Modular Forms

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Exercise 1

- We know $\#\,\mathrm{GL}_2(\mathbb{F}_p) = (p^2-1)(p^2-p)$ and

$$\Gamma(p) = \ker\left[\det : \mathrm{GL}_2(\mathbb{F}_p) \twoheadrightarrow \mathbb{F}_p^\times\right],$$

  therefore

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(p)] = \#\,\mathrm{SL}_2(\mathbb{F}_p) = \frac{(p^2-1)(p^2-p)}{p-1} = p^3 - p.$$

- Note that

$$\Gamma_1(p) \to \mathbb{F}_p, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \bmod p$$

  is a surjective group homomorphism with kernel $\Gamma(p)$, so

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(p)] = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(p)]}{[\Gamma_1(p) : \Gamma(p)]} = \frac{p^3-p}{p} = p^2 - 1.$$

- Note that

$$\Gamma_0(p) \to \mathbb{F}_p^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \bmod p$$

  is a surjective group homomorphism with kernel $\Gamma_1(p)$, so

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)] = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(p)]}{[\Gamma_0(p) : \Gamma_1(p)]} = \frac{p^2-1}{p-1} = p + 1.$$

Exercise 2

- *Existence.* From Exercise 1, we know that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(2)] = 6.$$

  Hence, finding such a Γ is equivalent to finding a level 2 congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ in which the index of $\Gamma(2)$ is 3.

  Let

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

  and Γ the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\Gamma(2)$ and A. Note that

$$A \notin \Gamma(2),\ A^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \notin \Gamma(2),\ A^3 = -I \in \Gamma(2),$$

  and $\Gamma(2) \triangleleft \Gamma$ since $\Gamma(2) \triangleleft \mathrm{SL}_2(\mathbb{Z})$. Therefore, $\Gamma/\Gamma(2)$ is generated by A and has cardinality 3. So Γ is a congruence subgroup of level 2 and index 2 in $\mathrm{SL}_2(\mathbb{Z})$.

- *Uniqueness.* Let Γ' be such a group. Since $[\Gamma' : \Gamma(2)] = 3$, the image of Γ under the map $\bmod 2 : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_2)$ must be a subgroup of $\mathrm{SL}_2(\mathbb{F}_2)$ of order 3.

  The group $\mathrm{SL}_2(\mathbb{F}_2)$ is a non-abelian group of order 6, so it is isomorphic to the symmtry group S_3, and it has a unique subgroup of order 3. This subgroup is generated by $B := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and must be the image of Γ'. Since B is the image of A under the map $\bmod 2$, we have $A \in \Gamma'$, and thus $\Gamma' \supset \langle \Gamma(2), A \rangle = \Gamma$. Comparing indexes shows that $\Gamma' = \Gamma$.

- As $-I \in \Gamma(2) \subset \Gamma$, we see $\bar{\Gamma} = \Gamma$.

- *Cusp.* The $\langle A \rangle$-orbit of ∞ is $\{\infty, 0, -1\}$. Every rational number can be written as $r = a/b$ with $a, b \in \mathbb{Z}$ satisfying one of the following three cases:

  - $2 \nmid a$ and $2 \mid b$, whence
  $$r \in \Gamma(2) \cdot \infty = \left\{ \frac{a}{c} \middle| a \in 1 + 2\mathbb{Z},\ c \in 2\mathbb{Z} \right\}$$
  - $2 \mid a$ and $2 \nmid b$, whence
  $$r \in \Gamma(2) \cdot 0 = \left\{ \frac{b}{d} \middle| b \in 2\mathbb{Z},\ d \in 1 + 2\mathbb{Z} \right\};$$
  - $2 \nmid a$ and $2 \nmid b$, whence
  $$r \in \Gamma(2) \cdot (-1) = \left\{ \frac{b-a}{d-c} \middle| a, c \in 1 + 2\mathbb{Z},\ b, d \in 2\mathbb{Z} \right\}.$$

  Therefore, $\Gamma \cdot \infty = \mathbb{Q} \cup \{\infty\}$, and Γ has only one cusp.

  Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma(2)$, the width is at most 2. If $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = AT^{-1} \in \Gamma$. But $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, so S and T cannot be in Γ together. Therefore, $T \notin \Gamma$, and the width of the only cusp for Γ is 2.