.

(2) The coach wants to have 11 players with at most 5 male players on the grass.

Solution:

(1) C(8, 6) • C(7, 5)
= 8!/(6!·2!) • 7!/(5!·2!)
= 28 • 21
= 588

(2) C(8，6)C(7，5)+C(8，7)C(7，4)+C(8，8) C(7，3)

58

Homework

第7版 Sec 6.3: 20, 42, 44

第8版 Sec 6.3: 20, 44, 46

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)\,(x + y)\,(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x\,y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x\,y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◄

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof (*using binomial theorem*): With x = 1 and y = 1, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}. \blacktriangleleft$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, ..., and $\binom{n}{n}$ with n elements. Therefore the total is $\sum_{k=0}^{n} \binom{n}{k}$.

Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \blacktriangleleft$$

【 Corollary 2 】Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

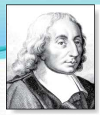
Proof:

Using the Binomial Theorem with $x = 1$ and $y = -1$.

Remark:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Blaise Pascal
(1623-1662)



Pascal's Identity

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence, $$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$ ◀

See Exercise 19 for an algebraic proof.

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\ldots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1\quad 1$$
$$1\quad 2\quad 1$$
$$1\quad 3\quad 3\quad 1$$
$$1\quad 4\quad 6\quad 4\quad 1$$
$$1\quad 5\quad 10\quad 10\quad 5\quad 1$$
$$1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1$$
$$1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1$$
$$1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1$$
$$\ldots$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

【 Theorem 3 】 *Vandermonde's Identity*

Let m, n and r be nonnegative integer with r not exceeding either m or n. Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$

Proof:

A and B are two disjoint sets. $|A|=m$, $|B|=n$,

$C(m+n, r)$ ---- the number of ways to pick r elements from $A \cup B$

Another way to pick r element from $A \cup B$ is to pick r-k elements from A and then k elements from B, where $0 \leq k \leq r$, which can be done in $C(m, r-k)\, C(n, r)$

69

【 Corollary 4 】 If n is a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof:

We use Vandermonde's Identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

70

【 Theorem 4 】Let n and r be nonnegative integer with $r \le n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof:

The left-hand side counts the bit strings of length n+1 containing r+1 1s.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with r+1 ones.

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

71

Homework

第7版 Sec.6.4: 14, 22, 26, 30

第8版 Sec.6.4: 18, 26, 30, 34