Notes on CFT

1 A Bit of p-adic Analysis

In this section, we consider some basic properties concerning powerseries over a closed subfield K of $\mathbb{C}_p$ as functions.

Let $f(X) = \sum_{i\geq 0} a_i X^i \in K[\![X]\!]$. We can evaluate f at $z \in \mathbb{C}_p$ iff $a_i z^i \to \infty$, so the **radius of convergence** is

$$\rho(f) := \sup\{\rho \in \mathbb{R} \mid a_i \rho^i \to \infty (i \to \infty)\}.$$

- If $|z| < \rho(f)$, then $f(z)$ converges in $\mathbb{C}_p$.
- If $|z| > \rho(f)$, then f diverges.
- $\rho(f(\alpha X)) = \rho(f) \cdot |\alpha|^{-1}$.

We are mainly interested in the power series converging on the unit disk, i.e.,

$$\begin{aligned} H_K &:= \{f \in K[\![X]\!] \mid \rho(f) > 1\} \\ &= \{f \in K[\![X]\!] \mid a_i \rho^i \to 0, \forall \rho < 1\} \\ &= \{f \in K[\![X]\!] \mid f \text{ converges on the open unit disk } \mathfrak{m}_{\mathbb{C}_p} = B(0,1)\}. \end{aligned}$$

Example 1. $K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![X]\!] =$ power series over K with bounded coefficients $\subsetneq H_K$.

Example 2. $\log(1+X) = \log_{\mathbb{G}_{\mathrm{m}}}(X) = X - \dfrac{X^2}{2} + \dfrac{X^3}{3} - \cdots \in H_K \setminus K \otimes_{\mathcal{O}_K} \mathcal{O}_K[\![X]\!]$.

1.1 The Gauss Norm

Theorem 1. Let $f(X) = \sum_{i\geq 0} a_i X^i \in K[\![X]\!]$ with $\rho(f) > 0$, a real number $\rho < \rho(f)$ s.t. $\rho \in |\mathbb{C}_p^\times|$. Then $\sup_{i\geq 1} |a_i|\rho^i$ is a maximum (i.e., $\sup_{i\geq 1} |a_i|\rho^i = |a_j|\rho^j$ for some j), and

$$\sup_{i\geq 1} |a_i|\rho^i = \sup_{|z|=\rho} |f(z)| =: |f|_\rho.$$

Proof.

- $\rho < \rho(f) \implies |a_i|\rho^i \to 0 \implies \sup_{i\geq 1} |a_i|\rho^i$ is a maximum.
- $|f(z)| = \left|\sum_{i\geq 1} a_i z^i\right| \leq \sup_{i\geq 1} |a_i||z|^i$, so $|f|_\rho \leq \sup_{i\geq 1} |a_i|\rho^i$.
- Take $\alpha \in \mathbb{C}_p$ with $|\alpha| = \rho$, and $j \in \mathbb{Z}_{\geq 0}$ s.t. $\sup_{i\geq 1} |a_i|\rho^i = |a_j|\rho^j$. Let $\beta := a_j \alpha^j$. Then

$$g(X) := \frac{f(\alpha X)}{\beta} \in \mathcal{O}_{\mathbb{C}_p}[\![X]\!].$$

□