p-adic Locally Analytic Representations

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1 Reductive Groups over Fields

2 Locally Analytic Representations

Let L be a p-adic field, and K/L an extension with valuation extended.

2.1 Definition

2.1.1 The groups

An **L-locally analytic group** (of dimension d?) is a topological group G such that:

- it is a **locally L-analytic manifold** admitting an atlas consists of charts $G \supset U \to K^d$ where the transition maps given by locally analytic functions (i.e. locally given by power series over L); and
- the multiplication is locally analytic.

2.1.2 The vector space

A **semi-norm** on a vector space is a "norm" that is not seperated. A vector space V over a topological field K is called **locally convex**, if its topology is given by a family of seminorms $\{q_i\}_{i\in I}$, i.e. V is equipped with the weakest topology, such that

- every semi-norm $q_i : V \to \mathbb{R}$ is continuous,
- for every $v \in V$, the translation map $v + (-) : V \to V$ is continuous.

(Well let's just say this is a kind of good space to make the function space more "complete". Normed spaces are locally convex).

Let

$$V' := \mathrm{Hom}_K^{\mathrm{cont}}(V, K)$$

be the continuous dual of V, on which we have the

- **strong topology**,

If M is a locally L-analytic manifold, we let $C^{\mathrm{an}}(G, V)$ be the set of locally analytic maps $G \to V$. It has the topology what?

2.1.3 Locally analytic representations

Let G be a L-analytic group, V be a locally convex Hausdorff vector space over K.

For a representation $\rho : G \to \mathrm{GL}(V)$, we say that $v \in V$ is a **locally analytic vector**, if

$$g \mapsto \rho(g)v : G \to V$$

is locally analytic. We collect locally analytic vectors in V by V^{an}, which is a subrepresentation of G.

Example 2.1. If $G = \mathbb{G}(K)$ for some algebraic group $\mathbb{G}$, then any algebraic representation is locally analytic. This is the case we will look at.

2.2 The distribution algebras

We assume that $L/\mathbb{Q}_p < \infty$, and K is spherically complete.

For a locally L-analytic manifold M, we define the **space of distributions on** M to be

$$D(M, K) := C^{\mathrm{an}}(M, K)'$$

the dual of locally analytic functions from M to K.

3 The BGG category with P-action

Let $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$ be over L. For each standard[1] parabolic $\mathbb{P}$, let $\mathbb{L}_{\mathbb{P}}$ be the Levi containing $\mathbb{T}$.

We write $G = \mathbb{G}(L)$. For a Lie algebra $\mathfrak{a}$ over L, we denote by $U(\mathfrak{a})$ the universal enveloping of $\mathfrak{a}$ **after base change to** K.

[1] Being standard means that $\mathbb{P} \supset \mathbb{B}$.