# Time Response of Second Order Systems

$$M\frac{d^2y(t)}{dt^2} = f(t) - B\frac{dy(t)}{dt} - Ky(t)$$

$$M(s^2Y(s) - sy(0) - y'(0)) = F(s) - B(sY(s) - y(0)) - KY(s)$$

$$Let\ y'(0) = 0$$

$$Ms^2Y(s) + BsY(s) + KY(s) = Msy_0 + By_0 + F(s)$$

$$Or$$

$$Y(s) = \frac{(Ms+B)y_0}{Ms^2 + Bs + K} + \frac{F(s)}{Ms^2 + Bs + K}$$

Consider the first term only:

$$Y(s) = \frac{(s + B/M)y_0}{s^2 + (B/M)s + K/M} = \frac{(s + 2\varsigma\omega_n)y_0}{s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}}$$

$\varsigma$ = dimensionless damping ratio

$\omega_n$ = the natural frequency

*Where*

$$\omega_n = \sqrt{\frac{K}{M}}; \quad \varsigma = \frac{B}{2\sqrt{KM}}$$

The characteristic equation $s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}$ has two roots:

$$s_1 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}$$

$$s_2 = -\varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2}$$

If $\varsigma > 1 \Rightarrow$ roots are real

If $\varsigma < 1 \Rightarrow$ roots are complex (under damped)

If $\varsigma = 1 \Rightarrow$ same roots and real (critically damped)

For unit step response:

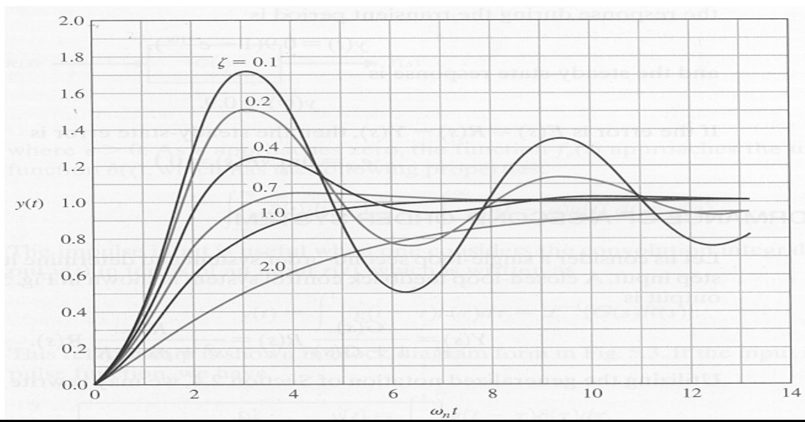
$$y(t) = 1 - \frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t + \theta)$$

$$\beta = \sqrt{1-\varsigma^2}, \quad \theta = \tan^{-1}\left(\frac{\beta}{\varsigma}\right) = \cos^{-1}\varsigma$$

Step response: $$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

where $$\beta = \sqrt{1-\zeta^2}, \quad \theta = \cos^{-1}\zeta$$

Showing the step response with different damping coefficients




## Standard Performance measures

- Performance measures are usually defined in terms of the step response of a system as below:
- Swiftness of the response is measured by **rise time** $T_r$ , and **peak time** $T_p$
- For underdamped system, the Rise time $T_r$ (0-100% rise time) is useful,
- For overdamped systems, the the Peak time is not defined, and the (10-90 % rise time) $T_{r1}$ is normally used

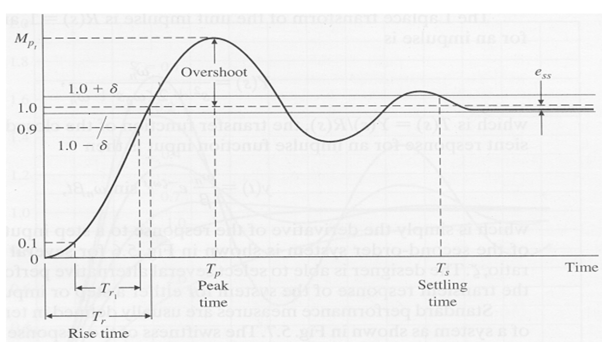
- Peak time: $T_p$
- Steady-state error: $e_{ss}$
- Settling time: $T_s$

- Percent of Overshoot:

$$\text{P.O.} = \frac{M_{pt} - fv}{fv} \times 100\ \%$$

$M_{pt}$ is the peak value

fv is the final value of the response



- Percentage overshoot measures the closeness of the response to the desired response.
- The **settling time** $T_s$ is the time required for the system to settle within a certain percentage $\delta$ of the input amplitude.
- For second order system, we seek $T_s$ for which the response remains within 2% of the final value. This occurs approximately when:

$$e^{-\zeta\omega_n T_s} < 0.02$$

or: $\zeta\omega_n T_s \cong 4$

Therefore:

$$T_s \cong 4\tau = \frac{4}{\zeta\omega_n}$$

- Hence the settling time is defined as 4 time constants.

- Explicit relations for $M_{p_t}$ and $T_p$ :
- To find $T_p$ we can either differentiate y(t) directly or indirectly through Laplace Transform of y'(t):

$$T_p = \frac{\pi}{\omega_n \beta} = \frac{\pi}{\omega_n\sqrt{1-\varsigma^2}}$$

The peak response is:

$$M_{p_t} = 1 + e^{-\zeta\pi/\sqrt{1-\varsigma^2}}$$

Percentage overshoot:

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\varsigma^2}}$$

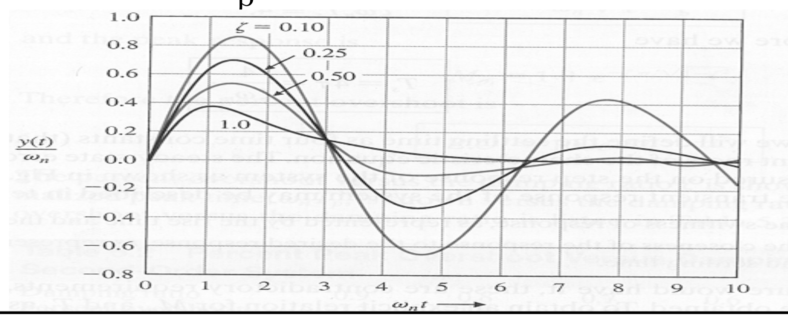
# Impulse response of the second order system:

- Laplace transform of the unit impulse is R(s)=1

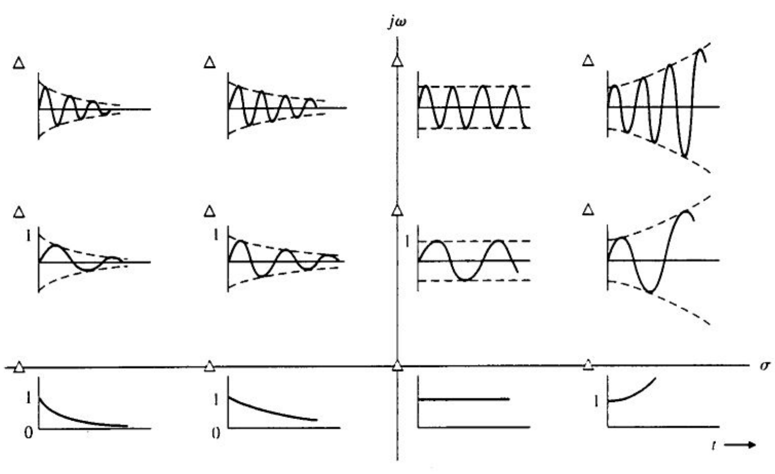
$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- Impulse response:
- Transient response for the impulse function, which is simply is the derivative of the response to the unit step:

$$y(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t)$$




- Responses and pole locations



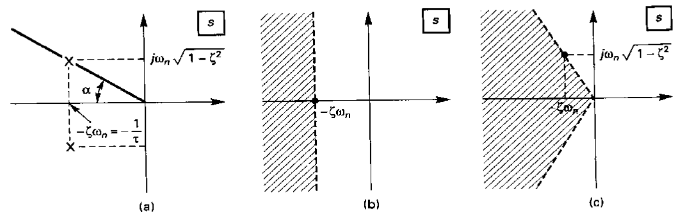
- Time Responses and Pole Locations:

The characteristic equation $s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}$ has two roots :

$$s_1 = -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2}$$

$$s_2 = -\varsigma\omega_n - j\omega_n\sqrt{1-\varsigma^2}$$

- $s_1, \quad s_2$ are the poles.




$$T_s \cong 4\tau = \frac{4}{\zeta\omega_n}$$

- Therefore the settling time is inversely proportional to the real part of the poles.