Mathematics & Statistics | Lancaster University

# An introduction to changepoints

## Using R

Rebecca Killick(r.killick@lancs.ac.uk)
eRum Workshop 2016

# Workshop Plan



- What are changepoints?
- Notation
- Likelihood based changepoints
    - Change in mean
    - Change in variance
    - (coffee break)
    - Change in mean & variance
- How many changes?
- Non-parametric changepoints
- Checking assumptions (if time allows)

There will be tasks throughout the sections

# What are Changepoints?



Changepoints are also known as:

- breakpoints
- segmentation
- structural breaks
- regime switching
- detecting disorder

and can be found in a wide range of literature including

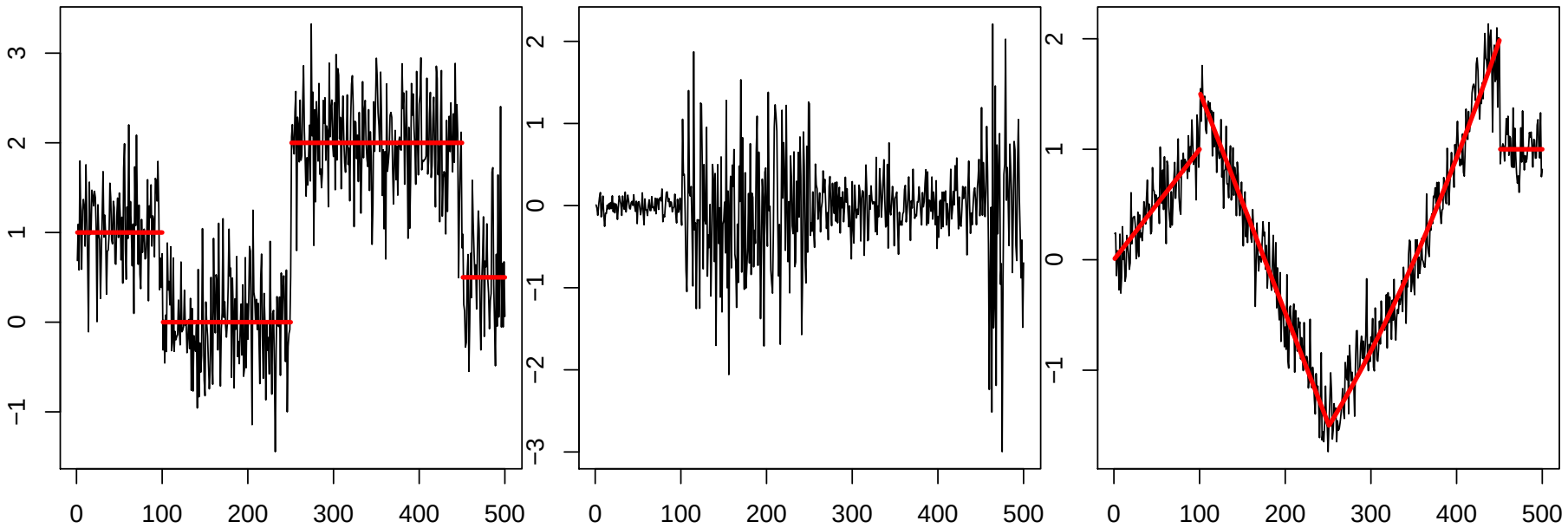
- quality control
- economics
- medicine
- environment
- linguistics
- ...

# What are changepoints?

Mathematics & Statistics | Lancaster University

For data $y_1, \ldots, y_n$, if a changepoint exists at $\tau$, then $y_1, \ldots, y_\tau$ differ from $y_{\tau+1}, \ldots, y_n$ in some way.

There are many different types of change.



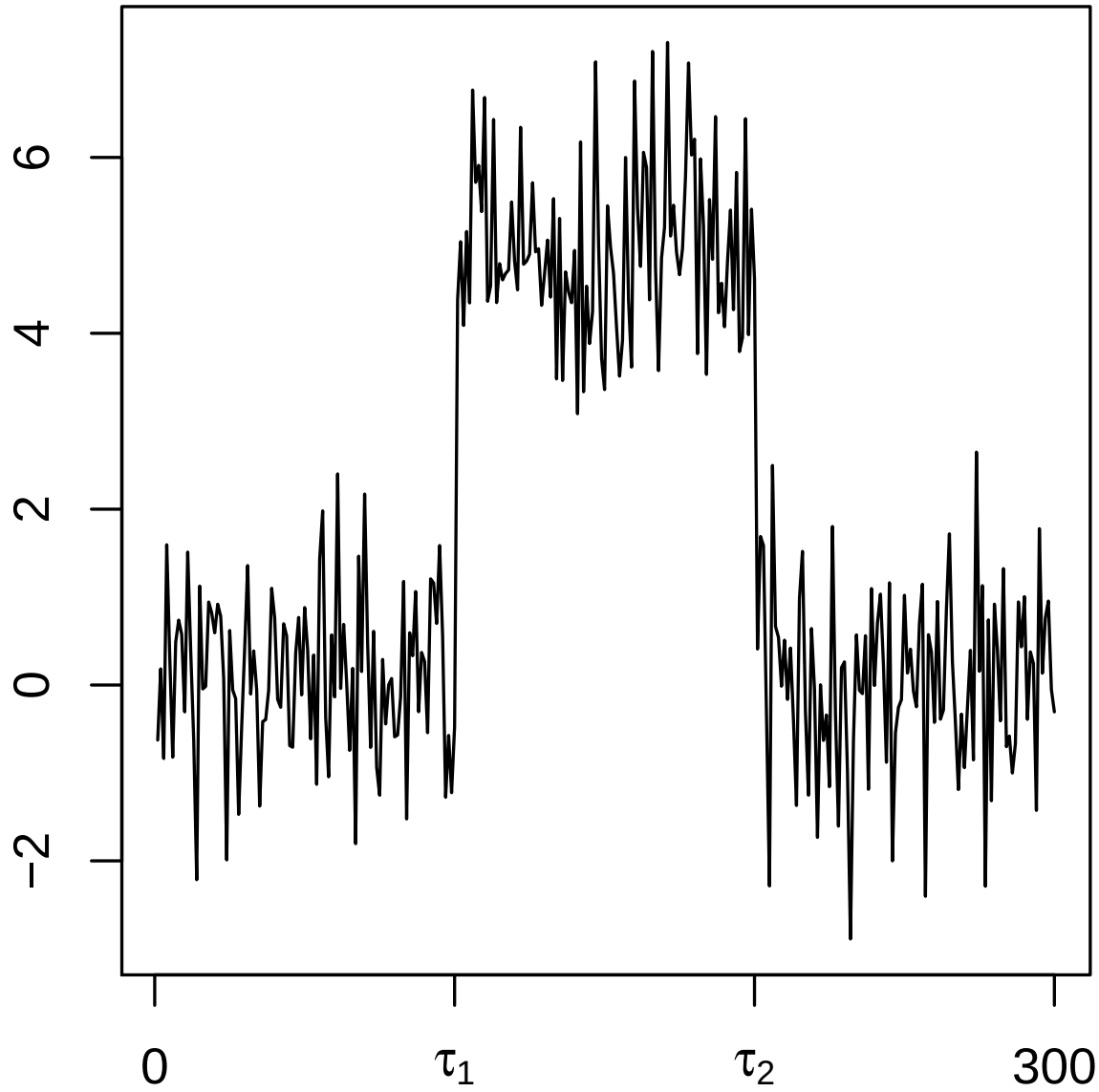
# What is the goal?



- Has a change occurred?
- If yes, where is the change?
- What is the difference between the pre and post change data?
  - Maybe this is the type of change
  - Maybe it is the parameter values before and after the change
- What is the probability that a change has occured?
- How certain are we of the changepoint location?
- How many changes have occurred (+ all the above for each change)?
- Why has there been a change?

# Notation and Concepts





# Notation and Concepts

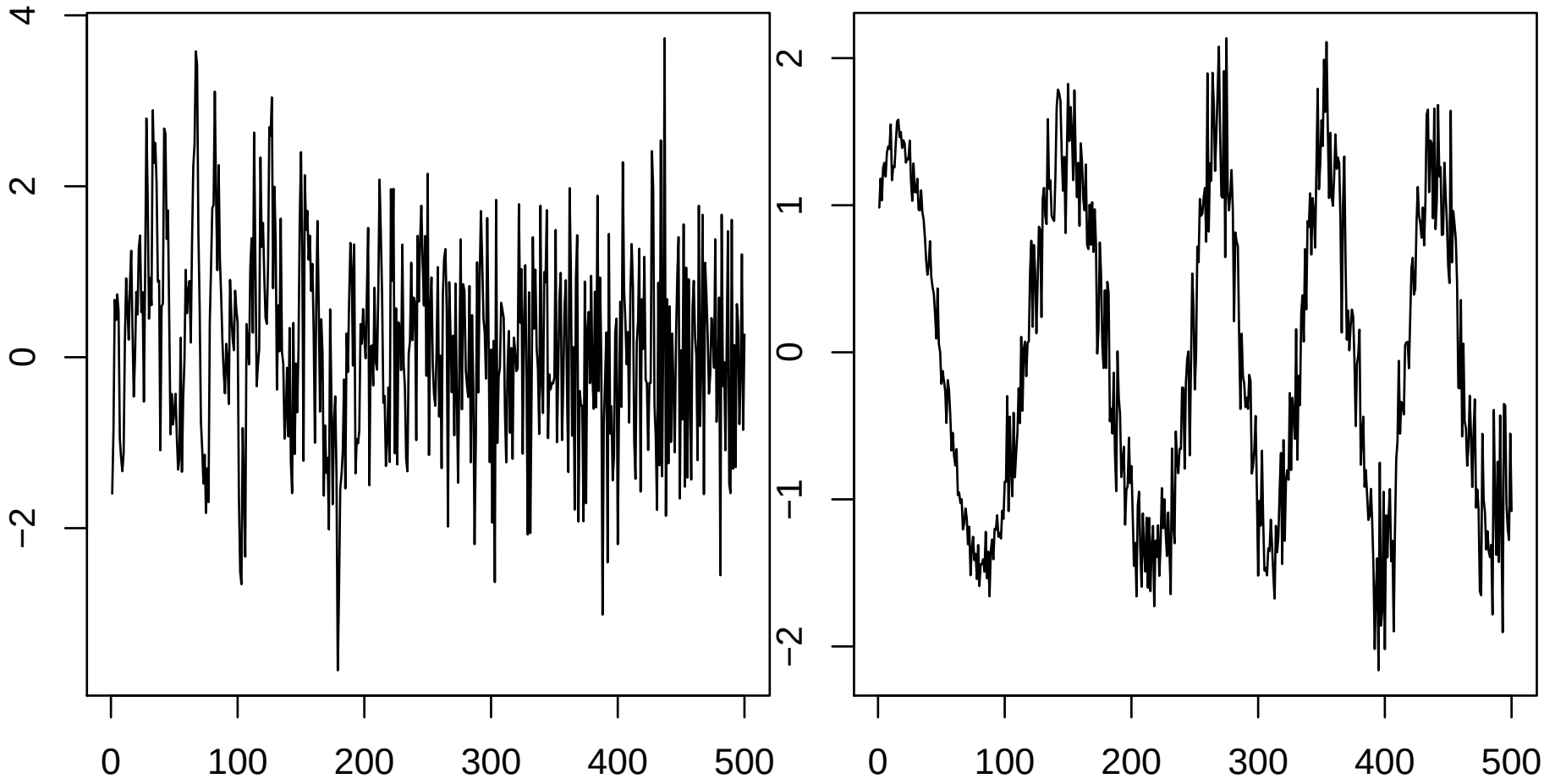


Thus a changepoint model for a change in mean has the following formulation:

$$y_t = \begin{cases} \mu_1 & \text{if} \quad 1 \leq t \leq \tau_1 \\ \mu_2 & \text{if} \quad \tau_1 < t \leq \tau_2 \\ \vdots & \quad \vdots \\ \mu_{m+1} & \text{if} \quad \tau_m < t \leq \tau_{m+1} = n \end{cases}$$

# More complicated changes





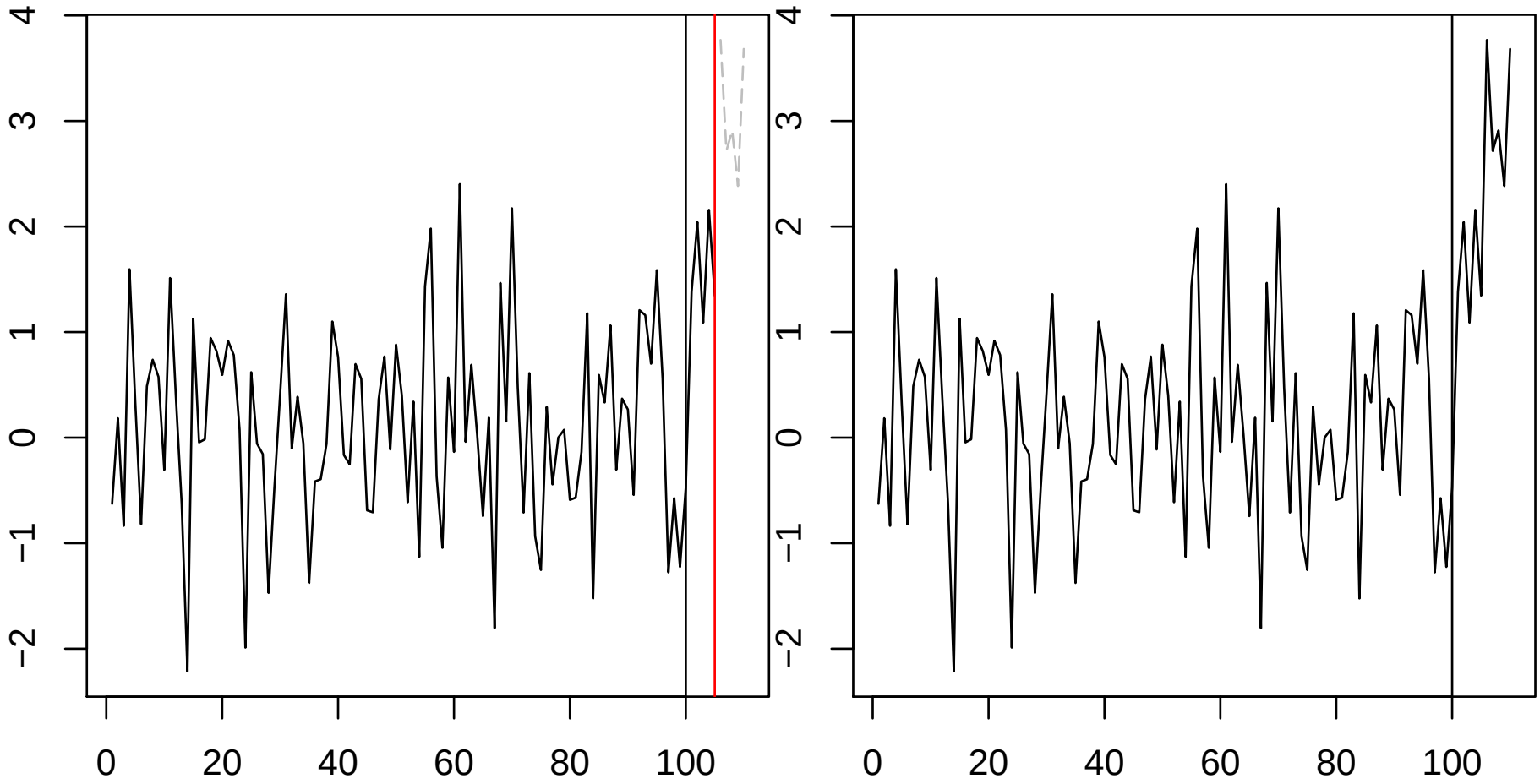
# Online vs Offline



- Online
  - Processes data as it arrives or in batches
  - Goal is quickest detection of a change
  - Often used in processing control, intrusion detection
- Offline
  - Processes all the data in one go
  - Goal is accurate detection of a change
  - Often used in genome analysis, audiology

# Online vs Offline





# Packages



Today we will use the

```
library(changepoint)
```

```
library(changepoint.np)
```

packages.

Other notable `R` packages are available for changepoint analysis including

- `strucchange` - for changes in regression
- `bcp` - if you want to be Bayesian
- `cpm` - for online changes (`changepoint.online` coming soon)

See my talk tomorrow in the **15:15** session for AR(1) and trend detection.

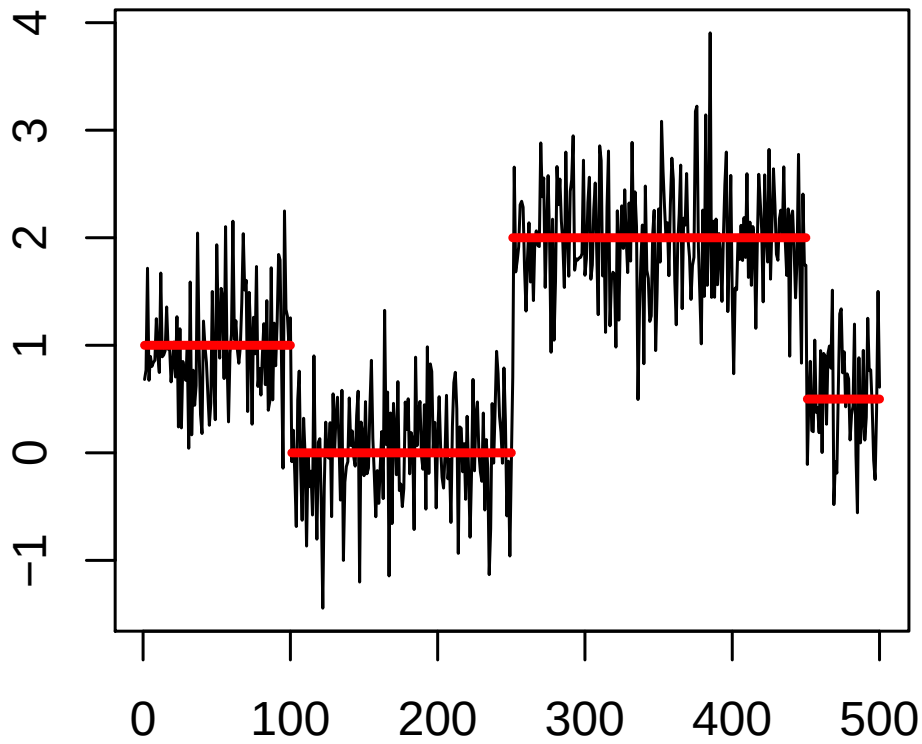
# Change in mean



Assume we have time-series data where

$$Y_t|\theta_t \sim \mathrm{N}(\theta_t, 1),$$

but where the means, $\theta_t$, are piecewise constant through time.



# Inferring Changepoints



We want to infer the number and position of the points at which the mean changes. One approach:

**Likelihood Ratio Test**

To detect a single changepoint we can use the likelihood ratio test statistic:

$$LR = \max_{\tau}\{\ell(y_{1:\tau}) + \ell(y_{\tau+1:n}) - \ell(y_{1:n})\}.$$

We infer a changepoint if $LR > \beta$ for some (suitably chosen) $\beta$. If we infer a changepoint its position is estimated as

$$\tau = \arg\max\{\ell(y_{1:\tau}) + \ell(y_{\tau+1:n}) - \ell(y_{1:n})\}.$$

# changepoint R package



The changepoint R package contains 3 wrapper functions:

- cpt.mean - mean only changes
- cpt.var - variance only changes
- cpt.meanvar - mean and variance changes

The package also contains:

- functions/methods for the cpt S4 class
- 5 data sets
- 4 other R functions that are made available for those who know what they are doing and might want to extend/modify the package.

# The `cpt` class

Mathematics & Statistics | Lancaster University

- S4 class
- Slots store all the information from the analysis
  - e.g. data.set, cpts, param.est, pen.value, ncpts.max
- Slots are accessed via their names e.g. `cpts(x)`
- Standard methods are available for the class e.g. plot, summary
- Additional generic functions are available e.g. seg.len, ncpts
- Each core function outputs a `cpt` object

# cpt.mean



```
cpt.mean(data, penalty="MBIC", pen.value=0,
method="AMOC", Q=5, test.stat="Normal", class=TRUE,
param.estimates=TRUE,minseglen=1)
```

- `data` - vector or `ts` object
- `penalty` - cut-off point, MBIC, SIC, BIC, AIC, Hannan-Quinn, Asymptotic, Manual.
- `pen.value` - Type I error for Asymptotic, number or character for manual.
- `method` - AMOC, PELT, SegNeigh, BinSeg.
- `Q` - max number of changes for SegNeigh or BinSeg.
- `test.stat` - Test statistic, Normal or CUSUM.
- `class` - return a `cpt` object or not.
- `param.estimates` - return parameter estimates or not.
- `minseglen` - minimum number of data points between changes.

# Single Change in Mean

Mathematics & Statistics | Lancaster University

```
set.seed(1)
m1=c(rnorm(100,0,1),rnorm(100,5,1))
m1.amoc=cpt.mean(m1)
cpts(m1.amoc)
```

```
## [1] 100
```

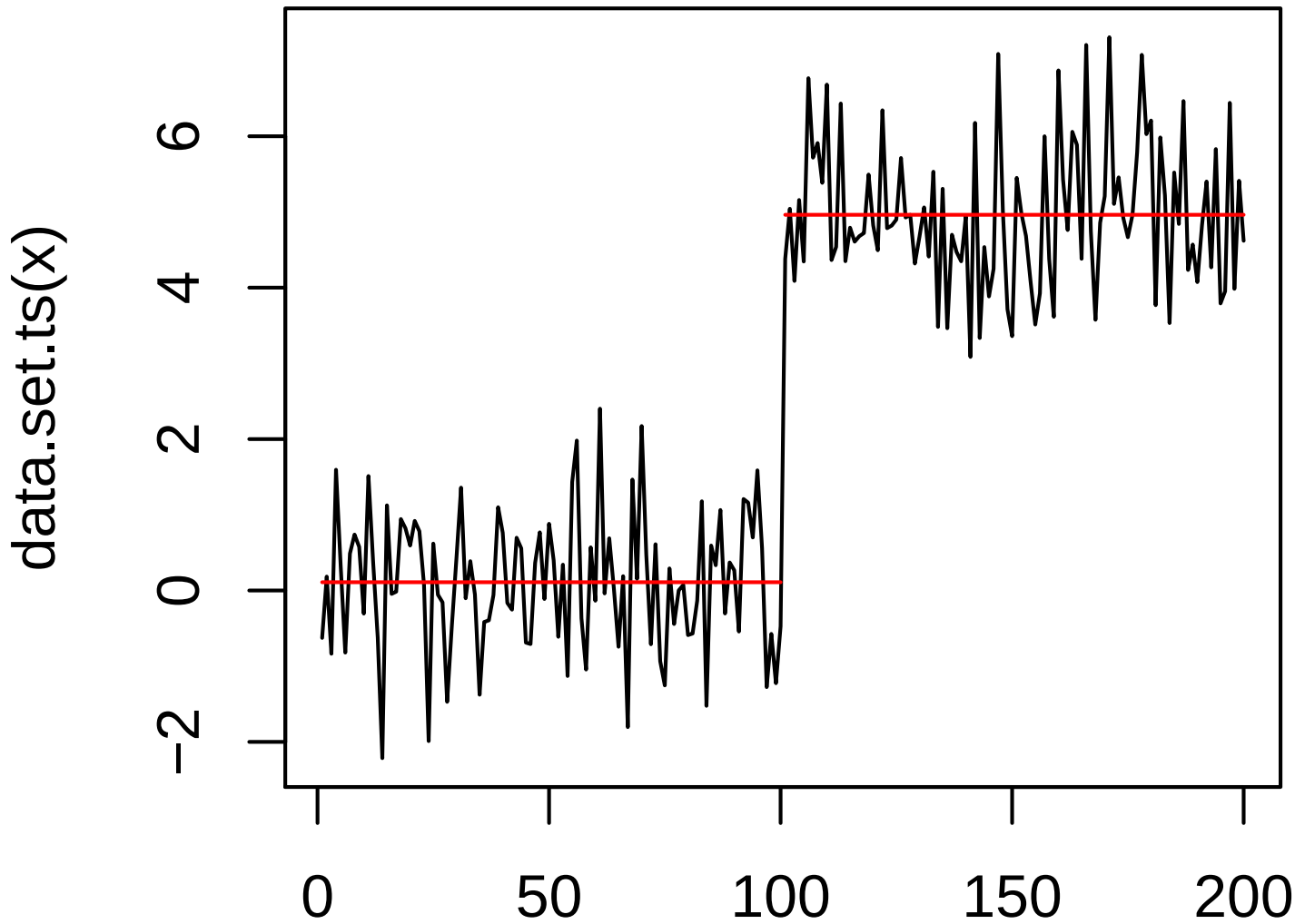
```
m1.cusum=cpt.mean(m1,pen.value=1,penalty='Manual',
                  test.stat='CUSUM')
```

```
## Warning in cpt.mean(m1, pen.value = 1, penalty = "Manual",
## "CUSUM"): Traditional penalty values are not appropriate f
## statistic
```

# Single Change in Mean



```
plot(m1.amoc)
```

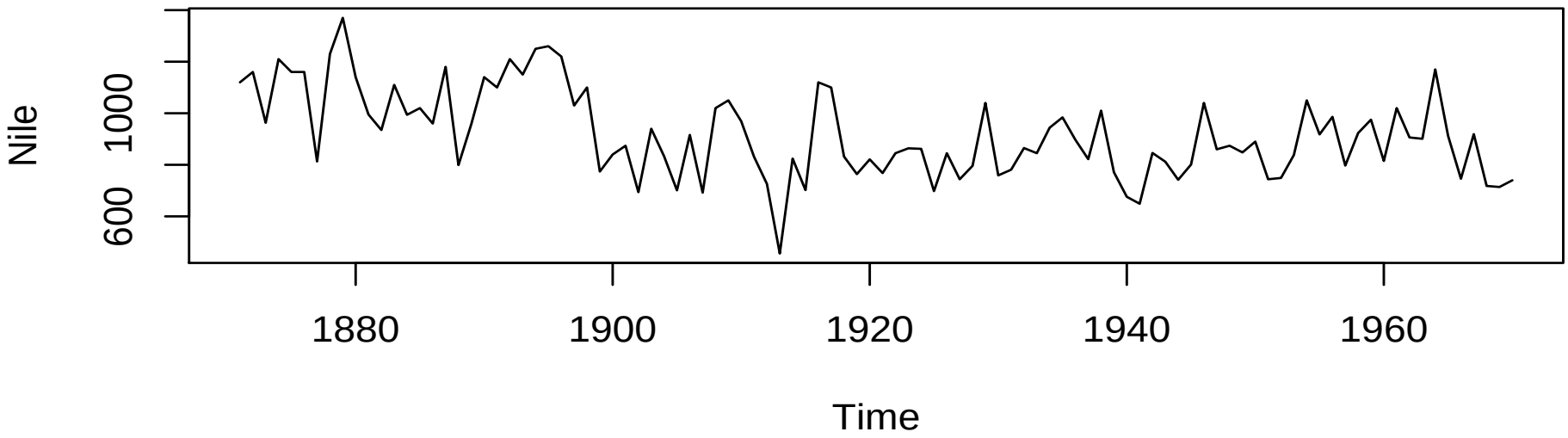


# Example: Nile



Data from Cobb (1978): readings of the annual flow volume of the Nile River at Aswan from 1871 to 1970.

```
data(Nile)
ts.plot(Nile)
```




Hypothesized that there was a change around the turn of the century.

# Task



Use the `cpt.mean` function to see if there is evidence for a change in mean in the Nile river data.

```
data(Nile)
```

If you identify a change, where is it and what are the pre and post change means?

# Example: Nile

Mathematics & Statistics | Lancaster University

Annual flow volume of the Nile River at Aswan from 1871 to 1970 studied in Cobb(1978).

```
nile.default=cpt.mean(Nile)
cpts(nile.default)
```

```
## [1] 28
```

```
cpts.ts(nile.default)
```

```
## [1] 1898
```

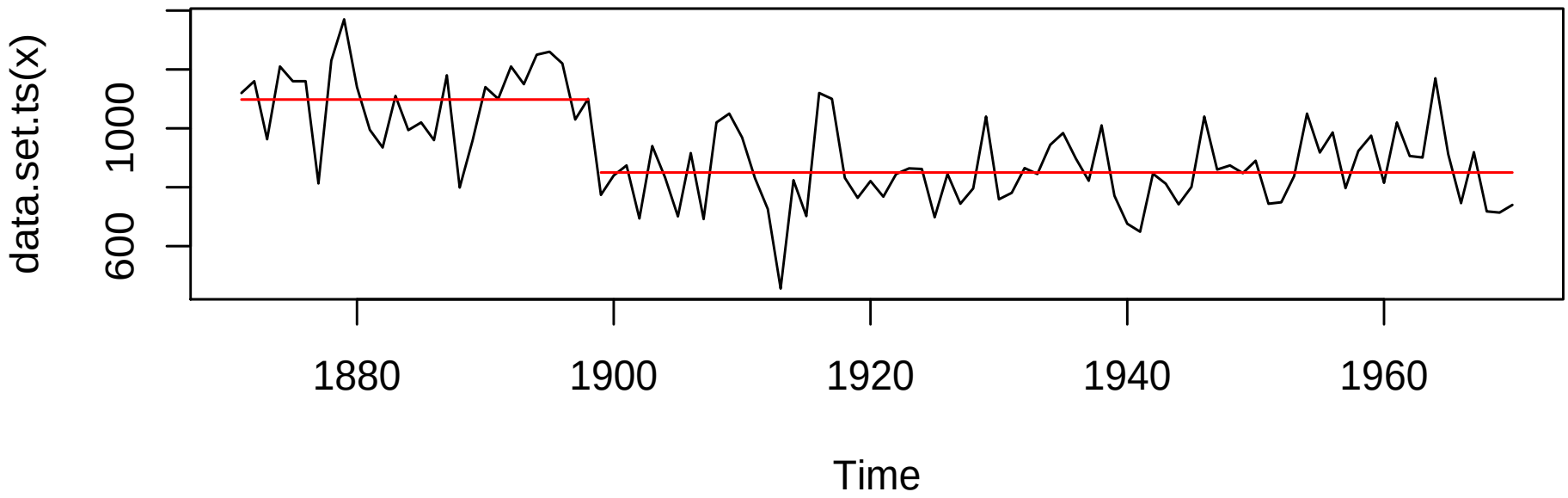
```
param.est(nile.default)
```

```
## $mean
## [1] 1097.7500  849.9722
```

# Example: Nile



```
plot(nile.default)
```



# Multiple Changepoints

# Likelihood Ratio Test



Define $m$ to be the number of changepoints, with positions $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{m+1})$ where $\tau_0 = 0$ and $\tau_{m+1} = n$.

Then one application of the Likelihood ratio test can be viewed as

$$\min_{m \in \{0,1\}, \boldsymbol{\tau}} \left\{ \sum_{i=1}^{m+1} \left[ -\ell(y_{\tau_{i-1}:\tau_i}) \right] + \beta m \right\}$$

Repeated application is thus aiming to minimise

$$\min_{m, \boldsymbol{\tau}} \left\{ \sum_{i=1}^{m+1} \left[ -\ell(y_{\tau_{i-1}:\tau_i}) \right] + \beta m \right\}$$

# Penalised Likelihood



The above can be viewed as a special case of penalised likelihood. Here the aim is to maximise the *likelihood* over the number and position of the changepoints, but *subject to* a penalty, that depends on the number of changepoints. The penalty is to avoid over-fitting.

This is equivalent to minimising

$$\min_{m,\boldsymbol{\tau}} \left\{ \sum_{i=1}^{m+1} \left[ -\ell(y_{\tau_{i-1}:\tau_i}) \right] + \beta f(m) \right\}$$

for a suitable penalty function $f(m)$ and penalty constant $\beta$.

# Identifying changes?



All these methods can be cast in terms of minimising a function of $m$ and $\tau$ of the form:

$$\sum_{i=1}^{m+1} \left[ \mathcal{C}(y_{(\tau_{i-1}+1):\tau_i}) \right] + \beta f(m).$$

This function depends on the data just through a sum of a *cost* for each segment. There is then a penalty term that depends on the number of segments.

## Open Research Question

What penalty should I use?

Several have attempted to answer this question, but in reality have added their own criteria to the list. At best, we have specific criteria shown to be optimal in very specific settings.

# The Challenge



- What are the values of $\tau_1, \ldots, \tau_m$?
- What is $m$?
- For $n$ data points there are $2^{n-1}$ possible solutions
- If $m$ is known there are still $\binom{n-1}{m-1}$ solutions
- If $n = 1000$ and $m = 10$, $2.634096 \times 10^{21}$ solutions
- How do we search the solution space efficiently?

# Methods in `changepoint`



- At Most One Change (`AMOC`)

*Approximate* but computationally *fast*:

- Binary Segmenation (`BinSeg`) (Scott and Knott (1974)) which is $\mathcal{O}(n \log n)$ in CPU time.

*Slower* but *exact*:

- Segment Neighbourhood (`SegNeigh`) (Auger and Lawrence (1989)) is $\mathcal{O}(Qn^2)$.

*Fast* and *exact*:

- Pruned Exact Linear Time (`PELT`) (Killick et al. (2012)) At worst $\mathcal{O}(n^2)$. For linear penalties $f(m) = m$, scaling changes, $\mathcal{O}(n)$.

# cpt.var



```
cpt.var(data, penalty, pen.value, know.mean=FALSE, mu=NA,
method, Q, test.stat="Normal", class, param.estimates,
minseglen=2)
```

Majority of arguments are the same as for `cpt.mean`

- `know.mean` - if known we don't count it as an estimated parameter when calculating penalties.
- `mu` - Mean if known.
- `test.stat` - Normal or CSS (cumulative sums of squares)
- `minseglen` - Default is 2

# Changes in Variance

Mathematics & Statistics | Lancaster University

```
set.seed(1)
v1=c(rnorm(100,0,1),rnorm(100,0,2),rnorm(100,0,10),
     rnorm(100,0,9))
v1.man=cpt.var(v1,method='PELT',penalty='Manual',
     pen.value='2*log(n)')
cpts(v1.man)
param.est(v1.man)
```

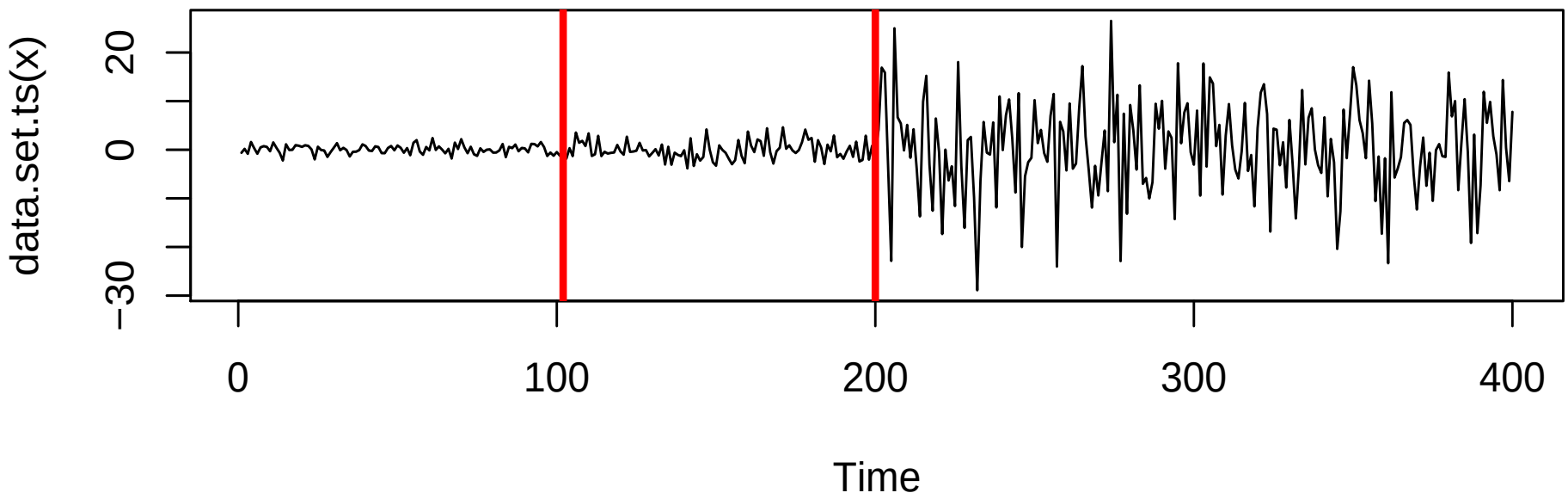
```
## [1] 102 200
## $variance
## [1]  0.8007158  3.6933616 92.3876410
##
## $mean
## [1] 0.1986058
```

# Changes in Variance

Mathematics & Statistics | Lancaster University

Ratios of true variances (4, 25, 0.81)

```
plot(v1.man,cpt.width=3)
```

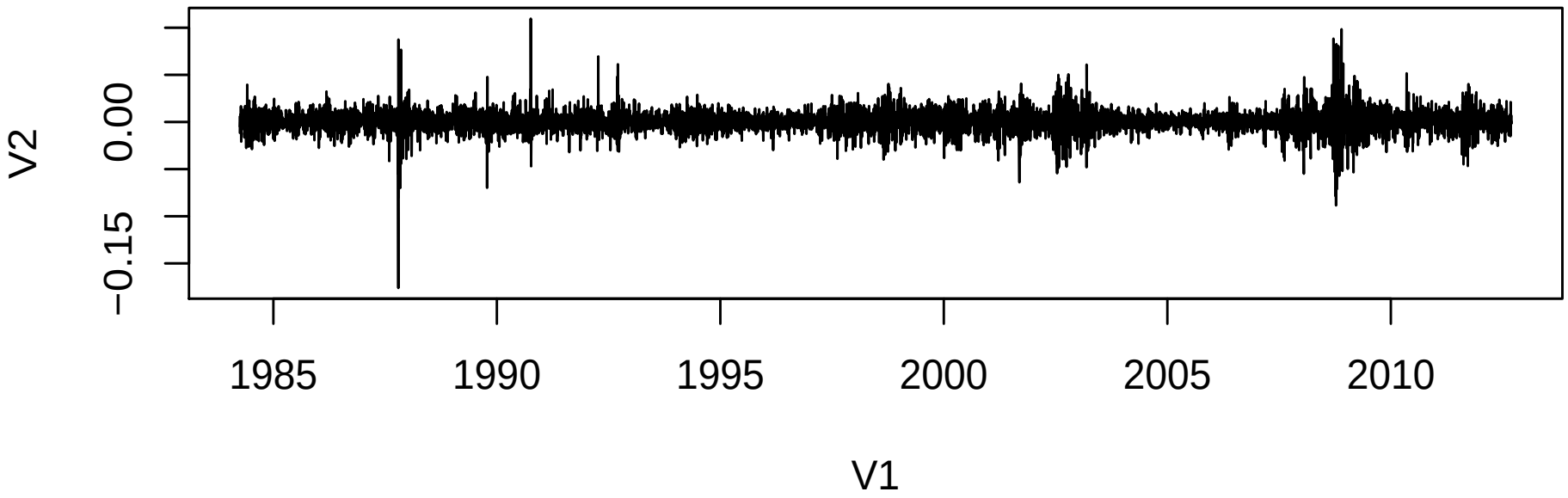


# FTSE100



Yahoo! Finance data, daily returns from FTSE100 index. 2nd April 1984 until the 13th September 2012

```
data(ftse100)
plot(ftse100,type='l')
```



# Task



Use the `cpt.var` function to see if there is evidence for changes in variance in the FTSE100 data.

```
data(ftse100)
```

If you identify changes, where are they and what are the variances in each segment?

# Example: FTSE100



Yahoo! Finance data, daily returns from FTSE100 index.

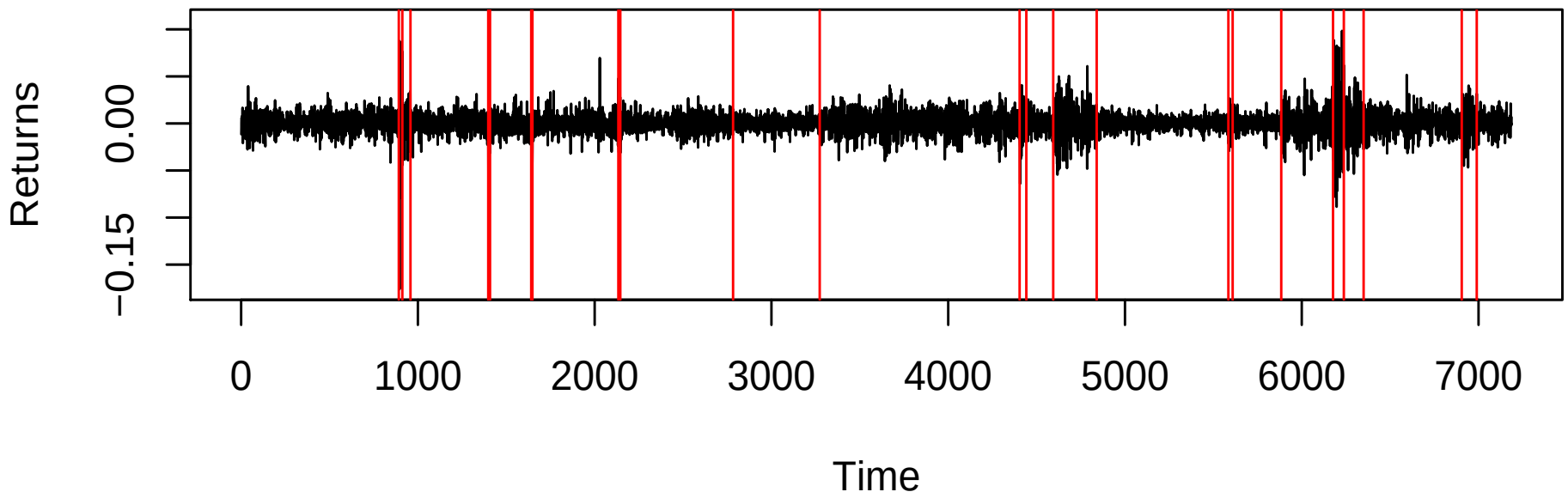
```
data(ftse100)
ftse.man=cpt.var(ftse100[,2],method='PELT',minseglen=7)
ncpts(ftse.man)
```

```
## [1] 23
```

# Example: FTSE100



```
plot(ftse.man,ylab='Returns')
```



# cpt.meanvar



```
cpt.meanvar(data, penalty, pen.value, method, Q,
test.stat="Normal", class, param.estimates,
shape=1,minseglen=2)
```

Again the same underlying structure as `cpt.mean`.

- `test.stat` - choice of Normal, Gamma, Exponential, Poisson.
- `shape` - assumed shape parameter for Gamma.
- `minseglen` - minimum segment length of 2

# Mean & Variance



```
set.seed(1)
mv1=c(rexp(50,rate=1),rexp(50,5),rexp(50,2),rexp(50,7))
mv1.pelt=cpt.meanvar(mv1,test.stat='Exponential',
      method='BinSeg',Q=10,penalty="SIC")
cpts(mv1.pelt)
```

```
## [1]  50 100 150
```

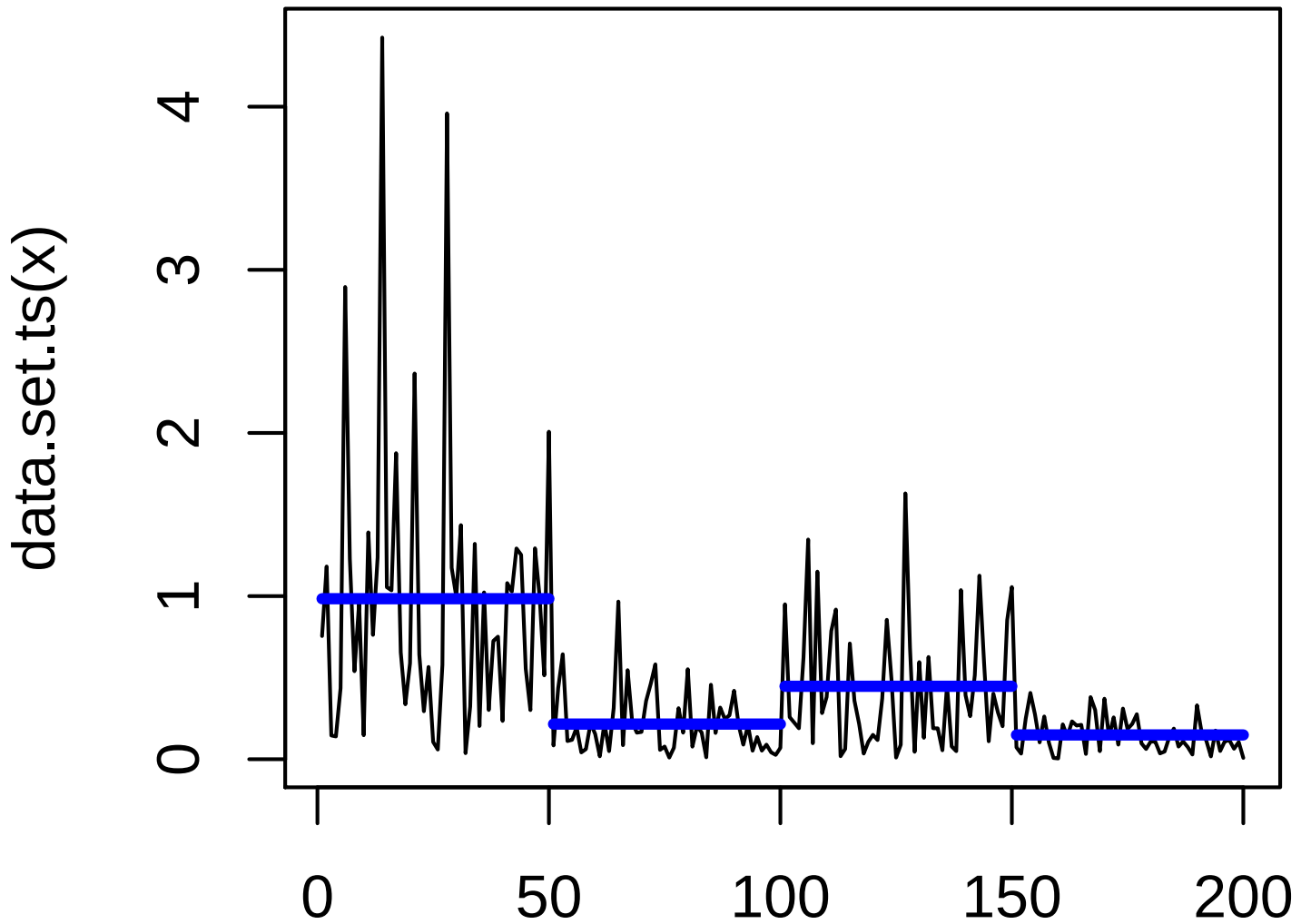
```
param.est(mv1.pelt)
```

```
## $rate
## [1] 1.016217 4.641184 2.235431 6.705612
```

# Mean & Variance



```
plot(mv1.pelt,cpt.width=3,cpt.col='blue')
```



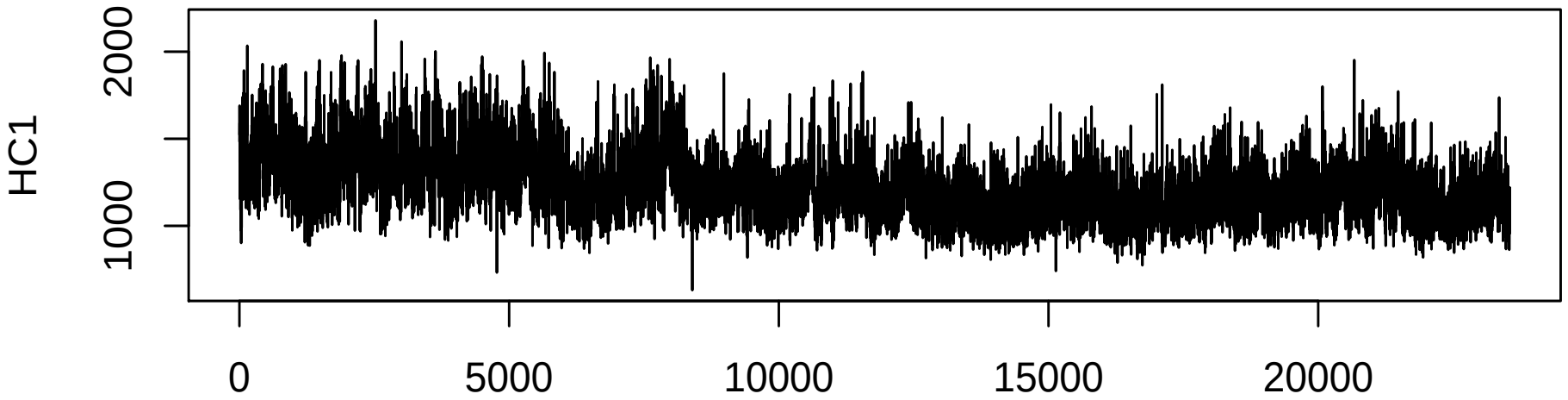
# Task



G+C content within part of Human Chromosome 1, data from NCBI. 3kb windows along the Human Chromosome from 10Mb to 33Mb.

Use the `cpt.meanvar` function to identify regions with different C+G content.

```
data(HC1)
ts.plot(HC1)
```



# Example: HC1



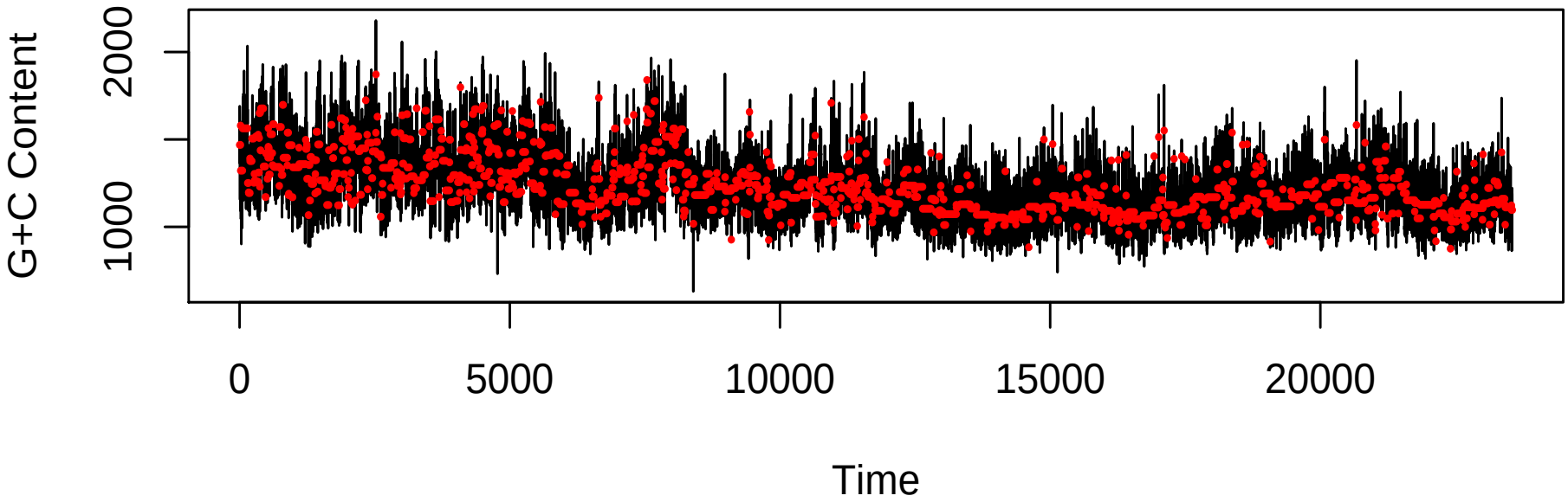
```
data(HC1)
hc1.pelt=cpt.meanvar(HC1,method='PELT',penalty='Manual',
        pen.value=14)
ncpts(hc1.pelt)
```

```
## [1] 805
```

# Example: HC1



```
plot(hc1.pelt,ylab='G+C Content',cpt.width=3)
```

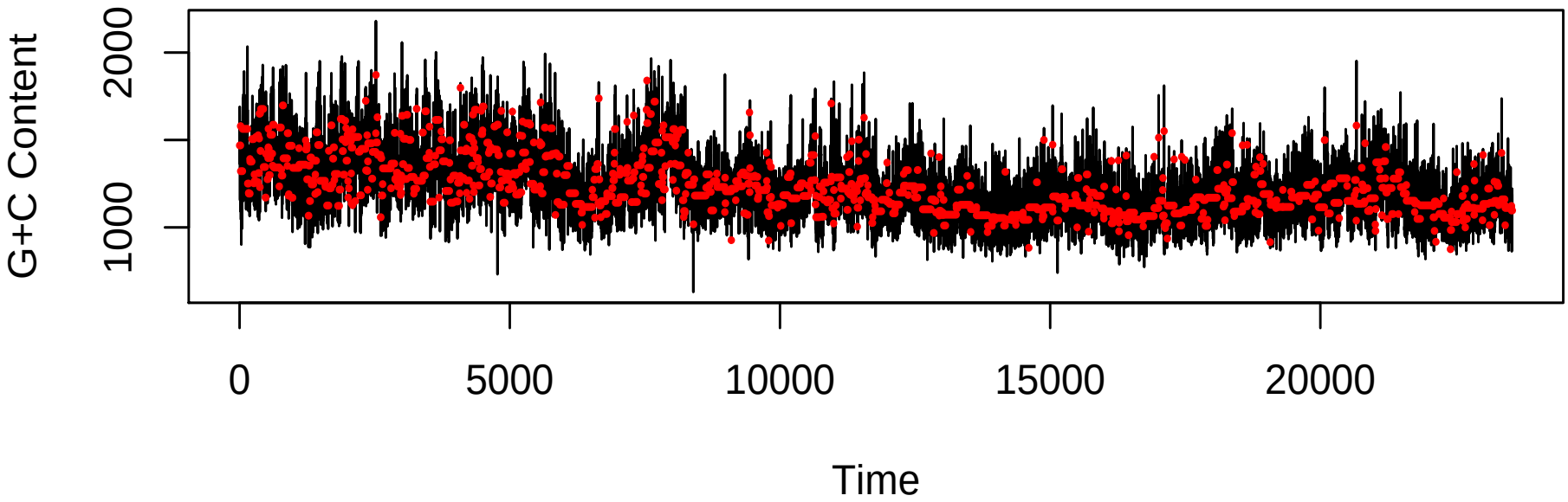


# Number of changes?

Mathematics & Statistics | Lancaster University

Does the number of changes appear reasonable?

```
plot(hc1.pelt,ylab='G+C Content',cpt.width=3)
```



# CROPS



**C**hangepoints for a **r**ange **o**f **p**enalti**es**

Use `penalty='CROPS'` with `method='PELT'` to get all segmentations for a range of penalty values.

```
v1.crops=cpt.var(v1,method="PELT",penalty="CROPS",
                 pen.value=c(5,500))
```

```
## [1] "Maximum number of runs of algorithm = 10"
## [1] "Completed runs = 2"
## [1] "Completed runs = 3"
## [1] "Completed runs = 4"
## [1] "Completed runs = 5"
## [1] "Completed runs = 6"
## [1] "Completed runs = 8"
## [1] "Completed runs = 9"
```

# CROPS



```
cpts.full(v1.crops)
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8]
## [1,]  102  114  133  201  206  213  375  379
## [2,]  102  114  133  201  206  375  379   NA
## [3,]  102  114  133  201  206   NA   NA   NA
## [4,]   96  133  201  206   NA   NA   NA   NA
## [5,]  102  201  206   NA   NA   NA   NA   NA
## [6,]  102  200   NA   NA   NA   NA   NA   NA
## [7,]  200   NA   NA   NA   NA   NA   NA   NA
## [8,]   NA   NA   NA   NA   NA   NA   NA   NA
```

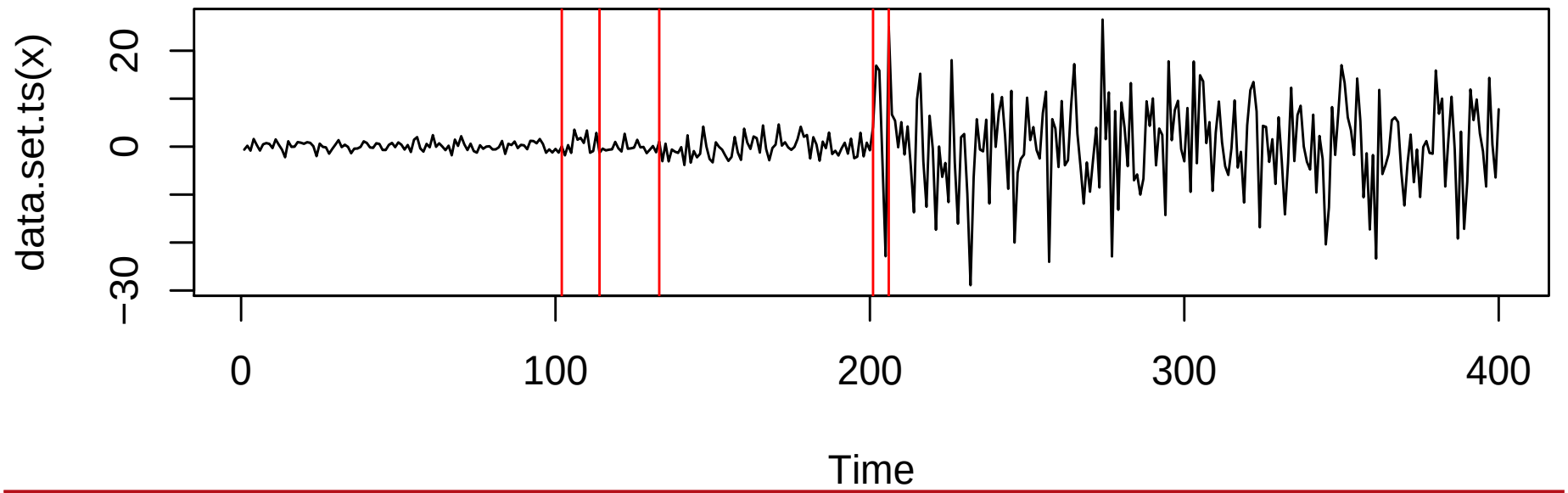
# CROPS



```
pen.value.full(v1.crops)
```

```
## [1]   5.000000   5.431360   6.151053   6.270164   6.314013
## [7]  54.317625 474.797364
```

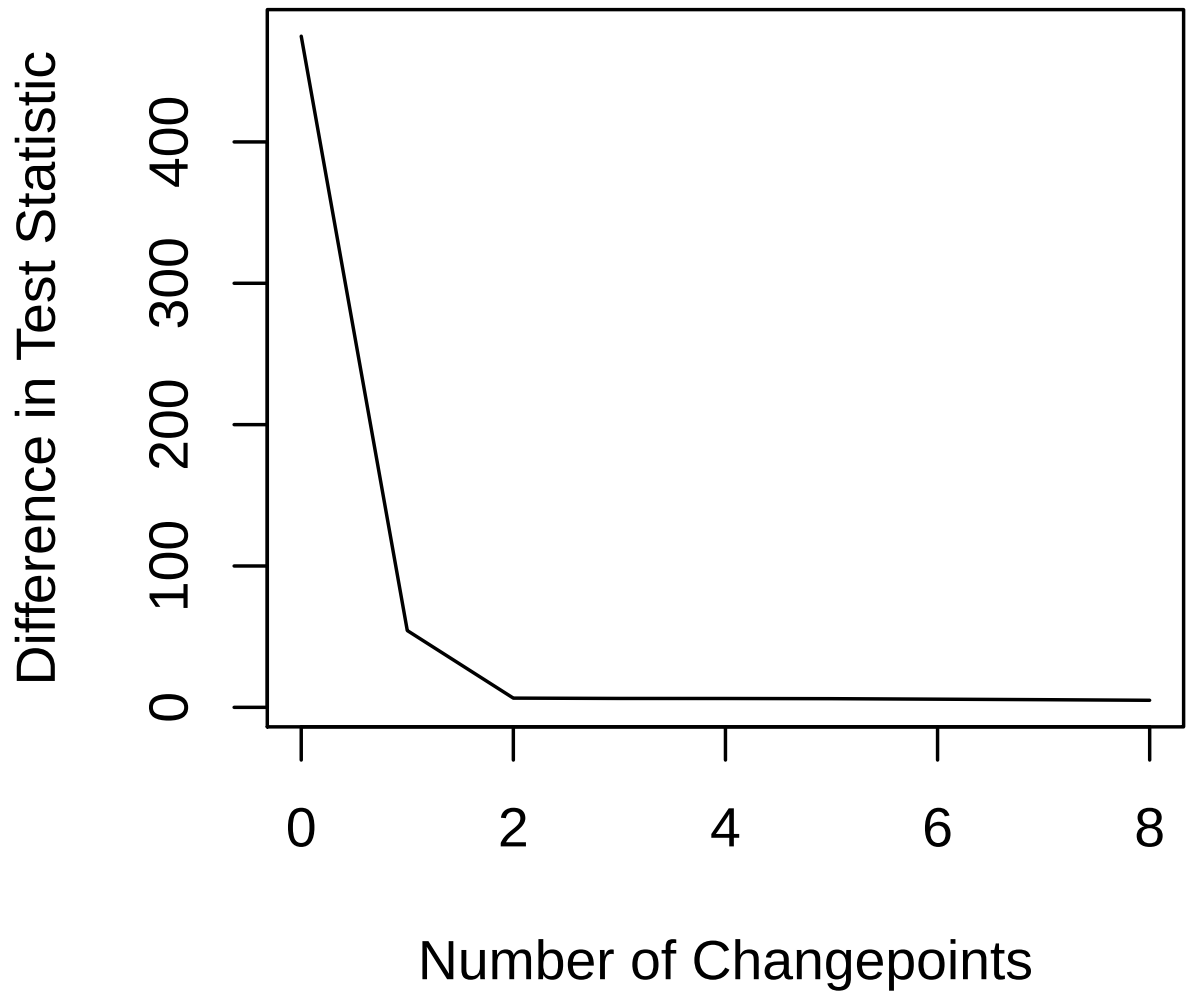
```
plot(v1.crops,ncpts=5)
```



# CROPS



```
plot(v1.crops,diagnostic=TRUE)
```



# Task



Look at the FTSE100 data again and use the CROPS technique to determine an appropriate number of changes.

# FTSE100-CROPS



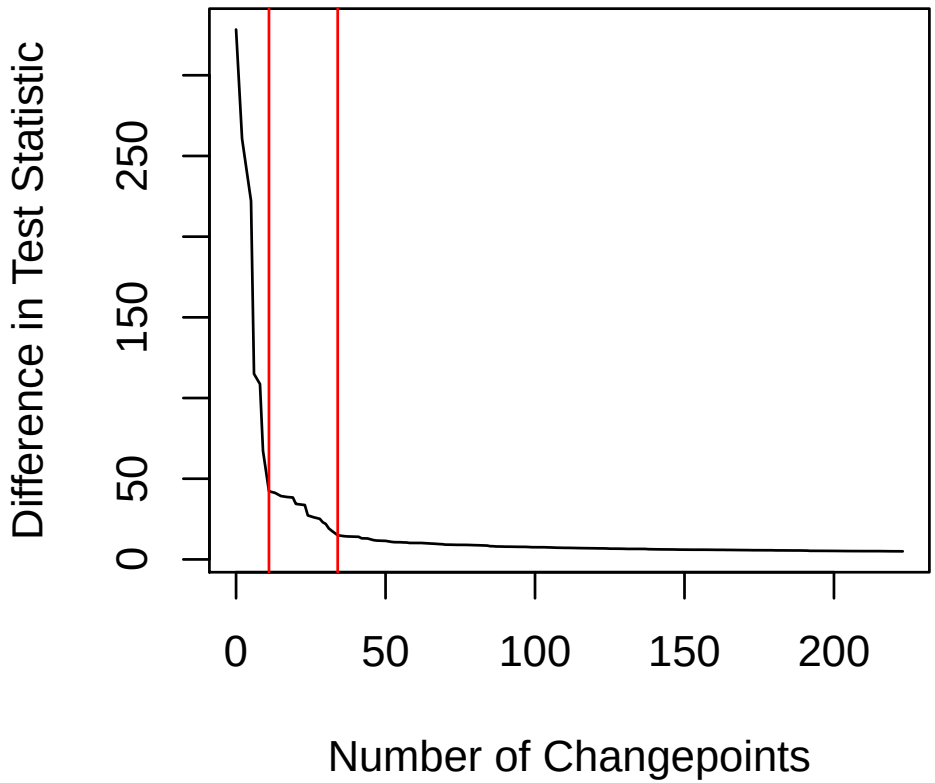
```
ftse.crops=cpt.var(ftse100[,2],method='PELT',
    penalty='CROPS',pen.value=c(5,1000))
```

```
## [1] "Maximum number of runs of algorithm = 225"
## [1] "Completed runs = 2"
## [1] "Completed runs = 3"
## [1] "Completed runs = 5"
## [1] "Completed runs = 9"
## [1] "Completed runs = 17"
## [1] "Completed runs = 32"
## [1] "Completed runs = 54"
## [1] "Completed runs = 93"
## [1] "Completed runs = 144"
## [1] "Completed runs = 188"
## [1] "Completed runs = 206"
## [1] "Completed runs = 207"
```

# FTSE100-CROPS



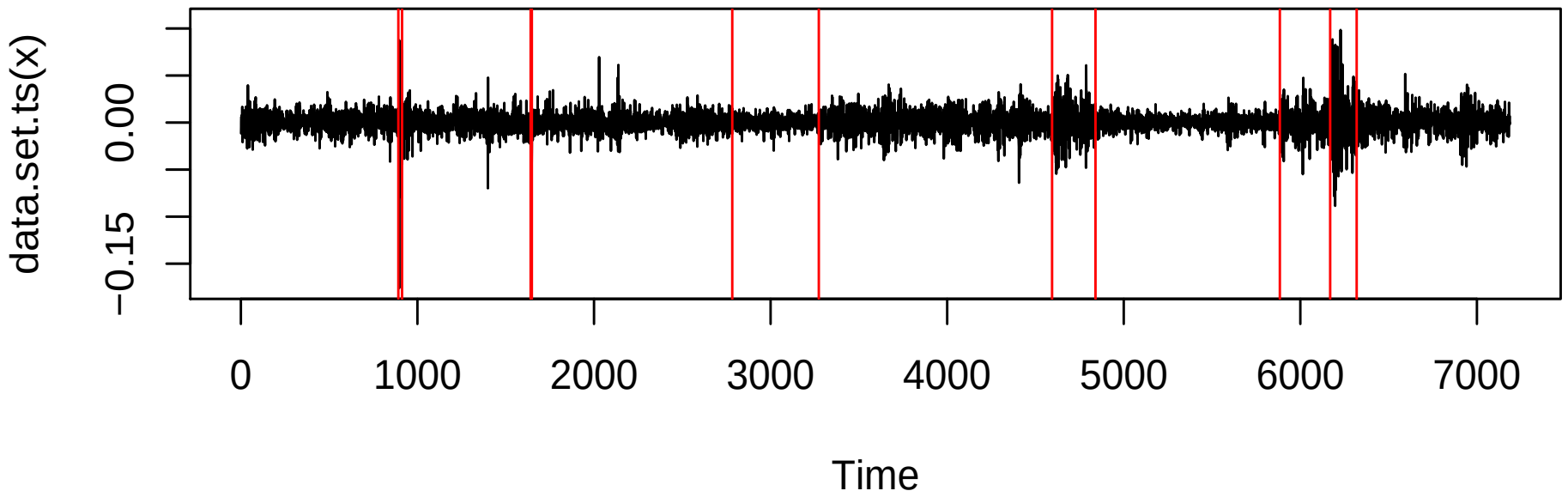
```
plot(ftse.crops,diagnostic=TRUE)
abline(v=11,col='red')
abline(v=34,col='red')
```



# FTSE100-CROPS



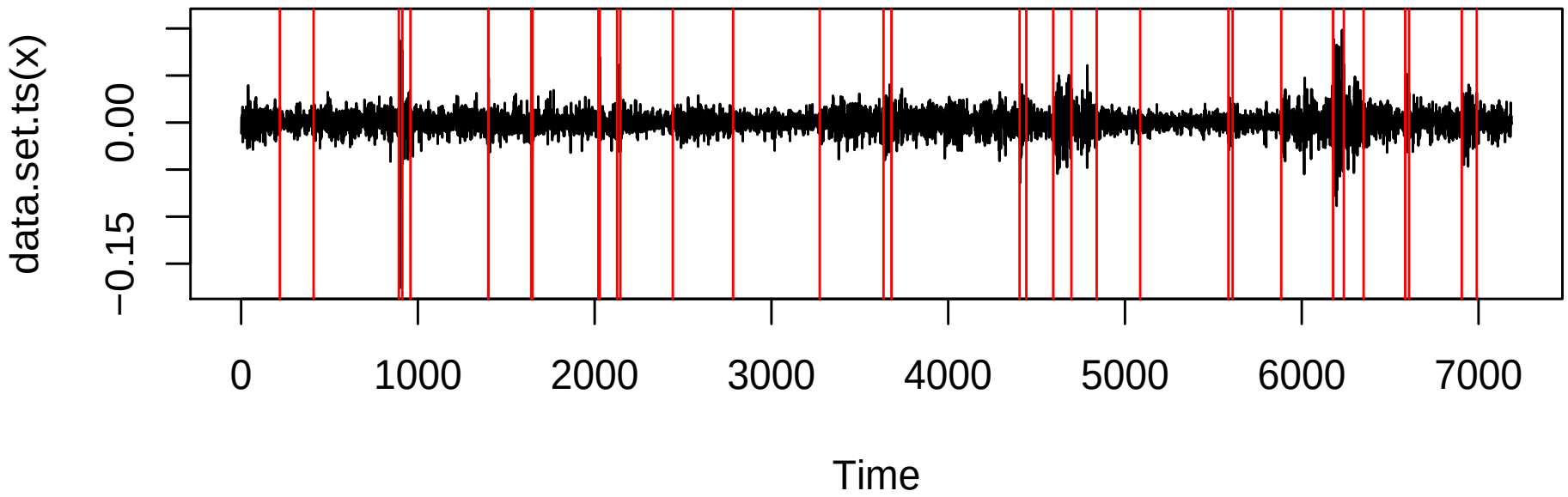
```
plot(ftse.crops,ncpts=11)
```



# FTSE100-CROPS



```
plot(ftse.crops,ncpts=34)
```



# cpt.np



```
cpt.np(data, penalty, pen.value, method,
test.stat="empirical_distribution", class, minseglen=1,
nquantiles=10)
```

Again the same underlying structure as `cpt.mean`.

- `test.stat` - choice of empirical_distribution
- `minseglen` - minimum segment length of 1
- `nquantiles` - number of quantiles to use

# Example



```
set.seed(12)
J <- function(x){(1+sign(x))/2}
n <- 1000
tau <- c(0.1,0.13,0.15,0.23,0.25,0.4,0.44,0.65,0.76,0.78,
      0.81)*n
h <- c(2.01, -2.51, 1.51, -2.01, 2.51, -2.11, 1.05, 2.16,
      -1.56, 2.56, -2.11)
sigma <- 0.5
t <- seq(0,1,length.out = n)
data <- array()
for (i in 1:n){
   data[i] <- sum(h*J(n*t[i] - tau)) + (sigma * rnorm(1))
}
```

# Example



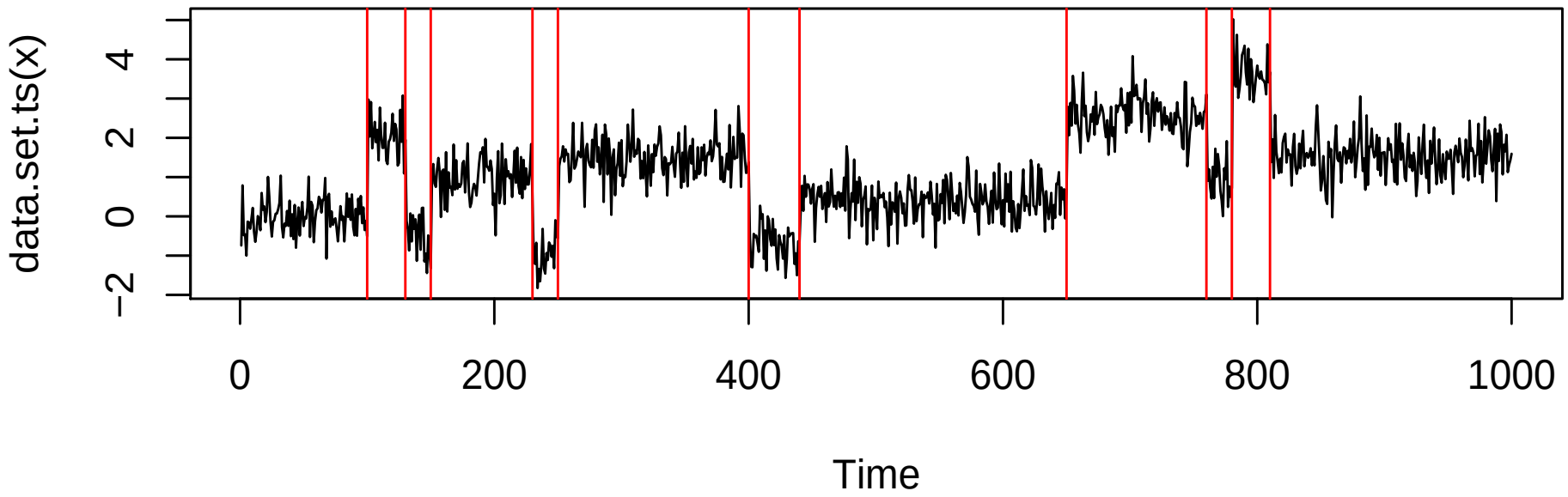
```
out <- cpt.np(data, method="PELT",minseglen=2,
        nquantiles =4*log(length(data)))
cpts(out)
```

```
##  [1] 100 130 150 230 250 400 440 650 760 780 810
```

# Example



```
plot(out)
```



# Task



Look at the `HeartRate` data from the `changepoint.np` package. Use one of the non-parametric functions to see if there is evidence for changes in heart rate.

```
data(HeartRate)
```

# HeartRate



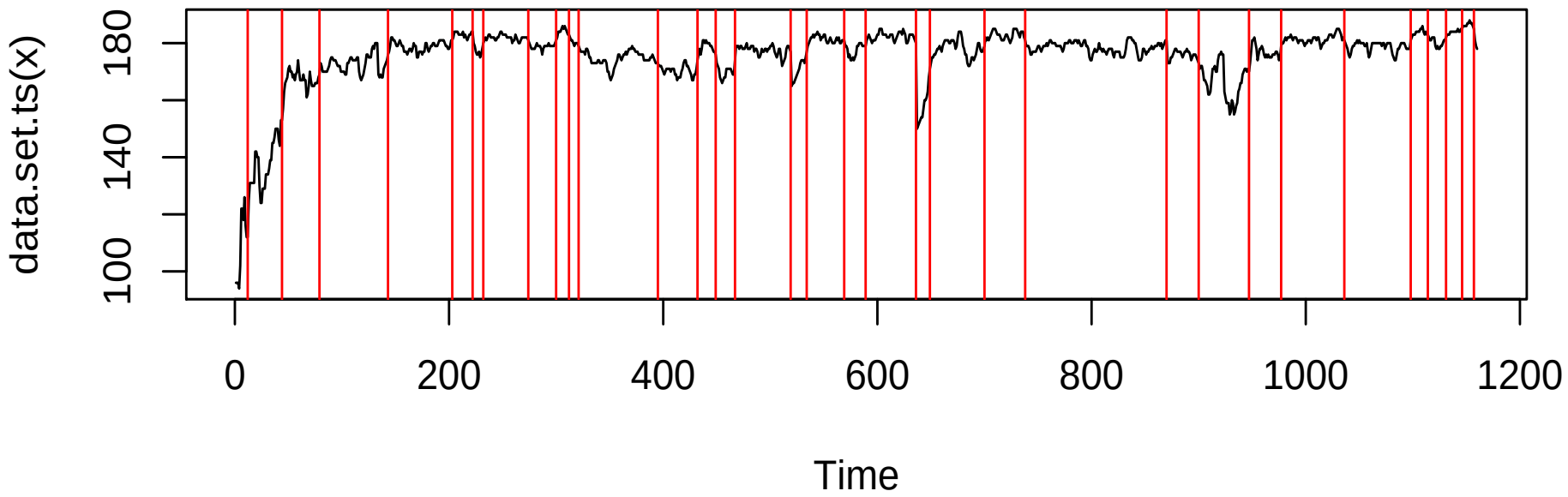
```
HR.pelt=cpt.np(HeartRate,method='PELT',
               nquantiles=4*log(length(HeartRate)))
ncpts(HR.pelt)
```

```
## [1] 33
```

# HeartRate



```
plot(HR.pelt)
```



# HeartRate



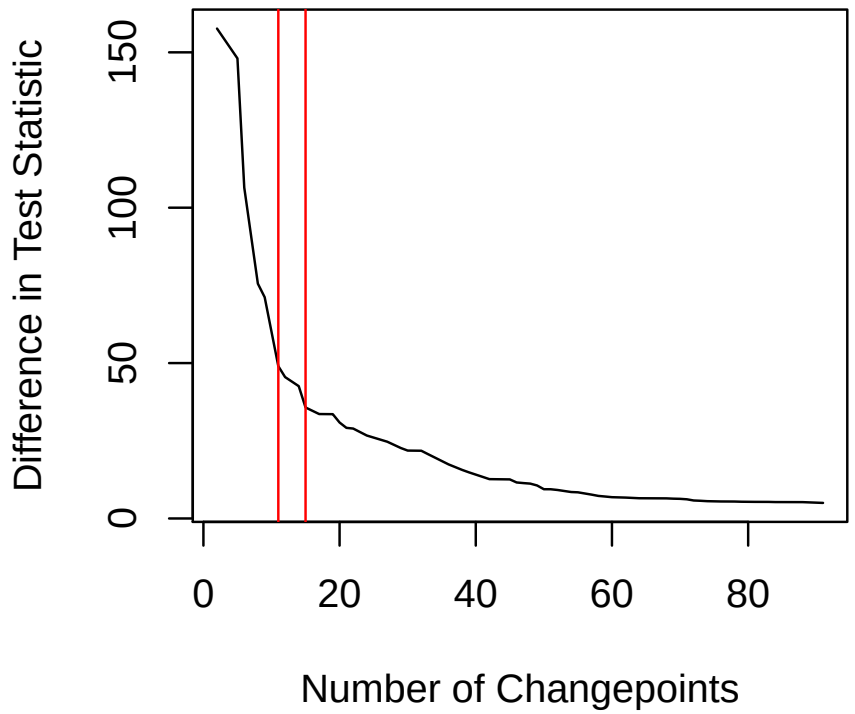
```
HR.crops=cpt.np(HeartRate, penalty = "CROPS",
        pen.value = c(5,200), method="PELT",minseglen=2,
        nquantiles =4*log(length(HeartRate)))
```

```
## [1] "Maximum number of runs of algorithm = 91"
## [1] "Completed runs = 2"
## [1] "Completed runs = 3"
## [1] "Completed runs = 5"
## [1] "Completed runs = 9"
## [1] "Completed runs = 17"
## [1] "Completed runs = 32"
## [1] "Completed runs = 52"
## [1] "Completed runs = 75"
## [1] "Completed runs = 84"
```

# HeartRate



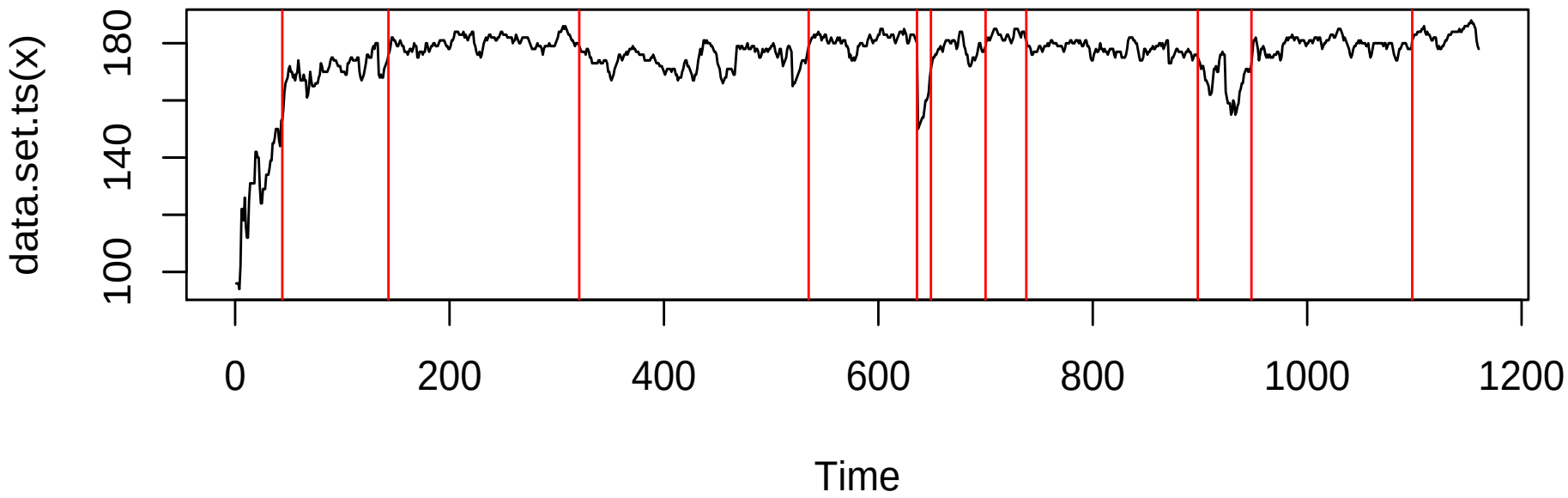
```
plot(HR.crops, diagnostic = TRUE)
abline(v=11,col='red')
abline(v=15,col='red')
```



# HeartRate



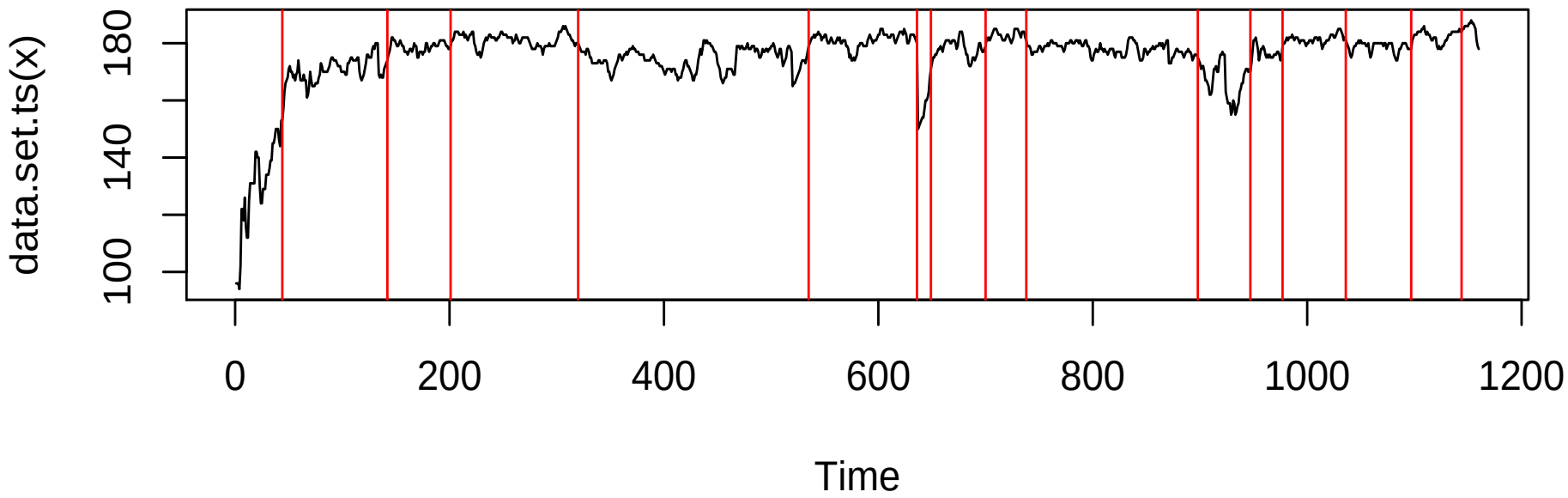
```
plot(HR.crops, ncpts = 11)
```



# HeartRate



```
plot(HR.crops, ncpts = 15)
```



# Checking Assumptions



The main assumptions for a Normal likelihood ratio test for a change in mean are:

- Independent data points;
- Normal distributed points pre and post change;
- Constant variance across the data.

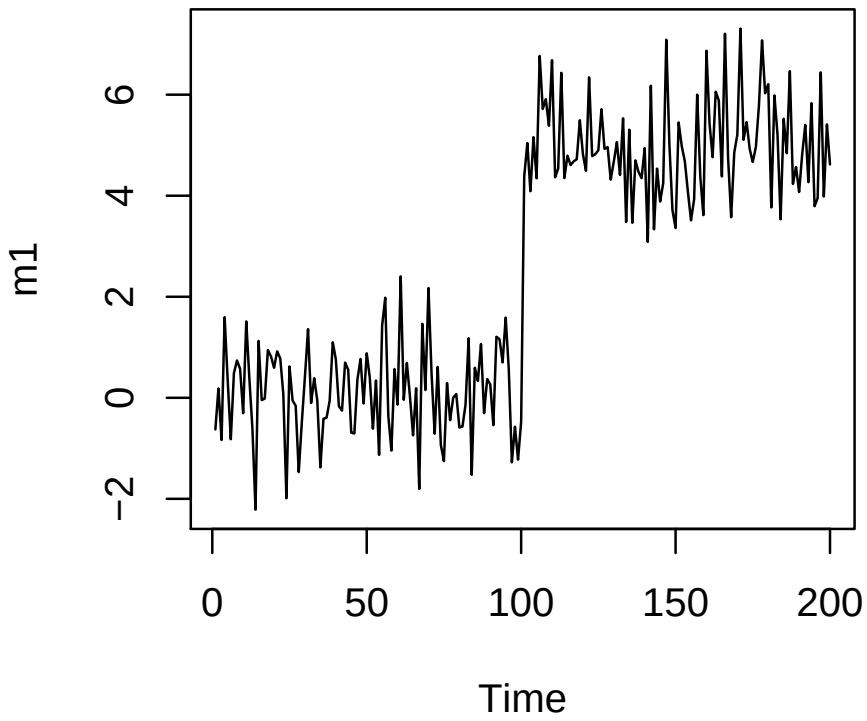
How can we check these?

# Checking Assumptions



In reality we can't check assumptions prior to analysis.

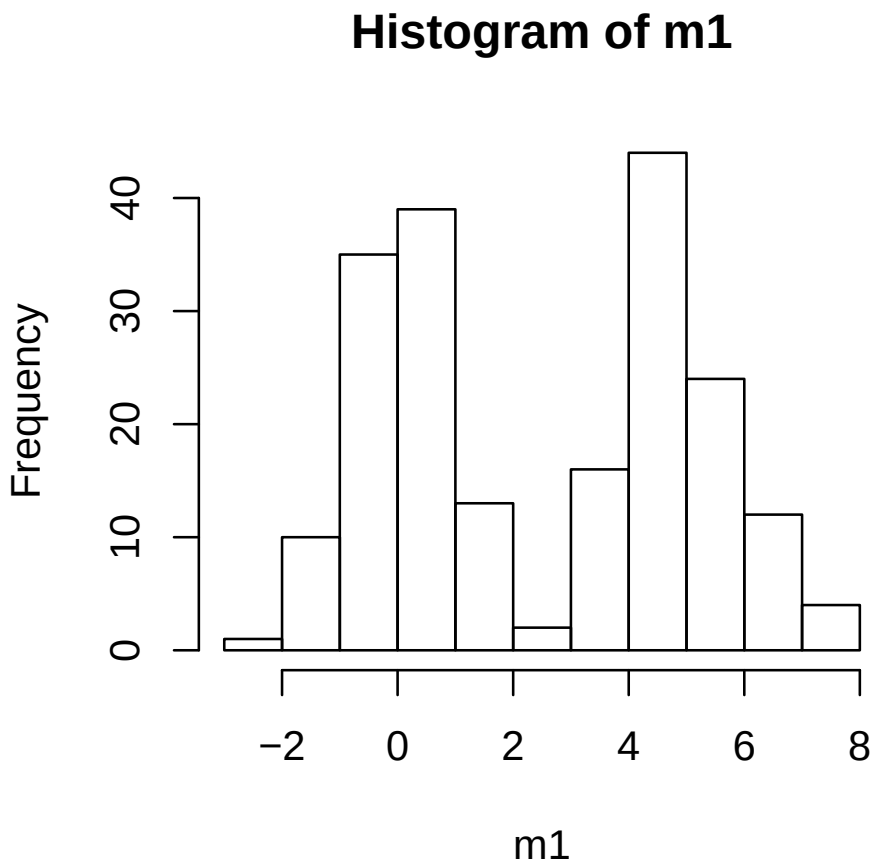
```
ts.plot(m1)
```



# Checking Assumptions



```
hist(m1)
```



# Checking Assumptions

Mathematics & Statistics | Lancaster University

```
shapiro.test(m1)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  m1
## W = 0.91086, p-value = 1.31e-09
```

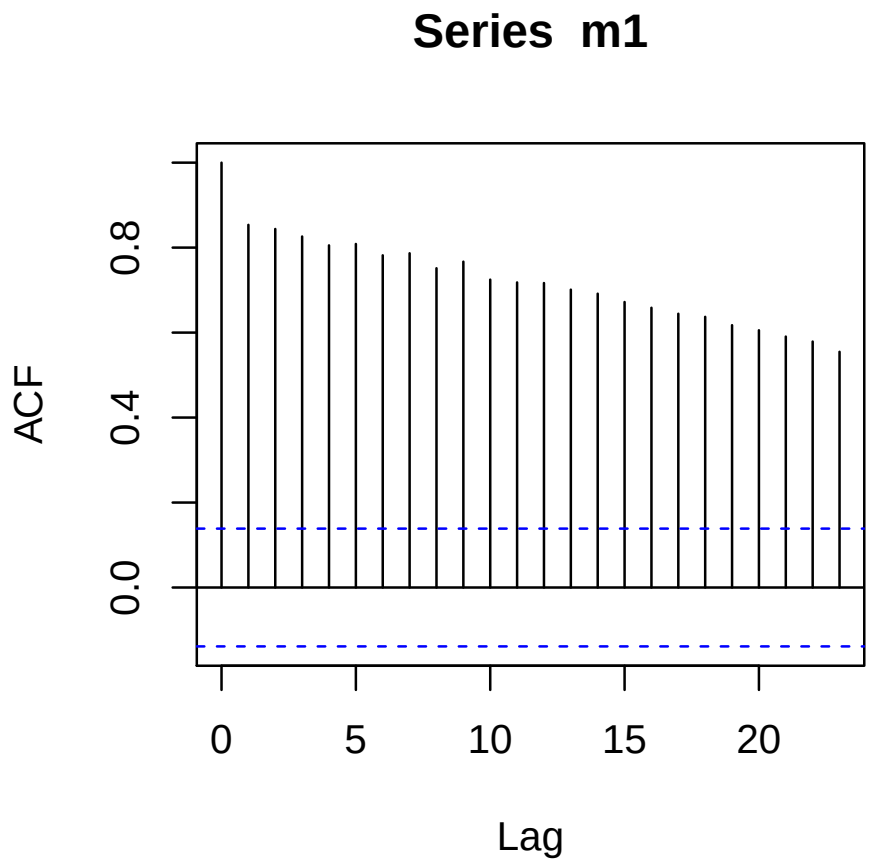
```
ks.test(m1,pnorm,mean=mean(m1),sd=sd(m1))
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  m1
## D = 0.15491, p-value = 0.0001355
## alternative hypothesis: two-sided
```

# Checking Assumptions



```
acf(m1)
```



# How to check



- Check each segment independently

```
cpt.seg=cbind(c(0,cpts(m1.amoc)),seg.len(m1.amoc))
data=data.set(m1.amoc)
shapiro.func=function(x){
  out=shapiro.test(data[(x[1]+1):(x[1]+x[2])])
  return(c(out$statistic,p=out$p.value))}
apply(cpt.seg,1,shapiro.func)
```

```
##         [,1]       [,2]
## W 0.9955979 0.97430709
## p 0.9876221 0.04762887
```

# Segment Check



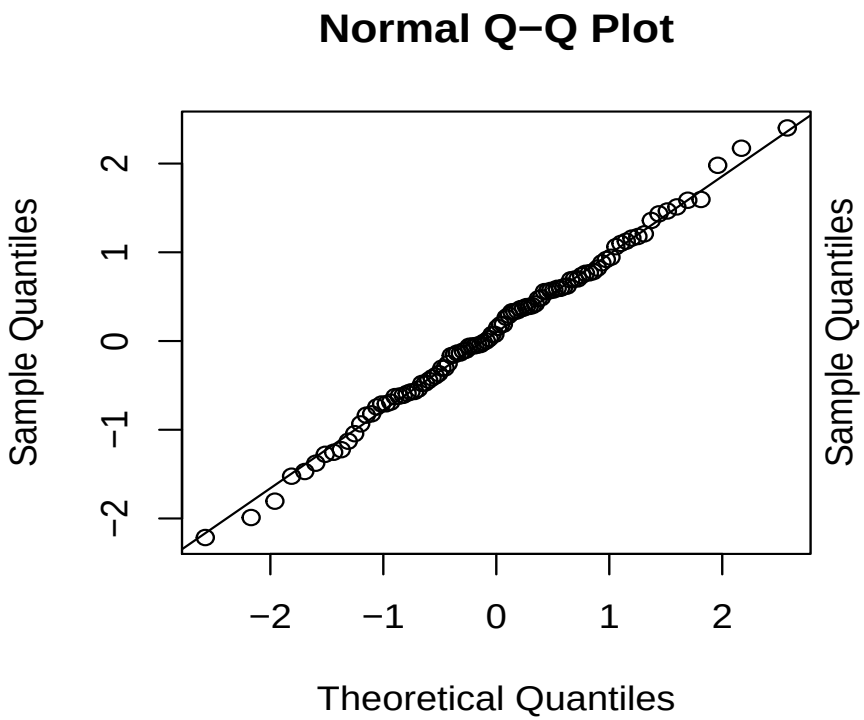
```
ks.func=function(x){
  tmp=data[(x[1]+1):(x[1]+x[2])]
  out=ks.test(tmp,pnorm,mean=mean(tmp),sd=sd(tmp))
  return(c(out$statistic,p=out$p.value))}
apply(cpt.seg,1,ks.func)
```

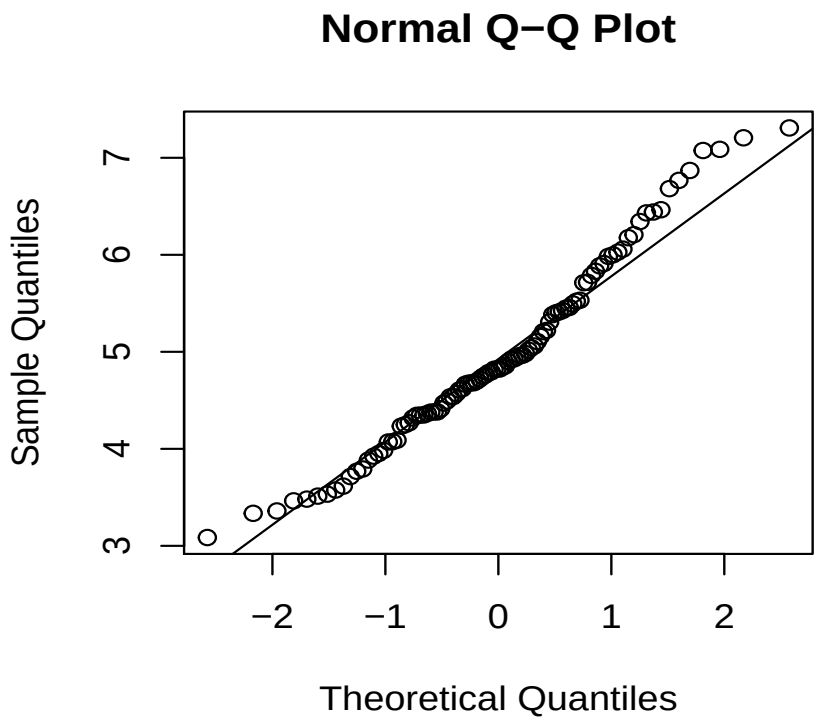
```
##          [,1]       [,2]
## D 0.04701381 0.09198393
## p 0.97991805 0.36593267
```

# Segment Check



```
qqnorm.func=function(x){
  qqnorm(data[(x[1]+1):(x[1]+x[2])])
  qqline(data[(x[1]+1):(x[1]+x[2])])}
out=apply(cpt.seg,1,qqnorm.func)
```

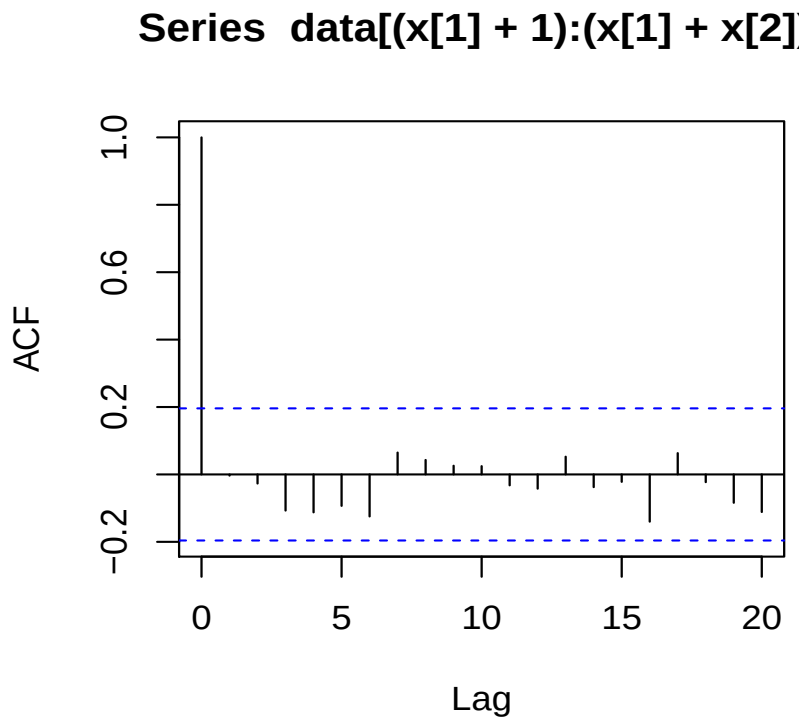


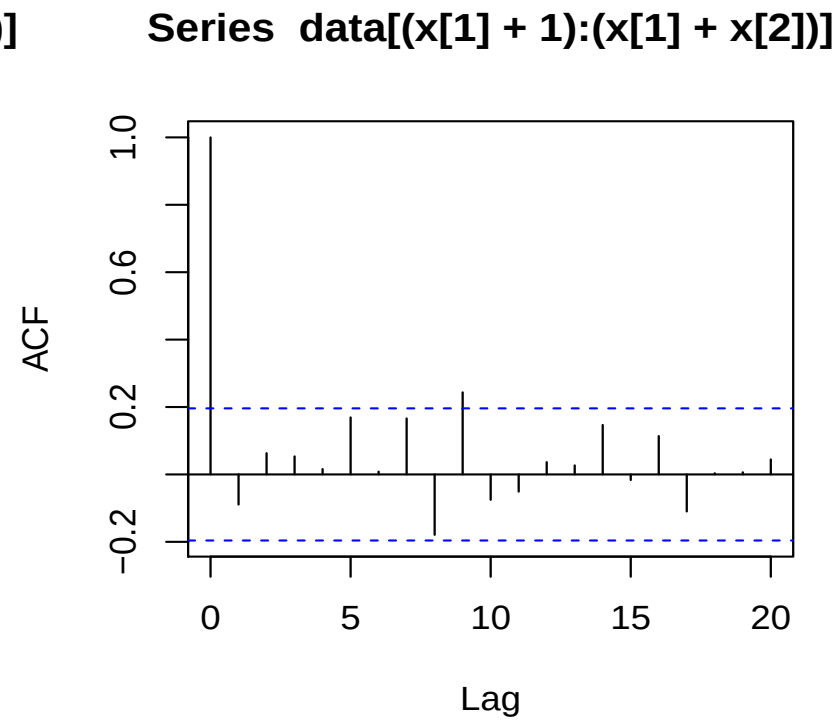



# Segment Check



```
acf.func=function(x){
  acf(data[(x[1]+1):(x[1]+x[2])])}
out=apply(cpt.seg,1,acf.func)
```



# How to check



- Check the residuals

```
means=param.est(m1.amoc)$mean
m1.resid=m1-rep(means,seg.len(m1.amoc))
shapiro.test(m1.resid)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  m1.resid
## W = 0.99228, p-value = 0.3721
```

# Residual Check



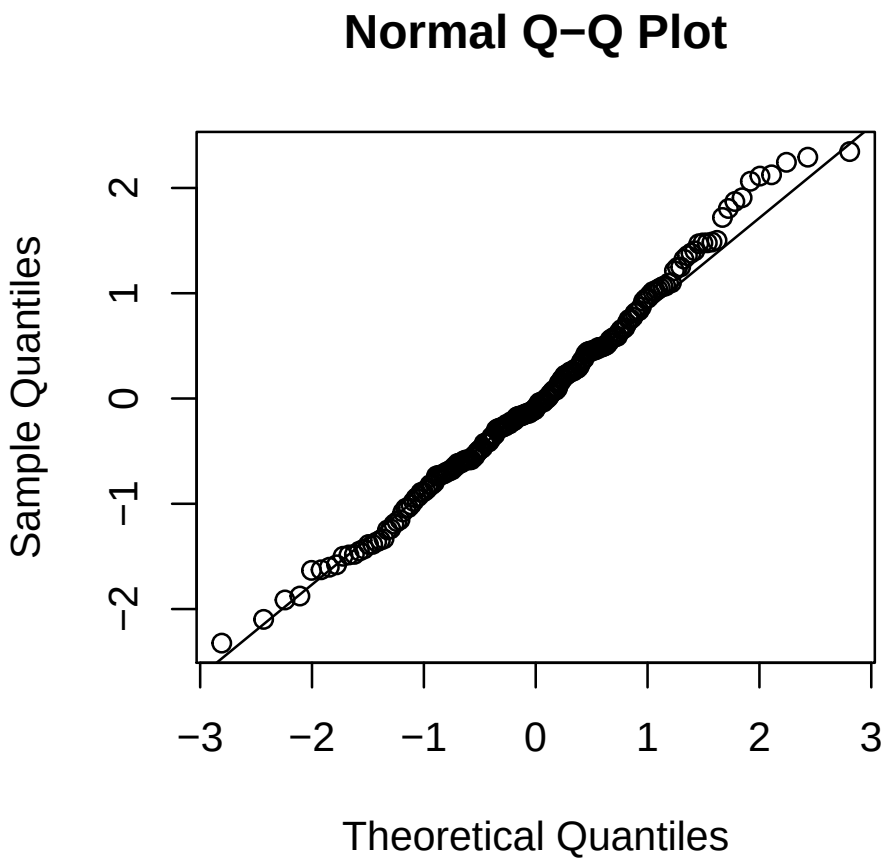
```
ks.test(m1.resid,pnorm,mean=mean(m1.resid),sd=sd(m1.resid))
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  m1.resid
## D = 0.045812, p-value = 0.7953
## alternative hypothesis: two-sided
```

# Residual Check



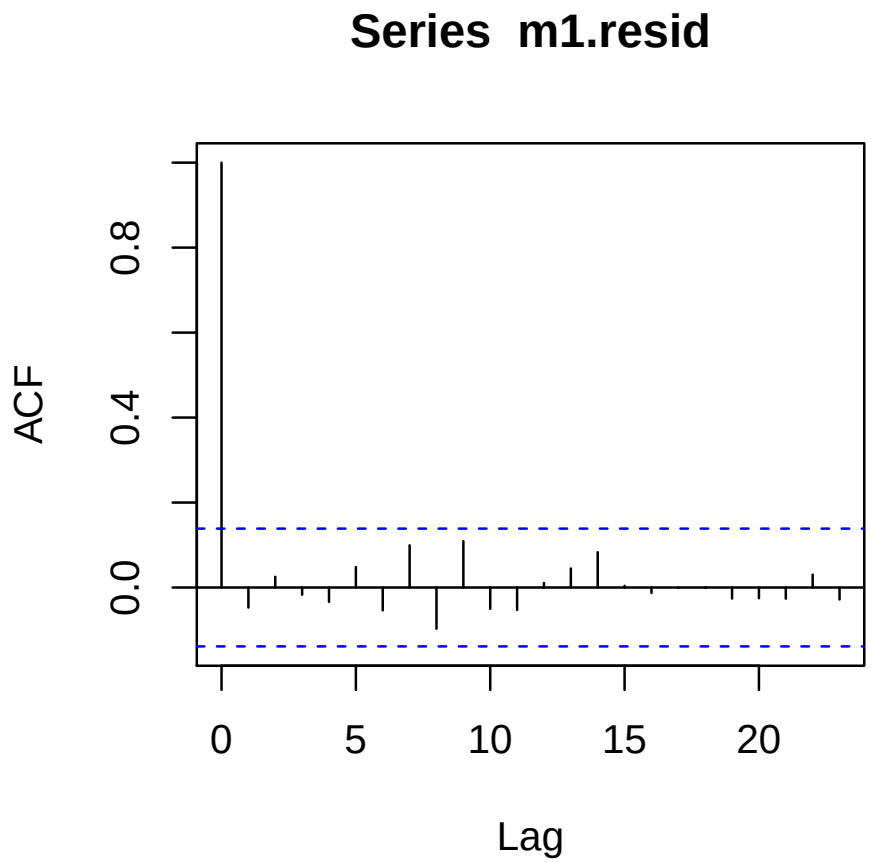
```
qqnorm(m1.resid)
qqline(m1.resid)
```



# Residual Check



```
acf(m1.resid)
```



# Task



Check the assumptions you have made on the simulated, Nile, FTSE100 and HeartRate data using either the segment or residual check.

What effect might any invalid assumptions have on the inference?

# Consolidating Task



Download the ratings for the following TV shows from the IMDB and analyze the series using some of the techniques you have learnt from today. For each series, do you identify any changes? Are the assumptions you are making valid? What effect might any invalid assumptions have on the inference?

- Doctor Who
- Grey's Anaytomy
- Mistresses
- The Simpsons
- Top Gear

(Understandably IMBD does not allow screen scraping nor downloads of information for redistribution so you will have to copy and paste the table into Excel, or equivalent, yourself in order to get the ratings data into R.)

# Bonus



Just from looking at the data, can you predict which shows have been cancelled?

# References



JSS: Killick, Eckley (2014)
PELT: Killick, Fearnhead, Eckley (2012)
CROPS: Kaynes, Eckley, Fearnhead (2015)
cpt.np: Haynes, Fearnhead, Eckley (2016)

# Coming soon



. . . to a changepoint package near you

- Join-pin regression
- FPOP (faster than binary segmentation but exact)
- Online PELT
- Multivariate changepoints
- Long memory or changepoint?