Question 4.- Prove that every odd natural number is of one of the forms 4n+ 1 or 4n+ 3, where n is an integer.

Let's previously recall what the properties of being odd and even mean:

A.- $(\forall p \in \mathbb{Z})(\exists m \in \mathbb{Z})$ s.t. Odd(p) => p = 2m+1
Odd(x) meaning that x has the property of being odd
B.- $(\forall p \in \mathbb{Z})(\exists m \in \mathbb{Z})$ s.t. Even(p) => p = 2m
Even(x) meaning that x has the property of being even

With that in mind, the statement can be written as follows:

$$(\forall p \in \mathbb{N})(\exists n \in \mathbb{Z})(Odd(p) \Rightarrow (p=4n+1) \vee (p=4n+3))$$

On that basis, we can proceed with the proof.

Proof: We will compare both possibilities with the form of odd numbers seen (2m+1)

1.- We equate the first form

2m + 1 = 4n+1 => 2m = 4n => m = 2n   Thus m is even

Provided n and m are integers, taking the above B, 4n+1 is the form that an odd natural number can take if, when expressed in the form 2m+1, m is an even integer.

2.- We equate the second form

2m + 1 = 4n+3 => 2m = 4n+2 => m = 2n+1   Thus m is odd

Provided n and m are integers, taking the above A, 4n+1 is the form that an odd natural number can take if, when expressed in the form 2m+1, m is an odd integer.

3.- Since an integer is either even or odd and all odd natural numbers can be expressed as 2m+1 where m is either an odd o an even integer, we have proven that all odd naturals can be expressed as 4n+1 or 4n+3, where n is an integer. QED