Demostraciones de PRYE

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Probabilidad de una unión finita

La probabilidad de una unión finita está dada por

$$P\left(A_1 \cup A_2\right) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P(A_{n+1}) + P\left(\bigcup_{i=1}^{n} A_i\right) - P\left(\bigcup_{i=1}^{n}(A_{n+1} \cap A_i)\right)$$

Probabilidad de una unión finita

$$P\left(\bigcup_{i=1}^{n} A_i\right)$$
$$=$$
$$\sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j)$$
$$+ \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^n P\left(\bigcap_{i=1}^{n} A_i\right)$$

Sobre eventos independientes

Dos eventos, A y B, de un espacio muestral, son llamados independientes cuando $P(A \cap B) = P(A)P(B)$.
Si A y B son independientes, entonces A^c y B son independientes; $A^c y B^c$ son independientes.

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Usando la igualdad $P(B) = P(B)\,P(A) + P(B \cap A^c)$

$$P(B \cap A^c) = P(B) - P(B)P(A) = P(B)(1 - P(A)) = P(B)\,P(A^c)$$

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$$P(A^c \cap B^c)$$
$$=$$
$$1 - P(A \cup B)$$
$$=$$
$$1 - [P(A) + P(B) - P(A \cap B)]$$
$$=$$
$$(1 - P(A))(1 - P(B))$$
$$=$$
$$P(A^c)\, P(B^c)$$

Binomial

$$X \sim B(n, p)$$

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad (0 \leq x \leq n)$$

1. Para todo $x \in \mathbb{Z}$ con $0 \leq x \leq n$, $P(X = x) \geq 0$.
2. $\sum_{x=0}^{n} P(X = x) = 1$.
3. $\mathrm{E}[X] = np$.
4. $\mathrm{Var}[X] = np(1-p)$.

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Binomial

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

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Geometrica

$$X \sim \mathrm{Geom}(p)$$

$$f(x) = P(X = x) = p(1-p)^{x-1} \qquad (x \in \mathbb{Z}^+)$$

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Geometrica

$$\sum_{k=0}^{n} p^k = \frac{1-p^{n+1}}{1-p}$$

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Hipergeométrica

$$X \sim Hg(N, K, n)$$

$$f(x) = P(X = x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$(\max\{0, n + K - N\} \leq x \leq \min\{K, n\})$$

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Hipergeométrica

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}$$

☕

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Poisson

$$X \sim \text{Pois}(\lambda)$$

☕

$$f(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad (x \in \mathbb{N})$$

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Poisson

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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Normal

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/\left(2\sigma^2\right)} \qquad (x \in \mathbb{R})$$

1. Para todo $x \in \mathbb{R}$, $f(x) \geq 0$.
2. $\int_{\mathbb{R}} f(x) \mathrm{d}x = 1$.
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Normal

$$\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x = \sqrt{\pi}$$

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Chi-Cuadrado

$$X \sim \chi^2(v)$$

$$f(x) = \frac{1}{2^{v/2}\Gamma\left(\frac{v}{2}\right)} x^{(v/2)-1} e^{-x/2} \qquad (x \geq 0)$$

1. para todo $x \in \mathbb{R}$, $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x \geq 0$
3. $\mathrm{E}[X] = v \qquad (v \geq 0)$
4. $\mathrm{Var}[X] = 2v \qquad (v \geq 0)$

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Chi-Cuadrado

$$\Gamma(x+1) = x\Gamma(x)$$

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f

$$F \sim \mathfrak{f}$$

$$f(x) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{u/2}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)} \frac{x^{(u/2)-1}}{\left(1+\frac{u}{v}x\right)^{(u+v)/2}} \qquad (x \in \mathbb{R}^+)$$

1. Para todo $x \in \mathbb{R}$, $f(x) \geq 0$.
2. $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$.
3. $\mathrm{E}[F] = \frac{v}{v-2} \qquad (v > 2)$
4. $\mathrm{Var}[F] = \frac{2v^2(u+v-2)}{u(v-2)^2(v-4)} \qquad (v > 4)$

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$$F \sim \mathfrak{f}$$

$$f(x) = \frac{\left(\frac{u}{v}\right)^{u/2}}{B\left(\frac{u}{2}, \frac{v}{2}\right)} \frac{x^{(u/2)-1}}{\left(1 + \frac{u}{v}x\right)^{(u+v)/2}} \qquad (x \in \mathbb{R}^+)$$

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$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

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Gamma

$$X \sim \mathrm{Gamma}(\alpha, \beta)$$

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad (x \in \mathbb{R}^{+})$$

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Gamma

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t$$

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Hipergeométrica a Binomial

Sea $X \sim Hg(N, K, n)$. Para n fijo, si N, K cumplen que

$$\frac{x-1}{K} < \epsilon_1$$
$$\frac{n-x-1}{N-K} < \epsilon_2$$
$$\frac{n-1}{N-n+1} < \epsilon_3$$

Entonces,

$$\left| \frac{Hg(N, K, n)(x)}{B\left(n, \frac{K}{N}\right)(x)} - 1 \right| < (\epsilon_1 + 1)^x (\epsilon_2 + 1)^{n-x} (\epsilon_3 + 1)^n - 1$$