# Homework

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*Proof.* If $\{x_n\}_n$ is eventually periodic, we may assume that $\{x_n\}_n$ is periodic; that is, $\exists t \geq 0$ s.t.

$$x_{n+t} = x_n,\ \forall n \in \mathbb{Z}, n \geq 0.$$

Otherwise we may just subtract the non-periodic part, which is an integer and doesn't affect the rationality of $x$. Let $a := \sum_{j=0}^{t-1} x_j p^j \in \mathbb{Z}$, then

$$x = \sum_{i\geq 0}\sum_{j=0}^{t-1} x_{it+j} p^{it} p^j = \sum_{i\geq 0} p^{it} \sum_{j=0}^{t-1} x_j p^j = \frac{a}{1-p^t} \in \mathbb{Q}.$$

Conversely, suppose that $x = \frac{a}{b} \in \mathbb{Q}$, where $a, b \in \mathbb{Z}$ are coprime and $b \geq 1$. Because $x \in \mathbb{Z}_p$, we have $p \nmid b$, and thus there is an integer $t \geq 1$ s.t. $b \mid p^t - 1$. Write $c := \frac{1-p^t}{b}$ [1], then

$$x = \frac{ac}{1-p^t},$$

and

$$ac = x(1-p^t) = \sum_{n\geq 0} x_n p^n - \sum_{n\geq 0} x_n p^{n+t} = \sum_{i=0}^{t-1} x_i p^i + \sum_{n\geq t}(x_n - x_{n-t})p^n. \quad .$$

It suffices to show that $x_n - x_{n-t} = 0$ for all $n$ large enough. Note that either

$$\sum_{n\geq t}(x_n - x_{n-t})p^n = ac - \sum_{i=1}^{t-1} x_i p^i$$

or

$$\sum_{n\geq t}(x_{n-t} - x_n)p^n = \sum_{i=1}^{t-1} x_i p^i - ac$$

is a positive integer, and thus have a fintie expansion in base $p$.

[1] I feels that a simpler way to proof that $\{x_n\}$ is eventually periodic by showing that: (1) $x$ has an eventually periodic expansion implies that

$$\begin{aligned} -x &= p^k - p^k - x \\ &= p^k + \sum_{n\geq k}(p-1-x_n)p^n \\ &= (p-x_k)p^k + \sum_{n\geq k+1}(p-1-x_n)p^n \end{aligned}$$

does, and (2) $x$ has an eventually periodic expansion implies that

$$x + \frac{1}{1-p^t} = x + \sum_{n\geq 0} p^{nt}$$

does (I think the period of $x + \frac{1}{1-p^t}$ should divides $\operatorname{lcm}(T, t)$, where $T$ is the period of $x$.). But, I have completed this tedious proof below so didn't write another...

Consider first the case of

$$\sum_{n\geq t}(x_n - x_{n-t})p^n = \sum_{i=0}^{r} y_i p^i \tag{1}$$

being postive, where $y_i \in \{0, 1, \ldots, p-1\}$.

(A) If $r < t$, then

$$v_p\left(\sum_{n\geq t}(x_n - x_{n-t})p^n\right) \geq t$$

and

$$v_p\left(\sum_{i=0}^{r} y_i p^i\right) \leq r < t,$$

if $\sum_{i=0}^{r} y_i p^i \neq 0$. Therefore

$$\sum_{n\geq t}(x_n - x_{n-t})p^n = \sum_{i=0}^{r} y_i p^i = 0.$$

Since

$$1 - p \leq x_{n-t} - x_n \leq p - 1, \quad v_p\left((x_n - x_{n-t})p^n\right) = \begin{cases} n, & x_n \neq x_{n-t}, \\ \infty, & x_n = x_{n-t} \end{cases}$$

for all $n \geq t$, we have

$$v_p\left(\sum_{n\geq t}(x_n - x_{n-t})p^n\right) = \min\{n \geq t | x_n \neq x_{n-t}\},$$

where $\min \varnothing = \infty$, and thus $x_n = x_{n-t}$ for all $n \geq t$.

(B) If $r \geq t$, then

$$\sum_{i=0}^{t-1} y_i p^i + \sum_{j=t}^{r}(y_j - x_j + x_{j-t})p^j + \sum_{n\geq r+1}(x_{n-t} - x_n)p^n = 0.$$

Again by computing $p$-adic valuation, we see that $y_0 = \cdots = y_{t-1} = 0$, and

$$p^t\left(\sum_{j=t}^{r}(y_j - x_j + x_{j-t})p^{j-t} + \sum_{n\geq r+1}(x_{n-t} - x_n)p^{n-t}\right) = 0.$$

Hence,

$$\sum_{j=t}^{r}(y_j - x_j + x_{j-t})p^{j-t} + \sum_{n\geq r+1}(x_{n-t} - x_n)p^{n-t} = 0. \tag{2}$$

For simplicity, put

$$a_n := \begin{cases} y_n - x_n + x_{n-t}, & t \leq n \leq r, \\ x_{n-t} - x_n, & n \geq r+1. \end{cases}$$

We have

$$1 - p \leq y_j - x_j + x_{j-t} \leq 2p - 2,$$

for all $t \leq j \leq r$, and

$$1 - p \leq x_{n-t} - x_n \leq p - 1,$$

for all $n \geq r + 1$.

From Eq. (2), we see that

$$a_t = -\sum_{n\geq t+1} a_n p^{n-t} = -p\sum_{n\geq t+1} a_j p^{n-t-1}$$

is divided by $p$ in $\mathbb{Z}_p$. Hence $a_t = 0$ or $p$, so

$$a_{t+1} + \sum_{n \geq t+2} a_n p^{n-t-1} = 0$$

or

$$(a_{t+1} + 1) + \sum_{n \geq t+2} a_n p^{n-t-1} = 0.$$

This procedure continues. More precisely, define $b_t := a_t$ and

$$b_j := \begin{cases} a_j, \ b_{j-1} = 0, \\ a_j + 1, \ b_{j-1} = p, \end{cases} \qquad j \geq t+1.$$

Then we can show inductively that

$$b_j + \sum_{n \geq j+1} a_n p^{n-j} = 0 \tag{3}$$

and $b_j \in \{0, p\}$ for all $j \geq t$.

Consider $j \geq r+1$. If $b_j = 0$, then $b_{j+1} = a_{j+1} \in \{1-p, \ldots, p-1\}$, so $b_{j+1} = 0$. Therefore, we get recursively that if $b_N = 0$ for some $N \geq r+1$, then $a_{j+1} = b_{j+1} = 0$, i.e., $x_{j-t} = x_j$ for all $j > N$.

Now suppose that $b_j = p$ for all $j \geq r+1$. Then

$$x_{j-t} - x_j = a_j = b_j - 1 = p - 1$$

for all $j \geq r+2$. However, $x_{j-t} - x_j = p-1$ if and only if $x_{j-t} = p-1$ and $x_j = 0$, which cannot be true for all $j \geq r+2$.

Therefore, we proved that $\{x_n\}_n$ is eventually periodic given $\sum_{n \geq t}(x_n - x_{n-t})p^n > 0$. But there is no hard to construct a similar proof in the case of $\sum_{n \geq t}(x_{n-t} - x_n)p^n > 0$. $\square$