Introduction

Euklidian Norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$

$\|x\|_2^2 = x^T \cdot x$

Weighting Eukl. Norm: $\|x\|_Q^2 = x^T Q \cdot x$

Frobenius Norm: $\|x\|_F^2 = \mathrm{trace}(AA^T) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} A_{ij}$

Jacobian: $\nabla f(x) = \frac{\partial f}{\partial x}(x)$ in $\mathbb{R}^{n \times m}$ Hessian: $\nabla^2 f(x)$

$(AB)^\top = B^\top A^\top \quad (AB)^{-1} = B^{-1} A^{-1} \quad (A^\top)^{-1} = (A^{-1})^\top$

Linear and non-linear models:

- linear if parameters linear i.e. $(\theta_1 x^2 + \theta_2 x + \theta_3)$
- nonliniar if i.e $(\sin(\theta_1)x + \theta_2)$ or derivatives in other orders than 1

Table of Derivatives:

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$g(x) \cdot h(x)$	$g'(x) \cdot h(x) + g(x) \cdot h'(x)$
$g(h(x))$	$g'(h(x)) \cdot h'(x)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = \sec^2(x)$
e^{kx}	$\frac{1}{k} e^{kx}$
$\ln(x)$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \cdot \ln a}$
Ax	A
$x^\top A$	$A^\top$
$x^\top Bx$	$x^\top (B^\top + B)$
$x^\top Ax$	$x^\top (A + A^\top)$
$x^\top A^\top Ax$	$2x^\top A^\top A$

Random Variables and Probability

Dependent Probability: $P(A \vee B) = P(A) + P(B)$

Independent Prob.: $P(A, B) = P(A \wedge B) = P(A) \cdot P(B)$

Conditional Prob.: $P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$ (Bayes' theorem)

$P(X \in [a,b]) = \int_a^b p_X(x) \mathrm{d}x \qquad p(x|y) = \frac{p(x,y)}{p(y)}$

Mean/Expectation value: $\mathbb{E}\{\mu_X\} := \mu_X = \int_{-\infty}^{\infty} x \cdot p_X(x) dx$

$\mathbb{E}\{a + bX\} := a + b\mathbb{E}\{X\}$

$\mathbb{E}\{XY\} = \mathbb{E}\{X\} \cdot \mathbb{E}\{X\} \Leftrightarrow X, Y$ independent

Variance: $\sigma_X^2 := \mathbb{E}\{(X - \mu_X)^2\} = \mathbb{E}\{X^2\} - \mu_X^2$

Standard deviation: $\sigma_X = \sqrt{\sigma_X^2}$

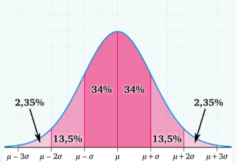
Distributions

Uniform distribution: $P_y(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{else} \end{cases}$, $\mathrm{var}(x) = \frac{(b-a)^2}{12}$

Mean: $\mu_X = \int_{-\infty}^{\infty} x\, p_X(x) \mathrm{d}x = \int_a^b \frac{x}{b-a} \mathrm{d}x = \frac{a+b}{2} =: \mu_X$

Normal distribution: $X \sim \mathcal{N}(\mu, \sigma^2) \quad \hat{\theta}_{LS} \sim \mathcal{N}(\theta_0, \Sigma_{\hat{\theta}})$

$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$



Multidimensional Normal Distribution:

$p(x) = \frac{1}{\sqrt{(2\pi)^n \cdot \det(\Sigma)}} \cdot \exp\left(-\frac{1}{2} \cdot (x-\mu)^T \cdot \Sigma^{-1} \cdot (x-\mu)\right)$

Weibull distribution: $F(x) = 1 - \exp\left(-(\lambda \cdot x)^k\right)$

Laplace distribution: $f(x|\mu, b) = \frac{1}{2b} \cdot \exp\left(-\frac{|x-\mu|}{b}\right)$

Useful statistic definitions

Covariance and Correlaton: $\sigma(X,Y) := \mathbb{E}(X - \mu_X)(Y - \mu_Y)$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) \cdot p_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y$

Covariance Matrix: $\Sigma_x = cov(X) = \mathbb{E}\{XX^T\} - \mu_x \mu_x^T$ is PSD

$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{yx} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \qquad \sigma_{xy} = \sigma yx = \rho_{xy} \cdot \sigma_x \cdot \sigma_y$ where ρ is correlation

x and y are i.i.d. $\Rightarrow \Sigma$ is diagonal

Multidimensional Random Variables:

$\mathbb{E}\{f(X)\} = \int_{\mathbb{R}^n} f(x) p_X(x) d^n x$

$\mathrm{cov}(X) = \mathbb{E}\{(X - \mu_X)(X - \mu_X)^T\}$

$\mathrm{cov}(X) = \mathbb{E}\{XX^T\} - \mu_X \mu_X^T$

$\mathrm{cov}(Y) = \Sigma_y = A \Sigma_x A^T \quad for \quad y = A \cdot x$

$\mathbb{E}\{AX\} = A \cdot \mathbb{E}\{X\}$

Rules for variance:

$\mathrm{var}(aX) = a^2 \cdot \mathrm{var}(X)$

$\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2 \cdot \mathrm{cov}(X, Y)$

Formula for variance: $\mathrm{var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Correlation:
uncorrelated if $\rho(X,Y) = 0$, $\rho(X,Y) := \frac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_y}$

Statistical estimators:

Biased- and Unbiasedness An estimator $\hat{\theta}_N$ is unbiased $\Leftrightarrow \mathbb{E}\{\hat{\theta}_N(y_N)\} = \theta_0$, where $\theta_0 \equiv$ "true" value of θ. Otherwise: biased.

Asymptotic Unbiasedness An estimator $\hat{\theta}_N$ is called asymptotically unbiased $\Leftrightarrow \lim_{n \to \infty} \mathbb{E}\{\hat{\theta}_N(y_N)\} = \theta_0$

Consistency An estimator $\hat{\theta}_N(y_N)$ is called consistent if, for any $\epsilon > 0$, the probability $P(\hat{\theta}_N(y_N) \in [\theta_0 - \epsilon, \theta_0 + \epsilon])$ tends to 1 for $N \to \infty$.

Unconstrainded Optimization

Theorem 1: (First Order Necessary Conditions)
If $x^* \in D$ is local minimizer of $f : D \to \mathbb{R}$ and $f \in C^1$ then $\nabla f(x^*) = 0$ Definition (Stationary Point) A point $\bar{x}$ with $\nabla f(\bar{x}) = 0$ is called a stationary point of f.

Theorem 2: (Second Order Necessary Conditions)
If $x^* \in D$ is local minimizer of $f : D \to R$ and $f \in C^2$ then $\nabla^2 f(x^*) \succeq 0$

Theorem 3: (Second Order Sufficient Conditions and Stability under Perturbations)
Assume that $f : D \to R$ is C^2. If $x^* \in D$ is a stationary point and $\nabla^2 f(x^*) \succ 0$ then x^* is a strict local minimizer of f. In addition, this minimizer is locally unique and is stable against small perturbations of f, i.e. there exists a constant C such that for sufficiently small $p \in \mathbb{R}^n$ holds

$\|x^* - \underset{x}{\mathrm{argmin}}(f(x) + p^T x)\| \leq C\|p\|$

Linear Least Squares Estimation

Preliminaries: i.i.d. and Gaussian noise

Overall Model: $y(k) = \phi(k)^T \theta + \varepsilon(k)$

LS cost function as sum: $\sum_{k=1}^{N} (y(k) - \phi(k)^T \theta)^2$

LS cost function: $f(\theta) = \|y_N - \Phi_N \theta\|_2^2$

Unique minimizers: $\hat{\theta}_{LS} = \underset{\theta \in \mathbb{R}}{\mathrm{argmin}}\, f(\theta) \theta^* = \underbrace{(\Phi^T \Phi)^{-1} \Phi^T}_{:= \Phi^+} y$

Pseudo Inverse: $\Phi^+ = (\Phi^T \Phi)^{-1} \Phi^T$

Weighted Least Squares (unitless)

For i.i.d noise: Unweight Least Squares is optimal: $W = I$

$f_{WLS}(\theta) = \sum_{k=1}^{N} \frac{(y(k) - \phi(k)^T \theta)^2}{\sigma_\epsilon^2(k)} = \|y_N - \Phi_N \theta\|_W^2$

$= \|W^{\frac{1}{2}} y - W^{\frac{1}{2}} \Phi_N \theta\|_2^2 = (y_N - \Phi \cdot \theta)^T \cdot W \cdot (y_N - \Phi \cdot \theta)$

Solution for WLS:

$\hat{\theta}_{WLS} = \tilde{\Phi}^+ \tilde{y} \qquad$ with $\tilde{\Phi} = W^{\frac{1}{2}} \Phi$ and $\tilde{y} = W^{\frac{1}{2}} y$

$= \underset{\theta \in \mathbb{R}}{\mathrm{argmin}}\, f_{\mathrm{WLS}}(\theta) = (\Phi^T W \Phi)^{-1} \Phi^T W y$

Ill-Posed Least Squares

Singular Value Decomposition: $A = USV^T \in \mathbb{R}^{mxn}$
with $U \in \mathbb{R}^{mxm}$, $V \in \mathbb{R}^{nxn}$ and $S \in \mathbb{R}^{mxn}$ where S is a diagonal Matrix with non-negative elements $(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$

Moore Penrose Pseudo Inverse:

$\Phi^+ = VS^+U^T = V(S^T S + \alpha I)^{-1} S^T U^T$

Φ^+ therefore selects $\theta^* \in S^*$ with minimal norm

Regularization for Least Squares:

$\lim_{a \to 0} (\Phi^T \Phi + \alpha I)^{-1} \Phi^T = \Phi^+ \quad$ with Φ^+ $MPPI$

$\theta^* = (\Phi^T \Phi + \alpha \mathbb{I})^{-1} \Phi^T y$

Statistical Analysis of WLS

Expectation of Least Squares Estimator:

$$E\{\hat\theta_{WLS}\} = E\{(\Phi_N^T W \Phi_N)^{-1}\Phi_N^T W y_N\} = \theta_0$$

Covariance of the least squares estimator:

$$\mathrm{cov}(\hat\theta_{\mathrm{WLS}}) = (\Phi^\top W\Phi)^{-1}\Phi^\top W\Sigma_\varepsilon^{-1}W\Phi(\Phi^\top W\Phi)^{-1} = \underbrace{(\Phi_N^\top\Sigma_\varepsilon^{-1}\Phi_N)^{-1}}_{\text{for } W=\Sigma_\varepsilon}$$

$$\mathrm{cov}(\hat\theta_{WLS}) \succeq (\Phi_N^T W\Phi_N)^{-1}$$

Example LLS

Example of the Linear Least Square Estimator for: $N = 2$

$\varepsilon(1) \sim \mathcal{N}(0|\sigma_1^2)$ $\qquad\varepsilon(2) \sim \mathcal{N}(0|\sigma_2^2)$

$$N=2;\quad \Sigma_{\varepsilon_N} = \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}\qquad W^{OPT} = \Sigma_{\varepsilon_N}^{-1} = \begin{bmatrix}\frac{1}{\sigma_1^2} & 0\\ 0 & \frac{1}{\sigma_2^2}\end{bmatrix}$$

$$\mathrm{cov}(\hat\theta_{WLS}) = (Y_N - \Phi_N\theta)^T\cdot W\cdot(Y_N-\Phi_N\theta)$$
$$= \sum_{k=1}^2 (y(k)-\phi(k)^T\theta)\cdot\frac{1}{\sigma_k^2}\cdot(y(k)-\phi(k)^T\theta)$$

Measuring the goodness of Fit using: $R^2 \quad (0\le R^2\le 1)$

$$R^2 = 1-\frac{\|y_N-\Phi_N\hat\theta\|_2^2}{\|y_N\|_2^2} = 1-\frac{\|\epsilon_N\|_2^2}{\|y_N\|_2^2}$$
$$= \frac{\|y_N\|_2^2-\|\epsilon_N\|_2^2}{\|y_N\|_2^2} = \frac{\|\hat y_N\|_2^2}{\|y_N\|_2^2}$$

Residual: $\epsilon_N \uparrow \to \quad R^2 \to 0$ ($\Rightarrow$ *bad*)

Covariance estimation with a single experiment

Estimate $\hat\sigma_\varepsilon$ (Assumption: ε i.i.d.)

$$\hat\sigma_\varepsilon^2 = \frac{1}{N-d}\|y-\Phi_N\hat\theta\|_2^2,\quad \text{with } \hat\theta\in\mathbb{R}^d$$

$$\hat\Sigma_{\hat\theta} = \hat\sigma_\varepsilon(\Phi_N^\top\Phi_N)^{-1} = \frac{\|y-\Phi_N\hat\theta\|_2^2}{N-d}(\Phi_N^\top\Phi_N)^{-1}$$

Bayesian Estimation and the Maximum a Posteriori Estimate

Assumptions:

- Measurement: $\quad y_N \in \mathbb{R}^N$ has i.i.d. noise
- Linear Model: $\quad M(\theta) = \phi_N\cdot\theta$ and $\theta\in\mathbb{R}$

$$p(\theta|y_N) = \frac{p(y_N,\theta)}{p(y_N)} = \frac{p(y_N|\theta)\cdot p(\theta)}{p(y_N)}$$

$$\hat\theta_{MAP} = \underset{\theta\in\mathbb{R}}{\mathrm{argmin}}\{\underbrace{-\log(p(y_N|\theta))}_{\text{Max. Likelihood}}\ \underbrace{-\log(p(\theta))}_{\text{prev. knowledge}}\}$$

MAP Example: Regularised Least Squares

$\theta = \bar\theta \pm \sigma_\theta \quad$ with $\quad \bar\theta = \theta_{\text{a-priori}}$

$$\hat\theta_{\mathrm{MAP}} = \underset{\theta\in\mathbb{R}}{\mathrm{argmin}}\frac{1}{2}\cdot\frac{1}{\sigma_\varepsilon{}^2}\cdot\|y_N-\Phi_N\cdot\theta\|_2^2+\frac{1}{2}\cdot\frac{1}{\sigma_\theta^2}\cdot(\theta-\bar\theta)^2$$

Maximum Likelihood Estimation

L_2 **Estimation: Maximum Likelihood Estimation (ML):**

- Measurement Errors assumed to be Normally distributed
- Model described by a non-linear function $M(\theta)$
- Every unbiased estimator needs to satisfy the Cramer-Rao inequality, which gives a lower bound on the covariance matrix.

Model: $y = M(\theta)+\varepsilon$

$$p(y|\theta) = C\prod_{i=1}^N \exp\left(\frac{-(y_i-M_i(\theta))^2}{2\cdot\sigma_i^2}\right)\qquad C=\prod_{i=1}^N\frac{1}{\sqrt{2\cdot\pi\sigma_i^2}}$$

Positive log-Likelihood: (log changes products into sums)

$$\log p(y|\theta) = \log(C)+\sum_{i=1}^N -\frac{(y_i-M_i(\theta))^2}{2\cdot\sigma_i^2}$$

Negative log-Likelihood:

$$\hat\theta_{ML} = \arg\max_{\theta\in\mathbb{R}^d} p(y|\theta) = \arg\min_{\theta\in\mathbb{R}^d}\sum_{i=1}^N\frac{(y_i-M_i(\theta))^2}{2\sigma_i^2}$$
$$=\arg\min_{\theta\in\mathbb{R}^d}\frac{1}{2}\sum_{i=1}^N\left(\frac{y_i-M_i(\theta)}{\sigma_i}\right)^2$$
$$=\arg\min_{\theta\in\mathbb{R}^d}\frac{1}{2}\|S^{-1}(y-M(\theta))\|_2^2\qquad\text{with: } S=\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_N\end{bmatrix}$$

L_1 **Estimation:**

- Measurement Errors assumed to be Laplace distributed and more robust against outliers.

$$\min_\theta\|y-M(\theta)\|_1 = \min_\theta\sum_{i=1}^N|y_i-M_i(\theta)|$$
$$\Rightarrow\ \text{median of } \{Y_1,\cdots,Y_N\}$$

Recursive Linear Least Squares

$$Q_{N+1} = \alpha\cdot Q_N+\varphi(N+1)\cdot\varphi(N+1)^T,\ \ \alpha \mathrel{\hat=} \text{"forgettingfactor'''}$$

$$\hat\theta_{ML}(N+1) = \hat\theta_{ML}(N)+Q_{N+1}^{-1}\cdot\varphi(N+1)$$
$$\cdot[\ \underbrace{y(N+1)}_{\text{new measurement}}\ \underbrace{-\varphi(N+1)^T\cdot\hat\theta_{ML}(N)}_{\text{old prediction}}]$$

Q_0 and $\hat\theta_0$ have to be chosen.
Q_0 should be **non-singular, small and positive definite**. (e.g. $10^{-3}\cdot I$)
$Q_N \approx \Sigma^{-1}_{\hat\theta_{\mathrm{ML}}(N)}$

Cramer-Rao-Inequality (Fisher information Matrix M)

$$\mathrm{cov}(\hat\theta(y_N)) = \Sigma_{\hat\theta}\succeq M^{-1}\ \underbrace{=(\Phi_N^T\cdot\Sigma^{-1}\cdot\Phi_N)^{-1}}_{\text{for lin. model w. }\varepsilon\sim\mathcal{N}(\mu,\sigma)}$$

$M=\int_{y_n}\nabla_\theta^2 L(\theta_0,y_n)\cdot p(y_n|\theta_0)\mathrm{d}y_n$

Assumptions:

- Minimising a Linear Model
- Gaussian Noise: $X\sim\mathcal{N}(0,\Sigma)$)

$$L(\theta,y_N) = -\log(p(y_N|\theta))$$
$$(\text{if lin. model, etc.}) = \frac{1}{2}\cdot(\Phi_N\cdot\theta-y_N)^T\cdot\Sigma^{-1}\cdot(\Phi_N\cdot\theta-y_N)$$
$$M=\mathbb{E}\{\nabla_\theta^2 L(\theta,y_N)\} = \nabla_\theta^2 L(\theta,y_N) = (\Phi_N^T\cdot\Sigma^{-1}\cdot\Phi_N)$$

$\Rightarrow W=\Sigma^{-1}$ is the optimal weighting Matrix for WLS.

Continuous Time Systems

Ordinary Differential Equations (ODE):

$\dot x = f(x(t),u(t),\epsilon(t),p)$

Differential Algebraic Equations(DAE):

$\dot x = f(x(t),u(t),\epsilon(t),p)$

$0 = g(x,z).$

LTI Sytem (ODE):

$$\dot x = Ax+Bu\quad y=Cx+Du$$
$$G(s) = C(sI-A)^{-1}B+D$$

Zero order hold

Constant control u_{const} for time steps of length Δt

$$x(t;x_0u_{\text{const}}) = e^{A\Delta t}\cdot x_0+\int_0^{\Delta t}e^{A(\Delta t-\tau)}Bu_{\text{const}}\mathrm{d}\tau$$

Numerical Integration Methods

Euler Integration Step

$$\tilde x(t;x_0,u_{\text{const}}) = x_0+tf(x_0,u_{\text{const}}),\quad t\in[0,\Delta t]$$
$$\tilde x_{j+1} = \tilde x_j+hf(\tilde x_j,u_{\text{const}}),\quad j=0,\ldots,M-1$$

- Approximation becomes better by decreasing the step size h.
- Concistency Error: h^2
- Total Number of steps: $\Delta t/h$
- Error in the final step of order $h\Delta t$
- Linear in step size $\to$ order one
- Taking more steps is more accurate but needs more computation

Runge-Kutta Method of Order Four (RK4)

$$k_1 = f(\tilde x_j,u_{\text{const}})$$
$$k_2 = f(\tilde x_j,\frac{h}{2}k_1,u_{\text{const}})$$
$$k_3 = f(\tilde x_j,\frac{h}{2}k_2,u_{\text{const}})$$
$$k_4 = f(\tilde x_j,hk_3,u_{\text{const}})$$
$$\tilde x_{j+1} = \tilde x_j+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

One Step of RK4 is thus as expensive as four steps of euler
accurency of final approximation is of order $h^4\Delta$ t
$\to$ RK4 needs fewer functions to obtain the same accuracy level as euler

Discrete Time Systems

Det. Model as State Space	Stoch. Model as State Space
Det. Model as Input-Output	Stoch. Model as Input-Output

State Space Model

$x_{k+1} = f_k(x_k,u_k), k=0,1,\ldots,N-1$ with input vector u_k and state vector x_k

Input-Output Model

$y(k) = h(u(k),\ldots,u(k-n),y(k-1),\ldots,y(k-n))$

LTI system as State-Space Model:

$$x_{k+1} = Ax_k+Bu_k,\quad k=0,1,\ldots,N-1.$$

LTI system as Input-Output Model:

$$G(s) = \frac{b_0+b_1s+\ldots+b_ns^n}{a_0+a_1s+\ldots+a_{n-1}s^{n-1}+s^n}\quad|\cdot s=z^{-1}$$
$$G(z) = \frac{b_0+b_1z^{-1}+\ldots+b_nz^{-n}}{a_0+a_1z^{-1}+\ldots+a_nz^{-n}}$$
$$=\frac{b_0z^n+b_nz^{n-1}+\ldots+b_n}{a_0z^n+a_1z^{n-1}+\ldots+a_n}$$

$\Rightarrow$ Also called "polynomial model".

Deterministic Model

The output of the system can be obtained with absolute certainty. The Output y or the state x, depend on the known inputs $u(1), \dots, u(N)$, the previous Outputs $y(1), \dots, y(N)$ or state $x(n-1)$ and initial conditions. All deterministic models are **time invariant**.

State Space Model:

$$x(k+1) = f(x(k), u(k))$$
$$y(k) = g(x(k), u(k))$$

Initial conditions: $x(1) = x_0$

Input-Output Model

$$y(k) = h(u(k), \dots, u(k-n), y(k-1), \dots, y(k-n))$$

Initial conditions: $y(1) = y_1, \dots, y(n) = y_n \quad u(1) = u_1, \dots, u(n) = u_n$

Finite Impulse Response (FIR):

$$y(k) = b_0 u(k) + \dots + b_{n_b} u(k-n_b)$$

$$G(z) = b_0 + b_1 z^{-1} + \dots + b_{n_b} z^{-n_b} \quad | \cdot \frac{z^{n_b}}{z^{n_b}}$$
$$= \frac{b_0 z^{n_b} + b_1 z^{n_b - 1} + \dots + b_{n_b}}{z^{n_b}}$$

Auto Regressive model with eXogenous inputs (ARX/IRR):

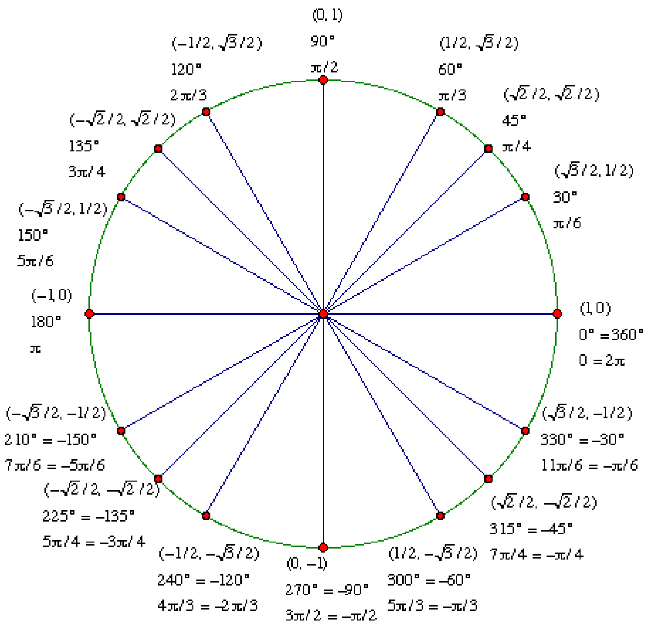
$$a_0 y(k) + \cdots + a_{n_a} y(k-n_a) = b_0 u(k) + \cdots + b_{n_b} u(k-n_b)$$

$$G(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}$$

The next output depends on the previous output. Also called **IIR** (infinite impulse response)

Auto Regressive model(AR):

$$y(k) = a_1 y(k-1) - \dots - a_{n_a} y(k-n_a)$$



Stochastic Model

Real systems are far from deterministic.

- there is stochastic noise $\varepsilon(k)$
- there are constant and unknown parameters p
- measured outputs $y(k)$ depend in both, $\varepsilon(k)$ and p

Assumptions: noise is **i.i.d** and enters the model like a normal input, but as a random variable

State Space Model

$$x(k+1) = f(x(k), u(k), \varepsilon(k))$$
$$y(k) = g(x(k), u(k), \varepsilon(k))$$

Input-Output Model

Only interested in input and output, not the whole model state

$$y(k) = h(u(k), \dots, u(k-n), y(k-1), \dots, y(k-n), \varepsilon(k), \dots, \varepsilon(k-n))$$
$$\text{for} \quad k = n+1, n+2, \dots$$

Measurement Noise (Output Error Model)

$$y(k) = M(k; U, x_0, p) + \varepsilon(k)$$

Stochastic Disturbance (Equation Errors)

$$y(k) = h(p, u(k), \dots, u(k-n), y(k-1), \dots, y(k-n)) + \varepsilon(k)$$
$$\text{for } k = n+1, n+2, \dots$$

Linear In the Parameters models (LIP):

$$y(k) = \sum_{i=1}^{d} \theta_i \phi_i (u(k), \dots, y(k-1), \dots) + \varepsilon(k)$$
$$y(k) = \varphi(k)^T \theta + \varepsilon(k) \quad \text{where } \varphi = (\phi_1(\cdot), \dots, \phi_d(\cdot))$$

LIP-LTI Models with Equation Errors (ARX)[a]

- Combining best of two worlds (LTI and LIP)

$$a_0 y(k) + \dots + a_{n_a} y(k-n_a) = b_0 u(k) + \dots + b_{n_b} u(k-n_b) + \varepsilon(k)$$

Auto-Regressive Moving Average with eXogeneous input (ARMAX):

$$a_0 y(k) + \dots + a_{n_a} y(k-n_a) = b_0 u(k) + \dots + b_{n_b} u(k-n_b) + \varepsilon(k) +$$
$$c_1 \varepsilon(k-1) + \dots + c_{n_x} \varepsilon(k-n_c)$$

Auto-Regressive Moving Average without inputs (ARMA):

$$a_0 y(k) + \dots + a_{n_a} y(k-n_a) = \varepsilon(k) + c_1 \varepsilon(k-1) + \dots + c_{n_x} \varepsilon(k-n_c)$$

Where c_i represent the noise coefficient, we have to use non-linear least squares with the unknown noise terms $\varepsilon(k-i)$

Difference between Deterministic and Stochastic Models

- stochastic noise $\varepsilon(k)$
- unknown but constant parameter p
- measured output $y(k)$ depend on both, $\varepsilon(k)$ and p

[a] additive noise is a special case

Example for State Space Model

$\ddot{a} = m \cdot \dot{a} + g \cdot a + c \cdot u$
$y = \dot{a}$

$$x = \begin{bmatrix} a \\ \dot{a} \end{bmatrix} \quad \dot{x} = \begin{bmatrix} \dot{a} \\ \ddot{a} \end{bmatrix} \quad \dot{x} = Ax + Bu \quad y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 \\ g & m \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Pure Output Error (OE) Minimization

Assume: i.i.d. gaussian noise only affecting output using non-linear least squares

$$\theta_{ML} = \min_{\theta} \sum_{k=1}^{N} (y(k) - M(k; U, x_0 p))^2$$

Output Error Minimization for FIR Models: lead to convex problems, therefore global minimum can be found

$$y(k) = (u(k), u(k-1), \dots, u(k-n_{n_b})) \cdot \theta + \varepsilon(k)$$
$$= \min_{\theta} \sum_{k=n_b+1}^{N} (y(k) - \underbrace{(u(k), u(k-1), \dots, u(k-n_{n_b})) \cdot \theta}_{\text{det. part is also } M(k; U, x_0, p)})^2$$

They often need a very high dimension n_b to obtain a reasonable fit. As a consequence ARX models are usually used instead.

Equation Error Minimization: Assume: i.i.d. $\varepsilon(k)$ noise enters the input-output equation as additive disturbance

$$y(k) = h(p, u(k), \dots, u(k-n), y(k-1), \dots, y(k-n)) + \varepsilon(k)$$
$$\text{for} \quad k = n+1, n+2$$

if the i.i.d noise is gaussian, a maximum likelihood formulation to estimate the unknown parameter vector $\theta = p$ is given:

$$\theta_{ML} = \min_{\theta} \sum_{k=n+1}^{N} (y(k) - h(p, u(k), \dots, y(k-1), \dots)))^2$$

u and k are known input and output measurements, and the algorithm minimises the so called **equation errors** or **prediction errors**.
This problem is also known as **Prediction error minimisation(PEM)**
Such a problem is convex if p enters linearly in f, i.e. if the model is **linear-in-the-parameters (LIP)**

PEM of LIP Models

$$y(k) = \varphi(k)^T \theta + \varepsilon(k)$$
$$\text{where } \varphi = (\phi_1(\cdot), \dots, \phi_d(\cdot))^T \text{ are the regressor variables}$$

considering this last expression, the prediction error minimisation(PEM) problem can be formulated as:

$$\min_{\theta} \underbrace{\sum_{k=1}^{N} (y(k) - \varphi(k)^{\mathrm{T}} \theta)^2}_{= \| y_N - \Phi_N \theta \|_2^2}$$

Which can be solved using LLS $\theta^* = \Phi_N^+ y_N$

Special Case: PEM of LIP-LTI Models with Equation Errors(ARX) General ARX model equation

$$a_0 y(k) + \dots + a_{n_a} y(k-n_a) = b_0 u(k) + \dots + b_{n_b} u(k-n_b) + \varepsilon(k)$$

In order to have a determined estimation problem, a_0 has to be fixed, otherwise the number of optimal solutions would be infinitive. Therefore we sually fix $a_0 = 1$ and use $\theta = (a_1, \dots, a_{n_a}, b_0, \dots, b_{n_b})^{\mathrm{T}}$ as the parameter estimation vector. The regressor vector is given by:

$$\varphi = (-y(k-1), \dots, -y(k-n_a), u(k), \dots, u(k-n_b))^{\mathrm{T}}$$

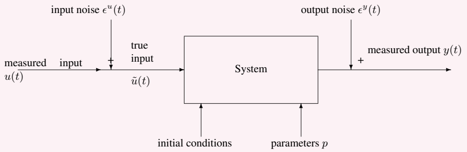
leading to the optimal solution provided by LLS:

$$y(k) = \varphi(k)^{\mathrm{T}} \theta + \varepsilon(k)$$

Pure Output Error (OE) Minimization

Models with Input and Output Errors:

$$y(k) = M(k; U + \varepsilon_N^u, x_0, p) + \varepsilon^y(k)$$



Assume: i.i.d. gaussian noise on both input and output with variance σ_u^2 for the input and σ_y^2 for the output

$$\hat{\theta} = \arg\min_{\theta} \sum_{k-1}^{N} \frac{1}{\sigma_y^2}(y(k) - M(k; U + \varepsilon_N^u, x_0, p))^2 + \frac{1}{\sigma_u^2}(\varepsilon_u(k))^2$$

$$\hat{\theta} = \arg\min_{\theta} \sum_{k-1}^{N} \frac{1}{\sigma_y^2}(y(k) - M(k; \tilde{U}, x_0, p))^2 + \frac{1}{\sigma_u^2}(u(k) - \tilde{u}(k))^2$$

$$\tilde{U} := U + \varepsilon_N^u$$

Fourier Transformation

FT:

$$F\{F\}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\mathrm{d}t$$

$$G(j\omega) = \frac{Y_0}{U_0}e^{j\omega t}$$

iFT:

$$f(t) = F^{-1}\{F\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega$$

DFT:

$$U(m) := \sum_{k=0}^{N-1} u(t)e^{-j\frac{2\pi mk}{N}}$$

iDFT:

$$u(n) := \frac{1}{N-1}\sum_{k=0}^{N-1} U(k)e^{j\frac{2\pi kn}{N}}$$

Useful frequency things

$\omega = 2\pi f = \frac{2\pi}{T}, \quad f_s > 2f_{\max}, \quad T = N\Delta t = \frac{N}{f_s} = (f_{\mathrm{res}})^{-1}$

$\sin(\varphi) = \frac{e^{j\varphi} - e^{-j\varphi}}{2}, \ \cos(\varphi) = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$

Aliasing and Leakage Errors

Aliasing Error: Due to sampling of continous signal to discrete signal. Avoid with Nyquist Theoreme:

$$f_{\mathrm{Nyquist}} = \frac{1}{2\Delta t}[\mathrm{Hz}] \quad or \quad \omega_{\mathrm{Nyquist}} = \frac{2\pi}{2\Delta t}[\mathrm{rad/s}]$$

Leakage Error: Due to windowing.

$$\omega_{\mathrm{base}} := \frac{2\pi}{N\cdot\Delta t} = \frac{2\pi}{T} \to \omega = m\cdot\frac{2\pi}{N\cdot\Delta t}$$

Crest Factor (ger. = Scheitelfaktor)

$$\text{Crest Factor} = \frac{u_{\max}}{u_{\mathrm{rms}}} \quad \text{with} : u_{\mathrm{rms}} := \sqrt{\frac{1}{T}\int_0^T u(t)^2\,\mathrm{d}t}\ \underbrace{\left(= \sqrt{\frac{u_1^2 + u_2^2}{2}}\right)}_{\text{symm. square wave}}$$

$$\text{and} \quad u_{\max} := \max_{t\in[0,T]} |u(t)|$$

Optimising Multisine for optimal crest factor

Frequency: Choose frequencies in logarithmic manner as multiples of the base frequency. $\omega_{k+1}/\omega_k \approx 1.05/1.01/1.03$ (round to $n\cdot\omega_{\mathrm{base}}$!)

Phase: To prevent high peaks (Crest Factor): random algorithm for phase shifts

Multisine Identification Implementation procedure

Window Length: Integer multiple of sampling time: $T = N\cdot\Delta t$

Harmonics of base frequency: Are contained in multisine $\omega_{base} = \frac{2\pi}{T}$

Highest contained Frequency: Is **half** of Nyquist frequency: $\omega_{\mathrm{Nyquist}} = \frac{2\pi}{4\Delta T}$

Experiment and Analysis: :

1. Insert Multisine periodically
2. Drop first Periods (till transients died out)
3. Record M Periods, with N samples, of input and output data
4. Average all windows and apply DFT (or vice versa)
5. Build transfer function: $\hat{G}_{j\omega_k} = \frac{\hat{Y}(k(p))}{\hat{U}(k(p))}$

Nonparametric and Frequency Domain Identification Models

Impulse response and transfer function:

$$y(t) = \int_0^{\infty} g(\tau)u(t-\tau)\,\mathrm{d}\tau$$

$$Y(s) = G(s)\cdot U(s)$$

$$G(s) = \int_0^{\infty} e^{-st}g(t)\,\mathrm{d}t$$

Bode diagram from frequency sweeps:

$$u(t) = A\cdot\sin(\omega\cdot t), \quad y(t) = \|G(j\cdot\omega)\|A\cdot\sin(\omega\cdot t + \alpha)$$

Bode Diagramm

Magnitude $\widehat{=}$ Amplitude $|G(j\omega)| = \sqrt{\mathrm{Re}(G)^2 + \mathrm{Im}(G)^2}$

Phase $\widehat{=}$ arg $G(j\omega) = \arctan\left({}^{\mathrm{Im}(G)}/{}_{\mathrm{Re}(G)}\right)$

When to use what in frequency response

- In general: Multisines are a good approach
- Very fast system and transients $\Rightarrow$ Frequency sweep
- **Very** slow system $\Rightarrow$ Step response
- In the middle $\Rightarrow$ Multisines

Kalman Filter

Valid for Discrete and Linear!

If recursive least squares: $x_{k+1} = A_k\cdot x_k$

$x_{k+1} = A_k\cdot x_k + \omega_k \quad \text{and} \quad y_k = C_k\cdot x_k + v_k$

Steps of Kalman Filter

1 Prediction

$$\hat{x}_{[k|k-1]} = A_{k-1}\cdot\hat{x}_{[k-1|k-1]}$$

$$P_{[k|k-1]} = A_{k-1}\cdot P_{[k-1|k-1]}\cdot A_{k-1}^T\cdot W_{k-1}$$

If RLS $\Rightarrow$ no W_{k-1}

2 Innovation update

$$P_{[k|k]} = (P_{[k|k-1]}^{-1} + C_k^T\cdot V^{-1}\cdot C_k)^{-1}$$

$$\hat{x}_{[k|k]} = \hat{x}_{[k|k-1]} + P_{[k|k]}\cdot C_k^T\cdot V^{-1}\cdot(y_k - C_k\cdot\hat{x}_{[k|k-1]})$$

Useful hints and practices

- Time invariant: t is only an argument of $u(*)$ or $y(*)$ i.e. **not**: $\dot{y}(t) = u(t) + \cos(t), \ \dot{y}(t) = t\cdot u(t) + y(t)$
- Linear: highest exponent of u and y is 1, i.e. **not**: $\dot{y}(t) = u(t)^2$
- Affine: linear and has additive term **independent** from u and y, i.e.: $\dot{y}(t) = y(t) + \cos(t), \ \dot{y}(t) = y(t) + C$
- Sample Mean: $\hat{\theta}_N(Y_N) = \frac{1}{N}\sum_{k=1}^{N} Y(k)$
- Sample Variance: $S^2 = \frac{1}{N-1}\sum_{k=1}^{N}(Y(k) - \mathbb{E}\{Y_N\})^2$

Magic Matrix for 8 point DFT:

$$W_r = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 1.0 & 0.7 & 0.0 & -0.7 & -1.0 & -0.7 & 0.0 & 0.7 \\ 1.0 & 0.0 & -1.0 & 0.0 & 1.0 & 0.0 & -1.0 & 0.0 \\ 1.0 & -0.7 & 0.0 & 0.7 & -1.0 & 0.7 & 0.0 & -0.7 \\ 1.0 & -1.0 & 1.0 & -1.0 & 1.0 & -1.0 & 1.0 & -1.0 \\ 1.0 & -0.7 & 0.0 & 0.7 & -1.0 & 0.7 & 0.0 & -0.7 \\ 1.0 & 0.0 & -1.0 & 0.0 & 1.0 & 0.0 & -1.0 & 0.0 \\ 1.0 & 0.7 & 0.0 & -0.7 & -1.0 & -0.7 & 0.0 & 0.7 \end{bmatrix}$$