Notes on Algebraic Number Theory

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Some Notations

Let F be a number field, then we denote by r_1 the number of real embeddings, r_2 the number of the pairs of complex embeddings, $\mathrm{Cl}(F)$ the class group, h_F the class number, R_F the regulator, w_F the number of roots of unity in F, $\mathfrak{d} = \mathfrak{d}_F$ the different ideal.

Always denote $\sqrt{-1} \in \mathbb{C}$ by i.

1 Adeles and Ideles

Note that the topology on $\mathbb{A}_F^\times$ (defined using natrual nbhd of 1 in $\mathbb{Q}_p^\times$) is different from (more precisely, finer than) that on $\mathbb{A}_F$ (defined using natrual nbhd of 0 in $\mathbb{Q}_p$), but the topology on $\mathbb{A}_F^{\times,1}$ induced from $\mathbb{A}_F$ and that from $\mathbb{A}_F^\times$ coincide.

Theorem 1. *The quotient $\mathbb{A}_F^{\times,1}/F^\times$ is compact.*

Proof. Let I_F be the group of fractional ideals. Observe that we have an epimorphism

$$\mathbb{A}_F^{\times,1} \twoheadrightarrow I_F,\ (x_v) \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x_\mathfrak{p})},$$

under which $x \in F^\times$ is send to the principle fractional ideal $(x) \in I_F$, and thus gives an epimorphism $\mathbb{A}_F^{\times,1}/F^\times \twoheadrightarrow \mathrm{Cl}(F)$. As $\mathrm{Cl}(F)$ is finite, it reduces to show that the kernel of this homomorphism is compact.

An element $(x_v) \in \ker$ iff it is mapped to a principle ideal, i.e., $\exists x \in F^\times$ s.t. $\forall \mathfrak{p}$, $x_\mathfrak{p}\mathcal{O}_\mathfrak{p} = x\mathcal{O}_\mathfrak{p}$, or say $x_\mathfrak{p} \in x^{-1}\mathcal{O}_\mathfrak{p}^\times$. Therefore the kernel is the image of

$$\left(\prod_{\mathfrak{p}} \mathcal{O}_\mathfrak{p}^\times \times \prod_{v|\infty} F_v^\times\right) \cap \mathbb{A}_F^{\times,1} = \prod_{\mathfrak{p}} \mathcal{O}_p^\times \times \left(\prod_{v|\infty} F_v^\times\right)^1$$

in $\mathbb{A}_F^{\times,1}/F^\times$, where $\left(\prod_{v|\infty} F_v^\times\right)^1$ denotes the set of element with norm 1. Because two elements in this set cannot differ by an element in $F^\times \setminus \mathcal{O}_F^\times$, we see that

$$\ker = \left(\prod_{\mathfrak{p}} \mathcal{O}_p^\times \times \left(\prod_{v|\infty} F_v^\times\right)^1\right) \Big/ \mathcal{O}_F^\times.$$

Now it suffices to prove that $\left(\prod_{v|\infty} F_v^\times\right)^1 / \mathcal{O}_F^\times$ is compact. Let $v_1, \cdots, v_r$ be the places of real embeddings and $v_{r_1+1}, \cdots, v_{r_1+r_2}$ the places of complex ones. The logarithm map

$$\left(\prod_{v|\infty} F_v^\times\right)^1 \to \mathbb{R}^{r_1+r_2},\ x \mapsto (\log|x_{v_1}|, \cdots, \log|x_{v_{r_1}}|, \log|x_{v_{r_1+1}}|_\mathbb{C}, \cdots, \log|x_{v_{r_1+r_2}}|_\mathbb{C})$$

is a homomorphism with kernel $T = \{\pm 1\}^{r_1} \times (S^1)^{r_2}$, which is compact and the intersection $T \cap \mathcal{O}_F^\times = W_F$, the roots of unity in F. So $T/T \cap \mathcal{O}_F^\times$ is compact. Its image is the hypersurface

$$\Sigma : x_1 + \cdots + x_{r_1+r_2} = 1$$

in $\mathbb{R}^{r_1+r_2}$. Dirichlet units theorem says that the image of $\mathcal{O}_F^\times$ in Σ is a lattice of full rank, so the quotient $\Sigma/\mathcal{O}_F^\times$ is also compact. Our goal follows. $\square$

Remark. This theorem is equivalent to the combination of the finiteness of class group and Dirichlet units theorem.

2 L-functions

2.1 Riemann Zeta Function

Recall that the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

converges on $\operatorname{Re} s > 1$ and can be extended to a meromorphic function on $\mathbb{C}$ with $s = 1$ the only simple pole. The functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s)$$

is satisfied.

2.2 Charaters

A **character** of a group G is a continuous homomorphism $G \to \mathbb{C}^\times$, and **trivial character** is the character $G \to \{1\}$. The charaters of a group G forms a group, denoted by $\widehat{G}$.

Lemma 2.1. Let G be a finite abelian group.

1. There exists an non-canonical isomorphism $G \simeq \widehat{G}$.
2. If χ is a non-trivial character, then
$$\sum_{g \in G} \chi(g) = 0.$$
Conversely, if $g \neq 1$, then
$$\sum_{\chi \in \widehat{G}} \chi(g) = 0.$$

$\square$

Let F be a number field. A **Hecke character** of F is a character of $\mathbb{A}_F^\times / F^\times$.

Proposition 2.1. Let χ be a character on $\mathbb{A}_F^\times$. Then χ is of the form $\prod_v \chi_v$, where $\chi_v \in \widehat{F_v^\times}$ and χ_v's are **unramified** (i.e., trivial on $\mathcal{O}_{F_v}^\times$) for almost all nonarchimedean places.

So we can go back to charaters on local fields. Let F be a local field and χ a character of $F^\times$. The character χ is called **unitary**, if $|\chi(F^\times)| = \{1\}$. We can describe χ explicitly.

- If $F = \mathbb{R}$, then
$$\chi(x) = \left(\frac{x}{|x|}\right)^\epsilon |x|^s, \ \epsilon = 0, 1, \ s \in \mathbb{C}.$$
It is unitary iff $s \in i\mathbb{R}$.

◇ If $F = \mathbb{C}$, then

$$\chi(x) = \left(\frac{x}{\sqrt{x\overline{x}}}\right)^m (x\overline{x})^s,\ m \in \mathbb{Z},\ s \in \mathbb{C}.$$

It is unitary iff $s \in i\mathbb{R}$.

◇ If F is nonarchimedean, then there exists a minimal integer N s.t. $\chi(1 + \varpi^N \mathcal{O}_F^\times) = \{1\}$, whence χ factors through the finite group $\mathcal{O}_F^\times/(1 + \varpi^N \mathcal{O}_F^\times)$, and thus

$$\chi(x) = |x|^s \chi_0(x),$$

where χ_0 is a character of $\mathcal{O}_F^\times/(1+\varpi^N \mathcal{O}_F^\times)$. It is unitary if $s \in i\mathbb{R}$. This integer is called the **conductor** of χ.

From now on, all multiplicative charaters of local fields are assumed to be unitary.

2.3 Lift a Dirichlet Charater to a Hecke Charater

Look at a character $\chi : (\mathbb{Z}/\ell^e\mathbb{Z})^\times \to \mathbb{C}^\times$ first. We define a character χ_p on $\mathbb{Q}_p^\times \simeq p^{\mathbb{Z}} \times \mathbb{Z}_p^\times$ and χ_∞ on $\mathbb{R}^\times$ as follows.

- If $p = \ell$, then the isomorphism $\mathbb{Z}_\ell^\times/(1 + \ell^e\mathbb{Z}_\ell) \simeq (\mathbb{Z}/\ell^e\mathbb{Z})^\times$ enable us to lift χ^{-1} (note the '-1'!) to a character χ_ℓ on $\mathbb{Q}_\ell^\times$ that is trivial on $\ell^{\mathbb{Z}}$ and $1 + \ell^e\mathbb{Z}_\ell$.
- If $p \neq \ell$, then p is invertible mod ℓ^e, so we can define $\chi_p(p) := \chi(p)$, then make it trivial on $\mathbb{Z}_p^\times$.
- Put $\chi_\infty := \mathrm{sgn}^{\chi(-1)}$.

Since χ_p are trivial on $\mathbb{Z}_p^\times$ only except for $p = \ell$, patching them together yields a character $\widetilde{\chi} := \prod_v \chi_v$ on $\mathbb{A}_Q^\times$.

Lemma 2.2. The character $\widetilde{\chi}$ is trivial on $\mathbb{Q}^\times$.

Proof. It suffices to check for every prime p and -1. If $p \neq \ell$, then $\widetilde{\chi}(p) = \chi_p(p)\chi_\ell(p) = 1$; otherwise $\chi_v(\ell) = 1$ for all places v. To conclude, $\widetilde{\chi}(-1) = \chi_\infty(-1)\chi_\ell(-1) = 1$. □

Now consider $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. The factorisation $N = p_1^{e_1} \cdots p_r^{e_r}$ gives

$$(\mathbb{Z}/N)^\times \simeq (\mathbb{Z}/p_1^{e_1})^\times \times \cdots \times (\mathbb{Z}/p_r^{e_r})^\times,$$

so we have $\chi = \chi_1 \cdots \chi_r$, where $\chi_i : (\mathbb{Z}/p_1^{e_1})^\times \to \mathbb{C}^\times$, and obtain a Hecke character $\widetilde{\chi} := \widetilde{\chi_1} \dots \widetilde{\chi_r}$.

Remark. The character $\widetilde{\chi}$ is

$$\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times\mathbb{R}_{>0} \simeq \widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Conversely, every Hecke character factors through $\widehat{\mathbb{Z}}^\times \to \mathbb{C}^\times$, and hence it is of finite order iff it comes from a Dirichlet character.

3 Fourier Analysis

3.1 Fourier analysis on local fields

Let F be a local field. We only need the Schwartz functions and consider their integrals. The space of Schwartz functions $F \to \mathbb{C}$ is denoted by $\mathcal{S}(F)$. We are familiar with $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{C})$: the space of function f that decreases faster than any polynomial, i.e.,

$$\lim_{x\to\infty} x^n \left(\frac{d}{dx}\right)^m = 0,\ \forall m, n.$$

As for a nonarchimedean local field F, $\mathcal{S}(F)$ is defined to be the space of locally constant compactly supported functions. Because the topology of F and $\mathbb{C}$ are "totally incompatible", these are actually all the continuous functions from F to $\mathbb{C}$ with compact supports. Note that every Schwartz function may be written as a finite linear combination of functions $1_{a+\varpi^n\mathcal{O}_F}$, where ϖ is an uniformizer.

Then we fix an addtive measure on F.

- ◇ If $F=\mathbb{R}$, then $\mathrm{d}x :=$ the Lebesgue measure.
- ◇ If $F=\mathbb{C}$, then $\mathrm{d}x :=$ two-times the Lebesgue measure.
- ◇ If $F/\mathbb{Q}_p<\infty$, then $\mathrm{d}x$ satisfies $\mathrm{vol}(\mathcal{O}_F)=(N\mathfrak{d})^{-\frac{1}{2}}$.

To define Fourier transformation, one need to fix an additive character ψ on F.

- ◇ If $F=\mathbb{R}$, then $\psi(x):=\mathrm{e}^{-2\pi i x}$.
- ◇ If $F=\mathbb{C}$, then $\psi(x):=\mathrm{e}^{-2\pi i(x+\overline{x})}$.
- ◇ If $F/\mathbb{Q}_p<\infty$, then $\psi(x):=\mathrm{e}^{2\pi i\{\mathrm{Tr}_{F/\mathbb{Q}_p}x\}}$, where $\{\cdot\}:\mathbb{Q}_p/\mathbb{Z}_p\xrightarrow{\sim}\mathbb{Z}[1/p]/\mathbb{Z}\subset\mathbb{Q}/\mathbb{Z}$.

Then we define the Fourier transformation for $f\in\mathcal{S}(F)$ to be

$$\mathcal{F}_\psi f(y):=\widehat{f}(y):=\int_F f(x)\psi(xy)\,\mathrm{d}x.$$

Under these choices, we have the following properties known for $\mathbb{R}$ and $\mathbb{C}$.

Theorem 2. *Let $f\in\mathcal{S}(F)$. Then $\widehat{f}\in\mathcal{S}(F)$, and*

$$\widehat{\widehat{f}}(x)=f(-x).$$

In particular, if F is nonarchimedean and unramified, then

$$\widehat{1_{\mathcal{O}_F}}=1_{\mathcal{O}_F}.$$

Proof. (An important example of computation!)

We may assume F to be a nonarchimedean local field with ϖ an uniformizer, $f=1_{a+\varpi^n\mathcal{O}_F}$.

We have

$$\widehat{1_{a+\varpi^n\mathcal{O}_F}}(y)=\int_{a+\varpi^n\mathcal{O}}\psi(xy)\,\mathrm{d}x=\psi(ay)\int_{\varpi^n\mathcal{O}}\psi(xy)\,\mathrm{d}x=|\varpi|^n\psi(ay)\int_{\mathcal{O}}\psi(\varpi^n xy)\,\mathrm{d}x.$$

Note that $\phi:x\mapsto\psi(\varpi^n xy)$ is an additive character, and

$$\phi|_{\mathcal{O}}=1\iff\varpi^n y\in\mathfrak{d}^{-1}$$

(by definition), hence

$$\int_{\mathcal{O}}\phi(x)\,\mathrm{d}x=\begin{cases}\mathrm{vol}(\mathcal{O}), & y\in\varpi^{-n}\mathfrak{d}^{-1},\\ 0, & y\notin\varpi^{-n}\mathfrak{d}^{-1}.\end{cases}$$

(In the second case, ϕ has conductor smaller than $\mathcal{O}$ and thus factors through a non-trivial character of a finite group.) So

$$\widehat{1_{a+\varpi^n\mathcal{O}}}(y)=|\varpi|^n\psi(ay)(N\mathfrak{d})^{-\frac{1}{2}}1_{\varpi^{-n}\mathcal{O}}(y).$$

Similarly,

$$\int_F\psi(ay)1_{\varpi^{-n}\mathfrak{d}^{-1}}(y)\psi(xy)\,\mathrm{d}y=\int_{\varpi^{-n}\mathfrak{d}^{-1}}\psi((a+x)y)\,\mathrm{d}y=\mathrm{vol}(\varpi^{-n}\mathfrak{d}^{-1})\cdot1_{-a+\varpi^n\mathcal{O}}(x),$$

where

$$\mathrm{vol}(\varpi^{-n}\mathfrak{d}^{-1})=|\varpi|^{-n}\cdot\mathrm{vol}(\mathfrak{d}^{-1})=|\varpi|^{-n}\cdot\mathrm{vol}(\mathcal{O})N\mathfrak{d}=|\varpi|^{-n}(N\mathfrak{d})^{\frac{1}{2}}.$$

The result follows. □

The multiplicative measure on $F^\times$ is chosen as follows.

- $\diamond$ If $F = \mathbb{R}$, then $\mathrm{d}^\times x := |x|^{-1}\,\mathrm{d}x$.
- $\diamond$ If $F = \mathbb{C}$, then $\mathrm{d}^\times x := |x|_{\mathbb{C}}^{-1}\,\mathrm{d}x$, where $|x|_{\mathbb{C}} := x\overline{x}$. (Reason?)
- $\diamond$ If $F/\mathbb{Q}_p < \infty$, then $\mathrm{vol}(\mathcal{O}_F^\times, \mathrm{d}^\times x) = 1$.

As an example, integration on local fields can give the factor of L-function at $\mathfrak{p}$.

Lemma 3.1. Let χ be an unramified character $F^\times \to \mathbb{C}^\times$. Then

$$\int_{F^\times} 1_{\mathcal{O}_F}(x)\chi(x)|x|^s\,\mathrm{d}^\times x = (1-\chi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}.$$

Proof. Since $\mathcal{O}_F = \bigsqcup_{n\geq 0}\varpi^n\mathcal{O}_F^\times$,

$$\int_{F^\times} 1_{\mathcal{O}_F}(x)\chi(x)|x|^s\,\mathrm{d}^\times x = \sum_{n\geq 0}(\chi(\varpi)^n\cdot 1)\cdot N\mathfrak{p}^{-ns} = \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}.$$ $\square$

3.2 Fourier analysis on adeles

Let F be a number field.

A **Schwartz-Bruhat function** is a finite linear combination of functions of the form

$$\prod_v f_v : \mathbb{A}_F \to \mathbb{C}, \quad f_v \in \mathcal{S}(F_v), \quad f_v = 1_{\mathcal{O}_{F_v}} \text{ a.e.},$$

and denote the space of Schwartz-Bruhat functions by $\mathcal{S}(\mathbb{A}_F)$. Then define the additive character on $\mathbb{A}_F$ by

$$\psi(x) := \prod_v \psi_v(x_v).$$

This is by definition a finite product and thus well-defined.

Lemma 3.2. $\psi|_F = 1$. $\square$

Then we need to define and fix measures on $\mathbb{A}_F, \mathbb{A}_F^\times$ and $\mathbb{A}_F^{\times,1}$. For $\mathbb{A}$ resp. $\mathbb{A}^\times$, simply multiply the measures on each places yields an additive resp. multiplicative measure, if $\mathrm{vol}(\mathcal{O}_F, \mathrm{d}x) = 1$ resp. $\mathrm{vol}(\mathcal{O}_F^\times, \mathrm{d}^\times x) = 1$ (which is ture for our choices). So for a Schwartz-Bruhat function $f = \prod_v f_v$,

$$\int_{\mathbb{A}_F} f(x)\,\mathrm{d}x = \prod_v \int_{F_v} f_v(x_v)\,\mathrm{d}x_v, \quad \int_{\mathbb{A}_F^\times} f(x)\,\mathrm{d}^\times x = \prod_v \int_{F_v^\times} f_v(x_v)\,\mathrm{d}^\times x_v.$$

Theorem 3. *The volume of the foundamental domain of $\mathbb{A}_F/F$ under the given measure is* 1.

For $\mathbb{A}^{\times,1}$, fix an archimedean place u first. Define a continuous homomorphism $\phi : \mathbb{A}_F^\times \to \mathbb{A}_F^{\times,1}$ by $\phi(x)_u := x_u/|x|$ and $\phi(x)_v := x_v$ for $v \neq u$. The multiplicative measure $\mathrm{d}^\times x$ on $\mathbb{A}_F^{\times,1}$ is defined s.t. for a measurable set $U \subset \mathbb{A}_F^{\times,1}$,

$$\mathrm{vol}_{\mathbb{A}^\times}(U, \mathrm{d}^\times x) := \mathrm{vol}_{\mathbb{A}^{\times,1}}(U', \mathrm{d}^\times x), \text{ where } U' := \{x \in \mathbb{A}_F^\times : \phi(x) \in U, 0 \leq \log|x| \leq 1\}.$$

For example, let $F = \mathbb{Q}$ and $U = \prod_p \mathbb{Z}_p^\times \times \{1\}$, then $U' = \prod_p \mathbb{Z}_p^\times \times [1, e]$, so

$$\mathrm{vol}(U) = \int_1^e \frac{\mathrm{d}x}{x} = 1.$$

Remark. This is the measure defined by the exact sequence

$$1 \to \mathbb{A}_F^{\times,1} \to \mathbb{A}_F^\times \to \mathbb{R}_{>0} \to 1.$$

For $U = U^1 \times I$, where $U \subset \mathbb{A}_F^{\times,1}$ and $I \subset \mathbb{R}_{>0}$, $\mathrm{vol}(U) = \mathrm{vol}(U^1)\,\mathrm{vol}(I)$.

Theorem 4. *The volume of the foundamental of* $\mathbb{A}_F^{\times,1}/F^\times$ *is*

$$\frac{2^{r_1}(2\pi)^{r_2}h_F R_F}{w_F},$$

Now take $f \in \mathcal{S}(\mathbb{A}_F)$. Define

$$\mathcal{F}_\psi f(y) := \widehat{f}(y) := \int_{\mathbb{A}_F} f(x)\psi(xy)\,\mathrm{d}x.$$

In particular,

$$\widehat{\prod_v f_v} = \prod_v \widehat{f}_v.$$

By the lemma above, $\widehat{f} \in \mathcal{S}(\mathbb{A}_F)$.

Theorem 5 (Poisson Summation Formula)**.** *Let* $f \in \mathcal{S}(\mathbb{A}_F)$*, then*

$$\sum_{x\in F} f(x) = \sum_{x\in F} \widehat{f}(x).$$

(The summation obviously converges.)

Corollary 3.1. Let $\alpha \in \mathbb{A}_F^\times$, then

$$|\alpha| \sum_{x\in F} f(\alpha x) = \sum_{x\in F} \widehat{f}(\alpha^{-1}x). \quad \square$$

4 Analytic Properties of Hecke L-functions

Let F be a number field, $\chi = \prod_v \chi_v : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ a Hecke character, S a finite set containing all infinite places and all places v s.t. χ_v is ramified.

Recall that

$$L(s,\chi_v) := (1-\chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1},$$

and the **partial Hecke L-function**

$$L^S(s,\chi) := \prod_{v\notin S} L(s,\chi_v).$$

Lemma 4.1. The Euler product $L^S(s,\chi)$ absolutely converges if $\operatorname{Re} s > 1$.

Proof. If $\mathfrak{p}\cap\mathbb{Z} = p$, then $N\mathfrak{p} \geq p$, and since χ is unitary,

$$|(1-\chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq (1-p^{-\operatorname{Re} s})^{-1}.$$

Since there are at most $n = [F:\mathbb{Q}]$ primes over p,

$$\prod_v |(1-\chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq \prod_p (1-p^{-\operatorname{Re} s})^{-n}. \qquad \square$$

Take $f \in \mathcal{S}(\mathbb{A}_F)$ s.t. $f_v = 1_{\mathcal{O}}$ for $v \notin S$. Define the zeta integral

$$Z(s,f,\chi) := \int_{\mathbb{A}_F^\times} f(x)\chi(x)|x|^s\,\mathrm{d}^\times x$$

and local zeta integral

$$Z_v(s,f_v,\chi_v) := \int_{F_v^\times} f_v(x)\chi_v(x)|x|^s\,\mathrm{d}^\times x_v.$$

By definition,

$$Z(s,f,\chi)=\prod_v Z_v(s,f_v,\chi_v).$$

For $v\notin S$, we have seen in Lemma 3.1 that

$$L(s,\chi_v)=Z_v(s,f_v,\chi_v),$$

so

$$Z(s,f,\chi)=L^S(s,\chi)\prod_{v\in S} Z_v(s,f_v,\chi_v),$$

and it is absolutely convergent on $\operatorname{Re} s>1$.

Theorem 6. *$Z(s,f,\chi)$ can be extended to a meromorphic function on $\mathbb{C}$, satisfying*

$$Z(s,f,\chi)=Z(1-s,\widehat{f},\chi^{-1}).$$

Moreover, if there does not exist $\lambda\in i\mathbb{R}$ s.t. $\chi(x)=|x|^\lambda$, then $Z(s,f,\chi)$ is entire; otherwise the only poles of $Z(s,f,\chi)$ are $s=1-\lambda$ and $s=-\lambda$, which are both simple poles with residue $\widehat{f}(0)\operatorname{vol}(\mathbb{A}_F^{\times,1}/F^\times)$ and $-f(0)\operatorname{vol}(\mathbb{A}_F^{\times,1}/F^\times)$.

Proof. Because $\{|x|=1\}$ is of measure zero in $\mathbb{A}_F^\times$, we have

$$Z(s,f,\chi)=\int_{\mathbb{A}_F^\times}=\int_{\mathbb{A}_F^{>1}}+\int_{\mathbb{A}_F^{<1}}=:Z^{>1}+Z^{<1}.$$

For all $s\in\mathbb{C}$, the integrand is continuous when $|x|>1$, so $Z^{>1}$ converges on $\mathbb{C}$.

Now we turn to $Z^{<1}$. Let Ω be a foundamental domain of $\mathbb{A}_F^{<1}/F^\times$. Assume that s is big enough, then

$$\begin{aligned}
Z^{<1}&=\sum_{\alpha\in F^\times}\int_{\alpha\Omega} f(x)\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_\Omega\left(\sum_{\alpha\in F^\times} f(\alpha x)\right)\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_\Omega\left(\sum_{\alpha\in F} f(\alpha x)\right)\chi(x)|x|^s\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_\Omega\left(\sum_{\alpha\in F} \widehat{f}(\alpha x^{-1})\right)\chi(x)|x|^{s-1}\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_\Omega\left(\sum_{\alpha\in F^\times} \widehat{f}(\alpha x^{-1})\right)\chi(x)|x|^{s-1}\,\mathrm{d}^\times x+\widehat{f}(0)\int_\Omega\chi(x)|x|^{s-1}\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_{\Omega^{-1}}\left(\sum_{\alpha\in F^\times} \widehat{f}(\alpha x)\right)\chi(x^{-1})|x|^{1-s}\,\mathrm{d}^\times x+\widehat{f}(0)\int_\Omega\chi(x)|x|^{s-1}\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_{\mathbb{A}_F^{>1}}\widehat{f}(x)\chi(x)^{-1}|x|^{1-s}\,\mathrm{d}^\times x+\widehat{f}(0)\int_\Omega\chi(x)|x|^{s-1}\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x.
\end{aligned}$$

We used "$\chi(\alpha x)=\chi(x)$ for $\alpha\in F^\times$", Poisson summation, "$\mathrm{d}^\times x$ is invariant under $x\mapsto x^{-1}$", and "Ω^{-1} is a foundamental domain of $\mathbb{A}_F^{>1}/F^\times$" in the above calculation. The integral over $\mathbb{A}_F^{>1}$ is again convergent on $\mathbb{C}$, so we look at the rest two integrals.

Write $\Omega=\Omega^1\times(0,1)$, where Ω^1 is a foundamental domain of $\mathbb{A}_F^{\times,1}/F^\times$. Then if χ is non-trivial on $\mathbb{A}_F^{\times,1}$, both integrals vanish (as in Theorem 2). Otherwise χ factors through $\mathbb{A}_F^\times/\mathbb{A}_F^{\times,1}\simeq\mathbb{R}_{>0}\to\mathbb{C}^\times$, hence $\chi(x)=|x|^\lambda$ for some $\lambda\in i\mathbb{R}$, and

$$\widehat{f}(0)\int_\Omega\chi(x)|x|^{s-1}\,\mathrm{d}^\times x-f(0)\int_\Omega\chi(x)|x|^s\,\mathrm{d}^\times x=\frac{\widehat{f}(0)\operatorname{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s+\lambda-1}-\frac{f(0)\operatorname{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s+\lambda}.$$

The theorem is easy to deduce from the expression. $\square$

Our next target is $Z_v(s,f,\chi_v)=\int_{F_v^\times} f_v(x)\chi_v(x)|x|^s\,\mathrm{d}^\times x_v$.

Lemma 4.2. $Z_v(s,f,\chi_v)$ converges on $\operatorname{Re} s>0$.

Proof. Consider only the nonarchimedean case.

Take ϵ small enough s.t. $f_v(x)=f_v(0)$ for $|x|<\epsilon$. Write

$$Z_v(s,f_v,\chi_v)=\int_{|x|>\epsilon}+\int_{|x|<\epsilon}.$$

Similarly, the first integral converges on $\mathbb{C}$. For the second one, $\{|x|<\epsilon\}=\bigcup_{n\geq N}\varpi^n\mathcal{O}_{F_v}^\times$ for an integer N. Thus we see that

$$\int_{|x|<\epsilon}|\chi_v(x)|x|^s|\,\mathrm{d}^\times x=\sum_{n\geq N}\int_{\varpi^n\mathcal{O}_{F_v}^\times}|\varpi|^{-n\operatorname{Re} s}\,\mathrm{d}^\times x$$

converges when $\operatorname{Re} s>0$. □

Theorem 7. *(1) $Z_v(s,f,\chi_v)$ can be extended to a meromorphic function on $\mathbb{C}$ which is holomorphic on* $\operatorname{Re} s>0$.

*(2) There exists a meromorphic function $\gamma_v(s,\chi_v,\psi_v)$, called **local γ-factor**, irrelevant to f_v, s.t. for any $f_v\in\mathcal{S}(F_v)$,*

$$Z_v(1-s,\widehat{f_v},\chi_v^{-1})=\gamma_v(s,\chi_v,\psi_v)Z_v(s,f_v,\chi_v).$$

Proof. Firstly, both sides of the equation converge on $0<\operatorname{Re} s<1$.

We need to show that $\frac{Z_v(1-s,\widehat{f_v},\chi_v^{-1})}{Z_v(s,f_v,\chi_v)}$ is irrelevant to f_v; i.e.,

$$Z_v(1-s,\widehat{f_v},\chi_v^{-1})Z_v(s,g_v,\chi_v)=Z_v(1-s,\widehat{g_v},\chi_v^{-1})Z_v(s,f_v,\chi_v),\ \forall g_v\in\mathcal{S}(F_v).$$

Assume that $\mathrm{d}^\times x_v=|x|^{-1}\,\mathrm{d}x$, then the LHS

$$\begin{aligned}
&=\int_{F_v^\times}\left(\int_F f_v(y)\psi_v(xy)\,\mathrm{d}y\right)\chi_v(x)^{-1}|x|^{1-s}\,\mathrm{d}^\times x\int_{F_v^\times}g_v(x)\chi_v(x)|x|^s\,\mathrm{d}^\times x\\
&=\int_{F_v^\times}\int_{F_v^\times}\int_{F_v^\times}f_v(y)g_v(z)\psi_v(xy)\chi_v(zx^{-1})|x|^{1-s}|z|^s\,\mathrm{d}^\times x\,\mathrm{d}y\,\mathrm{d}^\times z\\
&=\iiint f_v(y)g_v(z)\psi_v(xy)\chi_v(zx^{-1})|x|^{1-s}|z|^s\cdot|y|\,\mathrm{d}^\times x\,\mathrm{d}^\times y\,\mathrm{d}^\times z\\
&=\iiint f_v(y)g_v(z)\psi_v(x)\chi_v(zyx^{-1})|x|^{1-s}|zy|^s\,\mathrm{d}^\times x\,\mathrm{d}^\times y\,\mathrm{d}^\times z\qquad(x\mapsto y^{-1}x).
\end{aligned}$$

Hence LHS = RHS.

So γ_v is well-defined on $0<\operatorname{Re} s<1$. If γ_v can be a meromorphic function on $\mathbb{C}$, then the equation gives the analytic continuation of Z_v on $\operatorname{Re} s<1$. (The formula of γ-factor is only computed for archimedean place in this proof.)

(1) $F_v=\mathbb{R}$. Note that

$$Z_v(s,f_v,\chi_v|\cdot|^t)=Z_v(s+t,f_v,\chi_v),$$

so we only need to compute for χ_v trivial or $\chi_v=\operatorname{sgn}$ character. The result is

$$\gamma_v(s,\chi_v,\psi_v)=\begin{cases}\dfrac{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}, & \chi=1,\\[2ex] i\dfrac{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)}{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}, & \chi=\operatorname{sgn}.\end{cases}$$

For example, when $\chi_v = 1$, we take $f_v(x) = \mathrm{e}^{-\pi x^2}$, then $\widehat{f_v} = f_v$, and

$$\begin{aligned} Z_v(s, f_v, 1) &= \int_{\mathbb{R}^\times} \mathrm{e}^{-\pi x^2} |x|^{s-1} \,\mathrm{d}x \\ &= 2\int_0^{+\infty} \mathrm{e}^{-\pi x^2} x^{s-1} \,\mathrm{d}x \\ &= \pi^{-\frac{s}{2}} \int_0^{+\infty} y^{\frac{s}{2}-1} \mathrm{e}^{-y} \,\mathrm{d}y \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \end{aligned}$$

therefore

$$\gamma_v(s, \chi_v, \psi_v) = \frac{Z_v(1-s, f_v, 1)}{Z_v(s, f_v, 1)} = \frac{\pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)}, \chi = 1.$$

(2) $F_v = \mathbb{C}$. For $\chi_v(x) = \left(\frac{x}{\sqrt{x\overline{x}}}\right)^n$, $n \in \mathbb{Z}$, using

$$f_v(x) = \begin{cases} \overline{x}^n \mathrm{e}^{-2x\overline{x}}, & n \geq 0, \\ x^{-n} \mathrm{e}^{-2x\overline{x}}, & n < 0, \end{cases}$$

we obtain

$$\gamma_v(s, \chi^v, \psi_v) = i^{|n|} \frac{(2\pi)^{-\left(1-s+\frac{|n|}{2}\right)} \Gamma\left(1-s+\frac{|n|}{2}\right)}{(2\pi)^{-\left(-s+\frac{|n|}{2}\right)} \Gamma\left(s+\frac{|n|}{2}\right)}.$$

(3) v is nonarchimedean. We show that γ_v is defined on $\operatorname{Re} s > 1$. Let U be a sufficiently small open compact nbhd of -1 in F_v s.t. χ_v is trivial on $-U$, and put $f_v := \widehat{1_U}$. Then

$$Z_v(t, \widehat{f_v}, \chi_v^{-1}) = \int_{F_v^\times} 1_U(-x) \chi(x^{-1}) |x|^t \,\mathrm{d}^\times x = \mathrm{vol}(U) \neq 0$$

and is irrelevant to t. Therefore γ_v^{-1} can be defined on $\operatorname{Re} s > 0$. Similar for $\operatorname{Re} s < 1$. ☐

Finaly, we obtain the analytic continuation of Hecke L-functions and the main theorem of functional equations.

Theorem 8. *Let S be a finite set of places s.t. $\forall v \notin S$, v is archimedean with χ_v unramified, and $\mathfrak{d}_v = \mathcal{O}_{F_v}$. Then the partial Hecke L-function can be extended to a meromorphic function on $\mathbb{C}$, satisfying*

$$L^S(s, \chi) = \left(\prod_{v \in S} \gamma_v(s, \chi_v, \psi_v)\right) L^S(1-s, \chi^{-1}).$$

Moreover, if there does not exist $\lambda \in i\mathbb{R}$ s.t. $\chi(x) = |x|^\lambda$, then $L^S(s, \chi)$ is entire; otherwise only $s = 1-\lambda$ and $s = -\lambda$ have the possibility to be poles.

Proof. Take $f = \prod_v f_v$ s.t. $f_v = 1_{\mathcal{O}_{F_v}}$, $\forall v \notin S$. For $v \notin S$, the additional condition $\mathfrak{d}_v = \mathcal{O}_{F_v}$ implies that (by Lemma 3.1)

$$\widehat{f_v}(x) = (N\mathfrak{d})^{-\frac{1}{2}} 1_{\mathcal{O}}(x) = 1_{\mathcal{O}}(x),$$

and the functional equation follows.

It is left to show the property about poles. Suppose that $\chi(x) = |x|^\lambda$ with $\lambda \in i\mathbb{R}$ and $s = s_0$ is a pole of L^S other than $-\lambda$ or $1-\lambda$. Consider the equation

$$Z(s, f, \chi) = L^S(s, \chi) \prod_{v \in S} Z_v(s, f_v, \chi_v).$$

By Theorem 6, LHS is holomorphic at $s = s_0$.

We choose an f s.t. for all $v \in S$, f_v supports in a sufficiently small nbhd U_v of $1 \in F_v$. With a similar argument in the previous proof, one sees that $Z_v(s_0, f_v, \chi_v) \neq 0$. Therefore the RHS has a pole at $s = s_0$, which is a contradiction.

□

4.1 Exercise

Let $F = \mathbb{Q}$, $\chi = 1$ the trivial character. Repeat the calculation before to prove the analytic continuation and functional equation of Riemann zeta function, and compute its residue at $s = 1$.

Proof. The Riemann zeta function is

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Let $S = \{\infty\}$. The local unramified L-functions are

$$L(s, 1_p) = (1 - p^{-s})^{-1},$$

so $\zeta(s) = L^S(s, \chi)$.

Let $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ be defined by $f_p = 1_{\mathbb{Z}_p}$ and $f_\infty(x) = \mathrm{e}^{-\pi x^2}$. The zeta integral is

$$Z(s, f, 1) = \int_{\mathbb{A}_{\mathbb{Q}}^\times} f(x)|x|^s \,\mathrm{d}^\times x$$

and the local zeta integral at infinity is

$$Z_\infty(s, f_\infty, 1) = \int_{\mathbb{R}^\times} \mathrm{e}^{-\pi x^2}|x|^s \,\mathrm{d}^\times x.$$

We have

$$Z(s, f, 1) = \zeta(s) Z_\infty(s, f_\infty, 1).$$

(TBC......)

□

5 Dedekind Zeta Functions and Dirichlet L-functions

5.1 Dedekind Zeta Functions and the Analytic Class Number Formula

Let F be a number field, χ the trivial character, S the set of all archimedean places. The **Dedekind zeta function** of F is defined to be

$$\zeta_F(s) := L^S(s, \chi) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}.$$

In this subsection, we will compute the local γ-factors at ramified places to deduce the functional equation of Dedekind zeta functions and the analytic class number formula.

Theorem 9. *The Dedekind zeta function $\zeta_F(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with only poles at $s = 0$ and $s = 1$.*

1. *(Analytic class number formula.) $\zeta_F(s)$ has a simple pole at $s = 1$ with residue*

$$\operatorname{res}_1 \zeta_F = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{\sqrt{|\operatorname{disc} F|}\, w_F},$$

and is of order $r_1 + r_2 - 1$ at $s = 0$ with

$$\lim_{s \to 0} s^{r_1 + r_2 - 1} \zeta_F(s) = -\frac{h_F R_F}{w_F}$$

2. *Define the completed Dedekind zeta function*

$$\Lambda(s) := |\operatorname{disc} F|^{\frac{s}{2}} \left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2} \zeta_F(s).$$

Then

$$\Lambda_F(s) = \Lambda_F(1-s).$$

5.2 Dirichlet L-functions

Let $F=\mathbb{Q}$, χ a Dirichlet character with conductor N, $S=\{p : p|N\}\cup\{\infty\}$. Lifting χ to a Hecke character $\widetilde{\chi}$, we get an partial L-function

$$L^S(s,\widetilde{\chi}) = \prod_{p\nmid N}(1-\chi_p(p)Np^{-s})^{-1},$$

which is exactly the classic Dirichlet L-function

$$L(s,\chi) = \prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1} = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}, \quad \operatorname{Re} s > 1.$$

The functional equation

$$L(s,\chi) = \left(\prod_{p|N}\gamma_p(s,\chi_p,\psi_p)\right)\gamma_\infty(s,\chi_\infty,\psi_\infty)L(1-s,\chi^{-1})$$

has been proved, and γ_∞ have been computed. It is left to compute the γ_p's for $p \mid N$.

Suppose $p^e \parallel N$, then the conductor of χ_p is p^e (or $1+p^e\mathbb{Z}_p$). Take $f_p := 1_{1+p^e\mathbb{Z}_p}$, so

$$Z_p(s,f_p,\chi_p) = \int_{1+p^e\mathbb{Z}_p}|x|^s\,\mathrm{d}^\times x = \operatorname{vol}(1+p^e\mathbb{Z}_p)$$

is easy to compute. Then

$$\widehat{f_p}(x) = p^{-e}\psi_p(x)1_{p^{-e}\mathbb{Z}_p}(x),$$

hence

$$\begin{aligned}\gamma_p(s,\chi_p,\psi_p) &= p^{-e}\operatorname{vol}(1+p^e\mathbb{Z}_p)^{-1}\int_{p^{-e}\mathbb{Z}_p}\psi_p(x)\chi_p(x)|x|^{1-s}\,\mathrm{d}^\times x\\ &= p^{-e}\operatorname{vol}(1+p^e\mathbb{Z}_p)^{-1}\sum_{n\geq -e}p^{-n(1-s)}\int_{p^n\mathbb{Z}_p^\times}\psi_p(x)\chi_p(x)\,\mathrm{d}^\times x.\end{aligned}$$

Next, we show that $\int_{p^n\mathbb{Z}_p^\times} = 0$ whenever $n \neq -e$.

(1) $n \geq 0$. Then ψ_p is trivial on $p^n\mathbb{Z}_p^\times \subset \mathbb{Z}_p$, so the integral equals

$$\int_{p^n\mathbb{Z}_p^\times}\chi_p(x)\,\mathrm{d}^\times x = \chi_p(p^n)\int_{\mathbb{Z}_p^\times}\chi_p(x) = 0$$

as χ_p is nontrivial on $\mathbb{Z}_p^\times$.

(2) $-e < n < 0$. We can decompose $p^n\mathbb{Z}_p^\times$ into $p^{-n-1}(p-1)$ copies of $\mathbb{Z}_p$, i.e.,

$$p^n\mathbb{Z}_p^\times = \bigsqcup_{\alpha}(\alpha+\mathbb{Z}_p)\,,\ \alpha\in p^n\mathbb{Z}_p^\times,$$

then

$$\int_{p^n\mathbb{Z}_p^\times}\psi_p(x)\chi_p(x)\,\mathrm{d}^\times x = \sum_\alpha\psi_p(\alpha)\int_{\alpha+\mathbb{Z}_p}\chi_p(\alpha)\,\mathrm{d}^\times x = \sum_\alpha\psi_p(\alpha)\chi_p(\alpha)\int_{1+\alpha^{-1}\mathbb{Z}_p}\chi(x)\,\mathrm{d}^\times x.$$

Since $-n<e$, $1+\alpha^{-1}\mathbb{Z}_p \supset 1+p^e\mathbb{Z}_p$, so the integrals vanish agains.

So we get

$$\begin{aligned}
\gamma_p(s,\chi_p,\psi_p) &= p^{-es}\operatorname{vol}(1+p^e\mathbb{Z}_p)^{-1}\int_{p^{-e}\mathbb{Z}_p^\times}\psi_p(x)\chi_p(x)\,\mathrm{d}^\times x \\
&= p^{-es}\operatorname{vol}(1+p^e\mathbb{Z}_p)^{-1}\sum_{\alpha\in\mathbb{Z}_p^\times/(1+p^e\mathbb{Z}_p)}\psi_p(p^{-e}\alpha)\chi_p(p^{-e}\alpha)\int_{\alpha(1+p^e\mathbb{Z}_p)}\mathrm{d}^\times x \\
&= p^{-es}\sum_{\alpha\in(\mathbb{Z}/p^e\mathbb{Z})^\times}\psi_p(p^{-e}\alpha)\chi_p(p^{-e}\alpha).
\end{aligned}$$

(T.B.C.)

5.3 Quadratic Fields

Let $F=\mathbb{Q}(\sqrt{d})$ and $D=|\operatorname{disc}F|$. Define $\chi_d:(\mathbb{Z}/D\mathbb{Z})^\times\to\{\pm1\}$ by

$$\chi_d(p):=\begin{cases}1, & p \text{ splits in } F,\\ -1, & p \text{ is inert in } F.\end{cases}$$

Remark. Every primitive quadratic Dirichlet character is of the form above, which is just the Legendre symbol $\left(\frac{\cdot}{D}\right)$.

Lemma 5.1. $\zeta_F(s)=\zeta(s)L(s,\chi_d)$.

Proof. Check the equation

$$\prod_{\mathfrak{p}|p}\zeta_{\mathfrak{p}}(s)=\zeta_p(s)L_p(s,\chi_{d,p}).$$ □

Proposition 5.1 (Dirichlet)**.** If $d<0$, then

$$L(1,\chi_d)=\frac{2\pi h_F}{\sqrt{D}w_F}.$$

If $d>0$, then

$$L(1,\chi_d)=\frac{h_F\log\epsilon_F}{\sqrt{D}},$$

where $\epsilon_F>0$ is a foundamental unit of F.

This proposition explains why the class number of real quadratic fields are harder to study than imaginary ones, because there are powerful analytic methods to study $L(1,\chi_d)$, but we don't have a general method to seperate h_F and $\log\epsilon_F$ (when $d>0$).

The Gauss Class Number Problems

From now on, assume that $d<0$. Gauss had conjectured that (1) $h_F\to\infty$ as $d\to-\infty$, and (2) give a list of imaginary quadratic fields F with $h_F=1,2,3$, suspecting that these are all such fields. In particular, only 9 imaginary quadratic fields have class number 1:

$$d=-1,-2,-3,-7,-11,-19,-43,-67,-163.$$

Problem (1) was solved by C.Siegel in 1935.

Theorem 10 (Siegel)**.** *For all $\varepsilon>0$, there exists a constant $C(\varepsilon)>0$ s.t.*

$$L(1,\chi)\geq\frac{C(\varepsilon)}{N^\varepsilon}$$

for any primitive character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \{\pm 1\}$*. In particular, there exists a constant* $C'(\varepsilon)$ *s.t.*

$$h_F \geq C'(\varepsilon) D^{\frac{1}{2}-\varepsilon}.$$

This implies that there are only finite many imaginary quadratic fields F *with* $h_F = A$ *for any given constant* A*.*

In Siegel's theorem, the constant $C(\varepsilon)$ is not an *effective constant*, meaning that there is no explicit formula for $C(\varepsilon)$ using ε and N. Hence the problem (2) wasn't solved until 1983.