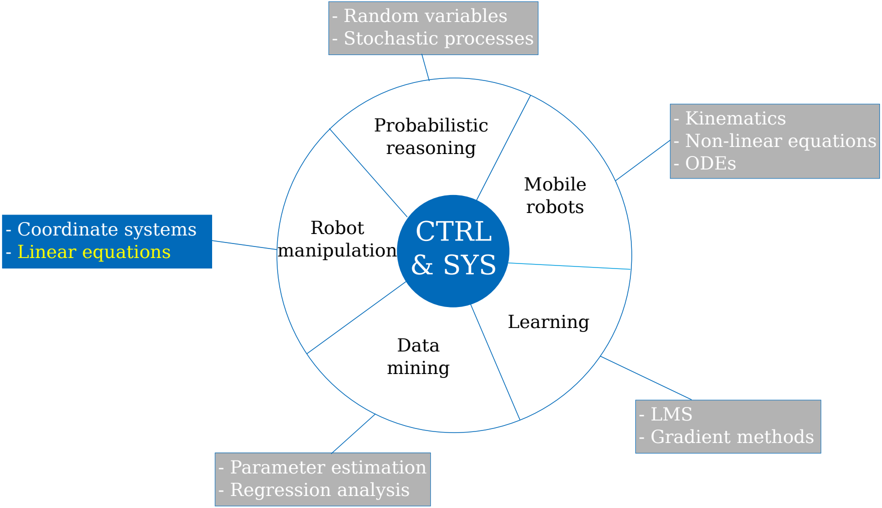
Norms, Eigenvalues, Singular Values, and Related Topics



Today's Topics

$\rightarrow$ Matrix norms

$\rightarrow$ Condition number

Rotations are norm preserving

Eigenvalues and eigenvectors

Spectral norm of $A ==$ max. eigenvalue of $A^T A$ (aka singular value)

What can go wrong when we solve $A\mathbf{x} = \mathbf{b}$ numerically?

Numerical representation of floating numbers in computers are imprecise $\Longrightarrow$?

Need to measure precision (how much an error in one side of an equation influences the other)

Use the condition number

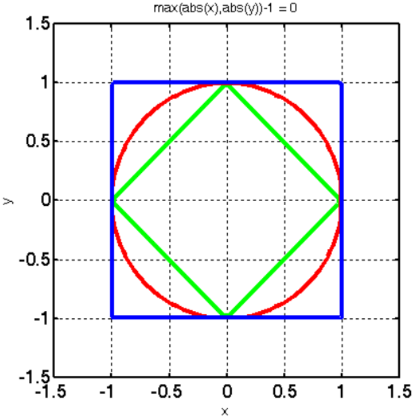
Calculated using vector and matrix norms

Vector Norms

$$\|\mathbf{x}\|_1 = norm(\mathbf{x}, 1)$$
$$\|\mathbf{x}\|_2 = norm(\mathbf{x})$$
$$\|\mathbf{x}\|_\infty = norm(\mathbf{x}, inf)$$

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$
$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$
$$\|\mathbf{x}\|_\infty = \max_i |x_i|$$

$$\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_\infty$$



Vector Norms - Definition

A **norm** is a function that assigns a strictly positive length or size to all vectors in a vector space

The vector norm for $p = 1, 2, \ldots$ is defined as

$$\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}$$

Matrix Norms - Definition

The matrix norm $\|A\|_p$ measures the maximum stretch of a vector for a given vector norm p

$$\|A\|_p = \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \implies \|A\mathbf{x}\|_p \leq \|A\|_p \|\mathbf{x}\|_p$$

Remark:
The maximum can be calculated by taking the maximum over the unit circle, i.e. $\|\mathbf{x}\|_p = 1$

Matrix Norms for $p = 1, 2, \infty$

$\|A\|_1 = \max\limits_j \sum_{i=1}^n |a_{ij}|$ **sum of columns**

$\|A\|_\infty = \max\limits_i \sum_{j=1}^n |a_{ij}|$ **sum of rows**

$$\|A\|_2 = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|_2 = \max_{\|\mathbf{x}\|=1} \{\mathbf{x}^t A^T A\mathbf{x}\}^{\frac{1}{2}}$$

$$= \max_i \{\sqrt{\mu_i} \mid \mu_i \text{ eigenvalue of } A^T A\}$$

spectral norm

Absolute and Relative Error

Given a vector $\mathbf{x} \neq \mathbf{0}$ and its approximation $\mathbf{y}$, we have

Absolute error

$$e = \|\mathbf{x} - \mathbf{y}\|$$

Relative error

$$\rho = \frac{\|\mathbf{x} - \mathbf{y}\|}{\|\mathbf{x}\|}$$

Relative Error Condition

Given $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y}$ vectors, we have

$$\rho = \frac{\|\mathbf{x} - \mathbf{y}\|}{\|\mathbf{x}\|}$$

In the scalar case, if the relative error in $\mathbf{y}$ is equal to ρ, $\mathbf{x}$ and $\mathbf{y}$ match for $\log_{10} \rho$ decimal places

UNFORTUNATELY(!)

Relative errors of vectors won't tell you about the relative errors in each component!

Relative Error Condition

$$\mathbf{x} = \begin{pmatrix} 1.0000 & 0.0100 & 0.0001 \end{pmatrix}^T \implies \|\mathbf{x}\|_\infty = 1$$

$$\mathbf{y} = \begin{pmatrix} 1.0002 & 0.0103 & 0.0002 \end{pmatrix}^T$$

$$\|\mathbf{x} - \mathbf{y}\|_\infty = \max\{|0.0002|, |0.0003|, |0.0001|\} = 3e^{-4}$$

$$\text{so } \rho = \frac{\|\mathbf{x} - \mathbf{y}\|_\infty}{\|\mathbf{x}\|_\infty} = \frac{|3e^{-4}|}{|1|} = 3e^{-4}$$

But

$$\rho_x = \frac{|2e^{-4}|}{|1|} = 2e^{-4}$$

$$\rho_y = \frac{|3e^{-4}|}{|1e^{-2}|} = 3e^{-2}$$

$$\rho_z = \frac{|1e^{-4}|}{|1e^{-4}|} = 1(!)$$

What can go wrong when we solve $A\mathbf{x} = \mathbf{b}$ numerically

Assume: the right side $\mathbf{b}$ is a bit imprecise

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.01 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2.01 \end{pmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $\tilde{\mathbf{b}} = \begin{pmatrix} 2 \\ 2.015 \end{pmatrix}$, then the solution $\mathbf{x} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}$

$$\rho(\mathbf{b}) = \frac{0.005}{2.01} = 0.0025 = 0.25\%$$

but

$$\rho(\mathbf{x}) = \frac{0.5}{1} = 0.5 = 50\%(!)$$

Estimation of Precision

Let $\tilde{\mathbf{x}}$ denote the distorted solution

Two sources of errors

1. Distortion in $A \implies \tilde{A}\tilde{\mathbf{x}} = \mathbf{b}$
2. Distortion in $\mathbf{b} \implies A\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$

Concentrate on case 2:

Estimate the relative error in the produced solution vector

$$\rho = \frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|}$$

Estimation of Precision

Assume case 2, so $\tilde{\mathbf{b}}$ is on RHS

$$
\begin{aligned}
&A\tilde{\mathbf{x}} = \mathbf{b} + \Delta\mathbf{b} \\
&\implies \Delta\mathbf{b} = A\tilde{\mathbf{x}} - \mathbf{b} = A\tilde{\mathbf{x}} - A\mathbf{x} = A(\tilde{\mathbf{x}} - \mathbf{x}) \\
&\implies A^{-1}\Delta\mathbf{b} = \tilde{\mathbf{x}} - \mathbf{x} \\
&\implies \|\tilde{\mathbf{x}} - \mathbf{x}\| \leq \left\|A^{-1}\right\| \|\Delta\mathbf{b}\|
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{b} = A\mathbf{x} \\
&\implies \|\mathbf{b}\| \leq \|A\| \|\mathbf{x}\| \\
&\implies \frac{1}{\|\mathbf{x}\|} \leq \frac{\|A\|}{\|\mathbf{b}\|}
\end{aligned}
$$

Estimation of Precision

Find how much an error in the right-hand side of the equation is amplified

$$\rho(\mathbf{x}) = \frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \frac{\left\|A^{-1}\right\| \|\Delta \mathbf{b}\|}{\|\mathbf{x}\|}$$
$$\leq \frac{\left\|A^{-1}\right\| \|A\| \|\Delta \mathbf{b}\|}{\|\mathbf{b}\|} = \operatorname{cond}(A)\rho(\mathbf{b})$$

$$\left\|A^{-1}\right\| \|A\| = \operatorname{cond}(A)$$

Example in Python

```
import numpy as np
from scipy.linalg import hilbert
for n in xrange(1, 11):
    print np.linalg.cond(hilbert(n))
```

Condition Number

Condition number

$$\left\|A^{-1}\right\| \|A\| = \mathrm{cond}(A)$$

A problem with a **low condition number** is said to be **well-conditioned**, while a problem with a **high condition number** is said to be **ill-conditioned**

Estimation of Precision - Example

Assume: Right side $\mathbf{b}$ is imprecise

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.01 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 2.01 \end{pmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

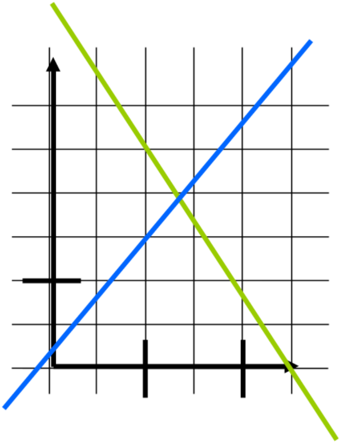
Let $\tilde{\mathbf{b}} = \begin{pmatrix} 2 \\ 2.015 \end{pmatrix}$, then the solution $\mathbf{x} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}$

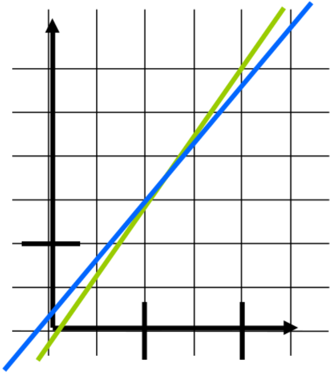
$$\rho(\mathbf{b}) = \frac{0.005}{2.01} = 0.0025 = 0.25\%$$

Estimation of Precision - Python Example

```
import numpy as np
A = np.array([[1., 1.], [1., 1.01]])

b = np.array([[2.], [2.01]])
x = np.linalg.lstsq(A, b)

bs = np.array([[2.], [2.015]])
xs = np.linalg.lstsq(A, bs)

relative_error = np.linalg.norm(bs - b) / np.linalg.norm(b)
condition_number = np.linalg.cond(A)
```

Geometric Interpretation





Each line represents one row of a linear system of equations
A little distortion can push the intersection far away

Fixes

Scale the matrix rows

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$$

$$\mathrm{cond}(A) = \frac{1}{\epsilon}$$

$\Longrightarrow$ divide the second row by ϵ

$\Longrightarrow$ get $\mathrm{cond}(A) = 1$(!)

Matrix norms

Condition number

$\rightarrow$ Rotations are norm preserving

Eigenvalues and eigenvectors

Spectral norm of $A ==$ max. eigenvalue of $A^T A$ (aka singular value)

Norm Preserving Transformations $\Longrightarrow$ Rotations

Definition

A transformation L of a vector space that preserves distances between every pair of points is called an **isometry**

If the transformation is linear and also preserves orientations, it is a **rotation**

More formal

1. L is linear

$$L(a\mathbf{x} + \mathbf{y}) = aL(\mathbf{x}) + L(\mathbf{y})$$

2. $\|L\mathbf{v}\| = \|\mathbf{v}\|$
3. $\det(L) = 1$

An Equation for Rotations

$$\begin{aligned} \|L\mathbf{v} - L\mathbf{w}\|_2^2 &=< (L\mathbf{v} - L\mathbf{w}), (L\mathbf{v} - L\mathbf{w}) > \\ &=< L(\mathbf{v} - \mathbf{w}), L(\mathbf{v} - \mathbf{w}) > \end{aligned}$$

Observe: $< \mathbf{a}, \mathbf{b} >= \mathbf{a}^T\mathbf{b}$

$$\begin{gathered} (L(\mathbf{v} - \mathbf{w}))^T L(\mathbf{v} - \mathbf{w}) \\ = (\mathbf{v} - \mathbf{w})^T L^T L(\mathbf{v} - \mathbf{w}) \end{gathered}$$

On the other hand

$$\|L\mathbf{v} - L\mathbf{w}\|_2^2 = \|\mathbf{v} - \mathbf{w}\|_2^2 = (\mathbf{v} - \mathbf{w})^T(\mathbf{v} - \mathbf{w})$$

So, it must hold that $L^T L = I$

An Equation for Rotations

$$\|L\mathbf{v} - L\mathbf{w}\|_2^2 = < (L\mathbf{v} - L\mathbf{w}), (L\mathbf{v} - L\mathbf{w}) >$$
$$= < L(\mathbf{v} - \mathbf{w}), L(\mathbf{v} - \mathbf{w}) >$$

Observe: $< \mathbf{a}, \mathbf{b} >= \mathbf{a}^T\mathbf{b}$

$$(L(\mathbf{v} - \mathbf{w}))^T L(\mathbf{v} - \mathbf{w})$$
$$= (\mathbf{v} - \mathbf{w})^T L^T L(\mathbf{v} - \mathbf{w})$$

On the other hand

$$\|L\mathbf{v} - L\mathbf{w}\|_2^2 = \|\mathbf{v} - \mathbf{w}\|_2^2 = (\mathbf{v} - \mathbf{w})^T(\mathbf{v} - \mathbf{w})$$

So, it must hold that $L^T L = I$.

Quick check: What is the condition number of a rotation matrix R using the 2 norm $\|\cdot\|_2$?

What is the Spectral Norm?

The Euclidean norm (a.k.a. 2-norm) is the most “natural” one.

If so, what is its respective matrix norm?

Answer:
The spectral norm $\Longrightarrow$ we need eigenvalues and eigenvectors

What **are** eigenvalues/eigenvectors?

Matrix norms

Condition number

Rotations are norm preserving

$\rightarrow$ Eigenvalues and eigenvectors

Spectral norm of $A ==$ max. eigenvalue of $A^T A$ (aka singular value)

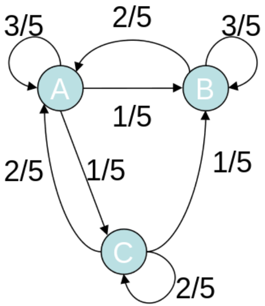
Markov chains - Example (1/2)

Nodes: $A, B, C =$ parties

Edges: probability of change from state to state, e.g. $A \to B$

Every full hour: shift

Question:
Is there a stable state $\mathbf{x}^*$ such that $T\mathbf{x}^* = \mathbf{x}^*$?



Markov chains - Example (2/2)

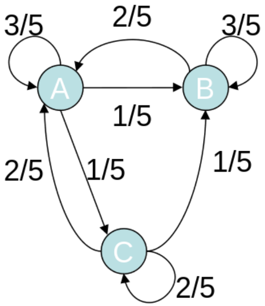
$$T\mathbf{x}_n = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} = \mathbf{x}_{n+1}$$

Boundary condition:

$$x_1 + x_2 + x_3 = 1$$

Observe:

$$\forall l \sum_{k=1}^{n} t_{kl} = 1$$



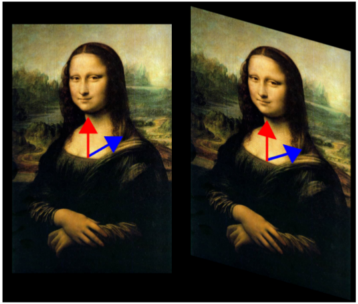
Eigenvalues and Eigenvectors

Vectors that don't change

Most of the time $A\mathbf{x} \neq \mathbf{x}$

But sometimes $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

This $\mathbf{x}$ is called an eigenvector and λ is its eigenvalue



www.wikipedia.org

After deformation of the picture, the central vertical axis (red vector) has not changed its direction; hence, the red vector is an eigenvector of the distortion map

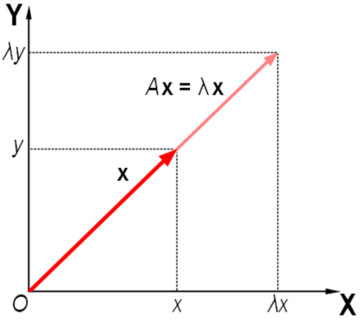
Eigenvectors

If $\mathbf{x}$ is an **eigenvector** of A, then $\alpha\mathbf{x}$ is also an **eigenvector**

Thus, any eigenvalue λ has many different eigenvectors

The direction of the vector $\mathbf{x}$ is not changed by the transformation A

Its direction is called an **eigenspace**



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Characteristic Polynomial

Eigenvalues obey the equation

$$\det(A - \lambda I) = 0 \text{ (also written as } |A - \lambda I| = 0)$$

This is a polynomial equation in λ and is called the **characteristic polynomial**

We can get the eigenvalues by solving the characteristic polynomial for λ

Example (2D):

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

Calculating a Determinant

2D

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (+)a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

3D (rule of Sarrus):

$$\left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right. = \begin{array}{l} (+)a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{array}$$

Calculating a Determinant

4D and more: The **Laplace expansion** of the determinant of an $n \times n$ square matrix $B = (b_{ij}), i, j = 1...n$ expresses the determinant $|B|$ as a sum of n determinants of $(n-1) \times (n-1)$ submatrices of B

Define the **i, j minor matrix** M_{ij} of B as the $(n-1) \times (n-1)$ matrix that results from deleting the i-th row and the j-th column of B, and C_{ij}, the cofactor of B, as

Cofactor

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

then

Determinant of a matrix

$$|B| = \sum_{i=1}^{n} b_{1j} C_{1j}$$

Determinant Multiplication Theorem

If A is a square matrix and A' is derived from A by taking **sums or scalar multiples** of some rows or columns in A, then

$$\det(A) = \det(A')$$

For two square matrices A and B, we also have the

Multiplication theorem

$$\det(AB) = \det(A) \cdot \det(B)$$

Determinant Example

Perform some simple transformations on A

$$A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{pmatrix}$$

use $1_{2,3}$:

$$R_{1_{new}} = R_1 + (-2) \cdot R_2$$
$$R_{3_{new}} = R_3 + (3) \cdot R_2$$
$$R_{4_{new}} = R_4 + (1) \cdot R_2$$

Now compute the determinant by the theorem

$$\det(A) = \det \begin{pmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix} = (-1)^{3+2} \det \begin{pmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$= (-1)\left[4 - 18 + 5 - 30 - 3 + 4\right] = 38$$

Determinant of Singular Matrices

Singular matrix

If A is a square matrix with $\det(A) = 0$, A is called **singular**.

For singular matrices, the equation $A\mathbf{x} = 0$ has a solution with $\mathbf{x} \neq \mathbf{0}$

This solution may be found by fixing/choosing one component in $\mathbf{x}$ to a constant a and then using this constant in the remaining equations (see the example below)

Eigenvalues/Eigenvectors Example (1/7)

$$A = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{2}{5} \end{pmatrix}$$

Find the eigenvalues

$$\det \begin{pmatrix} \frac{3}{5} - \lambda & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} - \lambda & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{2}{5} - \lambda \end{pmatrix} = \left. \begin{array}{c} \left(\frac{3}{5} - \lambda\right)\left(\frac{3}{5} - \lambda\right)\left(\frac{2}{5} - \lambda\right) + \frac{2}{125} \\ -\frac{2}{25}\left(\frac{3}{5} - \lambda\right) - \frac{2}{25}\left(\frac{2}{5} - \lambda\right) \end{array} \right\} = 0$$

$$-\lambda^3 + \frac{8}{5}\lambda^2 - \frac{21}{25}\lambda + \frac{18}{125} + \frac{2}{125} - \frac{6}{125} + \frac{2}{25}\lambda - \frac{4}{125} + \frac{4}{25}\lambda = 0$$

$$-\frac{1}{25}(\lambda - 1)(5\lambda - 2)(5\lambda - 1) = 0$$

$$\boldsymbol{\lambda_1 = 1, \lambda_2 = \frac{2}{5}, \lambda_3 = \frac{1}{5}}$$

Eigenvalues/Eigenvectors Example (2/7)

Finding the eigenvalues using Python

```
import numpy as np
A = np.array([[3/5., 2/5., 2/5.], [1/5., 3/5., 1/5.], [1/5., 0.,
    2/5.]])
eigenvalues = np.linalg.eigvals(A)
```

We could even use symbolic Python to achieve the same goal

```
import sympy as sp

# solving the characteristic polynomial
lam = sp.Symbol('\lambda')
A = sp.Matrix([[3/5. - lam, 2/5., 2/5.], [1/5., 3/5. - lam, 1/5.],
    [1/5., 0., 2/5. - lam]])
sp.init_printing(use_latex=True)
sp.factor(sp.det(A))

# finding the eigenvalues directly
A = sp.Matrix([[3/5., 2/5., 2/5.], [1/5., 3/5., 1/5.], [1/5., 0.,
    2/5.]])
eigenvalues = A.eigenvals()
```

Eigenvalues/Eigenvectors Example (3/7)

$$\lambda_1 = 1 : \begin{pmatrix} -2 & 2 & 2 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The system must be singular, choose $z = 1$

$$\left.\begin{array}{lcl} \text{eq. (3)} & \Longrightarrow & x = 3 \\ \text{eq. (2)} & \Longrightarrow & y = 2 \end{array}\right\} \Longrightarrow \mathbf{ev}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Eigenvalues/Eigenvectors Example (4/7)

$$\lambda_2 = \frac{2}{5} : \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choose $z = 1$

$$\left.\begin{array}{r} \text{eq. (3)} \implies x = 0 \\ \text{eq. (2)} \implies y = -1 \end{array}\right\} \implies \mathbf{ev}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Eigenvalues/Eigenvectors Example (5/7)

$$\lambda_3 = \frac{1}{5} : \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choose $z = 1$

$$\left.\begin{array}{l} \text{eq. (3)} \implies x = -1 \\ \text{eq. (2)} \implies y = 0 \end{array}\right\} \implies \mathbf{ev}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvalues/Eigenvectors Example (6/7)

Finding the eigenvectors using Python

```
import numpy as np
A = np.array([[3/5., 2/5., 2/5.], [1/5., 3/5., 1/5.], [1/5., 0.,
    2/5.]])
eigenvalues, eigenvectors = np.linalg.eig(A)
```

We could also use symbolic Python to achieve the same goal

```
import sympy as sp
A = sp.Matrix([[3/5., 2/5., 2/5.], [1/5., 3/5., 1/5.], [1/5., 0.,
    2/5.]])
eigenvectors = A.eigenvects()
```

Eigenvalues/Eigenvectors Example (7/7)

$\mathbf{ev}_1, \mathbf{ev}_2, \mathbf{ev}_3$ are linearly independent, so for every

$$\mathbf{x}_0 = \alpha \mathbf{ev}_1 + \beta \mathbf{ev}_2 + \gamma \mathbf{ev}_3$$

$$T\mathbf{x}_0 = \alpha \mathbf{ev}_1 + \frac{2}{5}\beta \mathbf{ev}_2 + \frac{1}{5}\gamma \mathbf{ev}_3$$

$$\implies \mathbf{x}_n = T^n \mathbf{x}_0 = \alpha \mathbf{ev}_1 + \frac{2}{5}^n \beta \mathbf{ev}_2 + \frac{1}{5}^n \gamma \mathbf{ev}_3$$

$$\implies \frac{1}{6}\mathbf{ev}_1 \text{ is a fixpoint } \left(\frac{1}{6} \text{ because the length must be } 1(!)\right)$$

$$\begin{pmatrix} \frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{9}{30} + \frac{4}{30} + \frac{2}{30} \\ \frac{3}{30} + \frac{6}{30} + \frac{1}{30} \\ \frac{3}{30} + \frac{2}{30} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

Fibonacci Numbers - Example (1/5)

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13...$ has the generating sequence $F_{k+2} = F_{k+1} + F_k$

Question: $F_{100} = ?$

$$\mathbf{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \implies \begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix} \implies \mathbf{u}_{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_k$$

$$\mathbf{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_{99} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \mathbf{u}_{98} = ... = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{100} \mathbf{u}_0$$

Fibonacci Numbers - Example (2/5)

Now problem solving is easy:

$$A^{100}\mathbf{x} = A^{99}A\mathbf{x} = A^{99}\lambda\mathbf{x} = \ldots = \lambda^{100}\mathbf{x}$$

so

1. Determine the eigenvalues
2. Determine the eigenvectors

Fibonacci Numbers - Example (3/5)

Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\left(\lambda - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\implies \lambda_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

Fibonacci Numbers - Example (4/5)

$A\mathbf{x} = \lambda\mathbf{x}$, so solve $(A - \lambda I)\mathbf{x} = 0$ for $\mathbf{x}$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is solvable because $A - \lambda I$ is not full rank $(\det(A - \lambda I) = 0)$

Guess $x_1 = \lambda_1$, find $x_2 = 1$, so the eigenvector is $\mathbf{x}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Guess $x_1 = \lambda_2$, find $x_2 = 1$, so the eigenvector is $\mathbf{x}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

Fibonacci Numbers - Example (5/5)

Determine $\mathbf{u}_0$ in terms of $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2}\left(\begin{pmatrix}\lambda_1\\1\end{pmatrix} - \begin{pmatrix}\lambda_2\\1\end{pmatrix}\right)$$

$$\begin{aligned}
\mathbf{u}_0 &= \frac{1}{\lambda_1 - \lambda_2}(\mathbf{x}_1 - \mathbf{x}_2)\\
\mathbf{u}_{100} &= A^{100}\mathbf{u}_0 = A^{100}\left(\frac{1}{\lambda_1 - \lambda_2}(\mathbf{x}_1 - \mathbf{x}_2)\right)\\
&= \frac{1}{\lambda_1 - \lambda_2}\left(A^{100}\mathbf{x}_1 - A^{100}\mathbf{x}_2\right)\\
&= \frac{\lambda_1^{100}\mathbf{x}_1 - \lambda_2^{100}\mathbf{x}_2}{\lambda_1 - \lambda_2}
\end{aligned}$$

$$F_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right] \approx 3.54 \cdot 10^{20}$$

Finding Fixpoints - Example (1/4)

Find all fixpoints of a reflection map along a line

Try to find all lines which are kept stable when reflected:

$$A = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

A stable line can be characterised by an equation like

$$A\mathbf{x} = \lambda\mathbf{x}$$

where λ is either a real or a complex number

Here, λ and $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ are unknown

Finding Fixpoints - Example (2/4)

$$A = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

Calculate the eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{aligned} &(\cos\alpha - \lambda)(-\cos\alpha - \lambda) - \sin\alpha\sin\alpha = 0 \\ &\implies -\cos^2\alpha - \lambda\cos\alpha + \lambda\cos\alpha + \lambda^2 - \sin^2\alpha = 0 \\ &\implies \lambda^2 = 1 \\ &\implies \lambda = \pm 1 \end{aligned}$$

Finding Fixpoints - Example (3/4)

Calculate the eigenvectors

$\lambda = 1$

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Set $x_2 = 1$

$$\Longrightarrow \; x_1(\cos\alpha - 1) + \sin\alpha \cdot 1 = 0$$

$$\Longrightarrow \; x_1 = \frac{-\sin\alpha}{\cos\alpha - 1}$$

Check: $x_1 \sin\alpha - (\cos\alpha + 1) \cdot 1 = 0$ is OK since

$$\frac{-\sin^2\alpha}{\cos\alpha - 1} - \frac{(\cos\alpha + 1)(\cos\alpha - 1)}{\cos\alpha - 1} = \frac{-1+1}{\cos\alpha - 1} = 0$$

Finding Fixpoints - Example (4/4)

$\lambda = -1$

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Set $x_2 = 1$

$$\implies x_1(\cos\alpha - 1) + \sin\alpha \cdot 1 = 0$$

$$\implies x_1 = \frac{-\sin\alpha}{\cos\alpha + 1}$$

Check: $x_1 \sin\alpha - (\cos\alpha + 1) \cdot 1 = 0$ is OK since

$$\frac{-\sin^2\alpha}{\cos\alpha + 1} - \frac{(\cos\alpha + 1)(\cos\alpha - 1)}{\cos\alpha + 1} = \frac{-1+1}{\cos\alpha + 1} = 0$$

System of ODEs - Example (1/6)

Consider a system of ODEs with constant coefficients

$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= 2y_1 - 2y_3 \\ y_3' &= 4y_1 - 2y_2 - y_3 \end{aligned} \qquad A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{pmatrix}$$

System of ODEs - Example (2/6)

Solve the characteristic polynomial

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 2 & -\lambda & -1 \\ 4 & -2 & -1-\lambda \end{vmatrix} \begin{matrix} 1-\lambda & -2 \\ 2 & -\lambda \\ 4 & -2 \end{matrix}$$

$$\begin{aligned}
&= (1-\lambda)(-\lambda)(-1-\lambda) + (-2 \cdot -1 \cdot 4) + (0 \cdot 2 \cdot 2) \\
&- (4 \cdot -\lambda \cdot 0) - (-2 \cdot -1) \cdot (1-\lambda)) - ((-1-\lambda) \cdot 2 \cdot 2) \\
&= (1-\lambda)(-\lambda)(-1-\lambda) + 8 - 2(1-\lambda) - 4(1+\lambda) \\
&= \lambda(1-\lambda)(1+\lambda) + 2(1-\lambda) \\
&= (1-\lambda)(\lambda^2 + \lambda + 2) = 0
\end{aligned}$$

Taking $\alpha = \frac{\sqrt{7}}{2}$

$$\lambda_1 = -\frac{-1}{2} + i\alpha,\ \lambda_2 = -\frac{1}{2} - i\alpha,\ \lambda_3 = 1$$

System of ODEs - Example (3/6)

Calculate the eigenvectors

$$\begin{pmatrix} \frac{3}{2} - i\alpha & -2 & 0 \\ 2 & \frac{1}{2} - i\alpha & -1 \\ 4 & -2 & -\frac{1}{2} - i\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Must be singular, choose cleverly $x = \frac{3}{2} + i\alpha$

$$(1) \implies \left(\frac{3}{2} - i\alpha\right) \cdot x = \frac{9}{4} + \frac{7}{4} = 4 \implies y = 2$$

$$(2) \implies 3 + i \cdot \sqrt{7} + 1 - i \cdot \sqrt{7} - z = 0 \implies z = 4$$

System of ODEs - Example (4/6)

$$\mathbf{e}_1 = \begin{pmatrix} \frac{3}{2} + i\alpha \\ 2 \\ 4 \end{pmatrix} \text{ is a complex eigenvector for } \lambda = -\frac{1}{2} + i\alpha$$

$$\mathbf{e}_2 = \begin{pmatrix} \frac{3}{2} - i\alpha \\ 2 \\ 4 \end{pmatrix} \text{ is a complex eigenvector for } \lambda = -\frac{1}{2} - i\alpha$$

$$\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ is a real eigenvector for } \lambda = 1$$

System of ODEs - Example (5/6)

Complex solutions

$$\mathbf{y}_{1,2}(t) = \begin{pmatrix} \frac{3}{2} \pm i\alpha \\ 2 \\ 4 \end{pmatrix} \cdot e^{\left(-\frac{1}{2} \pm i\alpha\right)t}$$

Real solution

$$\mathbf{y}_3(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot e^{t}$$

From a complex solution, get two real solutions by taking the real and imaginary parts

System of ODEs - Example (6/6)

$$\begin{pmatrix} \frac{3}{2} \pm i\alpha \\ 2 \\ 4 \end{pmatrix} \cdot e^{\left(-\frac{1}{2} \pm i\alpha\right)t} = \left(e^{-\frac{1}{2}t} \cdot e^{i\alpha t}\right) \left(\begin{pmatrix} \frac{3}{2} \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} i\alpha \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \left(e^{-\frac{1}{2}t} \cdot (\cos\alpha t + i \sin\alpha t)\right) \left(\begin{pmatrix} \frac{3}{2} \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} i\alpha \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\implies e^{-\frac{1}{2}t} \left(\begin{pmatrix} \frac{3}{2} \\ 2 \\ 4 \end{pmatrix} \cos\alpha t - \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \sin\alpha t \right)$$

$$\implies e^{-\frac{1}{2}t} \left(\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \cos\alpha t + \begin{pmatrix} \frac{3}{2} \\ 2 \\ 4 \end{pmatrix} \sin\alpha t \right)$$

Matrix norms

Condition number

Rotations are norm preserving

Eigenvalues and eigenvectors

$\rightarrow$ Spectral norm of $A ==$ max. eigenvalue of $A^T A$ (aka singular value)

Spectral Norm == Maximum Singular Value

$$\|A\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} < A\mathbf{x}, A\mathbf{x} >= \max_{\|\mathbf{x}\|_2=1} < A^T A\mathbf{x}, \mathbf{x} >$$

Since $A^T A$ is positive semidefinite, there is a “rotation” U that diagonalizes it, i.e. $U^T A^T A U = D$. Here, D is given by

$$\begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix}$$

where μ_i are the eigenvalues of $A^T A$. $\sigma_i = \sqrt{\mu_i}$ are called the **singular values** of A.

$$\begin{aligned} \|A\|_2^2 &= \max_{\|\mathbf{z}\|_2=1} < A^T A\mathbf{z}, \mathbf{z} >= \max_{\|U\mathbf{y}\|_2=1} < A^T A U\mathbf{y}, U\mathbf{y} > \\ &= \max_{\|\mathbf{y}\|_2=1} < D\mathbf{y}, \mathbf{y} >= \max_{\|\mathbf{y}\|_2=1} \left(\mu_1 |y_1|^2 + \ldots + \mu_n |y_n|^2\right) = \mu_{\max} \\ &\implies \|A\|_2 = \sqrt{\mu_{\max}} = \sigma_{\max} \end{aligned}$$