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Finite Arithmetic Axiomatization for the Basis of Hyperrational Non-Standard Analysis

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Abstract: The standard elementary number theory is not a finite axiomatic system due to the presence of the induction axiom scheme. Absence of a finite axiomatic system is not an obstacle for most tasks, but may be considered as imperfect since the induction is strongly associated with the presence of set theory external to the axiomatic system. Also in the case of logic approach to the artificial intelligence problems presence of a finite number of basic axioms and states is important. Axiomatic hyperrational analysis is the axiomatic system of hyperrational number field. The properties of hyperrational numbers and functions allow them to be used to model real numbers and functions of classical elementary mathematical analysis. However hyperrational analysis is based on well-known non-finite hyperarithmetic axiomatics. In the article we present a new finite first-order arithmetic theory designed to be the basis of the axiomatic hyperrational analysis and, as a consequence, mathematical analysis in general as a basis for all mathematical application including AI problems. It is shown that this axiomatics meet the requirements, i.e., it could be used as the basis of an axiomatic hyperrational analysis. The article in effect completes the foundation of axiomatic hyperrational analysis without calling in an arithmetic extension, since in the framework of the presented theory infinite numbers arise without invoking any new constants. The proposed system describes a class of numbers in which infinite numbers exist as natural objects of the theory itself. We also do not appeal to any "enveloping" set theory.

Keywords: axiomatic non-standard analysis; finite arithmetic; hyperrational numbers

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1. Introduction

The common elementary number theory, the axiomatic theory of arithmetic, is not a finite system of axioms due to the presence of an induction axiom scheme. Although the absence of a finite axiomatic system is not an obstacle for most problems, according to the authors, such axiomatic system could be considered as subject of discussion because the induction is closely related to the presence of a set theory external to the axiomatic theory under study. Also, in the logical approach to the artificial intelligence (AI) problems, it is important to have a finite number of basic states and axioms, on the basis of which the formalization and solution of the tasks are constructed.

We introduce a finite axiomatic system of arithmetic, designed to be the basis of axiomatic hyperrational non-standard analysis and, as a consequence, mathematical analysis in a general. This is important since the analysis is the basis of all mathematical applications including AI problems. The finiteness of the number of axioms and the absence of the induction axiom make the proposed system similar to R. Robinson's arithmetic [1]. However, we postulate the commutativity of operations, which are generally unprovable in Robinson's arithmetic. Of course, any axiomatic system is also interesting from a purely logical point of view. However in this paper, we do not conduct a complete study of the theory itself, but restrict ourselves to its applicability to the previously developed system of axiomatic hyperrational analysis.

2. Related Works

Axiomatic hyperrational analysis (originally dating back to A. G. Dragalin [2], the development task of which was defined by N. K. Kosovsky [3]), was studied in the works of the authors and E. V. Prazdnikova [4,5] (in Russian), [6], and others. It is an axiomatic theory of the field of hyperrational numbers based on the axiomatics of the hyperarithmetic, that is, the arithmetic, with the addition of a constant corresponding to an infinite number. In this theory, differential and integral calculus are constructed, with the proven properties close to the classical and constructive real mathematical analysis. The paper [6] also provides an overview of the main concepts and results of this theory, and shows that hyperrational numbers can be used to measure line segments and areas of figures. Thus the properties of hyperrational numbers and functions allow them to model real numbers and functions of classical elementary mathematical analysis. This makes development of hyperrational analysis an interesting and relevant problem. However, in all these works, axiomatic hyperrational analysis is based on the non-finite axiomatic theory of arithmetic.

When constructing both classical and hyperrational analysis, a class of natural numbers is required in which there are addition and multiplication operations, as well as an order consistent with them that satisfies the law of trichotomy. Various versions of non-standard analysis extend this class by introducing some (ideal) infinite numbers. The resulting system of hypernatural numbers generates a class of hyperrational numbers, which act in the same manner as rational numbers in classical analysis.

The objects described by the axiomatic system presented are designed to serve as the basis for hyperrational analysis in the same way as the set of natural numbers serve the classic analysis. It is the presentation of the basic properties of these objects that is the main goal of this work. Thus we show that the described finite axiomatic system for natural numbers satisfies the necessary requirements; that is, a hyperrational non-standard analysis can be built on the basis of it. Within the framework of the proposed theory, infinite numbers exist without invoking new constants. This approach is to some extent close to the ideas of P. Vopenka's alternative set theory [7] and E. Nelson's internal set theory [8]. Thus the article actually completes the rationale of hyperrational analysis without involving the extension of arithmetic.

Also, the work of Juan Pablo Ramírez [9], where the author contributes to this problem, is noteworthy. However, unlike this work, we do not assume an "enveloping" set theory, giving only descriptions of the introduced concepts and properties in the set-theoretic language if this contributes to clarity. Unlike an approach proposed in, for example, [10,11] and based on intuitionistic logic, our approach used in this paper is based on the classic first-order logic. Also, in contrast to the authors' report mentioned above, this article provides formal proofs of theorems in a deductive system.

3. Basic Notation and Deductive System

We have adopted the following notations in the article. Capital calligraphic letters denote formulas of the first-order predicate calculus. The Greek ones are for lists of formulas. Following classic Gentzen notation, the $\longrightarrow$ symbol is used as the separator of sequent's antecedent and succedent, symbols $\supset$ and $\equiv$ are for implication and equivalence correspondingly. The $\vdash$ symbol is used to denote the fact of a theorem, i.e., the fact that a sequent is derivable or a formula is provable in a theory. Theories are represented with capital boldface letters. We consider only finite theories. So $\vdash S$ indicates that the sequent S is derivable in the deductive system used, and $\mathbf{T} \vdash \mathcal{A}$ means that in theory $\mathbf{T}$, the formula $\mathcal{A}$ is proved (witch is the same as $\vdash \mathbf{T} \longrightarrow \mathcal{A}$), i.e., $\mathcal{A}$ is the theorem of the theory $\mathbf{T}$. The character $:=$ is used to define notations or reductions (for formulas, predicates, etc.). $[\mathcal{A}]^u_t$ denotes a (syntactic) result of substituting all free occurrences of a variable u to the term t, supposing that the term t is free for the variable u in $\mathcal{A}$. If x is the only free variable of $\mathcal{A}$, we denote it as $\mathcal{A}(x)$. We also use infix notation for binary predicate and binary function symbols.

Note that the authors avoid frequently used designations such as $\Longrightarrow$, $\Longleftrightarrow$, $\models$, which are related to semantics (the model theory), since, according to the authors, one should clearly distinguish between the syntactic approach (derivability, provability) and the semantic one, associated with the concept of truth, which requires the involvement of an external set theory.

Here is a list of inference rules. This inference system borrowed by the author from [12], where its completeness and semantic validity are proved. A similar deduction system was initially listed in [13].

$$\frac{}{\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Gamma_1\mathcal{A}\Gamma_2}\,(axiom) \qquad \frac{\begin{array}{c}\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta \\ \Gamma \longrightarrow \Delta_1\mathcal{A}\Delta_2\end{array}}{\Gamma_1\Gamma_2\Gamma \longrightarrow \Delta\Delta_1\Delta_2}\,(cut)$$

$$\frac{\Gamma_1\Gamma_2 \longrightarrow \Delta_1\mathcal{A}\Delta_2}{\Gamma_1\neg\mathcal{A}\Gamma_2 \longrightarrow \Delta_1\Delta_2}\,(\neg L) \qquad \frac{\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta_1\Delta_2}{\Gamma_1\Gamma_2 \longrightarrow \Delta_1\neg\mathcal{A}\Delta_2}\,(\neg R)$$

$$\frac{\Gamma_1\mathcal{A}\mathcal{B}\Gamma_2 \longrightarrow \Delta}{\Gamma_1\mathcal{A}\ \&\ \mathcal{B}\Gamma_2 \longrightarrow \Delta}\,(\&\ L) \qquad \frac{\begin{array}{c}\Gamma \longrightarrow \Delta_1\mathcal{A}\Delta_2 \\ \Gamma \longrightarrow \Delta_1\mathcal{B}\Delta_2\end{array}}{\Gamma \longrightarrow \Delta_1\mathcal{A}\ \&\ \mathcal{B}\Delta_2}\,(\&\ R)$$

$$\frac{\begin{array}{c}\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta \\ \Gamma_1\mathcal{B}\Gamma_2 \longrightarrow \Delta\end{array}}{\Gamma_1\mathcal{A}\vee\mathcal{B}\Gamma_2 \longrightarrow \Delta}\,(\vee L) \qquad \frac{\Gamma \longrightarrow \Delta_1\mathcal{A}\mathcal{B}\Delta_2}{\Gamma \longrightarrow \Delta_1\mathcal{A}\vee\mathcal{B}\Delta_2}\,(\vee R)$$

$$\frac{\begin{array}{c}\Gamma_1\mathcal{B}\Gamma_2 \longrightarrow \Delta_1\Delta_2 \\ \Gamma_1\Gamma_2 \longrightarrow \Delta_1\mathcal{A}\Delta_2\end{array}}{\Gamma_1\mathcal{A}\supset\mathcal{B}\Gamma_2 \longrightarrow \Delta_1\Delta_2}\,(\supset L) \qquad \frac{\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta_1\mathcal{B}\Delta_2}{\Gamma_1\Gamma_2 \longrightarrow \Delta_1\mathcal{A}\supset\mathcal{B}\Delta_2}\,(\supset R)$$

$$\frac{\begin{array}{c}\Gamma_1\mathcal{A}\mathcal{B}\Gamma_2 \longrightarrow \Delta_1\Delta_1 \\ \Gamma_1\Gamma_2 \longrightarrow \Delta_1\mathcal{B}\mathcal{A}\Delta_2\end{array}}{\Gamma_1\mathcal{A}\equiv\mathcal{B}\Gamma_2 \longrightarrow \Delta_1\Delta_2}\,(\equiv L) \qquad \frac{\begin{array}{c}\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta_1\mathcal{B}\Delta_2 \\ \Gamma_1\mathcal{B}\Gamma_2 \longrightarrow \Delta_1\mathcal{A}\Delta_2\end{array}}{\Gamma_1\Gamma_1 \longrightarrow \Delta_1\mathcal{A}\equiv\mathcal{B}\Delta_2}\,(\equiv R)$$

$$\frac{\Gamma_1\mathcal{A}\mathcal{B}\Gamma_2 \longrightarrow \Delta}{\Gamma_1\mathcal{B}\mathcal{A}\Gamma_2 \longrightarrow \Delta}\,(PL) \qquad \frac{\Gamma \longrightarrow \Delta_1\mathcal{A}\mathcal{B}\Delta_2}{\Gamma \longrightarrow \Delta_1\mathcal{B}\mathcal{A}\Delta_2}\,(PR)$$

$$\frac{\Gamma_1\mathcal{A}\mathcal{A}\Gamma_2 \longrightarrow \Delta}{\Gamma_1\mathcal{A}\Gamma_2 \longrightarrow \Delta}\,(CL) \qquad \frac{\Gamma \longrightarrow \Delta_1\mathcal{A}\mathcal{A}\Delta_2}{\Gamma \longrightarrow \Delta_1\mathcal{A}\Delta_2}\,(CR)$$

$$\frac{\Gamma_1[\mathcal{A}]^x_u\Gamma_2 \longrightarrow \Delta}{\Gamma_1\exists x\mathcal{A}\Gamma_2 \longrightarrow \Delta}\,(\exists L) \qquad \frac{\Gamma \longrightarrow \Delta_1[\mathcal{A}]^x_t\Delta_2}{\Gamma \longrightarrow \Delta_1\exists x\mathcal{A}\Delta_2}\,(\exists R)$$

$$\frac{\Gamma_1[\mathcal{A}]^x_t\Gamma_2 \longrightarrow \Delta}{\Gamma_1\forall x\mathcal{A}\Gamma_2 \longrightarrow \Delta}\,(\forall L) \qquad \frac{\Gamma \longrightarrow \Delta_1[\mathcal{A}]^x_u\Delta_2}{\Gamma \longrightarrow \Delta_1\forall x\mathcal{A}\Delta_2}\,(\forall R)$$

In the $(\forall L)$ and $(\exists R)$, the term t should be free for the variable x; in the rules $(\forall R)$ and $(\exists L)$, variable y must not occur free anywhere in the lower sequents.

For this deductive system the reverse rules $(\not\neg L)$, $(\not\& L)$, $(\not\vee L)$, $(\not\supset L)$, $(\not\equiv L)$, $(\not\neg R)$, $(\not\& R)$, $(\not\vee R)$, $(\not\supset R)$, $(\not\equiv R)$ and $(\not\exists L)$, $(\not\forall L)$, $(\not\exists R)$, $(\not\forall R)$ with corresponding restrictions are admissible.

4. The Finite Arithmetic

4.1. Axioms List

The formal language of the finite arithmetic is the first-order predicate calculus language, whose signature contains the constant "zero" denoted as 0, the equality binary predicate denoted by common symbol $=$ and functional symbols for addition and multiplication, $+$ and $\cdot$ correspondingly. To describe a formalized theory, the formal language of predicate calculus is used, described, for example, in [13,14]. We list the following formulas to be the axioms of finite arithmetic.

$\mathcal{A}_1 := \forall x(x = x);$
$\mathcal{A}_2 := \forall x \forall y(x = y \supset y = x);$
$\mathcal{A}_3 := \forall x \forall y \forall z(x = y \,\&\, y = z \supset x = z);$
$\mathcal{A}_4 := \forall x \forall y \forall u \forall v(x = u \,\&\, y = v \supset (x = y \equiv u = v));$
$\mathcal{A}_5 := \forall x \forall y(x' = y' \equiv x = y);$
$\mathcal{A}_6 := \forall x \forall y \forall u \forall v(x = u \,\&\, y = v \supset x + y = u + v);$
$\mathcal{A}_7 := \forall x \forall y \forall u \forall v(x = u \,\&\, y = v \supset x \cdot y = u \cdot v);$
$\mathcal{A}_8 := \forall x \forall y(x + y = y + x);$
$\mathcal{A}_9 := \forall x \forall y(x \cdot y = y \cdot x);$
$\mathcal{A}_{10} := \forall x \forall y \forall z((x + y) + z = x + (y + z));$
$\mathcal{A}_{11} := \forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z));$
$\mathcal{A}_{12} := \forall x \forall y \forall z(x \cdot (y + z) = (x \cdot y) + (x \cdot z));$
$\mathcal{A}_{13} := \forall x(x + 0 = x);$
$\mathcal{A}_{14} := \forall x(x \cdot 0 = 0);$
$\mathcal{A}_{15} := \forall x \forall y\Big(x + y' = (x + y)'\Big);$
$\mathcal{A}_{16} := \forall x \forall y(x \cdot y' = (x \cdot y) + x);$
$\mathcal{A}_{17} := \forall x \neg(x = x');$
$\mathcal{A}_{18} := \forall x \neg(x' = 0);$
$\mathcal{A}_{19} := \forall x(\neg(x = 0) \supset \exists y(x = y'));$
$\mathcal{A}_{20} := \forall x \forall y \exists z(x + z = y \vee y + z = x);$
$\mathcal{A}_{21} := \forall x \forall y \forall z(x + z = x + y \supset z = y).$

This axiomatic system represents the understanding of a natural number as the basis of practical arithmetic. In other words, we postulate the basic properties of operations (addition and multiplication) and order. Let's denote this theory as **A**.

4.2. Preliminary Results

Lemma 1. *The following statements are valid.*

1. *If* $\mathbf{T} \vdash \forall x\mathcal{A}$ *for any theory* $\mathbf{T}$, *then for any term t free for the variable x in* $\mathcal{A}$ *the* $\mathbf{T} \vdash [\mathcal{A}]^x_t$ *occurs.*
   Indeed, taking logical axiom $[\mathcal{A}]^x_t \longrightarrow [\mathcal{A}]^x_t$, *applying* $(\exists L)$ *we have* $\vdash \forall x\mathcal{A} \longrightarrow [\mathcal{A}]^x_t$. *Applying* (cut) *to it and provable by condition* $\mathbf{T} \longrightarrow \forall x\mathcal{A}$ *sequent we have that* $\mathbf{T} \vdash [\mathcal{A}]^x_t$.
2. *Since the rule* $(\not\forall R)$ *is valid, then from sequent* $\mathbf{A} \longrightarrow \forall x\mathcal{A}$ *provability we immediately have provability of* $(\mathbf{A} \longrightarrow \mathcal{A})$, *i.e., formula* $\mathcal{A}$ *is the theorem of finite arithmetic.*
3. *Applying* $(\not\forall R)$ *the required number of times to each sequent got from axioms* $\mathcal{A}_k$, $k = 1, 2, \ldots, 21$ *we get the following finite arithmetic theorems.*

   $\mathcal{B}_1 := (x = x);$
   $\mathcal{B}_2 := (x = y \supset y = x);$
   $\mathcal{B}_3 := (x = y \,\&\, y = z \supset x = z);$
   $\mathcal{B}_4 := (x = u \,\&\, y = v \supset (x = y \equiv u = v));$
   $\mathcal{B}_5 := (x' = y' \equiv x = y);$
   $\mathcal{B}_6 := (x = u \,\&\, y = v \supset x + y = u + v);$
   $\mathcal{B}_7 := (x = u \,\&\, y = v \supset x \cdot y = u \cdot v);$
   $\mathcal{B}_8 := (x + y = y + x);$
   $\mathcal{B}_9 := (x \cdot y = y \cdot x);$
   $\mathcal{B}_{10} := ((x + y) + z = x + (y + z));$
   $\mathcal{B}_{11} := ((x \cdot y) \cdot z = x \cdot (y \cdot z));$
   $\mathcal{B}_{12} := (x \cdot (y + z) = (x \cdot y) + (x \cdot z));$
   $\mathcal{B}_{13} := (x + 0 = x);$
   $\mathcal{B}_{14} := (x \cdot 0 = 0);$
   $\mathcal{B}_{15} := \Big(x + y' = (x + y)'\Big);$
   $\mathcal{B}_{16} := (x \cdot y' = (x \cdot y) + x);$
   $\mathcal{B}_{17} := \neg(x = x');$

$\mathcal{B}_{18} := \neg(x' = 0)$;
$\mathcal{B}_{19} := (\neg(x = 0) \supset \exists y(x = y'))$;
$\mathcal{B}_{20} := \exists z(x + z = y \vee y + z = x)$;
$\mathcal{B}_{21} := (x + z = x + y \supset z = y)$.

4. *Considering the first statement, all formulas obtained from the formulas $\mathcal{B}_1$–$\mathcal{B}_{21}$ by substituting arbitrary terms instead of free variables are theorems of finite arithmetic. In particular, any designation can be used since it was possible to proceed from any designations of bound variables when applying the backward rule of the universal quantifier of exclusion.*
5. *We have an axiom $[\mathcal{A}]^x_t \longrightarrow [\mathcal{A}]^x_t$ for a term t free for variable x. Using $(\exists R)$ we obtain $\vdash [\mathcal{A}]^x_t \longrightarrow \exists x \mathcal{A}$.*
6. *Let $\mathbf{T} \vdash \exists x \mathcal{A}$. Then, extending the theory $\mathbf{T}$ language by Skolem's constant z and the theory itself by an axiom $[\mathcal{A}]^x_z$ we can get a conservative Skolem's extension of the theory $\mathbf{T}$. This trick can be used in the proof, returning afterwards to the original theory by eliminating z as a free for substitution variable.*

The above lemma is technical. The facts noted in it facilitate formal proofs, allowing not to give a complete sequential inference.

4.3. Properties of the Order

Let us define predicates $x \leq y := \exists z(x + z = y)$ and $x < y := x \leq y \;\&\; \neg(x = y)$ and prove the natural properties of order in the finite arithmetic.

Theorem 1. *The following formulas are theorems of the theory* **A**.

1. $\forall x(x \leq x)$;
2. $\forall x \forall y \forall z(x \leq y \;\&\; y \leq z \supset x \leq z)$;
3. $\forall x \forall y(x \leq y \;\&\; y \leq x \supset x = y)$;
4. $\forall x \forall y(x < y \vee x = y \vee y < x)$;
5. $\forall x \forall y \forall u \forall v(x = u \;\&\; y = v \supset (x \leq y \equiv u \leq v))$;
6. $\forall x \forall y \forall z(x < y \supset x + z < y + z)$;
7. $\forall x \forall y \forall z(x < y \;\&\; 0 < z \supset x \cdot z < y \cdot z)$.

Proof. The first part of the theorem is proved trivially. To prove the second point, it is required to check the derivability of the sequent

$$\mathbf{A} \longrightarrow \forall x \forall y \forall (x \leq y \;\&\; y \leq z \supset x \leq z).$$

Here is the restoration of the inference (from bottom to top):

- $\mathbf{A} \longrightarrow x \leq y \;\&\; y \leq z \supset x \leq z \; (\not\forall R)$;
- $\mathbf{A},\; x \leq y \;\&\; y \leq z \longrightarrow x \leq z \; (\not\supset R)$
- $\mathbf{A},\; x \leq y \;\&\; y \leq z \longrightarrow x \leq z \; (\not\& L)$
- $\mathbf{A},\; x \leq y,\; y \leq z \longrightarrow x \leq z \; (order\ definition)$
- $\mathbf{A},\; \exists u(x + u = y) \; \exists v(y + v = z) \longrightarrow \exists w(x + w = z)$
- $\mathbf{A},\; x + u = y,\; y + v = z \longrightarrow \exists w(x + w = z) \; (\not\exists L)$ with restriction $u \neq v$.

We prove the last sequent in a following way. From $\mathcal{B}_3$ by $(\not\supset R)$ the derivability of the sequent

$$\mathbf{A},\; y = x + u,\; y + v = z \longrightarrow (x + u) + v = z$$

is got. From the axiom of addition's associativity $\mathcal{B}_{10}$ the sequent

$$\mathbf{A},\; (x + u) + v = z \longrightarrow x + (u + v) = z$$

is derived. From two last sequent by (cut) we get the sequent

$$\mathbf{A},\; y = x + u,\; y + v = z \longrightarrow x + (u + v) = z.$$

Now, using the equality symmetry

$$\mathbf{A}\ \ x+u=y\longrightarrow y=x+u,$$

and the cut rule we get

$$\mathbf{A}\ \ x+u=y\ y+v=z\longrightarrow x+(u+v)=z.$$

Applying $(\exists R)$ to the last sequent we obtain

$$\mathbf{A},\ x+u=y,\ y+v=z\longrightarrow \exists w(x+w=z),$$

i.e., the required sequent.

Then take into account the sequent

$$\mathbf{A}\longrightarrow \forall x\forall y\exists z(x+z=y\vee y+z=x).$$

It is an axiom due to $\mathcal{A}_{20}$. Using rules $(\not\forall R)$ we get

$$\vdash \mathbf{A}\longrightarrow \exists z(x+z=y\vee y+z=x).$$

So, for some z, possibly a constants from the Skolem's extension, of the theory **A**, the sequent

$$\mathbf{A}\ \ \longrightarrow x+z=y\vee y+z=x$$

is derivable. Applying $(\not\vee R)$ to which we get that

$$\vdash \mathbf{A}\ \ \longrightarrow x+z=y,\ y+z=x.$$

Applying consequently $(\exists R)$:

$$\mathbf{A}\ \ \longrightarrow \exists z(x+z=y),\ \exists z(y+z=x).$$

In other words, the sequent

$$\mathbf{A}\ \ \longrightarrow x\leq y\ \ y\leq x,$$

is derivable. From the last sequent by $(\vee R)$ we get

$$\mathbf{A}\ \ \longrightarrow x\leq y\vee y\leq x.$$

Applying consequently $(\forall R)$:

$$\vdash \mathbf{A}\longrightarrow \forall x\forall y(x\leq y\vee y\leq x).$$

From this result, statement 4 follows easily.

The proof of the third statement is given by reasoning without a complete presentation of the derivation of the sequents. If $x\leq y$ and $y\leq x$ then for some u, v we have

$$\mathbf{A}\longrightarrow (x+u=y)\ \&\ (y+v=x).$$

It is clear that

$$\mathbf{A}\longrightarrow x+(u+v)=y,$$

i.e., the sequent

$$\mathbf{A}\longrightarrow \exists x(x+z=x)$$

is derivable. Then for some z, perhaps from Skolem's extension of finite arithmetic, we have $x+z=x$. Suppose that $\neg(z=0)$, then $z=w'$ for some w and hence $x=x+w'=x'+w$, i.e., $x' \leq x$. On the other hand it is clear that $x' = (x+0)' = x + 0'$, i.e., $x < x'$. The last two inequalities obtained contradict the statement 4.

If now $x = u$ & $y = v$ & $x + z = y$, то $x + z = u + z$, hence $u + z = y$ & $u + z = v$. So, the sequent

$$\mathbf{A} \longrightarrow x = u \,\&\, y = v \supset (x \leq y \supset u \leq v)$$

is derivable. In the same way, the derivability of the sequent

$$\mathbf{A} \longrightarrow x = u \,\&\, y = v \supset (u \leq v \supset x \leq y)$$

is proven.

Excluding from this sequents' succedents the implication $(\not\supset R)$ and, then, applying to the derived sequents $(\& \, R)$ we get the derivability of the sequent

$$\mathbf{A},\ x = u \,\&\, y = v \longrightarrow x \leq y \equiv u \leq v,$$

or applying $(\supset R)$,

$$\mathbf{A} \longrightarrow x = u \,\&\, y = v \supset (x \leq y \equiv u \leq v),$$

which completes the proof of the fifth statement of the theorem.

The remaining statements are proved similarly. □

The proved theorem describes the properties of the order and its consistency with algebraic operations. This result means that the objects represented by the axioms of the finite arithmetic **A** satisfy the usual properties of natural numbers.

5. Definable Sets and Functions

The concept of a set makes sense only in axiomatic set theory. Since, in our context, we do not assume the existence of an "enveloping" set theory, the usage of the term "set" seems inappropriate. However, almost all modern mathematics is set-theoretic and set-theoretic terms, and often even the terms and notations of "Cantor's paradise" (the so-called naive set theory) are so ingrained that it seems inconceivable to present mathematical facts without them. In addition, these concepts are convenient, and their use significantly shortens the presentation itself, so it often makes it easier to understand. The author's education and a number of his works are also set-theoretic, but when presenting the foundations of both mathematics and its main part, the elementary analysis, the set-theoretic approach should be avoided. But, paying tribute to what was said earlier and considering the affection of most mathematicians to the term "set", we present a specific set theory. This set theory is internal to the finite arithmetic.

In this theory a set is understood as a property of terms of the finite arithmetic language. But, since these properties are not all-encompassing but are described by quite specific formulas, theoretical and multiple paradoxes do not arise. Here the situation is similar to considering formulas with bounded quantifiers since from the point of view of axiomatic set theory, all formulas are determined precisely by bounded quantifiers. That is why we are talking about *definable* sets: our sets are determined using finite arithmetic. This situation is analogous to the one that arises in A. Robinson's classical non-standard analysis [15], where internal sets appear (those that are elements of a non-standard universe). Opposite to them are external sets, they lie outside the universe and cannot be defined inside it. To work with both internal and external sets, it is necessary to expand the existing theories by adding special predicates and functional symbols. Looking ahead, we note that the author also resorts to extending the language of finite arithmetic to describe some properties inherent in natural numbers. In non-standard set-theoretic analysis, this role is played, for example, by Nelson's internal set theory [8].

5.1. Definable Sets

Definition 1. *Let $\mathcal{A}$ be a formula of some predicate calculus theory (here the* **A** *theory is considered but corresponding notions could be introduced for any arbitrary first-order predicate theory). We say that the formula $\mathcal{A}$ defines the set* A *if $\vdash \exists x\mathcal{A}$. Moreover, we will say that for some term t that is free for variable x substitution in $\mathcal{A}$, $t \in \mathrm{A}$ takes place if and only if $\vdash [\mathcal{A}]^x_t$. We also say that t is an element of set* A. *In this case, set* A *is called definable by the formula $\mathcal{A}$. In this case, we write* $\mathrm{A} = \mathrm{A}_{\mathcal{A}} := \{n : \vdash [\mathcal{A}]^x_n\}$.

For example, consider the set of all *natural values* N, an analogue of the set of natural numbers in set-theoretic analysis. This set is defined by the formula $\mathcal{N} := x = x$. In other words, $n \in \mathrm{N}$ if and only if $n = n$. Note that the definition by the formula $x = x$ does not mean the equality of two objects, but the provability within the framework of a certain axiomatic system of this equality. Therefore, in different axiomatic theories, the same formula will define different from the Cantor's paradise point of view sets.

Let us now prove those equivalent formulas define equal sets.

Theorem 2. *Let $\vdash \mathcal{A} \equiv \mathcal{B}$. Then $\vdash \exists x\mathcal{A} \equiv \exists x\mathcal{B}$. In particular, $t \in \mathrm{A}_{\mathcal{A}}$ if and only if $t \in \mathrm{B}_{\mathcal{B}}$.*

Proof. Let u be a variable that appears neither in formula $\mathcal{A}$ nor in formula $\mathcal{B}$. Consider the derivable sequent $\mathbf{A} \longrightarrow [\mathcal{A}]^x_u \supset [\mathcal{B}]^x_u$, excluding the implication from its succedent by $(\not\supset R)$ we get that the sequent $\mathbf{A}, [\mathcal{A}]^x_u \longrightarrow [\mathcal{B}]^x_u$ is derived, to which we apply $(\exists R)$.

Thus we have that the sequent $\mathbf{A}, [\mathcal{A}]^x_u \longrightarrow \exists x\mathcal{B}$ is derivable. The rule was applied correctly due to the choice of u. For the same reason, the rule $(\forall L)$ applies to the resulting sequent. Therefore, the derivable sequent $\mathbf{A}, \exists x\mathcal{A} \longrightarrow \exists x\mathcal{B}$, to which we apply the rule $(\supset R)$ and get that $\vdash \mathbf{A} \longrightarrow \exists x\mathcal{A} \supset \exists x\mathcal{B}$. The opposite implication is proved similarly, which allows us to conclude by $(\equiv R)$ that $\vdash \exists x\mathcal{A} \equiv \exists x\mathcal{B}$. □

The proved theorem allows us to introduce the concept of equality of sets.

Definition 2. *For (definable) sets* A *and* B, *we say* A = B *if and only if $\vdash \mathcal{A} \equiv \mathcal{B}$.*

In other words, as in the common set theory, two sets are equal if and only if they consist of the same elements. Similarly, introducing the concept of a subset, we obtain that sets are equal if and only if this sets are subsets of each other.

Definition 3. *For definable sets* A *and* B, *we say that $A \subset B$ if and only if from $a \in \mathrm{A}$ is derived that $a \in \mathrm{B}$.*

The following theorem could be easily proved.

Theorem 3.

- *for any set* A, $\mathrm{A} \subset \mathrm{A}$;
- *if* $\mathrm{A} \subset \mathrm{B}$ *and* $\mathrm{B} \subset \mathrm{C}$, *then* $\mathrm{A} \subset \mathrm{C}$;
- $\mathrm{A} = \mathrm{B}$ *if and only if* $\mathrm{A} \subset \mathrm{B}$ *and* $\mathrm{B} \subset \mathrm{A}$.

Proof. To prove it, note that $\mathrm{A} \subset \mathrm{B}$ means that $\vdash \mathcal{A} \supset \mathcal{B}$ for the set defining formulas. Then the first and third statements are apparent; the second follows trivially from the provability of the sequent

$$\mathbf{A} \longrightarrow (\mathcal{A} \supset \mathcal{B}) \supset ((\mathcal{B} \supset \mathcal{C}) \supset (\mathcal{A} \supset \mathcal{C})).$$

Consider a sequent $\mathbf{A} \longrightarrow \mathcal{A} \mathbin{\&} \neg\mathcal{A} \equiv \mathcal{B} \mathbin{\&} \neg\mathcal{B}$ for any formula $\mathcal{A}$ and $\mathcal{B}$. Since

$$\mathcal{A} \longrightarrow \mathcal{A}, \mathcal{B} \mathbin{\&} \neg\mathcal{B}$$

is an axiom, applying first $(\& \, L)$ then $(\supset R)$, we get that the sequent

$$\mathbf{A}\mathcal{A} \neg\mathcal{A} \longrightarrow \mathcal{B} \,\&\, \neg\mathcal{B}$$

is derivable. So

$$\mathbf{A} \longrightarrow \mathcal{A} \,\&\, \neg\mathcal{A} \supset \mathcal{B} \,\&\, \neg\mathcal{B}$$

is also derivable. The opposite implication is proved similarly. □

Considering the above, the following definition of an empty set is correct, and the empty set is uniquely defined.

Definition 4. *The empty set, denoted further by the symbol* Λ*, is defined by the formula* $\neg(x = x)$.

Definition 5. *The union of (definable) sets* A *and* B *is a (definable) set denoted as usual* $\mathrm{A} \cup \mathrm{B} := \{n : \vdash [\mathcal{A}]_n^x \vee [\mathcal{B}]_n^x\}$. *The intersection of (definable) sets* A *and* B *is a (definable) set denoted as usual* $\mathrm{A} \cap \mathrm{B} := \{n : \vdash [\mathcal{A}]_n^x \,\&\, [\mathcal{B}]_n^x\}$. *The subtraction of (definable) sets* A *and* B *is a (definable) set denoted as usual* $\mathrm{A} \setminus \mathrm{B} := \{n : \vdash [\mathcal{A}]_n^x \,\&\, \neg[\mathcal{B}]_n^x\}$.

Theorem 4. *All the usual set-theoretic relations are valid for the introduced set-theoretic operations.*

Proof. We will not give all set-theoretic formulas, but restrict ourselves to proving some. For example, the validity of one of de Morgan's laws is obtained by constructing the inference of the sequent

$$\mathbf{A} \longrightarrow \mathcal{A} \,\&\, \neg(\mathcal{B} \vee \mathcal{C}) \equiv (\mathcal{A} \,\&\, \neg\mathcal{B}) \,\&\, (\mathcal{A} \,\&\, \neg\mathcal{C}).$$

The derivability of this sequent is equivalent to the derivability of the following two sequents

$$\mathbf{A}, \mathcal{A} \,\&\, \neg(\mathcal{B} \vee \mathcal{C}) \longrightarrow \mathcal{A} \,\&\, \neg\mathcal{B} \,\&\, \mathcal{A} \,\&\, \neg\mathcal{C}$$

$$\mathbf{A}, \mathcal{A} \,\&\, \neg\mathcal{B} \,\&\, \mathcal{A} \,\&\, \neg\mathcal{C} \longrightarrow \mathcal{A} \,\&\, \neg(\mathcal{B} \vee \mathcal{C}).$$

The inference of the first one is obtained from axioms

$$\mathcal{A} \longrightarrow \mathcal{B}, \mathcal{C}, \mathcal{A} \quad \mathcal{A}, \mathcal{B} \longrightarrow \mathcal{B}, \mathcal{C} \quad \mathcal{A} \longrightarrow \mathcal{B}, \mathcal{C}, \mathcal{A} \quad \mathcal{A}, \mathcal{C} \longrightarrow \mathcal{B}, \mathcal{C}$$

using $(\vee R)$ first with obtaining formula $\mathcal{B} \vee \mathcal{C}$

$$\mathcal{A} \longrightarrow \mathcal{B} \vee \mathcal{C}, \mathcal{A} \quad \mathcal{A}, \mathcal{B} \longrightarrow \mathcal{B} \vee \mathcal{C} \quad \mathcal{A} \longrightarrow \mathcal{B} \vee \mathcal{C}\, \mathcal{A} \quad \mathcal{A}, \mathcal{C} \longrightarrow \mathcal{B} \vee \mathcal{C},$$

and $(\neg R)$ then for the second and the fourth sequent

$$\mathcal{A} \longrightarrow \mathcal{B} \vee \mathcal{C}, \mathcal{A} \quad \mathcal{A}, \longrightarrow \neg\mathcal{B}, \mathcal{B} \vee \mathcal{C} \quad \mathcal{A} \longrightarrow \mathcal{B} \vee \mathcal{C}, \mathcal{A} \quad \mathcal{A}, \longrightarrow \neg\mathcal{C}, \mathcal{B} \vee \mathcal{C}.$$

Now allying $(\neg L)$

$$\mathcal{A}, \neg(\mathcal{B} \vee \mathcal{C}) \longrightarrow \mathcal{A} \quad \mathcal{A}, \neg(\mathcal{B} \vee \mathcal{C}) \longrightarrow \neg\mathcal{B} \quad \mathcal{A}, \neg(\mathcal{B} \vee \mathcal{C}) \longrightarrow \mathcal{A} \quad \mathcal{A}, \neg(\mathcal{B} \vee \mathcal{C}) \longrightarrow \neg\mathcal{C}.$$

Then consequently applying the rule $(\& \, R)$ we obtain the required sequent.
The inference of the second sequent is constructed similarly. □

Note that the presence of axioms of the finite arithmetic in the antecedent of sequents is not necessary. That is, the corresponding properties are derivable without any theory, they are predicate tautologies. In other words, in the presence of an enveloping set theory, set-theoretic formulas for definable sets are absolute, independent of the theory.

5.2. Definable Functions

Definition 6. *By a definable function f we mean a formula $\mathcal{F}$ in the language of the theory* **A** *with the following properties:*

- $\vdash \forall x \Big(\exists y [\mathcal{F}]^{uv}_{xy} \supset \Big(\forall z \forall w \forall p \Big([\mathcal{F}]^{uv}_{zw} \ \& \ [\mathcal{F}]^{uv}_{zp} \supset w = p \Big) \Big) \Big)$;
- *formula $\exists y [\mathcal{F}]^{uv}_{xy}$ defines a set called function domain and denoted as* $\mathrm{dom}\, f$.

Just like in the traditional analysis, we say that a function f maps some (definable) set X containing $\mathrm{dom}\, f$ into a definable set Y, and there is a (definable) set $\mathrm{rng}\, f$, defined by the formula $\exists x [\mathcal{F}]^{uv}_{xy}$.

Moreover, if f is a definable function, then for every $x \in \mathrm{dom} f$ there is a unique element $y \in \mathrm{N}$ such then $\vdash [\mathcal{F}]^{uv}_{xy}$, that is called the value of the function f at the element x and denoted as $f(x)$. The uniqueness is followed from the fact that if $\mathbf{A} \vdash [\mathcal{F}]^{uv}_{xy_1} \ \& \ [\mathcal{F}]^{uv}_{xy_2}$, then by the function definition $\mathbf{A} \vdash y_1 = y_2$.

If $\vdash \exists y [\mathcal{F}]^{uv}_{xy}$, then we denote $y = f(x)$. At the same time, the case of *several* variables is not excluded. This will be the case when a *tuple* of free variables of the formula $\mathcal{F}$ is treated as the single variable y.

It is easy to see that the following functions are definable and defined everywhere.

1. $f(x, y) = x + y$;
2. $f(x, y) = x \cdot y$;
3. $f(x) = t$, where t is a some term;
4. $f(x) = \mathrm{rem}(x, y)$ (remainder of a division x to y);
5. $f(x) = \mathrm{quot}(x, y)$ (quotient of a division x to y;
6. $f(x, y) = \max(x, y)$;
7. $f(x, y) = \min(x, y)$.

Functions $f(x, y) = x - y$ and $g(x, y) = \dfrac{x}{y}$ are definable but not everywhere defined.

6. Finite and Infinite Natural Numbers

6.1. Algorithmically Reachable Natural Numbers

It is apparent that in the Zermelo-Fraenkel set theory, the set of natural numbers (i.e., the set of finite ordinals ω) is a model of finite arithmetic and even formalized number theory [3]. Earlier, we have introduced the set N defined by the formula $n = n$. This set is designed to be an analogue of natural numbers in the finite arithmetic. It is apparent that the class N does not represent all the properties of natural numbers, so we call these "defective" objects not numbers but values.

The axioms of finite arithmetic guarantee that the *addition of unity function* $x' = x + 1$ is defined on the set of all natural values. On the other hand, if n is a non-zero natural value, there is such m that $n = m'$. Moreover, this element n is uniquely determined because on $\mathcal{B}_4$.

Consider for any non-zero $n \in \mathrm{N}$ that the unique element $m \in \mathrm{N}$, such then $n = m'$. If $m = 0$, the procedure is considered as finished. If $m \neq 0$, then there exists a unique element $k \in \mathrm{N}$ such then $m = k'$, i.e., $n = k''$. In this case, again, if $k = 0$, the procedure is finished, otherwise it is continued. There arises a natural question, will this procedure ever stop? In other words, will zero be reached at some step?

The presence of the induction axiom in classical number theory allows us to prove that the process will stop and, therefore, any natural number has the form $0''^{\cdots\prime}$. In the case of finite arithmetic, this cannot be guaranteed. Therefore, there is a class of those natural values with the form $0''^{\cdots\prime}$. Clearly this class is not a definable set within the framework of the theory of finite arithmetic. On the other hand, the sum and product of values of the form $0''^{\cdots\prime}$ is also a value of the same form. However, as follows from the impossibility of defining this form using finite arithmetic, it is impossible to prove this within the theory **A**. The situation here exactly represents the fact that the set of natural numbers in the non-standard universe of A. Robinson is external.

6.2. Hyperarithmetic

All of the above induces us to expand the finite arithmetic by introducing a new unary predicate symbol $\mathfrak{N}$ into the language and the following new axioms:

$\mathcal{N}_0 := \mathfrak{N}(0)$;
$\mathcal{N}_1 := \forall x(\mathfrak{N}(x) \supset \mathfrak{N}(x'))$;
$\mathcal{N}_2 := \forall x \forall y(\mathfrak{N}(x) \ \& \ \mathfrak{N}(y) \equiv \mathfrak{N}(x+y))$;
$\mathcal{N}_3 := \forall x \forall y(\mathfrak{N}(x) \ \& \ \mathfrak{N}(y) \equiv \mathfrak{N}(x \cdot y))$;
$\mathcal{N}_4 := \forall x \forall y(x = y \supset (\mathfrak{N}(x) \equiv \mathfrak{N}(y)))$.

Definition 7. *The theory* **HA***, the union of* **A** *and axioms* $\mathcal{N}_0$*–*$\mathcal{N}_4$*, is called finite hyperarithmetic.*

Definition 8. *The set,* $\mathbb{N}$*, defined in the theory* **HA** *by the formula* $\mathfrak{N}(x)$*, is called the set of finite natural numbers. The complementary set, defined by the formula* $\neg\mathfrak{N}(x)$ *is called the set of infinite natural numbers.*

In the theory **HA** one could define a set of all natural values N and its subset $\mathbb{N}$ containing only finite values. From the set-theoretical point of view, the relationship between these sets is the same as between the set of hypernatural numbers and the set of (finite) natural numbers.

The axioms $\mathcal{N}_1$–$\mathcal{N}_4$ guarantee that the set of finite numbers is closed under all arithmetic operations.

From the axioms $\mathcal{N}_2$, $\mathcal{N}_3$ the following statement instantly follows.

Theorem 5. *The sum and product of infinite numbers are infinite. The sum of a finite and an infinite number is infinite.*

It is also easy to prove

Theorem 6.
$$\vdash \forall x \forall y(\mathfrak{N}(x) \ \& \ \neg\mathfrak{N}(y) \supset x < y).$$

Proof. Apparently there cannot be $x + z = y$ and $y + z = x$ simultaneously since this contradicts the belonging of x and y to different classes. On the other hand, since for the same reason, there cannot be $x = y$ due to the trichotomy of $x < y$. □

Note that if $x \neq 0$ is finite, then its predecessor is also finite, which is either a zero value or has a predecessor. So we get the chain $x = x_1 + 1 = x_2 + 2 = \ldots x_k + k$. Since x is finite, every x_i in this chain is finite. And, since zero is a finite number without a predecessor, this chain will terminate at a step m such that $x_m = 0$. In other words, finite numbers are *reachable*. Infinite numbers are unreachable. Here we identified the concept of a step with the addition of one; we brought the concept of finiteness from the *external world*.

Thus, from the external world point of view, there is a class N of all objects having the properties of natural numbers. It has a subclass of finite numbers distinguished as a class of objects satisfying the property $\mathbb{N}$. A complementary class represents those numbers that do not exist with the usual introduction of natural numbers. We owe its possible appearance to the absence of an induction axiom scheme, with the help of which it is proved that all numbers are reachable. The non-emptiness of the class of infinite numbers, on the one hand, is a payment for the finiteness of the system of axioms, and on the other hand, a bonus that allows, by extending finite hyperarithmetic to the theory of hyperrational numbers, to obtain infinitesimal as actual objects of analysis, as is done in [6].

The notion of reachability can also be described from the point of view of the existence of some algorithm, which, in a certain number of steps, matched in this context the act of adding one (taking a follower), allows one to construct a reachable number. Thus, the class of finite numbers is a class of all numbers that can be constructed using some algorithm. This makes our theory close to a constructive conception of the natural number [16].

6.3. Class N Structure

Theorem 7. *For any natural values x and y if $x > y$, then there exists a unique z such then $x + z = y$.*

The existence is followed from the inequality definition, and the uniqueness is obtained from $\mathcal{B}_{21}$.

Definition 9. *So the function f of* $\mathrm{dom} f = \{\langle x, y\rangle : x > y\}$ *is definable. For pairs from complementary to* $\mathrm{dom} f$ *set we put $f(x, y) = 0$. Let us denote it as $x \ominus y = f(x, y)$ and call it pseudosubstraction.*

The following statement could be easily proven.

Theorem 8. $\vdash \forall x \forall y (x > y \equiv x \ominus y > 0)$.

Proof. To the proof, note that if $x \ominus y = k > 0$, then it is clear that $y + k = x$ and, therefore, $x > y$ since $k \neq 0$. The deductibility of the backward implication could be proved similarly. □

Note that if w is an infinite number, then $2w = w + w$ is also infinite. On the other hand, $2w > w$. In other words, it is impossible to reach $2w$ from w by adding one identity. The chain $w < w' < w'' < \ldots$ does not end when it reaches the number $2w$. It is clear that decreasing the number w by one, i.e., building the chain $w > w \ominus 1 > w \ominus 2 > \ldots$, will never give a finite number. Following this reasoning, we introduce the following appropriate terminology.

Definition 10. *The block of infinite number w is the set $B(w)$, defined by the formula* $\exists k(\mathfrak{N}(k) \; \& \; (z = w + k \vee z = w \ominus k))$.

Apparently, a block is defined by any number that included in it.

Theorem 9. *Let w_1 be w_2 two infinite numbers. Then $B(w_1) = B(w_2)$ if and only if* $\vdash \exists k(\mathfrak{N}(k) \; \& \; (w_1 = w_2 + k \vee w_2 = w_1 + k))$.

The proof of this statement is trivial. Obviously, if two infinite numbers are not linked by the relation $w_1 = w_2 \pm k$ where k is a finite number, then their blocks do not intersect.

Thus, the class N from the external (descriptive) point of view consists of an initial segment of finite numbers, and then a class of disjoint blocks, each of which is isomorphic in the sense of order to integers from the point of view of set-theoretic mathematics. There is no "smallest" and "largest" block, because it is clear that $\vdash 2w > w$. On the other hand, if w_1 and w_2 are such that $w_1 < w_2$ and their blocks do not intersect, then by setting $w = \left\lfloor \dfrac{w_1 + w_2}{2} \right\rfloor$ (integer floor), we get that $w_1 < w < w_2$ and all three numbers are from different blocks. In other words, the order on the class of all blocks is dense. What is said in this paragraph also applies to the external description (structure) of the class $\mathcal{N}$. There is a similar reasoning in [17].

6.4. The Existence of Infinite Numbers

Note that the existence of infinite numbers as well as their absence does not follow from the axioms of finite hyperarithmetic. Therefore, for a meaningful theory in our applications, it is necessary to add the axiom of the non-emptiness of the class of infinite numbers. As such an axiom, we introduce the $\mathcal{W}$-axiom

$$\mathcal{W} := \exists x \neg \mathfrak{N}(x).$$

Finally, we will consider as an alternative arithmetic a theory that includes axioms $\mathcal{A}_1$–$\mathcal{A}_{21}$, $\mathcal{N}_0$–$\mathcal{N}_4$, $\mathcal{W}$. Further, basing on the adopted axiomatics, one can develop the theory of hyperrational numbers and functions.

7. Conclusions and Discussion

We have constructed a finite axiomatic theory of arithmetic that is sufficient for constructing an axiomatic hyperrational analysis.

Note that the author introduced a similar axiomatic system in his report at the conference at the A. I. Herzen Russian State Pedagogical University [18], but, firstly, this article does not appeal to the theory of models, added the results presented in Section 6 and the proofs of the main results, and also the axioms $\mathcal{N}_3$ and $\mathcal{N}_4$ were changed, and the axiom $\mathcal{A}_{21}$ was added.

Earlier at Section 6.2, it was noted that reachable numbers could be obtained by applying the Markov (normal) algorithm, which could be represented as an algorithm in the $\{|\}$ alphabet, described by the scheme

$$\langle \text{Word} \rangle := \langle \text{Empty word} \rangle \Bigg| \langle \text{Word} \rangle |$$

(note the difference in size between BNF expansion character and the actual alphabet letter). In other words, in the Zermelo-Fraenkel theory, there is an isomorphism between reachable numbers and words in the alphabet $\{|\}$. Thus, finite natural numbers in our theory from the external set theory point of view are constructive natural numbers.

In conclusion, let us focus on the issues not disclosed in this work, which sets relevant problems for further work.

1. Investigation of the consistency and completeness of theories **A** and $\mathbf{HA} + \mathcal{W}$. Note that there are models of these theories in the set theory, and the **HA** model is an elementary extension of the **A** model.
   Note that the finite arithmetic takes an intermediate position between formalized number theory (being its subtheory) and R. Robinson's arithmetic [1], which is incomplete (recursively incomplete) and undecidable subtheory of finite arithmetic.
2. The problem of minimizing the list of axioms of finite arithmetic and eliminating dependent axioms is also relevant. In the present work, the authors did not set this problem.
3. The development of axiomatics of arithmetic sufficient for constructing of a hyperrational analysis, allowing the modelling of hypernatural numbers using sets of finite natural numbers, is still an important relevant problem. For example, polynomials of the form $\sum_{i=0}^{n} a_i \mathrm{Y}^i$, where $n, a_i \in \mathbb{Z}$, and Y is some chosen infinite natural number, or other structures could be used. Solving this problem will create an algorithmically decidable hyperrational analysis.

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