

Notes on Algebraic Number Theory

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Some Notations

Let F be a number field, then we denote by r_1 the number of real embeddings, r_2 the number of the pairs of complex embeddings, $\text{Cl}(F)$ the class group, h_F the class number, R_F the regulator, w_F the number of roots of unity in F , $\mathfrak{d} = \mathfrak{d}_F$ the different ideal.

Always denote $\sqrt{-1} \in \mathbb{C}$ by i .

1 Adeles and Ideles

Note that the topology on \mathbb{A}_F^\times (defined using natural nbhd of 1 in \mathbb{Q}_p^\times) is different from (more precisely, finer than) that on \mathbb{A}_F (defined using natural nbhd of 0 in \mathbb{Q}_p), but the topology on $\mathbb{A}_F^{\times,1}$ induced from \mathbb{A}_F and that from \mathbb{A}_F^\times coincide.

Theorem 1. The quotient $\mathbb{A}_F^{\times,1}/F^\times$ is compact.

Proof. Let I_F be the group of fractional ideals. Observe that we have an epimorphism

$$\mathbb{A}_F^{\times,1} \twoheadrightarrow I_F, (x_v) \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})},$$

under which $x \in F^\times$ is sent to the principle fractional ideal $(x) \in I_F$, and thus gives an epimorphism $\mathbb{A}_F^{\times,1}/F^\times \twoheadrightarrow \text{Cl}(F)$. As $\text{Cl}(F)$ is finite, it reduces to show that the kernel of this homomorphism is compact.

An element $(x_v) \in \ker$ iff it is mapped to a principle ideal, i.e., $\exists x \in F^\times$ s.t. $\forall \mathfrak{p}, x_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} = x \mathcal{O}_{\mathfrak{p}}$, or say $x_{\mathfrak{p}} \in x^{-1} \mathcal{O}_{\mathfrak{p}}^\times$. Therefore the kernel is the image of

$$\left(\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \prod_{v|\infty} F_v^\times \right) \cap \mathbb{A}_F^{\times,1} = \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \left(\prod_{v|\infty} F_v^\times \right)^1$$

in $\mathbb{A}_F^{\times,1}/F^\times$, where $\left(\prod_{v|\infty} F_v^\times \right)^1$ denotes the set of element with norm 1. Because two elements in this set cannot differ by an element in $F^\times \setminus \mathcal{O}_F^\times$, we see that

$$\ker = \left(\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \left(\prod_{v|\infty} F_v^\times \right)^1 \right) / \mathcal{O}_F^\times.$$

Now it suffices to prove that $\left(\prod_{v|\infty} F_v^\times\right)^1 / \mathcal{O}_F^\times$ is compact. Let v_1, \dots, v_r be the places of real embeddings and $v_{r_1+1}, \dots, v_{r_1+r_2}$ the places of complex ones. The logarithm map

$$\left(\prod_{v|\infty} F_v^\times\right)^1 \rightarrow \mathbb{R}^{r_1+r_2}, \quad x \mapsto (\log |x_{v_1}|, \dots, \log |x_{v_{r_1}}|, \log |x_{v_{r_1+1}}|_{\mathbb{C}}, \dots, \log |x_{v_{r_1+r_2}}|_{\mathbb{C}})$$

is a homomorphism with kernel $T = \{\pm 1\}^{r_1} \times (S^1)^{r_2}$, which is compact and the intersection $T \cap \mathcal{O}_F^\times = W_F$, the roots of unity in F . So $T/T \cap \mathcal{O}_F^\times$ is compact. Its image is the hypersurface

$$\Sigma : x_1 + \dots + x_{r_1+r_2} = 1$$

in $\mathbb{R}^{r_1+r_2}$. Dirichlet units theorem says that the image of \mathcal{O}_F^\times in Σ is a lattice of full rank, so the quotient $\Sigma/\mathcal{O}_F^\times$ is also compact. Our goal follows. \square

Remark. This theorem is equivalent to the combination of the finiteness of class group and Dirichlet units theorem.

2 L -functions

2.1 Riemann Zeta Function

Recall that the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

converges on $\operatorname{Re} s > 1$ and can be extended to a meromorphic function on \mathbb{C} with $s = 1$ the only simple pole. The functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$

is satisfied.

2.2 Charaters

A **character** of a group G is a continuous homomorphism $G \rightarrow \mathbb{C}^\times$, and **trivial character** is the character $G \rightarrow \{1\}$. The charaters of a group G forms a group, denoted by \hat{G} .

Lemma 2.1. Let G be a finite abelian group.

1. There exists an non-canonical isomorphism $G \simeq \hat{G}$.
2. If χ is a non-trivial character, then

$$\sum_{g \in G} \chi(g) = 0.$$

Conversely, if $g \neq 1$, then

$$\sum_{\chi \in \hat{G}} \chi(g) = 0.$$

\square

Let F be a number field. A **Hecke character** of F is a character of $\mathbb{A}_F^\times / F^\times$.

Proposition 2.1. Let χ be a character on \mathbb{A}_F^\times . Then χ is of the form $\prod_v \chi_v$, where $\chi_v \in \hat{F}_v^\times$ and χ_v 's are **unramified** (i.e., trivial on $\mathcal{O}_{F_v}^\times$) for almost all nonarchimedean places.

So we can go back to characters on local fields. Let F be a local field and χ a character of F^\times . The character χ is called **unitary**, if $|\chi(F^\times)| = \{1\}$. We can describe χ explicitly.

◊ If $F = \mathbb{R}$, then

$$\chi(x) = \left(\frac{x}{|x|} \right)^\epsilon |x|^s, \quad \epsilon = 0, 1, \quad s \in \mathbb{C}.$$

It is unitary iff $s \in i\mathbb{R}$.

◊ If $F = \mathbb{C}$, then

$$\chi(x) = \left(\frac{x}{\sqrt{x\bar{x}}} \right)^m (x\bar{x})^s, \quad m \in \mathbb{Z}, \quad s \in \mathbb{C}.$$

It is unitary iff $s \in i\mathbb{R}$.

◊ If F is nonarchimedean and ramified, then there exists a minimal integer N s.t. $\chi(1 + \varpi^N \mathcal{O}_F^\times) = \{1\}$, whence χ factors through the finite group $\mathcal{O}_F^\times / (1 + \varpi^N \mathcal{O}_F^\times)$, and thus

$$\chi(x) = |x|^s \chi_0(x),$$

where χ_0 is a character of $\mathcal{O}_F^\times / (1 + \varpi^N \mathcal{O}_F^\times)$. It is unitary if $s \in i\mathbb{R}$. This integer is called the **conductor** of χ .

From now on, all multiplicative characters of local fields are assumed to be unitary.

2.3 Lift a Dirichlet Character to a Hecke Character

Look at a character $\chi : (\mathbb{Z}/\ell^e \mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ first. We define a character χ_p on $\mathbb{Q}_p^\times \simeq p^\mathbb{Z} \times \mathbb{Z}_p^\times$ and χ_∞ on \mathbb{R}^\times as follows.

- If $p = \ell$, then the isomorphism $\mathbb{Z}_\ell^\times / (1 + \ell^e \mathbb{Z}_\ell) \simeq (\mathbb{Z}/\ell^e \mathbb{Z})^\times$ enables us to lift χ^{-1} (note the ‘ -1 ’!) to a character χ_ℓ on \mathbb{Q}_ℓ^\times that is trivial on $\ell^\mathbb{Z}$ and $1 + \ell^e \mathbb{Z}_\ell$.
- If $p \neq \ell$, then p is invertible mod ℓ^e , so we can define $\chi_p(p) := \chi(p)$, then make it trivial on \mathbb{Z}_p^\times .
- Put $\chi_\infty := \text{sgn } \chi(-1)$.

Since χ_p are trivial on \mathbb{Z}_p^\times only except for $p = \ell$, patching them together yields a character $\tilde{\chi} := \prod_v \chi_v$ on \mathbb{A}_Q^\times .

Lemma 2.2. The character $\tilde{\chi}$ is trivial on \mathbb{Q}^\times .

Proof. It suffices to check for every prime p and -1 . If $p \neq \ell$, then $\tilde{\chi}(p) = \chi_p(p) \chi_\ell(p) = 1$; otherwise $\chi_v(\ell) = 1$ for all places v . To conclude, $\tilde{\chi}(-1) = \chi_\infty(-1) \chi_\ell(-1) = 1$. \square

Now consider $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. The factorisation $N = p_1^{e_1} \cdots p_r^{e_r}$ gives

$$(\mathbb{Z}/N)^\times \simeq (\mathbb{Z}/p_1^{e_1})^\times \times \cdots \times (\mathbb{Z}/p_r^{e_r})^\times,$$

so we have $\chi = \chi_1 \cdots \chi_r$, where $\chi_i : (\mathbb{Z}/p_i^{e_i})^\times \rightarrow \mathbb{C}^\times$, and obtain a Hecke character $\tilde{\chi} := \tilde{\chi}_1 \cdots \tilde{\chi}_r$.

Remark. The character $\tilde{\chi}$ is (the inverse of?)

$$\mathbb{A}_Q^\times / \mathbb{Q}^\times \rightarrow \mathbb{A}_Q^\times / \mathbb{Q}^\times \mathbb{R}_{>0} \simeq \hat{\mathbb{Z}}^\times \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} \mathbb{C}^\times.$$

Conversely, every (unitary) Hecke character factors through $\hat{\mathbb{Z}}^\times \rightarrow \mathbb{C}^\times$, and hence it is of finite order iff it comes from a Dirichlet character.

3 Fourier Analysis

3.1 Fourier analysis on local fields

Let F be a local field. We only need the Schwartz functions and consider their integrals. The space of Schwartz functions $F \rightarrow \mathbb{C}$ is denoted by $\mathcal{S}(F)$. We are familiar with $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{C})$: the space of function f that decreases faster than any polynomial, i.e.,

$$\lim_{x \rightarrow \infty} x^n \left(\frac{d}{dx} \right)^m = 0, \quad \forall m, n.$$

As for a nonarchimedean local field F , $\mathcal{S}(F)$ is defined to be the space of locally constant compactly supported functions. Because the topology of F and \mathbb{C} are “totally incompatible”, these are actually all the continuous functions from F to \mathbb{C} with compact supports. Note that every Schwartz function may be written as a finite linear combination of functions $1_{a+\varpi^n \mathcal{O}_F}$, where ϖ is an uniformizer.

Then we fix an additive measure on F .

- ◇ If $F = \mathbb{R}$, then $dx :=$ the Lebesgue measure.
- ◇ If $F = \mathbb{C}$, then $dx :=$ two-times the Lebesgue measure.
- ◇ If $F/\mathbb{Q}_p < \infty$, then dx satisfies $\text{vol}(\mathcal{O}_F) = (N\mathfrak{d})^{-\frac{1}{2}}$.

To define Fourier transformation, one need to fix an additive character ψ on F .

- ◇ If $F = \mathbb{R}$, then $\psi(x) := e^{-2\pi i x}$.
- ◇ If $F = \mathbb{C}$, then $\psi(x) := e^{-2\pi i(x+\bar{x})}$.
- ◇ If $F/\mathbb{Q}_p < \infty$, then $\psi(x) := e^{2\pi i \{\text{Tr}_{F/\mathbb{Q}_p} x\}}$, where $\{\cdot\} : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

Then we define the Fourier transformation for $f \in \mathcal{S}(F)$ to be

$$\mathcal{F}_\psi f(y) := \hat{f}(y) := \int_F f(x) \psi(xy) dx.$$

Under these choices, we have the following properties known for \mathbb{R} and \mathbb{C} .

Theorem 2. Let $f \in \mathcal{S}(F)$. Then $\hat{\hat{f}} \in \mathcal{S}(F)$, and

$$\hat{\hat{f}}(x) = f(-x).$$

In particular, if F is nonarchimedean and unramified, then

$$1_{\hat{\mathcal{O}}_F} = 1_{\mathcal{O}_F}.$$

Proof. (An important example of computation!)

We may assume F to be a nonarchimedean local field with ϖ an uniformizer, $f = 1_{a+\varpi^n \mathcal{O}_F}$.

We have

$$1_{a+\varpi^n \mathcal{O}_F}(y) = \int_{a+\varpi^n \mathcal{O}_F} \psi(xy) dx = \psi(ay) \int_{\varpi^n \mathcal{O}_F} \psi(xy) dx = |\varpi|^n \psi(ay) \int_{\mathcal{O}_F} \psi(\varpi^n xy) dx.$$

Note that $\phi : x \mapsto \psi(\varpi^n xy)$ is an additive character, and

$$\phi|_{\mathcal{O}} = 1 \iff \varpi^n y \in \mathfrak{d}^{-1}$$

(by definition), hence

$$\int_{\mathcal{O}} \phi(x) dx = \begin{cases} \text{vol}(\mathcal{O}), & y \in \varpi^{-n}\mathfrak{d}^{-1}, \\ 0, & y \notin \varpi^{-n}\mathfrak{d}^{-1}. \end{cases}$$

(In the second case, ϕ has conductor smaller than \mathcal{O} and thus factors through a non-trivial character of a finite group.) So

$$1_{a+\varpi^n\mathcal{O}}(y) = |\varpi|^n \psi(ay) (N\mathfrak{d})^{-\frac{1}{2}} 1_{\varpi^{-n}\mathcal{O}}(y).$$

Similarly,

$$\int_F \psi(ay) 1_{\varpi^{-n}\mathfrak{d}^{-1}}(y) \psi(xy) dy = \int_{\varpi^{-n}\mathfrak{d}^{-1}} \psi((a+x)y) dy = \text{vol}(\varpi^{-n}\mathfrak{d}^{-1}) \cdot 1_{-a+\varpi^n\mathcal{O}}(x),$$

where

$$\text{vol}(\varpi^{-n}\mathfrak{d}^{-1}) = |\varpi|^{-n} \cdot \text{vol}(\mathfrak{d}^{-1}) = |\varpi|^{-n} \cdot \text{vol}(\mathcal{O})N\mathfrak{d} = |\varpi|^{-n}(N\mathfrak{d})^{\frac{1}{2}}.$$

The result follows. □

The multiplicative measure on F^\times is chosen as follows.

- ◇ If $F = \mathbb{R}$, then $d^\times x := |x|^{-1} dx$.
- ◇ If $F = \mathbb{C}$, then $d^\times x := |x|_{\mathbb{C}}^{-1} dx$, where $|x|_{\mathbb{C}} := x\bar{x}$. (Reason?)
- ◇ If $F/\mathbb{Q}_p < \infty$, then $\text{vol}(\mathcal{O}_F^\times, d^\times x) = 1$.

As an example, integration on local fields can give the factor of L -function at \mathfrak{p} .

Lemma 3.1. Let χ be an unramified character $F^\times \rightarrow \mathbb{C}^\times$. Then

$$\int_{F^\times} 1_{\mathcal{O}_F}(x) \chi(x) |x|^s d^\times x = (1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}.$$

Proof. Since $\mathcal{O}_F = \bigsqcup_{n \geq 0} \varpi^n \mathcal{O}_F^\times$,

$$\int_{F^\times} 1_{\mathcal{O}_F}(x) \chi(x) |x|^s d^\times x = \sum_{n \geq 0} (\chi(\varpi)^n \cdot 1) \cdot N\mathfrak{p}^{-ns} = \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}. \quad \square$$

3.2 Fourier analysis on adeles

Let F be a number field.

A **Schwartz-Bruhat function** is a finite linear combination of functions of the form

$$\prod_v f_v : \mathbb{A}_F \rightarrow \mathbb{C}, \quad f_v \in \mathcal{S}(F_v), \quad f_v = 1_{\mathcal{O}_{F_v}} \text{ a.e.,}$$

and denote the space of Schwartz-Bruhat functions by $\mathcal{S}(\mathbb{A}_F)$. Then define the additive character on \mathbb{A}_F by

$$\psi(x) := \prod_v \psi_v(x_v).$$

This is by definition a finite product and thus well-defined.

Lemma 3.2. $\psi|_F = 1$. □

Then we need to define and fix measures on $\mathbb{A}_F, \mathbb{A}_F^\times$ and $\mathbb{A}_F^{\times,1}$. For \mathbb{A} resp. \mathbb{A}^\times , simply multiply the measures on each places yields an additive resp. multiplicative measure, if $\text{vol}(\mathcal{O}_F, dx) = 1$ resp. $\text{vol}(\mathcal{O}_F^\times, d^\times x) = 1$ (which is true for our choices). So for a Schwartz-Bruhat function $f = \prod_v f_v$,

$$\int_{\mathbb{A}_F} f(x) dx = \prod_v \int_{F_v} f_v(x_v) dx_v, \quad \int_{\mathbb{A}_F^\times} f(x) d^\times x = \prod_v \int_{F_v^\times} f_v(x_v) d^\times x_v.$$

Theorem 3. The volume of the fundamental domain of \mathbb{A}_F/F under the given measure is 1.

For $\mathbb{A}^{\times,1}$, fix an archimedean place u first. Define a continuous homomorphism $\phi : \mathbb{A}_F^\times \rightarrow \mathbb{A}_F^{\times,1}$ by $\phi(x)_u := x_u/|x|$ and $\phi(x)_v := x_v$ for $v \neq u$. The multiplicative measure $d^\times x$ on $\mathbb{A}_F^{\times,1}$ is defined s.t. for a measurable set $U \subset \mathbb{A}_F^{\times,1}$,

$$\text{vol}_{\mathbb{A}^\times}(U, d^\times x) := \text{vol}_{\mathbb{A}^{\times,1}}(U', d^\times x), \quad \text{where } U' := \{x \in \mathbb{A}_F^\times : \phi(x) \in U, 0 \leq \log |x| \leq 1\}.$$

For example, let $F = \mathbb{Q}$ and $U = \prod_p \mathbb{Z}_p^\times \times \{1\}$, then $U' = \prod_p \mathbb{Z}_p^\times \times [1, e]$, so

$$\text{vol}(U) = \int_1^e \frac{dx}{x} = 1.$$

Remark. This is the measure defined by the exact sequence

$$1 \rightarrow \mathbb{A}_F^{\times,1} \rightarrow \mathbb{A}_F^\times \rightarrow \mathbb{R}_{>0} \rightarrow 1.$$

For $U = U^1 \times I$, where $U \subset \mathbb{A}_F^{\times,1}$ and $I \subset \mathbb{R}_{>0}$, $\text{vol}(U) = \text{vol}(U^1) \text{vol}(I)$.

Theorem 4. The volume of the fundamental of $\mathbb{A}_F^{\times,1}/F^\times$ is

$$\frac{2^{r_1} (2\pi)^{r_2} h_F R_F}{w_F},$$

Now take $f \in \mathcal{S}(\mathbb{A}_F)$. Define

$$\mathcal{F}_\psi f(y) := \hat{f}(y) := \int_{\mathbb{A}_F} f(x) \psi(xy) dx.$$

In particular,

$$\prod_v \hat{f}_v = \prod_v \hat{f}_v.$$

By the lemma above, $\hat{f} \in \mathcal{S}(\mathbb{A}_F)$.

Theorem 5 (Poisson Summation Formula). Let $f \in \mathcal{S}(\mathbb{A}_F)$, then

$$\sum_{x \in F} f(x) = \sum_{x \in F} \hat{f}(x).$$

(The summation obviously converges.)

Corollary 3.1. Let $\alpha \in \mathbb{A}_F^\times$, then

$$|\alpha| \sum_{x \in F} f(\alpha x) = \sum_{x \in F} \hat{f}(\alpha^{-1} x). \quad \square$$

4 Analytic Properties of Hecke L -functions

Let F be a number field, $\chi = \prod_v \chi_v : \mathbb{A}_F^\times / F^\times \rightarrow \mathbb{C}^\times$ a Hecke character, S a finite set containing all infinite places and all places v s.t. χ_v is ramified.

Recall that

$$L(s, \chi_v) := (1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1},$$

and the **partial Hecke L -function**

$$L^S(s, \chi) := \prod_{v \notin S} L(s, \chi_v).$$

Lemma 4.1. The Euler product $L^S(s, \chi)$ absolutely converges if $\operatorname{Re} s > 1$.

Proof. If $\mathfrak{p} \cap \mathbb{Z} = p$, then $N\mathfrak{p} \geq p$, and since χ is unitary,

$$|(1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq (1 - p^{-\operatorname{Re} s})^{-1}.$$

Since there are at most $n = [F : \mathbb{Q}]$ primes over p ,

$$\prod_v |(1 - \chi_v(\mathfrak{p})N\mathfrak{p}^{-s})^{-1}| \leq \prod_p (1 - p^{-\operatorname{Re} s})^{-n}. \quad \square$$

Take $f \in \mathcal{S}(\mathbb{A}_F)$ s.t. $f_v = 1_{\mathcal{O}}$ for $v \notin S$. Define the zeta integral

$$Z(s, f, \chi) := \int_{\mathbb{A}_F^\times} f(x) \chi(x) |x|^s d^\times x$$

and local zeta integral

$$Z_v(s, f_v, \chi_v) := \int_{F_v^\times} f_v(x) \chi_v(x) |x|^s d^\times x_v.$$

By definition,

$$Z(s, f, \chi) = \prod_v Z_v(s, f_v, \chi_v).$$

For $v \notin S$, we have seen in Lemma 3.1 that

$$L(s, \chi_v) = Z_v(s, f_v, \chi_v),$$

so

$$Z(s, f, \chi) = L^S(s, \chi) \prod_{v \in S} Z_v(s, f_v, \chi_v),$$

and it is absolutely convergent on $\operatorname{Re} s > 1$.

Theorem 6. $Z(s, f, \chi)$ can be extended to a meromorphic function on \mathbb{C} , satisfying

$$Z(s, f, \chi) = Z(1 - s, \hat{f}, \chi^{-1}).$$

Moreover, if there does not exist $\lambda \in i\mathbb{R}$ s.t. $\chi(x) = |x|^\lambda$, then $Z(s, f, \chi)$ is entire; otherwise the only poles of $Z(s, f, \chi)$ are $s = 1 - \lambda$ and $s = -\lambda$, which are both simple poles with residue $\hat{f}(0) \operatorname{vol}(\mathbb{A}_F^{\times, 1} / F^\times)$ and $-f(0) \operatorname{vol}(\mathbb{A}_F^{\times, 1} / F^\times)$.

Proof. Because $\{|x| = 1\}$ is of measure zero in \mathbb{A}_F^\times , we have

$$Z(s, f, \chi) = \int_{\mathbb{A}_F^\times} = \int_{\mathbb{A}_F^{>1}} + \int_{\mathbb{A}_F^{\leq 1}} =: Z^{>1} + Z^{<1}.$$

For all $s \in \mathbb{C}$, the integrand is continuous when $|x| > 1$, so $Z^{>1}$ converges on \mathbb{C} .

Now we turn to $Z^{<1}$. Let Ω be a fundamental domain of $\mathbb{A}_F^{<1}/F^\times$. Assume that s is big enough, then

$$\begin{aligned}
Z^{<1} &= \sum_{\alpha \in F^\times} \int_{\alpha\Omega} f(x) \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left(\sum_{\alpha \in F^\times} f(\alpha x) \right) \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left(\sum_{\alpha \in F} f(\alpha x) \right) \chi(x) |x|^s d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left(\sum_{\alpha \in F} \hat{f}(\alpha x^{-1}) \right) \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega} \left(\sum_{\alpha \in F^\times} \hat{f}(\alpha x^{-1}) \right) \chi(x) |x|^{s-1} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\Omega^{-1}} \left(\sum_{\alpha \in F^\times} \hat{f}(\alpha x) \right) \chi(x^{-1}) |x|^{1-s} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x \\
&= \int_{\mathbb{A}_F^{>1}} \hat{f}(x) \chi(x)^{-1} |x|^{1-s} d^\times x + \hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x.
\end{aligned}$$

We used “ $\chi(\alpha x) = \chi(x)$ for $\alpha \in F^\times$ ”, Poisson summation, “ $d^\times x$ is invariant under $x \mapsto x^{-1}$ ”, and “ Ω^{-1} is a fundamental domain of $\mathbb{A}_F^{>1}/F^\times$ ” in the above calculation. The integral over $\mathbb{A}_F^{>1}$ is again convergent on \mathbb{C} , so we look at the rest two integrals.

Write $\Omega = \Omega^1 \times (0, 1)$, where Ω^1 is a fundamental domain of $\mathbb{A}_F^{\times,1}/F^\times$. Then if χ is non-trivial on $\mathbb{A}_F^{\times,1}$, both integrals vanish (as in Theorem 2). Otherwise χ factors through $\mathbb{A}_F^\times/\mathbb{A}_F^{\times,1} \simeq \mathbb{R}_{>0} \rightarrow \mathbb{C}^\times$, hence $\chi(x) = |x|^\lambda$ for some $\lambda \in i\mathbb{R}$, and

$$\hat{f}(0) \int_{\Omega} \chi(x) |x|^{s-1} d^\times x - f(0) \int_{\Omega} \chi(x) |x|^s d^\times x = \frac{\hat{f}(0) \text{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s + \lambda - 1} - \frac{f(0) \text{vol}(\mathbb{A}_F^{\times,1}/F^\times)}{s + \lambda}.$$

The theorem is easy to deduce from the expression. □

Our next target is $Z_v(s, f, \chi_v) = \int_{F_v^\times} f_v(x) \chi_v(x) |x|^s d^\times x_v$.

Lemma 4.2. $Z_v(s, f, \chi_v)$ converges on $\text{Re } s > 0$.

Proof. Consider only the nonarchimedean case.

Take ϵ small enough s.t. $f_v(x) = f_v(0)$ for $|x| < \epsilon$. Write

$$Z_v(s, f_v, \chi_v) = \int_{|x| > \epsilon} + \int_{|x| < \epsilon}.$$

Similarly, the first integral converges on \mathbb{C} . For the second one, $\{|x| < \epsilon\} = \bigcup_{n \geq N} \varpi^n \mathcal{O}_{F_v}^\times$ for an integer N . Thus we see that

$$\int_{|x| < \epsilon} |\chi_v(x) |x|^s| d^\times x = \sum_{n \geq N} \int_{\varpi^n \mathcal{O}_{F_v}^\times} |\varpi|^{-n \text{Re } s} d^\times x$$

converges when $\text{Re } s > 0$. □

Theorem 7. (1) $Z_v(s, f, \chi_v)$ can be extended to a meromorphic function on \mathbb{C} which is holomorphic on $\text{Re } s > 0$.

(2) There exists a meromorphic function $\gamma_v(s, \chi_v, \psi_v)$, called **local γ -factor**, irrelevant to f_v , s.t. for any $f_v \in \mathcal{S}(F_v)$,

$$Z_v(1-s, \hat{f}_v, \chi_v^{-1}) = \gamma_v(s, \chi_v, \psi_v) Z_v(s, f_v, \chi_v).$$

Proof. Firstly, both sides of the equation converge on $0 < \text{Re } s < 1$.

We need to show that $\frac{Z_v(1-s, \hat{f}_v, \chi_v^{-1})}{Z_v(s, f_v, \chi_v)}$ is irrelevant to f_v ; i.e.,

$$Z_v(1-s, \hat{f}_v, \chi_v^{-1}) Z_v(s, g_v, \chi_v) = Z_v(1-s, \hat{g}_v, \chi_v^{-1}) Z_v(s, f_v, \chi_v), \quad \forall g_v \in \mathcal{S}(F_v).$$

Assume that $d^\times x_v = |x|^{-1} dx$, then the LHS

$$\begin{aligned} &= \int_{F_v^\times} \left(\int_F f_v(y) \psi_v(xy) dy \right) \chi_v(x)^{-1} |x|^{1-s} d^\times x \int_{F_v^\times} g_v(x) \chi_v(x) |x|^s d^\times x \\ &= \int_{F_v^\times} \int_{F_v^\times} \int_{F_v^\times} f_v(y) g_v(z) \psi_v(xy) \chi_v(zx^{-1}) |x|^{1-s} |z|^s d^\times x dy d^\times z \\ &= \iiint f_v(y) g_v(z) \psi_v(xy) \chi_v(zx^{-1}) |x|^{1-s} |z|^s \cdot |y| d^\times x d^\times y d^\times z \\ &= \iiint f_v(y) g_v(z) \psi_v(x) \chi_v(zyx^{-1}) |x|^{1-s} |zy|^s d^\times x d^\times y d^\times z \quad (x \mapsto y^{-1}x). \end{aligned}$$

Hence LHS = RHS.

So γ_v is well-defined on $0 < \text{Re } s < 1$. If γ_v can be a meromorphic function on \mathbb{C} , then the equation gives the analytic continuation of Z_v on $\text{Re } s < 1$. (The formula of γ -factor is only computed for archimedean place in this proof.)

(1) $F_v = \mathbb{R}$. Note that

$$Z_v(s, f_v, \chi_v | \cdot |^t) = Z_v(s+t, f_v, \chi_v),$$

so we only need to compute for χ_v trivial or $\chi_v = \text{sgn}$ character. The result is

$$\gamma_v(s, \chi_v, \psi_v) = \begin{cases} \frac{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}, & \chi = 1, \\ i \frac{\pi^{-\frac{(1-s)+1}{2}} \Gamma(\frac{(1-s)+1}{2})}{\pi^{-\frac{s+1}{2}} \Gamma(\frac{s+1}{2})}, & \chi = \text{sgn}. \end{cases}$$

For example, when $\chi_v = 1$, we take $f_v(x) = e^{-\pi x^2}$, then $\hat{f}_v = f_v$, and

$$\begin{aligned} Z_v(s, f_v, 1) &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{s-1} dx \\ &= 2 \int_0^{+\infty} e^{-\pi x^2} x^{s-1} dx \\ &= \pi^{-\frac{s}{2}} \int_0^{+\infty} y^{\frac{s}{2}-1} e^{-y} dy \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \end{aligned}$$

therefore

$$\gamma_v(s, \chi_v, \psi_v) = \frac{Z_v(1-s, f_v, 1)}{Z_v(s, f_v, 1)} = \frac{\pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})}{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})}, \quad \chi = 1.$$

(2) $F_v = \mathbb{C}$. For $\chi_v(x) = \left(\frac{x}{\sqrt{x\bar{x}}}\right)^n$, $n \in \mathbb{Z}$, using

$$f_v(x) = \begin{cases} \bar{x}^n e^{-2x\bar{x}}, & n \geq 0, \\ x^{-n} e^{-2x\bar{x}}, & n < 0, \end{cases}$$

we obtain

$$\gamma_v(s, \chi^v, \psi_v) = i^{|n|} \frac{(2\pi)^{-(1-s+\frac{|n|}{2})} \Gamma\left(1-s+\frac{|n|}{2}\right)}{(2\pi)^{-(-s+\frac{|n|}{2})} \Gamma\left(s+\frac{|n|}{2}\right)}.$$

(3) v is nonarchimedean. We show that γ_v is defined on $\operatorname{Re} s > 1$. Let U be a sufficiently small open compact nbhd of -1 in F_v s.t. χ_v is trivial on $-U$, and put $f_v := 1_U$. Then

$$Z_v(t, \hat{f}_v, \chi_v^{-1}) = \int_{F_v^\times} 1_U(-x) \chi(x^{-1}) |x|^t d^\times x = \operatorname{vol}(U) \neq 0$$

and is irrelevant to t . Therefore γ_v^{-1} can be defined on $\operatorname{Re} s > 0$. Similar for $\operatorname{Re} s < 1$. \square

Finally, we obtain the analytic continuation of Hecke L -functions and the main theorem of functional equations.

Theorem 8. Let S be a finite set of places s.t. $\forall v \notin S$, v is archimedean with χ_v unramified, and $\mathfrak{d}_v = \mathcal{O}_{F_v}$. Then the partial Hecke L -function can be extended to a meromorphic function on \mathbb{C} , satisfying

$$L^S(s, \chi) = \left(\prod_{v \in S} \gamma_v(s, \chi_v, \psi_v) \right) L^S(1-s, \chi^{-1}).$$

Moreover, if there does not exist $\lambda \in i\mathbb{R}$ s.t. $\chi(x) = |x|^\lambda$, then $L^S(s, \chi)$ is entire; otherwise only $s = 1 - \lambda$ and $s = -\lambda$ have the possibility to be poles.

Proof. Take $f = \prod_v f_v$ s.t. $f_v = 1_{\mathcal{O}_{F_v}}$, $\forall v \notin S$. For $v \notin S$, the additional condition $\mathfrak{d}_v = \mathcal{O}_{F_v}$ implies that (by Lemma 3.1)

$$\hat{f}_v(x) = (N\mathfrak{d})^{-\frac{1}{2}} 1_{\mathcal{O}}(x) = 1_{\mathcal{O}}(x),$$

and the functional equation follows.

It is left to show the property about poles. Suppose that $\chi(x) = |x|^\lambda$ with $\lambda \in i\mathbb{R}$ and $s = s_0$ is a pole of L^S other than $-\lambda$ or $1 - \lambda$. Consider the equation

$$Z(s, f, \chi) = L^S(s, \chi) \prod_{v \in S} Z_v(s, f_v, \chi_v).$$

By Theorem 6, LHS is holomorphic at $s = s_0$.

We choose an f s.t. for all $v \in S$, f_v supports in a sufficiently small nbhd U_v of $1 \in F_v$. With a similar argument in the previous proof, one sees that $Z_v(s_0, f_v, \chi_v) \neq 0$. Therefore the RHS has a pole at $s = s_0$, which is a contradiction. \square

4.1 Exercise

Let $F = \mathbb{Q}$, $\chi = 1$ the trivial character. Repeat the calculation before to prove the analytic continuation and functional equation of Riemann zeta function, and compute its residue at $s = 1$.

Proof. The Riemann zeta function is

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Let $S = \{\infty\}$. The local unramified L -functions are

$$L(s, 1_p) = (1 - p^{-s})^{-1},$$

so $\zeta(s) = L^S(s, \chi)$.

Let $f \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$ be defined by $f_p = 1_{\mathbb{Z}_p}$ and $f_{\infty}(x) = e^{-\pi x^2}$. The zeta integral is

$$Z(s, f, 1) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x) |x|^s d^{\times} x$$

and the local zeta integral at infinity is

$$Z_{\infty}(s, f_{\infty}, 1) = \int_{\mathbb{R}^{\times}} e^{-\pi x^2} |x|^s d^{\times} x.$$

We have

$$Z(s, f, 1) = \zeta(s) Z_{\infty}(s, f_{\infty}, 1).$$

(TBC.....)

□

5 Dedekind Zeta Functions and Dirichlet L -functions

5.1 Dedekind Zeta Functions and the Analytic Class Number Formula

Let F be a number field, χ the trivial character, S the set of all archimedean places. The **Dedekind zeta function** of F is defined to be

$$\zeta_F(s) := L^S(s, \chi) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}.$$

In this subsection, we will compute the local γ -factors at ramified places to deduce the functional equation of Dedekind zeta functions and the analytic class number formula.

Theorem 9. The Dedekind zeta function $\zeta_F(s)$ can be extended to a meromorphic function on \mathbb{C} with only poles at $s = 0$ and $s = 1$.

1. (Analytic class number formula.) $\zeta_F(s)$ has a simple pole at $s = 1$ with residue

$$\text{res}_1 \zeta_F = \frac{2^{r_1} (2\pi)^{r_2} h_F R_F}{\sqrt{|\text{disc } F| w_F}},$$

and is of order $r_1 + r_2 - 1$ at $s = 0$ with

$$\lim_{s \rightarrow 0} s^{r_1 + r_2 - 1} \zeta_F(s) = -\frac{h_F R_F}{w_F}$$

2. Define the completed Dedekind zeta function

$$\Lambda(s) := |\text{disc } F|^{\frac{s}{2}} \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right)^{r_1} (2(2\pi)^{-s} \Gamma(s))^{r_2} \zeta_F(s).$$

Then

$$\Lambda_F(s) = \Lambda_F(1 - s).$$

5.2 Dirichlet L -functions

Let $F = \mathbb{Q}$, χ a Dirichlet character with conductor N , $S = \{p : p|N\} \cup \{\infty\}$. Lifting χ to a Hecke character $\tilde{\chi}$, we get an partial L -function

$$L^S(s, \tilde{\chi}) = \prod_{p \nmid N} (1 - \chi_p(p) N p^{-s})^{-1},$$

which is exactly the classic Dirichlet L -function

$$L(s, \chi) = \prod_{p \nmid N} (1 - \chi(p) p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \text{Re } s > 1.$$

The functional equation

$$L(s, \chi) = \left(\prod_{p|N} \gamma_p(s, \chi_p, \psi_p) \right) \gamma_{\infty}(s, \chi_{\infty}, \psi_{\infty}) L(1-s, \chi^{-1})$$

has been proved, and γ_{∞} have been computed. It is left to compute the γ_p 's for $p \mid N$.

Suppose $p^e \parallel N$, then the conductor of χ_p is p^e (or $1 + p^e \mathbb{Z}_p$). Take $f_p := 1_{1+p^e \mathbb{Z}_p}$, so

$$Z_p(s, f_p, \chi_p) = \int_{1+p^e \mathbb{Z}_p} |x|^s d^{\times} x = \text{vol}(1 + p^e \mathbb{Z}_p)$$

is easy to compute. Then

$$\hat{f}_p(x) = p^{-e} \psi_p(x) 1_{p^{-e} \mathbb{Z}_p}(x),$$

hence

$$\begin{aligned} \gamma_p(s, \chi_p, \psi_p) &= p^{-e} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \int_{p^{-e} \mathbb{Z}_p} \psi_p(x) \chi_p(x) |x|^{1-s} d^{\times} x \\ &= p^{-e} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \sum_{n \geq -e} p^{-n(1-s)} \int_{p^n \mathbb{Z}_p^{\times}} \psi_p(x) \chi_p(x) d^{\times} x. \end{aligned}$$

Next, we show that $\int_{p^n \mathbb{Z}_p^{\times}} = 0$ whenever $n \neq -e$.

(1) $n \geq 0$. Then ψ_p is trivial on $p^n \mathbb{Z}_p^{\times} \subset \mathbb{Z}_p$, so the integral equals

$$\int_{p^n \mathbb{Z}_p^{\times}} \chi_p(x) d^{\times} x = \chi_p(p^n) \int_{\mathbb{Z}_p^{\times}} \chi_p(x) d^{\times} x = 0$$

as χ_p is nontrivial on \mathbb{Z}_p^{\times} .

(2) $-e < n < 0$. We can decompose $p^n \mathbb{Z}_p^{\times}$ into $p^{-n-1}(p-1)$ copies of \mathbb{Z}_p , i.e.,

$$p^n \mathbb{Z}_p^{\times} = \bigsqcup_{\alpha} (\alpha + \mathbb{Z}_p), \quad \alpha \in p^n \mathbb{Z}_p^{\times},$$

then

$$\int_{p^n \mathbb{Z}_p^{\times}} \psi_p(x) \chi_p(x) d^{\times} x = \sum_{\alpha} \psi_p(\alpha) \int_{\alpha + \mathbb{Z}_p} \chi_p(x) d^{\times} x = \sum_{\alpha} \psi_p(\alpha) \chi_p(\alpha) \int_{1+\alpha^{-1} \mathbb{Z}_p} \chi(x) d^{\times} x.$$

Since $-n < e$, $1 + \alpha^{-1} \mathbb{Z}_p \supset 1 + p^e \mathbb{Z}_p$, so the integrals vanish agains.

So we get

$$\begin{aligned}
\gamma_p(s, \chi_p, \psi_p) &= p^{-es} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \int_{p^{-e} \mathbb{Z}_p^\times} \psi_p(x) \chi_p(x) d^\times x \\
&= p^{-es} \text{vol}(1 + p^e \mathbb{Z}_p)^{-1} \sum_{\alpha \in \mathbb{Z}_p^\times / (1 + p^e \mathbb{Z}_p)} \psi_p(p^{-e} \alpha) \chi_p(p^{-e} \alpha) \int_{\alpha(1 + p^e \mathbb{Z}_p)} d^\times x \\
&= p^{-es} \sum_{\alpha \in (\mathbb{Z}/p^e \mathbb{Z})^\times} \psi_p(p^{-e} \alpha) \chi_p(p^{-e} \alpha).
\end{aligned}$$

(T.B.C.)

5.3 Quadratic Fields

Let $F = \mathbb{Q}(\sqrt{d})$ and $D = |\text{disc } F|$. Define $\chi_d : (\mathbb{Z}/D\mathbb{Z})^\times \rightarrow \{\pm 1\}$ by

$$\chi_d(p) := \begin{cases} 1, & p \text{ splits in } F, \\ -1, & p \text{ is inert in } F. \end{cases}$$

Remark. Every primitive quadratic Dirichlet character is of the form above, which is just the Legendre symbol $(\frac{\cdot}{D})$.

Lemma 5.1. $\zeta_F(s) = \zeta(s) L(s, \chi_d)$.

Proof. Check the equation

$$\prod_{\mathfrak{p}|p} \zeta_{\mathfrak{p}}(s) = \zeta_p(s) L_p(s, \chi_{d,p}).$$

□

Proposition 5.1 (Dirichlet). If $d < 0$, then

$$L(1, \chi_d) = \frac{2\pi h_F}{\sqrt{D} w_F}.$$

If $d > 0$, then

$$L(1, \chi_d) = \frac{h_F \log \epsilon_F}{\sqrt{D}},$$

where $\epsilon_F > 0$ is a fundamental unit of F .

This proposition explains why the class number of real quadratic fields are harder to study than imaginary ones, because there are powerful analytic methods to study $L(1, \chi_d)$, but we don't have a general method to separate h_F and $\log \epsilon_F$ (when $d > 0$).

The Gauss Class Number Problems

From now on, assume that $d < 0$. Gauss had conjectured that (1) $h_F \rightarrow \infty$ as $d \rightarrow -\infty$, and (2) give a list of imaginary quadratic fields F with $h_F = 1, 2, 3$, suspecting that these are all such fields. In particular, only 9 imaginary quadratic fields have class number 1:

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

Problem (1) was solved by C.Siegel in 1935.

Theorem 10 (Siegel). For all $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ s.t.

$$L(1, \chi) \geq \frac{C(\varepsilon)}{N^\varepsilon}$$

for any primitive character $\chi : (\mathbb{Z}/N\mathbb{Z}) \rightarrow \{\pm 1\}$. In particular, there exists a constant $C'(\varepsilon)$ s.t.

$$h_F \geq C'(\varepsilon) D^{\frac{1}{2} - \varepsilon}.$$

This implies that there are only finite many imaginary quadratic fields F with $h_F = A$ for any given constant A .

In Siegel's theorem, the constant $C(\varepsilon)$ is not an *effective constant*, meaning that there is no explicit formula for $C(\varepsilon)$ using ε and N . Hence the problem (2) wasn't solved until 1983.