Homework 3
Due November 5

You can write your answers in English or in French, choose what works best for you.

Exercises 1 to 4 go together, exercise 5 is independent.

Exercice 1. — Let $f \in M_k(\Gamma_1(N), \chi)$ be a normalized Hecke eigenform for $\mathbf{T}_1^0(N)$, for some Dirichlet character χ modulo N. Let $f(z) = \sum_n a_n q^n$ be the q-expansion of f.

Show that for all n prime to N, we have $\overline{a_n} = \overline{\chi(n)} a_n$.

Exercice 2. — Let w_N be the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. If $f \in M_k(\Gamma_1(N))$, we set $w_N f = f|_k w_N$. Recall that $w_N f \in M_k(\Gamma_1(N))$.

1. Show that if $f \in M_k(\Gamma_1(N), \chi)$ then $w_N f \in M_k(\Gamma_1(N), \overline{\chi})$, where $\overline{\chi}$ is the character $a \mapsto \overline{\chi(a)}$.

2. Let $f \in M_k(\Gamma_1(N), \chi)$. Show that if f is an eigenvector for some T_n with n prime to N, with eigenvalue λ, then $w_N f$ is also an eigenvector for T_n, with eigenvalue $\overline{\lambda}$.

Exercice 3. — Let C be the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Let f be a function $\mathcal{H} \to \mathbf{C}$. We denote by $c(f)$ the function $z \mapsto \overline{f(-\overline{z})}$. Show that if $f \in M_k(\Gamma)$ then $c(f) \in M_k(C\Gamma C^{-1})$.

2. Show that C normalizes $\Gamma_1(N)$.

3. Let $f \in M_k(\Gamma_1(N))$ with q-expansion $f(z) = \sum_{n\geq 0} a_n q^n$. Show that $c(f)$ has q-expansion $\sum_{n\geq 0} \overline{a_n} q^n$.

4. Show that if $f \in M_k(\Gamma_1(N), \chi)$ for some Dirichlet character χ modulo N, then $c(f) \in M_k(\Gamma_1(N), \overline{\chi})$.

5. Let $f \in M_k(\Gamma_1(N), \chi)$. Show that if f is an eigenvector for some T_n, with eigenvalue λ, then $c(f)$ is also an eigenvector for T_n, with eigenvalue $\overline{\lambda}$.

6. Show that $c(f)$ is old if f is old, and $c(f)$ is new if f is new.

Exercice 4. — Let $f \in S_k(\Gamma_1(N))^{new}$ be a normalized eigenform.

1. Show that exists some $\eta_f \in \mathbf{C}^\times$ such that $w_N f = \eta_f c(f)$.

2. Prove the identities $\eta_f \eta_{c(f)} = (-N)^k$, $\eta_{c(f)} = (-1)^k \overline{\eta_f}$ and $|\eta_f| = N^{k/2}$.

Exercice 5. — Fix some integer N. Show that the subalgebra of $\mathbf{T}_1(N)$ generated by the T_n, n prime to N, contains all the diamond operators $\langle a \rangle$, $a \in (\mathbf{Z}/N\mathbf{Z})^\times$.

You can make use of the following theorem of Dirichlet on arithmetic progressions: let u, v be coprime positive integers. Then there exists an infinite number of primes that are congruent to u modulo v.