Modular Forms

Lei Bichang

Exercise 1. Let $(n, N) = 1$. Because $n \mapsto a_n$ and $n \mapsto \langle n \rangle$ are multiplicative, it suffices to prove the result for $n = p^e$ a power of a prime $p \nmid N$. We do induction on e.

If $n = p$, then since $T_p^* = \langle p^{-1} \rangle T_p$, we have $T_p^* f = \chi(p^{-1}) a_p f = \overline{\chi}(p) a_p f$, and thus

$$\overline{a_p} \langle f, f \rangle = \langle f, a_p f \rangle = \langle f, T_p f \rangle = \langle T_p^* f, f \rangle = \left\langle \overline{\chi(p)} a_p f, f \right\rangle = \overline{\chi(p)} a_p \langle f, f \rangle .$$

As $f \neq 0$, $\overline{a_p} = \overline{\chi(p)} a_p$.

Next, assume the result holds for $n = p^r$ with $1 \leq r \leq e$. For $n = p^{e+1}$,

$$\begin{aligned} \overline{a_{p^{e+1}}} &= \overline{a_p a_{p^e} - p^{k-1} \chi(p) a_{p^{e-1}}} \\ &= \overline{\chi(p)} a_p \overline{\chi(p^e)} a_{p^e} - p^{k-1} \overline{\chi(p) \chi(p^{e-1})} a_{p^{e-1}} \\ &= \overline{\chi(p^{e+1})} \left(a_p a_{p^e} - p^{k-1} \chi(p) a_{p^{e-1}} \right) = \overline{\chi(p^{e+1})} a_{p^{e+1}} . \end{aligned}$$

Exercise 2.

1. Let $(d, N) = 1$ and take

$$\gamma_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then

$$\tilde{\gamma} := w_N \gamma_d w_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix} .$$

Since $1 = ad - bc \equiv ad \pmod N$, the matrix $\tilde{\gamma} \in \Gamma_0(N)$ is a lift of $a \bmod N = (d \bmod N)^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\times$. Therefore,

$$\begin{aligned} \langle d \rangle (w_N f) &= f|_k (w_N \gamma_d) = f|_k (\tilde{\gamma} w_N) = w_N \left(\langle d^{-1} \rangle f \right) \\ &= w_N (\chi(d^{-1}) f) = w_N (\overline{\chi(d)} f) = \overline{\chi(d)} (w_N f). \end{aligned}$$

2. Assume that $T_n f = \lambda f$ for a specific n prime to N.

If $f = 0$, the statement is trivial. Otherwise, $a_1(f) \neq 0$. Without loss of generality, we may assume $a_1(f) = 1$, so $a_n(f) = \lambda$. By Exercise 1,

$$\bar{\lambda} = \overline{a_n(f)} = \overline{\chi(n)} a_n(f) = \overline{\chi(n)} \lambda.$$

Since $w_N f \in M_1(\Gamma_1(N), \bar{\chi})$,

$$\bar{\lambda} w_N f = \lambda \overline{\chi(n)} w_N f = \lambda \langle n \rangle w_N f = \langle n \rangle w_N (\lambda f) = \langle n \rangle w_N T_n f.$$

So we need to show that $\langle n \rangle w_N T_n f = T_n w_N f$.

Lemma 1. If $(n, N) = 1$, then $T_n^* = \langle n \rangle^{-1} T_n$.

Proof. It suffices to prove this for $n = p^e$ a prime power with $p \nmid N$. We do induction on e.

The case of $n = p$ is already known. Suppose the lemma holds for $n = p^r$ with $1 \leq r \leq e$. For $n = p^{e+1}$,

$$\begin{aligned} T_{p^{e+1}}^* &= \left(T_p T_{p^e} - p^{k-1} \langle p \rangle T_{p^{e-1}} \right)^* \\ &= T_{p^e}^* T_p^* - p^{k-1} T_{p^{e-1}}^* \overline{\langle p \rangle} \\ &= \langle p^e \rangle^{-1} T_{p^e} \langle p \rangle^{-1} T_p - p^{k-1} \left\langle p^{e-1} \right\rangle^{-1} T_{p^{e-1}} \langle p \rangle^{-1} \\ &= \left\langle p^{e+1} \right\rangle^{-1} \left(T_{p^e} T_p - p^{k-1} \langle p \rangle T_{p^{e-1}} \right) = \left\langle p^{e+1} \right\rangle^{-1} T_{p^{e+1}} . \end{aligned}$$

□

Therefore, $w_N T_n w_N^{-1} = T_n^* = \langle n \rangle^{-1} T_n$, and thus

$$\bar{\lambda} w_N f = \langle n \rangle w_N T_n f = T_n W_N f.$$

Exercise 3. 1. Let $f \in M_k(\Gamma)$.

- As f is holomorphic, $c(f) : z \mapsto \overline{f(-\bar{z})}$ is also holomorphic.
- If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then $C\gamma C^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. So for all $z \in \mathcal{H}$,

$$\begin{aligned} c(f)|_k(C\gamma C^{-1})(z) &= (-cz+d)^{-k} c(f)\left(\frac{az-b}{-cz+d}\right) \\ &= (-cz+d)^{-k}\overline{f\left(\frac{a\bar{z}-b}{c\bar{z}-d}\right)} \\ &= \overline{(-c\bar{z}+d)^{-k} f\left(\frac{-a\bar{z}+b}{-c\bar{z}+d}\right)} \\ &= \overline{f|_k\gamma(-\bar{z})}. \end{aligned}$$

  For $\gamma \in \Gamma$, we obtain

$$c(f)|_k(C\gamma C^{-1})(z) = \overline{f(-\bar{z})} = c(f)(z).$$

- Consider a cusp $g\infty$ of $C\Gamma C^{-1}$, where $g \in \mathrm{SL}_2(\mathbb{Z})$. Let $f|_k g(z) = \sum_{n \geq 0} a_n q_N^n$ be the q-expansion of $f|_k g$. Then by the computation in **1.** and $\overline{\mathrm{e}^s} = \mathrm{e}^{\bar{s}}$ for all $s \in \mathbb{C}$,

$$c(f)|_k(CgC^{-1})(z) = \overline{f|_k g(-\bar{z})} = \overline{\sum_{n\geq 0} a_n \mathrm{e}^{-\frac{2\pi i}{N}\bar{z}}} = \sum_{n \geq 0} \overline{a_n}\mathrm{e}^{\frac{2\pi i}{N}z} = \sum_{n\geq 0} \overline{a_n} q_N^n, \tag{1}$$

  which gives a q-expansion of $c(f)|_k(CgC^{-1})$. As $f|_k g$ is bounded at the cusp ∞, so is $c(f)|_k(CgC^{-1})$. Now CgC^{-1} permutes all elements of $\mathrm{SL}_2(\mathbb{Z})$ as g goes through $\mathrm{SL}_2(\mathbb{Z})$, so we see that $c(f)$ is bounded at every cusps.

In conclusion, $c(f) \in M_k(C\Gamma C^{-1})$.

2. As we have computed,

$$C\begin{pmatrix} a & b \\ c & d \end{pmatrix} C^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

   So $C\Gamma_1(N)C^{-1} = \Gamma_1(N)$.

3. Set $g = 1$ in Eq. (1).

4. Let $f \in M_k(\Gamma_1(N), \chi)$. If $n \in (\mathbb{Z}/N\mathbb{Z})^\times$ and γ_n is a lift of n in $\Gamma_0(N)$, then the computation in Exercise 3.2 shows that $C\gamma_n C^{-1}$ is also a lift of n. Hence

$$\begin{aligned} (\langle n \rangle c(f))(z) &= c(f)|_k(C\gamma_n C^{-1})(z) \\ &= \overline{f|_k\gamma_n(-\bar{z})} \\ &= \overline{(\langle n \rangle f)(-\bar{z})} \\ &= \overline{\chi(n) f(-\bar{z})} = \overline{\chi(n)} c(f)(z). \end{aligned}$$

   This means $c(f) \in M_k(\Gamma_1(N), \bar{\chi})$.

5. Assume that $T_n f = \lambda f$. By the formula of T_n action on q-expansion and Exercise 3.3,

$$\begin{aligned}
a_m(T_n c(f)) &= \sum_{d|(m,n)} \bar{\chi}(d) d^{k-1} a_{mn/d^2}(c(f)) \\
&= \sum_{d|(m,n)} \bar{\chi}(d) d^{k-1} \overline{a_{mn/d^2}(f)} \\
&= \overline{\sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f)} \\
&= \overline{a_m(T_n f)} = \overline{\lambda a_m(f)} = \bar{\lambda} a_m(c(f)).
\end{aligned}$$

Hence $c(f)$ is is an eigenvector for T_n with eigenvalue $\bar{\lambda}$.

6. We first show that, f being old $\implies$ $c(f)$ being old. This can be deduced via computation. Let $M \mid N$, $d \mid \frac{N}{M}$, and $h \in S_k(\Gamma_1(M))$. Then

$$\begin{aligned}
i_d(c(h))(z) &= d^{1-k} \left(c(h) \Big|_k \begin{pmatrix} d & \\ & 1 \end{pmatrix} \right)(z) \\
&= d^{1-k} \overline{\left(h \Big|_k C^{-1} \begin{pmatrix} d & \\ & 1 \end{pmatrix} C \right)(-\bar{z})} \\
&= d^{1-k} \overline{\left(h \Big|_k \begin{pmatrix} d & \\ & 1 \end{pmatrix} \right)(-\bar{z})} \\
&= \overline{i_d(h)(-\bar{z})} = c(i_d(h))(z).
\end{aligned}$$

Every form $f \in S_k(\Gamma_1(N))^{\mathrm{old}}$ is a finite sum of elements in the form $i_{d,M,N}(h)$, and note that $c(f_1+f_2) = c(f_1) + c(f_2)$, we can conclude that $c(f)$ is also old.

To prove that f being new $\implies$ $c(f)$ being new, we use the following result.

Lemma 2. $\langle c(f), c(g) \rangle = \langle g, f \rangle, \quad \forall f, g \in S_k(\Gamma_1(N))$.

Proof. Let D be a fundamental domain of $\Gamma_1(N)$. Let $f, g \in S_k(\Gamma_1(N))$, then

$$\langle c(f), c(g) \rangle = \frac{1}{\mathrm{vol}(\Gamma_1(N)\backslash\mathcal{H})} \int_D \overline{f(-\bar{z})} g(-\bar{z}) \,\mathrm{Im}(z)^k d\mu(z),$$

where $d\mu(z) = y^{-2} dx dy$. Under the change of variable $t := -\bar{z} = -x + iy$, D is converted to $D' = \{t \in \mathcal{H} \mid -\bar{t} \in D\}$ Put $\tau(z) := -\bar{z}$ so that $D' = \tau(D)$.

We know that $\mathrm{SL}_2(\mathbb{Z})$ has a fundamental domain D_0 that is mirror-symmetric along the y-axis, i.e.,

$$D_0 = \{-\bar{z} \mid z \in D_0\}.$$

Write $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_g g\Gamma_1(N)$ so that $D = \bigcup_g gD_0$ for finitely many $g \in \mathrm{SL}_2(\mathbb{Z})$. Then since

$$-\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix} z} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} (-\bar{z}) = C \begin{pmatrix} a & b \\ c & d \end{pmatrix} C^{-1}(-\bar{z}),$$

we find that

$$\tau(gD_0) = \{-\overline{gz} \mid z \in D_0\} = \left\{(CgC^{-1})(-\bar{z}) \mid z \in D_0\right\} = CgC^{-1}D_0.$$

Hence

$$D' = \bigcup_g \tau(gD_0) = \bigcup_g CgC^{-1}D_0.$$

By Exercise 4.2,

$$\mathrm{SL}_2(\mathbb{Z}) = C\,\mathrm{SL}_2(\mathbb{Z})C^{-1} = \bigsqcup_g Cg\Gamma_1(N)C^{-1} = \bigsqcup_g CgC\Gamma_1(N).$$

As $C = C^{-1}$, the above shows that D' is also a fundamental domain for $\Gamma_1(N)$.

Therefore, the integral becomes

$$\frac{1}{\operatorname{vol}(\Gamma_1(N)\backslash\mathcal{H})}\int_{D'} \overline{f(t)}g(t)\operatorname{Im}(t)^k d\mu(t) = \langle g, f\rangle . \qquad \square$$

Note that $c \circ c = \mathrm{id}$. Therefore, if f is new and g is old, then

$$\langle c(f), g\rangle = \langle c(g), f\rangle = 0$$

because $c(g)$ is also old. This implies that $c(f)$ is new.

Exercise 4. 1. Because f is a primitive form, Exercise 3 shows that $c(f)$ is also a primitive form, and Exercise 2 shows that $w_N f$ is an eigenform for $\mathbb{T}_1^{\circ}(N) = \mathbb{T}_1^{(N)}(N)$. Moreover, $c(f)$ and $w_N f$ have the same eigenvalues for $T \in \mathbb{T}_1^{\circ}(N)$.

By the weak multiplicity one theorem, once we verify that $w_N f$ is new, we shall see that $w_N f$ is a nonzero multiple of $c(f)$. Note that

$$w_N^2 f = (-1)^k N^{k-2} f,$$

so we use a strategy similar to Exercise 3.6.

Lemma 3. If $f \in S_k(\Gamma_1(N))$ is old, then $w_N(f)$ is old.

Proof. It suffices to show that w_N stabilises $S_k(\Gamma_1(N))^{p\text{-old}}$ for every $p \mid N$.

Let $h \in S_k(\Gamma_1(N/p))$. Then

$$\begin{aligned}
w_N(i_1 h) &= h\Big|_k \begin{pmatrix} & -1 \\ N & \end{pmatrix} \\
&= f\Big|_k \begin{pmatrix} & -1 \\ N/p & \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix} \\
&= p^{k-1} i_p(w_{N/p} h) \in i_p S_k(\Gamma_1(N)), \\
w_N(i_p h) &= p^{1-k} h\Big|_k \begin{pmatrix} p & \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ N & \end{pmatrix} = p^{1-k} h\Big|_k \begin{pmatrix} & -p \\ N & \end{pmatrix} \\
&= p^{1-k} h\Big|_k \begin{pmatrix} & -1 \\ N/p & \end{pmatrix}\begin{pmatrix} p & \\ & p \end{pmatrix} \\
&= p^{1-k} w_{N/p} h\Big|_k \begin{pmatrix} p & \\ & p \end{pmatrix} = p^{1-k}(p^2)^{k-1}p^{-k} w_{N/p} f \\
&= p^{-1} i_1(w_{N/p} h) \in i_1 S_k(\Gamma_1(N/p)).
\end{aligned}$$

We thus proved that $S_k(\Gamma_1(N))^{p\text{-old}} = i_1(S_k(\Gamma_1(N/p))) + i_p(S_k(\Gamma_1(N/p)))$ is stable under w_N. $\square$

Since $W_N = i^k N^{1-\frac{k}{2}} w_N$ is self-adjoint, we have $w_N^* = (-1)^k w_N$, and thus

$$\langle w_N f, g\rangle = \left\langle f, (-1)^k w_N g\right\rangle = 0$$

because f is new and $(-1)^k w_N g$ is old by Lemma 3. Hence $w_N(f)$ is new, and applying the weak multiplicity one theorem completes the proof.

2. By definition,

$$w_N^2 f = w_N(\eta_f c(f)) = \eta_f(w_N c(f)) = \eta_f \eta_{c(f)} c(c(f)) = \eta_f \eta_{c(f)} f.$$

As $w_N^2 f = (-1)^k N^{k-2} f = (-N)^{k-2} f$ and $f \neq 0$, we get $\eta_f \eta_{c(f)} = (-N)^{k-2}$.

We have seen that $w_N^* = (-1)^k w_N$, so

$$\begin{aligned} \eta_{c(f)} \langle f, f \rangle &= \left\langle \eta_{c(f)} f, f \right\rangle = \langle w_N c(f), f \rangle \\ &= \left\langle c(f), (-1)^k w_N f \right\rangle = \left\langle c(f), (-1)^k \eta_f c(f) \right\rangle = (-1)^k \overline{\eta_f} \langle c(f), c(f) \rangle . \end{aligned}$$

By Lemma 2, $\langle f, f \rangle = \langle c(f), c(f) \rangle \neq 0$, which implies $\eta_{c(f)} = (-1)^k \overline{\eta_f}$.

Since $|\eta_f|^2 = |\eta_f \eta_{c(f)}| = N^{k-2}$, we have $|\eta_f| = N^{k/2-1}$.

Exercise 5. Since $\langle \cdot \rangle$ is multiplicative, it suffices to show that every $\langle p \rangle$, in which $p \nmid N$ is a prime, can be generated by the T_n's with n prime to N.

For $p \nmid N$, we have

$$p^{k-1} \langle p \rangle = T_p^2 - T_{p^2}.$$

By Dirichlet's theorem on arithmetic progression, $\{p + Nk \mid k \in \mathbb{Z}_{\geq 1}\}$ contains infinitely many primes. In particular, there exists a prime $q \neq p$ s.t $q \equiv p \pmod N$, and hence

$$q^{k-1} \langle p \rangle = q^{k-1} \langle q \rangle = T_q^2 - T_{q^2}.$$

Since $(p^{k-1}, q^{k-1}) = 1$, there exists $u, v \in \mathbb{Z}$ s.t. $1 = up^{k-1} + vq^{k-1}$, which yields

$$\langle p \rangle = up^{k-1} \langle p \rangle + vq^{k-1} \langle p \rangle = u(T_p^2 - T_{p^2}) + v(T_q^2 - T_{q^2}).$$