Notes on Explicit CFT for Function Fields

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March 9, 2025

1 Review of CFT

Let F be a global field, $C_F = \mathbb{A}_F^\times / F^\times$ be its idele class group, and F^{ab} be its maximal abelian extension inside a separable closure in a fixed algebraic closure $\bar{F}$. The class field theory asserts that the Artin map

$$\theta_F : C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

is a continuous group homomorphism with dense image, establishing a bijection

$$\{\text{finite abelian extensions of } F\} \longleftrightarrow \{\text{finite index open subgroups of } C_F\}.$$

The direction "$\to$" is computable: for a finite abelian L/F, the corresponding open subgroup of C_F is the kernel U of $C_F \xrightarrow{\theta_F} \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(L/F)$, which can be computed as $U = N_{L/F}(C_L)$[1].

The goal of explictit class field theory is to find the construciton "$\leftarrow$", and to describe F^{ab}. Known cases for number fields inlcude $\mathbb{Q}$ and imaginary quadratic fields, and they all uses torsion points of some geometric object ($\mathbb{G}_{\mathrm{m}}$ and CM elliptic curves, respectively). In the article [Zyw11], the author constructed the inverse of Artin map for function fields using one distinguished "place at infinity" with a sign function as well as Drinfeld modules, a characteristic p analogue for $\mathbb{G}_{\mathrm{m}}$ and elliptic curves. In the end, he described explicitly the structure of $k(t)^{\mathrm{ab}}$, the maximal abelian extension of the field of rational functions over a finite field k. Most of the proofs for general fact about Drinfeld modules can be found in [Gos12], and those specific for function fields can be found in [Hay74] and [Zyw11].

2 Function Fields and Drinfeld Modules

Let $k = \mathbb{F}_q$ be a finite field, F be a global function field with a fixed place[2] ∞, and with field of constants k, i.e. F is a finite extension of the field of rational functions $k(t)$ over k.

If λ is a place of F, we denote by F_λ the completion at λ, by $\mathbb{C}_\lambda$ the completion of $\overline{F}_\lambda$, by $\mathcal{O}_\lambda \subset F_\lambda$ the valuation ring, by $\mathbb{F}_\lambda := \mathcal{O}_\lambda/\mathfrak{m}_\lambda$ the residue field at λ, and by ord_λ the normalized valuation on F_λ with value group $\mathbb{Z}$. We regard $\mathbb{F}_\lambda \subset \mathcal{O}_\lambda \subset F_\lambda$ as a subfield via the Teichmüller lifting.

For any extension L of k, we denote by $\bar{L}$ an algebraic closure. Let L^{sep} be the separable closure of L in $\bar{L}$, $\mathrm{Gal}_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ be the absolute Galois group.

[1] $N_{L/F} : C_L \to C_F$ is the norm map. The norm for an idele is just the multiplication of the norm at every places.

[2] A **place** of a function field is a valuation subring, or equivalently, an equivalence class of discrete valuations. Note that there are no archimedean places.

2.1 The holomorphy rings

Let $A := \{x \in F \mid \mathrm{ord}_\lambda(x) \geq 0, \forall \lambda \neq \infty\}$, the ring of functions that are regular away from ∞. By the general theory of holomorphy rings, A is a Dedekind domain with fractional field $\mathrm{Frac}(A) = F$, and there is a 1-1 correspondence between maximal ideals of A and the places of F except for ∞.

2.2 The Weil group

Let L be an extension of k. The algebraic closure $\bar{k}$ of k in $\bar{F}$ is contained in L^{sep}, and the absolute Galois group $\mathrm{Gal}_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ stabilizes $\bar{k}$. Therefore, we can construct Weil group for L just like for local fields. The **Weil group** is the subgroup W_L of Gal_L of elements σ that acts on $\bar{k}$ by an integral power of the Frobenius-q, i.e. $\sigma(x) = x^{q^{\deg(\sigma)}}$ for $\sigma \in W_L,\ x \in \bar{k}$. The kernel of the map $\deg : W_L \to \mathbb{Z}$ is still $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$. We endow W_L with the weakest topology for which

$$1 \longrightarrow \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k}) \longrightarrow W_L \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

is an exact sequence of topological groups, where $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$ has its usual profinite topology and $\mathbb{Z}$ has discrete topology[3]. The inclusion $W_L \hookrightarrow \mathrm{Gal}_L$ is still continuous with dense image.

2.3 Drinfeld modules and isogenies

Let L be an extension of k, $L[T]$ be the ring of polynomial over L. Consider the Frobenius-q map

$$\tau : L[T] \to L[T] \quad \sum_{i=0}^{n} a_i T^i \mapsto \sum_{i=0}^{n} a_i^q T^{iq}.$$

This is a k-linear endomorphism of $L[T]$, and we denote by $L[\tau]$ the sub-L-algebra of $\mathrm{End}_k(L[T])$ generated by τ. The ring $L[\tau]$ is a ring of **twisted polynomials**, because it is non-commutative: $\tau a = a^q \tau$, $\forall a \in L$.

Recall that $A = \{x \in F \mid \mathrm{ord}_\lambda(x) \geq 0, \forall \lambda \neq \infty\}$. Let L be an extension of F. A **Drinfeld A-module**[4] **over** L is a homomorphism

$$\phi : A \to L[\tau] \quad x \mapsto \phi(x) =: \phi_x$$

of k-algebras, such that $\phi(A)$ is *not contained* in $L \subset L[\tau]$, and the map

$$A \to L[\tau] \to L \quad x \mapsto \phi_x = a_0 + a_1\tau + \cdots + a_n\tau^n \mapsto a_0$$

is the restriction of the inclusion map $F \hookrightarrow L$ to A. In particular, $\phi : A \hookrightarrow L[\tau]$ is injective.

Let ϕ and ϕ' be two Drinfeld modules $A \to L[\tau]$, M be an extension of L. An **isogeny** over M from ϕ to ϕ' is an $f \in M[\tau] \smallsetminus \{0\}$ such that

$$f\phi_a = \phi'_a f, \quad \forall a \in A.$$

An **isomorphism** over M from ϕ to ϕ' is an invertible isogeny, namely an isogeny $f \in M[\tau]^\times$.

2.3.1 Torsion submodules and the rank

A Drinfeld module $\phi : A \to L[\tau]$ defines an A-module structure on $\bar{L}$ by

$$x \cdot b := \phi_x(b), \quad \forall x \in A, b \in \bar{L}.$$

[3]This is not the topology induced from $\mathbb{Z} \subset \hat{\mathbb{Z}}$.

[4]There is a more general definition, but we only need and consider Drinfeld modules of this kind.

Every ϕ_x acts by a polynomial $\phi_x(T) = a_0T + a_1T^q + \cdots + a_nT^{q^n}$ with $a_i \in L$. This polynomial is separable, because $x \mapsto \phi_x \mapsto a_0$ is injective. Therefore ϕ gives an A-module structure on L^{sep}.

For an ideal $\mathfrak{a}$ of A, we define the $\mathfrak{a}$-torsion submodule to be

$$\phi[\mathfrak{a}] := \left\{b \in \bar{L} \mid \phi_x(b) = 0, \forall x \in \mathfrak{a}\right\},$$

an A-submodule of L^{sep} with A-module structure from ϕ, carrying a natrual Gal_L-action.

Similar to elliptic curves, $\phi[\mathfrak{a}]$ is a finite free $A/\mathfrak{a}$-module, whose rank $r \in \mathbb{Z}$ is the same for all ideals $\mathfrak{a} \subset A$. We call this number r the **rank** of the Drinfeld module ϕ. It is an isogeneous invariant.

2.4 The sign functions and the ε-normalized Drinfeld modules

A **sign function for** F_∞ is a group homomorphism $F_\infty^\times \to \mathbb{F}_\infty^\times$ such that $\varepsilon|_{\mathbb{F}_\infty^\times} = \mathrm{id}_{\mathbb{F}_\infty^\times}$, and we write

$$F_\infty^+ := \{x \in F_\infty^\times \mid \varepsilon(x) = 1\} = \ker(\varepsilon : F_\infty \to \mathbb{F}_\infty^\times).$$

Such a function ε is determined by its value on any uniformizer[5].

We will fix a sign function ε for F_∞ and require our Drinfeld modules to be ε**-normalized**. This is a technical condition we don't need to worry much, because every Drinfeld module over L is isomorphic to some ε-normalized Drinfeld module of the *same rank* over the algebraic closure $\bar{L}$.

2.5 Hayes modules and group actions on it

Fix a sign function $\varepsilon : F_\infty^\times \to \mathbb{F}_\infty^\times$ for F_∞. A **Hayes module for** ε is a ε-normalized Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$ of rank 1. The Drinfeld modules of rank 1 over $\mathbb{C}_\infty$ exist and can be constructed analytically. Since $\mathbb{C}_\infty$ is algebraically closed, the Hayes modules must exist.

Let X_ε be the set of Hayes modules for ε. There is a natrual action of the group $\mathcal{I}_A$ of fractional ideals of A on X_ε, denoted by

$$(\mathfrak{a}, \phi) \mapsto \mathfrak{a} * \phi, \quad \mathfrak{a} \in \mathcal{I}_A,\ \phi \in X_\varepsilon.$$

This action has the following properties.

(i) If $\mathfrak{a} \subset A$ is an integral ideal, then there is a unique $\phi_\mathfrak{a} \in L[\tau]$, and $\mathfrak{a} * \phi$ is the unique Drinfeld module making $\phi_\mathfrak{a}$ an isogeny $\phi \to \mathfrak{a} * \phi$. In particular, $\phi_A = 1$ and $A * \phi = \phi$. These isogenies are important in later construcitons.

(ii) The subgroup $\mathcal{P}_A^+ := \{(x) \mid x \in F^\times \cap F_\infty^+\}$ of $\mathcal{I}_A$ acts trivially on X_ε.

We call $\mathrm{Pic}^+(A) := \mathcal{I}_A/\mathcal{P}_A^+$ the **narrow class group**, so that X_ε is a $\mathrm{Pic}^+(A)$-set.

Proposition 2.1. The set X_ε is a principal homogeneous space for $\mathrm{Pic}^+(A)$, i.e. $\mathrm{Pic}^+(A)$ acts freely and transitively on X_ε.

The group $\mathrm{Pic}^+(A)$ will be realized as the Galois group for an "almost" unramified extension. Define the **narrow Hilbert class field** or the **normalizing field for** (F, ∞, ε) to be the extension

$$H_A^+ := F\left(\{\text{coefficient of } \phi_x \mid \phi \in X_\varepsilon, x \in A\}\right)$$

of F in $\mathbb{C}_\infty$. This is the minimal extension of F on which all Hayes modules for ε are defined.

[5] Choosing a uniformizer π of F_∞ yields a decomposition $F_\infty^\times \simeq \mathbb{F}_\infty^\times \times (1 + \mathfrak{m}_\infty) \times \pi^\mathbb{Z}$. The value of ε on $\mathbb{F}_\infty^\times$ is fixed, and it must be trivial on the pro-q group $1 + \mathfrak{m}^\infty$.

Proposition 2.2. The extension H_A^+/F is finite abelian, and it is unramified away from ∞.

There is thus a natrual action of Gal_F on X_ε through $\mathrm{Gal}(H_A^+/F)$, given by

$$\sigma(\phi)_x := \sigma(\phi_x)^{6}, \quad \forall \sigma \in \mathrm{Gal}_F,\ \phi \in X_\varepsilon,\ x \in A.$$

Any $\phi \in X_\varepsilon$, by Proposition 2.1, induces an injective group homomorphism

$$\Psi : \mathrm{Gal}(H_A^+/F) \hookrightarrow \mathrm{Pic}^+(A),$$

such that $\sigma(\phi) = \Psi(\sigma) * \phi$ for all $\sigma \in \mathrm{Gal}_F$.

Proposition 2.3. $\Psi : \mathrm{Gal}(H_A^+/F) \to \mathrm{Pic}^+(A)$ is an isomorphism, independent of the choice of ϕ. For each non-zero prime $\mathfrak{p}$ of A, the class of $\Psi(\mathrm{Frob}_\mathfrak{p})$ in $\mathrm{Pic}^+(A)$ equals the class of $\mathfrak{p}$.

3 Construction of the Inverse to the Artin Map

We fix the tuple (F, ∞, ε) and a Hayes module $\phi \in X_\varepsilon$.

3.1 λ-adic representation

Let λ be a place of F. Take $\sigma \in \mathrm{Gal}_F$. By Proposition 2.3, pick an ideal $\mathfrak{a}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$.

- $\lambda \neq \infty$. Regarding λ as a prime ideal of A, we consider the rank 1 free A/λ^e-module $\phi[\lambda^e]$ for $e \in \mathbb{Z}_{\geq 1}$. Define the **$\lambda$-adic Tate module** to be

  $$T_\lambda(\phi) := \mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty]),$$

  which is a free $\mathcal{O}_\lambda$-module of rank 1. Hence $V_\lambda(\phi) := T_\lambda(\phi) \otimes_{\mathcal{O}_\lambda} F_\lambda$ is an 1-dimensional F_λ-vector space. We have the following two isomophisms between vector spaces.

  - σ induces $\phi[\lambda^e] \simeq (\sigma(\phi))[\lambda^e]$ for all $e \in \mathbb{Z}_{\geq 1}$, patching to an isomophism $V_\lambda(\sigma) : V_\lambda(\phi) \simeq V_\lambda(\sigma(\phi))$.
  - The isogeny $\phi_\mathfrak{a} : \phi \to \mathfrak{a} * \phi$ induces an isomorphism[7] $V_\lambda(\phi_\mathfrak{a}) : V_\lambda(\phi) \simeq V_\lambda(\mathfrak{a} * \phi)$.

  As $\mathfrak{a} * \phi = \sigma(\phi)$, we obtain an element $V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\phi) \in \mathrm{GL}_{F_\lambda}(V_\lambda(\sigma)) = F_\lambda^\times \cdot \mathrm{id}$, corresponding to an element $\rho_\lambda^\mathfrak{a}(\sigma) \in F_\lambda^\times$.

- $\lambda = \infty$. If $\sigma \in W_F$, the next Lemma 3.1 provides a unique element $\rho_\infty^\mathfrak{a}(\sigma) \in F_\infty^+$.

  Lemma 3.1. There exists some series $u \in F^{\mathrm{sep}}[\![\tau^{-1}]\!]^\times$, such that $u^{-1}\phi(F_\infty)u \subset \bar{k}(\!(\tau^{-1})\!)$.[8] For such a series u, if $\sigma \in W_F$, then there is a unique element $\rho_\infty^\mathfrak{a}(\sigma) \in F_\infty^+$, such that

  $$\phi_\mathfrak{a}^{-1} \cdot \sigma(u) \cdot \tau^{\deg(\sigma)} \cdot u^{-1} = \phi(\rho_\infty^\mathfrak{a}(\sigma)).$$

These elements $\rho_\lambda^\mathfrak{a}(\sigma)$ has the following properties.

Lemma 3.2. Let λ be a place of F, $\sigma, \gamma \in \mathrm{Gal}_F$ (in W_F if $\lambda = \infty$) and $\mathfrak{a}, \mathfrak{b}$ be ideals of A.

[6] Gal_F acts on $\bar{F}[\tau]$ by acting on the coefficients. It is direct to check that Gal_F stabilizes X_ε by definition.

[7] Since ϕ has rank 1, it is equivalent to that $V_\lambda(\phi_\mathfrak{a})$ is non-zero. This is true, because, parallel to elliptic curves, taking Tate module is a faithful functor; see [Gos12], §4.10.

[8] Any Drinfeld module $\phi : A \to H_A^+[\tau]$ extends to an injective homomorphism $\phi : F_\infty \to \left(H_A^+\right)^{\mathrm{perf}}(\!(\tau^{-1})\!)$.

(i) If $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$, then $(\sigma\gamma)(\phi) = (\mathfrak{ab}) * \phi$, and $\rho_\lambda^{\mathfrak{ab}}(\sigma\gamma) = \rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\gamma)$.

(ii) If $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$, then $\rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\sigma)^{-1} \in F^\times \cap F_\infty^+$ and $\mathfrak{b}^{-1}\mathfrak{a}$ is generated by $\rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\sigma)^{-1}$

(iii) If $\lambda \neq \infty$, and $\sigma(\phi) = \mathfrak{a} * \phi$, then $\mathrm{ord}_\lambda(\rho_\lambda^{\mathfrak{a}}(\sigma)) = -\mathrm{ord}_\lambda(\mathfrak{a})$, the largest power of λ dividing $\mathfrak{a}$.

If $\sigma \in \mathrm{Gal}_{H_A^+}$, then $\sigma(\phi) = \phi = A * \phi$. By Lemma 3.2 (i), we obtain homomorphisms

$$\rho_\lambda : \mathrm{Gal}_{H_A^+} \to \mathcal{O}_\lambda^\times \quad \sigma \mapsto \rho_\lambda^A(\sigma)$$

for $\lambda \neq \infty$, and the homomorphism

$$\rho_\infty : W_{H_A^+} \to F_\infty^+, \quad \sigma \mapsto \rho_\infty^A(\sigma).$$

In particular, $\phi_A = 1$, so the representation ρ_λ is the representation of $\mathrm{Gal}_{H_A^+}$ on $T_\lambda(\phi)$ and hence it takes value in $\mathcal{O}_\lambda^\times$. These representations ρ_λ are continuous and unramified at all places of H_A^+ not over λ or ∞.

3.2 The inverse of the Artin map

For each $\sigma \in W_F$, fix an ideal $\mathfrak{a}_\sigma$ of A, such that $\sigma(\phi) = \mathfrak{a}_\sigma * \phi$. By Lemma 3.2, $(\rho_\lambda^{\mathfrak{a}_\sigma}(\sigma))_\lambda$ is an idele of F, whose class $\rho(\sigma)$ in C_F is independent of the choice of $\mathfrak{a}_\sigma$, and the map

$$\rho : W_F \to C_F, \quad \sigma \mapsto \rho(\sigma)$$

is a group homomorphism. The restriction of $\rho : W_F \to C_F$ to $W_{H_A^+}$ is

$$W_{H_A^+} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \hookrightarrow \mathbb{A}_F^\times \twoheadrightarrow C_F.$$

This homomorphism is continuous since all ρ_λ are continuous. The group $W_{H_A^+}$ has finite index in W_F, so ρ is continuous on W_F. Taking profinite completion yields a continuous homomorphism

$$\hat{\rho} : \mathrm{Gal}_F \to \hat{C}_F.$$

that factors through the maximal abelian quotient $\mathrm{Gal}_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Theorem 1. The map $\hat{\rho} : \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F$ is a topological isomophism depends only on F, and the map

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F \quad \sigma \mapsto \hat{\rho}(\sigma)^{-1}$$

is the inverse of the Artin map $\hat{\theta}_F : \hat{C}_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Sketch of the proof. Let $U < C_F$ be an open subgroup of finite index. Consider the finite abelian extension $L_U := (F^{\mathrm{ab}})^{\rho^{-1}(U)}$ of F fixed by $\rho^{-1}(U) < W_F^{\mathrm{ab}}$, so that we have an injective continuous homomorphism

$$\rho_U : \mathrm{Gal}(L_U/F) \simeq \mathrm{Gal}_F^{\mathrm{ab}} / \mathrm{Gal}_{L_U}^{\mathrm{ab}} \simeq W_F^{\mathrm{ab}}/\rho^{-1}(U) \hookrightarrow C_F/U.$$

Using weak approximation and the description of ρ_λ on (almost all) Frobenius elements, one can show that there is a finite set of places S_U containing ∞ and all places ramified in L_U/F, such that:

- for each $\mathfrak{p} \notin S_U$, ρ_U sends $\mathrm{Frob}_\mathfrak{p}$ to the class of $(\cdots, 1, \pi_\mathfrak{p}^{-1}, 1, \cdots)$, where $\pi_\mathfrak{p}$ is an uniformizer of $F_\mathfrak{p}$;
- $\rho_U : \mathrm{Gal}(L_U/F) \to C_F/U$ is surjective and thus an isomophism.

Therefore the pointwise inverse of ρ_U^{-1} is $C_F/U \to \mathrm{Gal}(L_U/F)$, $\alpha \mapsto (\rho_U^{-1}(\alpha))^{-1} = \theta_F(\alpha)|_{L_U}$, the Artin map.

The result above together with class field theory shows that $F^{\mathrm{ab}} = \bigcup_U L_U$. Passing to the limit of these compatible isomophisms $\{\rho_U\}_U$, we get back to $\hat{\rho} : \mathrm{Gal}_F^{\mathrm{ab}} \to C_F$ and see that it is an isomophism, whose inverse is the point-wise inverse of the Artin map $\hat{\theta}_F$. $\square$

4 Example: the Rational Function Field

Let $F = k(t)$. We consider the usual place ∞, so that $A = k[t]$, $F_\infty = k(\!(t^{-1})\!)$, $\mathbb{F}_\infty = k$, $\mathfrak{m}_\infty = t^{-1}k[\![t^{-1}]\!]$, $\mathrm{ord}_\infty(t^{-1}) = 1$. Let $\varepsilon : F_\infty^\times \to k^\times$ be the sign function defined by $\varepsilon(t^{-1}) = 1$, so that $F_\infty^+ = t^{\mathbb{Z}} \cdot (1 + \mathfrak{m}_\infty)$.

The **Carlitz module** ϕ is a Hayes module for ε defined by

$$\phi : A = k[t] \to F[\tau] \quad t \mapsto \phi_t := t + \tau.$$

The normalizing field for (F, ∞, ε) is $H_A^+ = F$, so ϕ is the only Hayes module for ε.

We have defined the representations $\rho_\lambda : W_F^{\mathrm{ab}} \to F_\lambda^\times$. As a corollary of Theorem 1,

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \to C_F$$

is an isomophism. Similar to $\mathbb{Q}$, the second arrow above is an isomophism[9], and thus the first arrow

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \times t^{\mathbb{Z}} \times (1 + \mathfrak{m}_\infty)$$

is also an isomophism. Taking profinite completion, we got a decomposition

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \simeq \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \times t^{\hat{\mathbb{Z}}} \times (1 + \mathfrak{m}_\infty)$$

of $\mathrm{Gal}_F^{\mathrm{ab}}$, corresponding to three disjoint abelian extension of F whose compositum is F^{ab}.

The "cyclotomic" extension K_∞

For $\lambda \neq \infty$, the representation $\rho_\lambda : \mathrm{Gal}_F \to \mathcal{O}_\lambda^\times$ is precisely the Galois representation on $T_\lambda(\phi)$, where ϕ is the Carlitz module. The representation

$$\chi := \prod_{\lambda \neq \infty} \rho_\lambda : \mathrm{Gal}_F \to \prod_{\lambda \neq \infty} \mathcal{O}_\lambda = \hat{A}^\times$$

is the inverse limit of $\chi_m : \mathrm{Gal}_F \to (A/(m))^\times$, which are from the Gal_F-action on $\phi[m]$ for monic $m \in k[t]$, ordered by divisibility. Hence the field fixed by $\ker(\chi)$ is $K_\infty = \bigcup_m F(\phi[m])$. The extension K_∞/F is a geometric extension[10], tamely ramified at ∞[11].

The extension of constants $\bar{k}(t)$

For each $\sigma \in W_F$, the factor in $t^{\mathbb{Z}} \simeq \mathbb{Z}$ is $\mathrm{ord}_t(\rho_\infty(\sigma))$. One can show that this number is $\deg(\sigma)$. The field fixed by (the closure of) $\ker(\deg)$ is $\bar{k}(t)$, and the extension $\bar{k}(t)/k(t)$ is the maximal constant field extension.

The wildly ramified extension L_∞

By discussion above, the projection onto $1 + \mathfrak{m}_\infty$ is

$$W_F \to 1 + \mathfrak{m}_\infty \quad \sigma \mapsto \rho_\infty(\sigma)/\,\mathrm{ord}_t(\rho_\infty(\sigma)) = \rho_\infty(\sigma)/\deg(\sigma).$$

[9]Let $x \in \mathbb{A}_F^\times$, Every place $\lambda \neq \infty$ has a "canonical" uniformizer $\mathfrak{p} \in k[t]$, namely the unique monic irreducible polynomial, and we write $x_\mathfrak{p} = u_\mathfrak{p}\mathfrak{p}^{n_\mathfrak{p}}$ with $u_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$. Put $f := \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}} \in k(t)^\times$. We have $f^{-1}x_\infty = a_\infty t^n +$ terms with lower degree in t for some $a_\infty \in k$. Then $(a_\infty f)^{-1}x \in F_\infty^+ \times \prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times$. This decomposition of ideles provides the desired isomophism.

[10]A **geometric extension** is an extension of function fields that doesn't extend the field of constants.

[11]The ramification indexes are all $q - 1$; see [Hay74], §3.

Let $\beta : \mathrm{Gal}_F \to 1 + \mathfrak{m}_\infty$ be its profinite completion. Denote by L_∞ the fixed field of $\ker(\beta)$. The extension L_∞/F is a geometric extension, unramified away from ∞ and wildly ramified at ∞.

To describe this field explicitly, we need to look at the construciton of ρ_∞. Choose recursively a sequence of elements $\{a_i\}_{i\geq 0} \subset F^{\mathrm{sep}}$ by

$$a_0 := 1; \quad a_i^q - a_i = -ta_{i-1},\ i \geq 1.$$

Then $u := \sum_{i\geq 0} a_i \tau^{-i}$ verifies the condition of u in Lemma 3.1. For $\sigma \in W_F$, $\rho_\infty(\sigma) \in F_\infty^+$ is characterized by $\phi(\rho_\infty(\sigma)) = \sigma(u)\tau^{\deg(\sigma)}u^{-1}$. Every $\sigma \in \mathrm{Gal}(L_\infty/F)$ has representatives in W_F with $\deg = 0$ since it acts trivially on $\bar{k}$. Hence $\phi(\beta(\sigma)) = \sigma(u)u^{-1}$, which shows that[12] $\beta(\sigma) = 1$ if and only if $\sigma(u) = u$, and thus $L_\infty = F\left(\{a_i\}_{i\geq 0}\right)$.

References

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[12] Recall that ϕ is injective.