Chain Rule Assignment

1. Given, $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

$\Rightarrow$ If, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then, $x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \end{bmatrix}$

$$x^T x = [x_1^2 + x_2^2 + \cdots + x_d^2]$$

Applying chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\log_e(1+z)\right) \cdot \frac{d}{dx}\left(x^T x\right)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(z) \cdot \frac{d}{dx}\left(x_1^2 + x_2^2 + \cdots + x_d^2\right)$$

$$= \frac{1}{1+z} \cdot \left(2x_1 + 2x_2 + \cdots + 2x_d\right)$$

$$= \frac{1}{1+z} \cdot 2\left(x_1 + x_2 + \cdots + x_d\right)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

(Ans)

2. $f(z) = e^{-z/2}$; where $z = g(y)$, $g(y) = y^T s^{-1} y$, $y = h(x)$,
$h(x) = x - \mu$

$\Rightarrow$ using chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

here, $\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T s^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) s^{-1} (y+h) - y^T s^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T s^{-1} + h s^{-1})(y+h) - y^T s^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{y^T s^{-1} y} + y^T s^{-1} h + h s^{-1} y + h^2 s^{-1} - \cancel{y^T s^{-1} y}}{h}$$

$$= \lim_{h \to 0} \frac{h \left(y^T s^{-1} + s^{-1} y + h s^{-1}\right)}{h}$$

$$= \lim_{h \to 0} \left(y^T s^{-1} + s^{-1} y + h s^{-1}\right)$$

$$= y^T s^{-1} + s^{-1} y$$

$$\frac{dy}{dx} = \frac{d(x-\mu)}{dx} = 1$$

$$\therefore \frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= -\frac{e^{-z/2}}{2} \cdot \left(y^T S^{-1} + S^{-1} y\right) \cdot 1$$

$$= -\frac{e^{-\frac{z}{2}}}{2} \cdot \frac{1}{S}\left(y^T + y\right)$$

(Ans)