

Note on Modular Forms

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November 6, 2024

1 Hecke Operators

Throughout this section, we fix the following data:

- a group Ω ;
- a submonoid $\Delta \subset \Omega$;
- a nonempty collection \mathcal{X} of subgroups of Ω , in which all members are commensurable¹ to each other, and

$$\Gamma \subset \Delta \subset \tilde{\Gamma} := \{g \in \Omega \mid g\Gamma g^{-1} \approx \Gamma\}, \forall \Gamma \in \mathcal{X};$$

- a commutative ring \mathbb{K} ;
- a left \mathbb{K} -module M with a right Δ -action $(m, \delta) \mapsto m\delta$, i.e, a monoid homomorphism

$$\Delta \rightarrow \text{End}_{\mathbb{K}}(M) \quad \delta \mapsto m \mapsto m\delta.$$

1.1 Commensurability

Recall that two subgroups $\Gamma, \Gamma' < \Omega$ are commensurable if both $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are finite, and this is an equivalence relation.

Lemma 1.1. $\tilde{\Gamma}$ is a group and depends only on the commensurable class of Γ . □

Proposition 1.1. Let $\alpha \in \tilde{\Gamma}$ and $\Gamma \approx \Gamma'$. Then there is a bijection

$$\begin{aligned} \Gamma' \cap (\alpha^{-1}\Gamma\alpha) \backslash \Gamma' &\longleftrightarrow \Gamma \backslash \Gamma\alpha\Gamma' \\ \Gamma''^2 x &\longleftrightarrow \Gamma\alpha x \end{aligned}$$

and $\Gamma \backslash \Gamma\alpha\Gamma'$ is finite.

Proof. The map

$$\Gamma' \rightarrow \Gamma \backslash \Gamma\alpha\Gamma' \quad x \mapsto \Gamma\alpha x$$

is clearly surjective. Now $\forall x, y \in \Gamma'$,

$$\begin{aligned} \Gamma\alpha x = \Gamma\alpha y &\iff \exists g \in \Gamma, g\alpha x = \alpha y \\ &\iff \exists g' \in \Gamma'', g'x = y; \end{aligned}$$

so injective.

By definitions and the last lemma, $\Gamma' \cap (\alpha^{-1}\Gamma\alpha) \approx \Gamma'$, giving finiteness. □

¹Write $\Gamma \approx \Gamma'$ if Γ is commensurable to Γ' .

²Of course, $\Gamma'' = \Gamma' \cap (\alpha^{-1}\Gamma\alpha)$.

1.2 Double Coset Algebra

1.2.1 Double Cosets and Convolution

Recall that the \mathbb{K} -module $\mathcal{F}(\Omega, \mathbb{K})$ of all functions $\Omega \rightarrow \mathbb{K}$ admits a \mathbb{K} -linear left Ω -action

$$(\gamma f)(z) := f(\gamma^{-1}z)$$

and a right Ω -action

$$(f\gamma)(z) := f(z\gamma).$$

Def-Theorem 1. Let $\Gamma, \Gamma' \in \mathcal{X}$. Define $\mathcal{H}(\Gamma \backslash \Delta / \Gamma')$ to be the \mathbb{K} -module³ consists of functions $f : \Omega \rightarrow \mathbb{K}$ such that:

- $\text{supp } f \subset \Delta$ and $\Gamma \backslash \text{supp } f / \Gamma'$ is a finite set,
- f is left- Γ -invariant and right- Γ' -invariant.

Then $\mathcal{H}(\Gamma \backslash \Delta / \Gamma')$ is a *free* \mathbb{K} -module, with a basis given by the double cosets in $\Gamma \backslash \Delta / \Gamma'$, i.e.,

$$[\Gamma \gamma \Gamma'] := \mathbf{1}_{\Gamma \gamma \Gamma'}, \quad \gamma \in \Delta.$$

We thus identify $\mathcal{H}(\Gamma \backslash \Delta / \Gamma')$ with the free module $\mathbb{Z}[\Gamma \backslash \Delta / \Gamma']$ generated by $\Gamma \backslash \Delta / \Gamma'$, and we identify the function $[\Gamma \gamma \Gamma'] := \mathbf{1}_{\Gamma \gamma \Gamma'}$ with the double coset $\Gamma \gamma \Gamma'$.

Def-Theorem 2 (Convolution). Let $\Gamma, \Gamma', \Gamma'' \in \mathcal{X}$. We define an convolution operator

$$* : \mathcal{H}(\Gamma \backslash \Delta / \Gamma') \times \mathcal{H}(\Gamma' \backslash \Delta / \Gamma'') \rightarrow \mathcal{H}(\Gamma \backslash \Delta / \Gamma'')$$

via

$$(\alpha * \beta)(x) := \sum_{h \in \Gamma' \backslash \Omega} \alpha(xh^{-1})\beta(h) = \sum_{\Omega / \Gamma'} \alpha(h)\beta(h^{-1}x).$$

The above equation is well-defined and holds. Moreover,

- this convolution operator $*$ is *distributive* and *associative*,
- $\mathbf{1}_\Gamma \in \mathcal{H}(\Gamma \backslash \Delta / \Gamma)$ is both a left and right *identity* for $*$.

In particular, the operator $*$ makes

$$\mathcal{H}_\Delta(\Gamma) := \mathcal{H}(\Gamma \backslash \Delta / \Gamma)$$

a \mathbb{K} -algebra.

We then give a formula of $*$. For $\alpha, \beta, \gamma \in \Delta$, write

$$[\Gamma \alpha \Gamma'] * [\Gamma' \beta \Gamma''] = \sum_{\gamma \in \Delta \backslash \Gamma / \Gamma''} m(\alpha, \beta; \gamma) [\Gamma \gamma \Gamma''].$$

Apply RHS to γ , one checks $([\Gamma \alpha \Gamma'] * [\Gamma' \beta \Gamma'']) (\gamma) = m(\alpha, \beta; \gamma)$. To determine these quantities, write

$$\Gamma \alpha \Gamma' = \bigsqcup_{a \in A} \Gamma a, \quad \Gamma' \beta \Gamma'' = \bigsqcup_{b \in B} \Gamma' b.$$

Then

$$\begin{aligned} m(\alpha, \beta; \gamma) &= ([\Gamma \alpha \Gamma'] * [\Gamma' \beta \Gamma'']) (\gamma) \\ &= \sum_{h \in \Gamma' \backslash \Omega} [\Gamma \alpha \Gamma'](\gamma h^{-1}) \cdot [\Gamma' \beta \Gamma''](h) \\ &= \sum_{h \in \Gamma' \backslash (\Gamma' \beta \Gamma'')} [\Gamma \alpha \Gamma'](\gamma h^{-1}) = \sum_{b \in B} [\Gamma \alpha \Gamma'](\gamma b^{-1}). \end{aligned}$$

³A \mathbb{K} -submodule of $\mathcal{F}(\Omega, \mathbb{K})$

Note that

$$[\Gamma\alpha\Gamma'](x) = \begin{cases} 1, & \exists a \in A, x \in \Gamma a \\ 0, & \text{otherwise} \end{cases} = \#\{a \in A \mid \Gamma x = \Gamma a\},$$

so

$$m(\alpha, \beta; \gamma) = \#\{(a, b) \in A \times B \mid \Gamma\gamma = \Gamma ab\}.$$

Considering right cosets rather than left cosets gives a similar formula.

The following is a useful result in computation.

Proposition 1.2. If $\alpha, \gamma \in \Delta$, and γ normalises Γ , then

$$[\Gamma\alpha\Gamma] * [\Gamma\gamma\Gamma] = [\Gamma\alpha\gamma\Gamma],$$

$$[\Gamma\gamma\Gamma] * [\Gamma\alpha\Gamma] = [\Gamma\gamma\alpha\Gamma].$$

Proof. Write $\Gamma\alpha\Gamma = \bigsqcup_{a \in A} \Gamma a$. As $\Gamma\gamma\Gamma = \Gamma\gamma$ and

$$\Gamma\alpha\gamma\Gamma = \Gamma\alpha\Gamma\gamma = \bigsqcup_{a \in A} \Gamma a\gamma,$$

the structure constants

$$m(\alpha, \gamma; \delta) = \#\{a \in A \mid \Gamma\delta = \Gamma a\gamma\} = \begin{cases} 1, & \delta \in \Gamma\alpha\gamma\Gamma, \\ 0, & \delta \notin \Gamma\alpha\gamma\Gamma. \end{cases} \quad \square$$

1.2.2 Commutativity

An **anti involution** of a monoid Δ is a map $\tau : \Delta \rightarrow \Delta$ s.t.

$$\tau(xy) = \tau(y)\tau(x), \quad \tau(1) = 1, \quad \tau^2 := \tau \circ \tau = \text{id}.$$

Theorem 3. Let $\Gamma \in \mathcal{X}$. If there *exists* an anti involution $\tau : \Delta \rightarrow \Delta$ that stabilises every double coset of Γ , then $\mathcal{H}(\Gamma \backslash \Delta / \Gamma)$ is a commutative \mathbb{K} -algebra.

1.3 The Action of Double Coset Algebras

We consider the action of $\mathcal{H}(\Gamma \backslash \Delta / \Gamma)$ on

$$M^\Gamma = \{m \in M \mid m\gamma = m, \forall \gamma \in \Gamma\}.$$

Def-Theorem 4. For $f \in \mathcal{H}(\Gamma \backslash \Delta / \Gamma')$, define

$$\begin{aligned} \cdot f : M^\Gamma &\longrightarrow M^{\Gamma'} \\ m &\longmapsto mf := \sum_{\delta \in \Gamma \backslash \Delta} f(\delta)m\delta. \end{aligned}$$

This action is well-defined. Moreover, it is comptatible with convolution.

- If $f \in \mathcal{H}(\Gamma \backslash \Delta / \Gamma')$, $f' \in \mathcal{H}(\Gamma' \backslash \Delta / \Gamma'')$, then $m(f * f') = (mf)f'$.
- In case $\Gamma' = \Gamma$, $m\mathbf{1}_\Gamma = m$.

In particular, M^Γ admits a right $\mathcal{H}(\Gamma \backslash \Delta / \Gamma)$ -module, with action of a basis given by

$$\Gamma\gamma\Gamma = \bigsqcup_{i=1}^n \Gamma\gamma_i \implies m[\Gamma\gamma\Gamma] = \sum_{i=1}^n \gamma_i.$$

Corollary 1.1. If γ normalises Γ , then $m[\Gamma\gamma\Gamma] = m\gamma$. □

2 Hecke Operators for $\Gamma_0(N)$ and $\Gamma_1(N)$

We specialise our discussion in the last section to the case of modular forms. Let

- $\Omega := \mathrm{GL}(2, \mathbb{Q})^+$,
- $\mathbb{K} := \mathbb{Z}$,
- $\mathcal{X} =$ congruence subgroups,

Lemma 2.1. Any two congruence subgroups are commensurable.

Proof. Note that $\Gamma(N) \cap \Gamma(N') = \Gamma(\mathrm{lcm}(N, N'))$. □

Lemma 2.2. If Γ is a discrete subgroup of $\mathrm{SL}(2, \mathbb{Z})$, then in $\mathrm{GL}(2, \mathbb{Q})^+$, the group $\tilde{\Gamma} = \mathrm{GL}(2, \mathbb{Q})^+$.

Proof. □

Fix a weight k and consider all the modular forms

$$M := \bigcup_{\Gamma \in \mathcal{X}} M_k(\Gamma) = \sum_{\Gamma} M_k(\Gamma)$$

and its \mathbb{C} -subspace

$$S := \bigcup_{\Gamma \in \mathcal{X}} S_k(\Gamma) = \sum_{\Gamma} S_k(\Gamma).$$

- Note that we can write $\bigcup = \sum$, because

$$M_k(\Gamma) + M_k(\Gamma') \subset M_k(\Gamma \cap \Gamma').$$

Define $M \odot \mathrm{GL}(2, \mathbb{R})^+$ by

$$f|_k \gamma(z) := (\det \gamma)^{k-1} j(\gamma, z)^{-k} f(\gamma z).$$

Lemma 2.3. For all $\Gamma \in \mathcal{X}$ and $\gamma \in \mathrm{GL}(2, \mathbb{R})^+$,

$$f \in M_k(\Gamma) \implies f|_k \gamma \in M_k(\Gamma \cap \gamma^{-1} \Gamma \gamma).$$

It remains true for S_k .

Proof. Just don't forget to check the cusps! □

It is now straightforward to check that we defined an action on M which stabilises S .

Lemma 2.4. $M^\Gamma = M_k(\Gamma)$, $S^\Gamma = S_k(\Gamma)$.

Now we go to the case of $\Gamma_0(N)$ and $\Gamma_1(N)$.

2.1 The Algebras

We consider these monoids:

$$\begin{aligned} \Delta(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det A > 0, (a, N) = 1, N \mid c \right\} \\ &= \left\{ A \in \mathrm{GL}(2, \mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} (\mathbb{Z}/N\mathbb{Z})^\times & * \\ * & * \end{pmatrix} \right\}, \\ \Delta^\circ(N) &:= \{ A \in \Delta(N) \mid (\det A, N) = 1 \}, \\ \Delta_1(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta^1(N) \middle| a \equiv 1 \pmod{N} \right\} \\ &= \left\{ A \in \mathrm{GL}(2, \mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} 1 & * \\ * & * \end{pmatrix} \right\}. \end{aligned}$$

Define

$$\mathcal{H}_i(N) := \mathcal{H}_{\Delta(N)}(\Gamma_i(N)), \quad \mathcal{H}_i^\circ(N) := \mathcal{H}_{\Delta^\circ(N)}(\Gamma_i(N)), \quad i = 0, 1$$

and $\mathcal{H}_1(N) := \mathcal{H}_{\Delta_1(N)}(\Gamma_1(N))$.

Proposition 2.1. All the algebras mentioned above are commutative.

Proof. Check that

$$A = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ bN & d \end{pmatrix} = \left(\begin{pmatrix} 1 & \\ & N \end{pmatrix}^{-1} A \begin{pmatrix} 1 & \\ & N \end{pmatrix} \right)^t$$

verifies the conditions of Theorem 3. □

We are particularly interested in $\mathcal{H}_0(N)$ and $\mathcal{H}_1(N)$.

2.2 Product Formula for $\mathcal{H}_0(N)$

Theorem 5 (coset representative of $\mathcal{H}_0(N)$). $\Gamma_0(N) \backslash \Delta(N) / \Gamma_0(N)$ admits coset representative given by

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad u \mid v, \quad (u, N) = 1.$$

The double coset of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad \text{where } \begin{cases} uv = ad - bc \\ u = (a, b, c, d). \end{cases}$$

Proposition 2.2.

$$\Gamma_0(N) \begin{pmatrix} u & \\ & v \end{pmatrix} \Gamma_0(N) = \bigsqcup_{g \in M_{u,uv}} \Gamma_0(N)g,$$

where

$$M_{u,n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \left| \begin{array}{l} u = (a, b, d) \\ n = ad \\ (a, N) = 1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right. \right\}.$$

In particular,

$$\Gamma_0(N) \begin{pmatrix} 1 & \\ & n \end{pmatrix} \Gamma_0(N) = \bigsqcup_{g \in M_{1,n}} \Gamma_0(N)g$$

and

$$M_{1,n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \left| \begin{array}{l} (a, b, d) = 1 \\ ad = n \\ (a, N) = 1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right. \right\}.$$

Proposition 2.3 (multiplication formula). Write $[A] := [\Gamma_0(N)A\Gamma_0(N)]$. Let $n, m \in \mathbb{Z}$, p be a prime.

- If $(n, m) = 1$, then

$$\left[\begin{pmatrix} 1 & \\ & n \end{pmatrix} \right] \left[\begin{pmatrix} 1 & \\ & m \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & \\ & nm \end{pmatrix} \right].$$

- If $p \mid N$, then

$$\left[\begin{pmatrix} 1 & \\ & p \end{pmatrix} \right] \left[\begin{pmatrix} 1 & \\ & p^r \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & \\ & p^{r+1} \end{pmatrix} \right].$$

- If $p \nmid N$, then

$$\begin{bmatrix} 1 & \\ & p \end{bmatrix} \begin{bmatrix} 1 & \\ & p^r \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & \\ & p^2 \end{bmatrix} + (p+1) \begin{bmatrix} p & \\ & p \end{bmatrix}, & r = 1, \\ \begin{bmatrix} 1 & \\ & p^{r+1} \end{bmatrix} + p \begin{bmatrix} p & \\ & p \end{bmatrix} \begin{bmatrix} 1 & \\ & p^{r-1} \end{bmatrix}, & r \geq 2. \end{cases}$$

Proof. Note that $\text{diag}(u, u)$ lies in the centre of $\text{GL}(2, \mathbb{Q})^+$, so

$$\begin{bmatrix} u & \\ & v \end{bmatrix} = \begin{bmatrix} u & \\ & u \end{bmatrix} \begin{bmatrix} 1 & \\ & v/u \end{bmatrix},$$

and thus we need only to prove the formula for $\text{diag}(1, n)$.

To be continued.... □

2.3 From Γ_0 to Γ_1

Recall that

$$\Gamma_0(N) \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times \quad \begin{pmatrix} * & * \\ & d \end{pmatrix} \mapsto \bar{d}$$

induces a group isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

Definition 1 (diamond operator). For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define

$$\langle d \rangle := [\Gamma_1(N) \gamma_d \Gamma_1(N)],$$

where $\gamma_d \in \Gamma_0(N)$ is any lift of d .

- The operator $\langle d \rangle$ is independent to the choice of γ_d , because the γ_d 's differ by an element in $\Gamma_1(N)$.
- $\langle d \rangle \langle d' \rangle = \langle dd' \rangle$.

2.4 Another Basis

Definition 2 (operator $T(n)$). Let $n \in \mathbb{Z}_{\geq 1}$ and consider

$$\Delta^n(N) := \{A \in \Delta(N) \mid \det A = n\}.$$

Write $\Gamma_0(N) \backslash \Delta^n(N) / \Gamma_0(N) = \bigsqcup_i \Gamma_0(N) g_i \Gamma_0(N)$, we define

$$T(n) := \sum_i [\Gamma_0(N) g_i \Gamma_0(N)].$$

By Theorem 5, we may take

$$\begin{pmatrix} u & \\ & n/u \end{pmatrix} \text{ with } \begin{cases} (u, N) = 1, \\ u^2 \mid n, \end{cases}$$

yielding

$$\begin{aligned} T(n) &= \sum_u \left[\begin{pmatrix} u & \\ & n/u \end{pmatrix} \right] \\ &= \sum_u \left[\begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} 1 & \\ & n/u^2 \end{pmatrix} \right]. \end{aligned}$$

as the representative g_i 's, which in turn shows that $\Gamma_0(N) \backslash \Delta^n(N) / \Gamma_0(N)$ is a finite set and $T(n)$ is well-defined. In particular, for p prime,

$$T(p) = \left[\begin{pmatrix} 1 & \\ & p \end{pmatrix} \right].$$

2.5 Hecke Algebra

Define

$$\mathbb{T}_i(N) := \text{im}(\mathcal{H}_i(N) \rightarrow \text{End}_{\mathbb{C}}(M_k(N))) \quad (1)$$

$$T_n := \text{image of } T(n) \in \mathcal{H}_i(N) \quad (2)$$

for $i = 0, 1$.