Homework 2

Due October 15

You can write your answers in English or in French, choose what works best for you.

We fix an integer $k \geq 3$.
Let Γ be a congruence subgroup. We define

$$\Gamma_\infty^+ = \Gamma \cap \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbf{Z}\}$$

and

$$\Gamma_\infty = \Gamma \cap \{\pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbf{Z}\}$$

1. Show that $[\Gamma_\infty : \Gamma_\infty^+] = 1$ or 2.

2. Let $N > 2$. Show that $[\Gamma_1(N)_\infty : \Gamma_1(N)_\infty^+] = 1$ and $[\Gamma_0(N)_\infty : \Gamma_0(N)_\infty^+] = 2$.

3. Assume that k is odd and $[\Gamma_\infty : \Gamma_\infty^+] = 2$. Show that any $f \in M_k(\Gamma)$ satisfies $f(\infty) = 0$.

4. Show that the map $\Gamma_\infty^+ \backslash \Gamma \to \{(c, d) \in \mathbf{Z}^2, (c, d) \text{ coprime}\}$, sending a matrix to its second row, is well-defined and injective.

Recall that we have proved in the lecture: for all $k \geq 3$, the series $\sum_{(c,d) \in \mathbf{Z}^2 \backslash (0,0)} \frac{1}{(cz+d)^k}$ converges normally on any subset of $\mathcal{H}$ of the form $X_{A,B} = \{-A \leq x \leq A, B < y\}$ for $A, B > 0$.

We define

$$G_{k,\Gamma,\infty}(z) = \sum_{g \in \Gamma_\infty^+ \backslash \Gamma} j(g, z)^{-k}$$

so that the series converges normally on any $X_{A,B}$, and the sum is holomorphic.

5. Show that $G_{k,\Gamma,\infty}$ is a weakly modular form of weight k.

6. Assume that k is odd and $[\Gamma_\infty : \Gamma_\infty^+] = 2$. Show that $G_{k,\Gamma,\infty} = 0$.

7. Show that $G_{k,\Gamma,\infty}$ is bounded at infinity.
Compute $G_{k,\Gamma,\infty}$ (the result depends on the parity of k and the value of $[\Gamma_\infty : \Gamma_\infty^+]$).

8. Let c be a cusp of Γ that is not ∞, t a representative of c, and $g \in \mathrm{SL}_2(\mathbf{Z})$ such that $g\infty = t$. Show that $G_{k,\Gamma,\infty}|_k g$ is bounded at infinity and that $(G_{k,\Gamma,\infty}|_k g)(\infty) = 0$.

9. Deduce that $G_{k,\Gamma,\infty} \in M_k(\Gamma)$.

10. Show that $M_k(\Gamma)$ is generated by $S_k(\Gamma)$ and the $G_{k,g\Gamma g^{-1},\infty}|_k g^{-1}$, $g \in \mathrm{SL}_2(\mathbf{Z})$.

11. Assume that k is even. Show that $\dim M_k(\Gamma) = \dim S_k(\Gamma) + |C_\Gamma|$, where C_Γ is the set of cusps of Γ.

12. Let $C'_\Gamma \subset C_\Gamma$ be the subset of C_Γ defined as follows: let $c \in C_\Gamma$, and let $g \in \mathrm{SL}_2(\mathbf{Z})$ be such that $g\infty$ is a representative of c. Then $c \in C'_\Gamma$ if and only if $[g\Gamma_\infty g^{-1} : g\Gamma_\infty^+ g^{-1}] = 1$.
Assume that k is odd. Show that $\dim M_k(\Gamma) = \dim S_k(\Gamma) + |C'_\Gamma|$.

13. Let $f \in S_k(\Gamma)$. Show that

$$\langle f, G_{k,\Gamma,\infty} \rangle = \frac{1}{\mathrm{vol}(\Gamma \backslash \mathcal{H})} \int_{\Gamma_\infty^+ \backslash \mathcal{H}} f(z) y^{k-2} dx dy$$

and deduce that $\langle f, G_{k,\Gamma,\infty} \rangle = 0$.

14. How is $G_{k,\mathrm{SL}_2(\mathbf{Z}),\infty}$ related to the Eisenstein series defined in the lecture?