Notes on Modular Forms

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February 20, 2025

1 Hecke Operators

Thoughout this section, we fix the following data:

- a group Ω;
- a submonoid $\Delta \subset \Omega$;
- a nonempty collection $\mathscr{X}$ of subgroups of Ω, in which all members are commensurable[1] to each other, and
$$\Gamma \subset \Delta \subset \tilde{\Gamma} := \left\{ g \in \Omega \mid g\Gamma g^{-1} \approx \Gamma \right\}, \ \forall \Gamma \in \mathscr{X};$$
- a commutative ring $\mathbb{K}$'
- a left $\mathbb{K}$-module M with a right Δ-action $(m, \delta) \mapsto m\delta$, i.e, a monoid homomorphism
$$\Delta \to \mathrm{End}_{\mathbb{K}}(M) \quad \delta \mapsto m \mapsto m\delta.$$

1.1 Commensurability

Recall that two subgroups $\Gamma, \Gamma' < \Omega$ are commensurable if both $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are finite, and this is an equivalence relation.

Lemma 1.1. $\tilde{\Gamma}$ is a group and depends only on the commensurable class of Γ. □

Proposition 1.1. Let $\alpha \in \tilde{\Gamma}$ and $\Gamma \approx \Gamma'$. Then there is a bijection

$$\begin{aligned} \Gamma' \cap (\alpha^{-1}\Gamma\alpha)\backslash\Gamma' &\longleftrightarrow \Gamma\backslash\Gamma\alpha\Gamma' \\ \Gamma''^{2} x &\longleftrightarrow \Gamma\alpha x \end{aligned}$$

and $\Gamma\backslash\Gamma\alpha\Gamma'$ is finite.

Proof. The map
$$\Gamma' \to \Gamma\backslash\Gamma\alpha\Gamma' \quad x \mapsto \Gamma\alpha x$$
is clearly surjective. Now $\forall x, y \in \Gamma'$,
$$\begin{aligned} \Gamma\alpha x = \Gamma\alpha y &\iff \exists g \in \Gamma, g\alpha x = \alpha y \\ &\iff \exists g' \in \Gamma'', g' x = y; \end{aligned}$$
so injective.

By definitions and the last lemma, $\Gamma' \cap (\alpha^{-1}\Gamma\alpha) \approx \Gamma'$, giving finiteness. □

[1] Write $\Gamma \approx \Gamma'$ if Γ is commensurable to Γ'.

[2] Of course, $\Gamma'' = \Gamma' \cap (\alpha^{-1}\Gamma\alpha)$.

1.2 Double Coset Algebra

1.2.1 Double Cosets and Convolution

Recall that the $\mathbb{K}$-module $\mathcal{F}(\Omega, \mathbb{K})$ of all functions $\Omega \to \mathbb{K}$ admits a $\mathbb{K}$-linear left Ω-action

$$(\gamma f)(z) := f(\gamma^{-1} z)$$

and a right Ω-action

$$(f\gamma)(z) := f(z\gamma).$$

Def-Thm 1. Let $\Gamma, \Gamma' \in \mathscr{X}$. Define $\mathscr{H}(\Gamma\backslash\Delta/\Gamma')$ to be the $\mathbb{K}$-module[3] consists of functions $f : \Omega \to \mathbb{K}$ such that:

- $\operatorname{supp} f \subset \Delta$ and $\Gamma\backslash(\operatorname{supp} f)/\Gamma'$ is a finite set,
- f is left-Γ-invariant and right-Γ'-invariant.

Then $\mathscr{H}(\Gamma\backslash\Delta/\Gamma')$ is a *free* $\mathbb{K}$-module, with a basis given by the double cosets in $\Gamma\backslash\Delta/\Gamma'$, i.e.,

$$[\Gamma\gamma\Gamma'] := \mathbf{1}_{\Gamma\gamma\Gamma'},\ \gamma \in \Delta.$$

We thus identify $\mathscr{H}(\Gamma\backslash\Delta/\Gamma')$ with the free module $\mathbb{Z}[\Gamma\backslash\Delta/\Gamma']$ generated by $\Gamma\backslash\Delta/\Gamma'$, and we identify the function $[\Gamma\gamma\Gamma'] := \mathbf{1}_{\Gamma\gamma\Gamma'}$ with the double coset $\Gamma\gamma\Gamma'$.

Def-Thm 2 (Convolution)**.** Let $\Gamma, \Gamma', \Gamma'' \in \mathscr{X}$. We define an convolution operator

$$* : \mathscr{H}(\Gamma\backslash\Delta/\Gamma') \times \mathscr{H}(\Gamma'\backslash\Delta/\Gamma'') \to \mathscr{H}(\Gamma\backslash\Delta/\Gamma'')$$

via

$$(\alpha * \beta)(x) := \sum_{h \in \Gamma'\backslash\Omega} \alpha(xh^{-1})\beta(h) = \sum_{\Omega/\Gamma'} \alpha(h)\beta(h^{-1}x).$$

The above equation is well-defined and holds. Moreover,

- this convolution operator $*$ is *distributive* and *associative*,
- $1_\Gamma \in \mathscr{H}(\Gamma\backslash\Delta/\Gamma)$ is both a left and right *identity* for $*$.

In particular, the operator $*$ makes

$$\mathscr{H}_\Delta(\Gamma) := \mathscr{H}(\Gamma\backslash\Delta/\Gamma) = \mathbb{Z}[\Gamma\backslash\Delta/\Gamma]$$

a $\mathbb{K}$-algebra.

We then give a formula of $*$. For $\alpha, \beta, \gamma \in \Delta$, write

$$[\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma''] = \sum_{\gamma \in \Gamma\backslash\Delta/\Gamma''} m(\alpha, \beta; \gamma)[\Gamma\gamma\Gamma''].$$

Apply RHS to γ, one checks $\left([\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma'']\right)(\gamma) = m(\alpha, \beta; \gamma)$. To determine these quantities, write

$$\Gamma\alpha\Gamma' = \bigsqcup_{a \in A} \Gamma a,\ \Gamma'\beta\Gamma'' = \bigsqcup_{b \in B} \Gamma' b.$$

[3] A $\mathbb{K}$-submodule of $\mathcal{F}(\Omega, \mathbb{K})$

Then

$$\begin{aligned}
m(\alpha,\beta;\gamma) &= ([\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma''])\,(\gamma) \\
&= \sum_{h\in\Gamma'\backslash\Omega} [\Gamma\alpha\Gamma'](\gamma h^{-1}) \cdot [\Gamma'\beta\Gamma''](h) \\
&= \sum_{h\in\Gamma'\backslash(\Gamma'\beta\Gamma'')} [\Gamma\alpha\Gamma'](\gamma h^{-1}) = \sum_{b\in B} [\Gamma\alpha\Gamma'](\gamma b^{-1}).
\end{aligned}$$

Note that

$$[\Gamma\alpha\Gamma'](x) = \left\{\begin{matrix} 1, & \exists a \in A, x \in \Gamma a \\ 0, & \text{otherwise} \end{matrix}\right\} = \#\{a \in A \mid \Gamma x = \Gamma a\},$$

so

$$m(\alpha,\beta;\gamma) = \#\left\{(a,b) \in A \times B \mid \Gamma\gamma = \Gamma ab\right\}. \tag{1}$$

Considering right cosets rather than left cosets gives a similar formula.

The following is a useful result in computation.

Proposition 1.2. If $\alpha, \gamma \in \Delta$, and γ normalises Γ, then

$$[\Gamma\alpha\Gamma] * [\Gamma\gamma\Gamma] = [\Gamma\alpha\gamma\Gamma],$$

$$[\Gamma\gamma\Gamma] * [\Gamma\alpha\Gamma] = [\Gamma\gamma\alpha\Gamma].$$

Proof. Write $\Gamma\alpha\Gamma = \bigsqcup_{a\in A} \Gamma a$. As $\Gamma\gamma\Gamma = \Gamma\gamma$ and

$$\Gamma\alpha\gamma\Gamma = \Gamma\alpha\Gamma\gamma = \bigsqcup_{a\in A} \Gamma a\gamma,$$

the structure constants

$$m(\alpha,\gamma;\delta) = \#\left\{a \in A \mid \Gamma\delta = \Gamma a\gamma\right\} = \begin{cases} 1, & \delta \in \Gamma\alpha\gamma\Gamma, \\ 0, & \delta \notin \Gamma\alpha\gamma\Gamma. \end{cases}$$ □

1.2.2 Commutativity

An **anti-involution** of a monoid Δ is a map $\tau : \Delta \to \Delta$ s.t.

$$\tau(xy) = \tau(y)\tau(x), \quad \tau(1) = 1, \quad \tau^2 := \tau \circ \tau = \mathrm{id}\,.$$

Theorem 1. Let $\Gamma \in \mathscr{X}$. If there *exists* an anti-involution $\tau : \Delta \to \Delta$ that stabilises every double coset of Γ, then $\mathscr{H}_\Delta(\Gamma) = \mathscr{H}(\Gamma\backslash\Delta/\Gamma)$ is a commutative $\mathbb{K}$-algebra.

1.3 The Action of Double Coset Algebras

We consider the action of double cosets $\mathscr{H}(\Gamma\backslash\Delta/\Gamma')$ on

$$M^\Gamma = \{x \in M \mid x\gamma = x, \forall \gamma \in \Gamma\}.$$

Def-Thm 3. For $f \in \mathscr{H}(\Gamma\backslash\Delta/\Gamma')$, define

$$\begin{aligned}
\cdot f : M^\Gamma &\longrightarrow M^{\Gamma'} \\
x &\longmapsto xf := \sum_{\delta\in\Gamma\backslash\Delta} f(\delta) x\delta.
\end{aligned}$$

This action is well-defined. Moreover, it is comptatible with convolution.

- If $f \in \mathscr{H}(\Gamma\backslash\Delta/\Gamma')$, $f' \in \mathscr{H}(\Gamma'\backslash\Delta/\Gamma'')$, then $x(f * f') = (xf)f'$.
- In case $\Gamma' = \Gamma$, $x\mathbf{1}_\Gamma = x$.

In particular, M^Γ is a right $\mathscr{H}_\Delta(\Gamma)$-module, with the action of the basis $\{\Gamma\gamma\Gamma\}_{\gamma\in\Delta}$ given by

$$\Gamma\gamma\Gamma = \bigsqcup_{i=1}^n \Gamma\gamma_i \implies m[\Gamma\gamma\Gamma] = \sum_{i=1}^n m\gamma_i.$$

Corollary 1.1. If γ normalises Γ, then $m[\Gamma\gamma\Gamma] = m\gamma$. $\square$

2 Hecke Operators for $\Gamma_0(N)$ and $\Gamma_1(N)$

We specialise our discussion in the last section to the case of modular forms. Let

- $\Omega := \mathrm{GL}(2,\mathbb{Q})^+$,
- $\mathbb{K} := \mathbb{Z}$,
- $\mathscr{X}$ = congruence subgroups,

Lemma 2.1. Any two congruence subgroups are commensurable.

Proof. Note that $\Gamma(N) \cap \Gamma(N') = \Gamma(\mathrm{lcm}(N, N'))$. $\square$

Lemma 2.2. If Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then in $\mathrm{GL}(2,\mathbb{Q})^+$, the group $\tilde{\Gamma} = \mathrm{GL}(2,\mathbb{Q})^+$.

Fix a weight k and consider all the modular forms

$$M := \bigcup_{\Gamma\in\mathscr{X}} M_k(\Gamma) = \sum_\Gamma M_k(\Gamma)$$

and its $\mathbb{C}$-subspace

$$S := \bigcup_{\Gamma\in\mathscr{X}} S_k(\Gamma) = \sum_\Gamma S_k(\Gamma).$$

- Note that we have $\bigcup = \sum$, because

$$M_k(\Gamma) + M_k(\Gamma') \subset M_k(\Gamma \cap \Gamma').$$

Define a right-action of $\mathrm{GL}(2,\mathbb{R})^+$ on M by

$$f|_k\gamma(z) := (\det\gamma)^{k-1} j(\gamma,z)^{-k} f(\gamma z).$$

Lemma 2.3. For all $\Gamma \in \mathscr{X}$ and $\gamma \in \mathrm{GL}(2,\mathbb{R})^+$,

$$f \in M_k(\Gamma) \implies f|_k\gamma \in M_k(\Gamma \cap \gamma^{-1}\Gamma\gamma).$$

It remains true for S_k.

Proof. Just don't forget to check the cusps! $\square$

It is now straightforward to check that we defined an action on M which stabilises S.

Lemma 2.4. $M^\Gamma = M_k(\Gamma)$, $S^\Gamma = S_k(\Gamma)$.

Now we go to the case of $\Gamma_0(N)$ and $\Gamma_1(N)$.

2.1 The Algebras

We consider these monoids:

$$\begin{aligned}
\Delta(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det A > 0,\ (a,N)=1,\ N \mid c \right\} \\
&= \left\{ A \in \mathrm{GL}(2,\mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} (\mathbb{Z}/N\mathbb{Z})^\times & * \\ & * \end{pmatrix} \right\}, \\
\Delta^\circ(N) &:= \{A \in \Delta(N) \mid (\det A, N) = 1\}, \\
\Delta_1(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta^1(N) \middle| a \equiv 1 \pmod N \right\} \\
&= \left\{ A \in \mathrm{GL}(2,\mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} 1 & * \\ & * \end{pmatrix} \right\}.
\end{aligned}$$

Define

$$\mathscr{H}_i(N) := \mathscr{H}_{\Delta(N)}(\Gamma_i(N)), \quad \mathscr{H}_i^\circ(N) := \mathscr{H}_{\Delta^\circ(N)}(\Gamma_i(N)), \qquad i = 0,1$$

and $\mathscr{H}_1(N) := \mathscr{H}_{\Delta_1(N)}(\Gamma_1(N))$.

Proposition 2.1. All the algebras mentioned above are commutative.

Proof. Check that

$$A = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \longmapsto \begin{pmatrix} a & c \\ bN & d \end{pmatrix} = \left(\begin{pmatrix} 1 & \\ & N \end{pmatrix}^{-1} A \begin{pmatrix} 1 & \\ & N \end{pmatrix} \right)^{\mathrm{t}}$$

verifies the conditions of Theorem 1. □

We are particularly interested in $\mathscr{H}_0(N)$ and $\mathscr{H}_1(N)$.

2.2 Product Formula for $\mathscr{H}_0(N)$

Theorem 2 (A coset representative of $\mathscr{H}_0(N)$)**.** $\Gamma_0(N)\backslash\Delta(N)/\Gamma_0(N)$ admits coset representative given by

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad u \mid v,\ (u,N) = 1.$$

The double coset of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad \text{where } \begin{cases} uv = ad - bc \\ u = (a,b,c,d). \end{cases}$$

Proposition 2.2. The double coset

$$\Gamma_0(N) \begin{pmatrix} u & \\ & v \end{pmatrix} \Gamma_0(N) = \bigsqcup_{g \in M_{u,uv}} \Gamma_0(N) g,$$

where

$$M_{u,n} = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \middle| \begin{array}{c} u = (a,b,d) \\ n = ad \\ (a,N) = 1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right\}.$$

In particular,

$$\left[\begin{pmatrix}1 & \\ & n\end{pmatrix}\right]\Gamma_0(N)\begin{pmatrix}1 & \\ & n\end{pmatrix}\Gamma_0(N) = \bigsqcup_{g\in M_{1,n}} \Gamma_0(N)g$$

and

$$M_{1,n} = \left\{ \begin{pmatrix}a & b\\ & d\end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \;\middle|\; \begin{array}{c} (a,b,d)=1 \\ ad = n \\ (a,N)=1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right\}.$$

Example 2.1. Let p be a prime.

- If $p \mid N$, then
$$\left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right] = \bigsqcup_{i\in\mathbb{Z}/p\mathbb{Z}} \Gamma_0(N)\begin{pmatrix}1 & i\\ & p\end{pmatrix}.$$

- If $p \nmid N$, then
$$\left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right] = \bigsqcup_{i\in\mathbb{Z}/p\mathbb{Z}} \Gamma_0(N)\begin{pmatrix}1 & i\\ & p\end{pmatrix} \sqcup \Gamma_0(N)\begin{pmatrix}p & \\ & 1\end{pmatrix}.$$

Next, we must find the multiplication formula for these double cosets. Note that $\mathrm{diag}(u,u)$ lies in the centre of $\mathrm{GL}(2,\mathbb{Q})^+$, so $\mathrm{diag}(u,u)$ normalises $\Gamma_0(N)$. Hence

$$\left[\begin{pmatrix}u & \\ & v\end{pmatrix}\right] = \left[\begin{pmatrix}u & \\ & u\end{pmatrix}\right]\left[\begin{pmatrix}1 & \\ & v/u\end{pmatrix}\right],$$

and thus we need only to find the formula for $\mathrm{diag}(1,n)$'s.

Proposition 2.3 (Multiplication formulas)**.** Let $n,m \in \mathbb{Z}$, p be a prime.

- If $(n,m)=1$, then
$$\left[\begin{pmatrix}1 & \\ & n\end{pmatrix}\right]\left[\begin{pmatrix}1 & \\ & m\end{pmatrix}\right] = \left[\begin{pmatrix}1 & \\ & nm\end{pmatrix}\right].$$

- If $p \mid N$, then
$$\left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right]\left[\begin{pmatrix}1 & \\ & p^r\end{pmatrix}\right] = \left[\begin{pmatrix}1 & \\ & p^{r+1}\end{pmatrix}\right].$$

- If $p \nmid N$, then
$$\left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right]\left[\begin{pmatrix}1 & \\ & p^r\end{pmatrix}\right] = \begin{cases} \left[\begin{pmatrix}1 & \\ & p^2\end{pmatrix}\right] + (p+1)\left[\begin{pmatrix}p & \\ & p^r\end{pmatrix}\right], & r=1, \\[2ex] \left[\begin{pmatrix}1 & \\ & p^{r+1}\end{pmatrix}\right] + p\left[\begin{pmatrix}p & \\ & p^r\end{pmatrix}\right], & r\geq 2. \end{cases}$$

Proof. Just some elementary computation, but I would like to write them down as detailed as possible.

Write $\Gamma = \Gamma_0(N)$. Let $(n,m)=1$. We need to find

$$\#\{(A,B)\in M_{1,n}\times M_{1,m} \mid \Gamma AB = \Gamma\gamma\}, \quad \gamma = \begin{pmatrix}u & \\ & v\end{pmatrix},$$

so we investigate $M_{1,n}M_{1,m}$ first. Look at

$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \begin{pmatrix} e & f \\ & h \end{pmatrix} = \begin{pmatrix} ae & af+bh \\ & dh \end{pmatrix}$$

One checks directly that:

- $(ae, af+bh, dh) = 1$.
- ae permutes the factors of nm that are prime to N.
- When diaganol fixed, since $a \in (\mathbb{Z}/h\mathbb{Z})^\times$ and $h \in (\mathbb{Z}/d\mathbb{Z})^\times$, the upper-right $af+bh$ permutes $\mathbb{Z}/dh\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/h\mathbb{Z}$.

Therefore

$$M_{1,n}M_{1,m} = M_{1,nn}, \quad (n,m) = 1,$$

$$\begin{aligned} \left[\begin{pmatrix} 1 & \\ & n \end{pmatrix}\right]\left[\begin{pmatrix} 1 & \\ & m \end{pmatrix}\right] &= \sum_{u|v,(u,N)=1} \#\{A \in M_{1,nm} \mid \Gamma A = \Gamma \operatorname{diag}(u,v)\} \left[\begin{pmatrix} u & \\ & v \end{pmatrix}\right] \\ &= \sum_{u|v,(u,N)=1} \#\{A \in M_{1,nm} \mid \Gamma A = \Gamma \operatorname{diag}(u,v)\} \left[A\right]. \end{aligned}$$

For different $A \in M_{1,nm}$, the cosets ΓA are different, hence

$$\#\{A \in M_{1,nm} \mid \Gamma A = \Gamma \operatorname{diag}(u,v)\} \leq 1.$$

Actually, there is a unique $\operatorname{diag}(u,v)$ in each ΓA: in order

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} x & y \\ & z \end{pmatrix} = \begin{pmatrix} ax & ay+bz \\ Ncx & Ncy+dz \end{pmatrix}$$

being diaganol, c must be 0, so $a = d = \pm 1$, and $b = \pm y/z$. As $u = ax > 0$, the choice is unique, and we have proven that $[\operatorname{diag}(1,n)][\operatorname{diag}(1,m)] = [\operatorname{diag}(1,nm)]$.

(T.B.C.) $\square$

2.3 From Γ_0 to Γ_1

Proposition 2.4. Let $\Gamma_0 \supset \Gamma_1$ be congruence subgroups, $\Delta_0 \supset \Delta_1$ be monoids, satisfying the following conditions:

(a) $\Delta_i \supset \Gamma_i$, $i = 0, 1$.

(b) $\forall \alpha \in \Delta_1$, $\Gamma_0 \alpha \Gamma_0 = \Gamma_0 \alpha \Gamma_1$.

(c) $\forall \alpha \in \Delta_1$, $\Gamma_0 \alpha \cap \Delta_1 = \Gamma_1 \alpha$.

(d) $\Gamma_0 \Delta_1 = \Delta_0$.

Then the map

$$\Gamma_1 \backslash \Delta_1 / \Gamma_1 \to \Gamma_0 \backslash \Delta_0 / \Gamma_0, \quad \Gamma_1 \alpha \Gamma_1 \mapsto \Gamma_0 \alpha \Gamma_0$$

is bijective, and induces an isomorphism

$$\mathscr{H}_{\Delta_1}(\Gamma_1) \simeq \mathscr{H}_{\Delta_0}(\Gamma_0)$$

as $\mathbb{Z}$-algebras.

If $\alpha \in \Delta_1$, and the double coset

$$\Gamma_0\alpha\Gamma_0 = \bigsqcup_i \Gamma_0\alpha_i, \quad \text{with } \alpha_i \in \Gamma_1,$$

then

$$\Gamma_1\alpha\Gamma_1 = \bigsqcup_i \Gamma_1\alpha_i.$$

The conditions in Proposition 2.4 are satisfied when

$$\begin{aligned} \Gamma_0 &= \Gamma_0(N), & \Delta_0 &= \Delta(N), \\ \Gamma_1 &= \Gamma_1(N), & \Delta_1 &= \Delta_1(N), \end{aligned}$$

giving $\mathscr{H}_1(N) \simeq \mathscr{H}_0(N)$. Theorem 2 holds if we replace $\Gamma_0(N)$ by $\Gamma_1(N)$, while Proposition 2.2 needs a bit adjustment.

Recall that

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \qquad \begin{pmatrix} * & * \\ & d \end{pmatrix} \mapsto \bar{d}$$

induces a group isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Definition 4 (diamond operator)**.** For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define

$$\langle d \rangle := [\Gamma_1(N)\gamma_d\Gamma_1(N)],$$

where $\gamma_d \in \Gamma_0(N)$ is any lift of d.

- The operator $\langle d \rangle$ is independent to the choice of γ_d, because the γ_d's differ by an element in $\Gamma_1(N)$.
- $\langle d \rangle \langle d' \rangle = \langle dd' \rangle$.

Proposition 2.5. The double coset

$$\Gamma_1(N) \begin{pmatrix} u & \\ & v \end{pmatrix} \Gamma_1(N) = \bigsqcup_{g \in M_{u,v}} \Gamma_1(N)\gamma_a g, \quad g = \begin{pmatrix} a & * \\ & * \end{pmatrix}.$$

Proof. We can find γ_a s.t. $\gamma_a g \in \Gamma_1(N)$. As $\gamma_a \in \Gamma_0(N)$, the formula is true by Proposition 2.4. □

Moreover, the formulas in Proposition 2.3 holds for $\Gamma_1(N)$ after changing every $\begin{pmatrix} a & * \\ & * \end{pmatrix}$ to $\gamma_a \begin{pmatrix} a & * \\ & * \end{pmatrix}$.

2.4 Another Basis

Definition 5 (The operator $T(n)$)**.** Let $n \in \mathbb{Z}_{\geq 1}$ and consider

$$\Delta^n(N) := \{A \in \Delta(N) \mid \det A = n\}.$$

Write $\Gamma_0(N)\backslash\Delta^n(N)/\Gamma_0(N) = \bigsqcup_i \Gamma_0(N)g_i\Gamma_0(N)$, we define

$$T(n) := \sum_i [\Gamma_0(N)g_i\Gamma_0(N)] \in \mathscr{H}_0(N).$$

By Theorem 2, we may take g_i's to be

$$\begin{pmatrix} u & \\ & n/u \end{pmatrix} \text{ with } \begin{cases} (u, N) = 1, \\ u^2 \mid n, \end{cases}$$

yielding

$$\begin{aligned} T(n) &= \sum_u \left[\begin{pmatrix} u & \\ & n/u \end{pmatrix} \right] \\ &= \sum_u \left[\begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} 1 & \\ & n/u^2 \end{pmatrix} \right]. \end{aligned}$$

as the representative g_i's, which in turn shows that $\Gamma_0(N)\backslash\Delta^n(N)/\Gamma_0(N)$ is a finite set and $T(n)$ is well-defined. In particular, for p prime,

$$T(p) = \left[\begin{pmatrix} 1 & \\ & p \end{pmatrix} \right].$$

For $\Gamma_1(N)$, we consider $\Delta_1^n(N) := \Delta^n(N) \cap \Delta_1(N)$ and define $T(n) \in \mathscr{H}_1(N)$ using the same formula.

From Proposition 2.3, we deduce the formulas for $T(n)$'s.

Proposition 2.6 (Multiplication formulas for $T(n)$)**.** Let $n, m \in \mathbb{Z}$, p be a prime.

- The map $T : \mathbb{Z}_{\geq 1} \to \mathscr{H}_i(N)$ is multiplicative: if $(n, m) = 1$, then $T(nm) = T(n)T(m)$.
- If $p \mid N$, then $T(p)T(p^r) = T(p^{r+1})$, $r \in \mathbb{Z}_{\geq 1}$.
- If $p \nmid N$, then $T(p)T(p^r) = T(p^{r+1}) + p \left[\gamma_p \begin{pmatrix} p & \\ & p \end{pmatrix} \right] T(p^{r-1})$.

2.5 Hecke Algebra: the Hecke Action on Modular Forms

Fix a weight $k \in \mathbb{Z}_{\geq 1}$. Define the ring or $\mathbb{Z}$-algebra

$$\mathbb{T}_i(N) := \mathrm{im}\left(\mathscr{H}_i(N) \to \mathrm{End}_{\mathbb{C}}(M_k(N))\right) \tag{2}$$

and the operators

$$T_n := \text{image of } T(n) \in \mathscr{H}_i(N) \text{ in } \mathrm{End}_{\mathbb{C}}(M_k(N)).$$

for $i = 0, 1$.

Proposition 2.7. Let $m, n \in \mathbb{Z}_{\geq 1}$ and p a prime.

- $T_{mn} = T_m T_n$ if $(m, n) = 1$.
- $T_p T_{p^r} = T_{p^{r+1}}$ if $p \mid N$.
- $T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} \langle p \rangle T_{p^{r-1}}$, where the diamond operators act trivially on $M_k(\Gamma_0(N))$.

Proof. Let $f \in M_k(\Gamma_1(N))$ or $M_k(\Gamma_0(N))$. Since $\mathrm{diag}(p, p)$ normalises $\Gamma_1(N)$ and $\Gamma_0(N)$, we have

$$f \bigg|_k \left[\begin{pmatrix} p & \\ & p \end{pmatrix} \right] = f \bigg|_k \begin{pmatrix} p & \\ & p \end{pmatrix} = p^{k-2} f,$$

Since $\Gamma_1(N) \lhd \Gamma_0(N)$, we have

$$f|_k[\gamma_p] = f|_k \, \gamma_p = \langle p \rangle f.$$

The relations between these operators are now clear from Proposition 2.6. □

3 Group Cohomology

Recall that for a group G and a G-mod M, we define

$$H^1(G,M) = \frac{Z^1(G,M)}{B^1(G,M)} = \frac{\{f : G \to M \mid f(ab) = af(b) + f(a)\}}{\{g \mapsto gm - m \mid m \in M\}}.$$

We apply this construction to:

- $G =$ a congruence subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$,
- $M = V_n(R)$ as follows. Let R be a ring, $n \in \mathbb{Z}_{\geq 1}$. Define

  $$R[X,Y]_n := \{\text{homogeneous polynomials of degree } n\},$$

  a free R-module of rank $n+1$. The monoid $\mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})^+$ acts on $R[X,Y]_n$ by

  $$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\right)(X,Y) := P\left(\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = P(aX + cY, bX + dY);$$

  this is the *left* action on $R[X,Y] \hookrightarrow \{\text{function } R \times R \to R\}$ induced by the *right* action on $R \times R$. We set $V_n(R) := R[X,Y]_n$ with its $\mathrm{SL}_2(\mathbb{Z})$-action.

  Note that $V_n(R) \simeq \mathrm{Sym}^n R^2$, where R^2 is equipped with the standard $\mathrm{SL}_2(\mathbb{Z})$-action.

We will show that $H^1(\Gamma, V_n(\mathbb{C}))$ "resembles" a space of modular forms. It has an integral structure

$$H^1(\Gamma, V_n(\mathbb{Z})) \hookrightarrow H^1(\Gamma, V_n(\mathbb{C})),$$

which could give rise to the $\mathbb{Z}$-lattice we used in the last section.

Proposition 3.1. If S is flat over R, then as S-modules,

$$H^1(\Gamma, V_n(S)) \simeq H^1(\Gamma, V_n(R)) \otimes_R S.$$

3.1 The Eichler-Shimura map

Define the space of **anti-holomorphic cusp forms**

$$\overline{S_k(\Gamma)} := \{\bar{f} : z \mapsto \overline{f(z)} \mid f \in S_k(\Gamma)\}.$$

Definition 6. For $n \geq 0$, $u, v \in \mathcal{H}$, $f \in M_{n+2}(\Gamma)$, define

$$I_f(u,v) := \int_u^v f(z)(Xz + Y)^n dz$$
$$I_{\bar{f}}(u,v) := \int_u^v \overline{f(z)}(X\bar{z} + Y)^n dz.$$

These integrals take values in $V_n(\mathbb{C})$.

Lemma 3.1. Let $f \in M_{n+2}(\Gamma)$ or $S_{n+2}(\Gamma)$, $u, v, w \in \mathcal{H}$.

- $I_f(u,w) = I_f(u,v) + I_f(v,w)$.

- If $\gamma \in \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})^+$, then
$$I_f(\gamma u, \gamma v) = (\det g)^{-n}\, \gamma I_{f|_{n+2}\gamma}(u, v).$$
In particular, if $\gamma \in \Gamma$, then
$$I_f(\gamma u, \gamma v) = \gamma I_f(u, v).$$

Theorem 3. The map
$$M_{n+2}(\Gamma) \oplus \overline{S_{n+2}(\Gamma)} \longrightarrow H^1(\Gamma, V_n(\mathbb{C}))$$
$$(f, \bar{g}) \longmapsto (\gamma \mapsto I_f(a, \gamma a) + I_{\bar{g}}(b, \gamma b))$$
where $a, b \in \mathcal{H}$ are arbitarily chosen, is a well-defined isomorphism, called the **Eichler-Shimura map**.

It won't be proved in this course that this is an isomorphism.

Proof that this is well defined. □

4 Modular Curves

The purpose of this section is to introduce modular curves as Riemann surfaces and realize modular forms in the cohomologies of modular curves.

4.1 Modular Curves: Classical Version

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We assume that Γ contains no ellptic elements, so that
$$\pi_\Gamma : \mathcal{H} \to Y(\Gamma)$$
is a covering of group Γ, and any $\Gamma' \triangleleft \Gamma$ induces a Galois covering
$$\pi_{\Gamma,\Gamma'} : Y(\Gamma') \to Y(\Gamma)$$
of group Γ/Γ'.

4.1.1 Review: cohomologies of sheaves

4.1.2 Cohomologies of locally constant sheaves on $Y(\Gamma)$

Let V be a $\mathbb{Z}[\Gamma]$-module with discrete topology. Consider the diagonal action of Γ on $\mathcal{H} \times V$, which gives a covering
$$\Gamma\backslash(\mathcal{H} \times V) \to \Gamma\backslash\mathcal{H} = Y(\Gamma).$$
Define $\underline{V}_\Gamma$ to be the sheaf on $Y(\Gamma)$ of sections of the covering $\Gamma\backslash(\mathcal{H} \times V) \to Y(\Gamma)$. This is a locally constant sheaf with stalk V, and there is a natrual isomorphism $\underline{V}_\Gamma \simeq (\pi_{\Gamma *}\underline{V})^\Gamma$. Moreover, $\Gamma \mapsto \underline{V}_\Gamma$ is functorial: for $\Gamma' \triangleleft \Gamma$,
$$\underline{V}_\Gamma \simeq \left(\pi_{\Gamma,\Gamma'}{}_*\underline{V}_{\Gamma'}\right)^{\Gamma/\Gamma'},$$
which induces
$$H^i(Y(\Gamma), \underline{V}_\Gamma) = H^i\left(Y(\Gamma), \left(\pi_{\Gamma,\Gamma'}{}_*\underline{V}_{\Gamma'}\right)^{\Gamma/\Gamma'}\right) \to H^i(Y(\Gamma), \pi_{\Gamma,\Gamma'}{}_*\underline{V}_{\Gamma'})^{\Gamma/\Gamma'} = H^i(Y(\Gamma'), \underline{V}_{\Gamma'})^{\Gamma/\Gamma'}.$$
The kernels and cokernels of these maps are controlled by the Leray-Serre spectrual sequence. If, furthermore, V is a $\mathbb{Q}$-vector space, then we get isomorphisms $H^i(Y(\Gamma), \underline{V}_\Gamma) \simeq H^i(Y(\Gamma'), \underline{V}_{\Gamma'})^{\Gamma/\Gamma'}$.

4.1.3 The Hecke action on cohomologies

Let $G = \mathrm{GL}_2(\mathbb{Q})^+$, and V be a $\mathbb{Z}[G]$-module. We define

$$H^i(V) := \varinjlim_{\Gamma} H^i(Y(\Gamma), \underline{V}_\Gamma).$$

If V is a $\mathbb{Q}$-vector space, $H^i(V)^\Gamma = H^i(Y(\Gamma), \underline{V}_\Gamma)$. So our goal is to define a right G-action on $H^i(V)$ to obtain the Hecke operators on cohomologies, and we do this by defining

$$H^i(Y(\Gamma),\ \underline{V}_\Gamma) \to H^i(Y(g^{-1}\Gamma g),\ \underline{V}_{g^{-1}\Gamma g}), \quad g \in G$$

as the one induced from

$$l_g : Y(g^{-1}\Gamma g) \to Y(\Gamma) \quad z \mapsto gz$$

.

4.2 Modular Curves: Adelic Version

Let $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ be compact open. We define

$$Y(K) := \mathrm{GL}_2(\mathbb{Q})^+ \backslash \mathcal{H} \times (\mathrm{GL}_2(\mathbb{A}_f)/K)$$

4.2.1 Cohomologies

Let V be a $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Q})^+]$-module.

4.2.2 The Hecke action on cohomologies

This time we consider the right $G = \mathrm{GL}_2(\mathbb{A}_f)$-action on

$$H^i(V) = \varinjlim_{K} H^i(Y(K), \underline{V}_K)$$

given by

$$l_g : Y(g^{-1}Kg) \to Y(K) \quad (z,\ k \cdot g^{-1}Kg) \mapsto (z,\ kg^{-1} \cdot K).$$

4.3 The Eichler-Shimura Isomorphism