Notes on Drinfeld Modules and Explicit CFT for Function Fields

February 26, 2025

Pre-date: March 10! It is close!

1) Give a 30min (strict limit !!!) talk. Ideally more like 25min + 5 min for questions. The talks will be in March. I will try to reserve a room, and will give a more precise time/date when possible.

2) Write an "extended summary" (meaning around 5 pages NOT!!! >=10) of you article. It should summarise the article and its main ideas and be accessible to advanced Master students (i.e., the other students in this group).

1 Review on CFT

2 Drinfeld Modules

Let F be a global function field with a fixed place ∞ neccessarily at infinity?, and with field of constants $k = \mathbb{F}_q$. If λ is a place of F, we denote by F_λ the completion at λ, by $\mathcal{O}_\lambda \subset F_\lambda$ the valuation ring, by $\mathbb{F}_\lambda := \mathcal{O}_\lambda/\mathfrak{m}_\lambda$ the residue field at λ, and by $N(\lambda) := \#\mathbb{F}_\lambda$ its cardinality. Since we are working with function fields, the Teichmüller lifting $\mathbb{F}_\lambda \hookrightarrow \mathcal{O}_\lambda$ is a field homomorphism (Check this!); we regard $\mathbb{F}_\lambda \subset \mathcal{O}_\lambda \subset F_\lambda$ as a subfield via this embedding.

Let L be an arbitary extension of k with a fixed algebraic closure $\bar{L}$.

Function fields: holomorphy ring

Let S be a non-empty set of (not all the) places of F. Define

$$\mathcal{O}_S := \bigcap_{\lambda \notin S} \mathcal{O}_\lambda = \{x \in F \mid \operatorname{ord}_\lambda(x) \geq 0,\ \forall \lambda \notin S\}$$

to be the subring of F consisting of elements regular away from S. A **holomorphy ring** is a ring of this form. For example, our $A = \mathcal{O}_{\{\infty\}}$ is a holomorphy ring.

Proposition 2.1. Consider a holomorphy ring $\mathcal{O}_S$.

(1) $\operatorname{Frac}(\mathcal{O}_S) = F$.

(2) $\mathcal{O}_S$ is a Dedekind domain.

(3) There is a bijection

$$\{\text{place of } F \text{ not in } S\} \longleftrightarrow \operatorname{MaxSpec} \mathcal{O}_S$$

giving by $\lambda \mapsto \mathfrak{m}_\lambda \cap \mathcal{O}_S$, which induces isomorphisms

$$\mathbb{F}_\lambda = \mathcal{O}_\lambda/\mathfrak{m}_\lambda \simeq \mathcal{O}_S/(\mathfrak{m}_\lambda \cap \mathcal{O}_S)$$

So we can regard λ as a maximal ideal of A.

2.1 Definition

2.1.1 Endomorphisms of the additive group

Consider the additive group $\mathbb{G}_{\mathrm{a}/L}$ over L. Now the point is, $\mathbb{G}_{\mathrm{a}/L}$ is not only a group scheme, but a k-vector space scheme, and we consider the ring $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L})$ of all k-linear endomorphism of group schemes.

Proposition 2.2. $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) = L[\tau]$, where τ is the Frobenius-q endomorphism.

We explain the notation in the proof.

Proof. An endomorphism $\mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ of schemes over L is given by an L-algebra homomorphism $\Phi : L[X] \to L[X]$, hence it is determined by the image $\varphi(X) = \Phi(X)$[1] of X. It respects the group-scheme structure if it commutes with the co-multiplication map (also an L-algebra homomorphism)

$$\Delta : F[X] \to F[X] \otimes_L F[X], \quad X \mapsto X \otimes 1 + 1 \otimes X.$$

which amounts to

$$(\Phi \otimes \Phi)(\Delta(X)) = (\Phi \otimes \Phi)(X \otimes 1 + 1 \otimes X) = \Phi(X) \otimes 1 + 1 \otimes \Phi(X) = \varphi(X) \otimes 1 + 1 \otimes \varphi(X)$$

equals

$$\Delta(\Phi(X)) = \Delta(\varphi(X)) = \varphi(\Delta(X)) = \varphi(X \otimes 1 + 1 \otimes X).$$

This is to say that[2] φ is additive, i.e. $\varphi(X+Y) = \varphi(X) + \varphi(Y)$.

We require furthur that Φ respects the "co-k-scalar multiplication", which I don't have the formula right now. So let's use the functor point of view. Take $c \in k$. Yoneda tells us that

$$\mathrm{Hom}_{[k\text{-}\mathbf{Alg}^{\mathrm{op}},\mathbf{Grp}]}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{a}}) \simeq \mathbb{G}_{\mathrm{a}}(L[X]), \quad \phi \mapsto \phi(\mathrm{id}_{L[X]}),$$

so the co-c-multiplication is given by $X \mapsto cX$. Therefore Φ respects this map if $\varphi(cX) = c\varphi(X)$.

In conclusion,

$$\begin{aligned}
\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) &= \{k\text{-linear polynomials in } L[X]\} \\
&= \left\{ \sum_i a_i X^{p^i} \,\middle|\, a_i \in L,\ \sum a_i c X^{p^i} = \sum a_i c^{p^i} X^{p^i}, \forall c \in k = \mathbb{F}_q \right\} \\
&= \left\{ \sum_i a_i X^{q^i} \,\middle|\, a_i \in L \right\} = \left\{ \left(\sum_i a_i \tau^i \right)(X) \,\middle|\, a_i \in L \right\},
\end{aligned}$$

where $\tau(X) := X^q$.

Note that $\tau : L[X] \to L[X]$ is additive, but doesn't commutes with elements in L:

$$\tau a = a^q \tau, \quad \forall a \in L.$$

[1] Note that if $\varphi(X) = a_n X^n + \cdots + a_0$, then

$$\varphi(f(X)) = a_n f(X)^n + \cdots + a_0$$

and

$$\Phi(f(X)) = f(\Phi(X)) = f(\varphi(X))$$

are ***different*** in general.

[2] Recall that the multiplicative structure on $B \otimes_A C$ is given by

$$(b \otimes b') \cdot (c \otimes c') = bb' \otimes cc'.$$

Therefore $L[\tau]$ is a *non-commutative* subring of $\mathrm{End}(L[X])$, where multiplication is composition; it is a ring of **twisted polynomials**. And we have $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) \simeq L[\tau]$. □

Remark. τ corresponds to the Frobenius-q endomorphism of $\mathbb{G}_{\mathrm{a}/L}$. (What is this? $\mathbb{G}_{\mathrm{a}/L}$ is NOT over $\mathbb{F}_q = k$.)

2.1.2 Drinfeld modules and isogenies

Let A be a k-algebra. A **Drinfeld A-module**[3] **over** L is a homomorphism

$$\phi : A \to L[\tau] \quad x \mapsto \phi(x) =: \phi_x$$

of k-algebras such that $\phi(A)$ is *not contained* in $L \subset L[\tau]$.

Let ϕ and ϕ' be two Drinfeld modules $A \to L[\tau]$. An **isogeny** over L from ϕ to ϕ' is an $f \in L[\tau] \setminus \{0\}$ such that

$$f\phi_a = \phi'_a f, \quad \forall a \in A.$$

An **isomorphism** over L from ϕ to ϕ' is an invertible isogeny, namely an isogeny $f \in L[\tau]^\times$. If M/L is an extension, then a Drinfeld module over L induces natrually a Drinfeld module over M, and we can talk about isogenies over M for Drinfeld modules over L.

Remark (Another interpretation).

Let

$$\partial : L[\tau] \to L \quad \sum_i a_i \tau^i \mapsto a_0$$

be the homomorphism of taking the constant term. We say that a Drinfeld module $\phi : A \to L[\tau]$ **has generic characteristic**, if

$$\partial \circ \phi : A \to L[\tau] \twoheadrightarrow L$$

is *injective*. This implies that ϕ is injective.

2.2 The Drinfeld modules we need

In what follows, we take $A \subset F$ to be the subring consisting of functions that are regular away from ∞, and we assume that every Drinfeld modules $\phi : A \to L[\tau]$ is of generic characteristic, so that $\partial \circ \phi : A \hookrightarrow L$ extends to an embedding

$$F \hookrightarrow L.$$

Through the latter, we view F as a subfield of L.

Let L^{perf} be the purely inseperable closure of L in $\bar{L}$, then $L^{\mathrm{perf}}((\tau^{-1}))$ is a well-defined skew-field[4], containing $L[\tau]$ as a subring.

Under our assumption, $\phi : A \hookrightarrow L[\tau]$ is injective, so it extends to a unique embedding (Does $L^{\mathrm{perf}}(\tau)$ make sense?)

$$\phi : F \hookrightarrow L^{\mathrm{perf}}((\tau^{-1})).$$

The function

$$v_\phi : F \to \mathbb{Z} \cup \{\infty\} \quad x \mapsto \mathrm{ord}_{\tau^{-1}}(\phi_x)$$

[3] There is more general definition, but this one suffices.

[4] We need to have all p-th root, so that $\tau^{-1}a = a^{1/q}\tau$ is always valid.

is a nontrivial[5] valuation, and $v_\phi(x) \le 0$ for all $x \in A \smallsetminus \{0\}$. Therefore v_ϕ is equivalent to the valuation ord_∞ attached to the place ∞. We define the **rank of** ϕ to be the rational number $r \in \mathbb{Q}$ such that

$$\mathrm{ord}_{\tau^{-1}}(\phi_x) = rd_\infty \, \mathrm{ord}_\infty(x), \quad \forall x \in F,$$

where $d_\infty = [\mathbb{F}_\infty : k]$ is the inertia degree of F at ∞. The tank r is always an integer (by a proposition we may encounter later). Since $L^{\mathrm{perf}}((\tau^{-1}))$ is complete under $\mathrm{ord}_{\tau^{-1}}$, the homomorphism $\phi : F \to L^{\mathrm{perf}}((\tau^{-1}))$ gives rise to a unique homomorphism

$$\phi : F_\infty \to L^{\mathrm{perf}}((\tau^{-1}))$$

such that $\mathrm{ord}_{\tau^{-1}}(\phi_x) = rd_\infty \, \mathrm{ord}_\infty(x)$ for all $x \in F_\infty$.

Now the map ϕ restricts to a homomorphism

$$\phi : \mathbb{F}_\infty \subset \mathcal{O}_\infty \to L^{\mathrm{perf}}[\![\tau^{-1}]\!].$$

Composing with $\partial : L^{\mathrm{perf}}[\![\tau^{-1}]\!] \to L^{\mathrm{perf}}$ of taking constant term, we obtain an embedding

$$\partial \circ \phi|_{\mathbb{F}_\infty} : \mathbb{F}_\infty \hookrightarrow L^{\mathrm{perf}},$$

whose image lies in L (WHY???).

2.3 ε-normalized Drinfeld modules

Let $\phi : A \to L[\tau]$ be a Drinfeld module of rank r, extending to an embedding $\phi : F \to L^{\mathrm{perf}}((\tau^{-1}))$. For $x \in F_\infty^\times$, we define

$$\mu_\phi(x) := \text{first non-zero coefficient of } \phi_x \text{ as a Laurent series in } \tau^{-1},$$

so that $\mu_\phi(x) \in (L^{\mathrm{perf}})^\times$, and the first term, i.e. the term with *highest* τ-order, of ϕ_x is

$$\mu_\phi(x)\tau^{-rd_\infty \, \mathrm{ord}_\infty(x)}.$$

In particular, if $x \in A$, $\mu_\phi(x)$ is the leading coefficient of $\phi_x \in L[\tau]$, which is what we used before to define reduction type.

By definition, for $x, y \in F_\infty^\times$,

$$\mu_\phi(xy) = \mu_\phi(x)\mu_\phi(y)^{1/q^{rd_\infty \ \mathrm{ord}_\infty(x)}}.$$

Recall that ϕ gives us an embedding

$$\partial \circ \phi|_{\mathbb{F}_\infty} : \mathbb{F}_\infty \hookrightarrow L$$

With respect to this embedding, why???

$$\mu_\phi(x) = x, \quad \forall x \in \mathbb{F}_\infty$$

Definition 1. A **sign function for** F_∞ is a group homomorphism $F_\infty \to \mathbb{F}_\infty$ such that $\varepsilon|_{\mathbb{F}_\infty} = \mathrm{id}_{\mathbb{F}_\infty}$. Note that a sign function ε is trivial on $1 + \mathfrak{m}_\infty$, so it is determined by $\varepsilon(\pi)$ for a uniformizer $\pi \in \mathfrak{m}_\infty$.

Let $\varepsilon : F_\infty \to \mathbb{F}_\infty$ be a sign function for F_∞. We say that ϕ is

- **normalized**, if
$$\mu_\phi(x) \in \mathbb{F}_\infty, \quad \forall x \in F_\infty,$$
- **ε-normalized**, if
$$\exists \sigma \in \mathrm{Aut}_k(\mathbb{F}_\infty), \quad \phi = \sigma \circ \varepsilon.$$

Lemma 2.1. Let ε be a sign function for F_∞. Any Drinfeld module over L is isomorphic over $\bar{L}$ to some ε-normalized Drinfeld module.

[5] Because $\phi(A) \not\subset L$.

2.4 The action of an ideal on a Drinfeld module

Let $\phi : A \to L[\tau]$ be a Drinfeld module. For an ideal $\mathfrak{a}$ of A, Define

$$I_{\mathfrak{a},\phi} := \text{ ideal of } L[\tau] \text{ generated by } \{\phi_a \mid a \in \mathfrak{a}\}.$$

Every *left*-ideal of $L[\tau]$ is principal,[6] so

$$I_{\mathfrak{a},\phi} = L[\tau]\phi_{\mathfrak{a}}$$

for a *unique monic* $\phi_{\mathfrak{a}} \in L[\tau]$. It is a plain to verify that for every $x \in A$, $I_{\mathfrak{a},\phi}$ absorb ϕ_x also from the right, i.e. $I_{\mathfrak{a},\phi}\phi_x \subset I_{\mathfrak{a},\phi}$, and therefore gives us a *unique* Drinfeld module

$$\mathfrak{a} * \phi : A \to L[\tau] \quad x \mapsto (\mathfrak{a} * \phi)_x$$

together with an isogeny $\phi_{\mathfrak{a}}$ from ϕ to $\mathfrak{a} * \phi$, namely

$$\phi_{\mathfrak{a}} \cdot \phi_x = (\mathfrak{a} * \phi)_x \cdot \phi_{\mathfrak{a}},$$

Lemma 2.2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals of A, then

$$\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_{\mathfrak{a}} \cdot \phi_{\mathfrak{b}},$$

$$\mathfrak{a}\mathfrak{b} * \phi = \mathfrak{a} * (\mathfrak{b} * \phi).$$

Lemma 2.3. Let $\mathfrak{a} = Aw \neq 0$ be a principal ideal of A, then

$$\phi_{\mathfrak{a}} = \mu_\phi(w)^{-1} \cdot \phi_w,$$

$$(\mathfrak{a} * \phi)_x = \mu_\phi(w)^{-1} \cdot \phi_x \cdot \mu_\phi(w), \ \forall x \in A.$$

Lemma 2.4. Let $\sigma : L \hookrightarrow M$ be a field extension, inducing a Drinfeld module

$$\sigma(\phi) : A \to M[\tau], \ x \mapsto \sigma(\phi)_x = \sigma(\phi_x).$$

Then

$$\sigma(\mathfrak{a} * \phi) = \mathfrak{a} * \sigma(\phi),$$

$$\sigma(\phi_{\mathfrak{a}}) = \sigma(\phi)_{\mathfrak{a}}.$$

Fix a sign function $\varepsilon : F_\infty \to \mathbb{F}_\infty$ for F_∞. Consider

- $\mathcal{I}$, the group of fractional ideals of A,
- $\mathcal{P}^+$, a subgroup of the group $\mathcal{P}$ of principal fractional ideals of A, which is generated by $x \in F^\times$ with $\varepsilon(x) = 1$, and
- the **narrow class group** $\mathrm{Pic}^+(A) := \mathcal{I}/\mathcal{P}^+$.

We can define $\mathfrak{a} * \phi$ for every $\mathfrak{a} \in \mathcal{I}$ by Lemma 2.2, giving an action of $\mathcal{I}$ on the set of Drinfeld modules $A \to L[\tau]$. If, in addition, ϕ is ε-normalized, then $\mathcal{P}^+$ fixes ϕ by Lemma 2.3, giving an action of $\mathrm{Pic}^+(A)$.

[6] By an argument similar to $L[X]$, probably.

2.5 Torsion submodule

A Drinfeld module $\phi : A \to L[\tau]$ defines an A-module structure on $\bar{L}$ by

$$x \cdot b := \phi_x(b), \quad \forall x \in A, b \in \bar{L}.$$[7]

All ϕ_x has coefficient in L, so ϕ, in particular, gives an A-module structure on L^{sep}.

For an ideal $\mathfrak{a}$ of A, we define

$$\phi[\mathfrak{a}] := \left\{ b \in \bar{L} \mid \phi_{\mathfrak{a}}(b) = 0 \right\} = \left\{ b \in \bar{L} \mid \phi_x(b) = 0, \forall x \in \mathfrak{a} \right\},$$

an $A/\mathfrak{a}$-module and an A-submodule of $\bar{L}$ with A-module structure induced by ϕ.

Proposition 2.3. Let ϕ be a Drinfeld module of rank r, $\mathfrak{a}$ an ideal of A. Then $\phi[\mathfrak{a}]$ is a free $A/\mathfrak{a}$-module of rank r, and it is contained in F^{sep}.

Proof. Every ϕ_x acts by a polynomial of the form

$$\phi_x(T) = a_0 T + a_1 T^q + \cdots + a_n T^{q^n},$$

which is seperable, because $x \mapsto \phi_x \mapsto a_0$ is injective, which implies that $\phi_x'(T) = a_0 \neq 0$ if $\phi_x \neq 0$.

For the other claim, we use the structure of modules over Dedekind domains. □

2.6 Hayes modules

Let $\mathbb{C}_\infty$ be a completion of an algebraic closure of F_∞. It is ∞-adically complete and algebraically closed.

Fix a sign function $\varepsilon : F_\infty \to \mathbb{F}_\infty$ for F_∞. A **Hayes module for** ε is a Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$ over $\mathbb{C}_\infty$, such that

- It is of rank 1.
- It is ε-normalized.
- $\partial \circ \phi : A \hookrightarrow \mathbb{C}_\infty$ is the inclusion $A \subset F \subset F_\infty \subset \mathbb{C}_\infty$.

Let X_ε be the set of Hayes modules for ε.

If $\mathfrak{a}$ is an ideal of A, and $\phi \in X_\varepsilon$ then $\mathfrak{a} * \phi \in X_\varepsilon$. By some discussion before, this defines an action of $\text{Pic}^+(A) = \mathcal{I}/\mathcal{P}^+$ on X_ε.

Proposition 2.4. The set X_ε is a principal homogeneous space for $\text{Pic}^+(A)$, i.e. $\text{Pic}^+(A)$ acts *freely* and *transitively* on X_ε.

2.6.1 Galois action on X_ε

We define the **normalizing field for** (F, ∞, ε) to be the extension

$$H_A^+ := F\left(\text{coefficient of } \phi_x \mid \phi \in X_\varepsilon, x \in A\right)$$

of F in $\mathbb{C}_\infty$.

[7] Note that if $\phi_x = \sum_{a_i \tau^i}$, then

$$\phi_x(b) = \sum_i \tau^i(b) = \sum_i b^{q^i}.$$

At least I think so!

Theorem 1. (1) For any $\phi \in X_\varepsilon$ and $x \in A$,

$$H_A^+ = F\,(\text{coefficient of } \phi_x)$$

(2) Let B be the integral closure of A in H_A^+. For any $\phi \in X_\varepsilon$ and $x \in A$, $\phi_x \in H_A^+[\tau]$ has integral coefficient, i.e. ϕ_x has coefficient in B.

(3) The extension H_A^+/F is finite abelian, and it is unramified away from ∞.

By Lemma 2.4, there is a natrual action of $\mathrm{Gal}(H_A^+/F)$ on X_ε. For a fixed $\phi \in X_\varepsilon$, ϕ induces an injective group homomorphism

$$\psi : \mathrm{Gal}(H_A^+/F) \to \mathrm{Pic}^+(A).$$

(4) For each non-zero prime $\mathfrak{p}$ of A, the class of $\psi(\mathrm{Frob}_\mathfrak{p})$ in $\mathrm{Pic}^+(A)$ equals the class of $\mathfrak{p}$.

(5) $\psi : \mathrm{Gal}(H_A^+/F) \to \mathrm{Pic}^+(A)$ is an isomorphism.

2.6.2 Reduction of Hayes modules

Corollary 2.1. Every Hayes module ϕ has **good reduction** over H_A^+ at every finite place $\mathfrak{P}$ *not* over ∞, i.e. the composition of reduction modulo $\mathfrak{P}$ with ϕ is a Drinfeld module of rank 1 over $B/\mathfrak{P}$.

Proof. after finishing construciton of Artin^{-1}. □

3 Construction of the Inverse to the Artin Map

We fix the tuple (F, ∞, ε) and a Hayes module $\phi \in X_\varepsilon$. Let

$$F_\infty^+ := \{x \in F_\infty^\times \mid \varepsilon(x) = 1\} = \ker(\varepsilon : F_\infty \to \mathbb{F}_\infty^\times).$$

3.1 λ-adic representation

Let λ be a place of F different from ∞, corresponding to a *maximal ideal* λ of A.

Take $e \geq 1$ and consider $\phi[\lambda^e]$. By Proposition 2.3, $\phi[\lambda^e]$ is an A/λ^e-module of rank 1. Define the **λ-adic Tate module** to be

$$T_\lambda(\phi) := \mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty]).$$

Proposition 3.1. $T_\lambda(\phi)$ is a free $\mathcal{O}_\lambda$-module of rank 1.

Proof. The ring $\mathcal{O}_\lambda$ is a DVR, so

$$\mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(\mathcal{O}_\lambda/\mathfrak{m}_\lambda^e, \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(A/\lambda^e, \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(A/\lambda^e, \phi[\lambda^e]).$$

□

Hence

$$V_\lambda(\phi) := T_\lambda(\phi) \otimes_{\mathcal{O}_\lambda} F_\lambda$$

is a 1-dimensional F_λ-vector space.

Of course the next step is to try to find a Galois action. There is some ideal $\mathfrak{a}$ such that $\sigma(\phi) = \mathfrak{a} * \phi$, and thus we have two isomorphisms between this one and ϕ:

- σ induces an isomorphism $V_\lambda(\sigma) : V_\lambda(\phi) \simeq V_\lambda(\sigma(\phi))$,

- $\phi_{\mathfrak{a}}$ induces an isomorphism[8] $V_\lambda(\phi_{\mathfrak{a}}) : V_\lambda(\phi) \simeq V_\lambda(\mathfrak{a} * \phi)$.

So we obtain an element

$$V_\lambda(\phi_{\mathfrak{a}})^{-1} \circ V_\lambda(\sigma) \in \mathrm{GL}_{F_\lambda}(V_\lambda(\sigma)) = F_\lambda^\times \cdot \mathrm{id},$$

corresponding to an element $\rho_\lambda^{\mathfrak{a}}(\sigma) \in F_\lambda^\times$.

3.2 ∞-adic representation

3.3 The inverse of Artin map

4 Example: the Rational Function Field

Let $F = k(t)$.

5 Comparision with Elliptic Curves

6 Proof of (some) lemmas

[8]Because?