

# Notes on Algebraic Number Theory

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## Some Notations

Let  $F$  be a number field, then we denote by  $r_1$  the number of real embeddings,  $r_2$  the number of the pairs of complex embeddings,  $\text{Cl}(F)$  the class group,  $h_F$  the class number,  $R_F$  the regulator,  $w_F$  the number of roots of unity in  $F$ ,  $\mathfrak{d} = \mathfrak{d}_F$  the different ideal.

Always denote  $\sqrt{-1} \in \mathbb{C}$  by  $i$ .

## 1 Adeles and Ideles

Note that the topology on  $\mathbb{A}_F^\times$  (defined using natural nbhd of 1 in  $\mathbb{Q}_p^\times$ ) is different from (more precisely, finer than) that on  $\mathbb{A}_F$  (defined using natural nbhd of 0 in  $\mathbb{Q}_p$ ), but the topology on  $\mathbb{A}_F^{\times,1}$  induced from  $\mathbb{A}_F$  and that from  $\mathbb{A}_F^\times$  coincide.

**Theorem 1.** The quotient  $\mathbb{A}_F^{\times,1}/F^\times$  is compact.

*Proof.* Let  $I_F$  be the group of fractional ideals. Observe that we have an epimorphism

$$\mathbb{A}_F^{\times,1} \twoheadrightarrow I_F, (x_v) \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})},$$

under which  $x \in F^\times$  is sent to the principle fractional ideal  $(x) \in I_F$ , and thus gives an epimorphism  $\mathbb{A}_F^{\times,1}/F^\times \twoheadrightarrow \text{Cl}(F)$ . As  $\text{Cl}(F)$  is finite, it reduces to show that the kernel of this homomorphism is compact.

An element  $(x_v) \in \ker$  iff it is mapped to a principle ideal, i.e.,  $\exists x \in F^\times$  s.t.  $\forall \mathfrak{p}, x_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} = x \mathcal{O}_{\mathfrak{p}}$ , or say  $x_{\mathfrak{p}} \in x^{-1} \mathcal{O}_{\mathfrak{p}}^\times$ . Therefore the kernel is the image of

$$\left( \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \prod_{v|\infty} F_v^\times \right) \cap \mathbb{A}_F^{\times,1} = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \left( \prod_{v|\infty} F_v^\times \right)^1$$

in  $\mathbb{A}_F^{\times,1}/F^\times$ , where  $\left( \prod_{v|\infty} F_v^\times \right)^1$  denotes the set of element with norm 1. Because two elements in this set cannot differ by an element in  $F^\times \setminus \mathcal{O}_F^\times$ , we see that

$$\ker = \left( \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \left( \prod_{v|\infty} F_v^\times \right)^1 \right) / \mathcal{O}_F^\times.$$

Now it suffices to prove that  $\left(\prod_{v|\infty} F_v^\times\right)^1 / \mathcal{O}_F^\times$  is compact. Let  $v_1, \dots, v_r$  be the places of real embeddings and  $v_{r_1+1}, \dots, v_{r_1+r_2}$  the places of complex ones. The logarithm map

$$\left(\prod_{v|\infty} F_v^\times\right)^1 \rightarrow \mathbb{R}^{r_1+r_2}, \quad x \mapsto (\log |x_{v_1}|, \dots, \log |x_{v_{r_1}}|, \log |x_{v_{r_1+1}}|_{\mathbb{C}}, \dots, \log |x_{v_{r_1+r_2}}|_{\mathbb{C}})$$

is a homomorphism with kernel  $T = \{\pm 1\}^{r_1} \times (S^1)^{r_2}$ , which is compact and the intersection  $T \cap \mathcal{O}_F^\times = W_F$ , the roots of unity in  $F$ . So  $T/T \cap \mathcal{O}_F^\times$  is compact. Its image is the hypersurface

$$\Sigma : x_1 + \dots + x_{r_1+r_2} = 1$$

in  $\mathbb{R}^{r_1+r_2}$ . Dirichlet units theorem says that the image of  $\mathcal{O}_F^\times$  in  $\Sigma$  is a lattice of full rank, so the quotient  $\Sigma/\mathcal{O}_F^\times$  is also compact. Our goal follows.  $\square$

*Remark.* This theorem is equivalent to the combination of the finiteness of class group and Dirichlet units theorem.

## 2 $L$ -functions

### 2.1 Riemann Zeta Function

Recall that the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

converges on  $\operatorname{Re} s > 1$  and can be extended to a meromorphic function on  $\mathbb{C}$  with  $s = 1$  the only simple pole. The functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$

is satisfied.

### 2.2 Charaters

A **character** of a group  $G$  is a continuous homomorphism  $G \rightarrow \mathbb{C}^\times$ , and **trivial character** is the character  $G \rightarrow \{1\}$ . The charaters of a group  $G$  forms a group, denoted by  $\hat{G}$ .

**Lemma 2.1.** Let  $G$  be a finite abelian group.

1. There exists an non-canonical isomorphism  $G \simeq \hat{G}$ .
2. If  $\chi$  is a non-trivial character, then

$$\sum_{g \in G} \chi(g) = 0.$$

Conversely, if  $g \neq 1$ , then

$$\sum_{\chi \in \hat{G}} \chi(g) = 0.$$

$\square$

Let  $F$  be a number field. A **Hecke character** of  $F$  is a character of  $\mathbb{A}_F^\times / F^\times$ .

**Proposition 2.1.** Let  $\chi$  be a character on  $\mathbb{A}_F^\times$ . Then  $\chi$  is of the form  $\prod_v \chi_v$ , where  $\chi_v \in \hat{F}_v^\times$  and  $\chi_v$ 's are **unramified** (i.e., trivial on  $\mathcal{O}_{F_v}^\times$ ) for almost all nonarchimedean places.

So we can go back to characters on local fields. Let  $F$  be a local field and  $\chi$  a character of  $F^\times$ . The character  $\chi$  is called **unitary**, if  $|\chi(F^\times)| = \{1\}$ . We can describe  $\chi$  explicitly.

◊ If  $F = \mathbb{R}$ , then

$$\chi(x) = \left( \frac{x}{|x|} \right)^\epsilon |x|^s, \quad \epsilon = 0, 1, \quad s \in \mathbb{C}.$$

It is unitary iff  $s \in i\mathbb{R}$ .

◊ If  $F = \mathbb{C}$ , then

$$\chi(x) = \left( \frac{x}{\sqrt{x\bar{x}}} \right)^m (x\bar{x})^s, \quad m \in \mathbb{Z}, \quad s \in \mathbb{C}.$$

It is unitary iff  $s \in i\mathbb{R}$ .

◊ If  $F$  is nonarchimedean, then there exists a minimal integer  $N$  s.t.  $\chi(1 + \varpi^N \mathcal{O}_F^\times) = \{1\}$ , whence  $\chi$  factors through the finite group  $\mathcal{O}_F^\times / (1 + \varpi^N \mathcal{O}_F^\times)$ , and thus

$$\chi(x) = |x|^s \chi_0(x),$$

where  $\chi_0$  is a character of  $\mathcal{O}_F^\times / (1 + \varpi^N \mathcal{O}_F^\times)$ . It is unitary if  $s \in i\mathbb{R}$ . This integer is called the **conductor** of  $\chi$ .

**From now on, all multiplicative characters of local fields are assumed to be unitary.**

### 2.3 Lift a Dirichlet Character to a Hecke Character

Look at a character  $\chi : (\mathbb{Z}/\ell^e \mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  first. We define a character  $\chi_p$  on  $\mathbb{Q}_p^\times \simeq p^\mathbb{Z} \times \mathbb{Z}_p^\times$  and  $\chi_\infty$  on  $\mathbb{R}^\times$  as follows.

- If  $p = \ell$ , then the isomorphism  $\mathbb{Z}_\ell^\times / (1 + \ell^e \mathbb{Z}_\ell) \simeq (\mathbb{Z}/\ell^e \mathbb{Z})^\times$  enable us to lift  $\chi^{-1}$  (note the ‘-1!’) to a character  $\chi_\ell$  on  $\mathbb{Q}_\ell^\times$  that is trivial on  $\ell^\mathbb{Z}$  and  $1 + \ell^e \mathbb{Z}_\ell$ .
- If  $p \neq \ell$ , then  $p$  is invertible mod  $\ell^e$ , so we can define  $\chi_p(p) := \chi(p)$ , then make it trivial on  $\mathbb{Z}_p^\times$ .
- Put  $\chi_\infty := \text{sgn}^{\chi(-1)}$ .

Since  $\chi_p$  are trivial on  $\mathbb{Z}_p^\times$  only except for  $p = \ell$ , patching them together yields a character  $\tilde{\chi} := \prod_v \chi_v$  on  $\mathbb{A}_Q^\times$ .

**Lemma 2.2.** The character  $\tilde{\chi}$  is trivial on  $\mathbb{Q}^\times$ .

*Proof.* It suffices to check for every prime  $p$  and  $-1$ . If  $p \neq \ell$ , then  $\tilde{\chi}(p) = \chi_p(p) \chi_\ell(p) = 1$ ; otherwise  $\chi_v(\ell) = 1$  for all places  $v$ . To conclude,  $\tilde{\chi}(-1) = \chi_\infty(-1) \chi_\ell(-1) = 1$ .  $\square$

Now consider  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ . The factorisation  $N = p_1^{e_1} \cdots p_r^{e_r}$  gives

$$(\mathbb{Z}/N)^\times \simeq (\mathbb{Z}/p_1^{e_1})^\times \times \cdots \times (\mathbb{Z}/p_r^{e_r})^\times,$$

so we have  $\chi = \chi_1 \cdots \chi_r$ , where  $\chi_i : (\mathbb{Z}/p_i^{e_i})^\times \rightarrow \mathbb{C}^\times$ , and obtain a Hecke character  $\tilde{\chi} := \tilde{\chi}_1 \cdots \tilde{\chi}_r$ .

*Remark.* The character  $\tilde{\chi}$  is

$$\mathbb{A}_Q^\times / \mathbb{Q}^\times \rightarrow \mathbb{A}_Q^\times / \mathbb{Q}^\times \mathbb{R}_{>0} \simeq \hat{\mathbb{Z}}^\times \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\tilde{\chi}} \mathbb{C}^\times.$$

Conversely, every Hecke character factors through  $\hat{\mathbb{Z}}^\times \rightarrow \mathbb{C}^\times$ , and hence it is of finite order iff it comes from a Dirichlet character.

### 3 Fourier Analysis

#### 3.1 Fourier analysis on local fields

Let  $F$  be a local field. We only need the Schwartz functions and consider their integrals. The space of Schwartz functions  $F \rightarrow \mathbb{C}$  is denoted by  $\mathcal{S}(F)$ . We are familiar with  $\mathcal{S}(\mathbb{R})$  and  $\mathcal{S}(\mathbb{C})$ : the space of function  $f$  that decreases faster than any polynomial, i.e.,

$$\lim_{x \rightarrow \infty} x^n \left( \frac{d}{dx} \right)^m = 0, \quad \forall m, n.$$

As for a nonarchimedean local field  $F$ ,  $\mathcal{S}(F)$  is defined to be the space of locally constant compactly supported functions. Because the topology of  $F$  and  $\mathbb{C}$  are “totally incompatible”, these are actually all the continuous functions from  $F$  to  $\mathbb{C}$  with compact supports. Note that every Schwartz function may be written as a finite linear combination of functions  $1_{a+\varpi^n \mathcal{O}_F}$ , where  $\varpi$  is an uniformizer.

Then we fix an additive measure on  $F$ .

- ◇ If  $F = \mathbb{R}$ , then  $dx :=$  the Lebesgue measure.
- ◇ If  $F = \mathbb{C}$ , then  $dx :=$  two-times the Lebesgue measure.
- ◇ If  $F/\mathbb{Q}_p < \infty$ , then  $dx$  satisfies  $\text{vol}(\mathcal{O}_F) = (N\mathfrak{d})^{-\frac{1}{2}}$ .

To define Fourier transformation, one need to fix an additive character  $\psi$  on  $F$ .

- ◇ If  $F = \mathbb{R}$ , then  $\psi(x) := e^{-2\pi i x}$ .
- ◇ If  $F = \mathbb{C}$ , then  $\psi(x) := e^{-2\pi i(x+\bar{x})}$ .
- ◇ If  $F/\mathbb{Q}_p < \infty$ , then  $\psi(x) := e^{2\pi i \{\text{Tr}_{F/\mathbb{Q}_p} x\}}$ , where  $\{\cdot\} : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ .

Then we define the Fourier transformation for  $f \in \mathcal{S}(F)$  to be

$$\mathcal{F}_\psi f(y) := \hat{f}(y) := \int_F f(x) \psi(xy) dx.$$

Under these choices, we have the following properties known for  $\mathbb{R}$  and  $\mathbb{C}$ .

**Theorem 2.** Let  $f \in \mathcal{S}(F)$ . Then  $\hat{\hat{f}} \in \mathcal{S}(F)$ , and

$$\hat{\hat{f}}(x) = f(-x).$$

In particular, if  $F$  is nonarchimedean and unramified, then

$$1_{\hat{\mathcal{O}}_F} = 1_{\mathcal{O}_F}.$$

*Proof.* (An important example of computation!)

We may assume  $F$  to be a nonarchimedean local field with  $\varpi$  an uniformizer,  $f = 1_{a+\varpi^n \mathcal{O}_F}$ .

We have

$$1_{a+\varpi^n \mathcal{O}_F}(y) = \int_{a+\varpi^n \mathcal{O}_F} \psi(xy) dx = \psi(ay) \int_{\varpi^n \mathcal{O}_F} \psi(xy) dx = |\varpi|^n \psi(ay) \int_{\mathcal{O}_F} \psi(\varpi^n xy) dx.$$

Note that  $\phi : x \mapsto \psi(\varpi^n xy)$  is an additive character, and

$$\phi|_{\mathcal{O}} = 1 \iff \varpi^n y \in \mathfrak{d}^{-1}$$

(by definition), hence

$$\int_{\mathcal{O}} \phi(x) dx = \begin{cases} \text{vol}(\mathcal{O}), & y \in \varpi^{-n}\mathfrak{d}^{-1}, \\ 0, & y \notin \varpi^{-n}\mathfrak{d}^{-1}. \end{cases}$$

(In the second case,  $\phi$  has conductor smaller than  $\mathcal{O}$  and thus factors through a non-trivial character of a finite group.) So

$$1_{a+\varpi^n\mathcal{O}}(y) = |\varpi|^n \psi(ay) (N\mathfrak{d})^{-\frac{1}{2}} 1_{\varpi^{-n}\mathcal{O}}(y).$$

Similarly,

$$\int_F \psi(ay) 1_{\varpi^{-n}\mathfrak{d}^{-1}}(y) \psi(xy) dy = \int_{\varpi^{-n}\mathfrak{d}^{-1}} \psi((a+x)y) dy = \text{vol}(\varpi^{-n}\mathfrak{d}^{-1}) \cdot 1_{-a+\varpi^n\mathcal{O}}(x),$$

where

$$\text{vol}(\varpi^{-n}\mathfrak{d}^{-1}) = |\varpi|^{-n} \cdot \text{vol}(\mathfrak{d}^{-1}) = |\varpi|^{-n} \cdot \text{vol}(\mathcal{O})N\mathfrak{d} = |\varpi|^{-n}(N\mathfrak{d})^{\frac{1}{2}}.$$

The result follows. □

The multiplicative measure on  $F^\times$  is chosen as follows.

- ◇ If  $F = \mathbb{R}$ , then  $d^\times x := |x|^{-1} dx$ .
- ◇ If  $F = \mathbb{C}$ , then  $d^\times x := |x|_{\mathbb{C}}^{-1} dx$ , where  $|x|_{\mathbb{C}} := x\bar{x}$ . (Reason?)
- ◇ If  $F/\mathbb{Q}_p < \infty$ , then  $\text{vol}(\mathcal{O}_F^\times, d^\times x) = 1$ .

As an example, integration on local fields can give the factor of  $L$ -function at  $\mathfrak{p}$ .

**Lemma 3.1.** Let  $\chi$  be an unramified character  $F^\times \rightarrow \mathbb{C}^\times$ . Then

$$\int_{F^\times} 1_{\mathcal{O}_F}(x) \chi(x) |x|^s d^\times x = (1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}.$$

*Proof.* Since  $\mathcal{O}_F = \bigsqcup_{n \geq 0} \varpi^n \mathcal{O}_F^\times$ ,

$$\int_{F^\times} 1_{\mathcal{O}_F}(x) \chi(x) |x|^s d^\times x = \sum_{n \geq 0} (\chi(\varpi)^n \cdot 1) \cdot N\mathfrak{p}^{-ns} = \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}. \quad \square$$

## 3.2 Fourier analysis on adeles

Let  $F$  be a number field.

A **Schwartz-Bruhat function** is a finite linear combination of functions of the form

$$\prod_v f_v : \mathbb{A}_F \rightarrow \mathbb{C}, \quad f_v \in \mathcal{S}(F_v), \quad f_v = 1_{\mathcal{O}_{F_v}} \text{ a.e.,}$$

and denote the space of Schwartz-Bruhat functions by  $\mathcal{S}(\mathbb{A}_F)$ . Then define the additive character on  $\mathbb{A}_F$  by

$$\psi(x) := \prod_v \psi_v(x_v).$$

This is by definition a finite product and thus well-defined.

**Lemma 3.2.**  $\psi|_F = 1$ . □

Then we need to define and fix measures on  $\mathbb{A}_F, \mathbb{A}_F^\times$  and  $\mathbb{A}_F^{\times,1}$ . For  $\mathbb{A}$  resp.  $\mathbb{A}^\times$ , simply multiply the measures on each places yields an additive resp. multiplicative measure, if  $\text{vol}(\mathcal{O}_F, dx) = 1$  resp.  $\text{vol}(\mathcal{O}_F^\times, d^\times x) = 1$  (which is true for our choices). So for a Schwartz-Bruhat function  $f = \prod_v f_v$ ,

$$\int_{\mathbb{A}_F} f(x) dx = \prod_v \int_{F_v} f_v(x_v) dx_v, \quad \int_{\mathbb{A}_F^\times} f(x) d^\times x = \prod_v \int_{F_v^\times} f_v(x_v) d^\times x_v.$$

**Theorem 3.** The volume of the fundamental domain of  $\mathbb{A}_F/F$  under the given measure is 1.

For  $\mathbb{A}^{\times,1}$ , fix an archimedean place  $u$  first. Define a continuous homomorphism  $\phi : \mathbb{A}_F^\times \rightarrow \mathbb{A}_F^{\times,1}$  by  $\phi(x)_u := x_u/|x|$  and  $\phi(x)_v := x_v$  for  $v \neq u$ . The multiplicative measure  $d^\times x$  on  $\mathbb{A}_F^{\times,1}$  is defined s.t. for a measurable set  $U \subset \mathbb{A}_F^{\times,1}$ ,

$$\text{vol}_{\mathbb{A}^\times}(U, d^\times x) := \text{vol}_{\mathbb{A}^{\times,1}}(U', d^\times x), \quad \text{where } U' := \{x \in \mathbb{A}_F^\times : \phi(x) \in U, 0 \leq \log |x| \leq 1\}.$$

For example, let  $F = \mathbb{Q}$  and  $U = \prod_p \mathbb{Z}_p^\times \times \{1\}$ , then  $U' = \prod_p \mathbb{Z}_p^\times \times [1, e]$ , so

$$\text{vol}(U) = \int_1^e \frac{dx}{x} = 1.$$

*Remark.* This is the measure defined by the exact sequence

$$1 \rightarrow \mathbb{A}_F^{\times,1} \rightarrow \mathbb{A}_F^\times \rightarrow \mathbb{R}_{>0} \rightarrow 1.$$

For  $U = U^1 \times I$ , where  $U \subset \mathbb{A}_F^{\times,1}$  and  $I \subset \mathbb{R}_{>0}$ ,  $\text{vol}(U) = \text{vol}(U^1) \text{vol}(I)$ .

**Theorem 4.** The volume of the fundamental of  $\mathbb{A}_F^{\times,1}/F^\times$  is

$$\frac{2^{r_1} (2\pi)^{r_2} h_F R_F}{w_F},$$

Now take  $f \in \mathcal{S}(\mathbb{A}_F)$ . Define

$$\mathcal{F}_\psi f(y) := \hat{f}(y) := \int_{\mathbb{A}_F} f(x) \psi(xy) dx.$$

In particular,

$$\prod_v \hat{f}_v = \prod_v \hat{f}_v.$$

By the lemma above,  $\hat{f} \in \mathcal{S}(\mathbb{A}_F)$ .

**Theorem 5** (Poisson Summation Formula). Let  $f \in \mathcal{S}(\mathbb{A}_F)$ , then

$$\sum_{x \in F} f(x) = \sum_{x \in F} \hat{f}(x).$$

(The summation obviously converges.)

**Corollary 3.1.** Let  $\alpha \in \mathbb{A}_F^\times$ , then

$$|\alpha| \sum_{x \in F} f(\alpha x) = \sum_{x \in F} \hat{f}(\alpha^{-1} x). \quad \square$$

## 4 Analytic Properties of Hecke $L$ -functions

Let  $F$  be a number field,  $\chi = \prod_v \chi_v : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times$  a Hecke character,  $S$  a finite set containing all infinite places and all places  $v$  s.t.  $\chi_v$  is ramified.

Recall that

$$L(s, \chi_v) := (1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1},$$

and the **partial Hecke  $L$ -function**

$$L^S(s, \chi) := \prod_{v \notin S} L(s, \chi_v).$$

**Lemma 4.1.** The Euler product  $L^S(s, \chi)$  absolutely converges if  $\operatorname{Re} s > 1$ .

*Proof.* If  $\mathfrak{p} \cap \mathbb{Z} = p$ , then  $N\mathfrak{p} \geq p$ , and since  $\chi$  is unitary,

$$|(1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq (1 - p^{-\operatorname{Re} s})^{-1}.$$

Since there are at most  $n = [F : \mathbb{Q}]$  primes over  $p$ ,

$$\prod_v |(1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq \prod_p (1 - p^{-\operatorname{Re} s})^{-n}. \quad \square$$

Take  $f \in \mathcal{S}(\mathbb{A}_F)$  s.t.  $f_v = 1_{\mathcal{O}}$  for  $v \notin S$ . Define the zeta integral

$$Z(s, f, \chi) := \int_{\mathbb{A}_F^\times} f(x) \chi(x) |x|^s d^\times x$$

and local zeta integral

$$Z_v(s, f_v, \chi_v) := \int_{F_v^\times} f_v(x) \chi_v(x) |x|^s d^\times x_v.$$

By definition,

$$Z(s, f, \chi) = \prod_v Z_v(s, f_v, \chi_v).$$

For  $v \notin S$ , we have seen in Lemma 3.1 that

$$L(s, \chi_v) = Z_v(s, f_v, \chi_v),$$

so

$$Z(s, f, \chi) = L^S(s, \chi) \prod_{v \in S} Z_v(s, f_v, \chi_v),$$

and it is absolutely convergent on  $\operatorname{Re} s > 1$ .

**Theorem 6.**  $Z(s, f, \chi)$  can be extended to a meromorphic function on  $\mathbb{C}$ , satisfying

$$Z(s, f, \chi) = Z(1 - s, \hat{f}, \chi^{-1}).$$

Moreover, if there does not exist  $\lambda \in i\mathbb{R}$  s.t.  $\chi(x) = |x|^\lambda$ , then  $Z(s, f, \chi)$  is entire; otherwise the only poles of  $Z(s, f, \chi)$  are  $s = 1 - \lambda$  and  $s = -\lambda$ , which are both simple poles with residue  $\hat{f}(0) \operatorname{vol}(\mathbb{A}_F^{\times, 1} / F^\times)$  and  $-f(0) \operatorname{vol}(\mathbb{A}_F^{\times, 1} / F^\times)$ .

*Proof.* Because  $\{|x| = 1\}$  is of measure zero in  $\mathbb{A}_F^\times$ , we have

$$Z(s, f, \chi) = \int_{\mathbb{A}_F^\times} = \int_{\mathbb{A}_F^{>1}} + \int_{\mathbb{A}_F^{\leq 1}} =: Z^{>1} + Z^{<1}.$$

For all  $s \in \mathbb{C}$ , the integrand is continuous when  $|x| > 1$ , so  $Z^{>1}$  converges on  $\mathbb{C}$ .

Now we turn to  $Z^{<1}$ . Let  $\Omega$  be a fundamental domain of  $\mathbb{A}_F^{<1}/F^\times$ . Assume that  $s$  is big enough, then

$$\begin{aligned}
Z^{<1} &= \sum_{\alpha \in F^\times} \int_{\alpha\Omega} f(x) \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left( \sum_{\alpha \in F^\times} f(\alpha x) \right) \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left( \sum_{\alpha \in F} f(\alpha x) \right) \chi(x) |x|^s d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left( \sum_{\alpha \in F} \hat{f}(\alpha x^{-1}) \right) \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left( \sum_{\alpha \in F^\times} \hat{f}(\alpha x^{-1}) \right) \chi(x) |x|^{s-1} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega^{-1}} \left( \sum_{\alpha \in F^\times} \hat{f}(\alpha x) \right) \chi(x^{-1}) |x|^{1-s} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\mathbb{A}_F^{>1}} \hat{f}(x) \chi(x)^{-1} |x|^{1-s} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x.
\end{aligned}$$

We used “ $\chi(\alpha x) = \chi(x)$  for  $\alpha \in F^\times$ ”, Poisson summation, “ $d^\times x$  is invariant under  $x \mapsto x^{-1}$ ”, and “ $\Omega^{-1}$  is a fundamental domain of  $\mathbb{A}_F^{>1}/F^\times$ ” in the above calculation. The integral over  $\mathbb{A}_F^{>1}$  is again convergent on  $\mathbb{C}$ , so we look at the rest two integrals.

Write  $\Omega = \Omega^1 \times (0, 1)$ , where  $\Omega^1$  is a fundamental domain of  $\mathbb{A}_F^{\times,1}/F^\times$ . Then if  $\chi$  is non-trivial on  $\mathbb{A}_F^{\times,1}$ , both integrals vanish (as in Theorem 2). Otherwise  $\chi$  factors through  $\mathbb{A}_F^\times/\mathbb{A}_F^{\times,1} \simeq \mathbb{R}_{>0} \rightarrow \mathbb{C}^\times$ , hence  $\chi(x) = |x|^\lambda$  for some  $\lambda \in i\mathbb{R}$ , and

$$\hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x = \frac{\hat{f}(0) \text{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s + \lambda - 1} - \frac{f(0) \text{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s + \lambda}.$$

The theorem is easy to deduce from the expression. □

Our next target is  $Z_v(s, f, \chi_v) = \int_{F_v^\times} f_v(x) \chi_v(x) |x|^s d^\times x_v$ .

**Lemma 4.2.**  $Z_v(s, f, \chi_v)$  converges on  $\text{Re } s > 0$ .

*Proof.* Consider only the nonarchimedean case.

Take  $\epsilon$  small enough s.t.  $f_v(x) = f_v(0)$  for  $|x| < \epsilon$ . Write

$$Z_v(s, f_v, \chi_v) = \int_{|x| > \epsilon} + \int_{|x| < \epsilon}.$$

Similarly, the first integral converges on  $\mathbb{C}$ . For the second one,  $\{|x| < \epsilon\} = \bigcup_{n \geq N} \varpi^n \mathcal{O}_{F_v}^\times$  for an integer  $N$ . Thus we see that

$$\int_{|x| < \epsilon} |\chi_v(x) |x|^s| d^\times x = \sum_{n \geq N} \int_{\varpi^n \mathcal{O}_{F_v}^\times} |\varpi|^{-n \text{Re } s} d^\times x$$

converges when  $\text{Re } s > 0$ . □



**Theorem 7.** (1)  $Z_v(s, f, \chi_v)$  can be extended to a meromorphic function on  $\mathbb{C}$  which is holomorphic on  $\text{Re } s > 0$ .

(2) There exists a meromorphic function  $\gamma_v(s, \chi_v, \psi_v)$ , called **local  $\gamma$ -factor**, irrelevant to  $f_v$ , s.t. for any  $f_v \in \mathcal{S}(F_v)$ ,

$$Z_v(1-s, \hat{f}_v, \chi_v^{-1}) = \gamma_v(s, \chi_v, \psi_v) Z_v(s, f_v, \chi_v).$$

*Proof.* Firstly, both sides of the equation converge on  $0 < \text{Re } s < 1$ .

We need to show that  $\frac{Z_v(1-s, \hat{f}_v, \chi_v^{-1})}{Z_v(s, f_v, \chi_v)}$  is irrelevant to  $f_v$ ; i.e.,

$$Z_v(1-s, \hat{f}_v, \chi_v^{-1}) Z_v(s, g_v, \chi_v) = Z_v(1-s, \hat{g}_v, \chi_v^{-1}) Z_v(s, f_v, \chi_v), \quad \forall g_v \in \mathcal{S}(F_v).$$

Assume that  $d^\times x_v = |x|^{-1} dx$ , then the LHS

$$\begin{aligned} &= \int_{F_v^\times} \left( \int_F f_v(y) \psi_v(xy) dy \right) \chi_v(x)^{-1} |x|^{1-s} d^\times x \int_{F_v^\times} g_v(x) \chi_v(x) |x|^s d^\times x \\ &= \int_{F_v^\times} \int_{F_v^\times} \int_{F_v^\times} f_v(y) g_v(z) \psi_v(xy) \chi_v(zx^{-1}) |x|^{1-s} |z|^s d^\times x dy d^\times z \\ &= \iiint f_v(y) g_v(z) \psi_v(xy) \chi_v(zx^{-1}) |x|^{1-s} |z|^s \cdot |y| d^\times x d^\times y d^\times z \\ &= \iiint f_v(y) g_v(z) \psi_v(x) \chi_v(zyx^{-1}) |x|^{1-s} |zy|^s d^\times x d^\times y d^\times z \quad (x \mapsto y^{-1}x). \end{aligned}$$

Hence LHS = RHS.

So  $\gamma_v$  is well-defined on  $0 < \text{Re } s < 1$ . If  $\gamma_v$  can be a meromorphic function on  $\mathbb{C}$ , then the equation gives the analytic continuation of  $Z_v$  on  $\text{Re } s < 1$ . (The formula of  $\gamma$ -factor is only computed for archimedean place in this proof.)

(1)  $F_v = \mathbb{R}$ . Note that

$$Z_v(s, f_v, \chi_v | \cdot |^t) = Z_v(s+t, f_v, \chi_v),$$

so we only need to compute for  $\chi_v$  trivial or  $\chi_v = \text{sgn}$  character. The result is

$$\gamma_v(s, \chi_v, \psi_v) = \begin{cases} \frac{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}, & \chi = 1, \\ i \frac{\pi^{-\frac{(1-s)+1}{2}} \Gamma(\frac{(1-s)+1}{2})}{\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})}, & \chi = \text{sgn}. \end{cases}$$

For example, when  $\chi_v = 1$ , we take  $f_v(x) = e^{-\pi x^2}$ , then  $\hat{f}_v = f_v$ , and

$$\begin{aligned} Z_v(s, f_v, 1) &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{s-1} dx \\ &= 2 \int_0^{+\infty} e^{-\pi x^2} x^{s-1} dx \\ &= \pi^{-\frac{s}{2}} \int_0^{+\infty} y^{\frac{s}{2}-1} e^{-y} dy \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \end{aligned}$$

therefore

$$\gamma_v(s, \chi_v, \psi_v) = \frac{Z_v(1-s, f_v, 1)}{Z_v(s, f_v, 1)} = \frac{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}, \quad \chi = 1.$$

(2)  $F_v = \mathbb{C}$ . For  $\chi_v(x) = \left(\frac{x}{\sqrt{x\bar{x}}}\right)^n$ ,  $n \in \mathbb{Z}$ , using

$$f_v(x) = \begin{cases} \bar{x}^n e^{-2x\bar{x}}, & n \geq 0, \\ x^{-n} e^{-2x\bar{x}}, & n < 0, \end{cases}$$

we obtain

$$\gamma_v(s, \chi^v, \psi_v) = i^{|n|} \frac{(2\pi)^{-(1-s+\frac{|n|}{2})} \Gamma\left(1-s+\frac{|n|}{2}\right)}{(2\pi)^{-(-s+\frac{|n|}{2})} \Gamma\left(s+\frac{|n|}{2}\right)}.$$

(3)  $v$  is nonarchimedean. We show that  $\gamma_v$  is defined on  $\operatorname{Re} s > 1$ . Let  $U$  be a sufficiently small open compact nbhd of  $-1$  in  $F_v$  s.t.  $\chi_v$  is trivial on  $-U$ , and put  $f_v := 1_U$ . Then

$$Z_v(t, \hat{f}_v, \chi_v^{-1}) = \int_{F_v^\times} 1_U(-x) \chi(x^{-1}) |x|^t d^\times x = \operatorname{vol}(U) \neq 0$$

and is irrelevant to  $t$ . Therefore  $\gamma_v^{-1}$  can be defined on  $\operatorname{Re} s > 0$ . Similar for  $\operatorname{Re} s < 1$ .  $\square$

Finally, we obtain the analytic continuation of Hecke  $L$ -functions and the main theorem of functional equations.

**Theorem 8.** Let  $S$  be a finite set of places s.t.  $\forall v \notin S$ ,  $v$  is archimedean with  $\chi_v$  unramified, and  $\mathfrak{d}_v = \mathcal{O}_{F_v}$ . Then the partial Hecke  $L$ -function can be extended to a meromorphic function on  $\mathbb{C}$ , satisfying

$$L^S(s, \chi) = \left( \prod_{v \in S} \gamma_v(s, \chi_v, \psi_v) \right) L^S(1-s, \chi^{-1}).$$

Moreover, if there does not exist  $\lambda \in i\mathbb{R}$  s.t.  $\chi(x) = |x|^\lambda$ , then  $L^S(s, \chi)$  is entire; otherwise only  $s = 1 - \lambda$  and  $s = -\lambda$  have the possibility to be poles.

*Proof.* Take  $f = \prod_v f_v$  s.t.  $f_v = 1_{\mathcal{O}_{F_v}}$ ,  $\forall v \notin S$ . For  $v \notin S$ , the additional condition  $\mathfrak{d}_v = \mathcal{O}_{F_v}$  implies that (by Lemma 3.1)

$$\hat{f}_v(x) = (N\mathfrak{d})^{-\frac{1}{2}} 1_{\mathcal{O}}(x) = 1_{\mathcal{O}}(x),$$

and the functional equation follows.

It is left to show the property about poles. Suppose that  $\chi(x) = |x|^\lambda$  with  $\lambda \in i\mathbb{R}$  and  $s = s_0$  is a pole of  $L^S$  other than  $-\lambda$  or  $1 - \lambda$ . Consider the equation

$$Z(s, f, \chi) = L^S(s, \chi) \prod_{v \in S} Z_v(s, f_v, \chi_v).$$

By Theorem 6, LHS is holomorphic at  $s = s_0$ .

We choose an  $f$  s.t. for all  $v \in S$ ,  $f_v$  supports in a sufficiently small nbhd  $U_v$  of  $1 \in F_v$ . With a similar argument in the previous proof, one sees that  $Z_v(s_0, f_v, \chi_v) \neq 0$ . Therefore the RHS has a pole at  $s = s_0$ , which is a contradiction.  $\square$

## 4.1 Exercise

Let  $F = \mathbb{Q}$ ,  $\chi = 1$  the trivial character. Repeat the calculation before to prove the analytic continuation and functional equation of Riemann zeta function, and compute its residue at  $s = 1$ .

*Proof.* The Riemann zeta function is

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Let  $S = \{\infty\}$ . The local unramified  $L$ -functions are

$$L(s, 1_p) = (1 - p^{-s})^{-1},$$

so  $\zeta(s) = L^S(s, \chi)$ .

Let  $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$  be defined by  $f_p = 1_{\mathbb{Z}_p}$  and  $f_{\infty}(x) = e^{-\pi x^2}$ . The zeta integral is

$$Z(s, f, 1) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x) |x|^s d^{\times} x$$

and the local zeta integral at infinity is

$$Z_{\infty}(s, f_{\infty}, 1) = \int_{\mathbb{R}^{\times}} e^{-\pi x^2} |x|^s d^{\times} x.$$

We have

$$Z(s, f, 1) = \zeta(s) Z_{\infty}(s, f_{\infty}, 1).$$

(TBC.....)

□

## 5 Dedekind Zeta Functions and Dirichlet $L$ -functions

### 5.1 Dedekind Zeta Functions and the Analytic Class Number Formula

Let  $F$  be a number field,  $\chi$  the trivial character,  $S$  the set of all archimedean places. The **Dedekind zeta function** of  $F$  is defined to be

$$\zeta_F(s) := L^S(s, \chi) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}.$$

In this subsection, we will compute the local  $\gamma$ -factors at ramified places to deduce the functional equation of Dedekind zeta functions and the analytic class number formula.

**Theorem 9.** The Dedekind zeta function  $\zeta_F(s)$  can be extended to a meromorphic function on  $\mathbb{C}$  with only poles at  $s = 0$  and  $s = 1$ .

1. (Analytic class number formula.)  $\zeta_F(s)$  has a simple pole at  $s = 1$  with residue

$$\text{res}_1 \zeta_F = \frac{2^{r_1} (2\pi)^{r_2} h_F R_F}{\sqrt{|\text{disc } F| w_F}},$$

and is of order  $r_1 + r_2 - 1$  at  $s = 0$  with

$$\lim_{s \rightarrow 0} s^{r_1 + r_2 - 1} \zeta_F(s) = -\frac{h_F R_F}{w_F}$$

2. Define the completed Dedekind zeta function

$$\Lambda(s) := |\text{disc } F|^{\frac{s}{2}} \left( \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right)^{r_1} (2(2\pi)^{-s} \Gamma(s))^{r_2} \zeta_F(s).$$

Then

$$\Lambda_F(s) = \Lambda_F(1 - s).$$

## 5.2 Dirichlet $L$ -functions

Let  $F = \mathbb{Q}$ ,  $\chi$  a Dirichlet character with conductor  $N$ ,  $S = \{p : p|N\} \cup \{\infty\}$ . Lifting  $\chi$  to a Hecke character  $\tilde{\chi}$ , we get an partial  $L$ -function

$$L^S(s, \tilde{\chi}) = \prod_{p \nmid N} (1 - \chi_p(p) N p^{-s})^{-1},$$

which is exactly the classic Dirichlet  $L$ -function

$$L(s, \chi) = \prod_{p \nmid N} (1 - \chi(p) p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{Re } s > 1.$$

The functional equation

$$L(s, \chi) = \left( \prod_{p|N} \gamma_p(s, \chi_p, \psi_p) \right) \gamma_{\infty}(s, \chi_{\infty}, \psi_{\infty}) L(1-s, \chi^{-1})$$

has been proved, and  $\gamma_{\infty}$  have been computed. It is left to compute the  $\gamma_p$ 's for  $p \mid N$ .

Suppose  $p^e \parallel N$ , then the conductor of  $\chi_p$  is  $p^e$  (or  $1 + p^e \mathbb{Z}_p$ ). Take  $f_p := 1_{1+p^e \mathbb{Z}_p}$ , so

$$Z_p(s, f_p, \chi_p) = \int_{1+p^e \mathbb{Z}_p} |x|^s d^{\times} x = \text{vol}(1 + p^e \mathbb{Z}_p)$$

is easy to compute. Then

$$\hat{f}_p(x) = p^{-e} \psi_p(x) 1_{p^{-e} \mathbb{Z}_p}(x),$$

hence

$$\begin{aligned} \gamma_p(s, \chi_p, \psi_p) &= p^{-e} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \int_{p^{-e} \mathbb{Z}_p} \psi_p(x) \chi_p(x) |x|^{1-s} d^{\times} x \\ &= p^{-e} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \sum_{n \geq -e} p^{-n(1-s)} \int_{p^n \mathbb{Z}_p^{\times}} \psi_p(x) \chi_p(x) d^{\times} x. \end{aligned}$$

Next, we show that  $\int_{p^n \mathbb{Z}_p^{\times}} = 0$  whenever  $n \neq -e$ .

(1)  $n \geq 0$ . Then  $\psi_p$  is trivial on  $p^n \mathbb{Z}_p^{\times} \subset \mathbb{Z}_p$ , so the integral equals

$$\int_{p^n \mathbb{Z}_p^{\times}} \chi_p(x) d^{\times} x = \chi_p(p^n) \int_{\mathbb{Z}_p^{\times}} \chi_p(x) d^{\times} x = 0$$

as  $\chi_p$  is nontrivial on  $\mathbb{Z}_p^{\times}$ .

(2)  $-e < n < 0$ . We can decompose  $p^n \mathbb{Z}_p^{\times}$  into  $p^{-n-1}(p-1)$  copies of  $\mathbb{Z}_p$ , i.e.,

$$p^n \mathbb{Z}_p^{\times} = \bigsqcup_{\alpha} (\alpha + \mathbb{Z}_p), \quad \alpha \in p^n \mathbb{Z}_p^{\times},$$

then

$$\int_{p^n \mathbb{Z}_p^{\times}} \psi_p(x) \chi_p(x) d^{\times} x = \sum_{\alpha} \psi_p(\alpha) \int_{\alpha + \mathbb{Z}_p} \chi_p(x) d^{\times} x = \sum_{\alpha} \psi_p(\alpha) \chi_p(\alpha) \int_{1 + \alpha^{-1} \mathbb{Z}_p} \chi(x) d^{\times} x.$$

Since  $-n < e$ ,  $1 + \alpha^{-1} \mathbb{Z}_p \supset 1 + p^e \mathbb{Z}_p$ , so the integrals vanish agains.

So we get

$$\begin{aligned}
\gamma_p(s, \chi_p, \psi_p) &= p^{-es} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \int_{p^{-e} \mathbb{Z}_p^\times} \psi_p(x) \chi_p(x) d^\times x \\
&= p^{-es} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \sum_{\alpha \in \mathbb{Z}_p^\times / (1 + p^e \mathbb{Z}_p)} \psi_p(p^{-e} \alpha) \chi_p(p^{-e} \alpha) \int_{\alpha(1 + p^e \mathbb{Z}_p)} d^\times x \\
&= p^{-es} \sum_{\alpha \in (\mathbb{Z}/p^e \mathbb{Z})^\times} \psi_p(p^{-e} \alpha) \chi_p(p^{-e} \alpha).
\end{aligned}$$

(T.B.C.)

### 5.3 Quadratic Fields

Let  $F = \mathbb{Q}(\sqrt{d})$  and  $D = |\text{disc } F|$ . Define  $\chi_d : (\mathbb{Z}/D\mathbb{Z})^\times \rightarrow \{\pm 1\}$  by

$$\chi_d(p) := \begin{cases} 1, & p \text{ splits in } F, \\ -1, & p \text{ is inert in } F. \end{cases}$$

*Remark.* Every primitive quadratic Dirichlet character is of the form above, which is just the Legendre symbol  $(\frac{\cdot}{D})$ .

**Lemma 5.1.**  $\zeta_F(s) = \zeta(s) L(s, \chi_d)$ .

*Proof.* Check the equation

$$\prod_{\mathfrak{p}|p} \zeta_{\mathfrak{p}}(s) = \zeta_p(s) L_p(s, \chi_{d,p}).$$

□

**Proposition 5.1** (Dirichlet). If  $d < 0$ , then

$$L(1, \chi_d) = \frac{2\pi h_F}{\sqrt{D} w_F}.$$

If  $d > 0$ , then

$$L(1, \chi_d) = \frac{h_F \log \epsilon_F}{\sqrt{D}},$$

where  $\epsilon_F > 0$  is a fundamental unit of  $F$ .

This proposition explains why the class number of real quadratic fields are harder to study than imaginary ones, because there are powerful analytic methods to study  $L(1, \chi_d)$ , but we don't have a general method to separate  $h_F$  and  $\log \epsilon_F$  (when  $d > 0$ ).

#### The Gauss Class Number Problems

From now on, assume that  $d < 0$ . Gauss had conjectured that (1)  $h_F \rightarrow \infty$  as  $d \rightarrow -\infty$ , and (2) give a list of imaginary quadratic fields  $F$  with  $h_F = 1, 2, 3$ , suspecting that these are all such fields. In particular, only 9 imaginary quadratic fields have class number 1:

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

Problem (1) was solved by C.Siegel in 1935.

**Theorem 10** (Siegel). For all  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon) > 0$  s.t.

$$L(1, \chi) \geq \frac{C(\varepsilon)}{N^\varepsilon}$$

for any primitive character  $\chi : (\mathbb{Z}/N\mathbb{Z}) \rightarrow \{\pm 1\}$ . In particular, there exists a constant  $C'(\varepsilon)$  s.t.

$$h_F \geq C'(\varepsilon) D^{\frac{1}{2} - \varepsilon}.$$

This implies that there are only finite many imaginary quadratic fields  $F$  with  $h_F = A$  for any given constant  $A$ .

In Siegel's theorem, the constant  $C(\varepsilon)$  is not an *effective constant*, meaning that there is no explicit formula for  $C(\varepsilon)$  using  $\varepsilon$  and  $N$ . Hence the problem (2) wasn't solved until 1983.