Galois Deformations

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1 Review of Category Theory and Homological Algebra

All the set-theoreric issues are ignored for now.

1.1 Representability

Let $\mathfrak{C}$ be a category. We define the functors[1]

$$\begin{aligned} h^{\mathfrak{C}} : \mathfrak{C}^{\mathrm{op}} &\longrightarrow [\mathfrak{C}, \mathbf{Set}], & \mathrm{ev}^{\mathfrak{C}} : [\mathfrak{C}, \mathbf{Set}] \times \mathfrak{C} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \mathrm{Hom}_{\mathfrak{C}}(S, \cdot) & (F, S) &\longmapsto F(S). \end{aligned}$$

Theorem 1 (Yoneda)**.** There is an isomorphism

$$\mathrm{Hom}_{[\mathfrak{C},\mathbf{Set}]^{\mathrm{op}}}(-, h^{\mathfrak{C}}(-)) \simeq \mathrm{ev}^{\mathfrak{C}}$$

as functors $[\mathfrak{C}, \mathbf{Set}] \times \mathfrak{C} \to \mathbf{Set}$ given by

$$\begin{aligned} \mathrm{Hom}_{[\mathfrak{C},\mathbf{Set}]^{\mathrm{op}}}\left(F, h^{\mathfrak{C}}(S)\right) &\longrightarrow F(S) \\ \left(F \xleftarrow{\phi} \mathrm{Hom}_{\mathfrak{C}}(S, -)\right) &\longmapsto \phi_S(\mathrm{id}_S) \end{aligned}$$

for all $F : \mathfrak{C} \to \mathbf{Set}$ and $S \in \mathfrak{C}$, and the functor $h^{\mathfrak{C}} : \mathfrak{C}^{\mathrm{op}} \to [\mathfrak{C}, \mathbf{Set}]$ is fully faithful.

We say that a functor $F : \mathfrak{C} \to \mathbf{Set}$ is **representable**, if there is $X \in \mathfrak{C}$ along with an isomorphism

$$\phi : \mathrm{Hom}_{\mathfrak{C}}(X, -) \simeq F$$

as functors. Note that the functor ϕ is determined[2] by the universal element $u := \phi_X(\mathrm{id}_X) \in F(X)$, from which every thing in $F(T)$ is pushed forward, i.e. for any morphism $f : X \to T$ in $\mathfrak{C}$, the unique corresponding element in $F(T)$ is $\phi_T(f) = F(f)(\phi_X(\mathrm{id}_X)) = F(f)(u)$.

1.2 The Ext Functors

Let $\mathfrak{A}$ be an abelian category with enough projective and injective objects. We have

$$\mathrm{Ext}^i_{\mathfrak{A}}(X, Y) := \mathrm{R}^i \mathrm{Hom}_{\mathfrak{A}}(X, -)(Y) \simeq \mathrm{R}^i \mathrm{Hom}_{\mathfrak{A}}(-, Y)(X)$$

for $X, Y \in \mathfrak{A}$, $i \geq 0$.

[1] There is also the version for $h_{\mathfrak{C}} : \mathfrak{C} \to [\mathfrak{C}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathrm{ev}_{\mathfrak{C}} : [\mathfrak{C}^{\mathrm{op}}, \mathbf{Set}] \times \mathfrak{C} \to \mathbf{Set}$.

[2] This does *not* mean that we can decode ϕ from u without knowing ϕ a priori?

We will focus on Ext^1. An **extension of A by B**[3] is a short exact sequence

$$\xi : 0 \to B \to X \to A \to 0.$$

(I may denote ξ by X if there is no confusion.) An isomorphism of two extensions X and X' of A by B is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\simeq} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

An extension of A by B that is isomorphic to

$$0 \to B \hookrightarrow A \oplus B \to A \to 0$$

is said to be split.

Given an extension $\xi : 0 \to B \to X \to A \to 0$ of A by B, the cohomological functors $\mathrm{Ext}^*(A, -)$ induces the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \xrightarrow{\partial_\xi} \mathrm{Ext}^1(A, B).$$

Let's look at the class $\Theta(\xi) := \partial_\xi(\mathrm{id}_A) \in \mathrm{Ext}^1(A, B) = 0$. If $\Theta(\xi) = 0$, then there is a section $f : A \to X$ of $X \to A$ in ξ, i.e. ξ is split. This means that $\Theta(\xi) \in \mathrm{Ext}^1(A, B)$ is the *obstruction* for ξ to be split.

Theorem 2. Let R be a (possibly non-commutative) ring. For left R-modules A and B, there is a natural bijection

$$\left\{\begin{matrix} \text{isomorphism classes of} \\ \text{extensions of } A \text{ by } B \end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \mathrm{Ext}^1_R(A, B)$$

given by $\Theta : \xi \mapsto \partial_\xi(\mathrm{id}_A)$.

Example 1.1. Let k be a topological ring (field if necessary), G be a topological group, V be a continuous $k[G]$-module that is free of k-rank d. Then there is a canonical isomorphism

$$\mathrm{Ext}^1_{k[G]}(V, V) \simeq H^1(G, \mathrm{ad}\, V).$$

(There should be a constructive proof, but I failed...)

We propose another proof in the next subsection.

1.3 Universal δ-Functors

We concentrate on cohomological things.

Definition 1. A (covariant) **cohomological δ-functor** is a collection of additive functors

$$\{T^n : \mathfrak{A} \to \mathfrak{B}\}_{n \geq 0}$$

indexed by non-negative integers, which induces *functorially* a long exact sequences in $\mathfrak{B}$ from a short exact sequence in $\mathfrak{A}$. More precisely, for each exact sequence

$$0 \to A \to B \to C \to 0 \quad \text{in } \mathfrak{A},$$

there are fixed morphisms

$$\delta^n : T^n(C) \to T^{n+1}(A) \quad \text{in } \mathfrak{B}, \quad n \geq 0,$$

[3] In a category where these operations make sense.

s.t.

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta^0} T^1(A) \to \cdots$$

is exact[4]; moreover, a morphism of short exact sequences in $\mathfrak{A}$ induces a morphism of long exact sequences in $\mathfrak{B}$.

For instance, taking cohomology for chain complexes

$$H^* : \mathbf{Ch}_{\geq 0}(\mathfrak{A}) \to \mathfrak{A}$$

or taking right-derivation of a left-exact functor are cohomological δ-functors.

Definition 2. The cohomological δ-functors from $\mathfrak{A}$ to $\mathfrak{B}$ form a category, where morphisms are the natural tranformations commuting with the δ^n's. A **universal cohomological δ-functor** is a δ-functor $T = (T^n)$, such that for any δ-functor $S = (S^n)$ and a morphism $f^0 : T \to S$, there is a unique morphism $f : T \to S$ extending f^0.

So a universal δ-functor is like an initial object among δ-functors but it is "weaker".

Theorem 3. If $F : \mathfrak{A} \to \mathfrak{B}$ is a left-exact additive functor, then (if $\mathfrak{A}$ has enough injectives) the right derivations $R^*F : \mathfrak{A} \to \mathfrak{B}$ form a universal δ-functor.

Another proof of Example 1.1. Let k be a field. We show that both $H^*(G, V^\vee \otimes_k (-))$ and $\mathrm{Ext}^*_G(V, -)$ are universal δ-functors. Then since they agree at $i = 0$, they must agree everywhere.

The functors $\mathrm{Ext}^*_G(V, -)$ are derived from $\mathrm{Hom}_G(V, -)$, so they are universal. For $H^*(G, V^\vee \otimes_k (-))$, since $V^\vee \otimes_k (-)$ is exact, we have[5]

$$H^*(G, V^\vee \otimes_k (-)) = \mathrm{R}^* \mathrm{Hom}_G(k, -) \circ (V^\vee \otimes_k (-)) = R^* \left(\mathrm{Hom}_G(k, -) \circ (V^\vee \otimes_k (-))\right),$$

which is also a derived functor. $\square$

2 Deformation of Representations of Profinite Groups

2.1 The category of complete Noetherian algebras

Let $L/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}$, so that its ring of integers $\mathcal{O} := \mathcal{O}_L$ contains the ring of Witt vectors $W(\mathbb{F})$ of $\mathbb{F}$. We consider the category $\widehat{\mathfrak{Ar}_{\mathcal{O}}}$ of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$.

For $A \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$, we call

$$t_A^\vee := \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}} A)$$

the **Zariski** (or **relative**) **cotangent space of A over $\mathcal{O}$**, and

$$t_A := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}), \mathbb{F})$$

the **relative tangent space of A over $\mathcal{O}$**.

[4] In particular, T^0 is left-exact.

[5] I've never learnt this but I accept this for now.

Proposition 2.1. For any homomorphism $\mathcal{O} \to A$ of local rings (no need for completeness) with the same residue field $\mathbb{F}$, there is a perfect pairing

$$\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}} A) \times \operatorname{Der}_{\mathcal{O}}(A, \mathbb{F}) \to \mathbb{F}$$

of k-vector spaces, where[6]

$$\operatorname{Der}_{\mathcal{O}}(A, \mathbb{F}) = \{\delta \in \operatorname{Hom}_{\mathcal{O}}(A, \mathbb{F}) \mid \delta(ab) = \bar{a}\delta(b) + \bar{b}\delta(a),\ \forall a, b \in A\}$$

is the ring of $\mathcal{O}$-linear derivations from A to k. In particular,

$$\operatorname{Hom}_{\mathbb{F}}(\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}}), \mathbb{F}) \simeq \operatorname{Der}_{\mathcal{O}}(A, \mathbb{F}).$$

Before proving, note that $A = \operatorname{im}(\mathcal{O} \to A) + \mathfrak{m}_A$, and A is generated by $\operatorname{im}(\mathcal{O} \to A)$ and $\mathfrak{m}_A$ as an $\mathcal{O}$-module, because $A/\mathfrak{m}_A = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$.

Proof. Consider the natural pairing $\mathfrak{m}_A \times \operatorname{Der}_{\mathcal{O}}(A, \mathbb{F}) \to \mathbb{F}$ given by evaluation.
Right kernel. If $\delta(a) = 0$ for all $a \in \mathfrak{m}_A$, then $\delta \in \operatorname{Der}_{\mathcal{O}}(\mathbb{F}, \mathbb{F}) = 0$.
Left kernel. Both $\mathfrak{m}_A$ and $\mathfrak{m}_{\mathcal{O}}$ acts by 0 on $\mathbb{F}$, so any $\mathcal{O}$-linear derivation $\delta : A \to \mathbb{F}$ (T.B.C.) ☐

Lemma 2.1. A morphism $A \to B$ in $\widehat{\mathfrak{Ar}_{\mathcal{O}}}$ is surjective if and only if the induced map $t_A^* \to t_B^*$ is surjective.

Proof. For any morphism $A \to B$ in $\widehat{\mathfrak{Ar}_{\mathcal{O}}}$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}_{\mathcal{O}} A/(\mathfrak{m}_{\mathcal{O}} A \cap \mathfrak{m}_A^2) \arrow[r] \arrow[d] & \mathfrak{m}_A/\mathfrak{m}_A^2 \arrow[r] \arrow[d] & \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}} A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}_{\mathcal{O}} B/(\mathfrak{m}_{\mathcal{O}} B \cap \mathfrak{m}_B^2) \arrow[r] & \mathfrak{m}_B/\mathfrak{m}_B^2 \arrow[r] & \mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_{\mathcal{O}} B) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. The left column is surjective, because by the comments above,

(some commutative algebra...) ☐

2.2 Deformation functors

Let G be a profinite group, $\mathbb{F}$ be a finite field of characteristic p, V an $\mathbb{F}[G]$-module of $\mathbb{F}$-dimension d with G acting continuously[7]. We fix a $\mathbb{F}$-basis $\beta_{\mathbb{F}}$ of V, via which V is identified with a continuous representation $\bar{\rho} : G \to \mathrm{GL}_d(\mathbb{F})$.

Take $A \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$. A **deformation** of V to A is a pair (V_A, ι_A), where

- V_A is an $A[G]$-module that is free of finite rank over A, and
- $\iota_A : V_A \otimes_A \mathbb{F} \simeq V$ is an isomorphism of $\mathbb{F}[G]$-modules.

A **framed deformation** of (V, β) is a triple (V, ι_A, β_A), where

- (V, ι_A) is a deformation of V to A,

[6] For $a \in A$, denote by $\bar{a} \in k$ the residue class.
[7] This means that the map

$$G \times V \to V \quad (g, v) \mapsto gv$$

is continuous; or equivalently, $G \to \mathrm{GL}(V)$ is continuous.

- β_A is a basis of V_A over A that reduces to β_F via ι_A.

Define $D_V(A)$ (resp. $D_V^\square(A)$) to the set of isomorphism classes of deformations (resp. framed deformations) of V to A.

Remark. If we view (V, β) as the representation $\bar{\rho} : G \to \mathrm{GL}_d(V)$, then a framed deformation (V_A, ι_A, β_A) is a representation $\rho_A : G \to \mathrm{GL}_d(A)$ lifting $\bar{\rho}$, namely $G \overset{\rho_A}{\to} \mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})$ is exactly $\bar{\rho}$, and two framed deformations are isomorphic if the are the same representation $G \to \mathrm{GL}_d(A)$. Forgetting the basis, we see that two deformations are isomorphic if they are, as representations, conjugate by some element in $\ker\left(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F})\right)$

2.3 Representability

A profinite group G satisfies the **Mazur's finiteness condition** Φ_p, if for every *open* subgroup $G' \subset G$, the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{gp}}(G', \mathbb{F}_p)$ of continuous group homomorphisms is finite.

Theorem 4 (Mazur)**.** Assume that G satisfies condition Φ_p.

(a) $D_V^\square$ is representable by an $R_V^\square \in \widehat{\mathfrak{Ar}_\mathcal{O}}$.

(b) If Schur's lemma $\mathrm{End}_{\mathbb{F}[G]}(V) = \mathbb{F}$ is true, then D_V is representable by an $R_V \in \widehat{\mathfrak{Ar}_\mathcal{O}}$.

2.3.1 Construction of $R_V^\square$

We are looking for a universal representation $\rho^\square : G \to \mathrm{GL}_d\left(R_V^\square\right)$ in the sense that for any lift $\rho_A : G \to \mathrm{GL}_d(A)$ of $\bar{\rho}$ with $A \in \widehat{\mathfrak{Ar}_\mathcal{O}}$, there is a morphism $R_V^\square \to A$ s.t. $G \overset{\rho^\square}{\to} \mathrm{GL}_d\left(R_V^\square\right) \to \mathrm{GL}_d(A)$ equals ρ_A.

Suppose first that G is finite with presentation

$$G = \langle g_1, \ldots, g_s \mid r_1(g_1, \ldots, g_s) = \cdots = r_t(g_1, \ldots, g_s) \rangle .$$

Let

$$\mathcal{R} := \mathcal{O}\left[\left\{X_{ij}^k\right\}_{1 \le i,j \le d}^{1 \le k \le s}\right] \Big/ \mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by all *entries* of the matrices

$$r_l(X^1, \ldots, X^k) - \mathrm{id}, \quad X^k = \left(X_{ij}^k\right)_{i,j}, \ 1 \le k \le s, \ 1 \le l \le t.$$

Consider the kernel $\mathcal{J}$ of the homomorphism

$$\mathcal{R} \to \mathbb{F} \quad X_{ij}^k \mapsto \text{the } (i,j)\text{-entry of } \bar{\rho}(g_k)$$

and define $R_V^\square := \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$. Define $\rho^\square : G \to \mathrm{GL}_d(R_V^\square)$ by $\rho^\square(g_k) := X^k$.

- $R_V^\square$ is complete, and it is local (with residue field $\mathbb{F}$) because $\mathcal{J}$ is a maximal ideal.
- $\rho^\square$ is well-defined, because

$$\begin{tikzcd} & \mathrm{GL}_d\left(R_V^\square\right) \arrow[r, "\det"] \arrow[d] & R_V^\square \arrow[d] \\ G \arrow[ur, "\rho^\square"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_d(\mathbb{F}) \arrow[r, "\det"] & \mathbb{F} \end{tikzcd}$$

commutes, and $R_V^\square$ is local.

- One checks that $R_V^\square$ represents $D_V^\square$ and $\rho^\square$ is the **universal framed deformation** of $\bar\rho$ (if $R_V^\square$ is Noetherian).

In the general case of G being profinite, we write $G = \varprojlim_i G/H_i$ with $H_i \subset \ker\bar\rho$ open and normal in G and consider the universal lifts (R_i, ρ_i) of the representations $G/H_i \to \mathbb{F}$ from $\bar\rho$. For $G/H_i \to G/H_j$, the universality of ρ_i provides the dotted arrow in the commutative diagram

$$\begin{tikzcd} G/H_i \arrow[r, "\rho_i"] \arrow[d] & \mathrm{GL}_d(R_i) \arrow[d, dashed] \\ G/H_j \arrow[r, "\rho_j"] & \mathrm{GL}_d(R_j) \end{tikzcd}$$

Therefore we obtain $(R_V^\square, \rho^\square) := \varprojlim_i (R_i, \rho_i)$.

Note that this contruction requires not the condition Φ_p, which is needed to make $R_V^\square$ Noetherian.

2.3.2 The tangent space

Let $\mathbb{F}[\varepsilon] := \mathbb{F}[X]/(X^2)$, which is called the ring of **dual numbers**. For a functor $D : \widehat{\mathfrak{Ar}_{\mathcal{O}}} \to \mathbf{Set}$ sending the terminal object $\mathbb{F}$ to the terminal object $D(\mathbb{F}) = \{\bullet\}$, we call the set $t_D := D(\mathbb{F}[\varepsilon])$ the **Zariski tangent space** of D. If there is a fixed bijection $D(\mathbb{F}[\varepsilon] \oplus \mathbb{F}[\varepsilon]) \simeq D(\mathbb{F}[\varepsilon]) \times D(\mathbb{F}[\varepsilon])$, we equip t_D with the $\mathbb{F}$-linear structure give by this bijection.

- Assume that $D : \widehat{\mathfrak{Ar}_{\mathcal{O}}} \to \mathbf{Set}$ is representable by $R \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$. Then the tangent space

$$t_D \simeq \mathrm{Hom}_{\mathcal{O}}(R, \mathbb{F}[\varepsilon]) \simeq \mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}), \mathbb{F}[\varepsilon]) = t_R$$

  is the Zariski or relative tangent space of R over $\mathcal{O}$. (what is the last isomorphism (if there is one...)?)

Define $\mathrm{ad}\, V := \mathrm{End}_{\mathbb{F}}(V) \simeq V^\vee \otimes_{\mathbb{F}} V$ with the standard G-module structure $\mathrm{ad}\,\bar\rho = \bar\rho^\vee \otimes \bar\rho$.

Lemma 2.2. There are canonical isomorphisms[8]

$$D_V(\mathbb{F}[\varepsilon]) \simeq \mathrm{Ext}^1_{\mathbb{F}[G]}(V, V) \simeq H^1(G, \mathrm{ad}\, V).$$

Proof. (1) Given an extension

$$0 \longrightarrow V \xrightarrow{i} W \xrightarrow{\pi} V \longrightarrow 0$$

of $\mathbb{F}[G]$-modules, we define the $\mathbb{F}[G]$-linear action of ε on W by $\varepsilon|_W := i \circ \pi$, which endows W with an $\mathbb{F}[\varepsilon][G]$-module structure and an isomorphism

$$W \otimes_{\mathbb{F}[\varepsilon]} \mathbb{F} = W/\varepsilon W = W/i(V) \overset{\pi}{\simeq} V.$$

Conversely, for a deformation (W, ι) of V to $\mathbb{F}[\varepsilon]$, we get an extension of V by itself

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "\iota"'] & W \arrow[r] \arrow[d, equal] \arrow[dr, two heads] & V \arrow[r] & 0 \\ & W \otimes_{\mathbb{F}[\varepsilon]} \mathbb{F} \arrow[r, hook] & W \otimes_{\mathbb{F}[\varepsilon]} \mathbb{F}[\varepsilon] & W/\varepsilon W \simeq W \otimes_{\mathbb{F}[\varepsilon]} \mathbb{F} \arrow[u, "\iota"'] & \end{tikzcd}$$

as $\mathbb{F}[G]$-modules.[9] The first isomophism is thereby established.

[8] In Ext^1, we consider *continuous* extension classes.

[9] The fact $W \simeq V \oplus V$ as $\mathbb{F}[G]$-modules doesn't mean that the extension split.

(2) The second isomophism is a general fact that we have extracted as Example 1.1. ☐

We use the abbreviation $h^i(\cdots) := \dim_{\mathbb{F}} H^i(\cdots)$.

Lemma 2.3. If G satisfies condition Φ_p, then $D_V(\mathbb{F}[\varepsilon])$ is a finite dimensional $\mathbb{F}$-vector space, and

$$\dim_{\mathbb{F}} D_V^{\square}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_V(\mathbb{F}[\varepsilon]) + d^2 - h^0(G, \operatorname{ad} V)$$

is also finite.

Proof. Let $G' := \ker(G \to \mathrm{GL}(V))$. Since G acts continuously, G' is an open normal subgroup of G. Consider the inflation-restriction exact sequence

$$0 \to H^1(G/G', \operatorname{ad} V) \to H^1(G, \operatorname{ad} V) \to H^1(G', \operatorname{ad} V)^{G/G'}.$$

The left term is obviously finite. For the right term, G' acts trivially, so[10]

$$H^1(G', \operatorname{ad} V) = \operatorname{Hom}_{\mathrm{gp}}(G', \operatorname{ad} V) \simeq \operatorname{Hom}_{\mathrm{gp}}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \operatorname{ad} V$$

is finite by condition Φ_p. Therefore $\dim_{\mathbb{F}} D_V(\mathbb{F}[\varepsilon]) = h^1(G, \operatorname{ad} V) < \infty$.

(Do the equation later.) ☐

Lemma 2.4. Let A be a complete local $\mathcal{O}$-algebra with residue field $\mathbb{F}$. If $\{\alpha_i\}_{i \in I} \subset \mathfrak{m}_A$ generates the relative cotangent space $t_A^{\vee} = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}})$ of A over $\mathcal{O}$ as an $\mathcal{O}$-module, then the homomorphism

$$\mathcal{O}[\![X_i \mid i \in I]\!] \to A \quad X_i \mapsto \alpha_i$$

is surjective.

Proof. Cannot use Lemma 2.1 because Noetherianity of A is the goal! ☐

Corollary 2.1. The ring $R_V^{\square}$ is Noetherian if $H^1(G, \operatorname{ad} V)$ is $\mathbb{F}$-finite-dimensional.

Proof. Combine the lemmata above. ☐

This completes the proof of Theorem 4 (a).

2.3.3 Quotient by group action and the representability of D_V

Result is $\operatorname{Spf} R_V = \operatorname{Spf} R_V^{\square} \Big/ \widehat{\mathrm{PGL}_d}$.

2.3.4 Presentation of the universal deformation ring R_V

3 Taylor-Wiles Patching

Keep the notations $\mathcal{O} = \mathcal{O}_L$ for $L/\mathbb{Q}_p$, and let $k = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ and $\varpi \in \mathcal{O}$ be a uniformizer.

Fix a continuous absolutely irreducible modular representation $\rho : \mathrm{Gal}_{\mathbb{Q}, \{p, \infty\}} \to \mathrm{GL}_2(k)$ with determinant $\bar{\varepsilon}^{-1}$.

[10]We have

$$\operatorname{Hom}_{\mathrm{gp}}(G, V) \simeq \operatorname{Hom}_{\mathrm{gp}}(G, k) \otimes_k V$$

for any group G and any *finite* dimensional vector space V over a field k.