A. Proof of Soundness

Theorem A.1. *Let $s = s'$ be an instance of one of the tactics introduced in this section. let $a_i = b_i$, $i = 1..k$, be the proof obligations. If $[\![a_i]\!] = [\![b_i]\!]$ for all interpretations of the free variables of a_i and b_i, then $[\![s]\!] = [\![s']\!]$ for all interpretations of the free variables of s and s'.*

Proof. For the tactics with ***Obligations*:** tactic, the theorem is trivial.

> For Stratify, let f, g be partial functions such that

$$\forall \theta, \zeta. \quad f\,(g\,\zeta\,\theta) \;=\; f\,\zeta \quad \wedge \quad g\,(f\,\theta)\,\theta \;=\; h\,\theta$$

Assume that $\zeta = \mathrm{fix}\, f$ and $\theta = \mathrm{fix}(g\,\zeta)$. That is, $f\,\zeta = \zeta$ and $g\,\zeta\,\theta = \theta$. Then —

$$h\,\theta = g\,(f\,\theta)\,\theta = g\,(f\,(g\,\zeta\,\theta))\,\theta = g\,(f\,\zeta)\,\theta = \theta$$

So $\theta = \mathrm{fix}\, h$. We get $\mathrm{fix}\, h = \mathrm{fix}\,\big(g\,(\mathrm{fix}\, f)\big)$; equivalently,

$$\mathrm{fix}\, h = (\mathrm{fix}\, f) \,»\, \big(\psi \mapsto \mathrm{fix}(g\,\psi)\big)$$

Now instantiate h, f, and g, with $f \,»\, g$, f, and g' from (4.1), and we obtain the equality in the tactic.

> For Synth, (*i*) assume $f_i = \mathrm{fix}\, g$ and

$$h :: \mathcal{T} \to \mathcal{Y} = h :: \mathcal{Y} \to \mathcal{Y} = g :: \mathcal{Y} \to \mathcal{T}$$

Intuitively, $\mathcal{Y}$ "cuts out" a region of an array $\theta :: \mathcal{T}$ given as input to h and g. This area is self-contained, in the sense that only elements in $\mathcal{Y}$ are needed to compute elements in $\mathcal{Y}$, as indicated by the refined type $\mathcal{Y} \to \mathcal{Y}$.

Notice that from the premise follows $g :: \mathcal{Y} \to \mathcal{T} = g :: \mathcal{Y} \to \mathcal{Y}$. We use the following corollary:

Corollary. Let $f : \mathcal{T} \to \mathcal{T}$; if either $f :: \mathcal{T} \to \mathcal{Y} = f :: \mathcal{Y} \to \mathcal{Y}$ or $f :: \mathcal{Y} \to \mathcal{T} = f :: \mathcal{Y} \to \mathcal{Y}$, then $(\mathrm{fix}\, f) :: \mathcal{Y} = \mathrm{fix}(f :: \mathcal{Y} \to \mathcal{Y})$.

Proof follows later in this appendix.

From the corollary, and for the given h and g, we learn that $(\mathrm{fix}\, h) :: \mathcal{Y} = \mathrm{fix}(h :: \mathcal{Y} \to \mathcal{Y})$, and also $(\mathrm{fix}\, g) :: \mathcal{Y} = \mathrm{fix}(g :: \mathcal{Y} \to \mathcal{Y})$. Since $h :: \mathcal{Y} \to \mathcal{Y} = g :: \mathcal{Y} \to \mathcal{Y}$, we get $(\mathrm{fix}\, h) :: \mathcal{Y} = (\mathrm{fix}\, g) :: \mathcal{Y}$; now, $\mathcal{Y}$ is a supertype of $\mathcal{T}_i$, so $(\theta :: \mathcal{Y}) :: \mathcal{T}_i = \theta :: \mathcal{T}_i$:

$$\begin{aligned}(\mathrm{fix}\, h) :: \mathcal{T}_i &= ((\mathrm{fix}\, h) :: \mathcal{Y}) :: \mathcal{T}_i = ((\mathrm{fix}\, g) :: \mathcal{Y}) :: \mathcal{T}_i = \\ &= (\mathrm{fix}\, g) :: \mathcal{T}_i = f_i :: \mathcal{T}_i\end{aligned}$$

(ii) Assume $h\,(h\,\theta) :: \mathcal{T}_i = f_i :: \mathcal{T}_i$ holds for any $\theta : \mathcal{T}$, then in particlar, for $\theta = \mathrm{fix}\ h$, we get $h\,(h\ \mathrm{fix}\, h) :: \mathcal{T}_i = f_i :: \mathcal{T}_i$. Since $h\,(h\ \mathrm{fix}\, h) = \mathrm{fix}\, h$, we obtain the conjecture $(\mathrm{fix}\, h) :: \mathcal{T}_i = f_i :: \mathcal{T}_i$. □

Our reliance on the termination of fix expressions may seem conspicuous, since some of these expressions are generated automatically by the system. However, a closer look reveals that whenever such a computation is introduced, the set of the recursive calls it makes is a subset of those made by the existing one. Therefore, if the original recurrence terminates, so does the new one. In any case, all the recurrences in our development have a trivial termination argument (the indexes i,j change monotonically between calls), so practically, this should never become a problem.

We now prove the corollary from the proof of Synth.

Corollary. Let $f : \mathcal{T} \to \mathcal{T}$; if either $f :: \mathcal{T} \to \mathcal{Y} = f :: \mathcal{Y} \to \mathcal{Y}$ or $f :: \mathcal{Y} \to \mathcal{T} = f :: \mathcal{Y} \to \mathcal{Y}$, then $(\mathrm{fix}\, f) :: \mathcal{Y} = \mathrm{fix}(f :: \mathcal{Y} \to \mathcal{Y})$.

Proof.

For the first case, assume $\theta = \mathrm{fix}\, f$,

$$\begin{aligned}\theta :: \mathcal{Y} &= (\theta \,»\, f) :: \mathcal{Y} = \theta \,»\, (f :: \mathcal{T} \to \mathcal{Y}) = \\ &= \theta \,»\, (f :: \mathcal{Y} \to \mathcal{Y}) = (\theta :: \mathcal{Y}) \,»\, (f :: \mathcal{Y} \to \mathcal{Y})\end{aligned}$$

This means that $\theta :: \mathcal{Y} = \mathrm{fix}(f :: \mathcal{Y} \to \mathcal{Y})$, as desired. For the second case, from domain theory we know that $\mathrm{fix}\, f = f^k \bot$ for some $k \geq 1$. We prove by induction that $f^k \bot = (f :: \mathcal{Y} \to \mathcal{Y})^k \bot$.

For $k = 1$,

$$f\,\bot = f\,(\bot :: \mathcal{Y}) = (f :: \mathcal{Y} \to \mathcal{T})\,\bot = (f :: \mathcal{Y} \to \mathcal{Y})\,\bot$$

Assume $f^k \bot = (f :: \mathcal{Y} \to \mathcal{Y})^k \bot$, then definitely $f^k \bot = f^k \bot :: \mathcal{Y}$. Therefore,

$$\begin{aligned}f^{k+1} \bot &= (f^k \bot) \,»\, f = (f^k \bot :: \mathcal{Y}) \,»\, f = \\ &= (f^k \bot) \,»\, (f :: \mathcal{Y} \to \mathcal{T}) = \\ &= ((f :: \mathcal{Y} \to \mathcal{Y})^k \bot) \,»\, (f :: \mathcal{Y} \to \mathcal{Y}) = \\ &= (f :: \mathcal{Y} \to \mathcal{Y})^{k+1} \bot\end{aligned}$$

From this we learn that $\mathrm{fix}\, f = \mathrm{fix}(f :: \mathcal{Y} \to \mathcal{Y}) = (\mathrm{fix}\, f) :: \mathcal{Y}$.

B. More Tactics

Shrink

$$f \;=\; f :: \mathcal{T}$$

Used to specify tighter qualifiers for the type of a sub-term.

***Obligations*:** tactic.

For arrow-typed terms, this essentially requires to prove that f is only defined for arguments in the domain of $\mathcal{T}$, and that the values are in the range of $\mathcal{T}$. This can be seen as a special case of Slice with $r = 1$, with the additional feature of specifying the range as well.

Associativity

$$\mathrm{reduce}\left\langle\, \mathrm{reduce}\langle \overline{x}_1\rangle, \cdots, \mathrm{reduce}\langle \overline{x}_r\rangle \right\rangle \;=\; \mathrm{reduce}\langle \overline{x}_1, \cdots, \overline{x}_r\rangle$$

where reduce is a built-in aggregation ($\min$, $\max$, Σ), and $\overline{x}_i$ are lists of terms (of the same type). If any of $\overline{x}_i$ is of length one, $\mathrm{reduce}\langle \overline{x}_i\rangle$ can be replaced by $\overline{x}_i$.

***Obligations*:** none.

Distributivity

Let e be an expression with a hole, $e[\square] = (\cdots \square \cdots)$.

$$\begin{aligned} e[t_1/\cdots/t_r] &= e[t_1]/\cdots/e[t_r] \\ e[t_1/\cdots/t_r] &= \mathrm{reduce}\langle e[t_1], \cdots, e[t_r]\rangle \\ \mathrm{reduce}\, e[t_1/\cdots/t_r] &= \mathrm{reduce}\langle \mathrm{reduce}\, e[t_1], \cdots, \mathrm{reduce}\, e[t_r]\rangle \end{aligned}$$

This tactic provides several alternatives for different uses of aggregations. Clearly, $/$ does not distribute over any expression; we give just a few examples where this tactic is applicable.

- $(x/y) + 1 \;=\; (x+1)\,/\,(y+1)$
- $x/0 \;=\; \max\langle x, 0\rangle$ (for $x : \mathbb{N}$)
- $\min\left([f]_{J_0} \,/\, [f]_{J_1}\right) \;=\; \min\langle \min[f]_{J_0}\,,\; \min[f]_{J_1}\rangle$

***Obligations*:** tactic.

Elimination

$$e[t] \;=\; e[\bot]$$

Used to eliminate a sub-term that is either always undefined or has no effect in the context in which it occurs.

***Obligations*:** tactic.

Let Insertion

Let e be an expression with a hole, $e[\square] \;=\; (\cdots x_1 \;\mapsto\; \cdots x_k \;\mapsto\; \cdots \square \cdots)$, where $x_{1..k} \;\mapsto$ are abstraction terms enclosing $\square$. The bodies may contain arbitrary terms in addition to these abstractions.

$$\begin{aligned} e[t] \;&=\; (\overline{x} \mapsto t) \;\text{»}\; z \mapsto e[z\,\overline{x}] \\ e[\mathrm{reduce}\langle \overline{a}, \overline{b}\rangle] \;&=\; (\overline{x} \mapsto \mathrm{reduce}\langle \overline{a}\rangle) \;\text{»}\; z \mapsto e[\mathrm{reduce}\langle z\,\overline{x}, \overline{b}\rangle] \end{aligned}$$

where $\overline{x} = x_{1..k}$, and z is a fresh variable. This tactic also has a special version that involves reduce. The items in $\langle \overline{a}, \overline{b}\rangle$ may be interleaved, since $\min, \max, \Sigma$ all happen to be commutative.[6]

***Obligations*:** tactic, if z occurs free in e; otherwise none.

Let Insertion [reduce]

$$e[\mathrm{reduce}\langle \overline{a}, \overline{b}\rangle] \;=\; (\overline{x} \mapsto \mathrm{reduce}\langle \overline{a}\rangle) \;\text{»}\; z \mapsto e[\mathrm{reduce}\langle z\,\overline{x}, \overline{b}\rangle]$$

where $\overline{x} = x_{1..k}$, and z a fresh variable.

***Obligations*:** tactic, if z occurs free in e; otherwise none.

Padding

$$t \;=\; \left(t \,/\, f_1/\cdots/f_r\right) :: \mathcal{T}$$

where $\mathcal{T}$ is the type of t. This tactic is commonly used with Let insertion, to make the type of a sub-computation match the type of the entire term.

***Obligations*:** tactic.

Pull Out

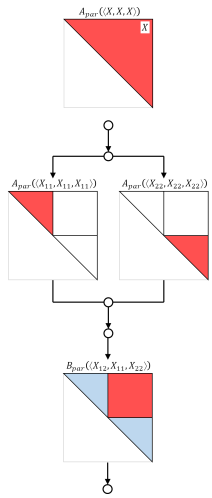
For $e[\square]$ as defined previously:

$$z \;=\; \overline{x} \mapsto t$$

where z is a fresh variable.

Similar to Let Insertion, but does not change the original term; instead, it is used to single out and name a particular expression t, preserving the context in which it occurs in $e[t]$. It is not a tactic *per se*, as it does not actually effect any transformation on $e[t]$; instead, it is designed to increase readability of the development and simplify successive human-computer interaction.

[6] If non-commutative functions get added in the future, then this will change into $\langle \overline{a}, \overline{b}, \overline{c}\rangle$ non-interleaving, with the right hand side being $(\overline{x} \mapsto \mathrm{reduce}\langle \overline{b}\rangle)$ » $z \mapsto e[\mathrm{reduce}\langle \overline{a}, z\,\overline{x}, \overline{c}\rangle]$.



Slice (find ($\theta \mapsto ?$)) (? $\langle J_0 \times J_0, J_0 \times J_1, J_1 \times J_1\rangle$)
Stratify "/" (fixee $\boxed{A}$) $\boxed{A}$ ψ
Stratify "/" (fixee $\boxed{A}$) $\boxed{A}$ ψ
$\boxed{A}$ $\boxed{B}$ $\boxed{C}$ $\mapsto$ SynthAuto . ... ψ

Figure 11. Development of subroutine A of the Parenthesis problem as conceptually described in [11] (top) and using Bellmania (bottom).

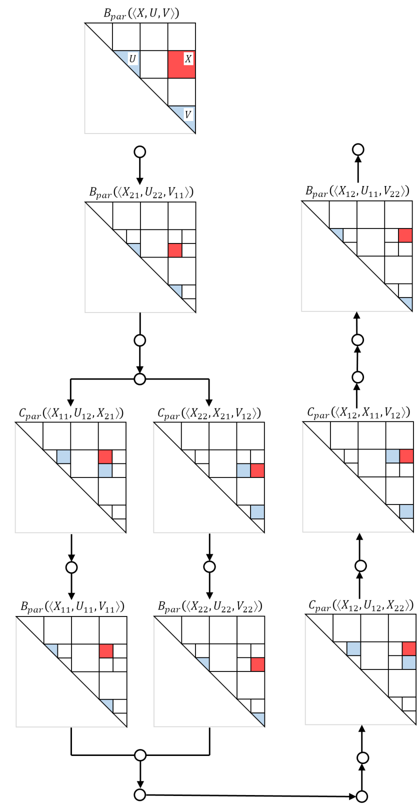
C. Examples

Many aspects of Bellmania are best illustrated via examples. While the main sections include many such examples, some more elaborated ones may prove useful and interesting.

C.1 Development of Parenthesis (with diagrams)

The running example from Sections 1 and 2 has a total of three subroutines, which we label A, B, and C. As promised, we include original design diagrams by the technique's authors taken from [11]. The blocks (triangles) in the diagrams represent intermediate steps of the computation. Below each diagram, we show a transcript of what the user has to type in when using Bellmania to carry out the same development.

Boxed letters in the scripts are used to refer to sub-terms of the current program (in Section 4 we used boxed digits, but in the actual UI we use letters because there are more of them). Reading the scripts in a non-interactive setting might be hard since the reader cannot observe the program; they are listed here just to give an idea of the size and structure. The supplementary material contains some screenshots of an interactive session.



Slice f $\langle ?\times K_0\times K_2, ?\times K_0\times K_3, ?\times K_1\times K_2, ?\times K_1\times K_3\rangle$
$\boxed{D}$ $\mapsto$ Stratify "/" (fixee .) . ψ
$\boxed{C}$ $\mapsto$ Stratify "/" (fixee .) . ψ
$\boxed{E}$ $\mapsto$ Stratify "/" (fixee .) . ψ

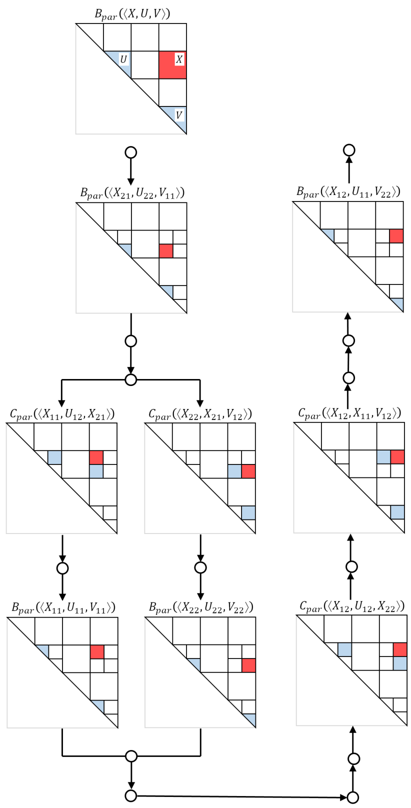
$\langle$ Slice (find ($k \mapsto ?$)) $\langle K_0, K_1, K_2, K_3\rangle$,
Slice (find ($k \mapsto ?$)) $\langle K_1, K_2, K_3\rangle$,
Slice (find ($k \mapsto ?$)) $\langle K_0, K_1, K_2\rangle$ $\rangle$

Distrib min
Assoc min

$\langle$ Stratify min (fixee $\boxed{A}$) $\langle\boxed{G},\boxed{J}\rangle$ ψ,
Stratify min (fixee $\boxed{B}$) $\langle\boxed{M},\boxed{O}\rangle$ ψ,
Stratify min (fixee $\boxed{C}$) $\langle\boxed{R},\boxed{T}\rangle$ ψ $\rangle$
Stratify min (fixee $\boxed{A}$) $\langle\boxed{I},\boxed{K}\rangle$ ψ

$\boxed{I}$ $\boxed{S}$ $\boxed{Z}$ $\boxed{G}$ $\boxed{M}$ $\boxed{P}$ $\boxed{W}$ $\boxed{D}$ $\mapsto$ SynthAuto . ... ψ

Figure 12. Same, for subroutine B of Parenthesis.



Slice (find ($i \mapsto ?$)) $\langle L_0 \times L_4, L_0 \times L, L_1 \times L_4, L_1 \times L_5 \rangle$
Let "/" (slasher $\boxed{A}$) $\boxed{A}$ ψ
Let "/" (slasher $\boxed{A}$) $\boxed{A}$ ψ
Let "/" (slasher $\boxed{A}$) $\boxed{A}$ ψ
Slice (findAll ($k \mapsto ?$)) $\langle L_2, L_3 \rangle$

Distrib min
Assoc min

$\langle$ Let min (slasher $\boxed{A}$) $\langle\boxed{E}, \boxed{G}\rangle$ ψ,
Let min (slasher $\boxed{B}$) $\langle\boxed{H}, \boxed{J}\rangle$ ψ,
Let min (slasher $\boxed{C}$) $\langle\boxed{K}, \boxed{M}\rangle$ ψ,
Let min (slasher $\boxed{D}$) $\langle\boxed{N}, \boxed{P}\rangle$ ψ $\rangle$

$\boxed{A}$ $\boxed{B}$ $\boxed{C}$ $\boxed{D}$ $\boxed{E}$ $\boxed{F}$ $\boxed{G}$ $\boxed{H}$ $\mapsto$ SynthAuto . ... ψ

Figure 13. Same, for subroutine C of Parenthesis

C.2 Qualified Type Inference

We provide an example of how qualifiers are inferred in program terms.

Example

Assume that:

- I, T are types
- $\widehat{I}_0 : I \to \mathbb{B}$ is a unary qualifier
- $0 : T$ is a constant
- $\mathcal{S}$ a type variable,

Consider the term $(f : I_0 \to \mathcal{S})\, i \mapsto f\, i\, i \,/\, 0$. The first step of Hindley-Milner inference will induce the following type shapes through unification:

$$\underbrace{\underbrace{(f : I \to \mathcal{S})^{0}}_{I \to I \to T} \quad \underbrace{i^{0}}_{I} \quad \mapsto \quad \underbrace{\underbrace{\underbrace{\underbrace{f^{1}}_{I \to I \to T} \quad \underbrace{i^{1}}_{I}}_{I \to T} \quad \underbrace{i^{2}}_{I}}_{T} \quad / \quad \underbrace{0}_{T}}_{T}}_{(I \to I \to T) \to I \to T}$$

Superscript numerals denote different occurrences of the same variable. In this case, the type variable $\mathcal{S}$ has been assigned $I \to T$.

The process would have stopped here if it weren't for the qualifier I_0 used in the type for f. At this point we can use type refinements to get more accurate types for f and i in the body of the function term.

$$\frac{f : I_0 \to I \to T,\ i : I \ \vdash\ f^{1} : I \to I \to T}{f : I_0 \to I \to T,\ i : I \ \vdash\ f^{1} : I_0 \to I \to T}$$

Notice that $(I \to I \to T) \sqcap (I_0 \to I \to T) = I_0 \to I \to T$. Truthfully, in this case this is quite a trivial result.

Let $\Gamma = \{f : I_0 \to I \to T,\ i : I\}$.

$$\frac{\Gamma \ \vdash\ (f^{1}\, i^{1}) : I \to T,\ \ f^{1} : I_0 \to I \to T,\ \ i^{1} : I}{\Gamma \ \vdash\ (f^{1}\, i^{1}) : I \to T,\ \ f^{1} : I_0 \to I \to T,\ \ i^{1} : I_0}$$

The types of f^{1} and $f^{1}\, i^{1}$ have not changed, but the type of i^{1} was lowered to I_0.

After applying the typing rules similarly to all the sub-terms, we get the inferred types as shown:

$$\underbrace{\underbrace{(f : I_0 \to \mathcal{S})}_{I_0 \to I \to T} \quad \underbrace{i}_{I} \quad \mapsto \quad \underbrace{\underbrace{\underbrace{\underbrace{f}_{I_0 \to I \to T} \quad \underbrace{i}_{I_0}}_{I_0 \to T} \quad \underbrace{i}_{I}}_{T} \quad / \quad \underbrace{0}_{T}}_{T}}_{(I_0 \to I \to T) \to I \to T}$$

C.3 Simplified Arbiter

To give another example of a development for a different problem, we provide a simplified version of the Gap problem (Section 7), which we call the Simplified Arbiter. Two processes x and y must be scheduled to run n and m seconds, respectively, on a single processor, using one-second slots. Execution starts at $t = 0$. The cost for scheduling the slots $[a..b)$ of x after having scheduled slots $[0..c)$ of y is given by w^x_{abc}, and the cost for schedulting the slots $[a..b)$ of y after scheduling $[0..c)$ of x is given by w^y_{abc}.

In Bellmania language, the Simplified Arbiter example is specified by a base case Ψ and a computation part A^{IJ}, where I and J are row and column index sets, respectively.

$$
\begin{aligned}
\Psi \;&=\; \mathrm{fix}\left(\theta\, i\, j \mapsto [0]_{i=j=0} \,/\, [w^y_{0j0}]_{i=0} \,/\, [w^x_{0i0}]_{j=0}\right) \\
A^{IJ} \;&=\; \psi \mapsto \mathrm{fix}\,\theta \underset{(I)}{i} \underset{(J)}{j} \mapsto \min\langle\, \psi_{ij} \\
&\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\
&\qquad\qquad \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \,\rangle
\end{aligned}
$$

Vertical typeset was used to save some horizontal space, but $\underset{(\mathcal{T})}{v}$ should be read as just $v : \mathcal{T}$.

After Richard applies Slice, he gets the four quadrants $I_0 \times J_0$, $I_0 \times J_1$, $I_1 \times J_0$, $I_1 \times J_1$ (Figure 2). The system defined unary qualifiers with the axioms:

$$
\begin{array}{ll}
\forall i{:}I.\;\; I_0(i) \lor I_1(i) & \forall i_0{:}I_0,\; i_1{:}I_1.\;\; i_0 < i_1 \\
\forall j{:}J.\;\; J_0(j) \lor J_1(j) & \forall j_0{:}J_0,\; j_1{:}J_1.\;\; j_0 < j_1
\end{array}
$$

The program is just about to grow quite large; to make such terms easy to read and refer to, we provide boxed letters as labels for sub-terms, using them as abbreviations where they occur in the larger expression

In addition, to allude to the reader's intuition, expressions of the form $a/b/c/d$ will be written as $\begin{array}{c|c} a & b \\ \hline c & d \end{array}$ when the slices represent quadrants.

Slice

$$
\begin{array}{ll}
f \;=\; \theta\, i\, j \mapsto \cdots & \\
X_1 \;=\; \_ \times I_0 \times J_0 & X_2 \;=\; \_ \times I_0 \times J_1 \\
X_3 \;=\; \_ \times I_1 \times J_0 & X_4 \;=\; \_ \times I_1 \times J_1
\end{array}
$$

(*each "_" is a fresh type variable*)

$$
\begin{aligned}
A^{IJ} \;&=\; \psi \mapsto \mathrm{fix} \begin{array}{c|c} \boxed{A} & \boxed{B} \\ \hline \boxed{C} & \boxed{D} \end{array} \\
\boxed{A} \;&=\; \theta \underset{(I_0)}{i} \underset{(J_0)}{j} \mapsto \min\langle\, \psi_{ij} \\
&\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\
&\qquad\qquad \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \,\rangle \\
\boxed{B} \;&=\; \theta \underset{(I_0)}{i} \underset{(J_1)}{j} \mapsto \min\langle\, \psi_{ij} \\
&\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\
&\qquad\qquad \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \,\rangle \\
\boxed{C} \;&=\; \theta \underset{(I_1)}{i} \underset{(J_0)}{j} \mapsto \min\langle\, \psi_{ij} \\
&\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\
&\qquad\qquad \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \,\rangle \\
\boxed{D} \;&=\; \theta \underset{(I_1)}{i} \underset{(J_1)}{j} \mapsto \min\langle\, \psi_{ij} \\
&\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\
&\qquad\qquad \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \,\rangle
\end{aligned}
\tag{C.1}
$$

Let

$$
e[\square] \;=\; \begin{array}{c|c} \square & \boxed{B} \\ \hline \boxed{C} & \boxed{D} \end{array} \qquad t \;=\; \boxed{A}
$$

$$
A^{IJ} \;=\; \psi \mapsto \mathrm{fix}\left(\boxed{A} \,»\, z \mapsto \begin{array}{c|c} z & \boxed{B} \\ \hline \boxed{C} & \boxed{D} \end{array} \right) \tag{C.2}
$$

Stratify[with Padding]

$$
\begin{array}{ll}
f \;=\; \begin{array}{c|c} \boxed{A} & \widehat{\psi} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} & (\textit{recall that } \widehat{\psi} = \theta \mapsto \psi) \\
g \;=\; z \mapsto \begin{array}{c|c} z & \boxed{B} \\ \hline \boxed{C} & \boxed{D} \end{array} & \psi = \psi
\end{array}
$$

$$
A^{IJ} \;=\; \psi \mapsto \mathrm{fix} \begin{array}{c|c} \boxed{A} & \widehat{\psi} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \,»\, \psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \boxed{B} \\ \hline \boxed{C} & \boxed{D} \end{array} \tag{C.3}
$$

Notice that an existing variable ψ is reused, rebinding any occurrences within $\boxed{B}$, $\boxed{C}$, $\boxed{D}$. This effect is useful, as it limits the context of the expression: the inner ψ shadows the outer ψ, meaning $\boxed{B}$, $\boxed{C}$, $\boxed{D}$ do not need to access the data that was input to $\boxed{A}$, only its output.

The sequence Let, Stratify[with Padding] is now applied in the same manner to $\boxed{B}$ and $\boxed{C}$ (see Figure 4). We do not list the applications as they are analogous to the previous ones.

$$A^{IJ} = \psi \mapsto \mathrm{fix} \begin{array}{c|c} \boxed{A} & \widehat{\psi} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \gg \psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \boxed{B} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \gg$$
$$\psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \boxed{C} & \widehat{\psi} \end{array} \gg \psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \widehat{\psi} & \boxed{D} \end{array} \tag{C.4}$$

Synth

$$\begin{aligned} & h_1 = \boxed{A} \\ & h_{2,3,4} = \widehat{\psi} \\ & f_1 = \theta\ \underset{(I_0)}{i}\ \underset{(J_0)}{j} \mapsto \min \langle\ \psi_{ij} \\ & \qquad\qquad \min \underset{(I_0)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \qquad\qquad \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}\ \rangle \\ & f_{2,3,4} = \widehat{\psi} \end{aligned}$$

$$A^{IJ} = \psi \mapsto \begin{array}{c|c} A_{\psi}^{I_0 J_0} & \psi \\ \hline \psi & \psi \end{array} \gg \psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \boxed{B} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \gg$$
$$\psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \boxed{C} & \widehat{\psi} \end{array} \gg \psi \mapsto \mathrm{fix} \begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \widehat{\psi} & \boxed{D} \end{array} \tag{C.5}$$

We note that $\mathrm{fix}\, f_1 = A_{\psi}^{I_0 J_0}$ are identical (up to β-reduction), which is the whole reason f_1 was chosen. Also, we took the liberty to simplify $\mathrm{fix}\,\widehat{\psi}$ into ψ — although this is not necessary — just to display a shorter term.

The next few tactics will focus on the subterm $\boxed{B}$ from (C.1).

$$\begin{aligned} \boxed{B} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min \langle\ & \psi_{ij} \\ & \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \min \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji}\ \rangle \end{aligned} \tag{C.6}$$

Slice

$$\begin{aligned} & f = \underset{(J)}{q} \mapsto \theta_{iq} + w'_{qji} \\ & X_1 = J_0 \to \_ \qquad X_2 = J_1 \to \_ \end{aligned}$$

$$\begin{aligned} \boxed{B} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min \Big\langle\ & \psi_{ij} \\ & \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \min \big(\big(\underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji} \big) \,/ \\ & \qquad \big(\underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji} \big) \big)\ \Big\rangle \end{aligned} \tag{C.7}$$

For the intuition behind this, see the top-right part of Figure 14. The colors represent cell ranges that will be read by different sub-routines (presumably running on different cores). The range of q is split into the part that lies within $\boxed{1}$ ($q \in J_0$) and the one that lies within $\boxed{2}$ ($q \in J_1$). The same reasoning is applied to the other quadrants.

Distributivity

$$\begin{aligned} & e[\square] = \min \square \\ & t_1 = \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji} \\ & t_2 = \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji} \end{aligned}$$

Associativity

$$\begin{aligned} & \mathrm{reduce} = \min \\ & \overline{x}_1 = \psi_{ij} \\ & \overline{x}_2 = \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij} \\ & \overline{x}_3 = \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}\ , \\ & \qquad \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji} \end{aligned}$$

$$\begin{aligned} \boxed{B} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min \Big\langle\ & \psi_{ij} \\ & \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}, \\ & \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji}\ \Big\rangle \end{aligned} \tag{C.8}$$

Let[reduce]

$$\begin{aligned} & e[\square] = \begin{array}{c|c} \widehat{\psi} & \theta\, i\, j \mapsto \square \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \\ & \overline{a} = \psi_{ij},\ \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji} \\ & \overline{b} = \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \qquad \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji} \end{aligned}$$

$$\begin{aligned} & \begin{array}{c|c} \widehat{\psi} & \boxed{B} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} = \mathrm{fix} \left(\boxed{E} \gg z \mapsto \begin{array}{c|c} \widehat{\psi} & \boxed{F} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \right) \\ & \boxed{E} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle \psi_{ij}, \\ & \qquad\qquad \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji} \rangle \\ & \boxed{F} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle z_{\theta ij}, \\ & \qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij}, \\ & \qquad\qquad \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji} \rangle \end{aligned} \tag{C.9}$$

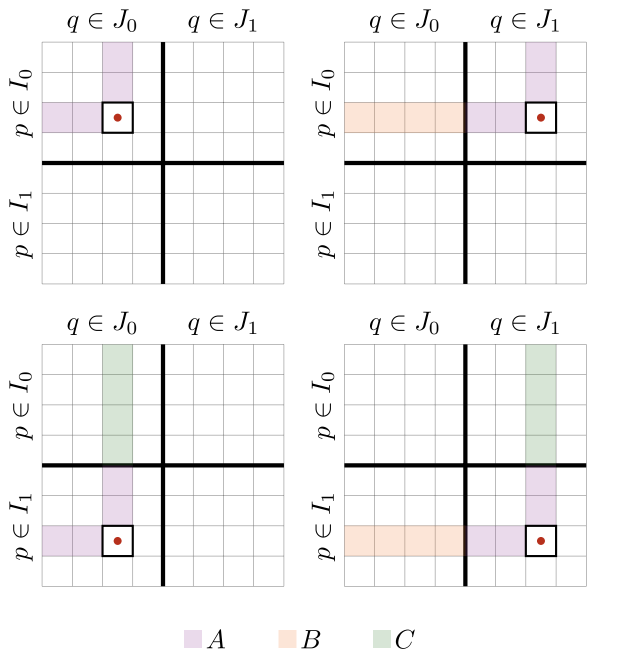


Figure 14. The strategy for applications of Slice in the case study.

$$
\boxed{
\begin{aligned}
&\underline{\textsf{Stratify[with Padding]}}\\
&f \;=\; \begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{E}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array}\\
&g \;=\; z \mapsto \begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{F}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \qquad \psi = \psi
\end{aligned}
}
$$

$$
\begin{aligned}
&\text{fix}\,\begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{B}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} = \text{fix}\,\begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{E}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} \text{»}\, \psi \mapsto \text{fix}\,\begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{F}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array}\\
&\boxed{\mathrm{E}} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad\qquad \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}\rangle\\
&\boxed{\mathrm{F}} = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad\qquad \min \underset{(I)}{p} \mapsto \theta_{pj} + w_{pij},\\
&\qquad\qquad\qquad \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji}\rangle
\end{aligned}
\tag{C.10}
$$

Define

$$
\begin{aligned}
B^{IJ_0J_1} \;=\; &\big(\psi \mapsto\\
&\text{fix}\,\theta\ \underset{(I)}{i}\ \underset{(J)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}\rangle\big)\\
&:: \big((I \times J_0) \to \mathbb{R}\big) \to \big((I \times J_1) \to \mathbb{R}\big)
\end{aligned}
\tag{C.11}
$$

$$
\boxed{
\begin{aligned}
&\underline{\textsf{Synth}}\\
&h_2 = \boxed{\mathrm{E}}\\
&h_{1,3,4} = \widehat{\psi}\\
&f_2 = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad\qquad \min \underset{(J_0)}{q} \mapsto \theta_{iq} + w'_{qji}\rangle\\
&f_{1,3,4} = \widehat{\psi}
\end{aligned}
}
$$

$$
\boxed{
\begin{aligned}
&\underline{\textsf{Synth}}\\
&h_2 = \boxed{\mathrm{F}}\\
&h_{1,3,4} = \widehat{\psi}\\
&f_2 = \theta\ \underset{(I_0)}{i}\ \underset{(J_1)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad\qquad \min \underset{(I_0)}{p} \mapsto \theta_{pj} + w_{pij},\\
&\qquad\qquad\qquad \min \underset{(J_1)}{q} \mapsto \theta_{iq} + w'_{qji}\rangle\\
&f_{1,3,4} = \widehat{\psi}
\end{aligned}
}
$$

$$
\text{fix}\,\begin{array}{c|c} \widehat{\psi} & \boxed{\mathrm{B}} \\ \hline \widehat{\psi} & \widehat{\psi} \end{array} = \begin{array}{c|c} \psi & B_{\psi}^{I_0J_0J_1} \\ \hline \psi & \psi \end{array} \text{»}\, \psi \mapsto \begin{array}{c|c} \psi & A_{\psi}^{I_0J_1} \\ \hline \psi & \psi \end{array}
\tag{C.12}
$$

In a similar manner, we will obtain the following:

$$
\text{fix}\,\begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \boxed{\mathrm{C}} & \widehat{\psi} \end{array} = \begin{array}{c|c} \psi & \psi \\ \hline C_{\psi}^{I_0I_1J_0} & \psi \end{array} \text{»}\, \psi \mapsto \begin{array}{c|c} \psi & \psi \\ \hline A_{\psi}^{I_1J_0} & \psi \end{array}
\tag{C.13}
$$

$$
\begin{aligned}
C^{I_0I_1J} \;=\; &\big(\psi \mapsto\\
&\text{fix}\,\theta\ \underset{(I)}{i}\ \underset{(J)}{j} \mapsto \min\langle \psi_{ij},\\
&\qquad\qquad \min \underset{(I_0)}{p} \mapsto \theta_{pj} + w_{pij}\rangle\big)\\
&:: \big((I_0 \times J) \to \mathbb{R}\big) \to \big((I_1 \times J) \to \mathbb{R}\big)
\end{aligned}
\tag{C.14}
$$

And —

$$
\begin{aligned}
\text{fix}\,\begin{array}{c|c} \widehat{\psi} & \widehat{\psi} \\ \hline \widehat{\psi} & \boxed{\mathrm{D}} \end{array} &= \begin{array}{c|c} \psi & \psi \\ \hline \psi & B_{\psi}^{I_1J_0J_1} \end{array} \text{»}\, \psi \mapsto \begin{array}{c|c} \psi & \psi \\ \hline \psi & C_{\psi}^{I_0I_1J_1} \end{array}\\
&\quad \text{»}\, \psi \mapsto \begin{array}{c|c} \psi & \psi \\ \hline \psi & A_{\psi}^{I_1J_1} \end{array}
\end{aligned}
\tag{C.15}
$$

This gives the stratified version as shown in Figure 15. The read and write regions are already encoded in the types of A, B, C in (C.5), (C.12), (C.13), and (C.15).

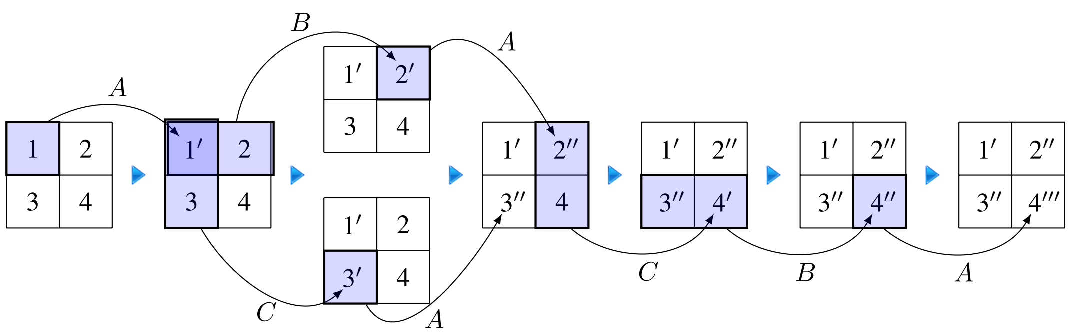


Figure 15. Fully divide-and-conquered version of A^{IJ} in the example development.