HOMEWORK

LOCAL FIELDS

1. Composition of ramified extensions

Let $p \geq 3$ be a prime number, and let ζ_p denote a root of unity of order p.

1.1. Give an example of a totally ramified finite extension $K/\mathbf{Q}_p$ such that $K(\zeta_p)/\mathbf{Q}_p$ is not totally ramified.

2. Multiplication by p

Let p be a prime number, and let F be a formal group over a (commutative) ring R.

2.1. Show that $[p](X) \in p \cdot R[\![X]\!] + R[\![X^p]\!]$.

3. The zeroes of the logarithm

Let K be a finite extension of $\mathbf{Q}_p$. If F is a formal group over $\mathcal{O}_K$, let $\mathrm{Tors}(F) = \{z \in \mathfrak{m}_{\mathbf{C}_p}$ such that there exists $n \geq 0$ with $[p^n](z) = 0\}$, and let $\log_F$ denote the logarithm of F.

3.1. Prove that $\log_F(X) \in \mathrm{H}_K$.

3.2. Take $z \in \mathfrak{m}_{\mathbf{C}_p}$. Prove that if $z \neq 0$, then $|[p](z)|_p < |z|_p$.

3.3. Take $z \in \mathfrak{m}_{\mathbf{C}_p}$ such that $\log_F(z) = 0$. What can you say about $\log_F([p](z))$? Prove that the set of zeroes of $\log_F$ is precisely $\mathrm{Tors}(F)$.

4. Torsion of some formal groups

We use the notation and results of exercise 4.

4.1. Take $\alpha \in \mathcal{O}_K$. Prove that $F_\alpha(X, Y) = X + Y + \alpha XY$ is a formal group.
Hint: compute $1 + \alpha F_\alpha$.

4.2. Compute the height of F_α and compute $\mathrm{Tors}(F_\alpha)$.

4.3. Assume that F is a formal group over $\mathcal{O}_K$ of infinite height, namely that $\overline{[p](X)} = 0$ in $k_K[\![X]\!]$. Prove that $\mathrm{Tors}(F)$ is finite.

4.4. Prove that if $\mathrm{Tors}(F) = \{0\}$, then $\log_F(X) \in \mathcal{O}_K[\![X]\!]$ and that F is then isomorphic over $\mathcal{O}_K$ to the additive formal group.

4.5. Prove that if $K/\mathbf{Q}_p$ is unramified, and F is of infinite height, then $\mathrm{Tors}(F) = \{0\}$.