Question 5.- Prove that for any integer n, at least one of the integers n, n+ 2, n+ 4 is divisible by 3.

As a preparation for the proof:

A.- Let's recall what does the property "to be divisible by 3" means:

$$(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})(3|n \Rightarrow n=3m)$$

B.- Let's find a simpler equivalent to the questions statement:

In terms of divisibility by 3, n+4 is equivalent to n+1

$$(\forall n \in \mathbb{Z})(\exists p \in \mathbb{Z})(n+4=n+1+3=p+3) \wedge (n+1=p)$$
$$(\forall m \in \mathbb{Z})(\exists q \in \mathbb{Z})(3m=3(m-1)+3=3q+3) \wedge (m-1=q)$$

n+4 = 3m => p+3 = 3q+3 => p=3q

Since p=n+1, n+1 is divisible by 3 and equivalent, in regard to that property, to n+4, we can rephrase the question statement as follows

$$(n \in \mathbb{Z})((3|n) \vee (3|n+1) \vee (3|n+2))$$

And we can work our proof with consecutive numbers

Proof: By contradiction, let's assume the statement is false so, for any integer, neither n, nor n+1 or n+2 are divisible by 3

1.- Let's check out if n+3 is divisible by 3. Since 3 is divisible by 3, for n+3 to be divisible by 3, n would have to be divisible by 3. So n+3 is not divisible by 3.
2.- Let's check out if n+4 is divisible by 3. n+4=n+1+3. Since 3 is divisible by 3, for n+4 to be divisible by 3, n+1 would have to be divisible by 3. So n+4 is not divisible by 3.
3.- Let's check out if n+5 is divisible by 3. n+5=n+2+3. Since 3 is divisible by 3, for n+5 to be divisible by 3, n+2 would have to be divisible by 3. So n+5 is not divisible by 3.
4.- Let's check out if n+6 is divisible by 3. n+6=n+3+3. Since 3 is divisible by 3, for n+6 to be divisible by 3, n would have to be divisible by 3. So n+6 is not divisible by 3.
5.- At this point, we have found that any integer can be shown in the form n + 3x or n + 1 + 3x or n + 2 + 3x, denoting 3x a multiple of 3. Since 3x will, by definition, be divisible by 3, we have found a contradiction: if the statement is false, there is no integer divisible by 3, which we know is false. QED