Notes on Explicit CFT for Function Fields

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1 Review on CFT

Let F be a global field, $C_F = \mathbb{A}_F^\times / F^\times$ be its idele class group, and F^{ab} be its maximal abelian extension inside a separable closure in a fixed algebraic closure $\bar{F}$. The class field theory asserts that the Artin map

$$\theta_F : C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

is a continuous group homomorphism with dense image, establishing a bijection

$$\{\text{finite abelian extensions of } F\} \longleftrightarrow \{\text{finite index open subgroups of } C_F\}.$$

The direction "$\to$" is computable: for a finite abelian L/F, the corresponding open subgroup of C_F is the kernel U of $C_F \xrightarrow{\theta_F} \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(L/F)$, which can be computed as $U = N_{L/F}(C_L)$[1]. But the other direction "$\leftarrow$" is not known in general: given a finite index open subgroup of C_F, the Artin map θ_F doesn't produce the generators of the corresponding extension L/F. It neither gives an explicit description of F^{ab}.

The goal of explictit class field theory is to find the construciton "$\leftarrow$", and to describe F^{ab}. In the article (give reference!), the author constructed the inverse of Artin map for function fields using one distinguished "place at infinity" with a sign function as well as Drinfeld modules, and described explicitly the structure of $k(t)^{\mathrm{ab}}$, the maximal abelian extension of the field of rational functions over a finite field k.

2 Function Fields and Drinfeld Modules

Let F be a global function field with a fixed place[2] ∞, and with field of constants $k = \mathbb{F}_q$, i.e. F is a finite extension of the field of rational functions $k(t)$ over k. If λ is a place of F, we denote by F_λ the completion at λ, by $\mathcal{O}_\lambda \subset F_\lambda$ the valuation ring, by $\mathbb{F}_\lambda := \mathcal{O}_\lambda/\mathfrak{m}_\lambda$ the residue field at λ, and by ord_λ the normalized valuation with value group $\mathbb{Z}$.

For each place λ, the Teichmüller lifting $\mathbb{F}_\lambda \hookrightarrow \mathcal{O}_\lambda$ is a field homomorphism; we regard $\mathbb{F}_\lambda \subset \mathcal{O}_\lambda \subset F_\lambda$ as a subfield via this embedding. This is the same as picking a uniformizer π so that $F_\lambda = \mathbb{F}_\infty(\!(\pi)\!)$ and then identifying $\mathbb{F}_\infty$ as the field of constants.

For any extension L of k, we denote by $\bar{L}$ an algebraic closure. Let L^{sep} be the separable closure of L in $\bar{L}$, and $\mathrm{Gal}_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ be the absolute Galois group.

2.1 Function fields

Here are some facts about function fields that are different from number fields.

[1] $N_{L/F} : C_L \to C_F$ is the norm map. The norm for an idele is just the multiplication of the norm at every places.

[2] A **place** of a function field is a valuation subring, or equivalently, an equivalence class of discrete valuations. Note that there are no archimedean places.

2.1.1 holomorphy ring

There are no "ring of integers" in function fields. Instead, we consider the holomorphy rings. A **holomorphy ring** is a ring of the form

$$\mathcal{O}^S := \bigcap_{\lambda \notin S} \mathcal{O}_\lambda = \{x \in F \mid \operatorname{ord}_\lambda(x) \geq 0,\ \forall \lambda \notin S\},$$

where S is a non-empty set of (not all the) places of F.

Proposition 2.1. Let S be a finite set of places of F. Then $\mathcal{O}^S$ is a Dedekind domain with fractional field $\operatorname{Frac}(\mathcal{O}^S) = F$, and there is a bijection between maximal ideals of $\mathcal{O}^S$ and places of F that are not in S.

We will consider $A := \mathcal{O}^{\{\infty\}}$, so that every place $\lambda \neq \infty$ of F cooresponds to a maximal ideal of A.

2.1.2 The Weil group

Let L be an extension of k. The algebraic closure $\bar{k}$ of k in $\bar{F}$ is contained in L^{sep}, and the absolute Galois group $\operatorname{Gal}_L = \operatorname{Gal}(L^{\mathrm{sep}}/L)$ stabilizes $\bar{k}$. Hence we have an exact sequence of topological groups

$$1 \longrightarrow \operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k}) \longrightarrow \operatorname{Gal}_L \xrightarrow{\deg} \hat{\mathbb{Z}} \to 0,$$

where $\deg : \operatorname{Gal}_L \to \operatorname{Gal}_k \simeq \hat{\mathbb{Z}}$ is defined by $\sigma(x) = \operatorname{Frob}_q^{\deg(\sigma)}(x)$ for $\sigma \in \operatorname{Gal}_L,\ x \in \bar{k}$.

Therefore, we can construct Weil group for L just like in the case of local fields. The **Weil group** is the subgroup W_L of Gal_L of elements that acts on $\bar{k}$ by an integral power of the Frobenius-q, i.e.

$$\sigma(x) = x^{q^{\deg(\sigma)}}, \quad \sigma \in W_L,\ x \in \bar{k}.$$

The kernel of the map $\deg : W_L \to \mathbb{Z}$ is still $\operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k})$. We endow W_L with the weakest topology for which

$$1 \longrightarrow \operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k}) \longrightarrow W_L \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

is an exact sequence of topological groups, where

- $\operatorname{Gal}(L^{\mathrm{sep}}/L\bar{k})$ has its usual profinite topology,
- $\mathbb{Z}$ has discrete topology[3].

With respect to this topology, the inclusion $W_L \hookrightarrow \operatorname{Gal}_L$ is continuous with dense image.

2.2 Definition of Drinfeld modules

2.2.1 Drinfeld modules and isogenies

Let L be an extension of k, $L[T]$ be the ring of polynomial over L. Consider the Frobenius-q map

$$\tau : L[T] \to L[T] \quad \sum_{i=0}^{n} a_i T^i \mapsto \sum_{i=0}^{n} a_i^q T^{iq}.$$

This is a k-linear endomorphism of $L[T]$, and we denote by $L[\tau]$ the sub-L-algebra of $\operatorname{End}_k(L[T])$ generated by τ. The ring $L[\tau]$ is a ring of **twisted polynomials**, because it is non-commutative:

$$\tau a = a^q \tau, \quad \forall a \in L.$$

[3]This is not the topology induced from $\mathbb{Z} \subset \hat{\mathbb{Z}}$.

Recall that $A = \{x \in F \mid \mathrm{ord}_\lambda(x) \geq 0, \forall \lambda \neq \infty\}$. Let L be an extension of F. A **Drinfeld A-module**[4] **over** L is a homomorphism

$$\phi : A \to L[\tau] \quad x \mapsto \phi(x) =: \phi_x$$

of k-algebras such that

- $\phi(A)$ is *not contained* in $L \subset L[\tau]$, and
- the map
$$A \to L[\tau] \to L \quad x \mapsto \phi_x = a_0 + a_1\tau + \cdots + a_n\tau^n \mapsto a_0$$
is the restriction of the inclusion map $F \hookrightarrow L$ to A.

In particular, $\phi : A \hookrightarrow L[\tau]$ is injective.

Let ϕ and ϕ' be two Drinfeld modules $A \to L[\tau]$. An **isogeny** over L from ϕ to ϕ' is an $f \in L[\tau] \setminus \{0\}$ such that

$$f\phi_a = \phi'_a f, \quad \forall a \in A.$$

An **isomorphism** over L from ϕ to ϕ' is an invertible isogeny, namely an isogeny $f \in L[\tau]^\times$.

If M/L is an extension, then a Drinfeld module over L induces natrually a Drinfeld module over M, so we can talk about isogenies over M for Drinfeld modules over L.

2.2.2 The rank

Let L^{perf} be the purely inseparable closure of L in $\bar{L}$, then $L^{\mathrm{perf}}(\!(\tau^{-1})\!)$ is a well-defined[5] skew-field, containing $L[\tau]$ as a subring. The injection $\phi : A \hookrightarrow L[\tau]$ extends uniquely to a embedding $\phi : F \hookrightarrow L^{\mathrm{perf}}(\!(\tau^{-1})\!)$. The function

$$v_\phi : F \to \mathbb{Z} \cup \{\infty\} \quad x \mapsto \mathrm{ord}_{\tau^{-1}}(\phi_x)$$

is a nontrivial valuation as $\phi(A) \not\subset L$, and $v_\phi(x) \leq 0$ for all $x \in A \setminus \{0\}$. Therefore v_ϕ is equivalent to the valuation ord_∞ attached to the place ∞. We define the **rank of** ϕ to be$r \in \mathbb{Q}$ such that

$$\mathrm{ord}_{\tau^{-1}}(\phi_x) = rd_\infty \,\mathrm{ord}_\infty(x), \tag{1}$$

for $x \in F$, where $d_\infty = [\mathbb{F}_\infty : k]$. The rank r is always an integer, and it is an isogeneous invariant.

2.2.3 Torsion submodule

A Drinfeld module $\phi : A \to L[\tau]$ defines an A-module structure on $\bar{L}$ by

$$x \cdot b := \phi_x(b), \quad \forall x \in A, b \in \bar{L}.$$

Every ϕ_x acts by a polynomial of the form $\phi_x(T) = a_0T + a_1T^q + \cdots + a_nT^{q^n}$, where $a_i \in L$. This polynomial is separable, because $x \mapsto \phi_x \mapsto a_0$ is injective. Therefore ϕ gives an A-module structure on L^{sep}.

For an ideal $\mathfrak{a}$ of A, we define

$$\phi[\mathfrak{a}] := \left\{b \in \bar{L} \mid \phi_x(b) = 0, \forall x \in \mathfrak{a}\right\},$$

an A-submodule of L^{sep} with A-module structure from ϕ, carrying a natrual Gal_L-action.

Lemma 2.1. The $A/\mathfrak{a}$-module $\phi[\mathfrak{a}]$ is free of rank r.

[4]There is a more general definition, but we only need and consider Drinfeld modules of this kind.

[5]We need to have all p-th root, so that $\tau^{-1}a = a^{1/q}\tau$ is always valid.

2.2.4 The sign functions and the ε-normalized Drinfeld modules

A **sign function for** F_∞ is a group homomorphism $F_\infty^\times \to \mathbb{F}_\infty^\times$ such that $\varepsilon|_{\mathbb{F}_\infty^\times} = \mathrm{id}_{\mathbb{F}_\infty^\times}$. These functions can be described as follows: choosing a uniformizer π of F_∞ yields a decomposition

$$F_\infty^\times \simeq \mathbb{F}_\infty^\times \times (1+\mathfrak{m}_\infty) \times \pi^{\mathbb{Z}}.$$

The factor $1+\mathfrak{m}_\infty$ is a pro-q group, but $\mathbb{F}_\infty^\times$ has order $q^f - 1$ for some integer f, so ε must be trivial on $1+\mathfrak{m}_\infty$. Therefore ε is determined by its value $\varepsilon(\pi)$.

Let $\phi : A \to L[\tau]$ be a Drinfeld module of rank r, $\varepsilon : F_\infty \to \mathbb{F}_\infty$ be a sign function for F_∞.

For $f = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots \in L^{\mathrm{perf}}(\!(\tau^{-1})\!)$, we write $\mathrm{lc}(f) := a_n$[6]. Using this notation, we say that ϕ is ε**-normalized**, if there exists some k-algebra homomorphism $\eta : \mathbb{F}_\infty \to L$ such that

$$\mathrm{lc}(\phi_x) = \eta(\varepsilon(x)), \quad \forall x \in A \smallsetminus \{0\}.$$

Lemma 2.2. Any Drinfeld module over L is isomorphic to some ε-normalized Drinfeld module over $\bar{L}$.

2.2.5 The action of ideals on a Drinfeld module

Let $\phi : A \to L[\tau]$ be a Drinfeld module. For an ideal $\mathfrak{a}$ of A, consider the *left* ideal $I_{\mathfrak{a},\phi}$ of $L[\tau]$ generated by $\phi(\mathfrak{a})$. Every left ideal of $L[\tau]$ is principal,[7] so $I_{\mathfrak{a},\phi} = L[\tau]\phi_{\mathfrak{a}}$ for a *unique monic* $\phi_{\mathfrak{a}} \in L[\tau]$. It is a plain to verify that for every $x \in A$, $I_{\mathfrak{a},\phi}$ absorb ϕ_x also from the right, i.e. $I_{\mathfrak{a},\phi}\phi_x \subset I_{\mathfrak{a},\phi}$, and therefore gives us a *unique* Drinfeld module

$$\mathfrak{a} * \phi : A \to L[\tau] \quad x \mapsto (\mathfrak{a} * \phi)_x,$$

characterized by the property that $\phi_{\mathfrak{a}}$ is an isogeny from ϕ to $\mathfrak{a} * \phi$, i.e.

$$\phi_{\mathfrak{a}} \cdot \phi_x = (\mathfrak{a} * \phi)_x \cdot \phi_{\mathfrak{a}}.$$

For example, if $\mathfrak{a} = A$, then $\phi_A = \phi_1 = 1$ and $A * \phi = \phi$. This construciton can be extend to fractional ideals by virtue of the following properties.

Lemma 2.3. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals of A, then

$$\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_{\mathfrak{a}} \cdot \phi_{\mathfrak{b}}, \quad \mathfrak{a}\mathfrak{b} * \phi = \mathfrak{a} * (\mathfrak{b} * \phi).$$

Lemma 2.4. Let $\mathfrak{a} = (w) \neq 0$ be a principal ideal of A, then

$$\phi_{(w)} = \mathrm{lc}(\phi_w)^{-1} \cdot \phi_w, \quad ((w) * \phi)_x = \mathrm{lc}(\phi_w)^{-1} \cdot \phi_x \cdot \mathrm{lc}(\phi_w), \ \forall x \in A.$$

In particular, $\phi \simeq (w) * \phi$ (not necessarily given by $\phi_{(w)}$).

For $w \in A \smallsetminus \{0\}$, Lemma 2.4 suggests us to define $\big((w^{-1}) * \phi\big)_x := \mathrm{lc}(\phi_w) \cdot \phi_x \cdot \mathrm{lc}(\phi_w)^{-1}$. For a general fractional ideal $w^{-1}\mathfrak{a}$ where $\mathfrak{a}$ is an integral ideal of A, we set

$$(w^{-1}\mathfrak{a}) * \phi := w^{-1} * (\mathfrak{a} * \phi) : x \mapsto \mathrm{lc}(\phi_w) \cdot (\mathfrak{a} * \phi)_x \cdot \mathrm{lc}(\phi_w)^{-1}.$$

Lemma 2.3 shows that these formulae indeed define an action of $\mathcal{I}_A$, the group of fractional ideals of A, on the set of Drinfeld modules $A \to L[\tau]$.

[6] The abbreviation lc stands for leading coefficient.
[7] By an argument similar to that of $L[X]$.

2.2.6 Hayes modules

Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of F_∞. Fix a sign function $\varepsilon : F_\infty \to \mathbb{F}_\infty$ for F_∞. A **Hayes module for** ε is a ε-normalized Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$ over $\mathbb{C}_\infty$ of rank 1. By Lemma 2.2, the existence of Hayes module for ε is equivalent to the existence of Drinfeld module of rank 1 over $\mathbb{C}_\infty$; (give reference!)

Let X_ε be the set of Hayes modules for ε. If $\mathfrak{a}$ is an ideal of A, and $\phi \in X_\varepsilon$ then $\mathfrak{a} * \phi \in X_\varepsilon$. Let $\mathcal{P}_A^+$ be the subgroup of $\mathcal{I}_A$ consisting of principal fractional ideals generated by $x \in F^\times$ with $\varepsilon(x) = 1$, and consider the **narrow class group** $\mathrm{Pic}^+(A) := \mathcal{I}_A/\mathcal{P}_A^+$. By Lemma 2.4, $\mathcal{P}_A^+$ acts trivially on X_ε, giving an action of $\mathrm{Pic}^+(A)$ on X_ε.

Proposition 2.2. The narrow class group $\mathrm{Pic}^+(A)$ acts *freely* and *transitively* on X_ε.

2.2.7 Galois action on X_ε

The **narrow Hilbert class field** or the **normalizing field for** (F, ∞, ε) is the extension

$$H_A^+ := F\left(\text{coefficient of } \phi_x \mid \phi \in X_\varepsilon, x \in A\right)$$

of F in $\mathbb{C}_\infty$. This is the minimal extension of F on which all Hayes modules for ε are defined.

Proposition 2.3. The extension H_A^+/F is finite abelian, and it is unramified away from ∞.

There is thus a natrual action of Gal_F on X_ε through $\mathrm{Gal}(H_A^+/F)$, given by

$$\sigma(\phi)_x := \sigma(\phi_x)^{8}, \quad \forall \sigma \in \mathrm{Gal}_F,\ \phi \in X_\varepsilon,\ x \in A.$$

Any $\phi \in X_\varepsilon$, by Proposition 2.2, induces an injective group homomorphism

$$\Psi : \mathrm{Gal}(H_A^+/F) \hookrightarrow \mathrm{Pic}^+(A),$$

such that $\sigma(\phi) = \Psi(\sigma) * \phi$ for all $\sigma \in \mathrm{Gal}_F$.

Proposition 2.4. $\Psi : \mathrm{Gal}(H_A^+/F) \to \mathrm{Pic}^+(A)$ is an isomorphism, independent to the choice of ϕ. For each non-zero prime $\mathfrak{p}$ of A, the class of $\Psi(\mathrm{Frob}_\mathfrak{p})$ in $\mathrm{Pic}^+(A)$ equals the class of $\mathfrak{p}$.

3 Construction of the Inverse to the Artin Map

We fix the tuple (F, ∞, ε) and a Hayes module $\phi \in X_\varepsilon$. Let $F_\infty^+ := \{x \in F_\infty^\times \mid \varepsilon(x) = 1\} = \ker(\varepsilon : F_\infty \to \mathbb{F}_\infty^\times)$.

3.1 λ-adic representation

Let λ be a place of F. Take $\sigma \in \mathrm{Gal}_F$. By Proposition 2.4, we pick an ideal $\mathfrak{a}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$.

- $\lambda \neq \infty$. Regarding λ as a prime ideal of A, we consider $\phi[\lambda^e]$ for $e \in \mathbb{Z}_{\geq 1}$. By Lemma 2.1, $\phi[\lambda^e]$ is an A/λ^e-module of rank 1. Define the **λ-adic Tate module** to be

  $$T_\lambda(\phi) := \mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty])$$

  which is a free $\mathcal{O}_\lambda$-module of rank 1. Hence $V_\lambda(\phi) := T_\lambda(\phi) \otimes_{\mathcal{O}_\lambda} F_\lambda$ is an 1-dimensional F_λ-vector space. We have the following two isomophisms between vector spaces.

[8] Gal_F acts on $\bar{F}[\tau]$ by acting on the coefficients. It is direct to check that Gal_F stabilizes X_ε.

– σ induces an isomorphism $\phi[\lambda^e] \simeq (\sigma(\phi))[\lambda^e]$ at each finite level, patching to an isomophism $V_\lambda(\sigma) : V_\lambda(\phi) \simeq V_\lambda(\sigma(\phi))$.
– The isogeny $\phi_{\mathfrak{a}} : \phi \to \mathfrak{a} * \phi$ induces an isomorphism[9] $V_\lambda(\phi_{\mathfrak{a}}) : V_\lambda(\phi) \simeq V_\lambda(\mathfrak{a} * \phi)$.

As $\mathfrak{a} * \phi = \sigma(\phi)$, we obtain an element

$$V_\lambda(\phi_{\mathfrak{a}})^{-1} \circ V_\lambda(\phi) \in \mathrm{GL}_{F_\lambda}(V_\lambda(\sigma)) = F_\lambda^\times \cdot \mathrm{id},$$

corresponding to an element $\rho_\lambda^{\mathfrak{a}}(\sigma) \in F_\lambda^\times$.

- $\lambda = \infty$. If $\sigma \in W_F$, the next Lemma 3.1 provides a unique element $\rho_\infty^{\mathfrak{a}}(\sigma) \in F_\infty^+$.

Lemma 3.1. There exists some series $u \in F^{\mathrm{sep}}[\![\tau^{-1}]\!]^\times$, such that

$$u^{-1}\phi(F_\infty)u \subset \bar{k}(\!(\tau^{-1})\!).$$[10]

For such a series u, if $\sigma \in W_F$, then there is a unique element $\rho_\infty^{\mathfrak{a}}(\sigma) \in F_\infty^+$, such that

$$\phi_{\mathfrak{a}}^{-1} \cdot \sigma(u) \cdot \tau^{\deg(\sigma)} \cdot u^{-1} = \phi(\rho_\infty^{\mathfrak{a}}(\sigma)).$$

These elements $\rho_\lambda^{\mathfrak{a}}(\sigma)$ has the following properties.

Lemma 3.2. Let λ be a place of F, $\sigma, \gamma \in \mathrm{Gal}_F$ (in W_F if $\lambda = \infty$) and $\mathfrak{a}, \mathfrak{b}$ be ideals of A.

(i) If $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$, then $(\sigma\gamma)(\phi) = (\mathfrak{a}\mathfrak{b}) * \phi$, and $\rho_\lambda^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\gamma)$.

(ii) If $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$, then $\mathfrak{b}^{-1}\mathfrak{a}$ is generated by a *unique* $w \in F_\infty^+ \cap F^\times$, and $\rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\sigma)^{-1} = w$.

(iii) If $\lambda \neq \infty$, and $\sigma(\phi) = \mathfrak{a} * \phi$, then $\mathrm{ord}_\lambda(\rho_\lambda^{\mathfrak{a}}(\sigma)) = -\mathrm{ord}_\lambda(\mathfrak{a})$[11].

If $\sigma \in \mathrm{Gal}_{H_A^+}$, then $\sigma(\phi) = \phi = A * \phi$. By Lemma 3.2 (i), we obtain homomorphisms

$$\rho_\lambda : \mathrm{Gal}_{H_A^+} \to \mathcal{O}_\lambda^\times \quad \sigma \mapsto \rho_\lambda^A(\sigma)$$

for $\lambda \neq \infty$, and homomorphism

$$\rho_\infty : W_{H_A^+} \to F_\infty^+, \quad \sigma \mapsto \rho_\infty^A(\sigma).$$

In particular, $\phi_A = 1$, so the representation ρ_λ is the representation of $\mathrm{Gal}_{H_A^+}$ on $T_\lambda(\phi)$ and hence it takes value in $\mathcal{O}_\lambda^\times$. These representations ρ_λ are continuous and unramified at all places of H_A^+ not over λ or ∞.

3.2 The inverse of the Artin map

For each $\sigma \in W_F$, fix an ideal $\mathfrak{a}_\sigma$ of A, such that $\sigma(\phi) = \mathfrak{a}_\sigma * \phi$. By Lemma 3.2, $(\rho_\lambda^{\mathfrak{a}_\sigma}(\sigma))_\lambda$ is an idele of F, whose class $\rho(\sigma)$ in C_F is independent to the choice of $\mathfrak{a}_\sigma$, and the map

$$\rho : W_F \to C_F, \quad \sigma \mapsto \rho(\sigma)$$

[9]Since ϕ has rank 1, it is equivalent to that $V_\lambda(\phi_{\mathfrak{a}})$ is non-zero. This is true, because, parallel to elliptic curves, taking Tate module is a faithful functor, i.e. for any two Drinfeld modules ϕ and ϕ' over L, the map

$$\mathrm{Hom}_L(\phi, \phi') \hookrightarrow \mathrm{Hom}_{\mathcal{O}_\lambda}(T_\lambda(\phi), T_\lambda(\phi'))$$

is injective.

[10]Recall that the Hayes module $\phi : A \to H_A^+[\tau]$ extends to an injective homomorphism $\phi : F_\infty \to \left(H_A^+\right)^{\mathrm{perf}}(\!(\tau^{-1})\!)$.

[11]Recall that we identify λ with a prime ideal of A. The number $\mathrm{ord}_\lambda(\mathfrak{a})$ is the largest power of λ dividing $\mathfrak{a}$.

is a group homomorphism.

The restriction of $\rho : W_F \to C_F$ to $W_{H_A^+}$ is

$$W_{H_A^+} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \hookrightarrow \mathbb{A}_F^\times \twoheadrightarrow C_F.$$

This homomorphism is continuous since all ρ_λ are continuous. The group $W_{H_A^+}$ has finite index in W_F, so ρ is continuous on W_F. Taking profinite completion yields a continuous homomorphism

$$\hat{\rho} : \mathrm{Gal}_F \to \hat{C}_F.$$

that factors through the maximal abelian quotient $\mathrm{Gal}_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Theorem 1. The map $\hat{\rho} : \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F$ is a topological isomophism independent to the choice of ∞, ε and ϕ, and the map

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F \quad \sigma \mapsto \hat{\rho}(\sigma)^{-1}$$

is the inverse of the Artin map $\hat{\theta}_F : \hat{C}_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Sketch of the proof. First, we need an arithmetic input.

Lemma 3.3. Let λ be a place of F, $\mathfrak{p}$ be another place of F that is not λ or ∞. Then $\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}) = 1$.

Remark (Explaination to the notation $\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}})$). Let λ and $\mathfrak{p}$ be places of F with $\mathfrak{p} \neq \infty$. By Proposition 2.3 and Proposition 2.4, the extension H_A^+/F is unramified at all places $\neq \infty$, and any $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}_F$ satisfies $\mathrm{Frob}_{\mathfrak{p}}(\phi) = \mathfrak{p} * \phi$.

Now we begin the construciton. Let $U < C_F$ be an open subgroup of finite index. The subgroup $\rho^{-1}(U) < W_F^{\mathrm{ab}}$ is open. Consider the finite abelian extension $L_U := (F^{\mathrm{ab}})^{\rho^{-1}(U)}$ of F fixed by this subgroup, so that $\mathrm{Gal}_{L_U}^{\mathrm{ab}} =$ the closure of $\rho^{-1}(U)$ in $\mathrm{Gal}_F^{\mathrm{ab}}$. We have an injective continuous homomorphism

$$\rho_U : \mathrm{Gal}(L_U/F) \simeq \mathrm{Gal}_F^{\mathrm{ab}} / \mathrm{Gal}_{L_U}^{\mathrm{ab}} \simeq W_F^{\mathrm{ab}}/\rho^{-1}(U) \hookrightarrow C_F/U.$$

Using weak approximation and Lemma 3.3, one can show that there is a finite set of places S_U containing ∞, such that:

- L_U/F is unramified at $\mathfrak{p} \notin S_U$,
- for each $\mathfrak{p} \notin S_U$, ρ_U sends $\mathrm{Frob}_{\mathfrak{p}}$ to the class of $\pi_{\mathfrak{p}}$, where $\pi_{\mathfrak{p}}$ is an uniformizer of $F_{\mathfrak{p}}$, and
- $\rho_U : \mathrm{Gal}(L_U/F) \to C_F/U$ is surjective and thus an isomophism.

Therefore the continuous isomophism

$$C_F/U \to \mathrm{Gal}(L_U/F) \quad \alpha \mapsto (\rho_U^{-1}(\alpha))^{-1}$$

is the Artin map

$$\theta_U : C_F/U \to \mathrm{Gal}(L_U/F) \quad \alpha \mapsto \theta_F(\alpha)|_{L_U}.$$

If L is a finite abelian extension of F, then the corresponding open subgroup U_L of C_F according to class field theory is the kernel of

$$C_F \to \mathrm{Gal}(L/F) \quad \alpha \mapsto \theta_F(\alpha)|_L.$$

Therefore $L = L_{U_L}$, and $F^{\mathrm{ab}} = \bigcup_U L_U$.

Now we can pass to the limit of the compatible isomophisms $\rho_{U\,U}$ and go back to see that $\hat{\rho} : \mathrm{Gal}_F^{\mathrm{ab}} \to C_F$ is an isomophism, whose inverse is the "multiplicative inverse" of the Artin map $\hat{\theta}_F$. $\square$

Corollary 3.1. The homomorphism $\rho : W_F^{\mathrm{ab}} \to C_F$ is a topological isomophism, and the map

$$W_F^{\mathrm{ab}} \to C_F \quad \sigma \mapsto \rho(\sigma)^{-1}$$

is the inverse of the Artin map $\theta_F : C_F \to W_F^{\mathrm{ab}}$.

4 Example: the Rational Function Field

Let $F = k(t)$. We consider the usual place ∞, so that $A = k[t]$, $F_\infty = k(\!(t^{-1})\!)$, $\mathbb{F}_\infty = k$, $\mathfrak{m}_\infty = t^{-1}k[\![t^{-1}]\!]$, $\mathrm{ord}_\infty(t^{-1}) = 1$. Let $\varepsilon : F_\infty^\times \to k^\times$ be the sign function defined by $\varepsilon(t^{-1}) = 1$, so that $F_\infty^+ = t^{\mathbb{Z}} \cdot (1 + \mathfrak{m}_\infty)$.

The **Carlitz module** ϕ is a Hayes module for ε defined by

$$\phi : A = k[t] \to F[\tau] \quad t \mapsto \phi_t := t + \tau.$$

The normalizing field for (F, ∞, ε) is $H_A^+ = F$, so ϕ is the only Hayes module for ε.

We have defined the representations $\rho_\lambda : W_F^{\mathrm{ab}} \to F_\lambda^\times$, and the isomophism between the (abelianized) Weil group and the idele class group factors as

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \to C_F. \tag{2}$$

Similar to $\mathbb{Q}$, the second arrow above is an isomophism[12], and thus the first arrow

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \times t^{\mathbb{Z}} \times (1 + \mathfrak{m}_\infty)$$

is also an isomophism. Taking profinite completion, we got a decomposition

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \simeq \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \times t^{\hat{\mathbb{Z}}} \times (1 + \mathfrak{m}_\infty)$$

of $\mathrm{Gal}_F^{\mathrm{ab}}$, corresponding to three disjoint abelian extension of F whose compositum is F^{ab}.

The "cyclotomic" extension K_∞

For $\lambda \neq \infty$, the representation $\rho_\lambda : \mathrm{Gal}_F \to \mathcal{O}_\lambda^\times$ is precisely the Galois representation on $T_\lambda(\phi)$, where ϕ is the Carlitz module. The representation

$$\chi := \prod_{\lambda \neq \infty} \rho_\lambda : \mathrm{Gal}_F \to \prod_{\lambda \neq \infty} \mathcal{O}_\lambda = \hat{A}^\times$$

is the inverse limit of $\chi_m : \mathrm{Gal}_F \to (A/(m))^\times$, which are from the Gal_F-action on $\phi[m]$ for all monic $m \in A = k[t]$, ordered by divisibility. Hence the field fixed by $\ker(\chi)$ is $K_\infty = \bigcup_m F(\phi[m])$. The extension K_∞/F is a geometric extension[13], tamely ramified at ∞.

The extension of constants $\bar{k}(t)$

For each $\sigma \in W_F$, the factor in $t^{\mathbb{Z}} \simeq \mathbb{Z}$ is $\mathrm{ord}_t(\rho_\infty(\sigma)) = -\mathrm{ord}_\infty(\rho_\infty(\sigma))$, which equals $-\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma)))$ by (1). By Lemma 3.1, $\phi(\rho_\infty(\sigma)) = \sigma(u)\tau^{\deg(\sigma)}u^{-1}$, so $-\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma))) = \deg(\sigma)$. This shows that the projection $W_F \to \mathbb{Z}$ is precisely the map deg. The field fixed by (the closure of) ker(deg) is $\bar{k}(t)$, and the extension $\bar{k}(t)/k(t)$ is the maximal constant field extension.

[12]Let $x \in \mathbb{A}_F^\times$, Every place $\lambda \neq \infty$ has a "canonical" uniformizer $\mathfrak{p} \in k[t]$, namely the unique monic irreducible polynomial, and we write $x_\mathfrak{p} = u_\mathfrak{p}\mathfrak{p}^{n_\mathfrak{p}}$ with $u_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$. Put $f := a_\infty \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}} \in k(t)^\times$. We have $f^{-1}x_\infty = a_\infty t^n +$ terms with lower degree in t for some $a_\infty \in k$. Then $(a_\infty f)^{-1}x \in F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times$. This decomposition of ideles provides the desired isomophism

[13]A **geometric extension** of a function field extension that doesn't extend the field of constants.

The wildly ramified extension L_∞

By discussion above, the projection onto $1+\mathfrak{m}_\infty$ is

$$W_F \to 1+\mathfrak{m}_\infty \quad \sigma \mapsto \rho_\infty(\sigma)/\operatorname{ord}_t(\rho_\infty(\sigma)) = \rho_\infty(\sigma)/\deg(\sigma).$$

Taking profinite completion, we get a Galois representation $\beta : \mathrm{Gal}_F \to 1+\mathfrak{m}_\infty$. Denote by L_∞ the fixed field of $\ker(\beta)$. The extension L_∞/F is a geometric extension, unramified away from ∞ and wildly ramified at ∞.

To describe this field explicitly, we need to look at the construciton of ρ_∞. Choose recursively a sequence of elements $\{a_i\}_{i\geq 0} \subset F^{\mathrm{sep}}$ by

$$a_0 := 1; \quad a_i^q - a_i = -ta_{i-1},\ i \geq 1.$$

Then $u := \sum_{i\geq 0} a_i \tau^{-i}$ verifies the condition of u in Lemma 3.1. For $\sigma \in W_F$, $\rho_\infty(\sigma) \in F_\infty^+$ is characterized by $\phi(\rho_\infty(\sigma)) = \sigma(u)\tau^{\deg(\sigma)}u^{-1}$. Every $\sigma \in \mathrm{Gal}(L_\infty/F)$ has representatives in W_F with $\deg = 0$ since it acts trivially on $\bar{k}$. Hence $\phi(\beta(\sigma)) = \sigma(u)u^{-1}$, which shows that[14] $\beta(\sigma) = 1$ if and only if $\sigma(u) = u$, and thus $L_\infty = F\left(\{a_i\}_{i\geq 0}\right)$.

[14]Recall that ϕ is injective.