Galois Deformations

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1 Review of Category Theory and Homological Algebra

All the set-theoretic issues are ignored for now.

1.1 Representability

Let $\mathfrak{C}$ be a category. We define the functors[1]

$$\begin{aligned} h^{\mathfrak{C}} : \mathfrak{C}^{\mathrm{op}} &\longrightarrow [\mathfrak{C}, \mathbf{Set}], \\ S &\longmapsto \mathrm{Hom}_{\mathfrak{C}}(S, \cdot) \end{aligned} \qquad \begin{aligned} \mathrm{ev}^{\mathfrak{C}} : [\mathfrak{C}, \mathbf{Set}] \times \mathfrak{C} &\longrightarrow \mathbf{Set} \\ (F, S) &\longmapsto F(S). \end{aligned}$$

Theorem 1 (Yoneda)**.** There is an isomorphism

$$\mathrm{Hom}_{[\mathfrak{C},\mathbf{Set}]^{\mathrm{op}}}(-, h^{\mathfrak{C}}(-)) \simeq \mathrm{ev}^{\mathfrak{C}}$$

as functors $[\mathfrak{C}, \mathbf{Set}] \times \mathfrak{C} \to \mathbf{Set}$ given by

$$\begin{aligned} \mathrm{Hom}_{[\mathfrak{C},\mathbf{Set}]^{\mathrm{op}}}\left(F, h^{\mathfrak{C}}(S)\right) &\longrightarrow F(S) \\ \left(F \xleftarrow{\phi} \mathrm{Hom}_{\mathfrak{C}}(S, -)\right) &\longmapsto \phi_S(\mathrm{id}_S) \end{aligned}$$

for all $F : \mathfrak{C} \to \mathbf{Set}$ and $S \in \mathfrak{C}$, and the functor $h^{\mathfrak{C}} : \mathfrak{C}^{\mathrm{op}} \to [\mathfrak{C}, \mathbf{Set}]$ is fully faithful.

We say that a functor $F : \mathfrak{C} \to \mathbf{Set}$ is **representable**, if there is $X \in \mathfrak{C}$ along with an isomorphism

$$\phi : \mathrm{Hom}_{\mathfrak{C}}(X, -) \simeq F$$

as functors. Note that the functor ϕ is determined[2] by the universal element $u := \phi_X(\mathrm{id}_X) \in F(X)$, from which every thing in $F(T)$ is pushed forward, i.e. for any morphism $f : X \to T$ in $\mathfrak{C}$, the unique corresponding element in $F(T)$ is $\phi_T(f) = F(f)(\phi_X(\mathrm{id}_X)) = F(f)(u)$.

1.2 The Ext Functors

Let $\mathfrak{A}$ be an abelian category with enough projective and injective objects. We have

$$\mathrm{Ext}^i_{\mathfrak{A}}(X, Y) := \mathrm{R}^i \mathrm{Hom}_{\mathfrak{A}}(X, -)(Y) \simeq \mathrm{R}^i \mathrm{Hom}_{\mathfrak{A}}(-, Y)(X)$$

for $X, Y \in \mathfrak{A}$, $i \geq 0$.

[1] There is also the version for $h_{\mathfrak{C}} : \mathfrak{C} \to [\mathfrak{C}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathrm{ev}_{\mathfrak{C}} : [\mathfrak{C}^{\mathrm{op}}, \mathbf{Set}] \times \mathfrak{C} \to \mathbf{Set}$.

[2] This does *not* mean that we can decode ϕ from u without knowing ϕ a priori?

We will focus on Ext^1. An **extension of A by B**[3] is a short exact sequence

$$\xi : 0 \to B \to X \to A \to 0.$$

(I may denote ξ by X if there is no confusion.) An isomorphism of two extensions X and X' of A by B is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow\simeq & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

An extension of A by B that is isomorphic to

$$0 \to B \hookrightarrow A \oplus B \to A \to 0$$

is said to be split.

Given an extension $\xi : 0 \to B \to X \to A \to 0$ of A by B, the cohomological functors $\mathrm{Ext}^*(A, -)$ induces the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \xrightarrow{\partial_\xi} \mathrm{Ext}^1(A, B).$$

Let's look at the class $\Theta(\xi) := \partial_\xi(\mathrm{id}_A) \in \mathrm{Ext}^1(A, B) = 0$. If $\Theta(\xi) = 0$, then there is a section $f : A \to X$ of $X \to A$ in ξ, i.e. ξ is split. This means that $\Theta(\xi) \in \mathrm{Ext}^1(A, B)$ is the *obstruction* for ξ to be split.

Theorem 2. Let R be a (possibly non-commutative) ring. For left R-modules A and B, there is a natural bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \xleftrightarrow{1:1} \mathrm{Ext}^1_R(A, B)$$

given by $\Theta : \xi \mapsto \partial_\xi(\mathrm{id}_A)$.

Example 1.1. Let k be a topological ring (field if necessary), G be a topological group, V be a continuous $k[G]$-module that is free of k-rank d. Then there is a canonical isomorphism

$$\mathrm{Ext}^1_{k[G]}(V, V) \simeq H^1(G, \mathrm{ad}\, V).$$

(There should be a constructive proof, but I failed...)

We propose another proof in the next subsection.

1.3 Universal δ-Functors

We concentrate on cohomological things.

Definition 1. A (covariant) **cohomological δ-functor** is a collection of additive functors

$$\{T^n : \mathfrak{A} \to \mathfrak{B}\}_{n \geq 0}$$

indexed by non-negative integers, which induces *functorially* a long exact sequences in $\mathfrak{B}$ from a short exact sequence in $\mathfrak{A}$. More precisely, for each exact sequence

$$0 \to A \to B \to C \to 0 \quad \text{in } \mathfrak{A},$$

there are fixed morphisms

$$\delta^n : T^n(C) \to T^{n+1}(A) \quad \text{in } \mathfrak{B}, \quad n \geq 0,$$

[3]In a category where these operations make sense.

s.t.

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta^0} T^1(A) \to \cdots$$

is exact[4]; moreover, a morphism of short exact sequences in $\mathfrak{A}$ induces a morphism of long exact sequences in $\mathfrak{B}$.

For instance, taking cohomology for chain complexes

$$H^* : \mathbf{Ch}_{\geq 0}(\mathfrak{A}) \to \mathfrak{A}$$

or taking right-derivation of a left-exact functor are cohomological δ-functors.

Definition 2. The cohomological δ-functors from $\mathfrak{A}$ to $\mathfrak{B}$ form a category, where morphisms are the natural tranformations commuting with the δ^n's. A **universal cohomological δ-functor** is a δ-functor $T = (T^n)$, such that for any δ-functor $S = (S^n)$ and a morphism $f^0 : T \to S$, there is a unique morphism $f : T \to S$ extending f^0.

So a universal δ-functor is like an initial object among δ-functors but it is "weaker".

Theorem 3. If $F : \mathfrak{A} \to \mathfrak{B}$ is a left-exact additive functor, then (if $\mathfrak{A}$ has enough injectives) the right derivations $R^*F : \mathfrak{A} \to \mathfrak{B}$ form a universal δ-functor.

Another proof of Example 1.1. Let k be a field. We show that both $H^*(G, V^\vee \otimes_k (-))$ and $\mathrm{Ext}^*_G(V, -)$ are universal δ-functors. Then since they agree at $i = 0$, they must agree everywhere.

The functors $\mathrm{Ext}^*_G(V, -)$ are derived from $\mathrm{Hom}_G(V, -)$, so they are universal. For $H^*(G, V^\vee \otimes_k (-))$, since $V^\vee \otimes_k (-)$ is exact (Really? In which category?), we have[5]

$$H^*(G, V^\vee \otimes_k (-)) = \mathrm{R}^* \mathrm{Hom}_G(k, -) \circ (V^\vee \otimes_k (-)) = R^* \left(\mathrm{Hom}_G(k, -) \circ (V^\vee \otimes_k (-))\right),$$

which is also a derived functor. □

2 Deformation of Representations of Profinite Groups

2.1 The category of complete Noetherian algebras

Let $\mathcal{O}$ be a Noetherian ring with residue field k. We consider the category $\widehat{\mathfrak{Ar}_{\mathcal{O}}}$ of *complete Noetherian local $\mathcal{O}$-algebras with residue field k*, where morphisms are continuous $\mathcal{O}$-homomorphisms. By "complete", we mean that $A \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$ is a topological ring isomorphic to a projective limit of local (finite?) Artinian $\mathcal{O}$-algebras. But since A is Noetherian, A is $\mathfrak{m}_A$-adically complete.

Proposition 2.1. For every $A \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$, the topology on A equals the $\mathfrak{m}_A$-adic topology. Moreover, every $\mathcal{O}$-algebra homomorphism $A \to B$ where B is a complete local $\mathcal{O}$-algebra[6] is continuous.

In practice(?), we take a finite extension $L/\mathbb{Q}_p$ with residue field k and set $\mathcal{O} := \mathcal{O}_L$, the ring of integers in L, which is complete and contains the ring $W(k)$ of Witt vectors of k.

[4] In particular, T^0 is left-exact.
[5] I've never learnt this, but it seems very true and I accept it for now.
[6] As before, B is a projective limit of local Artinian $\mathcal{O}$-algebras. But we don't require Noetherianity.

2.2 Deformation functors

Let G be a profinite group, k be a finite field of characteristic p, V an $k[G]$-module of k-dimension d with G acting continuously[7]. We fix a k-basis β_k of V.

Take $A \in \widehat{\mathfrak{Ar}_{\mathcal{O}}}$. A **deformation** of V to A is a pair (V_A, ι_A), where

- V_A is an $A[G]$-module that is free of finite rank over A, and
- $\iota_A : V_A \otimes_A k \simeq V$ is an isomorphism of $k[G]$-modules.

A **lift** or **framed deformation** of (V, β_k) to A is a triple (V, ι_A, β_A), where

- (V, ι_A) is a deformation of V to A,
- β_A is a basis of V_A over A that reduces to β_k via ι_A.

Define $D_V(A)$ (resp. $D_V^{\square}(A)$) to be the set of isomorphism classes of deformations (resp. framed deformations) of V to A.

Remark. We can define deformations in terms of representations. But be careful to the diffrence between "$=$" and "$\simeq$"!

Let A be a commutative ring. An $A[G]$-module V_A free of finite rank d over A is equivalent to a finite free A-module V_A^0 with a morphism $G \to \mathrm{GL}(V_A^0)$. An isomophism $\varphi : V_A \simeq W_A$ of $A[G]$-modules is equivalent to an isomophism $\varphi^0 : V_A^0 \simeq W_A^0$ such that

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathrm{GL}\left(V_A^0\right) \arrow[d, "\mathrm{GL}(\varphi^0)"] \\
& \mathrm{GL}\left(W_A^0\right)
\end{tikzcd}$$

Now we pick an A-basis β_A for V_A^0, then the datum (V_A, β_A) is equivalent to the datum $(V_A^0, \beta_A, (\rho_V : G \to \mathrm{GL}_d(A)))$, and the above diagram extends to

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & \mathrm{GL}\left(V_A^0\right) \arrow[r, "\simeq"] \arrow[d, "\mathrm{GL}(\varphi^0)"] & \mathrm{GL}_d(A) \arrow[d, "\simeq"] \\
& \mathrm{GL}\left(W_A^0\right) \arrow[r, "\simeq"'] & \mathrm{GL}_d(A)
\end{tikzcd}$$

The last vertical arrow is induced by changing the A-basis of A^d, so it is a conjugation by some element $C \in \mathrm{GL}_d(A)$, i.e. $\rho_W = C\rho_V C^{-1}$. Now it is easy to see that

$$D_V^{\square}(A) = \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{cont}}(G, \mathrm{GL}_d(A)), \quad D_V(A) = \frac{\mathrm{Hom}_{\mathrm{gp}}^{\mathrm{cont}}(G, \mathrm{GL}_d(A))}{\text{conjugation by an element in } \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(k))}.$$

2.3 Representability

A profinite group G satisfies the **Mazur's finiteness condition** Φ_p, if for every *open* subgroup $G' \subset G$, the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{gp}}(G', \mathbb{F}_p)$ of continuous group homomorphisms is finite.

Theorem 4 (Mazur)**.** Assume that G satisfies condition Φ_p.

[7]This means that the map

$$G \times V \to V \quad (g, v) \mapsto gv$$

is continuous; or equivalently, $G \to \mathrm{GL}(V)$ is continuous.

(a) $D_V^\square$ is representable by an $R_V^\square \in \widehat{\mathfrak{Ar}_\mathcal{O}}$.

(b) If Schur's lemma $\mathrm{End}_{k[G]}(V) = k$ is true, then D_V is representable by an $R_V \in \widehat{\mathfrak{Ar}_\mathcal{O}}$.

By universality, R and R_V are unique up to a unique isomophism in $\widehat{\mathfrak{Ar}_\mathcal{O}}$.

Remark. We explain how some conditions are applied.

- We may extend the deformation functors (to the pro-category of $\mathfrak{Ar}_\mathcal{O}$?) by dropping the condition of being Noetherian[8], and they are still representable by $R_V^\square$ and R_V.
- If V is *absolutely irreducible*, then $\mathrm{End}_{k[G]}(V) = k$. In this simpler setting, one can construct R_V as a subring of $R_V^\square$ directly.
- Without the condition Φ_p, the ring $R_V^\square$ and R_V exist (in the pro-category of $\mathfrak{Ar}_\mathcal{O}$) if we don't require them to be Noetherian. They are Noetherian if and only if $\dim_k H^1(G, \mathrm{ad}\, V) < +\infty$, and the latter condition is implied by G satisfying Φ_p.

2.3.1 Construction of $R_V^\square$

We are looking for a universal representation $\rho^\square : G \to \mathrm{GL}_d\left(R_V^\square\right)$, in the sense that for any lift $\rho_A : G \to \mathrm{GL}_d(A)$ of $\bar\rho$ with $A \in \widehat{\mathfrak{Ar}_\mathcal{O}}$, there is a unique morphism $R_V^\square \to A$ s.t. $G \xrightarrow{\rho^\square} \mathrm{GL}_d\left(R_V^\square\right) \to \mathrm{GL}_d(A)$ equals ρ_A.

Suppose first that G is finite with presentation given by s generators and t relations:

$$G = \langle g_1, \dots, g_s \mid r_1(g_1, \dots, g_s), \dots, r_t(g_1, \dots, g_s) \rangle \, .$$

Let

$$\mathcal{R} := \mathcal{O}\left[\left\{X_{ij}^k\right\}_{1\le i,j\le d}^{1\le k\le s}\right] \Big/ \mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by all *entries* of the matrices

$$r_l(X^1, \dots, X^k) - \mathrm{id}, \quad X^k = \left(X_{ij}^k\right)_{i,j}, \ 1 \le k \le s, \ 1 \le l \le t.$$

One sees immdediately that:

Lemma 2.1. The ring $\mathcal{R}$ is Noetherian. For every $A \in \widehat{\mathfrak{Ar}_\mathcal{O}}$, there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{R}, A) \longleftrightarrow \mathrm{Hom}_{\mathrm{gp}}(G, \mathrm{GL}_d(A))$$

given by $\left(X_{ij}^k \mapsto a_{ij}^k\right) \longmapsto \Big(g_k \mapsto \left(a_{ij}^k\right)_{1\le i,j\le d}\Big)$. □

The ring $\mathcal{R}$ carries no topology. Consider the kernel $\mathcal{J}$ of the homomorphism

$$\mathcal{R} \to k \quad X_{ij}^k \mapsto \text{the } (i,j)\text{-entry of } \bar\rho(g_k),$$

namely the one corresponding to $\bar\rho$. We define $R_V^\square := \varprojlim_n \mathcal{R}/\mathcal{J}^n$ to be the $\mathcal{J}$-adic completion of $\mathcal{R}$, and define $\rho^\square : G \to \mathrm{GL}_d(R_V^\square)$ by $\rho^\square(g_k) := X^k$.

Lemma 2.2. Let G be a finite group. The ring $R_V^\square$ constructed above is a complete local $\mathcal{O}$-algebra, and $\rho^\square$ is a well-defined framed deformation that is universal.

Proof. We verify that $\rho^\square$ is a (continuous) lift of $\bar\rho$ in the following steps.

[8] That is, define $D_V^\square$ and D_V on the category of complete local $\mathcal{O}$-algebras with residue field k. As before, "complete" means to be a projective limit of Artinian algebras.

- Every $\mathcal{R}/\mathcal{J}^n$ is local of dimension 0, because every prime ideal contains its nilradical $\mathcal{J}/\mathcal{J}^n$, which is maximal, and thus it can only be $\mathcal{J}/\mathcal{J}^n$. In particular, $\mathcal{R}/\mathcal{J}^n$ is Artinian.
- $R_V^\square$ is local with maximal ideal $\mathfrak{m}_{R_V^\square} = \ker(R_V^\square \to \mathcal{R}/\mathcal{J})$ and residue field k, because for any $x \in R_V^\square \setminus \mathfrak{m}_{R_V^\square}$, we can deduce inductively that the images x_n of x under $R_V^\square \to \mathcal{R}/\mathcal{J}^n$ is non-nilpotent and, by the previous step, invertible, yielding a series $\left(x_n^{-1}\right)_n$ whose compatibility is easy to check.
- $\rho^\square$ is well-defined (i.e. all matrices in $\rho^\square(G)$ are invertible) and lifts $\bar{\rho}$, because

$$\begin{tikzcd} & \mathrm{Mat}_d\left(R_V^\square\right) \arrow[r, "\det"] \arrow[d] & R_V^\square \arrow[d] \\ G \arrow[ur, "\rho^\square"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_d(k) \arrow[r, "\det"] & k \end{tikzcd}$$

  commutes.

For universality, take a continuous lift $\rho : G \to \mathrm{GL}_d(A)$ of $\bar{\rho}$ with

$$A \simeq \varprojlim_{\mathfrak{a}} A/\mathfrak{a}$$

where $\mathfrak{a} \subset A$ are open ideals and $A/\mathfrak{a}$ are Artinian. Let $f : \mathcal{R} \to A$ be the corresponding $\mathcal{O}$-homomorphism obtained from Lemma 2.1. Since ρ reduces to $\bar{\rho}$, we have $f(\mathcal{J}) \subset \mathfrak{m}_A$, and $f(\mathcal{J}^n) \subset \mathfrak{m}_A^n$ for all $n \geq 1$. For any $\mathfrak{a}$, since $A/\mathfrak{a}$ is Artinian, the chain $\mathfrak{m}_A/\mathfrak{a} \supset (\mathfrak{m}_A/\mathfrak{a})^2 \supset \cdots$ terminates. Hence there is some $n \geq 1$ such that $\mathfrak{m}_A^n \subset \mathfrak{a}$, i.e. the composition $R \xrightarrow{f} A \to A/\mathfrak{a}$ is continuous w.r.t. the $\mathcal{J}$-adic topology on $\mathcal{R}$. Therefore, the map f extends uniquely to a continuous homomorphism

$$\hat{f} : R_V^\square \to A,$$

such that $f = \hat{f} \circ (\mathcal{R} \to R_V^\square)$. Again by Lemma 2.1, the representation $\rho = \mathrm{GL}_d(\hat{f}) \circ \rho^\square : G \to \mathrm{GL}_d(R_V^\square) \to \mathrm{GL}_d(A)$. $\square$

In the general case of G being profinite, we write $G = \varprojlim_i G/H_i$ with $H_i \subset \ker \bar{\rho}$ open and normal in G and consider the universal lifts (R_i, ρ_i) of the representations $G/H_i \to k$ from $\bar{\rho}$. For $G/H_i \to G/H_j$, the universality of ρ_i provides the dotted arrow in the commutative diagram

$$\begin{tikzcd} G/H_i \arrow[r, "\rho_i"] \arrow[d] & \mathrm{GL}_d(R_i) \arrow[d, dashed] \\ G/H_j \arrow[r, "\rho_j"] & \mathrm{GL}_d(R_j) \end{tikzcd}$$

Therefore we obtain $(R_V^\square, \rho^\square) := \varprojlim_i (R_i, \rho_i)$.

Lemma 2.3. $R_V^\square$ is a complete local $\mathcal{O}$-algebra, and $\rho^\square$ is the universal framed deformation of V.

Proof. The ring $R_V^\square$ is a projective limit of complete local $\mathcal{O}$-algebras, so is it.

By definition, $D_V^\square = \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{cont}}(G, \mathrm{GL}_d(-))$. So for $A = \varprojlim_i A_i$ with A_i being Artinian quotients,

$$\begin{aligned} D_V^\square(A) &= \varprojlim_i \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{cont}}(G, \mathrm{GL}_d(A_i)) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{cont}}(G/H_j, \mathrm{GL}_d(A_i)) \\ &= \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{O}\text{-alg}}^{\mathrm{cont}}(R_j, A_i) = \varprojlim_i \mathrm{Hom}_{\mathcal{O}\text{-alg}}^{\mathrm{cont}}(R_V^\square, A_i) \\ &= \mathrm{Hom}_{\mathcal{O}\text{-alg}}^{\mathrm{cont}}(R_V^\square, A).\text{[9]} \end{aligned}$$

$\square$

[9] Might be some set-theoretic problems here...?

2.3.2 Construction of R_V for absolutely irreducible V

Assume that V is absolutely irreducible. Let R_V be the smallest closed sub-$\mathcal{O}$-algebra of $R_V^\square$ that contains $\operatorname{Tr}\rho^\square(g)$ for all $g \in G$. We prove that D_V is representable by R_V assuming the following proposition.

Proposition 2.2. Let $A \in \widehat{\mathfrak{Ar}_\mathcal{O}}$, W be an $A[G]$-module that is free of finite rank over A, $A' \subset A$ be a subring such that $A' \in \widehat{\mathfrak{Ar}_\mathcal{O}}$ with topology induced from A. If A' contains the traces of all the G-action on W, i.e.

$$\operatorname{Tr}(g|_W) \in A', \quad \forall g \in G,$$

and $W \otimes_A k$ is absolutely irreducible, then there is an $A'[G]$-module that is free of finite rank over A', such that $W' \otimes_{A'} A \simeq W$ as $A[G]$-modules.

Let $(V^\square, \iota^\square, \beta^\square)$ be a framed deformation given by $\rho^\square : G \to \mathrm{GL}_d(R_V^\square)$. By the previous Proposition 2.2, there is an $R_V[G]$-module $\tilde{V}$ such with two isomophisms $\tilde{V} \otimes_{R_V} R_V^\square \simeq V^\square$ and

$$\iota : \tilde{V} \otimes_{R_V} k \simeq \tilde{V} \otimes_{R_V} R_V^\square \otimes_{R_V^\square} k \simeq V^\square \otimes_{R_V^\square} k \overset{\iota^\square}{\simeq} V$$

Lemma 2.4. $(\tilde{V}, \iota)$ is a universal deformation of the absolutely irreducible $k[G]$-module V.

Proof. Let (V_A, ι_A) be a deformation of V to $A \in \widehat{\mathfrak{Ar}_\mathcal{O}}$, and (V_A, ι_A, β_A) be a framed deformation of V. By the universality of $V^\square$ and Proposition 2.2, there is a unique map $R_V \hookrightarrow R_V^\square \to A$,

$\square$

2.3.3 The tangent space

Let $k[\varepsilon] := k[X]/(X^2)$, which is called the ring of **dual numbers**. For a functor $D : \widehat{\mathfrak{Ar}_\mathcal{O}} \to \mathbf{Set}$ sending the terminal object k to the terminal object $D(k) = \{\bullet\}$, we call the set $t_D := D(k[\varepsilon])$[10] the **Zariski tangent space** of D. If there is a fixed bijection $D(k[\varepsilon] \oplus k[\varepsilon]) \simeq D(k[\varepsilon]) \times D(k[\varepsilon])$, we equip t_D with the k-linear structure given by this bijection.

- Assume that $D : \widehat{\mathfrak{Ar}_\mathcal{O}} \to \mathbf{Set}$ is representable by $R \in \widehat{\mathfrak{Ar}_\mathcal{O}}$. Then the tangent space

$$t_D \simeq \operatorname{Hom}_\mathcal{O}(R, k[\varepsilon]) \simeq \operatorname{Hom}_k(\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\mathcal{O}), k[\varepsilon]) = t_R$$

  is the Zariski or relative tangent space of R over $\mathcal{O}$. (what is the last isomorphism (if there is one...)?)

Define $\operatorname{ad} V := \operatorname{End}_k(V) \simeq V^\vee \otimes_k V$ with the standard G-module structure $\operatorname{ad}\bar{\rho} = \bar{\rho}^\vee \otimes \bar{\rho}$.

Lemma 2.5. There are canonical isomorphisms[11]

$$D_V(k[\varepsilon]) \simeq \operatorname{Ext}^1_{k[G]}(V, V) \simeq H^1(G, \operatorname{ad} V).$$

Proof. (1) Given an extension

$$0 \longrightarrow V \xrightarrow{i} W \xrightarrow{\pi} V \longrightarrow 0$$

of $k[G]$-modules, we define the $k[G]$-linear action of ε on W by $\varepsilon|_W := i \circ \pi$, which endows W with an $k[\varepsilon][G]$-module structure and an isomorphism

$$W \otimes_{k[\varepsilon]} k = W/\varepsilon W = W/i(V) \overset{\pi}{\simeq} V.$$

[10] $D(k)$ is a singleton, so $t_D =$ "$\ker(D(k[\varepsilon]) \to D(k))$".

[11] In Ext^1, we consider *continuous* extension classes.

Conversely, for a deformation (W, ι) of V to $k[\varepsilon]$, we get an extension of V by itself

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, "\iota"'] & W \arrow[r] \arrow[d, equal] \arrow[dr, two heads] & V \arrow[r] & 0 \\
& W \otimes_{k[\varepsilon]} k \arrow[r, hook] & W \otimes_{k[\varepsilon]} k[\varepsilon] & W/\varepsilon W \simeq W \otimes_{k[\varepsilon]} k \arrow[u, "\iota"'] &
\end{tikzcd}$$

as $k[G]$-modules.[12] The first isomophism is thereby established.

(2) The second isomophism is a general fact that we have extracted as Example 1.1. ☐

We use the abbreviation $h^i(\cdots) := \dim_k H^i(\cdots)$.

Lemma 2.6. If G satisfies condition Φ_p, then $D_V(k[\varepsilon])$ is a finite dimensional k-vector space, and

$$\dim_k D_V^{\square}(k[\varepsilon]) = \dim_k D_V(k[\varepsilon]) + d^2 - h^0(G, \operatorname{ad} V)$$

is also finite.

Proof. Let $G' := \ker(G \to \mathrm{GL}(V))$. Since G acts continuously, G' is an open normal subgroup of G. Consider the inflation-restriction exact sequence

$$0 \to H^1(G/G', \operatorname{ad} V) \to H^1(G, \operatorname{ad} V) \to H^1(G', \operatorname{ad} V)^{G/G'}.$$

The left term is obviously finite. For the right term, G' acts trivially, so[13]

$$H^1(G', \operatorname{ad} V) = \operatorname{Hom}_{\mathrm{gp}}(G', \operatorname{ad} V) \simeq \operatorname{Hom}_{\mathrm{gp}}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \operatorname{ad} V$$

is finite by condition Φ_p. Therefore $\dim_k D_V(k[\varepsilon]) = h^1(G, \operatorname{ad} V) < \infty$.

(Do the equation later.) ☐

Lemma 2.7. Let A be a complete local $\mathcal{O}$-algebra with residue field k. If $\{\alpha_i\}_{i \in I} \subset \mathfrak{m}_A$ generates the relative cotangent space $t_A^\vee = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_{\mathcal{O}})$ of A over $\mathcal{O}$ as an $\mathcal{O}$-module, then the homomorphism

$$\mathcal{O}[\![X_i \mid i \in I]\!] \to A \quad X_i \mapsto \alpha_i$$

is surjective.

Proof. Cannot use **??** because Noetherianity of A is the goal! ☐

Corollary 2.1. The ring $R_V^{\square}$ is Noetherian if $H^1(G, \operatorname{ad} V)$ is k-finite-dimensional.

Proof. Combine the lemmata above. ☐

This completes the proof of Theorem 4 (a).

2.3.4 Quotient by group action and the representability of D_V

Result is $\operatorname{Spf} R_V = \operatorname{Spf} R_V^{\square} \Big/ \widehat{\mathrm{PGL}_d}$.

[12] The fact $W \simeq V \oplus V$ as $k[G]$-modules doesn't mean that the extension split.

[13] We have

$$\operatorname{Hom}_{\mathrm{gp}}(G, V) \simeq \operatorname{Hom}_{\mathrm{gp}}(G, k) \otimes_k V$$

for any group G and any *finite* dimensional vector space V over a field k.

2.3.5 Presentation of the universal deformation ring R_V

3 Taylor-Wiles Patching

Keep the notations $\mathcal{O} = \mathcal{O}_L$ for $L/\mathbb{Q}_p$, and let $k = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ and $\varpi \in \mathcal{O}$ be a uniformizer.

Fix a continuous absolutely irreducible modular representation $\rho : \mathrm{Gal}_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(k)$ with determinant $\bar{\varepsilon}^{-1}$.