

Bruhat Decomposition for $\mathrm{SL}_n(k)$

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The goal would be to prove the affine Bruhat decomposition, and to deduce from this the Cartan decomposition. The affine Bruhat decomposition is a special case of the Bruhat decomposition, for groups with Tits systems whose associated Weyl group is an affine Coxeter group. In other words, the main question would be to prove the affine Bruhat decomposition (decomposition of the group as a partition of double classes modulo an alcove stabilizer - called an Iwahori subgroup), to interpret it geometrically and to explain how to deduce the Cartan decomposition from the affine Bruhat decomposition (by seeing a maximal compact subgroup as a finite union of double cosets modulo an Iwahori subgroup). Iwasawa decomposition is a different story ; one way to prove it consists in showing that $\mathrm{SL}(n, k^\circ)$ acts transitively on maximal flags.

Throughout this note, k will be a non-archimedean local field with residue field of size q and the absolute value normalized by $|\varpi| = q^{-1}$ for any uniformizer ϖ .

1 Buildings

We clarify some notations first. Combinatorically, we have the following definitions.

- A **simplex** = a point, a segment, a triangle, (just think of them as in Euclidean geometry) ... Formally, an **n -simplex** (or a simplex of **dimension** n) σ is a set with $(n + 1)$ -elements, in which the elements are called the **vertices** of σ .
- A **face** τ of a simplex σ is a subset $\tau \subset \sigma$. A face itself is a simplex, and the **codimension** of τ in σ is $\#(\sigma \setminus \tau)$.
- A **facet** or a **panel** is a face of codimension 1. Two simplices *of the same dimension* are **adjacent**, if they intersect at a common facet.
- A **simplicial complex** X with **vertices** V is a set of simplices $\sigma \subset V$, i.e., every $\sigma \in X$ is a finite subset of V , such that every face of $\sigma \in X$ is also in X . For convenience, we require further that $\bigcup X = V$.
- A morphism $X \rightarrow Y$ of simplicial complexes is a **cellular map**, i.e. a map of vertices $f : \bigcup X \rightarrow \bigcup Y$, such that for each simplex $\sigma \in X$, its image $f(\sigma) \in Y$ is a simplex of the same dimension.
- If σ is a simplex, the **closure** of σ is the simplicial complex $\bar{\sigma} :=$ the powerset of σ .
- An **alcove**¹ or a **chamber** c in a simplicial complex X is a maximal simplex in X , i.e. c is not a face of any other simplex.

¹The notion “alcove” may be specific to affine buildings, like a group action?

- A **gallery** connecting two alcoves c and c' is a sequence $c = c_0, c_1, \dots, c_n = c'$ such that c_i is adjacent to c_{i+1} . A **chamber complex** is a simplicial complex such that every two chambers are connected by a gallery.
- Let X be a chamber complex. Fix an alcove c in X . A **typing** or **colouring** of X is a morphism $\lambda : X \rightarrow c$. Equivalently², we fix a colouring of the panels of c , and propagate it to the whole complex X , so that: any panel of an alcove is coloured, and two adjacent panels (from different alcoves) have different colours.

However, in this note, we will (really? we need the interiors to talk about things like intervals or points (continuous parameter?) but it produces much trouble when talking combinatoric things) add in all the **interiors** of simplices/complexes, and define a simplex to be the interior, and a complex to be the union all vertices and interiors. In particular, a simplex σ of a simplicial complex X , in our sense, is a *subset* of X . For example, an 1-simplex is now literally a segment with two endpoints included.

1.1 Affine Buildings

Fix an affine tiling $\Sigma = (E, W)$ in a Euclidean space E with tiling group W . This means that E carries a tiling by one kind of polyhedron³ σ . For each facet of a polyhedron σ , the reflection along that facet is an automorphism of E preserving the tiling. We require $W = \langle S \rangle$ to be the group generated by the set S of such reflections for *one* polyhedrons (equivalent, W is generated by *all* reflections).

We could regard such a tiling as a chamber complex, where alcoves are the polyhedrons. Given an alcove $\sigma \subset E$, we can colour this alcove then propagate it using W to the whole complex E .

Lemma 1.1. The colouring of E is unique, and W is the group of type-preserving automorphisms.

This colouring can also be viewed as a map

$$\lambda : E \rightarrow \mathcal{P}(S)$$

defined as follows. Each facet τ is assigned with a unique reflection $s \in S$, and we define the colour of τ to be s . Any point $x \in X$ is in the intersection of facets with different type, so we can define $\lambda(x) \in \mathcal{P}(S)$ to be the collection of the colours of facets containing that point. (Need refinement!)

An **affine building** of type Σ is a poly-simplicial⁴ complex X , covered by sub-complexes \mathbb{A} isomorphic to Σ , called **apartments**, such that:

- (i) Any two points in X are contained in a same apartment.
- (ii) Given any two apartments \mathbb{A} and \mathbb{A}' , there is an isomorphism $\mathbb{A} \xrightarrow{\sim} \mathbb{A}'$ fixing $\mathbb{A} \cap \mathbb{A}'$.

(How to define global colouring?)

Theorem 1. Let X be a building of type (E, W) . Given a W -invariant distance d_E on E , there is a distance d on X as follows: for $x, y \in X$, pick an apartment \mathbb{A} containing x and y with an isomorphism $\mathbb{A} \simeq E$, and we define $d(x, y) := d_E(x, y)$. This map d is a well-defined distance with the following properties.

- (i) (X, d) is complete.

²It may require more condition on X for this equivalence?

³It is a simplex for SL_n .

⁴For SL_n , the building is a simplicial complex, so let's skip the definition of poly-simplicial complex for now.

(ii) Given an inclusion $c \subset \mathbb{A}$ of an alcove into an apartment, there is a *unique* 1-Lipschitz retraction

$$\rho = \rho_{\mathbb{A},c} : X \rightarrow \mathbb{A},$$

i.e. $\rho|_{\mathbb{A}} = \text{id}_{\mathbb{A}}$ and

$$d(\rho(x), \rho(y)) \leq d(x, y), \quad \forall x, y \in X,$$

with “=” holds if $x \in \bar{c}$. Moreover, for any apartment $\mathbb{A}' \supset c$,

$$\rho|_{\mathbb{A}'} : \mathbb{A}' \rightarrow \mathbb{A}$$

is the unique isomorphism fixing c .

(iii) For any two points $x, y \in X$,

$$[x, y] := \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\} \subset \mathbb{A}$$

for any apartment \mathbb{A} containing x and y .

(iv) For $x, y \in X$ and $0 \leq t \leq 1$, we denote by

$$(1-t)x + ty \in [x, y]$$

the unique point z such that $d(x, z) = t \cdot d(x, y)$. Then the map

$$X \times X \times [0, 1] \rightarrow X \quad (x, y, t) \mapsto (1-t)x + ty$$

is continuous, and X is contractible.

Let \mathbb{G} be a simply connected semi-simple linear group over k . The Bruhat-Tits building $X = X(\mathbb{G}, k)$ is an affine building, on which $G = \mathbb{G}(k)$ acts by cellular and isometric automorphisms, such that the G -action is

- type-preserving,
- proper: the stabilizer of any face⁵ is compact,
- strongly transitive⁶: the G -action is transitive on the set of all inclusion relations $c \subset \mathbb{A}$, where c is an alcove and \mathbb{A} is an apartment.

1.2 Bruhat decomposition

Fix an apartment \mathbb{A} and an alcove $c \subset \mathbb{A}$. We define

$$B := \{g \in G \mid gc = c\} = \text{Stab}_G(c),$$

$$N := \{g \in G \mid g\mathbb{A} = \mathbb{A}\} = \text{Stab}_G(\mathbb{A}),$$

$$T := \{g \in G \mid gx = x, \forall x \in \mathbb{A}\}.$$

Recall that $W = \langle S \rangle$ is the group of type preserving automorphisms of \mathbb{A} , where S is the set of reflections along the facets of c . So for any $n \in N$, there is some $w \in W$ such that $n|_{\mathbb{A}} = w$, and thus we have an

⁵Facets of alcoves?

⁶Note that for our “geometric” building, the G -action is not set-theoretically transitive. For example, it cannot send a vertex to a non-vertex point.

epimorphism $N \twoheadrightarrow W$ with kernel T . Also by the type-preserving property (not clear! why can't such an element send an alcove different from c to another?), $B \cap N = T$.

Take $S' \subset S$ and let a be the face of c with colour $S \setminus S'$, so that

$$W' := \text{Stab}_W(a) = \langle S' \rangle.$$

Let

$$P_{S'} := \text{Stab}_G(a).$$

For $g \in P_{S'}$, choose an apartment \mathbb{A}' containing c and gc . By strong transitivity, there is some $b \in B$ with $b\mathbb{A}' = \mathbb{A}$. The alcove $bgc \subset \mathbb{A}$, so $bgc = wc$ for a unique $w \in W$. Thus $\tilde{w}^{-1}bg \in B = \text{Stab}_G(c)$, where $\tilde{w} \in N$ is any lift of $w \in W$, and

$$g \in BwB := B\tilde{w}B.$$

The double coset BwB is well-defined because $T = \ker(N \rightarrow W) \subset B$. Note that

$$B = \{g \in G \mid gx = x, \forall x \in c\}$$

as G is type-preserving, so we have $B \subset P_{S'} = \text{Stab}_G(a)$. Hence $\tilde{w} \in P_{S'}$, and $w \in \text{Stab}_W(a) = W'$. This shows that

$$P_{S'} = \bigcup_{w \in W'} BwB = BW'B.$$

These double cosets are disjoint by Theorem 2, and we will focus on the three special cases:

- the stabilizer of the empty face, $G = P_S$,
- the stabilizer of a vertex, $K = P_{S \setminus \{s\}}$ for some $s \in S$, and
- the stabilizer of a facet, P_s for some $s \in S$. Since s is a reflection, $P_s = B \cup BsB$.

Theorem 2 (Bruhat decomposition). The double cosets BwB are disjoint for $w \in W$. In particular,

$$G = BWB = \bigsqcup_{w \in W} BwB.$$

Proof. In the above construction, $b \in B$ induces an isomorphism $b|_{\mathbb{A}'} : \mathbb{A}' \rightarrow \mathbb{A}$ fixing c . By the unicity of canonical retractions from Theorem 1,

$$b|_{\mathbb{A}'} = \rho|_{\mathbb{A}'}, \quad \rho = \rho_{\mathbb{A}, c}.$$

So for each $g \in G$, we have shown that $g \in B\bar{\rho}(g)B$, where $\bar{\rho}(g) \in W$ is the unique element such that $\bar{\rho}(g)c = \rho(gc)$.

Now we need to show that for any $w \in W$ and $g \in BwB$, $\bar{\rho}(g) = w$. Write $g = b\tilde{w}b'$ with $\tilde{w} \in N$ lifting w and $b, b' \in B$. Then $gc = bwc \subset b\mathbb{A}$. Now b^{-1} is an isomorphism $b\mathbb{A} \rightarrow \mathbb{A}$ fixing c , so $\rho(gc) = b^{-1}gc = wc$, i.e. $\bar{\rho}(g) = w$. *The proof seems to be tautologic...* \square

Our next goal is the Cartan decomposition $G = KAK$.

Theorem 3. Let $S_1, S_2 \subset S$, $W_i = \langle S_i \rangle$, $P_i = P_{S_i}$. Then there is a bijection

$$W_1 \backslash W / W_2 \leftrightarrow P_1 \backslash G / P_2$$

given by $W_1 w W_2 \mapsto P_1 w P_2$.

Theorem 4 (Cartan decomposition). $G = KAK$.

2 The Building for SL_n

Let V be a $(d+1)$ -dimensional k -vector space. Let $q := \#(k^\circ/k^{\circ\circ})$. We consider the group

$$G := \mathrm{SL}(V, k) \simeq \mathrm{SL}_{d+1}(k).$$

Define

$$\mathcal{N}(V, k) := \{\text{ultrametric norms } \|\cdot\| : V \rightarrow k\}$$

and

$$\mathcal{X}(V, k) := \frac{\mathcal{N}(V, k)}{\|\cdot\| \simeq \|\cdot\|' \iff \exists c \in \mathbb{R}_{>0}, \|\cdot\| = c \|\cdot\|'}.$$

Every norm on V takes the following form.

- Let $e = (e_0, \dots, e_d)$ be a basis of V , $c = (c_0, \dots, c_d) \in \mathbb{R}^{d+1}$ be parameters, then

$$\|\cdot\|_{e,c} : x = \sum_i \lambda_i e_i \mapsto \max_i q^{c_i} |\lambda_i|$$

is a norm on V . The norms $\|\cdot\|_{e,c}$ and $\|\cdot\|_{e,c'}$ are homothetic if and only if $c - c' \in \mathbb{R} \cdot \mathbf{1}$.

- In particular, a basis e of V corresponds to a k° -lattice $L = \sum_i k^\circ e_i$ in V , and the norm

$$\|\cdot\|_{e,0} = \|\cdot\|_L : x \mapsto \inf\{|\lambda| \mid \lambda \in k^\times, \lambda^{-1}x \in L\}.$$

Proposition 2.1. All (ultrametric) norms on V are of the form $\|\cdot\|_{e,c}$ for some basis e and parameters $c \in \mathbb{R}^{d+1}$. Moreover, for any two norms $\|\cdot\|$ and $\|\cdot\|'$ on V , there is some basis e of V and parameters $c, c' \in \mathbb{R}^{d+1}$, such that

$$\|\cdot\| = \|\cdot\|_{e,c}, \quad \|\cdot\|' = \|\cdot\|_{e,c'}.$$

For a basis e of V , we define

$$\begin{aligned} \tilde{\mathbb{A}}_e &:= \{\|\cdot\|_{e,c} \mid c \in \mathbb{R}^{d+1}\} \simeq \mathbb{R}^{d+1} \\ \mathbb{A}_e &:= \tilde{\mathbb{A}}_e / \text{homothety} \simeq \mathbb{R}^{d+1} / \mathbb{R} \cdot \mathbf{1}. \end{aligned}$$

Proposition 2.1 says that any two norms are contained in a same subset $\tilde{\mathbb{A}}_e$.

The group $\mathrm{GL}(V, k) \simeq \mathrm{GL}_{d+1}(k)$ acts on $\mathcal{N}(V, k)$ by

$$g \|\cdot\| := \|\cdot\| \circ g^{-1}.$$

Fix a basis e of V . It is plain to verify that

$$g \|\cdot\|_{e,c} = \|\cdot\|_{ge,c}.$$

We admit the following facts.

- The above action induces an action of $G = \mathrm{SL}(V, k)$ on the space $\mathcal{X}(V, k)$, making the latter the affine Bruhat-Tits building for G . The apartments are given by $\tilde{\mathbb{A}}_e$'s and vertices are given by (homothety classes of) lattices, or $\|\cdot\|_{e,0}$'s.
- The (affine) Weyl group is

$$W \simeq \mathfrak{S}_d \ltimes \mathbb{Z}^d,$$

and

$$N := \mathrm{Stab}_G(\tilde{\mathbb{A}}_e) = \{\text{monomial matrices in base } e \text{ of determinant } 1\}.$$

These implies that

$$T := \ker(N \rightarrow W) = \{\text{diagonal matrices in base } e \text{ in } \mathrm{SL}_{d+1}(k^\circ)\} \simeq ((k^\circ)^\times)^d.$$

- Pick a vertex $x = [\|\cdot\|_{e,0}] \in \mathcal{X}(V, k)$. The subset

$$C := \{[\|\cdot\|_{e,c}] \mid c_0 \leq c_1 \leq \dots \leq c_d \leq 1 + c_0\}$$

of the appartment $\tilde{\mathbb{A}}_e$ is an alcove. We call it the **standard alcove** attached to x and e .

Now we compute some special groups of G .

- The stabilizer of $x = [\|\cdot\|_{e,c}]$ is the maximal compact

$$K = \text{Stab}_G(x) = \text{Stab}_G(e) = \text{SL}_{d+1}(k^\circ) \text{ in base } e.$$

Then inclusion “ \supset ” is clear, so we only need to show that: an element $g \in \text{SL}_{d+1}(k)$ stabilizes a homothety class of a lattice if and only if it stabilizes the lattice itself. If

$$g \|\cdot\|_{e,0} = \|\cdot\|_{ge,0} = \|\cdot\|_{e,c \cdot 1}$$

for some $c \in \mathbb{R}$, then $c \in \mathbb{Z}$ (by hand or by type-preserving), so $ge = \varpi^c e$. In particular, $\det(g) = \det(\varpi^c)$, and hence $c = 0$.

- The stabilizer of C is the standard Iwahori

$$B = \text{Stab}_G(C) = \begin{pmatrix} (k^\circ)^\times & k^\circ & \dots & k^\circ \\ \varpi k^\circ & (k^\circ)^\times & \dots & k^\circ \\ \vdots & \vdots & \ddots & \vdots \\ \varpi k^\circ & \varpi k^\circ & \dots & (k^\circ)^\times \end{pmatrix} \text{ in base } e.$$

The group B contains precisely the elements fixing the vertices of C . Let $gx \in c$ be another vertex, i.e. $\|\cdot\|_{ge,0} = \|\cdot\|_{e,c}$ for a suitable $c \in \mathbb{R}^{d+1}$. Such c must in \mathbb{Z}^{d+1} ,

Cartan decomposition for SL_{d+1} : guess:

$$S_d \backslash (S_d \times \mathbb{Z}^d) / S_d = \mathbb{Z}^d / S_d = \left\{ \begin{pmatrix} \varpi^{a_1} & & \\ & \ddots & \\ & & \varpi^{a_d} \end{pmatrix} \mid a_1 \geq \dots \geq a_d \right\}.$$