Notes on Drinfeld Modules and Explicit CFT for Function Fields

March 8, 2025

Pre-date: March 10! It is close!

1) Give a 30min (strict limit !!!) talk. Ideally more like 25min + 5 min for questions. The talks will be in March. I will try to reserve a room, and will give a more precise time/date when possible.

2) Write an "extended summary" (meaning around 5 pages NOT!!! >=10) of you article. It should summarise the article and its main ideas and be accessible to advanced Master students (i.e., the other students in this group).

1 Review on CFT

Let F be a global field, $C_F = \mathbb{A}_F^\times / F^\times$ be its idele class group, and F^{ab} be its maximal abelian extension inside a separable closure in a fixed algebraic closure $\bar{F}$. The class field theory asserts that the Artin map

$$\theta_F : C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

is a continuous group homomorphism with dense image, establishing a bijection

$$\{\text{finite abelian extensions of } F\} \longleftrightarrow \{\text{finite index open subgroups of } C_F\}.$$

The direction "$\to$" is computable: for a finite abelian L/F, the composition $C_F \xrightarrow{\theta_F} \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(L/F)$ is surjective, and its kernel $U = N_{L/F}(C_L)$ is the corresponding open subgroup of C_F, where $N_{L/F} : C_L \to C_F$ is the norm map[1]. But the other direction "$\leftarrow$" is not known in general: given a finite index open subgroup of C_F, the Artin map θ_F doesn't produce the generators of the corresponding extension L/F.

The goal of explictit class field theory is to find this inverse.

2 Function Fields and Drinfeld Modules

Let F be a global function field with a fixed place ∞, and with field of constants $k = \mathbb{F}_q$. If λ is a place of F, we denote by F_λ the completion at λ, by $\mathcal{O}_\lambda \subset F_\lambda$ the valuation ring, by $\mathbb{F}_\lambda := \mathcal{O}_\lambda / \mathfrak{m}_\lambda$ the residue field at λ. Since we are working with function fields, the Teichmüller lifting $\mathbb{F}_\lambda \hookrightarrow \mathcal{O}_\lambda$ is a field homomorphism; we regard $\mathbb{F}_\lambda \subset \mathcal{O}_\lambda \subset F_\lambda$ as a subfield via this embedding.

For any extension L of k, we fixed an algebraic closure $\bar{L}$.

[1] The norm for a idele is just the multiplication of the norm at every places.

2.1 Function fields

2.1.1 holomorphy ring

Let S be a non-empty set of (not all the) places of F. Define

$$\mathcal{O}^S := \bigcap_{\lambda \notin S} \mathcal{O}_\lambda = \{x \in F \mid \mathrm{ord}_\lambda(x) \geq 0,\ \forall \lambda \notin S\}$$

to be the subring of F consisting of elements regular away from S. A **holomorphy ring** is a ring of this form. For example, our $A = \mathcal{O}^{\{\infty\}}$ is a holomorphy ring.

Proposition 2.1. Consider a holomorphy ring $\mathcal{O}^S$.

(1) $\mathrm{Frac}(\mathcal{O}^S) = F$.

(2) $\mathcal{O}^S$ is a Dedekind domain.

(3) There is a bijection

$$\{\text{place of } F \text{ not in } S\} \longleftrightarrow \mathrm{MaxSpec}\, \mathcal{O}^S$$

giving by $\lambda \mapsto \mathfrak{m}_\lambda \cap \mathcal{O}^S$, which induces isomorphisms

$$\mathbb{F}_\lambda = \mathcal{O}_\lambda / \mathfrak{m}_\lambda \simeq \mathcal{O}^S / (\mathfrak{m}_\lambda \cap \mathcal{O}^S)$$

So we can regard λ as a maximal ideal of A.

2.1.2 The Weil group

Let L be an extension of k. The field k is perfect, so the algebraic closure $\bar{k}$ of k in $\bar{F}$ is contained in L^{sep}, and the absolute Galois group $\mathrm{Gal}_L = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ stablizes $\bar{k}$. Hence we have an exact sequence of topological groups

$$1 \longrightarrow \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k}) \longrightarrow \mathrm{Gal}_L \stackrel{\deg}{\longrightarrow} \hat{\mathbb{Z}} \to 0,$$

where $\deg : \mathrm{Gal}_L \to \mathrm{Gal}_k \simeq \hat{\mathbb{Z}}$ is defined by

$$\sigma(x) = \mathrm{Frob}_q^{\deg(\sigma)}(x), \quad \sigma \in \mathrm{Gal}_L,\ x \in \bar{k}.$$

The **Weil group** is the subgroup W_L of Gal_L of elements that acts on $\bar{k}$ by an integral power of the Frobenius-q, i.e.

$$\sigma(x) = x^{q^{\deg(\sigma)}}, \quad \sigma \in W_L,\ x \in \bar{k}.$$

The kernel of the map $\deg : W_L \to \mathbb{Z}$ is still $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$. We endow W_L with the weakest topology for which

$$1 \longrightarrow \mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k}) \longrightarrow W_L \stackrel{\deg}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

is an exact sequence of topological groups, where

- $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$ has its usual profinite topology,
- $\mathbb{Z}$ has discrete topology[2].

With respect to this topology, the inclusion $W_L \hookrightarrow \mathrm{Gal}_L$ is continuous with dense image. (?)

[2] This is not the topology induced from $\mathbb{Z} \subset \hat{\mathbb{Z}}$.

2.2 Definition of Drinfeld modules

2.2.1 Endomorphisms of the additive group

Consider the additive group $\mathbb{G}_{\mathrm{a}/L}$ over L, which is not only a group scheme, but also a k-vector space scheme, and we consider the ring $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L})$ of all k-linear endomorphism.

Proposition 2.2. $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) = L[\tau]$, where τ is the Frobenius-q endomorphism.

We explain the notation in the proof.

Proof. An endomorphism $\mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ of schemes over L is given by an L-algebra homomorphism $\Phi : L[X] \to L[X]$, hence it is determined by the image $\varphi(X) = \Phi(X)$[3] of X. It respects the group-scheme structure if it commutes with the co-multiplication map (also an L-algebra homomorphism)

$$\Delta : F[X] \to F[X] \otimes_L F[X], \quad X \mapsto X \otimes 1 + 1 \otimes X.$$

which amounts to

$$(\Phi \otimes \Phi)(\Delta(X)) = (\Phi \otimes \Phi)(X \otimes 1 + 1 \otimes X) = \Phi(X) \otimes 1 + 1 \otimes \Phi(X) = \varphi(X) \otimes 1 + 1 \otimes \varphi(X)$$

equals

$$\Delta(\Phi(X)) = \Delta(\varphi(X)) = \varphi(\Delta(X)) = \varphi(X \otimes 1 + 1 \otimes X).$$

This is to say that[4] φ is additive, i.e. $\varphi(X+Y) = \varphi(X) + \varphi(Y)$.

We require furthur that Φ respects the "co-k-scalar multiplication", which I don't have the formula right now. So let's use the functor point of view. Take $c \in k$. Yoneda tells us that

$$\mathrm{Hom}_{[k\text{-}\mathbf{Alg}^{\mathrm{op}},\mathbf{Grp}]}(\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{a}}) \simeq \mathbb{G}_{\mathrm{a}}(L[X]), \quad \phi \mapsto \phi(\mathrm{id}_{L[X]}),$$

so the co-c-multiplication is given by $X \mapsto cX$. Therefore Φ respects this map if $\varphi(cX) = c\varphi(X)$.

In conclusion,

$$\begin{aligned} \mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) &= \{k\text{-linear polynomials in } L[X]\} \\ &= \left\{ \sum_i a_i X^{p^i} \,\middle|\, a_i \in L,\ \sum a_i c X^{p^i} = \sum a_i c^{p^i} X^{p^i}, \forall c \in k = \mathbb{F}_q \right\} \\ &= \left\{ \sum_i a_i X^{q^i} \,\middle|\, a_i \in L \right\} = \left\{ \left(\sum_i a_i \tau^i \right)(X) \,\middle|\, a_i \in L \right\}, \end{aligned}$$

where $\tau(X) := X^q$.

Note that $\tau : L[X] \to L[X]$ is additive, but doesn't commutes with elements in L:

$$\tau a = a^q \tau, \quad \forall a \in L.$$

[3] Note that if $\varphi(X) = a_n X^n + \cdots + a_0$, then

$$\varphi(f(X)) = a_n f(X)^n + \cdots + a_0$$

and

$$\Phi(f(X)) = f(\Phi(X)) = f(\varphi(X))$$

are ***different*** in general.

[4] Recall that the multiplicative structure on $B \otimes_A C$ is given by

$$(b \otimes b') \cdot (c \otimes c') = bb' \otimes cc'.$$

Therefore $L[\tau]$ is a *non-commutative* subring of $\mathrm{End}(L[X])$, where multiplication is composition; it is a ring of **twisted polynomials**. And we have $\mathrm{End}_k(\mathbb{G}_{\mathrm{a}/L}) \simeq L[\tau]$. □

Remark. τ corresponds to the Frobenius-q endomorphism of $\mathbb{G}_{\mathrm{a}/L}$. (What is this? $\mathbb{G}_{\mathrm{a}/L}$ is NOT over $\mathbb{F}_q = k$.)

2.2.2 Drinfeld modules and isogenies

Let A be a k-algebra. A **Drinfeld A-module**[5] **over** L is a homomorphism

$$\phi : A \to L[\tau] \quad x \mapsto \phi(x) =: \phi_x$$

of k-algebras such that $\phi(A)$ is *not contained* in $L \subset L[\tau]$.

Let ϕ and ϕ' be two Drinfeld modules $A \to L[\tau]$. An **isogeny** over L from ϕ to ϕ' is an $f \in L[\tau] \setminus \{0\}$ such that

$$f\phi_a = \phi'_a f, \quad \forall a \in A.$$

An **isomorphism** over L from ϕ to ϕ' is an invertible isogeny, namely an isogeny $f \in L[\tau]^\times$. If M/L is an extension, then a Drinfeld module over L induces natrually a Drinfeld module over M, and we can talk about isogenies over M for Drinfeld modules over L.

Let

$$\partial : L[\tau] \to L \quad \sum_i a_i \tau^i \mapsto a_0$$

be the homomorphism of taking the constant term. We say that a Drinfeld module $\phi : A \to L[\tau]$ **has generic characteristic**, if

$$\partial \circ \phi : A \to L[\tau] \twoheadrightarrow L$$

is *injective*. This implies that ϕ is injective.

2.3 The Drinfeld modules we need

In what follows, we take $A := \mathcal{O}^{\{\infty\}} \subset F$ to be the subring of F consisting of functions that are regular away from ∞, and we assume that every Drinfeld modules $\phi : A \to L[\tau]$ is of generic characteristic, so that $\partial \circ \phi : A \hookrightarrow L$ is injective and it extends to an embedding

$$F \hookrightarrow L.$$

Through the latter, we view F as a subfield of L.

Let L^{perf} be the purely inseparable closure of L in $\bar{L}$, then $L^{\mathrm{perf}}((\tau^{-1}))$ is a well-defined skew-field[6], containing $L[\tau]$ as a subring.

Under our assumption, $\phi : A \hookrightarrow L[\tau]$ is injective, so it extends to a unique embedding

$$\phi : F \hookrightarrow L^{\mathrm{perf}}((\tau^{-1})).$$

The function

$$v_\phi : F \to \mathbb{Z} \cup \{\infty\} \quad x \mapsto \mathrm{ord}_{\tau^{-1}}(\phi_x)$$

[5] There is more general definition, but this one suffices.

[6] We need to have all p-th root, so that $\tau^{-1}a = a^{1/q}\tau$ is always valid.

is a nontrivial[7] valuation, and $v_\phi(x) \leq 0$ for all $x \in A \smallsetminus \{0\}$. Therefore v_ϕ is equivalent to the valuation ord_∞ attached to the place ∞. We define the **rank of** ϕ to be the rational number $r \in \mathbb{Q}$ such that

$$\mathrm{ord}_{\tau^{-1}}(\phi_x) = r d_\infty \,\mathrm{ord}_\infty(x), \tag{1}$$

for $x \in F$, where $d_\infty = [\mathbb{F}_\infty : k]$ is the inertia degree of F at ∞. The tank r is always an integer (by a proposition we may encounter later). Since $L^{\mathrm{perf}}((\tau^{-1}))$ is complete under $\mathrm{ord}_{\tau^{-1}}$, the homomorphism $\phi : F \to L^{\mathrm{perf}}((\tau^{-1}))$ gives rise to a unique homomorphism

$$\phi : F_\infty \to L^{\mathrm{perf}}((\tau^{-1}))$$

such that $\mathrm{ord}_{\tau^{-1}}(\phi_x) = r d_\infty \,\mathrm{ord}_\infty(x)$ for all $x \in F_\infty$.

Now the map ϕ restricts to a homomorphism

$$\phi : \mathbb{F}_\infty \subset \mathcal{O}_\infty \to L^{\mathrm{perf}}[\![\tau^{-1}]\!].$$

Composing with $\partial : L^{\mathrm{perf}}[\![\tau^{-1}]\!] \to L^{\mathrm{perf}}$ of taking constant term, we obtain an embedding

$$\partial \circ \phi|_{\mathbb{F}_\infty} : \mathbb{F}_\infty \hookrightarrow L^{\mathrm{perf}},$$

whose image lies in L (why?).

2.4 ε-normalized Drinfeld modules

Let $\phi : A \to L[\tau]$ be a Drinfeld module of rank r, extending to an embedding $\phi : F \to L^{\mathrm{perf}}((\tau^{-1}))$. For $x \in F_\infty^\times$, we define

$$\mu_\phi(x) := \text{first non-zero coefficient of } \phi_x \text{ as a Laurent series in } \tau^{-1},$$

so that $\mu_\phi(x) \in (L^{\mathrm{perf}})^\times$, and the first term, i.e. the term with *highest* τ-order, of ϕ_x is

$$\mu_\phi(x)\tau^{-r d_\infty \,\mathrm{ord}_\infty(x)}.$$

In particular, if $x \in A$, $\mu_\phi(x)$ is the leading coefficient of $\phi_x \in L[\tau]$, which is what we used before to define reduction type.

By definition, for $x, y \in F_\infty^\times$,

$$\mu_\phi(xy) = \mu_\phi(x)\mu_\phi(y)^{1/q^{r d_\infty \ \mathrm{ord}_\infty(x)}}.$$

Recall that ϕ gives us an embedding

$$\partial \circ \phi|_{\mathbb{F}_\infty} : \mathbb{F}_\infty \hookrightarrow L$$

With respect to this embedding, why?

$$\mu_\phi(x) = x, \quad \forall x \in \mathbb{F}_\infty$$

A **sign function for** F_∞ is a group homomorphism $F_\infty^\times \to \mathbb{F}_\infty^\times$ such that $\varepsilon|_{\mathbb{F}_\infty^\times} = \mathrm{id}_{\mathbb{F}_\infty^\times}$. These functions can be described completely. A uniformizer π of F_∞, yields a decomposition

$$F_\infty^\times \simeq \mathbb{F}_\infty \times (1 + \mathfrak{m}_\infty) \times \pi^{\mathbb{Z}}.$$

If p^r = cardinality of $\mathbb{F}_\infty$, then $1 + \mathfrak{m}_\infty$ is a pro-p group, but $\mathbb{F}_\infty^\times$ has order $p^r - 1$, so ε must be trivial on $\mathfrak{m}_\infty$. Therefore ε is determined by its value $\varepsilon(\pi)$.

Let $\varepsilon : F_\infty \to \mathbb{F}_\infty$ be a sign function for F_∞. We say that ϕ is

[7] Because $\phi(A) \not\subset L$.

- **normalized**, if
$$\mu_\phi(x) \in \mathbb{F}_\infty, \quad \forall x \in F_\infty,$$
- **ε-normalized**, if
$$\exists \sigma \in \mathrm{Aut}_k(\mathbb{F}_\infty), \quad \phi = \sigma \circ \varepsilon.$$

Lemma 2.1. Let ε be a sign function for F_∞. Any Drinfeld module over L is isomorphic over $\bar{L}$ to some ε-normalized Drinfeld module.

2.5 The action of an ideal on a Drinfeld module

Let $\phi : A \to L[\tau]$ be a Drinfeld module. For an ideal $\mathfrak{a}$ of A, Define
$$I_{\mathfrak{a},\phi} := \text{ ideal of } L[\tau] \text{ generated by } \{\phi_a \mid a \in \mathfrak{a}\}.$$
Every *left*-ideal of $L[\tau]$ is principal,[8] so
$$I_{\mathfrak{a},\phi} = L[\tau]\phi_{\mathfrak{a}}$$
for a *unique monic* $\phi_{\mathfrak{a}} \in L[\tau]$. It is a plain to verify that for every $x \in A$, $I_{\mathfrak{a},\phi}$ absorb ϕ_x also from the right, i.e. $I_{\mathfrak{a},\phi}\phi_x \subset I_{\mathfrak{a},\phi}$, and therefore gives us a *unique* Drinfeld module
$$\mathfrak{a} * \phi : A \to L[\tau] \quad x \mapsto (\mathfrak{a} * \phi)_x,$$
which is characterized by
$$\phi_{\mathfrak{a}} \cdot \phi_x = (\mathfrak{a} * \phi)_x \cdot \phi_{\mathfrak{a}},$$
namely that $\phi_{\mathfrak{a}}$ is an isogeny from ϕ to $\mathfrak{a} * \phi$.

Lemma 2.2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals of A, then
$$\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_{\mathfrak{a}} \cdot \phi_{\mathfrak{b}},$$
$$\mathfrak{a}\mathfrak{b} * \phi = \mathfrak{a} * (\mathfrak{b} * \phi).$$

Lemma 2.3. Let $\mathfrak{a} = (w) \neq 0$ be a principal ideal of A, then
$$\phi_{(w)} = \mu_\phi(w)^{-1} \cdot \phi_w,$$
$$((w) * \phi)_x = \mu_\phi(w)^{-1} \cdot \phi_x \cdot \mu_\phi(w), \ \forall x \in A.$$
In particular, $\phi \simeq (w) * \phi$ (not given by $\phi_{(w)}$).

Lemma 2.4. Let $\sigma : L \hookrightarrow M$ be a field extension, inducing a Drinfeld module
$$\sigma(\phi) : A \to M[\tau], \ x \mapsto \sigma(\phi)_x = \sigma(\phi_x).$$
Then
$$\sigma(\mathfrak{a} * \phi) = \mathfrak{a} * \sigma(\phi),$$
$$\sigma(\phi_{\mathfrak{a}}) = \sigma(\phi)_{\mathfrak{a}}.$$

Now we can extend the action of ideals to

[8] By an argument similar to $L[X]$, probably.

- $\mathcal{I}_A$, the group of fractional ideals of A

More precisely, for $w \in A \setminus \{0\}$, Lemma 2.3 suggests us to define

$$\left((w^{-1}) * \phi\right)_x := \mu_\phi(w) \cdot \phi_x \cdot \mu_\phi(w)^{-1}.$$

For a general fractional ideal $w^{-1}\mathfrak{a}$ where $\mathfrak{a}$ is an integral ideal of A, we set

$$(w^{-1}\mathfrak{a}) * \phi := w^{-1} * (\mathfrak{a} * \phi) : x \mapsto \mu_\phi(w) \cdot (\mathfrak{a} * \phi)_x \cdot \mu_\phi(w)^{-1}.$$

Lemma 2.2 shows that these formulae define an action of $\mathcal{I}_A$ on the set of Drinfeld modules $A \to L[\tau]$.

Given a sign function $\varepsilon : F_\infty \to \mathbb{F}_\infty$ for F_∞, we can consider

- $\mathcal{P}_A^+$, a subgroup of the group $\mathcal{P}$ of principal fractional ideals of A, which is generated by $x \in F^\times$ with $\varepsilon(x) = 1$, and
- the **narrow class group** $\mathrm{Pic}^+(A) := \mathcal{I}_A / \mathcal{P}_A^+$.

If, in addition, ϕ is ε-normalized, then $\mathcal{P}^+$ fixes ϕ by Lemma 2.3, giving an action of $\mathrm{Pic}^+(A)$.

2.6 Torsion submodule

A Drinfeld module $\phi : A \to L[\tau]$ defines an A-module structure on $\bar{L}$ by

$$x \cdot b := \phi_x(b), \quad \forall x \in A, b \in \bar{L}.$$[9]

All ϕ_x has coefficient in L, so ϕ, in particular, gives an A-module structure on L^{sep}.

For an ideal $\mathfrak{a}$ of A, we define

$$\phi[\mathfrak{a}] := \left\{b \in \bar{L} \mid \phi_\mathfrak{a}(b) = 0\right\} = \left\{b \in \bar{L} \mid \phi_x(b) = 0, \forall x \in \mathfrak{a}\right\},$$

an $A/\mathfrak{a}$-module and an A-submodule of $\bar{L}$ with A-module structure induced by ϕ.

Proposition 2.3. Let ϕ be a Drinfeld module of rank r, $\mathfrak{a}$ an ideal of A. Then $\phi[\mathfrak{a}]$ is a free $A/\mathfrak{a}$-module of rank r, and it is contained in F^{sep}.

Proof. Every ϕ_x acts by a polynomial of the form

$$\phi_x(T) = a_0 T + a_1 T^q + \cdots + a_n T^{q^n}.$$

This polynomial is separable, because $x \mapsto \phi_x \mapsto a_0$ is injective, which implies that $\phi_x'(T) = a_0 \neq 0$ if $\phi_x \neq 0$.

For the other claim, we use the structure of modules over Dedekind domains. $\square$

2.7 Hayes modules

Let $\mathbb{C}_\infty$ be a completion of an algebraic closure of F_∞. It is ∞-adically complete and algebraically closed.

Fix a sign function $\varepsilon : F_\infty \to \mathbb{F}_\infty$ for F_∞. A **Hayes module for** ε is a Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$ over $\mathbb{C}_\infty$, such that

[9] Note that if $\phi_x = \sum_{a_i \tau^i}$, then

$$\phi_x(b) = \sum_i \tau^i(b) = \sum_i b^{q^i}.$$

At least I think so!

- it is of rank 1,
- it is ε-normalized,
- $\partial \circ \phi : A \hookrightarrow \mathbb{C}_\infty$ is the inclusion $A \subset F \subset F_\infty \subset \mathbb{C}_\infty$.

Let X_ε be the set of Hayes modules for ε.

If $\mathfrak{a}$ is an ideal of A, and $\phi \in X_\varepsilon$ then $\mathfrak{a} * \phi \in X_\varepsilon$. By some discussion before, this defines an action of $\mathrm{Pic}^+(A) = \mathcal{I}_A/\mathcal{P}_A^+$ on X_ε.

Proposition 2.4. The set X_ε is a principal homogeneous space for $\mathrm{Pic}^+(A)$, i.e. $\mathrm{Pic}^+(A)$ acts *freely* and *transitively* on X_ε.

2.7.1 Galois action on X_ε

We define the **narrow Hilbert class field** ot the **normalizing field for** (F, ∞, ε) to be the extension

$$H_A^+ := F\left(\text{coefficient of } \phi_x \mid \phi \in X_\varepsilon, x \in A\right)$$

of F in $\mathbb{C}_\infty$.

Theorem 1. (1) For any $\phi \in X_\varepsilon$ and $x \in A$,

$$H_A^+ = F\left(\text{coefficient of } \phi_x\right)$$

(2) Let B be the integral closure of A in H_A^+. For any $\phi \in X_\varepsilon$ and $x \in A$, $\phi_x \in H_A^+[\tau]$ has integral coefficient, i.e. ϕ_x has coefficient in B.

(3) The extension H_A^+/F is finite abelian, and it is unramified away from ∞.

By Lemma 2.4, there is a natrual action of $\mathrm{Gal}(H_A^+/F)$ on X_ε. For a fixed $\phi \in X_\varepsilon$, ϕ induces an injective group homomorphism

$$\Psi : \mathrm{Gal}(H_A^+/F) \hookrightarrow \mathrm{Pic}^+(A),$$

such that

$$\sigma(\phi) = \Psi(\sigma) * \phi, \quad \forall \sigma \in \mathrm{Gal}_F .$$

(4) For each non-zero prime $\mathfrak{p}$ of A, the class of $\Psi(\mathrm{Frob}_\mathfrak{p})$ in $\mathrm{Pic}^+(A)$ equals the class of $\mathfrak{p}$.

(5) $\Psi : \mathrm{Gal}(H_A^+/F) \to \mathrm{Pic}^+(A)$ is an isomorphism.

3 Construction of the Inverse to the Artin Map

We fix the tuple (F, ∞, ε) and a Hayes module $\phi \in X_\varepsilon$.

3.1 λ-adic representation

Let λ be a place of F different from ∞, and we denote the corresponding maximal ideal of A still by λ.

Take $e \geq 1$ and consider $\phi[\lambda^e]$. By Proposition 2.3, $\phi[\lambda^e]$ is an A/λ^e-module of rank 1. Define the **λ-adic Tate module** to be

$$T_\lambda(\phi) := \mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty]).$$

Proposition 3.1. $T_\lambda(\phi)$ is a free $\mathcal{O}_\lambda$-module of rank 1.

Proof. The ring $\mathcal{O}_\lambda$ is a DVR, so

$$\mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda,\ \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(\mathcal{O}_\lambda/\mathfrak{m}_\lambda^e, \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(A/\lambda^e, \phi[\lambda^\infty]) = \varprojlim_e \mathrm{Hom}_A(A/\lambda^e, \phi[\lambda^e]).$$

□

Hence

$$V_\lambda(\phi) := T_\lambda(\phi) \otimes_{\mathcal{O}_\lambda} F_\lambda$$

is a 1-dimensional F_λ-vector space.

Using the isomophism $\Psi : \mathrm{Gal}(H_A^+/F) \simeq \mathrm{Pic}^+(A)$ from Theorem 1, any ideal $\mathfrak{a} \in \Psi(\sigma)$ of A satisfies that $\sigma(\phi) = \mathfrak{a} * \phi$, and thus we have two isogenies between $\sigma(\phi)$ and ϕ, such that

- σ induces an isomorphism $V_\lambda(\sigma) : V_\lambda(\phi) \simeq V_\lambda(\sigma(\phi))$,
- $\phi_\mathfrak{a}$ induces an isomorphism[10] $V_\lambda(\phi_\mathfrak{a}) : V_\lambda(\phi) \simeq V_\lambda(\mathfrak{a} * \phi)$.

So we obtain an element

$$V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\sigma) \in \mathrm{GL}_{F_\lambda}(V_\lambda(\sigma)) = F_\lambda^\times \cdot \mathrm{id},$$

corresponding to an element $\rho_\lambda^\mathfrak{a}(\sigma) \in F_\lambda^\times$.

Lemma 3.1. Let $\sigma, \gamma \in \mathrm{Gal}_F$ and $\mathfrak{a}, \mathfrak{b}$ be ideals of A.

(i) If $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$, then $(\sigma\gamma)(\phi) = (\mathfrak{a}\mathfrak{b}) * \phi$, and $\rho_\lambda^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\lambda^\mathfrak{a}(\sigma)\rho_\lambda^\mathfrak{b}(\gamma)$.

(ii) If $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$, then $\mathfrak{b}^{-1}\mathfrak{a}$ is generated by a *unique* $w \in F_\infty^+ \cap F^\times$, and $\rho_\lambda^\mathfrak{a}(\sigma)\rho_\lambda^\mathfrak{b}(\sigma)^{-1} = w$.

(iii) If $\sigma(\phi) = \mathfrak{a} * \phi$, then $\mathrm{ord}_\lambda(\rho_\lambda^\mathfrak{a}(\sigma)) = -\mathrm{ord}_\lambda(\mathfrak{a})$[11].

If $\sigma \in \mathrm{Gal}_{H_A^+}$, then $\sigma(\phi) = \phi = A * \phi$. By Lemma 3.1 (i), we obtain a homomorphism

$$\rho_\lambda : \mathrm{Gal}_{H_A^+} \to \mathcal{O}_\lambda^\times \quad \sigma \mapsto \rho_\lambda^A(\sigma).$$

Lemma 3.2. $\rho_\lambda : \mathrm{Gal}_{H_A^+} \to \mathcal{O}_\lambda^\times$ is continuous and unramified at all places of H_A^+ not over λ or ∞.

3.2 ∞-adic representation

(Merge this and the last section in pre.) Let $F_\infty^+ := \{x \in F_\infty^\times \mid \varepsilon(x) = 1\} = \ker(\varepsilon : F_\infty \to \mathbb{F}_\infty^\times)$. Recall that the Hayes module $\phi : A \to H_A^+[\tau]$ extends to an injective homomorphism $\phi : F_\infty \to \left(H_A^+\right)^{\mathrm{perf}}(\!(\tau^{-1})\!)$.

Lemma 3.3. Let $\sigma, \gamma \in W_F$ and $\mathfrak{a}, \mathfrak{b}$ be ideals of A.

There exists some series $u \in F^{\mathrm{sep}}[\![\tau^{-1}]\!]^\times$, such that

$$u^{-1}\phi(F_\infty)u \subset \bar{k}(\!(\tau^{-1})\!).$$

For such a series u, if $\sigma(\phi) = \mathfrak{a} * \phi$, then there is a unique element $\rho_\infty^\mathfrak{a}(\sigma) \in F_\infty^+$, such that

$$\phi_\mathfrak{a}^{-1} \cdot \sigma(u) \cdot \tau^{\deg(\sigma)} \cdot u^{-1} = \phi(\rho_\infty^\mathfrak{a}(\sigma)).$$

This element satisfies the following properties:

[10]Since ϕ has rank 1, it is equivalent to that $V_\lambda(\phi_\mathfrak{a})$ is non-zero. This is true, because, parallel to elliptic curves, taking Tate module is a faithful functor, i.e. for any two Drinfeld modules ϕ and ϕ' over L, the map

$$\mathrm{Hom}_L(\phi, \phi') \hookrightarrow \mathrm{Hom}_{\mathcal{O}_\lambda}(T_\lambda(\phi), T_\lambda(\phi'))$$

is injective.

[11]Recall that we identify λ with a prime ideal of A. The number $\mathrm{ord}_\lambda(\mathfrak{a})$ is the largest power of λ dividing $\mathfrak{a}$.

(i) If $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$, then $(\sigma\gamma)(\phi) = (\mathfrak{a}\mathfrak{b}) * \phi$, and $\rho_\lambda^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\gamma)$.

(ii) If $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$, then $\mathfrak{b}^{-1}\mathfrak{a}$ is generated by a *unique* $w \in F_\infty^+ \cap F^\times$, and $\rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\sigma)^{-1} = w$.

Similar to the λ-adic case, restricting to the Weil group over H_A^+ gives us a homomorphism

$$\rho_\infty : W_{H_A^+} \to F_\infty^+, \quad \sigma \mapsto \rho_\infty^A(\sigma).$$

Lemma 3.4. $\rho_\infty : W_{H_A^+} \to F_\infty^+$ is continuous and unramified at all places of H_A^+ not over ∞.

3.3 The inverse of the Artin map

For each $\sigma \in W_F$, fix an ideal $\mathfrak{a}_\sigma$ of A, such that

$$\sigma(\phi) = \mathfrak{a}_\sigma * \phi.$$

By Lemma 3.1 (iii), $\rho_\lambda^{\mathfrak{a}_\sigma}(\sigma) \in \mathcal{O}_\lambda^\times$ for almost all places λ. Hence $(\rho_\lambda^{\mathfrak{a}_\sigma}(\sigma))_\lambda$ is an idele of F; we define $\rho(\sigma)$ to be its class in C_F. By Lemma 3.1 (ii) and Lemma 3.3 (ii), for different choices of $\mathfrak{a}_\sigma$, $\rho_\lambda^{\mathfrak{a}_\sigma}(\sigma)$ will differ by an element in $F^\times$. Therefore $\rho(\sigma)$ is independent to the choice of $\mathfrak{a}_\sigma$, and the map

$$\rho : W_F \to C_F, \quad \sigma \mapsto \rho(\sigma)$$

is a group homomorphism by Lemma 3.1 (i) and Lemma 3.3 (i).

The restriction of $\rho : W_F \to C_F$ to $W_{H_A^+}$ is

$$W_{H_A^+} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \hookrightarrow \mathbb{A}_F^\times \twoheadrightarrow C_F.$$

This homomorphism is continuous since all ρ_λ are continuous. The group $W_{H_A^+}$ has finite index in W_F, so ρ is continuous on W_F. The group C_F is abelian, so ρ factors through the maximal abelian quotient W_F^{ab}, and taking profinite completion yields a continuous homomorphism

$$\hat{\rho} : \mathrm{Gal}_F \to \hat{C}_F$$

that factors through the maximal abelian quotient $\mathrm{Gal}_F^{\mathrm{ab}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Recall that the Artin map $\theta_F : C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ extends to a topological isomophism

$$\hat{\theta}_F : \hat{C}_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

Theorem 2. The map $\hat{\rho} : \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F$ is a topological isomophism independent to the choice of ∞, ε and ϕ, and the map

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \hat{C}_F \quad \sigma \mapsto \hat{\rho}(\sigma)^{-1}$$

is the inverse of the Artin map $\hat{\theta}_F : \hat{C}_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Proof. First, we need an arithmetic input.

Lemma 3.5. Let λ be a place of F, $\mathfrak{p}$ be another place of F that is not λ or ∞. Then $\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}) = 1$.

Remark (Explaination to the notation $\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}})$). Let λ and $\mathfrak{p}$ be places of F with $\mathfrak{p} \neq \infty$. By Theorem 1, the extension H_A^+/F is unramified at all places $\neq \infty$, and the unique $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(H_A^+/F)$ satisfies $\mathrm{Frob}_{\mathfrak{p}}(\phi) = \mathfrak{p} * \phi$. Also by Theorem 1, that Gal_F-action on X_ε factors through $\mathrm{Gal}(H_A^+/F)$, hence any (non-unique) $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}_F$ satisfies $\mathrm{Frob}_{\mathfrak{p}}(\phi) = \mathfrak{p} * \phi$.

Now we begin the proof. We will denote the class of $\alpha \in \mathbb{A}_F^\times$ in C_F by $[\alpha]$.

Let $U < C_F$ be an open subgroup of finite index. The subgroup $\rho^{-1}(U) < W_F^{\mathrm{ab}}$ is open. Consider the finite abelian extension $L_U := (F^{\mathrm{ab}})^{\rho^{-1}(U)}$ of F fixed by this subgroup, so that $\mathrm{Gal}_{L_U}^{\mathrm{ab}}$ = the closure of $\rho^{-1}(U)$ in $\mathrm{Gal}_F^{\mathrm{ab}}$. Hence we have an injective continuous homomorphism [12]

$$\rho_U : \mathrm{Gal}(L_U/F) \simeq \mathrm{Gal}_F^{\mathrm{ab}} / \mathrm{Gal}_{L_U}^{\mathrm{ab}} \simeq W_F^{\mathrm{ab}}/\rho^{-1}(U) \hookrightarrow C_F/U.$$

Let S_U be the set of places consists of

- ∞, and
- $\mathfrak{p}$ for which there exists some idele $\alpha \in \mathcal{O}_\mathfrak{p}^\times \hookrightarrow$ [13] $\mathbb{A}_F^\times$ whose class in C_F is not in U.

Since U is open in C_F, the set S_U is finite.

For a place $\mathfrak{p} \notin S_U$, choose a uniformizer $\pi_\mathfrak{p}$ of $F_\mathfrak{p}$ and consider the idele $\pi_\mathfrak{p} = (\cdots, 1, \pi_\mathfrak{p}, 1, \cdots) \in \mathbb{A}_F^\times$.

Lemma 3.6. C_F/U is generated by $\{\pi_\mathfrak{p}\}_{\mathfrak{p}\notin S_U}$.

Proof of Lemma 3.6. Let V be the preimage of U in $\mathbb{A}_F^\times$, W be the subgroup of $\mathbb{A}_F^\times$ generated by V and $\{\pi_\mathfrak{p}\}_{\mathfrak{p}\notin S_U}$. We need to show that $W = \mathbb{A}_F^\times$.

Take an arbitary $\alpha \in \mathbb{A}_F^\times$. By definition of S_U, $\prod_{\mathfrak{p}\notin S_U} \mathcal{O}_\mathfrak{p}^\times \subset V$, so there is some integer $e \in \mathbb{Z}$ such that

$$\prod_{\mathfrak{p}\in S_U} (1+\mathfrak{m}_p^e) \times \prod_{\mathfrak{p}\notin S_U} \mathcal{O}_\mathfrak{p}^\times \subset V.$$

By weak approximation theorem, there is some $x \in F^\times$, such that $\mathrm{ord}_\mathfrak{p}(\alpha_\mathfrak{p} - x) > \max\{e, \mathrm{ord}_\mathfrak{p}(\alpha_p)\}$ for all $\mathfrak{p} \in S_U$. This implies that $x^{-1}\alpha_\mathfrak{p} \in 1 + \mathfrak{m}_\mathfrak{p}^e$, and thus

$$x^{-1}\alpha \in \prod_{\mathfrak{p}\notin S_U} F_\mathfrak{p}^\times \times \prod_{\mathfrak{p}\in S_U} (1+\mathfrak{m}_p^e) = \prod_{\mathfrak{p}\notin S_U} \mathcal{O}_\mathfrak{p}^\times \pi_\mathfrak{p}^\mathbb{Z} \times \prod_{\mathfrak{p}\in S_U} (1+\mathfrak{m}_\mathfrak{p}^e) \subset W.$$

As $x \in F^\times \subset V \subset W$, we have proved $\alpha \in W$. $\square$

Now consider the idele

$$\beta := \left(\rho_\lambda^\mathfrak{p}(\mathrm{Frob}_\mathfrak{p})\right)_\lambda \cdot \pi_\mathfrak{p} \in \mathbb{A}_F^\times$$

for some $\mathrm{Frob}_\mathfrak{p} \in W_F$. By Lemma 3.5, $\beta_\lambda = 1$ for all $\lambda \neq \mathfrak{p}$. By Lemma 3.1 (iii),

$$\mathrm{ord}_\mathfrak{p}(\beta_\mathfrak{p}) = -\,\mathrm{ord}_\mathfrak{p}(\mathfrak{p}) \cdot 1 = 0.$$

Hence the image of β in C_F is in U, namely $\rho_U(\mathrm{Frob}_\mathfrak{p}) = [\rho(\mathrm{Frob}_\mathfrak{p})] \cdot U = [\pi_\mathfrak{p}^{-1}] \cdot U \in C_F/U$. Consequently,

- L_U/F is unramified at $\mathfrak{p} \notin S_U$, since there is a unique $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(L_U/F)$ by the injectivity of ρ_U;
- $\rho_U : \mathrm{Gal}(L_U/F) \to C_F/U$ is surjective and thus an isomophism.

Next, we show that these L_U are all the finite abelian extensions of F. For each open $U < C_F$ of finite index, the continuous isomophism

$$C_F/U \to \mathrm{Gal}(L_U/F) \quad \alpha \mapsto (\rho_U^{-1}(\alpha))^{-1}$$

[12] I hope these are true..? i.e. if H is a dense subgroup of G and U is open in H, then $H/U \simeq G/\bar{U}$.

[13] $\alpha = (\cdots, 1, \alpha_\mathfrak{p}, 1, \cdots)$ for some $\alpha_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$.

maps $\pi_\mathfrak{p}$ to $\mathrm{Frob}_\mathfrak{p}$. This is Artin map[14]. So

$$\mathrm{Gal}(L_U/F) \to C_F/U \quad \sigma \mapsto \rho_U(\sigma)^{-1}$$

is the inverse to

$$\theta_U : C_F/U \to \mathrm{Gal}(L_U/F) \quad \alpha \mapsto \theta_F(\alpha)|_{L_U},$$

the Artin map at this finite level. If L is a finite abelian extension of F, then the corresponding open subgroup U_L of C_F according to class field theory is the kernel of

$$C_F \to \mathrm{Gal}(L/F) \quad \alpha \mapsto \theta_F(\alpha)|_L.$$

Therefore $L = L_{U_L}$, and $F^{\mathrm{ab}} = \bigcup_U L_U$.

Now we can pass to the limit of the compatible isomophisms ρ_{UU} and go back to see that $\hat{\rho} : \mathrm{Gal}_F^{\mathrm{ab}} \to C_F$ is an isomophism, whose inverse is the "multiplicative inverse" of the Artin map $\hat{\theta}_F$. □

Corollary 3.1. The homomorphism $\rho : W_F^{\mathrm{ab}} \to C_F$ is a topological isomophism, and the map

$$W_F^{\mathrm{ab}} \to C_F \quad \sigma \mapsto \rho(\sigma)^{-1}$$

is the inverse of the Artin map $\theta_F : C_F \to W_F^{\mathrm{ab}}$.

4 Example: the Rational Function Field

Let $F = k(t)$. We consider the usual place ∞ and $A = k[t]$, so that $F_\infty = k((t))$, $\mathbb{F}_\infty = k$, $\mathfrak{m}_\infty = t^{-1}k[[t^{-1}]]$, $\mathrm{ord}_\infty(t^{-1}) = 1$. Let $\varepsilon : F_\infty^\times \to k^\times$ be the unique sign function such that $\varepsilon(t^{-1}) = 1$, so that $F_\infty^+ = t^{\mathbb{Z}}\cdot(1+\mathfrak{m}_\infty)$.

The **Carlitz module** ϕ is defined by

$$\phi : A = k[t] \to F[\tau] \quad t \mapsto \phi_t := t + \tau.$$

It is a Hayes module for ε, and the normalizing field for (F, ∞, ε) is $H_A^+ = F$, so ϕ is the only Hayes module for ε.

We have defined the representations

$$\rho_\lambda : W_F^{\mathrm{ab}} \to F_\lambda^\times \quad \sigma \mapsto \rho_\lambda^A(\sigma)$$

for every place λ of F. For $\lambda \neq \infty$, the representation ρ_λ comes from a continuous Galois representation $\rho_\lambda : \mathrm{Gal}_F \to \mathcal{O}_\lambda^\times$. For ∞, ρ_∞ takes value in F_∞^+. So the isomophism between the (abelianized) Weil group and the idele class group factors as

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \to C_F. \tag{2}$$

Similar to $\mathbb{Q}$, we have an isomophism

$$\mathbb{A}_F^\times \simeq F^\times \times F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times$$

for $F = k(t)$ as follows. Every place $\lambda \neq \infty$ has a "canonical" uniformizer $\mathfrak{p} \in k[t]$, namely the unique monic irreducible polynomial, and we write $x_\mathfrak{p} = u_\mathfrak{p}\mathfrak{p}^{n_\mathfrak{p}}$ with $u_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$. Put

$$f := a_\infty \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}} \in k(t)^\times.$$

[14]See this post on MSE, for instance.

At the place ∞, we write $f^{-1}x_\infty = a_\infty t^n +$ terms with lower degree in t, where $a_\infty \in k$. Then $(a_\infty f)^{-1}x \in F_\infty^+ \times \prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times$. This gives the decomposition above, which implies that the second arrow in (2) is an isomophism, and thus so is the first arrow

$$W_F^{\mathrm{ab}} \xrightarrow{\prod_\lambda \rho_\lambda} \prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times \times t^{\mathbb{Z}} \times (1+\mathfrak{m}_\infty).$$

Taking profinite completion, we got a decomposition

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \simeq \prod_{\lambda\neq\infty} \mathcal{O}_\lambda^\times \times t^{\hat{\mathbb{Z}}} \times (1+\mathfrak{m}_\infty)$$

of $\mathrm{Gal}_F^{\mathrm{ab}}$, which gives three disjoint abelian extension of F whose compositum is F^{ab}.

Description of F^{ab}

Recall that if L/K is an extension of function fields with fields of constants k_L and k_K respectively, we say that:

- L/K is a **constant field extension**, if $L = Kk_L$;
- L/K is a **geometric extension**, if $k_L = k_K$.

The "cyclotomic" extension K_∞

For $\lambda \neq \infty$, the representation $\rho_\lambda : \mathrm{Gal}_F \to \mathcal{O}_\lambda^\times$ is precisely the Galois representation on $T_\lambda(\phi)$, where ϕ is the Carlitz module. The representation

$$\chi := \prod_{\lambda\neq\infty} \rho_\lambda : \mathrm{Gal}_F \to \prod_{\lambda\neq\infty} \mathcal{O}_\lambda = \hat{A}^\times$$

is the inverse limit of

$$\chi_m : \mathrm{Gal}_F \to (A/(m))^\times$$

from Gal_F-action on $\phi[m]$ for all monic irreducible $m \in A = k[t]$, ordered by divisibility. Hence the field fixed by $\ker(\chi)$ is

$$K_\infty = \bigcup_m F(\phi[m]).$$

The extension K_∞/F is a geometric extension, tamely ramified at ∞.

The extension of constants $\bar{k}(t)$

For each $\sigma \in W_F$, the factor in $t^{\mathbb{Z}} \simeq \mathbb{Z}$ is $\mathrm{ord}_t(\rho_\infty(\sigma)) = -\mathrm{ord}_\infty(\rho_\infty(\sigma))$, which equals $-\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma)))$ by (1). By Lemma 3.3, $\phi(\rho_\infty(\sigma)) = \sigma(u)\tau^{\deg(\sigma)}u^{-1}$, so $-\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma))) = \deg(\sigma)$. This shows that the projection $W_F \to \mathbb{Z}$ is precisely the map deg. The field fixed by (the closure of) $\ker(\deg)$ is $\bar{k}(t)$, and the extension $\bar{k}(t)/k(t)$ is the maximal constant field extension.

The wildly ramified extension L_∞

By discussion above, the projection onto $1+\mathfrak{m}_\infty$ is

$$W_F \to 1+\mathfrak{m}_\infty \quad \sigma \mapsto \rho_\infty(\sigma)/\,\mathrm{ord}_t(\rho_\infty(\sigma)) = \rho_\infty(\sigma)/\deg(\sigma).$$

Taking profinite completion, we get a Galois representation $\beta : \mathrm{Gal}_F \to 1+\mathfrak{m}_\infty$. Denote by L_∞ the fixed field of $\ker(\beta)$. The extension L_∞/F is unramified away from ∞ and wildly ramified at ∞.

5 Comparision with Elliptic Curves

6 Proof of (some) lemmas