# Note on Modular Forms

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November 3, 2024

# 1 Hecke Operators

Thoughout this section, we fix the following data:

- a group $\Omega$;
- a submonoid $\Delta \subset \Omega$;
- a nonempty collection $\mathscr{X}$ of subgroups of $\Omega$, in which all members are commensurable[1] to each other, and
$$\Gamma \subset \Delta \subset \tilde{\Gamma} := \left\{ g \in \Omega \mid g\Gamma g^{-1} \approx \Gamma \right\}, \ \forall \Gamma \in \mathscr{X};$$
- a commutative ring $\mathbb{K}$'
- a left $\mathbb{K}$-module $M$ with a right $\Delta$-action $(m, \delta) \mapsto m\delta$, i.e, a monoid homomorphism
$$\Delta \to \mathrm{End}_{\mathbb{K}}(M) \quad \delta \mapsto m \mapsto m\delta.$$

## 1.1 Commensurability

Recall that two subgroups $\Gamma, \Gamma' < \Omega$ are commensurable if both $[\Gamma : \Gamma \cap \Gamma']$ and $[\Gamma' : \Gamma \cap \Gamma']$ are finite, and this is an equivalence relation.

**Lemma 1.1.** $\tilde{\Gamma}$ is a group and depends only on the commensurable class of $\Gamma$. $\square$

**Proposition 1.1.** Let $\alpha \in \tilde{\Gamma}$ and $\Gamma \approx \Gamma'$. Then there is a bijection

$$\begin{aligned} \Gamma' \cap (\alpha^{-1}\Gamma\alpha) \backslash \Gamma' &\longleftrightarrow \Gamma \backslash \Gamma\alpha\Gamma' \\ \Gamma''{}^{2} x &\longleftrightarrow \Gamma\alpha x \end{aligned}$$

and $\Gamma \backslash \Gamma\alpha\Gamma'$ is finite.

*Proof.* The map
$$\Gamma' \to \Gamma \backslash \Gamma\alpha\Gamma' \quad x \mapsto \Gamma\alpha x$$
is clearly surjective. Now $\forall x, y \in \Gamma'$,

$$\begin{aligned} \Gamma\alpha x = \Gamma\alpha y &\iff \exists g \in \Gamma, g\alpha x = \alpha y \\ &\iff \exists g' \in \Gamma'', g'x = y; \end{aligned}$$

so injective.

By definitions and the last lemma, $\Gamma' \cap (\alpha^{-1}\Gamma\alpha) \approx \Gamma'$, giving finiteness. $\square$

[1] Write $\Gamma \approx \Gamma'$ if $\Gamma$ is commensurable to $\Gamma'$.

[2] Of course, $\Gamma'' = \Gamma' \cap (\alpha^{-1}\Gamma\alpha)$.

## 1.2 Double Coset Algebra

### 1.2.1 Double Cosets and Convolution

Recall that the $\mathbb{K}$-module $\mathcal{F}(\Omega, \mathbb{K})$ of all functions $\Omega \to \mathbb{K}$ admits a $\mathbb{K}$-linear left $\Omega$-action

$$(\gamma f)(z) := f(\gamma^{-1} z)$$

and a right $\Omega$-action

$$(f\gamma)(z) := f(z\gamma).$$

**Def-Theorem 1.** Let $\Gamma, \Gamma' \in \mathscr{X}$. Define $\mathcal{H}(\Gamma\backslash\Delta/\Gamma')$ to be the $\mathbb{K}$-module[3] consists of functions $f : \Omega \to \mathbb{K}$ such that:

- $\operatorname{supp} f \subset \Delta$ and $\Gamma\backslash \operatorname{supp} f/\Gamma'$ is a finite set,
- $f$ is left-$\Gamma$-invariant and right-$\Gamma'$-invariant.

Then $\mathcal{H}(\Gamma\backslash\Delta/\Gamma')$ is a *free* $\mathbb{K}$-module, with a basis given by the double cosets in $\Gamma\backslash\Delta/\Gamma'$, i.e.,

$$[\Gamma\gamma\Gamma'] := \mathbf{1}_{\Gamma\gamma\Gamma'}, \ \gamma \in \Delta.$$

We thus identify $\mathcal{H}(\Gamma\backslash\Delta/\Gamma')$ with the free module $\mathbb{Z}[\Gamma\backslash\Delta/\Gamma']$ generated by $\Gamma\backslash\Delta/\Gamma'$, and we identify the function $[\Gamma\gamma\Gamma'] := \mathbf{1}_{\Gamma\gamma\Gamma'}$ with the double coset $\Gamma\gamma\Gamma'$.

**Def-Theorem 2** (Convolution)**.** Let $\Gamma, \Gamma', \Gamma'' \in \mathscr{X}$. We define an convolution operator

$$* : \mathcal{H}(\Gamma\backslash\Delta/\Gamma') \times \mathcal{H}(\Gamma'\backslash\Delta/\Gamma'') \to \mathcal{H}(\Gamma\backslash\Delta/\Gamma'')$$

via

$$(\alpha * \beta)(x) := \sum_{h \in \Gamma'\backslash\Omega} \alpha(xh^{-1})\beta(h) = \sum_{\Omega/\Gamma'} \alpha(h)\beta(h^{-1}x).$$

The above equation is well-defined and holds. Moreover,

- this convolution operator $*$ is *distributive* and *associative*,
- $1_\Gamma \in \mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ is both a left and right *identity* for $*$.

In particular, the operator $*$ makes

$$\mathcal{H}_\Delta(\Gamma) := \mathcal{H}(\Gamma\backslash\Delta/\Gamma)$$

a $\mathbb{K}$-algebra.

We then give a formula of $*$. For $\alpha, \beta, \gamma \in \Delta$, write

$$[\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma''] = \sum_{\gamma \in \Delta\backslash\Gamma/\Gamma''} m(\alpha, \beta; \gamma)[\Gamma\gamma\Gamma''].$$

Apply RHS to $\gamma$, one checks $([\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma''])(\gamma) = m(\alpha, \beta; \gamma)$. To determine these quantities, write

$$\Gamma\alpha\Gamma' = \bigsqcup_{a \in A} \Gamma a, \ \Gamma'\beta\Gamma'' = \bigsqcup_{b \in B} \Gamma' b.$$

Then

$$\begin{aligned} m(\alpha, \beta; \gamma) &= ([\Gamma\alpha\Gamma'] * [\Gamma'\beta\Gamma''])(\gamma) \\ &= \sum_{h \in \Gamma'\backslash\Omega} [\Gamma\alpha\Gamma'](\gamma h^{-1}) \cdot [\Gamma'\beta\Gamma''](h) \\ &= \sum_{h \in \Gamma'\backslash(\Gamma'\beta\Gamma'')} [\Gamma\alpha\Gamma'](\gamma h^{-1}) = \sum_{b \in B} [\Gamma\alpha\Gamma'](\gamma b^{-1}). \end{aligned}$$

[3] A $\mathbb{K}$-submodule of $\mathcal{F}(\Omega, \mathbb{K})$

Note that

$$[\Gamma\alpha\Gamma'](x) = \begin{Bmatrix} 1, & \exists a \in A, x \in \Gamma a \\ 0, & \text{otherwise} \end{Bmatrix} = \#\{a \in A \mid \Gamma x = \Gamma a\},$$

so

$$m(\alpha, \beta; \gamma) = \#\left\{(a, b) \in A \times B \mid \Gamma\gamma = \Gamma ab\right\}.$$

Considering right cosets rather than left cosets gives a similar formula.

The following is a useful result in computation.

**Proposition 1.2.** If $\alpha, \gamma \in \Delta$, and $\gamma$ normalises $\Gamma$, then

$$[\Gamma\alpha\Gamma] * [\Gamma\gamma\Gamma] = [\Gamma\alpha\gamma\Gamma],$$

$$[\Gamma\gamma\Gamma] * [\Gamma\alpha\Gamma] = [\Gamma\gamma\alpha\Gamma].$$

*Proof.* Write $\Gamma\alpha\Gamma = \bigsqcup_{a \in A} \Gamma a$. As $\Gamma\gamma\Gamma = \Gamma\gamma$ and

$$\Gamma\alpha\gamma\Gamma = \Gamma\alpha\Gamma\gamma = \bigsqcup_{a \in A} \Gamma a\gamma,$$

the structure constants

$$m(\alpha, \gamma; \delta) = \#\left\{a \in A \mid \Gamma\delta = \Gamma a\gamma\right\} = \begin{cases} 1, & \delta \in \Gamma\alpha\gamma\Gamma, \\ 0, & \delta \notin \Gamma\alpha\gamma\Gamma. \end{cases}$$ □

### 1.2.2 Commutativity

An **anti involution** of a monoid $\Delta$ is a map $\tau : \Delta \to \Delta$ s.t.

$$\tau(xy) = \tau(y)\tau(x), \quad \tau(1) = 1, \quad \tau^2 := \tau \circ \tau = \mathrm{id}.$$

**Theorem 3.** Let $\Gamma \in \mathscr{X}$. If there *exists* an anti involution $\tau : \Delta \to \Delta$ that stabilises every double coset of $\Gamma$, then $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ is a commutative $\mathbb{K}$-algebra.

## 1.3 The Action of Double Coset Algebras

We consider the action of $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$ on

$$M^\Gamma = \{m \in M \mid m\gamma = m, \forall \gamma \in \Gamma\}.$$

**Def-Theorem 4.** For $f \in \mathcal{H}(\Gamma\backslash\Delta/\Gamma')$, define

$$\begin{aligned} \cdot f : M^\Gamma &\longrightarrow M^{\Gamma'} \\ m &\longmapsto mf := \sum_{\delta \in \Gamma\backslash\Delta} f(\delta) m\delta. \end{aligned}$$

This action is well-defined. Moreover, it is comptatible with convolution.

- If $f \in \mathcal{H}(\Gamma\backslash\Delta/\Gamma')$, $f' \in \mathcal{H}(\Gamma'\backslash\Delta/\Gamma'')$, then $m(f * f') = (mf)f'$.
- In case $\Gamma' = \Gamma$, $m\mathbf{1}_\Gamma = m$.

In particular, $M^\Gamma$ admits a right $\mathcal{H}(\Gamma\backslash\Delta/\Gamma)$-module, with action of a basis given by

$$\Gamma\gamma\Gamma = \bigsqcup_{i=1}^{n} \Gamma\gamma_i \implies m[\Gamma\gamma\Gamma] = \sum_{i=1}^{n} \gamma_i.$$

**Corollary 1.1.** If $\gamma$ normalises $\Gamma$, then $m[\Gamma\gamma\Gamma] = m\gamma$. □

# 2 Hecke Operators for $\Gamma_0(N)$ and $\Gamma_1(N)$

We specialise our discussion in the last section to the case of modular forms. Let

- $\Omega := \mathrm{GL}(2,\mathbb{Q})^+$,
- $\mathbb{K} := \mathbb{C}$,
- $\mathscr{X}$ = congruence subgroups,

**Lemma 2.1.** Any two congruence subgroups are commensurable.

*Proof.* Note that $\Gamma(N) \cap \Gamma(N') = \Gamma(\mathrm{lcm}(N,N'))$. □

**Lemma 2.2.** If $\Gamma$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then in $\mathrm{GL}(2,\mathbb{Q})^+$, the group $\tilde{\Gamma} = \mathrm{GL}(2,\mathbb{Q})^+$.

*Proof.* □

Fix a weight $k$ and consider all the modular forms

$$M := \bigcup_{\Gamma \in \mathscr{X}} M_k(\Gamma) = \sum_{\Gamma} M_k(\Gamma)$$

and its $\mathbb{C}$-subspace

$$S := \bigcup_{\Gamma \in \mathscr{X}} S_k(\Gamma) = \sum_{\Gamma} S_k(\Gamma).$$

- Note that we can write $\bigcup = \sum$, because
$$M_k(\Gamma) + M_k(\Gamma') \subset M_k(\Gamma \cap \Gamma').$$

Define $M \circlearrowleft \mathrm{GL}(2,\mathbb{R})^+$ by

$$f|_k\gamma(z) := (\det\gamma)^{k-1} j(\gamma,z)^{-k} f(\gamma z).$$

**Lemma 2.3.** For all $\Gamma \in \mathscr{X}$ and $\gamma \in \mathrm{GL}(2,\mathbb{R})^+$,

$$f \in M_k(\Gamma) \implies f|_k\gamma \in M_k(\Gamma \cap \gamma^{-1}\Gamma\gamma).$$

It remains true for $S_k$.

*Proof.* Just don't forget to check the cusps! □

It is now straightforward to check that we defined an action on $M$ which stabilises $S$.

**Lemma 2.4.** $M^\Gamma = M_k(\Gamma)$, $S^\Gamma = S_k(\Gamma)$.

Now we go to the case of $\Gamma_0(N)$ and $\Gamma_1(N)$.

## 2.1 The Algebras

We consider these monoids:

$$\begin{aligned}
\Delta(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \det A > 0,\ (a,N) = 1,\ N \mid c \right\} \\
&= \left\{ A \in \mathrm{GL}(2,\mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} (\mathbb{Z}/N\mathbb{Z})^\times & * \\ & * \end{pmatrix} \right\}, \\
\Delta^\circ(N) &:= \{A \in \Delta(N) \mid (\det A, N) = 1\}, \\
\Delta_1(N) &:= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta^1(N) \middle| a \equiv 1 \pmod N \right\} \\
&= \left\{ A \in \mathrm{GL}(2,\mathbb{Q})^+ \cap \mathrm{M}_2(\mathbb{Z}) \middle| A \bmod N \in \begin{pmatrix} 1 & * \\ & * \end{pmatrix} \right\}.
\end{aligned}$$

Define

$$\mathcal{H}_i(N) := \mathcal{H}_{\Delta(N)}(\Gamma_i(N)), \quad \mathcal{H}_i^\circ(N) := \mathcal{H}_{\Delta^\circ(N)}(\Gamma_i(N)), \qquad i = 0, 1$$

and $\mathcal{H}_1(N) := \mathcal{H}_{\Delta_1(N)}(\Gamma_1(N))$.

**Proposition 2.1.** All the algebras mentioned above are commutative.

*Proof.* Check that

$$A = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ bN & d \end{pmatrix} = \left( \begin{pmatrix} 1 & \\ & N \end{pmatrix}^{-1} A \begin{pmatrix} 1 & \\ & N \end{pmatrix} \right)^{\mathrm{t}}$$

verifies the conditions of Theorem 3. $\square$

We are particularly interested in $\mathcal{H}_0(N)$ and $\mathcal{H}_1(N)$.

## 2.2 Product Formula for $\mathcal{H}_0(N)$

**Theorem 5** (coset representative of $\mathcal{H}_0(N)$)**.** $\Gamma_0(N)\backslash\Delta(N)/\Gamma_0(N)$ admits coset representative given by

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad u \mid v, \ (u, N) = 1.$$

The double coset of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to

$$\begin{pmatrix} u & \\ & v \end{pmatrix}, \quad \text{where } \begin{cases} uv = ad - bc \\ u = (a, b, c, d). \end{cases}$$

**Proposition 2.2.**

$$\Gamma_0(N) \begin{pmatrix} u & \\ & v \end{pmatrix} \Gamma_0(N) = \bigsqcup_{g \in M_{u,uv}} \Gamma_0(N) g,$$

where

$$M_{u,n} = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \;\middle|\; \begin{array}{c} u = (a, b, d) \\ n = ad \\ (a, N) = 1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right\}.$$

In particular,

$$\Gamma_0(N) \begin{pmatrix} 1 & \\ & n \end{pmatrix} \Gamma_0(N) = \bigsqcup_{g \in M_{1,n}} \Gamma_0(N) g$$

and

$$M_{1,n} = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \;\middle|\; \begin{array}{c} (a, b, d) = 1 \\ ad = n \\ (a, N) = 1 \\ b \text{ permutes a representative of } \mathbb{Z}/d\mathbb{Z} \end{array} \right\}.$$

**Proposition 2.3** (multiplication formula)**.** Write $[A] := [\Gamma_0(N) A \Gamma_0(N)]$. Let $n, m \in \mathbb{Z}$, $p$ be a prime.

- If $(n, m) = 1$, then
$$\left[\begin{pmatrix} 1 & \\ & n \end{pmatrix}\right] \left[\begin{pmatrix} 1 & \\ & m \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \\ & nm \end{pmatrix}\right].$$
- If $p \mid N$, then
$$\left[\begin{pmatrix} 1 & \\ & p \end{pmatrix}\right] \left[\begin{pmatrix} 1 & \\ & p^r \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \\ & p^{r+1} \end{pmatrix}\right].$$

- If $p \nmid N$, then

$$\left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right]\left[\begin{pmatrix}1 & \\ & p^r\end{pmatrix}\right] = \begin{cases}\left[\begin{pmatrix}1 & \\ & p^2\end{pmatrix}\right] + (p+1)\left[\begin{pmatrix}p & \\ & p\end{pmatrix}\right], & r = 1, \\ \left[\begin{pmatrix}1 & \\ & p^{r+1}\end{pmatrix}\right] + p\left[\begin{pmatrix}p & \\ & p\end{pmatrix}\begin{pmatrix}1 & \\ & p^{r-1}\end{pmatrix}\right], & r \geq 2.\end{cases}$$

*Proof.* Note that $\operatorname{diag}(u, u)$ lies in the centre of $\mathrm{GL}(2, \mathbb{Q})^+$, so

$$\left[\begin{pmatrix}u & \\ & v\end{pmatrix}\right] = \left[\begin{pmatrix}u & \\ & u\end{pmatrix}\begin{pmatrix}1 & \\ & v/u\end{pmatrix}\right],$$

and thus we need only to prove the formula for $\operatorname{diag}(1, n)$.

To be continued.... □

## 2.3 From $\Gamma_0$ to $\Gamma_1$

Recall that

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \qquad \begin{pmatrix}* & * \\ & d\end{pmatrix} \mapsto \bar{d}$$

induces a group isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

**Definition 1** (diamond operator)**.** For $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define

$$\langle d \rangle := [\Gamma_1(N)\gamma_d\Gamma_1(N)],$$

where $\gamma_d \in \Gamma_0(N)$ is any lift of $d$.

- The operator $\langle d \rangle$ is independent to the choice of $\gamma_d$, because the $\gamma_d$'s differ by an element in $\Gamma_1(N)$.
- $\langle d \rangle \langle d' \rangle = \langle dd' \rangle$.

## 2.4 Another Basis

**Definition 2** (operator $T(n)$)**.** Let $n \in \mathbb{Z}_{\geq 1}$ and consider

$$\Delta^n(N) := \{A \in \Delta(N) \mid \det A = n\}.$$

Write $\Gamma_0(N)\backslash\Delta^n(N)/\Gamma_0(N) = \bigsqcup_i \Gamma_0(N)g_i\Gamma_0(N)$, we define

$$T(n) := \sum_i [\Gamma_0(N)g_i\Gamma_0(N)].$$

By Theorem 5, we may take

$$\begin{pmatrix}u & \\ & n/u\end{pmatrix} \text{ with } \begin{cases}(u, N) = 1, \\ u^2 \mid n,\end{cases}$$

yielding

$$\begin{aligned}T(n) &= \sum_u \left[\begin{pmatrix}u & \\ & n/u\end{pmatrix}\right] \\ &= \sum_u \left[\begin{pmatrix}u & \\ & u\end{pmatrix}\begin{pmatrix}1 & \\ & n/u^2\end{pmatrix}\right].\end{aligned}$$

as the representative $g_i$'s, which in turn shows that $\Gamma_0(N)\backslash\Delta^n(N)/\Gamma_0(N)$ is a finite set and $T(n)$ is well-defined. In particular, for $p$ prime,

$$T(p) = \left[\begin{pmatrix}1 & \\ & p\end{pmatrix}\right].$$

### 2.5 Hecke Algebra

Define

$$\mathbb{T}_i(N) := \mathrm{im}\,(\mathcal{H}_i(N) \to \mathrm{End}_{\mathbb{C}}(M_k(N))) \tag{1}$$
$$T_n := \text{image of } T(n) \in \mathcal{H}_i(N) \tag{2}$$

for $i = 0, 1$.