Final (2021)

Question 1a:

At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$? Please note: this says $y = x^2 + z^2$, not $z = x^2 + y^2$.

Question 1b:

Consider the integral

$$\int_0^4 \int_0^{x^2} f(x, y)\, dy\, dx$$

Rewrite this integral when the order of integration is switched, that is, fill in what each of the question marks are in the expression below (two different question marks can have two different values)

$$\int_{??}^{??} \int_{??}^{??} f(x, y)\, dx\, dy$$

Question 2a:

Find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the region in the xy plane bounded by $y = x^2$ and $x = y^2$ and located in the first quadrant.

Question 2b:

Suppose $\vec{F} = (2xy^3, xy^4z, z^3 + 8x)$. Let $\vec{G} = \nabla \times \vec{F}$.

Let S be the unit sphere. Find $\iint_S \vec{G} \cdot \vec{n}\, dS$. You must explain your answer.

Question 4:

Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $x^2 + 2y^2 + 3z^2 = 6$. At what points do these extremes occur? You must explain your work.

Question 5:

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left(-\frac{y^3}{3} + x^4, y^2 + \frac{x^3}{3}, z^2\right)$ and C is the curve given by the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$. The curve is oriented so that it goes counter-clockwise when viewed from above. You must explain your work.

Question 6:

Let the surface S_1 be the upper hemisphere $x^2+y^2+z^2=1$, with $z \geq 0$

Let the surface S_2 be the upper cylinder $x^2+y^2=1$, with $0 \leq z \leq 1$, with the top closed off (so at $z=1$, the top of the surface is a disk, thus at $z=1$ we have $x^2+y^2 \leq 1$), but the bottom of the surface is open.

Let $\vec{F}$ be the vector field $(xy, yz, xz+xyz)$

Let $\hat{n}$ be the normal oriented to point away from the origin.

Compute

$$\int_{S_1} [\nabla \times \vec{F}] \cdot \vec{n} \, dS_1 - \int_{S_2} [\nabla \times \vec{F}] \cdot \vec{n} \, dS_2$$

You must explain your work.

Question 7:

Let g be a differentiable function of one variable. Let $z = a \cdot g\left(\frac{b}{a}\right)$, where a and b are input variables and g is continuously differentiable. Let T be the tangent plane to the surface $z = a \cdot g\left(\frac{b}{a}\right)$ at the input point (a_1, b_1) . Assume that $a_1 \neq 0$. Show that T goes through the origin.

Question 8:

Use the transformation $x = u^3$, $y = v^3$, $z = w^3$, to find the volume of the region bounded by the surface $x^{1/3} + y^{1/3} + z^{1/3} = 1$ and the coordinate planes. Here, $x \geq 0$, $y \geq 0$, and $z \geq 0$. You must explain your work.

Question 9:

Let $\vec{F}$ be the vector field $\vec{F} = \left(14x - \frac{8}{3}x^3, -\frac{4}{3}y^3, -\frac{2}{3}z^3\right)$

Compute the normal at the point $(1, 1, 1)$ to the closed surface S in $\mathbb{R}^3$ which maximizes the value of the flux integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

Here, the surface is oriented so that its normal points outward. Make sure the normal you get has unit length.

Question 10:

Suppose $(x(s,t), y(s,t), z(s,t))$, $0 \leq t \leq 1$, $0 \leq s \leq 2$ gives a closed, simple oriented surface which we call S . (That is, the surface S in output space is closed, surrounds a volume, and does not touch itself more than once as you move along the parameterization: thus, $(x(0,0), y(0,0), z(0,0)) = (x(1,2), y(1,2), z(1,2))$.)

Show that the volume enclosed by S is given by

$$\frac{1}{3}\int_0^1 \int_0^2 [x(y_s z_t - z_s y_t) + y(x_t z_s - x_s z_t) + z(x_s y_t - x_t y_s)]\, ds\, dt$$