Modular Forms

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1. It is equivalent to $[\Gamma_\infty : \Gamma_\infty^+] \leq 2$. Let
$$L_1 := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}, L_2 = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{Z} \right\} = L_1 \cup (-1) \cdot L_1,$$
then both L_1 and L_2 are subgroups of $\mathrm{SL}_2(\mathbb{Z})$, and thus
$$[\Gamma_\infty : \Gamma_\infty^+] = [\Gamma \cap L_2 : \Gamma \cap L_1] \leq [L_2 : L_1] = 2.$$

2. Let $N > 2$. Then $-1 \not\equiv 1 \pmod N$, so $\Gamma_1(N) \cap (-1) \cdot L_1 = \varnothing$ and thus $\Gamma_1(N)_\infty = \Gamma_1(N)_\infty^+$. Since $-1 \in \Gamma_0(N)_\infty$ and $-1 \notin \Gamma_0(N)_\infty^+$, we know $[\Gamma_0(N)_\infty : \Gamma_0(N)_\infty^+] \neq 1$, so it equals 2.

3. If $[\Gamma_\infty : \Gamma_\infty^+] = 2$, then there exists a $t \in \mathbb{Z}$ s.t.
$$g := \begin{pmatrix} -1 & t \\ & -1 \end{pmatrix} \in \Gamma.$$
Let $f \in M_k(\Gamma)$, then
$$f(z) = f|_k g(z) = (-1)^{-k} f(z - t) = -f(z - t).$$
If $f = \sum_{n \geq 0} a_n q_N^n$ is the Fourier expansion of f at infinity, then
$$\sum_{n \geq 0} a_n e^{\frac{2\pi i n}{N} z} = -\sum_{n \geq 0} a_n e^{-\frac{2\pi i n t}{N}} e^{\frac{2\pi i n}{N} z}.$$
Comparing the terms gives
$$f(\infty) = a_0 = 0.$$

4. Let
$$\mathbb{Z}^2_{\text{prim}} := \left\{ (c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\} \middle| \gcd(c,d) = 1 \right\}$$
Take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Since $\det g = ad - bc = 1$, the integers c and d are coprime. Then because
$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} g = \begin{pmatrix} a + tc & b + td \\ c & d \end{pmatrix},$$
the map $\Gamma_\infty^+ \backslash \Gamma \to \mathbb{Z}^2_{\text{prim}}$ is well-defined.
If $g' = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then $a'd - b'c = 1$, so
$$(a' - a)d = (b' - b)c.$$
Since c, d are coprime, we have $c \mid (a' - a)$ and $d \mid (b' - b)$. Hence,
$$t' := \frac{a' - a}{c} = \frac{b' - b}{d} \in \mathbb{Z}.$$
If $g' \in \Gamma$, then
$$\begin{pmatrix} 1 & t' \\ & 1 \end{pmatrix} = g' g^{-1} \in \Gamma_\infty^+,$$
i.e., $g' \in \Gamma_\infty^+ g$. So the map $\Gamma_\infty^+ \backslash \Gamma \to \mathbb{Z}^2_{\text{prim}}$ is injective.

5. Let $G = G_{k,\Gamma,\infty}$. For all $g \in \Gamma$ and $z \in \mathcal{H}$,

$$\begin{aligned}(G|_k g)(z) &= j(g,z)^{-k} G(gz) \\ &= \sum_{h \in \Gamma_\infty^+ \backslash \Gamma} j(g,z)^{-k} j(h, gz)^{-k} \\ &= \sum_{h \in \Gamma_\infty^+ \backslash \Gamma} j(hg, z)^{-k} = G(z).\end{aligned}$$

6. Let $G = G_{k,\Gamma,\infty}$. If $[\Gamma_\infty : \Gamma_\infty^+] = 2$, then we can write $\Gamma_\infty = \Gamma_\infty^+ \sqcup \Gamma_\infty^+ \gamma$ with

$$\gamma = \begin{pmatrix} -1 & t \\ & -1 \end{pmatrix}$$

for some $t \in \mathbb{Z}$. Hence

$$\Gamma = \bigsqcup_h \Gamma_\infty h = \bigsqcup_h \left(\Gamma_\infty^+ h \sqcup \Gamma_\infty^+ \gamma h \right),$$

and

$$\begin{aligned}G(z) &= \sum_{g \in \Gamma_\infty^+ \backslash \Gamma} j(g,z)^{-k} \\ &= \sum_{h \in \Gamma_\infty \backslash \Gamma} \left(j(h,z)^{-k} + j(\gamma h, z)^{-k} \right) \\ &= \sum_{h \in \Gamma_\infty \backslash \Gamma} (1 + j(\gamma, hz)^{-k}) j(h,z)^{-k}.\end{aligned}$$

Since $j(\gamma, \tau) = -1$ for all $\tau \in \mathcal{H}$, we get $G(z) = 0$ for all $z \in \mathcal{H}$ once k were odd.

7. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, as $z \to i\infty$, $j(g,z) \to \infty$ if $c \neq 0$ and $j(g,z) = d = \pm 1$ if $c = 0$. If $g \in \Gamma$, then $c = 0$ if and only if $g \in \Gamma_\infty$. Hence

$$\begin{aligned}\lim_{z \to i\infty} G_{k,\Gamma,\infty}(z) &= \sum_{g \in \Gamma_\infty^+ \backslash \Gamma} \lim_{z \to i\infty} j(g,z)^{-k} = \sum_{g \in \Gamma_\infty^+ \backslash \Gamma_\infty} \lim_{z \to i\infty} j(g,z)^{-k} \\ &= \begin{cases} 1, & [\Gamma_\infty : \Gamma_\infty^+] = 1, \\ 0, & [\Gamma_\infty : \Gamma_\infty^+] = 2 \text{ and } k \text{ is odd;} \\ 2, & [\Gamma_\infty : \Gamma_\infty^+] = 2 \text{ and } k \text{ is even.} \end{cases}\end{aligned}$$

So $G_{k,\Gamma,\infty}$ is bounded at infinity.

8. We have

$$G_{k,\Gamma,\infty}|_k g(z) = \sum_{h \in \Gamma_\infty^+ \backslash \Gamma} j(hg, z)^{-k}.$$

As we see in **7.**, $\lim_{z \to i\infty} j(hg,z)^{-k} = 0$ if and only if the matrix hg has nonzero bottom-left term. For each $h \in \Gamma$, since $hg\infty \in c = \Gamma \cdot g\infty$ and the cusp $c \neq \infty$, we know that $hg\infty \neq \infty$. Therefore hg has nonzero bottom-left term, and thus

$$G_{k,\Gamma,\infty}|_k g(\infty) = \sum_{h \in \Gamma_\infty^+ \backslash \Gamma} \lim_{z \to i\infty} j(hg, z)^{-k} = 0.$$

9. This follows from **5.** ($G_{k,\gamma,\infty}$ is a weak modular form of weight k), **7.** ($G_{k,\gamma,\infty}$ is bounded at infinity) and **8.** ($G_{k,\gamma,\infty}$ is bounded at all the cusps different from infinity).

10. To begin with, we note that:

Lemma 1. If $f \in M_k(\Gamma)$ and $g \in \mathrm{SL}_2(\mathbb{Z})$, then $f|_k g \in M_k(g^{-1}\Gamma g)$.

Proof. • For all $\gamma \in \Gamma$, $(f|_k g)|_k(g^{-1}\gamma g) = f|_k(\gamma g) = (f|_k\gamma)|_k g = f|_k g$.

- For all $h \in \mathrm{SL}_2(\mathbb{Z})$,$(f|_k g)|_k h = f|_k(gh)$ is bounded at infinity.

Hence $f|_k g \in M_k(g^{-1}\Gamma g)$. □

For simplicity, we use the following notation.

Definition 1. For every $f \in M_k(\Gamma)$ and $g \in \mathrm{SL}_2(\mathbb{Z})$, define

$$f(g\infty) := (f|_k g)(\infty) = \lim_{z\to i\infty} f|_k g(z).$$

We can verify some basic properties.

Lemma 2. Let $f \in M_k(\Gamma)$ and $g, h \in \mathrm{SL}_2(\mathbb{Z})$.

(a) $(f|_k g)(h\infty) = f(gh\infty)$.

(b) If $g\infty$ and $h\infty$ represent the same cusp of Γ, then $f(g\infty)$ and $f(h\infty)$ only differ by a sign, which is independent of f. In particular, if $\{g_1\infty, \ldots, g_r\infty\}$ is a set of representatives of the cusps of Γ, then $f \in S_k(\Gamma)$ if and only if $f(g_1\infty) = \cdots = f(g_r\infty) = 0$.

Proof. Property (a) is straightforward. For (b), suppose that $g\infty = \gamma h\infty$ for some $\gamma \in \Gamma$. Then $g^{-1}\gamma h \in \mathrm{SL}_2(\mathbb{Z})_\infty$, so there is a $t \in \mathbb{Z}$ s.t.

$$T := g^{-1}\gamma h = \begin{pmatrix} \pm 1 & t \\ & \pm 1 \end{pmatrix}.$$

Now

$$\begin{aligned}(f|_k h)(z) &= (f|_k(\gamma^{-1} g T))(z) = ((f|_k g)|_k T)(z) \\ &= (\pm 1)^{-k}(f|_k g(z \pm t)).\end{aligned}$$

So $f(g\infty) = \pm f(h\infty)$, and the sign is determined by g and h. □

Now let $\{g_1\infty, \ldots, g_r\infty\}$ be fixed representatives of all the different cusps of Γ, where $g_1, \ldots, g_r \in \mathrm{SL}_2(\mathbb{Z})$. For each $i \in \{1, \ldots, r\}$, the function $G_{k, g_i^{-1}\Gamma g_i, \infty} \in M_k(g_i^{-1}\Gamma g_i)$, so

$$G_i := G_{k, g_i^{-1}\Gamma g_i, \infty}|_k g_i^{-1} \in M_k(\Gamma).$$

If $j \neq i$, then the cusp reprensented by $g_i^{-1} g_j\infty$ is not infinity, and thus

$$G_i(g_j\infty) = \left(G_{k, g_i^{-1}\Gamma g_i, \infty}|_k(g_i^{-1} g_j)\right)(\infty) = 0 \tag{1}$$

by **8.**

Now take $f \in M_k(\Gamma)$. We claim that

$$f_0 := f - \sum_{\substack{1\le i\le r \\ G_i(g_i\infty)\neq 0}} \frac{f(g_i\infty)}{G_i(g_i\infty)} G_i \in S_k(\Gamma), \tag{2}$$

and thereby proving that $S_k(\Gamma)$ together with $G_1, \ldots, G_r$ generates $M_k(\Gamma)$. By Lemma 2, it suffices to show for $1 \le i \le r$,

$$f_0(g_i\infty) = f(g_i\infty) - \sum_{\substack{1\le j\le r \\ G_j(g_j\infty)\neq 0}} \frac{f(g_j\infty)}{G_j(g_j\infty)} G_j(g_i\infty) = 0.$$

By Eq. (1), this is true if

$$f(g_i\infty) \neq 0 \implies G_i(g_i\infty) \neq 0.$$

Since $f|_k g_i \in M_k(g_i^{-1}\Gamma g_i)$, then by **3.** and **7.**,

$$\begin{aligned} G_i(g_i\infty) = \left(G_{k,g_i^{-1}\Gamma g_i,\infty}\right)(\infty) = 0 &\iff k \text{ is odd and } [(g_i^{-1}\Gamma g_i)_\infty : (g_i^{-1}\Gamma g_i)_\infty^+] = 2 \\ &\implies f(g_i\infty) = (f|_k g_i)(\infty) = 0, \end{aligned}$$

which completes the proof.

11. Keep our notations in **10.** Consider the $\mathbb{C}$-linear map

$$\iota : M_k(\Gamma) \to \mathbb{C}^{|C_\Gamma|}$$

given by

$$f \mapsto (f(g_1\infty), \ldots, f(g_r\infty)). \tag{3}$$

From Eq. (2), we deduce that $\ker\iota = S_k(\Gamma)$ and $\operatorname{im}\iota$ is generated by $\iota(G_1), \ldots, \iota(G_r)$ because $M_k(\Gamma)$ is generated by $S_k(\Gamma)$ and $G_1, \ldots, G_r$.

If k is even, then $G_i(g_i\infty) \neq 0$ for all $i \in \{1, \ldots, r\}$, and thus $\iota(G_i)$ is the vector in $\mathbb{C}^{|C_\Gamma|}$ whose i-th element is nonzero and other elements are zero. Therefore, $\iota(G_1), \ldots, \iota(G_r)$ form a basis of $\mathbb{C}^{|C_\Gamma|}$. Hence $\dim M_k(\Gamma) = \dim S_k(\Gamma) + |C_\Gamma|$.

12. Keep the notations in **11.** When k is odd, the image of ι is still generated by $\iota(G_i)$'s, but

$$\iota(G_i) \neq 0 \iff [(g_i^{-1}\Gamma g_i)_\infty : (g_i^{-1}\Gamma g_i)_\infty^+] = 1,$$

and those nonzero $\iota(G_i)$'s are linearly-independent. Therefore $\dim(\operatorname{im}\iota) = |C'_\Gamma|$, and $\dim M_k(\Gamma) = \dim S_k(\Gamma) + |C'_\Gamma|$.

13. Since the series $G_{k,\Gamma,\infty}$ is normally convergent on any $X_{A,B}$,

$$\begin{aligned} \operatorname{vol}(\Gamma\backslash\mathcal{H}) \langle f, G_{k,\Gamma,\infty}\rangle &= \int_{\Gamma\backslash\mathcal{H}} f(z) \sum_{g\in\Gamma_\infty^+\backslash\Gamma} \overline{j(g,z)^{-k}} y^{k-2} dxdy \\ &= \sum_{g\in\Gamma_\infty^+\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{j(g,z)^{-k}} y^{k-2} dxdy, \end{aligned}$$

where we write $z = x + iy$. Take a fundamental domain D_Γ for Γ. For each $g \in \Gamma$, since the volume form

$$d\mu(z) := \frac{dxdy}{y^2}$$

is $\mathrm{SL}_2(\mathbb{R})$-invariant, so under the change of variable $z \mapsto g^{-1}\tau$,

$$\begin{aligned} \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{j(g,z)^{-k}} y^{k-2} dxdy &= \int_{D_\Gamma} f(z)\overline{j(g,z)^{-k}} (\operatorname{Im} z)^k d\mu(z) \\ &= \int_{gD_\Gamma} f(g^{-1}\tau)\overline{j(g,g^{-1}\tau)^{-k}} (\operatorname{Im} g^{-1}z)^k d\mu(\tau) \\ &= \int_{gD_\Gamma} f(\tau) j(g^{-1},\tau)^k \overline{j(g^{-1},\tau)^k} |j(g^{-1},\tau)|^{-2k} (\operatorname{Im}\tau)^k d\mu(\tau) \\ &= \int_{gD_\Gamma} f(\tau)(\operatorname{Im}\tau)^k d\mu(\tau), \end{aligned}$$

where we used $1 = j(1,\tau) = j(g, g^{-1}\tau)j(g^{-1},\tau)$. Because $\bigcup_{g\in\Gamma_\infty^+\backslash\Gamma} gD_\Gamma$ is a fundamental domain for Γ_∞^+,

$$\begin{aligned}\langle f, G_{k,\Gamma,\infty}\rangle &= \frac{1}{\mathrm{vol}(\Gamma\backslash\mathcal{H})}\sum_{g\in\Gamma_\infty^+\backslash\Gamma}\int_{gD_\Gamma} f(\tau)(\mathrm{Im}\,\tau)^k d\mu(\tau) \\ &= \frac{1}{\mathrm{vol}(\Gamma\backslash\mathcal{H})}\int_{\Gamma_\infty^+\backslash\mathcal{H}} f(z)y^{k-2}dxdy.\end{aligned}$$

The group Γ_∞^+ is a subgroup of $\begin{pmatrix}1 & \mathbb{Z}\\ & 1\end{pmatrix}$, so it is generated by $\begin{pmatrix}1 & t\\ & 1\end{pmatrix}$ for some $t\in\mathbb{Z}$, and therefore $\{z\in\mathcal{H}|0\le\mathrm{Re}(z)\le t\}$ is a fundamental domain for Γ_∞^+. So

$$\begin{aligned}\int_{\Gamma_\infty^+\backslash\mathcal{H}} f(z)y^{k-2}dxdy &= \int_0^\infty y^{k-2}dy\int_0^N f(z)dx \\ &= \int_0^\infty y^{k-2}a_0 = 0,\end{aligned}$$

where $a_0 = 0$ is the constant term of the q-expansion of $f\in S_k(\Gamma)$. Hence $\langle f, G_{k,\Gamma,\infty}\rangle = 0$.

14. The injective map

$$\mathrm{SL}_2(\mathbb{Z})_\infty^+\backslash\mathrm{SL}_2(\mathbb{Z})\to\mathbb{Z}^2_{\mathrm{prim}}$$

is surjective, because for each $(c,d)\in\mathbb{Z}_{\mathrm{prim}}$, we can find $a,b\in\mathbb{Z}$ s.t. $ac-bd=1$, which gives a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$. Therefore,

$$G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty} = \sum_{g\in\mathrm{SL}_2(\mathbb{Z})_\infty^+\backslash\mathrm{SL}_2(\mathbb{Z})} j(g,z)^{-k} = \sum_{(c,d)\in\mathbb{Z}^2_{\mathrm{prim}}}(cz+d)^{-k}.$$

Note that the map

$$\mathbb{Z}^2_{\mathrm{prim}}\times\mathbb{Z}_{\ge1}\to\mathbb{Z}^2\setminus\{0,0\}\quad ((c,d),u)\mapsto(cu,du)$$

is bijective, whose inverse is given by $(c,d)\mapsto((c/\gcd(c,d), d/\gcd(c,d)), \gcd(c,d))$. Hence, the Eisenstein series

$$\begin{aligned}G_k(z) &= \sum_{(c,d)\in\mathbb{Z}^2\setminus\{0,0\}}(cz+d)^{-k} = \sum_{n\ge1}n^{-k}\sum_{(c,d)\in\mathbb{Z}^2_{\mathrm{prim}}}(cz+d)^{-k} \\ &= \zeta(k)G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty}.\end{aligned}$$

So $G_{k,\mathrm{SL}_2(\mathbb{Z}),\infty} = 2E_k(z)$.