Question 9.- Given an infinite collection A_n, n=1,2,… of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals A_n, n=1,2,…, such that $A_{n+1} \subset A_n$ for all *n* and

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Prove that your example has the stated property.

Part 1.- Give an example of a family of intervals A_n, n=1,2,…, such that $A_{n+1} \subset A_n$ for all *n*

I propose the following family of intervals: $A_n = (\ 1, 1+\frac{1}{n} \ ]$

The family of intervals open on the left side and closed on the right side, from 1 to $1+\frac{1}{n}$

The property $A_{n+1} \subset A_n$ is fulfilled because as *n* gets arbitrarily bigger, $\frac{1}{n}$ gets smaller. So

$$(\forall n \in \mathbb{N})\left(\frac{1}{(n+1)} < \frac{1}{n}\right) \Rightarrow A_n = A_{n+1} \cup \left\{\frac{1}{(n+1)}, \frac{1}{n}\right\} \Rightarrow A_{n+1} \subset A_n$$

Part 2.- Prove that the intersection of the family of interval has no elements

1.- All the intervals are open on the left side, thus 1 is not included by the left side.

2.- We have already proven in class that for the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$

Thus for $\left\{1+\frac{1}{n}\right\}_{n=1}^{\infty}$. $1+\frac{1}{n} \rightarrow 1$ as $n \rightarrow \infty$

3.- So the right side of the intervals get smaller and closer and closer to 1, without reaching it, as *n* grows.

4.- Given that 1 is not in the family of intervals by the left side and it gets smaller and closer to 1 without reaching it, the intersection of the family has no elements. QED