1 Local Units

(goal: $\mathcal{U}_\infty$?)

Let $\mathcal{K}_n := \mathbb{Q}_p(\mu_{p^{n+1}})$, $\mathcal{U}_n := \mathcal{O}_{\mathcal{K}_n}^\times = \mathcal{O}_{\mathbb{Q}_p(\mu_{p^{n+1}})}^\times$.

$\mathcal{O}_{\mathcal{K}_n} = \mathbb{Z}_p[\zeta_n] = \mathbb{Z}_p[\pi_n]$.

Fix $\zeta_n \in \mu_{p^{n+1}}$ a primitive p^{n+1}-th root of unity, and $\pi_n := \zeta_n - 1$ a uniformiser of $\mathcal{K}_n$.

For $n \geq m$, let $N_{n,m} := N_{\mathcal{K}_n/\mathcal{K}_m} : \mathcal{K}_n \to \mathcal{K}_m$ be the norm map. Define the $\mathcal{U}_\infty := \varprojlim_n \mathcal{U}_n$, where the inverse limit is taken w.r.t.(right words?) the norm maps. (Question: does $\mathcal{U}_\infty$ relate to the completion of $\mathcal{O}_{\mathbb{Q}_p(\mu_p^\infty)}^\times$? Like $\widehat{\mathcal{K}_\infty} = \varprojlim \mathcal{K}_n$ w.r.t. (normalized) trace.)

Let $\mathcal{G} := \mathrm{Gal}(\mathcal{K}_\infty|\mathbb{Q}_p)$, and $\chi : \mathcal{G} \to \mathbb{Z}_p^\times$ the cyclotomic character.

1.1 The actions on $\mathbb{Z}_p[\![T]\!]$

Let R be the ring $\mathbb{Z}_p[\![T]\!] = \varprojlim_n \mathbb{Z}_p[T]/T^n$ equipped with the $\mathfrak{m} := (p, T)$-adic topology. Coleman introduced several operators on this complete ring R, which have been generalized in Fontaine's theory of (φ, Γ)-modules. (These operators seems to have connection with the formal group $\mathbb{G}_\mathrm{m}$, hence it should be generalized directly to Lubin-Tate extensions?)

1.1.1 The operator φ

Recall that the formal group $\mathbb{G}_\mathrm{m}$ over $\mathbb{Z}_p$ is a formal $\mathbb{Z}_p$-module, with $\mathbb{Z}_p \to \mathrm{End}(\mathbb{G}_\mathrm{m})$ given by

$$[a](T) = (1+T)^a - 1, \quad a \in \mathbb{Z}_p.$$

For $f(T) \in R$, define

$$\varphi(f(T)) := f([p](T)) = f((1+T)^p - 1).$$

- $\varphi : R \to R$ is an injective $\mathbb{Z}_p$-algebra homomorphism.
- $\varphi(R^\times) \subset R^\times$.

1.1.2 The norm and trace

Consider $f \in R$ and $\xi \in \mu_p$. Then $f(\xi(T+1)-1) \in \mathcal{O}[\![T]\!]$, where $\mathcal{O} = \mathcal{O}_{\mathcal{K}_0} = \mathbb{Z}_p[\mu_p]$ because $\xi - 1 \in \mathcal{O}$ has strictly positive valuation.

Lemma 1.1. For $f \in R$, both $\prod_{\xi \in \mu_p} f(\xi(1+T)-1)$ and $\sum_{\xi \in \mu_p} f(\xi(1+T)-1)$ are in R.

Proof. (Not quite sure!) The group $\mathrm{Gal}(\mathcal{K}_0/\mathbb{Q}_p)$ acts on $\mathcal{O}[\![T]\!]$ by acting on the coefficients, and $R = \mathcal{O}[\![T]\!]^{\mathrm{Gal}(\mathcal{K}_0/\mathbb{Q}_p)}$. The lemma holds since for each $\sigma \in \mathrm{Gal}(\mathcal{K}_0/\mathbb{Q}_p)$, $\sigma(\xi)$ permutes μ_p as ξ runs over μ_p. $\square$

Proposition 1.1. There eixsts unique continuous maps $\mathcal{N}, \psi : R \to R$ s.t. for $f(T) \in R$,

$$(\varphi \circ \mathcal{N})(f(T)) = \prod_{\xi \in \mu_p} f(\xi(1+T)-1),$$
$$(\varphi \circ \psi)(f(T)) = \frac{1}{p} \sum_{\xi \in \mu_p} f(\xi(1+T)-1).$$

These maps verifies the following properties.

- $\mathcal{N}$ is multiplicative, and $\mathcal{N}(R^\times) \subset R^\times$;

- ψ is $\mathbb{Z}_p$-linear;
- $\psi \circ \varphi = \mathrm{id}_R$.

Since φ is injective, the maps $\mathcal{N}$ and ψ are unique if exists, and showing that RHSs in the above equations are in $\varphi(R)$ would prove the existence.

Lemma 1.2. $\varphi(R) = \{h \in R \mid h(\xi(1+T)-1) = h(T), \forall \xi \in \mu_p\}$.

Proof. "$\subset$" is obvious. For "$\supset$", take $h \in$ RHS, and we are to show that $h \in \varphi(R) = \mathbb{Z}_p[\![\varphi(T)]\!]$. By definition, $\xi - 1$ is a root of $h(T) - h(0)$ for every $\xi \in \mu_p$, hence $\varphi(T) = \prod_{\xi \in \mu_p}(T + 1 - \xi)$ divides $h(T)$ in R, and we get $h(T) = \varphi(T)h_1(T)$ with $h_1 \in R$. Now h_1 is again in RHS, so the induction goes on. $\square$

Proof of Proposition 1.1. Existence and unicity. Take $f \in R$. By Lemma 1.2, $\prod_{\xi\in\mu_p} f(\xi(1+T)-1) \in \varphi(R)$. For the existence of ψ, let

$$r(T) := \sum_{\xi \in \mu_p} f(\xi(1+T)-1).$$

We know that $r(T) \in \varphi(R)$ from Lemma 1.2 and we need to show that $r(T) \in pR$. Let $\mathfrak{p}_0$ be the maximal ideal of $\mathcal{K}_0 = \mathbb{Q}_p(\mu_p)$. For each $\xi \in \mu_p$, $\xi - 1 \in \mathfrak{p}_0$, so

$$\xi(1+T) - 1 \equiv T \mod \mathfrak{p}_0\mathcal{O}[\![T]\!],$$

$$\implies r(T) = \sum_{\xi\in\mu_p} f(\xi(1+T)-1) \equiv pf(T) \equiv 0 \mod (\mathfrak{p}_0\mathcal{O}[\![T]\!] \cap R = pR)$$

Properties. Straitforward. For example,

$$(\varphi \circ \psi \circ \varphi)(f(T)) = \frac{1}{p}\sum_{\xi\in\mu_p} f((\xi(1+T))^p - 1) = \varphi(f(T)),$$

so $\psi \circ \varphi = \mathrm{id}$. $\square$

Lemma 1.3. $\mathcal{N}(T) = T$.

Proof. Almost by definition. $\square$

$\log(T) = \log(1 + (T-1)) = \sum_{n\geq 1} \frac{(-1)^{n-1}(T-1)^n}{n}$

Lemma 1.4. For $n \in \mathbb{Z}_{\geq 1}$,

$$\psi\left(\frac{1+T}{T} \cdot \varphi(T)^n\right) = \frac{1+T}{T} \cdot T^n$$

Proof. Mei Kan Dong! About taking logarithmic derivative. $\square$

1.1.3 The action of $\mathcal{G}$

We let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})|\mathbb{Q}_p)$ act on $R = \mathbb{Z}_p[\![T]\!]$ via the cyclotomic character $\chi : \mathcal{G} \to \mathbb{Z}_p^\times$ (and the formal group $\mathbb{G}_{\mathrm{m}}$?), i.e, for $\sigma \in \mathcal{G}$ and $f(T) \in R$, define

$$\sigma(f(T)) := f([\chi(\sigma)](T)) = f((1+T)^{\chi(\sigma)} - 1).$$

- The above formula defines a $\mathcal{G}$-action on R that sending $R^\times$ to $R^\times$.
- This action of $\mathcal{G}$ commutes with φ, $\mathcal{N}$ and ψ.

1.2 The Interpolating Power Series

Theorem 1. For each $u = (u_n)_n \in \mathcal{U}_\infty$, there is a *unique* $f_u \in R$, s.t. $f_u(\pi_n) = u_n$ for all $n \geq 0$.

The unicity follows directly from the Weierstrass preparation theorem for $\mathbb{Z}_p[\![T]\!]$. More precisely, if both f_u and g_u satifies the condition above for $u \in \mathcal{U}_\infty$, then $f_u - g_u$ has infinitely many zeros $\pi_n \in \mathfrak{m}_{\bar{\mathbb{Q}}_p}$, so $f_u = g_u$.

Example 1.1. Let $a \in \mathbb{Z}$ be such that $(a, p) = 1$, then

$$w_a(T) := \frac{(1+T)^{-a/2} - (1+T)^{a/2}}{T} = -a + O(T) \in R^\times.$$

One checks that for $a, b \in \mathbb{Z}$ prime to p,

$$c_n(a, b) := \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}$$

defines an element $c(a, b) \in \mathcal{U}_\infty$, and that $f_{c(a,b)}(T) = w_a(T)/w_b(T)$.

The essential idea is to look at $W := (R^\times)^{\mathcal{N}=1}$. Let $f \in W$. Recall that we have fixed $\zeta_n \in \mu_{p^{n+1}}$ and set $\pi_n = \zeta_n - 1$, so that $\varphi(\pi_n) = [p](\pi_n) = \pi_{n-1}$. Since $f \in R^\times$, $f(\pi_n) \in \mathcal{U}_n$ for all $n \geq 0$. Moreover, since $f = \mathcal{N}(f)$, we have

$$\varphi(f(T)) = (\varphi \circ \mathcal{N})(f(T)) = \prod_{\xi \in \mu_p} f(\xi(1+T) - 1),$$

so

$$f(\pi_{n-1}) = \varphi(f)(\pi_n) = \prod_{\xi \in \mu_p} f(\xi\zeta_n - 1) = N_{n,n-1} f(\pi_n).$$

Hence $(f(\pi_n))_n \in \mathcal{U}_\infty$. We will show that every element in $\mathcal{U}_n$ is obtained in this manner.

(some lemmata and proofs)

Corollary 1.1. For any $f \in R^\times$, $\{\mathcal{N}^k(f)\}_{k \geq 0}$ converges in $R^\times$. ☐

(By the previous lemma.) This limit is in W.

(a lemma)

Proof of Theorem 1. Let $u \in \mathcal{U}_\infty$. For each $n \geq 0$, there is some $g_n \in R^\times$ s.t. $g_n(\pi_n) = u_n$. Consider the sequence $h_n(T) := \mathcal{N}^{2n}(g_n)$ in $R^\times$. (T.B.C.) ☐

The Galois group $\mathcal{G}$ act on $\mathcal{U}_\infty$ naturally via $\mathcal{G} \twoheadrightarrow \mathrm{Gal}(\mathcal{K}_n|\mathbb{Q}_p)$, where the latter acts on $\mathcal{U}_n$. (check the compactibility.)

Corollary 1.2. There is an isomorphism $\mathcal{U}_\infty \simeq W$ of $\mathcal{G}$-modules given by $u \mapsto f_u$.

Proof. For $\sigma \in \mathcal{G}$ and $f \in W$, we have

$$(\sigma f)(\pi_n) = f([\chi(\sigma)](\pi_n)) = f(\zeta_n^{\chi(\sigma)} - 1) = \sigma(f(\zeta_n - 1))$$

by the definition of the cyclotomic character. ☐

1.3 The Logarithm Derivative

For $f \in R^\times$, define

$$\Delta(f(T)) := (1+T)\frac{f'(T)}{f(T)}.$$

- $\Delta : R^\times \to R$ is a group homomorphism.
- $\Delta(W) \subset R^{\psi=1}$, and $\ker(\Delta|_W) = \mathbb{Z}_p^\times \cap W = \mu_{p-1}$.

Theorem 2. $\Delta(W) = R^{\psi=1}$.

Strategy: reduction mod p. Denote by $x \mapsto \tilde{x}$ the mod p map.

Lemma 1.5. If $\widetilde{\Delta(W)} = \widetilde{R^{\psi=1}}$, then $\Delta(W) = R^{\psi=1}$.

Proof. Take $g \in R^{\psi=1}$. Assume $\widetilde{\Delta(W)} = \widetilde{R^{\psi=1}}$, then $g_1 := g = \Delta(h_1) + pg_2$ for some $h_1 \in W$ and $g_2 \in R$, and we can find inductively

$$g_n = \Delta(h_n) + pg_{n+1}, \quad h_n \in W,\ g_{n+1} \in R.$$

Since Δ is a group homomorphism, and $R^{\psi=1}$ is a closed subset of the p-adic complete ring R, we thereby get a sequence $\{f_n\}_n \subset W$ such that $\Delta(f_n) \to g$ in $R^{\psi=1}$. Since $\ker(\Delta|_W) = \mu_{p-1}$, the sequence $\{f_n \bmod \mu_{p-1}\}_n$ converges in W/μ_{p-1} (this "limit" make sense??). Let $f \in W$ be a lift of this limit in W/μ_{p-1}, then $\Delta(f) = g$. $\square$

Let $\Omega := R/pR = \mathbb{F}_p[\![T]\!]$.

Lemma 1.6. $\widetilde{W} = \Omega^\times$.

Proof. For $f \in \Omega$, there is some $g \in R$ with $\tilde{g} = f$. By Corollary 1.1, $\mathcal{N}^n g \to h \in W$ as $n \to \infty$, and (by a nontyped lemma), $\tilde{h} = \tilde{g} = f$. $\square$

(many words to prove the hypothesis of the lemma, mostly T-adic approximation.)L

1.4 An Exact Sequence

For $f \in R^\times$, define

$$\mathcal{L}(f)(T) := \frac{1}{p}\log\left(\frac{f(T)^p}{\varphi(f)(T)}\right).$$

Lemma 1.7. $\mathcal{L}$ defines a group homomorphism $R^\times \to R$. In addition,

- $\mathcal{L}(W) \subset R^{\psi=0}$.
- $\mathcal{L}$ is $\mathcal{G}$-equivariant.

Proof. Let $f \in R^\times$. Since $\varphi(f) \equiv f^p \bmod p$,

$$\frac{f(T)^p}{\varphi(f)(T)} = 1 + ph(T)$$

for some $h \in R$, and thus

$$\mathcal{L}(f)(T) = \sum_{n \geq 1} \frac{(-1)^{n-1}p^{n-1}}{n} h(T)^n \in R$$

because $p^{n-1}/n \in \mathbb{Z}_p$ for all $n \geq 1$. It is easy to check that $\mathcal{L}$ is a $\mathcal{G}$-homomorphism.

Next we assume that $\mathcal{N}(f) = 1$ and show that $\psi(\mathcal{L}(f)) = 0$, or

$$\sum_{\xi \in \mu_p} \mathcal{L}(f)(\xi(1+T) - 1) = 0.$$

This is done be computation. (Really need to check convergence of $\log(f(T))$ in $\mathbb{Q}_p[\![T]\!]$?) $\square$

Theorem 3. There is a canonical exact sequence

$$1 \longrightarrow A \longrightarrow W \xrightarrow{\mathcal{L}} R^{\psi=0} \xrightarrow{\alpha} \mathbb{Z}_p \longrightarrow 0$$

of $\mathcal{G}$-modules, where

- $A := \{\xi(1+T)^a \mid \xi \in \mu_p,\ a \in \mathbb{Z}_p\}$,
- $D(f)(T) := f(T)\Delta(f(T)) = (1+T)f'(T)$,
- $\alpha(f) := (Df)(0)$

Preparations:

Lemma 1.8. $(1-\varphi)R = TR$.

Proof. "$\subset$" is trivial. For "$\supset$", take $h \in TR$. Consider the polynomial

$$\omega_n(T) := [p^n](T) = (1+T)^{p^n} - 1$$

which has Weierstrass degree p^n. Dividing h by ω_n yields

$$h = h_n + \omega_n q_n, \quad h_n \in R,\ q_n \in \mathbb{Z}_p[T],\ \deg q_n \leq p^n - 1.$$

Since $\omega_n \in (p,T)^n$, $h_n \to h$ as $n \to \infty$. Define

$$g_n := \sum_{k=0}^{n-1} \varphi^i(h_{n-i}),$$

so that

$$g_{n+1} - \varphi(g_n) = h_{n+1}.$$

Now it suffices to show that g_n converges in R. For $1 \leq k \leq n$,

$$\varphi^{n-k}(h) = \varphi^{n-k}(h_k) + \omega_n \varphi^{n-k}(q_k),$$

so

$$g_n = \sum_{i=0}^{n-1} \varphi^i(h) + \omega_n s_n$$

for some $s_n \in R$. Because $h \in TR$, $\varphi^n(h) = \omega_n$ is divided by $\varphi^n(T)$ and thus g_n converges. $\square$

Lemma 1.9 (This lemma is never used...?)**.** There is an exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow R^{\psi=1} \xrightarrow{1-\varphi} R^{\psi=0} \xrightarrow{\beta} \mathbb{Z}_p \longrightarrow 0,$$

where $\beta(f) := f(0)$ is the evaluation at $T = 0$. $\square$

Lemma 1.10. If $f \in R$ is such that $f(0) \equiv 1 \bmod p$ and $\log f(T) = 0$, then $f(T) = 1$.

Proof. Write $f = bg$ with $b \in 1 + p\mathbb{Z}_p$ and $g = 1 + c_r T^r + \cdots$. Then the identity

$$0 = \log f = \log b + \log g = \log b + c_r T^r + \cdots$$

implies that $\log b = 0$ and $c_r = 0$. Therefore $b = \exp(\log b) = 1$ (assuming $p \geq 3$?) and $g = 1$. $\square$

Proof of Theorem 3. (1) *Exactness at W.* Clearly $A \subset \ker \mathcal{L}$. Let $f \in \ker \mathcal{L} \cap W$. We may assume that $f(0) \equiv 1 \bmod p$ by multiplying an element in μ_{p-1}, so that $f(0)^p/\varphi(f)(0) \equiv 1 \bmod p$. As $\mathcal{L}(f) = 0$, Lemma 1.10 shows that

$$f(T)^p = \varphi(f)(T).$$

Therefore $f(\pi_n)^p = f(\pi_{n-1})$ and $f(0) \in \mu_{p-1}$. From the assumption $f(0) \equiv 1 \bmod p$, we get $f(0) = 1$. By Corollary 1.2, $u := (f(\pi_n))_n \in \mathcal{U}_\infty$. (WHY $f(0) = 1 \implies u = (f(\pi_n))_n \in T_p(\mu)$?)

(2) *Exactness at $R^{\psi=0}$.* (Used Lemma 1.8.)

(3) *Exactness at $\mathbb{Z}_p$.* Just observe that $\psi(1+T) = 0$ and $\alpha(1+T) = 1$.

$\square$

1.5 Logarithmic Derivatives of Cyclotomic Units

For $k \in \mathbb{Z}_{\geq 1}$ and $u \in \mathcal{U}_\infty$, define

$$\delta_k(u) := (D^{k-1} \circ \Delta)(f)(0) = D^{k-1}\left((1+T)\frac{f_u'(T)}{f_u(T)}\right)\bigg|_{T=0}$$

Lemma 1.11. For all $k \geq 1$, the map $\delta_k : \mathcal{U}_\infty \to \mathbb{Z}_p$ is a group homomorphism, satisfying

$$\delta_k(\sigma u) = \chi(\sigma)^k \delta_k(u), \quad \sigma \in \mathcal{G}, u \in \mathcal{U}_\infty.$$

Note that this lemma implies that $\delta_k(\mathcal{U}_\infty)$ is an ideal in $\mathbb{Z}_p$, since the cyclotomic character is an isomorphism.

Proof. By Corollary 1.2, $f_{\sigma u}(T) = f_u((1+T)^{\chi(\sigma)} - 1)$. Note that

$$D^m(g((1+T)^a - 1)) = a^m (D^m g)((1+T)^a - 1), \quad g \in R, m \in \mathbb{Z}_{\geq 0}, a \in \mathbb{Z}_p,$$

and the lemma follows. $\square$

Recall that

$$c_n(a,b) := \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}$$

defines an element $c(a,b) \in \mathcal{U}_\infty$ when both a and b are prime to p.

Proposition 1.2. Let $a, b \in \mathbb{Z}$ be prime to p, then

$$\delta_k(c(a,b)) = \begin{cases} 0, & 2 \nmid k, \\ (b^k - a^k)\zeta(1-k), & 2 \mid k. \end{cases}$$

(the value of zeta function at negative odd integers finally appeared!)

Proof. Write $u = c(a,b)$ and

$$f(T) = f_u(T) = \frac{(1+T)^{-a/2} - (1+T)^{a/2}}{(1+T)^{-b/2} - (1+T)^{b/2}}.$$

Put $T := e^z - 1$, so that $D = \frac{d}{dz}$, and

$$\delta_k(u) = D^{k-1} g(z)\big|_{z=0},$$

where

$$\begin{aligned} g(z) &= e^z \frac{f'(e^z-1)}{f(e^z-1)} = \frac{d}{dz} \log f(e^z - 1) \\ &= \frac{b}{2}\left(\frac{1}{e^{-bz}-1} - \frac{1}{e^{bz}-1}\right) - \frac{a}{2}\left(\frac{1}{e^{-az}-1} - \frac{1}{e^{az}-1}\right) \\ &= \sum_{k \geq 2} \frac{B_k(a^k - b^k)}{k!} z^{k-1}, \end{aligned}$$

and B_k's are the Bernoulli numbers, satisfying

$$\begin{cases} B_k = 0, & 2 \nmid k, \\ \zeta(1-k) = -\frac{B_k}{k}, & 2 \mid k \end{cases}$$

for $k \in \mathbb{Z}_{\geq 1}$. □

Theorem 4. For $1 \leq k \leq p-1$, $\delta_k(\mathcal{U}_\infty) = \mathbb{Z}_p$.

Proof. Since $\delta_k(\mathcal{U}_\infty)$ is an ideal in $\mathbb{Z}_p$, it suffices to show that $\delta_k(u) \in \mathbb{Z}_p^\times$ for some $u \in \mathcal{U}_\infty$. (T.B.C.) □

Summary:

1.

$$\begin{tikzcd} & & & 0 \arrow[d] & & \\ & & & \mathbb{Z}_p \arrow[d] & & \\ 1 \arrow[r] & \mu_{p-1} \arrow[r] & W \arrow[r, "\Delta"] & R^{\psi=1} \arrow[r] \arrow[d, "1-\varphi"] & 0 & \\ 1 \arrow[r] & \mathbb{Z}_p^\times \arrow[r] & W \arrow[r, "\mathcal{L}"] & R^{\psi=0} \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] & 0 \\ & & & \mathbb{Z}_p \arrow[d] & & \\ & & & 0 & & \end{tikzcd}$$

2.

$$\begin{aligned} \mathcal{U}_\infty &\simeq W \\ u &\mapsto f_u \\ (f(\pi_n))_n &\leftarrow\!\shortmid f \end{aligned}$$