On the Galois Groups of Some Recursive Polynomials: Magma codes

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Abstract

In this study, we investigate the Galois groups of Fibonacci and Lucas polynomials over both the rational field and the rational function field. Using computational experiments performed with MAGMA, we examine the structure of the Galois groups of generalized Fibonacci-like polynomials. Furthermore, we demonstrate that the roots of these generalized Fibonacci-like polynomials are not constructible using a ruler and compass.

In this note we provide the coding in MAGMA that is used to generate the experimental results for the Galois group of the polynomials. The information generated include the order, the numbers of the roots, the transitivity, the solvability, and the structure of the group for some values of n where $1 \leq n \leq 20$. Here n is the degree of the polynomial. These outputs are listed in the Table **??** and **??**.

Include your Magma code here

```
function FibonacciPoly(k, n, x)
    if n eq 0 then
        return PolynomialRing(Rationals())!1;
    elif n eq 1 then
        return x + 1;
    else
        P1 := FibonacciPoly(k, n-1, x);
        P2 := FibonacciPoly(k, n-2, x);
        return (x^Numerator(k)/Denominator(k)) * P1 + P2;
    end if;
end function;
k := 1,2,3;
for n in [1..20] do
    R<x> := PolynomialRing(Rationals());
    F_k_n := FibonacciPoly(k, n, x);
    roots := Roots(F_k_n, ComplexField(100));
    precision := 4;
    roots_numeric := [RealField(precision)!Re(r[1]) : r in roots];
    G := GaloisGroup(F_k_n);
    order := #G;
    transitive := IsTransitive(G);
    solvable := IsSolvable(G);
    print "Polynomials F_", k, "_", n, "(x):", F_k_n;
    print "Roots:", roots_numeric;
    print "Gallois group'structure:", G;
    print "Order:", order;
    print "Is G transitive?", transitive;
    print "Is G solvable?", solvable;
    print "\n";
end for;
```

Listing 1: Magma code of findings Galois groups for $\mathcal{F}_{k,n}(x)$ with k positive integer

SageMath code

Include your Magma code here

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```python
from sage.all import *
def fibonacci_poly(k, n, x):
    if n == 0:
        return 1
    elif n == 1:
        return x+1
    else:
        return x**k * fibonacci_poly(k, n-1, x) + %fibonacci_poly(k, n-2, x)

def get_non_negative_integer(prompt):
    while True:
        try:
            value = int(input(prompt))
            if value >= 0:
                return value
            else:
                print("Please enter a non-negative integer.")
        except ValueError:
            print("Invalid input. Please enter an integer.")
k = get_non_negative_integer("Enter the value of k (a non-negative integer): ")
n = get_non_negative_integer("Enter the value of n (a non-negative integer): ")
x = QQ['x'].gen()
F_k_n = fibonacci_poly(k, n, x)
G = F_k_n.galois_group()
char_table = G.character_table()
print(f"Character table of the Galois group of the polynomial F_{k},{n}(x):")
print(char_table)
```

Listing 2: SageMath code for the computation of character table for $\mathcal{F}_{k,n}(x)$

Character table of G_{10}

Example 1. *In this example, we would like to make use of the character table for the case where $k = 1, n = 10$ to show that the Galois group of the corresponding polynomials G_{10} is not solvable.*

```
    Character table of the Galois group of the polynomial F_1,10(x):
[  1  -1   1  -1   1  -1   1  -1   1  -1   1  -1   1   1  -1   1  -1   1  -1   1  -1   1  -1   1  -1
[  9  -7   5  -3   1   1   6  -4   2   0   3  -1  -1   0  -5   3  -1  -1  -2   0   1   1   1   4  -2
[ 35 -21  11  -5   3  -5  14  -6   2  -2   2   0   2  -1  -9   3  -1   3   0   0   0  -1  -1   5  -1
[ 75 -35  15  -7   3   5  15  -5   3  -1   0  -2   0   3  -5   1  -1  -3   1   1  -2  -1   1   0   0
[ 90 -34  14  -6   2 -10   6  -4   2   0   3  -1  -1   0   4   0   0   4  -2   0   1   2  -2  -5   1  -
[ 42 -14   6  -2   2  10   0  -2   0  -2   3   1   3  -3   4   0   0  -4  -2   0   1   2   2  -3   1
[ 36 -20   8   0  -4   4  15  -5  -1   3   3   1  -1   0 -10   2   2  -2  -1  -1  -1   0   0   6   0  -
[160 -64  16   0   0   0  34  -4  -2   0  -2   2  -2  -2 -16   0   0   0   2   0   2   0   0   5   1
[315 -91  19  -3  -5   5  21  -1   1   3  -3  -1   1   0   1  -1   1  -1   1  -1   1  -1   1  -5  -1  -
[288 -64  16   0   0   0  -6  -4  -2   0   6   2  -2   0  16   0   0   0  -2   0  -2   0   0  -7   1
[225 -55   5  -3   9 -15  15   5  -1  -3  -6   2   2   0  -5  -1  -1   3   1  -1  -2   1   1   0   0
[450 -70  10  -6   2  10 -15   5   1  -3  -3  -1   1   0  10  -2   2  -2   1   1   1  -2  -2   0   0
[252 -28   8   0   4 -20 -21  -1  -1   3   3  -1  -1   0  10  -2  -2   2   1   1   1   0   0   2   2  -
[210 -14   6  -2  -6  10 -21   1   3   1   0  -2   0   3   4  -4   0   0   1  -1  -2   2   2   5   1
[ 84 -28   0   8  -4  -4  21  -1  -3  -1   3  -1   3   3 -10  -2   2   2  -1   1  -1   0   0   4   2
[350 -70 -10  10  -2  10  35   5  -1   1  -1  -1  -1  -1 -10  -2  -2  -2  -1   1  -1   2  -2   0   0
[567 -63  -9   9  -9 -15   0   0   0   0   0   0   0   0   9   3   1   3   0   0   0  -1   1  -3  -3
[300 -20   0   8   4  20 -15  -5  -3  -1   3   1   3   3  10   2  -2  -2   1  -1   1   0   0   0   0
[525 -35 -15  -7   5   5   0  10   0   2  -3   1  -3   3  -5   3  -1  -1  -2   0   1   1   1   0   0
[768   0   0   0   0   0 -48   0   0   0   0   0   0  -6   0   0   0   0   0   0   0   0   0   8   0
[210  14   6   2  -6 -10 -21  -1   3  -1   0   2   0   3  -4  -4   0   0  -1  -1   2   2  -2   5  -1
[300  20   0  -8   4 -20 -15   5  -3   1   3  -1   3   3 -10   2   2  -2  -1  -1  -1   0   0   0   0
[252  28   8   0   4  20 -21   1  -1  -3   3   1  -1   0 -10  -2   2   2  -1   1  -1   0   0   2  -2  -
[126 -14 -14   6   6  -6  21   1   1  -3   6  -2  -2   0  -4  -4   0   0  -1  -1   2  -2   2   1   1
[448   0 -32   0   0   0  28   0   4   0   4   0   4  -2   0   0   0   0   0   0   0   0   0  -2   0  -
[525  35 -15   7   5  -5   0 -10   0  -2  -3  -1  -3   3   5   3   1  -1   2   0  -1   1  -1   0   0
[567  63  -9  -9  -9  15   0   0   0   0   0   0   0   0  -9   3  -1   3   0   0   0  -1  -1  -3   3
[450  70  10   6   2 -10 -15  -5   1   3  -3   1   1   0 -10  -2  -2  -2  -1   1  -1  -2   2   0   0
[288  64  16   0   0   0  -6   4  -2   0   6  -2  -2   0 -16   0   0   0   2   0   2   0   0  -7  -1
[ 42  14   6   2   2 -10   0   2   0   2   3  -1   3  -3  -4   0   0  -4   2   0  -1   2  -2  -3  -1
[126  14 -14  -6   6   6  21  -1   1   3   6   2  -2   0   4  -4   0   0   1  -1  -2  -2  -2   1  -1
```

```
[350  70 -10 -10  -2 -10  35  -5  -1  -1  -1   1  -1  -1  10  -2   2  -2   1   1   1   2   2   0   0
[225  55   5   3   9  15  15  -5  -1   3  -6  -2   2   0   5  -1   1   3  -1  -1   2   1  -1   0   0
[315  91  19   3  -5  -5  21   1   1  -3  -3   1   1   0  -1  -1  -1  -1  -1  -1  -1  -1  -1  -5   1  -
[ 90  34  14   6   2  10   6   4   2   0   3   1  -1   0  -4   0   0   4   2   0  -1   2   2  -5  -1  -
[ 84  28   0  -8  -4   4  21   1  -3   1   3   1   3   3  10  -2  -2   2   1   1   1   0   0   4  -2
[160  64  16   0   0   0  34   4  -2   0  -2  -2  -2  -2  16   0   0   0  -2   0  -2   0   0   5  -1
[ 75  35  15   7   3  -5  15   5   3   1   0   2   0   3   5   1   1  -3  -1   1   2  -1  -1   0   0
[ 36  20   8   0  -4  -4  15   5  -1  -3   3  -1  -1   0  10   2  -2  -2   1  -1   1   0   0   6   0  -
[ 35  21  11   5   3   5  14   6   2   2   2   0   2  -1   9   3   1   3   0   0   0  -1   1   5   1
[  9   7   5   3   1  -1   6   4   2   0   3   1  -1   0   5   3   1  -1   2   0  -1   1  -1   4   2

[  1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
```

Now the proof using the character table that G_{10} is not solvable goes as follow:For a solvable group, the degrees of the irreducible characters divide the order of the group, and the group cannot have an irreducible character of a degree higher than 1 which is not a power of a prime number. In other words, if any character degree is not a divisor of the group order, or if the character degrees suggest the existence of a simple non-abelian quotient, the group is not solvable.

Example 2. *In this example, we would like to make use of the character table for the case where $k = 1, n = 7$ to show that the Galois group of the corresponding polynomials $G_7 = \mathrm{Gal}(F_{1,7}(x)/\mathbb{Q})$ is not solvable.*

```
Character table of the Galois group of the polynomial F_1,7(x):
[ 1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1 -1  1]
[ 6 -4  2  0  3 -1 -1  0 -2  0  1  1  1  0 -1]
[14 -6  2 -2  2  0  2 -1  0  0  0 -1 -1  1  0]
[14 -4  2  0 -1 -1 -1  2  2  0 -1 -1  1  0  0]
[15 -5 -1  3  3  1 -1  0 -1 -1 -1  0  0  0  1]
[35 -5 -1 -1 -1  1 -1 -1  1  1  1  0  0 -1  0]
[21 -1  1  3 -3 -1  1  0  1 -1  1  1 -1  0  0]
[21  1  1 -3 -3  1  1  0 -1 -1 -1  1  1  0  0]
[20  0 -4  0  2  0  2  2  0  0  0  0  0  0 -1]
[35  5 -1  1 -1 -1 -1 -1 -1  1 -1  0  0  1  0]
[14  4  2  0 -1  1 -1  2 -2  0  1 -1 -1  0  0]
[15  5 -1 -3  3 -1 -1  0  1 -1  1  0  0  0  1]
[14  6  2  2  2  0  2 -1  0  0  0 -1  1 -1  0]
[ 6  4  2  0  3  1 -1  0  2  0 -1  1 -1  0 -1]
[ 1  1  1  1  1  1  1  1  1  1  1  1  1  1  1]
```

It is well known that for a solvable group, the degrees of the irreducible characters divide the order of the group, and the group cannot have an irreducible character of a degree higher than 1 which is not a power of a prime number. In other words, if any character degree is not a divisor of the group order, or if the character degrees suggest the existence of a simple non-abelian quotient are, the group is not solvable. In this case the degrees of characters are: $1, 6, 14, 15, 20, 21, 35$. We also notice that:

1. *The degrees of $6, 14, 15, 20, 21, 35$ are not powers of primes.*
2. *High Degrees: The presence of such high degrees indicates non-trivial structure, suggesting the group is not simple.*
3. *Non-divisors of Group Order: the degrees such as $20, 21, 35$ do not align with the typical solvable group properties.*

These degrees suggest the existence of non-abelian simple groups as quotients, which precludes the group from having a series of abelian factor groups as required for solvability. Therefore, Galois group G_n for $n = 7$ is non-solvable.

For the case where $k = 1, n = 4$ it is clear that the Galois group of the corresponding polynomials $\mathrm{Gal}(F_{1,4}(x)/\mathbb{Q})$ is solvable as illustrated below:

```
Character table of the Galois group of the polynomial F_1,4(x):
[ 1 -1  1  1 -1]
[ 3 -1 -1  0  1]
[ 2  0  2 -1  0]
[ 3  1 -1  0 -1]
[ 1  1  1  1  1]
```