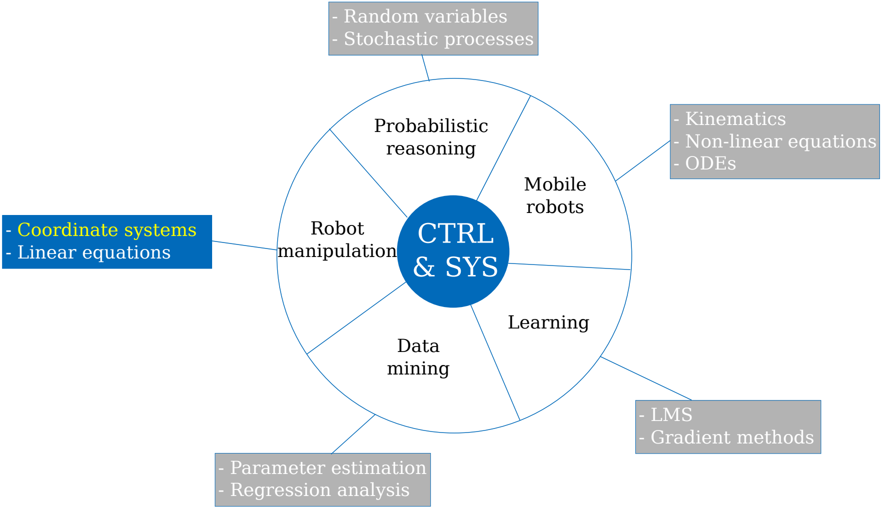
# Frames and Coordinate Transformations



# Today's Topics

→ Coordinate frames

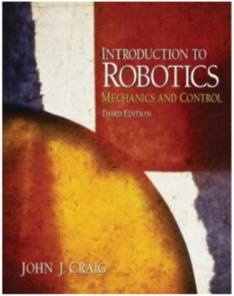
Transformation basics

Homogeneous coordinates

Inverse transforms

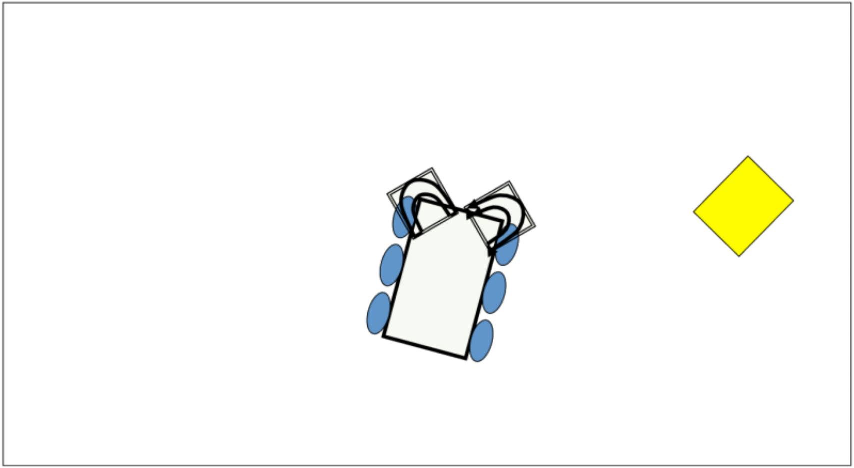
Euler angles

Efficiency



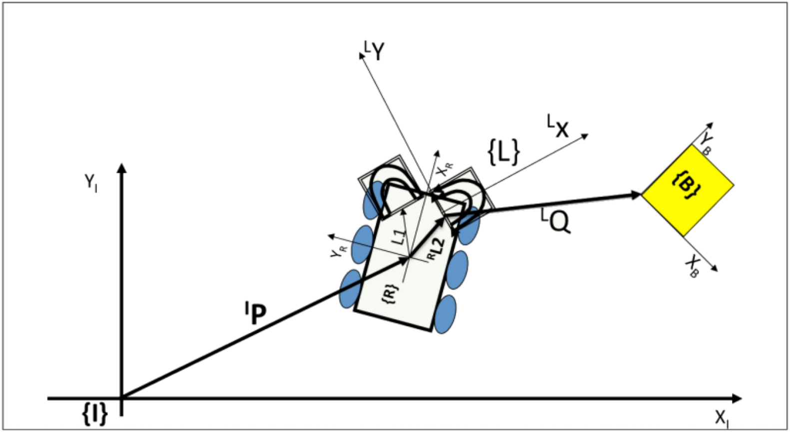
# Coordinate Systems - 2D Example (1/3)

- A robot in a room observes a box using its laser scanners
- Where in the room is the box?



# Coordinate Systems - 2D Example (2/3)

- $\{I\}$ - Inertial
- $\{R\}$ - Robot
- $\{L\}$ - Laser
- $\{B\}$ - Box



# Coordinate Systems - 2D Example (3/3)

**Calculate box position**

$$^{I}\mathbf{B} = {}^{I}\mathbf{P} + {}^{R}\mathbf{L} + {}^{L}\mathbf{Q}$$

Question:

What is the **orientation** of the box w.r.t the walls of the room?

# Coordinate Systems - 3D Example

How to grab a screw?

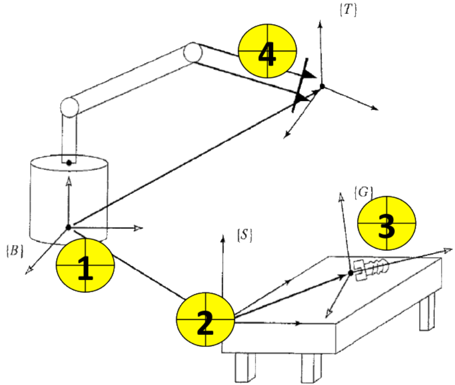



FIGURE 2.16: Manipulator reaching for a bolt.

Needed:

**Displacements**

Vectors
Translations

**Angles**

Orientations (frames)
Rotations

We mix 2D (easy explanation) yet also apply in 3D

Coordinate frames

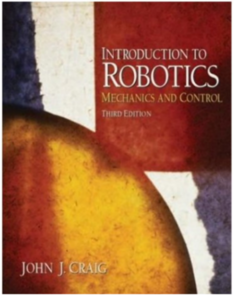
→ Transformation basics

Homogeneous coordinates

Inverse transforms

Euler angles

Efficiency



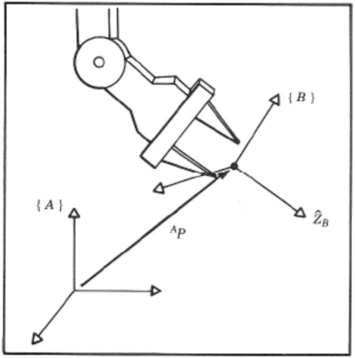
# Central Topic

## Problem

Robotic manipulation, by definition, implies that parts and tools will be moving around in space by the manipulator mechanism. This naturally leads to the need of representing positions and orientations of the parts, tools, and the mechanism itself

## Solution

Mathematical tools for representing position and orientation of objects/frames in a 3D space



# Description of a Position

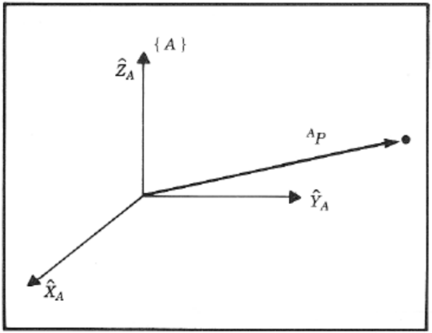
The location of any point in 3D can be described as a $3 \times 1$ **position vector** in a reference coordinate system

## Position vector

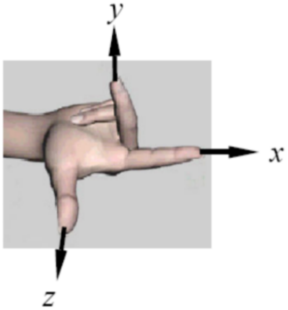
$$^{A}\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

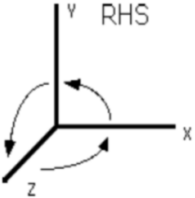
$$\xrightarrow{\text{coordinate system}} {}^{A}\mathbf{p} \xleftarrow{\text{position vector}}$$

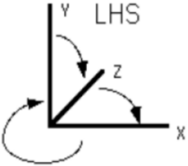
Observe: the notation in the “upper left” is the frame we are in



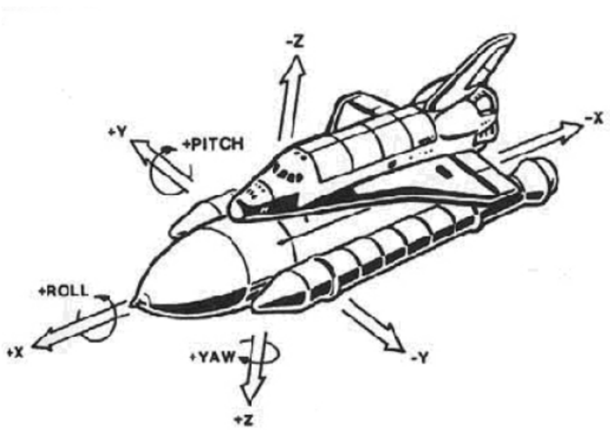
# Coordinate Systems (1/2)



# Coordinate Systems (2/2)



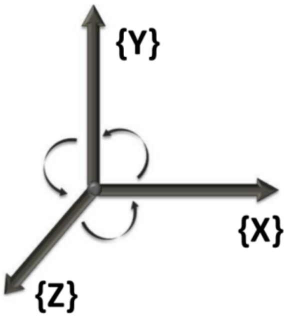
# Hand Out the Dice (5 min)

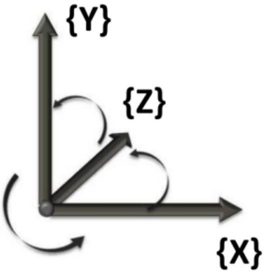
Assume:

- The $x$-axis goes through the middle of face "$1$"
- The $y$-axis goes through the middle of face "$3$"
- The $z$-axis goes through the middle of face "$5$"

Is the resulting system LHS or RHS?

# Coordinate Systems



# Description of an Orientation

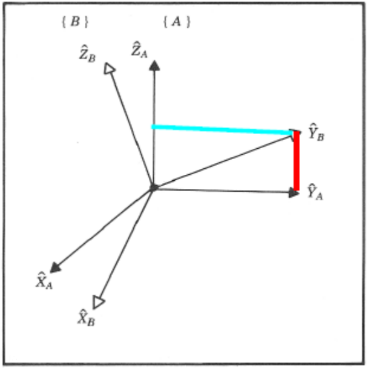
The orientation of a body is described by attaching a coordinate system to the body $\{B\}$ and then defining the relationship between the body frame and the reference frame $\{A\}$ using a rotation matrix

**Rotation matrix describing frame $\{B\}$ relative to frame $\{A\}$**

$$
{}^{A}_{B}R = [{}^{A}\hat{\mathbf{x}}_B, {}^{A}\hat{\mathbf{y}}_B, {}^{A}\hat{\mathbf{z}}_B] = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}
$$

$$
\begin{array}{l} \xrightarrow{\text{reference frame}} A \\ \xrightarrow{\text{body frame}} B \end{array} R
$$

where ${}^{A}\hat{\mathbf{x}}_B$ is the unit vector $\mathbf{x}$ of frame $\{B\}$ expressed in frame $\{A\}$



# Description of a Frame

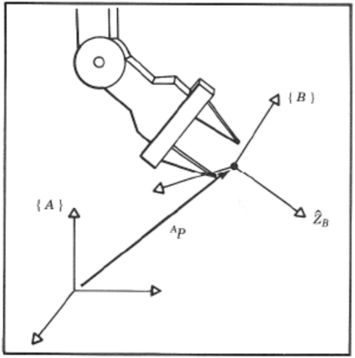
**A position and an orientation** is the information needed to completely specify where the manipulator hand is

## Frame description

$$\{B\} = \{{}^{A}_{B}R, {}^{A}\mathbf{p}_{B_{ORG}}\}$$

where

- ${}^{A}_{B}R$ is the rotation matrix describing frame $\{B\}$ relative to frame $\{A\}$
- ${}^{A}\mathbf{p}_{B_{ORG}}$ is the origin of frame $\{B\}$ relative to frame $\{A\}$

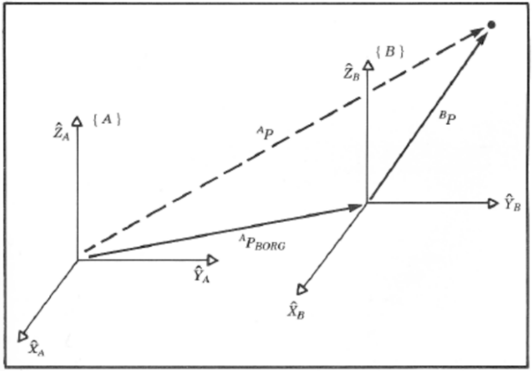


# Mapping - Translated Frames

Assuming that frame $\{B\}$ is only **translated (not rotated)** with respect to frame $\{A\}$, the position of a point can be expressed in frame $\{A\}$ as follows:

**Position in the case of translation only**

$$^{A}\mathbf{p} = {}^{B}\mathbf{p} + {}^{A}\mathbf{p}_{B_{ORG}}$$



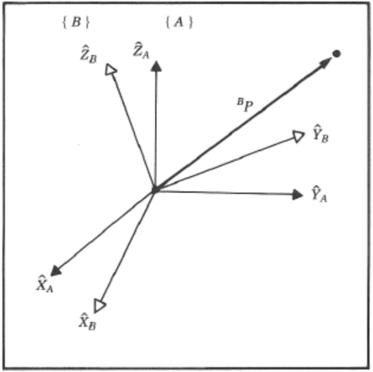
# Mapping - Rotated Frames

Assuming that frame $\{B\}$ is only **rotated** (**not translated**) with respect to frame $\{A\}$ (the origins of the two frames are located at the same point), the position of a point in frame $\{B\}$ can be expressed in frame $\{A\}$ using a rotation matrix as follows:

**Position in the case of rotation only**

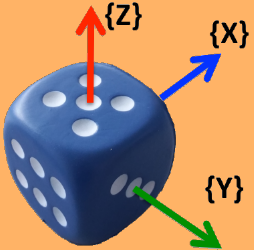
$$^{A}\mathbf{p} = {}^{A}_{B}R\,{}^{B}\mathbf{p}$$

$$^{B}\mathbf{p} = {}^{B}_{A}R\,{}^{A}\mathbf{p}$$



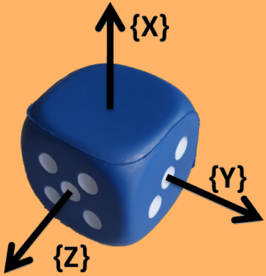
# Coordinate System Rotation Example

$\{B\}$ body frame, coordinate system is rotated by $+90^o$ around $\{Y\}$




assume $^B\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ given

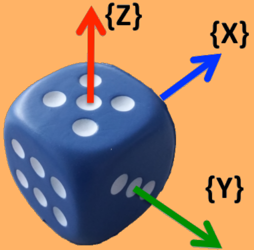
$\{A\}$ reference frame coordinate system




what is $^A\mathbf{p} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$

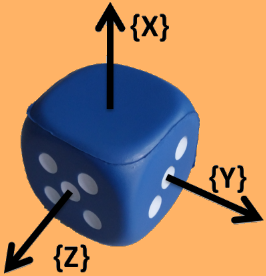
# Coordinate System Rotation Example

$\{B\}$ body frame, coordinate system is rotated by $+90^o$ around $\{Y\}$




assume $^B\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ given

$\{A\}$ reference frame coordinate system




what is $^A\mathbf{p} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

# Task: Map Column to Column

Apply this to all three vectors:

$$ {}^{B}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} $$

$$ {}^{B}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} $$

$$ {}^{B}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix} $$

# Task: Map Column to Column

Apply this to all three vectors:

$$
{}^{B}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
$$

$$
{}^{B}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}
$$

$$
{}^{B}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow {}^{A}\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}
$$

# Task: Map Column to Column

Apply this to all three vectors:

$$^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}?\\?\\?\end{pmatrix}$$

# Task: Map Column to Column

Apply this to all three vectors:

$$^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

# Task: Map Column to Column

Apply this to all three vectors:

$$^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \to {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \to {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \to {}^{A}\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Collect all three results columnwise:

$$\underbrace{\left( {}^{A}\begin{pmatrix}0\\0\\-1\end{pmatrix} \quad {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix} \right)}_{M}$$

$$= {}^{A}_{B}R \cdot \left( {}^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \quad {}^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \right)$$

$$\Longrightarrow \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 0\end{pmatrix} = {}^{A}_{B}R \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

# Task: Map Column to Column

Apply this to all three vectors:

$$^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow {}^{A}\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Collect all three results columnwise:

$$\underbrace{\left( {}^{A}\begin{pmatrix}0\\0\\-1\end{pmatrix} \quad {}^{A}\begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^{A}\begin{pmatrix}1\\0\\0\end{pmatrix} \right)}_{M}$$

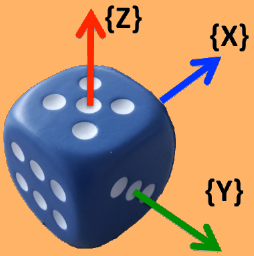
$$= {}^{A}_{B}R \cdot \left( {}^{B}\begin{pmatrix}1\\0\\0\end{pmatrix} \quad {}^{B}\begin{pmatrix}0\\1\\0\end{pmatrix} \quad {}^{B}\begin{pmatrix}0\\0\\1\end{pmatrix} \right)$$

$$\Longrightarrow \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 0\end{pmatrix} = {}^{A}_{B}R \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

so, $M = {}^{A}_{B}R$, or in words, $R$ maps unit vectors from the body frame to the reference frame

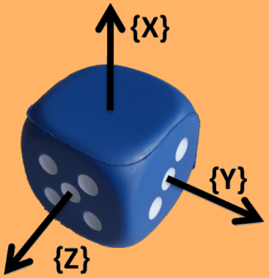
# Task NOW: Coordinate System Rotation

$\{B\}$ body frame, coordinate system is rotated by $+90^o$ around $\{Y\}$




assume $^B\mathbf{p} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$ given

$\{A\}$ reference frame coordinate system




what is $^A\mathbf{p} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$

# Coordinate System Rotation Solution

$$ {}^{A}\mathbf{p} = {}^{A}_{B}R\,{}^{B}\mathbf{p} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} $$

# Mapping - Rotated Frames Example (1/3)

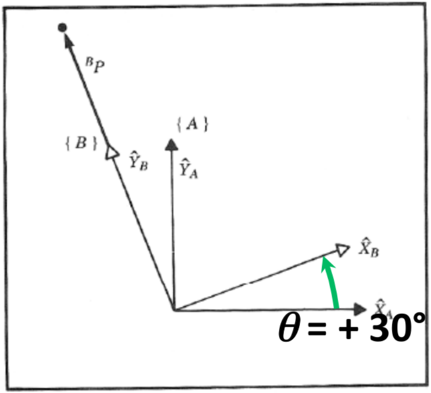
Given:

$$^B\mathbf{p} = \begin{pmatrix} 0 \\ ^Bp_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\theta = 30^o$$

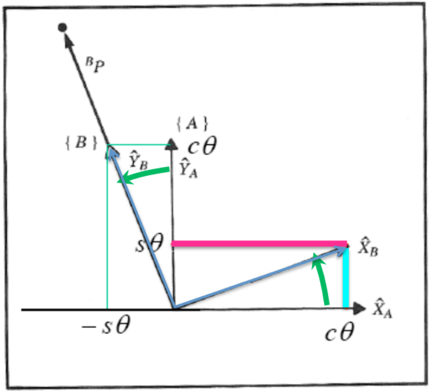
Compute: $^A\mathbf{p}$

Solution: $^A\mathbf{p} = {}^A_BR\,{}^B\mathbf{p}$



# Mapping - Rotated Frames Example (2/3)

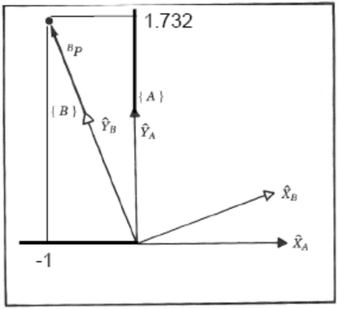
$$
\begin{aligned}
{}^A_B R &= \left({}^A\hat{\mathbf{x}}_B, \quad {}^A\hat{\mathbf{y}}_B, \quad {}^A\hat{\mathbf{z}}_B\right) \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$



# Mapping - Rotated Frames Example (3/3)

$$
{}^{A}\mathbf{p} = {}^{A}_{B}R\,{}^{B}\mathbf{p} = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ {}^{B}p_y \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} \begin{pmatrix} 0.000 \\ 2.000 \\ 0.000 \end{pmatrix} = \begin{pmatrix} -1.000 \\ 1.732 \\ 0.000 \end{pmatrix}
$$




What is the inverse of this rotation?

## Answer

$$
\begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
\begin{pmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

# Mapping - Rotated Frames - General Notation

## Rotation matrices with respect to the reference frame

$$R_{\mathbf{x}}(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix}$$

$$R_{\mathbf{y}}(\beta) = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}$$

$$R_{\mathbf{z}}(\alpha) = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Coordinate frames

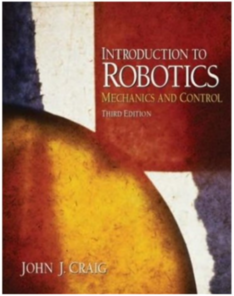
Transformation basics

→ Homogeneous coordinates

Inverse transforms

Euler angles

Efficiency



# Mapping - General Frames

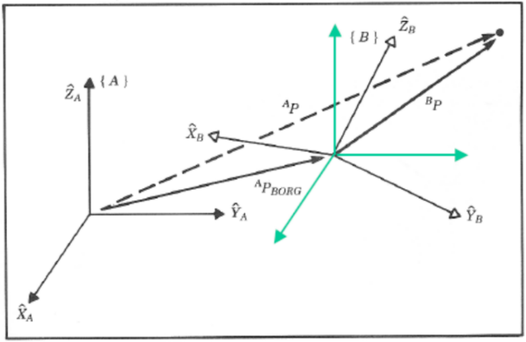
Assuming that frame $\{B\}$ is both **translated** and **rotated** with respect to frame $\{A\}$, the position of a point expressed in frame $\{B\}$ can be expressed in frame $\{A\}$ as follows:

**Position in the case of both translation and rotation**

$$\{B\} = \{{}^A_B R, {}^A\mathbf{p}_{B_{ORG}}\}$$

$${}^A\mathbf{p} = {}^A_B R\, {}^B\mathbf{p} + {}^A\mathbf{p}_{B_{ORG}}$$

$${}^A\mathbf{p} = {}^A_B T\, {}^B\mathbf{p}$$



# Mapping - Homogeneous Transform

The homogeneous transform is a $4 \times 4$ matrix casting the **rotation** and **translation** of a general transform into a single matrix. When the last row of the rotation matrix is other than $[0001]$ or the matrix is not orthonormal, the matrix can be used to compute perspective and scaling operations

$$
\begin{aligned}
{}^{A}\mathbf{p} &= {}^{A}_{B}R\,{}^{B}\mathbf{p} + {}^{A}\mathbf{p}_{B_{ORG}} \\
{}^{A}\mathbf{p} &= {}^{A}_{B}T\,{}^{B}\mathbf{p}
\end{aligned}
$$

$$
\left(\begin{array}{c} {}^{A}\mathbf{p} \\ \hdashline 1 \end{array}\right) = \left(\begin{array}{ccc:c} & {}^{A}_{B}R & & {}^{A}\mathbf{p}_{B_{ORG}} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c} {}^{B}\mathbf{p} \\ \hdashline 1 \end{array}\right)
$$

# Homogeneous Transform - Example (1/2)

Given:

$$ {}^{B}\mathbf{p} = \begin{pmatrix} 0 \\ {}^{B}p_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} $$

Frame $\{B\}$ is rotated relative to frame $\{A\}$ about $\hat{\mathbf{z}}$ by $30$ degrees and translated $10$ units in $\hat{\mathbf{x}}_A$ and $5$ units in $\hat{\mathbf{y}}_A$

Calculate: The vector ${}^{A}\mathbf{p}$ expressed in frame $\{A\}$

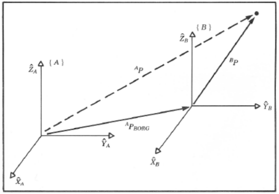
# Homogeneous Transform - Example (2/2)

$$
{}^{A}\mathbf{p} = {}^{A}_{B}T\,{}^{B}\mathbf{p} = \begin{pmatrix} {}^{A}\mathbf{p} \\ \\ 1 \end{pmatrix} = \begin{pmatrix} & {}^{A}_{B}R & & {}^{A}\mathbf{p}_{B_{ORG}} \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{B}\mathbf{p} \\ \\ 1 \end{pmatrix}
$$

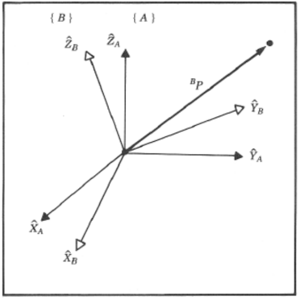
$$
{}^{A}\mathbf{p} = \begin{pmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.0 \\ 2.0 \\ 0.0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9.0 \\ 6.7 \\ 0.0 \\ 1 \end{pmatrix}
$$

# Homogeneous Transform - Special Cases

Translation $${}^A_BT = \begin{pmatrix} 1 & 0 & 0 & {}^A\mathbf{p}_{BORGx} \\ 0 & 1 & 0 & {}^A\mathbf{p}_{BORGy} \\ 0 & 0 & 1 & {}^A\mathbf{p}_{BORGz} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$




Rotation $${}^A_BT = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



# Operator - Rotating Vector

## Rotational operator

Operates on a vector ${}^{A}\mathbf{p}_1$ and changes that vector to a new vector ${}^{A}\mathbf{p}_2$ by means of a rotation $R$

$$ {}^{A}\mathbf{p}_2 = R\,{}^{A}\mathbf{p}_1 $$

Note: The rotation matrix that rotates vectors through a rotation $R$ is the same as the rotation that describes a frame rotated by $R$ relative to the reference frame

$$ \underbrace{{}^{A}\mathbf{p}_2 = R\,{}^{A}\mathbf{p}_1}_{\textbf{operator}} \Longleftrightarrow \underbrace{{}^{A}\mathbf{p} = {}^{A}_{B}R\,{}^{B}\mathbf{p}}_{\textbf{mapping}} $$

# Operator - Rotating Vector Example

Given:

$$^{A}\mathbf{p}_1 = \begin{pmatrix} 0 \\ ^{A}p_{1y} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Compute: The vector $^{A}\mathbf{p}_2$ obtained by rotating $^{A}\mathbf{p}_1$ about $\hat{\mathbf{z}}$ by 30 degrees

## Solution

$$^{A}\mathbf{p}_2 = R(30^o)\,^{A}\mathbf{p}_1 = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ ^{A}p_{1y} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{pmatrix} \begin{pmatrix} 0.000 \\ 2.000 \\ 0.000 \end{pmatrix} = \begin{pmatrix} -1.000 \\ 1.732 \\ 0.000 \end{pmatrix}$$

# Operator - Transforming Vector

## Transformation operator

Operates on a vector ${}^{A}\mathbf{p}_1$ and changes that vector to a new vector ${}^{A}\mathbf{p}_2$ by means of a rotation $R$ and translation $Q$

$$ {}^{A}\mathbf{p}_2 = T\,{}^{A}\mathbf{p}_1 $$

Note: The matrix of the transform operator $T$ that rotates vectors by $R$ and translates by $Q$ is the same as the transformation matrix that describes a frame rotated by $R$ and translated by $Q$ relative to the reference frame

$$ \underbrace{{}^{A}\mathbf{p}_2 = T\,{}^{A}\mathbf{p}_1}_{\textbf{operator}} \iff \underbrace{{}^{A}\mathbf{p} = {}^{A}_{B}T\,{}^{B}\mathbf{p}}_{\textbf{mapping}} $$

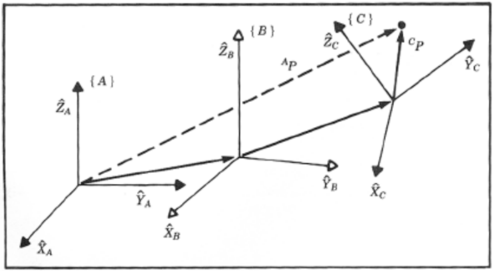
# Transformation Arithmetic - Compound Transformations

Given: Vector ${}^{C}\mathbf{p}$
Frame $\{C\}$ is known relative to frame $\{B\}$ - ${}^{B}_{C}T$
Frame $\{B\}$ is known relative to frame $\{A\}$ - ${}^{A}_{B}T$

Calculate: Vector ${}^{A}\mathbf{p}$

## Compound transformations

$$ {}^{B}\mathbf{p} = {}^{B}_{C}T\,{}^{C}\mathbf{p} $$
$$ {}^{A}\mathbf{p} = {}^{A}_{B}T\,{}^{B}\mathbf{p} $$
$$ {}^{A}\mathbf{p} = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}\mathbf{p} $$



Coordinate frames

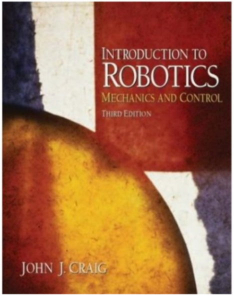
Transformation basics

Homogeneous coordinates

$\rightarrow$ Inverse transforms

Euler angles

Efficiency



# Rotated Frames Inversion

Given: The rotation matrix from frame $\{A\}$ to frame $\{B\}$ - ${}^A_BR$
Calculate: The rotation matrix from frame $\{B\}$ to frame $\{A\}$ - ${}^B_AR$

$${}^A\mathbf{p} = {}^A_BR\,{}^B\mathbf{p}$$

$${}^A_BR^{-1}\,{}^A\mathbf{p} = {}^A_BR^{-1}\,{}^A_BR\,{}^B\mathbf{p}$$

$${}^A_BR^{-1}\,{}^A_BR = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p} = I\mathbf{p}$$

$${}^A_BR^{-1}\,{}^A\mathbf{p} = {}^A_BR^{-1}\,{}^A_BR\,{}^B\mathbf{p} = I\,{}^B\mathbf{p} = {}^B\mathbf{p}$$

$${}^B\mathbf{p} = {}^A_BR^{-1}\,{}^A\mathbf{p}$$

$${}^B\mathbf{p} = {}^B_AR\,{}^A\mathbf{p}$$

$${}^B_AR = {}^A_BR^{-1} = {}^A_BR^T$$

$${}^A_BR = {}^B_AR^{-1} = {}^B_AR^T$$

Note: ${}^B_AR^{-1} = {}^B_AR^T \xleftarrow{\text{orthogonal coordinate system}}$

# Transformation Arithmetic - Inverted Transformation

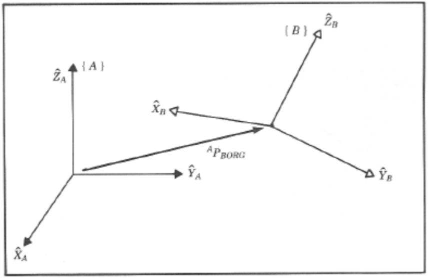
Given: The description of frame $\{B\}$ relative to frame $\{A\}$
- ${}^{A}_{B}T \; ({}^{A}_{B}R, {}^{A}\mathbf{p}_{B_{ORG}})$

Calculate: The description of frame $\{A\}$ relative to frame $\{B\}$
- Homogeneous transform ${}^{B}_{A}T \; ({}^{B}_{A}R, {}^{B}\mathbf{p}_{A_{ORG}})$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

$$ {}^{B}_{A}T = \left( \begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}\mathbf{p}_{B_{ORG}} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) $$

Note: ${}^{B}_{A}T = {}^{A}_{B}T^{-1}$



# Inverted Transformation Example (1/2)

Given: Description of frame $\{B\}$ relative to frame $\{A\}$ - ${}^{A}_{B}T\ ({}^{A}_{B}R, {}^{A}\mathbf{p}_{B_{ORG}})$

Frame $\{B\}$ is rotated relative to frame $\{A\}$ about $\hat{\mathbf{z}}$ by $30$ degrees and translated $4$ units in $\hat{\mathbf{x}}$ and $3$ units in $\hat{\mathbf{y}}$

Calculate: Homogeneous transform ${}^{B}_{A}T\ ({}^{B}_{A}R, {}^{B}\mathbf{p}_{A_{ORG}})$

# Inverted Transformation Example (2/2)

$$ {}^{A}_{B}T = \left( \begin{array}{ccc|c} c\theta & -s\theta & 0 & {}^{A}\mathbf{p}_{B_{ORGx}} \\ s\theta & c\theta & 0 & {}^{A}\mathbf{p}_{B_{ORGy}} \\ 0 & 0 & 1 & {}^{A}\mathbf{p}_{B_{ORGz}} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 4.000 \\ 0.500 & 0.866 & 0.000 & 3.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0.000 & 0.000 & 0.000 & 1.000 \end{array} \right) $$

$$ {}^{B}_{A}T = \left( \begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}\mathbf{p}_{B_{ORG}} \\ \hline 0 & 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{ccc|c} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0.000 & 0.000 & 0.000 & 1.000 \end{array} \right) $$

# Homogeneous Transform - Summary of Interpretations

As a general tool to represent a frame, we have introduced the **homogeneous transformation**, a $4 \times 4$ matrix containing orientation and position information

## Three interpretations of the **homogeneous transformation**

1. **Description of a frame**: ${}^A_BT$ describes the frame $\{B\}$ relative to frame $\{A\}$

$$
{}^A_BT = \begin{pmatrix} & {}^A_BR & & {}^A\mathbf{p}_{B_{ORG}} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

2. **Transform mapping**: ${}^A_BT$ maps ${}^B\mathbf{p} \rightarrow {}^A\mathbf{p}, \quad {}^A\mathbf{p} = {}^A_BT^B\mathbf{p}$
3. **Transform operator**: $T$ operates on ${}^A\mathbf{p}_1$ to create ${}^A\mathbf{p}_2, \quad {}^A\mathbf{p}_2 = T^A\mathbf{p}_1$

# Transform Equations

Given: ${}^U_AT, {}^A_DT, {}^U_BT, {}^C_DT$

Calculate: ${}^B_CT$

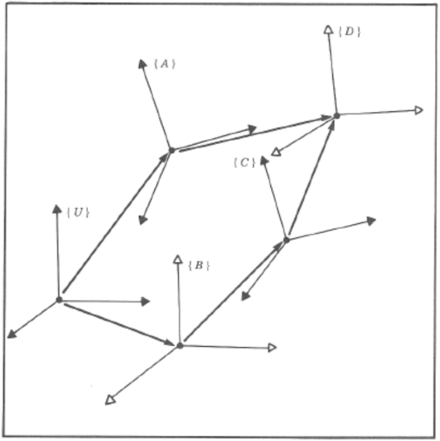
$${}^U_DT = {}^U_AT {}^A_DT$$

$${}^U_DT = {}^U_BT {}^B_CT {}^C_DT$$

$${}^U_AT {}^A_DT = {}^U_BT {}^B_CT {}^C_DT$$

$${}^U_BT^{-1} {}^U_AT {}^A_DT {}^C_DT^{-1} = {}^U_BT^{-1} {}^U_BT {}^B_CT {}^C_DT {}^C_DT^{-1}$$

$${}^B_CT = {}^U_BT^{-1} {}^U_AT {}^A_DT {}^C_DT^{-1}$$



Coordinate frames

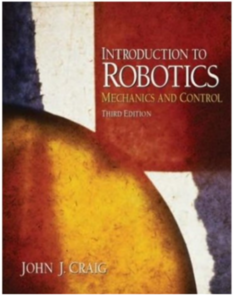
Transformation basics

Homogeneous coordinates

Inverse transforms

→ Euler angles

Efficiency



# Rotated Frames - General Notation

Recall that the rotation matrices with respect to the reference frame are defined as follows:

$$R_{\mathbf{x}}(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix}$$

$$R_{\mathbf{y}}(\beta) = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}$$

$$R_{\mathbf{z}}(\alpha) = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Rotated Frames - Methods

**$X - Y - Z$ fixed angles**

The rotations are performed about an axis of a **fixed** reference frame

**$Z - Y - X$ Euler angles**

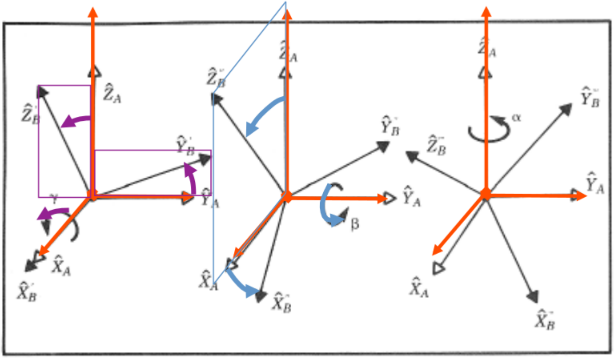
The rotations are performed about an axis of a **moving** reference frame

# Rotated Frames - $X - Y - Z$ Fixed Angles

Start with frame $\{B\}$ coincident with a known reference frame $\{A\}$

$$\left.\begin{array}{l}\text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{x}}_A \text{ by an angle } \gamma \\ \text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{y}}_A \text{ by an angle } \beta \\ \text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{z}}_A \text{ by an angle } \alpha\end{array}\right\} \textbf{Fixed angles}$$

**Note**: Each of the three rotations takes place about an axis in the **fixed reference frame** $\{\mathbf{A}\}$



# Rotated Frames - $X - Y - Z$ Fixed Angles

$$
\begin{aligned}
{}^A_B R_{\mathbf{xyz}}(\gamma, \beta, \alpha) &= R_{\mathbf{z}}(\alpha) R_{\mathbf{y}}(\beta) R_{\mathbf{x}}(\gamma) \\
&= \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} \\
&= \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}
\end{aligned}
$$

# Rotated Frames - $X - Y - Z$ Fixed Angles

$$
{}^{A}_{B}R_{\mathbf{xyz}}(\gamma,\beta,\alpha) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}
$$

$$\beta = atan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

for $-90^o \leq \beta \leq 90^o$

$$\gamma = atan2\left(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta}\right)$$

$$\alpha = atan2\left(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta}\right)$$

Special case when the denominator $= 0$:

$$\beta = \pm 90^o$$

$$\alpha = 0$$

$$\gamma = atan2\left(r_{12}, r_{22}\right)$$

# $atan2$ - Definition

Four-quadrant inverse tangent (arctangent) in the range of

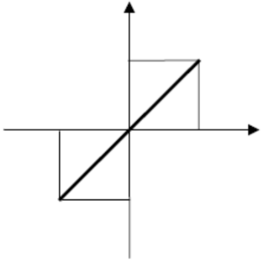
$$atan2(y, x) = tan^{-1}\left(\frac{y}{x}\right)$$

For example

$$[-\pi \ \pi]$$

$$atan(1, 1) = 45^o$$
$$atan2(1, 1) = 45^o$$

$$atan(-1, -1) = 45^o$$
$$atan2(-1, -1) = -135^o$$

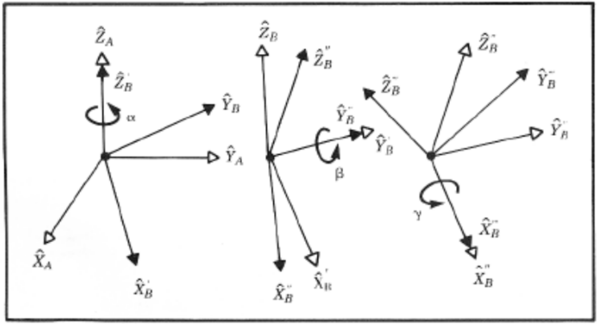


# Rotated Frames - $Z - Y - X$ Euler Angles

Start with frame $\{B\}$ coincident with a known reference frame $\{A\}$

$$\left.\begin{array}{l}\text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{z}}_A \text{ by an angle } \alpha \\ \text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{y}}_B \text{ by an angle } \beta \\ \text{Rotate frame } \{B\} \text{ about } \hat{\mathbf{x}}_B \text{ by an angle } \gamma\end{array}\right\} \textbf{Euler angles}$$

**Note**: Each rotation is performed about an axis of the **moving reference frame** $\{\mathbf{B}\}$ rather than a fixed reference frame $\{A\}$



# Rotated Frames - $Z - Y - X$ Euler Angles

$$
\begin{aligned}
{}^{A}_{B}R_{\mathbf{z'y'x'}}(\alpha,\beta,\gamma) &= R_{\mathbf{z}}(\alpha)R_{\mathbf{y}}(\beta)R_{\mathbf{x}}(\gamma) \\
&= \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix} \\
&= \begin{pmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{pmatrix}
\end{aligned}
$$

If $A$, $B$ are rotation matrices around the axes $G$ and $H$ (in the fixed frame), the concatenation of $(G, H')$ can be found via the observation that the working of $B$ on vectors which were already twisted by $A$ can be expressed as a transformation of $B$ into a new coordinate system basis, i.e. $B_{new} = ABA^{-1}$. The complete concatenation thus becomes $C = B_{new}A = ABA^{-1}A = AB$

# Rotated Frames

## Fixed angles versus Euler angles

$$ {}^{A}_{B}R_{\mathbf{xyz}}(\gamma, \beta, \alpha) = {}^{A}_{B}R_{\mathbf{z'y'x'}}(\alpha, \beta, \gamma) $$

Three rotations taken about fixed axes (fixed angles) yield the same final orientation as the same three rotations taken in an opposite order about the axes of the moving frame (Euler angles)

Coordinate frames

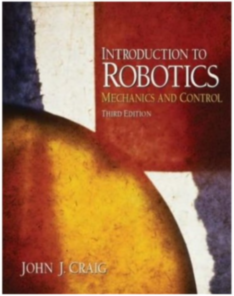
Transformation basics

Homogeneous coordinates

Inverse transforms

Euler angles

→ Efficiency



# Computational Considerations

$$^{A}\mathbf{p} = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}\mathbf{p}$$

**Blockwise multiplication**

$${}^{A}_{D}R = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R$$
$$^{A}\mathbf{p} = {}^{A}_{D}R\,{}^{D}\mathbf{p}$$

$54$ multiplications and $36$ additions

**Stepwise multiplication**

$$^{A}\mathbf{p} = {}^{A}_{B}R \circ \left({}^{B}_{C}R \circ \left({}^{C}_{D}R \circ {}^{D}\mathbf{p}\right)\right)$$

$27$ multiplications and $18$ additions

# Computational Considerations Example

## Direct product

$$R = {}^A_B R \, {}^B_C R$$

This method requires $27$ multiplications and $18$ additions

## Cross product

Take $\hat{\mathbf{l}}_i$ to be the columns of ${}^B_C R$ and $\hat{\mathbf{c}}_i$ to be the columns of the resulting $R$. We then have

$$\hat{\mathbf{c}}_1 = {}^A_B R \hat{\mathbf{l}}_1$$
$$\hat{\mathbf{c}}_2 = {}^A_B R \hat{\mathbf{l}}_2$$
$$\hat{\mathbf{c}}_3 = \hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2$$

The cross product-based method requires $24$ multiplications and $15$ additions