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The subspace structure of finite dimensional Beidleman near-vector spaces

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ABSTRACT
The subspace structure of Beidleman near-vector spaces is investigated. We characterize finite dimensional Beidleman near-vector spaces and classify their *R*-subgroups. We provide an algorithm to compute the smallest *R*-subgroup containing a given set of vectors. Finally, we classify the subspaces of finite dimensional Beidleman near-vector spaces.

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1. Introduction

The first notion of near-vector spaces was introduced by Beidleman in 1966 [1]. Subsequently, several researchers like Whaling, André, and Karzel introduced a similar notion in different ways. André near-vector spaces have been studied in many papers (for example [2–4]). In this paper, we add to the theory of near-vector spaces originally defined by Beidleman. As in [4] for André near-vector spaces, we investigate the subspace structure of Beidleman near-vector spaces and highlight differences and similarities between these two types of near-vector spaces.

In Section 2 we give some preliminary material needed for this paper. In Section 3 we characterize finite dimensional Beidleman near-vector spaces while in Section 4 we investigate some properties of the substructures of Beidleman near-vector spaces. In Section 5 we classify the R-subgroups of the near-vector space R^n containing a given set of vectors where R is a nearfield. We provide an algorithm and Sage program for computations. Finally in Section 6 we classify all the subspaces of R^n.

2. Preliminary material

In this section we give the basic definitions and results we will need.

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Definition 2.1 ([5]): The triple $(R, +, \cdot)$ is a (left) nearring if $(R, +)$ is a group, $(R, \cdot)$ is a semigroup, and $a(b + c) = ab + ac$ for all $a, b, c \in R$.

A nearfield is an algebraic structure similar to a skew-field, sometimes called a division ring, except that only one of the two distributive laws is required.

Definition 2.2 ([6]): Let R be nearring. If $(R^* = R \setminus \{0\}, \cdot)$ is a group then $(R, +, \cdot)$ is called nearfield.

We will make use of left nearfields and right nearring modules in this paper. Dickson, Zassenhauss, Neumann, Karzel and Zemmer have shown by different methods that the additive group of a nearfield is abelian.

Theorem 2.3 ([6]): *The additive group of a nearfield is abelian.*

To construct finite Dickson nearfields, we need two concepts:

Definition 2.4 ([6]): A pair of numbers $(q, n) \in \mathbb{N}^2$ is called a Dickson pair if q is some power p^l of a prime p, each prime divisor of n divides $q-1$, $q \equiv 3 \bmod 4$ implies 4 does not divide n.

Definition 2.5 ([6]): Let R be a nearfield and $Aut(R, +, \cdot)$ the set of all automorphisms of R. A map

$$\begin{aligned} \phi : R^* &\to Aut(R, +, \cdot) \\ n &\mapsto \phi_n \end{aligned}$$

is called a coupling map if for all $n, m \in R^*, \phi_n \circ \phi_m = \phi_{\phi_n(m) \cdot n}$.

The first proper finite nearfield was discovered by Dickson in 1905. He distorted the multiplication of a finite field. For all pairs of Dickson numbers (q, n), there exists some associated finite Dickson nearfields, of order q^n which arise by taking the Galois field $GF(q^n)$ and changing the multiplication. Thus $DN(q, n) = (GF(q^n), +, \cdot)^\phi = (GF(q^n), +, \circ)$. We will use $DN(q, n)$ to denote a Dickson nearfield arising from the Dickson pair (q, n). For more details on the construction of the new multiplication '$\circ$' we refer the reader to [6,7]. As a first example we have,

Example 2.6: Consider the field $(GF(3^2), +, \cdot)$ with

$$GF(3^2) := \{0, 1, 2, x, 1 + x, 2 + x, 2x, 1 + 2x, 2 + 2x\},$$

where x is a zero of $x^2 + 1 \in \mathbb{Z}_3[x]$. In [6, p. 257], it is observed that $(GF(3^2), +, \cdot)$ with the new multiplication

$$a \circ b := \begin{cases} a \cdot b & \text{if } b \text{ is a square in } (GF(3^2), +, \cdot) \\ a^3 \cdot b & \text{otherwise} \end{cases}$$

is a (right) nearfield but not a field. When the new multiplication is defined as

$$a \circ b := \begin{cases} a \cdot b & \text{if } a \text{ is a square in } (GF(3^2), +, \cdot) \\ a \cdot b^3 & \text{otherwise} \end{cases}$$

we obtain the smallest finite Dickson (left) nearfield $DN(3,2) := (GF(3^2), +, \circ)$, which is not a field. The table of the new operation ° for the (left) nearfield $DN(3,2)$ is given by:

$\circ$	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
2	0	2	1	$2x$	$2x+2$	$2x+1$	x	$x+2$	$x+1$
x	0	x	$2x$	2	$x+2$	$2x+2$	1	$x+1$	$2x+1$
$1+x$	0	$x+1$	$2x+2$	$2x+1$	2	x	$x+2$	$2x$	1
$x+2$	0	$x+2$	$2x+1$	$x+1$	$2x$	2	$2x+2$	1	x
$2x$	0	$2x$	x	1	$2x+1$	$x+1$	2	$2x+2$	$x+2$
$2x+1$	0	$2x+1$	$x+2$	$2x+2$	x	1	$x+1$	2	$2x$
$2x+2$	0	$2x+2$	$x+1$	$x+2$	1	$2x$	$2x+1$	x	2

We will refer to this example in later sections.

The concept of a ring module can be extended to a more general concept called a nearring module where the set of scalars is taken to be a nearring.

Definition 2.7: An additive group $(M, +)$ is called (right) nearring module over a (left) nearring R if there exists a mapping,

$$\eta : M \times R \to M$$
$$(m, r) \to mr$$

such that $m(r_1 + r_2) = mr_1 + mr_2$ and $m(r_1 r_2) = (mr_1)r_2$ for all $r_1, r_2 \in R$ and $m \in M$.
We write M_R to denote that M is a (right) nearring module over a (left) nearring R.

Submodules and R-subgroups of nearring modules are central to our discussion.

Definition 2.8 ([1]): A subset A of a nearring module M_R is called a R-subgroup if A is a subgroup of $(M, +)$, and $AR = \{ar | a \in A, r \in R\} \subseteq A$.

Definition 2.9 ([1]): A nearring module M_R is said to be irreducible if M_R contains no proper R-subgroups. In other words, the only R-subgroups of M_R are M_R and $\{0\}$.

Corollary 2.10 ([1]): *Let M_R be a unitary R-module. Then M_R is irreducible if and only if $mR = M_R$ for every non-zero element $m \in M$.*

Definition 2.11 ([1]): Let M_R be a nearring module. N is a submodule of M_R if :

- $(N, +)$ is normal subgroup of $(M, +)$,
- $(m + n)r - mr \in N$ for all $m \in M, n \in N$ and $r \in R$.

Proposition 2.12 ([1]): *Let N be a submodule of M_R. Then N is a R-subgroup of M_R.*

Note that the converse of this proposition is not true in general. In his thesis [1, page 14] Beidleman gives a counter example. However, it is not difficult to show that

Lemma 2.13: *If M_R is a ring module, then the notions of R-subgroup and submodule of M_R coincide.*

Theorem 2.14 ([1]): *Let R be a nearring that contains a right identity element $e \neq 0$. R is nearfield if and only if R contains no proper R-subgroups.*

Remark 2.15: Let R be a nearfield. By Theorem 2.14, R_R is an irreducible R-module. Thus R contains only $\{0\}$ and R as submodules of R_R.

In order to define Beidleman near-vector spaces we will need:

Definition 2.16 ([1]): Let $\{M_i | i \in I\}$ be a collection of submodules of the nearring module M_R. M_R is said to be a direct sum of the submodules M_i, for $i \in I$, if the additive group $(M, +)$ is a direct sum of the normal subgroups $(M_i, +)$, for $i \in I$. In this case we write $M_R = \bigoplus_{i \in I} M_i$.

Proposition 2.17 ([1]): *$M_R = \sum_{i \in I} M_i$ and every element of M_R has a unique representation as a finite sum of elements chosen from the submodules M_i if and only if $M_R = \sum_{i \in I} M_i$ and $M_k \cap \sum_{i \in I, i \neq k} M_i = \{0\}$ for each $k \in I$.*

We also have that

Proposition 2.18 ([1]): *Let $\{M_i \mid i \in I\}$ be a collection of submodules of the nearring module M_R. Then $M_R = \bigoplus_{i \in I} M_i$ implies that $M_R = \sum_{i \in I} M_i$ and the elements of any two distinct submodules M_i and M_j commute.*

According to the definition of a nearring module, we do not have distributivity of elements of R over the elements of M. If we consider M_R as direct sum of the collection of submodules $\{M_i \mid i \in I\}$ of the nearring module M_R, then the following result will allow us to distribute the elements of R over the elements which are contained in distinct submodules in the direct sum. The result is useful in the concept of Beidleman near-vector spaces.

Lemma 2.19 ([1]): *Let $M_R = \bigoplus_{i \in I} M_i$, M_i is a submodule of M_R. If $m = \sum_{i \in I} m_i$ where $m_i \in M_i$ and $r \in R$ then*

$$mr = \left(\sum_{i \in I} m_i \right) r = \sum_{i \in I} (m_i r).$$

Beidleman near-vector spaces

In [2], the concept of a vector space (or linear space) is generalized by André to a less linear structure which he called a near-vector space. André near-vector spaces use automorphisms in the construction, resulting in the right distributive law holding. However, for Beidleman near-vector spaces, we have the left distributive law holding and nearring modules are used in the construction.

Definition 2.20 ([1]): A nearring module M_R is called strictly semi-simple if M_R is a direct sum of irreducible submodules.

We now have,

Definition 2.21 ([1]): Let $(M, +)$ be a group. M_R is called Beidleman near-vector space if M_R is a strictly semi-simple R-module where R is a nearfield.

The simplest example of a Beidleman near-vector space is obtained when M_R itself is an irreducible R-module.

Lemma 2.22 ([1]): *Let R be a nearfield and M_R an irreducible R-module. Then M_R is a Beidleman near-vector space. Moreover,*

$$M_R \cong R_R.$$

As with vector spaces, we have the notion of a basis and dimension.

Definition 2.23 ([1]): A non-empty subset X of a near-vector space M_R is called a basis of M_R if X is a spanning set for M_R and the representation of the elements of M_R as a linear combinations of the elements of X is unique.

Theorem 2.24 ([1]): *If M_R is a near-vector space, then M_R has a basis.*

Finally we have,

Definition 2.25 ([1]): If M_R is a near-vector space over R, then the cardinality of any basis is called the dimension of M_R and is denoted by $\dim M_R$.

3. Finite dimensional Beidleman near-vector spaces

In [8] van der Walt characterized finite dimensional André near-vector spaces. In this section we do the same for finite dimensional Beidleman near-vector spaces. We will see that finite dimensional Beidleman near-vector spaces are closest (in terms of structure) to traditional finite dimensional vector spaces. Let $r_1, \dots, r_n \in R$. We will use $(r_1, \dots, r_n)$ to denote elements of R^n where n is a positive integer.

Theorem 3.1: *Let R be a (left) nearfield and M_R be a right nearring module. M_R is a finite dimensional near-vector space if and only if $M_R \cong R^n$ for some positive integer $n = \dim M_R$.*

Proof: Let us consider the R-modules M_R and R_R^n with the action given as $M \times R \to M$ such that $(m, r) \mapsto mr$. Since M_R is a Beidleman near-vector space, $M_R = \bigoplus_{i=1}^n M_i$ where M_i for $i \in \{1, \ldots, n\}$ are non-zero irreducible R-submodules of M_R. Then by Corollary 2.10, there exists $0 \neq m_i \in M_i$ such that $m_i R = M_i$ for $i \in \{1, \ldots, n\}$. Hence $M_R = \bigoplus_{i=1}^n m_i R$. Then it is not difficult to see that $B = \{m_1, \ldots, m_n\}$ is a basis of M_R. Let us consider the map

$$\phi : R^n \to M$$
$$(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n m_i r_i.$$

Let $(r_1, \ldots, r_n), (r'_1, \ldots, r'_n) \in R^n$. We have

$$\begin{aligned}\big((r_1, \ldots, r_n) + (r'_1, \ldots, r'_n)\big)\phi &= \sum_{i=1}^n m_i(r_i + r'_i) = \sum_{i=1}^n (m_i r_i + m_i r'_i) \\ &= \sum_{i=1}^n m_i r_i + \sum_{i=1}^n m_i r'_i \\ &= (r_1, \ldots, r_n)\phi + (r'_1, \ldots, r'_n)\phi.\end{aligned}$$

Let $r \in R$ and $(r_1, \ldots, r_n) \in R^n$. Using Lemma 2.19, we obtain

$$\big((r_1, \ldots, r_n)r\big)\phi = \sum_{i=1}^n (m_i r_i)r = \left(\sum_{i=1}^n m_i r_i\right) r = (r_1, \ldots, r_n)\phi r.$$

Since B is a basis of M_R, $(r_1, \ldots, r_n)\phi = 0 \Rightarrow \sum_{i=1}^n m_i r_i = 0 \Rightarrow r_1 = r_2 = \ldots = r_n = 0$. We deduce that,

$$Ker\, \phi = \big\{(r_1, \ldots, r_n) \in R^n | (r_1, \ldots, r_n)\phi = 0\big\} = \big\{(0, \ldots, 0)\big\}.$$

It follows that ϕ is injective.

Let $m \in M_R$. Since B is a basis of M_R, there exists $r_1, \ldots, r_n \in R$ such that $m = \sum_{i=1}^n m_i r_i = (r_1, \ldots, r_n)\phi$. It follows that ϕ is surjective. Hence ϕ is bijective map. ■

Remark 3.2: Let R be a nearfield.

- By van der Walt's Theorem [8], (R^n, R) is an André near-vector space with the scalar multiplication defined by

$$\alpha(x_1, \ldots, x_n) = \big(\psi_1(\alpha)x_1, \ldots, \psi_n(\alpha)x_n\big),$$

  for all $\alpha \in R$ and $(x_1, \ldots, x_n) \in R^n$ where the ψ_i for $i \in \{1, \ldots, n\}$ are multiplicative automorphisms of $R^* = R \setminus \{0\}$. Note that the action of the scalars on the vectors are on the left, whereas for Beidleman the action of the scalars on the vectors are on the right. Also by van der Walt's construction theorem [8] we can take different nearfields to construct finite dimensional André near-vector spaces, as long as the nearfields are multiplicatively isomorphic. To construct Beidleman near-vector spaces using this Theorem, n copies of the same nearfield can be used in the construction and we can use the nearfield automorphism to define the scalar multiplication. Later in this paper, we will only focus on the case where the scalar multiplication is defined by the identity automorphism.

- If R is a field then (R^n, R) is both an André and Beidleman near-vector space, and both coincide with a vector space. Here we are taking the ψ_i's equal to the identity for all $i \in \{1, \ldots, n\}$.

Remark 3.3: Let I be an index set and M_R a Beidleman near-vector space. Then $M_R = \bigoplus_{i\in I} M_i$, where the $M_i, i \in I$, are irreducible submodules of M_R. By Corollary 2.10 there exists $0 \neq m \in M_i$ such that $M_R = \bigoplus_{i\in I} m_i R$. Hence $\{m_i \mid i \in I\}$ is a basis for M_R. If I is infinite then M_R has infinite dimension and we can use the same procedure in the proof of Theorem 3.1 to show that $M \cong R^{\dim M_R}$.

4. The subspace structure of Beidleman near-vector spaces

In this section we investigate some properties of the subspace structure of Beidleman near-vector spaces. We find that with regard to subspace structure, André near-vector spaces are closest to traditional vector spaces.

As with André near-vector spaces, from Theorem 3.1 and Remark 3.3 we have:

Lemma 4.1: *Let M_R be a Beidleman near-vector space. Then $(M, +)$ is abelian.*

Let M_R be a Beidleman near-vector space. N is a subspace of M_R if $\emptyset \neq N \subset M$ and N is also a Beidleman near-vector space.

From Lemma 4.1, we can give an equivalent definition of a subspace as follows.

Definition 4.2: Let M_R be a Beidleman near-vector space. $\emptyset \neq N \subset M$ is a subspace of M_R if

(1) $(N, +)$ is a subgroup of $(M_R, +)$,
(2) $(m + n)r - mr \in N$ for all $m \in M, n \in N$ and $r \in R$.

Remark 4.3: According to [1], $(N, +)$ should be a normal subgroup of $(M, +)$. But since $(M, +)$ is abelian, we do not need the normality again.

It is not difficult to show that:

Lemma 4.4: *Let M_R be a Beidleman near-vector space. Let M_1 and M_2 be subspaces of M. Then $M_1 \cap M_2$ is also subspace of M.*

By Lemma 4.1, we now deduce the following.

Lemma 4.5: *Let M_R be a Beidleman near-vector space. Let M_1 and M_2 be subspaces of M. Then $M_1 + M_2$ is also a subspace of M where $M_1 + M_2 = \{m_1 + m_2 \mid m_1 \in M_1, m_2 \in M_2\}$.*

For vector spaces and André near-vector spaces, a non-empty subset is a subspace if and only if it is closed under addition and scalar multiplication [4]. For a Beidleman near-vector space we only have

Lemma 4.6: *If $\emptyset \neq N_R$ is a subspace of M_R then N_R is closed under addition and scalar multiplication.*

Note that every subspace is a R-subgroup but being closed under addition and scalar multiplication does not in general give a subspace. We now provide a counter example.

Example 4.7: Let $R = DN(3,2)$ be the finite Dickson nearfield that arises from the Dickson pair $(3,2)$. Then (R^2, R) is a Beidleman near-vector space. Let us consider

$$T = \Big\{(1,x)r \mid r \in R\Big\} = \langle (1,x) \rangle \,.$$

Note that T is a R-subgroup of R^2 but not a subspace of R^2. Indeed by Example 2.6, Let $(1, x+1) \in R^2, (1,x) \in T$ and $x \in R$. We have

$$((1,x+1) + (1,x)) \circ x - (1,x+1) \circ x = (x, (2x+1) \circ x - (x+1) \circ x) = (x,1) \notin T.$$

Hence T is not a subspace of R^2.

5. Classification of the *R*-subgroups of R^n

Let $(R, +, \circ, 0, 1)$ be a Dickson nearfield for the Dickson pair (q,m) with $m > 1$. We know that the distributive elements of R form a subfield of size q. Also, not all elements are distributive, thus there are elements $\lambda \in R$ such that $(\alpha + \beta) \circ \lambda \neq \alpha \circ \lambda + \beta \circ \lambda$ for some $\alpha, \beta \in R$. We will use R_d to denote the set of all distributive elements of R, i.e.

$$R_d = \{z \in R \mid (x+y) \circ z = x \circ z + y \circ z \text{ for all } x, y \in R\},$$

where from now on, we shall simply use concatenation instead of °.

Consider the R-module R^n (for some fixed $n \in \mathbb{N}$) with componentwise addition and scalar multiplication $R^n \times R \to R^n$ given by $(x_1, x_2, \ldots, x_n)r = (x_1 r, x_2 r, \ldots, x_n r)$ for $(x_1, x_2, \ldots, x_n) \in R^n$ and $r \in R$. A R-subgroup S of R^n is a subgroup of $(R^n, +)$ that is closed under scalar multiplication. Let $v_1, v_2, \ldots, v_k \in R^n$ be a finite number of vectors. The smallest subspace of R^n containing $\{v_1, v_2, \ldots, v_k\}$ is called the *span* of $v_1, v_2, \ldots, v_k$ and is denoted by $span(v_1, \ldots, v_k)$.

In analogy to span, we introduce the notion of *gen*.

Definition 5.1: Let $v_1, v_2, \ldots, v_k \in R^n$ (for some $k \in \mathbb{N}$) be a finite number of vectors. We define $gen(v_1, \ldots, v_k)$ to be the smallest R-subgroup of R^n containing $\{v_1, v_2, \ldots, v_k\}$.

Our first aim is to find an explicit description of $gen(v_1, \ldots, v_k)$.

Let $LC_0(v_1, v_2, \dots, v_k) := \{v_1, v_2, \dots, v_k\}$ and for $n \geq 0$, let LC_{n+1} be the set of all linear combinations of elements in $LC_n(v_1, v_2, \dots, v_k)$, i.e.

$$LC_{n+1}(v_1, v_2, \dots, v_k) = \left\{ \sum_{w \in LC_n} w\lambda_w \,\middle|\, \lambda_w \in R \right\}.$$

We will denote $LC_n(v_1, v_2, \dots, v_k)$ by LC_n for short when there is no ambiguity with regard to the initial set of vectors.

Theorem 5.2: *Let $v_1, v_2, \dots, v_k \in R^n$. We have*

$$gen(v_1, \dots, v_k) = \bigcup_{i=0}^{\infty} LC_i.$$

Proof: We need to show that $\bigcup_{i=0}^{\infty} LC_i$ is a R-subgroup of R^n and for any R-subgroup S containing $v_1, v_2, \dots, v_k$, we have $\bigcup_{i=0}^{\infty} LC_i \subseteq S$.

The zero vector 0_{R^n} of $(R^n, +)$ is an element of $\bigcup_{i=0}^{\infty} LC_i$. Let $x, y \in \bigcup_{i=0}^{\infty} LC_i$. We distinguish two cases:

Case 1: $x, y \in LC_n$ for some $n \in \mathbb{N}$. So $x - y \in LC_{n+1}$.

Case 2: $x \in LC_k$ and $y \in LC_n$ for some $k < n$. Since $k < n$, we have $LC_k \subseteq LC_n$ by definition. Hence $x - y \in LC_n$. So $x - y \in \bigcup_{i=0}^{\infty} LC_i$.

Therefore $(\bigcup_{i=0}^{\infty} LC_i, +)$ is a subgroup of $(R^n, +)$.

Let $x \in \bigcup_{i=0}^{\infty} LC_i$ and $r \in R$. Then $x \in LC_n$ for some $n \in \mathbb{N}$. For $n=0$, $x = v_i$ for some $i \in \{1, \dots, k\}$, so $v_i r \in LC_1$. For $n \geq 1$, we have $x = \sum_{w \in LC_{n-1}} w\lambda_w$, and thus $xr = (\sum_{w \in LC_{n-1}} w\lambda_w)r = \sum_{w \in LC_{n-1}} w\lambda_w r \in LC_n$.

It remains to show that for any R-subgroup S containing $v_1, \dots, v_k$ we have $\bigcup_{i=0}^{\infty} LC_i \subseteq S$. It is sufficient to show that for all $n \in \mathbb{N}$, $LC_n \subseteq S$. We use induction on n. For $n=0$ we have $LC_0 \subseteq S$. Assume that $LC_k \subseteq S$ for $k \in \mathbb{N}$. Let $x \in LC_{k+1}$. Then $x = \sum_{w \in LC_k} w\lambda_w$, where $\lambda_w \in R$. But $w \in LC_k \subseteq S$. So $w\lambda_w \in S$ since S is a R-subgroup (closed under addition and scalar multiplication). Therefore $\sum_{w \in LC_n} w\lambda_w \in S$. Hence $x \in S$. ■

In the following propositions we give some basic properties of *gen*.

Proposition 5.3: *Let $k \in \mathbb{N}$ and T be a finite set of vectors in R^n. We have,*

$$LC_n\big(LC_k(T)\big) = LC_{n+k}(T) \quad \textit{for all } n \in \mathbb{N}.$$

The proof is not difficult and uses induction on the positive integer n.

Proposition 5.4: *Let S and T be finite sets of vectors of R^n. The following hold*:

(1) $S \subseteq gen(S) \subseteq span(S)$,
(2) *If* $S \subseteq T$ *then* $gen(S) \subseteq gen(T)$,
(3) $gen(S \cap T) \subseteq gen(S) \cap gen(T)$,
(4) $gen(S) \cup gen(T) \subseteq gen(S \cup T)$,
(5) $gen(gen(T)) = gen(T)$.

Proof: (1)–(4) are straightforward. Indeed $gen(T) \subseteq gen(gen(T))$. Also for all $n \in \mathbb{N}$, we have that $LC_n(gen(T)) \subseteq gen(T)$. Hence $gen(gen(T)) \subseteq gen(T)$. ∎

We want to give a description of $gen(v_1, \ldots, v_n)$ in terms of the basis elements. In the following lemmas, we first derive analogous results of row-reduction in vector spaces. The first lemma follows directly from Theorem 5.2 and we state it without proof.

Lemma 5.5: *For any permutation σ of the indices $1, 2, \ldots, k$, we have*

$$gen(v_1, \ldots, v_k) = gen(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

We can also show that

Lemma 5.6: *If $0 \neq \lambda \in R$, then*

$$gen(v_1, \ldots, v_k) = gen(v_1\lambda, \ldots, v_k).$$

Proof: Let $gen(v_1, \ldots, v_k) = \bigcup_{i=0}^{\infty} LC_i$ and $gen(v_1\lambda, \ldots, v_k) = \bigcup_{i=0}^{\infty} LC_i'$. Let $x \in \bigcup_{i=0}^{\infty} LC_i$. Then $x \in LC_n$ for some $n \in \mathbb{N}$. Clearly $LC_0 \subseteq LC_1'$ and $LC_1 = \{v_1\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k \mid \alpha_1, \ldots, \alpha_k \in R\}$. Say $x \in LC_1$. Since $\lambda \neq 0$ there exists $\lambda' \in R$ such that $\lambda\lambda' = 1$. So $x = v_1\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k = v_1(\lambda\lambda')\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k = (v_1\lambda)\lambda'\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k$. Then $x \in LC_1'$. Thus we see if $x \in LC_n$ for some $n \neq 0$ in the expression of x we have

$$v_1\alpha_1 = v_1 1\alpha_1 = v_1(\lambda\lambda')\alpha_1 = (v_1\lambda)(\lambda'\alpha_1).$$

Thus $x \in LC_n'$. So $x \in \bigcup_{i=0}^{\infty} LC_i'$. Thus $x \in gen(v_1\lambda, \ldots, v_k)$.

Let $x \in \bigcup_{i=0}^{\infty} LC_i'$. Then $x \in LC_n'$ for some $n \in \mathbb{N}$. In fact $LC_0' = \{v_1\lambda, v_2, \ldots, v_k\}$ and we have that $LC_0' \subseteq LC_1$. Also $LC_1' = \{(v_1\lambda)\alpha_1 + v_2\alpha_2 + \cdots + v_k\alpha_k \mid \alpha_1, \ldots, \alpha_k \in R\}$. Let $x \in LC_1'$ then $x = (v_1\lambda)\alpha_1 + v_2\alpha + \cdots + v_k\alpha_k = v_1(\lambda\alpha_1) + v_2\alpha_2 + \cdots + v_k\alpha_k$. It follows that $x \in LC_1$. Thus we see that if $x \in LC_n'$ for some $n \neq 0$ in the expression of x we have,

$$(v_1\lambda)\alpha_1 = v_1(\lambda\alpha_1) \text{ by the associativity of the multiplication.}$$

It follows that $x \in LC_n$. Thus $x \in \bigcup_{i=0}^{\infty} LC_i$. Hence $x \in gen(v_1, \ldots, v_k)$. ∎

Lemma 5.7: *For any scalars $\lambda_2, \lambda_3, \ldots, \lambda_k \in R$, we have*

$$gen(v_1, \ldots, v_k) = gen\left(v_1 + \sum_{i=2}^{k} v_i\lambda_i, v_2, \ldots, v_k\right).$$

Proof: By Theorem 5.2 we can write

$$gen(v_1, \ldots, v_k) = \bigcup_{i=0}^{\infty} LC_i \quad \text{and} \quad gen\left(v_1 + \sum_{i=2}^{k} v_i\lambda_i, v_2, \ldots, v_k\right) = \bigcup_{i=0}^{\infty} LC_i'.$$

We have $LC_0 = \{v_1, \ldots, v_k\}$ and $LC_1 = \{\sum_{i=1}^{k} v_i\alpha_i \mid \alpha_i \in R\}$. Also $LC_0' = \{v_1 + \sum_{i=2}^{k} v_i\lambda_i, v_2, \ldots, v_k\}$ and $LC_1' = \{(v_1 + \sum_{i=2}^{k} v_i\lambda_i)\beta_1 + \sum_{i=2}^{k} v_i\beta_i \mid \alpha_i, \beta_i \in R\}$. We proceed by induction. Clearly $v_2, \dots, v_k \in LC_0' \subseteq LC_1'$. Since

$$v_1 = \left(v_1 + \sum_{i=2}^{k} v_i\lambda_i\right) - \sum_{i=2}^{k} v_i\lambda_i,$$

we have $v_1 \in LC_1'$. So $LC_0 \subseteq LC_1'$. Assume that $LC_m \subseteq LC_{m+1}'$ for some $m \in \mathbb{N}$. We need to show that $LC_{m+1} \subseteq LC_{m+2}'$. Let $x \in LC_{m+1}$, then

$$x = \sum_{w \in LC_m} w\lambda_w = \sum_{w \in LC_{m+1}'} w\lambda_w.$$

It follows that $x \in LC_{m+2}'$. For the other inclusion, we also reason by induction. Clearly $v_2, \dots, v_k \in LC_1$ and $v_1 + \sum_{i=2}^{k} v_i\lambda_i \in LC_1$, so $LC_0' \subseteq LC_1$. Let $x \in LC_1'$, Then $x = (v_1 + \sum_{i=2}^{k} v_i\lambda_i)\beta_1 + \sum_{i=2}^{k} v_i\beta_i$. Then $x \in LC_2$. Assume that $LC_m' \subseteq LC_{m+1}$ for some $m \in \mathbb{N}$. We need to show that $LC_{m+1}' \subseteq LC_{m+2}$. Let $x \in LC_{m+1}'$. So $x = \sum_{w \in LC_m'} w\lambda_w = \sum_{w \in LC_{m+1}} w\lambda_w$. Hence $x \in LC_{m+2}$. ■

We need one more lemma which follows directly from Theorem 5.2 and we state it without proof.

Lemma 5.8: *If $w \in gen(v_1, \dots, v_k)$, then*

$$gen(v_1, \dots, v_k) = gen(w, v_1, v_2, \dots, v_k).$$

Given a set of row vectors $v_1, \dots, v_k$ in R^n, arranged in a matrix U of size $k \times n$, $gen(v_1, \dots, v_k)$ constructed from the matrix U does not change under elementary row operations (swopping rows, scaling rows, adding multiples of a row to another). Recall that two matrices are said to be row equivalent if one can be transformed to the other by a sequence of elementary row operations.

Lemma 5.9: *Suppose that $k \times n$ matrices $V = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ and $W = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}$ are row equivalent (with the rows $v_1, \dots, v_k, w_1, \dots, w_k \in R^n$). Then*

$$gen(v_1, \dots, v_k) = gen(w_1, \dots, w_k)$$

Proof: This proof follows from Lemmas 5.5–5.8. ■

We will use $\{v_1, \dots, \hat{v}_i, \dots, v_k\}$ to denote the fact that the vector v_i has been removed from the set of vectors $\{v_1, \dots, v_k\}$.

Definition 5.10: Let M_R be a Beidleman near-vector space. Let $V = \{v_1, \dots, v_k\}$ be a finite set of vectors of M_R. We say V is R-linearly dependent if there exists $v_i \in V$ such that

$v_i \in gen(v_1, \dots, \hat{v}_i, \dots, v_k)$. We say that V is R-linearly independent if it is not R-linearly dependent.

A matrix M consisting of k rows and n columns will be denoted by $M = (m_i^j)_{\substack{1\le i\le k\\1\le j\le n}}$ where m_i^j is the entry in the ith row and jth column.

Definition 5.11: The R-row space of a matrix is the set of all possible R-linear combinations of its row vectors.

Thus the R-row space of a given matrix M is the same as the *gen* of the rows of M. We now turn to the classification of the smallest R-subgroup containing a given set of vectors in R^n, the main result of this section.

Theorem 5.12: *Let $v_1, \dots, v_k$ be vectors in R^n. Then*

$$gen(v_1, \dots, v_k) = \bigoplus_{i=1}^{k'} u_i R,$$

where the u_i (obtained from v_i by an explicit procedure) for $i \in \{1, \dots, k'\}$ are the rows of the matrix $U = (u_i^j) \in R^{k' \times n}$ whose columns have exactly one non-zero entry.

Proof: Given a particular set of vectors $v_1, \dots, v_k$, arrange them in a matrix V whose ith row is composed of the components of v_i, i.e. $V = (v_i^j)$ where $1 \le j \le n$. Then $gen(v_1, \dots, v_k)$ is the R-row space of V , which is a R-subgroup of R^n. We can then do the usual Gaussian elimination on the rows. According to the previous lemmas, the *gen* spanned by the rows will remain unchanged with each operation (swopping rows, scaling rows, adding multiples of a row to another). When the algorithm terminates, we obtain a matrix $W \in R^{k\times n}$ in reduced row-echelon form (denoted by $RREF(V)$). Let the non-zero rows of W be denoted by $w_1, w_2, \dots, w_t$ where $t \le k$.

Case 1: Suppose that every column has at most one non-zero entry, then

$$gen(v_1, \dots, v_k) = gen(w_1, \dots, w_t) = w_1R + w_2R + \cdots + w_tR$$

where the sum is direct. In this case we are done.

Case 2: Suppose that the jth column is the first column that has two non-zero entries, say $w_r^j \neq 0 \neq w_s^j$ with $r < s$, (we necessarily have $r, s \le j$) where w_r^j is the jth entry of row w_r and w_s^j the jth entry of row w_s. Let $\alpha, \beta, \gamma \in R$ such that $(\alpha + \beta)\lambda \neq \alpha\lambda + \beta\lambda$. We apply what we will call the '*distributivity trick*':

Let $\alpha' = (w_r^j)^{-1}\alpha$ and $\beta' = (w_s^j)^{-1}\beta$. Then consider the new row

$$\theta = (w_r\alpha' + w_s\beta')\lambda - w_r(\alpha'\lambda) - w_s(\beta'\lambda).$$

Since $\theta \in LC_2(w_r, w_s)$ we have $\theta \in gen(w_1, \dots, w_t)$.

For $1 \leq l < j$, either w_r^l or w_s^l is zero because the jth column is the first column that has two non-zero entries, thus $\theta^l = 0$. Note that by the choice of α, β, λ, we have

$$\begin{aligned}\theta^j &= (w_r^j)\alpha' + w_s^j\beta')\lambda - (w_r^j\alpha')\lambda - (w_s^j\beta')\lambda \\ &= \big(w_r^j(w_r^j)^{-1}\alpha + w_s^j(w_s^j)^{-1}\beta\big)\lambda - \big(w_r^j(w_r^j)^{-1}\alpha\big)\lambda - \big(w_s^j(w_s^j)^{-1}\beta\big)\lambda \\ &= (\alpha + \beta)\lambda - \alpha\lambda - \beta\lambda \neq 0.\end{aligned}$$

It follows that $\theta^j \neq 0$. Hence $\theta = (0, \ldots, 0, \theta^j, \theta^{j+1}, \ldots, \theta^n)$. We now multiply the row θ by $(\theta^j)^{-1}$, obtaining the row $\phi = (0, \ldots, 0, 1, \theta^{j+1}(\theta^j)^{-1}, \ldots, \theta^n(\theta^j)^{-1}) \in gen(w_1, \ldots, w_k)$ where $\phi^j = 1$ is the pivot that we have created.

As a next step, we form a new matrix of size $(t + 1) \times n$ by adding ϕ to the rows $w_1, \ldots, w_t$. On this augmented matrix we replace the rows w_r, w_s with $y_r = w_r - (w_r^j)\phi, y_s = w_r - (w_s^j)\phi$ respectively. This yields another new matrix composed of the rows $w_1, \ldots, w_{r-1}, y_r, \ldots, y_s, \phi, w_{s+1}, \ldots, w_t$ which has only one non-zero entry in the jth column. By Lemma 5.8, the *gen* of the rows of the augmented matrix is the *gen* of the rows of W (which in turn is $gen(v_1, \ldots, v_k)$). Hence

$$\begin{aligned}gen(v_1, \ldots, v_k) &= gen(w_1, \ldots, w_r, \ldots, w_s, \ldots, w_t) \\ &= gen(w_1, \ldots, y_r, \ldots, y_s, \phi, \ldots, w_t).\end{aligned}$$

Continuing this process, we can eliminate all columns with more than one non-zero entry. Let the final matrix have rows $u_1, u_2, \ldots, u_{k'}$. Then

$$gen(v_1, \ldots, v_k) = gen(w_1, \ldots, w_t) = gen(u_1, \ldots, u_{k'}) = u_1R + u_2R + \cdots + u_{k'}R,$$

where the sum is direct. ∎

Remark 5.13:

- The procedure described in the proof of Theorem 5.12 will be called expanded Gaussian elimination (eGe algorithm).
- Let $v_1, \ldots, v_k \in R^n$ be vectors arranged in a matrix $V \in R^{k\times n}$. Suppose $RREF(V) = W$ and let $N(w^j)$ be the number of non-zero entries of the jth column of W where $1 \leq j \leq n$. Let t be the number of non-zero rows of W. If $v_1, \ldots, v_k$ are R-linearly independent and $\sum_{j=1}^{n} N(w^j) \geq n + k - t$ then by Theorem 5.12, we shall add $k{-}t{+}1$ new rows to the t non-zero rows. Hence we necessarily have that $k \leq k' \leq n$.

Example 5.14: Let $R = DN(3, 2)$ be the finite Dickson nearfield that arises from the pair $(3, 2)$ and $v_1 = (1, x, 1, 2x + 2, 1), v_2 = (x, 2x, x, 0, x), v_3 = (1, x + 1, 1, 0, 2) \in R^5$. By Theorem 5.12, we have

$$gen(v_1, v_2, v_3) = \bigoplus_{i=1}^{4} u_iR,$$

where $u_1 = (1, 0, 1, 0, 0), u_2 = (0, 1, 0, 0, 0), u_3 = (0, 0, 0, 1, 0)$ and $u_4 = (0, 0, 0, 0, 1)$. Note that $gen(v_1, v_2, v_3) = \{(x, y, x, z, t) \mid x, y, z, t \in R\}$. However $gen(v_1, v_2, v_3)$ is not a subspace of R^5. To see this, let $h \in gen(v_1, v_2, v_3)$, $r \in R$ and $(r_1, r_2, r_3, r_4, r_5) \in R^5$ such that $r_1 \neq r_3$. We have,

$$\big((r_1, r_2, r_3, r_4, r_5) + \big(x, y, x, z, t\big)\big)\, r - (r_1, r_2, r_3, r_4, r_5)\, r = \big((r_1 + x)r - r_1 r, \\ (r_2 + y)r - r_2 r, (r_3 + x)r - r_3 r, (r_4 + z)r - r_4 r, (r_5 + t)r - r_5 r\big) \notin gen(v_1, v_2, v_3).$$

One fundamental property of *gen* is the following.

Theorem 5.15: *Let $n \in \mathbb{N}$ and R be a nearfield. There exists vectors $v_1, \dots, v_{n-1}$ of R^n such that*

$$gen(v_1, \dots, v_{n-1}) = R^n.$$

Proof: Let $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \lambda \in R$ such that $(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})\lambda \neq \alpha_1\lambda + \alpha_2\lambda + \cdots + \alpha_{n-1}\lambda$. We choose $v_1 = (1, 1, 0, 0, \dots, 0)$, $v_2 = (1, 0, 1, 0, \dots, 0), \dots, v_{n-1} = (1, 0, 0, \dots, 0, 1) \in R^n$. Then

$$\begin{aligned} v &= ((1,1,0,0,\dots,0)\alpha_1 + (1,0,1,0,\dots,0)\alpha_2 + \cdots + (1,0,0,\dots,0,1)\alpha_{n-1})\,\lambda \\ &\quad - (1,1,0,0,\dots,0)\alpha_1\lambda - (1,0,1,0,\dots,0)\alpha_2\lambda - \cdots - (1,0,0,\dots,0,1)\alpha_{n-1}\lambda \\ &= ((\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})\lambda - \alpha_1\lambda - \alpha_2\lambda - \cdots - \alpha_{n-1}\lambda, 0, 0, \dots, 0)\,. \end{aligned}$$

So $v = (\gamma, 0, 0, \dots, 0)$ where $\gamma = (\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})\lambda - \alpha_1\lambda - \alpha_2\lambda - \cdots - \alpha_{n-1}\lambda$. It follows that $v\gamma^{-1} = (1, 0, 0, 0, \dots, 0) \in gen(v_1, v_2, \dots, v_{n-1})$.

Let $(x_1, x_2, x_3, \dots, x_n) \in R^n$, then

$$\begin{aligned} (x_1, x_2, x_3, \dots, x_n) &= v_1x_2 + v_2x_3 + \cdots + v_{n-1}x_n - v\gamma^{-1} \\ &\qquad \times (x_n + x_{n-1} + x_{n-2} + \cdots + x_2 - x_1) \\ &= v_1x_2 + v_2x_3 + \cdots v_{n-1}x_n - \big((v_1\alpha_1 + v_2\alpha_2 + \cdots v_{n-1}\alpha_{n-1})\lambda \\ &\quad - v_1\alpha_1\lambda - v_2\alpha_2\lambda - \cdots - v_{n-1}\alpha_{n-1}\lambda\big)\gamma^{-1} \\ &\qquad \times (x_n + x_{n-1} + x_{n-2} + \cdots + x_2 - x_1). \end{aligned}$$

It follows that $(x_1, x_2, x_3, \dots, x_n) \in LC_2(v_1, \dots, v_{n-1})$, so $(x_1, x_2, x_3, \dots, x_n) \in gen(v_1, \dots, v_{n-1})$. Thus $gen(v_1, \dots, v_{n-1}) = R^n$. ■

Remark 5.16: Surprisingly, unlike in traditional vector spaces, by Theorem 5.15 there exists two vectors v_1 and v_2 such that $gen(v_1, v_2) = R^3$. By experimental computations, there exists no two vectors v_1 and v_2 such that $gen(v_1, v_2)$ is the whole space if it has high enough dimension, for instance $R^{n^{n^2}}$.

We end this section with a description of $gen(m)$ for $m \in M_R$.

Lemma 5.17: *Let M_R be a Beidleman near-vector space. Let $m \in M$. Then $gen(m) = mR$.*

Proof: $LC_n(m) = mR$ for all positive integers n. Using Theorem 5.12 we have that $gen(m) = mR$. ■

6. Classification of the subspaces of R^n

The extended Gaussian elimination algorithm can be used to determine the span of a given set of vectors, which is defined as the smallest submodule containing all the given vectors. From Example 4.7, we know that some R-subgroups are not necessarily subspaces of Beidleman near-vector spaces R^n, where R is a nearfield. In Theorem 6.1 we describe the smallest subspace of R^n containing a given set of vectors. This allows us to classify all the subspaces of R^n in Corollary 6.3.

Theorem 6.1: *Let $v_1, \dots, v_k$ be vectors of R^n. Then*

$$span(v_1, \dots, v_k) = \bigoplus_{i=1}^{k''} e_i R,$$

where e_i is a row vector with only one non-zero entry '1' in its ith position obtained from v_i by an explicit procedure.

Proof: Let $v_1, \dots, v_k \in R^n$. Since a submodule is an R-subgroup, we have

$$gen(v_1, \dots, v_k) \subseteq span(v_1, \dots, v_k).$$

Note that $gen(v_1, \dots, v_k)$ will be a subspace if $gen(v_1, \dots, v_k) = span(v_1, \dots, v_k)$.

Let $u_1, \dots, u_{k'}$ be determined as before in Theorem 5.12.

Case 1: Suppose that u_i for all $i \in \{1, \dots, k'\}$ has only one non-zero component. Then $u_1R + u_2R + \cdots + u_{k'}R$ is a submodule of R^n, hence it is the desired span. So

$$\begin{aligned} span(v_1, \dots, v_k) &= gen(v_1, \dots, v_k) \\ &= \bigoplus_{i=1}^{k'} u_i R. \end{aligned}$$

Case 2: Suppose that no row has more than two non-zero entries and u_i is the first ith row that has entries $u_i^{j_1} \neq 0 \neq u_i^{j_2}$ i.e. $u_i = (0, \dots, 0, u_i^{j_1}, 0, \dots, 0, u_i^{j_2}, 0, \dots, 0)$. We apply what we will call the '*adjustment trick*'. Let $\alpha, \beta, \lambda \in R$ such that $(\alpha + \beta)\lambda \neq \alpha\lambda + \beta\lambda$. Define $m \in R^n$ by $m^j = \alpha\delta_{jj_2}$ for $1 \leq j \leq n$ where δ_{ij} is the Kronecker function. So

$$m^j = \begin{cases} 0 & \text{if } j \neq j_2 \\ \alpha & \text{if } j = j_2 \end{cases}.$$

It follows that $m = (0, \dots, 0, \alpha, 0, \dots, 0)$.

Define $a = u_i((u_i^{j_2})^{-1}\beta) = (0, \dots, 0, u_i^{j_1}(u_i^{j_2})^{-1}\beta, 0, \dots, 0, \beta, 0, \dots, 0)$, so that $a^{j_2} = \beta$.

Then let $v \in R^n$ be defined as

$$\begin{aligned} v &= (m + a)\lambda - m\lambda \\ &= \big(0, \dots, 0, u_i^{j_1}(u_i^{j_2})^{-1}\beta\lambda, 0, \dots, 0, (\alpha + \beta)\lambda - \alpha\lambda, 0, \dots, 0\big). \end{aligned}$$

By the additional condition on submodules, we must have $v \in span(u_1, \dots, u_{k'})$. Hence we may add v without changing the span (it strictly increases the gen though). Note that by construction, $v \neq a\lambda$. In fact, the only non-zero entry of $v - a\lambda$ is the j_2 component. Hence we may reduce the j_2 entry of u_i to zero. Since $a \in span(u_1, \dots, u_{k'})$,

$$\begin{aligned} v - a\lambda &= \big(0, \dots, 0, (\alpha + \beta)\lambda - \alpha\lambda - \beta\lambda, 0, \dots, 0\big) \\ &= \big(0, 0, \dots, 0, \gamma, 0, \dots, 0\big) \in span(u_1, \dots, u_{k'}) \\ \text{where} \quad \gamma &= (\alpha + \beta)\lambda - \alpha\lambda - \beta\lambda \text{ is in the } j_2\text{th position of } v - a\lambda. \end{aligned}$$

So $(v - a\lambda)\gamma^{-1} = (0, \dots, 0, 1, 0 \dots, 0)$ is denoted by e_{j_2} (the row that only has a non-zero entry '1' in its j_2th column) and $e_{j_2} \in span(u_1, \dots, u_{k'})$. Also $(u_i(u_i^{j_2})^{-1} - e_{j_2})(u_i^{j_1}(u_i^{j_2})^{-1})^{-1} = (0, \dots, 0, u_i^{j_1}(u_i^{j_2})^{-1}, 0, \dots, 0)(u_i^{j_1}(u_i^{j_2})^{-1})^{-1}$ is denoted as e_{j_1} (the row that only has a non-zero entry '1' in its j_1th column) and $e_{j_1} \in span(u_1, \dots, u_{k'})$. We now add the new rows e_{j_1}, e_{j_2} to the rows $u_1, \dots, u_i, \dots, u_{k'}$ and remove the row u_i (the span is unchanged). We have that

$$\begin{aligned} span(v_1, \dots, v_k) &= span(u_1, \dots, u_i, \dots, u_{k'}) \\ &= span(u_1, \dots, e_{j_1}, e_{j_2}, u_i, \dots, u_{k'}) \\ &= span(u_1, \dots, e_{j_1}, e_{j_2}, \dots, u_{k'}). \end{aligned}$$

Continuing the implementations of the '*adjustment trick*' on the other rows u_t (which has also two non-entries) where $t > i$, we may eliminate occurrences of multiple non-zero entries in the u_t while appending new vectors with only one non-zero entry to make up for them. Thus

$$\begin{aligned} span(v_1, \dots, v_k) &= span(u_1, \dots, u_i, \dots, u_{k'}) \\ &= span(e_1, e_1, e_2, \dots, e_{k''}) \\ &= \bigoplus_{i=1}^{k''} e_i R, \end{aligned}$$

where e_i is the ith row that has a non-zero entry '1' in its ith position only.

Case 3: Suppose that there exists at least one row which has more than two non-zero entries and u_i is the first ith row with l non-zero entries where $l \geq 2$. Then continuing with the procedure in Case 2, we must apply the '*adjustment trick*' on the first two non-zero entries of u_i and repeating the procedure on the other non-zero entries. Thus we may eliminate occurrences of multiple non-zero entries in the u_i while appending new vectors with only one non-zero entry to make up for them. Hence

$$\begin{aligned} span(v_1, \dots, v_k) &= span(u_1, \dots, u_i, \dots, u_{k'}) \\ &= span(u_1, \dots, u_i, e_{j_1}, e_{j_2}, \dots, e_{j_l}, \dots, u_{k'}) \\ &= span(u_1, \dots, e_{j_1}, e_{j_2}, \dots, e_{j_l}, \dots, u_{k'}) \\ &= span(e_1, e_1, e_2, \dots, e_{k''}) \\ &= \bigoplus_{i=1}^{k''} e_i R, \end{aligned}$$

where e_i is the ith row that has a non-zero entry '1' in column i only. Thus we have proved that the span can always be written as a direct sum of vectors with only one non-zero entry. ■

Remark 6.2: Let $v_1, \dots, v_k$ be finite number of vectors in R^n arranged in a matrix V. Since $span(v_1, \dots, v_k)$ is the row space of the matrix V, by Theorem 6.1 the dimension of $span(v_1, \dots, v_k)$ is k''.

From the description of subspaces of R^n in Theorem 6.1, we deduce the following.

Corollary 6.3: *The subspaces of R^n are all of the form $S_1 \times S_2 \times \cdots \times S_n$ where $S_i = \{0\}$ or $S_i = R$ for $i = 1, \dots, n$.*

Proof: Let S be a subspace of R^n. If $S = \{0\}$ or $S = R^n$ then S is the trivial subspace of R^n. Let $v_1 \in S \setminus \{0\}$, then $span(v_1) \subseteq S$. Let $v_2 \in S \setminus span(v_1)$, then $span(v_1, v_2) \subseteq S$. We continue this process until for $v_k \in S \setminus span(v_1, \dots, v_{k-1})$ we have $span(v_1, \dots, v_k) = S$. Thus by Theorem 6.1, S is isomorphic (up to the reordering of coordinates) to $R^{k''}$ for $k'' \leq n$. It is follows that S is of the form $S_1 \times S_2 \times \cdots \times S_n$ where $S_i = R$ for $i = 1, \dots, k''$ and $S_i = \{0\}$ for $i = k'' + 1, \dots, n$. Hence each S_i for $i = 1, \dots, n$ is subspace of R. But by Theorem 2.14 the only subspaces of R is $\{0\}$ and R. Hence $S_i = \{0\}$ or $S_i = R$, for $i = 1, \dots, n$. ■

Remark 6.4: Let us consider $T = \{(r, r, \dots, r) \in R^n \mid r \in R\}$. T is not of the form prescribed in Corollary 6.3. To see that T is not a subspace, let $(r_1, 0, \dots, 0), (r, r, \dots, r) \in R^n$ and $\alpha \notin R_d$. Then

$$\begin{aligned}
&((r_1, 0, \dots, 0) + (r, r, \dots, r))\,\alpha - ((r_1, 0, \dots, 0))\,\alpha \\
&\quad = \big((r_1 + r)\alpha + r_1\alpha, r\alpha, \dots, r\alpha\big).
\end{aligned}$$

Since $\alpha \notin R_d$, there exists $x, y \in R$ such that $(x + y)\alpha \neq x\alpha + y\alpha$. We choose $r_1 = x$ and $r = y$. So $(r_1 + r)\alpha \neq r_1\alpha + r\alpha$, from which it follows that $(r_1 + r)\alpha - r_1\alpha \neq r\alpha$. Hence T is not subspace of R^n.

Let us illustrate Theorem 6.1 with the following two examples.

Example 6.5: Let $R = DN(3, 2)$ and $v_1, v_2, v_3 \in R^5$ such that $v_1 = (0, 1, 1, 0, 0), v_2 = (0, x + 1, 2, 0, x + 1)$ and $v_3 = (1, x + 1, 1, 0, x)$. By Theorem 5.12,

$$gen(v_1, v_2, v_3) = u_1R \oplus u_2R \oplus u_3R \oplus u_4R,$$

where $u_1 = (1, 0, 0, 0, 0), u_2 = (0, 1, 0, 0, 0), u_3 = (0, 0, 1, 0, 0)$ and $u_4 = (0, 0, 0, 0, 1)$. By Theorem 6.1, case 1, we have

$$\begin{aligned}
gen(v_1, v_2, v_3) &= span(v_1, v_2, v_3) \\
&= \bigoplus_{i=1}^{4} e_iR \cong R^4 \quad \text{where} \quad e_i = u_i, \quad \text{for } i \in \{1, \dots, 4\},
\end{aligned}$$

is a subspace of R^5.

Example 6.6: Let $R = DN(3,2)$ and $v_1, v_2, v_3 \in R^5$ such that $v_1 = (1,1,2,x+1,1)$, $v_2 = (0,0,0,2x+2,1)$ and $v_3 = (1,1,1,x+2,1)$. By Theorem 5.12

$$gen(v_1, v_2, v_3) = u_1R \oplus u_2R \oplus u_3R \oplus u_4R,$$

where $u_1 = (1,1,0,0,0)$, $u_2 = (0,1,0,0,0)$, $u_3 = (0,0,1,0,0)$ and $u_4 = (0,0,0,0,1)$. We wish to determine $span(v_1, v_2, v_3)$. By Theorem 6.1, case 2, we may apply the '*adjustment trick*' to reduce u_1. Define $m \in R^5$ such that

$$m^j = \begin{cases} 0 & \text{if } j \neq 1 \\ \alpha & \text{if } j = 1. \end{cases}.$$

Let $\alpha, \beta, \lambda \in R$ such that $(\alpha+\beta)\lambda \neq \alpha\lambda + \beta\lambda$. Let $a = u_1((u_1^1)^{-1})\beta = u_1\beta$ and $S = span(u_1, u_2, u_3, u_4)$. We have

$$\begin{aligned}(m+a)\lambda - m\lambda &= ((1,0,0,0,0)\alpha + (1,1,0,0,0)\beta)\,\lambda - (1,0,0,0,0)\alpha\lambda \\ &= ((\alpha+\beta)\lambda - \alpha\lambda, \beta\lambda, 0,0,0) \in S\end{aligned}$$

and

$$\begin{aligned}((\alpha+\beta)\lambda - \alpha\lambda, \beta\lambda, 0,0,0) - (\beta\lambda, \beta\lambda, 0,0,0) &= ((\alpha+\beta)\lambda - \alpha\lambda - \beta\lambda, 0,0,0,0) \\ &= (\gamma, 0,0,0,0) \in S,\end{aligned}$$

where $\gamma = (\alpha+\beta)\lambda - \alpha\lambda - \beta\lambda$. It follows that

$$(\gamma, 0,0,0,0)\gamma^{-1} = (1,0,0,0,0) = e_1 \in S.$$

Also

$$u_1 - e_1 = e_2 \in S.$$

Therefore

$$\begin{aligned}span(v_1, v_2, v_3) &= span(e_1, e_2, u_1, u_2, u_3, u_4) \\ &= span(e_1, e_2, u_2, u_3, u_4) \\ &= span(e_1, e_2, e_3, e_4, e_5) \\ &= \bigoplus_{i=1}^{5} e_iR \cong R^5,\end{aligned}$$

where $u_2 = e_3$, $u_3 = e_4$ and $u_4 = e_5$.

Surprisingly, unlike traditional vector spaces, $span(v)$ for $v \in R^n$ can be the whole space.

Proposition 6.7: *Let $v \in R^n$ and $k \leqslant n$. Then $span(v)$ is k-dimensional if and only if v contains k non-zero entries.*

Proof: Let $V \in R^{1\times n}$ be a matrix of size $1 \times n$ which contains the entries of v. Suppose $span(v)$ is k-dimensional. Then the row space of the matrix V has standard basis

$\{e_1, \ldots, e_k\}$. So $span(v) = \bigoplus_{i=1}^{k} e_i R$. Then $v = \sum_{i=1}^{k} e_i r_i$ where $r_i \in R$. Thus v contains k non-zero entries. Conversely suppose v contains k non-zero entries. We now implement the '*adjustment trick*' (see case 3, proof of Theorem 6.1). Hence we eliminate occurrences of multiples of non-zero entries in v while appending k new vectors with each containing only one non-zero entry. Thus $span(v) = \bigoplus_{i=1}^{k} e_i R$. Therefore $span(v)$ is k-dimensional. ∎

Corollary 6.8: *Let $v \in R^n$. Then $span(v) = vR$ if and only if v has at most one non-zero entry.*

Proof: If $span(v) = vR$ then $span(v)$ is one dimensional and by Proposition 6.7 v must contain only one non-zero entry. Suppose v has at most one non-zero entry. By Lemma 5.17 $gen(v) = vR$. Thus $span(v) = gen(v) = vR$. ∎

In [9,10] the authors derived an expression that evaluates the number of k-dimensional subspaces of the finite dimensional vector space (F^n, F) where F is a finite field. In the case of the finite dimensional Beidleman near-vector space (R^n, R), where R is nearfield we have that

Proposition 6.9: *The number of subspaces of dimension k of R^n is $\binom{n}{k}$.*

Proof: By Corollary 6.3 a k-dimensional subspace is isomorphic to R^k and $n-k$ is the number of zeros appearing in the n coordinates. Now the number of subspaces of dimension k corresponds to choosing k out of the n coordinates, hence it is $\binom{n}{k}$. ∎

Corollary 6.10: *There are 2^n subspaces of R^n.*

7. Conclusion and open problem

In future work, we suggest investigating the following question. Given a matrix M in expanded reduced row echelon form with non-zero rows, can we determine the minimal sets of non-zero row vectors that form a matrix N such that $gen($ rows of M$) = gen($ rows of $N)$? It might be also fruitful to investigate on subspaces of R^n from a geometrical perspective.

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Appendices

Appendix 1. The expanded Gaussian elimination

Algorithm A.1: The expanded Gaussian elimination (eGe) computes the smallest R-subgroup containing a given set of vectors in R^n. The algorithm implements the normal Gaussian elimination plus the distributivity trick.

Input: $v_1, v_2, \dots, v_k \in R^n$ for $R = DN(q, m)$ where $m > 1$ arranged in a matrix $V = (v_i^j)_{1\le i\le k, 1\le j\le n} \in R^{k\times n}$.

Step 1 $W = RREF(V)$ (reduced row echelon form of V by Gaussian elimination operations).

Step 2

Case 1 Suppose that every column of W has at most one non-zero entry. Then $gen(v_1, \dots, v_k) = \bigoplus_{i=1}^k w_i R$.

Case 2 Suppose that jth column is the first and the only column of W that has at least two non-zero entries denoted by $w_r^j, w_s^j, w_t^j, \dots$ where $r < s < t < \cdots$

Subcase 1 Consider the first two non-zero entries say $w_r^j \neq 0 \neq w_s^j$ with $r < s$. Apply the first ‘*distributivity trick*’ by creating the new row ϕ.

Subcase 2 Consider the second two non-zero entries say $\phi^j \neq 0 \neq w_t^j$.Apply the second ‘*distributivity trick*’.

$\vdots$

Continuing this manner until in the jth column we have only one non-zero entry.

Case 3 Suppose that there are more than one non-zero entries at each jth, $(j+1)$th, $(j+2)$th, $\dots$, nth columns of W. Then apply the ‘*distributivity trick*’ on every of the jth, $(j+1)$th, $(j+2)$th, $\dots$, nth column until we have only one non-zero entry in every column.

Output: The final matrix $U = (u_i^j)_{1\le i\le k', 1\le j\le n} \in R^{k'\times n}$ which has at most one non-zero entry in every column. Let $u_1, \dots, u_{k'}$ be the rows of U. We have, $gen(v_1, \dots, v_k) = \bigoplus_{i=1}^{k'} u_i R$.

Appendix 2. The adjustment expanded Gaussian elimination (aeGe)

Algorithm A.2: The adjustment expanded Gaussian elimination (aeGe) computes the smallest subspace containing a given set of vectors in R^n. The algorithm implement the expanded Gaussian elimination plus the adjustment trick.

Input: $v_1, v_2, \ldots, v_k \in R^n$ for $R = DN(q, m)$ where $m > 1$ arranged in a matrix $V = (v_i^j)_{1\leq i\leq k, 1\leq j\leq n} \in R^{k\times n}$.

Step 1 Apply eGe and we get $gen(v_1, \ldots, v_k) = \bigoplus_{i=1}^{k'} u_i R$.

Step 2

Case 1 Suppose that u_i for $i = 1, \ldots, k'$ have exactly one non-zero entry. Then $span(v_1, \ldots, v_k) = \bigoplus_{i=1}^{k'} u_i R$.

Case 2 Suppose that u_i is the only ith row that has more than one non-zero entries denoted $u_i^{j_1}, u_i^{j_2}, u_i^{j_3}, \ldots$. Apply the '*adjustment trick*' on the row u_i until we eliminate occurrences of multiple non-zero entries while appending new vectors with only one non-zero entry.

Case 3 Suppose that the rows $u_i, u_{i+1}, \ldots, u_{k'}$ have each more than one non-zero entry. Apply on each row the '*adjustment trick*'.

Output: The final matrix $E = (e_i^j)_{1\leq i\leq k'', 1\leq j\leq n} \in R^{k''\times n}$ which has at most one non-zero entry in every column. Let $e_1, \ldots, e_{k'}$ be the rows of E. We have, $gen(v_1, \ldots, v_k) = \bigoplus_{i=1}^{k''} e_i R$.