Notes on Drinfeld Modules and Explicit CFT for Function Fields

February 18, 2025

1) Give a 30min (strict limit !!!) talk. Ideally more like 25min + 5 min for questions. The talks will be in March. I will try to reserve a room, and will give a more precise time/date when possible.

2) Write an "extended summary" (meaning around 5 pages NOT!!! >=10) of you article. It should summarise the article and its main ideas and be accessible to advanced Master students (i.e., the other students in this group).

1 Review on CFT

2 Drinfeld Modules

Let F be a global function field with field of constants $k = \mathbb{F}_q$.

2.1 Definition

Consider the additive group $\mathbb{G}_{\mathrm{a}/L}$ over L. Now the point is, $\mathbb{G}_{\mathrm{a}/L}$ is not only a group scheme, but a k-vector space scheme, and we consider the ring $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L})$ of all k-linear endomorphism of group schemes.

Proposition 2.1. $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) = k\{\tau\}$, where τ is the Frobenius-q endomorphism of $F[X]$.

Proof. An endomorphism $\mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ of schemes over L is given by an L-algebra homomorphism $\Phi : L[X] \to L[X]$, hence it is determined by the image $\varphi(X) = \Phi(X)$[1] of X. It respects the group-scheme structure if it commutes with the co-multiplication map (also an L-algebra homomorphism)

$$\Delta : F[X] \to F[X] \otimes_L F[X], \quad X \mapsto X \otimes 1 + 1 \otimes X.$$

which amounts to

$$(\Phi \otimes \Phi)(\Delta(X)) = (\Phi \otimes \Phi)(X \otimes 1 + 1 \otimes X) = \Phi(X) \otimes 1 + 1 \otimes \Phi(X) = \varphi(X) \otimes 1 + 1 \otimes \varphi(X)$$

equals

$$\Delta(\Phi(X)) = \Delta(\varphi(X)) = \varphi(\Delta(X)) = \varphi(X \otimes 1 + 1 \otimes X).$$

[1] Note that if $\varphi(X) = a_n X^n + \cdots + a_0$, then

$$\varphi(f(X)) = a_n f(X)^n + \cdots + a_0$$

and

$$\Phi(f(X)) = f(\Phi(X)) = f(\varphi(X))$$

are ***different*** in general.

This is to say that[2] φ is additive, i.e. $\varphi(X+Y) = \varphi(X) + \varphi(Y)$.

We require furthur that Φ respects the "co-k-scalar multiplication", which I don't have the formula right now. So let's use the functor point of view. Take $c \in k$. Yoneda tells us that

$$\mathrm{Hom}_{[k\text{-}\mathbf{Alg}^{\mathrm{op}},\mathbf{Grp}]}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{a}}) \simeq \mathbb{G}_{\mathrm{a}}(L[X]), \quad \phi \mapsto \phi(\mathrm{id}_{L[X]}),$$

so the co-c-multiplication is given by $X \mapsto cX$. Therefore Φ respects this map if $\varphi(cX) = c\varphi(X)$.

In conclusion,

$$\begin{aligned} \mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) &= \{k\text{-linear polynomials in} L[X]\} \\ &= \end{aligned}$$

$\square$

[2] Recall that the multiplicative structure on $B \otimes_A C$ is given by

$$(b \otimes b') \cdot (c \otimes c') = bb' \otimes cc'.$$